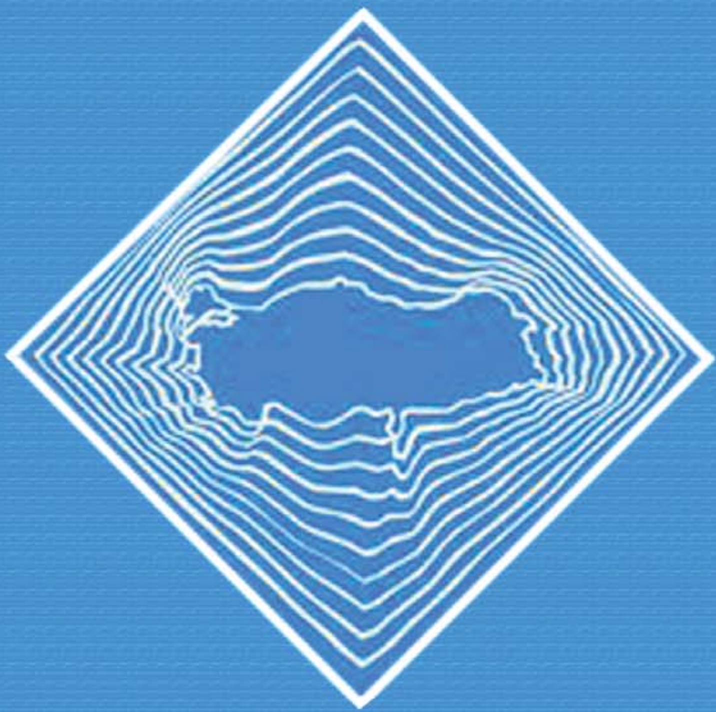
VOLUME/CİLT 11 NUMBER/SAYI 3 DECEMBER/ARALIK 2018

İSTATİSTİK

JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION

TÜRK İSTATİSTİK DERNEĞİ DERGİSİ

ISSN: 1300-4077



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ISSN: 1300-4077
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ABSTRACTED/INDEXED: İSTATİSTİK, Journal of the Turkish Statistical Association is indexed in indexed in ULAKBİM TR Dizin Database, MathSciNet, Zentralblatt MATH and EBSCO.

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İSTATİSTİK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION İSTATİSTİK
Vol. 11, No. 3, December 2018, pp. 46–52
ISSN 1300-4077 | 18 | 3 | 46 | 52

ON A COMBINATORIAL STRONG LAW OF LARGE NUMBERS

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Abstract: We derive strong laws of large numbers for combinatorial sums $\sum_i X_{ni\pi_n(i)}$, where $\|X_{nij}\|$ are $n \times n$ matrices of random variables with finite fourth moments and $(\pi_n(1), \dots, \pi_n(n))$ are uniformly distributed random permutations of $1, \dots, n$ independent with X's. We do not assume the independence of X's, but this case is included as well. Examples are discussed.

Key words: Combinatorial central limit theorem; combinatorial sums; strong law of large numbers
History: Submitted: 30 August 2018; Revised: 9 October 2018; Accepted: 15 November 2018

1. Introduction

Let $\{\|X_{nij}\|_{i,j=1}^n\}_{n=2}^\infty$ be a sequence of matrices of random variables and $\{\pi_n\}_{n=2}^\infty$ be a sequence of random permutations of $1, 2, \dots, n$. Put

$$S_n = \sum_{i=1}^n X_{ni\pi_n(i)}$$

for all $n \geqslant 2$, where $\pi_n = (\pi_n(1), \pi_n(2), \dots, \pi_n(n))$. Sums S_n are called the combinatorial sums.

If distributions of centered and normalized combinatorial sums converge weakly to the normal law, then one says that a combinatorial central limit theorem (CLT) holds true. If centered and normalized combinatorial sums converge almost surely (a.s.) to a constant, then one says that a combinatorial strong law of large numbers (SLLN) holds. Replacing strong convergence by convergence in probability, one arrives at a combinatorial weak law of large numbers (WLLN).

One cannot construct an interesting theory without additional assumptions on type of dependence of X's and π_n and their distributions. We follow a general line in which X's and π_n are independent and π_n has the uniform distribution.

Assume that for every n, components of $\|X_{nij}\|$ are independent, matrix $\|X_{nij}\|_{i,j=1}^n$ and permutation π_n are independent and π_n has the uniform distribution on the set of permutations of $1, 2, \dots, n$. Moreover, we also assume that $EX_{nij} = c_{nij}$ and

$$\sum_{j=1}^n c_{nij} = 0, \quad \sum_{i=1}^n c_{nij} = 0,$$

for all $1 \leqslant i, j \leqslant n$ and n. In results for combinatorial sums, the last condition provides that combinatorial sums S_n are centered at zero. Indeed,

$$EX_{ni\pi_n(i)} = \frac{1}{n} \sum_{j=1}^n c_{nij} = 0.$$

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* This investigation was partially supported by RFBR, research project N. 18–01–00393

If $\sigma^2_{nij} = DX_{nij} = EX^2_{nij} - (EX_{nij})^2$ for all $1 \leqslant i, j \leqslant n$ and $n \geqslant 2$, then we have

$$B_n = DS_n = \frac{1}{n-1}\sum_{i,j=1}^{n} c^2_{nij} + \frac{1}{n}\sum_{i,j=1}^{n} \sigma^2_{nij},$$

for all n. Hence, the norming sequence in combinatorial CLT is $\sqrt{B_n}$.

One can easy derive sufficient conditions for the combinatorial CLT from Esseen inequalities which give bounds for the accuracy of the normal approximation of distributions of $S_n/\sqrt{B_n}$. One can find such inequalities in von Bahr [1], Ho and Chen [2], Botlthausen [3], Goldstein [4], Neammanee and Suntornchost [5], Neammanee and Rattanawong [6], Chen and Fang [7] for X's with finite third moments. Earlier asymptotic results on combinatorial CLT may be found in references therein. Frolov [8,9] derived generalizations of Esseen bounds for combinatorial sums to the cases of finite moments of order $2+\delta$, $\delta \in (0,1]$ and infinite variations. Moderate deviations for combinatorial sums have been investigated in Frolov [10]. Esseen bounds for combinatorial random sums may be found in Frolov [11].

Together with CLT and large deviations, SLLN plays an important role in probability and statistics. In this paper, we derive the combinatorial SLLN. Note that properties of combinatorial sums S_n for independent X's are quite different from those of sums of independent random variables. First, combinatorial sums are sums of dependent random variables. Second, many summands of S_n and S_{n+1} can be different even when $\|X_{nij}\|$ is a sub-matrix of $\|X_{n+1,i,j}\|$ with $X_{nij} = X_{n+1,i,j}$ for all $1 \leqslant i, j \leqslant n$. This is the result of randomness of permutations π_n. It follows that we have no monotonicity of combinatorial sums for positive X's. Remember that monotonicity of sums of positive i.i.d. random variables are essentially used in the proof of the Kolmogorov SLLN. We also have no analogues of results on convergence of series of independent random variables. Moreover, we will not assume the independence of X's. This reduces our possibilities to prove strong limit theorems for combinatorial sums. Therefore, we obtain bounds for forth moments of combinatorial sums and apply the Borel–Cantelli lemma.

2. Combinatorial SLLN

Let $\{\|X_{nij}\|^n_{i,j=1}\}^\infty_{n=2}$ be a sequence of matrices of random variables with $EX_{nij} = c_{nij}$ for all $1 \leqslant i, j \leqslant n$ and $n \geqslant 2$ and $\{\pi_n\}^\infty_{n=2}$ be a sequence of random permutations of $1, 2, \ldots, n$. Assume that for every $n \geqslant 2$, relation

$$c_{ni.} = \sum_{j=1}^{n} c_{nij} = 0, \quad c_{n.j} = \sum_{i=1}^{n} c_{nij} = 0, \quad \text{for all} \quad 1 \leqslant i, j \leqslant n, \tag{2.1}$$

holds, π_n has the uniform distribution on the set of permutations of $1, 2, \ldots, n$ and $\|X_{nij}\|$ and π_n are independent. For all $n \geqslant 2$, put

$$S_n = \sum_{i=1}^{n} X_{ni\pi_n(i)},$$

where $\pi_n = (\pi_n(1), \pi_n(2), \ldots, \pi_n(n))$.

Note that if condition (2.1) is not satisfied, then one can center X's as follows. Put

$$X'_{nij} = X_{nij} - \frac{1}{n}c_{ni.} - \frac{1}{n}c_{n.j} + \frac{1}{n^2}c_{n..}, \quad \text{where} \quad c_{n..} = \sum_{i,j=1}^{n} c_{nij}.$$

It is not difficult to check that condition (2.1) holds with EX'_{nij} instead of c_{nij}.

The next result is the combinatorial SLLN.

THEOREM 1. *Suppose that the above assumptions hold and $EX_{nij}^4 < \infty$ for all i, j and n. For every n, put $C_n = \max_{1\leqslant i,j\leqslant n} EX_{nij}^4$ and $M_n = \max_{1\leqslant i\leqslant 4}\{m_{ni}\}$, where*

$$m_{n1} = \max_{1\leqslant i\neq j\neq k\neq l\leqslant n}\{|\sum_{p\neq q\neq r\neq s}(EX_{nip}X_{njq}X_{nkr}X_{nls} - c_{nip}c_{njq}c_{nkr}c_{nls})|\},$$
$$m_{n2} = \max_{1\leqslant i\neq j\neq k\leqslant n}\{|\sum_{p\neq q\neq r}(EX_{nip}^2X_{njq}X_{nkr} - EX_{nip}^2c_{njq}c_{nkr})|\},$$
$$m_{n3} = \max_{1\leqslant i\neq j\leqslant n}\{|\sum_{p\neq q}(EX_{nip}^2X_{njq}^2 - EX_{nip}^2EX_{njq}^2)|\},$$
$$m_{n4} = \max_{1\leqslant i\neq j\leqslant n}\{|\sum_{p\neq q}(EX_{nip}^3X_{njq} - EX_{nip}^3c_{njq})|\}.$$

Let $\{b_n\}_{n=2}^{\infty}$ be a sequence of positive constants. Assume that the series $\sum_n (C_n n^2 + M_n)b_n^{-4}$ converges.

Then

$$\frac{S_n}{b_n} \to 0 \quad a.s. \tag{2.2}$$

PROOF. For all natural n and k, denote $(n)_k = n(n-1)\cdot\ldots\cdot(n-k+1)$.

Put $\xi_i = X_{ni\pi_n(i)}$ for $1\leqslant i\leqslant n$. We have

$$ES_n^4 = \sum_{i=1}^n E\xi_i^4 + 4\sum_{i\neq j}E\xi_i^3\xi_j + 3\sum_{i\neq j}E\xi_i^2\xi_j^2 + 6\sum_{i\neq j\neq k}E\xi_i^2\xi_j\xi_k + \sum_{i\neq j\neq k\neq l}E\xi_i\xi_j\xi_k\xi_l, \tag{2.3}$$

where $1\leqslant i,j,k,l\leqslant n$ in the last four sums. Since X's and π_n are independent and π_n is uniformly distributed, we get

$$E\xi_i\xi_j\xi_k\xi_l = \frac{1}{(n)_4}\sum_{p\neq q\neq r\neq s}EX_{nip}X_{njq}X_{nkr}X_{nls}.$$

Hence,

$$|E\xi_i\xi_j\xi_k\xi_l| \leqslant \frac{1}{(n)_4}M_n + \frac{1}{(n)_4}|T_0|, \quad \text{where} \quad T_0 = \sum_{p\neq q\neq r\neq s}c_{nip}c_{njq}c_{nkr}c_{nls}.$$

It is clear that

$$T_0 = \sum_{p=1}^n\sum_{q:q\neq p}\sum_{r:r\neq p,q}\sum_{s:s\neq p,q,r}c_{nip}c_{njq}c_{nkr}c_{nls}.$$

By condition (2.1), we have

$$\sum_{s:s\neq p,q,r}c_{nls} = -(c_{nlp}+c_{nlq}+c_{nlr}).$$

It follows that

$$\begin{aligned}T_0 &= -\sum_{p=1}^n\sum_{q:q\neq p}\sum_{r:r\neq p,q}c_{nip}c_{njq}c_{nkr}\left(c_{nlp}+c_{nlq}+c_{nlr}\right)\\ &= -\sum_{p=1}^n\sum_{q:q\neq p}c_{nip}c_{njq}\left(c_{nlp}+c_{nlq}\right)\sum_{r:r\neq p,q}c_{nkr} - \sum_{p=1}^n\sum_{q:q\neq p}c_{nip}c_{njq}\sum_{r:r\neq p,q}c_{nkr}c_{nlr} = -T_1 - T_2\end{aligned}$$

for all $i \neq j \neq k \neq l$. Using again condition (2.1), we have

$$T_1 = -\sum_{p=1}^{n} \sum_{q:q\neq p} c_{nip} c_{njq} \left(c_{nlp} + c_{nlq}\right)\left(c_{nkp} + c_{nkq}\right).$$

