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Small supplements, weak supplements and proper classes

Rafail Alizade*, Engin Büyükaşık† and Yılmaz Durğun‡§

Abstract

Let $\mathcal{SS}$ denote the class of short exact sequences $\mathbb{E}: 0 \to A \xrightarrow{f} B \to C \to 0$ of R-modules and R-module homomorphisms such that $f(A)$ has a small supplement in B i.e. there exists a submodule K of M such that $f(A) + K = B$ and $f(A) \cap K$ is a small module. It is shown that, $\mathcal{SS}$ is a proper class over left hereditary rings. Moreover, in this case, the proper class $\mathcal{SS}$ coincides with the smallest proper class containing the class of short exact sequences determined by weak supplement submodules. The homological objects, such as, $\mathcal{SS}$-projective and $\mathcal{SS}$-coinjective modules are investigated. In order to describe the class $\mathcal{SS}$, we investigate small supplemented modules, i.e. the modules each of whose submodule has a small supplement. Besides proving some closure properties of small supplemented modules, we also give a complete characterization of these modules over Dedekind domains.

Keywords: Proper class of short exact sequences, weak supplement submodule, small module, small supplement submodule

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1. Introduction

All rings are associative with identity element and all modules are unitary left modules. We use the notation $E(M)$, $\mathrm{Soc}(M)$, $\mathrm{Rad}(M)$, for the injective hull, socle, radical of an R-module M respectively. Let M be any module and let N and K be submodules of M. N is said to be *small (or superfluous) in a module* M, denoted as $N \ll M$ if $N+K=M$ implies $K=M$ for any submodule K of M. N is said to be a *small module* if N is a small submodule of some R-module. N is a small module if and only if N is a small submodule of its injective envelope (see, [9]). A submodule N of M is called a supplement of K in M if N is minimal with respect to the property $M=K+N$, equivalently, $M=K+N$ and $K \cap N \ll N$. A submodule K of M has a supplement in M provided there exists a submodule N of M such that N is a supplement of K in M. A submodule N of M has (is) a weak supplement K in M if $M=K+N$ and $K \cap N \ll M$. If every submodule of M has a (weak) supplement in M, then M is called (weakly) supplemented.

Proper classes were introduced by Buchsbaum in order to axiomatize conditions under which a class of short exact sequences of modules can be computed as Ext groups corresponding to a certain relative homology. Let $\mathbb{E}: 0 \to A \xrightarrow{f} B \to C \to 0$ be a short exact sequence. It is well-known that the class of short exact sequences $\mathbb{E}$ such that $\mathrm{Im}(f)$ is a supplement in B, respectively pure in B is a proper class in the sense of Buchsbaum (see, [7, 20.7]). However, many other analogous classes of short exact sequences of modules do not form a proper class. For example, the classes $\mathcal{S}mall$, $\mathcal{S}$ or $\mathcal{WS}$ i.e. the classes of short exact sequences $\mathbb{E}$ such that $\mathrm{Im}\, f$ small in B, has a supplement in B, or has a weak supplement in B, respectively, are not proper classes. But, in this case, one may consider the least proper class containing a given class of short exact sequences, that is, the intersection of all proper classes containing them. Recently, in [3], the authors shows that, the least proper classes containing the classes $\mathcal{S}mall$, $\mathcal{S}$ or $\mathcal{WS}$ coincide over hereditary rings. They obtained this proper class by natural extension of the class $\mathcal{WS}$ and denoted it by $\overline{\mathcal{WS}}$.

At this point, the question which arises naturally is that, whether the class $\overline{\mathcal{WS}}$ can be described as a class of short exact sequences $\mathbb{E}$ such that $\mathrm{Im}(f)$ has a certain property in B. The answer of this question is affirmative over left hereditary rings. Over such rings the class $\overline{\mathcal{WS}}$ coincides with the class determined by small supplements.

The paper is organized as follows.

In section 3, weakening the notion of weak supplement we consider small supplement submodules. Namely, a submodule N of a module M has a small supplement in M if there exists a submodule K of M such that $N+K=M$ and $N \cap K$ is a small module. Let $\mathcal{SS}$ be the class of short exact sequences such that $\mathrm{Im}(f)$ has a small supplement in B. We prove that, $\mathcal{SS}$ is a subgroup of Ext, and over a hereditary ring $\mathcal{SS}$ is a proper class. Moreover, $\mathcal{SS}$ coincides with the proper class $\overline{\mathcal{WS}}$.

In section 4, we investigate $\mathcal{SS}$-projective modules which are projective relative to short exact sequences that belong to $\mathcal{SS}$. We show that an R-module F is $\mathcal{SS}$-projective if and only if $\mathrm{Ext}(F, S)=0$ for each small R-module S. Moreover, we prove that every $\mathcal{SS}$-projective module is flat if R is commutative C-ring (i.e. $\mathrm{Soc}(R/I) \neq 0$ for each essential proper left ideal I).

In section 5, we study on the properties of the modules whose submodules have small supplements. We call these modules small supplemented. Small supplemented modules are proper generalization of weakly supplemented modules. It is shown that, small supplemented modules are closed under submodules, factor modules, finite sums and extensions. An injective module is small supplemented if and only if it is weakly supplemented.

In section 6, we characterize small supplemented modules over Dedekind domains. We prove that, an R-module M is small supplemented if and only if every primary component of $T(M)$ is a direct sum of a bounded submodule and an artinian submodule, and $M/T(M)$ has finite uniform dimension, where $T(M)$ is the torsion submodule of M.

2. Proper Classes

Let us recall the definition of a proper class of short exact sequences (e.g., see [4], [7], [11], [19]).

2.1. Definition. Let $\mathcal{P}$ be a class of short exact sequences of R-modules and R-module homomorphisms. If a short exact sequence $\mathbb{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ belongs to $\mathcal{P}$, then f is said to be a $\mathcal{P}$-*monomorphism* and g is said to be an $\mathcal{P}$-*epimorphism*. Also, $\mathbb{E}$ is said to be a $\mathcal{P}$-exact sequence.

The class $\mathcal{P}$ is said to be *a proper class* (in the sense of Buchsbaum) if it has the following properties:

P-1) $\mathcal{P}$ is closed under isomorphisms;
P-2) $\mathcal{P}$ contains all splitting short exact sequences;
P-3) The class of $\mathcal{P}$-monomorphisms is closed under composition; if f, g are monomorphisms and gf is an $\mathcal{P}$-monomorphism, then f is an $\mathcal{P}$-monomorphism;
P-4) The class of $\mathcal{P}$-epimorphisms is closed under composition; if f, g are epimorphisms and gf is an $\mathcal{P}$-epimorphism, then g is an $\mathcal{P}$-epimorphism.

2.2. Example. Some examples of proper classes, which are interesting for the purpose of this paper are the following (e.g., see [7]).

(i) The class $\mathcal{S}plit$ of all splitting short exact sequences.
(ii) The class $\mathcal{P}$ of all short exact sequences on which the functor $\mathrm{Hom}(M, -)$ is exact for every $M \in \mathcal{M}$, where $\mathcal{M}$ is a class of modules. Its elements are called $\mathcal{P}$-pure exact sequences. For the class $\mathcal{M}$ of finitely presented modules, one has the classical pure exact sequences.
(iii) The classes of all short exact sequences $0 \to A \xrightarrow{f} B \to C \to 0$ with $\mathrm{Im}\, f$ is a supplement, or a closed submodule of B are proper classes.

The set $\mathrm{Ext}^1_{\mathcal{P}}(C, A)$ of all short exact sequence of $\mathrm{Ext}^1_R(C, A)$ that belongs to a proper class $\mathcal{P}$ is a subgroup of the group of the extensions $\mathrm{Ext}^1_R(C, A)$. Conversely given a class $\mathcal{P}$ of short exact sequences if $\mathrm{Ext}^1_{\mathcal{P}}(C,\ A)$ is a subfunctor of $\mathrm{Ext}^1_R(C,\ A)$, $\mathrm{Ext}^1_{\mathcal{P}}(C,\ A)$ is a subgroup of $\mathrm{Ext}^1_R(C,\ A)$ for every R-modules A, C and the composition of two $\mathcal{P}$-monomorphisms (or $\mathcal{P}$-epimorphisms) is a $\mathcal{P}$-monomorphism (a $\mathcal{P}$-epimorphism respectively) then $\mathcal{P}$ is a proper class (see Theorem 1.1 in [14]). For any class $\mathcal{P}$ of short exact sequences the intersection $\langle\mathcal{P}\rangle$ of all proper classes containing $\mathcal{P}$ is clearly a proper class. We say that $\langle\mathcal{P}\rangle$ is the proper class generated by $\mathcal{P}$ (see [15]). Clearly $\langle\mathcal{P}\rangle$ is the least proper class containing $\mathcal{P}$.

2.3. Definition. [3] A short exact sequence $\mathbb{E} : 0 \to A \to B \to C \to 0$ is said to be *extended weak supplement* if there is a short exact sequence $\mathbb{E}' : 0 \to A \xrightarrow{f} B' \to C' \to 0$ such that $\mathrm{Im}\, f$ has (is) a weak supplement in B' and there is a homomorphism $g : C \to C'$ such that $\mathbb{E} = g^*(\mathbb{E}')$, that is, there is a commutative diagram as follows:

$$
\begin{array}{ccccccccl}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 : \mathbb{E} \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B' & \longrightarrow & C' & \longrightarrow & 0 : \mathbb{E}'
\end{array}
$$

The class of all extended weak supplement short exact sequences will be denoted by $\overline{\mathcal{WS}}$. So $\mathrm{Ext}_{\overline{\mathcal{WS}}}(C,\, A) = \{\mathbb{E} : 0 \to A \to B \to C \to 0 \mid \mathbb{E} = g^*(\mathbb{E}')$ for some $\mathbb{E}' : 0 \to A \overset{f}{\to} B' \to C' \to 0 \in \mathcal{WS}$ and $g : C \to C'$ $\}$.

The class $\overline{\mathcal{WS}}$ is the least proper class containing the class $\mathcal{WS}$ (see, [3]).

3. The Proper Class $\mathcal{SS}$

3.1. Definition. A submodule L of an R-module M has a small supplement in M if there is a submodule K of M such that $L + K = M$ and $L \cap K$ is a small module.

Let $\mathcal{SS}$ be the class of all short exact sequences $\mathbb{E} : 0 \to A \overset{f}{\to} B \to C \to 0$ such that $\operatorname{Im} f$ has a small supplement in B. To prove that $\mathcal{SS}$ is a proper class we will use the result of [14, Theorem 1.1].

Firstly, we show that $\mathrm{Ext}_{\mathcal{SS}}(C, A)$ is a subgroup of $\mathrm{Ext}^1(C, A)$ for every R-modules A, C. The following lemma can be proved by using similar arguments as in [3, Lemma 3.3].

3.2. Lemma. *For every homomorphism* $f : A \to A'$, $f_* : \mathrm{Ext}(C,\, A) \to \mathrm{Ext}(C,\, A')$ *preserves short exact sequences from* $\mathcal{SS}$.

3.3. Lemma. *For every homomorphism* $g : C' \to C$, *the homomorphism* $g^* : \mathrm{Ext}(C,\, A) \to \mathrm{Ext}(C',\, A)$ *preserves short exact sequences from* $\mathcal{SS}$.

Proof. Let $\mathbb{E} : 0 \to A \to B \overset{h}{\to} C \to 0$ be a short exact sequence in $\mathcal{SS}$ and $g : C' \to C$ be a homomorphism. Then the following diagram is commutative with exact rows.

$$\begin{array}{ccccccccc}
\mathbb{E}_1 : 0 & \longrightarrow & A & \longrightarrow & B' & \xrightarrow{h'} & C' & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \\
\mathbb{E} : 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{h} & C & \longrightarrow & 0
\end{array}$$

where $g^*(\mathbb{E}) = \mathbb{E}_1$. Let V be a small supplement of $\operatorname{Ker} h$ in B. Then $\operatorname{Ker} h + V = B$ and $V \cap \operatorname{Ker} h$ is a small module. Then ${g'}^{-1}(V) + \operatorname{Ker} h' = B'$ by the pullback diagram. Since g' induces an isomorphism between ${g'}^{-1}(V) \cap \operatorname{Ker} h'$ and $V \cap \operatorname{Ker} h$, ${g'}^{-1}(V) \cap \operatorname{Ker} h'$ is a small module by [9, Theorem 2]. Therefore, $E_1 \in \mathcal{SS}$. □

The proof of the following is routine, hence we skip its proof.

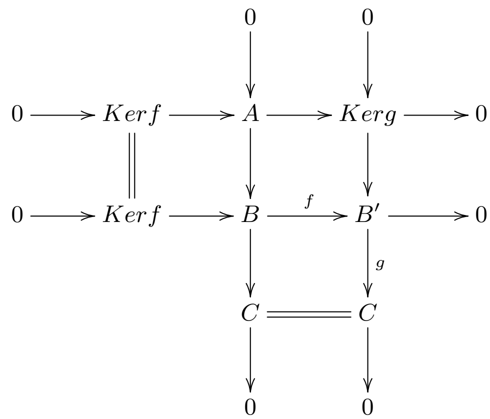
3.4. Proposition. *If* $\mathbb{E}_1,\, \mathbb{E}_2 \in \mathrm{Ext}_{\mathcal{SS}}(C,\, A)$, *then* $\mathbb{E}_1 \oplus \mathbb{E}_2 \in \mathrm{Ext}_{\mathcal{SS}}(C \oplus C,\, A \oplus A)$.

3.5. Corollary. $\mathrm{Ext}_{\mathcal{SS}}(C,\, A)$ *is a subgroup of* $\mathrm{Ext}(C,\, A)$ *for every modules* C *and* A.

