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The Fischer-Clifford matrices and character table of the split extension $2^6{:}S_8$

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Jamshid Moori†

Abstract

The sporadic simple group Fi_{22} is generated by a conjugacy class D of 3510 Fischer's 3-transpositions. In Fi_{22} there are 14 classes of maximal subgroups up to conjugacy as listed in the $\mathbb{ATLAS}$ [10] and Wilson [31]. The group $E = 2^6{:}Sp_6(2)$ is maximal subgroup of Fi_{22} of index 694980. In the present article we compute the Fischer-Clifford matrices and hence character table of a subgroup of the smallest Fischer group Fi_{22} of the form $2^6{:}S_8$ which sits maximally in E. The computations were carried out using the computer algebra systems $\mathbb{MAGMA}$ [9] and $\mathbb{GAP}$ [29].

Keywords: Fischer-Clifford matrix, extension, Fischer group Fi_{22}.

2000 AMS Classification: 20C15, 20D08.

1. Introduction

In recent years there has been considerable interest in the *Fischer-Clifford theory* for both split and non-split group extensions. Character tables for many maximal subgroups of the sporadic simple groups were computed using this technique. See for instance [1, 3, 4, 5, 7, 6], [11], [12], [16], [19], [20], [22, 23, 24] and [28]. In the present article we follow a similar approach as used in [1, 3, 4, 5, 7], [22] and [24] to compute the Fischer-Clifford matrices and character tables for many group extension.

Let $\bar{G} = N{:}G$ be the split extension of $N = 2^6$ by $G = S_8$ where N is the vector space of dimension 6 over $GF(2)$ on which G acts naturally. Let $E = 2^6{:}Sp_6(2)$ be a maximal subgroup of Fi_{22}. The group $\bar{G}$ sits maximally inside the group E. In the present article we aim to construct the character table of $\bar{G}$ by using the technique of *Fischer-Clifford matrices*. The character table of $\bar{G}$ can be constructed by using the Fischer-Clifford matrix $M(g)$ for each class representative g of G and

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the character tables of H_i's which are the inertia factor groups of the inertia groups $\bar{H}_i = 2^6{:}H_i$. We use the properties of the Fischer-Clifford matrices discussed in [1], [2], [3], [4], [5] and [22] to compute entries of these matrices.

The Fischer-Clifford matrix $M(g)$ will be partioned row-wise into blocks, where each block corresponds to an inertia group $\bar{H}_i$. Now using the columns of character table of the inertia factor H_i of $\bar{H}_i$ which correspond to the classes of H_i which fuse to the class $[g]$ in G and multiply these columns by the rows of the Fischer-Clifford matrix $M(g)$ that correspond to $\bar{H}_i$. In this way we construct the portion of the character table of $\bar{G}$ which is in the block corresponding to $\bar{H}_i$ for the classes of $\bar{G}$ that come from the coset Ng. For detailed information about this technique the reader is encouraged to consult [1], [3], [4], [5], [16] and [22].

We first use the method of coset analysis to determine the conjugacy classes of $\bar{G}$. For detailed information about the coset analysis method, the reader is referred to again [1], [4], [5] and [22]. The complete fusion of $\bar{G}$ into Fi_{22} will be fully determined.

The character table of $\bar{G}$ will be divided row-wise into blocks where each block corresponds to an inertia group $\bar{H}_i = N{:}H_i$. The computations have been carried out with the aid of computer algebra systems $\mathbb{MAGMA}$ [9] and $\mathbb{GAP}$ [29]. We follow the notation of $\mathbb{ATLAS}$ [10] for the conjugacy classes of the groups and permutation characters. For more information on character theory, see [15] and [17].

Recently, the representation theory of Hecke algebras of the generalized symmetric groups has received some special attention [8], and the computation of the Fischer-Clifford matrices in this context is also of some interest.

2. The Conjugacy Classes of $2^6{:}S_8$

The group S_8 is a maximal subgroup of $Sp_6(2)$ of index 36. From the conjugacy classes of $Sp_6(2)$, obtained using $\mathbb{MAGMA}$ [9], we generated S_8 by two elements α and β of $Sp_6(2)$ which are given by

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

where $o(\alpha) = 2$ and $o(\beta) = 7$.

Using $\mathbb{MAGMA}$, we compute the conjugacy classes of S_8 and observed that S_8 has 22 conjugacy classes of its elements. The action of S_8 on 2^6 gives rise to three orbits of lengths 1, 28 and 35 with corresponding point stabilizers S_8, $S_6 \times 2$ and $(S_4 \times S_4){:}2$ respectively. Let ϕ_1 and ϕ_2 be the permutation characters of S_8 of degrees 28 and 35. Then from $\mathbb{ATLAS}$ [10], we obtained that $\chi_{\phi_1} = 1a + 7a + 20a$ and $\chi_{\phi_2} = 1a + 14a + 20a$.

Suppose $\chi = \chi(S_8|2^6)$ is the permutation character of S_8 on 2^6. Then we obtain that

$$\chi = 1a + 1_{S_6\times 2}^{S_8} + 1_{(S_4\times S_4){:}2}^{S_8} = 3 \times 1a + 7a + 14a + 2 \times 20a,$$

where $1_{S_6\times 2}^{S_8}$ and $1_{(S_4\times S_4){:}2}^{S_8}$ are the characters of S_8 induced from identity characters of $S_6{\times}2$ and $(S_4{\times}S_4){:}2$ respectively. For each class representative $g \in S_8$, we calculate $k = \chi(S_8|2^6)(g)$, which is equal to the number of fixed points of g in 2^6. We list these values in the following table:

$[g]_{S_8}$	$1A$	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$4A$	$4B$	$4C$	$4D$
χ_{ϕ_1}	28	16	8	4	4	10	1	6	2	0	2
χ_{ϕ_2}	35	15	7	11	3	5	2	1	5	3	1
k	64	32	16	16	8	16	4	8	8	4	4
$[g]_{S_8}$	$5A$	$6A$	$6B$	$6C$	$6D$	$6E$	$7A$	$8A$	$10A$	$12A$	$15A$
χ_{ϕ_1}	3	1	4	2	1	1	0	0	1	0	0
χ_{ϕ_2}	0	0	3	1	0	2	0	1	0	1	0
k	4	2	8	4	2	4	1	2	2	2	1

We use the method of coset analysis, developed for computing the conjugacy classes of group extensions, to obtain the conjugacy classes of $2^6{:}S_8$. For detailed information and background material relating to coset analysis and the description of the parameters f_j, we encourage the readers to consult once again [1], [4], [5] and [22].

Now having obtained the values of the k's for each class representative $g \in S_8$, we use a computer programme for $2^6{:}S_8$ (see Programme A in [1]) written for $\mathbb{MAGMA}$ [9] to find the values of f_j's corresponding to these k's. From the programme output, we calculate the number f_j of orbits Q_i's ($1 \leq i \leq k$) of the action of $N = 2^6$ on Ng, which have come together under the action of $C_{S_8}(g)$ for each class representative $g \in S_8$. We deduce that altogether we have 64 conjugacy classes of the elements of $\bar{G} = 2^6{:}S_8$, which we list in Table 1. We also list the order of $C_{\bar{G}}(x)$ for each $[x]_{\bar{G}}$ in the last column of Table 1.

Table 1: The conjugacy classes of $\bar{G} = 2^6{:}S_8$

$[g]_{S_8}$	k	f_j	$[x]_{2^6:S_8}$	$\|[x]_{2^6:S_8}\|$	$\|C_{2^6:S_8}(x)\|$
$1A$	64	$f_1 = 1$	$1A$	1	2580480
		$f_2 = 28$	$2A$	28	92160
		$f_3 = 35$	$2B$	35	73728
$2A$	32	$f_1 = 1$	$2C$	56	46080
		$f_2 = 6$	$4A$	336	7680
		$f_3 = 10$	$4B$	560	4608
		$f_4 = 15$	$2D$	840	3072
$2B$	16	$f_1 = 1$	$2E$	420	6144
		$f_2 = 1$	$2F$	420	6144
		$f_3 = 2$	$2G$	840	3072
		$f_4 = 12$	$4C$	5040	512
$2C$	16	$f_1 = 1$	$2H$	840	3072
		$f_2 = 1$	$4D$	840	3072
		$f_3 = 3$	$2I$	2520	1024
		$f_4 = 3$	$4E$	2520	1024
		$f_5 = 8$	$4F$	6720	384
$2D$	8	$f_1 = 1$	$2J$	3360	768
		$f_2 = 1$	$4G$	3360	768
		$f_3 = 3$	$4H$	10080	256
		$f_4 = 3$	$4I$	10080	256
$3A$	16	$f_1 = 1$	$3A$	448	5760
		$f_2 = 5$	$6A$	2240	1152
		$f_3 = 10$	$6B$	4480	576

Table 1: The conjugacy classes of $\bar{G}$ (continued)

$[g]_{S_8}$	k	f_j	$[x]_{2^6:S_8}$	$\|[x]_{2^6:S_8}\|$	$\|C_{2^6:S_8}(x)\|$
$3B$	4	$f_1 = 1$	$3B$	17920	144
		$f_2 = 1$	$6C$	17920	144
		$f_3 = 2$	$6D$	35840	72
$4A$	8	$f_1 = 1$	$4J$	3360	768
		$f_2 = 3$	$4K$	10080	256
		$f_3 = 4$	$8A$	13440	192
$4B$	8	$f_1 = 1$	$4L$	10080	256
		$f_2 = 1$	$4M$	10080	256
		$f_3 = 2$	$4N$	20160	128
		$f_4 = 4$	$8B$	40320	64
$4C$	4	$f_1 = 1$	$4O$	20160	128
		$f_2 = 1$	$4P$	20160	128
		$f_3 = 2$	$4Q$	40320	64
$4D$	4	$f_1 = 1$	$4R$	40320	64
		$f_2 = 1$	$8C$	40320	64
		$f_3 = 1$	$8D$	40320	64
		$f_4 = 1$	$4S$	40320	64
$5A$	4	$f_1 = 1$	$5A$	21504	120
		$f_2 = 3$	$10A$	64512	40
$6A$	2	$f_1 = 1$	$6E$	35840	72
		$f_2 = 1$	$12A$	35840	72
$6B$	8	$f_1 = 1$	$6F$	8960	288
		$f_2 = 1$	$12B$	8960	288
		$f_3 = 3$	$12C$	26880	96
		$f_4 = 3$	$6G$	26880	96
$6C$	4	$f_1 = 1$	$6H$	26880	96
		$f_2 = 1$	$12D$	26880	96
		$f_3 = 2$	$12E$	53760	48
$6D$	2	$f_1 = 1$	$6I$	107520	24
		$f_2 = 1$	$12F$	107520	24
$6E$	4	$f_1 = 1$	$6J$	53760	48
		$f_2 = 1$	$6K$	53760	48
		$f_3 = 2$	$6L$	107520	24
$7A$	1	$f_1 = 1$	$7A$	368640	7
$8A$	2	$f_1 = 1$	$8E$	161280	16
		$f_2 = 1$	$8F$	161280	16
$10A$	2	$f_1 = 1$	$10B$	129024	20
		$f_2 = 1$	$20A$	129024	20
$12A$	2	$f_1 = 1$	$12G$	107520	24
		$f_2 = 1$	$24A$	107520	24
$15A$	1	$f_1 = 1$	$15A$	172032	15

3. The Inertia Groups of $\bar{G}$

The action of G on N produces three orbits of lengths 1, 28 and 35. Hence by Brauer's theorem (see Lemma 4.5.2 of [14]) G acting on $Irr(N)$ will also produce three orbits of lengths 1, s and t such that $s + t = 63$. From $\mathbb{ATLAS}$, by checking the indices of maximal subgroups of S_8, we can see that the only possibility is that $s = 28$ and $t = 35$. We deduce that the three inertia groups are $\bar{H}_i = 2^6{:}H_i$ of indices 1, 28 and 35 in $\bar{G}$ respectively where $i \in \{1, 2, 3\}$ and $H_i \leq S_8$ are the inertia factors. We also observe that $H_1 = S_8$, $H_2 = S_6{\times}2$ and $H_3 = (S_4{\times}S_4){:}2$.

The character tables and power maps of the elements of H_1, H_2 and H_3 are given in the $\mathbb{GAP}$ [29]. Using the permutation characters of S_8 on H_2 and H_3 of degrees 28 and 35 respectively we are able to obtain partial fusions of H_2 and H_3 into $H_1 = S_8$. We completed the fusions by using direct matrix conjugation in S_8. The complete fusion of H_2 and H_3 into H_1 are given in Tables 2 and 3 respectively.

Table 2: The fusion of H_2 into H_1

$[g]_{S_6\times 2}$	$\longrightarrow$	$[h]_{S_8}$	$[g]_{S_6\times 2}$	$\longrightarrow$	$[h]_{S_8}$
$1A$		$1A$	$2A$		$2A$
$2B$		$2A$	$2C$		$2D$
$2D$		$2B$	$2E$		$2C$
$2F$		$2C$	$2G$		$2D$
$3A$		$3A$	$3B$		$3B$
$4A$		$4D$	$4B$		$4A$
$4C$		$4B$	$4D$		$4D$
$5A$		$5A$	$6A$		$6B$
$6B$		$6A$	$6C$		$6B$
$6D$		$6E$	$6E$		$6D$
$6F$		$6C$	$10A$		$10A$

Table 3: The fusion of H_3 into H_1

$[g]_{S_4\times S_4}$	$\longrightarrow$	$[h]_{S_8}$	$[g]_{S_4\times S_4}$	$\longrightarrow$	$[h]_{S_8}$
$1A$		$1A$	$2A$		$2C$
$2B$		$2B$	$2C$		$2A$
$2D$		$2B$	$2E$		$2C$
$2F$		$2D$	$3A$		$3A$
$3B$		$3B$	$4A$		$4A$
$4B$		$4C$	$4C$		$4B$
$4D$		$4C$	$4E$		$4D$
$4F$		$4B$	$6A$		$6C$
$6B$		$6B$	$6C$		$6E$
$8A$		$8A$	$12A$		$12A$

4. The Fischer-Clifford Matrices of $\bar{G}$

For each conjugacy class $[g]$ of G with representative $g \in G$, we construct the corresponding Fischer-Clifford matrix $M(g)$ of $\bar{G} = 2^6{:}S_8$. We use properties of the Fischer-Clifford matrices (see [1], [3], [4], [5], [22]) together with fusions of H_2 and H_3 into H_1 (Tables 2 and 3) to compute the entries of the these matrices. The Fischer-Clifford matrix $M(g)$ will be partitioned row-wise into blocks, where each block corresponds to an inertia group $\bar{H}_i$. We list the Fischer-Clifford matrices of $\bar{G}$ in Table 4.

Table 4: The Fischer-Clifford matrices of $\bar{G}$

$M(g)$	$M(g)$	$M(g)$
$M(1A) = \begin{pmatrix} 1 & 1 & 1 \\ 28 & 4 & -4 \\ 35 & -5 & 3 \end{pmatrix}$	$M(2A) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 15 & 5 & -3 & -1 \\ 15 & -5 & 3 & -1 \end{pmatrix}$	$M(2B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 4 & -4 & 0 \\ 3 & 3 & 3 & -1 \\ 8 & -8 & 0 & 0 \end{pmatrix}$
$M(2C) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & -2 & -2 & 2 & 0 \\ 6 & 6 & -2 & -2 & 0 \\ 1 & 1 & 1 & 1 & -1 \\ 6 & -6 & 2 & -2 & 0 \end{pmatrix}$	$M(2D) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 3 & -3 & 1 & -1 \\ 3 & 3 & -1 & -1 \end{pmatrix}$	$M(3A) = \begin{pmatrix} 1 & 1 & 1 \\ 10 & 2 & -2 \\ 5 & -3 & 1 \end{pmatrix}$
$M(3B) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}$	$M(4A) = \begin{pmatrix} 1 & 1 & 1 \\ 6 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$	$M(4B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & -2 & 0 \\ 1 & 1 & 1 & -1 \\ 4 & -4 & 0 & 0 \end{pmatrix}$
$M(4C) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}$	$M(4D) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$	$M(5A) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$
$M(6A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(6B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 3 & 3 & -1 & -1 \\ 3 & -3 & 1 & -1 \end{pmatrix}$	$M(6C) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$
$M(6D) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(6E) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}$	$M(7A) = \begin{pmatrix} 1 \end{pmatrix}$
$M(8A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(10A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(12A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$M(15A) = \begin{pmatrix} 1 \end{pmatrix}$		

We use the above Fischer-Clifford matrices (Table 4) and the character tables of inertia factor groups $H_1 = S_8$, H_2 and H_3, together with the fusion of H_2 and H_3 into S_8, to obtain the character table of $\bar{G}$. The set of irreducible characters of $\bar{G} = 2^6{:}S_8$ will be partitioned into three blocks B_1, B_2 and B_3 corresponding to the inertia factors H_1, H_2 and H_3 respectively. In fact $B_1 = \{\chi_i \mid 1 \leq i \leq 22\}$, $B_2 = \{\chi_i \mid 23 \leq i \leq 44\}$ and $B_3 = \{\chi_i \mid 45 \leq i \leq 64\}$, where $Irr(2^6{:}S_8) = \bigcup_{i=1}^{3} B_i$. The character table of $\bar{G}$ is displayed in Table 5. Note that the centralizers of the elements of $\bar{G}$ are listed in the last column of Table 1.

The character table of $\bar{G} = 2^6{:}S_8$, which we computed in this paper and displayed in Table 5, has been incorporated into and available in the latest version of $\mathbb{GAP}$ [29] as well.

Table 5: The character table of $\bar{G}$

$[g]_{S_8}$	$1A$			$2A$				$2B$				$2C$				
$[x]_{2^6:S_8}$	$1A$	$2A$	$2B$	$2C$	$4A$	$4B$	$2D$	$2E$	$2F$	$2G$	$4C$	$2H$	$4D$	$2I$	$4E$	$4F$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1
χ_3	7	7	7	5	5	5	5	-1	-1	-1	-1	3	3	3	3	3
χ_4	7	7	7	-5	-5	-5	-5	-1	-1	-1	-1	3	3	3	3	3
χ_5	14	14	14	4	4	4	4	4	6	6	6	6	2	2	2	2
χ_6	14	14	14	-4	-4	-4	-4	4	6	6	6	6	2	2	2	2
χ_7	20	20	20	10	10	10	10	4	4	4	4	4	4	4	4	4
χ_8	20	20	20	-10	-10	-10	-10	4	4	4	4	4	4	4	4	4
χ_9	21	21	21	9	9	9	9	-3	-3	-3	-3	1	1	1	1	1
χ_{10}	21	21	21	-9	-9	-9	-9	-3	-3	-3	-3	1	1	1	1	1
χ_{11}	42	42	42	0	0	0	0	-6	-6	-6	-6	2	2	2	2	2
χ_{12}	28	28	28	10	10	10	10	-4	-4	-4	-4	4	4	4	4	4
χ_{13}	28	28	28	-10	-10	-10	-10	-4	-4	-4	-4	4	4	4	4	4
χ_{14}	35	35	35	5	5	5	5	3	3	3	3	-5	-5	-5	-5	-5
χ_{15}	35	35	35	-5	-5	-5	-5	3	3	3	3	-5	-5	-5	-5	-5
χ_{16}	90	90	90	0	0	0	0	-6	-6	-6	-6	-6	-6	-6	-6	-6
χ_{17}	56	56	56	4	4	4	4	8	8	8	8	0	0	0	0	0
χ_{18}	56	56	56	-4	-4	-4	-4	8	8	8	8	0	0	0	0	0
χ_{19}	64	64	64	16	16	16	16	0	0	0	0	0	0	0	0	0
χ_{20}	64	64	64	-16	-16	-16	-16	0	0	0	0	0	0	0	0	0
χ_{21}	70	70	70	10	10	10	10	-2	-2	-2	-2	2	2	2	2	2
χ_{22}	70	70	70	-10	-10	-10	-10	-2	-2	-2	-2	2	2	2	2	2
χ_{23}	28	4	-4	16	4	-4	0	4	4	-4	0	8	4	0	-4	0
χ_{24}	28	4	-4	14	6	-2	-2	-4	-4	4	0	4	8	-4	0	0
χ_{25}	28	4	-4	-16	-4	4	0	4	4	-4	0	8	4	0	-4	0
χ_{26}	28	4	-4	-14	-6	2	2	-4	-4	4	0	4	8	-4	0	0
χ_{27}	140	20	-20	-40	-20	4	8	4	4	-4	0	0	12	-8	4	0
χ_{28}	140	20	-20	40	20	-4	-8	4	4	-4	0	0	12	-8	4	0
χ_{29}	140	20	-20	50	10	-14	2	-4	-4	4	0	12	0	4	-8	0
χ_{30}	140	20	-20	-50	-10	14	-2	-4	-4	4	0	12	0	4	-8	0
χ_{31}	140	20	-20	20	0	-8	4	-12	-12	12	0	8	4	0	-4	0
χ_{32}	140	20	-20	-20	0	8	-4	-12	-12	12	0	8	4	0	-4	0

Table 5: The character table of $\bar{G}$ (continued)

$[g]_{S_8}$	$2D$				$3A$			$3B$			$4A$			$4B$			
$[x]_{2^6:S_8}$	$2J$	$4G$	$4H$	$4I$	$3A$	$6A$	$6B$	$3B$	$6C$	$6D$	$4J$	$4K$	$8A$	$4L$	$4M$	$4N$	$8B$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	-1	-1	1	1	1	1	1	1	-4	-4	-4	-4	-4	-4	-4
χ_3	1	1	1	1	4	4	4	1	1	1	3	3	3	-1	-1	-1	-1
χ_4	-1	-1	-1	-1	4	4	4	1	1	1	-3	-3	-3	1	1	1	1
χ_5	0	0	0	0	-1	-1	-1	2	2	2	-2	-2	-2	2	2	2	2
χ_6	0	0	0	0	-1	-1	-1	2	2	2	2	2	2	-2	-2	-2	-2
χ_7	2	2	2	2	5	5	5	-1	-1	-1	2	2	2	2	2	2	2
χ_8	-2	-2	-2	-2	5	5	5	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2
χ_9	-3	-3	-3	-3	6	6	6	0	0	0	3	3	3	-1	-1	-1	-1
χ_{10}	3	3	3	3	6	6	6	0	0	0	-3	-3	-3	1	1	1	1
χ_{11}	0	0	0	0	-6	-6	-6	0	0	0	0	0	0	0	0	0	0
χ_{12}	2	2	2	2	1	1	1	1	1	1	-2	-2	-2	-2	-2	-2	-2
χ_{13}	-2	-2	-2	-2	1	1	1	1	1	1	2	2	2	2	2	2	2
χ_{14}	-3	-3	-3	-3	5	5	5	2	2	2	1	1	1	1	1	1	1
χ_{15}	3	3	3	3	5	5	5	2	2	2	-1	-1	-1	-1	-1	-1	-1
χ_{16}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{17}	4	4	4	4	-4	-4	-4	-1	-1	-1	0	0	0	0	0	0	0
χ_{18}	-4	-4	-4	-4	-4	-4	-4	-1	-1	-1	0	0	0	0	0	0	0
χ_{19}	0	0	0	0	4	4	4	-2	-2	-2	0	0	0	0	0	0	0
χ_{20}	0	0	0	0	4	4	4	-2	-2	-2	0	0	0	0	0	0	0
χ_{21}	-2	-2	-2	-2	-5	-5	-5	1	1	1	-4	-4	-4	0	0	0	0
χ_{22}	2	2	2	2	-5	-5	-5	1	1	1	4	4	4	0	0	0	0
χ_{23}	4	-4	0	0	10	2	-2	1	1	-1	6	-2	0	2	2	-2	0
χ_{24}	-2	2	-2	2	10	2	-2	1	1	-1	6	-2	0	-2	-2	2	0
χ_{25}	-4	4	0	0	10	2	-2	1	1	-1	-6	2	0	-2	-2	2	0
χ_{26}	2	-2	2	-2	10	2	-2	1	1	-1	-6	2	0	2	2	-2	0
χ_{27}	4	-4	0	0	20	4	-4	-1	-1	1	-6	2	0	-2	-2	2	0
χ_{28}	-4	4	0	0	20	4	-4	-1	-1	1	6	-2	0	2	2	-2	0
χ_{29}	2	-2	2	-2	20	4	-4	-1	-1	1	6	-2	0	-2	-2	2	0
χ_{30}	-2	2	-2	2	20	4	-4	-1	-1	1	-6	2	0	2	2	-2	0
χ_{31}	0	0	4	-4	-10	-2	2	2	2	-2	-6	2	0	-2	-2	2	0
χ_{32}	0	0	-4	4	-10	-2	2	2	2	-2	6	-2	0	2	2	-2	0

Table 5: The character table of $\bar{G}$ (continued)

$[g]_{S_8}$	$4C$			$4D$				$5A$		$6A$		$6B$			
$[x]_{2^6:S_{S_8}}$	$4O$	$4P$	$4Q$	$4R$	$8C$	$8D$	$4S$	$5A$	$10A$	$6E$	$12A$	$6F$	$12B$	$12C$	$6G$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ_3	-1	-1	-1	1	1	1	1	2	2	-1	-1	2	2	2	2
χ_4	-1	-1	-1	1	1	1	1	2	2	1	1	-2	-2	-2	-2
χ_5	2	2	2	0	0	0	0	-1	-1	-2	-2	1	1	1	1
χ_6	2	2	2	0	0	0	0	-1	-1	2	2	-1	-1	-1	-1
χ_7	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
χ_8	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1
χ_9	1	1	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0
χ_{10}	1	1	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0
χ_{11}	2	2	2	-2	-2	-2	-2	2	2	0	0	0	0	0	0
χ_{12}	0	0	0	0	0	0	0	-2	-2	1	1	1	1	1	1
χ_{13}	0	0	0	0	0	0	0	-2	-2	-1	-1	-1	-1	-1	-1
χ_{14}	-1	-1	-1	-1	-1	-1	-1	0	0	2	2	-1	-1	-1	-1
χ_{15}	-1	-1	-1	-1	-1	-1	-1	0	0	-2	-2	1	1	1	1
χ_{16}	2	2	2	2	2	2	2	0	0	0	0	0	0	0	0
χ_{17}	0	0	0	0	0	0	0	1	1	1	1	-2	-2	-2	-2
χ_{18}	0	0	0	0	0	0	0	1	1	-1	-1	2	2	2	2
χ_{19}	0	0	0	0	0	0	0	-1	-1	-2	-2	-2	-2	-2	-2
χ_{20}	0	0	0	0	0	0	0	-1	-1	2	2	2	2	2	2
χ_{21}	-2	-2	-2	0	0	0	0	0	0	1	1	1	1	1	1
χ_{22}	-2	-2	-2	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1
χ_{23}	0	0	0	2	0	0	-2	3	-1	1	-1	4	2	-2	0
χ_{24}	0	0	0	0	-2	2	0	3	-1	-1	1	2	4	0	-2
χ_{25}	0	0	0	2	0	0	-2	3	-1	-1	1	-4	-2	2	0
χ_{26}	0	0	0	0	-2	2	0	3	-1	1	-1	-2	-4	0	2
χ_{27}	0	0	0	-2	0	0	2	0	0	-1	1	2	-2	-2	2
χ_{28}	0	0	0	-2	0	0	2	0	0	1	-1	-2	2	2	-2
χ_{29}	0	0	0	0	2	-2	0	0	0	-1	1	2	-2	-2	2
χ_{30}	0	0	0	0	2	-2	0	0	0	1	-1	-2	2	2	-2
χ_{31}	0	0	0	-2	0	0	-2	0	0	2	-2	2	4	0	-2
χ_{32}	0	0	0	-2	0	0	-2	0	0	-2	2	-2	-4	0	2

Table 5: The character table of $\bar{G}$ (continued)

$[g]_{S_8}$	$6C$			$6D$		$6E$			$7A$	$8A$		$10A$		$12A$		$15A$
$[x]_{2^6:S_{S_8}}$	$6H$	$12D$	$12E$	$6I$	$12F$	$6J$	$6K$	$6L$	$7A$	$8E$	$8F$	$10B$	$20A$	$12G$	$24A$	$15A$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1	1
χ_3	0	0	0	1	1	-1	-1	-1	0	-1	-1	0	0	0	0	-1
χ_4	0	0	0	-1	-1	-1	-1	-1	0	1	1	0	0	0	0	-1
χ_5	-1	-1	-1	0	0	0	0	0	0	0	0	-1	-1	1	1	-1
χ_6	-1	-1	-1	0	0	0	0	0	0	0	0	1	1	-1	-1	-1
χ_7	1	1	1	-1	-1	1	1	1	-1	0	0	0	0	-1	-1	0
χ_8	1	1	1	1	1	1	1	1	-1	0	0	0	0	1	1	0
χ_9	-2	-2	-2	0	0	0	0	0	0	1	1	-1	-1	0	0	1
χ_{10}	-2	-2	-2	0	0	0	0	0	0	-1	-1	1	1	0	0	1
χ_{11}	2	2	2	0	0	0	0	0	0	0	0	0	0	0	0	-1
χ_{12}	1	1	1	-1	-1	-1	-1	-1	0	0	0	0	0	1	1	1
χ_{13}	1	1	1	1	1	-1	-1	-1	0	0	0	0	0	-1	-1	1
χ_{14}	1	1	1	0	0	0	0	0	0	-1	-1	0	0	1	1	0
χ_{15}	1	1	1	0	0	0	0	0	0	1	1	0	0	-1	-1	0
χ_{16}	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0
χ_{17}	0	0	0	1	1	-1	-1	-1	0	0	0	-1	-1	0	0	1
χ_{18}	0	0	0	-1	-1	-1	-1	-1	0	0	0	1	1	0	0	1
χ_{19}	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	-1
χ_{20}	0	0	0	0	0	0	0	0	1	0	0	-1	-1	0	0	-1
χ_{21}	-1	-1	-1	1	1	1	1	1	0	0	0	0	0	-1	-1	0
χ_{22}	-1	-1	-1	-1	-1	1	1	1	0	0	0	0	0	1	1	0
χ_{23}	2	-2	0	1	-1	1	1	-1	0	0	0	1	-1	0	0	0
χ_{24}	-2	2	0	1	-1	-1	-1	1	0	0	0	-1	1	0	0	0
χ_{25}	2	-2	0	-1	1	1	1	-1	0	0	0	-1	1	0	0	0
χ_{26}	-2	2	0	-1	1	-1	-1	1	0	0	0	1	-1	0	0	0
χ_{27}	0	0	0	1	-1	1	1	-1	0	0	0	0	0	0	0	0
χ_{28}	0	0	0	-1	1	1	1	-1	0	0	0	0	0	0	0	0
χ_{29}	0	0	0	-1	1	-1	-1	1	0	0	0	0	0	0	0	0
χ_{30}	0	0	0	1	-1	-1	-1	1	0	0	0	0	0	0	0	0
χ_{31}	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{32}	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5: The character table of $\bar{G}$ (continued)

$[g]_{S_8}$	$1A$			$2A$				$2B$				$2C$				
$[x]_{2^6:S_8}$	$1A$	$2A$	$2B$	$2C$	$4A$	$4B$	$2D$	$2E$	$2F$	$2G$	$4C$	$2H$	$4D$	$2I$	$4E$	$4F$
χ_{33}	140	20	-20	10	10	2	-6	12	12	-12	0	4	8	-4	0	0
χ_{34}	140	20	-20	-10	-10	-2	6	12	12	-12	0	4	8	-4	0	0
χ_{35}	452	36	-36	-36	-24	0	12	-12	-12	12	0	0	12	-8	4	0
χ_{36}	452	36	-36	36	24	0	-12	-12	-12	12	0	0	12	-8	4	0
χ_{37}	452	36	-36	54	6	-18	6	12	12	-12	0	12	0	4	-8	0
χ_{38}	452	36	-36	-54	-6	18	-6	12	12	-12	0	12	0	4	-8	0
χ_{39}	280	40	-40	40	0	-16	8	-8	-8	8	0	-8	-16	8	0	0
χ_{40}	280	40	-40	-40	0	16	-8	-8	-8	8	0	-8	-16	8	0	0
χ_{41}	280	40	-40	20	20	4	-12	8	8	-8	0	-16	-8	0	8	0
χ_{42}	280	40	-40	-20	-20	-4	12	8	8	-8	0	-16	-8	0	8	0
χ_{43}	448	64	-64	16	-16	-16	16	0	0	0	0	0	0	0	0	0
χ_{44}	448	64	-64	-16	16	16	-16	0	0	0	0	0	0	0	0	0
χ_{45}	35	-5	3	15	-5	3	-1	11	-5	3	-1	7	-5	-1	3	-1
χ_{46}	35	-5	3	-15	5	-3	1	-5	11	3	-1	7	-5	-1	3	-1
χ_{47}	35	-5	3	-15	5	-3	1	11	-5	3	-1	7	-5	-1	3	-1
χ_{48}	35	-5	3	15	-5	3	-1	-5	11	3	-1	7	-5	-1	3	-1
χ_{49}	70	-10	6	0	0	0	0	6	6	6	-2	-10	14	6	-2	-2
χ_{50}	140	-20	12	-30	10	-6	2	12	12	12	-4	4	4	4	4	-4
χ_{51}	140	-20	12	30	-10	6	-2	12	12	12	-4	4	4	4	4	-4
χ_{52}	140	-20	12	0	0	0	0	-4	28	12	-4	4	4	4	4	-4
χ_{53}	140	-20	12	0	0	0	0	28	-4	12	-4	4	4	4	4	-4
χ_{54}	210	-30	18	-30	10	-6	2	-6	-6	-6	2	-10	14	6	-2	-2
χ_{55}	210	-30	18	30	-10	6	-2	-6	-6	-6	2	-10	14	6	-2	-2
χ_{56}	210	-30	18	-60	20	-12	4	-6	-6	-6	2	14	-10	-2	6	-2
χ_{57}	210	-30	18	60	-20	12	-4	-6	-6	-6	2	14	-10	-2	6	-2
χ_{58}	315	-45	27	-45	15	-9	3	-21	27	3	-1	3	-9	-5	-1	3
χ_{59}	315	-45	27	-45	15	-9	3	27	-21	3	-1	3	-9	-5	-1	3
χ_{60}	315	-45	27	45	-15	9	-3	-21	27	3	-1	3	-9	-5	-1	3
χ_{61}	315	-45	27	45	-15	9	-3	27	-21	3	-1	3	-9	-5	-1	3
χ_{62}	420	-60	36	-30	10	-6	2	-12	-12	-12	4	4	4	4	4	-4
χ_{63}	420	-60	36	30	-10	6	-2	-12	-12	-12	4	4	4	4	4	-4
χ_{64}	630	-90	54	0	0	0	0	6	6	6	-2	-18	6	-2	-10	6

Table 5: The character table of $\bar{G}$ (continued)

$[g]_{S_8}$	$2D$				$3A$			$3B$			$4A$			$4B$			
$[x]_{2^6:S_8}$	$2J$	$4G$	$4H$	$4I$	$3A$	$6A$	$6B$	$3B$	$6C$	$6D$	$4J$	$4K$	$8A$	$4L$	$4M$	$4N$	$8B$
χ_{33}	-6	6	2	-2	-10	-2	2	2	2	-2	-6	2	0	2	2	-2	0
χ_{34}	6	-6	-2	2	-10	-2	2	2	2	-2	6	-2	0	-2	-2	2	0
χ_{35}	0	0	4	-4	0	0	0	0	0	0	6	-2	0	2	2	-2	0
χ_{36}	0	0	-4	4	0	0	0	0	0	0	-6	2	0	-2	-2	2	0
χ_{37}	6	-6	-2	2	0	0	0	0	0	0	-6	2	0	2	2	-2	0
χ_{38}	-6	6	2	-2	0	0	0	0	0	0	6	-2	0	-2	-2	2	0
χ_{39}	-8	8	0	0	10	2	-2	1	1	-1	0	0	0	0	0	0	0
χ_{40}	8	-8	0	0	10	2	-2	1	1	-1	0	0	0	0	0	0	0
χ_{41}	4	-4	4	-4	10	2	-2	1	1	-1	0	0	0	0	0	0	0
χ_{42}	-4	4	-4	4	10	2	-2	1	1	-1	0	0	0	0	0	0	0
χ_{43}	0	0	0	0	-20	-4	4	-2	-2	2	0	0	0	0	0	0	0
χ_{44}	0	0	0	0	-20	-4	4	-2	-2	2	0	0	0	0	0	0	0
χ_{45}	3	3	-1	-1	5	-3	1	2	-2	0	1	1	-1	5	-3	1	-1
χ_{46}	-3	-3	1	1	5	-3	1	2	-2	0	-1	-1	1	3	-5	-1	1
χ_{47}	-3	-3	1	1	5	-3	1	2	-2	0	-1	-1	1	-5	3	-1	1
χ_{48}	3	3	-1	-1	5	-3	1	2	-2	0	1	1	-1	-3	5	1	-1
χ_{49}	0	0	0	0	10	-6	2	4	-4	0	0	0	0	0	0	0	0
χ_{50}	-6	-6	2	2	5	-3	1	-4	4	0	-2	-2	2	-2	-2	-2	2
χ_{51}	6	6	-2	-2	5	-3	1	-4	4	0	2	2	-2	2	2	2	-2
χ_{52}	0	0	0	0	-10	6	-2	2	-2	0	0	0	0	0	0	0	0
χ_{53}	0	0	0	0	-10	6	-2	2	-2	0	0	0	0	0	0	0	0
χ_{54}	6	6	-2	-2	15	-9	3	0	0	0	-4	-4	4	0	0	0	0
χ_{55}	-6	-6	2	2	15	-9	3	0	0	0	4	4	-4	0	0	0	0
χ_{56}	0	0	0	0	15	-9	3	0	0	0	-2	-2	2	2	2	2	-2
χ_{57}	0	0	0	0	15	-9	3	0	0	0	2	2	-2	-2	-2	-2	2
χ_{58}	3	3	-1	-1	0	0	0	0	0	0	3	3	-3	3	-5	-1	1
χ_{59}	3	3	-1	-1	0	0	0	0	0	0	3	3	-3	-5	3	-1	1
χ_{60}	-3	-3	1	1	0	0	0	0	0	0	-3	-3	3	-3	5	1	-1
χ_{61}	-3	-3	1	1	0	0	0	0	0	0	-3	-3	3	5	-3	1	-1
χ_{62}	-6	-6	2	2	-15	9	-3	0	0	0	2	2	-2	2	2	2	-2
χ_{63}	6	6	-2	-2	-15	9	-3	0	0	0	-2	-2	2	-2	-2	-2	2
χ_{64}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5: The character table of $\bar{G}$ (continued)

$[g]_{S_8}$	$4C$			$4D$				$5A$		$6A$		$6B$			
$[x]_{2^6:S_{S_8}}$	$4O$	$4P$	$4Q$	$4R$	$8C$	$8D$	$4S$	$5A$	$10A$	$6E$	$12A$	$6F$	$12B$	$12C$	$6G$
χ_{33}	0	0	0	0	2	-2	0	0	0	-2	2	4	2	-2	0
χ_{34}	0	0	0	0	2	-2	0	0	0	2	-2	-4	-2	2	0
χ_{35}	0	0	0	2	0	0	-2	-3	1	0	0	0	0	0	0
χ_{36}	0	0	0	2	0	0	-2	-3	1	0	0	0	0	0	0
χ_{37}	0	0	0	0	-2	2	0	-3	1	0	0	0	0	0	0
χ_{38}	0	0	0	0	-2	2	0	-3	1	0	0	0	0	0	0
χ_{39}	0	0	0	0	0	0	0	0	0	1	-1	-2	-4	0	2
χ_{40}	0	0	0	0	0	0	0	0	0	-1	1	2	4	0	-2
χ_{41}	0	0	0	0	0	0	0	0	0	-1	1	-4	-2	2	0
χ_{42}	0	0	0	0	0	0	0	0	0	1	-1	4	2	-2	0
χ_{43}	0	0	0	0	0	0	0	3	-1	-2	2	-2	2	2	-2
χ_{44}	0	0	0	0	0	0	0	3	-1	2	-2	2	-2	-2	2
χ_{45}	3	-1	-1	1	-1	-1	1	0	0	0	0	3	-3	1	-1
χ_{46}	-1	3	-1	1	-1	-1	1	0	0	0	0	-3	3	-1	1
χ_{47}	3	-1	-1	1	-1	-1	1	0	0	0	0	-3	3	-1	1
χ_{48}	-1	3	-1	1	-1	-1	1	0	0	0	0	3	-3	1	-1
χ_{49}	-2	-2	2	-2	2	2	-2	0	0	0	0	0	0	0	0
χ_{50}	0	0	0	0	0	0	0	0	0	0	0	3	-3	1	-1
χ_{51}	0	0	0	0	0	0	0	0	0	0	0	-3	3	-1	1
χ_{52}	-4	4	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{53}	4	-4	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{54}	2	2	-2	0	0	0	0	0	0	0	0	3	-3	1	-1
χ_{55}	2	2	-2	0	0	0	0	0	0	0	0	-3	3	-1	1
χ_{56}	-2	-2	2	0	0	0	0	0	0	0	0	-3	3	-1	1
χ_{57}	-2	-2	2	0	0	0	0	0	0	0	0	3	-3	1	-1
χ_{58}	3	-1	-1	-1	1	1	-1	0	0	0	0	0	0	0	0
χ_{59}	-1	3	-1	-1	1	1	-1	0	0	0	0	0	0	0	0
χ_{60}	3	-1	-1	-1	1	1	-1	0	0	0	0	0	0	0	0
χ_{61}	-1	3	-1	-1	1	1	-1	0	0	0	0	0	0	0	0
χ_{62}	0	0	0	0	0	0	0	0	0	0	0	3	-3	1	-1
χ_{63}	0	0	0	0	0	0	0	0	0	0	0	-3	3	-1	1
χ_{64}	-2	-2	2	2	-2	-2	2	0	0	0	0	0	0	0	0

Table 5: The character table of $\bar{G}$ (continued)

$[g]_{S_8}$	$6C$			$6D$		$6E$			$7A$	$8A$		$10A$		$12A$		$15A$
$[x]_{2^6:S_{S_8}}$	$6H$	$12D$	$12E$	$6I$	$12F$	$6J$	$6K$	$6L$	$7A$	$8E$	$8F$	$10B$	$20A$	$12G$	$24A$	$15A$
χ_{33}	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{34}	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{35}	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
χ_{36}	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0
χ_{37}	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
χ_{38}	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0
χ_{39}	-2	2	0	1	-1	1	1	-1	0	0	0	0	0	0	0	0
χ_{40}	-2	2	0	-1	1	1	1	-1	0	0	0	0	0	0	0	0
χ_{41}	2	-2	0	1	-1	-1	-1	1	0	0	0	0	0	0	0	0
χ_{42}	2	-2	0	-1	1	-1	-1	1	0	0	0	0	0	0	0	0
χ_{43}	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0
χ_{44}	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
χ_{45}	1	1	-1	0	0	2	-2	0	0	1	-1	0	0	1	-1	0
χ_{46}	1	1	-1	0	0	-2	2	0	0	1	-1	0	0	-1	1	0
χ_{47}	1	1	-1	0	0	2	-2	0	0	-1	1	0	0	-1	1	0
χ_{48}	1	1	-1	0	0	-2	2	0	0	-1	1	0	0	1	-1	0
χ_{49}	2	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{50}	1	1	-1	0	0	0	0	0	0	0	0	0	0	1	-1	0
χ_{51}	1	1	-1	0	0	2	-2	0	0	1	-1	0	0	-1	1	0
χ_{52}	-2	-2	2	0	0	2	-2	0	0	0	0	0	0	0	0	0
χ_{53}	-2	-2	2	0	0	-2	2	0	0	0	0	0	0	0	0	0
χ_{54}	-1	-1	1	0	0	0	0	0	0	0	0	0	0	-1	1	0
χ_{55}	-1	-1	1	0	0	0	0	0	0	0	0	0	0	1	-1	0
χ_{56}	-1	-1	1	0	0	0	0	0	0	0	0	0	0	1	-1	0
χ_{57}	-1	-1	1	0	0	0	0	0	0	0	0	0	0	-1	1	0
χ_{58}	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0
χ_{59}	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0
χ_{60}	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0
χ_{61}	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0
χ_{62}	1	1	-1	0	0	0	0	0	0	0	0	0	0	-1	1	0
χ_{63}	1	1	-1	0	0	0	0	0	0	0	0	0	0	1	-1	0
χ_{64}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

5. The Fusion of $\bar{G}$ into Fi_{22}

We use the results of the conjugacy classes of $\bar{G}$ which are given in Section 2, to compute the power maps of the elements of $\bar{G}$ which we list in Table 6.

Table 6: The power maps of the elements of $\bar{G}$

$[g]_{S_8}$	$[x]_{2^6:S_8}$	2	3	5	7	$[g]_{S_8}$	$[x]_{2^6:S_8}$	2	3	5	7
1A	1A					2A	2C	1A			
	2A	1A					4A	2A			
	2B	1A					4B	2A			
							2D	1A			
2B	2E	1A				2C	2H	1A			
	2F	1A					4D	2B			
	2G	1A					2I	1A			
	4C	2B					4E	2B			
							4F	2A			
2D	2J	1A				3A	3A		1A		
	4G	2A					6A	3A	2B		
	4H	2A					6B	3A	2A		
	4I	2B									
3B	3B		1A			4A	4J	2H			
	6C	3B	2A				4K	2H			
	6D	3B	2B				8A	4D			
4B	4L	2H				4C	4O	2E			
	4M	2H					4P	2F			
	4N	2H					4Q	2G			
	8B	4E									
4D	4R	2H				5A	5A			1A	
	8C	4D					10A	5A		2A	
	8D	4E									
	4S	2I									
6A	6E	3B	2D			6B	6F	3A	2C		
	12A	6C	4B				12B	6B	4B		
							12C	6B	4A		
							6G	3A	2D		
6C	6H	3A	2H			6D	6I	3B	2J		
	12D	6A	4D				12F	6C	4G		
	12E	6B	4F								
6E	6J	3B	2E			7A	7A				1A
	6K	3B	2F								
	6L	3B	2G								
8A	8E	4O				10A	10B	5A		2C	
	8F	4P					20A	10A		4A	
12A	12G	6H	4J			15A	15A		5A	3A	
	24A	12D	8A								

Our group $\bar{G} = 2^6{:}S_8$ sits maximally inside the group $E = 2^6{:}Sp_6(2)$. Moori and Mpono in [22] computed the character table of E, which is also available in $\mathbb{GAP}$ [29]. The fusion of $\bar{G}$ into E will help us to determine the fusion of $\bar{G}$ into Fi_{22}. We give the fusion map of $\bar{G}$ into E in Table 7.

The power maps of Fi_{22} are given in the $\mathbb{ATLAS}$ and $\mathbb{GAP}$. In order to complete the fusion of $\bar{G}$ into Fi_{22} we sometimes use the technique of set intersection. For detailed information regarding the technique of set intersection we refer to [1], [4], [5], [21] and [25]. We give the complete list of class fusions of $\bar{G}$ into Fi_{22} in Table 8.

Table 7: The fusion of $\bar{G}$ into E

$[g]_{\bar{G}}$	$\longrightarrow$	$[h]_E$	$[g]_{\bar{G}}$	$\longrightarrow$	$[h]_E$	$[g]_{\bar{G}}$	$\longrightarrow$	$[h]_E$	$[g]_{\bar{G}}$	$\longrightarrow$	$[h]_E$
1A		1A	2A		2A	2B		2A	2C		2B
4A		4A	4B		4A	2D		2C	2E		2D
2F		2E	2G		2E	4C		4B	2H		2F
4D		4C	2I		2G	4E		4C	4F		4D
2J		2H	4G		4E	4H		4F	4I		4G
3A		3A	6A		6A	6B		6A	3B		3C
6C		6B	6D		6B	4J		4L	4K		4M
8A		8B	4L		4J	4M		4K	4N		4K
8B		8A	4O		4N	4P		4O	4Q		4P
4R		4Q	8C		8D	8D		8C	4S		4R
5A		5A	10A		10A	6E		6H	12A		12E
6F		6D	12B		12B	12C		12B	6G		6E
6H		6G	12D		12C	12E		12D	6I		6I
12F		12F	6J		6J	6K		6K	6L		6K
7A		7A	8E		8E	8F		8F	10B		10B
20A		20A	12G		12H	24A		24B	15A		15A

Table 8: The fusion of $\bar{G}$ into Fi_{22}

$[g]_{S_8}$	$[x]_{2^6:S_8}$	$\longrightarrow$	$[h]_{Fi_{22}}$	$[g]_{S_8}$	$[x]_{2^6:S_8}$	$\longrightarrow$	$[h]_{Fi_{22}}$
1A	1A		1A	2A	2C		2A
	2A		2B		4A		4B
	2B		2B		4B		4B
					2D		2C
2B	2E		2B	2C	2H		2B
	2F		2C		4D		4A
	2G		2B		2I		2C
	4C		4A		4E		4A
					4F		4E
2D	2J		2C	3A	3A		3A
	4G		4B		6A		6D
	4H		4E		6B		6D
	4I		4C				
3B	3B		3C	4A	4J		4B
	6C		6I		4K		4E
	6D		6I		8A		8A
4B	4L		4B	4C	4O		4A
	4M		4E		4P		4D
	4N		4B		4Q		4E
	8B		8B				
4D	4R		4E	5A	5A		5A
	8C		8B		10A		10B
	8D		8A				
	4S		4D				
6A	6E		6E	6B	6F		6A
	12A		12J		12B		12D
					12C		12D
					6G		6F
6C	6H		6D	6D	6I		6J
	12D		12B		12F		12J
	12E		12I				
6E	6J		6I	7A	7A		7A
	6K		6H				
	6L		6I				
8A	8E		8B	10A	10B		10A
	8F		8D		20A		20A
12A	12G		12D	15A	15A		15A
	24A		24B				

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Generalized uniformly close-to-convex functions of order γ and type β

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Abstract

In this paper, a class of analytic functions f defined on the open unit disc satisfying

$$\operatorname{Re}\left\{\frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)}\right\} > \beta\left|\frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)} - 1\right| + \gamma,$$

is studied, where $\beta \geq 0, \quad -1 \leq \gamma < 1, \quad \beta + \gamma \geq 0.$ and g is a certain analytic function associated with conic domains.
Among other results, inclusion relations and the coefficients bound are studied. Various known special cases of these results are pointed out. A subclass of uniformly quasi-convex functions is also studied.

Keywords: Univalent functions, uniformly close-to-convex, uniformly quasi-convex, fractional differential operator.

2000 AMS Classification: 30C45.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

analytic in the unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$, and let S denote the class of functions $f \in A$ which are univalent on E. Denote by $CV(\gamma), ST(\gamma), CC(\gamma)$, and $QC(\gamma)$, where $0 \leq \gamma < 1$, the well-known subclasses of S which are convex, starlike, close-to-convex and quasi-convex functions of order γ, respectively, and by CV, ST, CC, and QC, the corresponding classes when $\gamma = 0$.

Define the function $\varphi(a, c; z)$ by

$$\varphi(a, c; z) = z_2F_1(1, a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k-1}, c \neq 0, -1, -2, \ldots, z \in E,$$

where $(\sigma)_k$ is Pochhammer symbol defined in terms of Gamma function.

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Owa and Srivastava [18] introduced the operator $\Omega^\alpha : A \to A$ where

$$\Omega^\alpha f(z) = \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z), \quad \alpha \neq 2,3,\ldots$$

$$= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k, \tag{1.2}$$

$$= \varphi(2, 2-\alpha; z) * f(z). \tag{1.3}$$

Note that $\Omega^0 f(z) = f(z)$.

The linear fractional differential operator $D_\lambda^{n,\alpha} f : A \to A, \quad 0 \leq \alpha < 1, \quad \lambda \geq 0, \quad n \in N_0 = N \cup \{0\}$ is defined [5] as follows

$$D_\lambda^{n,\alpha} f(z) = z + \sum_{k=2}^{\infty} \psi_{k,n}(\alpha, \lambda) a_k z^k, \quad n \in N_0, \tag{1.4}$$

where

$$\psi_{k,n}(\alpha, \lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} (1 + \lambda(k-1)) \right]^n.$$

From (1.3), and (1.4), $D_\lambda^{n,\alpha} f(z)$ can be written, in terms of convolution, as

$$D_\lambda^{n,\alpha} f(z) = \underbrace{[\varphi(2, 2-\alpha; z) * h_\lambda(z) * \cdots * \varphi(2, 2-\alpha; z) * h_\lambda(z)]}_{n\text{-times}} * f(z), \tag{1.5}$$

where

$$h_\lambda(z) = \frac{z - (1-\lambda) z^2}{(1-z)^2} = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)] z^k.$$

Note that $D_\lambda^{n,0} = D_\lambda^n$ (Al-Oboudi differential operator [4]), $D_1^{n,0} = D^n$ (Salagean differential operator [23]) and $D_0^{1,\alpha} = \Omega^\alpha$ (Owa-Srivastava fractional differential operator [18]).

Using the operator $D_\lambda^{n,\alpha}$, the following classes are defined [5].

The classes $UCV_\lambda^{n,\alpha}(\beta, \gamma)$, $\beta \geq 0$, $-1 \leq \gamma < 1$, $\beta + \gamma \geq 0$, and $SP_\lambda^{n,\alpha}(\beta, \gamma)$, satisfying

$$f \in UCV_\lambda^{n,\alpha}(\beta, \gamma) \text{ if and only if } zf' \in SP_\lambda^{n,\alpha}(\beta, \gamma).$$

Note that $f \in UCV_\lambda^{n,\alpha}(\beta, \gamma)(SP_\lambda^{n,\alpha}(\beta, \gamma))$ if and only if $D_\lambda^{n,\alpha} f \in UCV(\beta, \gamma)(SP(\beta, \gamma))$, where $UCV(\beta, \gamma)$, is the class of uniformly convex functions of order β and type γ and $SP(\beta, \gamma)$, is the class of functions of conic domains and related with$UCV(\beta, \gamma)$ by Alexander-type relation [7].

These classes generalize various other classes investigated earlier by Goodman [9], Ronning [20], [21], Kanas and Wisniowska [10], [11] Srivastava and Mishra [26] and others. Several basic and interesting results have been studied for these classes [5], [6], such as inclusion relations, convolution properties, coefficient bounds, subordination results.

The class $UCC(\beta, \gamma)$, of uniformly close-to-convex functions of order γ and type β is defined [3] as

$$\text{Re}\left\{ \frac{zf'(z)}{g(z)} \right\} > \beta \left| \frac{zf'(z)}{g(z)} - 1 \right| + \gamma,$$

where $g \in SP(\beta,\gamma)$, $\beta \geq 0, -1 \leq \gamma < 1$, and $\beta+\gamma \geq 0$. It is clear that $UCC(0,\gamma) = CC(\gamma)$.

Since these functions are related to the uniformly convex functions UCV and with the class SP, they are called uniformly close-to-convex functions [8].

Denote by $UQC(\beta,\gamma)$, the class of uniformly quasi-convex functions of order γ and type β [3], where

$$f \in UQC(\beta,\gamma), \text{ if and only if } zf' \in UCC(\beta,\gamma).$$

Note that

$$UCV(\beta,\gamma) \subset UQC(\beta,\gamma) \subset UCC(\beta,\gamma).$$

The classes of uniformly close-to-convex and quasi-convex functions of order γ and type β had been studied by a number of authors under different operators, for example Acu [1], Acu and Blezu [2], Blezu [8], Kumar and Ramesha [13], Noor et al [16], Srivastava and Mishra [25] and Srivastava et al [26].

In the following, we use the operator $D_\lambda^{n,\alpha}$ to define generalized classes of uniformly close-to-convex functions and uniformly quasi-convex functions of order γ and type β.

1.1. Definition. A function $f \in A$ is in the class $UCC_\lambda^{n,\alpha}(\beta,\gamma)$ if and only if, there exist a function $g \in SP_\lambda^{n,\alpha}(\beta,\gamma)$ such that $z \in E$,

$$\text{(1.6)} \qquad \mathrm{Re}\left\{\frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)}\right\} > \beta\left|\frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)} - 1\right| + \gamma,$$

where $\beta \geq 0,\ -1 \leq \gamma < 1,\ \beta+\gamma \geq 0$. Note that $D_\lambda^{n,\alpha}f \in UCC(\beta,\gamma)$, and that $SP_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\alpha}(\beta,\gamma)$.

1.2. Definition. A function $f \in A$ is in the class $U_\lambda^{n,\alpha}QC(\beta,\gamma)$ if and only if, there exists a function $g \in UCV_\lambda^{n,\alpha}(\beta,\gamma)$ such that for $z \in E$,

$$\text{(1.7)} \qquad \mathrm{Re}\left\{\frac{(z(D_\lambda^{n,\alpha}f(z))')'}{(D_\lambda^{n,\alpha}g(z))'}\right\} > \beta\left|\frac{(z(D_\lambda^{n,\alpha}f(z))')'}{(D_\lambda^{n,\alpha}g(z)} - 1\right| + \gamma,$$

where $\beta \geq 0,\ -1 \leq \gamma < 1,\ \beta+\gamma \geq 0$. Note that $D_\lambda^{n,\alpha}f \in UQC(\beta,\gamma)$.

It is clear that

$$\text{(1.8)} \qquad f \in UQC_\lambda^{n,\alpha}(\beta,\gamma) \text{ if and only if } zf' \in UCC_\lambda^{n,\alpha}(\beta,\gamma),$$

and that

$$\text{(1.9)} \qquad UCV_\lambda^{n,\alpha}(\beta,\gamma) \subset UQC_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\alpha}(\beta,\gamma).$$

We may rewrite the condition (1.6)((1.7)), in the form

$$\text{(1.10)} \qquad p \prec P_{\beta,\gamma},$$

where $p(z) = \frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)} \left(\frac{(z(D_\lambda^{n,\alpha}f(z))')'}{D(_\lambda^{n,\alpha}g(z))'}\right)$ and the function $P_{\beta,\gamma}$ is given in [5].

By virtue of (1.6), (1.7) and the properties of the domain $R_{\beta,\gamma}$, we have respectively

$$\text{(1.11)} \qquad \mathrm{Re}\left\{\frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)}\right\} > \frac{\beta+\gamma}{1+\beta}\ ,$$

and

$$\text{(1.12)}\quad \operatorname{Re}\left\{\frac{(z(D_\lambda^{n,\alpha}f(z))')'}{D(_\lambda^{n,\alpha}g(z))'}\right\} > \frac{\beta+\gamma}{1+\beta}\ ,$$

which means that

$$f \in UCC(\beta,\gamma) \text{ implies } D_\lambda^{n,\alpha}f \in CC\left(\frac{\beta+\gamma}{1+\beta}\right) \subseteq CC,$$

and

$$f \in UQC(\beta,\gamma) \text{ implies } D_\lambda^{n,\alpha}f \in QC\left(\frac{\beta+\gamma}{1+\beta}\right) \subseteq QC.$$

Definitions 1.1, and 1.2, includes various classes introduced earlier by Al-Oboudi and Al-Amoudi [4], Blezu [8], Acu and Bezu [2], Aghalary and Azadi [3], Subramanian et al [27],.Kumar and Ramesha [13],.Kaplan [12], and Noor and Thomas [15]

In this paper, basic results for the classes $UCC_\lambda^{n,\alpha}(\beta,\gamma)$ and $UQC_\lambda^{n,\alpha}(\beta,\gamma)$ such as inclusion relations, the coefficients bound and sufficient condition, will be studied. Various known special cases of these results are pointed out.

2. Inclusion Relations

The inclusion relations of the classes $UCC_\lambda^{n,\alpha}(\beta,\gamma)$ and $UQC_\lambda^{n,\alpha}(\beta,\gamma)$ for different values of the parameters n,α,β and γ will be studied. It will also be shown that the classes $UQC_\lambda^{n,\alpha}(\beta,\gamma)$ and $SP_\lambda^{n,\alpha}(\beta,\gamma)$ are not related with set inclusion. To derive our results we need the following.

2.1. Lemma. *[22] Let $f,g \in A$ be univalent starlike of order $\frac{1}{2}$. Then, for every function $F \in A$, we have*

$$\frac{f(z)*g(z)F(z)}{f(z)*g(z)} \in \overline{co}F(z), \quad z \in E,$$

where $\overline{co}$ denotes the closed convex hull.

2.2. Lemma. [14] *Let P be analytic function in E, with Re $P(z) > 0$ for $z \in E$, and let h be a convex function in E. If p is analytic in E, with $p(0) = h(0)$ and if $p(z) + P(z)zp'(z) \prec h(z)$, then $p(z) \prec h(z)$.*

Following the same method of [5, Lemma 2.5], we obtain.

2.3. Lemma. *Let $\Omega^\alpha f$ be in the class $UCC_\lambda^{n,\alpha}(\beta,\gamma)(UQC_\lambda^{n,\alpha}(\beta,\gamma))$, then so is f.*

2.4. Theorem. *Let $0 \le \lambda \le \dfrac{1+\beta}{1-\gamma}$. Then*

$$UCC_\lambda^{n+1,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\alpha}(\beta,\gamma).$$

Proof. Let $f \in UCC_\lambda^{n+1,\alpha}(\beta,\gamma)$. Then by (1.10)

$$\text{(2.1)}\quad \frac{z(D_\lambda^{n+1,\alpha}f(z))'}{D_\lambda^{n+1,\alpha}g(z)} \prec P_{\beta,\gamma}(z),$$

where the function $P_{\beta,\gamma}$ is given in [5], and $g \in SP_\lambda^{n+1,\alpha}(\beta,\gamma)$. From [5, proof of Theorem 2.4], $\Omega^\alpha g(z) \in SP_\lambda^{n,\alpha}(\beta,\gamma)$, for $0 \le \lambda < \dfrac{1+\beta}{1-\gamma}$. Hence

$$\frac{z(D_\lambda^{n,\alpha}\Omega^\alpha g(z))'}{D_\lambda^{n,\alpha}\Omega^\alpha g(z)} = q(z), \tag{2.2}$$

where $q(z) \prec P_{\beta,\gamma}(z)$.

By the definition of $D_\lambda^{n,\alpha} f$, we get

$$D_\lambda^{n+1,\alpha} f(z) = (1-\lambda) D_\lambda^{n,\alpha}\Omega^\alpha f(z) + \lambda z(D_\lambda^{n,\alpha}\Omega^\alpha f(z))'$$

and

$$D_\lambda^{n+1,\alpha} g(z) = (1-\lambda) D_\lambda^{n,\alpha}\Omega^\alpha g(z) + \lambda z(D_\lambda^{n,\alpha}\Omega^\alpha g(z))'.$$

Using (2.1), (2.2) and the above equalities, with the notation $p(z) = \dfrac{z(D_\lambda^{n,\alpha}\Omega^\alpha f(z))'}{D_\lambda^{n,\alpha}\Omega^\alpha g(z)}$, we obtain

$$\frac{z(D_\lambda^{n+1,\alpha} f(z))'}{D_\lambda^{n+1,\alpha} g(z)} = p(z) + \frac{\lambda z p'(z)}{(1-\lambda)q(z)} \ . \tag{2.3}$$

For $\lambda = 0$, $\Omega^\alpha f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$, from (2.1) and (2.3). Hence by Lemma 2.2 $f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$.

For $\lambda \ne 0$, (2.3) can be written, using (2.1), as

$$p(z) + \frac{zp'(z)}{\frac{(1-\lambda)}{\lambda} q(z)} \prec P_{\beta,\gamma}. \tag{2.4}$$

Hence by Lemma 2.2 and (1.11), we have $p(z) \prec P_{\beta,\gamma}(z)$ for $0 < \lambda \le \dfrac{1+\beta}{1-\gamma}$. Thus $\Omega^\alpha f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$, which implies that $f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$, using Lemma 2.3. □

2.5. Corollary. *Let* $0 \le \lambda \le \dfrac{1+\beta}{1-\gamma}$. *Then*

$$UQC_\lambda^{n+1,\alpha}(\beta,\gamma) \subset UQC_\lambda^{n,\alpha}(\beta,\gamma).$$

Proof. Let $f \in UQC_\lambda^{n+1,\alpha}(\beta,\gamma),\ 0 \le \lambda \le \dfrac{1+\beta}{1-\gamma}$. Then by (1.8) $z\acute{f} \in UCC_\lambda^{n+1,\alpha}(\beta,\gamma)$. Which implies, by Theorem 2.4, that

$$z\acute{f} \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$$

Hence, by (1.8), $f \in UQC_\lambda^{n,\alpha}(\beta,\gamma)$. □

2.6. Corollary. *Let* $0 \le \lambda \le \dfrac{1+\beta}{1-\lambda}$. *Then*

$$UCC_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{0,\alpha}(\beta,\gamma) \equiv UCC(\beta,\gamma) \subset CC,$$

and

$$UQC_\lambda^{n,\alpha}(\beta,\gamma) \subset UQC_\lambda^{0,\alpha}(\beta,\gamma) \equiv UQC(\beta,\gamma) \subset CC.$$

This means that, for $0 < \lambda \leq \dfrac{1+\beta}{1-\gamma}$ *functions in* $UCC_\lambda^{n,\alpha}(\beta,\gamma)$ *and* $UQC_\lambda^{n,\alpha}(\beta,\gamma)$*, are close-to-convex and hence univalent.*

2.7. Remark. If we put $\lambda = 1$ and $\alpha = 0$, in Theorem 2.4, then we get the result of Blezu [8].

In view of the relations

$$UCV_\lambda^{n,\alpha}(\beta,\gamma) \subset SP_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\alpha}(\beta,\gamma),$$

and

$$UCV_\lambda^{n,\alpha}(\beta,\gamma) \subset UQC_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\alpha}(\beta,\gamma),$$

one may ask whether the classes $SP_\lambda^{n,\alpha}(\beta,\gamma)$ and $UQC_\lambda^{n,\alpha}(\beta,\gamma)$ are related with set inclusion? The answer is negative. The function f_0, defined by

$$f_0(z) = \frac{1-i}{2}\frac{z}{1-z} - \frac{1+i}{2}\log(1-z).$$

belongs to $UQC_\lambda^{n,\alpha}(\beta,\gamma)$, but not to $SP_\lambda^{n,\alpha}(\beta,\gamma)$. In fact, Silverman and Telage [24], have shown that $f_0 \notin ST \equiv SP_\lambda^{0,\alpha}(1,0)$ and that $f_0 \in QC \equiv UQC_\lambda^{0,\alpha}(1,0)$..Also, the Koebe function $K(z) = \dfrac{z}{(1-z)^2} \in SP_\lambda^{0,\alpha}(1,0)$ and $K(z) \notin UQC_\lambda^{0,\alpha}(1,0)$.

In the following we prove the inclusion relation with respect to α.

2.8. Theorem. *Let* $0 \leq \mu \leq \alpha < 1$. *Then*

$$UCC_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\mu}(\beta,\gamma),$$

where $\left(0 \leq \beta < 1 \text{ and } \dfrac{1}{2} \leq \gamma < 1\right)$ *or* $(\beta \geq 1$ *and* $0 \leq \gamma < 1)$.

Proof. Let $f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$. Then by (1.5) and the convolution properties, we have

$$z(D_\lambda^{n,\mu} f(z))' = \underbrace{\varphi(2-\alpha,2-\mu;z) * \cdots * \varphi(2-\alpha,2-\mu;z)}_{n\text{-times}} * z(D_\lambda^{n,\alpha} f(z))'.$$

Hence

$$\frac{z(D_\lambda^{n,\mu} f(z))'}{D_\lambda^{n,\mu} g(z)} = \frac{\underbrace{\varphi(2-\alpha,2-\mu;z) * \cdots * \varphi(2-\alpha,2-\mu;z)}_{n\text{-times}} * \dfrac{z(D_\lambda^{n,\alpha} f(z))'}{D_\lambda^{n,\alpha} g(z)} D_\lambda^{n,\alpha} g(z)}{\underbrace{\varphi(2-\alpha,2-\mu;z) * \cdots * \varphi(2-\alpha,2-\mu;z)}_{n\text{-times}} * D_\lambda^{n,\alpha} g(z)}.$$

It has been shown [5] that the function $\underbrace{\varphi(2-\alpha,2-\mu;z) * \cdots * \varphi(2-\alpha,2-\mu;z)}_{n\text{-times}} \in ST\left(\dfrac{1}{2}\right)$ and $D_\lambda^{n,\alpha} g(z)$ is a starlike function of order $\dfrac{1}{2}$ for $\left(0 \leq \beta < 1 \text{ and } \dfrac{1}{2} \leq \gamma < 1\right)$ or $(\beta \geq 1$ and $0 \leq \gamma < 1)$. Applying Lemma 2.1, we get the required result. □

The next result follows using (1.8).

2.9. Corollary. *Let $0 \le \mu \le \alpha < 1$. Then*

$$UQC_\lambda^{n,\alpha}(\beta,\gamma) \subset UQC_\lambda^{n,\mu}(\beta,\gamma),$$

where $\left(0 \le \beta < 1 \text{ and } \frac{1}{2} \le \gamma < 1\right)$ *or* $(\beta \ge 1 \text{ and } 0 \le \gamma < 1)$.

The inclusion relation with respect to β and γ follows directly by (1.6) and (1.7).

2.10. Theorem. *Let $\beta_1 \ge \beta_2$, and $\gamma_1 \ge \gamma_2$. Then*

(i) $UCC_\lambda^{n,\alpha}(\beta_1,\gamma_1) \subset UCC_\lambda^{n,\alpha}(\beta_2,\gamma_2)$.

(ii) $UQC_\lambda^{n,\alpha}(\beta_1,\gamma_1) \subset UQC_\lambda^{n,\alpha}(\beta_2,\gamma_2)$.

2.11. Remark. If we put $\lambda = 1$ and $\alpha = 0$, in Theorem 2.10 (i), we get the result of Blezu [8].

3. Coefficients Bound

To derive our results we need the folowing.

3.1. Lemma. *[5] If a function $f \in A$, of the form (1.1) is in $SP_\lambda^{n,\alpha}(\beta,\gamma)$, then*

$$|a_k| \le \frac{1}{\psi_{k,n}(\alpha,\lambda)} \cdot \frac{(P_1)_{k-1}}{(1)_{k-1}}\ , \quad k \ge 2,$$

where

$$(3.1) \qquad P_1 = P_1(\beta,\gamma) = \begin{cases} \dfrac{8(1-\gamma)(\cos^{-1}\beta)^2}{\pi^2(1-\beta^2)}\ , & 0 \le \beta < 1, \\ \dfrac{8}{\pi^2}(1-\gamma)\ , & \beta = 1 \\ \dfrac{\pi^2(1-\gamma)}{4 \subseteq t(\beta^2-1)k^2(t)(1+t)}\ , & \beta > 1,\ 0 < t < 1, \end{cases}$$

3.2. Lemma. *[19] Let $h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ be subordinate to $H(z) = 1 + \sum_{k=1}^{\infty} C_k z^k$ in E. If $H(z)$ is univalent in E and $H(E)$ is convex, then $|c_k| \le |C_1|$, $\quad k \ge 1$.*

3.3. Theorem. *Let $f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$, and given by (1.1). Then*

$$|a_k| \le \frac{1}{\psi_{k,n}(\alpha,\lambda)} \cdot \frac{(P_1)_{k-1}}{(1)_{k-1}}\ , \quad k \ge 2,$$

where P_1 is given by (3.1).

Proof. Since $f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$, then

$$(3.2) \qquad \frac{z(D_\lambda^{n,\alpha} f(z))'}{D_\lambda^{n,\alpha} g(z)} = p(z) \prec P_{\beta,\gamma}\ ,$$

where $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$, $g \in SP_{\lambda}^{n,\alpha}(\beta,\gamma)$, and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. The function $P_{\beta,\gamma}$ is univalent in E and $P_{\beta,\gamma}(E)$, the conic domain is a convex domain, hence, applying Lemma 3.2, we obtain

$$|c_k| \le P_1, \quad k \ge 1.$$

where P_1 is given by (3.1).

From (3.2) and (1.4), we get

$$z + \sum_{k=2}^{\infty} \psi_{k,n}(\alpha,\lambda) k a_k z^k = \left(z + \sum_{k=2}^{\infty} \psi_{k,n}(\alpha,\lambda) b_k z^k \right) \left(1 + \sum_{k=1}^{\infty} c_k z^k \right). \tag{3.3}$$

Equating the coefficients of z^k in (3.3), we get

$$\begin{aligned} \psi_{k,n}(\alpha,\lambda) k a_k &= \sum_{j=1}^{k-1} [c_{k-j} b_j \psi_{j,n}(\alpha,\lambda)] + b_k \psi_{k,n}(\alpha,\lambda), \quad c_0 = 1 \\ &= c_{k-1} + \sum_{j=2}^{k-1} [c_{k-j} b_j \psi_{j,n}(\alpha,\lambda)] + b_k \psi_{k,n}(\alpha,\lambda), b_1 = \psi_{1,n}(\alpha,\lambda) = 1. \end{aligned}$$

Hence

$$\psi_{k,n}(\alpha,\lambda) k |a_k| \le |c_{k-1}| + \sum_{j=2}^{k-1} [|c_{k-j}|\ |b_j| \psi_{j,n}(\alpha,\lambda)] + |b_k| \psi_{k,n}(\alpha,\lambda).$$

Using Lemmas 3.1 and 3.2, we obtain

$$\psi_{k,n}(\alpha,\lambda) k |a_k| \le P_1 \left\{ 1 + \sum_{j=2}^{k-1} \left[\frac{(P_1)_{j-1}}{(1)_{j-1}} \right] \right\} + \frac{(P_1)_{k-1}}{(1)_{k-1}}. \tag{3.4}$$

Applying mathematical induction, we can see that

$$1 + \sum_{j=2}^{k-1} \left[\frac{(P_1)_{j-1}}{(1)_{j-1}} \right] = \frac{(P_1)_{k-1}}{P_1 (1)_{k-2}}. \tag{3.5}$$

Using (3.5) in (3.4), we get

$$\begin{aligned} \psi_{k,n}(\alpha,\lambda) k |a_k| &\le \frac{(P_1)_{k-1}}{(1)_{k-2}} + \frac{(P_1)_{k-1}}{(1)_{k-1}} \\ &= \frac{(P_1)_{k-1}}{(1)_{k-1}} k, \end{aligned}$$

which is the required result. □

From (1.8) and Theorem 3.3, we immediately have

3.4. Corollary. *Let* $f \in UQC_{\lambda}^{n,\alpha}(\beta,\gamma)$. *Then*

$$|a_k| \le \frac{1}{\psi_{k,n}(\alpha,\lambda)} \cdot \frac{(P_1)_{k-1}}{(1)_k}, \quad k \ge 2,$$

where P_1 *is given by (3.1).*

3.5. Remark. The results of Theorem 3.3 and Corollary 3.4 are sharp for $k = 2$.

3.6. Remark. In special cases, Theorem 3.1 reduces to the results of Acu and Blezu [2], Subramanian et al [27], Kaplan [12] and Noor and Thomas [15].