By the Lyapunov inequality, it follows that $|c_{nij}| \leqslant (E|X_{nij}|^4)^{1/4} \leqslant C_n^{1/4}$. It yields that

$$|T_1| \leqslant 4n^2 C_n.$$

Furthermore,

$$\begin{aligned} T_2 &= \sum_{p=1}^{n} \sum_{q:q\neq p} c_{nip} c_{njq} \left(\sum_{r=1}^{n} c_{nkr} c_{nlr} - (c_{nkp} c_{nlp} + c_{nkq} c_{nlq}) \right) \\ &= \left(\sum_{r=1}^{n} c_{nkr} c_{nlr} \right) \sum_{p=1}^{n} c_{nip} \sum_{q:q\neq p} c_{njq} - \sum_{p=1}^{n} \sum_{q:q\neq p} c_{nip} c_{njq} (c_{nkp} c_{nlp} + c_{nkq} c_{nlq}) \\ &= -\left(\sum_{r=1}^{n} c_{nkr} c_{nlr} \right) \sum_{p=1}^{n} c_{nip} c_{njp} - \sum_{p=1}^{n} \sum_{q:q\neq p} c_{nip} c_{njq} (c_{nkp} c_{nlp} + c_{nkq} c_{nlq}). \end{aligned}$$

In the last equality, we have applied condition (2.1). Using again inequalities $|c_{nij}| \leqslant C_n^{1/4}$, we get

$$|T_2| \leqslant 3n^2 C_n.$$

Therefore, for all $i \neq j \neq k \neq l$, inequalities

$$|E\xi_i \xi_j \xi_k \xi_l| \leqslant \frac{1}{(n)_4}(7n^2 C_n + M_n) \tag{2.4}$$

hold. For all $i \neq j \neq k$, we have

$$E\xi_i^2 \xi_j \xi_k = \frac{1}{(n)_3} \sum_{p\neq q\neq r} EX_{nip}^2 X_{njq} X_{nkr}$$

and

$$|E\xi_i^2 \xi_j \xi_k| \leqslant \frac{1}{(n)_3} M_n + \frac{1}{(n)_3}|T_3|, \quad \text{where} \quad T_3 = \frac{1}{(n)_3} \sum_{p\neq q\neq r} EX_{nip}^2 c_{njq} c_{nkr}.$$

We get by condition (2.1) that

$$T_3 = \frac{1}{(n)_3} \sum_{p=1}^{n} \sum_{q:q\neq p} EX_{nip}^2 c_{njq} \sum_{r:r\neq p,q} c_{nkr} = -\frac{1}{(n)_3} \sum_{p=1}^{n} \sum_{q:q\neq p} EX_{nip}^2 c_{njq} \left(c_{nkp} + c_{nkq}\right).$$

Since $EX_{nij}^2 \leqslant (EX_{nij}^4)^{1/2} \leqslant \sqrt{C_n}$ and $|c_{nij}| \leqslant C_n^{1/4}$ for all i and j, the latter implies that

$$|E\xi_i^2 \xi_j \xi_k| \leqslant \frac{1}{(n)_3}(2n^2 C_n + M_n) \tag{2.5}$$

for all $i \neq j \neq k$.

For all $i \neq j$, we get

$$E\xi_i^2 \xi_j^2 = \frac{1}{(n)_2} \sum_{p\neq q} EX_{nip}^2 X_{njq}^2, \quad E\xi_i^3 \xi_j = \frac{1}{(n)_2} \sum_{p\neq q} EX_{nip}^3 X_{njq},$$

and

$$E\xi_i^2\xi_j^2 \leqslant \frac{1}{(n)_2}M_n + \frac{1}{(n)_2}\sum_{p\neq q} EX_{nip}^2 EX_{njq}^2,$$
$$|E\xi_i^3\xi_j| \leqslant \frac{1}{(n)_2}M_n + \frac{1}{(n)_2}\sum_{p\neq q} EX_{nip}^3 EX_{njq}.$$

Applying inequalities $|c_{nij}| \leqslant C_n^{1/4}$, $EX_{nij}^2 \leqslant \sqrt{C_n}$ and $|EX_{nij}^3| \leqslant (EX_{nij}^4)^{4/3} \leqslant \sqrt{C_n}$ for all i and j, we have

$$E\xi_i^2\xi_j^2 \leqslant \frac{1}{(n)_2}(n^2C_n + M_n), \quad |E\xi_i^3\xi_j| \leqslant \frac{1}{(n)_2}(n^2C_n + M_n), \tag{2.6}$$

for all $i \neq j$.

Finally, for every i, we get

$$E\xi_i^4 = \frac{1}{n}\sum_{p=1}^{n} EX_{nip}^4 \leqslant C_n. \tag{2.7}$$

Substituting bounds (2.4)–(2.7) in equality (2.3), we have

$$ES_n^4 \leqslant nC_n + (4+3+12+7)n^2C_n + 4M_n \leqslant 27(n^2C_n + M_n).$$

It follows that

$$\sum_{n=1}^{\infty} P\left(|S_n| \geqslant \varepsilon b_n\right) \leqslant \sum_{n=1}^{\infty}\frac{ES_n^4}{\varepsilon^4 b_n^4} \leqslant 27\sum_{n=1}^{\infty}\frac{n^2C_n + M_n}{\varepsilon^4 b_n^4} < \infty$$

for all $\varepsilon > 0$. By the Borel–Cantelly lemma, we obtain

$$\frac{S_n}{b_n} \to 0 \quad \text{a.s.}$$

Note that $\sum_n DS_n b_n^{-2} < \infty$ is a sufficient condition for relation (2.2). For independent X's, $M_n = 0$ and, using formula for DS_n from Section 1, we see that DS_n has an order n in various partial cases. So, if $b_n = n$, the last series always diverges while the series from Theorem 1 can converge. For example, the latter holds for bounded (uniformly over n) random variables.

Remark 1. One can find further conditions sufficient for combinatorial SLLN by applications of bounds for ES_n^{2k} with $k \geqslant 3$ which may be derived in the same way as before.

Condition (2.1) is symmetric relatively to rows and columns of matrices of means. Substituting in (2.3) the formulae for the expectations, we can interchange sums over numbers of rows and columns. Further, we can apply the second equality in (2.1) instead of the first one. Hence, we arrive at the next remark.

Remark 2. In Theorem 1, one can interchange indices in maxima and sums in the definitions of $m_{n1}, \ldots, m_{n4}$.

Theorem 1 yields the following result.

Corollary 1. *If the conditions of Theorem 1 hold and series* $\sum_n (n^2C_n + M_n)n^{-4p}$ *converges for some* $p > 0$, *then*

$$\frac{S_n}{n^p} \to 0 \quad \textit{a.s.}$$

Note that if the series from Corollary 1 diverges, then its conclusion can fail. Indeed, let $\{\eta_n\}$ be a sequence of independent random variables such that $P(\eta_n = n^p) = P(\eta = -n^p) = 1/2$ for all n. Put $X_{nij} = \eta_i$ for all i, j and n. Then $C_n = E\eta_n^4 = n^{4p}$, $M_n = 0$ and

$$S_n = \eta_1 + \eta_2 + \cdots + \eta_n.$$

Assuming that $n^{-p}S_n \to 0$ a.s., we have

$$\frac{\eta_n}{n^p} = \frac{S_n}{n^p} - \frac{S_{n-1}}{(n-1)^p} \cdot \frac{(n-1)^p}{n^p} \to 0 \quad \text{a.s.}$$

that contradicts to relation $P(n^{-p}\eta_n = 1) = 1/2$ for all n.

It is clear that $M_n = 0$ provided every quadruple of different elements of matrices $\|X_{nij}\|$ is a set of independent random variables. Moreover, $M_n = 0$ when rows of $\|X_{nij}\|$ are independent while elements of one row may be dependent.

These conditions are much more less than mutual independence, but it is useful to have an example with positive M_n.

To this end, we consider matrices $\|X_{nij}\|$ with m-dependent rows, where m is a fixed natural number. The latter means that i-th and k-th rows are independent when $|i-k| > m$. (The case $m = 0$ correspond to independence.) At the same time, we do not assume that random variables of one rows are independent.

Note that there is a simple way to construct such matrix. Take matrix $\|X_{nij}\|$ of independent random variables and replace every even row by previous odd ones. Then rows will be 1-dependent. The construction for $m > 1$ follows the same pattern.

For simplicity, put $m = 1$ and assume that $C_n = C$ for all n. It is clear, that all items of sums in the definitions of m_{ni} are bounded by $2C$ and many of them equal to zero by independence of "far" rows. The number of zero items in m_{n1} is bounded from below by $n(n-2)(n-4)(n-6)$. Hence, the number of non-zero items is less than $(n)_4 - n(n-2)(n-4)(n-6) = O(n^3)$ as $n \to \infty$. It follows that $m_{n1} = O(n^3)$ as $n \to \infty$. Maxima m_{n2}, m_{n3} and m_{n4} have the same or smaller order. So, the series in Theorem 1 converges provided series $\sum_n n^3 b_n^{-4}$ converges. By Theorem 1, we have

$$\frac{S_n}{n(\ln n)^q} \to 0 \quad \text{a.s.}$$

for all $q > 1/4$.

Acknowledgement

The author thanks the anonymous referee for helpful remarks which improved the text of the paper.

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İSTATİSTİK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION İSTATİSTİK
Vol. 11, No. 3, December 2018, pp. 53–64
ISSN 1300-4077 | 18 | 3 | 53 | 64

WRAPPED FLEXIBLE SKEW LAPLACE DISTRIBUTION

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Abstract: We introduce a new circular distribution named as wrapped flexible skew Laplace distribution. This distribution is the generalization of wrapped Laplace which was introduced by Jammalamadaka and Kozubowski 2003 and has more flexibility properties in terms of skewness, kurtosis, unimodality or bimodality. We also derive expressions for characteristic function, trigonometric moments, coefficients of skewness and kurtosis. We analyzed two popular datasets from the literature to show the good modeling ability of the WFSL distribution.

Key words: Circular distribution; flexible skew laplace distribution; wrapped distribution; laplace distribution; skew-symmetric distribution
History: Submitted: 23 October 2018; Revised: 18 November 2018; Accepted: 5 December 2018

1. Introduction

Circular or directional data is encountered in various fields of science such as meteorology, astronomy, medicine, biology, geology, physics and sociology. The first studies on the modeling of directional data are very old. The book "Statistics for Circular Data" written by Mardia 1972 can be regarded as the first work in this area. Other important works on this subject can be listed as "Statistical Analysis of Circular Data" [4], "Directional statistics" [10], "Topics in Circular Statistics" [7]. In the following years, many authors have proposed models and statistical methods for the analysis of circular data.

The von Mises distribution, also known as the circular normal or the Tikhonov distribution, is one of the principal symmetric distributions on the circle. However, most of the classical models such as Von-Mises, cardioid and wrapped Cauchy are symmetric-unimodal distributions and rarely applied in practice, since circular data is very often asymmetric and multimodal. Therefore, several new unimodal/multimodal circular distributions are capable modeling symmetry as well as asymmetry has been proposed, for example asymmetric Laplace distribution [5], nonnegative trigonometric sums distribution [3], asymmetric version of the von Mises distribution [13] and stereographic extreme–value distribution [12].

In recent years, studies on obtaining circular models have generally focused on wrapping linear probability models on a circle. In the literature, there are many wrapped models obtained by various well-known linear distributions. Pewsey 2000 obtained the wrapped skew normal distribution by using the Azzalini's skew normal distribution 1985. Jammalamadaka and Kozlowski 2004 studied the circular distributions obtained by exponential and Laplace distributions. Rao et al 2007 derived new circular models by wrapping the lognormal, logistic, Weibull, and extreme-value distributions.

In a previous paper [14] we introduced the flexible skew Laplace (FSL) distribution. This distribution is a member of skew-symmetric distribution family, and that means it has a pdf form that $h(x) = 2f(x)F(g(x))$ where f and F are the pdf and cdf of Laplace distribution and

$$g(x) = \left(\lambda_1 x + \lambda_3 x^3\right)\left(1 + \lambda_2 x^2\right)^{-\frac{1}{2}}, \quad \lambda_1, \lambda_3 \in \mathbb{R}, \lambda_2 \geq 0. \tag{1.1}$$

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We showed that, this distribution has remarkable flexibility properties in data modelling via contained parameters such as unimodality-bimodality, skewness or kurtosis. In this paper, the wrapped version of the flexible skew Laplace distribution will be presented.

2. Definition

A well-known approach to obtain circular distributions is wrapping method. In this approach, a known distribution is taken on the real line and wrapped around a unit circle. Namely, taking a real random variable (say Y) and wrapping it around the circle by transformation $Y(mod\ 2\pi)$. The new random variable $Y(mod\ 2\pi)$ can be named as the corresponding wrapped version of Y and has a probability density function (pdf) form that

$$f_{Y(mod\ 2\pi)}(\theta) = \sum_{r=-\infty}^{\infty} f_Y(\theta + 2\pi r),$$

where f_Y is the pdf of random variable Y.