Proof. Let $\mathbb{E}_1,\, \mathbb{E}_2 \in \mathrm{Ext}_{\mathcal{SS}}(C,\, A)$. $\mathbb{E}_1 \oplus \mathbb{E}_2$ is $\mathcal{SS}$-element by Proposition 3.4. Since $\mathbb{E}_1 + \mathbb{E}_2 = \nabla_A(\mathbb{E}_1 \oplus \mathbb{E}_2)\Delta_C$ where $\Delta_C : c \mapsto (c,\, c)$ is the the diagonal map and $\nabla_A : (a_1,\, a_2) \mapsto a_1 + a_2$ is the codiagonal map, $\mathbb{E}_1 + \mathbb{E}_2$ is in $\mathcal{SS}$ by Lemma 3.2 and Lemma 3.3. □

3.6. Theorem. *If* R *is a left hereditary ring, then* $\mathcal{SS}$ *is a proper class.*

Proof. By Lemma 3.2, Lemma 3.3, Corollary 3.5, $\mathrm{Ext}^1_{\mathcal{SS}}(C,\, A)$ is a subfunctor of $\mathrm{Ext}^1_R(C,\, A)$ and $\mathrm{Ext}^1_{\mathcal{SS}}(C,\, A)$ is a subgroup of $\mathrm{Ext}^1_R(C,\, A)$ for every R-modules A and C. By [14, Theorem 1.1], we only need to show that the composition of two $\mathcal{SS}$-epimorphisms is an $\mathcal{SS}$-epimorphism. Let $f : B \to B'$ and $g : B' \to C$ be $\mathcal{SS}$-epimorphisms. We have the following commutative diagram with exact rows and columns:

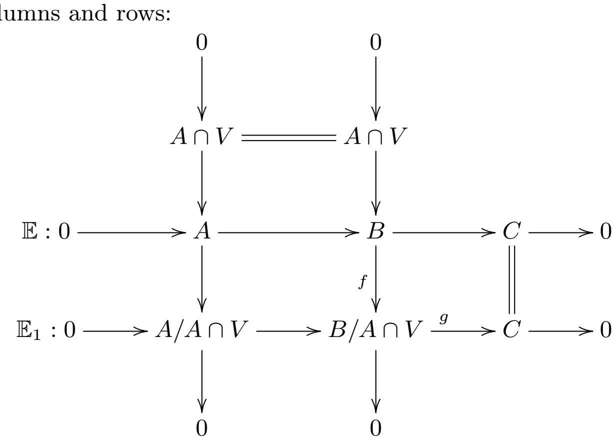


where $A/\operatorname{Ker} f \cong \operatorname{Ker} g$, $B/\operatorname{Ker} f \cong B'$. Therefore there exist a submodule V in B such that $\operatorname{Ker} f + V = B$ and $\operatorname{Ker} f \cap V$ is a small module and there exist a submodule $U/\operatorname{Ker} f$ in $B/\operatorname{Ker} f$ such that $(U/\operatorname{Ker} f) + (A/\operatorname{Ker} f) = B/\operatorname{Ker} f$ and $(A \cap U)/\operatorname{Ker} f$ is a small module. By modular law, $A = \operatorname{Ker} f + (A \cap V)$, $U = \operatorname{Ker} f + (U \cap V)$, $A \cap U = \operatorname{Ker} f + (A \cap V \cap U)$. Therefore, $B = A + U = A + (U \cap V)$ and $(A \cap U)/\operatorname{Ker} f \cong (A \cap U \cap V)/(\operatorname{Ker} f \cap V)$. Since $\operatorname{Ker} f \cap V$ and $(A \cap U \cap V)/(\operatorname{Ker} f \cap V)$ are small modules and R is a hereditary ring, $A \cap U \cap V$ is small by [9, Theorem 3]. Hence $g \circ f$ is a $\mathcal{SS}$-epimorphism. □

A module M is said to be $\overline{\mathcal{WS}}$-coinjective if every extension of M is extended weak supplement.

3.7. Theorem. *The classes $\mathcal{SS}$ and $\overline{\mathcal{WS}}$ coincide over left hereditary rings.*

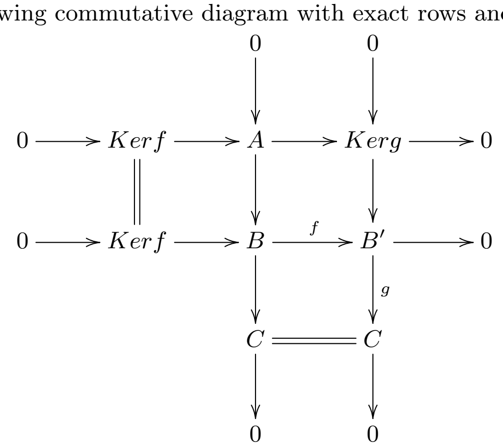
Proof. Let $0 \to A \to B \to C \to 0 \in \mathcal{SS}$. Then there is a submodule V of B such that $B = A + V$ and $A \cap V$ is a small module. So we have the following commutative diagram with exact columns and rows:



Clearly g is a $\mathcal{S}plit$-epimorphism, and since small modules are $\overline{\mathcal{WS}}$-coinjective by [3, Theorem 4.1], f is an $\overline{\mathcal{WS}}$-epimorphism. Since R is hereditary, $\overline{\mathcal{WS}}$ is a proper class by [3, Theorem 3.12], and hence the composition $g \circ f$ is a $\overline{\mathcal{WS}}$-epimorphism. Then, E is in $\overline{\mathcal{WS}}$. Conversely, since $\mathcal{WS} \subseteq \mathcal{SS}$ and $\overline{\mathcal{WS}}$ is the smallest proper class containing $\mathcal{WS}$, we have $\overline{\mathcal{WS}} \subseteq \mathcal{SS}$. □

3.8. Lemma. *The composition $g \circ f$ of a $\mathcal{S}plit$-epimorphism f and an $\mathcal{SS}$-epimorphism g is an $\mathcal{SS}$-epimorphism.*

Proof. Let $f : B \to B'$ be a $\mathcal{S}plit$-epimorphism and $g : B' \to C$ an $\mathcal{SS}$-epimorphism. We have the following commutative diagram with exact rows and columns:

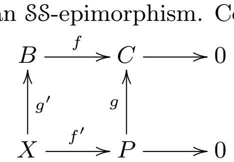


where $A/\operatorname{Ker} f \cong \operatorname{Ker} g$, $B/\operatorname{Ker} f \cong B'$. Therefore there exist a submodule V in B such that $\operatorname{Ker} f \oplus V = B$ and there exist a submodule $U/\operatorname{Ker} f$ in $B/\operatorname{Ker} f$ such that $(U/\operatorname{Ker} f) + (A/\operatorname{Ker} f) = B/\operatorname{Ker} f$ and $(A \cap U)/\operatorname{Ker} f$ is a small module. By modular law, $U = \operatorname{Ker} f \oplus (U \cap V)$, $A \cap U = \operatorname{Ker} f \oplus (A \cap V \cap U)$. Therefore, $B = A + U = A + (U \cap V)$. Since $(A \cap U)/\operatorname{Ker} f$ is a small module and $(A \cap U)/\operatorname{Ker} f \cong (A \cap U \cap V)$, $A \cap U \cap V$ is a small module. Hence $g \circ f$ is an $\mathcal{SS}$-epimorphism. □

An epimorphism $f : N \to M$ is said to be a *small cover* of M if $\operatorname{Ker} f \ll N$. Moreover, if N is projective, then f is called a *projective cover.* A ring R is said to be left (semi) perfect if every (finitely generated) module has a projective cover.

3.9. Corollary. *If R is a left perfect ring, then every short exact sequence is an $\mathcal{SS}$-exact. In particular, $\mathcal{SS}$ is a proper class.*

Proof. Let $0 \to A \to B \xrightarrow{f} C \to 0$ be a short exact sequence. Since R is left perfect ring, there exists an epimorphism $g : P \to C$ where P is a projective R-module and $\operatorname{Ker} g \ll P$. Therefore, g is an $\mathcal{SS}$-epimorphism. Consider the pullback diagram:



Since P is projective, f' is a $\mathcal{S}plit$-epimorphism. Then $g \circ f'$ is an $\mathcal{SS}$-epimorphism by Lemma 3.8, and hence f is an $\mathcal{SS}$-epimorphism by $P-4$). □

4. Homological Objects of The Class $\mathcal{SS}$

We begin with the following definition.

4.1. Definition. An R-module F is called $\mathcal{SS}$-projective if it is projective relative to the short exact sequences that belong to $\mathcal{SS}$ i.e., for each $\mathbb{E}$ in $\mathcal{SS}$ the sequence $\operatorname{Hom}(F, \mathbb{E})$ is exact.

4.2. Proposition. *The following are equivalent for an R-module F.*

(1) *F is $\mathcal{SS}$-projective.*

(2) $\mathrm{Ext}(F, S) = 0$ *for each small R-module S.*

Proof. (1) $\Rightarrow$ (2) is clear, since every short exact sequence starting with a small module is in $\mathcal{SS}$.

(2) $\Rightarrow$ (1) Let $0 \to A \to B \to C \to 0$ be a short exact sequence belongs to $\mathcal{SS}$. Then there is a submodule V of B such that $B = A + V$ and $A \cap V$ is a small module. So we have the following commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{g\pi} & C & \longrightarrow & 0 \\ \downarrow{\scriptstyle \pi} & & \| & & \\ B/(A \cap V) & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

where π is the canonical epimorphism and g a split epimorphism. Applying the functor $\mathrm{Hom}(F, .)$, we have the following diagram

$$\begin{array}{ccccc} \mathrm{Hom}(F, B) & \xrightarrow{g_*\pi_*} & \mathrm{Hom}(F, C) & & \\ \downarrow{\scriptstyle \pi_*} & & \| & & \\ \mathrm{Hom}(F, B/(A \cap V)) & \xrightarrow{g_*} & \mathrm{Hom}(F, C) & \longrightarrow & 0 \\ \downarrow & & & & \\ \mathrm{Ext}(F, A \cap V) & & & & \end{array}$$

Since g is a split epimorphism, g_* is an epimorphism. Then $\mathrm{Ext}(F, A \cap V) = 0$ by (2), and so π_* is an epimorphism. Therefore, $g_*\pi_*$ is an epimorphism. Thus F is an $\mathcal{SS}$-projective module.

□

Note that every (finitely generated) $\mathcal{SS}$-projective module is projective if R is left (semi) perfect by Proposition 4.2.

A ring R is said to be left C-ring if $\mathrm{Soc}(R/I) \neq 0$ for each proper essential left ideal I of R, (see [16]). Right perfect rings and left semiartinian rings are left C-rings. One of the characterization of left C-rings is the following: R is a left C-ring if and only if $\mathrm{Ext}(S, M) = 0$ for each simple R-module S implies M is injective R-module, ([20, Lemma 4]).

4.3. Theorem. *Let R be a commutative C-ring. Then $\mathcal{SS}$-projective R-modules are flat.*

Proof. Let M be an $\mathcal{SS}$-projective R-module. Since every simple R-module is either small or injective, for each simple R-module S, $\mathrm{Ext}(M, S) = 0$ by Proposition 4.2. Note that if R is commutative and E is an injective cogenerator, then $\mathrm{Hom}(S, E) \cong S$ for each simple R-module S. Then $\mathrm{Ext}(M, \mathrm{Hom}(S, \mathbb{Q}/\mathbb{Z}) = 0$. By the standart adjoint isomorphism $\mathrm{Hom}(Tor(M, S), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}(S, \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})) = 0$. Hence, $\mathrm{Hom}(Tor(M, S), \mathbb{Q}/\mathbb{Z}) = 0$ by [17, Theorem 2.75]. But R is C-ring, and so $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is injective. Therefore, M is flat by [17, Proposition 3.54].

□

Renault [16] proved that a left Noetherian ring is a C-ring if and only if for every essential left ideal I of R, R/I has finite length. If R is a left and right Noetherian, and left and right hereditary ring, then for every essential (proper) left ideal I of R, the left R-module R/I has finite length, ([12, Proposition 5.4.5]). Therefore, hereditary Noetherian commutative rings are C-rings.

In [15], it is shown that, for a class of short exact sequences $\mathcal{E}$, modules which are relatively projective with respect to the classes $\mathcal{E}$ and $\langle\mathcal{E}\rangle$ coincide. Therefore, by Theorem 3.7, we get:

4.4. Corollary. *Let R be a commutative hereditary Noetherian ring. Then every R-module which is relatively projective with respect to the short exact sequences in $\mathcal{WS}$ is flat.*

4.5. Remark. Let M be a left R-module. M is called $\mathcal{SS}$-coinjective if every short exact sequence starting with M is in $\mathcal{SS}$. Every small module is $\mathcal{SS}$-coinjective. M is called almost injective if M is a supplement submodule in each module that contains M as a submodule see [6]. It is easy to see that almost injective modules are $\mathcal{SS}$-coinjective, but the converse is not true in general. For example, $\mathbb{Z}$ is a small module, and so $\mathbb{Z}$ is $\mathcal{SS}$-coinjective. On the other hand, $\mathbb{Z}$ has no supplement in $\mathbb{Q}$, hence it is not almost injective.

Recall that a ring R is called a left V-ring if every simple R-module is injective or, equivalently, $\mathrm{Rad}(M) = 0$ for every R-module M (see [8, Theorem 3.75]).

4.6. Proposition. *The ring R is a left V-ring if and only if every $\mathcal{SS}$-coinjective R-module is injective.*

Proof. Let M be an $\mathcal{SS}$-coinjective R-module. Then M is small supplement in $E(M)$ i.e., there is a submodule V of $E(M)$ such that $E(M) = A + V$ and $A \cap V \ll E(M)$. But R is V-ring, hence $A \cap V = 0$. Then A is direct summand of $E(M)$, and so it is injective. The converse follows easily since every simple R-module is either small or injective. □

5. Small Supplemented Modules

An R-module M is called *small supplemented* if every submodule of M has a small supplement. In this section, we shall prove some properties of small supplemented modules. The proof of the following proposition is standard. We shall use it in the sequel.

5.1. Proposition. *Let M_1, U be submodules of M with M_1 small supplemented. If there is a small supplement for $M_1 + U$ in M, then U also has a small supplement in M.*

Proof. Let V be a small supplement of $M_1 + U$ in M, i.e. $V + (M_1 + U) = M$ and $V \cap (M_1 + U)$ is small. Since M_1 is small supplemented, there exist a submodule T of M_1 such that $T + [M_1 \cap (V + U)] = M_1$ and $T \cap [M_1 \cap (V + U)] = T \cap (V + U)$ is a small module. Then $M = V + T + [M_1 \cap (V + U)] + U = V + T + U$. Hence $U \cap (V + T) \subseteq T \cap (V + U) + V \cap (T + U)$, and so $U \cap (V + T)$ is a small module by [9, Theorem 2]. □

5.2. Corollary. *If $M = M_1 + M_2$ with M_1, M_2 small supplemented modules, then M is also small supplemented.*

Proof. For every submodule $N \subseteq M$, $M_1 + (M_2 + N)$ has the trivial small supplement and so, by Proposition 5.1, $M_2 + N$ has a small supplement. Then, again by Proposition 5.1, N has a small supplement. □

5.3. Proposition. *The class of small supplemented modules is closed under submodules, homomorphic images and finite sums.*

Proof. Let L be a submodule of a small supplemented module M. Suppose $T \subseteq L$ and N is a small supplement of T in M. Then $L = T + (N \cap L)$ and $T \cap N \cap L \subseteq N \cap T \ll E(N \cap T)$ showing that L is small supplemented.