Next we give a sufficient condition for a function to be in the class $UCC_{\lambda}^{n,\alpha}(\beta,\gamma)$.

3.7. Theorem. *If*

$$\sum_{k=2}^{\infty} k|a_k|\psi_{k,n}(\alpha,\lambda) \leq \frac{(1-\gamma)}{1+\beta}\ , \tag{3.6}$$

then a function f, given by (1.1), is in $UCC_{\lambda}^{n,\alpha}(\beta,\gamma)$.

Proof. Let $g(z)=z$. Then $D_{\lambda}^{n,\alpha}g(z)=z$, and

$$\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)} = z(D_{\lambda}^{n,\alpha}f(z))' = \sum_{k=2}^{\infty} k\psi_{k,n}(\alpha,\lambda)a_k z^k.$$

It is sufficient to show that

$$\beta\left|\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)}-1\right| - \mathrm{Re}\left\{\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)}-1\right\} < (1-\gamma).$$

Now

$$\begin{aligned}\beta\left|\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)}-1\right| &- \mathrm{Re}\left\{\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)}-1\right\}\\ &\leq (1+\beta)\left|\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)}-1\right|\\ &\leq (1+\beta)\left|\sum_{k=2}^{\infty} k\psi_{k,n}(\alpha,\lambda)a_k z^{k-1}\right|\\ &\leq (1+\beta)\sum_{k=2}^{\infty} k\psi_{k,n}(\alpha,\lambda)a_k.\end{aligned}$$

The last expression is bounded above by $(1-\gamma)$, if (3.6) is satisfied. □

From (1.8) and Theorem 3.7, we get

3.8. Corollary. *A function f of the form (1.1) is in $UQC_{\lambda}^{n,\alpha}(\beta,\gamma)$ if*

$$\sum_{k=2}^{\infty} k^2|a_k|\psi_{k,n}(\alpha,\lambda) \leq \frac{(1-\gamma)}{1+\beta}\ .$$

3.9. Remark. Theorem 3.7 and Corollary 3.8, reduces to a result of Subramanian et al [27].

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Orientable small covers over the product of 2-cube with n-gon

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Abstract

We calculate the number of D-J equivalence classes and equivariant homeomorphism classes of all orientable small covers over the product of 2-cube with n-gon.

Keywords: Small cover; D-J equivalence; Equivariant homeomorphism

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1. Introduction

As defined by Davis and Januszkiewicz [5], a small cover is a smooth closed manifold M^n with a locally standard $(\mathbb{Z}_2)^n-$action such that its orbit space is a simple convex polytope. For instance, the real projective space $\mathbb{R}P^n$ with a natural $(\mathbb{Z}_2)^n-$action is a small cover over an n-simplex. This gives a direct connection between equivariant topology and combinatorics, making research on the topology of small covers possible through the combinatorial structure of quotient spaces.

Lü and Masuda [7] showed that the equivariant homeomorphism class of a small cover over a simple convex polytope P^n agrees with the equivalence class of its corresponding $(\mathbb{Z}_2)^n-$coloring under the action of the automorphism group of the face poset of P^n. This finding also holds true for orientable small covers by the orientability condition in [8] (see Theoerem 2.5). However, general formulas for calculating the number of equivariant homeomorphism classes of (orientable) small covers over an arbitrary simple convex polytope do not exist.

In recent years, several studies have attempted to enumerate the number of equivalence classes of all small covers over a specific polytope. Garrison and Scott [6] used a computer program to calculate the number of homeomorphism classes of all small covers over a dodecahedron. Cai, Chen and Lü [2] calculated the

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number of equivariant homeomorphism classes of small covers over prisms (an n-sided prism is the product of 1-cube and n-gon). Choi [3] determined the number of equivariant homeomorphism classes of small covers over cubes. However, little is known about orientable small covers. Choi [4] calculated the number of D-J equivalence classes of orientable small covers over cubes. This paper aims to determine the number of D-J equivalence classes and equivariant homeomorphism classes of all orientable small covers over $I^2 \times P_n$ (see Theorem 3.1 and Theorem 4.1), where I^2 and P_n denote 2-cube and n-gon, respectively.

The paper is organized as follows. In Section 2, we review the basic theory on orientable small covers and calculate the automorphism group of the face poset of $I^2 \times P_n$. In Section 3, we determine the number of D-J equivalence classes of orientable small covers over $I^2 \times P_n$. In Section 4, we obtain a formula for the number of equivariant homeomorphism classes of orientable small covers over $I^2 \times P_n$.

2. Preliminaries

A convex polytope P^n of dimension n is simple if every vertex of P^n is the intersection of n facets (i.e., faces of dimension $(n-1)$) [9]. An n-dimensional smooth closed manifold M^n is a small cover if it admits a smooth $(\mathbb{Z}_2)^n$−action such that the action is locally isomorphic to a standard action of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$ and the orbit space $M^n/(\mathbb{Z}_2)^n$ is a simple convex polytope of dimension n.

Let P^n be a simple convex polytope of dimension n and $\mathcal{F}(P^n) = \{F_1, \cdots, F_\ell\}$ be the set of facets of P^n. Assuming that $\pi : M^n \to P^n$ is a small cover over P^n, then there are ℓ connected submanifolds $\pi^{-1}(F_1), \cdots, \pi^{-1}(F_\ell)$. Each submanifold $\pi^{-1}(F_i)$ is fixed pointwise by a $\mathbb{Z}_2$−subgroup $\mathbb{Z}_2(F_i)$ of $(\mathbb{Z}_2)^n$. Obviously, the $\mathbb{Z}_2$−subgroup $\mathbb{Z}_2(F_i)$ agrees with an element ν_i in $(\mathbb{Z}_2)^n$ as a vector space. For each face F of codimension u, given that P^n is simple, there are u facets $F_{i_1}, \cdots, F_{i_u}$ such that $F = F_{i_1} \cap \cdots \cap F_{i_u}$. Then, the corresponding submanifolds $\pi^{-1}(F_{i_1}), \cdots, \pi^{-1}(F_{i_u})$ intersect transversally in the $(n-u)$-dimensional submanifold $\pi^{-1}(F)$, and the isotropy subgroup $\mathbb{Z}_2(F)$ of $\pi^{-1}(F)$ is a subtorus of rank u generated by $\mathbb{Z}_2(F_{i_1}), \cdots, \mathbb{Z}_2(F_{i_u})$ (or is determined by $\nu_{i_1}, \cdots, \nu_{i_u}$ in $(\mathbb{Z}_2)^n$). This gives a characteristic function [5]

$$\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$$

which is defined by $\lambda(F_i) = \nu_i$ such that whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_u}$ is non-empty, $\lambda(F_{i_1}), \cdots, \lambda(F_{i_u})$ are linearly independent in $(\mathbb{Z}_2)^n$. Assuming that each nonzero vector of $(\mathbb{Z}_2)^n$ is a color, then the characteristic function λ means that each facet is colored. Hence, we also call λ a $(\mathbb{Z}_2)^n$-coloring on P^n.

In fact, Davis and Januszkiewicz gave a reconstruction process of a small cover by using a $(\mathbb{Z}_2)^n$-coloring $\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$. Let $\mathbb{Z}_2(F_i)$ be the subgroup of $(\mathbb{Z}_2)^n$ generated by $\lambda(F_i)$. Given a point $p \in P^n$, we denote the minimal face containing p in its relative interior by $F(p)$. Assuming that $F(p) = F_{i_1} \cap \cdots \cap F_{i_u}$ and $\mathbb{Z}_2(F(p)) = \bigoplus_{j=1}^{u} \mathbb{Z}_2(F_{i_j})$, then $\mathbb{Z}_2(F(p))$ is a u-dimensional subgroup of $(\mathbb{Z}_2)^n$. Let $M(\lambda)$ denote $P^n \times (\mathbb{Z}_2)^n / \sim$, where $(p, g) \sim (q, h)$ if $p = q$ and $g^{-1}h \in \mathbb{Z}_2(F(p))$. The free action of $(\mathbb{Z}_2)^n$ on $P^n \times (\mathbb{Z}_2)^n$ descends to an action on $M(\lambda)$ with quotient P^n. Thus, $M(\lambda)$ is a small cover over P^n [5].

Two small covers M_1 and M_2 over P^n are called weakly equivariantly homeomorphic if there is an automorphism $\varphi : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n$ and a homeomorphism $f : M_1 \to M_2$ such that $f(t \cdot x) = \varphi(t) \cdot f(x)$ for every $t \in (\mathbb{Z}_2)^n$ and $x \in M_1$. If φ is an identity, then M_1 and M_2 are equivariantly homeomorphic. Following [5], two small covers M_1 and M_2 over P^n are called Davis-Januszkiewicz equivalent (or simply, D-J equivalent) if there is a weakly equivariant homeomorphism $f : M_1 \to M_2$ covering the identity on P^n.

By $\Lambda(P^n)$, we denote the set of all $(\mathbb{Z}_2)^n$-colorings on P^n. We have

2.1. Theorem. ([5]) *All small covers over P^n are given by $\{M(\lambda) | \lambda \in \Lambda(P^n)\}$, i.e., for each small cover M^n over P^n, there is a $(\mathbb{Z}_2)^n$-coloring λ with an equivariant homeomorphism $M(\lambda) \longrightarrow M^n$ covering the identity on P^n.*

Nakayama and Nishimura [8] found an orientability condition for a small cover.

2.2. Theorem. *For a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$, a homomorphism $\varepsilon : (\mathbb{Z}_2)^n \longrightarrow \mathbb{Z}_2 = \{0, 1\}$ is defined by $\varepsilon(e_i) = 1 (i = 1, \cdots, n)$. A small cover $M(\lambda)$ over a simple convex polytope P^n is orientable if and only if there exists a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$ such that the image of $\varepsilon\lambda$ is $\{1\}$.*

A $(\mathbb{Z}_2)^n$-coloring that satisfies the orientability condition in Theorem 2.2 is an orientable coloring of P^n. We know that there exists an orientable small cover over every simple convex 3-polytope [8]. Similarly, we know the existence of orientable small cover over $I^2 \times P_n$ by the existence of orientable colorings and determine the number of D-J equivalence classes and equivariant homeomorphism classes.

By $O(P^n)$, we denote the set of all orientable colorings on P^n. There is a natural action of $GL(n, \mathbb{Z}_2)$ on $O(P^n)$ defined by the correspondence $\lambda \longmapsto \sigma \circ \lambda$, and the action on $O(P^n)$ is free. We assume that $F_1, \cdots, F_n$ of $\mathcal{F}(P^n)$ meet at one vertex p of P^n. Let $e_1, \cdots, e_n$ be the standard basis of $(\mathbb{Z}_2)^n$ and $B(P^n) = \{\lambda \in O(P^n) | \lambda(F_i) = e_i, i = 1, \cdots, n\}$. Then $B(P^n)$ is the orbit space of $O(P^n)$ under the action of $GL(n, \mathbb{Z}_2)$.

2.3. Remark. We have $B(P^n) = \{\lambda \in O(P^n) | \lambda(F_i) = e_i, i = 1, \cdots, n$ and for $n+1 \le j \le \ell, \lambda(F_j) = e_{j_1} + e_{j_2} + \cdots + e_{j_{2h_j+1}}, 1 \le j_1 < j_2 < \cdots < j_{2h_j+1} \le n\}$. Below, we show that $\lambda(F_j) = e_{j_1} + e_{j_2} + \cdots + e_{j_{2h_j+1}}$ for $n + 1 \le j \le \ell$. If $\lambda \in O(P^n)$, there exists a basis $\{e'_1, \cdots, e'_n\}$ of $(\mathbb{Z}_2)^n$ such that for $1 \le i \le \ell$, $\lambda(F_i) = e'_{i_1} + \cdots + e'_{i_{2f_i+1}}, 1 \le i_1 < \cdots < i_{2f_i+1} \le n$. Given that $\lambda(F_i) = e_i, i = 1, \cdots, n$, then $e_i = e'_{i_1} + \cdots + e'_{i_{2f_i+1}}$. Thus, for $n + 1 \le j \le \ell$, $\lambda(F_j)$ is not of the form $e_{j_1} + \cdots + e_{j_{2k}}, 1 \le j_1 < \cdots < j_{2k} \le n$.

Given that $B(P^n)$ is the orbit space of $O(P^n)$, then we have

2.4. Lemma. $|O(P^n)| = |B(P^n)| \times |GL(n, \mathbb{Z}_2)|$.

Note that $|GL(n, \mathbb{Z}_2)| = \prod_{k=1}^{n} \left(2^n - 2^{k-1}\right)$ [1]. Two orientable small covers $M(\lambda_1)$ and $M(\lambda_2)$ over P^n are D-J equivalent if and only if there is $\sigma \in GL(n, \mathbb{Z}_2)$ such that $\lambda_1 = \sigma \circ \lambda_2$. Thus the number of D-J equivalence classes of orientable small covers over P^n is $|B(P^n)|$.

Let P^n be a simple convex polytope of dimension n. All faces of P^n form a poset (i.e., a partially ordered set by inclusion). An automorphism of $\mathcal{F}(P^n)$ is a bijection from $\mathcal{F}(P^n)$ to itself that preserves the poset structure of all faces of P^n. By $Aut(\mathcal{F}(P^n))$, we denote the group of automorphisms of $\mathcal{F}(P^n)$. We define the right action of $Aut(\mathcal{F}(P^n))$ on $O(P^n)$ by $\lambda \times h \longmapsto \lambda \circ h$, where $\lambda \in O(P^n)$ and $h \in Aut(\mathcal{F}(P^n))$. By improving the classifying result on unoriented small covers in [7], we have

2.5. Theorem. *Two orientable small covers over an n-dimensional simple convex polytope P^n are equivariantly homeomorphic if and only if there is $h \in Aut(\mathcal{F}(P^n))$ such that $\lambda_1 = \lambda_2 \circ h$, where λ_1 and λ_2 are their corresponding orientable colorings on P^n.*

Proof. Theorem 2.5 is proven true by combining Lemma 5.4 in [7] with Theorem 2.2. $\square$

According to Theorem 2.5, the number of orbits of $O(P^n)$ under the action of $Aut(\mathcal{F}(P^n))$ is the number of equivariant homeomorphism classes of orientable small covers over P^n. Thus, we count the number of orbits. Burnside Lemma is very useful in enumerating the number of orbits.

Burnside Lemma *Let G be a finite group acting on a set X. Then the number of orbits X under the action of G equals $\frac{1}{|G|}\sum_{g\in G}|X_g|$, where $X_g = \{x \in X | gx = x\}$.*

Burnside Lemma suggests that, to determine the number of the orbits of $O(P^n)$ under the action of $Aut(\mathcal{F}(P^n))$, the structure of $Aut(\mathcal{F}(P^n))$ should first be understood. We shall particularly be concerned when the simple convex polytope is $I^2 \times P_n$.

For convenience, we introduce the following marks. By F_1', F_2', F_3', and F_4' we denote four edges of the 2-cube I^2 in their general order (here I^2 is considered as a 4-gon). Similarly, by $F_5', F_6', \cdots$, and F_{n+4}', we denote all edges of n-gon P_n in their general order. Set $\mathcal{F}' = \{F_i = F_i' \times P_n | 1 \leq i \leq 4\}$, and $\mathcal{F}'' = \{F_i = I^2 \times F_i' | 5 \leq i \leq n+4\}$. Then $\mathcal{F}(I^2 \times P_n) = \mathcal{F}' \bigcup \mathcal{F}''$.

Next, we determine the automorphism group of face poset of $I^2 \times P_n$.

2.6. Lemma. *When n=4, the automorphism group $Aut(\mathcal{F}(I^2 \times P_n))$ is isomorphic to $(\mathbb{Z}_2)^4 \times S_4$, where S_4 is the symmetric group on four symbols. When $n \neq 4$, $Aut(\mathcal{F}(I^2 \times P_n))$ is isomorphic to $D_4 \times D_n$, where D_n is the dihedral group of order $2n$.*

Proof. When n=4, $I^2 \times P_n$ is a 4-cube I^4. Obviously, the automorphism group $Aut(\mathcal{F}(I^4))$ contains a symmetric group S_4 because there is exactly one automorphism for each permutation of the four pairs of opposite sides of I^4. All elements of $Aut(\mathcal{F}(I^4))$ can be written in a simple form as $\chi_1^{e_1}\chi_2^{e_2}\chi_3^{e_3}\chi_4^{e_4} \cdot u$, where $e_1, e_2, e_3, e_4 \in \mathbb{Z}_2$, with reflections $\chi_1, \chi_2, \chi_3, \chi_4$ and $u \in S_4$. Thus, the automorphism group $Aut(\mathcal{F}(I^4))$ is isomorphic to $(\mathbb{Z}_2)^4 \times S_4$.

When$n \neq 4$, the facets of $\mathcal{F}'$ and $\mathcal{F}''$ are mapped to $\mathcal{F}'$ and $\mathcal{F}''$, respectively, under the automorphisms of $Aut(\mathcal{F}(I^2 \times P_n))$. Given that the automorphism group $Aut(\mathcal{F}(I^2))$ is isomorphic to D_4 and $Aut(\mathcal{F}(P_n))$ is isomorphic to D_n, $Aut(\mathcal{F}(I^2 \times P_n))$ is isomorphic to $D_4 \times D_n$. $\square$

2.7. Remark. Let x, y, x', y' be the four automorphisms of $Aut(\mathcal{F}(I^2 \times P_n))$ with the following properties:

(a) $x(F_i) = F_{i+1}(1 \leq i \leq 3), x(F_4) = F_1, x(F_j) = F_j, 5 \leq j \leq n+4$;

(b) $y(F_i) = F_{5-i}(1 \leq i \leq 4), y(F_j) = F_j, 5 \leq j \leq n+4$;

(c) $x'(F_i) = F_i(1 \leq i \leq 4), x'(F_j) = F_{j+1}(5 \leq j \leq n+3), x'(F_{n+4}) = F_5$;

(d) $y'(F_i) = F_i(1 \leq i \leq 4), y'(F_j) = F_{n+9-j}, 5 \leq j \leq n+4$.

Then, when $n \neq 4$, all automorphisms of $Aut(\mathcal{F}(I^2 \times P_n))$ can be written in a simple form as follows:

(1) $\qquad x^u y^v x'^{u'} y'^{v'},\ u \in \mathbb{Z}_4, u' \in \mathbb{Z}_n, v, v' \in \mathbb{Z}_2$

with $x^4 = y^2 = x'^n = y'^2 = 1, x^u y = y x^{4-u}$, and $x'^{u'} y' = y' x'^{n-u'}$.

3. Orientable colorings on $I^2 \times P_n$

This section is devoted to calculating the number of all orientable colorings on $I^2 \times P_n$. We also determine the number of D-J equivalence classes of orientable small covers over $I^2 \times P_n$.

3.1. Theorem. *By $\mathbb{N}$, we denote the set of natural numbers. Let a, b, c be the functions from $\mathbb{N}$ to $\mathbb{N}$ with the following properties:*

(1) $a(j) = 2a(j-1) + 8a(j-2)$ *with* $a(1) = 1,\ a(2) = 2$;

(2) $b(j) = b(j-1) + 4b(j-2)$ *with* $b(1) = b(2) = 1$;

(3) $c(j) = 2c(j-1)+4c(j-2)-6c(j-3)-3c(j-4)+4c(j-5)$ *with* $c(1) = c(2) = 1$, $c(3) = 3,\ c(4) = 7,\ c(5) = 17$.

Then, the number of all orientable colorings on $I^2 \times P_n$ is

$$|O(I^2 \times P_n)| = \prod_{k=1}^{4}\left(2^4 - 2^{k-1}\right) \cdot [a(n-1) + 4b(n-1) + 2c(n-1) + 5 \cdot \tfrac{1+(-1)^n}{2}].$$

Proof. Let e_1, e_2, e_3, e_4 be the standard basis of $(\mathbb{Z}_2)^4$, then $(\mathbb{Z}_2)^4$ contains 15 nonzero elements (or 15 colors). We choose F_1, F_2 from $\mathcal{F}'$ and F_5, F_6 from $\mathcal{F}''$ such that F_1, F_2, F_5, F_6 meet at one vertex of $I^2 \times P_n$. Then

$$B(I^2 \times P_n) = \{\lambda \in O(I^2 \times P_n) | \lambda(F_1) = e_1, \lambda(F_2) = e_2, \lambda(F_5) = e_3, \lambda(F_6) = e_4\}.$$

By Lemma 2.4, we have

$$|O(I^2 \times P_n)| = |B(I^2 \times P_n)| \times |GL(4, \mathbb{Z}_2)| = \prod_{k=1}^{4}\left(2^4 - 2^{k-1}\right) \cdot |B(I^2 \times P_n)|.$$

Write

$B_0(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1, e_1 + e_3 + e_4\}$,

$B_1(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_2 + e_3, e_1 + e_2 + e_4\}$.

By the definition of $B(P^n)$ and Remark 2.3, we have $|B(I^2 \times P_n)| = |B_0(I^2 \times P_n)| + |B_1(I^2 \times P_n)|$. Then, our argument proceeds as follows.

(I) Calculation of $|B_0(I^2 \times P_n)|$.

In this case, no matter which value of $\lambda(F_3)$ is chosen, $\lambda(F_4) = e_2, e_2 + e_1 + e_3, e_2 + e_1 + e_4, e_2 + e_3 + e_4$. Write

$B_0^0(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2\}$,

$B_0^1(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_1 + e_3\},$
$B_0^2(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_1 + e_4\},$
$B_0^3(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_3 + e_4\},$
$B_0^4(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2\},$
$B_0^5(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_1 + e_3\},$
$B_0^6(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_1 + e_4\},$
$B_0^7(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_3 + e_4\}.$

By the definition of $B_0(I^2 \times P_n)$ and Remark 2.3, we have $|B_0(I^2 \times P_n)| = \sum_{i=0}^{7} |B_0^i(I^2 \times P_n)|$. Then, our argument is divided into the following cases.

Case 1. Calculation of $|B_0^0(I^2 \times P_n)|$.

By the definition of $B(P^n)$ and Remark 2.3, we have $\lambda(F_{n+4}) = e_4, e_4 + e_1 + e_2, e_4 + e_1 + e_3, e_4 + e_2 + e_3$. Set $B_0^{0,0}(I^2 \times P_n) = \{\lambda \in B_0^0(I^2 \times P_n) | \lambda(F_{n+3}) = e_3, e_1+e_2+e_3\}$ and $B_0^{0,1}(I^2 \times P_n) = B_0^0(I^2 \times P_n) - B_0^{0,0}(I^2 \times P_n)$. Take an orientable coloring λ in $B_0^{0,0}(I^2 \times P_n)$. Then, $\lambda(F_{n+2}), \lambda(F_{n+4}) \in \{e_4, e_4 + e_1 + e_2, e_4 + e_1 + e_3, e_4+e_2+e_3\}$. In this case, the values of λ restricted to F_{n+3} and F_{n+4} have eight possible choices. Thus, $|B_0^{0,0}(I^2 \times P_n)| = 8|B_0^0(I^2 \times P_{n-2})|$. Take an orientable coloring λ in $B_0^{0,1}(I^2 \times P_n)$. Then, $\lambda(F_{n+3}) = e_4, e_4+e_1+e_2, e_4+e_1+e_3, e_4+e_2+e_3$. If we fix any value of $\lambda(F_{n+3})$, then $\lambda(F_{n+4})$ has only two possible values. Thus, $|B_0^{0,1}(I^2 \times P_n)| = 2|B_0^0(I^2 \times P_{n-1})|$. Furthermore, we have that

$$|B_0^0(I^2 \times P_n)| = 2|B_0^0(I^2 \times P_{n-1})| + 8|B_0^0(I^2 \times P_{n-2})|.$$

A direct observation shows that $|B_0^0(I^2 \times P_2)| = 1$ and $|B_0^0(I^2 \times P_3)| = 2$. Thus, $|B_0^0(I^2 \times P_n)| = a(n-1)$.

Case 2. Calculation of $|B_0^1(I^2 \times P_n)|$.

Set $B_0^{1,0}(I^2 \times P_n) = \{\lambda \in B_0^1(I^2 \times P_n) | \lambda(F_{n+3}) = e_3\}$ and $B_0^{1,1}(I^2 \times P_n) = B_0^1(I^2 \times P_n) - B_0^{1,0}(I^2 \times P_n)$. Take an orientable coloring λ in $B_0^{1,0}(I^2 \times P_n)$. Then, $\lambda(F_{n+2}), \lambda(F_{n+4}) \in \{e_4, e_4+e_1+e_2, e_4+e_1+e_3, e_4+e_2+e_3\}$, so $|B_0^{1,0}(I^2 \times P_n)| = 4|B_0^1(I^2 \times P_{n-2})|$. Take an orientable coloring λ in $B_0^{1,1}(I^2 \times P_n)$. Then, $\lambda(F_{n+3}) = e_4, e_4+e_1+e_2, e_4+e_1+e_3, e_4+e_2+e_3$. However, $\lambda(F_{n+4})$ has only one possible value whichever of the four possible values of $\lambda(F_{n+3})$ is chosen. Thus, $|B_0^{1,1}(I^2 \times P_n)| = |B_0^1(I^2 \times P_{n-1})|$. We easily determine that $|B_0^1(I^2 \times P_2)| = |B_0^1(I^2 \times P_3)| = 1$. Thus, $|B_0^1(I^2 \times P_n)| = b(n-1)$.

Case 3. Calculation of $|B_0^2(I^2 \times P_n)|$.

If we interchange e_3 and e_4, then the problem is reduced to Case 2. Thus, $|B_0^2(I^2 \times P_n)| = b(n-1)$.

Case 4. Calculation of $|B_0^3(I^2 \times P_n)|$.

In this case, $\lambda(F_{n+4}) = e_4, e_4 + e_1 + e_3$. Set $B_0^{3,0}(I^2 \times P_n) = \{\lambda \in B_0^3(I^2 \times P_n) | \lambda(F_{n+3}) = e_3\}$, $B_0^{3,1}(I^2 \times P_n) = \{\lambda \in B_0^3(I^2 \times P_n) | \lambda(F_{n+3}) = e_4, e_4+e_1+e_3\}$, and $B_0^{3,2}(I^2 \times P_n) = \{\lambda \in B_0^3(I^2 \times P_n) | \lambda(F_{n+3}) = e_1 + e_2 + e_3, e_1 + e_2 + e_4\}$.

Then, $|B_0^3(I^2\times P_n)| = |B_0^{3,0}(I^2\times P_n)| + |B_0^{3,1}(I^2\times P_n)| + |B_0^{3,2}(I^2\times P_n)|$. An easy argument shows that $|B_0^{3,0}(I^2 \times P_n)| = 2|B_0^3(I^2 \times P_{n-2})|$ and $|B_0^{3,1}(I^2 \times P_n)| = |B_0^3(I^2 \times P_{n-1})|$. Thus,

(2) $\quad |B_0^3(I^2 \times P_n)| = |B_0^3(I^2 \times P_{n-1})| + 2|B_0^3(I^2 \times P_{n-2})| + |B_0^{3,2}(I^2 \times P_n)|.$

Set $B(n) = \{\lambda \in B_0^{3,2}(I^2 \times P_n) | \lambda(F_{n+2}) = e_1 + e_3 + e_4\}$. Then,

(3) $\quad |B_0^{3,2}(I^2 \times P_n)| = |B_0^{3,2}(I^2 \times P_{n-1})| + |B(n)|$

and

(4) $\quad |B(n)| = 2|B_0^3(I^2 \times P_{n-4})| + 2|B_0^3(I^2 \times P_{n-5})| + |B(n-2)| + 2|B_0^{3,2}(I^2 \times P_{n-2})|$.

Combining Eqs. (2), (3) and (4), we obtain

$$|B_0^3(I^2 \times P_n)| = 2|B_0^3(I^2 \times P_{n-1})| + 4|B_0^3(I^2 \times P_{n-2})| - 6|B_0^3(I^2 \times P_{n-3})| - 3|B_0^3(I^2 \times P_{n-4})| + 4|B_0^3(I^2 \times P_{n-5})|.$$

A direct observation shows that $|B_0^3(I^2 \times P_2)| = |B_0^3(I^2 \times P_3)| = 1, |B_0^3(I^2 \times P_4)| = 3, |B_0^3(I^2\times P_5)| = 7$, and $|B_0^3(I^2\times P_6)| = 17$. Thus, $|B_0^3(I^2\times P_n)| = c(n-1)$.

Case 5. Calculation of $|B_0^4(I^2 \times P_n)|$.

If we interchange e_1 and e_2, then the problem is reduced to Case 4; thus, $|B_0^4(I^2 \times P_n)| = c(n-1)$.

Case 6. Calculation of $|B_0^5(I^2 \times P_n)|$.

In this case, $\lambda(F_7) = e_3, \lambda(F_8) = e_4, \cdots, \lambda(F_{7+2i}) = e_3, \lambda(F_{7+2i+1}) = e_4, \cdots$. Thus, $|B_0^5(I^2 \times P_n)| = \frac{1+(-1)^n}{2}$.

Case 7. Calculation of $|B_0^6(I^2 \times P_n)|$.

Similar to Case 6, we have $|B_0^6(I^2 \times P_n)| = \frac{1+(-1)^n}{2}$.

Case 8. Calculation of $|B_0^7(I^2 \times P_n)|$.

Similar to Case 6, we have $|B_0^7(I^2 \times P_n)| = \frac{1+(-1)^n}{2}$.

Thus, $|B_0(I^2 \times P_n)| = a(n-1) + 2b(n-1) + 2c(n-1) + 3 \cdot \frac{1+(-1)^n}{2}$.

(II) Calculation of $|B_1(I^2 \times P_n)|$.

In this case, no matter which value of $\lambda(F_3)$ is chosen, $\lambda(F_4) = e_2, e_2 + e_3 + e_4$. Write

$B_1^0(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_2 + e_3, \lambda(F_4) = e_2\}$,
$B_1^1(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_2 + e_3, \lambda(F_4) = e_2 + e_3 + e_4\}$,
$B_1^2(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_2 + e_4, \lambda(F_4) = e_2\}$,
$B_1^3(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_2 + e_4, \lambda(F_4) = e_2 + e_3 + e_4\}$.

By the definition of $B_1(I^2 \times P_n)$ and Remark 2.3, we have $|B_1(I^2 \times P_n)| = \sum_{i=0}^{3} |B_1^i(I^2 \times P_n)|$. Then, our argument is divided into the following cases.

Case 1. Calculation of $|B_1^0(I^2 \times P_n)|$.

If we interchange e_1 and e_2, then the problem is reduced to Case 2 in (I); thus, $|B_1^0(I^2 \times P_n)| = b(n-1)$.

Case 2. Calculation of $|B_1^1(I^2 \times P_n)|$.

Similar to Case 6 in (I), we have $|B_1^1(I^2 \times P_n)| = \frac{1+(-1)^n}{2}$.

Case 3. Calculation of $|B_1^2(I^2 \times P_n)|$.

If we interchange e_1 and e_2, then the problem is reduced to Case 3 in (I); thus $|B_1^2(I^2 \times P_n)| = b(n-1)$.

Case 4. Calculation of $|B_1^3(I^2 \times P_n)|$.

Similar to Case 6 in (I), we have $|B_1^3(I^2 \times P_n)| = \frac{1+(-1)^n}{2}$.

Thus, $|B_1(I^2 \times P_n)| = 2b(n-1) + 1 + (-1)^n$. □

3.2. Remark. By using the above method, we prove that

$$|O(P_2 \times P_n)| = \prod_{k=1}^{4} \left(2^4 - 2^{k-1}\right) \cdot a(n-1).$$

Based on Theorem 3.1, we know that the number of D-J equivalence classes of orientable small covers over $I^2 \times P_n$ is $a(n-1) + 4b(n-1) + 2c(n-1) + 5 \cdot \frac{1+(-1)^n}{2}$.

4. Number of equivariant homeomorphism classes

In this section, we determine the number of equivariant homeomorphism classes of all orientable small covers over $I^2 \times P_n$.

Let φ denote the Euler's totient function, i.e., $\varphi(1) = 1$, $\varphi(N)$ for a positive integer N $(N \geq 2)$ is the number of positive integers both less than N and coprime to N. We have

4.1. Theorem. *Let $E_o(I^2 \times P_n)$ denote the number of equivariant homeomorphism classes of orientable small covers over $I^2 \times P_n$. Then, $E_o(I^2 \times P_n)$ is equal to*

(1) $\frac{1}{16n}\{\sum_{t'>1, t'|n} \varphi(\frac{n}{t'})[|O(P_2 \times P_{t'})| + |O(P_4 \times P_{t'})|] + 40320 \sum_{t'>1, t'|n} \varphi(\frac{n}{t'})[a(t'-1) + 2b(t'-1) + c(t'-1)]\}$ *for n odd,*

(2) $\frac{1}{16n}\{\sum_{t'>1, t'|n} \varphi(\frac{n}{t'})[|O(P_2 \times P_{t'})| + |O(P_4 \times P_{t'})|] + 40320 \sum_{t'>1, t'|n} \varphi(\frac{n}{t'})[a(t'-1) + 2b(t'-1) + c(t'-1)] + 40320n[\tilde{a}(n) + \tilde{c}(n) + \tilde{d}(n) + \tilde{e}(n) + \frac{5}{4}]\}$ *for n even and $n \neq 4$,*

(3) *12180 for $n = 4$,*

where $\tilde{a}(j), \tilde{b}(j), \tilde{c}(j), \tilde{d}(j)$, and $\tilde{e}(j)$ are defined as follows

$$\tilde{a}(j) = \begin{cases} 0, & j \text{ odd}, \\ 1, & j = 2, \\ 4, & j = 4, \\ 2\tilde{a}(j-2) + 8\tilde{a}(j-4), & j \text{ even and } j \geq 6, \end{cases}$$

$$\tilde{b}(j) = \begin{cases} 4, & j = 6, \\ 8, & j = 8, \\ \tilde{b}(j-2) + 4\tilde{b}(j-4), & j \textit{ even and } j \geq 10, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\tilde{c}(j) = \begin{cases} 0, & j \ \textit{odd}, \\ 1, & j = 2, \\ 2, & j = 4, \\ 6, & j = 6, \\ \tilde{b}(j) + \tilde{b}(j-2) + \tilde{c}(j-4), & j \textit{ even and } j \geq 8, \end{cases}$$

$$\tilde{d}(j) = \begin{cases} 0, & j \ \textit{odd}, \\ 1, & j = 2, \\ 4, & j = 4, \\ \tilde{d}(j-2) + 4\tilde{d}(j-4), & j \textit{ even and } j \geq 6, \end{cases}$$

and

$$\tilde{e}(j) = \begin{cases} 0, & j \ \textit{odd}, \\ 1, & j = 2, \\ 2, & j = 4, \\ 6, & j = 6, \\ 14, & j = 8, \\ 38, & j = 10, \\ 2\tilde{e}(j-2) + 4\tilde{e}(j-4) - 6\tilde{e}(j-6) - 3\tilde{e}(j-8) + 4\tilde{e}(j-10), & \\ & j \textit{ even and } j \geq 12. \end{cases}$$

Proof. Based on Theorem 2.5, Burnside Lemma, and Lemma 2.6, we have

$$E_o(I^2 \times P_n) = \begin{cases} \frac{1}{16n} \sum_{g \in Aut(\mathcal{F}(I^2 \times P_n))} |\Lambda_g|, & n \neq 4, \\ \frac{1}{384} \sum_{g \in Aut(\mathcal{F}(I^4))} |\Lambda_g|, & n = 4, \end{cases}$$

where $\Lambda_g = \{\lambda \in O(I^2 \times P_n) | \lambda = \lambda \circ g\}$.

The argument is divided into three cases: (I) n odd, (II) n even and $n \neq 4$, (III) $n = 4$.

(I) n odd

Given that n is odd, by Remark 2.7, each automorphism g of $Aut(\mathcal{F}(I^2 \times P_n))$ can be written as $x^u y^v x'^{u'} y'^{v'}$.

Case 1. $g = x^u x'^{u'}$.

Let $t = gcd(u, 4)$ (the greatest common divisor of u and 4) and $t' = gcd(u', n)$. Then all facets of $\mathcal{F}'$ are divided into t orbits under the action of g, and each orbit contains $\frac{4}{t}$ facets. Thus, each orientable coloring of Λ_g gives the same coloring on all $\frac{4}{t}$ facets of each orbit. Similarly, all facets of $\mathcal{F}''$ are divided into t' orbits under the action of g, and each orbit contains $\frac{n}{t'}$ facets. Thus, each orientable coloring of Λ_g gives the same coloring on all $\frac{n}{t'}$ facets of each orbit. Hence, if $t \neq 1$ and $t' \neq 1$, then $|\Lambda_g| = |O(P_t \times P_{t'})|$. If t=1 (or $t' = 1$), then all facets of $\mathcal{F}'$ (or $\mathcal{F}''$) have the same coloring, which is impossible by the definition of orientable colorings. For every $t > 1$, there are exactly $\varphi(\frac{4}{t})$ automorphisms of the form x^u, each of which divides all facets of $\mathcal{F}'$ into t orbits. Similarly, for every $t' > 1$, there are exactly $\varphi(\frac{n}{t'})$ automorphisms of the form $x'^{u'}$, each of which divides all facets of $\mathcal{F}''$ into t' orbits. Thus, when $g = x^u x'^{u'}$,

$$\begin{aligned}\sum_{g=x^u x'^{u'}} |\Lambda_g| &= \sum_{t,t'>1,t|4,t'|n} \varphi(\tfrac{4}{t})\varphi(\tfrac{n}{t'})|O(P_t \times P_{t'}| \\ &= \sum_{t'>1,t'|n} \varphi(\tfrac{n}{t'})[|O(P_2 \times P_{t'})| + |O(P_4 \times P_{t'})|].\end{aligned}$$

Case 2. $g = x^u x'^{u'} y'$ or $x^u y x'^{u'} y'$.

Given that n is odd, each automorphism always gives an interchange between two neighborly facets of $\mathcal{F}''$. Thus, the two neighborly facets have the same coloring, which contradicts the definition of orientable colorings. Hence, Λ_g is empty.

Case 3. $g = x^u y x'^{u'}$ with u even.

Let $l = \frac{4-u}{2}$. Such an automorphism gives an interchange between two neighborly facets F_l and F_{l+1}. Hence, both facets F_l and F_{l+1} have the same coloring, which contradicts the definition of orientable colorings. Thus, in this case Λ_g is also empty.

Case 4. $g = x^u y x'^{u'}$ with u odd.

Let $t' = gcd(u', n)$. All facets of $\mathcal{F}''$ are divided into t' orbits under the action of g, and each orbit contains $\frac{n}{t'}$ facets. Hence, each orientable coloring of Λ_g gives the same coloring on all $\frac{n}{t'}$ facets of each orbit. If we choose an arbitrary facet from each orbit, it suffices to color t' chosen facets for $\mathcal{F}''$. Moreover, given that each automorphism $g = x^u y x'^{u'}$ contains y as its factor and u is odd, it suffices to color only three neighborly facets of $\mathcal{F}'$ for $\mathcal{F}'$. In fact, it suffices to consider the case $g = xyx'^{u'}$ because there is no essential difference between this case and other cases. Based on the argument of Theorem 3.1, we have

$$|\Lambda_g| = 20160[a(t'-1) + 2b(t'-1) + c(t'-1)],$$

where $a(t'-1), b(t'-1)$ and $c(t'-1)$ are stated as in Theorem 3.1. Given that u is odd and $u \in \mathbb{Z}_4$, u=1, 3. For every $t' > 1$, there are exactly $\varphi(\frac{n}{t'})$ automorphisms of the form $x'^{u'}$, each of which divides all facets of $\mathcal{F}''$ into t' orbits. Thus, when $g = x^u y x'^{u'}$,

$$\sum_{g=x^u y x'^{u'}} |\Lambda_g| = 2 \sum_{t'>1,t'|n} \varphi(\tfrac{n}{t'})20160[a(t'-1) + 2b(t'-1) + c(t'-1)].$$

Combining Cases 1 to 4, we complete the proof in (I).

(II) n **even and** $n \neq 4$

Given that $n \neq 4$, by Remark 2.7, each automorphism g of $Aut(\mathcal{F}(I^2 \times P_n))$ can be written as $x^u y^v x'^{u'} y'^{v'}$.

Case 1. $g = x^u x'^{u'}$.

Similar to Case 1 in (I), we have $\sum\limits_{g=x^u x'^{u'}} |\Lambda_g| = \sum\limits_{t'>1, t'|n} \varphi(\frac{n}{t'})[|O(P_2 \times P_{t'})| + |O(P_4 \times P_{t'})|]$.

Case 2. $g = x^u y x'^{u'}$ with u even.

Similar to Case 3 in (I), Λ_g is empty.

Case 3. $g = x^u y x'^{u'}$ with u odd.

Similar to Case 4 in (I), $\sum\limits_{g=x^u y x'^{u'}} |\Lambda_g| = 2 \sum\limits_{t'>1, t'|n} \varphi(\frac{n}{t'}) 20160[a(t'-1) + 2b(t'-1) + c(t'-1)]$.

Case 4. $g = x^u x'^{u'} y'$ with u' even.

Similar to Case 3 in (I), Λ_g is also empty.

Case 5. $g = x^u x'^{u'} y'$ with u' odd.

Let $t = gcd(u, 4)$. Then, all facets of $\mathcal{F}'$ are divided into t orbits under the action of g, and each orbit contains $\frac{4}{t}$ facets. Thus, each orientable coloring of Λ_g gives the same coloring on all $\frac{4}{t}$ facets of each orbit. If we choose an arbitrary facet from each orbit, it suffices to color t chosen facets for $\mathcal{F}'$. When t=1 (i.e., u=1, 3), all facets of $\mathcal{F}'$ have the same coloring, which is impossible by the definition of orientable colorings. Moreover, given that each automorphism $g = x^u x'^{u'} y'$ contains y' as its factor and u' is odd, it suffices to color only $\frac{n}{2} + 1$ neighborly facets of $\mathcal{F}''$ for $\mathcal{F}''$. First, we consider the case t=4 (i.e., u=4).

The argument of Theorem 3.1 can still be carried out. It suffices to consider the case $g = x'y'$ because no essential difference exists between this case and other cases. Set

$$C(n) = \{\lambda \in \Lambda_g | \lambda(F_1) = e_1, \lambda(F_2) = e_2, \lambda(F_5) = e_3, \lambda(F_6) = e_4\}.$$

We have $|\Lambda_g| = 20160|C(n)|$. Write

$C_0(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1, e_1 + e_3 + e_4\}$,
$C_1(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_2 + e_3, e_1 + e_2 + e_4\}$.

Based on the definition of $B(P^n)$ and Remark 2.3, we have $|C(n)| = |C_0(n)| + |C_1(n)|$. Next, we calculate $|C_0(n)|$ and $|C_1(n)|$.

(5.1). Calculation of $|C_0(n)|$.

Write

$C_0^0(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2\}$,

$C_0^1(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_1 + e_3\}$,
$C_0^2(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_1 + e_4\}$,
$C_0^3(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_3 + e_4\}$,
$C_0^4(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2\}$,
$C_0^5(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_1 + e_3\}$,
$C_0^6(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_1 + e_4\}$,
$C_0^7(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_3 + e_4\}$.

By the definition of $C_0(n)$ and Remark 2.3, we have $|C_0(n)| = \sum_{i=0}^{7} |C_0^i(n)|$. Then, our argument proceeds as follows.

(5.1.1). Calculation of $|C_0^0(n)|$.

Using a similar argument of Case 1 in (I) of Theorem 3.1, we have $|C_0^0(n)| = 2|C_0^0(n-2)| + 8|C_0^0(n-4)|$ with initial values of $|C_0^0(2)| = 1$ and $|C_0^0(4)| = 4$. Thus, $|C_0^0(n)| = \tilde{a}(n)$, where $\tilde{a}(n)$ is stated in Theorem 4.1.

(5.1.2). Calculation of $|C_0^1(n)|$.

In this case, $\lambda(F_7) = e_3, e_3+e_1+e_4$. Set $C_0^{1,0}(n) = \{\lambda \in C_0^1(n) | \lambda(F_7) = e_3\}$ and $C_0^{1,1}(n) = C_0^1(n) - C_0^{1,0}(n)$. Using a similar argument of Case 2 in (I) of Theorem 3.1, when $n \geq 10$, $|C_0^{1,0}(n)| = |C_0^{1,0}(n-2)| + 4|C_0^{1,0}(n-4)|$ with initial values of $|C_0^{1,0}(6)| = 4$ and $|C_0^{1,0}(8)| = 8$. Thus, $|C_0^{1,0}(n)| = \tilde{b}(n)$ for $n \geq 6$, where $\tilde{b}(n)$ is stated in Theorem 4.1.

Take an orientable coloring λ in $C_0^{1,1}(n)$. Then $\lambda(F_8) = e_3, e_4$ and $|C_0^{1,1}(n)| = \tilde{b}(n-2) + |C_0^1(n-4)|$ for $n \geq 8$. Therefore, when $n \geq 8, |C_0^1(n)| = \tilde{b}(n) + \tilde{b}(n-2) + |C_0^1(n-4)|$ with initial values of $|C_0^1(2)| = 1, |C_0^1(4)| = 2$ and $|C_0^1(6)| = 6$. Thus, $|C_0^1(n)| = \tilde{c}(n)$.

(5.1.3). Calculation of $|C_0^2(n)|$.

Similar to Case 2 in (I) of Theorem 3.1, we have $|C_0^2(n)| = |C_0^2(n-2)| + 4|C_0^2(n-4)|$ with initial values of $|C_0^2(2)| = 1$ and $|C_0^2(4)| = 4$. Thus, $|C_0^2(n)| = \tilde{d}(n)$.

(5.1.4). Calculation of $|C_0^3(n)|$.

Similar to Case 4 in (I) of Theorem 3.1, we have $|C_0^3(n)| = 2|C_0^3(n-2)| + 4|C_0^3(n-4)| - 6|C_0^3(n-6)| - 3|C_0^3(n-8)| + 4|C_0^3(n-10)|$. A direct observation shows that $|C_0^3(2)| = 1, |C_0^3(4)| = 2, |C_0^3(6)| = 6, |C_0^3(8)| = 14$, and $|C_0^3(10)| = 38$. Thus, $|C_0^3(n)| = \tilde{e}(n)$.

(5.1.5). Calculation of $|C_0^4(n)|$.

If we interchange e_1 and e_2, then the problem is reduced to (5.1.4). Thus, $|C_0^4(n)| = \tilde{e}(n)$.

(5.1.6). Calculation of $|C_0^5(n)|$.

In this case, $\lambda(F_7) = e_3, \lambda(F_8) = e_4, \cdots, \lambda(F_{\frac{n+10}{2}}) = \begin{cases} e_3, & n = 4k, \\ e_4, & n = 4k+2. \end{cases}$

Thus, $|C_0^5(n)| = 1$.

(5.1.7). Calculation of $|C_0^6(n)|$.

Similar to (5.1.6), $|C_0^6(n)| = 1$.

(5.1.8). Calculation of $|C_0^7(n)|$.

Similar to (5.1.6), $|C_0^7(n)| = 1$.

Thus, $|C_0(n)| = \tilde{a}(n) + \tilde{c}(n) + \tilde{d}(n) + 2\tilde{e}(n) + 3$.

(5.2). Calculation of $|C_1(n)|$.

Set

$C_1^0(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_2 + e_3, \lambda(F_4) = e_2\}$,
$C_1^1(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_2 + e_3, \lambda(F_4) = e_2 + e_3 + e_4\}$,
$C_1^2(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_2 + e_4, \lambda(F_4) = e_2\}$,
$C_1^3(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_2 + e_4, \lambda(F_4) = e_2 + e_3 + e_4\}$.

Based on the definition of $C_1(n)$ and Remark 2.3, we have $|C_1(n)| = \sum_{i=0}^{3} |C_1^i(n)|$. Then, the argument proceeds as follows.

(5.2.1). Calculation of $|C_1^0(n)|$.

If we interchange e_1 and e_2, then the problem is reduced to (5.1.2). Thus, $|C_1^0(n)| = \tilde{c}(n)$.

(5.2.2). Calculation of $|C_1^1(n)|$.

Similar to (5.1.6), $|C_1^1(n)| = 1$.

(5.2.3). Calculation of $|C_1^2(n)|$.

If we interchange e_1 and e_2, then the problem is reduced to (5.1.3). Thus, $|C_1^2(n)| = \tilde{d}(n)$.

(5.2.4). Calculation of $|C_1^3(n)|$.

Similar to (5.1.6), $|C_1^3(n)| = 1$.

Thus, $|C_1(n)| = \tilde{c}(n) + \tilde{d}(n) + 2$.

Hence, the number of all orientable colorings in Λ_g is just

$$|\Lambda_g| = 20160[\tilde{a}(n) + 2\tilde{c}(n) + 2\tilde{d}(n) + 2\tilde{e}(n) + 5].$$

There are exactly $\frac{n}{2}$ such automorphisms $g = x'^{u'}y'$ because n is even and u' is odd. Thus,

$$\sum_{g=x'^{u'}y'} |\Lambda_g| = 20160 \cdot \tfrac{n}{2}[\tilde{a}(n) + 2\tilde{c}(n) + 2\tilde{d}(n) + 2\tilde{e}(n) + 5].$$

When t=2 (i.e., u=2), we have

$$\sum_{g=x^2x'^{u'}y'} |\Lambda_g| = 20160 \cdot \tfrac{n}{2}\tilde{a}(n).$$

Thus, $\sum\limits_{g=x^u x'^{u'} y'} |\Lambda_g| = 20160[n\tilde{a}(n) + n\tilde{c}(n) + n\tilde{d}(n) + n\tilde{e}(n) + \frac{5}{2}n].$

Case 6. $g = x^u y x'^{u'} y'$ with u even or u' even.

Similar to Case 3 in (I), Λ_g is empty.

Case 7. $g = x^u y x'^{u'} y'$ with u odd and u' odd.

Similar to Case 5 , we have

$$\sum_{g=x^u y x'^{u'} y'} |\Lambda_g| = 20160n[\tilde{a}(n) + \tilde{c}(n) + \tilde{d}(n) + \tilde{e}(n)].$$

Combining Cases 1 to 7, we complete the proof in (II).

(III) n**=4**

When n=4, $I^2 \times P_n$ is a 4-cube I^4, and the automorphism group $Aut(\mathcal{F}(I^4))$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_4$. As before, let χ_1, χ_2, χ_3, and χ_4 denote generators of the first subgroup $\mathbb{Z}_2$, the second subgroup $\mathbb{Z}_2$, the third subgroup $\mathbb{Z}_2$, and the fourth subgroup $\mathbb{Z}_2$ of $Aut(\mathcal{F}(I^4))$ respectively. If $g = \chi_1$ and $\lambda \in \Lambda_g$, then $\lambda(F_1) = \lambda(F_3)$. Based on Theorem 3.1, we have $|\Lambda_g| = 20160[a(3) + 2b(3) + c(3)]$. Similarly, we also have $|\Lambda_g| = 20160[a(3) + 2b(3) + c(3)]$ for $g = \chi_2$, χ_3 or χ_4. If g=$\chi_1\chi_2$ and $\lambda \in \Lambda_g$, then $\lambda(F_1) = \lambda(F_3)$ and $\lambda(F_2) = \lambda(F_4)$. Based on Case 1 in (I) of Theorem 3.1, we obtain $|\Lambda_g| = 20160a(3)$. Similarly, we also obtain $|\Lambda_g| = 20160a(3)$ for $g = \chi_1\chi_3, \chi_1\chi_4, \chi_2\chi_3, \chi_2\chi_4$ or $\chi_3\chi_4$. If g=$\chi_1\chi_2\chi_3$ and $\lambda \in \Lambda_g$, then $\lambda(F_i) = \lambda(F_{i+2})$ for $i = 1, 2, 5$. We obtain $|\Lambda_g| = 20160 \cdot 4$. Similarly, we also obtain $|\Lambda_g| = 20160 \cdot 4$ for $g = \chi_1\chi_2\chi_4, \chi_1\chi_3\chi_4$ or $\chi_2\chi_3\chi_4$. If g=$\chi_1\chi_2\chi_3\chi_4$ and $\lambda \in \Lambda_g$, then $\lambda(F_i) = \lambda(F_{i+2})$ for $i = 1, 2, 5, 6$. We obtain $|\Lambda_g| = 20160$. Thus

$$\begin{aligned} E_o(I^4) = \tfrac{1}{384}\{&20160 \cdot 4[a(3) + 2b(3) + c(3)] + 20160 \cdot 6a(3) + 20160 \cdot 16 + 20160 + \\ &20160[a(3) + 4b(3) + 2c(3) + 5]\} \\ = 12180&. \end{aligned}$$

□

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An introduction to fuzzy soft topological spaces

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Abstract

The aim of this study is to define fuzzy soft topology which will be compatible to the fuzzy soft theory and investigate some of its fundamental properties. Firstly, we recall some basic properties of fuzzy soft sets and then we give the definitions of cartesian product of two fuzzy soft sets and projection mappings. Secondly, we introduce fuzzy soft topology and fuzzy soft continuous mapping. Moreover, we induce a fuzzy soft topology after given the definition of a fuzzy soft base. Also, we obtain an initial fuzzy soft topology and give the definition of product fuzzy soft topology. Finally, we prove that the category of fuzzy soft topological spaces **FSTOP** is a topological category over **SET**.

Keywords: fuzzy soft set, fuzzy soft topology, fuzzy soft base, initial fuzzy soft topology, product fuzzy soft topology.

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1. Introduction

Most of the existing mathematical tools for formal modeling, reasoning and computing are crisp, deterministic, and precise in character. But, in real life situation, the problems in economics, engineering, environment, social science, medical science etc. do not always involve crisp data. For this reason, we cannot successfully use the traditional classical methods because of various types of uncertainties presented in these problems. To exceed these uncertainties, some kinds of theories were given like theory of fuzzy sets [21], intuitionistic fuzzy sets [4], rough sets [16],i.e., which we can use as mathematical tools for dealing with uncertainties. But all these theories have their inherent difficulties as what were pointed out by Molodtsov in [15]. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theories. Consequently, Molodtsov [15] initiated the concept of soft set theory as a new mathematical tool for dealing with vagueness and uncertainties which is free from the above difficulties.

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Applications of Soft Set Theory in other disciplines and real life problems are now catching momentum. Molodtsov [15] successfully applied the soft set theory into several directions, such as smoothness of functions, game theory, Riemann integration, Perron integration, theory of measurement, and so on. Maji et al. [14] gave first practical application of soft sets in decision making problems. They have also introduced the concept of fuzzy soft set, a more generalized concept, which is a combination of fuzzy set and soft set and also studied some of its properties. Ahmad and Kharal [2, 11] also made further contributions to the properties of fuzzy soft sets and fuzzy soft mappings. Soft set and fuzzy soft set theories have a rich potential for applications in several directions, a few of which have been shown by some authors [15, 18].

The algebraic structure of soft set and fuzzy soft set theories dealing with uncertainties has also been studied by some authors. Aktaş and Çağman [3] have introduced the notion of soft groups. Jun [7] applied soft sets to the theory of BCK/BCI-algebras, and introduced the concept of soft BCK/BCI algebras. Jun and Park [8] and Jun et al. [9, 10] reported the applications of soft sets in ideal theory of BCK/BCI-algebras and d-algebras. Feng et al. [6] defined soft semirings and several related notions to establish a connection between soft sets and semirings. Sun et al. [20] presented the definition of soft modules and construct some basic properties using modules and Molodtsov's definition of soft sets. Aygünoğlu and Aygün [5] introduced the concept of fuzzy soft group and in the meantime, discussed some properties and structural characteristic of fuzzy soft group.

In this study, we consider the topological structure of fuzzy soft set theory. First of all, we give the definition of fuzzy soft topology τ which is a mapping from the parameter set E to $[0,1]^{\widetilde{(X,E)}}$ which satisfies the three certain conditions. With respect to this definition the fuzzy soft topology τ is a fuzzy soft set on the family of fuzzy soft sets $\widetilde{(X,E)}$. Also, since the value of a fuzzy soft set f_A under the mapping τ_e gives the degree of openness of the fuzzy soft set with respect to the parameter $e \in E$, τ_e can be thought as a fuzzy soft topology in the sense of Šostak [19]. In this manner, we introduce fuzzy soft cotopology and give the relations between fuzzy soft topology and fuzzy soft cotopology. Then we define fuzzy soft base. Moreover, we induce a fuzzy soft topology by using a fuzzy soft base on the same set. Also, we obtain an initial fuzzy soft topology and then we give the definition of product fuzzy soft topology. Finally, we show that the category of fuzzy soft topological spaces **FSTOP** is a topological category over **SET** with respect to the forgetful functor.

2. Preliminaries

Throughout this paper, X refers to an initial universe, E is the set of all parameters for X, I^X is the set of all fuzzy sets on X (where, $I = [0,1]$) and for $\lambda \in [0,1]$, $\overline{\lambda}(x) = \lambda$, for all $x \in X$.

2.1. Definition. [2, 13] f_A is called a fuzzy soft set on X, where f is a mapping from E into I^X,i.e., $f_e \triangleq f(e)$ is a fuzzy set on X, for each $e \in A$ and $f_e = \overline{0}$, if $e \notin A$, where $\overline{0}$ is zero function on X. f_e, for each $e \in E$, is called an element of the fuzzy soft set f_A.

$\widetilde{(X,E)}$ denotes the collection of all fuzzy soft sets on X and is called a fuzzy soft universe ([13]).

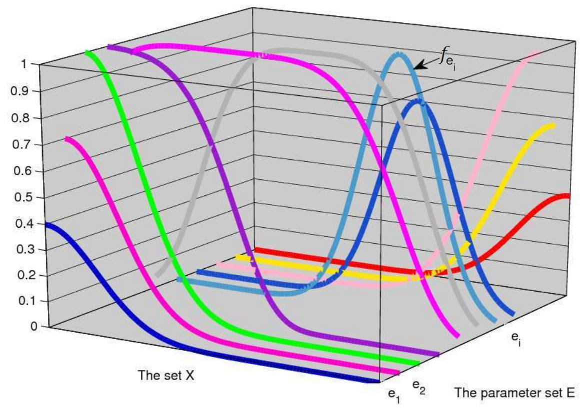

FIGURE 1. A fuzzy soft set f_E

2.2. Definition. [13] For two fuzzy soft sets f_A and g_B on X, we say that f_A is a fuzzy soft subset of g_B and write $f_A \sqsubseteq g_B$ if $f_e \leq g_e$, for each $e \in E$.

2.3. Definition. [13] Two fuzzy soft sets f_A and g_B on X are called equal if $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$.

2.4. Definition. [13] Union of two fuzzy soft sets f_A and g_B on X is the fuzzy soft set $h_C = f_A \sqcup g_B$, where $C = A \cup B$ and $h_e = f_e \vee g_e$, for each $e \in E$. That is, $h_e = f_e \vee \overline{0} = f_e$ for each $e \in A - B$, $h_e = \overline{0} \vee g_e = g_e$ for each $e \in B - A$ and $h_e = f_e \vee g_e$, for each $e \in A \cap B$.

2.5. Definition. [2, 13] Intersection of two fuzzy soft sets f_A and g_B on X is the fuzzy soft set $h_C = f_A \sqcap g_B$, where $C = A \cap B$ and $h_e = f_e \wedge g_e$, for each $e \in E$.

2.6. Definition. The complement of a fuzzy soft set f_A is denoted by f_A^c, where $f^c : E \longrightarrow I^X$ is a mapping given by $f_e^c = \overline{1} - f_e$, for each $e \in E$.

Clearly $(f_A^c)^c = f_A$.

2.7. Definition. [13] (Null fuzzy soft set) A fuzzy soft set f_E on X is called a null fuzzy soft set and denoted by Φ, if $f_e = \overline{0}$, for each $e \in E$.

2.8. Definition. (Absolute fuzzy soft set) A fuzzy soft set f_E on X is called an absolute fuzzy soft set and denoted by $\widetilde{E}$, if $f_e = \overline{1}$, for each $e \in E$. Clearly $(\widetilde{E})^c = \Phi$ and $\Phi^c = \widetilde{E}$.

2.9. Definition. (λ-absolute fuzzy soft set) A fuzzy soft set f_E on X is called a λ-absolute fuzzy soft set and denoted by $\widetilde{E}^{\lambda}$, if $f_e = \overline{\lambda}$, for each $e \in E$. Clearly, $(\widetilde{E}^{\lambda})^c = \widetilde{E}^{1-\lambda}$.

2.10. Proposition. [2] *Let Δ be an index set and $f_A, g_B, h_C, (f_A)_i \triangleq (f_i)_{A_i}, (g_B)_i \triangleq (g_i)_{B_i} \in \widetilde{(X,E)}$,$\forall i \in \Delta$, then we have the following properties:*

(1) $f_A \sqcap f_A = f_A$, $f_A \sqcup f_A = f_A$.
(2) $f_A \sqcap g_B = g_B \sqcap f_A$, $f_A \sqcup g_B = g_B \sqcup f_A$.
(3) $f_A \sqcup (g_B \sqcup h_C) = (f_A \sqcup g_B) \sqcup h_C$, $f_A \sqcap (g_B \sqcap h_C) = (f_A \sqcap g_B) \sqcap h_C$.
(4) $f_A = f_A \sqcup (f_A \sqcap g_B)$, $f_A = f_A \sqcap (f_A \sqcup g_B)$.
(5) $f_A \sqcap \left(\bigsqcup_{i\in\Delta}(g_B)_i\right) = \bigsqcup_{i\in\Delta}(f_A \sqcap (g_B)_i)$.
(6) $f_A \sqcup \left(\bigsqcap_{i\in\Delta}(g_B)_i\right) = \bigsqcap_{i\in\Delta}(f_A \sqcup (g_B)_i)$.
(7) $\Phi \sqsubseteq f_A \sqsubseteq \widetilde{E}$.
(8) $(f_A^c)^c = f_A$.
(9) $\left(\bigsqcap_{i\in\Delta}(f_A)_i\right)^c = \bigsqcup_{i\in\Delta}(f_A)_i^c$.
(10) $\left(\bigsqcup_{i\in\Delta}(f_A)_i\right)^c = \bigsqcap_{i\in\Delta}(f_A)_i^c$.
(11) If $f_A \sqsubseteq g_B$, *then* $g_B^c \sqsubseteq f_A^c$.

2.11. Definition. [5, 11] Let $\varphi : X \longrightarrow Y$ and $\psi : E \longrightarrow F$ be two mappings, where E and F are parameter sets for the crisp sets X and Y, respectively. Then the pair φ_ψ is called a fuzzy soft mapping from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ and denoted by $\varphi_\psi : \widetilde{(X,E)} \longrightarrow \widetilde{(Y,F)}$.

2.12. Definition. [5, 11] Let f_A and g_B be two fuzzy soft sets over X and Y, respectively and let φ_ψ be a fuzzy soft mapping from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$.

(1) The image of f_A under the fuzzy soft mapping φ_ψ, denoted by $\varphi_\psi(f_A)$, is the fuzzy soft set on Y defined by $\varphi_\psi(f_A) = \varphi(f)_{\psi(A)}$, where

$$\varphi(f)_k(y) = \begin{cases} \bigvee\limits_{x\in\varphi^{-1}(y)} \left(\bigvee\limits_{a\in\psi^{-1}(k)\cap A} f_a(x) \right), & \text{if } \varphi^{-1}(y) \neq \emptyset, \psi^{-1}(k)\cap A \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases},$$

$\forall k \in F$, $\forall y \in Y$.

(2) The pre-image of g_B under the fuzzy soft mapping φ_ψ, denoted by $\varphi_\psi^{-1}(g_B)$, is the fuzzy soft set on X defined by $\varphi_\psi^{-1}(g_B) = \varphi^{-1}(g)_{\psi^{-1}(A)}$, where

$$\varphi^{-1}(g)_a(x) = \begin{cases} g_{\psi(a)}(\varphi(x)), & \text{if } \psi(a) \in B; \\ 0, & \text{otherwise.} \end{cases}, \qquad \forall a \in E, \forall x \in X.$$

If φ and ψ is injective (surjective), then φ_ψ is said to be injective (surjective).

2.13. Definition. Let φ_ψ be a fuzzy soft mapping from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ and $\varphi^*_{\psi^*}$ be a fuzzy soft mapping from $\widetilde{(Y,F)}$ into $\widetilde{(Z,K)}$. Then the composition of these mappings from $\widetilde{(X,E)}$ into $\widetilde{(Z,K)}$ is defined as follows: $\varphi^*_{\psi^*} \circ \varphi_\psi \triangleq (\varphi^* \circ \varphi)_{\psi^*\circ\psi}$, where $\psi : E \longrightarrow F$ and $\psi^* : F \longrightarrow K$.

2.14. Proposition. [11] *Let* X *and* Y *be two universes* $f_A, (f_A)_1, (f_A)_2, (f_A)_i \in \widetilde{(X,E)}$, $g_B, (g_B)_1, (g_B)_2, (g_B)_i \in \widetilde{(Y,F)}$ $\forall i \in \Delta$, *where* Δ *is an index set, and* φ_ψ *be a fuzzy soft mapping from* $\widetilde{(X,E)}$ *into* $\widetilde{(Y,F)}$.

(1) If $(f_A)_1 \sqsubseteq (f_A)_2$, *then* $\varphi_\psi((f_A)_1) \sqsubseteq \varphi_\psi((f_A)_2)$.
(2) If $(g_B)_1 \sqsubseteq (g_B)_2$, *then* $\varphi_\psi^{-1}((g_B)_1) \sqsubseteq \varphi_\psi^{-1}((g_B)_2)$.
(3) $f_A \sqsubseteq \varphi_\psi^{-1}(\varphi_\psi(f_A))$, *the equality holds if* φ_ψ *is injective.*
(4) $\varphi_\psi\left(\varphi_\psi^{-1}(f_A)\right) \sqsubseteq f_A$, *the equality holds if* φ_ψ *is surjective.*

(5) $\varphi_\psi\left(\bigsqcup_{i\in\Delta}(f_A)_i\right)=\bigsqcup_{i\in\Delta}\varphi_\psi((f_A)_i)$.
(6) $\varphi_\psi\left(\bigsqcap_{i\in\Delta}(f_A)_i\right)\sqsubseteq\bigsqcap_{i\in\Delta}\varphi_\psi((f_A)_i)$, *the equality holds if* φ_ψ *is injective.*
(7) $\varphi_\psi^{-1}\left(\bigsqcup_{i\in\Delta}(g_B)_i\right)=\bigsqcup_{i\in\Delta}\varphi_\psi^{-1}((g_B)_i)$.
(8) $\varphi_\psi^{-1}\left(\bigsqcap_{i\in\Delta}(g_B)_i\right)=\bigsqcap_{i\in\Delta}\varphi_\psi^{-1}((g_B)_i)$.
(9) $\varphi_\psi^{-1}(g_B^c)=\left(\varphi_\psi^{-1}(g_B)\right)^c$.
(10) $\varphi_\psi^{-1}\left(\widetilde{E}_Y\right)=\widetilde{E}_X,\quad \varphi_\psi^{-1}(\Phi_Y)=\Phi_X$.
(11) $\varphi_\psi\left(\widetilde{E}_X\right)=\widetilde{E}_Y$ *if* φ_ψ *is surjective.*
(12) $\varphi_\psi(\Phi_X)=\Phi_Y$.

2.15. Definition. (Cartesian product of two fuzzy soft sets) Let X_1 and X_2 be nonempty crisp sets. $f_A\in\widetilde{(X_1,E_1)}$ and $g_B\in\widetilde{(X_2,E_2)}$. The cartesian product $f_A\times g_B$ of f_A and g_B is defined by $(f\times g)_{A\times B}$, where, for each $(e,\mathrm{f})\in E_1\times E_2$,
$(f\times g)_{(e,\mathrm{f})}(x,y)=f_e(x)\wedge g_{\mathrm{f}}(y)$, for all $(x,y)\in X\times Y$.
According to this definition the fuzzy soft set $(f\times g)_{A\times B}$ is a fuzzy soft set on $X_1\times X_2$ and the universal parameter set is $E_1\times E_2$.

2.16. Definition. Let $(f_A)_1\times(f_A)_2$ be a fuzzy soft set on $X_1\times X_2$. The projection mappings $(p_q)_i$, $i\in\{1,2\}$, are defined as follows:
$(p_q)_i((f_A)_1\times(f_A)_2)=p_i(f_1\times f_2)_{q_i(A_1\times A_2)}=(f_A)_i$ where $p_i:X_1\times X_2\longrightarrow X_i$ and $q_i:E_1\times E_2\longrightarrow E_i$ are projection mappings in classical meaning.

3. Fuzzy soft topological spaces

To formulate our program and general ideas more precisely, recall first the concept of fuzzy topological space, that is of a pair (X,τ) where X is a set and $\tau:I^X\longrightarrow I$ is a mapping (satisfying some axioms) which assigns to every fuzzy subset of X the real number, which shows "to what extent" this set is open. According to this idea a fuzzy topology τ is a fuzzy set on I^X. This approach has lead us to define fuzzy soft topology which is compatible to the fuzzy soft theory. By our definition, a fuzzy soft topology is a fuzzy soft set on the set of all fuzzy soft sets $\widetilde{(X,E)}$ which denotes "to what extent" this set is open according to the parameter set.

3.1. Definition. A mapping $\tau:E\longrightarrow[0,1]^{\widetilde{(X,E)}}$ is called a fuzzy soft topology on X if it satisfies the following conditions for each $e\in E$.
(O1) $\tau_e(\Phi)=\tau_e(\widetilde{E})=1$.
(O2) $\tau_e(f_A\sqcap g_B)\geq\tau_e(f_A)\wedge\tau_e(g_B),\quad\forall f_A,g_B\in\widetilde{(X,E)}$.
(O3) $\tau_e(\bigsqcup_{i\in\Delta}(f_A)_i)\geq\bigwedge_{i\in\Delta}\tau_e((f_A)_i),\forall(f_A)_i\in\widetilde{(X,E)},i\in\Delta$.
A fuzzy soft topology is called enriched if it provides that
(O1)′ $\tau_e(\widetilde{E}^\lambda)=1$.
Then the pair (X,τ_E) is called a fuzzy soft topological space. The value $\tau_e(f_A)$ is interpreted as the degree of openness of a fuzzy soft set f_A with respect to parameter $e\in E$.

Let τ_E^1 and τ_E^2 be fuzzy soft topologies on X. We say that τ_E^1 is finer than τ_E^2 (τ_E^2 is coarser than τ_E^1), denoted by $\tau_E^2 \sqsubseteq \tau_E^1$, if $\tau_e^2(f_A) \leq \tau_e^1(f_A)$ for each $e \in E, f_A \in \widetilde{(X,E)}$.

Example Let $\mathcal{T}$ be a fuzzy topology on X in Šostak's sense, that is, $\mathcal{T}$ is a mapping from I^X to I. Take $E = I$ and define $\overline{\mathcal{T}} : E \longrightarrow I^X$ as $\overline{\mathcal{T}}(e) \triangleq \{\mu : \mathcal{T}(\mu) \geq e\}$ which is levelwise fuzzy topology of $\mathcal{T}$ in Chang's sense, for each $e \in I$. However, it is well known that each Chang's fuzzy topology can be considered as Šostak fuzzy topology by using fuzzifying method. Hence, $\mathcal{T}(e)$ satisfies (O1), (O2) and (O3).

According to this definition and by using the decomposition theorem of fuzzy sets [12], if we know the resulting fuzzy soft topology, then we can find the first fuzzy topology. Therefore, we can say that a fuzzy topology can be uniquely represented as a fuzzy soft topology.

3.2. Definition. Let $\widetilde{(X,\tau)}$ and $\widetilde{(Y,\tau^*)}$ be fuzzy soft topological spaces. A fuzzy soft mapping φ_ψ from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ is called a fuzzy soft continuous map if $\tau_e(\varphi_\psi^{-1}(g_B)) \geq \tau^*_{\psi(e)}(g_B)$ for all $g_B \in \widetilde{(Y,F)}, e \in E$.

The category of fuzzy soft topological spaces and fuzzy soft continuous mappings is denoted by **FSTOP**.

3.3. Proposition. *Let $\{\tau_k\}_{k\in\Gamma}$ be a family of fuzzy soft topologies on X. Then $\tau = \bigwedge_{k\in\Gamma} \tau_k$ is also a fuzzy soft topology on X, where $\tau_e(f_A) = \bigwedge_{k\in\Gamma} (\tau_k)_e(f_A), \forall e \in E, f_A \in \widetilde{(X,E)}$.*

Proof. It is straightforward and therefore is omitted. □

3.4. Definition. A mapping $\eta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ is called a fuzzy soft cotopology on X if it satisfies the following conditions for each $e \in E$:

(C1) $\eta_e(\Phi) = \eta_e(\widetilde{E}) = 1$.

(C2) $\eta_e(f_A \sqcup g_B) \geq \eta_e(f_A) \wedge \eta_e(g_B), \quad \forall f_A, g_B \in \widetilde{(X,E)}$.