Let Y be a $FSL(\mu,\sigma,\lambda_1,\lambda_2,\lambda_3)$ random variable, i.e. has a pdf

$$f_Y(y;\underline{\upsilon}) = \frac{1}{2\sigma} e^{-\frac{|y-\mu|}{\sigma}} \left[1 + \operatorname{sgn}\left(\frac{\lambda_1(y-\mu) + \frac{\lambda_3}{\sigma^2}(y-\mu)^3}{\sqrt{\sigma^2 + \lambda_2 (y-\mu)^2}}\right)\left(1 - e^{-\left|\frac{\lambda_1(y-\mu)+\frac{\lambda_3}{\sigma^2}(y-\mu)^3}{\left(\sigma^2+\lambda_2(y-\mu)^2\right)^{0.5}}\right|}\right)\right],$$

where $\underline{\upsilon} = (\mu,\sigma,\lambda_1,\lambda_2,\lambda_3)$. Then the corresponding circular random variable is defined as

$$\Theta = Y(mod\ 2\pi),$$

and has the density

$$f_\Theta(\theta;\underline{\upsilon}) = \frac{1}{2\sigma}\left[e^{-\frac{|\theta-\mu|}{\sigma}} + \frac{e^{\frac{\theta-\mu}{\sigma}} + e^{\frac{\mu-\theta}{\sigma}}}{e^{\frac{2\pi}{\sigma}} - 1} + A(\theta,\underline{\upsilon})\right] \tag{2.1}$$

where

$$A(\theta,\underline{\upsilon}) = \sum_{r=-\infty}^{\infty} e^{-\frac{\left|\theta_\mu^r\right|}{\sigma}} \operatorname{sgn} g\left(\frac{\theta_\mu^r}{\sigma}\right)\left(1 - e^{-\left|g\left(\frac{\theta_\mu^r}{\sigma}\right)\right|}\right),$$

and $0 \le \theta < 2\pi$, $\theta_\mu^r = \theta + 2\pi r - \mu$. The parameters $\mu \in \mathbb{R}$ location, $\sigma > 0$ scale parameter and $\lambda_1, \lambda_3 \in \mathbb{R}$, $\lambda_2 \ge 0$ are shape parameters. The random variable Θ having wrapped flexible skew Laplace distribution is denoted by $\Theta \sim WFSL(\mu,\sigma,\lambda_1,\lambda_2,\lambda_3)$. Illustrations of the pdf of WFSL distribution for several values of parameters are shown in Figure 1.

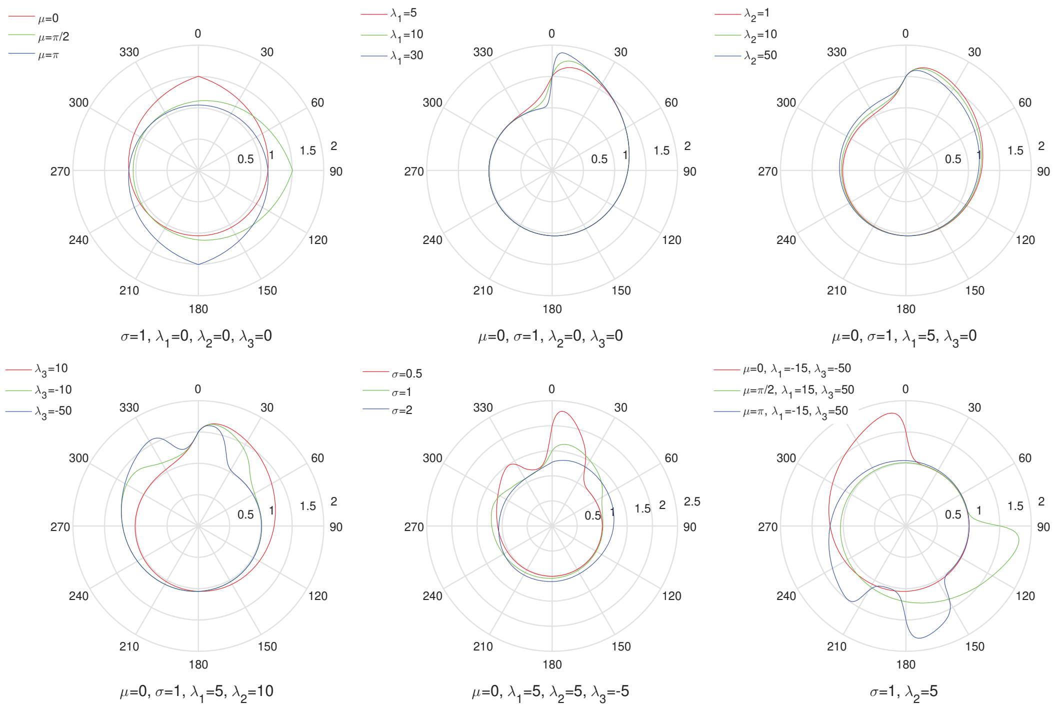

Figure 1. The pdf of WFSL distribution for several values of parameters.

The following sections of this article are organized as follows: In Section 3 we give the characteristic function of wrapped flexible skew Laplace distribution and some moments properties, i.e. location, dispersion, skewness and kurtosis. We also provide some results of limiting cases of parameters, and a simulation study in this section. In last section we will analyze two popular datasets from the literature.

3. Basic Properties

In this section, we obtain the equations for characteristic function, trigonometric moments, location, dispersion and coefficients of skewness and kurtosis. We also provide some properties and relations with other known distributions.

3.1. Trigonometric Moments

The characteristic function defines the entire probability distribution in the circular models as well as in the models defined on the real line. Note that, since the random variables with such distributions are periodic, have the same distribution when shifted by 2π. So if we consider $\Theta \stackrel{dist}{=} \Theta + 2\pi$, it must be

$$\varphi_\Theta(p) = E(e^{ip\Theta}) = E(e^{ip(\Theta+2\pi)}) = e^{ip2\pi}\varphi_\Theta(p).$$

Hence p must be an integer. The value of the characteristic function at an integer p is called the pth trigonometric moment of Θ. One can also write pth trigonometric moments in terms of α_p and β_p

$$\varphi_p = \varphi_\Theta(p) = \alpha_p + i\beta_p, \quad p = 0, \pm 1, \pm 2, \ldots.$$

where α_p is pth cosine moment and defined as $\alpha_p = E(\cos p\Theta)$, β_p is pth sine moment and defined as $\beta_p = E(\sin p\Theta)$. In order to obtain pth cosine and sine moments of $WFSL(\mu,\sigma,\lambda_1,\lambda_2,\lambda_3)$ distribution, we define two signum functions

$$\nabla = \begin{cases} \operatorname{sgn}(\lambda_1) & ,\text{if } \lambda_1\lambda_3 < 0 \\ 0 & ,\text{if } \lambda_1\lambda_3 \geq 0 \end{cases} \text{ and } \Delta = \begin{cases} 0 & ,\text{if } \lambda_1 = 0 \text{ and } \lambda_3 = 0 \\ \operatorname{sgn}(\lambda_3) & ,\text{if } \lambda_1 = 0 \text{ and } \lambda_3 \neq 0 \\ \operatorname{sgn}(\lambda_1) & ,\text{if } \lambda_1 \neq 0 \end{cases},$$

and quantities

$$C_p = \frac{1}{2\sigma}\int_0^{2\pi} \cos p\theta \sum_{r=-\infty}^{\infty} e^{-\left|\theta_\mu^r\sigma^{-1}\right|} \operatorname{sgn} g\left(\theta_\mu^r\sigma^{-1}\right) e^{-\left|g\left(\theta_\mu^r\sigma^{-1}\right)\right|} d\theta$$

and

$$S_p = \frac{1}{2\sigma}\int_0^{2\pi} \sin p\theta \sum_{r=-\infty}^{\infty} e^{-\left|\theta_\mu^r\sigma^{-1}\right|} \operatorname{sgn} g\left(\theta_\mu^r\sigma^{-1}\right) e^{-\left|g\left(\theta_\mu^r\sigma^{-1}\right)\right|} d\theta.$$

It follows that the pth cosine and sine moments are

$$\alpha_p = \frac{\cos p\mu + 2\nabla e^{-k}\sin(p\mu)\,\xi_p}{p^2\sigma^2+1} - \frac{\Delta p\sigma\sin p\mu}{p^2\sigma^2+1} - \Delta^2 C_p, \tag{3.1}$$

$$\beta_p = \frac{\sin p\mu - 2\nabla e^{-k}\cos(p\mu)\,\xi_p}{p^2\sigma^2+1} + \frac{\Delta p\sigma\cos p\mu}{p^2\sigma^2+1} - \Delta^2 S_p, \tag{3.2}$$

where

$$\xi_p = \sin kp\sigma + p\sigma\cos kp\sigma,$$

and

$$k = \begin{cases} \left(-\lambda_1\lambda_3^{-1}\right)^{0.5} & ,\text{if } \lambda_1\lambda_3 < 0 \\ 0 & ,\text{if } \lambda_1\lambda_3 \geq 0 \end{cases}.$$

Using these trigonometric values, an alternative representation for the density of Θ can be written as

$$f_\Theta(\theta;\underline{\upsilon}) = \frac{1}{2\pi} - \frac{1}{\pi}\sum_{p=1}^{\infty}\left\{ \begin{array}{c} \left(\frac{2\nabla e^{-k}\sin p(\theta-\mu)\xi_p}{p^2\sigma^2+1}\right) \\ -\frac{\cos p(\theta-\mu)+\Delta p\sigma\sin p(\theta-\mu)}{p^2\sigma^2+1} \\ +\Delta^2\left(C_p\cos p\theta + S_p\sin p\theta\right) \end{array} \right\}.$$

Thus, the first two trigonometric moments of $WFSL(0,1,\lambda_1,\lambda_2,\lambda_3)$ are

$$\varphi_1 = \frac{1+i\Delta}{2} - \nabla i e^{-k}\left[\cos k + \sin k\right] - i\Delta^2 S_1,$$

$$\varphi_2 = \frac{1+2i\Delta}{5} - \frac{2}{5}\nabla i e^{-k}\left[2\cos 2k + \sin 2k\right] - i\Delta^2 S_2.$$

Since the clear analytical form of both C_p and S_p cannot be found, they need to be evaluated numerically. However, the following two lemmas provide the values of C_p and S_p in some special cases of parameters.

LEMMA 1. *When $\mu = 0$ for each integer p,*
(a) $C_p = 0$.
(b) $S_p = 0$ when $\Delta = 0$.

PROOF. (a) When $\Delta = 0$, it is immediate $g\left(\theta_0^r \sigma^{-1}\right) = 0$ and thus $C_p = 0$. When $\Delta \neq 0$, denote

$$\Lambda(\theta) = \sum_{r=-\infty}^{\infty} e^{-\left|\frac{\theta-\pi+2\pi r}{\sigma}\right|} \operatorname{sgn} g\left(\theta_0^r \sigma^{-1}\right) e^{-\left|g\left(\theta_0^r \sigma^{-1}\right)\right|}.$$

It's easy to see $\Lambda(\theta)$ is an odd function, i.e. $\Lambda(\theta) + \Lambda(-\theta) = 0$, $\theta \in (-\pi, \pi)$. One can rewrite

$$\begin{aligned} C_p &= \frac{1}{2\sigma} \int_0^{2\pi} \cos p\theta \sum_{r=-\infty}^{\infty} e^{-\left|\frac{\theta+2\pi r}{\sigma}\right|} \operatorname{sgn} g\left(\theta_0^r \sigma^{-1}\right) e^{-\left|g\left(\theta_0^r \sigma^{-1}\right)\right|} d\theta \\ &= \frac{(-1)^p}{2\sigma} \int_{-\pi}^{\pi} \Lambda(\theta) \cos p\theta d\theta = 0. \end{aligned}$$

(b) Proof is clear since $g\left(\theta_0^r \sigma^{-1}\right) = 0$ for $\Delta = 0$.

LEMMA 2. *For each integer p,*
(a) When $\lambda_1 \to \infty$ or $\lambda_3 \to \infty$, $C_p = 0$ and $S_p = 0$.
(b) When $\nabla = 0$ and $\lambda_2 \to \infty$,

$$C_p = -\frac{\Delta p \sigma \sin p\mu}{p^2\sigma^2 + 1} \text{ and } S_p = \frac{\Delta p \sigma \cos p\mu}{p^2\sigma^2 + 1}.$$

PROOF. (a) Let's just consider $\lambda_3 \to \infty$

$$\begin{aligned} \lim_{\lambda_3 \to \infty} C_p &= \lim_{\lambda_3 \to \infty} \frac{1}{2\sigma} \int_0^{2\pi} \cos p\theta \sum_{r=-\infty}^{\infty} e^{-\left|\theta_\mu^r \sigma^{-1}\right|} \operatorname{sgn} g\left(\theta_\mu^r \sigma^{-1}\right) e^{-\left|g\left(\theta_\mu^r \sigma^{-1}\right)\right|} d\theta \\ &= \frac{1}{2\sigma} \int_0^{2\pi} \cos p\theta \sum_{r=-\infty}^{\infty} \left[e^{-\left|\theta_\mu^r \sigma^{-1}\right|} \left\{ \lim_{\lambda_3 \to \infty} \operatorname{sgn} g\left(\theta_\mu^r \sigma^{-1}\right) e^{-\left|g\left(\theta_\mu^r \sigma^{-1}\right)\right|} \right\} \right] d\theta \\ &= 0. \end{aligned}$$

The situation is the same for $\lambda_1 \to \infty$ or $\lambda_1 \to \infty$, $\lambda_3 \to \infty$ and the proof is similar for S_p.