Let N be a submodule of a small supplemented module M. Given a submodule K/N of M/N, let L be a small supplement of K in M. Then $M = K+L$ and $K \cap L \ll E(K \cap L)$. Thus $K/N + (L+N)/N = M/N$ and $(K/N) \cap ((L+N)/N) = ((K \cap L) + N)/N \cong (K \cap L)/(N \cap L)$ is a small module by [9, Theorem 2]. Thus $(L+N)/N$ is a small supplement of K/N in M/N. So M/N is small supplemented. The rest of the proof hold by Corollary 5.2. □

Note that small modules are closed under extensions over left hereditary rings (see, [9]).

5.4. Lemma. *Let R be a hereditary ring and M be a small supplemented R-module. If $f : N \to M$ is an epimorphism with* $\operatorname{Ker} f$ *a small module, then N is small supplemented.*

Proof. Let $K = \operatorname{Ker} f$. Since $M \cong N/K$, N/K is small supplemented by Proposition 5.3. Let L be a submodule of N. Then $(L+K)/K$ has a small supplement, say T/K, in N/K. So that $((L+K)/K) + (T/K) = N/K$ and $[(L+K)/K] \cap (T/K) = ((T \cap L) + K)/K \cong (T \cap L)/(K \cap L)$ is a small module. Then $N = L+T$ and $L \cap T$ is a small module by [9, Theorem 3]. Therefore N is small supplemented. □

5.5. Proposition. *Let R be a hereditary ring and $0 \to L \to M \to N \to 0$ be a short exact sequence from $\mathcal{SS}$. Then L and N are small supplemented if and only if M is small supplemented.*

Proof. Without restriction of generality we will assume that $L \subseteq M$. Let S be a small supplement of L in M i.e. $L + S = M$ and $L \cap S \ll E(L \cap S)$. Then we have,

$$M/(L \cap S) = L/(L \cap S) \oplus S/(L \cap S)$$

$L/(L \cap S)$ is small supplemented as a factor module of L by Proposition 5.3. On the other hand, $S/(L \cap S) \cong M/L \cong N$ is small supplemented. Then $M/(L \cap S)$ is small supplemented as a sum of small supplemented modules by Proposition 5.3. Therefore M is small supplemented by Lemma 5.4. The converse holds by Proposition 5.3. □

A submodule $L \leq M$ is called coclosed in M, if for any proper submodule $K \leq L$, there is a submodule N of M such that $L + N = M$ but $K + N \neq M$. A module M is called *weakly injective* if for every extension X of M, M is coclosed in X. The properties of weakly injective modules are studied in [23].

5.6. Proposition. *Let M be a weakly injective module. Then M is small supplemented if and only if M is weakly supplemented.*

Proof. ($\Rightarrow$) Suppose M is small supplemented and let L be any submodule of M. Then $M = L + T$ and $L \cap T \ll E(M)$ where T is small supplement of L in M. Since M is weakly injective module, it is coclosed in its injective hull $E(M)$ and so $L \cap T \ll M$ by [23, Lemma A.2].

($\Leftarrow$) Clear. □

5.7. Proposition. *Every R-module is small supplemented if and only if every injective R-module is weakly supplemented.*

Proof. Suppose that I is an injective R-module. Let L be any submodule of I. By the assumption, there is a submodule T of I such that $I = L+T$ and $L \cap T \ll I$. Conversely, $E(M)$ is weakly supplemented for any R-module M by the assumption. Then M is small supplemented by Proposition 5.3. □

6. Small Supplemented Modules Over Dedekind Domains

In this section, we shall describe the structure of small supplemented modules over Dedekind domains. Recall that, a local Dedekind domain is called a discrete valuation ring (or, DVR). If R is a DVR, then the unique maximal ideal of R is of the form pR, for some $p \in R$ and every nonzero ideal of R is of the form $p^n R$ for some $n \in \mathbb{Z}^+$. For a Dedekind domain R, Ω and Q will stand for the set of maximal ideals of R, and the quotient ring of R respectively.

A module M is called coatomic, if $\mathrm{Rad}(M/N) \neq M/N$ for every proper submodule N of M, equivalently every proper submodule of M is contained in a maximal submodule, (see [22]). Recall that a module M over a Dedekind domain is divisible if and only if it is injective if and only if it has no maximal submodules (see, [1], [18]).

The following lemma can be obtained from [22, Section 4]. We include it for completeness.

6.1. Lemma. *Let R be a Dedekind domain and M be an R-module. Then M is coatomic if and only if M is a small module.*

Proof. Let M be a coatomic module and suppose $M + K = E(M)$ for some submodule K of M. Then $M/M \cap K \cong E(M)/K$ is injective and so $E(M)/K$ has no maximal submodules. As M is coatomic, we must have $M/(M \cap K) = 0$, i.e. $M \subseteq K$. So that $K = E(M)$, and hence M is a small module.

Conversely, if M is small and $\mathrm{Rad}(M/K) = M/K$ for some $K \subseteq M$, then M/K is injective. So that M/K is a direct summand of $E(M)/K$. On the other hand M/K is a small module as a factor of the small module M, a contradiction. Hence $K = M$ and so M is coatomic. □

6.2. Lemma. *Let R be a Dedekind domain and M be a small supplemented R-module. Then* $\mathrm{Rad}(M)$ *has a weak supplement in M.*

Proof. Since M is small supplemented, $\mathrm{Rad}(M) + L = M$ and $\mathrm{Rad}(M) \cap L$ is a small module, for some $L \subseteq M$. Let $A = \mathrm{Rad}(M) \cap L$ and suppose that $A + Y = M$ for some $Y \subsetneqq M$. Then A is coatomic by Lemma 6.1, and so $A/(A \cap Y) \cong (A + Y)/Y = M/Y$ is also coatomic. So there is a maximal submodule Z of M containing Y. Now, we have $M = \mathrm{Rad}(M) + Y = \mathrm{Rad}(M) + Z \subseteq Z$, a contradiction. Therefore $A \ll M$, and so L is a weak supplement of $\mathrm{Rad}(M)$ in M. □

6.3. Lemma. [22, Lemma 4.1] *Let R be a commutative noetherian ring and M be an R-module. Then a submodule U of M is small in M if and only if $U_{\mathfrak{m}}$ is small in $M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R.*

6.4. Lemma. *Let R be a commutative noetherian ring and M be an R-module. If a submodule V of M is small supplement of a submodule U of M, then $V_{\mathfrak{m}}$ is a small supplement of $U_{\mathfrak{m}}$ in $M_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R.*

Proof. Suppose $U+V = M$ and $U \cap V$ is a small module. Then $U_{\mathfrak{m}} + V_{\mathfrak{m}} = (U+V)_{\mathfrak{m}} = M_{\mathfrak{m}}$, and $(U \cap V)_{\mathfrak{m}} = U_{\mathfrak{m}} \cap V_{\mathfrak{m}}$ is a small module by Lemma 6.3. So that $V_{\mathfrak{m}}$ is a small supplement of $U_{\mathfrak{m}}$ in $M_{\mathfrak{m}}$. □

6.5. Proposition. *Let R be a Dedekind domain and M be a torsion free R-module. Then M is small supplemented if and only if M has finite uniform dimension.*

Proof. Assume that the uniform dimension of M is not finite. Then M has a submodule L such that $L \cong R^{(\mathbb{N})}$. Then $R^{(\mathbb{N})}$ is small supplemented by Proposition 5.3. Set $N = R^{(\mathbb{N})}$.

Let $\mathfrak{m}$ be a maximal ideal of R. Then

$$N_{\mathfrak{m}} = (R^{(\mathbb{N})})_{\mathfrak{m}} \cong (R_{\mathfrak{m}})^{(\mathbb{N})}$$

and

$$\text{Rad}(N_{\mathfrak{m}}) = \mathfrak{m}((R)^{(\mathbb{N})})_{\mathfrak{m}} \cong (\mathfrak{m}R_{\mathfrak{m}})^{(\mathbb{N})} = (\mathfrak{m}_{\mathfrak{m}})^{(\mathbb{N})} = \text{Rad}(R_{\mathfrak{m}})^{(\mathbb{N})}.$$

Now $(\mathfrak{m})^{(\mathbb{N})}$ has a small supplement in $R^{(\mathbb{N})}$. Then $(\mathfrak{m}_{\mathfrak{m}})^{(\mathbb{N})}$ has a weak supplement in $(R_{\mathfrak{m}})^{(\mathbb{N})}$ by Lemma 6.2 Lemma 6.3 and Lemma 6.4. Therefore $R_{\mathfrak{m}}$ is a perfect ring by [5, Theorem 1]. This contradicts with the fact that $R_{\mathfrak{m}}$ is a domain. Therefore M has a finite uniform dimension. Conversely, suppose M has finite uniform dimension. Then $E(M) \cong Q^n$, where Q is the quotient ring of R and $n \in \mathbb{Z}^+$. Then $E(M)$ is weakly supplemented by [2, Lemma 2.8] and [10, Proposition 2.5]. So that $E(M)$ is small supplemented and so M is small supplemented by Proposition 5.3. □

6.6. Lemma. *Let R be a DVR and M be a torsion and reduced R-module. Then M is small supplemented if and only if M is bounded.*

Proof. Suppose M is small supplemented. Then $\text{Rad}(M) = pM$ has a weak supplement by Lemma 6.2. Hence $L + pM = M$ and $L \cap pM$ is small for some $L \subseteq M$. Since $\frac{L}{L \cap pM} \cong \frac{L+pM}{pM} = \frac{M}{pM}$ is semisimple, it is coatomic. So that, L is coatomic by [21, Lemma 1.5]. Then L is bounded by [21, Lemma 2.1], that is, $p^n L = 0$ for some $n \in \mathbb{Z}^+$. Hence we get $p^n M = p^n(pM + L) = p^{n+1}M = p(p^n M)$, and so $p^n M$ is divisible by [1, Lemma 4.4]. But M is reduced, so that we must have $p^n M = 0$.

The converse is clear, because bounded modules are small and small modules are small supplemented. □

6.7. Lemma. *Let R be a DVR and M be a divisible(injective) and torsion R-module. Then M is small supplemented if and only if $M \cong (Q/R)^n$, for some $n \in \mathbb{Z}^+$.*

Proof. Since M is divisible and torsion, $M \cong (Q/R)^{(I)}$ for some index set I. Suppose M is small supplemented. If I is finite then we are done. Otherwise, M has a submodule which is isomorphic to $L = \oplus_{i=1}^{\infty} < \dfrac{1}{p^i} + R >$. Then L is small supplemented by Proposition 5.3 and so L is bounded by Lemma 6.6, a contradiction. Hence I is finite.

Conversely, if $M \cong (Q/R)^n$, then M is weakly supplemented by [2, Lemma 2.8] and [10], and so M is small supplemented. □

6.8. Theorem. *Let R be a Dedekind domain and M be a torsion R-module. Then M is small supplemented if and only if $T_P(M)$ is small supplemented for every $P \in \Omega$.*

Proof. ($\Rightarrow$) Since M is torsion, $M = \oplus_{P \in \Omega} T_P(M)$. Then $T_P(M)$ is small supplemented by Proposition 5.3.

($\Leftarrow$) Let N be a submodule of M. As M is a torsion module, $N = \oplus_{P \in \Omega} N_P$, where $N_P = N \cap T_P(M)$. Let K_P be a small supplement of N_P in $T_P(M)$. Then it is straightforward to check that, for the submodule $K = \oplus_{p \in \Omega} K_p$, we have $N + K = M$ and $N \cap K$ is a small module. That is, K is a small supplement of N. Hence M is small supplemented. □

6.9. Lemma. *Let R be a Dedekind domain and M be an R-module. If $T(M)$ is small supplemented then $T(M)$ has a small supplement in M.*

Proof. Let $T(M) = N \oplus D$, where N is the reduced part and D is the divisible part of $T(M)$. Write $N = \oplus_{P \in \Omega} T_P(N)$. Since $T(M)$ is small supplemented N is small supplemented by Proposition 5.3. So that $T_P(N)$ is bounded, and so $T_P(N)$ is small in $E(N)$. Hence $N = \oplus_{P \in \Omega} T_P(N)$ is small in $E(N)$. Now as N is small in $E(M)$ and D

is an injective module, N and D have small supplements in $E(M)$. If $E(M) = D \oplus D'$, then $T(M) = D \oplus T(M) \cap D'$. So that $N \cong T(M) \cap D'$ is small, and hence D' is a small supplement of $T(M)$ in $E(M)$. Then $D' \cap M$ is small supplement of $T(M)$ in M. This completes the proof. □

6.10. Corollary. *Let R be a Dedekind domain and M be an R-module. Then M is small supplemented if and only if $T(M)$ and $M/T(M)$ are small supplemented.*

Proof. (⇒) By Proposition 5.3.
(⇐) By Proposition 5.5 and Lemma 6.9. □

Summing up, Lemma 6.6, Lemma 6.7, Theorem 6.8 and Corollary 6.10, we get:

6.11. Corollary. *Let R be a Dedekind domain and M be an R-module. Then M is small supplemented if and only if*

(1) *$M/T(M)$ has finite uniform dimension.*
(2) *For every $P \in \Omega$, the reduced part of $T_P(M)$ is bounded and the divisible part has finite uniform dimension.*

We finish the paper by showing that every small supplemented module is $\mathcal{SS}$-coinjective over Dedekind domains. Recall that, every module M over a Dedekind domain can be written as $M = N \oplus D$, where D is divisible (equivalently, injective) and N is reduced. Since injective modules are coinjective, M is $\mathcal{SS}$-coinjective if and only if N is $\mathcal{SS}$-coinjective.