(C3) $\eta_e(\bigsqcap_{i\in\Delta}(f_A)_i) \geq \bigwedge_{i\in\Delta} \eta_e((f_A)_i), \forall (f_A)_i \in \widetilde{(X,E)}, i \in \Delta$.

The pair (X,η) is called a fuzzy soft cotopological space.

Let τ be a fuzzy soft topology on X, then the mapping $\eta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ defined by $\eta_e(f_A) = \tau_e(f_A^c), \forall e \in E$ is a fuzzy soft cotopology on X. Let η be a fuzzy soft cotopology on X, then the mapping $\tau : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ defined by $\tau_e(f_A) = \eta_e(f_A^c), \forall e \in E$, is a fuzzy soft topology on X.

3.5. Definition. A mapping $\beta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ is called a fuzzy soft base on X if it satisfies the following conditions for each $e \in E$:

(B1) $\beta_e(\Phi) = \beta_e(\widetilde{E}) = 1$.

(B2) $\beta_e(f_A \sqcap g_B) \geq \beta_e(f_A) \wedge \beta_e(g_B), \quad \forall f_A, g_B \in \widetilde{(X,E)}$.

3.6. Theorem. *Let β be a fuzzy soft base on X. Define a map $\tau_\beta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ as follows:*

$$(\tau_\beta)_e(f_A) = \bigvee \left\{ \bigwedge_{j\in\Lambda} \beta_e((f_A)_j) \mid f_A = \bigsqcup_{j\in\Lambda} (f_A)_j \right\}, \quad \forall e \in E.$$

Then τ_β is the coarsest fuzzy soft topology on X for which $(\tau_\beta)_e(f_A) \geq \beta_e(f_A)$, for all $e \in E, f_A \in \widetilde{(X,E)}$.

Proof. (O1) It is trivial from the definition of τ_β.

(O2) Let $e \in E$. For all families $\{(f_A)_j \mid f_A = \bigsqcup_{j\in\Lambda}(f_A)_j\}$ and $\{(g_B)_k \mid g_B = \bigsqcup_{k\in\Gamma}(g_B)_k\}$, there exists a family $\{(f_A)_j \sqcap (g_B)_k\}$ such that

$$f_A \sqcap g_B = \left(\bigsqcup_{j\in\Lambda}(f_A)_j\right) \sqcap \left(\bigsqcup_{k\in\Gamma}(g_B)_k\right) = \bigsqcup_{j\in\Lambda, k\in\Gamma} ((f_A)_j \sqcap (g_B)_k).$$

It implies the followings:

$$\begin{aligned}
(\tau_\beta)_e(f_A \sqcap g_B) &\geq \bigwedge_{j\in\Lambda,k\in\Gamma} \beta_e((f_A)_j \sqcap (g_B)_k) \\
&\geq \bigwedge_{j\in\Lambda,k\in\Gamma} (\beta_e((f_A)_j) \wedge \beta_e((g_B)_k)) \\
&\geq (\bigwedge_{j\in\Lambda} \beta_e((f_A)_j)) \wedge (\bigwedge_{k\in\Gamma} \beta_e((g_B)_k)) \\
&\geq (\tau_\beta)_e(f_A) \wedge (\tau_\beta)_e(g_B).
\end{aligned}$$

(O3) Let $e \in E$ and $\wp_i$ be the collection of all index sets K_i such that $\{(f_A)_{i_k} \in \widetilde{(X,E)} \mid (f_A)_i = \bigsqcup_{k\in K_i}(f_A)_{i_k}\}$ with $f_A = \bigsqcup_{i\in\Gamma}(f_A)_i = \bigsqcup_{i\in\Gamma}\bigsqcup_{k\in K_i}(f_A)_{i_k}$. For each $i \in \Gamma$ and each $\Psi \in \Pi_{i\in\Gamma}\wp_i$ with $\Psi(i) = K_i$, we have $(\tau_\beta)_e(f_A) \geq \bigwedge_{i\in\Gamma}(\bigwedge_{k\in K_i} \beta_e((f_A)_{i_k}))$.

Put $a_{i,\Psi_i} = \bigwedge_{k\in K_i} (\beta_e((f_A)_{i_k}))$. Then we have the following:

$$\begin{aligned}
(\tau_\beta)_e(f_A) &\geq \bigvee_{\Psi\in\Pi_{i\in\Gamma}\wp_i} \left(\bigwedge_{i\in\Gamma} a_{i,\Psi(i)}\right) \\
&= \bigwedge_{i\in\Gamma}\left(\bigvee_{M_i\in\wp_i} a_{i,M_i}\right) \\
&= \bigwedge_{i\in\Gamma}\left(\bigvee_{M_i\in\wp_i}\left(\bigwedge_{m\in M_i}(\beta_e((f_A)_{i_m}))\right)\right) \\
&= \bigwedge_{i\in\Gamma}(\tau_\beta)_e((f_A)_i).
\end{aligned}$$

Thus, τ_β is a fuzzy soft topology on X. Let $\tau \sqsupseteq \beta$, then for every $e \in E$ and $f_A = \bigsqcup_{j\in\Lambda}(f_A)_j$, we have

$$\tau_e(f_A) \geq \bigwedge_{j\in\Lambda} \tau_e((f_A)_j) \geq \bigwedge_{j\in\Lambda} \beta_e((f_A)_j).$$

If we take supremum over the family $\{(f_A)_j \in \widetilde{(X,E)} \mid f_A = \bigsqcup_{j\in\Lambda} (f_A)_j\}$, then we obtain that $\tau \sqsupseteq \tau_\beta$. □

3.7. Lemma. *Let τ be a fuzzy soft topology on X and β be a fuzzy soft base on Y. Then a fuzzy soft mapping φ_ψ from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ is fuzzy soft continuous if and only if $\tau_e(\varphi_\psi^{-1}(g_B)) \geq \beta_{\psi(e)}(g_B)$, for each $e \in E, g_B \in \widetilde{(Y,F)}$.*

Proof. ($\Rightarrow$) Let $\varphi_\psi : (X,\tau) \longrightarrow (Y,\tau_\beta)$ be a fuzzy soft continuous mapping and $g_B \in \widetilde{(Y,F)}$. Then,

$$\tau_e(\varphi_\psi^{-1}(g_B)) \geq (\tau_\beta)_{\psi(e)}(g_B) \geq \beta_{\psi(e)}(g_B).$$

($\Leftarrow$) Let $\tau_e(\varphi_\psi^{-1}(g_B)) \geq \beta_{\psi(e)}(g_B)$, for each $g_B \in \widetilde{(Y,F)}$. Let $h_C \in \widetilde{(Y,F)}$. For every family of $\{(h_C)_j \in \widetilde{(Y,F)} \mid h_C = \bigsqcup_{j\in\Gamma} (h_C)_j\}$, we have

$$\begin{aligned}
\tau_e(\varphi_\psi^{-1}(h_C)) &= \tau_e\left(\varphi_\psi^{-1}\left(\bigsqcup_{j\in\Gamma} (h_C)_j\right)\right) \\
&= \tau_e\left(\bigsqcup_{j\in\Gamma} \varphi_\psi^{-1}((h_C)_j)\right) \\
&\geq \bigwedge_{j\in\Gamma} \tau_e(\varphi_\psi^{-1}((h_C)_j)) \\
&\geq \bigwedge_{j\in\Gamma} \beta_{\psi(e)}((h_C)_j).
\end{aligned}$$

If we take supremum over the family of $\{(h_C)_j \in \widetilde{(Y,F)} \mid h_C = \bigsqcup_{j\in\Gamma} (h_C)_j\}$, we obtain
$\tau_e(\varphi_\psi^{-1}(h_C)) \geq (\tau_\beta)_{\psi(e)}(h_C)$. □

3.8. Theorem. *Let $\{(X_i, (\tau_i)_{E_i})\}_{i\in\Gamma}$ be a family of fuzzy soft topological spaces, X be a set, E be a parameter set and for each $i \in \Gamma$, $\varphi_i : X \to X_i$ and $\psi_i : E \to E_i$ be maps. Define $\beta : E \to [0,1]^{\widetilde{(X,E)}}$ on X by:*

$$\beta_e(f_A) = \bigvee \left\{ \bigwedge_{j=1}^{n} (\tau_{k_j})_{\psi_{k_j}(e)}((f_A)_{k_j}) \mid f_A = \prod_{j=1}^{n} (\varphi_\psi)_{k_j}^{-1}((f_A)_{k_j}) \right\},$$

where $\bigvee$ is taken over all finite subsets $K = \{k_1, k_2, ..., k_n\} \subset \Gamma$. Then,

(1) β is a fuzzy soft base on X.

(2) The fuzzy soft topology τ_β generated by β is the coarsest fuzzy soft topology on X for which all $(\varphi_\psi)_i, i \in \Gamma$ are fuzzy soft continuous maps.

(3) A map $\varphi_\psi : (Y, \delta_F) \to (X, (\tau_\beta)_E)$ is fuzzy soft continuous iff for each $i \in \Gamma$, $(\varphi_\psi)_i \circ \varphi_\psi : (Y, \delta_F) \to (X_i, (\tau_i)_{E_i})$ is a fuzzy soft continuous map.

Proof. (1) (B1) Since $f_A = (\varphi_\psi)_i^{-1}(f_A)$ for each $f_A \in \{\Phi, \widetilde{E}\}$, $\beta_e(\Phi) = \beta_e(\widetilde{E}) = 1$, for each $e \in E$.

(B2) For all finite subsets $K = \{k_1, k_2, ..., k_n\}$ and $J = \{j_1, j_2, ..., j_m\}$ of Γ such that $f_A = \prod_{i=1}^{n} (\varphi_\psi)^{-1}_{k_i}((f_A)_{k_i})$ and $g_B = \prod_{i=1}^{m} (\varphi_\psi)^{-1}_{j_i}((g_B)_{j_i})$, we have $f_A \sqcap g_B = \left(\prod_{i=1}^{n} (\varphi_\psi)^{-1}_{k_i}((f_A)_{k_i})\right) \sqcap \left(\prod_{i=1}^{m} (\varphi_\psi)^{-1}_{j_i}((g_B)_{j_i})\right)$.

Furthermore, we have for each $k \in K \cap J$,

$$(\varphi_\psi)^{-1}_{k}((f_A)_k) \sqcap (\varphi_\psi)^{-1}_{k}((g_B)_k) = (\varphi_\psi)^{-1}_{k}((f_A)_k \sqcap (g_B)_k).$$

Put $f_A \sqcap g_B = \prod_{m_i \in K \cup J} (\varphi_\psi)^{-1}_{m_i}((h_C)_{m_i})$ where

$$(h_C)_{m_i} = \begin{cases} (f_A)_{m_i}, & \text{if } m_i \in K - (K \cap J); \\ (g_B)_{m_i}, & \text{if } m_i \in J - (K \cap J); \\ (f_A)_{m_i} \sqcap (g_B)_{m_i}, & \text{if } m_i \in (K \cap J). \end{cases}$$

So we have

$$\begin{aligned} \beta_e((f_A) \sqcap (g_B)) &\geq \bigwedge_{j \in K \cup J} (\tau_j)_{\psi_j(e)}((h_C)_j) \\ &\geq \left(\bigwedge_{i=1}^{n} (\tau_{k_i})_{\psi_{k_i}(e)}((f_A)_{k_i})\right) \wedge \left(\bigwedge_{i=1}^{m} (\tau_{j_i})_{\psi_{j_i}(e)}((g_B)_{j_i})\right). \end{aligned}$$

If we take supremum over the families $\{f_A = \prod_{i=1}^{n} (\varphi_\psi)^{-1}_{k_i}((f_A)_{k_i})\}$ and $\{g_B = \prod_{i=1}^{m} (\varphi_\psi)^{-1}_{j_i}((g_B)_{j_i})\}$, then we have,

$$\beta_e(f_A \sqcap g_B) \geq \beta_e(f_A) \wedge \beta_e(g_B), \forall e \in E.$$

(2) For each $(f_A)_i \in \widetilde{(X_i, E_i)}$, one family $\{(\varphi_\psi)^{-1}_{i}((f_A)_i)\}$ and $i \in \Gamma$, we have

$(\tau_\beta)_e((\varphi_\psi)^{-1}_{i}((f_A)_i)) \geq \beta_e((\varphi_\psi)^{-1}_{i}((f_A)_i)) \geq (\tau_i)_{\psi_i(e)}((f_A)_i)$, for each $e \in E$.

Therefore, for all $i \in \Gamma$, $(\varphi_\psi)_i : (X, (\tau_\beta)_E) \longrightarrow (X_i, (\tau_i)_{E_i})$ is fuzzy soft continuous.

Let $(\varphi_\psi)_i : (X, \zeta_E) \longrightarrow (X_i, (\tau_i)_{E_i})$ be fuzzy soft continuous, that is, for each $i \in \Gamma$ and $(f_A)_i \in \widetilde{(X_i, E_i)}$, $\zeta_e((\varphi_\psi)^{-1}_{i}((f_A)_i)) \geq (\tau_i)_{\psi_i(e)}((f_A)_i)$.

For all finite subsets $K = \{k_1, ..., k_n\}$ of Γ such that $f_A = \prod_{i=1}^{n} (\varphi_\psi)^{-1}_{k_i}((f_A)_{k_i})$ we have

$$\zeta_e(f_A) \geq \bigwedge_{i=1}^{n} \zeta_e((\varphi_\psi)^{-1}_{k_i}((f_A)_{k_i})) \geq \bigwedge_{i=1}^{n} (\tau_{k_i})_{\psi_{k_i}(e)}((f_A)_{k_i}).$$

It implies $\zeta_e(f_A) \geq \beta_e(f_A)$, for all $e \in E, f_A \in \widetilde{(X, E)}$. By Theorem 3.6, $\zeta \sqsupseteq \tau_\beta$.

(3) ($\Rightarrow$) Let $\widetilde{\varphi_\psi} : (Y, \delta_F) \to (X, (\tau_\beta)_E)$ be fuzzy soft continuous. For each $i \in \Gamma$ and $(f_A)_i \in \widetilde{(X_i, E_i)}$ we have

$$\delta_{\mathrm{f}}((\varphi_i\circ\varphi)^{-1}_{\psi_i\circ\psi}((f_A)_i))=\delta_{\mathrm{f}}(\varphi^{-1}_{\psi}((\varphi_\psi)^{-1}_i((f_A)_i)))\geq(\tau_\beta)_{\psi(\mathrm{f})}((\varphi_\psi)^{-1}_i((f_A)_i))\geq (\tau_i)_{(\psi_i\circ\psi)(\mathrm{f})}((f_A)_i).$$

Hence, $(\varphi_i\circ\varphi)_{\psi_i\circ\psi}:(Y,\delta_F)\to(X_i,(\tau_i)_{E_i})$ is fuzzy soft continuous.

($\Leftarrow$) For all finite subsets $K=\{k_1,...,k_n\}$ of Γ such that $f_A=\prod_{i=1}^{n}(\varphi_\psi)^{-1}_{k_i}((f_A)_{k_i})$, since $(\varphi_{k_i}\circ\varphi)_{\psi_{k_i}\circ\psi}:(Y,\delta_F)\to(X_{k_i},(\tau_{k_i})_{E_{k_i}})$ is fuzzy soft continuous, $\delta_{\mathrm{f}}(\varphi^{-1}_{\psi}((\varphi_\psi)^{-1}_{k_i}((f_A)_{k_i})))\geq(\tau_{k_i})_{(\psi_i\circ\psi)(\mathrm{f})}((f_A)_{k_i}),\forall\ \mathrm{f}\in F$.

Hence we have

$$\begin{aligned}\delta_{\mathrm{f}}(\varphi^{-1}_{\psi}(f_A)) &= \delta_{\mathrm{f}}(\varphi^{-1}_{\psi}(\prod_{i=1}^{n}(\varphi_\psi)^{-1}_{k_i}((f_A)_{k_i})))\\ &= \delta_{\mathrm{f}}(\prod_{i=1}^{n}(\varphi^{-1}_{\psi}((\varphi_\psi)^{-1}_{k_i}(((f_A)_{k_i})))))\\ &\geq \bigwedge_{i=1}^{n}\delta_{\mathrm{f}}(\varphi^{-1}_{\psi}((\varphi_\psi)^{-1}_{k_i}(((f_A)_{k_i}))))\\ &\geq \bigwedge_{i=1}^{n}(\tau_{k_i})_{(\psi_{k_i}\circ\psi)(\mathrm{f})}((f_A)_{k_i}).\end{aligned}$$

This inequality implies that $\delta_{\mathrm{f}}(\varphi^{-1}_{\psi}(f_A))\geq\beta_{\psi(\mathrm{f})}(f_A)$ for each $f_A\in\widetilde{(X,E)},\mathrm{f}\in F$.

By Lemma 3.7, $\varphi_\psi:(Y,\delta_F)\to(X,(\tau_\beta)_E)$ is fuzzy soft continuous.

Let $\{(X_i,(\tau_i)_{E_i})\}_{i\in\Gamma}$ be a family of fuzzy soft topological spaces, X be a set, E be a parameter set and for each $i\in\Gamma$, $\varphi_i:X\to X_i$ and $\psi_i:E\to E_i$ be maps. The initial fuzzy soft topology τ_β on X is the coarsest fuzzy soft topology on X for which all $(\varphi_\psi)_i, i\in\Gamma$ are fuzzy soft continuous maps. □

3.9. Definition. [1] A category **C** is called a topological category over **SET** with respect to the usual forgetful functor from **C** to **SET** if it satisfies the following conditions:

(TC1) *Existence of initial structures:* For any X, any class J, and any family $((X_j,\xi_j))_{j\in J}$ of **C**-object and any family $(f_j:X\longrightarrow X_j)_{j\in J}$ of maps, there exists a unique **C**-structure ξ on X which is initial with respect to the source $(f_j:X\longrightarrow (X_j,\xi_j))_{j\in J}$, this means that for a **C**-object (Y,η), a map $g:(Y,\eta)\longrightarrow(X,\xi)$ is a **C**-morphism if and only if for all $j\in J$, $f_j\circ g:(Y,\eta)\longrightarrow(X_j,\xi_j)$ is a **C**-morphism.

(TC2) *Fibre smallness:* For any set X, the **C**-fibre of X, i.e., the class of all **C**-structure on X, which we denote **C**(X), is a set.

3.10. Theorem. *The category* **FSTOP** *is a topological category over* **SET** *with respect to the forgetful functor* $V:$ **FSTOP**$\longrightarrow$ **SET** *which is defined by* $V(X,\tau_E)=X$ *and* $V(\varphi_\psi)=\varphi$.

Proof. The proof is straightforward and follows from Theorem 3.8. □

3.11. Definition. Let $\{(X_i,(\tau_i)_{E_i})\}_{i\in\Gamma}$ be a family of fuzzy soft topological spaces, for each $i\in\Gamma$, E_i be parameter sets, $X=\Pi_{i\in\Gamma}X_i$ and $E=\Pi_{i\in\Gamma}E_i$. Let $p_i:X\longrightarrow X_i$ and $q_i:E\longrightarrow E_i$ be projection maps, for all $i\in\Gamma$. The product of fuzzy soft topologies (X, τ_E) with respect to parameter set E is the coarsest fuzzy soft topology on X for which all $(p_q)_i, i \in \Gamma$, are fuzzy soft continuous maps.

4. Conclusion

In this paper, we have considered the topological structure of fuzzy soft set theory. We have given the definition of fuzzy soft topology τ which is a mapping from the parameter set E to $[0,1]^{\widetilde{(X,E)}}$ which satisfy the three certain conditions. Since the value of a fuzzy soft set f_A under the mapping τ_e gives us the degree of openness of the fuzzy soft set with respect to the parameter $e \in E$, τ_e can be thought of as a fuzzy soft topology in the sense of Šostak. In this sense, we have introduced fuzzy soft cotopology and given the relations between fuzzy soft topology and fuzzy soft cotopology. Then we have defined fuzzy soft base and by using a fuzzy soft base we have obtained a fuzzy soft topology on the same set. Also, we have introduced an initial fuzzy soft topology and then we have given the definition of product fuzzy soft topology. Further, we have proved that the category of fuzzy soft topological spaces **FSTOP** is a topological category over **SET** with respect to the forgetful functor.

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Convergence to common fixed points of multi-step iteration process for generalized asymptotically quasi-nonexpansive mappings in convex metric spaces

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Abstract

In this paper, we study strong convergence of multi-step iterations with errors for a finite family of generalized asymptotically quasi-nonexpansive mappings in the framework of convex metric spaces. The new iteration scheme includes modified Mann and Ishikawa iterations with errors, the three-step iteration scheme of Xu and Noor as special cases in Banach spaces. Our results extend and generalize many known results from the current literature.

Keywords: Generalized asymptotically quasi-nonexpansive mapping, multi-step iterations with errors, common fixed point, strong convergence, convex metric

2000 AMS Classification: 47H09, 47H10.

1. Introduction and Preliminaries

Let T be a self map on a nonempty subset C of a metric space (X, d). Denote the set of fixed points of T by $F(T) = \{x \in C : T(x) = x\}$. We say that T is:

(1) nonexpansive if

$$d(Tx, Ty) \leq d(x, y) \tag{1.1}$$

for all $x, y \in C$;

(2) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p) \tag{1.2}$$

for all $x \in C$ and $p \in F(T)$;

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(3) asymptotically nonexpansive [5] if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$d(T^n x, T^n y) \quad \leq \quad (1 + r_n) d(x, y), \tag{1.3}$$

for all $x, y \in C$ and $n \geq 1$;

(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$d(T^n x, p) \quad \leq \quad (1 + r_n) d(x, p), \tag{1.4}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$;

(5) generalized asymptotically quasi-nonexpansive [6] if $F(T) \neq \emptyset$ and there exist two sequences of real numbers $\{r_n\}$, $\{s_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty} s_n$ such that

$$d(T^n x, p) \quad \leq \quad (1 + r_n) d(x, p) + s_n, \tag{1.5}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$;

(6) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \quad \leq \quad L \ d(x, y), \tag{1.6}$$

for all $x, y \in C$ and $n \geq 1$;

(7) semi-compact if for any bounded sequence $\{x_n\}$ in C with $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there is a convergent subsequence of $\{x_n\}$.

Let $\{x_n\}$ be a sequence in a metric space (X, d), and let C be a subset of X. We say that $\{x_n\}$ is:

(8) of monotone type [22] with respect to C if for each $p \in C$, there exist two sequences $\{a_n\}$ and $\{b_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$ and

$$d(x_{n+1}, p) \quad \leq \quad (1 + a_n) d(x_n, p) + b_n. \tag{$*$}$$

1.1. Remark. (1) It is clear that the nonexpansive mappings with the nonempty fixed point set $F(T)$ are quasi-nonexpansive.

(2) The linear quasi-nonexpansive mappings are nonexpansive, but it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive; for example, define $T(x) = (x/2) sin(1/x)$ for all $x \neq 0$ and

$T(0) = 0$ in $\mathbb{R}$.

(3) It is obvious that if T is nonexpansive, then it is asymptotically nonexpansive with the constant sequence $\{1\}$.

(4) If T is asymptotically nonexpansive, then it is uniformly Lipschitzian with the uniform Lipschitz constant $L = \sup\{1 + r_n : n \geq 1\}$. However, the converse of this claim is not true.

(5) If in definition (5), $s_n = 0$ for all $n \geq 1$, then T becomes asymptotically quasi-nonexpansive, and hence the class of generalized asymptotically quasi-nonexpansive maps includes the class of asymptotically quasi-nonexpansive maps.

In 1991, Schu [16, 17] introduced the following iterative scheme: let X be a normed linear space, let C be a nonempty convex subset of X, and let $T\colon C \to C$ be a given mapping. Then, for arbitrary $x_1 \in C$, the modified Ishikawa iterative scheme $\{x_n\}$ is defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n, \ n \geq 1, \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are some suitable sequences $[0,1]$. With X, C, $\{\alpha_n\}$, and x_1 as above, the modified Mann iterative scheme $\{x_n\}$ is defined by

$$\begin{aligned} & x_1 \in C, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1. \end{aligned} \tag{1.8}$$

In 1998, Xu [21] introduced the following iterative scheme: let X be a normed linear space, let C be a nonempty convex subset of X, and let $T\colon C \to C$ be a given mapping. Then, for arbitrary $x_1 \in C$, the Ishikawa iterative scheme $\{x_n\}$ with errors is defined by

$$\begin{aligned} y_n &= \bar{a_n}x_n + \bar{b_n}Tx_n + \bar{c_n}v_n \\ x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \ n \geq 1, \end{aligned} \tag{1.9}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in C and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a_n}\}$, $\{\bar{b_n}\}$, $\{\bar{c_n}\}$ are sequences $[0,1]$ with $a_n + b_n + c_n = \bar{a_n} + \bar{b_n} + \bar{c_n} = 1$. With X, C, $\{u_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and x_1 as above, the Mann iterative scheme $\{x_n\}$ with errors is defined by

$$\begin{aligned} & x_1 \in C, \\ x_{n+1} &= a_n x_n + b_n T x_n + c_n u_n, \ n \geq 1. \end{aligned} \tag{1.10}$$

Based on the iterative scheme with errors introduced by Xu [21], the following iteration schemes have been used and studied by several authors (see [1, 3, 12]).

Let X be a normed linear space, let C be a nonempty convex subset of X, and let $T\colon C \to C$ be a given mapping. Then, for arbitrary $x_1 \in C$, the modified Ishikawa iteration scheme $\{x_n\}$ with errors is defined by

$$\begin{aligned} y_n &= \bar{a_n} x_n + \bar{b_n} T^n x_n + \bar{c_n} v_n \\ x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n,\ \ n \geq 1, \end{aligned} \tag{1.11}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in C and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a_n}\}$, $\{\bar{b_n}\}$, $\{\bar{c_n}\}$ are sequences $[0,1]$ with $a_n + b_n + c_n = \bar{a_n} + \bar{b_n} + \bar{c_n} = 1$. With X, C, $\{u_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and x_1 as above, the modified Mann iteration scheme $\{x_n\}$ with errors is defined by

$$\begin{aligned} & x_1 \in C, \\ x_{n+1} &= a_n x_n + b_n T^n x_n + c_n u_n,\ \ n \geq 1. \end{aligned} \tag{1.12}$$

Recently, Imnang and Suantai [6] studied multi-step Noor iterations with errors for a finite family of generalized asymptotically quasi-nonexpansive mappings and established some strong convergence theorems in the framework of uniformly convex Banach spaces. The scheme of [6] is as follows: Let $T_i\colon C \to C$ $(i = 1, 2, \ldots, k)$ be mappings and $F = \bigcap_{i=1}^{k} F(T_i)$. For a given $x_1 \in C$, and a fixed $k \in \mathbb{N}$ ($\mathbb{N}$ denote the set of all positive integers), compute the iterative sequences $\{x_n\}$ and $\{y_{in}\}$ by

$$\begin{aligned} x_{n+1} &= y_{kn} = \alpha_{kn} T_k^n y_{(k-1)n} + \beta_{kn} x_n + \gamma_{kn} u_{kn}, \\ y_{(r-1)n} &= \alpha_{(k-1)n} T_{k-1}^n y_{(k-2)n} + \beta_{(k-1)n} x_n + \gamma_{(k-1)n} u_{(k-1)n}, \\ &\vdots \\ y_{3n} &= \alpha_{3n} T_3^n y_{2n} + \beta_{3n} x_n + \gamma_{3n} u_{3n}, \\ y_{2n} &= \alpha_{2n} T_2^n y_{1n} + \beta_{2n} x_n + \gamma_{2n} u_{2n}, \\ y_{1n} &= \alpha_{1n} T_1^n y_{0n} + \beta_{1n} x_n + \gamma_{1n} u_{1n}, \qquad n \geq 1. \end{aligned} \tag{1.13}$$

where $y_{0n} = x_n$ and $\{u_{1n}\}, \{u_{2n}\}, \ldots, \{u_{kn}\}$ are bounded sequences in C with $\{\alpha_{in}\}$, $\{\beta_{in}\}$, and $\{\gamma_{in}\}$ are appropriate real sequences in $[0,1]$ such that $\alpha_{in} + \beta_{in} + \gamma_{in} = 1$ for all $i = 1, 2, \ldots, k$ and all n. This iteration scheme includes the modified Mann iteration scheme (1.12), the modified Ishikawa iteration scheme (1.11) and extends the three-step iteration by Xu and Noor [20].

One of the most interesting aspects of metric fixed point theory is to extend a linear version of a known result to the nonlinear case in metric spaces. To achieve this, Takahashi [18] introduced a convex structure in a metric space (X, d) and the properties of the space.

1.2. Definition. Let (X, d) be a metric space and $I = [0,1]$. A mapping $W\colon X^3 \times I^3 \to X$ is said to be a convex structure on X if it satisfies the following condition:

$$d(u, W(x, y, z; \alpha, \beta, \gamma)) \leq \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z),$$

for any $u, x, y, z \in X$ and for any $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$.

If (X, d) is a metric space with a convex structure W, then (X, d) is called a *convex metric space* and denotes it by (X, d, W).

1.3. Remark. It is easy to prove that every linear normed space is a convex metric space with a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for all $x, y, z \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$. But there exist some convex metric spaces which can not be embedded into any linear normed spaces (see, Takahashi [18]).

1.4. Example. Let $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0\}$. For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$, we define a mapping $W \colon X^3 \times I^3 \to X$ by

$$W(x, y, z; \alpha, \beta, \gamma) = (\alpha x_1 + \beta y_1 + \gamma z_1, \alpha x_2 + \beta y_2 + \gamma z_2, \alpha x_3 + \beta y_3 + \gamma z_3)$$

and define a metric $d \colon X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 y_1 + x_2 y_2 + x_3 y_3|.$$

Then we can show that (X, d, W) is a convex metric space, but it is not a normed space.

Denote the indexing set $\{1, 2, \ldots, k\}$ by I. We now translate the scheme (1.13) from the normed space setting to the more general setup of convex metric space as follows:

$$x_1 \in C, \quad x_{n+1} = U_{n(k)} x_n, \quad n \geq 1, \tag{1.14}$$

where

$$\begin{aligned}
U_{n(0)} &= I, \text{ the identity map}, \\
U_{n(1)} x &= W(T_1^n U_{n(0)} x, x, u_{n(1)};\ \alpha_{n(1)}, \beta_{n(1)}, \gamma_{n(1)}), \\
U_{n(2)} x &= W(T_2^n U_{n(1)} x, x, u_{n(2)};\ \alpha_{n(2)}, \beta_{n(2)}, \gamma_{n(2)}), \\
&\vdots \\
U_{n(k-1)} x &= W(T_{k-1}^n U_{n(k-2)} x, x, u_{n(k-1)};\ \alpha_{n(k-1)}, \beta_{n(k-1)}, \gamma_{n(k-1)}), \\
U_{n(k)} x &= W(T_k^n U_{n(k-1)} x, x, u_{n(k)};\ \alpha_{n(k)}, \beta_{n(k)}, \gamma_{n(k)}), \qquad n \geq 1,
\end{aligned}$$

where $\{u_{n(1)}\}, \{u_{n(2)}\}, \ldots, \{u_{n(k)}\}$ are bounded sequences in C with $\{\alpha_{n(i)}\}$, $\{\beta_{n(i)}\}$, and $\{\gamma_{n(i)}\}$ are appropriate real sequences in $[0, 1]$ such that $\alpha_{n(i)} + \beta_{n(i)} + \gamma_{n(i)} = 1$ for all $i \in I$ and all n.

In a convex metric space, the scheme (1.14) provides analogues of:

(i) the scheme (1.12) if $k = 1$ and $T_1 = T$;

(ii) the scheme (1.11) if $k = 2$ and $T_1 = T_2 = T$.

This scheme becomes the scheme (1.13) if we choose a special convex metric space, namely, a normed space.

In this paper, we establish strong convergence theorem for the iteration scheme (1.14) to converge to common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in the framework of convex metric spaces. Our result extends and as well as refines the corresponding results of [2], [4], [6]-[17], [20] and many others.

We need the following useful lemma to prove our convergence results.

1.5. Lemma. *(see* [19]*) Let* $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ *be three sequences of nonnegative real numbers satisfying the following conditions:*

$$p_{n+1} \leq (1 + q_n)p_n + r_n, \quad n \geq 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty.$$

Then

(1) $\lim_{n\to\infty} p_n$ *exists.*

(2) In addition, if $\liminf_{n\to\infty} p_n = 0$*, then* $\lim_{n\to\infty} p_n = 0$.

2. Main Results

In this section, we prove strong convergence theorems of multi-step iteration scheme (1.14) for a finite family of generalized asymptotically quasi-nonexpansive mappings in convex metric spaces.

2.1. Lemma. *Let* (X, d) *be a complete convex metric space, and let* C *be a nonempty closed convex subset of* X*. Let* $\{T_i : i \in I\}$ *be a finite family of generalized asymptotically quasi-nonexpansive self-maps on* C *with sequences* $\{r_{n(i)}\}, \{s_{n(i)}\} \subset [0, \infty)$ *for each* $i \in I$*, respectively, such that* $\sum_{n=1}^{\infty} r_{n(i)} < \infty$ *and* $\sum_{n=1}^{\infty} s_{n(i)} < \infty$*. Assume that* $F = \cap_{i=1}^{k} F(T_i)$ *is a nonempty set. Let* $\{x_n\}$ *be the multi-step iteration scheme defined by (1.14) with* $\sum_{n=1}^{\infty} \gamma_{n(i)} < \infty$ *for each* $i \in I$*. Then*

(i)

$$d(x_{n+1}, p) \leq (1 + B_{n(k)})d(x_n, p) + A_{n(k)},$$

with $\sum_{n=1}^{\infty} B_{n(k)} < \infty$ *and* $\sum_{n=1}^{\infty} A_{n(k)} < \infty$*.*

(ii)

$$d(x_{n+m},p) \leq Qd(x_n,p) + Q\sum_{j=n}^{n+m-1} A_{j(k)},$$

for $m \geq 1$, $n \geq 1$, $p \in F$ *and for some* $Q > 0$.

Proof. (i) For any $p \in F$, from (1.14), we have

$$\begin{aligned} d(U_{n(1)}x_n,p) &= d(W(T_1^n x_n, x_n, u_{n(1)};\ \alpha_{n(1)},\beta_{n(1)},\gamma_{n(1)}),p) \\ &\leq \alpha_{n(1)}d(T_1^n x_n,p) + \beta_{n(1)}d(x_n,p) + \gamma_{n(1)}d(u_{n(1)},p) \\ &\leq \alpha_{n(1)}[(1+r_{n(1)})d(x_n,p) + s_{n(1)}] + \beta_{n(1)}d(x_n,p) + \gamma_{n(1)}d(u_{n(1)},p) \\ &\leq [\alpha_{n(1)} + \beta_{n(1)}](1+r_{n(1)})d(x_n,p) + \alpha_{n(1)}s_{n(1)} + \gamma_{n(1)}d(u_{n(1)},p) \\ &= [1-\gamma_{n(1)}](1+r_{n(1)})d(x_n,p) + A_{n(1)} \\ \text{(2.1)} \qquad &\leq (1+r_{n(1)})d(x_n,p) + A_{n(1)}, \end{aligned}$$

where $A_{n(1)} = \alpha_{n(1)}s_{n(1)} + \gamma_{n(1)}d(u_{n(1)},p)$, since by assumption $\sum_{n=1}^{\infty} s_{n(1)} < \infty$ and $\sum_{n=1}^{\infty} \gamma_{n(1)} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{n(1)} < \infty$.

Again from (1.14) and using (2.1), we have

$$\begin{aligned} d(U_{n(2)}x_n,p) &= d(W(T_2^n U_{n(1)}x_n, x_n, u_{n(2)};\ \alpha_{n(2)},\beta_{n(2)},\gamma_{n(2)}),p) \\ &\leq \alpha_{n(2)}d(T_2^n U_{n(1)}x_n,p) + \beta_{n(2)}d(x_n,p) + \gamma_{n(2)}d(u_{n(2)},p) \\ &\leq \alpha_{n(2)}[(1+r_{n(2)})d(U_{n(1)}x_n,p) + s_{n(2)}] + \beta_{n(2)}d(x_n,p) + \gamma_{n(2)}d(u_{n(2)},p) \\ &\leq \alpha_{n(2)}(1+r_{n(2)})[(1+r_{n(1)})d(x_n,p) + A_{n(1)}] + \alpha_{n(2)}s_{n(2)} + \beta_{n(2)}d(x_n,p) \\ &\quad +\gamma_{n(2)}d(u_{n(2)},p) \\ &\leq [\alpha_{n(2)} + \beta_{n(2)}](1+r_{n(1)})(1+r_{n(2)})d(x_n,p) + \alpha_{n(2)}(1+r_{n(2)})A_{n(1)} \\ &\quad +\alpha_{n(2)}s_{n(2)} + \beta_{n(2)}d(x_n,p) + \gamma_{n(2)}d(u_{n(2)},p) \\ &= [1-\gamma_{n(2)}](1+r_{n(1)}+r_{n(2)}+r_{n(1)}r_{n(2)})d(x_n,p) + \alpha_{n(2)}(1+r_{n(2)})A_{n(1)} \\ &\quad +\alpha_{n(2)}s_{n(2)} + \gamma_{n(2)}d(u_{n(2)},p) \\ \text{(2.2)} \qquad &\leq (1+B_{n(2)})d(x_n,p) + A_{n(2)}, \end{aligned}$$

where $B_{n(2)} = r_{n(1)} + r_{n(2)} + r_{n(1)}r_{n(2)}$ and $A_{n(2)} = \alpha_{n(2)}(1+r_{n(2)})A_{n(1)} + \alpha_{n(2)}s_{n(2)} + \gamma_{n(2)}d(u_{n(2)},p)$, since by assumptions $\sum_{n=1}^{\infty} r_{n(1)} < \infty$, $\sum_{n=1}^{\infty} r_{n(2)} < \infty$, $\sum_{n=1}^{\infty} s_{n(2)} < \infty$, $\sum_{n=1}^{\infty} A_{n(1)} < \infty$ and $\sum_{n=1}^{\infty} \gamma_{n(2)} < \infty$, it follows that $\sum_{n=1}^{\infty} B_{n(2)} < \infty$ and $\sum_{n=1}^{\infty} A_{n(2)} < \infty$.

Further using (1.14) and (2.2), we have

$$\begin{aligned}
d(U_{n(3)}x_n,p) &= d(W(T_3^nU_{n(2)}x_n,x_n,u_{n(3)};\ \alpha_{n(3)},\beta_{n(3)},\gamma_{n(3)}),p)\\
&\le \alpha_{n(3)}d(T_3^nU_{n(2)}x_n,p)+\beta_{n(3)}d(x_n,p)+\gamma_{n(3)}d(u_{n(3)},p)\\
&\le \alpha_{n(3)}[(1+r_{n(3)})d(U_{n(2)}x_n,p)+s_{n(3)}]+\beta_{n(3)}d(x_n,p)+\gamma_{n(3)}d(u_{n(3)},p)\\
&\le \alpha_{n(3)}(1+r_{n(3)})[(1+B_{n(2)})d(x_n,p)+A_{n(2)}]+\alpha_{n(3)}s_{n(3)}+\beta_{n(3)}d(x_n,p)\\
&\quad +\gamma_{n(3)}d(u_{n(3)},p)\\
&\le [\alpha_{n(3)}+\beta_{n(3)}](1+r_{n(3)})(1+B_{n(2)})d(x_n,p)+\alpha_{n(3)}(1+r_{n(3)})A_{n(2)}\\
&\quad +\alpha_{n(3)}s_{n(3)}+\beta_{n(3)}d(x_n,p)+\gamma_{n(3)}d(u_{n(3)},p)\\
&= [1-\gamma_{n(3)}](1+r_{n(3)}+B_{n(2)}+r_{n(3)}B_{n(2)})d(x_n,p)+\alpha_{n(3)}(1+r_{n(3)})A_{n(2)}\\
&\quad +\alpha_{n(3)}s_{n(3)}+\gamma_{n(3)}d(u_{n(3)},p)\\
(2.3)&\le (1+B_{n(3)})d(x_n,p)+A_{n(3)},
\end{aligned}$$

where $B_{n(3)} = r_{n(3)} + B_{n(2)} + r_{n(3)}B_{n(2)}$ and $A_{n(3)} = \alpha_{n(3)}(1 + r_{n(3)})A_{n(2)} + \alpha_{n(3)}s_{n(3)} + \gamma_{n(3)}d(u_{n(3)}, p)$, since by assumptions $\sum_{n=1}^{\infty} r_{n(3)} < \infty$, $\sum_{n=1}^{\infty} B_{n(2)} < \infty$, $\sum_{n=1}^{\infty} s_{n(3)} < \infty$, $\sum_{n=1}^{\infty} A_{n(2)} < \infty$ and $\sum_{n=1}^{\infty} \gamma_{n(3)} < \infty$, it follows that $\sum_{n=1}^{\infty} B_{n(3)} < \infty$ and $\sum_{n=1}^{\infty} A_{n(3)} < \infty$. Continuing in this process, we get

$$(2.4)\qquad d(x_{n+1},p) \le (1+B_{n(k)})d(x_n,p)+A_{n(k)},$$

where $B_{n(k)} = r_{n(k)}+B_{n(k-1)}+r_{n(k)}B_{n(k-1)}$ and $A_{n(k)} = \alpha_{n(k)}(1+r_{n(k)})A_{n(k-1)}+\alpha_{n(k)}s_{n(k)} + \gamma_{n(k)}d(u_{n(k)}, p)$ with $\sum_{n=1}^{\infty} B_{n(k)} < \infty$ and $\sum_{n=1}^{\infty} A_{n(k)} < \infty$.

The conclusion (i) holds.

(ii) Note that when $x > 0$, $1 + x \le e^x$. It follows from conclusion (i) that for $m \ge 1$, $n \ge 1$ and $p \in F$, we have

$$
\begin{aligned}
d(x_{n+m},p) &\le (1+B_{n+m-1(k)})d(x_{n+m-1},p)+A_{n+m-1(k)}\\
&\le e^{B_{n+m-1(k)}}d(x_{n+m-1},p)+A_{n+m-1(k)}\\
&\le e^{B_{n+m-1(k)}}[e^{B_{n+m-2(k)}}d(x_{n+m-2},p)+A_{n+m-2(k)}]\\
&\quad +A_{n+m-1(k)}\\
&\le e^{\{B_{n+m-1(k)}+B_{n+m-2(k)}\}}d(x_{n+m-2},p)\\
&\quad +e^{B_{n+m-1(k)}}[A_{n+m-2(k)}+A_{n+m-1(k)}]\\
&\le \dots\\
&\le \Big\{e^{\sum_{j=n}^{n+m-1}B_{j(k)}}\Big\}d(x_n,p)+\Big\{e^{\sum_{j=n+1}^{n+m-1}B_{j(k)}}\Big\}\Big(\sum_{j=n}^{n+m-1}A_{j(k)}\Big)\\
&\le \Big\{e^{\sum_{j=n}^{n+m-1}B_{j(k)}}\Big\}d(x_n,p)+\Big\{e^{\sum_{j=n}^{n+m-1}B_{j(k)}}\Big\}\Big(\sum_{j=n}^{n+m-1}A_{j(k)}\Big).
\end{aligned}
$$

(2.5)

Let $Q=e^{\sum_{j=n}^{n+m-1}B_{j(k)}}$. Then $0<Q<\infty$ and

$$d(x_{n+m},p) \le Qd(x_n,p)+Q\Big(\sum_{j=n}^{n+m-1}A_{j(k)}\Big). \tag{2.6}$$

Thus, the conclusion (ii) holds.

□

We now state and prove the main theorem of this section.

2.2. Theorem. *Let (X,d) be a complete convex metric space, and let C be a nonempty closed convex subset of X. Let $\{T_i : i\in I\}$ be a finite family of generalized asymptotically quasi-nonexpansive self-maps on C with sequences $\{r_{n(i)}\},\{s_{n(i)}\}\subset[0,\infty)$ for each $i\in I$, respectively, such that $\sum_{n=1}^{\infty}r_{n(i)}<\infty$ and $\sum_{n=1}^{\infty}s_{n(i)}<\infty$. Assume that $F=\cap_{i=1}^{k}F(T_i)$ is a nonempty set. Let $\{x_n\}$ be the multi-step iteration scheme defined by (1.14) with $\sum_{n=1}^{\infty}\gamma_{n(i)}<\infty$ for each $i\in I$. Then the iterative sequence $\{x_n\}$ converges strongly to a point in F if and only if $\liminf_{n\to\infty}d(x_n,F)=0$, where $d(x,F)=\inf_{p\in F}\{d(x,p)\}$.*

Proof. If $\{x_n\}$ converges to $p\in F$, then $\liminf_{n\to\infty}d(x_n,p)=0$. Since $0\le d(x_n,F)\le d(x_n,p)$, we have $\liminf_{n\to\infty}d(x_n,F)=0$.

Conversely, suppose that $\liminf_{n\to\infty}d(x_n,F)=0$. From (2.4), we have that

$$d(x_{n+1},p)\le(1+B_{n(k)})d(x_n,p)+A_{n(k)}$$

with $\sum_{n=1}^{\infty}B_{n(k)}<\infty$ and $\sum_{n=1}^{\infty}A_{n(k)}<\infty$, which shows that the sequence $\{x_n\}$ is of monotone type, so $\lim_{n\to\infty}d(x_n,F)$ exists by Lemma 1.5. Now $\liminf_{n\to\infty}d(x_n,F)=$

0 reveals that $\lim_{n\to\infty} d(x_n, F) = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists an integer n_0 such that $d(x_n, F) < \varepsilon/6Q$ and $\sum_{j=n}^{n+m-1} A_{j(k)} < \varepsilon/4Q$ for all $n \geq n_0$. So, we can find $p^* \in F$ such that $d(x_{n_0}, p^*) < \varepsilon/4Q$. Hence, for all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\leq Qd(x_{n_0}, p^*) + Q\sum_{j=n_0}^{n+m-1} A_{j(k)} \\ &\quad +Qd(x_{n_0}, p^*) + Q\sum_{j=n_0}^{n+m-1} A_{j(k)} \\ &= 2Q\Big(d(x_{n_0}, p^*) + \sum_{j=n_0}^{n+m-1} A_{j(k)}\Big) \\ &\leq 2Q\big(\frac{\varepsilon}{4Q} + \frac{\varepsilon}{4Q}\big) = \varepsilon. \end{aligned} \tag{2.7}$$

This proves that $\{x_n\}$ is a Cauchy sequence. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = z$. Since C is closed, therefore $z \in C$. Next, we show that $z \in F$. Now, the following two inequalities:

$$\begin{aligned} d(z, p) &\leq d(z, x_n) + d(x_n, p) \quad \forall p \in F,\ n \geq 1, \\ d(z, x_n) &\leq d(z, p) + d(x_n, p) \quad \forall p \in F,\ n \geq 1, \end{aligned} \tag{2.8}$$

give

$$-d(z, x_n) \leq d(z, F) - d(x_n, F) \leq d(z, x_n),\ n \geq 1. \tag{2.9}$$

That is,

$$|d(z, F) - d(x_n, F)| \leq d(z, x_n),\ n \geq 1. \tag{2.10}$$

As $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, F) = 0$, we conclude that $z \in F$, that is, $\{x_n\}$ converges strongly to a point in F. This completes the proof.

□

We deduce some results from Theorem 2.2 as follows.

2.3. Corollary. *Let (X, d) be a complete convex metric space, and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of generalized asymptotically quasi-nonexpansive self-maps on C with sequences $\{r_{n(i)}\}, \{s_{n(i)}\} \subset [0, \infty)$ for each $i \in I$, respectively, such that $\sum_{n=1}^{\infty} r_{n(i)} < \infty$*

and $\sum_{n=1}^{\infty} s_{n(i)} < \infty$*. Assume that* $F = \cap_{i=1}^{k} F(T_i)$ *is a nonempty set. Let* $\{x_n\}$ *be the general iteration scheme defined by (1.14) with* $\sum_{n=1}^{\infty} \gamma_{n(i)} < \infty$ *for each* $i \in I$*. Then the sequence* $\{x_n\}$ *converges strongly to a point* p *in* F *if and only there exists some subsequence* $\{x_{n_j}\}$ *of* $\{x_n\}$ *which converges to a point* $p \in F$*.*

2.4. Corollary. *Let* (X, d) *be a complete convex metric space, and let* C *be a nonempty closed convex subset of* X*. Let* $\{T_i : i \in I\}$ *be a finite family of asymptotically quasi-nonexpansive self-maps on* C *with sequences* $\{r_{n(i)}\} \subset [0, \infty)$ *for each* $i \in I$*, such that* $\sum_{n=1}^{\infty} r_{n(i)} < \infty$*. Assume that* $F = \cap_{i=1}^{k} F(T_i)$ *is a nonempty set. Let* $\{x_n\}$ *be the general iteration scheme defined by (1.14) with* $\sum_{n=1}^{\infty} \gamma_{n(i)} < \infty$ *for each* $i \in I$*. Then the sequence* $\{x_n\}$ *converges strongly to a point in* F *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$*, where* $d(x, F) = \inf_{p\in F}\{d(x, p)\}$*.*

Proof. Follows from Theorem 2.2 with $s_{n(i)} = 0$ for each $i \in I$ and for all $n \geq 1$. This completes the proof.

□

2.5. Theorem. *Let* (X, d) *be a complete convex metric space, and let* C *be a nonempty closed convex subset of* X*. Let* $\{T_i : i \in I\}$ *be a finite family of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive self-maps on* C *with sequences* $\{r_{n(i)}\}$*,* $\{s_{n(i)}\} \subset [0, \infty)$ *for each* $i \in I$*, respectively, such that* $\sum_{n=1}^{\infty} r_{n(i)} < \infty$ *and* $\sum_{n=1}^{\infty} s_{n(i)} < \infty$*. Assume that* $F = \cap_{i=1}^{k} F(T_i) \neq \emptyset$*. Let* $\{x_n\}$ *be the general iteration scheme defined by (1.14) with* $\sum_{n=1}^{\infty} \gamma_{n(i)} < \infty$ *for each* $i \in I$ *and* $0 < \delta \leq \alpha_{n(i)} < 1 - \delta$ *for some* $\delta \in (0, \frac{1}{2})$*. Then the sequence* $\{x_n\}$ *converges to* $p \in F$ *provided* $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$*, for each* $i \in I$*, and one member of the family* $\{T_i : i \in I\}$ *is semi-compact.*

Proof. Without loss of generality, we assume that T_1 is semi-compact. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q \in C$. Hence, for any $i \in I$, we have

$$\begin{aligned} d(q, T_i q) &\leq d(q, x_{n_j}) + d(x_{n_j}, T_i x_{n_j}) + d(T_i x_{n_j}, T_i q) \\ &\leq (1 + L) d(q, x_{n_j}) + d(x_{n_j}, T_i x_{n_j}) \to 0. \end{aligned}$$

Thus $q \in F$. By Lemma 1.5 and Theorem 2.2, $x_n \to q$. This completes the proof.

□

2.6. Theorem. *Let* (X, d) *be a complete convex metric space, and let* C *be a nonempty closed convex subset of* X*. Let* $\{T_i : i \in I\}$ *be a finite family of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive self-maps on* C *with sequences* $\{r_{n(i)}\}$*,* $\{s_{n(i)}\} \subset [0, \infty)$ *for each* $i \in I$*, respectively, such that* $\sum_{n=1}^{\infty} r_{n(i)} < \infty$ *and* $\sum_{n=1}^{\infty} s_{n(i)} < \infty$*. Assume that* $F = \cap_{i=1}^{r} F(T_i) \neq \emptyset$*. Let* $\{x_n\}$ *be the general iteration scheme defined by (1.14) with* $\sum_{n=1}^{\infty} \gamma_{n(i)} < \infty$ *for each* $i \in I$ *and* $0 < \delta \leq \alpha_{n(i)} < 1 - \delta$ *for some* $\delta \in (0, \frac{1}{2})$*. Suppose that the mappings*

$\{T_i : i \in I\}$ for each $i \in I$ satisfy the following conditions:

(i) $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for each $i \in I$;

(ii) there exists a constant $K > 0$ such that $d(x_n, T_i x_n) \geq K d(x_n, F)$, for each $i \in I$ and for all $n \geq 1$.

Then $\{x_n\}$ converges strongly to a point in F.

Proof. From conditions (i) and (ii), we have $\lim_{n\to\infty} d(x_n, F) = 0$, it follows as in the proof of Theorem 2.2, that $\{x_n\}$ must converges strongly to a point in F. This completes the proof.

□

3. Application

In this section we give an application of Theorem 2.2.

3.1. Theorem. *Let X be a Banach space, and let C be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of generalized asymptotically quasi-nonexpansive self-maps on C with sequences $\{r_{n(i)}\}, \{s_{n(i)}\} \subset [0, \infty)$ for each $i \in I$, respectively, such that $\sum_{n=1}^{\infty} r_{n(i)} < \infty$ and $\sum_{n=1}^{\infty} s_{n(i)} < \infty$. Assume that $F = \cap_{i=1}^{k} F(T_i)$ is a nonempty set. Let $\{x_n\}$ be the multi-step iteration scheme defined as*

$$
\begin{aligned}
x_{n+1} &= y_{nk} = \alpha_{nk} T_k^n y_{n(k-1)} + \beta_{nk} x_n + \gamma_{nk} u_{nk}, \\
y_{n(r-1)} &= \alpha_{n(k-1)} T_{k-1}^n y_{n(k-2)} + \beta_{n(k-1)} x_n + \gamma_{n(k-1)} u_{n(k-1)}, \\
&\vdots \\
y_{n3} &= \alpha_{n3} T_3^n y_{n2} + \beta_{n3} x_n + \gamma_{n3} u_{n3}, \\
y_{n2} &= \alpha_{n2} T_2^n y_{n1} + \beta_{n2} x_n + \gamma_{n2} u_{n2}, \\
y_{n1} &= \alpha_{n1} T_1^n y_{n0} + \beta_{n1} x_n + \gamma_{n1} u_{n1}, \qquad n \geq 1,
\end{aligned} \tag{3.1}
$$

where $y_{n0} = x_n$ and $\{u_{n1}\}, \{u_{n2}\}, \ldots, \{u_{nk}\}$ are bounded sequences in C with $\{\alpha_{ni}\}$, $\{\beta_{ni}\}$, and $\{\gamma_{ni}\}$ are appropriate real sequences in $[0, 1]$ such that $\alpha_{ni} + \beta_{ni} + \gamma_{ni} = 1$ for all $i = 1, 2, \ldots, k$ and all n with $\sum_{n=1}^{\infty} \gamma_{ni} < \infty$ for each $i \in I$. If $\liminf_{n\to\infty} d(x_n, F) = 0$, then the iterative sequence $\{x_n\}$ converges strongly to a point $p \in F$.

Proof. Since $\{u_{ni},\ i = 1, 2, \ldots, k,\ n \geq 1\}$ are bounded sequences in C, so we can set

$$M = \max\left\{\sup_{n\geq 1} \|u_{ni} - p\|,\ \ i = 1, 2, \ldots, k\right\}.$$

Let $p \in F$, $r_n = \max\{r_{n(i)} : i = 1,2,\dots,k\}$ and $s_n = \max\{s_{n(i)} : i = 1,2,\dots,k\}$ for all n. Since $\sum_{n=1}^{\infty} r_{n(i)} < \infty$ and $\sum_{n=1}^{\infty} s_{n(i)} < \infty$, for all $i = 1,2,\dots,k$, therefore $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Then by using (3.1), we have

$$
\begin{aligned}
\|y_{n1} - p\| &= \|\alpha_{n1} T_1^n x_n + \beta_{n1} x_n + \gamma_{n1} u_{n1} - p\| \\
&\leq \alpha_{n1} \|T_1^n x_n - p\| + \beta_{n1} \|x_n - p\| + \gamma_{n1} \|u_{n1} - p\| \\
&\leq \alpha_{n1}[(1 + r_{n1}) \|x_n - p\| + s_{n1}] + \beta_{n1} \|x_n - p\| + \gamma_{n1} \|u_{n1} - p\| \\
&\leq \Big(\alpha_{n1} + \beta_{n1}\Big)(1 + r_{n1}) \|x_n - p\| + \alpha_{n1} s_{n1} + \gamma_{n1} \|u_{n1} - p\| \\
&\leq \Big(\alpha_{n1} + \beta_{n1}\Big)(1 + r_n) \|x_n - p\| + \alpha_{n1} s_n + \gamma_{n1} M \\
&= \Big(1 - \gamma_{n1}\Big)(1 + r_n) \|x_n - p\| + \alpha_{n1} s_n + \gamma_{n1} M \\
&\leq (1 + r_n) \|x_n - p\| + s_n + \gamma_{n1} M \\
&= (1 + r_n) \|x_n - p\| + A_{n1} \qquad (3.2)
\end{aligned}
$$

where $A_{n1} = s_n + \gamma_{n1} M$, since by assumptions $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_{n1} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{n1} < \infty$.

Again from (3.1) and (3.2), we obtain that

$$
\|y_{n2} - p\| \leq (1 + r_n)^2 \|x_n - p\| + A_{n2} \qquad (3.3)
$$

where $A_{n2} = (1 + r_n)A_{n1} + s_n + \gamma_{n2} M$, since by assumptions $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} \gamma_{n2} < \infty$ and $\sum_{n=1}^{\infty} A_{n1} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{n2} < \infty$.

Continuing the above process, using (3.1), we get

$$
\begin{aligned}
\|x_{n+1}-p\| &= \left\|\alpha_{nk}(T_k^n y_{n(k-1)}-p)+\beta_{nk}(x_n-p)+\gamma_{nk}(u_{nk}-p)\right\| \\
&\le \alpha_{nk}\left\|T_k^n y_{n(k-1)}-p\right\|+\beta_{nk}\|x_n-p\|+\gamma_{nk}\|u_{nk}-p\| \\
&\le \alpha_{nk}[(1+r_{nk})\left\|y_{n(k-1)}-p\right\|+s_{nk}]+\beta_{nk}\|x_n-p\| \\
&\quad +\gamma_{nk}\|u_{nk}-p\| \\
&\le \alpha_{nk}[(1+r_n)\left\|y_{n(k-1)}-p\right\|+s_n]+\beta_{nk}\|x_n-p\| \\
&\quad +\gamma_{nk}\|u_{nk}-p\| \\
&\le \alpha_{nk}(1+r_n)\left\|y_{n(k-1)}-p\right\|+\alpha_{nk}s_n+\beta_{nk}\|x_n-p\| \\
&\quad +\gamma_{nk}\|u_{nk}-p\| \\
&\le \alpha_{nk}(1+r_n)[(1+r_n)^{k-1}\|x_n-p\|+A_{n(k-1)}]+\alpha_{nk}s_n \\
&\quad +\beta_{nk}\|x_n-p\|+\gamma_{nk}\|u_{nk}-p\| \\
&\le \Big(\alpha_{nk}+\beta_{nk}\Big)(1+r_n)^k\|x_n-p\|+\alpha_{nk}(1+r_n)A_{n(k-1)} \\
&\quad +\alpha_{nk}s_n+\gamma_{nk}M \\
&= \big(1-\gamma_{nk}\big)(1+r_n)^k\|x_n-p\|+\alpha_{nk}(1+r_n)A_{n(k-1)} \\
&\quad +\alpha_{nk}s_n+\gamma_{nk}M \\
&\le (1+r_n)^k\|x_n-p\|+(1+r_n)A_{n(k-1)}+s_n+\gamma_{nk}M \\
&= (1+r_n)^k\|x_n-p\|+A_{nk} \qquad (3.4)
\end{aligned}
$$

where $A_{nk}=(1+r_n)A_{n(k-1)}+s_n+\gamma_{nk}M$, since by assumptions $\sum_{n=1}^{\infty} r_n<\infty$, $\sum_{n=1}^{\infty} s_n<\infty$, $\sum_{n=1}^{\infty}\gamma_{nk}<\infty$ and $\sum_{n=1}^{\infty} A_{n(k-1)}<\infty$, it follows that $\sum_{n=1}^{\infty} A_{nk}<\infty$. Therefore, by our assumptions, we know that the sequence $\{x_n\}$ is of monotone type and so the conclusion follows from Theorem 2.2. This completes the proof.

□

3.2. Remark. (1) If $\gamma_{n(i)}=0$ for each $i\in I$ and for all $n\ge 1$, then the approximation results about

(i) modified Mann iterations in [16] in Hilbert spaces,

(ii) modified Mann iterations in [17] in uniformly convex Banach spaces,

(iii) modified Ishikawa iterations in Banach spaces [4, 9, 11], and

(iv) the three-step iteration scheme in uniformly convex Banach spaces from [7, 20] are immediate consequences of our results.

(2) The approximation results about

(i) modified Ishikawa iterations with errors in Banach spaces [12], and

(ii) the two-step and three-step iteration scheme with errors in uniformly convex Banach spaces from [13, 15] are immediate consequences of our results.

(3) Our results also extend the results of Khan et al. [8] to the case of more general class of asymptotically quasi-nonexpansive mappings and iteration scheme with errors consider in this paper.

(4) Our results also generalize the results of [6] in the setup of convex metric spaces.

(5) Our results also extend the corresponding results of [2, 10] to the case of more general class of asymptotically nonexpansive and asymptotically nonexpansive type mappings and multi-step iteration scheme with errors considered in this paper.

3.3. Remark. Every uniformly convex Banach spaces are uniformly convex metric spaces as shown in the following example:

3.4. Example. Let H be a Hilbert space and let X be a nonempty closed subset of $\{x \in H : \|x\| = 1\}$ such that if x, $y \in X$ and α, $\beta \in [0,1]$ with $\alpha + \beta = 1$, then $(\alpha\, x + \beta\, y)/\, \|\alpha\, x + \beta\, y\| \in X$ and $\delta(X) \leq \sqrt{2}/2$; see [14], where δ is a modulus of convexity of X. Let $d(x,y) = cos^{-1}\{(x,y)\}$ for every x, $y \in X$, where $(.,.)$ is the inner product of H. When we define a convex structure W for (X,d) properly, it is easily seen that (X,d) becomes a complete and uniformly convex metric space.

Also, the following example shows that the generalized asymptotically quasi-nonexpansive mappings includes the class of asymptotically quasi-nonexpansive mappings:

3.5. Example. Let E be the real line with the usual metric and $K = [0,1]$. Define $T\colon K \to K$ by

$$T(x) = \begin{cases} \text{x}/2, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Obviously $T(0) = 0$, i.e., 0 is a fixed point of the mapping T. Thus, T is quasi-nonexpansive. It follows that T is uniformly quasi-1 Lipschitzian and asymptotically quasi-nonexpansive with the constant sequence $\{k_n\} = \{1\}$ for each $n \geq 1$ and hence it is generalized asymptotically quasi-nonexpansive mapping with constant sequences $\{k_n\} = \{1\}$ and $\{s_n\} = \{0\}$ for each $n \geq 1$ but the converse is not true in general.

Conclusion.

According to the Examples 3.4 and 3.5, we come to a conclusion that if the results are true in uniformly convex Banach spaces then the results are also true in complete convex metric spaces. Thus our results are good improvement and generalization of corresponding results of [2, 4, 6, 7, 9, 11, 12, 15, 16, 17, 20].

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A shorter proof of the Smith normal form of skew-Hadamard matrices and their designs

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Abstract

We provide a shorter algebraic proof for the Smith normal form of skew-hadamard matrices and the related designs.

Keywords: p-rank, Hadamard design, Smith normal form.

2000 AMS Classification: 20C08, 51E12, 05B20

1. Introduction

Smith normal forms and *p-ranks* of designs can help distinguish non-isomorphic designs with the same parameters. So it is interesting to know their Smith normal form explicitly. Smith normal forms of some designs were computed in [2],[3] and [5]. In this article we give a shorter proof for the Smith normal form of skew-hadamard matrices and their designs.

A *Hadamard matrix* H of order n is an n by n matrix whose elements are ± 1 and which satisfies $HH^T = nI_n$. It is *skew-Hadamard matrix* if, it also satisfies $H + H^T = 2I_n$. For more information about the Hadamard matrices please see [1], [9]. Similar definitions stated below can be found in [4], [5], [6], [7], [8], [9].

The *incidence matrix* of a Hadamard $(4m-1, 2m, m)$ design D is a $4m-1$ by $4m-1$ $(0,1)$-matrix A that satisfies

$$AA^T = A^T A = mI + mJ.$$

The complementary design $\overline{D}$ is a $(4m-1, 2m-1, m-1)$ design with incidence matrix $J - A$. A skew-hadamard $(4m-1, 2m, m)$ design is a hadamard design that satisfies(after some row and column permutations)

$$A + A^T = I + J$$

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Integral Equivalence: If A and B are matrices over the ring Z of integers, A and B are called *equivalent* ($A \sim B$) if there are $Z-$matrices P and Q, of determinant ± 1, such that

$$B = PAQ$$

which means that one can be obtained from the other by a sequence of the following operations:

- Reorder the rows,
- Negate some row,
- Add an integer multiple of one row to another,

and the corresponding column operations.

Smith Normal Form: If A is any n by n, $Z-$ matrix, then there is a unique $Z-$matrix

$$S = diag(a_1, a_2, ..., a_n)$$

such that $A \sim S$ and

$$a_1|a_2|...|a_r, a_{r+1} = \ldots = a_n = 0,$$

where the a_i are non-negative. The greatest common divisor of i by i subdeterminants of A is

$$a_1 a_2 a_3 \ldots a_i.$$

The a_i are called *invariants factors* of A and S is the Smith normal form($SNF(A)$) of A.

***p*-Rank:** The *p-rank* of an n by n, $Z-$ matrix A is the rank of A over a field of characteristic p and is denoted by $rank_p(A)$. The p-rank of A is related to the invariant factors $a_1, a_2, ..., a_n$ by

$$rank_p(A) = max\{i : p \text{ does not divide } a_i\}$$

2. Proof of the main theorem

2.1. Proposition. ([6] or [8])*: Let H be a Hadamard matrix of order $4m$ with invariant factors $h_1, ..., h_{4m}$. Then $h_1 = 1$, $h_2 = 2$, and $h_i h_{4m+1-i} = 4m$ ($i = 1, \ldots, 4m$).*

2.2. Theorem. ([7])*: Let A, B, $C = A+B$, be n by n matrices over Z, with invariant factors $h_1(A)|\ldots|h_n(A)$, $h_1(B)|\ldots|h_n(B)$, $h_1(C)|\ldots|h_n(C)$, respectively. Then*

$$gcd(h_i(A), h_j(B))|h_{i+j-1}(A+B)$$

for any indices i, j with $1 \leq i, j \leq n$, $i + j - 1 \leq n$, where gcd denotes greatest common divisor.

2.3. Theorem. ([4])*: Let D be a skew-Hadamard $(4m-1, 2m, m)$ design. Suppose that p divides m. Then $rank_p(D) = 2m - 1$ and $rank_p(\overline{D}) = 2m$.*

The author in [5] proves the following theorem by using completely different method. Here we provide a shorter algebraic proof for this theorem and the corollary following it.

2.4. Theorem. *A skew-Hadamard matrix of order* $4m$ *has Smith normal form*

$$diag[1, \underbrace{2, \ldots, 2}_{2m-1}, \underbrace{2m, \ldots, 2m}_{2m-1}, 4m].$$

Proof. Applying Theorem 2.2 with $A = H$ and $B = H^T$ we get $gcd(h_i(H), h_j(H^T))|2$ which means that $gcd(h_i(H), h_j(H^T)) = 1$ or 2 where $1 \leq i, j \leq 4m$, $i+j-1 \leq 4m$. If $m = 1$ then we have a skew-Hadamard matrix of order 4 and by proposition 1 the result follows. Assume that $m > 1$ then by proposition 1 we know that $h_1(H) = 1$, $h_2(H) = 2$, $h_{4m-1}(H) = 2m$ and $h_{4m}(H) = 4m$. Since $SNF(H) = SNF(H^T)$ assume that $h_{2m}(H) = 2k$ and $h_{2m}(H^T) = 2k$ where $k \neq 1$ and k is a divisor of m. In this case $i = j = 2m$ and Theorem 2.2 gives us $gcd(h_i(H), h_j(H^T)) = 2k|2$. But this is a contradiction since $k \neq 1$. So $k = 1$ which means that $h_{2m}(H) = h_{2m}(H^T) = 2$. So all the first $2m$ elements except the first one have to be 2. Since we found the first $2m$ elements, using proposition 1 again we obtain the remaining elements namely $h_{2m+1}(H) = h_{2m+2}(H) = \ldots = h_{4m-1}(H) = 2m$ and $h_{4m}(H) = 4m$. □

2.5. Corollary. *The Smith normal form of the incidence matrix of a skew-Hadamard* $(4m-1, 2m, m)$ *design is*

$$diag[\underbrace{1, \ldots, 1}_{2m-1}, \underbrace{m, \ldots, m}_{2m-1}, 2m].$$

Proof. By [5] any skew-Hadamard matrix of order $4m$ is integrally equivalent to $[1] \oplus (2A)$. This means that all the invariant factors of A are half of the corresponding invariant factors of H except the first one. So the result follows. □

Note that we know from Theorem 2.3 that $rank_p A = 2m-1$ which agrees with our result.

By using similar techniques that we used above we get the Smith normal form of the complementary skew-Hadamard design:

2.6. Corollary. *The Smith normal form of the incidence matrix of a skew-Hadamard* $(4m-1, 2m-1, m-1)$ *design is*

$$diag[\underbrace{1, \ldots, 1}_{2m}, \underbrace{m, \ldots, m}_{2m-2}, m(2m-1)].$$

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Base and subbase in intuitionistic I-fuzzy topological spaces

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Abstract

In this paper, the concepts of the base and subbase in intuitionistic I-fuzzy topological spaces are introduced, and use them to discuss fuzzy continuous mapping and fuzzy open mapping. We also study the base and subbase in the product of intuitionistic I-fuzzy topological spaces, and T_2 separation in product intuitionistic I-fuzzy topological spaces. Finally, the relation between the generated product intuitionistic I-fuzzy topological spaces and the product generated intuitionistic I-fuzzy topological spaces are studied.

Keywords: Intuitionistic I-fuzzy topological space; Base; Subbase; T_2 separation; Generated Intuitionistic I-fuzzy topological spaces.

2000 AMS Classification: 54A40

1. Introduction

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was first introduced by Atanassov [1]. From then on, this theory has been studied and applied in a variety areas ([4, 14, 18], etc). Among of them, the research of the theory of intuitionistic fuzzy topology is similar to the the theory of fuzzy topology. In fact, Çoker [4] introduced the concept of intuitionistic fuzzy topological spaces, this concept is originated from the fuzzy topology in the sense of Chang [3](in this paper we call it intuitionistic I-topological spaces). Based on Çoker's work [4], many topological properties of intuitionistic I-topological spaces has been discussed ([5, 10, 11, 12, 13]). On the other hand, Šostak [17] proposed a new notion of fuzzy topological spaces, and this new fuzzy topological structure has been accepted widely. Influenced by Šostak's work [17], Çoker [7] gave the notion of intuitionistic fuzzy topological spaces in the sense of Šostak. By the standardized terminology introduced in [16], we will call it intuitionistic I-fuzzy

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topological spaces in this paper. In [15], the authors studied the compactness in intuitionistic I-fuzzy topological spaces.

Recently, Yan and Wang [19] generalized Fang and Yue′s work ([8, 21]) from I-fuzzy topological spaces to intuitionistic I-fuzzy topological spaces. In [19], they introduced the concept of intuitionistic I-fuzzy quasi-coincident neighborhood systems of intuitiostic fuzzy points, and construct the notion of generated intuitionistic I-fuzzy topology by using fuzzifying topologies. As an important result, Yan and Wang proved that the category of intuitionistic I-fuzzy topological spaces is isomorphic to the category of intuitionistic I-fuzzy quasi-coincident neighborhood spaces in [19].

It is well known that base and subbase are very important notions in classical topology. They also discussed in I-fuzzy topological spaces by Fang and Yue [9]. As a subsequent work of Yan and Wang [19], the main purpose of this paper is to introduce the concepts of the base and subbase in intuitionistic I-fuzzy topological spaces, and use them to discuss fuzzy continuous mapping and fuzzy open mapping. Then we also study the base and subbase in the product of intuitionistic I-fuzzy topological spaces, and T_2 separation in product intuitionistic I-fuzzy topological spaces. Finally, we obtain that the generated product intuitionistic I-fuzzy topological spaces is equal to the product generated intuitionistic I-fuzzy topological spaces.

Throughout this paper, let $I = [0, 1]$, X a nonempty set, the family of all fuzzy sets and intuitionistic fuzzy sets on X be denoted by I^X and ζ^X, respectively. The notation $\mathrm{pt}(I^X)$ denotes the set of all fuzzy points on X. For all $\lambda \in I$, $\underline{\lambda}$ denotes the fuzzy set on X which takes the constant value λ. For all $A \in \zeta^X$, let $A =< \mu_A, \gamma_A >$. (For the relating to knowledge of intuitionistic fuzzy sets and intuitionistic I-fuzzy topological spaces, we may refer to [1] and [19].)

2. Some preliminaries

2.1. Definition. ([20]) A fuzzifying topology on a set X is a function $\tau : 2^X \to I$, such that

(1) $\tau(\emptyset) = \tau(X) = 1$;

(2) $\forall A, B \subseteq X$, $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$;

(3) $\forall A_t \subseteq X$, $t \in T$, $\tau(\bigvee_{t\in T} A_t) \geq \bigwedge_{t\in T} \tau(A_t)$.

The pair (X, τ) is called a fuzzifying topological space.

2.2. Definition. ([1, 2]) Let a, b be two real numbers in $[0, 1]$ satisfying the inequality $a + b \leq 1$. Then the pair $< a, b >$ is called an intuitionistic fuzzy pair.