(b) Just consider C_p since the proof is similar for S_p. While $\lambda_2 \to \infty$, $e^{-\left|g\left(\theta_\mu^r \sigma^{-1}\right)\right|}$ tends to 1. Thus,

$$\begin{aligned} \lim_{\lambda_2 \to \infty} C_p &= \frac{1}{2\sigma} \int_0^{2\pi} \cos p\theta \lim_{\lambda_2 \to \infty} \sum_{r=-\infty}^{\infty} e^{-\left|\theta_\mu^r \sigma^{-1}\right|} \operatorname{sgn} g\left(\theta_\mu^r \sigma^{-1}\right) e^{-\left|g\left(\theta_\mu^r \sigma^{-1}\right)\right|} d\theta \\ &= \frac{1}{2\sigma} \left[\int_0^{2\pi} \cos p\theta \lim_{\lambda_2 \to \infty} \sum_{r=-\infty}^{\infty} e^{-\left|\theta_\mu^r \sigma^{-1}\right|} \operatorname{sgn} g\left(\theta_\mu^r \sigma^{-1}\right) \right] \\ &= -\left(\Delta p \sigma \sin p\mu\right) \left(p^2\sigma^2 + 1\right)^{-1}. \end{aligned}$$

3.2. Location and Dispersion

Resultant vector length and direction for pth trigonometric moment of a circular distribution are

$$\rho_p = \sqrt{\alpha_p^2 + \beta_p^2} \text{ and } \mu_p = \operatorname{atan}\left(\alpha_p \beta_p^{-1}\right) \tag{3.3}$$

respectively, where atan(.) is quadrant inverse tangent function and defined as

$$\operatorname{atan}(y/x)=\begin{cases}\tan^{-1}(x/y) & ,y>0,x\geq 0\\ \pi/2 & ,y=0,x>0\\ \tan^{-1}(x/y)+\pi & ,y<0\\ \tan^{-1}(x/y)+2\pi & ,y\geq 0,x<0\\ \text{undefined} & ,y=0,x=0\end{cases}.$$

The pth trigonometric moment can be expressed in $\varphi_p=\rho_p e^{i\mu_p}$ and has a special meaning for $p=1$. The values of ρ_1 and μ_1 obtained from (3.3) are called the angular concentration and the mean direction, respectively. Mean direction of $WFSL(0,\sigma,\lambda_1,\lambda_2,\lambda_3)$ distribution is

$$\begin{aligned}\mu_1&=\operatorname{atan}\left[\left(\sigma\Delta-\Delta^2 S_1\left(1+\sigma^2\right)-2\nabla e^{-k}\left[\sin k\sigma+\sigma\cos k\sigma\right]\right)^{-1}\right]\\&=\operatorname{atan}\left[\left(\sigma\Delta-\Delta^2 S_1\left(1+\sigma^2\right)-2\nabla e^{-k}\xi_1\right)^{-1}\right].\end{aligned}\tag{3.4}$$

The mean direction vector gives information about the mean of the distribution as an analogy of the mean in the linear models. The length of this vector is a measure of its dispersion around the mean and corresponds to the usual standard deviation or variance in linear models. Square of angular concentration for $WFSL$ distribution is

$$\begin{aligned}\rho_1^2=&\frac{\nabla^2}{\varsigma}\xi_1^2\left(4e^{-2k}\right)+\frac{\nabla}{\varsigma}\xi_1\left(4\Delta e^{-k}\right)\left(\Delta S_1-\sigma+\sigma^2\Delta S_1\right)\\&+\frac{1}{\varsigma}\left(\Delta S_1\left(\sigma^2+1\right)\left(\Delta S_1-2\sigma+\sigma^2\Delta S_1\right)+\sigma^2\Delta^2+1\right),\end{aligned}$$

or with value of μ_1

$$\begin{aligned}\rho_1&=-\Delta^2 S_1\sin\mu_1+\frac{\Delta\sigma\sin\mu_1}{\sigma^2+1}-\frac{2\nabla e^{-k}\xi_1\sin\mu_1}{\sigma^2+1}+\frac{\cos\mu_1}{\sigma^2+1}\\&=\left[\Delta\frac{\sigma}{\sigma^2+1}-\Delta^2 S_1-\nabla\frac{2e^{-k}\xi_1}{\sigma^2+1}\right]\sin\mu_1+\frac{\cos\mu_1}{\sigma^2+1},\end{aligned}\tag{3.5}$$

where $\varsigma=\sigma^4+2\sigma^2+1$.

Corollary 1. *When $\mu=0$ and $\Delta=0$, $\varphi_p=\left(p^2\sigma^2+1\right)^{-1}e^{ip\mu}$, for each integer p. Hence, $\mu_1=\mu$ and $\rho_1=\left(\sigma^2+1\right)^{-1}$ for $WFSL(\mu,\sigma,0,\lambda_2,0)$ distribution.*

Corollary 2. *When $\lambda_1\to\infty$ or $\lambda_3\to\infty$, μ_1 tends to $[\mu+\operatorname{atan}(\sigma)]\bmod(2\pi)$ and ρ_1 tends to $\left(\sigma^2+1\right)^{-0.5}$ for $WFSL(\mu,\sigma,\lambda_1,\lambda_2,\lambda_3)$ distribution.*

Corollary 3. *When $\lambda_2\to\infty$, μ_1 tends to μ and ρ_1 tends to $\left(\sigma^2+1\right)^{-1}$ for $WFSL(\mu,\sigma,\lambda_1,\lambda_2,\lambda_3)$ distribution.*

It is clear that $0\leq\rho_1\leq 1$ and tends to maximum value when the concentration increases around the mean. The effect of μ and σ parameters on the angular concentration and the mean direction of the Θ random variable are shown in Figure 2. If Θ is rotated by θ_0 degrees, the value of angular concentration does not change but mean direction is shifted by θ_0.

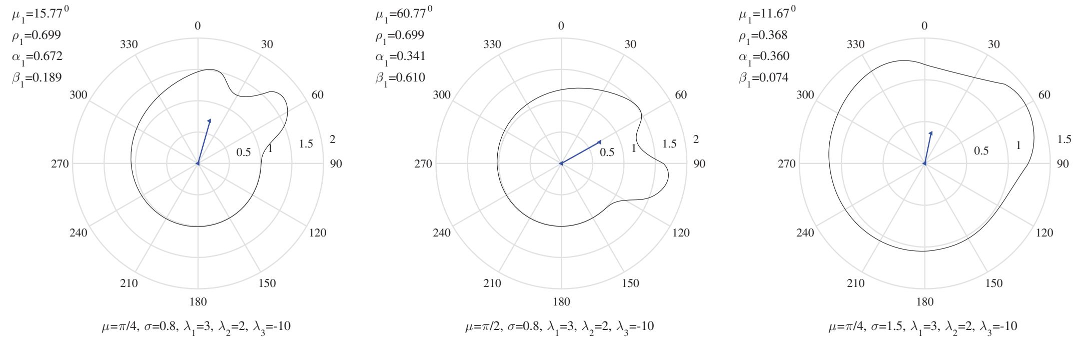

Figure 2. Amount of concentration and average direction change.

Another circular dispersion measure is the circular variance and defined as $V = 1-\rho_1$. Using (3.4) and (3.5), the circular variance of $WFSL(0,\sigma,\lambda_1,\lambda_2,\lambda_3)$ is

$$\begin{aligned} V &= 1-\beta_1 \sin\mu_1 - \alpha_1 \cos\mu_1 \\ &= 1 + S_1 \sin\mu_1 \Delta^2 - \frac{\Delta\sigma \sin\mu_1}{\sigma^2+1} + \frac{2\nabla e^{-k}\xi_1 \sin\mu_1}{\sigma^2+1} - \frac{\cos\mu_1}{\sigma^2+1}. \end{aligned} \tag{3.6}$$

Circular variance is interpreted as the opposite of angular concentration. That is, the circular variance decreases while the concentration around the mean direction increases and vice versa. In Figure 3, it can be seen (σ, V) plots for different $\lambda_1, \lambda_2, \lambda_3$ values $(\mu = 0)$. In generally, according to Figure 3 it can be said that the circular variance increases with the increase of σ. When $\lambda_1\lambda_3 \geq 0$, the circular variance decreases with increasing λ_1 or λ_3 and increases with λ_2. But this is only valid for some values of σ when $\lambda_1\lambda_3 < 0$.

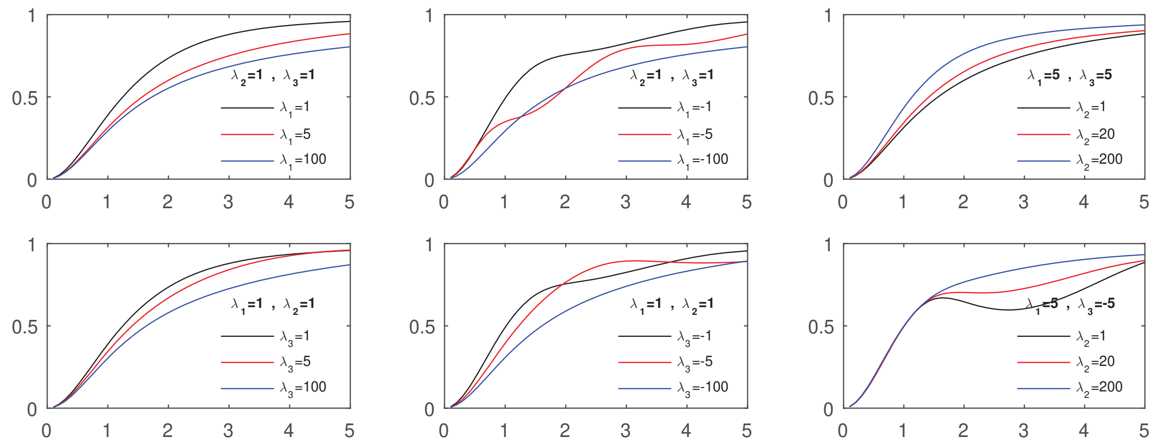

Figure 3. (σ, V) plots for different $\lambda_1, \lambda_2, \lambda_3$ values $(\mu = 0)$.

3.3. Skewness and Kurtosis

In a circular model, the pth central cosine moment and sine moments are $\bar{\alpha}_p = E[\cos p(\theta-\mu_1)]$ and $\bar{\beta}_p = E[\sin p(\theta-\mu_1)]$ respectively. Both of coefficients kurtosis and skewness are obtained using the second central moment and aren't affected by parameter μ. So, it is enough to calculate both coefficients according to $\mu = 0$. Thus,

$$\bar{\alpha}_2 = E[\cos 2(\theta-\mu_1)] \tag{3.7}$$

$$= \left[\Delta \frac{2\sigma}{4\sigma^2+1} - \Delta^2 S_2 - \nabla \frac{2e^{-k}\xi_2}{4\sigma^2+1} \right] \sin 2\mu_1 + \frac{\cos 2\mu_1}{4\sigma^2+1}$$

and

$$\begin{aligned} \bar{\beta}_2 &= E\left[\sin 2(\theta-\mu_1)\right] \\ &= \left[\Delta \frac{2\sigma}{4\sigma^2+1} - \Delta^2 S_2 - \nabla \frac{2e^{-k}\xi_2}{4\sigma^2+1} \right] \cos 2\mu_1 - \frac{\sin 2\mu_1}{4\sigma^2+1}. \end{aligned} \tag{3.8}$$

As a measure of asymmetry, skewness coefficient of a circular distribution is calculated by $\gamma_1 = \bar{\beta}_2 V^{-3/2}$ [9]. Using the values of (3.6) and (3.8) the skewness of $WFSL$ distribution is

$$\gamma_1 = \frac{\left(\Delta 2\sigma - \Delta^2 S_2 \left(4\sigma^2+1\right) - 2\nabla e^{-k}\xi_2\right)\cos 2\mu_1 - \sin 2\mu_1}{\left(4\sigma^2+1\right) V^{3/2}}.$$

If the distribution is symmetric and unimodal, the skewness coefficient will be zero.

Corollary 4. *$WFSL(\mu,\sigma,0,\lambda_2,0)$ is unimodal and symmetric about μ. From Corollary 2 mean direction is $\mu_1 = 0$, when $\mu = \lambda_1 = \lambda_3 = 0$, since $\Delta = \nabla = 0$ and $\mu_1 = 0$, $\gamma_1 = 0$.*

Kurtosis of a circular distribution is $\gamma_2 = \left(\bar{\alpha}_2 - \rho_1^4\right)\left(1-\rho_1\right)^{-2}$ [9]. Using the given values (3.5), (3.6) and (3.7) kurtosis of $WFSL(0,\sigma,\lambda_1,\lambda_2,\lambda_3)$ is

$$\gamma_2 = \frac{\left(\Delta 2\sigma - \Delta^2 S_2 \left(4\sigma^2+1\right) - 2\nabla e^{-k}\xi_2\right)\sin 2\mu_1 + \cos 2\mu_1 - \left(4\sigma^2+1\right)\rho_1^4}{\left(4\sigma^2+1\right) V^2}.$$

Corollary 5. *Kurtosis of $WFSL(\mu,\sigma,0,\lambda_2,0)$ is $\gamma_2 = \left(\sigma^4+4\sigma^2+6\right)\left(4\sigma^2+1\right)^{-1}\left(\sigma^2+1\right)^{-2}$.*

3.4. Some Propositions

In this section, we provide some properties related to the introduced distribution and the relationships with other distributions.

Proposition 1. *$WFSL(0,\sigma,0,\lambda_2,0) \stackrel{dist}{=} WL(\sigma^{-1},1)$ where $WL(\lambda,\kappa)$ denotes the wrapped Laplace distribution [5].*

Proof. Let denote φ_p^{WFSL} and φ_p^{WL} are characteristic functions of $WFSL(0,\sigma,0,\lambda_2,0)$ and $WL(\sigma^{-1},1)$ respectively. When $\lambda_1 = 0$, $\lambda_3 = 0$ and $\mu = 0$, it's easy to see for each integer p, $\varphi_p^{WFSL} = \varphi_p^{WL} = \left(p^2\sigma^2+1\right)^{-1}$.

Proposition 2. *Let $WE(\lambda)$ denotes the wrapped exponential distribution and $WL(\lambda)$ denotes the wrapped Laplace distribution [5].*

(a) $\lim_{\lambda_1\to\infty} WFSL(0,\sigma,\lambda_1,\lambda_2,\lambda_3) \stackrel{dist}{=} WE(\sigma^{-1})$.
(b) $\lim_{\lambda_3\to\infty} WFSL(0,\sigma,\lambda_1,\lambda_2,\lambda_3) \stackrel{dist}{=} WE(\sigma^{-1})$.
(c) $\lim_{\lambda^*\to\infty} WFSL(0,\sigma,\lambda^*,\lambda_2,\lambda^*) \stackrel{dist}{=} WE(\sigma^{-1})$.
(d) $\lim_{\lambda_2\to\infty} WFSL(0,\sigma,\lambda_1,\lambda_2,\lambda_3) \stackrel{dist}{=} WL(\sigma^{-1},1)$.