6.12. Theorem. *Over a Dedekind domain R, every small supplemented R-module is $\mathcal{SS}$-coinjective.*

Proof. Let M be a small supplemented module. Without loss of generality we may assume that M is a reduced R-module. We shall prove that both $T(M)$ and $M/T(M)$ are $\mathcal{SS}$-coinjective. Since M is reduced and small supplemented, in the decomposition $T(M) = \oplus_{P \in \Omega} T_P(M)$ each $T_P(M)$ is bounded by Corollary 6.11. Every bounded module is small and so $T(M)$ is a small module by Lemma 6.3. Therefore $T(M)$ is $\mathcal{SS}$-coinjective by [3, Theorem 4.1]. On the other hand, $M/T(M)$ has finite uniform dimension by Corollary 6.11. Then $M/T(M)$ is $\mathcal{SS}$-coinjective by [3, Corollary 4.4]. By [13], coinjective modules are closed under extensions. Hence M is $\mathcal{SS}$-coinjective, as $T(M)$ and $M/T(M)$ both are $\mathcal{SS}$-coinjective. □

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Crossed modules of hypergroups associated with generalized actions

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Abstract

In this article, by using the notion of generalized action, we introduce the concept of crossed module of hypergroups, in the sense of Marty, and its related structures from the light of crossed polymodules. Hypergroups in the sense of Marty are more different than polygroups since they have not identity element or inverse element in general. Examples of crossed modules of hypergroups are originally presented. These examples illustrate the structure and behavior of crossed modules of hypergroups. Moreover, we obtain a crossed module in the sense of Whitehead from a crossed module of hypergroups by applying the notion of fundamental relation.

Keywords: action, crossed module, hypergroup, fundamental relation.

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1. Introduction

The crossed module is a very powerful applications tools for mathematicians. The importance of crossed modules are: crossed modules may be thought of as 2-dimensional objects (Groups, polygroups, etc), a number of improvements in group theory are better seen from a crossed module point of view and crossed modules occur geometrically as $\pi_2(X, A) \to \pi_1 A$ when A is a subspace of X or as $\pi_1 F \to \pi_1 E$ where $F \to E \to B$ is a fibration.

Crossed modules were defined by J. H. C. Whitehead in [25]. The important constructions of crossed modules are induced crossed module [8], actor of a crossed module

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[23] and pullback crossed modules of algebroids [3]. A new application of crossed module is the crossed module of polygroups [4]. Polygroups application can be taught as generalization of crossed module on groups. Cat^1-structures are defined and proved that the category of crossed modules is equivalent to the category of cat^1- structures by Loday [20]. So, many crossed module applications related to cat^1-structure were given by mathematicians after the definition of cat^1-structures such as pullback cat^1-commutative algebra [2] and cat^1-polygroups[13]. Also computations of these two categories play very important role to solve specific problems and construct examples to well known theories. GAP [16] provides a high level programming language with so many kind advantages. A GAP share package XMOD [6] was improved by taking these advantages. As example, [5] and [1] can be considered to this share package usage. Another important application of crossed module is the crossed module of hypergroups and is presented in this paper. When we defined a crossed module of hypergroups we thought normal subgroup condition $gN = Ng$ since hypergroup does not have inverse element. The importance of this application comes from this point of view. Polygroups and hypergroups studies can give a new direction to the different studies such as equivalent categories of simplicial polygroups and cat^1-polygroups. Therefore, properties of crossed module of hypergroups are given very cletailed in this paper.

Hypergroup theory was born in 1934, when Marty [22] gave the definition of hypergroup and illustrated some applications and showed its utility in the study of groups, algebraic functions and relational fractions. Nowadays the hypergroups are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, ethnology, etc. (see [10, 11]).

An outline of the paper is as follows. After the introduction, in Section 2, we give the very well known definition of crossed module and its examples. Definition, properties and examples of hypergroups are presented in Section 3. To define crossed module of hypergroups we need hypergroup action and a strong homomorphism. Two important needs are presented. Specially, hypergroup action and its examples are given in Section 4 due to [24] and [21]. Crossed module of hypergroups and its components such as examples and properties are given in Section 5.

2. Crossed modules

In this section we recall the definition of crossed module.

2.1. Definition. Let G be a group and X be a non-empty set. A *(left) group action* is a binary operator $\tau : G \times X \to X$ that satisfies the following two axioms:

(1) $\tau(gh, x) = \tau(g, \tau(h, x))$, for all $g, h \in G$ and $x \in X$,
(2) $\tau(e, x) = x$, for all $x \in X$.

For $x \in X$ and $g \in G$, we write ${}^{g}x := \tau(g, x)$.

2.2. Definition. A *crossed module* $X = (M, G, \partial, \tau)$ consists of groups M and G together with a homomorphism $\partial : M \to G$ and a (left) action $\tau : G \times M \to M$ on M, satisfying the conditions:

(1) $\partial({}^{g}m) = g\partial(m)g^{-1}$, for all $m \in M$ and $g \in G$,
(2) ${}^{\partial(m)}m' = mm'm^{-1}$, for all $m, m' \in M$.

The standard examples of crossed modules are inclusion $M \hookrightarrow G$ of a normal subgroup M of G, the zero homomorphism $M \to G$ when M is a G-module, and any surjection $M \to G$ with central kernel, i.e., the kernel is a subset of center. There is also an important topological example: if $F \to E \to B$ is a fibration sequence of pointed spaces, then the induced homomorphism $\pi_1 F \to \pi_1 E$ of fundamental groups is naturally a crossed module [7].

In the next sections of the paper we present a very powerful application of crossed module due to [25]. The importance of this application comes from the fact that hypergroups do not have inverse element. From this reason we have to pay more attention to define hypergroup action and crossed module of hypergroup.

3. Hypergroups

Let H be a non-empty set and $\star : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation. The couple $(H, \star)$ is called a *hypergroupoid*. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \star B = \bigcup_{\substack{a \in A \\ b \in B}} a \star b, \;\; A \star x = A \star \{x\} \text{ and } x \star B = \{x\} \star B.$$

A hypergroupoid $(H, \star)$ is called a *semihypergroup* if for all a, b, c of H we have $(a \star b) \star c = a \star (b \star c)$, which means that

$$\bigcup_{u \in a \star b} u \star c = \bigcup_{v \in b \star c} a \star v.$$

A hypergroupoid $(H, \star)$ is called a *quasihypergroup* if for all a of H we have $a \star H = H \star a = H$. This condition is also called the *reproduction axiom*.

3.1. Definition. A hypergroupoid $(H, \star)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup*.

3.2. Remark. Every group is a hypergroup.

In a hypergroup $(H, \star)$, an element $e \in H$ is called a *scalar identity element* if $e \star x = x \star e = \{x\} := x$, for all $x \in H$.

There exist many examples of hypergroups in [9, 11]. Here, we present two examples of hypergroups.

3.3. Example. (1) [9, 11] Let $(G, \cdot)$ be a group and H be a non-normal subgroup of it. If we denote $G/H = \{xH \mid x \in G\}$, then $(G/H, \star)$ is a hypergroup, where for all xH, yH of G/H, we have $xH \star yH = \{zH \mid z \in xHy\}$.

(2) [14] Let $H = \{1,\ 2,\ 3,\ 4\}$ with the hyperoperation defined in the following table:

$\star$	1	2	3	4
1	1	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 4\}$
2	$\{1, 2, 3\}$	$\{2, 3\}$	$\{2, 3\}$	$\{2, 3, 4\}$
3	$\{1, 2, 3\}$	$\{2, 3\}$	$\{2, 3\}$	$\{2, 3, 4\}$
4	$\{1, 4\}$	$\{2, 3, 4\}$	$\{2, 3, 4\}$	4

Then, $(H, \star)$ is a hypergroup.

3.4. Definition. Let $(C, \star)$ and $(H, \circ)$ be two hypergroups. Let ∂ be a map from C into H. Then, ∂ is called

(1) an *inclusion homomorphism* if

$$\partial(x \star y) \subseteq \partial(x) \circ \partial(y), \text{ for all } x, y \in C;$$

(2) a *strong homomorphism* or a *good homomorphism* if

$$\partial(x \star y) = \partial(x) \circ \partial(y), \text{ for all } x, y \in C.$$

3.5. Example. In Example 3.3(1), suppose that G is the symmetric group of degree 3, $H = \langle(1\ 2)\rangle$ and $C = \langle(2\ 3)\rangle$. Then, we have

$$H = (1\ 2)H = \{e,\ (1\ 2)\}$$
$$(1\ 3)H = (1\ 3\ 2)H = \{(1\ 3),\ (1\ 3\ 2)\}$$
$$(2\ 3)H = (1\ 2\ 3)H = \{(2\ 3),\ (1\ 2\ 3)\}$$

Hence, $G/H = \{H,\ (1\ 3)H,\ (2\ 3)H\}$. By easy calculation we obtain the following multiplication table on G/H.

$\circ$	H	$(1\ 3)H$	$(2\ 3)H$
H	H	$\{(1\ 3)H,\ (2\ 3)H\}$	$\{(1\ 3)H,\ (2\ 3)H\}$
$(1\ 3)H$	$(1\ 3)H$	$\{H,\ (2\ 3)H\}$	$\{H,\ (2\ 3)H\}$
$(2\ 3)H$	$(2\ 3)H$	$\{H,\ (1\ 3)H\}$	$\{H,\ (1\ 3)H\}$

Similarly, we have

$$C = (2\ 3)C = \{e,\ (2\ 3)\}$$
$$(1\ 2)C = (1\ 3\ 2)C = \{(1\ 2),\ (1\ 3\ 2\}$$
$$(1\ 3)C = (1\ 2\ 3)C = \{(1\ 3),\ (1\ 2\ 3\}$$

Hence, $G/C = \{C,\ (1\ 2)C,\ (1\ 3)C\}$. Again, by easy calculation we obtain the following multiplication table on G/C.

$\star$	C	$(1\ 2)C$	$(1\ 3)C$
C	C	$\{(1\ 2)C,\ (1\ 3)C\}$	$\{(1\ 2)C,\ (1\ 3)C\}$
$(1\ 2)C$	$(1\ 3)C$	$\{C,\ (1\ 3)C\}$	$\{C,\ (1\ 3)C\}$
$(1\ 3)C$	$(2\ 3)C$	$\{C,\ (1\ 2)C\}$	$\{C,\ (1\ 2)C\}$

Now, we define the map $\partial : G/C \to G/H$ by $\partial(C) = H$, $\partial((1\ 2)C) = (1\ 3)H$ and $\partial((1\ 3)C) = (2\ 3)H$. It is straightforward to that ∂ is a strong homomorphism.

4. Hypergroup action

According to [17, 24], we can consider a *generalized permutation* on a non-empty set X as a map $f : X \to \mathcal{P}^*(X)$ such that the *reproductive axiom* holds, i.e.,

$$\bigcup_{x\in X} f(x) = f(X) = X.$$

We denote the set of all generalized permutations by M_X. A generalized permutation f is said to satisfy the condition θ if $x \in X$ and $z \in f(x)$, then $f(z) = f(x)$ [24]. We denote the set of all generalized permutations that satisfies the condition θ by M_θ.

4.1. Proposition. [24] *Let $f \in M_\theta$ and $M_f = \{g \in M_X \mid g \subseteq f\}$. Then, M_f is a hypergroup with respect to the hyperoperation $\star$ defined by $f_1 \star f_2 = \{p \in M_X \mid p \subseteq f_1 \circ f_2\}$, where $f_1 \circ f_2$ is defined by $f_1 \circ f_2 = \bigcup\limits_{y\in f_2(x)} f_1(y)$.*

Several mathematicians considered actions of algebraic hyperstructures, for example see [21, 12, 26]. In [21], Madanshekaf and Ashrafi considered a generalized action of a hypergroup H on a non-empty set X and obtained some results in this respect. For the definition of crossed modules of hypergroups, we need the notion of hypergroup action. So, we recall the following definition from [21].

4.2. Definition. Let $(H, \star)$ be a hypergroup and X be a non-empty set. A map $\alpha : H \times X \to \mathcal{P}^*(X)$ is called a *generalized action* of H on X, if the following axiom hold:

(1) $\alpha(g \star h, x) \subseteq \alpha(g, \alpha(h, x))$, for all $g, h \in H$ and $x \in X$, where

$$\alpha(g \star h, x) = \bigcup_{k\in g\star h} \alpha(k, x).$$

(2) For all $h \in H$, $\alpha(h, X) = X$, where
$$\alpha(h, X) = \bigcup_{x \in X} \alpha(h, x).$$

If the equality holds in axiom (1) of Definition 4.2, the action is called *strong generalized action*. Moreover, if H has the scalar identity element e, then the following condition must holds too,

(3) $\alpha(e, x) = \{x\} := x$, for all $x \in X$.

4.3. Example. [21]

(1) For any hypergroup $(H, \star)$ and any non-empty set X, the map $\alpha : H \times X \to \mathcal{P}^*(X)$, given by $\alpha(h, x) = X$ is a strong generalized action of H on X. If we define $\alpha(h, x) = \{x\}$, then this map is also a strong generalized action of H on X.

(2) Let $(H, \star)$ be a hypergroup. Then, the map $\alpha : H \times H \to \mathcal{P}^*(H)$, given by $\alpha(h, x) = h \star x$ is a strong generalized action of H on H.

4.4. Example. [21] Let X be a non-empty set, $f \in M_\theta$ and $H = M_f$. Then, the map $\alpha : H \times X \to \mathcal{P}^*(X)$, defined by $\alpha(h, x) = h(x)$ is a strong generalized action of H on X.

For $x \in X$, we put ${}^h x := \alpha(h, x)$. Then, for a strong generalized action, we have

(1) ${}^g({}^h x) = {}^{g \star h} x$, for all $g, h \in H$ and $x \in X$.

(2) $\bigcup_{x \in X} {}^h x = X$, for all $h \in H$.

4.5. Example. Consider Example 3.3(1). We define the map $\alpha : G/H \times G \to \mathcal{P}^*(G)$ by ${}^{yH}x := yHx$. Then, α is a strong generalized action.

4.6. Example. Suppose that G/H and G/C are the hypergroups defined in Example 3.5 and ∂ is the homomorphism between them. We define $\alpha : G/H \times G/C \to \mathcal{P}^*(G/C)$ by
$${}^{gH}xC := \{zC \mid z \in gHx\}.$$
We show that α is a strong generalized action.