Let $< a_1, b_1 >$, $< a_2, b_2 >$ be two intuitionistic fuzzy pairs, then we define

(1) $< a_1, b_1 >\leq< a_2, b_2 >$ if and only if $a_1 \leq a_2$ and $b_1 \geq b_2$;

(2) $< a_1, b_1 >=< a_2, b_2 >$ if and only if $a_1 = a_2$ and $b_1 = b_2$;

(3) if $< a_j, b_j >_{j\in J}$ is a family of intuitionistic fuzzy pairs, then $\bigvee_{j\in J} < a_j, b_j >=< \bigvee_{j\in J} a_j, \bigwedge_{j\in J} b_j >$, and $\bigwedge_{j\in J} < a_j, b_j >=< \bigwedge_{j\in J} a_j, \bigvee_{j\in J} b_j >$;

(4) the complement of an intuitionistic fuzzy pair $< a, b >$ is the intuitionistic fuzzy pair defined by $\overline{< a, b >} =< b, a >$;

In the following, for convenience, we will use the symbols $1^\sim$ and $0^\sim$ denote the intuitionistic fuzzy pairs $< 1, 0 >$ and $< 0, 1 >$. The family of all intuitionistic fuzzy pairs is denoted by $\mathcal{A}$. It is easy to find that the set of all intuitionistic fuzzy pairs with above order forms a complete lattice, and $1^\sim, 0^\sim$ are its top element and bottom element, respectively.

2.3. Definition. ([4]) Let X, Y be two nonempty sets and $f : X \to Y$ a function, if $B = \{< y, \mu_B(y), \gamma_B(y) >:\ y \in Y\} \in \zeta^Y$, then the preimage of B under f, denoted by $f^\leftarrow(B)$, is the intuitionistic fuzzy set defined by

$$f^\leftarrow(B) = \{< x, f^\leftarrow(\mu_B)(x), f^\leftarrow(\gamma_B)(x) >:\ x \in X\}.$$

Here $f^\leftarrow(\mu_B)(x) = \mu_B(f(x)),\ f^\leftarrow(\gamma_B)(x) = \gamma_B(f(x))$. (This notation is from [16]).

If $A = \{< x, \mu_A(x), \gamma_A(x) >:\ x \in X\} \in \zeta^X$, then the image A under f, denoted by $f^\rightarrow(A)$ is the intuitionistic fuzzy set defined by

$$f^\rightarrow(A) = \{< y, f^\rightarrow(\mu_A)(y), (\underline{1} - f^\rightarrow(\underline{1} - \gamma_A))(y) >:\ y \in Y\}.$$

Where

$$f^\rightarrow(\mu_A)(y) = \begin{cases} \sup_{x \in f^\leftarrow(y)} \mu_A(x), & \text{if } f^\leftarrow(y) \neq \emptyset, \\ 0, & \text{if } f^\leftarrow(y) = \emptyset. \end{cases}$$

$$\underline{1} - f^\rightarrow(\underline{1} - \gamma_A)(y) = \begin{cases} \inf_{x \in f^\leftarrow(y)} \gamma_A(x), & \text{if } f^\leftarrow(y) \neq \emptyset, \\ 1, & \text{if } f^\leftarrow(y) = \emptyset. \end{cases}$$

2.4. Definition. ([7]) Let X be a nonempty set, $\delta : \zeta^X \to \mathcal{A}$ satisfy the following:

(1) $\delta(< \underline{0}, \underline{1} >) = \delta(< \underline{1}, \underline{0} >) = 1^\sim$;

(2) $\forall A,\ B \in \zeta^X,\ \delta(A \bigwedge B) \geq \delta(A) \bigwedge \delta(B)$;

(3) $\forall A_t \in \zeta^X,\ t \in T,\ \delta(\bigvee_{t \in T} A_t) \geq \bigwedge_{t \in T} \delta(A_t)$.

Then δ is called an intuitionistic I-fuzzy topology on X, and the pair (X, δ) is called an intuitionistic I-fuzzy topological space. For any $A \in \zeta^X$, we always suppose that $\delta(A) =< \mu_\delta(A),\ \gamma_\delta(A) >$ later, the number $\mu_\delta(A)$ is called the openness degree of A, while $\gamma_\delta(A)$ is called the nonopenness degree of A. A fuzzy continuous mapping between two intuitionistic I-fuzzy topological spaces (ζ^X, δ_1) and (ζ^Y, δ_2) is a mapping $f : X \to Y$ such that $\delta_1(f^\leftarrow(A)) \geq \delta_2(A)$. The category of intuitionistic I-fuzzy topological spaces and fuzzy continuous mappings is denoted by II-**FTOP**.

2.5. Definition. ([6, 11, 12]) Let X be a nonempty set. An intuitionistic fuzzy point, denoted by $x_{(\alpha,\beta)}$, is an intuitionistic fuzzy set $A = \{< y, \mu_A(y), \gamma_A(y) >:\ y \in X\}$, such that

$$\mu_A(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

and

$$\gamma_A(y) = \begin{cases} \beta, & \text{if } y = x, \\ 1, & \text{if } y \neq x. \end{cases}$$

Where $x \in X$ is a fixed point, the constants $\alpha \in I_0,\ \beta \in I_1$ and $\alpha + \beta \leq 1$. The set of all intuitionistic fuzzy points $x_{(\alpha,\beta)}$ is denoted by $\mathrm{pt}(\zeta^X)$.

2.6. Definition. ([12]) Let $x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$ and $A,\ B \in \zeta^X$. We say $x_{(\alpha,\beta)}$ quasi-coincides with A, or $x_{(\alpha,\beta)}$ is quasi-coincident with A, denoted $x_{(\alpha,\beta)}\hat{q}A$, if $\mu_A(x) + \alpha > 1$ and $\gamma_A(x) + \beta < 1$. Say A quasi-coincides with B at x, or say A is quasi-coincident with B at x, $A\hat{q}B$ at x, in short, if $\mu_A(x) + \mu_B(x) > 1$ and $\gamma_A(x) + \gamma_B(x) < 1$. Say A quasi-coincides with B, or A is quasi-coincident with B, if A is quasi-coincident with B at some point $x \in X$.
Relation"does not quasi-coincides with" or "is not quasi-coincident with " is denoted by $\neg\hat{q}$.

It is easily to know for $\forall x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$, $x_{(\alpha,\beta)}\hat{q} < \underline{1}, \underline{0} >$ and $x_{(\alpha,\beta)}\neg\hat{q} < \underline{0}, \underline{1} >$.

2.7. Definition. ([19]) Let (X, δ) be an intuitionistic I-fuzzy topological space. For all $x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X), U \in \zeta^X$, the mapping $Q^{\delta}_{x_{(\alpha,\beta)}} : \zeta^X \to \mathcal{A}$ is defined as follows

$$Q^{\delta}_{x_{(\alpha,\beta)}}(U) = \begin{cases} \bigvee\limits_{x_{(\alpha,\beta)}\widehat{q}\ V \leq U} \delta(V), & x_{(\alpha,\beta)}\widehat{q}\ U; \\ 0^{\sim}, & x_{(\alpha,\beta)}\neg\widehat{q}\ U. \end{cases}$$

The set of $Q^{\delta} = \{Q^{\delta}_{x_{(\alpha,\beta)}} : x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)\}$ is called intuitionistic I-fuzzy quasi-coincident neighborhood system of δ on X.

2.8. Theorem. ([19]) *Let (X, δ) be an intuitionistic I-fuzzy topological space, $Q^{\delta} = \{Q^{\delta}_{x_{(\alpha,\beta)}} :\ x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)\}$ of maps $Q^{\delta}_{x_{(\alpha,\beta)}} : \zeta^X \to \mathcal{A}$ defined in Definition 2.7 satisfies: $\forall U, V \in \zeta^X$,*

(1) $Q^{\delta}_{x_{(\alpha,\beta)}}(\langle\underline{1}, \underline{0}\rangle) = 1^{\sim}, Q^{\delta}_{x_{(\alpha,\beta)}}(\langle\underline{0}, \underline{1}\rangle) = 0^{\sim}$;

(2) $Q^{\delta}_{x_{(\alpha,\beta)}}(U) > 0^{\sim} \Rightarrow x_{(\alpha,\beta)}\widehat{q}\ U$;

(3) $Q^{\delta}_{x_{(\alpha,\beta)}}(U \wedge V) = Q^{\delta}_{x_{(\alpha,\beta)}}(U) \wedge Q^{\delta}_{x_{(\alpha,\beta)}}(V)$;

(4) $Q^{\delta}_{x_{(\alpha,\beta)}}(U) = \bigvee\limits_{x_{(\alpha,\beta)}\widehat{q}\ V \leq U}\ \bigwedge\limits_{y_{(\lambda,\rho)}\widehat{q}\ V} Q^{\delta}_{y_{(\lambda,\rho)}}(V)$;

(5) $\delta(U) = \bigwedge\limits_{x_{(\alpha,\beta)}\widehat{q}\ U} Q^{\delta}_{x_{(\alpha,\beta)}}(U)$.

2.9. Lemma. ([21]) *Suppose that (X, τ) is a fuzzifying topological space, for each $A \in I^X$, let $\omega(\tau)(A) = \bigwedge_{r \in I} \tau(\sigma_r(A))$, where $\sigma_r(A) = \{x : A(x) > r\}$. Then $\omega(\tau)$ is an I-fuzzy topology on X, and $\omega(\tau)$ is called induced I-fuzzy topology determined by fuzzifying topology τ.*

2.10. Definition. ([19]) Let (X, τ) be a fuzzifying topological space, $\omega(\tau)$ is an induced I-fuzzy topology determined by fuzzifying topology τ. For each $A \in \zeta^X$, let $\mathrm{I}\omega(\tau)(A) =< \mu^{\tau}(A), \gamma^{\tau}(A) >$, where $\mu^{\tau}(A) = \omega(\tau)(\mu_A) \wedge \omega(\tau)(\underline{1} - \gamma_A)$, $\gamma^{\tau}(A) = 1 - \mu^{\tau}(A)$.We say that $(\zeta^X, \mathrm{I}\omega(\tau))$ is a generated intuitionistic I-fuzzy topological space by fuzzifying topological space (X, τ).

2.11. Lemma. ([19]) *Let (X, τ) be a fuzzifying topological space, then*

(1) $\forall A \subseteq X$, $\mu^{\tau}(< 1_A, 1_{A^c} >) = \tau(A)$.

(2) $\forall A =< \underline{\alpha}, \underline{\beta} >\in \zeta^X$, $\mathrm{I}\omega(\tau)(A) = 1^{\sim}$.

2.12. Lemma. ([19]) *Suppose that* $(\zeta^X,\ \delta)$ *is an intuitionistic* I*-fuzzy topological space, for each* $A \subseteq X$*, let* $[\delta](A) = \mu_\delta(< 1_A, 1_{A^c} >)$*. Then* $[\delta]$ *is a fuzzifying topology on* X*.*

2.13. Lemma. ([19]) *Let* (X, τ) *be a fuzzifying topological space and* $(X, \mathrm{I}\omega(\tau))$ *a generated intuitionistic* I*-fuzzy topological space. Then* $[\mathrm{I}\omega(\tau)] = \tau$*.*

3. Base and subbase in Intuitionistic I-fuzzy topological spaces

3.1. Definition. Let (X, τ) be an intuitionistic I-fuzzy topological space and $\mathcal{B} : \zeta^X \to \mathcal{A}$. $\mathcal{B}$ is called a base of τ if $\mathcal{B}$ satisfies the following condition

$$\tau(U) = \bigvee_{\bigvee\limits_{\lambda\in K} B_\lambda = U} \bigwedge_{\lambda\in K} \mathcal{B}(B_\lambda), \forall\, U \in \zeta^X.$$

3.2. Definition. Let (X, τ) be an intuitionistic I-fuzzy topological space and $\varphi : \zeta^X \to \mathcal{A}$, φ is called a subbase of τ if $\varphi^{(\sqcap)} : \zeta^X \to \mathcal{A}$ is a base, where $\varphi^{(\sqcap)}(A) = \bigvee\limits_{\sqcap\{B_\lambda : \lambda\in E\} = A} \bigwedge\limits_{\lambda\in E} \varphi(B_\lambda)$, for all $A \in \zeta^X$ with $(\sqcap)$ standing for "finite intersection".

3.3. Theorem. *Suppose that* $\mathcal{B} : \zeta^X \to \mathcal{A}$*. Then* $\mathcal{B}$ *is a base of some intuitionistic* I*-fuzzy topology, if* $\mathcal{B}$ *satisfies the following condition*

(1) $\mathcal{B}(0_\sim) = \mathcal{B}(1_\sim) = 1^\sim$,

(2) $\forall\ U, V \in \zeta^X$, $\mathcal{B}(U \wedge V) \geq \mathcal{B}(U) \wedge \mathcal{B}(V)$.

Proof. For $\forall\ A \in \zeta^X$, let $\tau(A) = \bigvee\limits_{\bigvee\limits_{\lambda\in K} B_\lambda = A} \bigwedge\limits_{\lambda\in K} \mathcal{B}(B_\lambda)$. To show that $\mathcal{B}$ is a base of τ, we only need to prove τ is an intuitionistic I-fuzzy topology on X. For all $U, V \in \zeta^X$,

$$\begin{aligned}
\tau(U) \wedge \tau(V) &= \Big(\bigvee_{\bigvee\limits_{\alpha\in K_1} A_\alpha = U} \bigwedge_{\alpha\in K_1} \mathcal{B}(A_\alpha)\Big) \wedge \Big(\bigvee_{\bigvee\limits_{\beta\in K_2} B_\beta = V} \bigwedge_{\beta\in K_2} \mathcal{B}(B_\beta)\Big) \\
&= \bigvee_{\bigvee\limits_{\alpha\in K_1} A_\alpha = U,\ \bigvee\limits_{\beta\in K_2} B_\beta = V} \Big(\Big(\bigwedge_{\alpha\in K_1} \mathcal{B}(A_\alpha)\Big) \wedge \Big(\bigwedge_{\beta\in K_2} \mathcal{B}(B_\beta)\Big)\Big) \\
&\leq \bigvee_{\bigvee\limits_{\alpha\in K_1, \beta\in K_2} (A_\alpha \wedge B_\beta) = U \wedge V} \Big(\bigwedge_{\alpha\in K_1, \beta\in K_2} \mathcal{B}(A_\alpha \wedge B_\beta)\Big) \\
&\leq \bigvee_{\bigvee\limits_{\lambda\in K} C_\lambda = U \wedge V} \bigwedge_{\lambda\in K} \mathcal{B}(C_\lambda) \\
&= \tau(U \wedge V).
\end{aligned}$$

For all $\{A_\lambda : \lambda \in K\} \subseteq \zeta^X$, Let $\mathcal{B}_\lambda = \{\{B_{\delta_\lambda} : \delta_\lambda \in K_\lambda\} : \bigvee\limits_{\delta_\lambda\in K_\lambda} B_{\delta_\lambda} = A_\lambda\}$, then

$$\tau\Big(\bigvee_{\lambda\in K} A_\lambda\Big) = \bigvee_{\bigvee\limits_{\delta\in K_1} B_\delta = \bigvee\limits_{\lambda\in K} A_\lambda} \bigwedge_{\delta\in K_1} \mathcal{B}(B_\delta).$$

For all $f \in \prod_{\lambda \in K} \mathcal{B}_\lambda$, we have

$$\bigvee_{\lambda \in K} \bigvee_{B_{\delta_\lambda} \in f(\lambda)} B_{\delta_\lambda} = \bigvee_{\lambda \in K} A_\lambda.$$

Therefore,

$$\begin{aligned}
\mu_{\tau(\bigvee_{\lambda \in K} A_\lambda)} &= \bigvee_{\bigvee_{\delta \in K_1} B_\delta = \bigvee_{\lambda \in K} A_\lambda} \bigwedge_{\delta \in K_1} \mu_{\mathcal{B}(B_\delta)} \\
&\geq \bigvee_{f \in \prod_{\lambda \in K} \mathcal{B}_\lambda} \bigwedge_{\lambda \in K} \bigwedge_{B_{\delta_\lambda} \in f(\lambda)} \mu_{\mathcal{B}(B_{\delta_\lambda})} \\
&= \bigwedge_{\lambda \in K} \bigvee_{\{B_{\delta_\lambda} : \delta_\lambda \in K_\lambda\} \in \mathcal{B}_\lambda} \bigwedge_{\delta_\lambda \in K_\lambda} \mu_{\mathcal{B}(B_{\delta_\lambda})} \\
&= \bigwedge_{\lambda \in E} \mu_{\tau(A_\lambda)}.
\end{aligned}$$

Similarly, we have

$$\gamma_{\tau(\bigvee_{\lambda \in K} A_\lambda)} \leq \bigvee_{\lambda \in K} \gamma_{\tau(A_\lambda)}.$$

Hence

$$\tau(\bigvee_{\lambda \in K} A_\lambda) \geq \bigwedge_{\lambda \in K} \tau(A_\lambda).$$

This means that τ is an intuitionistic I-fuzzy topology on X and $\mathcal{B}$ is a base of τ. □

3.4. Theorem. *Let $(X, \tau), (Y, \delta)$ be two intuitionistic I-fuzzy topology spaces and δ generated by its subbase φ. The mapping $f : (X, \tau) \to (Y, \delta)$ satisfies $\varphi(U) \leq \tau(f^{\leftarrow}(U))$, for all $U \in \zeta^Y$. Then f is fuzzy continuous, i.e., $\delta(U) \leq \tau(f^{\leftarrow}(U)), \forall\, U \in \zeta^Y$.*

Proof. $\forall\, U \in \zeta^Y$,

$$\begin{aligned}
\delta(U) &= \bigvee_{\bigvee_{\lambda \in K} A_\lambda = U} \bigwedge_{\lambda \in K} \bigvee_{\sqcap\{B_\mu : \mu \in K_\lambda\} = A_\lambda} \bigwedge_{\mu \in K_\lambda} \varphi(B_\mu) \\
&\leq \bigvee_{\bigvee_{\lambda \in K} A_\lambda = U} \bigwedge_{\lambda \in K} \bigvee_{\sqcap\{B_\mu : \mu \in K_\lambda\} = A_\lambda} \bigwedge_{\mu \in K_\lambda} \tau(f^{\leftarrow}(B_\mu)) \\
&\leq \bigvee_{\bigvee_{\lambda \in K} A_\lambda = U} \bigwedge_{\lambda \in K} \tau(f^{\leftarrow}(A_\lambda)) \\
&\leq \bigvee_{\bigvee_{\lambda \in K} A_\lambda = U} \tau(f^{\leftarrow}(\bigvee_{\lambda \in K} A_\lambda)) \\
&= \tau(f^{\leftarrow}(U)).
\end{aligned}$$

This completes the proof. □

3.5. Theorem. *Suppose that* (X,τ), (Y,δ) *are two intuitionistic* I*-fuzzy topology spaces and* τ *is generated by its base* $\mathcal{B}$*. If the mapping* $f:(X,\tau)\to(Y,\delta)$ *satisfies* $\mathcal{B}(U)\le\delta(f^{\to}(U))$*, for all* $U\in\zeta^X$*. Then* f *is fuzzy open, i.e.,* $\forall\ W\in\zeta^X,\tau(W)\le\delta(f^{\to}(W))$.

Proof. $\forall\ W\in\zeta^X$,

$$\begin{aligned}\tau(W) &= \bigvee_{\bigvee\limits_{\lambda\in K}A_\lambda=W}\ \bigwedge_{\lambda\in K}\mathcal{B}(A_\lambda)\\ &\le \bigvee_{\bigvee\limits_{\lambda\in K}A_\lambda=W}\ \bigwedge_{\lambda\in K}\delta(f^{\to}(A_\lambda))\\ &\le \bigvee_{\bigvee\limits_{\lambda\in K}A_\lambda=W}\ \delta(f^{\to}(\bigvee_{\lambda\in K}A_\lambda))\\ &= \delta(f^{\to}(W)).\end{aligned}$$

Therefore, f is open. □

3.5. Theorem. *Let* $(X,\tau),(Y,\delta)$ *be two intuitionistic* I*-fuzzy topology spaces and* $f:(X,\tau)\to(Y,\delta)$ *intuitionistic* I*-fuzzy continuous,* $Z\subseteq X$*. Then* $f|_Z:(Z,\tau|_Z)\to(Y,\delta)$ *is continuous, where* $(f|_Z)(x)=f(x),(\tau|_Z)(A)=\vee\{\tau(U):U|_Z=A\}$*, for all* $x\in Z,A\in\zeta^Z$*.*

Proof. $\forall\ W\in\zeta^Z,(f|_Z)^{\leftarrow}(W)=f^{\leftarrow}(W)|_Z$, we have

$$\begin{aligned}(\tau|_Z)((f|_Z)^{\leftarrow}(W)) &= \vee\{\tau(U):U|_Z=(f|_Z)^{\leftarrow}(W)\}\\ &\ge \tau(f^{\leftarrow}(W))\\ &\ge \delta(W).\end{aligned}$$

Then $f|_Z$ is intuitionistic I-fuzzy continuous. □

3.6. Theorem. *Let* (X,τ) *be an intuitionistic* I*-fuzzy topology space and* τ *generated by its base* $\mathcal{B}$*,* $\mathcal{B}|_Y(U)=\vee\{\mathcal{B}(W):W|_Y=U\}$*, for* $Y\subseteq X,U\in\zeta^Y$*. Then* $\mathcal{B}|_Y$ *is a base of* $\tau|_Y$*.*

Proof. For $\forall\ U\in\zeta^X,(\tau|_Y)(U)=\bigvee\limits_{V|_Y=U}\tau(V)=\bigvee\limits_{V|_Y=U}\ \bigvee\limits_{\bigvee\limits_{\lambda\in K}A_\lambda=V}\ \bigwedge\limits_{\lambda\in K}\mathcal{B}(A_\lambda)$. It remains to show the following equality

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee\limits_{\lambda\in K}A_\lambda=V}\ \bigwedge_{\lambda\in K}\mathcal{B}(A_\lambda)=\bigvee_{\bigvee\limits_{\lambda\in K}B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda}\mathcal{B}(W).$$

In one hand, for all $V\in\zeta^X$ with $V|_Y=U$, and $\bigvee\limits_{\lambda\in K}A_\lambda=V$, we have $\bigvee\limits_{\lambda\in K}A_\lambda|_Y=U$. Put $B_\lambda=A_\lambda|_Y$, clearly $\bigvee\limits_{\lambda\in K}B_\lambda=U$. Then

$$\bigvee_{\bigvee\limits_{\lambda\in K}B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda}\mathcal{B}(W)\ge\bigwedge_{\lambda\in K}\mathcal{B}(A_\lambda).$$

Thus,

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee\limits_{\lambda\in K} A_\lambda=V}\ \bigwedge_{\lambda\in K} \mathcal{B}(A_\lambda) \leq \bigvee_{\bigvee\limits_{\lambda\in K} B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda} \mathcal{B}(W).$$

On the other hand, $\forall\ a \in (0,1], a < \bigvee\limits_{\bigvee\limits_{\lambda\in K} B_\lambda=U}\ \bigwedge\limits_{\lambda\in K}\ \bigvee\limits_{W|_Y=B_\lambda} \mu_{\mathcal{B}(W)}$, there exists a family of $\{B_\lambda : \lambda \in K\} \subseteq \zeta^Y$, such that

(1) $\bigvee\limits_{\lambda\in K} B_\lambda = U$;

(2) $\forall\ \lambda \in K$, there exists $W_\lambda \in \zeta^X$ with $W_\lambda|_Y = B_\lambda$ such that $a < \mu_{\mathcal{B}(W_\lambda)}$.

Let $V = \bigvee\limits_{\lambda\in E} W_\lambda$, it is clear $V|_Y = U$ and $\bigwedge\limits_{\lambda\in K} \mu_{\mathcal{B}(W_\lambda)} \geq a$. Then

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee\limits_{\lambda\in K} A_\lambda=V}\ \bigwedge_{\lambda\in K} \mu_{\mathcal{B}(A_\lambda)} \geq a.$$

By the arbitrariness of a, we have

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee\limits_{\lambda\in K} A_\lambda=V}\ \bigwedge_{\lambda\in K} \mu_{\mathcal{B}(A_\lambda)} \geq \bigvee_{\bigvee\limits_{\lambda\in K} B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda} \mu_{\mathcal{B}(W)}.$$

Similarly, we may obtain that

$$\bigwedge_{V|_Y=U}\ \bigwedge_{\bigvee\limits_{\lambda\in K} A_\lambda=V}\ \bigvee_{\lambda\in K} \gamma_{\mathcal{B}(A_\lambda)} \leq \bigwedge_{\bigvee\limits_{\lambda\in K} B_\lambda=U}\ \bigvee_{\lambda\in K}\ \bigwedge_{W|_Y=B_\lambda} \gamma_{\mathcal{B}(W)}.$$

So we have

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee\limits_{\lambda\in K} A_\lambda=V}\ \bigwedge_{\lambda\in K} \mathcal{B}(A_\lambda) \geq \bigvee_{\bigvee\limits_{\lambda\in K} B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda} \mathcal{B}(W).$$

Therefore,

$$\bigvee_{V|_Y=U}\ \bigvee_{\bigvee\limits_{\lambda\in K} A_\lambda=V}\ \bigwedge_{\lambda\in K} \mathcal{B}(A_\lambda) = \bigvee_{\bigvee\limits_{\lambda\in K} B_\lambda=U}\ \bigwedge_{\lambda\in K}\ \bigvee_{W|_Y=B_\lambda} \mathcal{B}(W).$$

This means that $\mathcal{B}|_Y$ is a base of $\tau|_Y$. □

3.7. Theorem. *Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in J}$ be a family of intuitionistic I-fuzzy topology spaces and $P_\beta : \prod\limits_{\alpha\in J} X_\alpha \to X_\beta$ the projection. For all $W \in \zeta^{\prod_{\alpha\in J} X_\alpha}$, $\varphi(W) = \bigvee\limits_{\alpha\in J}\ \bigvee\limits_{P_\alpha^{\leftarrow}(U)=W} \tau_\alpha(U)$. Then φ is a subbase of some intuitionistic I-fuzzy topology τ, here τ is called the product intuitionistic I-fuzzy topologies of $\{\tau_\alpha : \alpha \in J\}$ and denoted by $\tau = \prod\limits_{\alpha\in J} \tau_\alpha$.*

Proof. We need to prove $\varphi^{(\sqcap)}$ is a subbase of τ.

$$\begin{aligned}\varphi^{(\sqcap)}(1_\sim) &= \bigvee_{\sqcap\{B_\lambda:\lambda\in E\}=1_\sim}\bigwedge_{\lambda\in E}\varphi(B_\lambda)\\ &= \bigvee_{\sqcap\{B_\lambda:\lambda\in E\}=1_\sim}\bigwedge_{\lambda\in E}\bigvee_{\alpha\in J}\bigvee_{P_\alpha^{\leftarrow}(U)=B_\lambda}\tau_\alpha(U)\\ &= 1^\sim.\end{aligned}$$

Similarly, $\varphi^{(\sqcap)}(0_\sim)=1^\sim$. For all $U,V\in\zeta^{\prod_{\alpha\in J}X_\alpha}$, we have

$$\begin{aligned}\varphi^{(\sqcap)}(U)\wedge\varphi^{(\sqcap)}(V) &= \Big(\bigvee_{\sqcap\{B_\alpha:\alpha\in E_1\}=U}\bigwedge_{\alpha\in E_1}\varphi(B_\alpha)\Big)\wedge\Big(\bigvee_{\sqcap\{C_\beta:\beta\in E_2\}=V}\bigwedge_{\beta\in E_2}\varphi(C_\beta)\Big)\\ &= \bigvee_{\sqcap\{B_\alpha:\alpha\in E_1\}=U}\bigvee_{\sqcap\{C_\beta:\beta\in E_2\}=V}\Big(\big(\bigwedge_{\alpha\in E_1}\varphi(B_\alpha)\big)\wedge\big(\bigwedge_{\beta\in E_2}\varphi(C_\beta)\big)\Big)\\ &\le \bigvee_{\sqcap\{B_\lambda:\lambda\in E\}=U\wedge V}\bigwedge_{\lambda\in E}\varphi(B_\lambda)\\ &= \varphi^{(\sqcap)}(U\wedge V).\end{aligned}$$

Hence, $\varphi^{(\sqcap)}$is a base of τ, i.e., φ is a subbase of τ. And by Theorem 3.3 we have

$$\begin{aligned}\tau(A) &= \bigvee_{\bigvee_{\lambda\in K}B_\lambda=A}\bigwedge_{\lambda\in K}\varphi^{(\sqcap)}(B_\lambda)\\ &= \bigvee_{\bigvee_{\lambda\in K}B_\lambda=A}\bigwedge_{\lambda\in K}\bigvee_{\sqcap\{C_\rho:\rho\in E\}=B_\lambda}\bigwedge_{\rho\in E}\varphi(C_\rho)\\ &= \bigvee_{\bigvee_{\lambda\in K}B_\lambda=A}\bigwedge_{\lambda\in K}\bigvee_{\sqcap\{C_\rho:\rho\in E\}=B_\lambda}\bigwedge_{\rho\in E}\bigvee_{\alpha\in J}\bigvee_{P_\alpha^{\leftarrow}(V)=C_\rho}\tau_\alpha(V).\end{aligned}$$

□

By the above discussions, we easily obtain the following corollary.

3.8. Corollary. *Let* $(\prod_{\alpha\in J}X_\alpha,\prod_{\alpha\in J}\tau_\alpha)$ *be the product space of a family of intuitionistic I-fuzzy topology spaces* $\{(X_\alpha,\tau_\alpha)\}_{\alpha\in J}$*. Then* $P_\beta:(\prod_{\alpha\in J}X_\alpha,\prod_{\alpha\in J}\tau_\alpha)\to(X_\beta,\tau_\beta)$ *is continuous, for all* $\beta\in J$.

Proof. $\forall\ U\in\zeta^{X_\beta}$,

$$\begin{aligned}\tau(P_\beta^{\leftarrow}(U)) &= \bigvee_{\bigvee_{\lambda\in K}B_\lambda=P_\beta^{\leftarrow}(U)}\bigwedge_{\lambda\in K}\bigvee_{\sqcap\{C_\rho:\rho\in E\}=B_\lambda}\bigwedge_{\rho\in E}\bigvee_{\alpha\in J}\bigvee_{P_\alpha^{\leftarrow}(V)=C_\rho}\tau_\alpha(V)\\ &\ge \tau_\beta(U)\end{aligned}$$

Therefore, P_β is continuous. □

4. Applications in product Intuitionistic I-fuzzy topological space

4.1. Definition. Let (X,τ) be an intuitionistic I-fuzzy topology space. The degree to which two distinguished intuitionistic fuzzy points $x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X)(x \neq y)$ are T_2 is defined as follows

$$T_2(x_{(\alpha,\beta)}, y_{(\lambda,\rho)}) = \bigvee_{U\wedge V=0_\sim} (Q_{x_{(\alpha,\beta)}}(U) \wedge Q_{y_{(\lambda,\rho)}}(V)).$$

The degree to which (X,τ) is T_2 is defined by

$$T_2(X,\tau) = \bigwedge \big\{T_2(x_{(\alpha,\beta)}, y_{(\lambda,\rho)}) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\}.$$

4.2. Theorem. *Let $(X, \mathrm{I}\omega(\tau))$ be a generated intuitionistic I-fuzzy topological space by fuzzifying topological space (X,τ) and $T_2(X,\mathrm{I}\omega(\tau)) \triangleq \langle \mu_{T_2(X,\mathrm{I}\omega(\tau))}, \gamma_{T_2(X,\mathrm{I}\omega(\tau))}\rangle$. Then $\mu_{T_2(X,\mathrm{I}\omega(\tau))} = T_2(X,\tau)$.*

Proof. For all $x, y \in X, x \neq y$, and each $a < \bigwedge\big\{ \bigvee_{U\wedge V=0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\}$, there exists $U, V \in \zeta^X$ with $U \wedge V = 0_\sim$ such that $a < \mu_{Q_{x_{(1,0)}}(U)}, a < \mu_{Q_{y_{(1,0)}}(V)}$. Then there exists $U_1, V_1 \in \zeta^X$, such that

$$x_{(1,0)}\widehat{q}\, U_1 \leq U,\ a < \omega(\tau)(\mu_{U_1}),$$
$$y_{(1,0)}\widehat{q}\, V_1 \leq V,\ a < \omega(\tau)(\mu_{V_1}).$$

Denote $A = \sigma_0(\mu_{U_1}), B = \sigma_0(\mu_{V_1})$, it is clear that $x \in A, y \in B$. From the fact $U \wedge V = 0_\sim$, it implies $\mu_{U_1} \wedge \mu_{V_1} = \underline{0}$. Then we have $\sigma_0(\mu_{U_1}) \wedge \sigma_0(\mu_{V_1}) = \emptyset$, i.e.,$A \wedge B = \emptyset$.

$$a < \omega(\tau)(\mu_{U_1}) = \bigwedge_{r\in I} \tau(\sigma_r(\mu_{U_1})) \leq \tau(\sigma_0(\mu_{U_1})) = \tau(A).$$

Thus

$$a < \bigvee_{x\in U\subseteq A} \tau(U) = N_x(A).$$

Similarly, we have $a < N_y(B)$. Hence

$$a < \bigvee_{A\cap B=\emptyset} (N_x(A) \wedge N_y(B)).$$

Then

$$a \leq \bigwedge\big\{ \bigvee_{A\cap B=\emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y\big\}.$$

Therefore,

$$\bigwedge\big\{ \bigvee_{U\wedge V=0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\}$$
$$\leq \bigwedge\big\{ \bigvee_{A\cap B=\emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y\big\}.$$

On the other hand, for all $x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y$, and $a < \bigwedge\big\{ \bigvee_{A\cap B=\emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y\big\}$, there exists $A, B \in 2^X, A \wedge B = \emptyset$, such that $a < N_x(A), a < N_y(B)$. Then there exists $A_1, B_1 \in 2^X$, such that

$$x \in A_1 \subseteq A,\ a < \tau(A_1),$$

$$y \in B_1 \subseteq B,\ a < \tau(B_1).$$

Let $U = \langle 1_{A_1}, 1_{A_1^c}\rangle, V = \langle 1_{B_1}, 1_{B_1^c}\rangle$, where A_1^c is the complement of A_1, then $x_{(\alpha,\beta)}\widehat{q}\ U, y_{(\lambda,\rho)}\widehat{q}\ V$. In fact, $1_{A_1}(x) = 1 > 1-\alpha, 1_{A_1^c}(x) = 0 < 1-\beta$. Thus $x_{(\alpha,\beta)}\widehat{q}\,U$. Similarly, we have $y_{(\lambda,\rho)}\widehat{q}\,V$. By $A \wedge B = \emptyset$, we have $A_1 \wedge B_1 = \emptyset$. Then for all $z \in X$, we obtain

$$(1_{A_1} \wedge 1_{B_1})(z) = 1_{A_1}(z) \wedge 1_{B_1}(z) = 0,$$
$$(1_{A_1^c} \vee 1_{B_1^c})(z) = 1_{A_1^c}(z) \vee 1_{B_1^c}(z) = 1.$$

Hence

$$1_{A_1} \wedge 1_{B_1} = \underline{0},\ 1_{A_1^c} \vee 1_{B_1^c} = \underline{1}.$$

Since $\forall\ r \in I_1, \sigma_r(1_{A_1}) = A_1$, we have

$$\omega(\tau)(1_{A_1}) = \bigwedge_{r\in I_1} \tau(\sigma_r(1_{A_1})) = \tau(A_1).$$

By $\underline{1} - 1_{A_1^c} = 1_{A_1}$, and $a < \tau(A_1)$, we have

$$\begin{array}{rcl} a & < & \omega(\tau)(1_{A_1}) \wedge \omega(\tau)(\underline{1} - 1_{A_1^c}) \\ & = & \omega(\tau)(\mu_U) \wedge \omega(\tau)(\underline{1} - \gamma_U). \end{array}$$

So,

$$a < \bigvee_{x_{(\alpha,\beta)}\widehat{q}\ W\subseteq U} (\omega(\tau)(\mu_W) \wedge \omega(\tau)(\underline{1} - \gamma_W)) = \mu_{Q_{x_{(\alpha,\beta)}}(U)}.$$

Similarly, we have $a < \mu_{Q_{y_{(\lambda,\rho)}}(V)}$. This deduces that

$$a < \bigvee_{U\wedge V = 0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big).$$

Furthermore, we may obtain

$$a \leq \bigwedge \big\{ \bigvee_{U\wedge V = 0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\}.$$

Hence

$$\begin{array}{l} \bigwedge \big\{ \bigvee_{U\wedge V = 0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\} \\ \geq \bigwedge \big\{ \bigvee_{A\cap B = \emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y \big\}. \end{array}$$

This means that $\bigwedge \big\{ \bigvee_{U\wedge V = 0_\sim} \big(\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}\big) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \mathrm{pt}(\zeta^X), x \neq y\big\} = \bigwedge \big\{ \bigvee_{A\cap B = \emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y\big\}$. Therefore we have

$$\mu_{T_2(X, \mathrm{I}\omega(\tau))} = T_2(X, \tau).$$

□

4.3. Lemma. *Let* $(\prod_{j\in J} X_j, \prod_{j\in J} \tau_j)$ *be the product space of a family of intuitionistic* I*-fuzzy topology spaces* $\{(X_j, \tau_j)\}_{j\in J}$*. Then* $\tau_j(A_j) \leq (\prod_{j\in J} \tau_j)(P_j^{\leftarrow}(A_j))$*, for all* $j \in J, A_j \in \zeta^{X_j}$.

Proof. Let $\prod_{j\in J}\tau_j=\delta$, $x_{(\alpha,\beta)}\widehat{q}\ f^{\leftarrow}(U)\Leftrightarrow f^{\rightarrow}(x_{(\alpha,\beta)})\widehat{q}\ U$. Then for all $j\in J, A_j\in\zeta^{X_j}$, we have

$$\begin{aligned}\delta(P_j^{\leftarrow}(A_j)) &= \bigwedge_{x_{(\alpha,\beta)}\widehat{q}\ P_j^{\leftarrow}(A_j)} Q^{\delta}_{x_{(\alpha,\beta)}}(P_j^{\leftarrow}(A_j))\\ &\geq \bigwedge_{x_{(\alpha,\beta)}\widehat{q}\ P_j^{\leftarrow}(A_j)} Q^{\tau_j}_{P_j^{\rightarrow}(x_{(\alpha,\beta)})}(A_j)\\ &= \bigwedge_{P_j^{\rightarrow}(x_{(\alpha,\beta)})\widehat{q}\ A_j} Q^{\tau_j}_{P_j^{\rightarrow}(x_{(\alpha,\beta)})}(A_j)\\ &\geq \bigwedge_{x^j_{(\alpha,\beta)}\widehat{q}\ A_j} Q^{\tau_j}_{x^j_{(\alpha,\beta)}}(A_j)\\ &= \tau_j(A_j).\end{aligned}$$

This completes the proof. □

4.4. Theorem. *Let* $(\prod_{j\in J}X_j,\prod_{j\in J}\tau_j)$ *be the product space of a family of intuitionistic I-fuzzy topology spaces* $\{(X_j,\tau_j)\}_{j\in J}$. *Then* $\bigwedge_{j\in J}T_2(X_j,\tau_j)\leq T_2(\prod_{j\in J}X_j,\prod_{j\in J}\tau_j)$.

Proof. For all $g_{(\alpha,\beta)},h_{(\lambda,\rho)}\in \mathrm{pt}(\zeta^{\prod_{j\in J}X_j})$ and $g\neq h$. Then there exists $j_0\in J$ such that $g(j_0)\neq h(j_0)$, where $g(j_0),h(j_0)\in X_{j_0}$.

For all $U_{j_0},V_{j_0}\in\zeta^{X_{j_0}}$ with $U_{j_0}\wedge V_{j_0}=\underset{\sim}{0}^{X_{j_0}}$, we have

$$P_{j_0}^{\leftarrow}(U_{j_0})\wedge P_{j_0}^{\leftarrow}(V_{j_0})=P_{j_0}^{\leftarrow}(U_{j_0}\wedge V_{j_0})=\underset{\sim}{0}^{\prod_{j\in J}X_j}.$$

Then $Q_{g(j_0)_{(\alpha,\beta)}}(U_{j_0})\leq Q_{g_{(\alpha,\beta)}}(P_{j_0}^{\leftarrow}(U_{j_0}))$. In fact, if $g(j_0)_{(\alpha,\beta)}\widehat{q}\,U_{j_0}$, then $g_{(\alpha,\beta)}\widehat{q}\,P_{j_0}^{\leftarrow}(U_{j_0})$. For all $V\leq U_{j_0}$, we have $P_{j_0}^{\leftarrow}(V)\leq P_{j_0}^{\leftarrow}(U_{j_0})$. On account of Lemma 4.3, we have

$$\begin{aligned}\bigvee_{g(j_0)_{(\alpha,\beta)}\widehat{q}\ V\leq U_{j_0}}\tau_{j_0}(V) &\leq \bigvee_{g_{(\alpha,\beta)}\widehat{q}\ P_{j_0}^{\leftarrow}(V)\leq P_{j_0}^{\leftarrow}(U_{j_0})}(\prod_{j\in J}\tau_j)(P_{j_0}^{\leftarrow}(V))\\ &\leq \bigvee_{g_{(\alpha,\beta)}\widehat{q}\ G\leq P_{j_0}^{\leftarrow}(U_{j_0})}(\prod_{j\in J}\tau_j)(G),\end{aligned}$$

i.e., $Q_{g(j_0)_{(\alpha,\beta)}}(U_{j_0})\leq Q_{g_{(\alpha,\beta)}}(P_{j_0}^{\leftarrow}(U_{j_0}))$. Thus,

$$\begin{aligned}&\bigvee_{U\wedge V=\underset{\sim}{0}^{X_{j_0}}}(Q_{g(j_0)_{(\alpha,\beta)}}(U)\wedge Q_{h(j_0)_{(\lambda,\rho)}}(V))\\ \leq&\bigvee_{P_{j_0}^{\leftarrow}(U)\wedge P_{j_0}^{\leftarrow}(V)=\underset{\sim}{0}^{\prod_{j\in J}X_j}}(Q_{g_{(\alpha,\beta)}}(P_{j_0}^{\leftarrow}(U))\wedge Q_{h_{(\lambda,\rho)}}(P_{j_0}^{\leftarrow}(V)))\\ \leq&\bigvee_{G\wedge H=\underset{\sim}{0}^{\prod_{j\in J}X_j}}(Q_{g_{(\alpha,\beta)}}(G)\wedge Q_{h_{(\lambda,\rho)}}(H)).\end{aligned}$$

So we have

$$T_2(g(j_0)_{(\alpha,\beta)}, h(j_0)_{(\lambda,\rho)}) \leq T_2(g_{(\alpha,\beta)}, h_{(\lambda,\rho)}).$$

Thus

$$T_2(X_{j_0}, \tau_{j_0}) \leq T_2(\prod_{j\in J} X_j, \prod_{j\in J} \tau_j).$$

Therefore,

$$\bigwedge_{j\in J} T_2(X_j, \tau_j) \leq T_2(\prod_{j\in J} X_j, \prod_{j\in J} \tau_j).$$

□

4.5. Lemma. *Let $(X, \mathrm{I}\omega(\tau))$ be a generated intuitionistic I-fuzzy topological space by fuzzifying topological space (X, τ). Then*

(1) $\mathrm{I}\omega(\tau)(A) = 1^{\sim}$, *for all* $A = \langle \underline{\alpha}, \underline{\beta} \rangle \in \zeta^X$;

(2) $\forall\ B \subseteq X, \tau(B) = \mu_{\mathrm{I}\omega(\tau)}(\langle 1_B, 1_{B^c} \rangle)$.

Proof. By Lemma 2.11, 2.12 and 2.13, it is easy to prove it. □

4.6. Lemma. *Let (X, δ) be a stratified intuitionistic I-fuzzy topological space (i.e., for all $< \alpha, \beta > \in \mathcal{A}, \delta(< \underline{\alpha}, \underline{\beta} >) = 1^{\sim}$). Then for all $A \in \zeta^X$*

$$\bigwedge_{r\in I} \mu_\delta(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c} \rangle) \leq \mu_\delta(A).$$

Proof. For all $A \in \zeta^X$, and for any $a < \bigwedge_{r\in I} \mu_\delta(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c} \rangle), y_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$ with $y_{(\alpha,\beta)} \widehat{q}\ A$, clearly $\mu_A(y) > 1 - \alpha$. Then there exists $\delta > 0$ such that $\mu_A(y) > 1 - \alpha + \delta$. Thus $y \in \sigma_{1-\alpha+\delta}(\mu_A)$. So we have

$$y_{(\alpha,\beta)} \widehat{q}\ \langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle.$$

Then

$$\begin{aligned} a &< \mu_\delta(\langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle) \\ &= \bigwedge_{z_{(\alpha,\beta)} \widehat{q}\ \langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle} \mu(Q_{z_{(\alpha,\beta)}}(\langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle)). \end{aligned}$$

Therefore,

$$a < \mu(Q_{y_{(\alpha,\beta)}}(\langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle)).$$

Since (X, δ) is a stratified intuitionistic I-fuzzy topological space, we have $Q_{y_{(\alpha,\beta)}}(\underline{1 - \alpha + \delta}, \underline{\alpha - \delta} \rangle) = 1^{\sim}$. Moreover, it is well known that the following relations hold

$$\underline{1 - \alpha + \delta} \wedge 1_{\sigma_{1-\alpha+\delta}(\mu_A)} \leq \mu_A,$$

$$\underline{\alpha - \delta} \vee 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \geq 1 - \mu_A \geq \gamma_A.$$

So we have

$$a < \mu(Q_{y_{(\alpha,\beta)}}(\langle \underline{1 - \alpha + \delta} \wedge 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, \underline{\alpha - \delta} \vee 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle)) \leq \mu(Q_{y_{(\alpha,\beta)}}(A)).$$

Then $a \le \mu_\delta(A)$. Therefore,

$$\bigwedge_{r\in I} \mu_\delta(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c}\rangle) \le \mu_\delta(A).$$

□

4.7. Theorem. *Let* $(\prod_{\alpha\in J} X_\alpha, \prod_{\alpha\in J} \tau_\alpha)$ *be the product space of a family of fuzzifying topological space* $\{(X_\alpha,\tau_\alpha)\}_{\alpha\in J}$. *Then* $(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A) = \mathrm{I}\omega(\prod_{\alpha\in J}\tau_\alpha)(A)$.

Proof. Let $(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A) = \langle \mu_{\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))}(A), \gamma_{\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))}(A)\rangle$. For all $a < \mu_{\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))}(A)$, there exists $\{U_j^a\}_{j\in K}$ such that $\bigvee_{j\in K} U_j^a = A$, for each U_j^a, there exists $\{A^a_{\lambda,j}\}_{\lambda\in E}$ such that $\bigwedge_{\lambda\in E} A^a_{\lambda,j} = U_j^a$, where E is an finite index set. In addition, for every $\lambda\in E$, there exists $\alpha \triangleq \alpha(\lambda)\in J$ and $W_\alpha \in \zeta^{X_\alpha}$ with $P_\alpha^{\leftarrow}(W_\alpha) = A^a_{\lambda,j}$ such that $a < \mu(\mathrm{I}\omega(\tau_\alpha)(W_\alpha))$. Then we have

$$a < \omega(\tau_\alpha)(\mu_{W_\alpha}),$$

$$a < \omega(\tau_\alpha)(\underline{1} - \gamma_{W_\alpha}).$$

Thus for all $r\in I$, we have

$$\begin{aligned} a &< \tau_\alpha(\sigma_r(\mu_{W_\alpha})) \\ &\le (\prod_{\alpha\in J}\tau_\alpha)(P_\alpha^{\leftarrow}(\sigma_r(\mu_{W_\alpha}))) \\ &= (\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(P_\alpha^{\leftarrow}(\mu_{W_\alpha}))) \\ &= (\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(\mu_{A^a_{\lambda,j}})). \end{aligned}$$

Hence

$$\begin{aligned} a &\le (\prod_{\alpha\in J}\tau_\alpha)(\bigwedge_{\lambda\in E}\sigma_r(\mu_{A^a_{\lambda,j}})) \\ &= (\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(\bigwedge_{\lambda\in E}\mu_{A^a_{\lambda,j}})) \\ &= (\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(\mu_{U_j^a})). \end{aligned}$$

Furthermore

$$\begin{aligned} a &\le (\prod_{\alpha\in J}\tau_\alpha)(\bigvee_{j\in K}\sigma_r(\mu_{U_j^a})) \\ &= (\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(\bigvee_{j\in K}\mu_{U_j^a})) \\ &= (\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(\mu_A)). \end{aligned}$$

So

$$\begin{aligned} a &\leq \bigwedge_{r\in I}(\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(\mu_A)) \\ &= \omega(\prod_{\alpha\in J}\tau_\alpha)(\mu_A). \end{aligned}$$

Similarly, we have

$$a \leq \omega(\prod_{\alpha\in J}\tau_\alpha)(\underline{1}-\gamma_A).$$

Hence $a \leq \mu(\mathrm{I}\omega(\prod_{\alpha\in J}\tau_\alpha)(A))$. By the arbitrariness of a, we have $\mu((\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(A)) \leq \mu(\mathrm{I}\omega(\prod_{\alpha\in J}\tau_\alpha)(A))$.

On the other hand, for $\forall\ a < \mu(\mathrm{I}\omega(\prod_{\alpha\in J}\tau_\alpha)(A))$, we have

$$a < \omega(\prod_{\alpha\in J}\tau_\alpha)(\mu_A) = \bigwedge_{r\in I}(\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(\mu_A))$$

and

$$a < \omega(\prod_{\alpha\in J}\tau_\alpha)(\underline{1}-\gamma_A).$$

Then for all $r \in I$, we have

$$a < (\prod_{\alpha\in J}\tau_\alpha)(\sigma_r(\mu_A)).$$

Thus there exists $\{U^a_{j,r}\}_{j\in K} \subseteq X$ satisfies $\bigvee_{j\in K} U^a_{j,r} = \sigma_r(\mu_A)$, and for all $j \in K$, there exists $\{A^a_{\lambda,j,r}\}_{\lambda\in E}$, where E is an finite index set, such that $\bigwedge_{\lambda\in E} A^a_{\lambda,j,r} = U^a_{j,r}$. For all $\lambda \in E$, there exists $\alpha(\lambda) \in J, W_\alpha \in \zeta^{X_\alpha}$, such that $P^{\leftarrow}_\alpha(W_\alpha) = A^a_{\lambda,j,r}$. By Lemma 4.5 we have

$$\begin{aligned} a < \tau_\alpha(W_\alpha) &= \mu_{\mathrm{I}\omega(\tau_\alpha)}(\langle 1_{W_\alpha}, 1_{W^c_\alpha}\rangle) \\ &\leq \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(P^{\leftarrow}_\alpha(\langle 1_{W_\alpha}, 1_{W^c_\alpha}\rangle)) \\ &= \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(\langle 1_{P^{\leftarrow}_\alpha(W_\alpha)}, 1_{P^{\leftarrow}_\alpha(W^c_\alpha)}\rangle) \\ &= \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(\langle 1_{A^a_{\lambda,j,r}}, 1_{(A^a_{\lambda,j,r})^c}\rangle) \\ &\leq \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(\langle \bigwedge_{\lambda\in E} 1_{A^a_{\lambda,j,r}}, \bigvee_{\lambda\in E} 1_{(A^a_{\lambda,j,r})^c}\rangle) \\ &= \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(\langle 1_{\bigwedge_{\lambda\in E} A^a_{\lambda,j,r}}, 1_{\bigvee_{\lambda\in E}(A^a_{\lambda,j,r})^c}\rangle) \\ &= \mu(\prod_{\alpha\in J}\mathrm{I}\omega(\tau_\alpha))(\langle 1_{U^a_{j,r}}, 1_{(U^a_{j,r})^c}\rangle). \end{aligned}$$

Then

$$
\begin{aligned}
a &\le \mu(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(\langle 1_{\bigvee\limits_{j\in K} U^{a}_{j,r}}, 1_{(\bigvee\limits_{j\in K} U^{a}_{j,r})^c}\rangle) \\
&= \mu(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c}\rangle).
\end{aligned}
$$

By Lemma 4.6 we have

$$
\begin{aligned}
a &\le \bigwedge_{r\in I} \mu(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c}\rangle) \\
&\le \mu((\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A)).
\end{aligned}
$$

Then

$$\mu((\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A)) \ge \mu(\mathrm{I}\omega(\prod_{\alpha\in J} \tau_\alpha)(A)).$$

Hence

$$\mu((\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A)) = \mu(\mathrm{I}\omega(\prod_{\alpha\in J} \tau_\alpha)(A)).$$

Then

$$\gamma((\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A)) = \gamma(\mathrm{I}\omega(\prod_{\alpha\in J} \tau_\alpha)(A)).$$

Therefore,

$$(\prod_{\alpha\in J} \mathrm{I}\omega(\tau_\alpha))(A) = \mathrm{I}\omega(\prod_{\alpha\in J} \tau_\alpha)(A).$$

□

5. Further remarks

As we have shown, the notions of the base and subbase in intuitionistic I-fuzzy topological spaces are introduced in this paper, and some important applications of them are obtained. Specially, we also use the concept of subbase to study the product of intuitionistic I-fuzzy topological spaces. In addition, we have proved that the functor Iω preserves the product.

There are two categories in our paper, the one is the category **FYTS** of fuzzifying topological spaces, and the other is the category **IFTS** of intuitionistic I-fuzzy topological spaces. It is easy to find that Iω is the functor from **FYTS** to **IFTS**. We discussed the property of the functor Iω in Theorem 4.7. A direction worthy of further study is to discuss the the properties of the functor Iω in detail. Moreover, we hope to point out that another continuation of this paper is to deal with other topological properties of intuitionistic I-fuzzy topological spaces.

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Fuzzy integro-differential equations with compactness type conditions

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Abstract

In the paper fuzzy integro-differential equations with almost continuous right hand sides are studied. The existence of solution is proved under compactness type conditions.

Keywords: Fuzzy integro-differential equation; Measure of noncompactness.

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1. Introduction

Many problems in modeling as well as in medicines are described by fuzzy integro-differential equations, which are helpful in studying the observability of dynamical control systems. This is the main reason to study these equations extensively. We mention the papers [1] and [2], where nonlinear integro-differential equations are studied in Banach spaces and in fuzzy space respectively. In [3], existence result for nonlinear fuzzy Volterra-Fredholm integral equation is proved. In [14], fuzzy Volterra integral equations are studied using fixed point theorem, while in [10], the method of successive approximation is used, when the right hand side satisfies Lipschitz condition. In [15] Kuratowski measure of noncompactness as well as imbedding map from fuzzy to Banach space is used to prove existence of solutions. In [11] existence and uniqueness result for fuzzy Volterra integral equation with Lipschitz right hand side and with infinite delay is proved using successive approximations method. We also refer to [4] where existence of solution of functional integral equation under compactness condition is proved.

In the paper we study the following fuzzy integro-differential equation:

$$\dot{x}(t) = F(t, x(t), (Vx)(t)),\ x(0) = x_0,\ t \in I = [0, T], \tag{1.1}$$

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where $(Vx)(t) = \int_0^t K(t,s)x(s)ds$ is an integral operator of Volterra type.

2. Preliminaries

In this section we give our main assumptions and preliminary results needed in the paper.

The fuzzy set space is denoted by $\mathbb{E}^n = \{x : \mathbb{R}^n \to [0,1];\ x \text{ satisfies } 1) - 4)\}$.

1) x is normal i.e. there exists $y_0 \in \mathbb{R}^n$ such that $x(y_0) = 1$,

2) x is fuzzy convex i.e. $x(\lambda y + (1-\lambda)z) \geq \min\{x(y), x(z)\}$ whenever $y, z \in \mathbb{R}^n$ and $\lambda \in [0,1]$,

3) x is upper semicontinuous i.e. for any $y_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta(y_0, \varepsilon) > 0$ such that $x(y) < x(y_0) + \varepsilon$ whenever $|y - y_0| < \delta$ and $y \in \mathbb{R}^n$,

4) The closure of the set $\{y \in \mathbb{R}^n;\ x(y) > 0\}$ is compact.

The set $[x]^\alpha = \{y \in \mathbb{R}^n;\ x(y) \geq \alpha\}$ is called α-level set of x.

It follows from 1) – 4) that the α-level sets $[x]^\alpha$ are convex compact subsets of $\mathbb{R}^n$ for all $\alpha \in (0,1]$. The fuzzy zero is

$$\hat{0}(y) = \begin{cases} 0 \ if \ y \neq 0, \\ 1 \ if \ y = 0. \end{cases}$$

Evidently $\mathbb{E}^n$ is a complete metric space equipped with metric

$$D(x,y) = \sup_{\alpha \in (0,1]} D_H([x]^\alpha, [y]^\alpha),$$

where $D_H(A,B) = \max\{\max_{a\in A} \min_{b \in B} |a-b|, \max_{b \in B} \min_{a \in A} |a-b|\}$ is the Hausdorff distance between the convex compact subsets of $\mathbb{R}^n$. From Theorem 2.1 of [7], we know that $\mathbb{E}^n$ can be embedded as a closed convex cone in a Banach space $\mathbb{X}$. The embedding map $j : \mathbb{E}^n \to \mathbb{X}$ is isometric and isomorphism.

The function $g : I \to \mathbb{E}^n$ is said to be simple function if there exists a finite number of pairwise disjoint measurable subsets $I_1, \ldots, I_n$ of I with $I = \bigcup_{k=1}^{n} I_k$ such that $g(\cdot)$ is constant on every I_k.

The map $f : I \to \mathbb{E}^n$ is said to be strongly measurable if there exists a sequence $\{f_m\}_{m=1}^{\infty}$ of simple functions $f_m : I \to \mathbb{E}^n$ such that $\lim_{m\to\infty} D(f_m(t), f(t)) = 0$ for a.a $t \in I$.

In the fuzzy set literature starting from [12] the integral of fuzzy functions is defined levelwise, i.e. there exists $g(t) \in \mathbb{E}^n$ such that $[g]^\alpha(t) = \int_0^t [f]^\alpha(s)ds$.

Now if $g(\cdot) : I \to \mathbb{E}^n$ is strongly measurable and integrable then $j(g)(\cdot)$ is strongly measurable and Bochner integrable and

$$j\left(\int_0^t g(s)ds\right) = \int_0^t j\,(g)\,(s)ds \text{ for all } t \in I. \tag{2.1}$$

We recall some properties of integrable fuzzy set valued mapping from [7].

2.1. Theorem. *Let $G, K : I \to \mathbb{E}^n$ be integrable and $\lambda \in \mathbb{R}$ then*

(i) $\int_I (G(t) + K(t))dt = \int_I G(t)dt + \int_I K(t)dt$,
(ii) $\int_I \lambda G(t)dt = \lambda \int_I G(t)dt$,
(iii) $D(G, K)$ *is integrable,*
(iv) $D(\int_I G(t)dt, \int_I K(t)dt) \leq \int_I D(G(t), K(t))dt$.

A mapping $F : I \to \mathbb{E}^n$ is said to be differentiable at $t \in I$ if there exists $\dot{F}(t) \in \mathbb{E}^n$ such that the limits $\lim\limits_{h\to 0^+} \frac{F(t+h)-F(t)}{h}$ and $\lim\limits_{h\to 0^+} \frac{F(t)-F(t-h)}{h}$ exist, and are equal to $\dot{F}(t)$. At the end point of I we consider only the one sided derivative.

Notice that $\mathbb{E}^n$ is not locally compact (cf. [13]). Consequently we need compactness type assumptions to prove existence of solution, we refer the interested reader to [5] and the references therein.

Let Y be complete metric space with metric $\varrho_Y(\cdot, \cdot)$. The Hausdorff measure of noncompactness $\beta : Y \to \mathbb{R}$ for the bounded subset A of Y is defined by

$$\beta(A) := \inf\{d > 0 : A \text{ can be covered by finite many balls with radius } \leq d\}$$

and "Kuratowski measure" of noncompactness $\rho : Y \to \mathbb{R}$ for the bounded subset A of Y is defined by

$$\rho(A) := \inf\{d > 0 : A \text{ can be covered by finite many sets with diameter } \leq d\},$$

where for any bounded set $A \subset Y$, we denote $\operatorname{diam}(A) = \sup\limits_{a,b\in A} \varrho_Y(a, b)$. It is well known that $\rho(A) \leq \beta(A) \leq 2\rho(A)$ (cf. [8] p.116).

Let $\gamma(\cdot)$ represent the both $\rho(\cdot)$ and $\beta(\cdot)$, then some properties of $\gamma(\cdot)$ are listed below:

(i) $\gamma(A) = 0$ if and only if A is precompact, i.e. its closure $\bar{A}$ is compact,
(ii) $\gamma(A + B) = \gamma(A) + \gamma(B)$ and $\gamma(\overline{co}A) = \gamma(A)$,
(iii) If $A \subset B$ then $\gamma(A) \leq \gamma(B)$,
(iv) $\gamma(A \bigcup B) = max(\gamma(A), \gamma(B))$,
(v) $\gamma(\cdot)$ is continuous with respect to the Hausdorff distance.

The following theorem of Kisielewicz can be found e.g. in [8].

2.2. Theorem. *Let X be separable Banach space and let $\{g_n(\cdot)\}_{n=1}^{\infty}$ be an integrally bounded sequence of measurable functions from I into X, then $t \to \beta\{g_n(t), n \geq 1\}$ is measurable and*

$$\beta\left(\int_t^{t+h} \left\{\bigcup_{i=1}^{\infty} g_i(s)\right\} ds\right) \leq \int_t^{t+h} \beta\left\{\bigcup_{i=1}^{\infty} g_i(s)\right\} ds, \tag{2.2}$$

where $t, t + h \in I$.

The map $t \to \{\bigcup_{i=1}^{\infty} g_i(t)\}$ is a set valued (multifunction). The integral is defined in Auman sense, i.e. union of the values of the integrals of all (strongly) measurable selections.

2.3. Remark. Since the imbedding map $j : \mathbb{E}^n \to \mathbb{X}$ is isometry and isomorphism, one has that it preserve diameter of any closed subset i.e. $\rho(A) = \rho(j(A))$, for any closed and bounded set $A \in \mathbb{E}^n$.

2.4. Theorem. *Let $\{f_n(\cdot)\}_{n=1}^{\infty}$ be a (integrally bounded) sequence of strongly measurable fuzzy functions defined from I into $\mathbb{E}^n$. Then $t \to \rho(\{f_m(t), m \geq 1\})$ is measurable and*

$$\rho\left(\int_a^b \bigcup_{m=1}^{\infty} f_m(s)ds\right) \leq 2\int_a^b \rho\left(\bigcup_{m=1}^{\infty} f_m(s)\right) ds. \tag{2.3}$$

Proof. Since f_m are strongly measurable, one has that $j(f_m)(\cdot)$ are also strongly measurable and hence almost everywhere separably valued.

Thus there exists a separable Banach space $Y \subset X$ such that $j(f_m)(I \setminus N) \subset Y$, where $N \subset I$ is a null set.

Furthermore without loss of generality from Theorem 2.2 and Remark 2.3, we have

$$\begin{aligned}
\rho\left(\int_a^b \left(\bigcup_{m=1}^{\infty} f_m(s)\right) ds\right) &= \rho\left(\int_a^b \left(\bigcup_{m=1}^{\infty} j(f_m(s))\right) ds\right) \\
\leq \beta\left(\int_a^b \left(\bigcup_{m=1}^{\infty} j\left(f_m(s)\right)\right) ds\right) &= \int_a^b \beta\left(\bigcup_{m=1}^{\infty} j(f_m(s))\right) ds \\
\leq 2\int_a^b \rho\left(\bigcup_{m=1}^{\infty} j(f_m(s))\right) ds &= 2\int_a^b \rho\left(\bigcup_{m=1}^{\infty} f_m(s)\right) ds.
\end{aligned}$$

Consequently, we get (2.3). □

2.5. Remark. Evidently one can replace $\rho(\cdot)$ by $\beta(\cdot)$ in *(2.3)*. It would be interesting to see is it possible to replace 2 in the right hand side by smaller constant, using the special structure of the fuzzy set space, i.e. is it true that

$$\beta\left(\int_a^b \bigcup_{m=1}^{\infty} f_m(s)ds\right) \leq C\int_a^b \beta\left(\bigcup_{m=1}^{\infty} f_m(s)\right) ds,$$

for some $1 \leq C < 2$?

3. Main Results

In this section we prove the existence of solution of (1.1). The following hypotheses will be used;

(H1) $F : I \times \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{E}^n$ is such that
(i) $t \to F(t, x, y)$ is strongly measurable for all $x, y \in \mathbb{E}^n$,
(ii) $(x, y) \to F(t, x, y)$ is continuous for almost all $t \in I$.

Suppose there exist $a(\cdot), b(\cdot) \in L^1(I, \mathbb{R}_+)$ such that:

(H2) $\rho(F(t, A, B)) \leq \lambda(t)(\rho(A)+\rho(B))$, for all non empty bounded subsets $A, B \in \mathbb{E}^n$ and $\lambda(\cdot) \in L^1(I, \mathbb{R}_+)$,

(H3) $D\left(F(t, x, y), \hat{0}\right) \leq a(t) + b(t)\left[D(x, \hat{0}) + D(y, \hat{0})\right]$,

(H4) $K : \triangle = \{(t, s); 0 \leq s \leq t \leq a\} \to \mathbb{R}_+$ is a continuous function.

3.1. Theorem. *If **(H1)**– **(H4)** hold, then problem (1.1) has at least one solution on $[0, T]$.*

Proof. First, we will show that a solution of (1.1) is bounded. Indeed, we have

$$\begin{aligned} D(x(t),\hat{0}) &= D(x_0,\hat{0}) + D\left(\int_0^t F(s,x(s),(Vx)(s))ds,\hat{0}\right) \\ &\le D(x_0,\hat{0}) + \int_0^t D\left(F(s,x(s),(Vx)(s)),\hat{0}\right)ds \\ &\le D(x_0,\hat{0}) + \int_0^t \left(a(s) + b(s)\left[D(x(s),\hat{0}) + D(\int_0^s K(s,\tau)x(\tau)d\tau,\hat{0})\right]\right)ds \\ &\le D(x_0,\hat{0}) + \int_0^t \left(a(s) + b(s)D(x(s),\hat{0}) + K_\Delta b(s)\int_0^s D(x(\tau)d\tau,\hat{0})\right)ds, \end{aligned}$$

where $K_\Delta = \max\limits_{(t,s)\in\Delta} |K(t,s)|$.

Therefore, if we denote $m(t) = D(x(t),\hat{0})$, then we obtain

$$m(t) = m(0) + \int_0^t \left(a(s) + b(s)m(s) + K_\Delta b(s)\int_0^s m(\tau)d\tau\right)ds.$$

By Pachpatte's inequality (see Theorem 1 in [9]), we get that there exists $M_0 > 0$ such that $m(t) = D(x(t),\hat{0}) \le M_0$ for all $t \in [0,T]$.

Moreover, we obtain that

$$\begin{aligned} D((Vx)(t),\hat{0}) &= D(\int_0^t K(t,s)x(s)ds,\hat{0}) \\ &\le \int_0^t D(K(t,s)x(s),\hat{0})ds \\ &\le K_\Delta \int_0^t D(x(s),\hat{0})ds \le K_\Delta M_0 T \doteq M_1. \end{aligned}$$

It follows that

$$D\left(F(t,x(t),(Vx)(t)),\hat{0}\right) \le a(t) + Mb(t) \doteq \mu(t),$$

where $M = M_0 + M_1$. Since $a(\cdot), b(\cdot) \in L^1(I,\mathbb{R}_+)$, one has that $\mu(\cdot) \in L^1(I,\mathbb{R}_+)$ Let $c = \int_0^T \mu(s)ds + 1$. We define

$$\Omega = \left\{ x(\cdot) \in C([0,T],\mathbb{E}^n) : \sup_{t\in[0,T]} D(x(t),x_0) \le c \right\}.$$

Clearly, Ω closed, bounded and convex set. We also define the operator $P : C[[0,T],\mathbb{E}^n] \to C[[0,T],\mathbb{E}^n]$ by

$$(Px)(t) = x_0 + \int_0^t F(s,x(s),(Vx)(s))ds,\ t \in [0,T].$$

Therefore

$$\begin{aligned} D((Px)(t),x_0) &= D\left(\int_0^t F(s,x(s),(Vx)(s))ds,\hat{0}\right) \\ &\le \int_0^t D\left(F(s,x(s),(Vx)(s)),\hat{0}\right)ds \\ &\le \int_0^T \mu(s)ds < c \end{aligned}$$

for $x \in \Omega$ and $t \in [0,T]$. Thus $P(\Omega) \subset \Omega$ and $P(\Omega)$ is uniformly bounded on $[0,T]$.

Next we have to show that P is a continuous operator on Ω. For this, let $x_n(\cdot) \in \Omega$ such that $x_n(\cdot) \to x(\cdot)$. Then

$$D((Px_n)(t),(Px)(t)) = D\left(\int_0^t F(s,x_n(s),(Vx_n)(s))ds, \int_0^t F(s,x(s),(Vx)(s))ds\right)$$
$$\leq \int_0^t D\left(F(s,x_n(s),(Vx_n)(s)), F(s,x(s),(Vx)(s))\right) ds.$$

Also, $V : \Omega \to \mathbb{E}^n$ defined by $(Vx)(t) = \int_0^t K(t,s)x(s)ds$ is a continuous operator, because

$$D((Vx_n)(t),(Vx)(t)) = D\left(\int_0^t K(t,s)x_n(s)ds, \int_0^t K(t,s)x(s)ds\right)$$
$$\leq \int_0^t D\left(K(t,s)x_n(s), K(t,s)x(s)\right) ds$$
$$\leq K_\Delta \int_0^t D(x_n(s),x(s))ds \to 0 \text{ as } n \to \infty.$$

Thus by **(H1)**, it follows that $D((Px_n)(t),(Px)(t)) \to 0$ as $n \to \infty$ uniformly on $[0,T]$, so P is a continuous operator on $[0,T]$.

The function $t \to \int\limits_0^t \mu(\cdot)ds$ is uniformly continuous on the closed set $[0,T]$, i.e. there exist $\eta > 0$ such that $\left|\int_s^t \mu(\tau)d\tau\right| \leq \frac{\varepsilon}{2}$ for all $t,s \in [0,T]$ with $|t-s| < \eta$.

Further, for each $m \geq 1$, we divide $[0,T]$ into m subintervals $[t_i,t_{i+1}]$ with $t_i = \frac{iT}{m}$.

$$x_m(t) = \begin{cases} x_0 & \text{if t} \in [0,\text{t}_1], \\ (Px_m)(t-t_i) & \text{if t} \in [\text{t}_\text{i},\text{t}_{\text{i}+1}]. \end{cases}$$

Then $x_m(\cdot) \in \Omega$ for every $m \geq 1$. Moreover, for $t \in [0,t_1]$, we have

$$D\left((Px_m)(t), x_m(t)\right) = D\left(\int_0^t F(s,x_m(s),(Vx_m)(s)),\hat{0}\right) ds$$
$$\leq \int_0^{t_1} D\left(F(s,x_m(s),(Vx_m)(s)),\hat{0}\right) ds \leq \int_0^{t_1} \mu(s)ds,$$

and for $t \in [t_i,t_{i+1}]$, we have $t - t_i \leq \frac{T}{m}$ and hence

$$D\left((Px_m)(t), x_m(t)\right) = D\left((Px_m)(t),(Px_m)(t-t_i)\right)$$
$$= D\left(\int_0^t F(s,x_m(s),(Vx_m)(s))ds, \int_0^{t_i} F(s,x_m(s),(Vx_m)(s))ds\right)$$
$$= D\left(\int\limits_{t-t_i}^t F(s,x_m(s),(Vx_m)(s))ds, \hat{0}\right)$$
$$\leq \int\limits_{t-T/m}^t D\left(F(s,x_m(s),(Vx_m)(s))ds, \hat{0}\right) ds$$

$$\leq \int\limits_{t-T/m}^{t} \mu(s)ds.$$

Therefore $\lim\limits_{m\to\infty} D((Px_m)(t), x_m(t)) = 0$ on [0,T]. Let $A = \{x_m(\cdot); m \geq 1\}$. We claim that A is equicontinuous on $[0, T]$. If $t, s \in [0, T/m]$, then $D(x_m(t), x_m(s)) = 0$. If $0 \leq s \leq T/m \leq t \leq T$, then

$$\begin{aligned} D\left(x_m(t), x_m(s)\right) &= D\left(x_0 + \int_0^{t-T/m} F(\sigma, x_m(\sigma), (Vx_m)(\sigma))d\sigma, x_0\right) \\ &\leq \int_0^{t-T/m} D\left(F(\sigma, x_m(\sigma), (Vx_m)(\sigma)), \hat{0}\right) d\sigma \\ &\leq \int_0^{t-T/m} \mu(\sigma)d\sigma \leq \int_0^{t} \mu(\sigma)d\sigma < \epsilon/2, \end{aligned}$$

for $|t - s| < \eta$. If $T/m \leq s \leq t \leq T$, then

$$D\left(x_m(t), x_m(s)\right) < \epsilon/2 \quad \text{when} \quad |t - s| < \epsilon.$$

Therefore A is equicontinuous on [0,T]. Set $A(t) = \{x_m(t); m \geq 1\}$ for $t \in [0, T]$. We are to show that $A(t)$ is precompact for each $t \in [0, T]$. We have

$$\rho(A(t)) \leq \rho\left(\int_0^{t-T/m} F(s, A(s), (VA)(s))ds\right) + \rho\left(\int_{t-T/m}^{t} F(s, A(s), (VA)(s))ds\right).$$

Given $\epsilon > 0$, we can find $m(\epsilon) > 0$, such that $\int_{t-T/m}^{t} \mu(s)ds < \epsilon/2$, for all $t \in [0, T]$ and $m \geq m(\epsilon)$. Hence

$$\begin{aligned} &\rho\left(\int_{t-T/m}^{t} F(s, A(s), (VA)(s))ds\right) \\ &= \rho\left(\left\{\int_{t-T/m}^{t} F(s, x_m(s), (Vx_m))ds; m \geq n(\epsilon)\right\}\right) \\ &\leq 2\int_{t-T/m}^{t} \mu(s)ds < \epsilon. \end{aligned}$$

It follows that

$$\begin{aligned} \rho(A(t)) \leq \rho\left(\int_0^t F(s, A(s), (VA)(s))ds\right) &\leq 2\int_0^t \rho\left(F(s, A(s), (VA)(s))\right) ds \\ &\leq 2\int_0^t \lambda(s)[\rho(A(s)) + \rho((VA)(s))]ds. \end{aligned}$$

However,

$$\rho(VA(s)) = \rho\left(\int_0^t K(t,s)A(s)ds\right) = \rho\left(\left\{\int_0^t K(t,s)x_m(s)ds;\, m \geq 1\right\}\right)$$
$$\leq 2\int_0^t \rho\left(\{K(t,s)x_m(s);\, m \geq 1\}\right) ds \leq 2\int_0^t K_\Delta \rho\left(\{x_m(s);\, m \geq 1\}\right) ds$$
$$= 2\int_0^t K_\Delta \rho(A(s))ds$$

and

$$\int_0^t \rho\left(VA(s)\right) ds \leq \int_0^t 2\int_0^s K_\Delta \rho\left(A(\tau)\right) d\tau\, ds$$
$$= 2\int_0^t \int_\tau^t K_\Delta \rho\left(A(\tau)\right) ds d\tau$$
$$= 2\int_0^t K_\Delta (t-\tau)\rho(A(\tau))d\tau \leq K_\Delta T\int_0^t \rho(A(\tau))d\tau.$$

Therefore we obtain that

$$\rho(A(t)) \leq 2\int_0^t \lambda(s)[\rho\left(A(s)\right) + K_\Delta T \rho\left(A(s)\right)]ds.$$

Let $R = e^{2(1+K_\Delta T)\int_0^T \lambda(t)dt}$. Due to Gronwall inequality

$$\rho(A(t)) \leq R\int_0^t \rho\left(A(s)\right) ds.$$

Therefore $\rho(A(t)) = 0$ and hence $A(t)$ is precompact for every $t \in [0,T]$. Since A is equicontinuous and $A(t)$ is precompact, one has that Arzela-Ascoli theorem holds true in our case. Thus (passing to subsequences if necessary) the sequence $\{x_n(t)\}_{n=1}^\infty$ converges uniformly on $[0,T]$ to a continuous function $x(\cdot) \in \Omega$. Due to the triangle inequality

$$D\left((Px)(t), x(t)\right) \leq D\left((Px)(t), (Px_n)(t)\right)$$
$$+D\left((Px_n)(t), x_n(t)\right) + D\left(x_n(t), x(t)\right) \to 0,$$

we have $(Px)(t) = x(t)$ for all $t \in [0,T]$, i.e. $x(\cdot)$ is a solution of (1.1). □

3.2. Remark. From Theorem 3.1 it is easy to see that the solution set of (1.1) denoted by

$$\Omega = \left\{x(\cdot) \in C([0,T], \mathbb{E}^n) : \sup_{t\in[0,T]} D(x(t), x_0) \leq c\right\}$$

is compact.