Proof. (a) The characteristic function of $WE(\sigma^{-1})$ distribution is

$$\varphi_p^{WE} = (1+ip\sigma)\left(p^2\sigma^2+1\right)^{-1}.$$

Using Lemma 3a pth cosine moment of $WFSL(0,\sigma,\lambda_1,\lambda_2,\lambda_3)$ is

$$\alpha_p = \left(p^2\sigma^2+1\right)^{-1},$$

when $\lambda_1 \to \infty$. If $\lambda_3 < 0$, e^{-k} tends to 0 otherwise $\nabla = 0$. So in both cases pth sine moment will be equal to

$$\beta_p = p\sigma \left(p^2\sigma^2 + 1\right)^{-1}.$$

Thus

$$\varphi_p^{WFSL} = \varphi_p^{WE} = (1 + ip\sigma)\left(p^2\sigma^2 + 1\right)^{-1}.$$

(b) In a similar way, when λ_3 tends to ∞, α_p tends to

$$\alpha_p = \left(p^2\sigma^2 + 1\right)^{-1}.$$

If $\lambda_1 < 0$, e^{-k} tends to 1, $\nabla = \Delta = -1$ and $\xi_p \to p\sigma$. Otherwise if $\lambda_1 > 0$, ∇ equals to 0. So in both cases

$$\beta_p = p\sigma \left(p^2\sigma^2 + 1\right)^{-1}.$$

(c) Proof is clear, since $\nabla = 0$ when λ_1 and λ_3 tends to ∞.

(d) When $\lambda_2 \to \infty$, from Lemma 3b and (3.1) it can be seen immediately that $\alpha_p = (p^2\sigma^2 + 1)^{-1}$ and

$$\begin{aligned}\beta_p &= -\frac{p\sigma\Delta(1-\Delta)(1+\Delta)}{p^2\sigma^2 + 1} \\ &= 0,\end{aligned}$$

for all cases of Δ. Thus

$$\varphi_p^{WFSL} = \alpha_p = \left(p^2\sigma^2 + 1\right)^{-1} = \varphi_p^{WL}.$$

Proposition 3. $\Theta \sim WFSL(0, \sigma, \lambda_1, \lambda_2, \lambda_3) \Leftrightarrow -\Theta \sim WFSL(0, \sigma, -\lambda_1, \lambda_2, -\lambda_3)$.

Proof. Let's use $g_{\lambda_1,\lambda_2,\lambda_3}(x)$ notation instead of $g(x)$ notation in equation (1.1). It's easy to see

$$g_{-\lambda_1,\lambda_2,-\lambda_3}(x) = g_{\lambda_1,\lambda_2,\lambda_3}(-x) = -g_{\lambda_1,\lambda_2,\lambda_3}(x).$$

Thus,

$$A(\theta, \mu, \sigma, -\lambda_1, \lambda_2, -\lambda_3) = A(-\theta, \mu, \sigma, \lambda_1, \lambda_2, \lambda_3)$$

and

$$f_\Theta(\theta; 0, \sigma, -\lambda_1, \lambda_2, -\lambda_3) = f_\Theta(-\theta; 0, \sigma, \lambda_1, \lambda_2, \lambda_3).$$

Corollary 6. *Mean direction of* $-\Theta \sim WFSL(0, \sigma, -\lambda_1, \lambda_2, -\lambda_3)$ *is* $2\pi - \mu_1$, *where* μ_1 *is the mean direction of* $\Theta \sim WFSL(0, \sigma, \lambda_1, \lambda_2, \lambda_3)$.

Proposition 4. *Let* X *and* Y *be independent Laplace* (η) *random variables. Define the random variable* Θ *as*

$$\Theta = \{X | Y < g(X)\} \pmod{2\pi}$$

where $g(.)$ *defined as* (1.1). *Then* $\Theta \sim WFSL(0, \eta, \lambda_1, \lambda_2, \lambda_3)$.

Proof. Proof is clear since $X | Y < g(X) \sim FSL(\eta, \lambda_1, \lambda_2, \lambda_3)$ [14].

Corollary 7. $\{-X | Y > g(X)\} \pmod{2\pi} \sim WFSL(0, \eta, \lambda_1, \lambda_2, \lambda_3)$.

Corollary 8. $\Theta = [IX - (1-I)X] \pmod{2\pi} \sim WFSL(0, \eta, \lambda_1, \lambda_2, \lambda_3)$ *where*

$$I = \begin{cases} 1, & Y < g(X) \\ 0, & Y \geq g(X) \end{cases}.$$

3.5. Simulation

The result of last proposition can be used to generate random numbers from $WFSL$ distribution. The following algorithm is based on this result and generates n random numbers.

Step 1. Generate $n \times 1$ random vectors $X \sim Laplace\,(1)$ and $Y \sim Laplace\,(1)$.
Step 2. Calculate $g\,(X) = (\lambda_1 X + \lambda_3 X^3)\,(1 + \lambda_2 X^2)^{-0.5}$.
Step 3. Calculate $T = [Y < g\,(X)]\,X - [Y \geq g\,(X)]\,X$.
Step 4. Calculate $Z = [\mu + \sigma T]\,(\bmod 2\pi)$.

Obtaining clear forms of maximum likelihood (ml) estimators is an insurmountable problem because of complex likelihood function. Therefore, likelihood function must be maximized by an iterative method. In this simulation study we used Matlab's mle function to obtain ml estimates of parameters. We ran the above algorithm 100 times for different values of n and $\mu = 0.79$, $\sigma = 1.5$, $\lambda_1 = 3$, $\lambda_2 = 3$ and $\lambda_3 = -5$. The bias and MSE(in parentheses) values of the parameters calculated with the ml estimates obtained in each step, are shown in Table 1.

Table 1. Average values of bias and MSE (in parentheses) of parameters.

n	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$
100	0.1187(0.0113)	0.0067(0.0388)	0.4875(3.0552)	0.2682(11.491)	-1.3720(19.436)
250	0.0006(0.0032)	0.0029(0.0200)	0.2574(1.0001)	0.7384(7.7087)	-0.5679(4.9759)
500	0.0070(0.0017)	-0.0134(0.0113)	0.0164(0.3399)	0.2394(5.7541)	-0.0891(1.6226)
1000	0.0034(0.0009)	-0.0030(0.0049)	0.0105(0.1427)	0.0440(1.6380)	-0.0269(0.7009)

From Table 1 it can be seen that, as the sample size increases, the bias and MSE values of parameters decrease to zero.

4. Application to Real Data

In order to demonstrate the modelling behavior of the $WFSL$ distribution, we will analyze two popular data sets from the literature. Both data sets in this section have been discussed in many studies and used for fitting the distributions proposed by the authors. Table 2 shows estimates of the parameters, estimated average direction and resultant length. We also provide maximized log likelihood value (L), Akaike information criterion (AIC), Bayesian information criterion (BIC) and Watson's U^2 (W^2) value in the same table.

Table 2. Summary of fits for the turtle data and ant data.

Turtle Data				Ant Data			
Estimates				Estimates			
$\widehat{\mu}$	1.7104	-L	107.7552	$\widehat{\mu}$	3.67	-L	128.936
$\widehat{\sigma}$	1.1761	AIC	225.5104	$\widehat{\sigma}$	0.92	AIC	267.8722
$\widehat{\lambda}_1$	-3.029	BIC	237.1641	$\widehat{\lambda}_1$	-2033	BIC	280.8980
$\widehat{\lambda}_2$	0.0127	W^2	0.0375	$\widehat{\lambda}_2$	$2.86E6$	W^2	0.2050
$\widehat{\lambda}_3$	1.1393			$\widehat{\lambda}_3$	280.5		
Mean Direction			1.09($\sim 62.6^\circ$)	Mean Direction			3.15($\sim 180.63^\circ$)
Res. Lenght			0.5146	Res. Lenght			0.6209

Turtle Data: The first dataset in this section refers to the orientations of 76 turtles after laying eggs [7]. Left panel of Figure 4 represents the circular data plot over rose diagram and fitted $WFSL$ distribution. The arrow and its length represents the sample mean resultant vector m_1 and resultant length r_1, respectively. Calculated sample statistics are $a_1 = 0.2166,\ b_1 = 0.4474,\ m_1 = 1.12\ (\sim 64.2^\circ)$ and $r_1 = 0.4971$. Maximum likelihood estimation of parameters are obtained by maximizing the likelihood function in Matlab via mle function. In order to avoid localmaxima, parameter intervals have been kept as wide as possible. The maximum likelihood estimates are seen

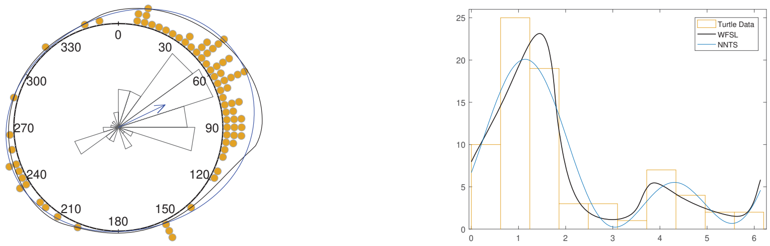

Figure 4. Plots for turtle data. Circular data plot, fitted circular pdf and rose diagram (left), linear histogram and fitted pdf (right).

This dataset was recently used by Joshi and Jose 2018 as an application of the wrapped Lindley ($\mathcal{WL}$) distribution. The authors reported the AIC value for $\mathcal{WL}$ distribution is 243.29, BIC value is 243.75 and maximized log likelihood value is 119.71. In the same study, the AIC value for the wrapped exponential distribution is 243.29, the BIC is 245.63 and maximized log likelihood value is 120.65. Yilmaz and Biçer 2018 modeled this data set using the transmuted version of wrapped exponential (TWE) distribution and they obtained the AIC, W^2 and maximized log likelihood value values as 239.89, 0.25 and 117.95, respectively. Also, Fernandez-Duran 2004 used this data set as an application for non-negative trigonometric sums ($NNTS$) distribution and obtained the lowest AIC value is 225.94. Based on the AIC, BIC, maximized log likelihood and W^2 statistics values reported by these authors, the $WFSL$ distribution gives better fit to turtle data than the mentioned alternatives.

Ant Data: The second data set consist of the directions chosen by 100 ants which have been analyzed by Fisher 1995 with the aim of fitting a von Mises distribution. Ants were placed into an arena one by one, and the directions they chose relative to an evenly illuminated black light source placed at 180° were recorded. Calculated sample statistics are $a_1 = -0.6091$, $b_1 = -0.0334$, $m_1 = 3.1964$ ($\sim 183.14°$) and $r_1 = 0.6101$. The maximum likelihood estimates obtained via Matlab are seen in Table 2

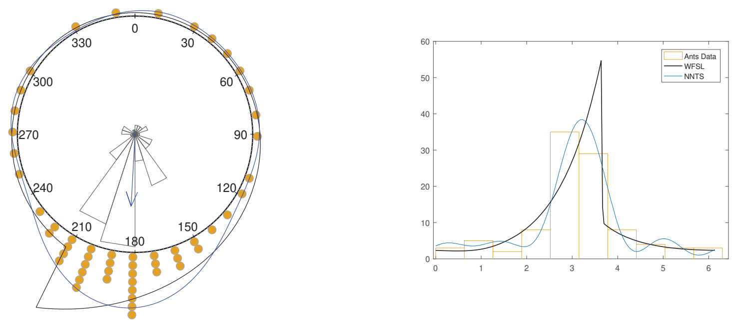

Figure 5. Plots for ant data. Circular data plot, fitted circular pdf and rose diagram (left), linear histogram and fitted pdf (right).

Fisher concludes that the von Mises distribution is not a suitable model for this data. The AIC and BIC values for this model was equal to 288.24 and 293.4, respectively. Fernandez-Duran 2004 reported the AIC values as 276.64 for the $NNTS$ distribution, 276.84 for the skewed wrapped Laplace and 275.74 for the symmetric wrapped Laplace distribution. Based on the AIC and BIC values reported by these authors, the $WFSL$ distribution gives better fit to ant data than the mentioned alternatives.

5. Conclusion

In this study, wrapped version of the flexible skew Laplace (FSL) distribution is introduced. The proposed distribution inherits the flexibility properties of FSL distribution. We also discussed characteristic function, trigonometric moments, location, dispersion and coefficients of skewness and kurtosis of proposed distribution. As we mentioned about, it is not possible to find explicit forms of maximum likelihood (ML) estimators of parameters. However, as can also be seen from many studies in recent years, this problem can be overcome with the help of computer softwares. Therefore, in last section, the mle function of Matlab used for obtaining the estimation of the parameters. Based on the AIC, BIC, maximized log likelihood and W^2 statistics values, the results showed that the proposed model is better fits to these datasets than the recently published wrapped Lindley distribution [8], transmuted wrapped exponential distribution [15] and non-negative trigonometric sums distribution [3].

Acknowledgement

The author is grateful to the anonymous referee for his helpful comments. This study was funded by Kirikkale University BAP coordination unit under grant number 2016/137.