(1) For all $g_1H, g_2H \in G/H$ and $xC \in G/C$ we have
$$\begin{aligned} {}^{g_2H}\left({}^{g_1H}xC\right) &= {}^{g_2H}\left(\{zC \mid z \in g_1Hx\}\right) \\ &= \{aC \mid a \in g_2Hz,\ z \in g_1Hx\} \\ &= \{aC \mid a \in g_2Hg_1Hx\}, \end{aligned}$$
$$\begin{aligned} {}^{g_2H \circ g_1H}xC &= {}^{\{zH \mid z \in g_2Hg_1\}}xC \\ &= \{bC \mid b \in zHx,\ z \in g_2Hg_1\} \\ &= \{bC \mid b \in g_2Hg_1Hx\}. \end{aligned}$$
Thus, ${}^{g_2H}\left({}^{g_1H}xC\right) = {}^{g_2H \circ g_1H}xC$.

(2) Clearly, for all $gH \in G/H$ we have $\bigcup_{xC \in G/C} {}^{gH}xC = G/C$.

5. Crossed module of hypergroups

Now, in this section, we give the notion of crossed module of hypergroups. To define a crossed module of hypergroups, we need the notion of hypergroup action and boundary strong homomorphism.

5.1. Definition. A *crossed module of hypergroups* $\mathcal{X} = (C, H, \partial, \alpha)$ consists of hypergroups $(C, \star)$ and $(H, \circ)$ together with a strong homomorphism $\partial : C \to H$ and a strong generalized action $\alpha : H \times C \to \mathcal{P}^*(C)$ on C, satisfying the conditions:

(1) $h \circ \partial(c) \subseteq \partial({}^{h}c) \circ h$, for all $c \in C$ and $h \in H$.
(2) $c \star c' \subseteq {}^{\partial(c)}c' \star c$, for all $c, c' \in C$.

5.2. Example. Suppose that H is a non-empty set. We define the hyperoperation $\circ$ on H by

$$h_1 \circ h_2 = \{h_1,\ h_2\},\ \text{ for all } h_1, h_2 \in H.$$

Then, $(H, \circ)$ is a hypergroup. Suppose that C is a subhypergroup of H and $\partial : C \to H$ is the identity map. The map $\alpha : H \times C \to \mathcal{P}^*(C)$ is defined by ${}^{h}c := C$ is a strong generalized action. Moreover,

(1) For all $c \in C$ and $h \in H$, we have

$$h \circ \partial(c) = h \circ c = \{h,\ c\} \subseteq C \cup \{h\} = C \circ h = \partial(C) \circ h = \partial({}^{h}c) \circ h.$$

(2) For all $c, c' \in C$, we have

$$c \circ c' = \{c,\ c'\} \subseteq C = C \circ c = {}^{c}c' \circ c = {}^{\partial(c)}c' \circ c.$$

Therefore, $\mathcal{X} = (C, H, \partial, \alpha)$ is a crossed module of hypergroups.

5.3. Example. Suppose that G is an abelian group and P a non-empty subset of G. We consider the P-hyperoperation $\star_P$ on G as follows:

$$x \star_P y = xyP,\ \text{ for all } x, y \in G.$$

Then, $(G, \star_P)$ is a hypergroup. Suppose that $\partial : G \to G$ is the identity map. The map $\alpha : G \times G \to \mathcal{P}^*(G)$ is defined by ${}^{g}x := \{x\}$ is a strong generalized action. Moreover,

(1) For all $x, y \in G$, we have

$$g \star_P \partial(x) = g \star_P x = gxP = xgP = x \star_P g = \partial(x) \star_P g = \partial({}^{g}x) \star_P g$$

(2) For all $x, y \in G$, we have

$$x \star_P y = xyP = yxP = y \star_P x = {}^{x}y \star_P x = {}^{\partial(x)}y \star_P x.$$

Therefore, $\mathcal{X} = ((G, \star_P), (G, \star_P), \partial, \alpha)$ is a crossed module of hypergroups.

5.4. Example. The direct product of $\mathcal{X}_1 \times \mathcal{X}_2$ of two crossed modules of hypergroups has source $C_1 \times C_2$, range $H_1 \times H_2$ and boundary homomorphism $\partial_1 \times \partial_2$ with $H_1 \times H_2$ acting obviously on $C_1 \times C_2$.

5.5. Theorem. *Every crossed module is a crossed module of hypergroups.*

Proof. By using Remark 3.2, the proof is straightforward. □

Let $(H\circ)$ be a hypergroup. We define the relation β^*_H as the smallest equivalence relation on H such that the quotient H/β^*_H, the set of all equivalence classes, is a group. In this case β^*_H is called the *fundamental equivalence relation* on H and H/β^*_H is called the *fundamental group.* The product $\odot$ in H/β^*_H is defined as follows: $\beta^*_H(x) \odot \beta^*_H(y) = \beta^*_H(z)$, for all $z \in \beta^*_H(x) \circ \beta^*_H(y)$. This relation is introduced by Koskas [18] and studied mainly by Corsini [9], Leoreanu-Fotea [19] and Freni [15] concerning hypergroups, Vougiouklis [24] concerning H_v-groups, Davvaz concerning polygroups [11], and many others. We consider the relation β_H as follows:

$$x\ \beta_H\ y \ \Leftrightarrow\ \text{there exist } z_1, \ldots z_n \text{ such that } \{x, y\} \subseteq \circ \prod_{i=1}^{n} z_i.$$

Freni proved that for hypergroups $\beta = \beta^*$ i in [15]. The kernel of the *canonical map* $\varphi_H : H \longrightarrow H/\beta^*_H$ is called the *heart* of H and is denoted by ω_H. Here we also denote by ω_H the unit of H/β^*_H. The heart of a hypergroup H is the intersection of all subhypergroups of H, which are complete parts.

5.6. Lemma. [9] *ω_P is a subhypergroup of H.*

Throughout the paper, we denote the binary operations of the fundamental groups H/β_H^* and C/β_C^* by $\odot$ and $\otimes$, respectively.

Now, we consider the notion of kernel of a strong homomorphism of hypergroups.

5.7. Definition. Let $(H, \circ)$ and $(C, \star)$ be two hypergroups and $\partial : C \to H$ be a strong homomorphism. The *core-kernel* of ∂ is defined by

$$ker^*\partial = \{x \in C \mid \partial(x) \in \omega_H\}.$$

5.8. Theorem. *$ker^*\partial$ is a subhypergroup of C.*

Proof. Suppose that $x, y \in ker^*\partial$ are arbitrary. Then, $\partial(x), \partial(y) \in \omega_H$ and so

$$\beta_H^*((\partial(x \star y))) = \beta_H^*(\partial(x) \circ \partial(y)) = \beta_H^*(\partial(x)) \otimes \beta_H^*(\partial(x)) = \omega_H \otimes \omega_H = \omega_H.$$

Therefore, $\partial(x \star y) \subseteq \omega_H$. This implies that $x \star y \subseteq ker^*\partial$. Now, we show that $x \star ker^*\partial = ker^*\partial \star x = ker^*\partial$, for all $x \in ker^*\partial$. Clearly, according to the above proof, we have $x \star ker^*\partial \subseteq ker^*\partial$. So, we show that $ker^*\partial \subseteq x \star ker^*\partial$. Suppose that $x, y \in ker^*\partial$. Then, there exists $z \in C$ such that $y \in x \star z$. Hence,

$$\partial(y) \in \partial(x \star z) = \partial(x) \circ \partial(z).$$

This implies that

$$\beta_H^*(\partial(y)) = \beta_H^*(\partial(x) \circ \partial(z)) = \beta_H^*(\partial(x)) \odot \beta_H^*(\partial(z))$$

and so we obtain $\omega_H = \omega_H \odot \beta_H^*(\partial(z))$. Hence, $z \in ker^*\partial$. Thus, $y \in x \star ker^*\partial$. Similarly, we can show that $ker^*\partial \star x = ker^*\partial$. □

5.9. Definition. We say that $(A, B, \partial', \alpha')$ is a *subcrossed module* of the crossed module of hypergroups (C, H, ∂, α) if

(1) A is a subhypergroup of C, and B is a subhypergroup of H,
(2) ∂' is the restriction of ∂ to A,
(3) the action of B on A is induced by the action of H on C.

5.10. Definition. Let $\mathcal{X} = (C, P, \partial, \alpha)$ and $\mathcal{X}' = (C', P', \partial', \alpha')$ be two crossed modules of hypergroups. A *crossed module of hypergroups morphisms*

$$< \theta, \phi >: (C, H, \partial, \alpha) \to (C', H', \partial', \alpha')$$

is a commutative diagram of strong homomorphisms of hypergroups

$$\begin{array}{ccc} C & \xrightarrow{\theta} & C' \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ H & \xrightarrow[\phi]{} & H' \end{array}$$

such that for all $h \in H$ and $c \in C$, we have

$$\theta(\ {}^{h}c) = \ {}^{\phi(h)}\theta(c).$$

We say that $< \theta, \phi >$ is an *isomorphism* if θ and ϕ are both isomorphisms. Similarly, we can define *monomorphism*, *epimorphism* and *automorphism* of crossed modules of hypergroups.

5.11. Proposition. *Let $(C, \star)$ and $(H, \circ)$ be two hypergroups and let $\partial : C \to H$ be a strong homomorphism. Then, ∂ induces a group homomorphism $\mathcal{D} : C/\beta_C^* \to H/\beta_H^*$ by setting*

$$\mathcal{D}(\beta_C^*(c)) = \beta_H^*(\partial(c)), \text{ for all } c \in C.$$

Proof. First, we prove that $\mathcal{D}$ is well defined. Suppose that $\beta^*_C(c_1) = \beta^*_C(c_2)$. Then, there exist $a_1, \ldots, a_n$ such that $\{c_1, c_2\} \subseteq \star \prod_{i=1}^{n} a_i$. So,

$$\{\partial(c_1), \partial(c_2)\} \subseteq \partial\left(\star \prod_{i=1}^{n} a_i\right) = \circ \prod_{i=1}^{n} \partial(a_i).$$

Hence, $\partial(c_1)\ \beta^*_H\ \partial(c_2)$, which implies that $\mathcal{D}\left(\beta^*_C(c_1)\right) = \mathcal{D}\left(\beta^*_C(c_2)\right)$. Now, we have

$$\begin{aligned}\mathcal{D}(\beta^*_C(c_1) \otimes \beta^*_C(c_2)) &= \mathcal{D}(\beta^*_C(c_1 \star c_2)) = \beta^*_H(\partial(c_1 \star c_2)) \\ &= \beta^*_H(\partial(c_1) \circ \partial(c_2)) = \beta^*_H(\partial(c_1)) \odot \beta^*_H(\partial(c_2)) \\ &= \mathcal{D}(\beta^*_C(c_1)) \odot \mathcal{D}(\beta^*_C(c_2)).\end{aligned}$$

□

We say the action of H on C is *productive*, if for all $c \in C$ and $h \in H$ there exist $c_1, \ldots, c_n$ in C such that ${}^{h}c = c_1 \star \ldots \star c_n$.

Let $(C, \star)$ and $(H, \circ)$ be two hypergroups and let $\alpha : H \times C \to \mathcal{P}^*(C)$ be a productive action on C. We define the map $\psi : H/\beta^*_H \times H/\beta^*_C \to \mathcal{P}^*(H/\beta^*_C)$ as usual manner:

$$\psi(\beta^*_H(h), \beta^*_C(c)) = \{\beta^*_C(x) \mid x \in \bigcup_{\substack{y\, \in\, \beta^*_C(c) \\ z\, \in\, \beta^*_H(h)}} {}^{z}y\}.$$

By the definition of β^*_C, since the action of H on C is productive, we conclude that $\psi(\beta^*_H(h), \beta^*_C(c))$ is singleton, i.e., we have

$$\psi : H/\beta^*_H \times H/\beta^*_C \to H/\beta^*_C,$$
$$\psi(\beta^*_H(h), \beta^*_C(c)) = \beta^*_C(x), \text{ for all } x \in \bigcup_{\substack{y\, \in\, \beta^*_C(c) \\ z\, \in\, \beta^*_H(h)}} {}^{z}y.$$

We denote $\psi(\beta^*_H(h), \beta^*_C(c)) = {}^{[\beta^*_H(h)]}[\beta^*_C(c)]$.