4. Conclusion

We pay our attention to find existence of solution of fuzzy integro-differential equations under mild assumption as compared with the already existing results in the literature, To overcome some difficulties as lack of compactness and other restrictive properties of fuzzy space $\mathbb{E}^n$, we use Kuratowski measure of non compactness, which enables us to use Arzela-Ascoli theorem.

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α-separation axioms based on Łukasiewicz logic

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Abstract

In the present paper, we introduce topological notions defined by means of α-open sets when these are planted into the framework of Ying's fuzzifying topological spaces (by Łukasiewicz logic in [0, 1]). We introduce $T_0^\alpha-$, $T_1^\alpha-$, T_2^α (α- Hausdorff)-, T_3^α (α-regular)- and T_4^α (α-normal)-separation axioms. Furthermore, the $R_0^\alpha-$ and $R_1^\alpha-$ separation axioms are studied and their relations with the $T_1^\alpha-$ and $T_2^\alpha-$ separation axioms are introduced. Moreover, we clarify the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms.

Keywords: Łukasiewicz logic, semantics, fuzzifying topology, fuzzifying separation axioms, α-separation axioms.

2000 AMS Classification: 54A40

1. Introduction and Preliminaries

In the last few years fuzzy topology, as an important research field in fuzzy set theory, has been developed into a quite mature discipline [7-9, 14-15, 27]. In contrast to classical topology, fuzzy topology is endowed with richer structure, to a certain extent, which is manifested with different ways to generalize certain classical concepts. So far, according to Ref. [8], the kind of topologies defined by Chang [4] and Goguen [5] is called the topologies of fuzzy subsets, and further is naturally called L-topological spaces if a lattice L of membership values has been chosen. Loosely speaking, a topology of fuzzy subsets (resp. an L-topological space) is a family τ of fuzzy subsets (resp. L-fuzzy subsets) of nonempty set X, and τ satisfies the basic conditions of classical topologies [11]. On the other hand, Höhle in [6] proposed the terminology L-fuzzy topology to be an L-valued mapping on the traditional powerset $P(X)$ of X. The authors in [10, 23] defined an L-fuzzy topology to be an L-valued mapping on the L-powerset L^X of X.

In 1952, Rosser and Turquette [25] proposed emphatically the following problem: If there are many-valued theories beyond the level of predicates calculus, then what are the detail of such theories ? As an attempt to give a partial answer

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to this problem in the case of point set topology, Ying in 1991-1993 [28-30] used a semantical method of continuous-valued logic to develop systematically fuzzifying topology. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. In fact, fuzzifying topologies are a special case of the L-fuzzy topologies in [10, 23] since all the t-norms on $I = [0,1]$ are included as a special class of tensor products in these paper. Ying uses one particular tensor product, namely Łukasiewicz conjunction. Thus his fuzzifying topologies are a special class of all the I- fuzzy topologies considered in the categorical frameworks [10, 23]. Roughly speaking, the semantical analysis approach transforms formal statements of interest, which are usually expressed as implication formulas in logical language, into some inequalities in the truth value set by truth valuation rules, and then these inequalities are demonstrated in an algebraic way and the semantic validity of conclusions is thus established. So far, there has been significant research on fuzzifying topologies [12-13, 20-21, 26]. For example, Shen [26] introduced and studied T_0-, T_1-, T_2 (Hausdorff)-, T_3(regular)- and T_4(normal)- separation axioms in fuzzifying topology. In [13], the concepts of the R_0- and R_1- separation axioms in fuzzifying topology were added and their relations with the T_1- and T_2- separation axioms, were studied. Also, in [12] the concepts of fuzzifying α-open set and fuzzifying α-continuity were introduced and studied. In classical topology, α-separation axioms have been studied in [2-3, 16-17, 19, 22]. As well as, they have been studied in fuzzy topology in [1,18, 24]. In the present paper, we explore the problem proposed by Rosser and Turquette [25] in fuzzy α-separation axioms.

A basic structure of the present paper is as follows. First, we offer some definitions and results which will be needed in this paper. Afterwards, in Section 2, in the framework of fuzzifying topology, the concept of α-separation axioms $T_0^\alpha-$, $T_1^\alpha-$, T_2^α (α-Hausdorff)-, T_3^α (α-regular)- and T_4^α (α-normal) are discussed. In Section 3, on the bases of fuzzifying topology the $R_0^\alpha-$ and $R_1^\alpha-$ separation axioms are introduced and their relations with the T_1^α and $T_{2^-}^\alpha-$ separation axioms are studied. Furthermore , we give the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms. Finally, in a conclusion, we summarize the main results obtained and raise some related problems for further study. Thus we fill a gap in the existing literature on fuzzifying topology. We will use the terminologies and notations in [12-13, 26, 28, 29] without any explanation. We will use the symbol $\otimes$ instead of the second "AND" operation $\dot\wedge$ as dot is hardly visible. This mean that $[\alpha] \leq [\varphi \to \psi] \Leftrightarrow [\alpha] \otimes [\varphi] \leq [\psi]$.

A fuzzifying topology on a set X [6, 28] is a mapping $\tau \in \Im(P(X))$ such that:

(1) $\tau(X) = 1, \tau(\phi) = 1$;

(2) for any $A, B, \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(3) for any $\{A_\lambda : \lambda \in \Lambda\}, \tau\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) \geq \bigwedge_{\lambda\in\Lambda} \tau(A_\lambda)$.

The family of all fuzzifying α-open sets [12], denoted by $\tau_\alpha \in \Im(P(X))$, is defined as

$$A \in \tau_\alpha := \forall x(x \in A \to x \in Int(Cl(Int(A)))), \text{ i. e., } \tau_\alpha(A) = \bigwedge_{x\in A} Int(Cl(Int(A)))(x)$$

The family of all fuzzifying α-closed sets [12], denoted by $F_\alpha \in \Im(P(X))$, is defined as $A \in F_\alpha := X - A \in \tau_\alpha$. The fuzzifying α-neighborhood system of a point $x \in X$

[12] is denoted by $N_x^\alpha \in \Im(P(X))$ and defined as $N_x^\alpha(A) = \bigvee_{x\in B\subseteq A} \tau_\alpha(B)$.The fuzzifying α-closure of a set $A \subseteq X$ [12], denoted by $Cl_\alpha \in \Im(X)$, is defined as $Cl_\alpha(A)(x) = 1 - N_x^\alpha(X - A)$.

Let (X, τ) be a fuzzifying topological space. The binary fuzzy predicates $K, H, M \in \Im(X\times X)$, $V \in \Im(X\times P(X))$ and $W \in \Im(P(X)\times P(X))$ [13] are defined as follows:

(1) $K(x, y) := \exists A((A \in N_x \wedge y \notin A) \vee (A \in N_y \wedge x \notin A))$;
(2) $H(x, y) := \exists B\exists C((B \in N_x \wedge y \notin B) \wedge (C \in N_y \wedge x \notin C))$;
(3) $M(x, y) := \exists B\exists C(B \in N_x \wedge C \in N_y \wedge B \cap C \equiv \emptyset)$;
(4) $V(x, D) := \exists A\exists B(A \in N_x \wedge B \in \tau \wedge D \subseteq B \wedge A \cap B \equiv \emptyset)$;
(5) $W(A, B) := \exists G\exists H(G \in \tau \wedge H \in \tau \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H \equiv \emptyset)$.

Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates $T_i \in \Im(\Omega), i = 0, 1, 2, 3, 4$ [26] (see the rewritten form in [13]) and $R_i \in \Im(\Omega), i = 0, 1$ [13] are defined as follows:

(1) $(X, \tau) \in T_0 := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow K(x, y)$;
(2) $(X, \tau) \in T_1 := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow H(x, y)$;
(3) $(X, \tau) \in T_2 := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow M(x, y)$;
(4) $(X, \tau) \in T_3 := \forall x\forall D(x \in X \wedge D \in F \wedge x \notin D) \longrightarrow V(x, D)$;
(5) $(X, \tau) \in T_4 := \forall A\forall B(A \in F \wedge B \in F \wedge A \cap B = \emptyset) \longrightarrow W(A, B)$;
(6) $(X, \tau) \in R_0 := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K(x, y) \longrightarrow H(x, y))$;
(7) $(X, \tau) \in R_1 := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K(x, y) \longrightarrow M(x, y))$.

2. Fuzzifying α- separation axioms and their equivalents

For simplicity we give the following definition.

2.1. Definition. Let (X, τ) be a fuzzifying topological space. The binary fuzzy predicates $K^\alpha, H^\alpha, M^\alpha \in \Im(X \times X)$, $V^\alpha \in \Im(X \times P(X))$ and $W^\alpha \in \Im(P(X) \times P(X))$ are defined as follows:

(1) $K^\alpha(x, y) := \exists A((A \in N_x^\alpha \wedge y \notin A) \vee (A \in N_y^\alpha \wedge x \notin A))$;
(2) $H^\alpha(x, y) := \exists B\exists C((B \in N_x^\alpha \wedge y \notin B) \wedge (C \in N_y^\alpha \wedge x \notin C))$;
(3) $M^\alpha(x, y) := \exists B\exists C(B \in N_x^\alpha \wedge C \in N_y^\alpha \wedge B \cap C \equiv \emptyset)$;
(4) $V^\alpha(x, D) := \exists A\exists B(A \in N_x^\alpha \wedge B \in \tau_\alpha \wedge D \subseteq B \wedge A \cap B \equiv \emptyset)$;
(5) $W^\alpha(A, B) := \exists G\exists H(G \in \tau_\alpha \wedge H \in \tau_\alpha \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H \equiv \emptyset)$.

2.2. Definition. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates $T_i^\alpha \in \Im(\Omega), i = 0, 1, 2, 3, 4$ and $R_i^\alpha \in \Im(\Omega), i = 0, 1$ are defined as follows:

(1) $(X, \tau) \in T_0^\alpha := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow K^\alpha(x, y)$;
(2) $(X, \tau) \in T_1^\alpha := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow H^\alpha(x, y)$;
(3) $(X, \tau) \in T_2^\alpha := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow M^\alpha(x, y)$;
(4) $(X, \tau) \in T_3^\alpha := \forall x\forall D(x \in X \wedge D \in F \wedge x \notin D) \longrightarrow V^\alpha(x, D)$;
(5) $(X, \tau) \in T_4^\alpha := \forall A\forall B(A \in F \wedge B \in F \wedge A \cap B = \emptyset) \longrightarrow W^\alpha(A, B)$;
(6) $(X, \tau) \in T_3^{\alpha'} := \forall x\forall D(x \in X \wedge D \in F_\alpha \wedge x \notin D) \longrightarrow V(x, D)$;
(7) $(X, \tau) \in T_4^{\alpha'} := \forall A\forall B(A \in F_\alpha \wedge B \in F_\alpha \wedge A \cap B = \emptyset) \longrightarrow W(A, B)$;
(8) $(X, \tau) \in R_0^\alpha := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K^\alpha(x, y) \longrightarrow H^\alpha(x, y))$;
(9) $(X, \tau) \in R_1^\alpha := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K^\alpha(x, y) \longrightarrow M^\alpha(x, y))$.

2.3. Theorem. *Let (X, τ) be a fuzzifying topological space. Then we have*

$\models (X,\tau) \in T_0^\alpha \longleftrightarrow \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \longrightarrow (\neg(x \in Cl_\alpha(\{y\})) \vee \neg(y \in Cl_\alpha(\{x\})))).$

Proof. Since for any x, A, B, $\models A \subseteq B \to (A \in N_x^\alpha \to B \in N_x^\alpha)$ (see [12, Theorem 4.2 (2)]), we have

$$\begin{aligned}
[(X,\tau) \in T_0^\alpha] &= \bigwedge_{x \neq y} \max(\bigvee_{y \notin A} N_x^\alpha(A), \bigvee_{x \notin A} N_y^\alpha(A)) \\
&= \bigwedge_{x \neq y} \max(N_x^\alpha(X - \{y\}), N_y^\alpha(X - \{x\})) \\
&= \bigwedge_{x \neq y} \max(1 - Cl_\alpha(\{y\})(x), 1 - Cl_\alpha(\{x\})(y)) \\
&= \bigwedge_{x \neq y} (\neg(Cl_\alpha(\{y\})(x)) \vee \neg(Cl_\alpha(\{x\})(y))) \\
&= [\forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \longrightarrow (\neg(x \in Cl_\alpha(\{y\})) \vee \neg(y \in Cl_\alpha(\{x\}))))].
\end{aligned}$$

□

2.4. Theorem. *For any fuzzifying topological space (X,τ) we have*
$\models \forall x(\{x\} \in F_\alpha) \leftrightarrow (X,\tau) \in T_1^\alpha$.

Proof. Since $\tau_\alpha(A) = \bigwedge_{x \in A} N_x^\alpha(A)$ (Corollary 4.1 in [12]), for any x_1, x_2 with $x_1 \neq x_2$, we have

$$\begin{aligned}
[\forall x(\{x\} \in F_\alpha)] &= \bigwedge_{x \in X} F_\alpha(\{x\}) = \bigwedge_{x \in X} \tau_\alpha(X - \{x\}) \leq \bigwedge_{x \in X} \bigwedge_{y \in X - \{x\}} N_y^\alpha(X - \{x\}) \\
&\leq \bigwedge_{y \in X - \{x_2\}} N_y^\alpha(X - \{x_2\}) \leq N_{x_1}^\alpha(X - \{x_2\}) = \bigvee_{x_2 \notin A} N_{x_1}^\alpha(A).
\end{aligned}$$

Similarly, we have, $[\forall x(\{x\} \in F_\alpha)] \leq \bigvee_{x_1 \notin B} N_{x_2}^\alpha(B)$. Then

$$\begin{aligned}
[\forall x(\{x\} \in F_\alpha)] &\leq \bigwedge_{x_1 \neq x_2} \min(\bigvee_{x_2 \notin A} N_{x_1}^\alpha(A), \bigvee_{x_1 \notin B} N_{x_2}^\alpha(B)) \\
&= \bigwedge_{x_1 \neq x_2} \bigvee_{x_1 \notin B,\ x_2 \notin A} \min(N_{x_1}^\alpha(A), N_{x_2}^\alpha(B)) \\
&= [(X,\tau) \in T_1^\alpha].
\end{aligned}$$

On the other hand

$$
\begin{aligned}
[(X,\tau)\in T_1^{\alpha}] &= \bigwedge_{x_1\neq x_2} \min\Big(\bigvee_{x_2\notin A} N^{\alpha}_{x_1}(A), \bigvee_{x_1\notin B} N^{\alpha}_{x_2}(B)\Big)\\
&= \bigwedge_{x_1\neq x_2} \min(N^{\alpha}_{x_1}(X-\{x_2\}), N^{\alpha}_{x_2}(X-\{x_1\}))\\
&\leq \bigwedge_{x_1\neq x_2} N^{\alpha}_{x_1}(X-\{x_2\}) = \bigwedge_{x_2\in X}\ \bigwedge_{x_1\in X-\{x_2\}} N^{\alpha}_{x_1}(X-\{x_2\})\\
&= \bigwedge_{x_2\in X} \tau_{\alpha}(X-\{x_2\}) = \bigwedge_{x\in X} \tau_{\alpha}(X-\{x\})\\
&= [\forall x(\{x\}\in F_{\alpha})].
\end{aligned}
$$

Therefore $[\forall x(\{x\}\in F_{\alpha})] = [(X,\tau)\in T_1^{\alpha}]$. □

2.5. Definition. Let (X,τ) be a fuzzifying topological space. The fuzzifying α-derived set $D_{\alpha}(A)$ of A is defined as follows: $x\in D_{\alpha}(A) := \forall B(B\in N^{\alpha}_x \to B\cap(A-\{x\})\neq\phi)$.

2.6. Lemma. $D_{\alpha}(A)(x) = 1 - N^{\alpha}_x((X-A)\cup\{x\})$.

Proof. From Theorem 4.2 (2) [12] we have

$$
D_{\alpha}(A)(x) = 1 - \bigvee_{B\cap(A-\{x\})=\phi} N^{\alpha}_x(B) = 1 - N^{\alpha}_x((X-A)\cup\{x\}).
$$

□

2.7. Theorem. *For any finite set $A\subseteq X$, $\models T_1^{\alpha}(X,\tau)\to D_{\alpha}(A)\equiv\phi$.*

Proof. From Theorem 4.2 (2) [12] we have

$$
\begin{aligned}
\bigwedge_{y\in X-A} N^{\alpha}_y((X-A)\cup\{y\}) &\geq \bigwedge_{y\in X-A} N^{\alpha}_y(X-A) = \bigwedge_{y\in X-A} N^{\alpha}_y\Big(\bigcap_{x\in A}(X-\{x\}\Big)\\
&\geq \bigwedge_{y\in X-A}\ \bigwedge_{x\in A} N^{\alpha}_y(X-\{x\}) \geq \bigwedge_{x\neq y} N^{\alpha}_y(X-\{x\}).
\end{aligned}
$$

Also

$$
\begin{aligned}
\bigwedge_{y\in A} N^{\alpha}_y((X-A)\cup\{y\}) &= \bigwedge_{y\in A} N^{\alpha}_y(X-(A-\{y\})) = \bigwedge_{y\in A} N^{\alpha}_y\Big(\bigcap_{x\in A-\{y\}}(X-\{x\}\Big)\\
&\geq \bigwedge_{y\in A}\ \bigwedge_{x\in A-\{y\}} N^{\alpha}_y(X-\{x\}) \geq \bigwedge_{x\neq y} N^{\alpha}_y(X-\{x\}).
\end{aligned}
$$

Therefore

$$\begin{aligned}[D_\alpha(A)\equiv\phi] &= \bigwedge_{x\in X} N_x^\alpha((X-A)\cup\{x\})\\ &= \min\Big(\bigwedge_{y\in X-A} N_y^\alpha((X-A)\cup\{y\}), \bigwedge_{y\in A} N_y^\alpha((X-A)\cup\{y\})\Big)\\ &\geq \bigwedge_{x\neq y} N_y^\alpha(X-\{x\}) = \bigwedge_{x\in X}\ \bigwedge_{x\in X-\{y\}} N_y^\alpha(X-\{x\})\\ &= \bigwedge_{x\in X}\tau_\alpha(X-\{x\}) = \bigwedge_{x\in X} F_\alpha(\{x\}) = T_1^\alpha(X,\tau).\end{aligned}$$

□

2.8. Definition. The fuzzifying α-local basis β_x^α of x is a function from $P(X)$ into $I=[0,1]$ satisfying the following conditions:
(1) $\models \beta_x^\alpha \subseteq N_x^\alpha$, and (2) $\models A\in N_x^\alpha \longrightarrow \exists B(B\in\beta_x^\alpha \wedge x\in B\subseteq A)$.

2.9. Lemma. $\models A\in N_x^\alpha \longleftrightarrow \exists B(B\in\beta_x^\alpha\wedge x\in B\subseteq A)$.

Proof. From condition (1) in Definition 2.8 and Theorem 4.2 (2) in [12] we have $N_x^\alpha(A)\geq N_x^\alpha(B)\geq\beta_x^\alpha(B)$ for each $B\in P(X)$ such that $x\in B\subseteq A$. So $N_x^\alpha(A)\geq \bigvee_{x\in B\subseteq A}\beta_x^\alpha(B)$. From condition (2) in Definition 2.8 we have $N_x^\alpha(A)\leq \bigvee_{x\in B\subseteq A}\beta_x^\alpha(B)$. Hence $N_x^\alpha(A)=\bigvee_{x\in B\subseteq A}\beta_x^\alpha(B)$. □

2.10. Theorem. *If β_x^α is a fuzzifying α-local basis of x, then*
$\models (X,\tau)\in T_1^\alpha \longleftrightarrow \forall x\forall y(x\in X\wedge y\in X\wedge x\neq y\longrightarrow \exists A(A\in\beta_x^\alpha\wedge y\notin A))$.

Proof. For any x,y with $x\neq y$, $\bigvee_{y\notin A}\beta_x^\alpha(A)\leq\bigvee_{y\notin A}N_x^\alpha(A)$, $\bigvee_{x\notin B}\beta_y^\alpha(B)\leq\bigvee_{x\notin B}N_y^\alpha(B)$. So $\min(\bigvee_{y\notin A}\beta_x^\alpha(A),\bigvee_{x\notin B}\beta_y^\alpha(B))\leq\min(\bigvee_{y\notin A}N_x^\alpha(A),\bigvee_{x\notin B}N_y^\alpha(B))=\bigvee_{y\notin A,x\notin B}\min(N_x^\alpha(A),N_y^\alpha(B))$, i.e., $\bigwedge_{x\neq y}\bigvee_{y\notin A}\beta_x^\alpha(A)\leq\bigwedge_{x\neq y}\bigvee_{y\notin A,x\notin B}\min(N_x^\alpha(A),N_y^\alpha(B))=[(X,\tau)\in T_1^\alpha]$. On the other hand, for any B with $x\in B\subseteq X-\{y\}$ we have $y\notin B$. So $\bigvee_{y\notin A}\beta_x^\alpha(A)\geq\beta_x^\alpha(B)$. According to Definition 2.8 we have $\bigvee_{y\notin A}\beta_x^\alpha(A)\geq\bigvee_{x\in B\subseteq X-\{y\}}\beta_x^\alpha(B)=N_x^\alpha(X-\{y\})$. Furthermore, from Corollary 4.1 [12] we have $\bigwedge_{x\neq y}\bigvee_{y\notin A}\beta_x^\alpha(A)\geq\bigwedge_{x\neq y}N_x(X-\{y\})=\bigwedge_{y\in X}\bigwedge_{x\in X-\{y\}}N_x(X-\{y\})=\bigwedge_{y\in X}\tau_\alpha(X-\{y\})=\bigwedge_{y\in X}F_\alpha(\{y\})=[(X,\tau)\in T_1^\alpha]$. □

2.11. Theorem. *If β_x^α is a fuzzifying α-local basis of x, then*
$\models (X,\tau)\in T_2^\alpha\longleftrightarrow\forall x\forall y(x\in X\wedge y\in X\wedge x\neq y\longrightarrow\exists B(B\in\beta_x^\alpha\wedge y\in\neg(Cl_\alpha(B))))$.

Proof.

$$\begin{aligned}
&[\forall x\forall y(x\in X\wedge y\in X\wedge x\neq y\longrightarrow \exists B(B\in\beta_x^\alpha\wedge y\in\neg(Cl_\alpha(B))))]\\
&=\bigwedge_{x\neq y}\bigvee_{B\in P(X)}\min(\beta_x^\alpha(B),\neg(1-N_y^\alpha(X-B)))\\
&=\bigwedge_{x\neq y}\bigvee_{B\in P(X)}\min(\beta_x^\alpha(B),N_y^\alpha(X-B))\\
&=\bigwedge_{x\neq y}\bigvee_{B\in P(X)}\bigvee_{y\in C\subseteq X-B}\min(\beta_x^\alpha(B),\beta_y^\alpha(C))\\
&=\bigwedge_{x\neq y}\bigvee_{B\cap C=\emptyset}\bigvee_{x\in D\subseteq B,\ y\in E\subseteq C}\min(\beta_x^\alpha(D),\beta_y^\alpha(E))\\
&=\bigwedge_{x\neq y}\bigvee_{B\cap C=\emptyset}\min(\bigvee_{x\in D\subseteq B}\beta_x^\alpha(D),\bigvee_{y\in E\subseteq C}\beta_y^\alpha(E))\\
&=\bigwedge_{x\neq y}\bigvee_{B\cap C=\emptyset}\min(N_x^\alpha(B),N_y^\alpha(C))=[(X,\tau)\in T_2^\alpha].
\end{aligned}$$

□

2.12. Definition. The binary fuzzy predicate $\rhd^\alpha\in\Im(N(X)\times X)$, is defined as $S\rhd^\alpha x:=\forall A(A\in N_x^\alpha\longrightarrow S\subsetneqq A)$, where $N(X)$ is the set of all nets of X, $[S\rhd^\alpha x]$ stands for the degree to which S α-converges to x and " $\subsetneqq$ " is the binary crisp predicates "almost in ".

2.13. Theorem. *Let (X,τ) be a fuzzifying topological space and $S\in N(X)$.*
$\models(X,\tau)\in T_2^\alpha\longleftrightarrow\forall S\forall x\forall y((S\subseteq X)\wedge(x\in X)\wedge(y\in X)\wedge(S\rhd^\alpha x)\wedge(S\rhd^\alpha y)\longrightarrow x=y)$.

Proof. $[(X,\tau)\in T_2^\alpha]=\bigwedge_{x\neq y}\bigvee_{A\cap B=\emptyset}(N_x^\alpha(A)\wedge N_y^\alpha(B))$,

$[\forall S\forall x\forall y((S\subseteq X)\wedge(x\in X)\wedge(y\in X)\wedge(S\rhd^\alpha x)\wedge(S\rhd^\alpha y)\longrightarrow x=y)]$
$=\bigwedge_{x\neq y}\bigwedge_{S\subseteq X}(\bigvee_{S\not\supsetneqq A}N_x^\alpha(A)\vee\bigvee_{S\not\supsetneqq B}N_y^\alpha(B))$
$=\bigwedge_{x\neq y}\bigwedge_{S\subseteq X}\bigvee_{S\not\supsetneqq A}\bigvee_{S\not\supsetneqq B}(N_x^\alpha(A)\vee N_y^\alpha(B))$.

(1) If $A\cap B=\emptyset$, then for any $S\in N(X)$, we have $S\not\subsetneqq A$ or $S\not\subsetneqq B$. Therefore, we obtain $N_x^\alpha(A)\wedge N_y^\alpha(B)\leq\bigvee_{S\not\supsetneqq A}N_x^\alpha(A)$ or $N_x^\alpha(A)\wedge N_y^\alpha(B)\leq\bigvee_{S\not\supsetneqq B}N_x^\alpha(B)$.
Consequently, $\bigvee_{A\cap B=\emptyset}(N_x^\alpha(A)\wedge N_y^\alpha(B))\leq\bigwedge_{S\subseteq X}(\bigvee_{S\not\supsetneqq A}N_x^\alpha(A)\vee\bigvee_{S\not\supsetneqq B}N_y^\alpha(B))$,
and
$[(X,\tau)\in T_2^\alpha]\leq[\forall S\forall x\forall y((S\subseteq X)\wedge(x\in X)\wedge(y\in X)\wedge(S\rhd^\alpha x)\wedge(S\rhd^\alpha y)\rightarrow x=y)]$.

(2) First, for any $x,\ y$ with $x\neq y$, if $\bigvee_{A\cap B=\emptyset}(N_x^\alpha(A)\wedge N_y^\alpha(B))<t$, then $N_x^\alpha(A)<t$ or $N_y^\alpha(B)<t$ provided $A\cap B=\emptyset$, i.e., $A\cap B\neq\emptyset$ when $A\in(N_x^\alpha)_t$ and $B\in(N_y^\alpha)_t$. Now, set a net $S^*:(N_x^\alpha)_t\times(N_y^\alpha)_t\longrightarrow X,\ (A,B)\longmapsto x_{(A,B)}\in A\cap B$. Then for any $A\in(N_x^\alpha)_t,\ B\in(N_y^\alpha)_t$, we have $S^*\subsetneqq A$ and $S^*\subsetneqq B$. Therefore, if $S^*\not\subsetneqq A$ and

$S^* \not\precsim B$, then $A \notin (N_x^\alpha)_t,\ B \notin (N_y^\alpha)_t$, i.e., $N_x^\alpha(A) \vee N_y^\alpha(B)) < t$. Consequently $\bigvee_{S^* \not\supseteq A} \bigvee_{S^* \not\supseteq B} (N_x^\alpha(A) \vee N_y^\alpha(B)) \le t$. Moreover $\bigwedge_{S\subseteq X} \bigvee_{S \not\supseteq A} \bigvee_{S \not\supseteq B} (N_x^\alpha(A) \vee N_y^\alpha(B)) \le t$.

Second, for any positive integer i, there exists $x_i,\ y_i$ with $x_i \ne y_i$, and

$$\bigvee_{A\cap B=\emptyset} (N_{x_i}^\alpha(A) \wedge N_{y_i}^\alpha(B)) < [(X,\tau) \in T_2^\alpha] + 1/i,$$

and hence

$$\bigwedge_{S\subseteq X} \bigvee_{S \not\supseteq A} \bigvee_{S \not\supseteq B} (N_{x_i}^\alpha(A) \vee N_{y_i}^\alpha(B)) < [(X,\tau) \in T_2^\alpha] + 1/i.$$

So we have

$$\begin{aligned}&[\forall S \forall x \forall y((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \rhd^\alpha x) \wedge (S \rhd^\alpha y) \longrightarrow x = y)] \\ &\qquad = \bigwedge_{x\ne y} \bigwedge_{S\subseteq X} \bigvee_{S \not\supseteq A} \bigvee_{S \not\supseteq B} (N_x^\alpha(A) \vee N_y^\alpha(B)) \le [(X,\tau) \in T_2^\alpha].\end{aligned}$$

□

2.14. Lemma. *Let (X,τ) be a fuzzifying topological space.*

(1) If $D \subseteq B$, then $\bigvee_{A\cap B=\emptyset} N_x^\alpha(A) = \bigvee_{A\cap B=\emptyset,\ D\subseteq B} N_x^\alpha(A)$,

(2) $\bigvee_{A\cap B=\emptyset} \bigwedge_{y\in D} N_y^\alpha(X - A) = \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \tau_\alpha(B)$.

Proof. (1) Since $D \subseteq B$ then

$$\bigvee_{A\cap B=\emptyset} N_x^\alpha(A) = \bigvee_{A\cap B=\emptyset} N_x^\alpha(A) \wedge [D \subseteq B] = \bigvee_{A\cap B=\emptyset,\ D\subseteq B} N_x^\alpha(A).$$

(2) Let $y \in D$ and $A \cap B = \emptyset$. Then

$$\begin{aligned}\bigvee_{A\cap B=\emptyset,\ D\subseteq B} \tau_\alpha(B) &= \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \tau_\alpha(B) \wedge [y \in D] \\ &= \bigvee_{y\in D\subseteq B\subseteq X-A} \tau_\alpha(B) = \bigvee_{y\in B\subseteq X-A} \tau_\alpha(B) \\ &= N_y^\alpha(X - A) = \bigwedge_{y\in D} N_y^\alpha(X - A) \\ &= \bigvee_{A\cap B=\emptyset} \bigwedge_{y\in D} N_y^\alpha(X - A).\end{aligned}$$

□

2.15. Definition. Let (X,τ) be a fuzzifying topological space.

$$\alpha T_3^{(1)}(X,\tau) := \forall x \forall D(x \in X \wedge D \in F \wedge x \notin D \longrightarrow \exists A(A \in N_x^\alpha \wedge (D \subseteq X - Cl_\alpha(A)))).$$

2.16. Theorem. $\models (X,\tau) \in T_3^\alpha \longleftrightarrow (X,\tau) \in \alpha T_3^{(1)}$.

Proof.

$$\alpha T_3^{(1)}(X,\tau) = \bigwedge_{x\notin D} \min(1, 1-\tau(X-D) + \bigvee_{A\in P(X)} \min(N_x^\alpha(A), \bigwedge_{y\in D}(1-Cl_\alpha(A)(y))))$$
$$= \bigwedge_{x\notin D} \min(1, 1-\tau(X-D) + \bigvee_{A\in P(X)} \min(N_x^\alpha(A), \bigwedge_{y\in D} N_y^\alpha(X-A)))$$

and $$T_3^\alpha(X,\tau) = \bigwedge_{x\notin D} \min(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B))).$$

So, the result holds if we prove that

$$\bigvee_{A\in P(X)} \min(N_x^\alpha(A), \bigwedge_{y\in D} N_y^\alpha(X-A)) = \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B)) \qquad (*)$$

It is clear that, on the left-hand side of $(*)$ in the case of $A\cap D \neq \emptyset$ there exists $y \in X$ such that $y \in D$ and $y \notin X - A$. So, $\bigwedge_{y\in D} N_y^\alpha(X-A) = 0$ and thus $(*)$ becomes

$$\bigvee_{A\in P(X),\ A\cap B=\emptyset} \min(N_x^\alpha(A), \bigwedge_{y\in D} N_y^\alpha(X-A)) = \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B)),$$

which is obtained from Lemma 2.14. □

2.17. Definition. Let (X,τ) be a fuzzifying topological space.
$\alpha T_3^{(2)}(X,\tau) := \forall x\forall B(x\in B \wedge B\in\tau \longrightarrow \exists A(A\in N_x^\alpha \wedge Cl_\alpha(A)\subseteq B))$.

2.18. Theorem. $\models (X,\tau)\in T_3^\alpha \longleftrightarrow (X,\tau)\in \alpha T_3^{(2)}$.

Proof. From Theorem 2.16 we have

$$T_3^\alpha(X,\tau) = \bigwedge_{x\notin D} \min(1, 1-\tau(X-D) + \bigvee_{A\in P(X)} \min(N_x^\alpha(A), \bigwedge_{y\in D} N_y^\alpha(X-A))).$$

Now,

$$\alpha T_3^{(2)}(X,\tau) = \bigwedge_{x\in B} \min(1, 1-\tau(B) + \bigvee_{A\in P(X)} \min(N_x^\alpha(A), \bigwedge_{y\in X-B}(1-Cl_\alpha(A)(y))))$$
$$= \bigwedge_{x\in B} \min(1, 1-\tau(B) + \bigvee_{A\in P(X)} \min(N_x^\alpha(A), \bigwedge_{y\in X-B}(1-(1-N_y^\alpha(X-A)))))$$
$$= \bigwedge_{x\in B} \min(1, 1-\tau(B) + \bigvee_{A\in P(X)} \min(N_x^\alpha(A), \bigwedge_{y\in X-B} N_y^\alpha(X-A))).$$

Put $B = X - D$ we have

$$\alpha T_3^{(2)}(X,\tau) = \bigwedge_{x\notin D} \min(1, 1-\tau(X-D) + \bigvee_{A\in P(X)} \min(N_x^\alpha(A), \bigwedge_{y\in D} N_y^\alpha(X-A)))$$
$$= T_3^\alpha(X,\tau).$$

□

2.19. Definition. Let (X,τ) be a fuzzifying topological space and φ be a subbase of τ then

$$\alpha T_3^{(3)}(X,\tau) := \forall x \forall D(x \in D \wedge D \in \varphi \longrightarrow \exists B(B \in N_x^\alpha \wedge Cl_\alpha(B) \subseteq D)).$$

2.20. Theorem. $\models (X,\tau) \in T_3^\alpha \longleftrightarrow (X,\tau) \in \alpha T_3^{(3)}$.

Proof. Since $[\varphi \subseteq \tau] = 1$, from Theorems 2.16 we have

$$\alpha T_3^{(3)}(X,\tau) \geq \alpha T_3^{(2)}(X,\tau) = T_3^\alpha(X,\tau).$$

So, it suffices to prove that $\alpha T_3^{(3)}(X,\tau) \leq \alpha T_3^{(2)}(X,\tau)$ and this is obtained if we prove for any $x \in A$,

$$\min(1, 1-\tau(A) + \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X-A} N_y^\alpha(X-B))) \geq \alpha T_3^{(3)}(X,\tau).$$

Set $\alpha T_3^{(3)}(X,\tau) = \delta$. Then for any $x \in X$ and any $D_{\lambda_i} \in P(X), x \in D_{\lambda_i}, \lambda_i \in I_\lambda$ (I_λ denotes a finite index set), $\lambda \in \Lambda$, $\bigcup_{\lambda \in \Lambda} \bigcap_{\lambda_i \in I_\lambda} D_{\lambda_i} = A$ we have

$$1 - \varphi(D_{\lambda_i}) + \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X - D_{\lambda_i}} N_y^\alpha(X-B)) \geq \delta > \delta - \epsilon,$$

where ϵ is any positive number. Thus

$$\bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X - D_{\lambda_i}} N_y^\alpha(X-B)) > \varphi(D_{\lambda_i}) - 1 + \delta - \epsilon.$$

Set $\gamma_{\lambda_i} = \{B : B \subseteq D_{\lambda_i}\}$. From the completely distributive law we have

$$\begin{aligned}
&\bigwedge_{\lambda_i \in I_\lambda} \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X - D_{\lambda_i}} N_y^\alpha(X-B)) \\
&\quad = \bigvee_{f \in \Pi\{\gamma_{\lambda_i} : \lambda_i \in I_\lambda\}} \bigwedge_{\lambda_i \in I_\lambda} \min(N_x^\alpha(f(\lambda_i)), \bigwedge_{y \in X - D_{\lambda_i}} N_y^\alpha(X - f(\lambda_i))) \\
&\quad = \bigvee_{f \in \Pi\{\gamma_{\lambda_i} : \lambda_i \in I_\lambda\}} \min(\bigwedge_{\lambda_i \in I_\lambda} N_x^\alpha(f(\lambda_i)), \bigwedge_{\lambda_i \in I_\lambda} \bigwedge_{y \in X - D_{\lambda_i}} N_y^\alpha(X - f(\lambda_i))) \\
&\quad = \bigvee_{f \in \Pi\{\gamma_{\lambda_i} : \lambda_i \in I_\lambda\}} \min(\bigwedge_{\lambda_i \in I_\lambda} N_x^\alpha(f(\lambda_i)), \bigwedge_{y \in \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\alpha(X - f(\lambda_i))) \\
&\quad = \bigvee_{B \in P(X)} \min(\bigwedge_{\lambda_i \in I_\lambda} N_x^\alpha(B), \bigwedge_{y \in \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\alpha(X-B)) \\
&\quad = \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\alpha(X-B)),
\end{aligned}$$

where $B = f(\lambda_i)$.
Similarly, we can prove

$$\bigwedge_{\lambda\in\Lambda}\bigvee_{B\in P(X)}\min(N_x^\alpha(B), \bigwedge_{y\in\bigcup_{\lambda_i\in I_\lambda}X-D_{\lambda_i}} N_y^\alpha(X-B))$$

$$= \bigvee_{B\in P(X)}\min(N_x^\alpha(B), \bigwedge_{y\in\bigcup_{\lambda\in\Lambda}\bigcup_{\lambda_i\in I_\lambda}X-D_{\lambda_i}} N_y^\alpha(X-B))$$

$$\leq \bigvee_{B\in P(X)}\min(N_x^\alpha(B), \bigwedge_{y\in\bigcap_{\lambda\in\Lambda}\bigcup_{\lambda_i\in I_\lambda}X-D_{\lambda_i}} N_y^\alpha(X-B))$$

$$\leq \bigvee_{B\in P(X)}\min(N_x^\alpha(B), \bigwedge_{y\in X-A} N_y^\alpha(X-B)),$$

so we have

$$\bigvee_{B\in P(X)}\min(N_x^\alpha(B), \bigwedge_{y\in X-A} N_y^\alpha(X-B))$$

$$\geq \bigwedge_{\lambda\in\Lambda}\bigwedge_{\lambda_i\in I_\lambda}\bigvee_{B\in P(X)}\min(N_x^\alpha(B), \bigwedge_{y\in X-D_{\lambda_i}} N_y^\alpha(X-B))$$

$$\geq \bigwedge_{\lambda\in\Lambda}\bigwedge_{\lambda_i\in I_\lambda}\varphi(D_{\lambda_i}) - 1 + \delta - \epsilon.$$

For any I_λ and Λ that satisfy $\bigcup_{\lambda\in\Lambda}\bigcap_{\lambda_i\in I_\lambda} D_{\lambda_i} = A$ the above inequality is true. So,

$$\bigvee_{B\in P(X)}\min(N_x^\alpha(B), \bigwedge_{y\in X-A} N_y^\alpha(X-B))$$

$$\geq \bigvee_{\cup_{\lambda\in\Lambda}D_\lambda=A}\bigwedge_{\lambda\in\Lambda}\bigvee_{\cap_{\lambda_i\in I_\lambda}D_{\lambda_i}=D_\lambda}\bigwedge_{\lambda_i\in I_\lambda}\varphi(D_{\lambda_i}) - 1 + \delta - \epsilon$$

$$= \tau(A) - 1 + \delta - \epsilon.$$

$$\text{i.e., } \min(1, 1-\tau(A) + \bigvee_{B\in P(X)}\min(N_x^\alpha(B), \bigwedge_{y\in X-A} N_y^\alpha(X-B))) \geq \delta - \epsilon.$$

Because ϵ is any arbitrary positive number, when $\epsilon \longrightarrow 0$ we have
$\alpha T_3^{(2)}(X,\tau) \geq \delta = \alpha T_3^{(3)}(X,\tau)$. So, $\models (X,\tau) \in T_3^\alpha \longleftrightarrow (X,\tau) \in \alpha T_3^{(3)}$. □

2.21. Definition. Let (X,τ) be any fuzzifying topological space.
(1) $\alpha' T_3^{(1)}(X,\tau) := \forall x \forall D(x \in X \wedge D \in F_\alpha \wedge x \notin D \longrightarrow \exists A(A \in N_x \wedge (D \subseteq X - Cl(A))))$;
(2) $\alpha' T_3^{(2)}(X,\tau) := \forall x \forall B(x \in B \wedge B \in \tau_\alpha \longrightarrow \exists A(A \in N_x \wedge Cl(A) \subseteq B))$;
(3) $\alpha T_4^{(1)}(X,\tau) := \forall A \forall B(A \in \tau \wedge B \in F \wedge A \cap B \equiv \emptyset \rightarrow \exists G(G \in \tau \wedge A \subseteq G \wedge Cl_\alpha(G) \cap B \equiv \phi))$;
(4) $\alpha T_4^{(2)}(X,\tau) := \forall A \forall B(A \in F \wedge B \in \tau \wedge A \subseteq B \rightarrow \exists G(G \in \tau \wedge A \subseteq G \wedge Cl_\alpha(G) \subseteq B))$;
(5) $\alpha' T_4^{(1)}(X,\tau) := \forall A \forall B(A \in \tau \wedge B \in F_\alpha \wedge A \cap B \equiv \emptyset \rightarrow \exists G(G \in \tau \wedge A \subseteq G \wedge Cl(G) \cap B \equiv \phi))$;

(6) $\alpha' T_4^{(2)}(X,\tau) := \forall A \forall B (A \in F \wedge B \in \tau_\alpha \wedge A \subseteq B \to \exists G (G \in \tau \wedge A \subseteq G \wedge Cl(G) \subseteq B))$.

By a similar proof of Theorem 2.16 and 2.18 we have the following theorem.

2.22. Theorem. *Let (X,τ) be a fuzzifying topological space.*
(1) $\models (X,\tau) \in T_3^{\alpha'} \longleftrightarrow (X,\tau) \in \alpha' T_3^{(i)}$;
(2) $\models (X,\tau) \in T_4^{\alpha} \longleftrightarrow (X,\tau) \in \alpha T_4^{(i)}$;
(3) $\models (X,\tau) \in T_4^{\alpha'} \longleftrightarrow (X,\tau) \in \alpha' T_4^{(i)}$, *where* $i = 1, 2$.

3. Relation among fuzzifying separation axioms

3.1. Lemma. *(1)* $\models K(x,y) \to K^\alpha(x,y)$,
(2) $\models H(x,y) \to H^\alpha(x,y)$,
(3) $\models M(x,y) \to M^\alpha(x,y)$,
(4) $\models V(x,D) \to V^\alpha(x,D)$,
(5) $\models W(A,B) \to W^\alpha(A,B)$.

Proof. Since $\models \tau \subseteq \tau_\alpha$, $N_x(A) \leq N_x^\alpha(A)$ for any $A \in P(X)$. Then the proof is immediate. □

3.2. Theorem. $\models (X,\tau) \in T_i \longrightarrow (X,\tau) \in T_i^\alpha$, *where* $i = 0, 1, 2, 3, 4$.

Proof. It is obtained from Lemma 3.1. □

3.3. Theorem. *If* $T_0(X,\tau) = 1$, *then*
(1) $\models (X,\tau) \in R_0 \longrightarrow (X,\tau) \in R_0^\alpha$,
(2) $\models (X,\tau) \in R_1 \longrightarrow (X,\tau) \in R_1^\alpha$,

Proof. Since $T_0(X,\tau) = 1$, for each $x, y \in X$ and $x \neq y$, we have $[K(x,y)] = 1$ and so $[K^\alpha(x,y)] = 1$.

(1) Using Lemma 3.1 (1) and (2) we obtain

$$[(X,\tau) \in R_0] = \bigwedge_{x \neq y} [K(x,y) \to H(x,y)] \leq \bigwedge_{x \neq y} [K(x,y) \to H^\alpha(x,y)]$$
$$\leq \bigwedge_{x \neq y} [K^\alpha(x,y) \to H^\alpha(x,y)] = R_0^\alpha(X,\tau).$$

(2) Using Lemma 3.1 (1) and (3) the proof is similar to (1). □

3.4. Lemma. *(1)* $\models M^\alpha(x,y) \longrightarrow H^\alpha(x,y)$;
(2) $\models H^\alpha(x,y) \longrightarrow K^\alpha(x,y)$;
(3) $\models M^\alpha(x,y) \longrightarrow K^\alpha(x,y)$.

Proof. (1) Since $\{B, C \in P(X) : B \cap C \equiv \emptyset\} \subseteq \{B, C \in P(X) : y \notin B \text{ and } x \notin C\}$, then

$$[M^\alpha(x,y)] = \bigvee_{B \cap C = \emptyset} \min(N_x^\alpha(B), N_y^\alpha(C)) \leq \bigvee_{y \notin B,\, x \notin C} \min(N_x^\alpha(B), N_y^\alpha(C)) = [H^\alpha(x,y)].$$

(2) $$[K^\alpha(x,y)] = \max\Big(\bigvee_{y \notin A} N_x^\alpha(A), \bigvee_{x \notin A} N_y^\alpha(A)\Big) \geq \bigvee_{y \notin A} N_x^\alpha(A) \geq \bigvee_{y \notin A,\, x \notin B} (N_x^\alpha(A) \wedge N_y^\alpha(B))$$
$$= [H^\alpha(x,y)].$$

(3) From (1) and (2) it is obvious. □

3.5. Theorem. *Let (X,τ) be a fuzzifying topological space. Then we have*
(1) $\models (X,\tau) \in T_1^\alpha \longrightarrow (X,\tau) \in T_0^\alpha$;
(2) $\models (X,\tau) \in T_2^\alpha \longrightarrow (X,\tau) \in T_1^\alpha$;
(3) $\models (X,\tau) \in T_2^\alpha \longrightarrow (X,\tau) \in T_0^\alpha$.

Proof. The proof of (1) and (2) are obtained from Lemma 3.4 (2) and (1), respectively.

(3) From (1) and (2) above the result is obtained. □

3.6. Theorem. $\models (X,\tau) \in R_1^\alpha \longrightarrow (X,\tau) \in R_0^\alpha$.

Proof. From Lemma 3.4 (2), the proof is immediate. □

3.7. Theorem. *For any fuzzifying topological space (X,τ) we have*
(1) $\models (X,\tau) \in T_1^\alpha \longrightarrow (X,\tau) \in R_0^\alpha$;
(2) $\models (X,\tau) \in T_1^\alpha \longrightarrow (X,\tau) \in R_0^\alpha \wedge (X,\tau) \in T_0^\alpha$;
(3) If $T_0^\alpha(X,\tau) = 1$, *then* $\models (X,\tau) \in T_1^\alpha \longleftrightarrow (X,\tau) \in R_0^\alpha \wedge (X,\tau) \in T_0^\alpha$.

Proof. (1) $T_1^\alpha(X,\tau) = \bigwedge_{x\neq y}[H^\alpha(x,y)] \leq \bigwedge_{x\neq y}[K^\alpha(x,y) \longrightarrow H^\alpha(x,y)] = R_0^\alpha(X,\tau)$.

(2) It is obtained from (1) and from Theorem 3.5 (1).

(3) Since $T_0^\alpha(X,\tau) = 1$, for every $x, y \in X$ such that $x \neq y$, then we have $[K^\alpha(x,y)] = 1$. Therefore

$$\begin{aligned}
[(X,\tau) \in R_0^\alpha \wedge (X,\tau) \in T_0^\alpha] &= [(X,\tau) \in R_0^\alpha] \\
&= \bigwedge_{x\neq y} \min(1, 1 - [K^\alpha(x,y)] + [H^\alpha(x,y)]) \\
&= \bigwedge_{x\neq y} [H^\alpha(x,y)] = T_1^\alpha(X,\tau).
\end{aligned}$$

□

3.8. Theorem. *Let (X,τ) be a fuzzifying topological space.*
(1) $\models (X,\tau) \in R_0^\alpha \otimes (X,\tau) \in T_0^\alpha \longrightarrow (X,\tau) \in T_1^\alpha$, *and*
(2) If $T_0^\alpha(X,\tau) = 1$, *then* $\models (X,\tau) \in R_0^\alpha \otimes (X,\tau) \in T_0^\alpha \longleftrightarrow (X,\tau) \in T_1^\alpha$.

Proof. (1)

$$\begin{aligned}
&[(X,\tau) \in R_0^\alpha \otimes (X,\tau) \in T_0^\alpha] \\
&= \max(0, R_0^\alpha(X,\tau) + T_0^\alpha(X,\tau) - 1) \\
&= \max(0, \bigwedge_{x\neq y} \min(1, 1 - [K^\alpha(x,y)] + [H^\alpha(x,y)]) + \bigwedge_{x\neq y} [K^\alpha(x,y)] - 1) \\
&\leq \max(0, \bigwedge_{x\neq y} \{\min(1, 1 - [K^\alpha(x,y)] + [H^\alpha(x,y)]) + [K^\alpha(x,y)]\} - 1) \\
&= \bigwedge_{x\neq y} [H^\alpha(x,y)] = T_1^\alpha(X,\tau).
\end{aligned}$$

(2)

$$\begin{aligned}[(X,\tau)\in R_0^\alpha \otimes (X,\tau)\in T_0^\alpha] &= [(X,\tau)\in R_0^\alpha]\\ &= \bigwedge_{x\neq y} \min(1, 1-[K^\alpha(x,y)]+[H^\alpha(x,y)])\\ &= \bigwedge_{x\neq y} [H^\alpha(x,y)] = T_1^\alpha(X,\tau),\end{aligned}$$

because $T_0^\alpha(X,\tau) = 1$, implies that for each x, y such that $x \neq y$ we have $[K^\alpha(x,y)] = 1$.

□

3.9. Theorem. *Let (X,τ) be a fuzzifying topological space.*
(1) $\models (X,\tau)\in T_0^\alpha \longrightarrow ((X,\tau)\in R_0^\alpha \longrightarrow (X,\tau)\in T_1^\alpha)$, *and*
(2) $\models (X,\tau)\in R_0^\alpha \longrightarrow ((X,\tau)\in T_0^\alpha \longrightarrow (X,\tau)\in T_1^\alpha)$.

Proof. It obtained From Theorems 3.7 (1) and 3.8 (1) and the fact that $[\alpha] \leq [\varphi \to \psi] \Leftrightarrow [\alpha]\otimes[\varphi] \leq [\psi]$. □

3.10. Theorem. *Let (X,τ) be a fuzzifying topological space.*
(1) $\models (X,\tau)\in T_2^\alpha \longrightarrow (X,\tau)\in R_1^\alpha$;
(2) $\models (X,\tau)\in T_2^\alpha \longrightarrow (X,\tau)\in R_i^\alpha \wedge (X,\tau)\in T_i^\alpha$, *where* $i = 0, 1$;
(3) If $T_0^\alpha(X,\tau) = 1$, *then*
(i) $\models (X,\tau)\in T_2^\alpha \longleftrightarrow (X,\tau)\in R_1^\alpha \wedge (X,\tau)\in T_0^\alpha$.
(ii) $\models (X,\tau)\in T_2^\alpha \longleftrightarrow (X,\tau)\in R_1^\alpha \wedge (X,\tau)\in T_1^\alpha$.

Proof. It is similar to the proof of Theorem 3.7. □

3.11. Theorem. *Let (X,τ) be a fuzzifying topological space.*
(1) $\models (X,\tau)\in R_1^\alpha \otimes (X,\tau)\in T_0^\alpha \longrightarrow (X,\tau)\in T_2^\alpha$, *and*
(2) If $T_0^\alpha(X,\tau) = 1$, *then* $\models (X,\tau)\in R_1^\alpha \otimes (X,\tau)\in T_0^\alpha \longleftrightarrow (X,\tau)\in T_2^\alpha$.

Proof. It is similar to the proof of Theorem 3.8. □

3.12. Theorem. *Let (X,τ) be a fuzzifying topological space.*
(1) $\models (X,\tau)\in T_0^\alpha \longrightarrow ((X,\tau)\in R_1^\alpha \longrightarrow (X,\tau)\in T_2^\alpha)$, *and*
(2) $\models (X,\tau)\in R_1^\alpha \longrightarrow ((X,\tau)\in T_0^\alpha \longrightarrow (X,\tau)\in T_2^\alpha)$.

Proof. It is similar to the proof of Theorem 3.9. □

3.13. Theorem. *If* $T_0^\alpha(X,\tau) = 1$, *then*
(1) $\models ((X,\tau)\in T_0^\alpha \longrightarrow ((X,\tau)\in R_0^\alpha \longrightarrow (X,\tau)\in T_1^\alpha)) \wedge ((X,\tau)\in T_1^\alpha \longrightarrow \neg((X,\tau)\in T_0^\alpha \longrightarrow \neg((X,\tau)\in \alpha_0^\alpha)))$;
(2) $\models ((X,\tau)\in R_0^\alpha \longrightarrow ((X,\tau)\in T_0^\alpha \longrightarrow (X,\tau)\in T_1^\alpha)) \wedge ((X,\tau)\in T_1^\alpha \longrightarrow \neg((X,\tau)\in T_0^\alpha \longrightarrow \neg((X,\tau)\in \alpha_0^\alpha)))$;
(3) $\models ((X,\tau)\in T_0^\alpha \longrightarrow ((X,\tau)\in R_0^\alpha \longrightarrow (X,\tau)\in T_1^\alpha)) \wedge ((X,\tau)\in T_1^\alpha \longrightarrow \neg((X,\tau)\in R_0^\alpha \longrightarrow \neg((X,\tau)\in T_0^\alpha)))$;
(4) $\models ((X,\tau)\in R_0^\alpha \longrightarrow ((X,\tau)\in T_0^\alpha \longrightarrow (X,\tau)\in T_1^\alpha)) \wedge ((X,\tau)\in T_1^\alpha \longrightarrow \neg((X,\tau)\in R_0^\alpha \longrightarrow \neg((X,\tau)\in T_0^\alpha)))$.

Proof. For simplicity we put, $T_0^\alpha(X,\tau)=\alpha$, $R_0^\alpha(X,\tau)=\beta$ and $T_1^\alpha(X,\tau)=\gamma$. Now, applying Theorem 3.8 (2), the proof is obtained with some relations in fuzzy logic as follows:

$$
\begin{aligned}
(1)\qquad 1 &= (\alpha\otimes\beta\longleftrightarrow\gamma)=(\alpha\otimes\beta\longrightarrow\gamma)\wedge(\gamma\longrightarrow\alpha\otimes\beta)\\
&=\neg((\alpha\otimes\beta)\otimes\neg\gamma)\wedge\neg(\gamma\otimes\neg(\alpha\otimes\beta))\\
&=\neg(\alpha\otimes\neg(\neg(\beta\otimes\neg\gamma)))\wedge\neg(\gamma\otimes(\alpha\longrightarrow\neg\beta))\\
&=(\alpha\longrightarrow\neg(\beta\otimes\neg\gamma))\wedge(\gamma\longrightarrow\neg(\alpha\longrightarrow\neg\beta))\\
&=(\alpha\longrightarrow(\beta\longrightarrow\gamma)\wedge(\gamma\longrightarrow\neg(\alpha\longrightarrow\neg\beta))),
\end{aligned}
$$

since $\otimes$ is commutative one can have the proof of statements (2) - (4) in a similar way as (1). □

By a similar procedure to Theorem 3.13 one can have the following theorem.

3.14. Theorem. *If* $T_0^\alpha(X,\tau)=1$, *then*

(1) $\models((X,\tau)\in T_0^\alpha\longrightarrow((X,\tau)\in R_1^\alpha\longrightarrow(X,\tau)\in T_2^\alpha))\wedge((X,\tau)\in T_2^\alpha\longrightarrow\neg((X,\tau)\in T_0^\alpha\longrightarrow\neg((X,\tau)\in R_1^\alpha)))$;

(2) $\models((X,\tau)\in R_1^\alpha\longrightarrow((X,\tau)\in T_0^\alpha\longrightarrow(X,\tau)\in T_2^\alpha))\wedge((X,\tau)\in T_2^\alpha\longrightarrow\neg((X,\tau)\in T_0^\alpha\longrightarrow\neg((X,\tau)\in\alpha_1^\alpha)))$;

(3) $\models((X,\tau)\in T_0^\alpha\longrightarrow((X,\tau)\in R_1^\alpha\longrightarrow(X,\tau)\in T_2^\alpha))\wedge((X,\tau)\in T_2^\alpha\longrightarrow\neg((X,\tau)\in R_1^\alpha\longrightarrow\neg((X,\tau)\in T_0^\alpha)))$;

(4) $\models((X,\tau)\in R_1^\alpha\longrightarrow((X,\tau)\in T_0^\alpha\longrightarrow(X,\tau)\in T_2^\alpha))\wedge((X,\tau)\in T_2^\alpha\longrightarrow\neg((X,\tau)\in R_1^\alpha\longrightarrow\neg((X,\tau)\in T_0^\alpha)))$.

3.15. Lemma. *For any* $\alpha,\beta\in I$ *we have,* $(1\wedge(1-\alpha+\beta))+\alpha\le 1+\beta$.

3.16. Theorem. $\models(X,\tau)\in T_3^\alpha\otimes(X,\tau)\in T_1\longrightarrow(X,\tau)\in T_2^\alpha$.

Proof. From Theorem 2.2 [26] we have, $T_1(X,\tau)=\bigwedge_{y\in X}\tau(X-\{y\})$ and applying Lemma 3.5 we have

$$
\begin{aligned}
&T_3^\alpha(X,\tau)+T_1(X,\tau)\\
&=\bigwedge_{x\notin D}\min\Big(1,1-\tau(X-D)+\bigvee_{A\cap B=\emptyset,\ D\subseteq B}\min(N_x^\alpha(A),\tau_\alpha(B))\Big)+\bigwedge_{y\in X}\tau(X-\{y\})\\
&\le\bigwedge_{x\in X,\ x\ne y}\ \bigwedge_{y\in X}\min\Big(1,1-\tau(X-\{y\})+\bigvee_{A\cap B=\emptyset}\min(N_x^\alpha(A),N_y^\alpha(B))\Big)+\bigwedge_{y\in X}\tau(X-\{y\})\\
&=\bigwedge_{x\in X,\ x\ne y}\Big(\bigwedge_{y\in X}\min(1,1-\tau(X-\{y\})+\bigvee_{A\cap B=\emptyset}\min(N_x^\alpha(A),N_y^\alpha(B)))+\bigwedge_{y\in X}\tau(X-\{y\})\Big)\\
&\le\bigwedge_{x\in X,\ x\ne y}\ \bigwedge_{y\in X}\Big(\min(1,1-\tau(X-\{y\})+\bigvee_{A\cap B=\emptyset}\min(N_x^\alpha(A),N_y^\alpha(B)))+\tau(X-\{y\})\Big)\\
&\le\bigwedge_{x\ne y}\Big(1+\bigvee_{A\cap B=\emptyset}\min(N_x^\alpha(A),N_y^\alpha(B))\Big)\\
&=1+\bigwedge_{x\ne y}\ \bigvee_{A\cap B=\emptyset}\min(N_x^\alpha(A),N_y^\alpha(B))=1+T_2^\alpha(X,\tau),
\end{aligned}
$$

namely, $T_2^\alpha(X,\tau) \geq T_3^\alpha(X,\tau) + T_1(X,\tau) - 1$. Thus $T_2^\alpha(X,\tau) \geq \max(0, T_3^\alpha(X,\tau) + T_1(X,\tau) - 1)$. □

3.17. Theorem. $\models (X,\tau) \in T_4^\alpha \otimes (X,\tau) \in T_1 \longrightarrow (X,\tau) \in T_3^\alpha$.

Proof. It is equivalent to prove that $T_3^\alpha(X,\tau) \geq T_4^\alpha(X,\tau) + T_1(X,\tau) - 1$. In fact,

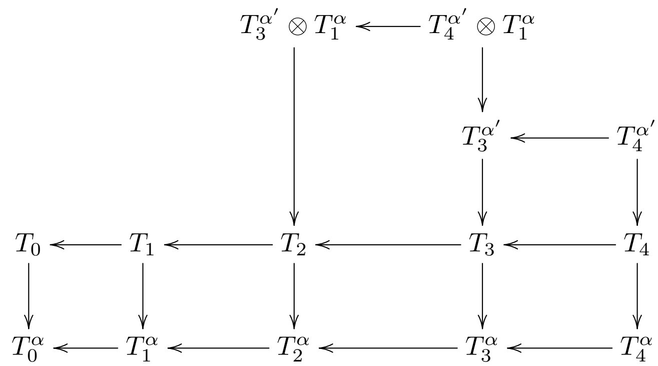
$$\begin{aligned}
&T_4^\alpha(X,\tau) + T_1(X,\tau)\\
&= \bigwedge_{E\cap D=\emptyset} \min\Big(1, 1 - \min(\tau(X-E), \tau(X-D))\\
&\qquad + \bigvee_{A\cap B=\emptyset,\ E\subseteq A,\ D\subseteq B} \min(\tau_\alpha(A), \tau_\alpha(B))\Big) + \bigwedge_{z\in X} \tau(X - \{z\})\\
&\leq \bigwedge_{x\notin D} \min\Big(1, 1 - \min(\tau(X-\{x\}), \tau(X-D))\\
&\qquad + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B))\Big) + \bigwedge_{z\in X} \tau(X-\{z\})\\
&= \bigwedge_{x\notin D} \min\Big(1, \max\Big(1 - \tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B)), 1 - \tau(X-\{x\})\\
&\qquad + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B))\Big)\Big) + \bigwedge_{z\in X} \tau(X-\{z\})\\
&= \bigwedge_{x\notin D} \max\Big(\min\Big(1, 1 - \tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B))\Big), \min\Big(1, 1 - \tau(X-\{x\})\\
&\qquad + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B))\Big)\Big) + \bigwedge_{z\in X} \tau(X-\{z\})\\
&\leq \bigwedge_{x\notin D} \max\Big(\min\Big(1, 1 - \tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B))\Big) + \tau(X-\{x\}),\\
&\qquad \min\Big(1, 1 - \tau(X-\{x\}) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B))\Big) + \tau(X-\{x\})\Big)\\
&\leq \bigwedge_{x\notin D} \max\Big(\min\Big(1, 1 - \tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B))\Big) + \tau(X-\{x\}), 1\Big)\\
&\qquad \leq \bigwedge_{x\notin D} \Big(\min\Big(1, 1 - \tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B))\Big) + 1\Big)\\
&\qquad = \bigwedge_{x\notin D} \min\Big(1, 1 - \tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B))\Big) + 1\\
&\qquad = T_3^\alpha(X,\tau) + 1.
\end{aligned}$$

□

By a similar procedures of Theorems 3.16 and 3.17 we have the following theorems

3.18. Theorem. *Let* (X,τ) *be a fuzzifying topological space.*
(1) $\models (X,\tau) \in T_3^{\alpha'} \otimes (X,\tau) \in T_1^{\alpha} \longrightarrow (X,\tau) \in T_2$.
(2) $\models (X,\tau) \in T_4^{\alpha'} \otimes (X,\tau) \in T_1^{\alpha} \longrightarrow (X,\tau) \in T_3^{\alpha'}$.

From the above discussion one can have the following diagram:



Conclusion: The present paper investigates topological notions when these are planted into the framework of Ying's fuzzifying topological spaces (in semantic method of continuous valued-logic). It continue various investigations into fuzzy topology in a legitimate way and extend some fundamental results in general topology to fuzzifying topology. An important virtue of our approach (in which we follow Ying) is that we define topological notions as fuzzy predicates (by formulae of Łukasiewicz fuzzy logic) and prove the validity of fuzzy implications (or equivalences). Unlike the (more wide-spread) style of defining notions in fuzzy mathematics as crisp predicates of fuzzy sets, fuzzy predicates of fuzzy sets provide a more genuine fuzzification; furthermore the theorems in the form of valid fuzzy implications are more general than the corresponding theorems on crisp predicates of fuzzy sets. The main contributions of the present paper are to study α-separation axioms in fuzzifying topology and give the relations of these axioms with each other as well as the relations with other fuzzifying separation axiom. The role or the meaning of each theorem in the present paper is obtained from its generalization to a corresponding theorem in crisp setting. For example: in crisp setting, a topological space (X,τ) is T_1^{α} if and only if for each $z \in X, z \in F_{\alpha}$, where F_{α} is the family of α-closed sets. This fact can be rewritten as follows: the truth value of a topological space (X,τ) to be T_1^{α} equal the infimum of the truth values of its singletons to be α-closed, where the set of truth values is $\{0,1\}$. Now, is this theorem still valid in fuzzifying settings, i.e., if the set of truth values is $[0,1]$? The answer of this question is positive and is given in Theorem 2.4 above.

There are some problems for further study:
(1) One obvious problem is: our results are derived in the Lukasiewicz continuous logic. It is possible to generalize them to more general logic setting, like residuated lattice-valued logic considered in [31-32].
(2) What is the justification for fuzzifying α-separation axioms in the setting of

(2, L) topologies.

(3) Obviously, fuzzifying topological spaces in [23] form a fuzzy category. Perhaps, this will become a motivation for further study of the fuzzy category.
(4) What is the justification for fuzzifying α-separation axioms in (M, L)-topologies etc.

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Erratum and notes for *near groups on nearness approximation spaces*

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2000 AMS Classification: 03E75, 03E99, 20A05, 20E99

Erratum and notes for: "İnan, E., Öztürk, M. A. Near groups on nearness approximation spaces, Hacet J Math Stat, 41(4), 2012, 545–558."

The authors would like to write some notes and correct errors in the original publication of the article [1]. The notes are given below:

0.1. Remark. In page 550, in Definition 3.1., (1) and (2) properties have to hold in $N_r(B)^* G$. Sometimes they may be hold in $\mathcal{O} \setminus N_r(B)^* G$, then G is not a near group on nearness approximation space.

Example 3.3. and 3.4. are nice examples of this case. In Example 3.3., if we consider associative property $(b \cdot e) \cdot b = b \cdot (e \cdot b)$ for $b, e \in H \subset G$, we obtain $\imath = \imath$, but $\imath \in \mathcal{O} \setminus N_r(B)^* H$. Hence, we can observe that if the associative property holds in $\mathcal{O} \setminus N_r(B)^* H$, then H can not be a subnear group of near group G. Consequently, Example 3.3. and 3.4. are incorrect, i.e., they are not subnear groups of near group G.

0.2. Remark. Multiplying of finite number of elements in G may not always belongs to $N_r(B)^* G$. Therefore always we can not say that $x^n \in N_r(B)^* G$, for all $x \in G$ and some positive integer n. If $\left(N_r(B)^* G, \cdot\right)$ is groupoid, then we can say that $x^n \in N_r(B)^* G$, for all $x \in R$ and all positive integer n.

In Example 3.2., the properties (1) and (2) hold in $N_r(B)^* G$. Hence G is a near group on nearness approximation space.

The corrections are given below:

In page 548, in subsection 2.4.1., definition of B-lower approximation of $X \subseteq \mathcal{O}$ must be

$$B_* X = \bigcup_{[x]_B \subseteq X} [x]_B.$$

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In page 554, Theorem 3.8. must be as in Theorem 0.3:

0.3. Theorem. *Let G be a near group on nearness approximation space, H a nonempty subset of G and $N_r(B)^* H$ a groupoid. $H \subseteq G$ is a subnear group of G if and only if $x^{-1} \in H$ for all $x \in H$.*

Proof. Suppose that H is a subnear group of G. Then H is a near group and so $x^{-1} \in H$ for all $x \in H$. Conversely, suppose $x^{-1} \in H$ for all $x \in H$. By the hypothesis, since $N_r(B)^* H$ is a groupoid and $H \subseteq G$, then closure and associative properties hold in $N_r(B)^* H$. Also we have $x \cdot x^{-1} = e \in N_r(B)^* H$. Hence H is a subnear group of G. □

In page 554, Theorem 3.9. must be as in Theorem 0.4:

0.4. Theorem. *Let H_1 and H_2 be two near subgroups of the near group G and $N_r(B)^* H_1$, $N_r(B)^* H_2$ groupoids. If*

$$(N_r(B)^* H_1) \cap (N_r(B)^* H_2) = N_r(B)^* (H_1 \cap H_2),$$

then $H_1 \cap H_2$ is a near subgroup of near group G.

Proof. Suppose H_1 and H_2 be two near subgroups of the near group G. It is obvious that $H_1 \cap H_2 \subset G$. Since $N_r(B)^* H_1$, $N_r(B)^* H_2$ are groupoids and $\left(N_r(B)^* H_1\right) \cap \left(N_r(B)^* H_2\right) = N_r(B)^* (H_1 \cap H_2)$, $N_r(B)^* (H_1 \cap H_2)$ is a groupoid. Consider $x \in H_1 \cap H_2$. Since H_1 and H_2 are near subgroups, we have $x^{-1} \in H_1$ and $x^{-1} \in H_2$, i.e., $x^{-1} \in H_1 \cap H_2$. Thus from Theorem 0.3 $H_1 \cap H_2$ is a near subgroup of G. □

In page 555, proof of Theorem 5.3. has some typos. It must be as in Theorem 0.5:

0.5. Theorem. *Let G be a near group on nearness approximation space and N a subnear group of G. N is a subnear normal group of G if and only if $a \cdot n \cdot a^{-1} \in N$ for all $a \in G$ and $n \in N$.*

Proof. Suppose N is a near normal subgroup of near group G. We have $a \cdot N \cdot a^{-1} = N$ for all $a \in G$. For any $n \in N$, therefore we have $a \cdot n \cdot a^{-1} \in N$. Suppose N is a near subgroup of near group G. Suppose $a \cdot n \cdot a^{-1} \in N$ for all $a \in G$ and $n \in N$. We have $a \cdot N \cdot a^{-1} \subset N$. Since $a^{-1} \in G$, we get $a \cdot \left(a^{-1} \cdot N \cdot a\right) \cdot a^{-1} \subset a \cdot N \cdot a^{-1}$, i.e., $N \subset a \cdot N \cdot a^{-1}$. Since $a \cdot N \cdot a^{-1} \subset N$ and $N \subset a \cdot N \cdot a^{-1}$, we obtain $a \cdot N \cdot a^{-1} = N$. Therefore N is a subnear normal group of G. □

In page 556, Theorem 6.6. must be as in Theorem 0.6:

0.6. Theorem. *Let $G_1 \subset \mathcal{O}_1, G_2 \subset \mathcal{O}_2$ be near groups that are near homomorphic, N near homomorphism kernel and $N_r(B)^* N$ a groupoid. Then N is a near normal subgroup of G_1.*

In page 557, Theorem 6.7. must be as in Theorem 0.7:

0.7. Theorem. *Let $G_1 \subset \mathcal{O}_1$, $G_2 \subset \mathcal{O}_2$ be near homomorphic groups, H_1 and N_1 a near subgroup and a near normal subgroup of G_1, respectively and $N_{r_1}(B)^* H_1$ groupoid. Then we have the following.*

(1) *If* $\varphi\left(N_{r_1}(B)^* H_1\right) = N_{r_2}(B)^* \varphi(H_1)$, *then* $\varphi(H_1)$ *is a near subgroup of* G_2.