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İSTATİSTİK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION
Vol. 11, No. 3, December 2018, pp. 65–76
ISSN 1300-4077 | 18 | 3 | 65 | 76

İSTATİSTİK

THE QUASI XGAMMA-POISSON DISTRIBUTION: PROPERTIES AND APPLICATION

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Abstract: In this work, we introduce a new xgamma-Poisson lifetime model called the quasi xgamma-Poisson distribution. Some of its mathematical properties are derived. The proposed model can be motivated with a physical motivation by compounding the quasi xgamma construction with the truncated Poisson distribution. The quasi xgamma-Poisson model also motivated by the wide use of the xgamma distribution in many applied areas as well as for the fact that the new generalization provides more flexibility to analyze real data. We discuss the maximum likelihood estimation of the quasi xgamma-Poisson model parameters. An application to illustrate that the proposed quasi xgamma-Poisson model provides consistently better fit than the other competitive models.

Key words: Xgamma-Poisson life distributions; zero-truncated poisson distribution; maximum likelihood estimation; order statistics.
History: Submitted: 3 October 2017; Revised: 1 December 2018; Accepted: 15 December 2018

1. Introduction

When the lifetime data present a bathtub shaped hazard rate function, such as human mortality and machine life cycles, practical problems generally require a wider range of possibilities in the medium risk. Researchers in the last years developed various extensions and modified forms of the xgamma distribution to obtain more flexible models with different number of parameters. A state-of-the-art survey on the class of such distributions can be found in Sen et al. (2016) and Sen and Chandra (2017). The xgamma distribution with its delegate structural and distributional properties serves as a potential survival model among the other popular lifetime models in the literature, more details can be seen in Sen et al. (2018). Recently, Sen et al. (2017) have introduced and studied a weighted version of xgamma distribution along with its length biased version for modeling time-to-event data sets. The quasi xgamma distribution, a two-parameter extension or generalization of xgamma distribution, shows superiority over many more life distributions when applied to real life survival and/or reliability data set.

In this present investigation, our aim is to introduce and study a three parameter extension of quasi xgamma distribution for modeling lifetime data. This extension is proposed by mixing quasi xgamma and zero-truncated Poisson distributions similarly Lindley-Poisson distribution (Gui et

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al., 2014). We can interpret the proposed model as follows: A situation where failure of a unit or system (be it mechanical or biological) occurs due to the presence of some unknown number of initial defects of similar kind. If we assume these unknown number of initial defects follow a zero-truncated Poisson distribution and their respective lives follow a quasi xgamma distribution, then the first failure distribution leads to what we call quasi xgamma-Poisson distribution. we aim, in this article, sythesis of the proposed model, its essential properties, method of estimating model parameters and real life application of the model.

The rest of the article is organized as follows. The proposed distribution is synthesized in Section 2. Different properties, such as, survival function, hazard rate function, moments and related measures, distributions of extreme order statistics and stochastic ordering, are discussed and studied in Section 3 and in its deliberate subsections. In Section 4, method of maximum likelihood is proposed for estimating the unknown parameters of the proposed distribution. Algorithm of a simulation is proposed along with a simulation study in Section 5. Section 6 deals with an application of the model with a real data illustration and comparison. Finally, Section 7 concludes.

2. Synthesis

If Y is a random variable (rv) following quasi xgamma (QXG) model with parameters α and θ (Sen and Chandra, 2017), then it has pdf as

$$f(y) = \frac{\theta}{(1+\alpha)}\left(\alpha + \frac{1}{2}\theta^2 y^2\right) e^{-\theta y}|_{(y>0,\alpha,\theta>0)}. \tag{2.1}$$

Let us denote it by $Y \sim QXG(\alpha, \theta)$, corresponding cdf is given by

$$F(y) = 1 - \frac{\left(1+\alpha+\theta y + \frac{1}{2}\theta^2 y^2\right)}{(1+\alpha)} e^{-\theta y}|_{(y>0,\alpha,\theta>0)}.$$

The new xgamma-Poisson distribution can be synthesized as follows:
Suppose that the life of a unit (be it mechanical or biological) fails due to the presence of M (an unknown number) initial defects for some kind. Let $Y_1, Y_2, \ldots, Y_M$ denote the lives of the initial defects, then the life of the unit, say X, can be expressed as

$$X = Min\{Y_1, Y_2, \ldots, Y_M\}.$$

Suppose that the lives of the initial defects, $Y_1, Y_2, \ldots, Y_M$, follow identically and independently distributed (i.i.d) $QXG(\alpha, \theta)$ and the number of initial defects M follows a zero-truncated Poisson distribution with parameter λ. Then, the probability mass function (pmf) of M is

$$\Pr(M = m|_{\lambda>0, m=1,2,\ldots}) = p(m) = \frac{\lambda^m e^{-\lambda}}{m!\,(1-e^{-\lambda})} = \frac{\lambda^m}{m!\,(-1+e^{\lambda})}.$$

Assuming that the rvs $Y_i (i = 1, 2, \ldots, M)$ and M are independent, the conditional density of X given $M = m$ is

$$f(x|m) = \frac{m\theta\left(\frac{1}{2}\theta^2 x^2 + \alpha\right)\left(\frac{1}{2}\theta^2 x^2 + \theta x + \alpha + 1\right)^{-1+m}}{(1+\alpha)^m e^{m\theta y}}|_{(x>0)}.$$

Then, the marginal pdf of X can be obtained as

$$f(x) = \sum_{m=1}^{\infty} f(x|m) p(m)$$

$$= \sum_{m=1}^{\infty} m\theta \frac{(1+\alpha)^{-m}\left(\frac{1}{2}\theta^2x^2+\alpha\right)e^{-m\theta y}}{\left(\frac{1}{2}\theta^2x^2+\theta x+\alpha+1\right)^{1-m}} \cdot \frac{\lambda^m}{m!\,(-1+e^{\lambda})}$$
$$= \frac{\lambda\theta e^{-\theta x}\left(\frac{1}{2}\theta^2x^2+\alpha\right)}{(-1+e^{\lambda})(1+\alpha)} \sum_{m=1}^{\infty} \lambda^{-1+m} \frac{\left(\frac{1}{2}\theta^2x^2+\theta x+\alpha+1\right)^{-1+m}}{(1+\alpha)^{-1+m}(-1+m)!e^{\theta(-1+m)x}}$$
$$= \frac{\lambda\theta\left(\frac{1}{2}\theta^2x^2+\alpha\right)}{(-1+e^{\lambda})(1+\alpha)} \exp\left[\frac{\lambda\left(\frac{1}{2}\theta^2x^2+\theta x+\alpha+1\right)}{(1+\alpha)e^{\theta x}} - \theta x\right]\Big|_{(x>0,\alpha,\theta,\lambda>0)}.$$

2.1. The quasi xgamma-Poisson distribution

We have the following definition for the new distribution obtained from the above synthesis:

Definition 1. An absolutely continuous rv X will be said to follow quasi xgamma-Poisson (QXGP) distribution with parameters α, θ and λ if its pdf is of the form

$$f(x) = K(\alpha,\theta,\lambda)\left(\frac{1}{2}\theta^2x^2+\alpha\right)e^{\frac{\lambda e^{-\theta x}\left(1+\alpha+\theta x+\frac{1}{2}\theta^2x^2\right)}{(1+\alpha)}-\theta x}, x>0, \alpha,\theta,\lambda>0, \tag{2.2}$$

where $K(\alpha,\theta,\lambda) = \frac{\lambda\theta}{(e^{\lambda}-1)(1+\alpha)}$, a function of α, θ and λ.

We denote it by $X \sim QXGP(\alpha,\theta,\lambda)$.

Special cases:

(i) When $\alpha = \theta$ in (2.2), we obtain a new family of probability distributions, can be termed as xgamma-Poisson (XGP) distribution, with the following pdf:

$$f_1(x) = K_1(\theta,\lambda)\left(1+\frac{\theta}{2}x^2\right)e^{\frac{\lambda e^{-\theta x}\left(1+\theta+\theta x+\frac{1}{2}\theta^2x^2\right)}{(1+\theta)}-\theta x}, x>0, \theta,\lambda>0,$$

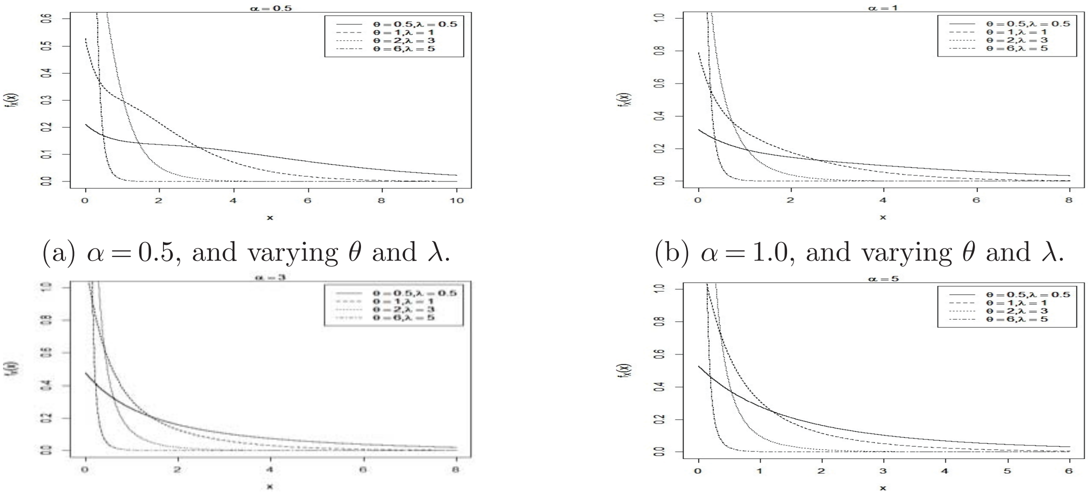
where $K_1(\theta,\lambda) = \frac{\lambda\theta^2}{\left(-1+e^{\lambda}\right)(1+\theta)}$, a function of θ and λ.

We can denote it by $X \sim XGP(\theta,\lambda)$.

(ii) While $\lambda \to 0$ in (2.2), the QXG model is obtained.

(iii) When $\alpha = \theta$ and $\lambda \to 0$ in (2.2), we obtain xgamma distribution with parameter θ (see for more details Sen et al., 2016).

The pdf curves for different values of α, θ and λ are shown in Figure 1.



(a) $\alpha = 0.5$, and varying θ and λ.

(b) $\alpha = 1.0$, and varying θ and λ.

(c) $\alpha = 3.0$, and varying θ and λ.

(d) $\alpha = 5.0$, and varying θ and λ.

FIGURE 1. The pdf curves of QXGP distribution for various values of α, θ and λ.

3. Properties

The cdf of $QXGP(\alpha,\theta,\lambda)$ is obtained as

$$F(x) = \frac{e^{\lambda} - e^{\frac{\lambda e^{-\theta x}\left(1+\alpha+\theta x+\frac{1}{2}\theta^2 x^2\right)}{(1+\alpha)}}}{e^{\lambda}-1}\,|_{(x>0)}. \tag{3.1}$$

The corresponding survival function (or reliability function) is given by

$$S(x) = \frac{e^{\frac{\lambda e^{-\theta x}\left(\frac{1}{2}\theta^2 x^2+\theta x+\alpha+1\right)}{(1+\alpha)}} - 1}{e^{\lambda}-1}\,|_{(x>0)}.$$

The failure rate function (or hazard rate function(hrf)) is, then, derived as

$$h(x) = \frac{f(x)}{S(x)} = \frac{\lambda\theta\left(\frac{1}{2}\theta^2x^2+\alpha\right)e^{\frac{\lambda e^{-\theta x}\left(\frac{1}{2}\theta^2 x^2+\theta x+\alpha+1\right)}{(1+\alpha)}-\theta x}}{(1+\alpha)\left[e^{\frac{\lambda e^{-\theta x}\left(\frac{1}{2}\theta^2 x^2+\theta x+\alpha+1\right)}{(1+\alpha)}}-1\right]}\,|_{(x>0)}.$$

The failure rate curves for different values of α, θ and λ are shown in Figure 2.

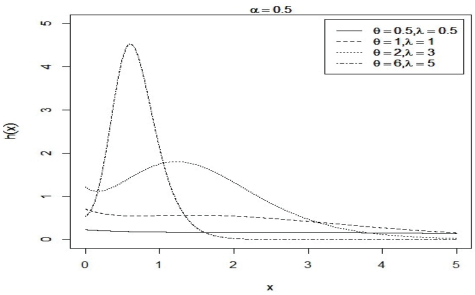
3.1. Moments and related measures

When $X \sim QXGP(\alpha,\theta,\lambda)$, the k^{th} raw moment of X is given by

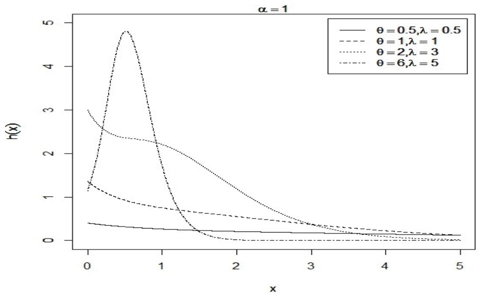
$$\begin{aligned} \mu_k^{'} &= k\int_0^{\infty} x^{k-1}S(x)dx \\ &= \frac{k}{e^{\lambda}-1}\int_0^{\infty} x^{k-1}\left[e^{\frac{\lambda e^{-\theta x}\left(\frac{1}{2}\theta^2 x^2+\theta x+\alpha+1\right)}{(1+\alpha)}}-1\right]dx\,|_{(k=1,2,\cdots)}. \end{aligned} \tag{3.2}$$

μ_k's can not expressed in a closed form and hence numerical integration can be applied to fine the mean and other important related measures. The j^{th} order central moment can be obtained by the following relationship.