5.12. Proposition. *Let $(C, \star)$ and $(H, \circ)$ be two hypergroups and let $\alpha : H \times C \to \mathcal{P}^*(C)$ be a productive action on C. Then, ψ is an action of the group H/β^*_H on the group C/β^*_C.*

Proof. Suppose that $g, h \in H$ and $c \in C$. Then, we have

$$\psi(\beta^*_H(h) \odot \beta^*_H(g), \beta^*_C(c)) = \psi(\beta^*_H(h \circ g), \beta^*_C(c)) = {}^{[\beta^*_H(h \circ g)]}[\beta^*_C(c)],$$

and

$$\psi(\beta^*_H(h), \psi(\beta^*_H(g), \beta^*_C(c))) = \psi\left(\beta^*_H(h), {}^{[\beta^*_H(g)]}[\beta^*_C(c)]\right) = {}^{[\beta^*_H(h)]}\left({}^{[\beta^*_H(g)]}[\beta^*_C(c)]\right).$$

By the condition (1) of Definition 4.2, we have ${}^{h}({}^{g}c) = {}^{h \circ g}c$. Now, it is not difficult to see that

$${}^{[\beta^*_H(h \circ g)]}[\beta^*_C(c)] = {}^{[\beta^*_H(h)]}\left({}^{[\beta^*_H(g)]}[\beta^*_C(c)]\right).$$

□

5.13. Theorem. *Let $\mathcal{X} = (C, H, \partial, \alpha)$ be a crosed module of hypergroups such that the action of H on C is productive. Then, $X = (C/\beta^*_C, H/\beta^*_H, \mathcal{D}, \psi)$ is a crossed module.*

Proof. By Propositions 5.11 and 5.12, it is enough to show that the conditions of Definition 2.2 hold. Suppose that $c \in C$ and $h \in H$ are arbitrary. Then, we have

$$\begin{aligned}
\mathcal{D}\left({}^{[\beta_H^*(h)]}([\beta_C^*(c)])\right) \odot \beta_H^*(h) &= \mathcal{D}\left([\beta_C^*(z]) \odot \beta_H^*(h), \text{ for all } z \in {}^{h}c\right. \\
&= \beta_H^*(\partial(z)) \odot \beta_H^*(h), \text{ for all } z \in {}^{h}c \\
&= \beta_H^*(\partial({}^{h}c)) \circ h)) \\
&= \beta_H^*(h \circ \partial(c)) \\
&= \beta_H^*(h) \odot \beta_H^*(\partial(c)) \\
&= \beta_H^*(h) \odot \mathcal{D}(\beta_C^*(c)),
\end{aligned}$$

which implies that $\mathcal{D}\left({}^{[\beta_H^*(h)]}([\beta_C^*(c)])\right) = \beta_H^*(h) \odot \mathcal{D}(\beta_C^*(c)) \odot \beta_H^*(h)^{-1}$. So, the first condition of Definition 2.2 holds. For the second condition, suppose that $c, c' \in C$ are arbitrary. Then, we have

$$\begin{aligned}
{}^{[\mathcal{D}(\beta_C^*(c))]}[\beta_C^*(c')] \otimes \beta_C^*(c) &= {}^{[\beta_P^*(\partial(c))]}[\beta_C^*(c')] \otimes \beta_C^*(c) \\
&= \beta_C^*(z) \otimes \beta_C^*(c), \text{ for all } z \in {}^{\partial(c)}c' \\
&= \beta_C^*\left({}^{\partial(c)}c' \star c\right) \\
&= \beta_C^*(z), \text{ for all } z \in c \star c' \\
&= \beta_C^*(c \star c') \\
&= \beta_C^*(c) \otimes \beta_C^*(c'),
\end{aligned}$$

which implies that ${}^{[\mathcal{D}(\beta_C^*(c))]}[\beta_C^*(c')] = \beta_C^*(c) \otimes \beta_C^*(c') \otimes \beta_C^*(c)^{-1}$. □

5.14. Theorem. *Let $\mathcal{X} = (C, H, \partial, \alpha)$ be a crossed module of hypergroups, φ_C and φ_P be canonical maps. Then, $< \varphi_C, \varphi_H >$ is a crossed modules of hypergroups morphisms.*

Proof. Note that according to Theorem 5.13, we can consider $(C/\beta_C^*, P/\beta_P^*, \mathcal{D}, \psi)$ as a crossed module of hypergroups. We show that the following diagram is commutative.

$$\begin{array}{ccc}
C & \xrightarrow{\varphi_C} & C/\beta_C^* \\
\partial \downarrow & & \downarrow \mathcal{D} \\
H & \xrightarrow[\varphi_H]{} & H/\beta_H^*
\end{array}$$

Indeed, we have $\mathcal{D}\varphi_C(c) = \mathcal{D}\left(\beta_C^*(c)\right) = \beta_H^*(\partial(c)) = \varphi_H \partial(c)$, for all $c \in C$. Moreover,

$$\varphi_C({}^{h}c) = \beta_C^*({}^{h}c) = {}^{[\beta_H^*(h)]}[\beta_C^*(c)] = {}^{\varphi_H(h)}\varphi_C(c),$$

for all $c \in C$ and $h \in H$. Therefore, $< \varphi_C, \varphi_H >$ is a crossed module of hypergroup morphism. □

The following example give us another crossed module structure on the fundamental groups.

5.15. Example. Suppose that $(H, \circ)$ is a hypergroup. Then, H/β_H^* is a group. Suppose that $Aut\,(H/\beta_H^*)$ its group of automorphisms. There is an obvious action α of $Aut\,(H/\beta_H^*)$ on H/β_H^*, and a group homomorphism $\partial : H/\beta_H^* \to Aut\,(H/\beta_H^*)$ sending each $\beta_H^*(h) \in P/\beta_P^*$ to the inner automorphism of conjugation by $\beta_P^*(p)$. These together form a crossed module $(H/\beta_H^*, Aut\,(H/\beta_H^*)\,, \partial, \alpha)$.

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On the P-interiors of submodules of Artinian modules

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Abstract

Let R be a commutative ring and M an Artinian R-module. In this paper, we study the dual notion of saturations (that is, P-interiors) of submodules of M and obtain some related results.

Keywords: Second submodule, saturation, P-interior.

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1. Introduction

Throughout this paper, R will denote a commutative ring with identity and "$\subset$" will denote the strict inclusion. We write $N \leq M$ to indicate that N is a submodule of an R-module M. Also $Spec(R)$ and $\mathbb{Z}$ will denote the set of all prime ideals of R and the ring of integers respectively.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the endomorphism $S \overset{a}{\to} S$ is either surjective or zero (see [13]). A submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M. Thus, the intersection of all completely irreducible submodule of M is zero (see [6]).

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The *saturation* of $N \leq M$ with respect to $P \in Spec(R)$ is the contraction of N_P in M and designated by $S_P(N)$. It is well known that

$$S_P(N) = \{e \in M : es \in N \ for\ some\ s \in R - P\}.$$

In [1], H. Ansari-Toroghy and F. Farshadifar, introduced the dual notions of saturations of submodules, that is, P-interiors of submodules and investigated some related results (see [1] and [3]). Let N be a submodule of M. *The P-interior of N relative to M* is defined [1, 2.7] as the set

$$I_P^M(N) = \cap\{L \mid L\ is\ a\ completely\ irreducible\ submodule\ of\ M\ and$$
$$rN \subseteq L\ for\ some\ r \in R - P\}.$$

There are considerable results about saturation of a module with respect to a prime ideal in literature (see, for example, [7], [8], and [9]). It is natural to ask that to what extent the dual of these results hold. The purpose of this paper is to answer this question and provide more information about the P-interiors of submodules in case that our module is an Artinian module.

2. P-interiors of submodules and related properties

Recall that an R-module L is said to be *cocyclic* if L is a submodule of $E(R/m)$ for some maximal ideal m of R, where $E(R/m)$ is the injective envelope of R/m (see [14]).

2.1. Lemma. Let L be a completely irreducible submodule of an R-module M and $a \in R$. Then $(L :_M a)$ is a completely irreducible submodule of M.

Proof. This follows from the fact that a submodule L of M is a completely irreducible submodule of M if and only if M/L is a cocyclic R-module by [6] and that $M/(L :_M a) \cong (aM + L)/L$.

□

We use the following basic fact without further comment.

2.2. Remark. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

2.3. Lemma. Let $P \in Spec(R)$ and N be a submodule of an R-module M. If $M/I_P^M(N)$ is a finitely cogenerated R-module, then there exists $r \in R - P$ such that $rN \subseteq I_P^M(N)$.

Proof. Since $M/I_P^M(N)$ is finitely cogenerated, there exists a finite number of completely irreducible submodules $L_1, L_2, ..., L_n$ of M such that $I_P^M(N) = \cap_{i=1}^n L_i$ and $r_i N \subseteq L_i$ for some $r_i \in R - P$. Set $r = r_1 ... r_n$. Then $rN \subseteq I_P^M(N)$. □

2.4. Theorem. Let $P \in Spec(R)$ and N be a submodule of an R-module M. Then we have the following.

(a) If M is an Artinian R-module, then $I_P^M(I_P^M(N)) = I_P^M(N)$.
(b) If M is an Artinian R-module, then $Hom_R(R_P, I_P^M(N)) = Hom_R(R_P, N)$.
(c) $Ann_R(N) \subseteq S_P(Ann_R(N)) \subseteq Ann_R(I_P^M(N))$.
(d) If M is an Artinian R-module, then $Ann_R(I_P^M(N)) = S_P(Ann_R(I_P^M(N)))$.

Proof. (a) Clearly, $I_P^M(I_P^M(N)) \subseteq I_P^M(N)$. To prove the opposite inclusion, let L be a completely irreducible submodule of M such that $I_P^M(I_P^M(N)) \subseteq L$. By Lemma 2.3, there exists $r \in R - P$ such that $rI_P^M(N) \subseteq I_P^M(I_P^M(N))$. Therefore, $rI_P^M(N) \subseteq L$. Again by Lemma 2.3, there exists $s \in R - P$ such that $sN \subseteq I_P^M(N)$. Hence $rsN \subseteq L$. It follows that $I_P^M(N) \subseteq L$, as required.

(b) By Lemma 2.3, there exists $r \in R - P$ such that $rN \subseteq I^M_P(N)$. Now $rN \subseteq I^M_P(N) \subseteq N$ implies that

$$Hom_R(R_P, rN) \subseteq Hom_R(R_P, I^M_P(N)) \subseteq Hom_R(R_P, N).$$

As $r \in R - P$, one can see that $Hom_R(R_P, rN) = Hom_R(R_P, N)$. Therefore,** $Hom_R(R_P, N) = Hom_R(R_P, I^M_P(N))$.

(c) Clearly, $Ann_R(N) \subseteq S_P(Ann_R(N))$. Now let $r \in S_P(Ann_R(N))$. Then there exists $s \in R - P$ such that $rs \in Ann_R(N)$ and so $rsN = (\mathbf{0})$. Thus for each $i \in I$, $rsN \subseteq L_i$, where $\{L_i\}_{i\in I}$ is the collection of all completely irreducible submodules of M. Hence $sN \subseteq (L_i :_M r)$ for each $i \in I$. This implies that $I^M_P(N) \subseteq (L_i :_M r)$ for each $i \in I$ because $(L_i :_M r)$ is a completely irreducible submodule of M by Lemma 2.1. Therefore, $rI^M_P(N) \subseteq \cap_{i\in I} L_i = (\mathbf{0})$. Thus $r \in Ann_R(I^M_P(N))$.

(d) Clearly, $Ann_R(I^M_P(N)) \subseteq S_P(Ann_R(I^M_P(N))$. Now let $r \in S_P(Ann_R(I^M_P(N))$. Then there exists $s \in R - P$ such that $rs \in Ann_R(I^M_P(N))$ and so $rsI^M_P(N) = (\mathbf{0})$. As M is an Artinian R-module, there exists $t \in R - P$ such that $tN \subseteq I^M_P(N)$ by Lemma 2.3. Therefore, $strN = (\mathbf{0})$. This implies that for each $i \in I$, $stN \subseteq (L_i :_M r)$, where $\{L_i\}_{i\in I}$ is the collection of all completely irreducible submodules of M. Hence $I^M_P(N) \subseteq (L_i :_M r)$. Therefore, $rI^M_P(N) \subseteq \cap_{i\in I} L_i = (\mathbf{0})$. Hence $r \in Ann_R(I^M_P(N))$, as required. □

2.5. Definition. We say that a submodule N of an R-module M is *cotorsion-free with respect to (w.r.t.)* P if $I^M_P(N) = N$, where $P \in Spec(R)$.

2.6. Lemma. Let N ba a submodule of an R-module M and $P \in Spec(R)$. If N is cotorsion-free w.r.t. P, then N is cotorsion-free w.r.t. Q for each $Q \in V(P)$.

Proof. Since $P \subseteq Q$, $I^M_P(N) \subseteq I^M_Q(N)$. Therefore, $N = I^M_P(N) \subseteq I^M_Q(N) \subseteq N$. Hence $N = I^M_P(N) = I^M_Q(N)$ for each $Q \in V(P)$. □

A non-zero R-module M is said to be *secondary* if for each $a \in R$, the endomorphism $M \xrightarrow{a} M$ is either surjective or nilpotent (see [10]). Clearly, every second module is a secondary module.

2.7. Example. (1) If $P \in Spec(R)$, then every P-secondary submodule of an R-module M is cotorsion-free w.r.t. P by [4, 2.8].

(2) The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is cotorsion-free w.r.t. (0).

2.8. Corollary. Let $P \in Spec(R)$ and N be a submodule of an R-module M. If N is cotorsion-free w.r.t. P, then $Ann_R(I^M_P(N)) = S_P(Ann_R(I^M_P(N)))$.

Proof. The results follows from part (c) of Theorem 2.4 because $N = I^M_P(N)$. □

The *cosupport* of an R-module M [12] is denoted by $Cosupp(M)$ and it is defined by

$$Cosupp(M) = \{P \in Spec(R) | P \supseteq Ann_R(L) \ for\ some\ cocyclic$$
$$homomorphic\ image\ L\ of\ M\}.$$

2.9. Theorem. Let $P \in Spec(R)$ and N be a submodule of an Artinian R-module M. Then we have the following.

(1) $Ann_{R_P}(Hom_R(R_P, N)) = (Ann_R(I^M_P(N)))_P$.
(2) The following statements are equivalent.
 (a) $Hom_R(R_P, N) \neq (\mathbf{0})$.
 (b) $Ann_R(I^M_P(N)) \subseteq P$.
 (c) $I^M_P(N) \neq (\mathbf{0})$.
 (d) $P \in Cosupp_R(N)$.

Proof. (1) By Theorem 2.4 (b), $Hom_R(R_P, I_P^M(N)) = Hom_R(R_P, N)$. It is easy to see that

$$(Ann_R(I_P^M(N)))_P \subseteq Ann_{R_P}(Hom_R(R_P, I_P^M(N)).$$

To see the reverse inclusion, we note that $I_P^M(I_P^M(N)) = \phi(Hom_R(R_P, I_P^M(N)))$ by [2, 2.15], where $\phi : Hom_R(R_P, I_P^M(N)) \to I_P^M(N)$ is the natural homomorphism defined by $\phi(f) = f(1_{R_P})$ for any $f \in Hom_R(R_P, I_P^M(N))$. Now by Theorem 2.4 (a), $I_P^M(N) = \phi(Hom_R(R_P, I_P^M(N)))$. But always we have

$$Ann_R(Hom_R(R_P, I_P^M(N))) \subseteq Ann_R(\phi(Hom_R(R_P, I_P^M(N))).$$

Hence $Ann_R(Hom_R(R_P, I_P^M(N))) \subseteq Ann_R(I_P^M(N))$. Therefore,

$$Ann_{R_P}(Hom_R(R_P, I_P^M(N))) \subseteq (Ann_R(I_P^M(N)))_P,$$

as required.

(2) $(a) \Leftrightarrow (d)$. By [12, 2.3], $Cosupp_R(N) = V(Ann_R(N))$ and by [11, p. 130], $Cos_R(N) = V(Ann_R(N))$, where $Cos_R(N) = \{P \in Spec(R) : Hom_R(R_P, N) \neq (\mathbf{0})\}$. Hence we get the equivalence (a) and (d).