(2) *if* $\varphi(G_1) = G_2$ *and* $\varphi\left(N_{r_1}(B)^* N_1\right) = N_{r_2}(B)^* \varphi(N_1)$, *then* $\varphi(N_1)$ *is a near normal subgroup of* G_2.

In page 557, Theorem 6.8. must be as in Theorem 0.8:

0.8. Theorem. *Let* $G_1 \subset \mathcal{O}_1$, $G_2 \subset \mathcal{O}_2$ *be near homomorphic groups,* H_2 *and* N_2 *a near subgroup and a near normal subgroup of* G_2, *respectively and* $N_{r_1}(B)^* H_1$ *groupoid. Then we have the following.*

(1) *if* $\varphi\left(N_{r_1}(B)^* H_1\right) = N_{r_2}(B)^* H_2$, *then* H_1 *is a near subgroup of* G_1 *where* H_1 *is the inverse image of* H_2.

(2) *if* $\varphi(G_1) = G_2$ *and* $\varphi\left(N_{r_1}(B)^* N_1\right) = N_{r_2}(B)^* N_2$, *then* N_1 *is a near normal subgroup of* G_1 *where* N_1 *is the inverse image of* N_2.

We apologize to the readers for any inconvenience of these errors might have caused.

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[1] İnan, E. and Öztürk, M. A. *Near groups on nearness approximation spaces*, Hacet J Math Stat **41** (4), 545–558, 2012.

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On estimating population parameters in the presence of censored data: overview of available methods

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Abstract

This paper examines recent results presented on estimating population parameters in the presence of censored data with a single detection limit (DL). The occurrence of censored data due to less than detectable measurements is a common problem with environmental data such as quality and quantity monitoring applications of water, soil, and air samples. In this paper, we present an overview of possible statistical methods for handling non-detectable values, including maximum likelihood, simple substitution, corrected biased maximum likelihood, and EM algorithm methods. Simple substitution methods (e.g. substituting 0, $DL/2$, or DL for the non-detected values) are the most commonly used. It has been shown via simulation that if population parameters are estimated through simple substitution methods, this can cause significant bias in estimated parameters. Maximum likelihood estimators may produce dependable estimates of population parameters even when 90% of the data values are censored and can be performed using a computer program written in the R Language. A new substitution method of estimating population parameters from data contain values that are below a detection limit is presented and evaluated. Worked examples are given illustrating the use of these estimators utilizing computer program. Copies of source codes are available upon request.

Keywords: detection limits, censored data, normal and lognormal distributions, likelihood function, maximum likelihood estimators.

1. Introduction

Environmental data frequently contain values that are below detection limits. Values that are below DL are reported as being less than some reported limit of detection, rather than as actual values. A data set for which all observations may be identified and counted, with some observations falling into the restricted interval of measurements and the remaining observations being fully measured, is said to be censored. A situation where observations may be censored would

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be chemical measurements where some observations have a concentration below the detection limit of the analytical method. A sample for which some observations are known only to fall below a known detection limit, while the remaining observations falling above the detection limit are fully measured and reported is called left-singly censored or simply left censored. Detection limits are usually determined and justified in terms of the uncertainties that apply to a single routine measurement. Left-censored data commonly arise in environmental contexts. Left-censored observations (observations reported as $< DL$) can occur when the substance or attribute being measured is either absent or exists at such low concentrations that the substance is not present above the DL. In type I censoring, the detection limit is fixed a priori for all observations and the number of the censored observations varies. In type II censoring, the number of censored observations is fixed a priori, and the detection limit vary.

The estimation of the parameters of normal and lognormal populations in the presence of censored data has been studied by several authors in the context of environmental data. There has been a corresponding increase in the amount of attention devoted to the most proper analysis of data which have been collected in related to environmental issues such as monitoring water and air quality, and monitoring groundwater quality. The lognormal is frequently the parametric probability distribution of choice used in fitting environmental data Gilbert (1987). However, Shumway et al. (1989) examined transformations to normality from among the Box and Cox (1964) family of transformations: $Y = \frac{X^\lambda - 1}{\lambda}$ for $\lambda \neq 0$, and $Y = ln(X)$ for $\lambda = 0$. The transformed variable Y is assumed to be normally distributed with mean μ and standard deviation σ . Cohen (1959) used the method of maximum likelihood to derive estimators for the μ and σ parameters from left censored samples. Cohen (1959) also provided tables that are needed to report these maximum likelihood estimates ($MLEs$). Aboueissa and Stoline (2004) introduced a new algorithm for computing Cohen (1959) MLE estimators of normal population parameters from censored data with a single detection limit. Estimators obtained via this algorithm required no tables and more easily computed than the ($MLEs$) of Cohen (1959). Hass and Scheff (1990) compared methodologies for the estimation of the averages in truncated samples. Saw (1961) derived the first-order term in the bias of the Cohen (1959) MLE estimators for μ and σ, and proposed bias-corrected MLE estimators. Based on the bias-corrected tables in Saw (1961b), Schneider (1984,1986) performed a least-squares fit to produce computational formulas for normally distributed singly-censored data. Dempster et. al. (1977) proposed an iterative method, called the expectation maximization algorithm (EM algorithm), for obtaining the maximum likelihood estimates for these censored normal samples. The procedure consists of alternately estimating the censored observations from the current parameter estimates and estimating the parameters from the actual and estimated observations.

In practice, probably due to computational ease, simple substitution methods are commonly used in many environmental applications. One of the most commonly used replacement method is to substitute each left censored observation by half of the detection limit DL, Helsel et al. (1986) and Helsel et al. (1988). Two simple substitution methods were suggested by Gilliom and Helsel (1986). In one method, all left censored observations are replaced by zero. In the other method, all left censored observations are replaced by the detection limit DL. Aboueissa and Stoline (2004) developed closed form estimators for estimating normal population parameters from singly-left censored data based on a new replacement method. It has been shown that via simulation if left-censored observations are estimated through these substitution methods, this can cause significant bias in estimated parameters. In this article, a new substitution method, called weighted substitution method, is introduced and examined. This method is based on assigning different weights for each left-censored observation. These weights are estimated from the sample data prior to computing estimates of population parameters. It has been shown that via simulation if left-censored data are estimated through the weighted substitution method, this will reduce the bias in estimated parameters. Other suggested methods are discussed in Gibbons (1994), Gleit (1985), Schneider (1986), Gupta (1952), Stoline (1993), El-Shaarawi A. H. and Dolan D. M. (1989), El-Shaarawi and Esterby (1992), USEPA (1989), NCASI (1985, 1991), Gilliom and Helsel (1986), Helsel and Gilliom (1986), Helsel and Hirsch (1988), Schmee et. al. (1985), and Wolynetz (1979).

The objective of this article is to develop a new substitution method which yield reliable estimates of population parameters from left-censored data, and also to compare the performances of the various estimation procedures. In addition, a simple-to-use computer program is introduced and described for estimating the population parameters of normally or lognormally distributed left-censored data sets with a single detection limit using the eight parameter estimation methods described in this article. The authors of this article performed a simulation study to asses the performance of various estimate procedures in terms of bias and mean squared error (MSE). Several methods, including MLE, bias-corrected MLE ($UMLE$), and EM algorithm (EMA), have been considered.

2. Methods Used for Estimation

To simplify the presentation in this section, the method is described and illustrated by reference to the analysis of normally distributed data, though this condition occurs infrequently in typical environmental data analysis. However, it is frequently necessary to transform real environmental data before analysis; typically the logarithmic transformation of $x_i = log(y_i)$ is used, although other transformations are possible. When the logarithmic or other transformation is used prior to censored data set analysis, it is necessary to transform the analysis results back to the original scale of measurement following parameter estimation. Let $\underbrace{\overbrace{x_1, ..., x_{m_c}}^{m_c-observations}}_{left-censored}, \underbrace{\overbrace{x_{m_c+1}, ..., x_n}^{m-observations}}_{non-censored}$ be a random sample of n observations of which m_c are left-censored while $m = n - m_c$ are non-censored (or fully measured) from a normal population with mean μ and standard deviation σ . For censored observations, it is only known that $x_j < DL$ for $j = 1, ..., m_c$.

Let

$$(2.1) \qquad \bar{x}_m = \frac{1}{m}\sum_{i=m_c+1}^{n} x_i \ , \quad and \quad s_m^2 = \frac{1}{m}\sum_{i=m_c+1}^{n}(x_i - \bar{x}_m)^2$$

be the sample mean and sample variance of the m non-censored observations $x_{m_c+1}, ..., x_n$.

2.1. MLE **Estimators of Cohen.** Cohen (1959) employed the method of maximum likelihood to the normally distributed left-censored samples, and developed the following MLE estimators for the mean and standard deviation in terms of a tabulated function of two arguments:

$$(2.2) \qquad \hat{\mu} = \bar{x}_m - \hat{\lambda}(\bar{x}_m - DL) \ ,$$

$$(2.3) \qquad \hat{\sigma} = \sqrt{s_m^2 + \hat{\lambda}(\bar{x}_m - DL)^2} \ ,$$

where

$$(2.4) \qquad \hat{\lambda} = \lambda(h, \gamma), \ h = \frac{m_c}{n} \ \ and \ \ \gamma = \frac{s_m^2}{(\bar{x}_m - DL)^2}$$

Cohen (1959) provided tables of the function $\hat{\lambda} = \lambda(\gamma, h)$ restricted to values of $\gamma = 0.00(0.05)1.00$, and values of $h = 0.01(0.01)0.10(0.05)0.70(0.10)0.90$. The Cohen (1959) method requires use of these tables. Schneider (1986) extended these tables to include values of γ up to 1.48. Schmee et. al. (1985) extended these tables further to include values of $\gamma = 0.00(0.10)1.00(1.00)10.00$ and values of $h = 0.10(0.10)0.90$. However, interpolations for h and γ values are often required for most applications.

2.2. Aboueissa and Stoline Algorithm for Computing MLE **of Cohen.** Aboueissa and Stoline (2004) introduced an algorithm for computing the Cohen MLE estimators. This algorithm is based on solving the estimating equation

$$(2.5) \qquad \gamma = \frac{\left(1 - \frac{h}{1-h}\frac{\phi(\xi)}{\Phi(\xi)}\left(\frac{h}{1-h}\frac{\phi(\xi)}{\Phi(\xi)} - \xi\right)\right)}{\left(\frac{h}{1-h}\frac{\phi(\xi)}{\Phi(\xi)} - \xi\right)^2} \ ,$$

numerically for ξ ($say\ \hat{\xi}$). With $\hat{\xi}$ obtained via this algorithm, the exact value of the λ-parameter is then given by:

$$(2.6) \qquad \hat{\lambda}_{as} = \lambda(h, \hat{\xi}) = \frac{Y(h, \hat{\xi})}{Y(h, \hat{\xi}) - \hat{\xi}},$$

where

$$Y = Y(h, \xi) = \left(\frac{h}{1-h}\right) Z(\xi),$$

$$Z(\xi) = \frac{\phi(-\xi)}{1-\Phi(-\xi)} \ , \quad and \quad h = \frac{m_c}{n} = CL = \ censoring\ level\ .$$

The functions $\phi(\xi)$ and $\Phi(\xi)$ are the *pdf* and *cdf* of the standard unit normal. with $\hat{\lambda}_{as}$ obtained from (2.6), the MLE estimators obtained via this algorithm are obtained from (2.2) and (2.3) as:

$$(2.7) \qquad \hat{\mu}_{as} = \bar{x}_m - \hat{\lambda}_{as}(\bar{x}_m - DL) \ ,$$

and

$$(2.8) \qquad \hat{\sigma}_{as} = \sqrt{s_m^2 + \hat{\lambda}_{as}(\bar{x}_m - DL)^2} \ .$$

MLE estimators obtained via this method are labeled the $ASAMLEOC$ method in this article. It should be noted that the $ASAMLEOC$ method can be used to obtain the MLE estimators of population parameters from censored samples for all values of h and γ without any restrictions, and for all censoring levels including censoring levels greater than 0.90. The $ASAMLEOC$ estimators $\hat{\mu}_{as}$ and $\hat{\sigma}_{as}$ given by (2.7) and (2.8) are essentially Cohen's (1959) MLE estimators, which are obtained without the use of any auxiliary tables. It should also be noted that Cohen's (1959) method can not be used to obtain the maximum likelihood estimates from censored samples that have a censoring level higher than 90% ($h > 0.90$).

2.3. Bias-Corrected MLE Estimators. Saw (1961) derived the first-order term in the bias of the MLE estimators of μ and σ and proposed bias-corrected MLE estimators. Based on the bias-corrected tables in Saw (1961), Schneider (1986) performed a least-squares fit to produce computational formulas for the unbiased MLE estimators of μ and σ from normally distributed singly-censored data. These formulas, for the singly left-censored samples can be written as

$$(2.9) \qquad \hat{\mu}_u = \hat{\mu} - \frac{\hat{\sigma}B_u}{n+1}, \quad and \quad \hat{\sigma}_u = \hat{\sigma} - \frac{\hat{\sigma}B_\sigma}{n+1} \ ,$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the MLE estimators of Cohen (1959) or equivalently the $ASAMLE$ estimators $\hat{\mu}_{as}$ and $\hat{\sigma}_{as}$, and

$$(2.10) \quad B_u = -e^{2.692 - \frac{5.439m}{n+1}} \quad and \quad B_\sigma = -\left(0.312 + \frac{0.859m}{n+1}\right)^{-2} .$$

This method will be referred to as the $UMLE$ method in this paper.

2.4. Haas and Scheff Estimators(1990). Haas and Scheff (1990)developed a power series expansion that fits the tabled values of the auxiliary function $\lambda(\gamma, h)$ to within 6% for Cohen's (1959) estimates. This power series expansion is given by:

(2.11)

$$\log\lambda = 0.182344 - \frac{0.3256}{\gamma+1} + 0.10017\gamma + 0.78079\omega - 0.00581\gamma^2 - 0.06642\omega^2$$
$$- 0.0234\gamma\omega + 0.000174\gamma^3 + 0.001663\gamma^2\omega - 0.00086\gamma\omega^2 - 0.00653\omega^3,$$
$$where \quad \omega = \log\left(\frac{h}{1-h}\right).$$

This method will be referred to as the HS method in this paper.

2.5. Expectation Maximization Algorithm. Dempster et. al. (1977) proposed an iterative method, called the expectation maximization algorithm, for obtaining the $MLE's$ for the mean μ and the standard deviation σ of the normal distribution from censored samples. The procedure used in expectation maximization algorithm is based on replacing the censored observations and their squares in the complete data likelihood function by their conditional expectations given the data and the current estimates of μ and σ. This method will be referred to as the EMA method here.

2.6. Substitution Methods. Replacement methods are easier to use and consist of calculating the usual estimates of the mean and standard deviation by assigning a constant value to observations that are less than the censoring limit. Two simple substitution methods were suggested by Gilliom and Helsel (1986). In one method, all censored observations are replaced by zero. This is the ZE method. In the other method, all censored observations are replaced by the detection limit (DL). This is the DL method. One of the most commonly used substitution method, suggested by Helsel et.al. (1988), is to substitute each censored observations by half of its detection limit ($\frac{DL}{2}$). This is the HDL method.

3. Weighted Substitution Method for Left-Censored Data

The common replacement methods are based on replacing censored observations that are less than DL by a single constant. Three existing substitution methods were discussed in Section 2 based on replacing all left-censored observations with a single value either 0, $DL/2$, *or* DL. To avoid tightly grouped replaced values in cases where there are several left-censored values that share a common detection limit, left-censored observations may be spaced from zero to the detection limit according to some specified weights assigned for these left-censored observations. In the suggested weighted substitution method left-censored observations that are less than DL are replaced by non-constant different values based on assigning a different weight for each left-censored observation.More details are now given in the proposed weighted substitution method yielding estimates for μ and σ. The following weights are assigned to the m_c left-censored observations $x_1, ..., x_{m_c}$:

$$(3.1) \qquad w_j = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} \quad (P(U \geq DL))^{\ln(m+j-1)} \quad , \ for \ \ j = 1, 2, ..., m_c \ ,$$

where the probability $P(U \geq DL)$ is estimated from the sample data by:

$$P(\widehat{U \geq D}L) = 1 - \Phi\left(\frac{DL - \bar{x}_m}{s_m}\right) \tag{3.2}$$

An extensive simulation study was conducted on several weights. The simulation results (shown in the appendix) indicate that the proposed estimators using (3.1) are superior to those using the other weights in the sense of mean square error (variance of the estimator plus the square of the bias) in addition to the ability to recover the true mean and standard deviation as well as the existing methods such as maximum likelihood and EM algorithm estimators.

Estimates of the weights given in (3.1) are given by:

$$\widehat{w_j} = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} \left(P(\widehat{U \geq D}L)\right)^{\ln(m+j-1)} . \tag{3.3}$$

where the distribution of U is approximated by a normal distribution with an estimated mean $\bar{x}_m$ and an estimated variance s_m^2.

These weights are selected on a trial and error basis by means of simulations to yield estimators of population parameters that perform nearly as well as estimators obtained via the existing methods such as MLE estimators and EMA method. Left-censored observations $x_1, x_2, ..., x_{m_c}$ are then replaced by the following weighted m_c observations:

$$(x_1^w, x_2^w, ..., x_{m_c}^w) \equiv (\widehat{w_1}DL, \widehat{w_2}DL, ..., \widehat{w_{m_c}}DL) \tag{3.4}$$

Let

$$\bar{x}_{m_c} = \frac{1}{m_c}\sum_{i=1}^{m_c} x_i^w \ , \quad and \quad s_{m_c}^2 = \frac{1}{m_c}\sum_{i=1}^{m_c}(x_i^w - \bar{x}_{m_c})^2 \tag{3.5}$$

be the sample mean and sample variance of the weighted m_c observations $x_1^w, x_2^w, ..., x_{m_c}^w$. The corresponding weighted substitution method estimators $\hat{\mu}_w$ and $\hat{\sigma}_w$ of μ and σ are given by, respectively:

$$\begin{aligned} \hat{\mu}_w &= \frac{1}{n}\left(\sum_{i=1}^{m_c} x_i^w \ + \sum_{i=m_c+1}^{n} x_i\right) \\ &= \bar{x}_m - \hat{\lambda}_{\mu_w}\ (\bar{x}_m - \bar{x}_{m_c}) \ , \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \hat{\sigma}_w &= \sqrt{\frac{1}{n}\left(\sum_{i=1}^{m_c}(x_i^w - \hat{\mu}_w)^2 \ + \sum_{i=m_c+1}^{n}(x_i - \hat{\mu}_w)^2\right)} \\ &= \sqrt{\frac{m\ s_m^2 + m_c\ s_{m_c}^2}{n} + \hat{\lambda}_{\sigma_w}\ (\bar{x}_m - \bar{x}_{m_c})^2} \ , \end{aligned} \tag{3.7}$$

where

$$(3.8) \qquad \hat{\lambda}_{\mu_w} = \frac{m_c}{n} \quad and \quad \hat{\lambda}_{\sigma_w} = \frac{m\, m_c}{n^2}\,.$$

It should be noted that $\hat{\mu}_w$ in (3.6) can be written as:

$$(3.9) \qquad \hat{\mu}_w = \frac{m\,\bar{x}_m + m_c\,\bar{x}_{m_c}}{n}\,,$$

which is the weighted average of the sample means $\bar{x}_m$ and $\bar{x}_{m_c}$ of fully measured and weighted observations, respectively. It should also be observed that $\hat{\sigma}_w$ in (3.7) can be written as:

$$(3.10) \quad \hat{\sigma}_w = \sqrt{s_w^2 + \hat{\lambda}_{\sigma_w}\,(\bar{x}_m - \bar{x}_{m_c})^2}$$

where $s_w^2 = \frac{m\,s_m^2 + m_c\,s_{m_c}^2}{n}$ is the weighted average of the sample variances s_m^2 and $s_{m_c}^2$ of fully measured and weighted observations, respectively. Extensive simulation results show that use of the WSM method leads to estimators that have the ability to recover the true population parameters as well as the maximum likelihood estimators, and are generally superior to the constant replacement methods. In environmental sciences such as applied medical and environmental studies most of the data sets include non-detected (or left-censored) data values. The use of statistical methods such as the proposed one allows estimates of population parameters from data under consideration.

Asymptotic Variances of Estimates: The asymptotic variance-covariance matrix of the maximum likelihood estimates $(\hat{\mu}, \hat{\sigma})$ is obtained by inverting the Fisher information matrix $\mathbf{I}$ with elements that are negatives of expected values of the second-order partial derivatives of the log-likelihood function with respect to the parameters evaluated at the estimates $\hat{\mu}$ and $\hat{\sigma}$. The asymptotic variance-covariance matrix showed by Cohen (1991, 1959), will be used to obtain the estimated asymptotic variances of both $\hat{\mu}$ and $\hat{\sigma}$. Cohen (1959) describes the estimated asymptotic variance-covariance matrix of $(\hat{\mu}, \hat{\sigma})$ by

$$Cov(\hat{\mu}, \hat{\sigma}) = \begin{pmatrix} \left(\frac{\hat{\sigma}^2}{n[1-\Phi(\hat{\xi})]}\right)\frac{\hat{\varphi}_{22}}{\hat{\varphi}_{11}\hat{\varphi}_{22}-\hat{\varphi}_{12}^2} & \left(\frac{\hat{\sigma}^2}{n[1-\Phi(\hat{\xi})]}\right)\frac{-\hat{\varphi}_{12}}{\hat{\varphi}_{11}\hat{\varphi}_{22}-\hat{\varphi}_{12}^2} \\ \left(\frac{\hat{\sigma}^2}{n[1-\Phi(\hat{\xi})]}\right)\frac{-\hat{\varphi}_{12}}{\hat{\varphi}_{11}\hat{\varphi}_{22}-\hat{\varphi}_{12}^2} & \left(\frac{\hat{\sigma}^2}{n[1-\Phi(\hat{\xi})]}\right)\frac{\hat{\varphi}_{11}}{\hat{\varphi}_{11}\hat{\varphi}_{22}-\hat{\varphi}_{12}^2} \end{pmatrix}$$

where

$$\hat{\varphi}_{11} = \varphi_{11}(\hat{\xi}) = 1 + Z(\hat{\xi})[Z(-\hat{\xi}) + \hat{\xi}]$$
$$\hat{\varphi}_{12} = \varphi_{12}(\hat{\xi}) = Z(\hat{\xi})\left(1 + \hat{\xi}[Z(-\hat{\xi}) + \hat{\xi}]\right)$$
$$\hat{\varphi}_{22} = \varphi_{22}(\hat{\xi}) = 2 + \hat{\xi}\hat{\varphi}_{12}$$

For the $ASAMLEOC$ $\hat{\xi}$ is the solution of (2.5) as described in the previous section. For all other methods, without loss of generality, $\hat{\xi} = \frac{DL-\hat{\mu}}{\hat{\sigma}}$.

4. Computer Programs

To facilitate the application of parameter estimation methods described in this article, a computer programs is presented to automate parameters estimation from left-censored data sets that are normally or lognormally distributed. This computer program is called "$SingleLeft.Censored.Normal.Lognormal.estimates$", and is written in the R language. The EM Algorithm method has been programmed in the R language. The program is called "$EMA.Method$", and is presented as a part of the main computer program "$SingleLeft.Censored.Normal.Lognormal.estimates$". Copies of source codes are available upon request.

5. Worked Example

The guidance document Statistical Analysis of Ground-Water Monitoring Data at $RCRA$ Facilities, Interim Final Guidance ($USEPA$, 1989b) contains an example involving a set of sulfate concentrations (mg/L) in which three values are reported as ($< 1450 = DL$). The sulfate concentrations are assumed to come from a normal distribution. These 24 sulfate concentration values are:

$<1,450$	$1,800$	$1,840$	$1,820$	$1,860$	$1,780$	$1,760$	$1,800$
$1,900$	$1,770$	$1,790$	$1,780$	$1,850$	$1,760$	$<1,450$	$1,710$
$1,575$	$1,475$	$1,780$	$1,790$	$1,780$	$<1,450$	$1,790$	$1,800$

For this sample $n = 24$, $m = 21$, $m_c = 3$, $h = \frac{3}{24}$. The sample mean and the sample variance of the non-censored sample values are $\bar{x}_m = 1771.905$ and $s_m^2 = 8184.467$.

WSM Method: From (3.3) and (3.4) we obtain the estimate weights and the weighted data as follows:

$$(\hat{w}_1 \ , \ \hat{w}_2 \ , \ \hat{w}_3) = (0.9348828 \ , \ 0.9430983 \ , \ 0.9680175),$$

and

$$(x_1^w \ , \ x_2^w \ , \ x_3^w) = (1355.580 \ , \ 1367.493 \ , \ 1403.625).$$

The updated data set (fully measured and weighted data) is given by:

$\mathbf{1,355.580}$	$1,800$	$1,840$	$1,820$	$1,860$	$1,780$	$1,760$	$1,800$
$1,900$	$1,770$	$1,790$	$1,780$	$1,850$	$1,760$	$\mathbf{1,367.493}$	$1,710$
$1,575$	$1,475$	$1,780$	$1,790$	$1,780$	$\mathbf{1,403.625}$	$1,790$	$1,800$

The sample mean $\bar{x}_{m_c}$ and sample variance $s_{m_c}^2$ of the weighted data x_1^w , x_2^w , x_3^w are given by:

$$\bar{x}_{m_c} = 1375.566 \ \ and \ \ s_{m_c}^2 = 417.3153$$

From (3.8) we obtain

$$\hat{\lambda}_{\mu_w} = \frac{m_c}{n} = \frac{3}{24} = 0.125 \ \ and \ \ \hat{\lambda}_{\sigma_w} = \frac{m\, m_c}{n} = \frac{(21)(3)}{24^2} = 0.109375 \ .$$

Accordingly, using estimators (3.6) - (3.7) we calculate the WSM method estimators $\hat{\mu}_w$ and $\hat{\sigma}_w$ as:

$$\hat{\mu}_w = 1771.905 - 0.125(1771.905 - 1375.566) = 1722.3626 \ ,$$

Table 1. Estimates for μ and σ from Sulfate Data

Method of Estimation	$\hat{\mu}$	$\hat{\sigma}$
$ASAMLEOC$	1723.9951	153.6451
$UMLE$	1723.0543	159.3983
HS	1719.8363	157.9416
EMA	1723.9951	153.6451
ZE	1550.4167	592.0813
HDL	1641.0417	356.4231
DL	1731.6667	135.9968
WSM	**1722.3624**	**156.1880**

and

$$\hat{\sigma}_w = \sqrt{\frac{21(8184.467) + 3(417.3153)}{24} + 0.109375(1771.905 - 1375.566)^2} = 156.1880 .$$

Applying the computer program "$SingleLeft.Censored.Normal$" for these data as shown in the Appendix, yields estimates for μ and σ parameters via eight methods of estimation including the WSM method. The results are summarized in Table 1.

Discussion: An inspection of Table 1 reveals that the $ASAMLEOC$, $UMLE$, HS, EMA and WSM methods yield quite similar estimates for both μ and σ. The DL method estimate for μ is close to those obtained by $ASAMLEOC$, EMA, WSM, $UMLE$ and HS methods. The DL method estimate for σ seems to be underestimated comparing to those estimates obtained by $ASAMLEOC$, EMA, WSM, $UMLE$ and HS methods. The ZE and HDL methods yield estimates which are different from those produced by $ASAMLEOC$, EMA, WSM, $UMLE$ and HS methods. The estimates of σ obtained by the ZE and HDL methods are highly overestimated, while the estimates of μ are underestimated comparing to estimates obtained by $ASAMLEOC$, EMA, WSM, $UMLE$ and HS methods. Overall, the WSM method performs similar to $ASAMLEOC$, EMA, $UMLE$ and HS methods, and superior to the common substitution ZE, HDL and DL methods.

For more investigations of the performance of the parameter estimation methods described in section 2, the sulfate concentrations data are artificially censored at censoring levels (0.25 , 0.50 , 0.625 , 0.75 , 0.875 , 0.917) with a single detection limit of $1,450$. The corresponding number of left-censored observations for each of these censoring levels are 6, 12, 15, 18, 21 and 22, respectively. Then the estimates of μ and σ are computed using the computer program "$SingleLeft.Censored.Normal$". Results are summarized in Table 2. The following observations are made from an examination of the results reported in Table 2. The WSM estimates for μ and σ are similar to those reported by $ASAMLEOC$, EMA, $UMLE$ and HS for cases with censoring levels less than or equal to 0.75.

TABLE 2. Estimates for μ and σ from Sulfate Data with artificial censoring levels

	$m_c = 6,\ CL = 0.25$		$m_c = 12,\ CL = 0.50$		$m_c = 15,\ CL = 0.625$	
Method of Estimation	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$
$ASAMLEOC$	1658.6581	205.1465	1497.0883	313.4529	1367.1360	361.7728
$UMLE$	1656.2454	214.6244	1483.4885	337.3514	1336.9885	399.2681
HS	1651.2191	210.7608	1484.2813	320.0817	1351.8316	368.3523
EMA	1658.6581	205.1465	1497.1077	313.4304	1367.8667	361.0457
ZE	1322.9166	768.2189	888.9583	890.6903	661.4583	855.3307
HDL	1504.1667	457.3376	1251.4583	529.3775	1114.5833	505.3091
DL	1685.4167	158.9478	1613.9583	173.1027	1567.7083	159.5957
WSM	**1647.8992**	**218.9194**	**1456.3776**	**341.5274**	**1314.6978**	**382.7615**
	$m_c = 18,\ CL = 0.75$		$m_c = 21,\ CL = 0.875$		$m_c = 22,\ CL = 0.917$	
$ASAMLEOC$	1191.5900	412.5770	815.9108	564.2837	623.1800	605.8733
$UMLE$	1125.5549	474.0433	642.4414	695.2905	391.6580	773.0710
HS	1170.5134	420.0236	801.0685	568.6071	633.4851	603.1457
EMA	1204.8421	401.5795	996.7565	443.5979	996.0501	381.4442
ZE	436.0417	756.5147	222.5000	588.7080	147.5000	489.2107
HDL	979.7917	443.4793	856.875	348.9562	812.0833	288.8372
DL	1523.5417	134.6947	1491.2500	109.2898	1476.6667	88.4904
WSM	**1149.1283**	**406.4031**	**968.9359**	**419.4892**	**897.6092**	**410.0749**

For cases with censoring levels above 0.75, the WSM and EMA methods yield similar results. For cases with censoring levels less than 0.75, μ is underestimated by both ZE and HDL Methods, while σ is overestimated comparing to estimates obtained by $ASAMLEOC$, EMA, $UMLE$ and HS methods. The DL method yield similar estimate for μ for cases with censoring levels less than 0.75, while σ is underestimated for all censoring levels via this method comparing to estimates obtained by $ASAMLEOC$ of Cohen, EMA, $UMLE$ and HS methods. Overall, the WSM method yields similar estimates to those obtained by $ASAMLEOC$, EMA, $UMLE$ and HS methods, and superior to the existing substitution methods ZE, HDL and DL for all censoring levels.

6. Comparison of Methods

In this section the estimation methods described above were compared by a simulation study. We shall assess the performance of estimators obtained via these methods in terms of the mean squared error MSE (variance of the estimator plus the square of the bias). The simulation study was performed with ten thousand repetitions ($N = 10000$) of samples from a normal distribution for each combination of n, μ, σ, and the censoring level $CL = h$. Simulations were conducted with censoring levels $0.15,\ 0.25,\ 0.50,\ 0.75,\ and\ 0.90$. The selected combinations of (n, μ, σ, CL) are:

$$\begin{array}{c} n = 10, 25, 50, 75, 100\ ,\ \mu = 25\ ,\ \ \sigma = 10, CL = 0.15 \\ n = 10, 25, 50, 75, 100\ ,\ \mu = 25\ ,\ \ \sigma = 10, CL = 0.25 \\ n = 10, 25, 50, 75, 100\ ,\ \mu = 25\ ,\ \ \sigma = 10, CL = 0.50 \\ n = 10, 25, 50, 75, 100\ ,\ \mu = 10\ ,\ \ \sigma = 5, CL = 0.75 \\ n = 10, 25, 50, 75, 100\ ,\ \mu = 10\ ,\ \ \sigma = 5, CL = 0.90 \end{array} \tag{6.1}$$

Given the censoring level CL, the detection limit is computed from the relation $DL = CL^{th}$ percentile. The data sets were then artificially censored at DL. Any value falling below DL was considered to be left-censored. These simulated data sets ($N = 10000$ for each combination of n, μ, σ and CL) were then utilized by these estimators to obtain estimates of μ and σ. The average of the $N = 10000$ estimates are reported as $\hat{\mu}$ and $\hat{\sigma}$ in Table 1 and 2. The MSE based on $N = 10000$ simulation runs are also reported in each table. The MSE of $\hat{\mu}$ is defined by:

$$(6.2) \qquad MSE(\hat{\mu}, \mu) = Var(\hat{\mu}) + (b(\hat{\mu}, \mu))^2 \ ,$$

where

$$(6.3) \qquad b(\hat{\mu}, \mu) = \hat{\mu} - \mu \ ,$$

is the bias of $\hat{\mu}$, where

$$(6.4) \qquad \hat{\mu} = \frac{1}{N}\sum_{i=1}^{N}\hat{\mu}_i \quad and \quad Var(\hat{\mu}) = \frac{1}{N-1}\sum_{i=1}^{N}(\hat{\mu}_i - \hat{\mu})^2.$$

The MSE of $\hat{\sigma}$ can be defined in a similar way.

Estimation Methods: The methods used for the estimation of the normal population parameters from singly-left-censored samples are:

- $ASAMLEOC$: Aboueissa and Stoline Algorithm for Calculating MLE of Cohen,
- $UMLE$: Bias-Corrected MLE Estimators,
- HS: Haas and Scheff method,
- EMA: Expectation Maximization algorithm method,
- ZE: Replacing all left-censored data by zero method,
- HDL: Replacing all left-censored data by half of the detection limit method,
- DL: Replacing all left-censored data by the detection limit method,
- WSM: The new Weighted Substitution Method.

Tables 4 and 5 are partitioned into 5 subgroups by increasing censoring level: $CL = 0.15, 0.25, 0.50$, 0.75 and 0.90. The simulation results within each subgroup are further partitioned by increasing sample size $n = 10, 25, 50$ and 75. Two simulation results are given for each method and for each combination of n, μ, σ and CL. These are the average value of the estimate and the MSE.

6.1. Comparison of Methods: μ Parameter.

WSM to existing methods: The following observations and conclusions are made from an examination of the simulation results reported for the mean μ.

For the $\mu = 25$ parameter value: the reported new WSM method estimates are all in the range $24.8722 - 25.2874$, the reported HDL method estimates are all in the range $23.7069 - 24.6108$, the reported EMA method estimates are all in the range $24.7206 - 25.002$ and the reported $ASAMLEOC$ method estimates are all in the range $24.5856 - 25.0355$ for cases with censoring level less than 50%. For cases with censoring levels less than 50%: (1) the MSE values for HDL method are larger than those reported by the new WSM method, and (2) the MSE values for WSM method are nearly equal to those reported by the new EMA and $ASAMLEOC$ methods. For cases with censoring level 50%: the reported new WSM method estimates are all in the range $23.4630 - 24.6600$, the reported HDL method estimates are all in the range $22.5559 - 22.9255$, the reported EMA method estimates are all in the range $24.8221 - 25.0050$ and the reported $ASAMLEOC$ method estimates are all in the range $25.0019 - 25.4548$. The MSE values for HDL method are larger than those reported by the new WSM method. The MSE values for the new WSM method are nearly equal to those reported by both EMA and $ASAMLEOC$ methods except for cases with sample sizes 50, 75, and 100.

For the $\mu = 10$ parameter value and for cases with censoring level greater than or equal to 75%: the reported new WSM method estimates are all in the range $8.5887 - 9.8231$, the reported HDL method estimates are all in the range $8.6633 - 9.2154$, the reported EMA method estimates are all in the range $10.1079 - 12.7350$ and the reported $ASAMLEOC$ method estimates are all in the range $9.8679 - 11.2427$. The MSE values for HDL and EMA methods are quite similar and smaller than those reported by EMA and $ASAMLEOC$ methods except for cases with sample sizes 75 and 100. For cases with censoring level 90% and sample size 10, it has been noted that estimates for the μ parameter are not available via EMA method.

Overall, the new WSM method appears to be superior to the existing methods for cases with censoring levels less than 50%, and superior to EMA and $ASAMLEOC$ methods for cases with censoring levels greater than or equal to 50% except for cases with sample sizes 75 and 100. The new WSM and HDL methods yield quite similar estimates for the μ parameter for cases with censoring levels greater than or equal to 50%.

6.2. Comparison of Methods: σ Parameter.

WSM to existing methods: The following observations and conclusions are made from an examination of the simulation results reported for the standard deviation σ.

For the $\sigma = 10$ parameter value: the reported new WSM method estimates are all in the range $9.2886 - 9.8007$, the reported HDL method estimates are all in the range $10.2680 - 10.7815$, the reported EMA method estimates are all in the range $9.6781 - 10.0459$ and the reported $ASAMLEOC$ method estimates are all in the range $9.5468 - 10.0068$ for cases with censoring level less than 50%. The MSE values for EMA and $ASAMLEOC$ methods are larger than those reported by the new WSM method for cases with censoring levels less than 50%. The MSE values reported by HDL and the new WSM methods are quite similar for cases with censoring levels less than 50%. For cases with censoring level 50%: the reported new WSM method estimates are all in the range $9.3601 - 10.5496$, the reported HDL method estimates are all in the range $10.7585 - 11.0397$, the reported EMA method estimates are all in the range $9.5578 - 9.9383$ and the reported $ASAMLEOC$ method estimates are all in the range $9.1672 - 9.8955$. The MSE values for HDL and the new WSM methods are quite similar, and smaller than those reported by both EMA and $ASAMLEOC$ methods except for cases with sample sizes 100.

For the $\sigma = 5$ parameter value and for cases with censoring level greater than or equal to 75%: the reported new WSM method estimates are all in the range $4.3496 - 5.0428$, the reported HDL method estimates are all in the range $3.0071 - 4.3463$, the reported EMA method estimates are all in the range $3.0289 - 4.8167$ and the reported $ASAMLEOC$ method estimates are all in the range $3.8187 - 4.9756$. The MSE values for EMA, EMA and $ASAMLEOC$ methods are larger than those reported by the new WSM method. For cases with censoring level 90% and sample size 10, it has been noted that estimates for the σ parameter are not available via EMA method. It should be noted that the $\sigma = 5$ parameter value for most cases is highly under estimated by EMA, EMA and $ASAMLEOC$ methods.

Overall, the new WSM method appears to be superior to HDL method for cases with censoring levels greater than or equal to 50%, and superior to EMA and $ASAMLEOC$ methods for all censoring cases. The HDL and the new WSM methods perform similarly for cases with censoring levels less than 50%.

In summary, the maximum likelihood estimators (ASAMLEOC), the new weighted substitution method estimators (WSM), and the EM algorithm estimators (EMA) perform similarly, and all are generally superior to the existing substitution method estimators.

6.3. Additional Simulation Results.

The following simulation results are obtained using the following combinations of n, μ, σ, and censoring level CL.

TABLE 3. Estimates for μ and σ from Sulfate Data

(n, μ, σ)	k	CL
$(k, 25, 10)$	$k = 10, 25, 50, 75, 100$	0.75 - 0.90
$(k, 10, 5)$	$k = 10, 25, 50, 75, 100$	0.15 - 0.50
$(k, 20, 3)$	$k = 10, 25, 50, 75, 100$	0.10 - 0.90

Tables 6, 7 and 8 are partitioned into two subgroups. Each subgroup has a different censoring level. The simulation results within each subgroup are given for both population mean μ and standard deviation σ. Two simulation results are given for each method and for each combination of n, μ, σ and CL. These simulation results are the average value of the estimate and the MSE.

Table 4. Simulation Estimates of the Mean μ from Normally Distributed Left-Censored Samples with a Single Detection Limit

		Methods Of Estimation							
			MLE			Replacement			
$(\mathbf{n}, \mu, \sigma)$		**EMA**	**ASAMLEOC**	**UMLE**	**HS**	**WSM**	**ZE**	**HDL**	**DL**
				$CL = 0.15$					
(10, 25, 10)	$\hat{\mu}$	24.7206	24.5856	24.3367	24.4160	**25.0303**	22.4497	24.0517	25.6536
	MSE	12.9903	12.1390	12.4803	12.3479	**10.2139**	14.1331	10.3785	12.1721
(25, 25, 10)	$\hat{\mu}$	25.0022	25.0047	24.9302	24.8702	**25.2874**	23.3820	24.6056	25.8292
	MSE	4.0587	4.0515	4.0600	4.0765	**3.6776**	5.5471	3.5844	4.7208
(50, 25, 10)	$\hat{\mu}$	24.9873	24.9610	24.9221	24.8229	**25.2175**	23.4054	24.6045	25.8036
	MSE	1.9815	1.9524	1.9576	1.9836	**1.7494**	3.9807	1.8237	2.5930
(75, 25, 10)	$\hat{\mu}$	24.9569	24.9167	24.8892	24.7757	**25.1520**	23.3772	24.5654	25.7536
	MSE	1.3018	1.2937	1.2997	1.3415	**1.2036**	3.5491	1.2649	1.8417
(100, 25, 10)	$\hat{\mu}$	24.9303	24.9455	24.9308	24.8173	**25.1187**	23.5041	24.6108	25.7176
	MSE	1.0832	1.0840	1.0861	1.1177	**1.0118**	3.0278	1.0706	1.5900
				$CL = 0.25$					
(10, 25, 10)	$\hat{\mu}$	24.9314	24.7705	24.3606	24.5564	**25.1387**	20.8147	23.7069	26.5991
	MSE	11.1167	10.1121	10.6242	10.2893	**8.6504**	22.7221	8.7693	12.2048
(25, 25, 10)	$\hat{\mu}$	24.8567	25.0355	24.9278	24.8842	**25.0651**	21.9426	24.1728	26.4031
	MSE	4.7220	4.4562	4.4708	4.4865	**3.8785**	11.9771	4.0882	6.3499
(50, 25, 10)	$\hat{\mu}$	24.9379	24.9031	24.8405	24.7176	**24.9884**	21.6906	24.0783	26.4659
	MSE	2.3519	2.2546	2.2750	2.3375	**1.9278**	12.1852	2.4918	4.3222
(75, 25, 10)	$\hat{\mu}$	24.9199	24.9175	24.8798	24.7403	**24.8745**	21.7923	24.1097	26.4272
	MSE	1.3578	1.3316	1.3406	1.3983	**1.1832**	11.0518	1.7847	3.3294
(100, 25, 10)	$\hat{\mu}$	24.9792	25.0024	24.9770	24.8292	**24.8722**	21.9016	24.1936	26.4857
	MSE	1.0849	1.0738	1.0749	1.1061	**0.9745**	10.2353	1.4676	3.2606
				$CL = 0.50$					
(10, 25, 10)	$\hat{\mu}$	24.8221	25.1868	24.1506	24.9091	**24.6600**	16.2848	22.5559	28.8270
	MSE	18.5532	15.2003	17.3134	15.4780	**10.1436**	79.3801	12.9385	27.0316
(25, 25, 10)	$\hat{\mu}$	24.9778	25.4548	25.0936	25.2066	**23.8873**	16.9032	22.9255	28.9479
	MSE	6.7314	6.3569	6.3417	6.3287	**5.2874**	67.0511	7.2413	20.7299
(50, 25, 10)	$\hat{\mu}$	24.9382	25.0019	24.8038	24.7091	**23.8758**	16.4341	22.6824	28.9307
	MSE	3.2269	2.9649	3.0511	3.1225	**4.0171**	74.0480	6.7341	17.8859
(75, 25, 10)	$\hat{\mu}$	25.0050	25.1557	25.0237	24.8749	**23.76042**	16.6278	22.8010	28.9742
	MSE	1.9961	1.9971	1.9946	2.0344	**4.2894**	70.5353	5.7310	17.3938
(100, 25, 10)	$\hat{\mu}$	24.9496	24.9960	24.8980	24.7015	**23.4630**	16.4471	22.6953	28.9434
	MSE	1.4499	1.3884	1.4097	1.5089	**5.1825**	73.4808	5.9611	16.6976
				$CL = 0.75$					
(10, 10, 5)	$\hat{\mu}$	11.1600	10.9294	9.7694	10.7134	**9.8231**	4.6079	9.1331	13.6582
	MSE	4.9783	6.9497	8.5266	6.9843	**2.3121**	29.4568	2.5634	16.9481
(25, 10, 5)	$\hat{\mu}$	10.3701	10.7352	10.1216	10.4767	**9.2815**	4.4301	9.2023	13.9745
	MSE	3.2304	3.9538	4.0427	3.8897	**2.3455**	31.1690	2.3161	17.4969
(50, 10, 5)	$\hat{\mu}$	10.2091	10.2622	9.8291	9.9475	**9.1792**	4.1868	9.1008	14.0148
	MSE	1.6294	1.7626	1.9279	1.8441	**1.2663**	33.8541	1.1847	16.8751
(75, 10, 5)	$\hat{\mu}$	10.1761	10.0857	9.7393	9.7467	**9.0131**	4.1183	9.0916	14.0649
	MSE	1.0370	1.2607	1.4362	1.4306	**1.6401**	34.6373	1.0413	17.0802
(100, 10, 5)	$\hat{\mu}$	10.1079	9.9587	9.6655	9.6094	**8.9862**	4.0600	9.0399	14.0197
	MSE	0.8523	0.9697	1.1543	1.2111	**1.2560**	35.3154	1.0860	16.5823
				$CL = 0.90$					
(10, 10, 5)	$\hat{\mu}$	NAN	9.9275	6.4233	9.8992	**9.7546**	1.7684	8.6633	15.5582
	MSE	NAN	28.3201	78.0182	28.5537	**2.1788**	67.8434	3.4499	36.3779
(25, 10, 5)	$\hat{\mu}$	12.7350	11.2427	9.8571	10.8905	**9.0188**	2.1579	9.1766	16.1952
	MSE	11.2545	10.9418	13.8752	11.3257	**2.8197**	7.8420	1.3721	40.5871
(50, 10, 5)	$\hat{\mu}$	12.4702	9.8679	9.0350	9.4993	**9.0459**	1.8560	9.1220	16.3880
	MSE	8.8839	8.3572	11.1032	9.3857	**5.7839**	66.3434	2.1813	42.1725
(75, 10, 5)	$\hat{\mu}$	12.3973	10.5135	9.9385	10.1106	**9.1537**	1.9673	9.2154	16.4635
	MSE	7.6331	4.9507	5.4244	5.2242	**6.9610**	64.5361	4.8869	42.6704
(100, 10, 5)	$\hat{\mu}$	12.2278	9.8990	9.4768	9.4882	**8.5887**	1.8662	9.1851	16.5041
	MSE	6.3274	4.2162	4.9141	4.8985	**6.3863**	66.1671	3.8660	42.9826

Table 5. Simulation Estimates of the Standard Deviation σ from Normally Distributed Left-Censored Samples with a Single Detection Limit

		Methods Of Estimation							
			MLE			Replacement			
$(\mathbf{n}, \mu, \sigma)$		**EMA**	**ASAMLEOC**	**UMLE**	**HS**	**WSM**	**ZE**	**HDL**	**DL**
$CL = 0.15$									
(10, 25, 10)	$\hat{\sigma}$	10.0459	9.6976	10.7021	9.8076	**9.4377**	13.0290	10.3384	8.0643
	MSE	6.3001	6.4146	8.1939	6.4824	**5.3325**	11.5996	4.9412	8.1432
(25, 25, 10)	$\hat{\sigma}$	9.8113	9.7730	10.1485	9.8612	**9.6837**	12.4614	10.2680	8.4608
	MSE	2.5130	2.5439	2.7096	2.5525	**2.2003**	7.0256	1.8620	4.2423
(50, 25, 10)	$\hat{\sigma}$	9.9661	10.0068	10.2000	10.0965	**9.7549**	12.5478	10.4218	8.6617
	MSE	1.3647	1.3279	1.4193	1.3572	**1.0455**	7.0106	0.9788	2.7903
(75, 25, 10)	$\hat{\sigma}$	9.9220	9.9845	10.1138	10.0765	**9.4640**	12.5021	10.4922	8.6389
	MSE	0.8380	0.8363	0.8708	0.8575	**0.7558**	6.5641	0.6781	2.4788
(100, 25, 10)	$\hat{\sigma}$	9.9273	9.9035	9.9950	9.9874	**9.7880**	12.3037	10.3035	8.6555
	MSE	0.6693	0.6685	0.6715	0.6704	**0.6262**	5.5621	0.5424	2.3117
$CL = 0.25$									
(10, 25, 10)	$\hat{\sigma}$	9.6782	9.7455	10.9469	9.8648	**9.2886**	14.7549	10.7656	7.2926
	MSE	7.6926	7.8342	10.6996	7.9515	**4.5755**	25.2537	3.3651	11.7217
(25, 25, 10)	$\hat{\sigma}$	9.7791	9.5468	9.9658	9.6299	**9.5273**	13.8585	10.4642	7.6557
	MSE	2.8727	2.7838	2.8110	2.7655	**2.0599**	15.9031	1.7138	7.1502
(50, 25, 10)	$\hat{\sigma}$	9.9417	9.9884	10.2161	10.0907	**9.5583**	14.2765	10.7815	7.8224
	MSE	1.5469	1.4527	1.5662	1.4891	**1.0459**	18.8102	1.1838	5.6361
(75, 25, 10)	$\hat{\sigma}$	9.9461	9.9494	10.0974	10.0468	**9.6981**	14.1626	10.7359	7.8518
	MSE	0.9648	0.9589	0.9944	0.9761	**0.6769**	17.6624	0.9304	5.2130
(100, 25, 10)	$\hat{\sigma}$	9.9602	9.9296	10.0385	10.0245	**9.8007**	14.1353	10.7264	7.8658
	MSE	0.7465	0.7489	0.7619	0.7601	**0.4713**	17.3686	0.8366	5.0213
$CL = 0.50$									
(10, 25, 10)	$\hat{\sigma}$	9.5578	9.1672	10.8553	9.2823	**9.3601**	16.7716	10.7585	5.3312
	MSE	15.6057	12.0724	16.6869	12.1633	**3.7688**	49.5103	3.1956	25.6978
(25, 25, 10)	$\hat{\sigma}$	9.7338	9.2809	9.9360	9.3811	**9.9686**	16.7956	10.8394	5.5985
	MSE	5.2199	5.2880	5.4728	5.2627	**1.6210**	47.6687	1.8370	21.1074
(50, 25, 10)	$\hat{\sigma}$	9.9095	9.8507	10.2101	9.9679	**10.2462**	16.9698	11.0231	5.7495
	MSE	2.6241	2.5035	2.7096	2.5395	**1.3816**	49.3191	1.6188	18.9206
(75, 25, 10)	$\hat{\sigma}$	9.9029	9.7586	9.9950	9.8706	**10.3847**	16.9536	11.0084	5.7604
	MSE	1.6039	1.5602	1.5756	1.5542	**1.4119**	48.8112	1.3613	18.4996
(100, 25, 10)	$\hat{\sigma}$	9.9383	9.8955	10.0758	10.0129	**10.5496**	16.9832	11.0397	5.7767
	MSE	1.2036	1.2086	1.2475	1.2261	**1.7183**	16.9832	1.3615	18.2478
$CL = 0.75$									
(10, 10, 5)	$\hat{\sigma}$	4.2618	3.8187	4.9812	3.8871	**4.3496**	7.1417	4.2411	1.5442
	MSE	5.2597	5.6734	7.2788	5.6663	**1.1665**	5.5348	1.5443	12.6638
(25, 10, 5)	$\hat{\sigma}$	4.5483	4.2664	4.8400	4.3438	**4.6328**	7.2150	4.3111	1.6639
	MSE	2.7109	2.7569	2.8810	2.7305	**0.4991**	5.3049	0.8201	11.4708
(50, 10, 5)	$\hat{\sigma}$	4.7104	4.6859	5.0415	4.7798	**4.8445**	7.1681	4.3241	1.7189
	MSE	1.3499	1.2885	1.3782	1.2854	**0.2456**	4.8861	0.5780	10.9302
(75, 10, 5)	$\hat{\sigma}$	4.8058	4.8796	5.1431	4.9806	**4.9818**	7.1746	4.3463	1.7528
	MSE	0.8648	0.8804	0.9823	0.9026	**0.1611**	4.8646	0.5148	10.6573
(100, 10, 5)	$\hat{\mu}$	4.8167	4.9337	5.1437	5.0375	**5.0428**	7.1359	4.3290	1.7547
	MSE	0.6827	0.6806	0.7556	0.7065	**0.1283**	4.6661	0.5177	10.6186
$CL = 0.90$									
(10, 10, 5)	$\hat{\sigma}$	NAN	4.2813	6.8391	4.2891	**4.3553**	5.3054	3.0071	0.7088
	MSE	NAN	13.5159	36.5542	13.5523	**1.5075**	1.463764	4.4102	18.7701
(25, 10, 5)	$\hat{\sigma}$	3.5159	4.0208	4.9838	4.1082	**4.4259**	5.8762	3.3071	0.8551
	MSE	5.3528	5.5123	6.9890	5.5572	**0.8011**	1.1049	3.0605	17.3969
(50, 10, 5)	$\hat{\sigma}$	3.4667	4.9175	5.5312	5.0063	**4.8617**	5.6002	3.2012	0.9177
	MSE	4.5614	3.7233	4.9842	3.8528	**0.4854**	0.5449	3.3510	16.7972
(75, 10, 5)	$\hat{\sigma}$	3.4707	4.5939	4.9854	4.6899	**4.8792**	5.7293	3.2513	0.9179
	MSE	3.8156	2.2375	2.4406	2.2567	**0.3493**	0.65004	3.1282	16.7499
(100, 10, 5)	$\hat{\sigma}$	3.5912	4.9756	5.2876	5.0733	**4.8105**	5.6338	3.2202	0.9437
	MSE	3.0289	1.7452	2.0531	1.8204	**0.2995**	0.4841	3.2185	16.5173

Table 6. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.75, 0.90$: $(k, 25, 10)$, $(k = 10, 25, 50, 75, 100)$

		Methods Of Estimation							
			MLE			Replacement			
$(\mathbf{n}, \mu, \sigma)$		**EMA**	**ASAMLEOC**	**UMLE**	**HS**	**WSM**	**ZE**	**HDL**	**DL**
				$CL = 0.75$					
(10, 25, 10)	$\hat{\mu}$	27.565	26.815	24.464	26.380	**27.349**	10.737	21.543	32.349
	MSE	21.380	34.594	42.150	34.912	**26.146**	205.140	19.480	72.169
(25, 25, 10)	$\hat{\mu}$	25.492	26.063	24.800	25.528	**25.797**	10.235	21.480	32.725
	MSE	11.982	14.436	16.016	14.680	**11.949**	218.58	14.862	65.793
(50, 25, 10)	$\hat{\mu}$	25.321	25.493	24.620	24.857	**25.407**	9.689	21.376	33.063
	MSE	6.356	6.912	7.578	7.245	**6.216**	234.71	14.493	68.410
(75, 25, 10)	$\hat{\mu}$	25.294	25.148	24.457	24.471	**25.221**	9.487	21.283	33.080
	MSE	4.253	5.072	5.826	5.801	**4.392**	240.82	14.641	67.417
(100, 25, 10)	$\hat{\mu}$	25.223	24.947	24.354	24.242	**25.086**	9.412	21.281	33.151
	MSE	3.434	3.905	4.632	4.850	**3.474**	243.12	14.450	68.053
(10, 25, 10)	$\hat{\sigma}$	8.464	7.742	10.099	7.880	**8.114**	16.589	9.635	3.133
	MSE	20.263	21.496	27.905	21.453	**18.758**	47.473	12.165	49.949
(25, 25, 10)	$\hat{\sigma}$	9.259	8.782	9.963	8.943	**9.040**	16.612	9.735	3.416
	MSE	11.333	11.516	12.918	11.534	**10.770**	45.296	7.948	44.854
(50, 25, 10)	$\hat{\sigma}$	9.548	9.451	10.167	9.640	**9.499**	16.532	9.734	3.469
	MSE	5.479	5.080	5.556	5.100	**5.083**	43.540	5.599	43.307
(75, 25, 10)	$\hat{\sigma}$	9.601	9.726	10.457	11.471	**9.664**	16.467	9.721	3.494
	MSE	3.588	3.637	5.826	5.801	**3.485**	42.344	2.422	42.789
(100, 25, 10)	$\hat{\sigma}$	9.748	9.970	10.394	10.179	**9.859**	16.483	9.756	3.543
	MSE	2.733	2.791	3.187	2.940	**2.666**	42.417	2.320	42.050
				$CL = 0.90$					
(10, 25, 10)	$\hat{\mu}$	NAN	24.304	16.853	24.244	**23.178**	4.080	20.178	36.276
	MSE		110.97	8.147	111.98	**29.204**	437.99	19.204	146.64
(25, 25, 10)	$\hat{\mu}$	30.247	27.918	25.236	27.246	**28.366**	4.916	21.211	37.506
	MSE	44.689	44.648	53.141	45.532	**42.684**	403.54	17.366	165.91
(50, 25, 10)	$\hat{\mu}$	29.900	24.606	22.926	23.866	**27.251**	4.214	20.983	37.751
	MSE	34.130	31.013	41.799	34.897	**21.481**	432.14	17.709	167.83
(75, 25, 10)	$\hat{\mu}$	29.811	25.967	24.815	25.164	**26.881**	4.465	21.179	37.894
	MSE	30.283	27.901	19.642	18.856	**18.244**	421.76	17.674	169.76
(100, 25, 10)	$\hat{\mu}$	29.637	24.944	24.116	24.138	**27.290**	4.213	21.053	37.892
	MSE	27.767	17.654	20.304	20.268	**15.268**	432.14	16.340	168.765
(10, 25, 10)	$\hat{\sigma}$	NAN	8.587	13.718	8.603	**11.177**	12.239	6.873	1.507
	MSE		52.124	141.74	52.263	**3.453**	8.032	11.582	73.601
(25, 25, 10)	$\hat{\sigma}$	7.251	7.784	9.649	7.951	**7.816**	13.367	7.390	1.654
	MSE	21.416	22.564	27.230	22.670	**19.561**	12.790	17.606	70.487
(50, 25, 10)	$\hat{\sigma}$	6.941	9.915	11.153	10.094	**8.475**	12.697	7.150	1.848
	MSE	17.354	13.288	18.132	13.751	**11.657**	7.945	8.526	66.917
(75, 25, 10)	$\hat{\sigma}$	6.935	9.207	9.991	9.398	**8.791**	12.984	7.257	1.842
	MSE	24.960	8.208	8.925	8.264	**9.549**	9.373	9.790	66.862
(100, 25, 10)	$\hat{\sigma}$	6.970	9.755	10.367	9.947	**8.363**	12.698	7.132	1.848
	MSE	13.718	7.583	8.631	7.832	**8.131**	7.606	8.736	66.734

The following observations and conclusions are made from an examination of the simulation results reported in Tables $6-8$. The new WSM method appears to be superior to existing substitution methods for all censoring cases, and yields quite similar estimates to EMA and $ASAMLEOC$ methods. The HDL and the new WSM methods perform similarly for cases with censoring levels less than 50%.

In summary, the maximum likelihood estimators (ASAMLEOC), the new weighted substitution method estimators (WSM), and the EM algorithm estimators (EMA)

TABLE 7. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.15, 0.50$: $(k, 10, 5)$, $(k = 10, 25, 50, 75, 100)$

		Methods Of Estimation							
			MLE			Replacement			
$(\mathbf{n}, \mu, \sigma)$		**EMA**	**ASAMLEOC**	**UMLE**	**HS**	**WSM**	**ZE**	**HDL**	**DL**
					$CL = 0.15$				
(10, 10, 5)	$\hat{\mu}$	10.078	10.103	9.888	9.928	**10.045**	9.405	9.976	10.547
	MSE	2.836	2.669	2.689	2.667	**2.437**	2.195	2.203	2.951
(25, 10, 5)	$\hat{\mu}$	10.046	10.047	10.010	9.980	**10.046**	9.626	10.043	10.460
	MSE	1.019	1.013	1.102	1.014	**1.011**	1.140	1.272	1.256
(50, 10, 5)	$\hat{\mu}$	9.977	9.964	9.945	9.894	**9.979**	9.580	9.978	10.380
	MSE	0.508	0.492	0.494	0.502	**0.498**	0.538	0.507	0.633
(75, 10, 5)	$\hat{\mu}$	9.959	9.939	9.925	9.868	**9.949**	9.585	9.973	10.362
	MSE	0.373	0.373	0.375	0.387	**0.371**	0.438	0.351	0.496
(100, 10, 5)	$\hat{\mu}$	9.993	9.999	9.991	9.934	**9.996**	9.643	10.213	10.382
	MSE	0.256	0.255	0.255	0.259	**0.254**	0.316	0.238	0.399
(10, 10, 5)	$\hat{\sigma}$	5.019	4.856	5.359	4.911	**5.027**	5.746	4.825	4.048
	MSE	1.524	1.703	2.178	1.723	**1.526**	1.727	1.852	2.194
(25, 10, 5)	$\hat{\sigma}$	4.917	4.886	5.073	4.929	**4.911**	5.520	4.819	4.230
	MSE	0.640	0.617	0.656	0.618	**0.596**	0.612	0.597	1.046
(50, 10, 5)	$\hat{\sigma}$	4.931	4.952	5.047	4.997	**4.956**	5.514	4.849	4.285
	MSE	0.337	0.328	0.341	0.332	**0.329**	0.398	0.321	0.755
(75, 10, 5)	$\hat{\sigma}$	5.015	5.045	5.111	5.092	**5.030**	5.553	4.913	4.366
	MSE	0.236	0.237	0.253	0.248	**0.234**	0.400	0.213	0.578
(100, 10, 5)	$\hat{\sigma}$	4.931	4.923	4.968	4.965	**4.927**	5.449	4.829	4.302
	MSE	0.120	0.161	0.159	0.159	**0.158**	0.267	0.175	0.605
					$CL = 0.50$				
(10, 10, 5)	$\hat{\mu}$	9.990	10.093	9.588	9.955	**10.061**	6.853	9.357	11.861
	MSE	3.691	3.433	3.878	3.485	**3.240**	10.724	2.066	6.368
(25, 10, 5)	$\hat{\mu}$	10.014	10.228	10.047	10.102	**10.121**	7.162	9.571	11.980
	MSE	1.548	1.533	1.531	1.528	**1.453**	8.392	1.183	5.131
(50, 10, 5)	$\hat{\mu}$	9.976	10.020	9.921	9.874	**9.983**	7.294	9.947	12.880
	MSE	0.754	0.711	0.728	0.743	**0.579**	5.436	0.441	4.692
(75, 10, 5)	$\hat{\mu}$	9.962	10.053	9.988	9.924	**10.007**	7.031	9.494	11.958
	MSE	0.571	0.522	0.525	0.538	**0.532**	8.934	0.490	4.252
(100, 10, 5)	$\hat{\mu}$	10.017	10.029	9.980	9.882	**10.022**	6.982	9.487	11.992
	MSE	0.329	0.339	0.340	0.359	**0.327**	9.196	0.432	4.265
(10, 10, 5)	$\hat{\sigma}$	4.626	4.464	5.287	4.522	**4.555**	7.115	4.736	2.590
	MSE	3.317	2.744	3.527	2.744	**2.785**	5.343	1.679	6.650
(25, 10, 5)	$\hat{\sigma}$	4.866	4.659	4.988	4.780	**4.763**	7.207	4.862	2.809
	MSE	1.316	1.280	1.333	1.272	**1.224**	5.217	1.097	5.230
(50, 10, 5)	$\hat{\sigma}$	4.975	4.934	5.114	4.992	**4.954**	7.294	4.947	2.880
	MSE	0.612	0.586	0.638	0.596	**0.579**	5.436	0.541	4.692
(75, 10, 5)	$\hat{\sigma}$	4.955	4.868	4.986	4.924	**4.917**	7.257	4.923	2.873
	MSE	0.431	0.427	0.430	0.425	**0.417**	5.217	0.302	4.666
(100, 10, 5)	$\hat{\sigma}$	4.936	4.924	5.024	4.983	**4.930**	7.290	4.924	2.873
	MSE	0.230	0.273	0.278	0.275	**0.265**	5.338	0.217	4.614

perform similarly, and all are generally superior to the existing substitution method estimators.

7. Conclusions and Recommendations

This article has dealt with the problem of estimating the mean and standard deviation of a normal and/or lognormal populations in the presence of left-censored data. To avoid clumping of replaced values in cases where there are several left-censored observations that share a common detection limit, a new replacement

TABLE 8. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.10, 0.90$: $(k, 20, 3)$, $(k = 10, 25, 50, 75, 100)$

		Methods Of Estimation							
			MLE			Replacement			
$(\mathbf{n}, \mu, \sigma)$		**EMA**	**ASAMLEOC**	**UMLE**	**HS**	**WSM**	**ZE**	**HDL**	**DL**
					$CL = 0.10$				
(10, 20, 3)	$\hat{\mu}$	19.984	20.026	19.982	20.004	**20.005**	18.482	19.323	20.164
	MSE	0.948	0.895	0.896	0.895	**0.915**	3.034	1.264	0.919
(25, 20, 3)	$\hat{\mu}$	19.962	19.948	19.929	19.914	**19.955**	18.151	19.137	20.123
	MSE	0.370	0.363	0.366	0.370	**0.365**	3.694	1.058	0.374
(50, 20, 3)	$\hat{\mu}$	19.973	19.980	19.973	19.952	**19.976**	18.496	19.309	20.122
	MSE	0.177	0.175	0.176	0.178	**0.176**	2.404	0.635	0.190
(75, 20, 3)	$\hat{\mu}$	19.900	19.983	19.977	19.952	**19.986**	18.405	19.271	20.137
	MSE	0.125	0.124	0.124	0.126	**0.124**	2.646	0.643	0.143
(100, 20, 3)	$\hat{\mu}$	19.990	19.992	19.989	19.964	**19.991**	18.513	19.324	19.991
	MSE	0.087	0.087	0.087	0.088	**0.087**	2.286	0.538	0.087
(10, 20, 3)	$\hat{\sigma}$	3.143	2.780	3.026	2.796	**3.068**	6.595	3.319	2.549
	MSE	0.537	0.554	0.600	0.552	**0.474**	13.047	1.948	0.629
(25, 20, 3)	$\hat{\sigma}$	2.988	2.967	3.075	2.992	**2.993**	7.090	4.644	2.666
	MSE	0.214	0.220	0.241	0.222	**0.211**	16.783	2.786	0.289
(50, 20, 3)	$\hat{\sigma}$	2.977	2.961	3.012	2.982	**2.970**	6.614	3.427	2.711
	MSE	0.104	0.102	104	0.102	**0.101**	13.083	2.080	0.168
(75, 20, 3)	$\hat{\sigma}$	2.988	2.999	3.035	3.022	**2.994**	6.789	4.526	2.728
	MSE	0.066	0.067	0.069	0.068	**0.065**	14.377	2.358	0.129
(100, 20, 3)	$\hat{\sigma}$	2.986	2.983	3.008	3.004	**2.985**	6.621	4.042	2.730
	MSE	0.053	0.052	0.053	0.052	**0.052**	13.129	2.103	0.116
					$CL = 0.90$				
(10, 20, 3)	$\hat{\mu}$	NAN	19.896	17.761	19.879	**18.866**	2.462	12.894	23.327
	MSE		11.399	32.365	11.512	**12.444**	307.61	51.034	12.850
(25, 20, 3)	$\hat{\mu}$	21.756	20.830	20.032	20.627	**21.385**	2.965	13.325	23.685
	MSE	4.333	4.200	5.098	4.317	**3.816**	290.20	44.812	14.411
(50, 20, 3)	$\hat{\mu}$	21.420	19.862	19.354	19.634	**20.631**	2.517	13.180	23.843
	MSE	2.985	3.364	4.479	3.796	**2.119**	305.67	46.661	15.274
(75, 20, 3)	$\hat{\mu}$	21.423	20.217	19.866	19.972	**20.716**	2.672	13.255	23.839
	MSE	2.733	1.646	1.881	1.787	**1.634**	300.27	45.576	15.018
(100, 20, 3)	$\hat{\mu}$	21.382	19.976	19.725	19.733	**20.678**	2.518	13.207	23.896
	MSE	2.393	1.468	1.696	1.691	**1.255**	305.62	46.219	15.425
(10, 20, 3)	$\hat{\sigma}$	NAN	2.609	4.167	2.613	**6.895**	7.387	3.909	0.432
	MSE		5.860	15.924	5.879	**6.173**	19.540	1.013	6.751
(25, 20, 3)	$\hat{\sigma}$	1.997	2.317	2.872	2.367	**2.320**	8.038	4.220	0.495
	MSE	2.061	2.027	2.411	2.034	**1.735**	25.496	1.546	6.353
(50, 20, 3)	$\hat{\sigma}$	2.128	3.002	3.377	3.057	**2.693**	7.560	4.012	0.561
	MSE	1.496	1.395	1.907	1.451	**1.086**	20.850	1.763	5.999
(75, 20, 3)	$\hat{\sigma}$	2.082	2.797	3.036	2.856	**2.444**	7.742	4.093	0.557
	MSE	1.404	0.811	0.908	0.824	**0.917**	22.529	1.220	5.999
(100, 20, 3)	$\hat{\sigma}$	2.120	2.954	3.139	3.011	**2.537**	7.563	4.008	0.559
	MSE	1.157	0.628	0.726	0.651	**0.665**	20.856	1.035	5.983

method called weighted substitution method is introduced. In this method left-censored observations are spaced from zero to the detection limit according to weights assigned to these non-detected data. To facilitate the application of estimation methods described in this article, a computer program is presented. The computer program "$SingleLeft.Censored.Normal$", written in the R language, is an easy-to-use computerized tool for obtaining estimates and their standard deviations of population parameters of singly left-censored data using either a normal or lognormal distribution. The simulation results presented in Tables 3-4 show that the new WSM and HDL methods perform similarly for cases where the censoring levels is less than 50%. The new WSM method perform better than EMA and $ASAMLEOC$ methods for cases where the censoring levels is less than 50%. For estimating the σ parameter the new WSM method perform better than the existing methods for cases where the censoring levels is greater than or equal to 75%. Taken together, the suggested new WSM method appear to work best for normally distributed censored samples, and lognormal versions of the estimator can be obtained simply by taking natural logarithm of the data and the detection limit.

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Appendix

The suggested weighted substitution method is based on replacing the left-censored observations that are less than the detection limit DL by non-constant different values based on assigning a different weight for each observation. Some of the choices of the weights that were examined are:

$$w1_j(=w_j) = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} \left(P(U \geq DL)\right)^{\ln(m+j-1)}, \qquad (3.1 \;\; given \;\; above)$$

$$w2_j = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} [P(U \geq DL)]$$

$$w3_j = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} \left(P(U \geq DL)\right)^{m+j-1},$$

$$w4_j = \left(\frac{(m+j-1)}{n}\right)^{(\frac{j}{j+1})} \left(P(U \leq DL)\right)^{\ln(m+j-1)},$$

$$w5_j = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} [P(U \leq DL)]^{(m+j-1)},$$

$$w6_j = \left(\frac{(m+j-1)}{n}\right) \left(P(U \geq DL)\right)^{\ln(m+j-1)},$$

$$w7_j = \left(\frac{(m+j-1)}{n}\right) \left(P(U \geq DL)\right),$$

$$for \;\; j = 1, 2, ..., m_c$$

where the probability $P(U \geq DL)$ is estimated from the sample data by:

$$P(\widehat{U \geq D}L) = 1 - \Phi\left(\frac{DL - \bar{x}_m}{s_m}\right)$$

An extensive simulation study was conducted on these weights in addition to other weights (not shown here). The simulation results indicate that the suggested weight in (3.1) leads to estimators that have the ability to recover the true mean and standard deviation as well as the existing methods such as maximum likelihood and EM algorithm estimators. More simulation results will be available in the web page of the author later on if needed.