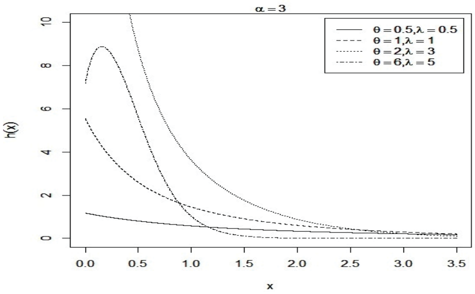
$$\mu_j = E[(X-\mu)^j] = \sum_{r=0}^{j}\binom{j}{r}\mu_r{}'(-\mu)^{j-r}\,|_{(j=2,3,\ldots)},$$



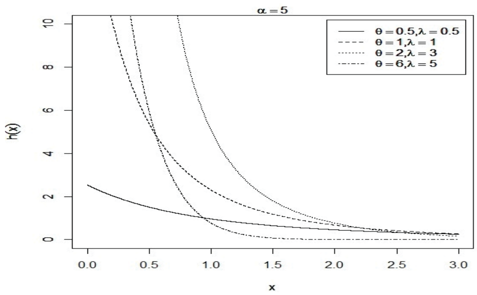
(a) $\alpha = 0.5$, and varying θ and λ.



(b) $\alpha = 1.0$, and varying θ and λ.



(c) $\alpha = 3.0$, and varying θ and λ.



(d) $\alpha = 5.0$, and varying θ and λ.

FIGURE 2. Failure rate curves of QXGP distribution for various values of α, θ and λ.

where $\mu = E(X)$.
With above formula, the skewness and kurtosis coefficients are respectively given by

$$\sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2}.$$

The values for mean, variance, $\sqrt{\beta_1}$ and β_2 for selected values of α, θ and λ are shown in Table 1. We note that for fixed values of α and λ, the values of $\sqrt{\beta_1}$ and β_2 do not depend on varying θ.

3.2. Asymptotic distributions of order statistics

Let $X_1, X_2, \ldots, X_{n-1}, X_n$ be a random sample (rs) of size n from $QXGP(\alpha, \theta, \lambda)$, then by the central limit theorem, the mean $(X_1 + X_2 + \ldots + X_n)/n$ approaches to normal distribution as $n \to \infty$.
Sometimes one might be interested in the asymptotics of the extreme order statistics. Let us denote:

$$X_{1:n} = Min\{X_1, X_2, \ldots, X_n\} := \text{Smallest order statistic}$$

and

$$X_{n:n} = Max\{X_1, X_2, \ldots, X_n\} := \text{Largest order statistic.}$$

These extreme order statistics represent the lives of series and parallel systems respectively and have important applications in reliability engineering and system sciences. We have the following theorem on the distributions of extreme order statistics.

THEOREM 1. *If $X_{n:n}$ and $X_{1:n}$ denote, respectively, the largest and smallest order statistics from $QXGP(\alpha, \theta, \lambda)$, then*
(1) $\lim_{n\to\infty} \Pr(X_{n:n} \le tb_n) = e^{-t^{-1}}, t > 0 \mid \left[F^{-1}\left(1 - \frac{1}{n}\right) = b_n\right].$

TABLE 1. Mean, variance, coefficients of skewness and kurtosis for different values of parameters

(α,θ,λ)	μ	$Var(X)$	$\sqrt{\beta_1}$	β_2
(0.5,0.5,0.5)	4.1906	11.764	1.3027	5.3126
(1.0,0.5,0.5)	3.5504	10.795	1.5115	5.9875
(2.0,0.5,0.5)	2.9321	9.0414	1.8055	7.3156
(5.0,0.5,0.5)	2.3348	6.5830	2.1681	9.5956
(0.5,0.5,1.0)	3.7410	10.523	1.4526	5.9288
(0.5,0.5,2.0)	2.9497	7.9725	1.7761	7.6041
(0.5,0.5,5.0)	1.5026	2.7438	2.5810	14.719
(0.5,1.0,0.5)	2.0953	2.9409	1.3027	5.3126
(0.5,2.0,0.5)	1.0476	0.7352	1.3027	5.3126
(0.5,5.0,0.5)	0.4190	0.1176	1.3027	5.3126
(0.5,0.05,1.0)	37.410	1052.3	1.4526	5.9288
(1.0,5.0,0.5)	0.3550	0.1079	1.5115	5.9875
(5.0,5.0,5.0)	0.0654	0.0092	4.9895	42.528

(2) $\lim_{n\to\infty} \Pr(X_{1:n} \le b_n^* t) = 1 - e^{-t}, t > 0 \mid \left[F^{-1}\left(\frac{1}{n}\right) = b_n^*\right]$.

PROOF. We apply the following asymptotic results (see Arnold et al., 2008) for $X_{1:n}$ and $X_{n:n}$.
(1) For the largest order statistic $X_{n:n}$, we have

$$\lim_{n\to\infty} \Pr(X_{n:n} \le a_n + b_n t) = e^{-t^{-d}}, t > 0, c > 0 \text{ (Fréchet type)},$$

where $a_n = 0$ and $b_n = F^{-1}(1 - 1/n)$ iff $F^{-1}(1) = \infty$ and $\exists$ a constant $d > 0$ such that,

$$\lim_{x\to\infty} \frac{1 - F(xt)}{1 - F(x)} - t^{-d}.$$

From the cdf of $QXGP(\alpha,\theta,\lambda)$ distribution as given in (3.1), letting $F(x) = 1$, we can easily see that $F^{-1}(1) = \infty$ and

$$\lim_{x\to\infty} \frac{1 - F(xt)}{1 - F(x)} - t^{-1}.$$

Therefore, we obtain $d = 1, a_n = 0$ and $b_n = F^{-1}(1 - 1/n)$.
(2) For the smallest order statistic $X_{1:n}$, we have

$$\lim_{n\to\infty} \Pr(X_{1:n} \le a_n^* + b_n^* t) = 1 - e^{-t^c}, t > 0, c > 0 \text{ (Weibull type)},$$

where $a_n^* = F^{-1}(0)$ and $b_n^* = F^{-1}(1/n) - F^{-1}(0)$ iff $F^{-1}(0)$ is finite,

$$\lim_{\epsilon\to 0^+} \frac{F(F^{-1}(0) + \epsilon t)}{F(F^{-1}(0) + \epsilon)} = t^c \quad \forall t > 0, c > 0.$$

Letting $F(x) = 0$ we see that $F^{-1}(0) = 0$ and finite. Moreover,

$$\lim_{\epsilon\to 0^+} \frac{F(0 + \epsilon t)}{F(0 + \epsilon)} = t.$$

Finally, we have $c = 1, a_n^* = 0$ and $b_n^* = F^{-1}(1/n)$.
Hence the proof is completed.

3.3. Stochastic ordering

For a positive continuous rv, stochastic ordering is an important tool for judging the comparative behavior. Let us denote the pdf, cdf, hrf amd mean residual life function (mrl) of a positive continuous rv X by $f_X(\cdot), F_X(\cdot), h_X(\cdot)$ and $m_X(\cdot)$, respectively, and those of another positive continuous rv Y by $f_Y(\cdot), F_Y(\cdot), h_Y(\cdot)$ and $m_Y(\cdot)$, respectively. We recall some basic definitions.

Definition 2. A rv X is said to be smaller than a rv Y in the

(i) The stochastic order $(X \leq_{(sto)} Y)$ if $F_X(x) \geq F_Y(x)$, $\forall\, x$.

(ii) The hazard rate order $(X \leq_{(hro)} Y)$ if $h_X(x) \geq h_Y(x)$, $\forall\, x$.

(iii) The mean residual life order $(X \leq_{(mrlo)} Y)$ if $m_X(x) \leq m_Y(x)$, $\forall\, x$.

(iv) The ikelihood ratio order $(X \leq_{(lro)} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The below given implications (see Shaked and Shanthikumar, 1994) are well justified:

$$\left[X \leq_{(lro)} Y\right] \Rightarrow \left[X \leq_{(hro)} Y\right] \Rightarrow [X \leq_{mrl} Y] \text{ and } \left[X \leq_{(hro)} Y\right] \Rightarrow \left[X \leq_{(sto)} Y\right] \tag{3.3}$$

The following theorem shows that the QXGP distributions are ordered with respect to different stochastic orderings.

Theorem 2. *Let $X \sim QXGP(\alpha,\theta,\lambda_1)$ and $Y \sim QXGP(\alpha,\theta,\lambda_2)$. If $\lambda_1 > \lambda_2$ then $\left[X \leq_{(lro)} Y\right]$ and $\left[X \leq_{(hro)} Y\right]$, $\left[X \leq_{(mrlo)} Y\right]$, $\left[X \leq_{(sto)} Y\right]$.*

Proof. For any $x > 0$, the ratio of the densities is given by

$$g(x) = \frac{f_X(x)}{f_Y(x)} = \frac{\lambda_1\left(e^{\lambda_2}-1\right)}{\lambda_2\left(e^{\lambda_1}-1\right)} exp\left\{\frac{(\lambda_1-\lambda_2)e^{-\theta x}\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right\}$$

Taking derivative with respect to x, we have

$$g'(x) = -\frac{\theta\lambda_1(\lambda_1-\lambda_2)\left(e^{\lambda_2}-1\right)e^{-\theta x}\left(\alpha+\frac{\theta^2}{2}x^2\right)}{\lambda_2\left(e^{\lambda_1}-1\right)(1+\alpha)} exp\left\{\frac{(\lambda_1-\lambda_2)e^{-\theta x}\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right\}$$

Now, $g'(x) < 0$ if $\lambda_1 > \lambda_2$ and hence $X \leq_{lr} Y$ if $\lambda_1 > \lambda_2$. The other orderings are immediate by (3.3). Hence the proof is completed.

4. Maximum likelihood estimation (MLE)

Let $x_1, x_2, \ldots, x_n$ be a rs from the QXGP. Let $\varphi = (\alpha,\theta,\lambda)^T$ be the parameter vector. Then, the log likelihood (**LL**) function for φ, say $\ell(\varphi) = \ell$,

$$\begin{aligned}\ell = &-n\log(1+\alpha) + n\log\theta + n\log\lambda - n\log(e^{\lambda}-1)\\ &+\sum_{i=1}^{n}\log\left(\frac{1}{2}\theta^2x^2+\alpha\right) + \sum_{i=1}^{n}\left[\frac{\lambda e^{-\theta x_i}\left(1+\alpha+\theta x_i+\frac{1}{2}\theta^2x_i^2\right)}{(1+\alpha)} - \theta x_i\right].\end{aligned} \tag{4.1}$$

Equation (10) can be maximized directly via some sub-routine in any packet programs. The score vector components, say $\mathbf{U}(\varphi) = \frac{\partial \ell}{\partial \varphi} = \left(U_\alpha, U_\theta, U_\lambda\right)^T$, are given by λ

$$U_\alpha = -\frac{n}{1+\alpha} + \sum_{i=1}^{n}\left(\frac{1}{2}\theta^2x_i^2+\alpha\right)^{-1} - \lambda\sum_{i=1}^{n} e^{-\theta x_i}\frac{\theta x_i+\frac{1}{2}\theta^2x_i^2}{(1+\alpha)^2},$$

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n}\frac{\theta x_i^2}{\frac{1}{2}\theta^2x^2+\alpha} - \sum_{i=1}^{n}\left[\frac{e^{-\theta x_i}}{(1+\alpha)}\left(\alpha x_i+\frac{1}{2}\theta^2x_i^3\right)+x_i\right]$$

and

$$U_\lambda = \frac{n}{\lambda} - \frac{ne^\lambda}{e^\lambda - 1} + \sum_{i=1}^{n} \frac{e^{-\theta x_i}\left(1+\alpha+\theta x_i + \frac{1}{2}\theta^2 x_i^2\right)}{(1+\alpha)}.$$

Setting $U_\alpha = U_\theta = U_\lambda = 0$ and solving them simultaneously we get the MLE $\widehat{\varphi} = (\widehat{\alpha}, \widehat{\theta}, \widehat{\lambda})^T$ of $\varphi = (\alpha, \theta, \lambda)^T$. The likelihood ratio (LR) statistic can be used for comparing the QXGP model with XGP model, which is equivalently to test $H_0 : \alpha = \theta$. For this situaiton, the LR statistic is computed with $w = 2[\ell(\hat{\alpha}, \hat{\theta}, \hat{\lambda}) - \ell(\tilde{\theta}, \tilde{\lambda})]$, where $(\hat{\alpha}, \hat{\theta}, \hat{\lambda})$ are the unrestricted MLEs and $(\tilde{\theta}, \tilde{\lambda})$ are the restricted estimates under H_0. The statistic w is asymptotically (as $n \to \infty$) distributed as χ^2_υ, where υ is difference of two parameter vectors of nested models. For example, $\upsilon = 1$ for above hypothesis test.

5. Simulation study

We can generate a random data from the $QXGP(\alpha, \theta, \lambda)$ distribution using the following simulation algorithm:

1. Generate $M \sim$ zero-truncated Poisson (λ);
2. Generate $U_i \sim uniform\ [Uni(0,1)], i = 1, 2, \ldots, M$;
3. Generate $V_i \sim exponential\ [Exp(\theta)], i = 1, 2, \ldots, M$;
4. Generate $W_i \sim gamma\,[Gam(3, \theta)], i = 1, 2, \ldots, M$;
5. If $U_i \leq \alpha/(1+\alpha)$, then set $Y_i = V_i$, otherwise, set $Y_i = W_i, i = 1, 2, \ldots, M$;
6. Set $X = min(Y_1, Y_2, \ldots, Y_M)$, then X is the required sample.