$(b) \Rightarrow (c)$. This is clear.

$(a) \Rightarrow (b)$. $Hom_R(R_P, N) \neq (\mathbf{0}) \Leftrightarrow Ann_{R_P}(Hom_R(R_P, N)) \neq R_P$. Thus by using part (1), we have

$$Hom_R(R_P, N) \neq (\mathbf{0}) \Leftrightarrow (Ann_R(I_P^M(N)))_P \neq R_P \Leftrightarrow Ann_R(I_P^M(N)) \subseteq P.$$

$(c) \Rightarrow (a)$. If $Hom_R(R_P, N) = (\mathbf{0})$, then $Hom_R(R_P, I_P^M(N)) = (\mathbf{0})$. Thus by [2, 2.15],

$$I_P^M(N) = I_P^M(I_P^M(N)) = \phi(Hom_R(R_P, I_P^M(N)) = (\mathbf{0}),$$

where $\phi : Hom_R(R_P, I_P^M(N)) \to I_P^M(N)$ is the natural homomorphism defined by $\phi(f) = f(1_{R_P})$ for any $f \in Hom_R(R_P, I_P^M(N))$. This contradiction completes the proof. □

We need the following lemma.

2.10. Lemma. [7, 2.2] Let I be an ideal of R and $P \in Spec(R)$. Then the following statements are equivalent.

(a) $S_P(I)$ is a P-primary ideal of R.
(b) $\sqrt{S_P(I)} = P$.
(c) P is a minimal prime ideal of I.

2.11. Theorem. Let $P \in Spec(R)$ and N be a submodule of an Artinian R-module M. Then the following statements are equivalent.

(a) $I_P^M(N)$ is a P-secondary submodule of M.
(b) $Ann_R(I_P^M(N))$ is a P-primary ideal of R.
(c) $\sqrt{Ann_R(I_P^M(N))} = P$.

In particular, $I_P^M(N)$ is P-second if and only if $Ann_R(I_P^M(N)) = P$.

Proof. $(a) \Rightarrow (b)$. This is clear.

$(b) \Rightarrow (a)$. Since $Ann_R(I_P^M(N))$ is a P-primary ideal of R and $I_P^M(I_P^M(N)) = I_P^M(N)$ by Theorem 2.4 (a), $I_P^M(N)$ is a P-secondary submodule of M by [4, 2.2].

$(b) \Rightarrow (c)$. This is elementary.

$(c) \Rightarrow (b)$. Put $I = Ann_R(I_P^M(N))$. Then by Theorem 2.4 (d), $S_P(I) = I$. Now, we have $\sqrt{I} = P = \sqrt{S_P(I)}$ by the hypothesis. It follows from Lemma 2.10 that $S_P(I)$ is a P-primary ideal of R. Hence $I = S_P(I) = Ann_R(I_P^M(N))$ is a P-primary ideal of R, as required. □

2.12. Definition. Let M be an R-module, $(\mathbf{0}) \neq N \leq M$ and $P \in Spec(R)$. We say the pair (N, P) satisfies *property* $(**)$ if $S_P(Ann_R(N)) = Ann_R(I_P^M(N)) \neq R$. We say the module M satisfies *property* $(**)$ if for every $(\mathbf{0}) \neq N \leq M$ and $P \in V(Ann_R(N))$ the pair (N, P) satisfies *property* $(**)$.

2.13. Remark. (a) For every $N \leq M$ and $P \in Spec(R)$, if $Ann_R(N) \not\subseteq P$, then $I_P^M(N) = (\mathbf{0})$ because there exists $r \in R - P$ such that $rN = (\mathbf{0})$. Hence for each $i \in I$, $rN \subseteq L_i$, where $\{L_i\}_{i \in I}$ is the set of all completely irreducible submodules of M. Therefore, $I_P^M(N) \subseteq \cap_{i \in I} L_i = (\mathbf{0})$. However, the converse is not true in general. As a counter example, take the $\mathbb{Z}$-module $\mathbb{Z}$ as M, $N = \mathbb{Z}$, and $P = (0)$.

(b) Let M be an R-module, $(\mathbf{0}) \neq N \leq M$ and $P \in Spec(R)$. If a pair (N, P) satisfies property $(**)$, then by part (a), we have $Ann_R(N) \subseteq P$.

2.14. Example. (a) The $\mathbb{Z}$-module $\mathbb{Z}$ does not satisfy property $(**)$ because $(\mathbb{Z}, (0))$ does not satisfy this property.

(b) Let N be a non-zero submodule of an R-module M and let P be a prime ideal of R. If N is cotorsion-free w.r.t. P, then (N, P) satisfies property $(**)$. This is because $I_P^M(N) = N \neq (\mathbf{0})$ implies that $Ann_R(I_P^M(N)) = Ann_R(N) \neq R$ and hence by Corollary 2.8, we have

$$Ann_R(N) = S_P(Ann_R(N)) = Ann_R(I_P^M(N)) \neq R.$$

Moreover, not only (N, P), but also (N, Q) for each $Q \in V(P)$ satisfies property $(**)$ by Lemma 2.6. In particular, every P-secondary submodule S of M and each $Q \in V(P) = V(Ann_R(S))$ satisfies property $(**)$ by Example 2.7.

2.15. Theorem. Every non-zero Artinian R-module M satisfies property $(**)$.

Proof. Let $(\mathbf{0}) \neq N \leq M$ and $P \in V(Ann_R(N))$. By Lemma 2.3, there exists $t \in R - P$ such that $tN \subseteq I_P^M(N)$. Now let $r \in Ann_R(I_P^M(N))$. Then $rtN = (\mathbf{0})$. Hence $r \in S_P(Ann_R(N))$. Thus $R \neq Ann_R(I_P^M(N)) \subseteq S_P(Ann_R(N))$. The reverse inclusion follows from Theorem 2.4 (c). □

2.16. Remark. Those modules M which satisfy property $(**)$ are not necessarily Artinian. For example, every vector space W satisfies property $(**)$ even it is of infinite dimensional. This is due to that every non-zero subspace U of W is (0)-second with $V(Ann_R(U)) = \{(0)\}$.

2.17. Corollary. Let M be an Artinian R-module, $(\mathbf{0}) \neq N \leq M$ and $P \in Spec(R)$.

(1) The following statements are equivalent.
   (a) $I_P^M(N)$ is a P-secondary submodule of M.
   (b) $\sqrt{S_P(Ann_R(N))} = P$.
   (c) P is a minimal prime ideal of $Ann_R(N)$.

(2) $I_P^M(N)$ is a P-second submodule of M if and only if $S_P(Ann_R(N)) = P$.

In particular, if $Ann_R(N) = P$, then $I_P^M(N)$ is a P-second submodule of M.

Proof. The proof is straightforward from Theorem 2.11, Lemma 2.10, and Theorem 2.4. □

3. Maximal second submodules

A submodule N of an R-module M is said to be a *maximal second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a second submodule L of M such that $N \subset L \subset K$ (see [1]).

3.1. Lemma. Let R be an integral domain and let M be an Artinian non-zero R-module.

(a) If $I^M_{(0)}(M) \neq (\mathbf{0})$, then $I^M_{(0)}(M)$ is a maximal (0)-second submodule of M and it contains every (0)-second submodule of M.
(b) $I^M_{(0)}(M) = M$ if and only if M is a (0)-second submodule of M.

Proof. (a) This follows from [1, 2.9] and [3, 2.10].
(b) This follows from part (a) and [3, 2.10]. □

3.2. Theorem. Let R be an integral domain of dimension 1, M be a non-zero Artinian R-module and $(0) \neq P \in V(Ann_R(M))$. Then $I^M_P((0 :_M P))$ is a maximal second submodule of M if and only if $I^M_P((0 :_M P)) \not\subseteq I^M_{(0)}(M)$.

Proof. Since $(0) \subset P \subseteq Ann_R((0 :_M P))$, $dimR = 1$, and R is a domain, it follows that if $Ann_R((0 :_M P)) \neq R$, then $Ann_R((0 :_M P)) = P$. Hence $I^M_P((0 :_M P))$ is a second submodule of M by [1, 2.8].
Suppose that $I^M_P((0 :_M P))$ is a maximal second submodule of M. Then there are two cases:

(i) $I^M_P((0 :_M P)) = M$ and
(ii) $I^M_P((0 :_M P)) \neq M$.

In case (i), M is a P-second submodule for $P \neq (0)$. Consequently, $I^M_{(0)}(M) \neq M$ by Lemma 3.1 (b). Hence $I^M_P((0 :_M P)) \not\subseteq I^M_{(0)}(M)$.
In case (ii), $I^M_P((0 :_M P))$ is a proper maximal second submodule of M. Hence M is not a second submodule, in particular, it is not a (0)-second submodule so that $I^M_{(0)}(M) \neq M$ by Lemma 3.1 (b) again. Thus if $I^M_{(0)}(M) \neq (\mathbf{0})$, then $I^M_{(0)}(M)$ is a proper maximal (0)-second submodule of M by Lemma 3.1 (a). Consequently, $I^M_P((0 :_M P)) \not\subseteq I^M_{(0)}(M)$ by the maximality of $I^M_P((0 :_M P))$ in M. On the other hand, if $I^M_{(0)}(M) = (\mathbf{0})$, then obviously, $I^M_P((0;_M P)) \not\subseteq I^M_{(0)}(M)$.

Conversely, suppose that $I^M_P((0;_M P)) \not\subseteq I^M_{(0)}(M)$. Then clearly $I^M_{(0)}(M) \neq M$. Thus by Lemma 3.1 (b), M is not a (0)-second submodule. To see that $I^M_P((0 :_M P))$ is a maximal second submodule of M, let K be a second submodule of M such that $I^M_P((0 :_M P)) \subseteq K \subseteq M$. Then

$$(0) \subseteq Ann_R(M) \subseteq Ann_R(K) \subseteq Ann_R(I^M_P((0 :_M P))) = P.$$

Since $dimR = 1$, the prime ideal $Ann_R(K) = (0)$ or P. If $Ann_R(K) = (0)$, then K is a (0)-second submodule. However, $K \neq M$ because M is not a (0)-second submodule as we have seen above. Since every proper (0)-second submodule contained in $I^M_{(0)}(M)$, we have that $I^M_P((0 :_M P)) \subseteq K \subseteq I^M_{(0)}(M) \neq (\mathbf{0})$ which contradicts to $I^M_P((0 :_M P)) \not\subseteq I^M_{(0)}(M)$. Therefore, $Ann_R(K) = P$, i.e., K is a P-second submodule. Thus $K = I^M_P(K) \subseteq I^M_P((0 :_M P))$. Therefore, $K = I^M_P((0 :_M P))$. This proves that $I^M_P((0 :_M P))$ is a maximal second submodule of M. □

3.3. Proposition. Let Y be a set of prime ideals of R which contains all the maximal ideals, M be an Artinian R-module, and N be a non-zero submodule of M. Then $N = \sum_{P\in Y} I^M_P(N)$.

Proof. Let L be a completely irreducible submodule of M such that $\sum_{P\in Y} I^M_P(N) \subseteq L$ so that $I^M_P(N) \subseteq L$ for every $P \in Y$. Hence by Lemma 2.3, we have $(L :_R N) \not\subseteq P$ for every $P \in Y$. This implies that $(L :_R N) \not\subseteq m$ for every maximal ideal $m \in Y$. This in turn implies that $(L :_R N) = R$ and hence $N \subseteq L$. Thus $N \subseteq \sum_{P\in Y} I^M_P(N)$. The reverse inclusion is clear. □

3.4. Corollary. Let (R, m) be a local ring, M an Artinian R-module, and $(\mathbf{0}) \neq N \leq M$. Then N is cotorsion-free w.r.t. m.

Proof. Take $Y = \{m\}$ in Proposition 3.3. Then we have $I_m^M(N) = N$. □

Let N be a submodule of an R-module M. The (*second*) *socle* of N is defined as the sum of all second submodules of M contained in N and it is denoted by $soc(N)$ or $sec(N)$ (see [1] and [5]). In case N does not contain any second submodule, the socle of N is defined to be $(\mathbf{0})$.

3.5. Proposition. Let M be an Artinian R-module, $P \in Spec(R)$, and $(\mathbf{0}) \neq N \leq M$. If P is a minimal prime ideal of $Ann_R(N)$ and $I_P^M((0 :_N P)) \neq (\mathbf{0})$, then $I_P^M((0 :_N P))$ is a maximal second submodule of $K \leq M$ with $I_P^M((0 :_N P)) \subseteq K \subseteq N$. In particular $I_P^M((0 :_N P))$ is a maximal P-second submodule of $sec(N)$.

Proof. Since $I_P^M((0 :_N P)) \neq (\mathbf{0})$, $I_P^M((0 :_N P))$ is a maximal P-second submodule of $(0 :_N P)$ by [1, 2.9]. Now suppose that K is a submodule of M such that $I_P^M((0 :_N P)) \subseteq K \subseteq N$ and S is a Q-second submodule of M such that $I_P^M((0 :_N P)) \subseteq S \subseteq K \subseteq N$. Then as P is a minimal prime ideal of $Ann_R(N)$, we have $Q = P$. Thus $S \subseteq (0 :_N P)$. It follows that $S = I_P^M((0 :_N P))$ as desired. The last assertion follows from the fact that $I_P^M((0 :_N P)) \subseteq sec(N) \subseteq N$. So the proof is completed.

□

The following example shows that the condition $I_P^M((0 :_N P)) \neq (\mathbf{0})$ in the statement of Proposition 3.5 can not be dropped.

3.6. Example. Consider $M = N = \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module, where p is a prime number. Let $q \neq p$ be an another prime number. Then clearly, $q\mathbb{Z}$ is a minimal prime ideal of $Ann_{\mathbb{Z}}(M)$ and $I_{(q)}^M((0 :_N q\mathbb{Z})) = (\mathbf{0})$.

The next theorem gives an important information on the maximal second submodules of an Artinian R-modules.

3.7. Theorem. Let N be a non-zero submodule of an Artinian R-module M. Then every maximal second submodule of N must be of the form $I_P^M((0 :_N P))$ for some $P \in V(Ann_R(N))$.