TABLE 9. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.75, 0.90$: $(k, 25, 10)$, $(k = 10, 25, 50, 75, 100)$

		Methods Of Estimation							
$(\mathbf{n}, \mu, \sigma)$		MLE	$W1_j (= W_j)$	$W2_j$	$W3_j$	$W4_j$	$W5_j$	$W6_j$	$W7_j$
				$CL = 0.75$					
(10, 25, 10)	$\hat{\mu}$	26.820	24.234	23.252	19.329	11.108	10.741	21.515	22.402
	MSE	34.258	12.401	16.047	51.433	194.777	205.259	21.974	15.600
(25, 25, 10)	$\hat{\mu}$	26.063	23.465	20.810	13.064	10.320	10.235	19.899	22.433
	MSE	14.436	6.016	22.861	148.885	216.079	218.583	30.475	9.668
(50, 25, 10)	$\hat{\mu}$	25.493	24.206	19.842	10.692	9.706	9.689	19.339	21.544
	MSE	6.912	5.299	30.481	206.662	234.184	234.705	35.300	7.859
(75, 25, 10)	$\hat{\mu}$	25.148	24.521	19.296	9.883	9.493	9.487	18.920	21.625
	MSE	5.072	5.233	35.658	229.296	240.630	240.824	40.013	8.410
(100, 25, 10)	$\hat{\mu}$	24.980	24.113	18.714	9.571	9.416	9.413	18.997	22.970
	MSE	3.789	4.305	42.200	238.248	242.978	243.065	38.541	6.051
(10, 25, 10)	$\hat{\sigma}$	7.490	8.921	8.536	11.143	16.354	16.586	10.111	3.023
	MSE	22.999	3.664	4.216	6.978	44.455	47.441	2.020	51.492
(25, 25, 10)	$\hat{\sigma}$	8.782	10.168	9.342	14.902	16.560	16.612	11.266	10.379
	MSE	11.516	1.413	5.342	27.034	44.618	45.296	5.969	7.088
(50, 25, 10)	$\hat{\sigma}$	9.451	9.745	11.873	15.960	16.522	16.532	12.475	11.987
	MSE	5.080	0.576	3.077	36.846	43.411	43.540	5.094	4.726
(75, 25, 10)	$\hat{\sigma}$	9.726	9.912	11.267	16.245	16.463	16.467	11.611	11.945
	MSE	3.637	0.314	2.292	39.594	42.298	42.344	4.972	5.237
(100, 25, 10)	$\hat{\sigma}$	9.950	10.486	11.732	16.397	16.484	16.485	12.464	10.997
	MSE	2.791	1.468	4.555	41.318	42.428	42.448	3.102	2.250
				$CL = 0.90$					
(10, 25, 10)	$\hat{\mu}$	24.891	24.215	22.568	7.982	5.174	4.045	20.009	20.099
	MSE	108.321	99.875	114.827	387.340	393.543	439.447	121.432	132.093
(25, 25, 10)	$\hat{\mu}$	27.918	23.327	20.050	12.317	5.044	4.918	18.748	20.927
	MSE	44.648	42.724	45.861	191.703	398.446	403.496	48.107	41.091
(50, 25, 10)	$\hat{\mu}$	24.606	23.938	18.526	8.797	4.245	4.214	17.937	20.844
	MSE	31.013	29.925	52.094	289.135	430.836	432.141	59.028	31.088
(75, 25, 10)	$\hat{\mu}$	25.967	23.983	16.584	5.541	4.485	4.465	15.983	20.569
	MSE	17.901	16.502	78.491	382.210	420.910	421.758	88.275	21.601
(100, 25, 10)	$\hat{\mu}$	24.944	23.896	16.544	5.135	4.222	4.213	16.188	21.690
	MSE	17.655	16.520	78.921	398.059	431.778	432.136	84.747	20.197
(10, 25, 10)	$\hat{\sigma}$	8.587	9.071	8.672	12.014	13.322	13.366	14.071	13.510
	MSE	52.124	50.071	55.982	67.803	84.007	58.602	55.341	52.762
(25, 25, 10)	$\hat{\sigma}$	7.784	9.771	9.585	11.906	16.560	16.612	12.647	12.993
	MSE	22.567	15.847	24.087	16.094	44.618	45.296	25.442	23.087
(50, 25, 10)	$\hat{\sigma}$	9.915	9.964	10.730	12.510	12.687	12.997	11.604	12.106
	MSE	13.288	10.487	11.522	13.951	14.890	13.944	15.604	12.106
(75, 25, 10)	$\hat{\sigma}$	9.207	10.156	11.415	12.656	12.977	12.984	10.938	11.048
	MSE	8.208	5.371	7.468	9.784	9.332	9.373	8.034	7.997
(100, 25, 10)	$\hat{\sigma}$	9.755	10.143	11.544	12.423	12.696	12.699	11.479	11.029
	MSE	7.583	5.264	6.514	7.486	8.591	8.606	6.479	8.029

Table 10. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.15, 0.50$: $(k, 10, 5)$, $(k = 10, 25, 50, 75, 100)$

$(\mathbf{n}, \mu, \sigma)$		MLE	$W1_j (= W_j)$	$W2_j$	$W3_j$	$W4_j$	$W5_j$	$W6_j$	$W7_j$
					$CL = 0.15$				
(10, 10, 5)	$\hat{\mu}$	10.013	10.327	10.388	11.072	9.001	9.105	10.704	11.264
	MSE	2.669	2.617	2.988	2.899	3.120	3.195	3.560	3.626
(25, 10, 5)	$\hat{\mu}$	10.047	10.073	10.560	9.661	9.326	9.034	10.544	10.703
	MSE	1.013	1.008	1.788	1.843	1.848	1.901	1.196	1.934
(50, 10, 5)	$\hat{\mu}$	9.964	10.084	10.286	9.570	9.380	9.294	10.363	10.565
	MSE	0.492	0.490	0.553	0.804	0.638	0.701	0.781	0.739
(75, 10, 5)	$\hat{\mu}$	9.939	10.164	10.372	9.618	9.585	9.275	10.357	10.470
	MSE	0.373	0.367	0.426	0.621	0.438	0.509	0.470	0.478
(100, 10, 5)	$\hat{\mu}$	9.991	10.082	10.298	9.654	9.542	9.343	10.165	10.380
	MSE	0.255	0.270	0.334	0.375	0.416	0.493	0.273	0.326
(10, 10, 5)	$\hat{\sigma}$	4.856	4.609	4.241	4.065	5.974	6.746	4.221	4.244
	MSE	1.703	1.544	1.973	2.164	2.218	2.228	1.986	2.507
(25, 10, 5)	$\hat{\sigma}$	4.886	4.772	4.356	5.209	5.520	5.728	4.522	4.409
	MSE	0.617	0.690	0.842	0.973	0.908	0.937	0.690	0.798
(50, 10, 5)	$\hat{\sigma}$	4.952	4.885	4.404	5.456	5.513	5.743	4.270	4.417
	MSE	0.329	0.302	0.585	0.604	0.698	0.599	0.603	0.957
(75, 10, 5)	$\hat{\sigma}$	5.045	4.829	4.482	5.496	5.553	5.729	4.645	4.510
	MSE	0.237	0.293	0.434	0.450	0.500	0.564	0.375	0.609
(100, 10, 5)	$\hat{\sigma}$	4.923	4.772	4.412	5.431	5.449	5.793	4.589	4.430
	MSE	0.161	0.189	0.458	0.354	0.377	0.386	0.306	0.414
					$CL = 0.50$				
(10, 10, 5)	$\hat{\mu}$	10.093	10.114	10.490	9.245	6.897	6.853	9.706	10.655
	MSE	3.433	2.506	2.339	3.189	10.452	10.724	3.213	3.233
(25, 10, 5)	$\hat{\mu}$	10.228	9.975	10.593	7.873	7.169	7.162	9.560	10.445
	MSE	1.533	0.827	1.230	5.163	8.352	8.392	0.889	0.989
(50, 10, 5)	$\hat{\mu}$	10.020	9.969	10.493	7.200	6.980	6.979	9.620	10.464
	MSE	0.711	0.561	0.677	8.130	9.288	9.288	0.608	0.664
(75, 10, 5)	$\hat{\mu}$	10.054	9.779	10.573	7.078	7.031	7.957	9.483	10.950
	MSE	0.522	0.517	0.547	8.670	8.931	8.652	0.690	0.472
(100, 10, 5)	$\hat{\mu}$	10.029	9.827	10.469	6.996	6.982	6.982	9.465	10.482
	MSE	0.339	0.366	0.466	9.112	9.194	9.196	0.627	0.769
(10, 10, 5)	$\hat{\sigma}$	4.464	4.310	3.741	4.633	7.074	7.115	4.370	4.179
	MSE	2.744	1.864	3.279	2.321	5.166	5.343	2.190	2.320
(25, 10, 5)	$\hat{\sigma}$	4.659	4.599	3.968	6.502	7.200	7.207	4.705	10.534
	MSE	1.280	0.668	1.407	2.879	5.184	5.217	0.998	1.093
(50, 10, 5)	$\hat{\sigma}$	4.934	4.866	4.118	7.082	7.293	7.294	9.520	10.964
	MSE	0.586	0.267	0.933	4.565	5.565	5.436	0.703	0.864
(75, 10, 5)	$\hat{\sigma}$	4.868	4.868	4.116	7.211	7.257	7.946	4.658	4.242
	MSE	0.427	0.179	0.895	5.021	5.215	6.012	0.258	0.683
(100, 10, 5)	$\hat{\sigma}$	4.924	4.932	4.148	7.276	7.290	7.290	5.311	4.261
	MSE	0.273	0.118	0.800	5.275	5.337	5.338	0.216	0.617

TABLE 11. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.10, 0.90$: $(k, 20, 3)$, $(k = 10, 25, 50, 75, 100)$

$(\mathbf{n}, \mu, \sigma)$		Methods Of Estimation							
		MLE	$W1_j(= W_j)$	$W2_j$	$W3_j$	$W4_j$	$W5_j$	$W6_j$	$W7_j$
					$CL = 0.10$				
(10, 20, 3)	$\hat{\mu}$	20.026	20.008	19.881	19.067	18.488	18.482	19.752	19.375
	MSE	0.895	0.881	0.892	1.319	3.016	3.034	0.913	0.870
(25, 20, 3)	$\hat{\mu}$	19.948	19.937	19.762	18.863	18.151	18.151	19.474	19.683
	MSE	0.363	0.353	0.408	1.735	3.693	3.694	0.538	0.483
(50, 20, 3)	$\hat{\mu}$	19.980	19.984	19.799	18.736	18.496	17.968	19.672	19.754
	MSE	0.176	0.172	0.217	1.794	2.404	2.725	0.238	0.282
(75, 20, 3)	$\hat{\mu}$	19.983	19.990	19.781	18.531	18.405	17.998	19.667	19.873
	MSE	0.124	0.122	0.175	2.277	2.646	2.763	0.0.238	0.195
(100, 20, 3)	$\hat{\mu}$	19.992	19.999	19.783	18.559	18.513	18.092	19.761	19.072
	MSE	0.087	0.087	0.139	2.157	2.286	14.109	0.147	0.185
(10, 20, 3)	$\hat{\sigma}$	2.780	3.026	2.789	4.218	6.579	6.595	3.204	2.772
	MSE	0.554	0.432	0.455	2.324	12.932	13.047	0.543	0.493
(25, 20, 3)	$\hat{\sigma}$	2.967	2.953	3.274	5.311	7.090	7.390	3.367	3.245
	MSE	0.220	0.173	0.292	6.297	16.777	15.638	0.427	0.258
(50, 20, 3)	$\hat{\sigma}$	2.961	2.928	3.285	5.951	6.614	7.025	3.348	2.790
	MSE	0.102	0.081	0.187	9.016	13.083	12.573	0.218	0.276
(75, 20, 3)	$\hat{\sigma}$	2.999	2.960	3.359	6.447	6.789	6.993	3.408	3.209
	MSE	0.067	0.054	0.204	12.022	14.377	12.948	0.241	0.187
(100, 20, 3)	$\hat{\sigma}$	2.983	2.945	3.365	6.492	6.621	6.904	3.405	2.789
	MSE	0.052	0.045	0.194	12.238	13.129	12.839	0.225	0.098
					$CL = 0.90$				
(10, 20, 3)	$\hat{\mu}$	19.398	18.993	17.859	14.982	5.676	4.462	12.836	13.908
	MSE	15.410	16.107	17.703	30.444	282.441	307.606	51.858	47.054
(25, 20, 3)	$\hat{\mu}$	20.830	18.759	17.983	12.467	11.050	13.966	11.658	13.133
	MSE	4.200	6.295	8.054	77.596	83.965	92.837	72.274	57.946
(50, 20, 3)	$\hat{\mu}$	19.862	18.699	15.795	10.277	6.537	6.517	13.125	12.042
	MSE	3.364	3.109	5.895	8.973	12.948	56.666	10.102	41.033
(75, 20, 3)	$\hat{\mu}$	20.217	18.649	16.972	9.683	8.375	7.047	13.196	12.972
	MSE	1.646	2.017	3.896	11.874	13.874	15.266	11.551	16.801
(100, 20, 3)	$\hat{\mu}$	19.976	19.274	14.280	14.168	8.523	7.518	13.138	14.973
	MSE	3.706	4.003	8.604	16.173	21.403	23.619	33.287	49.818
(10, 20, 3)	$\hat{\sigma}$	2.609	3.546	4.013	5.546	7.470	7.387	4.975	4.998
	MSE	5.860	6.027	6.627	7.182	15.271	16.539	5.048	6.192
(25, 20, 3)	$\hat{\sigma}$	2.317	3.402	3.869	5.678	5.678	6.038	4.812	5.091
	MSE	2.027	2.377	3.094	4.289	7.286	11.494	3.750	10.700
(50, 20, 3)	$\hat{\sigma}$	2.936	3.078	4.948	5.826	6.553	6.560	5.329	4.958
	MSE	1.392	1.973	3.275	5.749	9.788	10.849	8.188	7.854
(75, 20, 3)	$\hat{\sigma}$	2.797	3.306	3.972	8.522	7.738	6.803	5.028	6.145
	MSE	2.811	2.913	4.870	10.611	17.492	15.529	10.722	9.321
(100, 20, 3)	$\hat{\sigma}$	2.594	3.514	6.014	7.378	7.562	7.563	6.235	6.663
	MSE	4.628	5.023	6.286	13.307	15.009	14.721	8.517	7.452

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Parameter estimation by anfis where dependent variable has outlier

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Abstract

Regression analysis is investigation the relation between dependent and independent variables. And, the degree and functional shape of this relation is determinate by regression analysis. In case that dependent variable has outlier, the robust regression methods are proposed to make smaller the effect of the outlier on the parameter estimates. In this study, an algorithm has been suggested to define the unknown parameters of regression model, which is based on ANFIS (Adaptive Network based Fuzzy Inference System). The proposed algorithm, expressed the relation between the dependent and independent variables by more than one model and the estimated values are obtained by connected this model via ANFIS. In the solving process, the proposed method is not to be affected the outliers which are to exist in dependent variable. So, to test the activity of the proposed algorithm, estimated values obtained from this algorithm and some robust methods are compared.

Keywords: Adaptive network, fuzzy inference, robust regression.

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1. Introduction

In a regression analysis, it is assumed that the observations come from a single class in a data cluster and the simple functional relationship between the dependent and independent variables can be expressed using the general model; $Y = f(X) + \varepsilon$. However; a data set may consist of a combination of observations that have different distributions that are derived from different clusters. When faced with issues of estimating a regression model for fuzzy inputs that have been derived from different distributions, this regression model has been termed the 'switching regression model' and it is expressed with $Y^L = f^L(X) + \varepsilon^L$ $(L = \prod_{i=1}^{p} l_i)$. Here l_i indicates the class number of each independent variable and p is indicative of the number of independent variables [18, 19, 21]. In case that, the class numbers of the data and the number of the independent variables are more than two, simultaneously the numbers of sub-models are increased. At this stage, the method attempts to utilize the neural networks, which are intended to solve complex problems and systems. When faced with issues in which the data belong to an indefinite or fuzzy class, the neural network, termed the adaptive network, is used for establishing the regression model. In this study, adaptive networks have been used to construct a model that has been formed by gathering obtained models. There are methods that suggest the class numbers of independent variables heuristically. Alternatively, in defining the optimal class number of independent variables, the use of suggested validity criterion for fuzzy clustering has been aimed. There are many studies on the use of the adaptive network for parameter estimation. In a study by Chi-Bin, C. and Lee, E. S. a fuzzy adaptive network approach was established for fuzzy regression analysis [4] and it was studied on both fuzzy adaptive networks and the switching regression model [5]. Jang, J. R. studied the adaptive networks based on a fuzzy inference system [16]. In a study of Takagi, T. and Sugeno, M., the method for identifying a system using it's input-output data was presented [23]. James, P. D. and Donalt, W., were studied fuzzy regression using neural networks [15]. In a study by Cichocki, A. and Unbehauen, R., the different neural networks for optimization were explained [2]. There are different studies about fuzzy clustering and the validity criterion. In the study of Mu-Song, C. and Wang, S.W. the analysis of fuzzy clustering was done for determining fuzzy memberships and in this study a method was suggested for indicating the optimal class numbers that belong to the variables [20]. Bezdek, J.C. has conducted important studies on the fuzzy clustering topic [1]. One such study is by Hathaway R.J. and Bezdek J.C. were studied on switching regression and fuzzy clustering [7]. In 1991, Xie, X.L. and Beni, G. suggested a validity criterion for fuzzy clustering [24]. In this study we used the Xie-Beni validity criterion for determining optimal class numbers. Over the years, the least squares method(LSM) has commonly been used for the estimation of regression parameters. If a data set conforms to LSM assumptions, LSM estimates are known to be the best. However, if outliers exist in the data set, the LSM can yield bad results. In the conventional approach, outliers are removed from the data set, after which the classical method can be applied. However, in some research, these observations are not removed from the data set. In such cases, robust methods are preferred to the LSM [17]. The remainder of the paper is organized as follows. Section 2 explores the fuzzy if-then rules and the use of these rules will be introduced using adaptive networks for analysis. In Section 3 an algorithm for parameter estimation based ANFIS is given.In Section 4, we provide definitions of M methods of Huber, Hampel, Andrews and Tukey, which are commonly used in the literature. In Section 5, a numerical application examining the work and validity of the suggested algorithm as well as a comparison of the algorithm with these robust methods and LSM is provided.In the last part, a discussion and conclusion are provided.

2. ANFIS: Adaptive Network based Fuzzy Inference System

The most popular application of fuzzy methodology is known as fuzzy inference systems. This system forms a useful computing framework based on the concepts of fuzzy set theory, fuzzy reasoning and fuzzy if-then rules. Fuzzy inference systems usually perform on input-output relation, as in control applications where the inputs correspond to system state variables, and the outputs are control signals [3, 5, 16]. The fuzzy inference system is a powerful function approximater. The basic structure of a fuzzy inference system consist of five conceptual components;a rule base which contains a selection of fuzzy rules, a database which defines the membership functions of the fuzzy sets used in the fuzzy rules, a decision-making unit which performs inference operations on the rules, a fuzzification interface which transforms the crisp inputs into degrees of match with linguistic values, and a defuzzification interface which transform the fuzzy results of the inference into a crisp output [3, 15, 16]. The adaptive network used to estimate the unknown parameters of regression model is based on fuzzy if-then rules and fuzzy inference system. When issues of estimating a regression model to fuzzy inputs from different distributions arose, the Sugeno Fuzzy Inference System is appropriate and the proposed fuzzy rule in this case is indicated as

$$R^L = If; \left(x_1 = F_1^L \text{ and } x_2 = F_2^L \text{ and } \quad \ldots \quad x_p = F_p^L\right).$$

Then; $Y = Y^L = c_0^L + c_1^L x_1 + ... + c_p^L x_p$.

Here, F_i^L stands for fuzzy cluster and Y^L stands for system output according to the R^L rule [16, 23].

The weighted mean of the models obtained according to fuzzy rules is the output of Sugeno Fuzzy Inference System and a common regression model for data from different classes is indicated with this weighted mean as follows,

$$\widehat{Y} = \frac{\sum\limits_{L=1}^{m} w^L Y^L}{\sum\limits_{L=1}^{m} w^L}.$$

Here; w^L weight is indicated as,

$$w^L = \prod_{i=1}^{p} \mu_{F_i^L(x_i)}.$$

$\mu_{F_i^L}(x_i)$ is a membership function defined on the fuzzy set F_i^L, and m is fuzzy rule number [13, 14].

Neural networks that enable the use of fuzzy inference systems for fuzzy regression analysis is known as adaptive network and called ANFIS. An adaptive network is a multilayer feed forward network in which each node performs a particular function on incoming signals as well as a set of parameters pertaining to this node. The formulas for the node functions may wary from node to node and the choice of each node function depends on the overall input-output function of the network. Neural networks are used to obtain a good approach to regression functions and were formed via neural and adaptive network connections consistingof five layers $[4, 12-14, 15]$.

Fuzzy rule number of the system depends on numbers of independent variables and fuzzy class number forming independent variables. When independent variable number is indicated with p and if the fuzzy class number associated with each variable is indicated by l_i $(i = 1, ..., p)$, the fuzzy rule number indicated by

$$L = \prod_{i=1}^{p} l_i.$$

To illustrate how a fuzzy inference system can be represented by ANFIS, let us consider the following example. Suppose a data set has two-dimensional input $X = (x_1, x_2)$. For input x_1, there are two fuzzy sets "tall" and "short" and for input x_2, three fuzzy set "thin", "normal" and "fat". In this case a fuzzy inference system contains the following six rules:

$$\begin{aligned}
R^1 &: If(x_1 \; is \; tall \; and \; x_2 \; is \; thin), then; (Y^1 = c_0^1 + c_1^1 x_1 + c_2^1 x_2),\\
R^2 &: If(x_1 \; is \; tall \; and \; x_2 \; is \; normal), then; (Y^2 = c_0^2 + c_1^2 x_1 + c_2^2 x_2),\\
R^3 &: If(x_1 \; is \; tall \; and \; x_2 \; is \; fat), then; (Y^3 = c_0^3 + c_1^3 x_1 + c_2^2 x_2),\\
R^4 &: If(x_1 \; is \; short \; and \; x_2 \; is \; thin), then; (Y^4 = c_0^4 + c_1^4 x_1 + c_2^4 x_2),\\
R^5 &: If(x_1 \; is \; short \; and \; x_2 \; is \; normal), then; (Y^5 = c_0^5 + c_1^5 x_1 + c_2^5 x_2),\\
R^6 &: If(x_1 \; is \; short \; and \; x_2 \; is \; fat), then; (Y^6 = c_0^6 + c_1^6 x_1 + c_2^6 x_2).
\end{aligned}$$

This fuzzy system is represented by the ANFIS as shown in Figure 1. The functions of each node in Figure 1 defined as follows.

Layer 1: The output of node h in this layer is defined by the membership function on F_h

$$f_{1,h} = \mu_{F_h}(x_1) \quad for \quad h = 1, 2$$
$$f_{1,h} = \mu_{F_h}(x_2) \quad for \quad h = 3, 4, 5$$

where fuzzy cluster related to fuzzy rules are indicated with $F_1, F_2, ..., F_h$ and μ_{F_h} is the membership function relates to F_h. Different membership functions are can be define for F_h. In this example, the Gaussian membership function will be used whose parameters can be represented by $\{v_h, \sigma_h\}$.

]

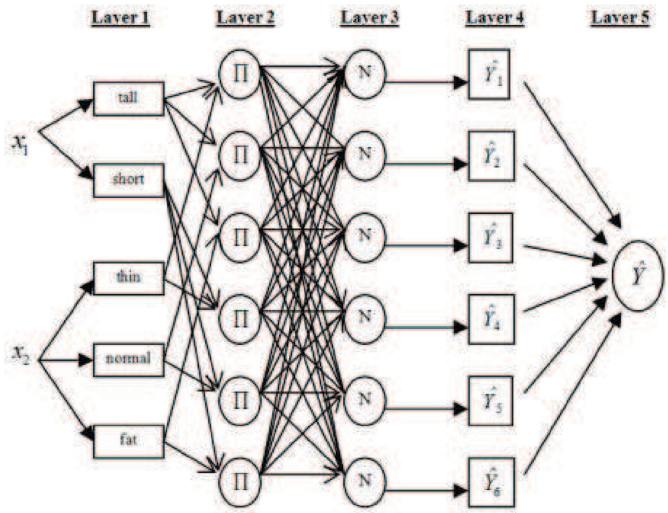

FIGURE 1. The ANFIS architecture

$$\mu_{F_h}(x_1) = \exp\left[-\left(\frac{x_1 - v_h}{\sigma_h}\right)^2\right] \quad for \quad h = 1, 2$$

$$\mu_{F_h}(x_2) = \exp\left[-\left(\frac{x_2 - v_h}{\sigma_h}\right)^2\right] \quad for \quad h = 3, 4, 5.$$

The parameter set $\{v_h, \sigma_h\}$ in this layer is referred to as the premise parameters.

Layer 2: Each nerve in the second layer has input signals coming from the first layer and they are defined as multiplication of these input signals. This multiplied output forms the firing strength w^l for rule l:

$$f_{2,1} = w^1 = \mu_{F_1}(x_1) \times \mu_{F_3}(x_2),$$
$$f_{2,2} = w^2 = \mu_{F_1}(x_1) \times \mu_{F_4}(x_2),$$
$$f_{2,3} = w^3 = \mu_{F_1}(x_1) \times \mu_{F_5}(x_2),$$
$$f_{2,4} = w^4 = \mu_{F_2}(x_1) \times \mu_{F_3}(x_2),$$
$$f_{2,5} = w^5 = \mu_{F_2}(x_1) \times \mu_{F_4}(x_2),$$
$$f_{2,6} = w^6 = \mu_{F_2}(x_1) \times \mu_{F_5}(x_2).$$

Layer 3: The output of this layer is a normalization of the outputs of the second layer and nerve function is defined as

$$f_{3,L} = \overline{w}^L = \frac{w^L}{\sum_{L=1}^{6} w^L}.$$

Layer 4: The output signals of the fourth layer are also connected to a function and this function is indicated with

$$f_{4,L} = \overline{w}^L Y^L$$

where, Y^L stands for conclusion part of fuzzy if-then rule and it is indicated with

$$Y^L = c_0^L + c_1^L x_1 + c_2^L x_2,$$

where c_i^L are fuzzy numbers and stands for posteriori parameters.

Layer 5: There is only one node which computes the overall output as the summation of all the incoming signals

$$f_{5,1} = \widehat{Y} = \sum_{L=1}^{6} \overline{w}^L Y^L.$$

3. An Algorithm for Parameter Estimation Based ANFIS

The estimation of parameters with an adaptive network is based on the principle of the minimizing of error criterion. There are two significant steps in the process of estimation. First, we must determine the a priori parameter set characterizing the class from which the data comes and then update these parameters within the process. The second step is to determine a posteriori parameters belonging to the regression models to be formed. The process of determining parameters for the switching regression model begins with determining class numbers of independent variables and a priori parameters [6]. The algorithm related to the proposed method for determining the switching regression model in the case of independent variables coming from a normal distribution is defined as follows.

Step 1: Optimal class numbers related to the data set associated with the independent variables are determined. Optimal value of class number l_i, ($l_i = 2$, $l_i = 3$ . . . $l_i =$ max) can be obtained by minimizing the fuzzy clustering validity function S_i. This function is expressed by

$$S_i = \frac{\frac{1}{n} \sum_{i=1}^{l_i} \sum_{j=1}^{n} (\ u_{ij})^m \left\| v_i - x_j \right\|^2}{\min_{i \neq j} \left\| v_i - v_j \right\|^2}.$$

As it can be seen in this statement, cluster centers, which are well-separated produce a high value of separation such that a smaller S_i value is obtained. When the lowest S_i value is observed, class number (l_i) with the lowest value is defined as an optimal class number.

Step 2: A priori parameters are determined. Spreading is determined intuitively according to the space in which input variables gain value and to the fuzzy class numbers of the variables. Center parameters are based on the space in which variables gain value and fuzzy class numbers and it is defined by

$$v_i = (\min X_i) + \frac{\max(X_i) - \min(X_i)}{l_i - 1}(i-1), \qquad i = 1, 2, ..., p.$$

Step 3: $\overline{w}^L$ weights are counted which are used to form matrix B to be used in counting the a posteriori parameter set. L is the fuzzy rule number. The $\overline{w}^L$ weights are outputs of the nerves in the third layer of the adaptive network, and they are counted based on a membership function related to the distribution family to which independent variable belongs. Nerve functions in the first layer of the adaptive network are defined by

$$f_{1,h} = \mu_{F_h}(x_i) \qquad h = 1, 2, ..., \sum_{i=1}^{p} l_i.$$

$\mu_{F_h}(x_i)$ is called the membership function. Here, when the normal distribution function which has the parameter set of $\{v_h, \sigma_h\}$ is considered, membership functions are defined as

$$\mu_{F_h}(x_i) = \exp\left[-\left(\frac{x_i - v_h}{\sigma_h}\right)^2\right].$$

From the defined membership functions, membership degrees related to each class forming independent variables are determined. The w^L weights are indicated as

$$w^L = \mu_{F_L}(x_i).\mu_{F_L}(x_j).$$

They are obtained via mutual multiplication of membership degrees at an amount depending on the number of independent variables and the fuzzy class numbers of these variables. $\overline{w}^L$ weight is a normalization of the weight defined as $\overline{w}^L$ and they are counted with

$$\overline{w}^L = \frac{w^L}{\sum\limits_{L=1}^{m} w^L}.$$

Step 4: On the condition that the independent variables are fuzzy and the dependent variables are crisp, a posteriori parameter set $c_i^L = (a_i^L, b_i^L)$ is obtained as crisp numbers in the shape of, $c_i^L = a_i^L$ $(i = 1, ..., p)$. In that condition, $Z = \left(B^T B\right)^{-1} B^T Y$ equation is used to determine the a posteriori parameter set. Here B is the data matrix which is weighted by membership degree and its dimension is $[(p+1) \times m \times n]$, Y dependent variable vector and Z is posterior parameter vector which is defined by

$$Z = \left[a_0^1, ..., a_0^m, a_1^1, ..., a_1^m, ..., a_p^1, ..., a_p^m\right]^T$$

Step 5: By using a posteriori parameter set $c_i^L = a_i^L$ obtained in Step 4, the regression model indicated by

$$Y^L = c_0^L + c_1^L x_1 + c_2^L x_2 + ... + c_p^L x_p$$

are constituted. Setting out from the models and weights specified in Step 1, the estimation values are obtained using

$$\widehat{Y} = \sum_{L=1}^{m} \overline{w}^L Y^L.$$

Step 6: The error related to model is counted as

$$\varepsilon_k = \sum_{k=1}^{n} \left(y_k - \widehat{y}_k\right)^2.$$

If $\varepsilon < \phi$, then the a posteriori parameters have been obtained as parameters of regression models to be formed, and the process is determinate. If $\varepsilon < \phi$, then, Step 6 begins. Here ϕ, is a law stable value determinated by the decision maker.
Step 7: Central a priori parameters specified in Step 2 are updated with

$$v_i^{'} = v_i \pm t$$

in a way that it increases from the lowest value to the highest and it decreases from the highest value to the lowest. Here, t is the size of the step;

$$t = \frac{\max\left(x_{ji}\right) - \min\left(x_{ji}\right)}{a} \qquad j = 1, 2, ..., n; \qquad i = 1, 2, ..., p$$

and a is a stable value, which is determinant by the size of the step, and is therefore an iteration number.

Step 8: Estimations for each a priori parameter obtained by change and the error criteria related to these estimations are counted. The lowest of the error criterion is defined. A priori parameters giving the lowest error specified, and the estimation obtained via the models related to these parameters is taken as output.

In the proposed algorithm, the estimated values which are obtained from the fuzzy adaptive network are not to be affected by the outliers that may exist in the dependent variable. This is because in this algorithm, all of the independent variables are weighted. Consequently, the proposed method has a robust method's properties, and, it is comparable to robust methods that are commonly used in literature.

4. M methods

The classical LSM is widely used in regression analysis because computing its estimate is easy and traditional. However, least square estimators are very sensitive to outliers and to deviations from basic assumptions of normal theory [11, 25]. The importance of eachobservation should therefore be recognized, and the data should be tested in detail whenit is analyzed. This is important because sometimes even a single observation can changethe value of the parameter estimates, and omitting this observation from the data maylead to totally different estimates. If there exist outliers in the data set, robust methods are preferred to estimate parameter values [22]. Now, we discuss the widely used methods of the Huber, Hampel, Andrews and Tukey M estimators. The M estimator utilizes minimizing of residual functions much more thanminimizing the sum of the squared residuals. Regression coefficients are obtained by theminimizing sum:

$$\sum_{i=1}^{n}\rho\left[\left(y_i-\sum_{j=1}^{p}x_{ij}\widehat{\beta}_j\right)\diagup d\right]. \tag{4.1}$$

By taking the first partial derivative of the sum in Equation (4.1) with respect to each $\widehat{\beta}_j$ and setting it to zero, it may be found regression coefficient that p equations:

$$\sum_{i=1}^{n}x_{ij}\Psi\left[\left(y_i-\sum_{j=1}^{p}x_{ij}\widehat{\beta}_j\right)\diagup d\right]=0\quad j=1,2,...,p$$

where $\Psi(z)=\rho^{'}(z)$. When the data contains outliers, standard deviations are not good measures of variability, and other robust measures of variability are therefore required. One robust measure of variability is d. In the case where r_i is the residual of i^{th} observation, $d=median\left|r_i-median\left(r_i\right)\right|\diagup 0.6745,\ i=1,2,...,n$. Therefore, the standardized residuals may be defined as $z=r_i\diagup d$. Inaddition $r_i=y_i-\sum\limits_{j=1}^{p}x_{ij}\widehat{\beta}_j$.

Huber's Ψ function is defined as:

$$\Psi\left(z\right)=\begin{cases}-k & z<-k\\ z & |z|\le k\\ k & z>k\end{cases}$$

with k=1.5.

The Hampel Ψ function is defined as:

$$\Psi(z)=\begin{cases}|z| & 0<|z|\le a\\ a\,sgn(z) & a<|z|\le b\\ a\left(\frac{c-|z|}{c-b}\right)sgn(z) & b<|z|\le c\\ 0 & c<|z|\end{cases}\qquad sgn(z)=\begin{cases}+1 & z>0\\ 0 & z=0\\ -1 & z<0\end{cases}$$

Reasonably good values of the constants are $a=1.7$, $b=3.4$ and $c=8.5$.

Andrews (sine estimate) Ψ function is defined as

$$\Psi\left(z\right)=\begin{cases}\sin\left(z\diagup k\right) & |z|\le k\pi\\ 0 & |z|>k\pi\end{cases}$$

with k =1.5 or $k=2.1$.

The Tukey (biweight estimate) Ψ function is defined as:

$$\Psi\left(z\right)=\begin{cases}\left(z\left(1-\left(z\diagup k\right)^2\right)^2\right) & |z|\le k\\ 0 & |z|>k\end{cases}$$

with k $=5.0$ or 6.0 $[8-11]$.

5. Numerical Example

The values related to the data set having three independent variables and one dependent variable is shown in Table 1. The values in the data set have been generated from normal distribution such that $X_1 \sim (\mu = 20; \sigma = 3)$, $X_2 \sim (\mu = 50; \sigma = 12)$, $X_3 \sim (\mu = 32; \sigma = 13)$, and dependent variable Y is depend on independent variables value. 5^{th} observation of the dependent variable is changed with $(y_{15} + 50)$ to work up this observation into outlier. The regression models and estimations for this model are obtained via the proposed algorithm for this data set. Moreover, estimations have been obtained using the robust regression methods are used for comparison. The proposed algorithm was executed with a program written in MATLAB. From the initial step of the proposed algorithm, fuzzy class numbers for each variable are defined as two. Number of fuzzy inference rules to be formed depending on these class numbers is obtained as

$$L = \prod_{i=1}^{p=3} l_i = l_1 \times l_2 \times l_3 = 8.$$

TABLE 1. Data set having three independent variables and one dependent variable

No	X_1	X_2	X_3	Y	No	X_1	X_2	X_3	Y
1	21.8101	50.5397	49.8319	125.4057	**16**	25.2815	50.2143	54.2714	128.9194
2	19.8248	78.9993	35.1925	137.4526	**17**	20.2663	30.6749	40.9755	60.9541
3	16.6740	46.2813	33.5450	97.0719	**18**	27.7867	64.8650	33.4679	130.0444
4	26.4327	52.2510	37.0010	116.3851	**19**	17.9736	58.2030	17.8756	87.9184
5	15.9415	61.3724	31.0880	107.0015	**20**	28.3604	40.6314	11.7420	78.4568
6	21.3711	43.6916	24.4820	92.0244	**21**	19.9495	56.3718	40.2863	116.6304
7	21.1735	36.6127	38.1010	100.6000	**22**	20.8150	75.6140	26.7405	118.1328
8	26.2190	30.8922	48.8959	90.8950	**23**	17.2577	54.2523	26.7568	103.9698
9	19.0300	64.0981	53.2524	136.5460	**24**	14.1459	52.7804	33.0930	101.7193
10	24.4044	55.8217	22.8635	104.7410	**25**	19.0477	65.4558	26.3405	113.7832
11	18.4928	69.7458	42.4943	133.6250	**26**	21.7650	49.8381	24.6859	93.9008
12	20.6288	44.5492	18.6431	83.1755	**27**	22.4870	33.9999	43.4148	101.4928
13	22.2644	62.1052	48.8284	134.8870	**28**	14.9754	43.3239	21.4096	85.7995
14	17.1554	74.5928	32.1941	126.0083	**29**	14.2331	59.0672	28.6413	105.1197
15	21.8395	57.2242	34.8432	**166.4707**	**30**	18.6900	39.0578	38.4129	99.5382

Models obtained via eight fuzzy inference rules are;

$$\begin{aligned}
\widehat{y}_1 &= 1308 + 346x_1 - 84x_2 - 314x_3 \\
\widehat{y}_2 &= 10896 - 145x_1 + 175x_2 - 230x_3 \\
\widehat{y}_3 &= 9022 - 211x_1 - 126x_2 + 263x_3 \\
\widehat{y}_4 &= -27061 - 24x_1 + 202x_2 + 207x_3 \\
\widehat{y}_5 &= -20670 + 701x_1 - 51x_2 + 436x_3 \\
\widehat{y}_6 &= -6201 - 405x_1 - 155x_2 + 341x_3 \\
\widehat{y}_7 &= 18219 - 610x_1 + 19x_2 - 316x_3 \\
\widehat{y}_8 &= 25742 + 283x_1 - 204x_2 - 283x_3 \qquad (5.1)
\end{aligned}$$

Regression model estimates, which are obtained from robust regression methods and the LSM, are located in Table 2.

Table 2. The estimation of regression parameters

	Constant	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_3$
LMS	−10.4360	1.0404	1.2420	0.9412
Huber	3.0366	0.8125	1.0329	1.0085
Hampel	5.5338	0.7794	0.9778	1.0412
Tukey	5.3224	0.8127	0.9625	1.0563
Andrews	5.2896	0.7775	0.9809	1.0430

The weights related to the observations that are used in estimation methods for regression models, are located in Table 3. The weights for robust methods are expression of that observation's effect on one model for each of the outlier observations of the robust method. On the other hand, weight obtained from the network is an expression of that observation's effect on more than one model, which are expressed in Equation (5.1). For this reason, eight different weights, which are called membership degrees of observation, are located in Table 3.

Table 3. The weight related to observation for all methods

No	**LMS**	**Huber**	**Hampel**	**Tukey**	**Andrews**	The membership degrees of the observation to belong to the models in Equation (5.1)							
						w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
1	1	1	1	0.9892	0.4710	0.2558	0.8399	0.1927	0.6327	0.2563	0.8714	0.1931	0.6341
2	1	1	1	0.9274	0.4632	0.1331	0.1553	0.6541	0.7634	0.1323	0.1544	0.6504	0.7590
3	1	1	1	0.9704	0.4713	0.4513	0.4688	0.2568	0.2667	0.4431	0.4603	0.2521	0.2619
4	1	1	1	0.9996	0.4752	0.2955	0.3919	0.2492	0.3305	0.3017	0.4001	0.2544	0.3374
5	1	1	1	0.9068	0.4545	0.2619	0.2287	0.4029	0.3518	0.2564	0.2239	0.3944	0.3444
6	1	1	1	0.9812	0.4702	0.9279	0.5080	0.4451	0.2437	0.9283	0.5082	0.4453	0.2438
7	1	0.9901	1	0.9383	0.4559	0.6285	0.9008	0.1891	0.2710	0.6283	0.9005	0.1890	0.2709
8	1	0.2780	0.1984	0	0.1023	0.1777	0.5464	0.0367	0.1128	0.1813	0.5573	0.0374	0.1150
9	1	1	1	0.9555	0.4685	0.1053	0.4404	0.1939	0.8109	0.1044	0.4365	0.1922	0.8037
10	1	1	1	0.9729	0.4692	0.5702	0.2784	0.6084	0.2971	0.5774	0.2820	0.6161	0.3009
11	1	1	1	0.9853	0.4732	0.1575	0.3079	0.4207	0.8225	0.1557	0.3045	0.4160	0.8134
12	1	1	1	0.9794	0.4728	0.9492	0.3440	0.4818	0.1746	0.9468	0.3431	0.4806	0.1742
13	1	1	1	0.9999	0.4758	0.1794	0.5488	0.2896	0.8859	0.1801	0.5510	0.2908	0.8895
14	1	1	1	0.9917	0.4754	0.1503	0.1419	0.5525	0.5216	0.1478	0.1396	0.5435	0.5132
15	1	0.0793	0	0	0	0.4992	0.5684	0.5843	0.6652	0.5004	0.5697	0.5856	0.6668
16	1	1	1	0.9371	0.4691	0.1247	0.5603	0.0919	0.4131	0.1267	0.5694	0.0934	0.4198
17	1	0.1292	0	0	0	0.5070	0.8905	0.1032	0.1812	0.5050	0.8869	0.1027	0.1804
18	1	1	1	0.8298	0.4345	0.1445	0.1493	0.2798	0.2891	0.1483	0.1532	0.2872	0.2967
19	1	0.5182	0.7229	0.6013	0.3816	0.5293	0.1817	0.6608	0.2268	0.5224	0.1793	0.6521	0.2238
20	1	1	1	0.9810	0.4742	0.3005	0.0669	0.1178	0.0262	0.3091	0.0688	0.1212	0.0270
21	1	1	1	0.9724	0.4718	0.3892	0.6510	0.4306	0.7203	0.3872	0.6476	0.4284	0.7165
22	1	0.5921	0.9567	0.7679	0.4197	0.2313	0.1485	0.9096	0.5841	0.2309	0.1483	0.9080	0.5831
23	1	1	1	0.8455	0.4425	0.5032	0.3235	0.4841	0.3113	0.4952	0.3184	0.4764	0.3063
24	1	1	1	0.9931	0.4747	0.2068	0.2081	0.1806	0.1817	0.2010	0.2022	0.1755	0.1766
25	1	1	1	0.9569	0.4682	0.4008	0.2502	0.8070	0.5038	0.3973	0.2480	0.7999	0.4994
26	1	1	1	0.9086	0.4589	0.8261	0.4589	0.5943	0.3301	0.8278	0.4598	0.5955	0.3307
27	1	1	1	0.9955	0.4762	0.4479	0.9437	0.1135	0.2367	0.4501	0.9393	0.1140	0.2379
28	1	0.9584	1	0.8559	0.4441	0.4073	0.1794	0.1907	0.0840	0.3971	0.1750	0.1860	0.0819
29	1	1	1	0.9882	0.4744	0.1948	0.1431	0.2575	0.1891	0.1894	0.1392	0.2504	0.1839
30	1	1	1	0.9931	0.4726	0.5378	0.7880	0.1901	0.2785	0.5323	0.7799	0.1882	0.2757

The residuals, which belong to estimates fromregression models in Equation (5.1) and belong to estimates for models from robust regression methods, are located in Table 4. The proposed algorithm was executed with a program written in MATLAB. In the stage of step operating, data sets have one dependent variables and this variable has an outlier observation.

TABLE 4. The residuals belong to observations for all methods

No	LMS Residual	Huber Residual	Hampel Residual	Tukey Residual	Andrews Residual	ANFIS Residual
1	3.4767	2.1904	1.5721	1.0752	1.6086	-17.6759
2	-3.9789	1.2178	2.5822	2.8061	2.5519	-7.8254
3	1.1051	-1.1466	-1.6369	-1.7818	-1.5673	-1.5879
4	-0.4020	0.5861	0.6348	0.2043	0.6978	-11.5606
5	-4.6338	-3.7318	-3.3337	-3.1872	-3.3084	-2.26604
6	2.9175	1.8044	1.6230	1.4196	1.7260	1.1031
7	7.6721	4.1178	3.0940	2.5835	3.1943	-7.6847
8	-10.3378	-14.6644	-16.1899	-17.1185	-16.0816	-28.9371
9	-2.5497	-1.8646	-1.9390	-2.1887	-1.9565	2.7615
10	-1.0641	1.1593	1.8002	1.7052	1.8737	-6.1937
11	-1.8004	0.6667	1.2379	1.2547	1.2210	-1.0513
12	-0.7286	-1.4388	-1.4064	-1.4838	-1.2970	1.2672
13	-0.9346	0.3685	0.4359	0.1152	0.4387	-13.3710
14	-4.3509	-0.4822	0.6489	0.9398	0.6329	-4.3802
15	50.3167	51.4430	51.6845	51.5151	51.7272	-10.2161
16	-0.3957	-1.2575	-1.9241	-2.6085	-1.8880	-29.6824
17	-26.3607	-31.5567	-33.0319	-33.6463	-32.9199	-9.3339
18	-0.4926	3.6793	4.5840	4.3540	4.6162	-11.3668
19	-9.4589	-7.8677	-7.1450	-6.9146	-7.0821	-5.1477
20	-2.1298	-1.4330	-1.1352	-1.4250	-0.9864	10.9992
21	-1.6218	-1.4706	-1.5166	-1.7185	-1.4846	-7.6727
22	-12.1691	-6.8861	-5.3992	-5.1320	-5.4017	-11.5501
23	3.8849	3.8895	4.0801	4.1399	4.1380	-3.2610
24	0.7363	-0.7023	-0.9030	-0.8580	-0.8578	0.0676
25	-1.6870	1.0962	1.9775	2.1547	2.0043	-7.9256
26	-3.4418	-3.1937	-3.0297	-3.1557	-2.9457	-2.9142
27	5.4420	1.2833	-0.0150	-0.6893	0.0864	-16.3709
28	6.6952	4.2542	3.9416	3.9915	4.0392	0.8268
29	0.4278	0.6231	0.9175	1.1227	0.9512	-1.3372
30	5.8638	2.2337	1.2525	0.8570	1.3398	-5.0809
Sum of Square Residual	3867.3	4087.8	4228.2	4278.2	4222.3	3580.9
Mean	128.9112	136.2609	140.9394	142.6050	140.7425	119.3650

The defined methods M (Huber, Hampel, Tukey, Andrews) were executed with programs written in MATLAB.The residuals of from the robust methods and LSM are large, but the residuals from the proposed algorithm based network are small. This is because, this method depend on fuzzy clustering.

As it can be seen in a numerical example, error related to estimations obtained via the network according to error criterion is lower than errors obtained via all the other methods.

6. Conclusion

In the study, we have proposed a method for obtaining optimal estimation values and compared various methods. Estimation values, which are obtained from the

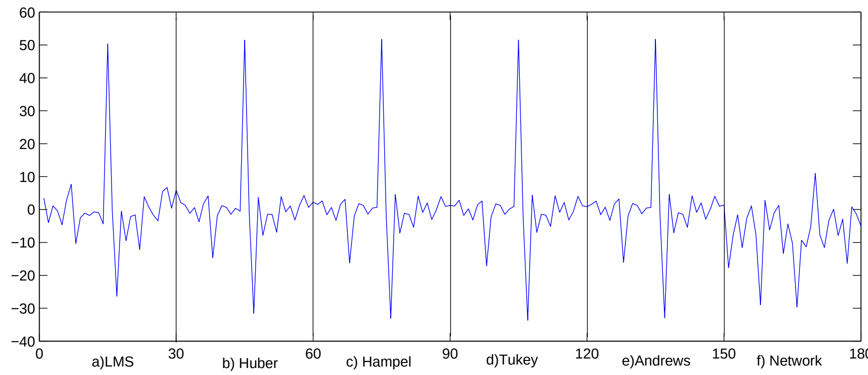

Figure 2. Graphs for errors related to data set in Table 1

proposed algorithm, have the lowest error values. Recently, in our field as well as others, adaptive networks that fall under the heading of neural networks and yield efficient estimations related to data are being used more frequently. In the proposed algorithms, the fuzzy class number of the independent variable is defined intuitively at first, and within the on going process, these class numbers are taken as the basis. In this study, it has been thought to use validity criterion based on fuzzy clustering at the stage of defining level numbers of independent variables. Moreover, as it can be observed in the algorithm in Section 3, an algorithm different from other proposed algorithms has been used for updating central parameters. The difference between the obtained estimation values and the observed values, that is, the network that decreases the errors to the minimum level, is formed based on the adaptive network architecture that includes a fuzzy inference system based on the fuzzy rules. The process followed in the proposed method can be accepted as an ascendant from other methods since it does not allow intuitional estimations and it brings us to the smallest error. At the same time, this method is robust, since it is not affected by the contradictory observations that can occur at dependent variables. Finally, the estimation values obtained from the networks that are formed through the proposed algorithm are compared with the estimation values obtained from the robust regression methods. According to the indicated error criterion, the errors related to the estimations that are obtained from the network are lower than the errors that are obtained from the robust regression methods and LSM.The figures of errors obtained from the six methods are given in Figure 2. Figure 2(a) shows the errors related to the estimations that are obtained from the LSM, (b,c,d,e) are show the errors related to the estimations that are obtained from M Methods, and (f) shows the errors related to the estimations that are obtained from the proposed algorithm based ANFIS.

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Complete qth moment convergence of weighted sums for arrays of row-wise extended negatively dependent random variables

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Abstract

In this paper, the complete qth moment convergence of weighted sums for arrays of row-wise extended negatively dependent (abbreviated to END in the following) random variables is investigated. By using Hoffmann-Jϕrgensen type inequality and truncation method, some general results concerning complete qth moment convergence of weighted sums for arrays of row-wise END random variables are obtained. As their applications, we extend the corresponding result of Wu (2012) to the case of arrays of row-wise END random variables. The complete qth moment convergence of moving average processes based on a sequence of END random variables is obtained, which improves the result of Li and Zhang (2004). Moreover, the Baum-Katz type result for arrays of row-wise END random variables is also obtained.

Keywords: END random variables; Weighted sums; Complete moment convergence; Complete convergence.

2000 AMS Classification: 60F15

1. Introduction and Lemmas

The concept of complete convergence was given by Hsu and Robbins[1] in the following way. A sequence of random variables $\{X_n, n \geq 1\}$ is said to converge completely to a constant θ if for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \epsilon) < \infty.$$

In view of the Borel-Cantelli lemma, the above result implies that $X_n \to \theta$ almost surely. Hence the complete convergence is very important tool in establishing almost sure convergence. When $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables, Baum and Katz[2] proved the following

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remarkable result concerning the convergence rate of the tail probabilities $P(|S_n| > \epsilon n^{1/p})$ for any $\epsilon > 0$, where $S_n = \sum_{i=1}^n X_i$.

1.1. Theorem. *$\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables, $r > 1/2$ and $p > 1$. Then*

$$\sum_{n=1}^{\infty} n^{p-2} P(|S_n| > \epsilon n^r) < \infty \quad for \quad all \quad \epsilon > 0,$$

if and only if $E|X|^{p/r} < \infty$, where $EX = 0$ whenever $1/2 < r \leq 1$.

Many useful linear statistics based on a random sample are weighted sums of independent and identically distributed random variables, see, for example, least-squares estimators, nonparametric regression function estimators and jackknife estimates, among others. However, in many stochastic model, the assumption that random variables are independent is not plausible. Increases in some random variables are often related to decreases in other random variables, so an assumption of dependence is more appropriate than an assumption of independence. The concept of END random variables was firstly introduced by Liu[3] as follows.

1.2. Definition. Random variables $\{X_i, i \geq 1\}$ are said to be END if there exists a constant $M > 0$ such that both

$$P\left(\bigcap_{i=1}^{n}(X_i \leq x_i)\right) \leq M \prod_{i=1}^{n} P(X_i \leq x_i) \tag{1.1}$$

and

$$P\left(\bigcap_{i=1}^{n}(X_i > x_i)\right) \leq M \prod_{i=1}^{n} P(X_i > x_i) \tag{1.2}$$

hold for each $n \geq 1$ and all real numbers $x_1, x_2, \cdots, x_n$.

In the case $M = 1$ the notion of END random variables reduces to the well-known notion of so-called negatively dependent (ND) random variables which was introduced by Lehmann[4]. Recall that random variables $\{X_i, i \geq 1\}$ are said to be positively dependent (PD) if the inequalities (1.1) and (1.2) hold both in the reverse direction when $M = 1$. Not looking that the notion of END random variables seems to be a straightforward generalization of the notion of ND, the END structure is substantially more comprehensive. As it is mentioned in Liu[3], the END structure can reflect not only a negative dependent structure but also a positive one, to some extend. Joag-Dev and Proschan[5] also pointed out that negatively associated (NA) random variables must be ND, therefore NA random variables are also END. Some applications for sequences of END random variables have been found. We refer to Shen[6] for the probability inequalities, Liu[3] for the precise large deviations, Chen[7] for the strong law of large numbers and applications to risk theory and renewal theory.

Recently, Baek et al.[8] discussed the complete convergence of weighted sums for arrays of row-wise NA random variables and obtained the following result:

1.3. Theorem. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of row-wise* NA *random variables with $EX_{ni} = 0$ and for some random variable X and constant $C > 0$,*

$P(|X_{ni}| > x) \leq CP(|X| > x)$ for all $i \geq 1, n \geq 1$ and $x \geq 0$. Suppose that $\beta \geq -1$, and that $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of constants such that

$$\sup_{i\geq 1} |a_{ni}| = O(n^{-r}) \quad for \ \ some \ \ r > 0 \tag{1.3}$$

and

$$\sum_{i=1}^{\infty} |a_{ni}| = O(n^{\alpha}) \quad for \ \ some \ \ \alpha \in [0, r). \tag{1.4}$$

(i) *If $\alpha + \beta + 1 > 0$ and there exists some $\delta > 0$ such that $\frac{\alpha}{r} + 1 < \delta \leq 2$, and $s = \max(1 + \frac{\alpha + \beta + 1}{r}, \delta)$, then, under $E|X|^s < \infty$, we have*

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right| > \epsilon\right) < \infty \ \ for \ \ all \ \ \epsilon > 0. \tag{1.5}$$

(ii) *If $\alpha + \beta + 1 = 0$, then, under $E|X|\log(1 + |X|) < \infty$, (1.5) remains true.*

If $\beta < -1$, then (1.5) is immediate. Hence Theorem 1.3 is of interest only for $\beta \geq -1$. Baek and Park [9] extended Theorem 1.3 to the case of arrays of row-wise pairwise negatively quadrant dependent (NQD) random variables. However, there is a question in the proofs of Theorem 1.3(i) in Baek and Park [9]. The Rosenthal type inequality plays a key role in this proof, but it is still an open problem to obtain Rosenthal type inequality for pairwise NQD random variables.

When $\beta > -1$, Wu [10] dealt with more general weight and proved the following complete convergence for weighted sums of arrays of row-wise ND random variables. But, the proof of Wu[10] does not work for the case of $\beta = -1$.

1.4. Theorem. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of row-wise* ND *random variables and for some random variable X and constant $C > 0$, $P(|X_{ni}| > x) \leq CP(|X| > x)$ for all $i \geq 1, n \geq 1$ and $x \geq 0$. Let $\beta > -1$ and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of constants satisfying (1.3) and*

$$\sum_{i=1}^{\infty} |a_{ni}|^{\theta} = O(n^{\alpha}) \quad for \ \ some \ \ 0 < \theta < 2 \ \ and \ some \ \ \alpha \ \ such \ that \ \ \theta + \alpha/r < 2. \tag{1.6}$$

Denote $s = \theta + (\alpha + \beta + 1)/r$. When $s \geq 1$, further assume that $EX_{ni} = 0$ for any $i \geq 1, n \geq 1$.

(i) *If $\alpha + \beta + 1 > 0$ and $E|X|^s < \infty$, then (1.5) holds.*

(ii) *If $\alpha + \beta + 1 = 0$ and $E|X|^{\theta} \log(1 + |X|) < \infty$, then (1.5) holds.*

The concept of complete moment convergence was introduced firstly by Chow [11]. As we know, the complete moment convergence implies complete convergence. Morover, the complete moment convergence can more exactly describe the convergence rate of a sequence of random variables than the complete convergence. So, a study on complete moment convergence is of interest. Liang et al. [12] obtained the complete qth moment convergence theorems of sequences of identically distributed NA random variables. Sung [13] proposed sets of sufficient conditions for complete qth moment convergence of arrays of random variables satisfying Marcinkiewicz-Zygmund and Rosenthal type inequalities. Guo [14] provided some sufficient conditions for complete moment convergence of row-wise NA arrays of random variables. Li and Zhang [15] established the complete moment convergence of moving average processes based on a sequence of identically distributed NA random variables as follows.

1.5. Theorem. *Suppose that* $Y_n = \sum_{i=-\infty}^{\infty} a_{i+n} X_i,\ n \geq 1,$ *where* $\{a_i, -\infty < i < \infty\}$ *is a sequence of real numbers with* $\sum_{-\infty}^{\infty} |a_i| < \infty$ *and* $\{X_i, -\infty < i < \infty\}$ *is a sequence of identically distributed and negatively associated random variables with* $EX_1 = 0, EX_1^2 < \infty$. *Let* $1/2 < r \leq 1, p \geq 1 + 1/(2r)$. *Then* $E|X_1|^p < \infty$ *implies that*

$$\sum_{n=1}^{\infty} n^{rp-2-r} l(n) E \left(\left| \sum_{i=1}^{n} Y_i \right| - \epsilon n^r \right)^+ < \infty \ \ for \ \ all \ \ \epsilon > 0.$$

The aim of this paper is to give a sufficient condition concerning complete qth moment convergence for arrays of row-wise END random variables. As an application, we not only generalize and extend the corresponding results of Baek et al. [8] and Wu [10] under some weaker conditions, but also greatly simplify their proof. Moreover, the complete qth moment convergence of moving average processes based on a sequence of END random variables is also obtained, which improves the result of Li and Zhang [15]. The Baum-Katz type result for arrays of row-wise END random variables is also established.

Before we start our main results, we firstly state some definitions and lemmas which will be useful in the proofs of our main results. Throughout this paper, the symbol C stands for a generic positive constant which may differ from one place to another. The symbol $I(A)$ denotes the indicator function of A. Let $a_n \ll b_n$ denote that there exists a constant $C > 0$ such that $a_n \leq C b_n$ for all $n \geq 1$. Denote $(x)_+^q = (\max(x, 0))^q$, $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$, $\log x = \ln \max(\mathrm{e}, x)$.

1.6. Definition. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C, such that $P(|X_n| > x) \leq CP(|X| > x)$ for all $x \geq 0$ and $n \geq 1$.

The following lemma establish the fundamental inequalities for stochastic domination, the proof is due to Wu [16].

1.7. Lemma. *Let the sequence* $\{X_n, n \geq 1\}$ *of random variables be stochastically dominated by a random variable* X. *Then for any* $n \geq 1, p > 0, x > 0$, *the following two statements hold:*

$$E|X_n|^p I(|X_n| \leq x) \leq C \left(E|X|^p I(|X| \leq x) + x^p P(|X| > x) \right),$$
$$E|X_n|^p I(|X_n| > x) \leq CE|X|^p I(|X| > x).$$

The following lemma is the Hoffmann-Jϕrgensen type inequality for sequences of END random variables and is obtained by Shen [6].

1.8. Lemma. *Let* $\{X_i, i \geq 1\}$ *be a sequence of* END *random variables with* $EX_i = 0$ *and* $EX_i^2 < \infty$ *for every* $i \geq 1$ *and set* $B_n = \sum_{i=1}^{n} EX_i^2$ *for any* $n \geq 1$. *Then for all* $y > 0, t > 0, n \geq 1$,

$$P\left(\left| \sum_{i=1}^{n} X_i \right| \geq y \right) \leq P\left(\max_{1 \leq k \leq n} |X_k| > t \right) + 2M \cdot \exp\left\{ \frac{y}{t} - \frac{y}{t} \log\left(1 + \frac{yt}{B_n} \right) \right\}.$$

1.9. Definition. A real-valued function $l(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be slowly varying if $\lim_{x\to\infty} \dfrac{l(x\lambda)}{l(x)} = 1$ for each $\lambda > 0$.

1.10. Lemma. *Let X be a random variable and $l(x) > 0$ be a slowly varying function. Then*

(i) $\sum_{n=1}^{\infty} n^{-1} E|X|^{\alpha} I(|X| > n^{\gamma}) \leq CE|X|^{\alpha} \log(1 + |X|)$ *for any* $\alpha \geq 0, \gamma > 0$,

(ii) $\sum_{n=1}^{\infty} n^{\beta} l(n) E|X|^{\alpha} I(|X| > n^{\gamma}) \leq CE|X|^{\alpha+(\beta+1)/\gamma} l(|X|^{1/\gamma})$ *for any* $\beta > -1$, $\alpha \geq 0, \gamma > 0$,

(iii) $\sum_{n=1}^{\infty} n^{\beta} l(n) E|X|^{\alpha} I(|X| \leq n^{\gamma}) \leq CE|X|^{\alpha+(\beta+1)/\gamma} l(|X|^{1/\gamma})$ *for any* $\beta < -1$, $\alpha \geq 0, \gamma > 0$.

Proof. We only prove (ii). Noting that $\beta > -1$, we have by Lemma 1.5 in Guo[14] that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta} l(n) E|X|^{\alpha} I(|X| > n^{\gamma}) = \sum_{n=1}^{\infty} n^{\beta} l(n) \sum_{k=n}^{\infty} E|X|^{\alpha} I(k^{\gamma} < |X| \leq (k+1)^{\gamma}) \\
=&\sum_{k=1}^{\infty} E|X|^{\alpha} I(k^{\gamma} < |X| \leq (k+1)^{\gamma}) \sum_{n=1}^{k} n^{\beta} l(n) \\
\leq& C \sum_{k=1}^{\infty} k^{\beta+1} l(k) E|X|^{\alpha} I(k^{\gamma} < |X| \leq (k+1)^{\gamma}) \leq CE|X|^{\alpha+(\beta+1)/\gamma} l(|X|^{1/\gamma}).
\end{aligned}$$

□

2. Main Results and the Proofs

In this section, let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of row-wise END random variables with the same M in each row. Let $\{k_n, n \geq 1\}$ be a sequence of positive integers and $\{a_n, n \geq 1\}$ be a sequence of positive constants. If $k_n = \infty$ we will assume that the series $\sum_{i=1}^{\infty} X_{ni}$ converges a.s. For any $x \geq 1, q > 0$, set

$$X'_{ni}(x) = x^{1/q} I(X_{ni} > x^{1/q}) + X_{ni} I(|X_{ni}| \leq x^{1/q}) - x^{1/q} I(X_{ni} < -x^{1/q}),$$

$1 \leq i \leq k_n, n \geq 1$. For any $x \geq 1, q > 0$, it is clear that $\{X'_{ni}(x), 1 \leq i \leq k_n, n \geq 1\}$ is an array of row-wise END random variables, since it is a sequence of monotone transformations of $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$.

2.1. Theorem. *Suppose that $q > 0$ and the following three conditions hold:*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} E|X_{ni}|^q I(|X_{ni}| > \epsilon) < \infty$ *for all* $\epsilon > 0$,

(ii) *there exist* $0 < r \leq 2$ *and* $s > q/r$ *such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E|X_{ni}|^r \right)^s < \infty,$$

(iii) $\sup_{x\geq 1} x^{-1/q} \sum_{i=1}^{k_n} |EX'_{ni}(x)| \to 0,$ *as* $n \to \infty$. *Then for all* $\epsilon > 0$,

$$\sum_{n=1}^{\infty} a_n E\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| - \epsilon\right)_+^q < \infty. \tag{2.1}$$

Proof. By Fubini's theorem, we get that

$$\begin{aligned}&\sum_{n=1}^{\infty} a_n E\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| - \epsilon\right)_+^q = \sum_{n=1}^{\infty} a_n \int_0^{\infty} P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \epsilon + x^{1/q}\right) \mathrm{d}x \\ \leq &\sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \epsilon\right) + \sum_{n=1}^{\infty} a_n \int_1^{\infty} P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > x^{1/q}\right) \mathrm{d}x =: I_1 + I_2.\end{aligned}$$

We prove only $I_2 < \infty$, the proof of $I_1 < \infty$ is analogous. Using a simple integral and Fubini's theorem, we obtain that for any $q > 0$ and a random variable X,

$$\int_1^{\infty} P(|X| > x^{1/q})\mathrm{d}x \leq E|X|^q I(|X| > 1). \tag{2.2}$$

Then by (2.2) and the subadditivity of probability measure we obtain the estimate

$$\begin{aligned}I_2 &\leq \sum_{n=1}^{\infty} a_n \int_1^{\infty} P\left(\left|\sum_{i=1}^{k_n} X'_{ni}(x)\right| > x^{1/q}\right) \mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_1^{\infty} \sum_{i=1}^{k_n} P\left(|X_{ni}| > x^{1/q}\right) \mathrm{d}x \\ &\leq \sum_{n=1}^{\infty} a_n \int_1^{\infty} P\left(\left|\sum_{i=1}^{k_n} X'_{ni}(x)\right| > x^{1/q}\right) \mathrm{d}x + \sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} E|X_{ni}|^q I(|X_{ni}| > 1) \\ &=: I_3 + I_4.\end{aligned}$$

By assumption (i), we have $I_4 < \infty$. By assumption (iii), we deduce that

$$I_3 \ll \sum_{n=1}^{\infty} a_n \int_1^{\infty} P\left(\left|\sum_{i=1}^{k_n} (X'_{ni}(x) - EX'_{ni}(x))\right| > x^{1/q}/2\right) \mathrm{d}x. \tag{2.3}$$

Set $B_n = \sum_{i=1}^{k_n} E(X'_{ni}(x) - EX'_{ni}(x))^2$, $y = x^{1/q}/2, t = x^{1/q}/(2s)$, we have by assumption (iii) and Lemma 1.8 that

$$\begin{aligned}&P\left(\left|\sum_{i=1}^{k_n} (X'_{ni}(x) - EX'_{ni}(x))\right| > x^{1/q}/2\right) \\ \leq &P\left(\max_{1\leq i\leq k_n} |X'_{ni}(x) - EX'_{ni}(x)| > x^{1/q}/(2s)\right) + 2Me^s \cdot \left(1 + \frac{x^{2/q}}{4sB_n}\right)^{-s} \\ \leq &P\left(\max_{1\leq i\leq k_n} |X'_{ni}(x)| > x^{1/q}/(4s)\right) + 2Me^s (4s)^s x^{-2s/q} B_n^s \qquad (2.4) \\ \leq &\sum_{i=1}^{k_n} P\left(|X'_{ni}(x)| > x^{1/q}/(4s)\right) + 2Me^s (4s)^s x^{-2s/q} \left(\sum_{i=1}^{k_n} E(X'_{ni}(x))^2\right)^s \\ \ll &\sum_{i=1}^{k_n} P\left(|X'_{ni}(x)| > x^{1/q}/(4s)\right) + x^{-2s/q} \left(\sum_{i=1}^{k_n} E(X'_{ni}(x))^2\right)^s.\end{aligned}$$

By (2.3) and (2.4), we obtain that

$$I_3 \ll \sum_{n=1}^{\infty} a_n \int_1^{\infty} \sum_{i=1}^{k_n} P\left(|X'_{ni}(x)| > x^{1/q}/(4s)\right) \mathrm{d}x \tag{2.1}$$
$$+ \sum_{n=1}^{\infty} a_n \int_1^{\infty} x^{-2s/q} \left(\sum_{i=1}^{k_n} E(X'_{ni}(x))^2\right)^s \mathrm{d}x$$
$$= I_4 + I_5.$$

Since $|X'_{ni}(x)| \le |X_{ni}|$, we have $P\left(|X'_{ni}(x)| > x^{1/q}/(4s)\right) \le P\left(|X_{ni}| > x^{1/q}/(4s)\right)$. By (2.2) and assumption (i), we conclude that

$$I_4 \le \sum_{n=1}^{\infty} a_n \int_1^{\infty} \sum_{i=1}^{k_n} P\left(|X_{ni}| > x^{1/q}/(4s)\right) \mathrm{d}x$$
$$\le \sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} (4s)^q E|X_{ni}|^q I(|X_{ni}| > 1/(4s)) < \infty.$$

Hence, to complete the proof, it suffices to show that $I_5 < \infty$. From the definition of $X'_{ni}(x)$, since $0 < r \le 2$, we have by C_r-inequality that
(2.5)

$$E(X'_{ni}(x))^2 \ll EX_{ni}^2 I(|X_{ni}| \le x^{1/q}) + x^{2/q} P(|X_{ni}| > x^{1/q}) \le 2x^{(2-r)/q} E|X_{ni}|^r.$$

Noting that $s > q/r$, it is clear that $\int_1^{\infty} x^{-sr/q}\mathrm{d}x < \infty$. Then we have by (2.5) and assumption (ii) that

$$I_5 \ll \sum_{n=1}^{\infty} a_n \int_1^{\infty} x^{-2s/q} \left(\sum_{i=1}^{k_n} x^{(2-r)/q} E|X_{ni}|^r\right)^s \mathrm{d}x$$
$$\le \sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E|X_{ni}|^r\right)^s \int_1^{\infty} x^{-sr/q}\mathrm{d}x \ll \sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E|X_{ni}|^r\right)^s < \infty.$$

Therefore, (2.1) holds. □

2.2. Remark. Note that

$$\sum_{n=1}^{\infty} a_n E\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| - \epsilon\right)_+^q = \int_0^{\infty} \sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \epsilon + x^{1/q}\right) \mathrm{d}x.$$

Thus, we obtain that the complete qth moment convergence implies the complete convergence, i.e., (2.1) implies

$$\sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \epsilon\right) < \infty \ for\ all\ \epsilon > 0.$$

2.3. Theorem. *Suppose that $\beta > -1$, $p > 0, q > 0$. Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of row-wise END random variables which are stochastically dominated by* *a random variable X. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of constants satisfying (1.3) and*

$$\sum_{i=1}^{\infty} |a_{ni}|^t \ll n^{-1-\beta+r(p-t)} \text{ for some } 0 < t < p. \tag{2.6}$$

Furthermore, assume that

$$\sum_{i=1}^{\infty} a_{ni}^2 \ll n^{-\mu} \text{ for some } \mu > 0 \tag{2.7}$$

if $p \geq 2$. Assume further that $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$ when $p \geq 1$. Then

$$\begin{cases} E|X|^q < \infty, & if \;\; q > p, \\ E|X|^p \log(1 + |X|) < \infty, & if \;\; q = p, \\ E|X|^p < \infty, & if \;\; q < p, \end{cases} \tag{2.8}$$

implies

$$\sum_{n=1}^{\infty} n^{\beta} E\left(\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right| - \epsilon\right)_+^q < \infty \;\; for \;\; all \;\; \epsilon > 0. \tag{2.9}$$

Proof. We will apply Theorem 2.1 with $a_n = n^{\beta},\ k_n = \infty$ and $\{X_{ni}, i \geq 1, n \geq 1\}$ replaced by $\{a_{ni}X_{ni}, i \geq 1, n \geq 1\}$. Without loss of generality, we can assume that $a_{ni} > 0$ for all $i \geq 1, n \geq 1$(otherwise, we use a_{ni}^+ and a_{ni}^- instead of a_{ni}, respectively, and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$). From (1.3) and (2.6), we can assume that

$$\sup_{i \geq 1} |a_{ni}| \leq n^{-r}, \;\; \sum_{i=1}^{\infty} |a_{ni}|^t \leq n^{-1-\beta+r(p-t)}. \tag{2.10}$$

Hence for any $q \geq t$, we obtain by (2.10) that

$$\sum_{i=1}^{\infty} |a_{ni}|^q = \sum_{i=1}^{\infty} |a_{ni}|^t |a_{ni}|^{q-t} \leq n^{-r(q-t)} \sum_{i=1}^{\infty} |a_{ni}|^t \leq n^{-1-\beta+r(p-q)}. \tag{2.11}$$

For all $\epsilon > 0$, we have by (1.3), (2.8), (2.11), Lemma 1.7 and Lemma 1.10 that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^q I(|a_{ni}X_{ni}| > \epsilon) \\
\ll &\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} |a_{ni}|^q E|X|^q I(|X| > \epsilon n^r) \\
\leq &\sum_{n=1}^{\infty} n^{-1+r(p-q)} E|X|^q I(|X| > \epsilon n^r) \\
\leq &\begin{cases} \sum_{n=1}^{\infty} n^{-1+r(p-q)} E|X|^q, & if\ q > p, \\ \sum_{n=1}^{\infty} n^{-1} E|X|^p I(|X| > \epsilon n^r), & if\ q = p, \end{cases} \\
\ll &\begin{cases} \sum_{n=1}^{\infty} n^{-1+r(p-q)}, & if\ q > p, \\ E|X|^p \log(1+|X|), & if\ q = p, \end{cases} \\
< &\infty.
\end{aligned} \tag{2.12}
$$

When $q < p$, taking q' such that $\max(q, t) < q' < p$, we have by (1.3), (2.8), (2.11), Lemma 1.7 and Lemma 1.10 that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^q I(|a_{ni}X_{ni}| > \epsilon) \\
\leq &\epsilon^{q-q'} \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^{q'} I(|a_{ni}X_{ni}| > \epsilon) \\
\ll &\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} |a_{ni}|^{q'} E|X|^{q'} I(|X| > \epsilon n^r) \\
\leq &\sum_{n=1}^{\infty} n^{-1+r(p-q')} E|X|^{q'} I(|X| > \epsilon n^r) \\
\ll &E|X|^p < \infty.
\end{aligned} \tag{2.13}
$$

It is obvious that (2.8) implies $E|X|^p < \infty$. When $p \geq 2$, It is clear that $EX^2 < \infty$. Noting that $\mu > 0$, we can choose sufficiently large s such that $\beta - \mu s < -1$ and $s > q/2$. Then, by Lemma 1.7, (2.7) and $EX^2 < \infty$ we get that

$$
\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{\infty} E a_{ni}^2 X_{ni}^2 \right)^s \ll \sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{\infty} a_{ni}^2 \right)^s \ll \sum_{n=1}^{\infty} n^{\beta - \mu s} < \infty. \tag{2.14}
$$

When $p < 2$, since $\beta > -1$, we can choose sufficiently large s such that $\beta + s(-1-\beta) < -1$ and $s > q/p$, we have by (2.11), $E|X|^p < \infty$ and Lemma 1.7 that

(2.15)
$$\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^p\right)^s \ll \sum_{n=1}^{\infty} n^{\beta}\left(\sum_{i=1}^{\infty} |a_{ni}|^p\right)^s \leq \sum_{n=1}^{\infty} n^{\beta+s(-1-\beta)} < \infty.$$

When $p < 1$, combining (2.11), $E|X|^p < \infty$, $\beta > -1$, C_r-inequality and Lemma 1.7, we obtain that

$$\begin{aligned}
&\sup_{x\geq 1} x^{-1/q}\sum_{i=1}^{\infty}|EX'_{ni}(x)| \leq \\
&\sum_{i=1}^{\infty} P(|a_{ni}X_{ni}| > 1) + \sup_{x\geq 1} x^{-1/q}\sum_{i=1}^{\infty} E\,|a_{ni}X_{ni}|\, I(|a_{ni}X_{ni}| \leq x^{1/q}) \\
&\leq \sum_{i=1}^{\infty} P(|a_{ni}X_{ni}| > 1) + \sup_{x\geq 1} x^{-p/q}\sum_{i=1}^{\infty} E\,|a_{ni}X_i|^p\, I(|a_{ni}X_{ni}| \leq x^{1/q}) \qquad (2.16)\\
&\leq 2\sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^p \ll \sum_{i=1}^{\infty}|a_{ni}|^p \\
&\leq n^{-1-\beta} \to 0,\ \text{as } n\to\infty.
\end{aligned}$$

When $p \geq 1$, since $EX_{ni} = 0$, we get that

$$Ea_{ni}X_{ni}I(|a_{ni}X_{ni}| \leq x^{1/q}) = -Ea_{ni}X_{ni}I(|a_{ni}X_{ni}| > x^{1/q}).$$

Thus, we have by $E|X|^p < \infty$, $\beta > -1$, C_r-inequality and Lemma 1.7 that

(2.17)
$$\begin{aligned}
&\sup_{x\geq 1} x^{-1/q}\sum_{i=1}^{\infty}|EX'_{ni}(x)| \\
&\leq \sum_{i=1}^{\infty} P(|a_{ni}X_{ni}| > 1) + \sup_{x\geq 1} x^{-1/q}\sum_{i=1}^{\infty}\left|Ea_{ni}X_{ni}I(|a_{ni}X_{ni}| > x^{1/q})\right| \\
&\leq 2\sum_{i=1}^{\infty} E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}| > 1) \ll \sum_{i=1}^{\infty}|a_{ni}|^p E|X|^p I(|X| > n^r) \ll \sum_{i=1}^{\infty}|a_{ni}|^p \\
&\leq n^{-1-\beta} \to 0,\ \text{as } n\to\infty.
\end{aligned}$$

Thus, by (2.12)–(2.17), we see that assumptions (i), (ii) and (iii) in Theorem 2.1 are fulfilled. Therefore (2.9) holds by Theorem 2.1. □

2.4. Remark. When $1+\alpha+\beta > 0$, the conditions (1.3), (2.6) and (2.7) are weaker than the conditions (1.3) and (1.6). In fact, taking $t = \theta,\ p = \theta + (1+\alpha+\beta)/r$, we immediately get (2.6) by (1.6). Noting that $\theta < 2$, we obtain by (1.3) and (1.6) that

$$\sum_{i=1}^{\infty} a_{ni}^2 \leq \sup_{i\geq 1}|a_{ni}|^{2-\theta}\sum_{i=1}^{\infty}|a_{ni}|^{\theta} \ll n^{-(r(2-\theta)-\alpha)}.$$

Since $\theta < 2 - \alpha/r$, we have $\mu =: r(2-\theta) - \alpha > 0$. Therefore (2.7) holds. So, Theorem 2.3 not only extends the result of Wu [10] for ND random variables to

END case, but also obtains the weaker sufficient condition of complete qth moment convergence of weighted sums for arrays of row-wise END random variables. It is worthy to point out that the method used in this article is novel, which differs from that of Wu [10]. Our method greatly simplify the proof of Wu [10].