Here, we give the simulation study based on graphical results to see performance of the maximum likelihood estimations of parameters. We generate $N = 1000$ samples of sizes $n = 20, 21, \ldots, 250$ from QXGP model with the true parameters values $\alpha = 2.2$, $\theta = 1$ and $\lambda = 0.5$. Random numbers procedure has been obtained by using inverse of QXGP cdf. We obtain the empirical mean (em), standard deviations (sd), bias and mean square error (MSE) of the MLEs for this simulation study. The empirical bias and MSE are calculated by (for $h = \alpha, \theta, \lambda$)

$$\widehat{Bias_h} = \tfrac{1}{N} \sum\nolimits_{i=1}^{N} \left(\hat{h}_i - h\right)$$

and

$$\widehat{MSE_h} = \tfrac{1}{N} \sum\nolimits_{i=1}^{N} \left(\hat{h}_i - h\right)^2,$$

respectively. All results and estimations have been calculated by optim-CG routine in the R programme. We give results of this simulation study in Figure 3. This Figure shows that that when the sample size (n) increases, all estimated means approach to true parameter value as well as empirical biases approach to 0. The sds and MSEs also decrease in all the cases, while sample size increases.

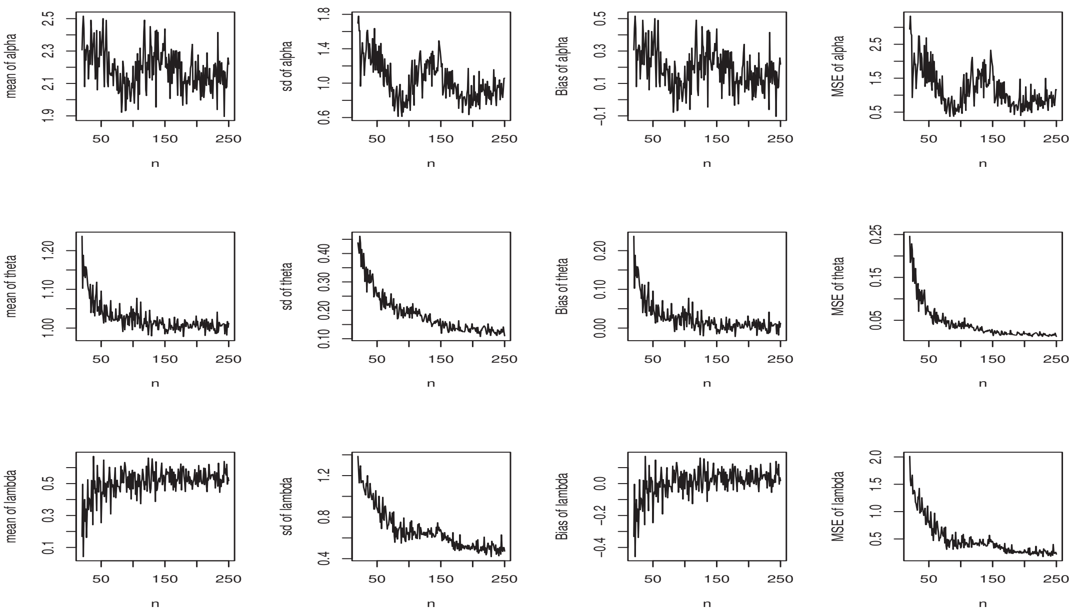


Figure 3. The empirical means, sds, biases and MSEs of the estimated parameters versus n

6. Application with real data illustration

We illustrate the flexibility of the QXGP model on the real data set. We also compare this model with the QXG model, XGP model, XG model, exponential Poisson (EP) model (see Kuş (2007)), exponentiated Weibull (EW) model (see Mudholkar and Srivasta (1993)), Weibull Poisson (WP) model (see Lu and Shi (2012)), exponentiated exponential (EE) model (see Gupta and Kundu (1999)) and exponentiated Nadarajah-Haghighi (ENH) model (see Lemonte (2013)) under the estimated log-likelihood values $\hat{\ell}$, Akaike Information Criteria (AIC), corrected Akaike information criterion (CAIC), Cramer von Mises (W^*) and Anderson-Darling (A^*) goodness of-fit statistics for all distribution models. We note that The AIC and CAIC are by given by $AIC = -2\widehat{\ell} + 2p$ and $CAIC = -2\widehat{\ell} + 2pn\left(n-k-1\right)^{-1}$, where p is the number of the estimated model parameters and n is sample size. The W^* and A^* statistics have been described as

$$W^* = \sum_{i=1}^{n} \left(\hat{F}\left(x_{(i)}\right) - \frac{i-0.5}{n} \right)^2 + \frac{1}{12n}$$

and

$$A^* = -\sum_{i=1}^{n} \frac{2i-1}{n} \left[\ln \hat{F}\left(x_{(i)}\right) + \ln \hat{\bar{F}}\left(x_{(n+1-i)}\right) \right] - n$$

by Evans et al. (2008). Also, one may see Chen and Balakrishnan (1995) for W^* and A^* in detail. It can be seen as the best model which has the smaller the values of the AIC, CAIC, W^* and A^* statistics and the larger the values of $\hat{\ell}$. The real data set is the stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all had failed. This data set was studied by Andrews and Herzberg (1985), Cooray and Ananda (2008), and Paraniaba et al. (2013).

In the applications, the information about the hazard shape can help in selecting a particular model. For this aim, a device called the total time on test (TTT) plot (Aarset, 1987) is useful. The TTT plot is obtained by plotting $T\left(\frac{r}{n}\right)$ against r/n where $T\left(\frac{r}{n}\right) =$

$\left[\sum_{i=1}^{n} y_{(i)} + (-r+n)\, y_{(r)}\right] / \sum_{i=1}^{n} y_{(i)} \;|_{(r=1,\ldots,n)}$ and y_i are the order statistics of the sample. It is convex shape for decreasing hrf and is concave shape for increasing hrf. The TTT plot for the kevlar data in Figure 4 deals with convex-concave-convex shaped. That is it has a firstly bathtub-shaped then decreasing shaped on the other words down-and-up shaped failure rate function. The MLEs

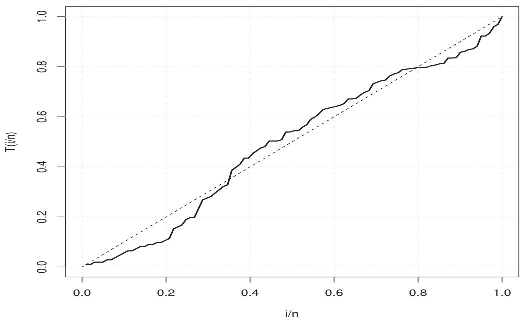


FIGURE 4. TTT plot for the kevlar data

of all models parameters, their standard erros, AIC, CAIC, W^* and A^* statistics are listed in Table 2 for data set. Table 2 shows that the QXGP model could be chosen as the best model among the fitted models under the AIC, CAIC, HQIC, and W^* statistics. We note that to show the likelihood equations have a unique solution, we plot the profiles of the **LL** of α, θ and λ in Figure 5. The WP model is better than QXGP model according to A^* statistics. In this case, the WP model can be choose as the best model.

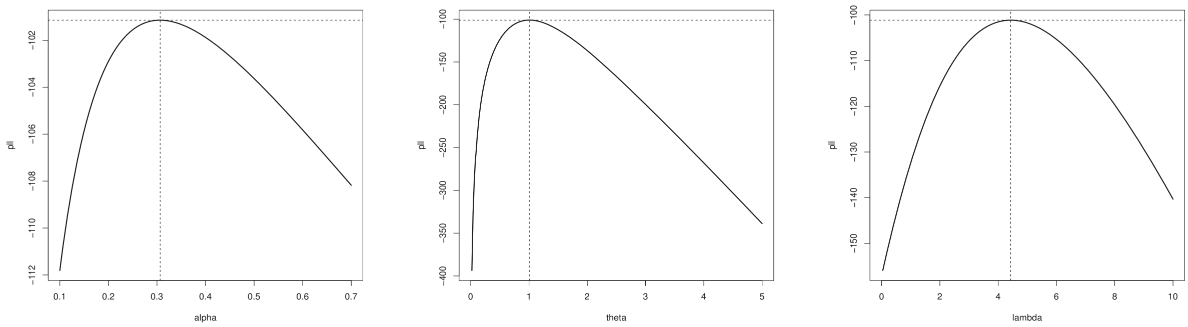


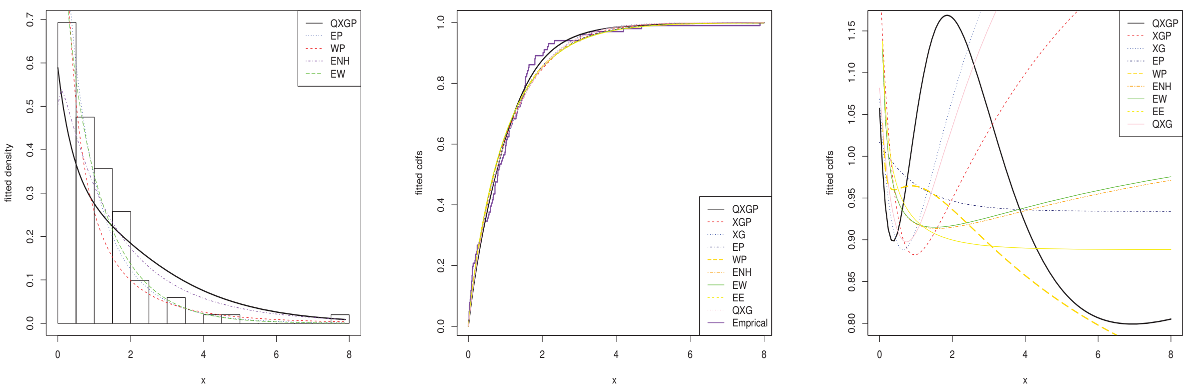
FIGURE 5. The profile of the **LL** function plots

The plots of the fitted densities, cdfs and hrfs of all models are displayed in Figure 6. These plots also shows that the QXGP model provides the good fit to these data compared to the other models. The fitted hrf shape both QXGP and WP models have firstly bathtub-shaped then decreasing shaped (convex-concave-convex).

A comparison of the proposed distribution with some of its sub-models using LR statistics is performed in Table 3. Table 3 shows that QXGP model provides a better representation of the data than the their sub-model based on the LR test at the 6% significance level. Hence, we reject all H_0 hypotheses in favour of the QXGP model.

TABLE 2. MLEs, standard erros of the estimates (in parentheses), $\hat{\ell}$, AIC, $CAIC$, A^* and W^* statistics for the applications models

Model	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$	$-\hat{\ell}$	AIC	$CAIC$	A^*	W^*
QXGP	0.3065	1.0051	4.4307	101.1425	208.2849	208.5324	0.9527	0.1168
	(0.1099)	(0.2652)	(1.7112)					
QXG	1.9343	1.6408		104.0904	212.1807	212.1807	1.0947	0.1397
	(2.0215)	(0.4662)						
XGP		1.4839	0.6804	103.7876	211.5751	211.6976	1.0761	0.1673
		(0.3296)	(0.9887)					
XG		1.6978		104.1007	210.2015	210.2419	1.0916	0.1322
		(0.1248)						
EP	0.9340		0.1720	103.4497	210.8994	211.0218	1.2332	0.1742
	(0.1963)		(0.7079)					
WP	0.8059	1.4042	-1.2719	102.3688	210.7376	210.9851	0.9303	0.1514
	(0.1273)	(0.3703)	(1.0984)					
ENH	1.0717	0.7860	0.8473	102.7904	211.5808	211.82834	0.9633	0.1668
	(0.3093)	(0.4094)	(0.1308)					
EW	0.7929	0.8210	1.0604	102.7872	211.5743	211.8218	0.9554	0.1648
	(0.2873)	(0.2654)	(0.2400)					
EE	0.8660		0.8883	102.8200	209.6369	209.7624	1.0215	0.1812
	(0.1097)		(0.1201)					



(a) Fitted pdfs for data set (b) Fitted cdfs for data set (c) Fitted hrfs for data set

FIGURE 6. Fitted pdfs, cdfs and hrfs for data set

TABLE 3. LR statistics for data set

Model	Hypothesis	Test statistics	p-value
QXGP vs XGP	$H_0 : \alpha = \theta$ & $H_1 : H_0$ false	5.2902	0.0214
QXGP vs QXG	$H_0 : \lambda = 0$ & $H_1 : H_0$ false	5.8958	0.0151
QXGP vs XG	$H_0 : \alpha = \theta, \lambda = 0$ & $H_1 : H_0$ false	5.9164	0.0520

7. Conclusions

In this paper, we propose a new three-parameter xgamma-Poisson model, called the quasi xgamma-Poisson (QXGP) distribution, which extends the xgamma-Poisson (XGP), QXG and xgamma distributions . In fact, the QXGP model is motivated by the wide use of the xgamma distribution in many applied areas and also for the fact that the new generalization provides more flexibility to analyze real data. We discuss the MLE of the model parameters. An applications illustrate that the proposed model provides consistently better fit than the other competitive models like QXG, XGP, XG, EW, EE, EP, WP and ENH models.

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VOLUME/CİLT 11 NUMBER/SAYI 3 DECEMBER/ARALIK 2018

İSTATİSTİK

JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION

TÜRK İSTATİSTİK DERNEĞİ DERGİSİ

ISSN: 1300-4077