Note that conditions (1.3) and (2.6) together imply

$$\sum_{i=1}^{\infty} |a_{ni}|^p \ll n^{-1-\beta}. \tag{2.18}$$

From the proof of Theorem 2.3, we can easily see that if $q > 0$ of Theorem 2.3 is replaced by $q \geq p$, then condition (2.6) can be replaced by the weaker condition (2.18).

2.5. Theorem. *Suppose that $\beta > -1$, $p > 0$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of row-wise END random variables which are stochastically dominated by a random variable X. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of constants satisfying (1.3) and (2.18). Furthermore, assume that (2.7) holds if $p \geq 2$. Assume further that $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$ when $p \geq 1$. Then*

$$\begin{cases} E|X|^q < \infty, & if \;\; q > p, \\ E|X|^p \log(1 + |X|) < \infty, & if \;\; q = p, \end{cases} \tag{2.19}$$

implies that (2.9) holds.

2.6. Remark. As in Remark 2.4 , when $1 + \alpha + \beta = 0$, the conditions (1.3), (2.7) and (2.18) are weaker than the conditions (1.3) and (1.6).

Take $q < p$ in Theorem 2.3 and $q = p$ in Theorem 2.5, by Remark 2.2 we can immediately obtain the following corollary:

2.7. Corollary. *Suppose that $\beta > -1$, $p > 0$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of row-wise END random variables which are stochastically dominated by a random variable X. Assume further that $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$ when $p \geq 1$. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of constants satisfying (1.3), (2.7) and*

$$\sum_{i=1}^{\infty} |a_{ni}|^t \ll n^{-1-\beta+r(p-t)} \; for\ some\ 0 < t \leq p. \tag{2.20}$$

(i) *If $t < p$, then $E|X|^p < \infty$ implies (1.5).*
(ii) *If $t = p$, then $E|X|^p \log(1 + |X|) < \infty$ implies (1.5).*

The following corollary establish complete qth moment convergence for moving average processes under a sequence of END non-identically distributed random variables, which extends the corresponding results of Li and Zhang [15] to the case of sequences of END non-identically distributed random variables. Moreover, our result covers the case of $r > 1$, which was not considered by Li and Zhang [15].

2.8. Corollary. *Suppose that $Y_n = \sum_{i=-\infty}^{\infty} a_{i+n} X_i$, $n \geq 1$, where $\{a_i, -\infty < i < \infty\}$ is a sequence of real numbers with $\sum_{-\infty}^{\infty} |a_i| < \infty$ and $\{X_i, -\infty < i < \infty\}$ is a sequence of END random variables with mean zero which are stochastically* *dominated by a random variable X. Let $r > 1/2,\ p \geq 1 + 1/(2r),\ q > 0$. Then*

$$(2.21)\qquad \begin{cases} E|X|^q < \infty, & if\ \ q > p, \\ E|X|^p \log(1 + |X|) < \infty, & if\ \ q = p, \\ E|X|^p < \infty, & if\ \ q < p, \end{cases}$$

implies that

$$(2.22)\qquad \sum_{n=1}^{\infty} n^{rp-2} E\left(\left|n^{-r}\sum_{i=1}^{n} Y_i\right| - \epsilon\right)_+^q < \infty,\ \ for\ \ all\ \ \epsilon > 0.$$

Proof. Note that

$$n^{-r}\sum_{i=1}^{n} Y_i = \sum_{i=-\infty}^{\infty}\left(n^{-r}\sum_{j=1}^{n} a_{i+j}\right) X_i.$$

We will apply Theorem 2.3 with $\beta = rp - 2,\ t = 1,\ a_{ni} = n^{-r}\sum_{j=1}^{n} a_{i+j}$ and $\{X_{ni}, i \geq 1, n \geq 1\}$ replaced by $\{X_i, -\infty < i < \infty\}$. Noting that $\sum_{-\infty}^{\infty} |a_i| < \infty$, $r > 1/2$ and $p \geq 1 + 1/(2r)$, we can easily see that the conditions (1.3) and (1.6) hold for $\theta = 1,\ \alpha = 1 - r$. Therefore (2.22) holds by (2.21), Theorem 2.3 and Remark 2.2. □

Similar to the proof of Corollary 2.8, we can get the following Baum-Katz type result for arrays of row-wise END random variables as follows.

2.9. Corollary. *Let $\{X_{ni}, 1 \leq i \leq n,\ n \geq 1\}$ be an array of row-wise END random variables which are stochastically dominated by a random variable X. Let $r > 1/2,\ p > 1,\ q > 0$. Assume further that $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$ when $p \geq r$. Then*

$$\begin{cases} E|X|^q < \infty, & if\ \ q > p/r, \\ E|X|^{p/r} \log(1 + |X|) < \infty, & if\ \ q = p/r, \\ E|X|^{p/r} < \infty, & if\ \ q < p/r, \end{cases}$$

implies that

$$\sum_{n=1}^{\infty} n^{p-2-rq} E\left(\left|\sum_{i=1}^{n} X_{ni}\right| - \epsilon n^r\right)_+^q < \infty,\ \ for\ \ all\ \ \epsilon > 0.$$

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A new calibration estimator in stratified double sampling

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Abstract

In the present article, we consider a new calibration estimator of the population mean in the stratified double sampling. We get more efficient calibration estimator using new calibration weights compared to the straight estimator. In addition, the estimators derived are analyzed for different populations by a simulation study. The simulation study shows that new calibration estimator is highly efficient than the existing estimator.

Keywords: Calibration, Auxiliary information, Stratified double sampling.

2000 AMS Classification: 62D05

1. Introduction

When the auxiliary information is available, the calibration estimator is widely used in the sampling literature to improve the estimates. Many authors, such as Deville and Sarndal [2], Estevao and Sarndal [3], Arnab and Singh [1], Farrell and Singh [4], Kim et al.[5], Kim and Park [6], Koyuncu and Kadilar etc.[8], defined some calibration estimators using different constraints. In the stratified random sampling, calibration approach is used to get optimum strata weights. Tracy et al.[9] defined calibration estimators in the stratified random sampling and stratified double sampling. In this study, we try to improve the calibration estimator in the stratified double sampling.

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2. Notations

Consider a finite population of N units consists of L strata such that the hth stratum consists of N_h units and $\sum_{h=1}^{L} N_h = N$. From the hth stratum of N_h units, draw a preliminary large sample of m_h units by the simple random sampling without replacement (SRSWOR) and measure the auxiliary character,x_{hi}, only. Select a sub-sample of n_h units from the given preliminary large sample of m_h units by SRSWOR and measure both the study variable, y_{hi} and auxiliary variable, x_{hi}. Let $\overline{x}_h^* = \frac{1}{m_h}\sum_{i=1}^{m_h} x_{hi}$ and $s_{hx}^{*2} = \frac{1}{m_h-1}\sum_{i=1}^{m_h}(x_{hi} - \overline{x}_h^*)^2$ denote the first phase sample mean and variance, respectively. Besides, assume that $\overline{x}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} x_{hi}$, $s_{hx}^2 = \frac{1}{n_h-1}\sum_{i=1}^{n_h}(x_{hi} - \overline{x}_h)^2$ and $\overline{y}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} y_{hi}$, $s_{hy}^2 = \frac{1}{n_h-1}\sum_{i=1}^{n_h}(y_{hi} - \overline{y}_h)^2$ denote the second phase sample means and variances for the auxiliary and study characters, respectively.

Calibration estimator, defined by Tracy et al.[9], is given by

$$\overline{y}_{st}(d) = \sum_{h=1}^{L} W_h^* \overline{y}_h, \tag{2.1}$$

where W_h^* are calibration weights minimizing the chi-square distance measure

$$\sum_{h=1}^{L} \frac{(W_h^* - W_h)^2}{Q_h W_h} \tag{2.2}$$

subject to calibration constraints defined by

$$\sum_{h=1}^{L} W_h^* \overline{x}_h = \sum_{h=1}^{L} W_h \overline{x}_h^*, \tag{2.3}$$

$$\sum_{h=1}^{L} W_h^* s_{hx}^2 = \sum_{h=1}^{L} W_h s_{hx}^{*2}. \tag{2.4}$$

The Lagrange function using calibration constraints and chi-square distance measure is given by

$$\triangle = \sum_{h=1}^{L} \frac{(W_h^* - W_h)^2}{Q_h W_h} - 2\lambda_1\left(\sum_{h=1}^{L} W_h^* \overline{x}_h - \sum_{h=1}^{L} W_h \overline{x}_h^*\right) - 2\lambda_2\left(\sum_{h=1}^{L} W_h^* s_{hx}^2 - \sum_{h=1}^{L} W_h s_{hx}^{*2}\right), \tag{2.5}$$

where λ_1 and λ_2 are Lagrange multipliers. Setting the derivative of Δ with respect to W_h^* equals to zero gives

(2.6) $W_h^* = W_h + Q_h W_h(\lambda_1 \overline{x}_h + \lambda_2 s_{hx}^2).$

Substituting (2.6) in (2.3) and (2.4) respectively, we get

$$\begin{bmatrix} \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2\right) & \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2\right) \\ \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2\right) & \left(\sum_{h=1}^{L} Q_h W_h s_{hx}^4\right) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \sum_{h=1}^{L} W_h \overline{x}_h^* - \sum_{h=1}^{L} W_h \overline{x}_h \\ \sum_{h=1}^{L} W_h s_{hx}^{*2} - \sum_{h=1}^{L} W_h s_{hx}^2 \end{bmatrix}$$

Solving the system of equations for lambdas, we obtain

$$\lambda_1 = \frac{(\sum_{h=1}^{L} Q_h W_h s_{hx}^4)(\sum_{h=1}^{L} W_h \overline{x}_h^* - \sum_{h=1}^{L} W_h \overline{x}_h) - (\sum_{h=1}^{L} W_h s_{hx}^{*2} - \sum_{h=1}^{L} W_h s_{hx}^2)(\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)}{(\sum_{h=1}^{L} Q_h W_h s_{hx}^4)(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2) - (\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)^2},$$

$$\lambda_2 = \frac{(\sum_{h=1}^{L} W_h s_{hx}^{*2} - \sum_{h=1}^{L} W_h s_{hx}^2)(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2) - (\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)(\sum_{h=1}^{L} W_h \overline{x}_h^* - \sum_{h=1}^{L} W_h \overline{x}_h)}{(\sum_{h=1}^{L} Q_h W_h s_{hx}^4)(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2) - (\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)^2}.$$

Substituting these values into (2.6), we get the weights as given by

$$W_h^* = W_h + \frac{(Q_h W_h \overline{x}_h)[(\sum_{h=1}^{L} Q_h W_h s_{hx}^4)(\sum_{h=1}^{L} W_h(\overline{x}_h^* - \overline{x}_h)) - (\sum_{h=1}^{L} W_h(s_{hx}^{*2} - s_{hx}^2))(\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)]}{(\sum_{h=1}^{L} Q_h W_h s_{hx}^4)(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2) - (\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)^2}$$

$$+ \frac{(Q_h W_h s_{hx}^2)[(\sum_{h=1}^{L} W_h(s_{hx}^{*2} - s_{hx}^2))(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2) - (\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)(\sum_{h=1}^{L} W_h(\overline{x}_h^* - \overline{x}_h))]}{(\sum_{h=1}^{L} Q_h W_h s_{hx}^4)(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2) - (\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)^2}$$

Writing these weights in (2.1), we get the calibration estimator as

$$\overline{y}_{st}(d) = (\sum_{h=1}^{L} W_h \overline{y}_h) + \beta_{1(d)}(\sum_{h=1}^{L} W_h(\overline{x}_h^* - \overline{x}_h)) + \beta_{2(d)}(\sum_{h=1}^{L} W_h(s_{hx}^{*2} - s_{hx}^2)),$$

where betas are given by

$$\beta_{1(d)} = \frac{(\sum_{h=1}^{L} Q_h W_h s_{hx}^4)(\sum_{h=1}^{L} Q_h W_h \overline{x}_h \overline{y}_h) - (\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)(\sum_{h=1}^{L} Q_h W_h \overline{y}_h s_{hx}^2)}{(\sum_{h=1}^{L} Q_h W_h s_{hx}^4)(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2) - (\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)^2},$$

$$\beta_{2(d)} = \frac{(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2)(\sum_{h=1}^{L} Q_h W_h \overline{y}_h s_{hx}^2) - (\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)(\sum_{h=1}^{L} Q_h W_h \overline{x}_h \overline{y}_h)}{(\sum_{h=1}^{L} Q_h W_h s_{hx}^4)(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2) - (\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2)^2}.$$

3. Suggested Estimator

Motivated by Tracy et al.[9], we consider a new calibration estimator as

$$\overline{y}_{st}(dnew) = \sum_{h=1}^{L} \Omega_h \overline{y}_h. \tag{3.1}$$

Using the chi-square distance

$$\sum_{h=1}^{L} \frac{(\Omega_h - W_h)^2}{Q_h W_h}, \tag{3.2}$$

and subject to calibration constraints defined by Koyuncu[7]

$$\sum_{h=1}^{L} \Omega_h \overline{x}_h = \sum_{h=1}^{L} W_h \overline{x}_h^*, \tag{3.3}$$

$$\sum_{h=1}^{L} \Omega_h s_{hx}^2 = \sum_{h=1}^{L} W_h s_{hx}^{*2}, \tag{3.4}$$

$$\sum_{h=1}^{L} \Omega_h = \sum_{h=1}^{L} W_h, \tag{3.5}$$

we can write the Lagrange function given by

$$\triangle = \sum_{h=1}^{L} \frac{(\Omega_h - W_h)^2}{Q_h W_h} - 2\lambda_1(\sum_{h=1}^{L} \Omega_h \overline{x}_h - \sum_{h=1}^{L} W_h \overline{x}_h^*) - 2\lambda_2(\sum_{h=1}^{L} \Omega_h s_{hx}^2 - \sum_{h=1}^{L} W_h s_{hx}^{*2}) - 2\lambda_3(\sum_{h=1}^{L} \Omega_h - \sum_{h=1}^{L} W_h),$$

Setting $\frac{\partial \triangle}{\triangle \Omega_h} = 0$, we obtain

$$\Omega_h = W_h + Q_h W_h(\lambda_1 \overline{x}_h + \lambda_2 s_{hx}^2 + \lambda_3). \tag{3.6}$$

Substituting (3.6) in (3.3)-(3.5), respectively, we get the following system of equations

$$\begin{bmatrix} \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2\right) & \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2\right) & \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\right) \\ \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2\right) & \left(\sum_{h=1}^{L} Q_h W_h s_{hx}^4\right) & \left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\right) \\ \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\right) & \left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\right) & \left(\sum_{h=1}^{L} Q_h W_h\right) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \sum_{h=1}^{L} W_h \overline{x}_h^* - \sum_{h=1}^{L} W_h \overline{x}_h \\ \sum_{h=1}^{L} W_h s_{hx}^{*2} - \sum_{h=1}^{L} W_h s_{hx}^2 \\ 0 \end{bmatrix}$$

Solving the system of equations for lambdas, we obtain

$$\lambda_1 = \frac{A}{D}, \lambda_2 = \frac{B}{D}, \lambda_3 = \frac{C}{D},$$

where

$$\begin{aligned} A = &\left(\sum_{h=1}^{L} W_h(\overline{x}_h^* - \overline{x}_h)\right) \left[\left(\sum_{h=1}^{L} Q_h W_h\right)\left(\sum_{h=1}^{L} Q_h W_h s_{hx}^4\right) - \left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\right)^2\right] \\ &+ \left(\sum_{h=1}^{L} W_h(s_{hx}^{*2} - s_{hx}^2)\right) \left[\left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\right)\left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\right) \right. \\ &\left. - \left(\sum_{h=1}^{L} Q_h W_h\right)\left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2 \overline{x}_h\right)\right] \end{aligned}$$

$$\begin{aligned} B = &\left(\sum_{h=1}^{L} W_h(s_{hx}^{*2} - s_{hx}^2)\right) \left[\left(\sum_{h=1}^{L} Q_h W_h\right)\left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2\right) - \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\right)^2\right] \\ &- \left(\sum_{h=1}^{L} W_h(\overline{x}_h^* - \overline{x}_h)\right) \left[\left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2 \overline{x}_h\right)\left(\sum_{h=1}^{L} Q_h W_h\right) \right. \\ &\left. -\left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\right)\left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\right)\right] \end{aligned}$$

$$\begin{aligned} C = &\left(\sum_{h=1}^{L} W_h(\overline{x}_h^* - \overline{x}_h)\right) \left[\left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\right)\left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2 \overline{x}_h\right) - \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\right)\left(\sum_{h=1}^{L} Q_h W_h s_{hx}^4\right)\right] \\ &+ \left(\sum_{h=1}^{L} W_h(s_{hx}^{*2} - s_{hx}^2)\right) \left[\left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\right)\left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2\right) \right. \\ &\left. -\left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2\right)\left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\right)\right] \end{aligned}$$

$$D = \Big(\sum_{h=1}^{L} Q_h W_h\Big)\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^4\Big)\Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2\Big) - \Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\Big)^2\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^4\Big)$$
$$- \Big(\sum_{h=1}^{L} Q_h W_h\Big)\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^2 \overline{x}_h\Big)^2 - \Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\Big)^2\Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2\Big)$$
$$+ 2\Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\Big)\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\Big)\Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2\Big)$$

Substituting these lambdas in (3.6) and then (3.1), we get

$$\overline{y}_{st}(dnew) = \overline{y}_{st} + \beta_{1(dnew)}\Big(\sum_{h=1}^{L} W_h(\overline{x}_h^* - \overline{x}_h)\Big) + \beta_{2(dnew)}\Big(\sum_{h=1}^{L} W_h(s_{hx}^{*2} - s_{hx}^2)\Big),$$

where $\beta_{1(dnew)} = \frac{A^*}{D}$ and $\beta_{2(dnew)} = \frac{B^*}{D}$

$$A^* = \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h \overline{y}_h\right)\left[\Big(\sum_{h=1}^{L} Q_h W_h\Big)\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^4\Big) - \Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\Big)^2\right]$$
$$- \left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2 \overline{y}_h\right)\left[\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^2 \overline{x}_h\Big)\Big(\sum_{h=1}^{L} Q_h W_h\Big) - \Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\Big)\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\Big)\right]$$
$$+ \left(\sum_{h=1}^{L} Q_h W_h \overline{y}_h\right)\left[\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\Big)\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^2 \overline{x}_h\Big) - \Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\Big)\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^4\Big)\right]$$

$$B^* = \left(\sum_{h=1}^{L} Q_h W_h \overline{x}_h \overline{y}_h\right)\left[\Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\Big)\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\Big) - \Big(\sum_{h=1}^{L} Q_h W_h\Big)\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^2 \overline{x}_h\Big)\right]$$
$$+ \left(\sum_{h=1}^{L} Q_h W_h s_{hx}^2 \overline{y}_h\right)\left[\Big(\sum_{h=1}^{L} Q_h W_h\Big)\Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2\Big) - \Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\Big)^2\right]$$
$$+ \left(\sum_{h=1}^{L} Q_h W_h \overline{y}_h\right)\left[\Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h\Big)\Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h s_{hx}^2\Big) - \Big(\sum_{h=1}^{L} Q_h W_h \overline{x}_h^2\Big)\Big(\sum_{h=1}^{L} Q_h W_h s_{hx}^2\Big)\right]$$

4. Theoretical Variance

We can write the estimators $\overline{y}_{st}(d)$ and $\overline{y}_{st}(dnew)$ as follows:

$$\overline{y}_{st}(\alpha) = \sum_{h=1}^{L} W_h \overline{y}_h + \beta_{1(\alpha)} \sum_{h=1}^{L} W_h(\overline{x}_h - \overline{x}_h^*) + \beta_{2(\alpha)} \sum_{h=1}^{L} W_h(s_{hx}^2 - s_{hx}^{*2}) \tag{4.1}$$

where $\alpha = d, dnew$. To find the variance of estimators, let us define following equations:

$e_{0h} = \dfrac{(\overline{y}_h - \overline{Y}_h)}{\overline{Y}_h}$, $e_{1h} = \dfrac{(\overline{x}_h - \overline{X}_h)}{\overline{X}_h}$, $e^*_{1h} = \dfrac{(\overline{x}^*_h - \overline{X}_h)}{\overline{X}_h}$, $e_{2h} = \dfrac{(s^2_{hx} - S^2_{hx})}{S^2_{hx}}$
and $e^*_{2h} = \dfrac{(s^{*2}_{hx} - S^2_{hx})}{S^2_{hx}}$ $\overline{y}_h = \overline{Y}_h(1+e_{0h})$, $\overline{x}_h = \overline{X}_h(1+e_{1h})$, $\overline{x}^*_h = \overline{X}_h(1+e^*_{1h})$,
$s^2_{hx} = S^2_{hx}(1+e_{2h})$,$s^{*2}_{hx} = S^2_{hx}(1+e^*_{2h})$,

$E(e^2_{0h}) = \lambda_{nh}C^2_{yh}$, $E(e^2_{1h}) = \lambda_{nh}C^2_{xh}$, $E(e_{0h}e_{1h}) = \lambda_{nh}C_{yxh}$, $E(e_{0h}e^*_{1h}) = \lambda_{mh}C_{yxh}$, $E(e^2_{2h}) = \lambda_{nh}(\lambda_{04h}-1)$, $E(e^{*2}_{2h}) = \lambda_{mh}(\lambda_{04h}-1)$,$E(e_{2h}e^*_{2h}) = \lambda_{mh}(\lambda_{04h}-1)$, $E(e_{0h}e_{2h}) = \lambda_{nh}C_{yh}\lambda_{12h}$, $E(e_{0h}e^*_{2h}) = \lambda_{mh}C_{yh}\lambda_{12h}$,$E(e^{*2}_{1h}) = \lambda_{mh}C^2_{xh}$, $E(e_{1h}e^*_{1h}) = \lambda_{mh}C^2_{xh}$,$E(e_{1h}e_{2h}) = \lambda_{nh}C_{xh}\lambda_{03h}$, $E(e^*_{1h}e_{2h}) = \lambda_{mh}C_{xh}\lambda_{03h}$,$E(e_{1h}e^*_{2h}) = \lambda_{mh}C_{xh}\lambda_{03h}$,$E(e^*_{1h}e^*_{2h}) = \lambda_{mh}C_{xh}\lambda_{03h}$

where $\lambda_{nh} = \frac{1}{n_h} - \frac{1}{N_h}$, $\lambda_{mh} = \frac{1}{m_h} - \frac{1}{N_h}$, $C_{yh} = \frac{S_{yh}}{\overline{Y}_h}$, $C_{xh} = \frac{S_{xh}}{\overline{X}_h}$, $C_{yxh} = \frac{S_{yxh}}{\overline{Y}_h\overline{X}_h}$,
$\lambda_{rsh} = \frac{\mu_{rsh}}{\mu^{\frac{r}{2}}_{20h}\mu^{\frac{s}{2}}_{02h}}$ and $\mu_{rsh} = \frac{=\sum_{i=1}^{N_h}(Y_{hi}-\overline{Y}_h)^r(X_{hi}-\overline{X}_h)^s}{N_h-1}$.

Expressing (4.1) in terms of e's, we have

(4.2)
$$\overline{y}_{st}(\alpha) = \sum_{h=1}^{L} W_h[\overline{Y}_h(1+e_{0h})+\beta_{1(\alpha)}\overline{X}_h((1+e_{1h})-(1+e^*_{1h}))+\beta_{2(\alpha)}S^2_{hx}((1+e_{2h})-(1+e^*_{2h})]$$

(4.3)
$$\overline{y}_{st}(\alpha)-\sum_{h=1}^{L} W_h\overline{Y}_h = \sum_{h=1}^{L} W_h[\overline{Y}_he_{0h}+\beta_{1(\alpha)}\overline{X}_h(e_{1h}-e^*_{1h})+\beta_{2(\alpha)}S^2_{hx}(e_{2h}-e^*_{2h})]$$

Squaring both sides of (4.3),

$$(4.4)\quad \left(\overline{y}_{st}(\alpha) - \sum_{h=1}^{L} W_h\overline{Y}_h\right)^2 = \sum_{h=1}^{L} W_h^2\left[\overline{Y}_he_{0h} + \beta_{1(\alpha)}\overline{X}_h(e_{1h}-e^*_{1h}) + \beta_{2(\alpha)}S^2_{hx}(e_{2h}-e^*_{2h})\right]^2$$
$$= \sum_{h=1}^{L} W_h^2\Big[\overline{Y}^2_he^2_{0h} + \beta^2_{1(\alpha)}\overline{X}^2_h(e_{1h}-e^*_{1h})^2 + \beta^2_{2(\alpha)}S^4_{hx}(e_{2h}-e^*_{2h})^2$$
$$+ 2\beta_{1(\alpha)}\overline{X}_h\overline{Y}_he_{0h}(e_{1h}-e^*_{1h}) + 2\beta_{2(\alpha)}\overline{Y}_hS^2_{hx}e_{0h}(e_{2h}-e^*_{2h})$$
$$+ 2\beta_{1(\alpha)}\beta_{2(\alpha)}\overline{X}_hS^2_{hx}(e_{1h}-e^*_{1h})(e_{2h}-e^*_{2h})\Big]$$

and taking expectations, we get the variance of $\overline{y}_{st}(\alpha)$ as

$$(4.5)\quad Var(\overline{y}_{st}(\alpha)) = \sum_{h=1}^{L} W_h^2[\overline{Y}^2_h\lambda_{nh}C^2_{yh} + (\lambda_{nh}-\lambda_{mh})\beta^2_{1(\alpha)}\overline{X}^2_hC^2_{xh} + (\lambda_{nh}-\lambda_{mh})\beta^2_{2(\alpha)}S^4_{hx}(\lambda_{04h}-1)$$
$$+ 2(\lambda_{nh}-\lambda_{mh})\beta_{1(\alpha)}\overline{Y}_h\overline{X}_hC_{yxh} + 2(\lambda_{nh}-\lambda_{mh})\beta_{2(\alpha)}\overline{Y}_hS^2_{hx}C_{yh}\lambda_{12h}$$
$$+ 2(\lambda_{nh}-\lambda_{mh})\beta_{1(\alpha)}\beta_{2(\alpha)}S^2_{hx}\overline{X}_hC_{xh}\lambda_{03h}]$$

The variance of $\overline{y}_{st}(\alpha)$ in (4.5) is minimized for

$$\frac{Var(\overline{y}_{st}(\alpha))}{\partial\beta_{1(\alpha)}} = 0,$$

$$\beta_{1(\alpha)} = \frac{-\beta_{2(\alpha)}S^2_{hx}C_{xh}\lambda_{03h} - \overline{Y}_h C_{yxh}}{\overline{X}_h C^2_{xh}}, \tag{4.6}$$

$$\frac{Var(\overline{y}_{st}(\alpha))}{\partial\beta_{2(\alpha)}} = 0,$$

$$\beta_{2(\alpha)} = \frac{-\beta_{1(\alpha)}\overline{X}_h C_{xh}\lambda_{03h} - \overline{Y}_h C_{yh}\lambda_{12h}}{S^2_{hx}(\lambda_{04h} - 1)}, \tag{4.7}$$

Substituting (4.6) in (4.7) or vice versa, we have optimum betas as given by

$$\beta_{1(\alpha)} = \frac{S_{yh}}{S_{xh}}\frac{\lambda_{12h}\lambda_{03h} - \lambda_{11h}(\lambda_{04h} - 1)}{(\lambda_{04h} - 1) - \lambda^2_{03h}}, \beta_{2(\alpha)} = \frac{S_{yh}}{S^2_{xh}}\frac{\lambda_{11h}\lambda_{03h} - \lambda_{12h}}{(\lambda_{04h} - 1) - \lambda^2_{03h}}$$

The resulting (minimum) variance of $\overline{y}_{st}(\alpha)$ is given by

$$\begin{aligned} Var(\overline{y}_{st}(\alpha)) &= \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left[\lambda_{nh}C^2_{yh} - (\lambda_{nh} - \lambda_{mh})\frac{C^2_{yxh}(\lambda_{04h} - 1) + C^2_{xh}C^2_{yh}\lambda^2_{12h} - 2C_{yxh}C_{yh}C_{xh}\lambda_{03h}\lambda_{12h}}{C^2_{xh}[(\lambda_{04h} - 1) - \lambda^2_{03h}]}\right] \\ &= \sum_{h=1}^{L} W_h^2\overline{Y}_h^2 C^2_{yh}\left[\lambda_{mh} + (\lambda_{nh} - \lambda_{mh})\left[1 - \lambda^2_{11h} - \frac{(\lambda_{12h} - \lambda_{11h}\lambda_{03h})^2}{(\lambda_{04h} - 1) - \lambda^2_{03h}}\right]\right] \end{aligned} \tag{4.8}$$

5. Simulation Study

To study the properties of the proposed calibration estimator, we perform a simulation study by generating four different artificial populations where $\overline{x}^*_{hi}$ and $\overline{y}^*_{hi}$ values are from different distributions given in Table 1. To get different level of correlations between study and auxiliary variables, we apply some transformations given in Table 2. Each population consists of three strata having 500 units. After selecting a preliminary sample of size 300 from each stratum, we select 5000 times for the second sample whose sample of sizes are 30 and 50. The correlation coefficients between study and auxiliary variables for each stratum are taken as $\rho_{xy1} = 0.5$,$\rho_{xy2} = 0.7$ and $\rho_{xy3} = 0.9$. The quantities,$S_{1x} = 4.5$, $S_{2x} = 6.2$, $S_{3x} = 8.4$ and $S_{1y} = S_{2y} = S_{3y}$ are taken as fixed in each stratum as in Tracy et al. [9]. We calculate the empirical mean square error and percent relative efficiency, respectively, using following formulas:

$$MSE(\overline{y}_{st}(\alpha)) = \frac{\sum_{k=1}^{\binom{N}{n}} [\overline{y}_{st}(\alpha) - \overline{Y}]^2}{\binom{N}{n}}, \alpha = d, dnew$$

$$PRE = \frac{MSE(\overline{y}_{st}(d))}{MSE(\overline{y}_{st}(dnew))} * 100$$

From Table 3, the simulation study shows that new calibration estimator is quite efficient than the existing estimator.

6. Conclusion

In this study we derived new calibration weights in stratified double sampling. The performance of the weights are compared with a simulation study. We found that suggested weights perform better than existing weights.

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Table 1. Parameters and Distributions of Study and Auxiliary Variables

Parameters and distributions of study variable	**Parameters and distributions of auxiliary variable**
	I. Population, h=1,2,3
$f(y^*_{hi}) = \frac{1}{\Gamma(1.5)} y^{*1.5-1}_{hi} e^{-y^*_{hi}}$	$f(x^*_{hi}) = \frac{1}{\Gamma(0.3)} x^{*0.3-1}_{hi} e^{-x^*_{hi}}$
	II. Population, h=1,2,3
$f(y^*_{hi}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^{*2}_{hi}}{2}}$	$f(x^*_{hi}) = \frac{1}{\Gamma(0.3)} x^{*0.3-1}_{hi} e^{-x^*_{hi}}$

Table 2. Properties of Strata

Strata	**Study Variable**	**Auxiliary Variable**
1. Stratum	$y_{1i} = 50 + y^*_{1i}$	$x_{1i} = 15 + \sqrt{(1-\rho^2_{xy1})} x^*_{1i} + \rho_{xy1} \frac{S_{1x}}{S_{1y}} y^*_{1i}$
2. Stratum	$y_{2i} = 150 + y^*_{2i}$	$x_{2i} = 100 + \sqrt{(1-\rho^2_{xy2})} x^*_{2i} + \rho_{xy2} \frac{S_{2x}}{S_{2y}} y^*_{2i}$
3. Stratum	$y_{3i} = 50 + y^*_{3i}$	$x_{3i} = 200 + \sqrt{(1-\rho^2_{xy3})} x^*_{3i} + \rho_{xy3} \frac{S_{3x}}{S_{3y}} y^*_{3i}$

Table 3. Empirical Mean Square Error (MSE) and Percent Relative Efficiency (PRE) of Estimators

Population	**Empirical MSE**($\overline{y}_{st}(d)$)	**Empirical MSE**($\overline{y}_{st}(dnew)$)	**PRE**
I ($m_h = 30$)	**61205495678**	**65730798**	**93115.4**
I ($m_h = 50$)	**626914412**	**38472456**	**1629.515**
II ($m_h = 30$)	**6.8404e+11**	**50343013**	**1358759**
II ($m_h = 50$)	**245901177**	**35046173**	**701.6491**

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Bayesian estimation of Marshall–Olkin extended exponential parameters under various approximation techniques

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Abstract

In this paper, we propse Bayes estimators of the parameters of Marshall Olkin extended exponential distribution (MOEED) introduced by Marshall-Olkin [2] for complete sample under squared error loss function (SELF). We have used different approximation techniques to obtain the Bayes estimate of the parameters. A Monte Carlo simulation study is carried out to compare the performance of proposed estimators with the corresponding maximum likelihood estimator (MLE's) on the basis of their simulated risk. A real data set has been considered for illustrative purpose of the study.

Keywords: Bayes estimator, Squared error loss function, Lindley's approximation method, T-K approximation, MCMC method.

2000 AMS Classification: 62F15, 62C10

1. Introduction

Due to simple, elegant and closed form of distribution function, Exponential distribution is most popular distribution for life time data analysis. Further Borlow and Proschan [22] have discussed the justification regarding the use of exponential distribution as the failure law of complex equipment. However its uses are restricted to constant hazard rate, which is difficult to justify in many real situations. Thus one can think to develop alternative model which has non-constant hazard rate. In the literature, various methods may be used to generalise exponential distributions and these generalized models have the property of non-constant hazard rate like Weibull, gamma and exponentiated exponential distribution etc. These generalized models are frequently used to analyse the life time data. In addition Marshall and Olkin [2] introduced a method of adding a new parameter to a specified distribution. The resulting distribution is known as Marshall Olkin extended distribution. The general methodology regarding the introducing a new

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parameters is as follows:
Let $\bar{F}(x)$ be the survival function of existing or specified distribution then, the survival function of new distribution can be obtained by using following relation

$$\bar{S}(x) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)}; \quad -\infty < x < \infty, \alpha > 0$$

where $\bar{\alpha} = 1 - \alpha$ and $\bar{S}(x)$ is the survival function of new distribution. Note that, when $\alpha = 1$, $\bar{S}(x) = \bar{F}(x)$. Thus, the form of density corresponding to the survival function $\bar{S}(x)$ is obtained as,

$$f(x, \alpha) = \frac{\alpha f(x)}{\left\{1 - \bar{\alpha}\bar{F}(x)\right\}^2}$$

Further more, Marshall and Olkin derived a distribution by introducing the survival function of exponential distribution say $(\bar{F}(x) = e^{-\lambda x})$. The resulting distribution is known as Marshall Olkin extended exponential distribution (MOEED) with increasing and decreasing failure rate functions see [2]. The probability density function (pdf) and cumulative distribution function (cdf) of this distribution are given as:

$$f(x, \alpha, \lambda) = \frac{\alpha \lambda e^{-\lambda x}}{(1 - \bar{\alpha} e^{-\lambda x})^2}; \quad x, \alpha, \lambda \geq 0 \tag{1.1}$$

$$F(x, \alpha, \lambda) = \frac{1 - e^{-\lambda x}}{1 - \bar{\alpha} e^{-\lambda x}}; \quad x, \alpha, \lambda \geq 0 \tag{1.2}$$

respectively. The considered distribution is very useful in life testing problem and it may be used as a good alternative to the gamma, Weibull and other exponentiated family of distributions. The basic properties related to this distribution have been discussed in [2]. The density function (1) has increasing failure rate for $\alpha \geq 1$, decreasing failure rate for $\alpha \leq 1$ and constant failure rate for $\alpha = 1$ similar to one parameter exponential distribution. G. Srinivasa Rao et al [3] used this distribution for making reliability test plan with sampling point of view. Shape of this distribution is presented bellow see figure 1. for different choices of shape and scale parameter.

In this paper, we mainly consider both the informative and non-informative priors under squared error loss function to compute the Bayes estimators of parameters. It has been noticed that the Bayes estimators of the parameters cannot be expressed in a nice closed form. Thus the different numerical approximation procedures are used to obtain Bayes estimator. Here we use the Lindley's, Tierney and Kadane (T-K) approximation methods and Markov Chain Monte Carlo (MCMC) technique to compute the Bayes estimators of the parameters.

The rest of the paper is organized as follows: In section 2.1, we describe the classical estimation with maximum likelihood estimator (MLE) of parameters. In section 2.2, we compute Bayes estimator of parameters with gamma prior and in section 2.2.1, 2.2.2 and 2.2.3 we describe different Bayesian approaches like Lindley Approximation, Tierney and Kadane approximation and Monte Carlo Markov

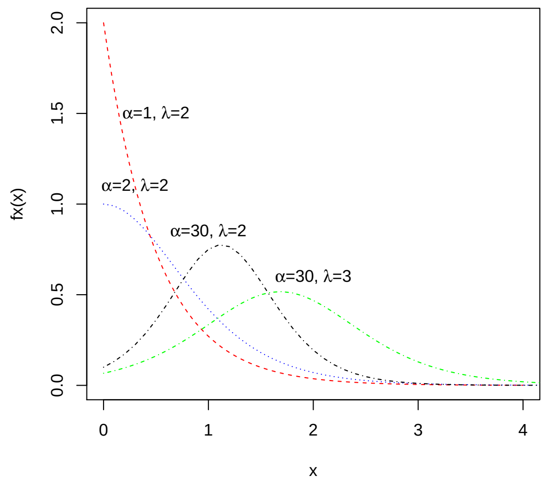

FIGURE 1. Density plot with different choice of α and λ

chain (MCMC) method for estimating the unknown parameters respectively. Section 3 provides the simulation and numerical result and one real data set has been analysed in section 4. Finally conclusion of the paper is provided in section 5.

2. Estimation of the parameters

2.1. Maximum likelihood estimators. Suppose $\{x_1, x_2, ..., x_n\}$ be a independently identically distributed (iid) random sample of size n from Marshall Olkin extended exponential distribution (MOEED) defined in (1). Thus the likelihood function of α and λ for the samples is,

$$L(x|\alpha,\lambda) = \alpha^n \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1-\bar{\alpha} e^{-\lambda x_i})^{-2}; \quad x, \alpha, \lambda \geq 0 \tag{2.1}$$

The maximum likelihood estimators of the parameters have obtained by differentiating the log of likelihood function w.r.t.to parameters and equating to zero. Thus two normal equations have been obtained as,

$$\frac{n}{\alpha} - 2\sum_{i=1}^{n} e^{-\lambda x_i}(1-\bar{\alpha} e^{-\lambda x_i})^{-1} = 0 \tag{2.2}$$

and

$$\frac{n}{\lambda} - \sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} \bar{\alpha} x_i e^{-\lambda x_i}(1-\bar{\alpha} e^{-\lambda x_i})^{-1} = 0 \tag{2.3}$$

Above normal equation of α and λ form an implicit system and does not exist an unique root for above system of equations, so they can not be solved analytically.

Thus maximum likelihood estimators (MLE) have been obtained By using Newton-Raphson (N-R) method.

2.2. Bayesian Estimation of the parameters. The Bayesian estimation procedure of the parameters related to various life time models has been extensively discussed the literature (see in[5],[6],[8] and so on). It may be mentioned here, that most of the discussions on Bayes estimator are confined to quadratic loss function because this loss function is most widely used as symmetrical loss function which has been justified in classical method on the ground of minimum variance unbiased estimation procedure and associates equal importance to the losses for overestimation and underestimation of equal magnitudes. This may be defined as,

$$L(\hat{\theta}, \theta) \propto (\hat{\theta} - \theta)^2$$

where $\hat{\theta}$ is the estimate of the parameter θ.

Under the above mentioned loss function, Bayes estimators are the posterior mean of the distributions. In Bayesian analysis, parameters of the models are considered to be a random variable and following certain distribution. This distribution is called prior distribution. If prior information available to us which may be used for selection of prior distribution. But in many real situation it is very difficult to select a prior distribution. Therefore selection of prior distribution plays an important role in estimation of the parameters. A natural choice for the prior of α and λ would be two independent gamma distributions i.e. $gamma(a, b)$ and $gamma(c, d)$ respectively . It is important to mention that Gamma prior has flexible nature as a non-informative prior in particular when the values of hyper parameters are considered to be zero. Thus the proposed prior for α and λ may be considered as,

$$\nu_1(\alpha) \propto \alpha^{a-1} e^{-b\alpha} \qquad and \qquad \nu_2(\lambda) \propto \lambda^{c-1} e^{-d\lambda}$$

respectively. Where a, b, c and d are the hyper-parameters of the prior distributions. Thus, the joint prior of α and λ may be taken as;

$$\nu(\alpha, \lambda) \propto \alpha^{a-1} \lambda^{c-1} e^{-d\lambda - b\alpha} \quad ; \quad \alpha, \lambda, a, b, c, d \geq 0 \tag{2.4}$$

Substituting $L(x|\alpha, \lambda)$ and $\nu(\alpha, \lambda)$ form equation no. (3) and (6) respectively then we can find the posterior distribution of α and λ i.e.$p(\alpha, \lambda|\underline{x})$ is given as,

$$p(\alpha, \lambda|\underline{x}) = K \alpha^{n+a-1} \lambda^{n+c-1} e^{-d\lambda - b\alpha - \lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - \bar{\alpha} e^{-\lambda x_i})^{-2} \tag{2.5}$$

where,

$$K^{-1} = \int_{\alpha} \int_{\lambda} \alpha^{n+a-1} \lambda^{n+c-1} e^{-d\lambda - b\alpha - \lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - \bar{\alpha} e^{-\lambda x_i})^{-2} d\alpha d\lambda \tag{2.6}$$

Here, we see that the posterior distribution involves an integral in the denominator which is not solvable and consequently the Bayes estimators of the parameters are the ratio of the integral, which are not in explicit form. Hence the determination of posterior expectation for obtaining the Bayes estimator of α and λ will be tedious. There are several methods available in literature to solve such type of integration problem. Among the entire methods we consider T-K, Lindley's and Monte Carlo Markov Chain (MCMC) approximation method, which approach the ratio of the integrals as a whole and produce a single numerical result. These methods are described bellow:

2.2.1. *Bayes estimator using Lindley's Approximation.* We consider the Lindley's approximation method to obtain the Bayes estimates of the parameters, which includes the posterior expectation is expressible in the form of ratio of integral as follow:

$$I(x) = E(\alpha, \lambda|\underline{x}) = \frac{\int u(\alpha,\lambda)e^{L(\alpha,\lambda)+G(\alpha,\lambda)}d(\alpha,\lambda)}{\int e^{L(\alpha,\lambda)+G(\alpha,\lambda)}d(\alpha,\lambda)} \tag{2.7}$$

where,
$u(\alpha, \lambda)$= is a function of α and λ only
$L(\alpha, \lambda)$= Log- likelihood function
$G(\alpha, \lambda)$= Log of joint prior density
According to D. V. Lindley [1], if ML estimates of the parameters are available and n is sufficiently large then the above ratio of the integral can be approximated as:

$$I(x) = u(\hat{\alpha},\hat{\lambda}) + 0.5[(\hat{u}_{\lambda\lambda} + 2\hat{u}_{\lambda}\hat{\tau}_{\lambda})\hat{\sigma}_{\lambda\lambda} + (\hat{u}_{\alpha\lambda} + 2\hat{u}_{\alpha}\hat{\tau}_{\lambda})\hat{\sigma}_{\alpha\lambda} + (\hat{u}_{\lambda\alpha} + 2\hat{u}_{\lambda}\hat{\tau}_{\alpha})\hat{\sigma}_{\lambda\alpha} + (\hat{u}_{\alpha\alpha} + 2\hat{u}_{\alpha}\hat{\tau}_{\alpha})\hat{\sigma}_{\alpha\alpha}] + \frac{1}{2}[(\hat{u}_{\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{u}_{\alpha}\hat{\sigma}_{\lambda\alpha})(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}) + (\hat{u}_{\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha})(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha})]$$

where $\hat{\alpha}$ and $\hat{\lambda}$ is the MLE of α and λ respectively, and

$$\hat{u}_{\alpha} = \frac{\partial u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}}, \hat{u}_{\lambda} = \frac{\partial u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}}, \hat{u}_{\alpha\lambda} = \frac{\partial u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\lambda}}, \hat{u}_{\lambda\alpha} = \frac{\partial u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\alpha}}, \hat{u}_{\alpha\alpha} = \frac{\partial^2 u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}^2},$$
$$\hat{u}_{\lambda\lambda} = \frac{\partial^2 u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}^2}, \hat{L}_{\alpha\alpha} = \frac{\partial^2 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}^2}, \hat{L}_{\lambda\lambda} = \frac{\partial^2 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}^2}, \hat{L}_{\alpha\alpha\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}^3},$$
$$\hat{L}_{\alpha\alpha\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\alpha}\partial\hat{\lambda}}, \hat{L}_{\lambda\lambda\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\lambda}\partial\hat{\alpha}}, \hat{L}_{\lambda\alpha\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\alpha}\partial\hat{\lambda}}, \hat{L}_{\alpha\alpha\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\alpha}\partial\hat{\lambda}},$$
$$\hat{L}_{\alpha\lambda\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\lambda}\partial\hat{\lambda}}, \hat{L}_{\lambda\alpha\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\alpha}\partial\hat{\alpha}}, \hat{p}_{\alpha} = \frac{\partial G(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}}, \hat{p}_{\lambda} = \frac{\partial G(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}}$$

After substitution of $p(\alpha, \lambda|\underline{x})$ from (7) in above equation (9) then this integral must be reduces like Lindley's integral, where:

$$u(\alpha,\lambda) = \alpha$$
$$L(\alpha,\lambda) = n\ln\alpha + n\ln\lambda - \lambda\sum_{i=1}^{n}x_i - 2\sum_{i=1}^{n}\ln(1-\bar{\alpha}e^{-\lambda x_i}) \text{ and}$$
$$G(\alpha,\lambda) = (a-1)\ln\alpha + (c-1)\ln\lambda - (b\alpha + d\lambda)$$

it may verified that,

$$u_{\alpha} = 1, \quad u_{\alpha\alpha} = u_{\lambda\lambda} = u_{\alpha\lambda} = u_{\lambda\alpha} = 0, \quad p_{\alpha} = \frac{a-1}{\alpha} - b, \; p_{\lambda} = \frac{c-1}{\lambda} - d$$
$$L_{\alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n}\frac{e^{-\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})}, \qquad L_{\alpha\alpha} = \frac{-n}{\alpha^2} + 2\sum_{i=1}^{n}\frac{e^{-2\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^2},$$
$$L_{\alpha\alpha\alpha} = \frac{2n}{\alpha^3} - 4\sum_{i=1}^{n}\frac{e^{-3\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^3}, \qquad L_{\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}x_i - 2\sum_{i=1}^{n}\frac{x_i\bar{\alpha}e^{-\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})},$$

$$L_{\lambda\lambda} = \frac{-n}{\lambda^2} + 2\sum_{i=1}^{n} \frac{x_i^2 \bar{\alpha} e^{-\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})} + 2\sum_{i=1}^{n} \frac{x_i^2 \bar{\alpha}^2 e^{-2\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^2},$$

$$L_{\lambda\lambda\lambda} = \frac{2n}{\lambda^3} - 2\sum_{i=1}^{n} \frac{x_i^3 \bar{\alpha} e^{-\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})} - 6\sum_{i=1}^{n} \frac{x_i^3 \bar{\alpha}^2 e^{-2\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^2} - 4\sum_{i=1}^{n} \frac{x_i^3 \bar{\alpha}^3 e^{-3\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^3},$$

$$L_{\alpha\alpha\lambda} = L_{\lambda\alpha\alpha} = -4\sum_{i=1}^{n} \frac{x_i e^{-2\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^2} - 4\sum_{i=1}^{n} \frac{x_i \bar{\alpha} e^{-3\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^3},$$

$$L_{\alpha\lambda\lambda} = L_{\lambda\lambda\alpha} = -2\sum_{i=1}^{n} \frac{x_i^2 e^{-\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})} - 6\sum_{i=1}^{n} \frac{x_i^2 \bar{\alpha} e^{-2\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^2} - 4\sum_{i=1}^{n} \frac{x_i^2 \bar{\alpha}^2 e^{-3\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^3}$$

If α and λ are orthogonal then $\sigma_{ij} = 0$ for $i \neq j$ and $\sigma_{ij} = \left(-\frac{1}{L_{ij}}\right)$ for $i = j$ After evaluation of all U-terms, L-terms, and p- terms at the point $(\hat{\alpha}, \hat{\lambda})$ and using the above expression, the approximate Bayes estimator of α under SELF is,

$$\hat{\alpha}_S^L = \hat{\alpha} + \hat{u}_\alpha \hat{p_\alpha} \hat{\sigma}_{\alpha\alpha} + 0.5\left(\hat{u}_\alpha \hat{\sigma}_{\alpha\alpha} \hat{\sigma}_{\lambda\lambda} \hat{L}_{\alpha\lambda\lambda} + \hat{u}_\alpha \hat{\sigma}_{\alpha\alpha}^2 \hat{L}_{\alpha\alpha\alpha}\right) \tag{2.8}$$

and similarly the Bayes estimate for λ under SELF is,
$u_\lambda = 1, \quad u_{\alpha\alpha} = u_{\lambda\lambda} = u_{\alpha\lambda} = u_{\lambda\alpha} = 0$ and remaining L-terms and -terms will be same as above thus we have,

$$\hat{\lambda}_S^L = \hat{\lambda} + \hat{u}_\lambda \hat{p_\lambda} \hat{\sigma}_{\lambda\lambda} + 0.5\left(\hat{u}_\lambda \hat{\sigma}_{\lambda\lambda}^2 \hat{L}_{\lambda\lambda\lambda} + \hat{u}_\lambda \hat{\sigma}_{\alpha\alpha} \hat{\sigma}_{\lambda\lambda} \hat{L}_{\alpha\alpha\lambda}\right) \tag{2.9}$$

2.2.2. *Bayes estimators using Tierney and Kadane's (T-K) Approximation.* Lindley's method of solving integral is accurate enough but one of the problems of this method is that it requires evaluation of third order partial derivatives and in p-parameters case the total number of derivatives is $\frac{p(p+1)(p+2)}{6}$ then this approximation will be quite complicated. thus one can think about T-K approximation method and this method may be used as an alternative to Lindley's method. According to the Tierney and Kadane's approximation any ratio of the integral of the form,

$$\hat{u}(\alpha,\lambda) = E_{p(\alpha,\lambda|\underline{x})}[u(\alpha,\lambda|\underline{x})] = \frac{\int_{\alpha,\lambda} e^{nL_*(\alpha,\lambda)} d(\alpha,\lambda)}{\int_{\alpha,\lambda} e^{nL_0(\alpha,\lambda)} d(\alpha,\lambda)} \tag{2.10}$$

where,

$$L_0(\alpha,\lambda) = \frac{1}{n}[L(\alpha,\lambda) + \ln \nu(\alpha,\lambda)] \quad and \quad L_*(\alpha,\lambda) = L_0(\alpha,\lambda) + \frac{1}{n}\ln u(\alpha,\lambda) \tag{2.11}$$

Thus estimate can be obtained as,

$$\hat{u}(\alpha,\lambda) = \sqrt{\frac{|\Sigma_*|}{|\Sigma_0|}} e^{[n\{L_*(\alpha_*,\lambda_*) - L_0(\alpha_0,\lambda_0)\}]} \tag{2.12}$$

where (α_*, λ_*) and (α_0, λ_0) maximize $L_*(\alpha, \lambda)$ and $L_0(\alpha, \lambda)$ respectively, and Σ_* and Σ_0 are the negative of the inverse of the matrices of second derivatives of $L_*(\alpha, \lambda)$ and $L_0(\alpha, \lambda)$ at the point (α_*, λ_*) and (α_0, λ_0) respectively. In our study, based on (14) the function $L_0(\alpha, \lambda)$ is given as,

(2.13)

$$L_0(\alpha,\lambda)=\frac{1}{n}[(n{+}a{-}1)\ln\alpha-b\alpha+(n{+}c{-}1)\ln\lambda-\lambda(d+\sum_{i=1}^{n}x_i)-2\sum_{i=1}^{n}\ln(1-\bar{\alpha}e^{-\lambda x_i})]$$

and thus for the Bayes estimator of α and λ under SELF using this approximation (17) can be written as,

$$\hat{\alpha}_S^{T-K}(\alpha,\lambda)=\sqrt{\frac{|\Sigma_*|}{|\Sigma_0|}}e^{[n\{L_*^{\alpha}(\alpha_*,\lambda_*)-L_0(\alpha_0,\lambda_0)\}]} \tag{2.14}$$

$$\hat{\lambda}_S^{T-K}(\alpha,\lambda)=\sqrt{\frac{|\Sigma_*|}{|\Sigma_0|}}e^{\left[n\left\{L_*^{\lambda}(\alpha_*,\lambda_*)-L_0(\alpha_0,\lambda_0)\right\}\right]} \tag{2.15}$$

where

$$L_*^{\alpha}(\alpha,\lambda)=L_0^{\alpha}(\alpha,\lambda)+\frac{1}{n}\ln\alpha \quad \textit{and} \quad L_*^{\lambda}(\alpha,\lambda)=L_0^{\lambda}(\alpha,\lambda)+\frac{1}{n}\ln\lambda$$

2.2.3. *Bayes estimator using Monte Carlo Markov Chain (MCMC) method.* In this section, we propose Monte Carlo Markov Chain (MCMC) method for obtaining the Bayes estimates of the parameters. Thus we consider the MCMC technique namely Gibbs sampler and Metropolis-Hastings algorithm to generate sample from the posterior distribution and then compute the Bayes estimate. The Gibbs sampler is best applied on problems where the marginal distributions of the parameters of interest are difficult to calculate, but the conditional distributions of each parameter given all the other parameters and the data have nice forms. If the conditional distributions of the parameters have standard forms, then they can be simulated easily. But generating samples from full conditionals corresponding to joint posterior is not easily manageable. Therefore we considered the Metropolis-Hastings algorithm. Metropolis step is used to extract samples from some of the full conditional to complete a cycle in Gibbs chain . For more detail about MCMC method see for example Gelfand and Smith [23], Upadhya and Gupta [24] . Thus utilizing the concept of Gibbs sampling procedure as mentioned above, generates sample from the posterior density function (7) under the assumption that parameters α and λ have independent Gamma density function with hyper parameters a, b and c, d respectively. To incorporate this technique we consider full conditional posterior densities of α and λ are written as ,

$$\pi(\alpha|\lambda,\underline{\mathrm{x}})\propto\alpha^{n+a-1}e^{-b\alpha}\prod_{i=1}^{n}(1-\bar{\alpha}e^{-\lambda x_i})^{-2} \tag{2.16}$$

$$\pi(\lambda|\alpha,\underline{\mathrm{x}})\propto\lambda^{n+c-1}e^{-\lambda(d+\sum_{i=1}^{n}x_i)}\prod_{i=1}^{n}(1-\bar{\alpha}e^{-\lambda x_i})^{-2} \tag{2.17}$$

The Gibbs algorithm consist the following steps

- Start with k=1 and initial values (α^0,λ^0)
- Using M-H algorithm generate posterior sample for α and λ from (18) and (19) respectively, where asymptotic normal distribution of full conditional densities are considered as the proposal.
- Repeat step 2, for all $k=1,2,3,\ldots,M$ and obtain $(\alpha_1,\lambda_1),(\alpha_2,\lambda_2),...(\alpha_M,\lambda_M)$
- After obtaining the posterior sample the Bayes estimates of α and λ with respect to the SELF are as follows:

$$(2.18) \quad \hat{\alpha}^{MC} = [E_\pi(\alpha|\underline{x})] \approx \left(\frac{1}{M-M_0} \sum_{i=1}^{M-M_0} \alpha_i \right)$$

$$(2.19) \quad \hat{\lambda}^{MC} = [E_\pi(\lambda|\underline{x})] \approx \left(\frac{1}{M-M_0} \sum_{i=1}^{M-M_0} \lambda_i \right)$$

Where, M_0 is the burn-in-period of Markov Chain.

3. Simulation Study

This section, consists of simulation study to compare the performance of the various estimation techniques described in the previous section 2. Comparison of the estimators have been made on the basis of simulated risk (average loss over whole sample space). It is not easy to obtain the risk of the estimators directly. Therefore the risk of the estimators are obtained on the basis of simulated sample. For this purpose, we generate 1000 samples of size n (small sample size n = 20, moderate sample size n = 30, and large sample size n = 50) from Mrshall-Olkin Extended exponential distribution. In order to consider MCMC method for obtaining the Bayes estimate of the parameters, we generate 20000 deviates for the parameters α and λ using algorithm discussed in section 2.2.3. First five hundred MCMC iterations (Burn-in period) have discarded from the generated sequence. We have also checked the convergence of the sequences of α and λ for their stationary distributions through different starting values. It was observed that all the Markov chains reached to the stationary condition very quickly. Further, in Bayes estimation choice of hyper-parameters have great importance. Therefore the values of hyper- parameters have been considered as follows:

- The values of hyper parameters are assumed in such a way that prior mean is equal to the guess value of the parameters when prior variances are taken as small (see Table 1), large (see Table 2) along with variation of sample size and for fixed value of parameters.
- The value of hyper parameters are assumed to be zero (i.e. non-informative case) along with variation of sample sizes and for fixed value of parameters (see Tables 3).

Here, we know that the Gamma prior provides flexible approach to handle estimation procedure in both scenarios i.e. informative and non-informative. The case of non-informative prior has been obtained by assuming the values of hyper parameters as zero i.e.$a = b = c = d = 0$. For informative prior, we take prior mean (say,μ) to be equal to the guess value of the parameter with varying prior variance (say,ν). The prior variance indicates the confidence of our prior guess. A large prior variance shows less confidence in prior guess and resulting prior distribution is relatively flat. On the other hand, small prior variance indicates greater confidence in prior guess. Several variations of sample size and hyper-parameters have been obtained and due to similar patterns some of them are presented below. In Table 1 the variation of various sample sizes has been observed through fixing the value of shape and scale parameter i.e $\alpha = \lambda = 2$ and choice of hyper-parameter is assumed as a=4, b=2 and c=4, d=2, such that, prior mean is 2 and prior variance is small (say 1). Table 2 shows the same patterns described as above for different choice of hyper-parameters which is assumed as a=0.4, b=0.2 and c=0.4, d=0.2, such that prior mean is 2 but prior variance is very large (say 10). Table 3 exhibits similar results under consideration of non-informative prior scenario. It is also observed that the risks of all the estimators decrease as sample size increases in all the considered cases. As we expected, it is also observed that when we consider informative prior, the proposed Bayes estimators behave better than the classical maximum likelihood estimators. But in case of non-informative prior, their behaviour are almost same as MLE, which may be seen in the following connected tables (see Table 1,2 and 3).

4. Real Illustration

In this section; we analyze a real data set from A. Wood [21] to illustrate our estimation procedure. The data is based on the failure times of the release of software given in terms of hours with average life time be 1000 hours from the starting of the execution of the software. This data can be regarded as an ordered sample of size 16 are given as,

0.519	0.968	1.430	1.893	2.490	3.058	3.625	4.442
5.218	5.823	6.539	7.083	7.485	7.846	8.205	8.564

Given data set have been already considered by Rao et al.[3] to construct a sampling plan only if the life time has Marshall-Olkin extended exponential distribution. To identify the validity of proposed model criterion of log-likelihood, Akaike information criterion (AIC) and Bayesian information criterion (BIC) have been discussed. It has been verified that the given data set provides better fit than other exponetiated family such as exponential, Generalized exponential and gamma distributions see Table (5) and empirical cumulative distribution function (ECDF) plot of this data is represented in figure (2).
To calculate the Bayes estimates of the parameters in absence of prior information, we consider the non-informative prior. Further we calculate the Maximum likelihood estimates of the parameter and also Bayes estimates of the parameters under different considered estimation methods which are presented in Table 4. The MCMC iterations of α and λ are plotted respectively. Density and Trace plots are indicating that the MCMC samples are well mixed and stationary achieved see figure 3.

5. Conclusion

In this paper, we have considered the classical as well as Bayesian estimation of the unknown parameters of the Marshall- Olkin extended exponential distribution under various approximation techniques. On the basis of extensive study we may conclude the followings:

- Under informative setup the performance of Bayes estimators of the parameters is better than the maximum likelihood estimators (MLE's) in all considered approximation techniques and also Lindley's approximation technique works quite well than rest of other methods such as T-K and MCMC.

- Under non-informative set up, we observed that T-K approximation method behaves like maximum likelihood estimators (MLE's) and performs well than Lindleys and MCMC approximation methods.

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Table 1. This table represents the estimates of the parameters obtained through various estimation techniques when prior mean is 2 and prior variance is 1 i.e.$\mu = 2$,$\nu = 1$ and also the quantity in second row exhibits the average expected loss over sample space i.e. risks of corresponding estimators.

Size	MLE		T-K		Lindley's		MCMC	
n	$\hat{\alpha}_M$	$\hat{\lambda}_M$	$\hat{\alpha}_S^{T-K}$	$\hat{\lambda}_S^{T-K}$	$\hat{\alpha}_S^{L}$	$\hat{\lambda}_S^{L}$	$\hat{\alpha}_S^{MC}$	$\hat{\lambda}_S^{MC}$
20	2.23773	2.06460	2.23737	2.06445	1.77297	1.98913	2.28136	2.07794
	1.39995	0.39985	1.40010	0.39962	0.26508	0.26972	1.15025	0.30349
30	2.21472	2.04999	2.21469	2.04986	1.95451	2.00332	2.23035	2.05050
	1.19207	0.26914	1.19255	0.26902	0.24508	0.21117	0.96397	0.19768
50	2.26295	2.06143	2.26278	2.06138	2.11909	2.03343	2.25792	2.05326
	1.00657	0.19669	1.00657	0.19668	0.39802	0.16998	0.88427	0.16388

Table 2. This table represents the estimates of the parameters obtained through various estimation techniques when prior mean is 2 and prior variance is 10 i.e.$\mu = 2$,$\nu = 10$ and also the quantity in second row exhibits the average expected loss over sample space i.e. risks of corresponding estimators.

Size	MLE		T-K		Lindley's		MCMC	
n	$\hat{\alpha}_M$	$\hat{\lambda}_M$	$\hat{\alpha}_S^{T-K}$	$\hat{\lambda}_S^{T-K}$	$\hat{\alpha}_S^{L}$	$\hat{\lambda}_S^{L}$	$\hat{\alpha}_S^{MC}$	$\hat{\lambda}_S^{MC}$
20	2.23773	2.06460	2.23834	2.06459	2.28362	2.00365	2.18074	2.00827
	1.39995	0.39985	1.40243	0.40017	1.32136	0.36720	1.40399	0.42792
30	2.21472	2.04999	2.21525	2.05007	2.24926	2.00974	2.14724	1.99070
	1.19207	0.26914	1.19239	0.26918	1.14257	0.25448	1.18310	0.28731
50	2.26295	2.06143	2.26265	2.06135	2.28564	2.03720	2.21927	2.02727
	1.00657	0.19669	1.00658	0.19668	0.98506	0.18914	0.99635	0.20304

TABLE 3. Table represents the estimates of the parameters obtained through various estimation techniques and also the quantity in square bracketed exhibits the average expected loss over sample space i.e. risks under non-informative prior.

Size	MLE		T-K		Lindley's		MCMC	
n	$\hat{\alpha}_M$	$\hat{\lambda}_M$	$\hat{\alpha}_S^{T-K}$	$\hat{\lambda}_S^{T-K}$	$\hat{\alpha}_S^{L}$	$\hat{\lambda}_S^{L}$	$\hat{\alpha}_S^{MC}$	$\hat{\lambda}_S^{MC}$
20	2.23773	2.06460	2.23750	2.06448	2.34036	2.00526	2.15852	1.98538
	1.39995	0.39985	1.40114	0.40005	1.60352	0.37905	1.46574	0.47475
30	2.21472	2.04999	2.21514	2.05005	2.28201	2.01046	2.12848	1.97225
	1.19207	0.26914	1.19233	0.26916	1.30748	0.25955	1.23519	0.32177
50	2.26295	2.06143	2.26262	2.06137	2.30415	2.03762	2.21263	2.02207
	1.00657	0.19669	1.00659	0.19669	1.07019	0.19134	1.01519	0.21145

TABLE 4. This table represents the estimates of the parameters obtained by various methods of estimation for real data set under the assumption that prior information assume to be non-informative.

Size	MLE		T-K		Lindley's		MCMC	
n	$\hat{\alpha}_M$	$\hat{\lambda}_M$	$\hat{\alpha}_S^{T-K}$	$\hat{\lambda}_S^{T-K}$	$\hat{\alpha}_S^{L}$	$\hat{\lambda}_S^{L}$	$\hat{\alpha}_S^{MC}$	$\hat{\lambda}_S^{MC}$
16	8.62532	0.50074	8.62534	0.50074	9.12253	0.48963	8.62581	0.49910

TABLE 5. This Table represents the values of Log-likelihood, AIC and BIC for different models in real data set.

Distribution	-log L	Information Criterion	
		AIC	BIC
Exponential	40.762	83.524	84.296
Generalized Exponential	38.836	81.673	83.218
Gamma	38.629	81.258	82.803
MOEED	38.044	80.089	81.634

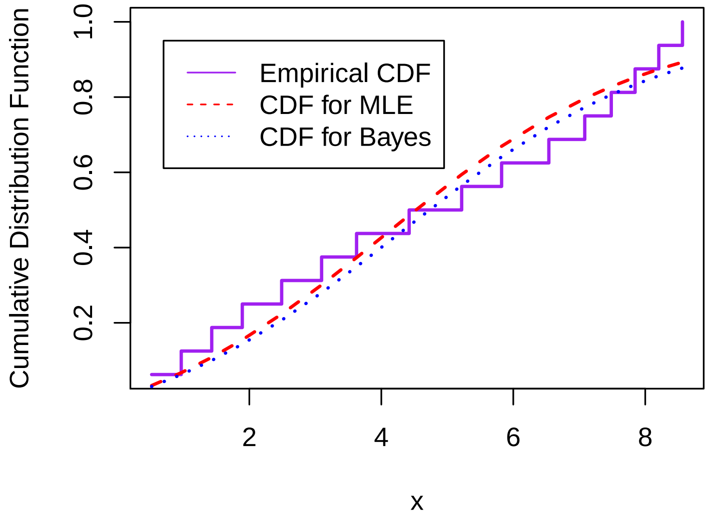

FIGURE 2. CDF plot for considered real data set

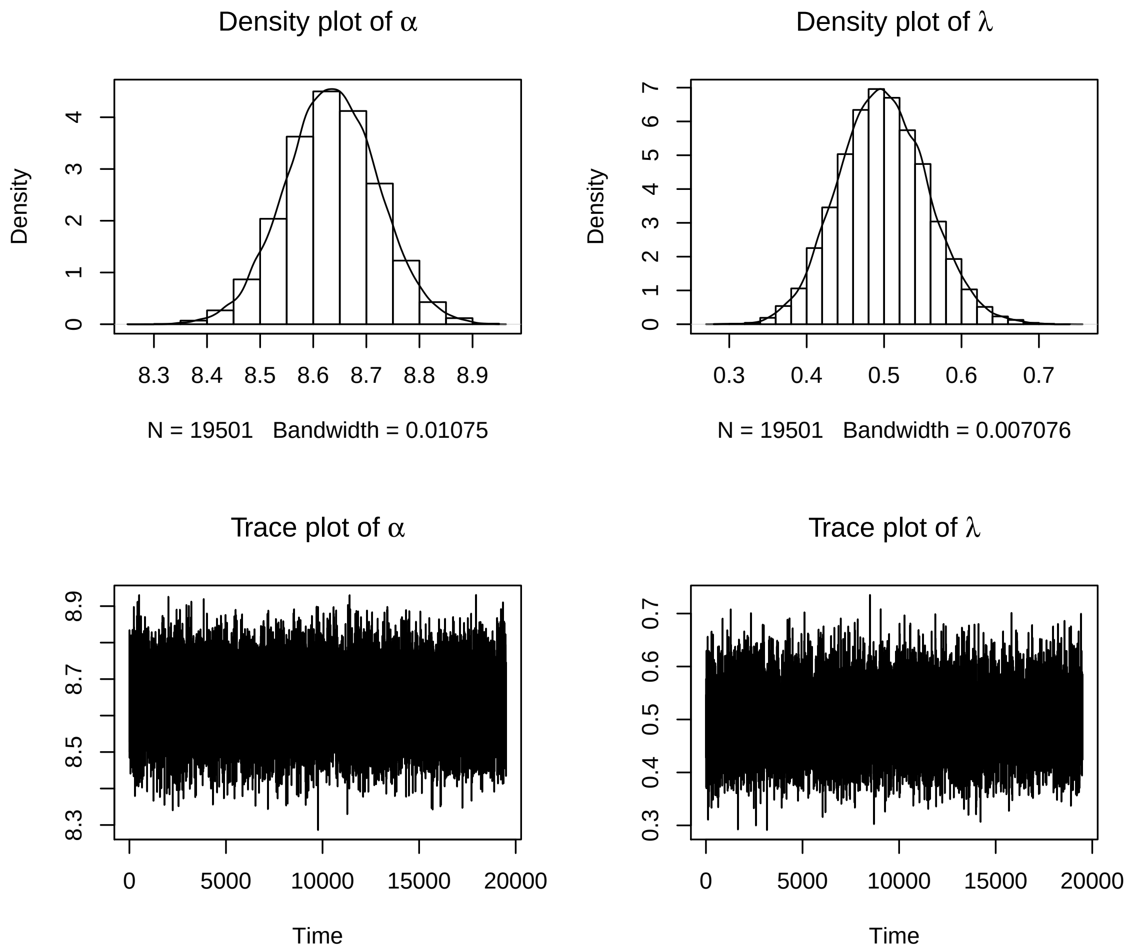

FIGURE 3. Posterior density and trace plot for considered real data set.

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