Proof. Let S be a maximal P-second submodule of N. Then $S \subseteq N$ and $Ann_R(S) = P$ so that $S \subseteq (0 :_N P)$. Therefore, $S = I_P^M(S) \subseteq I_P^M((0 :_N P)) \subseteq N$ by [3, 2.10]. Since $P \in V(Ann_R(N))$, $I_P^M((0 :_N P))$ is a P-second submodule , as we have seen in the proof of Proposition 3.5. Thus $S = I_P^M((0 :_N P))$. □

3.8. Corollary. Let M be an Artinian R-module and $(\mathbf{0}) \neq N \leq M$. Then $sec(N) = \sum_{P \in Y} I_P^M((0 :_N P))$, where Y is a finite subset of $V(Ann_R(N))$.

Proof. By [1, 2.6, 2.2], there exists $n \in \mathbb{Z}$ such that $sec(N) = \sum_{i=1}^n S_i$, where for $1 \leq i \leq n$, S_i is a maximal second submodule of N. Now the proof follows from Theorem 3.7. We remark that this corollary is also a direct consequence of [3, Proposition 2.7 (a)]. □

3.9. Corollary. Let N be a non-zero submodule of an Artinian R-module M. If $I_P^M((0 :_N P)) \neq (\mathbf{0})$ and N is a P-secondary submodule of an R-module M for some $P \in Spec(R)$, then we have the following.

(a) $I_P^M((0 :_N P))$ is a maximal P-second submodule of $sec(N)$.
(b) If P is a maximal ideal of R, then $sec(N) = I_P^M((0 :_N P))$ so that $sec(N)$ is a P-second submodule of M.

Proof. (a) This follows from Proposition 3.5 because P is a minimal prime ideal of $Ann_R(N)$.

(b) By Corollary 3.8, $sec(N) = \sum_{Q \in V(Ann_R(N))} I_Q^M((0 :_N Q))$. Since P is maximal and $\sqrt{Ann_R(N)} = P$, $V(Ann_R(N)) = \{P\}$. Thus $sec(N) = I_P^M((0 :_N P))$ as required. □

3.10. Corollary. Let I be an ideal of R and M be an Artinian R-module such that $(0 :_M I) \neq (\mathbf{0})$. Then $sec((0 :_M I)) = \sum_{P \in V(Ann_R((0:_M I)))} I_P^M((0 :_M P))$.

Proof. Set $N = (0 :_M I)$. Then this follows from Corollary 3.8 since, $(0 :_{(0:_M I)} P) = (0 :_M P)$ for every $P \in V(Ann_R((0 :_M I)))$. □

3.11. Example. For any prime integer p, let $M = (\mathbb{Z}/p\mathbb{Z}) \times \mathbb{Z}_{p^\infty}$. Then M is an Artinian faithful $\mathbb{Z}$-module and $V(Ann_{\mathbb{Z}}(M)) = V((0)) = Spec(\mathbb{Z})$. Hence $sec(M) = \sum_{(q) \in V((0))} I_{(q)}^M((0 :_M q\mathbb{Z}))$ by Corollary 3.10. Since $I_{(q)}^M((0 :_M q\mathbb{Z})) = I_{(q)}^M(\mathbf{0}) = (\mathbf{0})$ for each prime number $p \neq q$,

$$\begin{aligned} sec(M) &= I_{(0)}^M(M) + I_{(p)}^M((0 :_M p\mathbb{Z})) \\ &= ((0) \times \mathbb{Z}_{p^\infty}) + ((\mathbb{Z}/p\mathbb{Z}) \times < 1/p + \mathbb{Z} >) \\ &= M. \end{aligned}$$

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Strong summation process in locally integrable function spaces

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Abstract

In this paper, using the concept of strong summation process, we give a Korovkin type approximation theorem for a sequence of positive linear operators acting from $L_{p,q}\,(loc)$ into itself. We also study modulus of continuity for $L_{p,q}\,(loc)$ approximation and give the rate of convergence of these operators.

Keywords: $\mathcal{A}-$summability, positive linear operators, locally integrable functions, Korovkin type theorem, modulus of continuity, rate of convergence.

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1. Introduction

The classical theorem of Korovkin [7] on approximation of continuous functions on a compact interval gives conditions in order to decide whether a sequence of positive linear operators converges to the identity operator. Some results concerning the Korovkin type approximation theorem in the space $L_p[a, b]$ of the Lebesgue integrable functions on a compact interval may be found in [4]. If the sequence of positive linear operators does not converge then it might be benefical to use matrix summability methods.

Approximation theory has important applications in the theory of polynomial approximation, in functional analysis, numerical solutions of differential and integral equations [1], [8].

The purpose this paper is to study a Korovkin type approximation theorem of a function f by means of sequence of positive linear operators from the space of locally integrable functions into itself with the use of a matrix summability method which includes both convergence and almost convergence. We also obtain rate of convergence in $L_{p,q}\,(loc)$ approximation with positive linear operators by means of modulus of continuity.

Now we recall some information of locally integrable functions given in [6].

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Let $q(x)=1+x^2$; $-\infty<x<\infty$. For $h>0$, by $L_{p,q}(loc)$ we will denote the space of measurable functions f satisyfing the inequality,

$$\left(\frac{1}{2h}\int\limits_{x-h}^{x+h}|f(t)|^p\,dt\right)^{1/p}\leq M_f\;q\left(x\right)\quad,-\infty<x<\infty \tag{1}$$

where $p\geq 1$ and M_f is a positive constant which depends on the function f.

It is known [6] that $L_{p,q}(loc)$ is a linear normed space with norm,

$$\|f\|_{p,q}=\sup_{-\infty<x<\infty}\frac{\left(\frac{1}{2h}\int\limits_{x-h}^{x+h}|f(t)|^p\,dt\right)^{1/p}}{q\left(x\right)} \tag{2}$$

where $\|f\|_{p,q}$ may also depend on $h>0$. To simplify the notation, we need the following. For any real numbers a and b put

$$\|f;L_p\left(a,b\right)\|_{p,q}:=\left(\frac{1}{b-a}\int\limits_{a}^{b}|f(t)|^p\,dt\right)^{1/p},$$

$$\|f;L_{p,q}\left(a,b\right)\|_{p,q}=\sup_{a<x<b}\frac{\|f;L_p\left(x-h,x+h\right)\|_{p,q}}{q(x)},$$

$$\|f;L_{p,q}\left(|x|\geq a\right)\|_{p,q}=\sup_{|x|\geq a}\frac{\|f;L_p\left(x-h,x+h\right)\|_{p,q}}{q(x)}.$$

With this notation the norm in $L_{p,q}\left(loc\right)$ may be written in the form

$$\|f\|_{p,q}=\sup_{x\in\mathbb{R}}\frac{\|f;L_p\left(x-h,x+h\right)\|}{q(x)}.$$

It is known [6] that $L^k_{p,q}\left(loc\right)$ is the subspace of all functions $f\in L_{p,q}\left(loc\right)$ for which there exists a constant k_f such that

$$\lim_{|x|\to\infty}\frac{\|f-k_fq;L_p\left(x-h,x+h\right)\|}{q(x)}=0.$$

As usual, if T is a positive linear operator from $L_{p,q}\left(loc\right)$ into $L_{p,q}\left(loc\right)$, then the operator norm $\|T\|$ is given by $\|T\|:=\sup\limits_{f\neq 0}\frac{\|Tf\|_{p,q}}{\|f\|_{p,q}}$.

2. Strong $A-$summation process in $L_{p,q}\left(loc\right)$

The main aim of the present work is to study a Korovkin type approximation theorem for a sequence of positive linear operators acting on the weighted space $L_{p,q}\left(loc\right)$ by using matrix summability method which includes both convergence and almost convergence. We also give an example of positive linear operators which verifies our Theorem 2.5. but does not verify the classical one (see Theorem 2.1 below).

Let $\mathcal{A}:=\left\{A^{(n)}\right\}=\left\{a^{(n)}_{kj}\right\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}\left(loc\right)$ into itself. If

$$\lim_k\sum_j a^n_{kj}\left\|T_jf-f\right\|_{p,q}=0,\quad\text{uniformly in } n, \tag{3}$$

then we say that $\{T_jf\}$ is strongly $\mathcal{A}-$*summable* to f for every f in $L_{p,q}\left(loc\right)$ where it is assumed that the series converges for each k,n and f. Some results concerning summation processes on some other spaces may be found in [2], [9] and [10].

We recall the following result of [6] that we need in the sequel.

2.1. Theorem. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself and satisfy the conditions

$i)$ The sequence (T_j) is uniformly bounded, that is, $\|T_j\| \leq C < \infty$, where C is a constant independent of j,

$ii)$ $\lim_j \|T_j(f_i;x) - f_i(x)\|_{p,q} = 0$ where $f_i(y) = y^i$, $i = 0,1,2$.Then

$$\lim_j \|T_j f - f\|_{p,q} = 0$$

for each function $f \in L^k_{p,q}(loc)$, $(see$ [6]$)$.

The next result shows that Korovkin type theorem does not hold in the whole space $L_{p,q}(loc)$.

2.2. Theorem. Let $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a^{(n)}_{kj}\right\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself satisfying

$$\lim_k \sup_n \sum_j a^{(n)}_{kj} \|T_j(f_i;x) - f_i(x)\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0,1,2$. Then there exists a function f^* in $L_{p,q}(loc)$ for which

$$\lim_k \sup_n \sum_j a^{(n)}_{kj} \|T_j f^* - f^*\|_{p,q} \geq 2^{1-\frac{1}{p}}. \tag{4}$$

Proof. We consider the sequence of operators T_j given in [6] :

$$T_j(f;x) = \begin{cases} \frac{x^2}{(x+h)^2} f(x+h) & , x \in [2(j-1)h, (2j+1)h) \\ f(x) & , \ otherwise. \end{cases}$$

It is shown in [6] that

$$\|T_j f\|_{p,q} \leq 4\|f\|_{p,q}.$$

Assume now that $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a^{(n)}_{kj}\right\}$ is a sequence of infinite matrices defined by

$$a^{(n)}_{kj} = \begin{cases} \frac{1}{k+1} & , n \leq j \leq n+k \\ 0 & , \ otherwise. \end{cases}$$

Consider the following function f^* given in [6] :

$$f^*(x) = \begin{cases} x^2 & , \ if \ x \in \bigcup_{k=1}^{\infty} [(2k-1)h, 2kh) \\ -x^2 & , \ if \ x \in \bigcup_{k=1}^{\infty} [2kh, (2k+1)h) \\ 0 & , \ if \ x < 0. \end{cases}$$

Then $f^* \in L_{p,q}(loc)$ and it is shown in [6] that

$$\|T_j f^* - f^*\|_{p,q} \geq 2^{1-\frac{1}{p}} \frac{(2j-1)^2 h^2}{1+4j^2h^2}.$$

Hence

$$\frac{1}{k+1} \sum_{j=n}^{k+n} \|T_j f^* - f^*\|_{p,q} \geq \frac{1}{k+1} \sum_{j=n}^{k+n} 2^{1-\frac{1}{p}} \frac{(2j-1)^2 h^2}{1+4j^2h^2}.$$

On applying the operator $\lim_k \sup_n$ on both sides one can see that

$$\lim_k \sup_n \frac{1}{k+1}\sum_{j=n}^{k+n}\|T_j f^* - f^*\|_{p,q} \geq 2^{1-1/p}.$$

Therefore the theorem is proved.

Now we show that the above mentioned problem has a positive solution in the subspace $L^k_{p,q}(loc)$. First we give the following

2.3. Lemma. Let $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a^{(n)}_{kj}\right\}$ be a sequence of infinite matrices with non-negative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself satisfying

$$\lim_k \sup_n \sum_j a^{(n)}_{kj} \|T_j(f_i; x) - f_i(x)\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. Assume that

(5) $$H^{'} = \sup_{n,k} \sum_j a^{(n)}_{kj} < \infty.$$

Then, for any continuous and bounded function f on the real axis,

$$\lim_k \sup_n \sum_j a^{(n)}_{kj} \|T_j(f; x) - f(x); L_{p,q}(a,b)\| = 0$$

holds, where a and b are any real numbers.

Proof. Since f is uniformly continuous function on any closed interval, given $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that if $|t-x| < \delta$ implies that

(6) $$|f(t) - f(x)| < \varepsilon\ , \text{ for all } x \in [a,b], t \in \mathbb{R}.$$

Also, setting $M = \sup_{x\in\mathbb{R}} |f(x)|$, we can write if $|t-x| \geq \delta$ that

(7) $$|f(t) - f(x)| < 2M\ , \text{ for all } x \in [a,b], t \in \mathbb{R}.$$

Combining (6) and (7) we have

(8) $$|f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2}(t-x)^2\,,$$

where -$\infty < t < \infty$; $x \in [a,b]$. Let $c := maks\{|a|, |b|\}$ and using the positivity and linearity of operators T_j we obtain from (8) that

$$\begin{aligned}
&\sum_j a^{(n)}_{kj} \|T_j(f(t); x) - f(x); L_{p,q}(a,b)\| \\
&\leq \sum_j a^{(n)}_{kj} \|T_j(|f(t) - f(x)|; x)\|_{p,q} + |f(x)| \sum_j a^{(n)}_{kj} \|T_j(1; x) - 1\|_{p,q} \\
&< \sum_j a^{(n)}_{kj} \left\|T_j(\varepsilon + \frac{2M}{\delta^2}(t-x)^2; x)\right\|_{p,q} + M\sum_j a^{(n)}_{kj} \|T_j(1; x) - 1\|_{p,q} \\
&= \varepsilon \sum_j a^{(n)}_{kj} + \frac{2M}{\delta^2}\sum_j a^{(n)}_{kj} \left\|T_j(t^2; x) - x^2\right\|_{p,q} + \frac{4Mc}{\delta^2}\sum_j a^{(n)}_{kj} \|T_j(t; x) - x\|_{p,q} \\
&+ \left(\frac{2Mc^2}{\delta^2} + \varepsilon + M\right)\sum_j a^{(n)}_{kj} \|T_j(1; x) - 1\|_{p,q}\,.
\end{aligned}$$

Hence the proof is completed.

2.4. Theorem. Let $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Assume that

$$H := \sup_{n,k} \sum_j a_{kj}^{(n)} \|T_j\| < \infty \tag{9}$$

and

$$H' := \sup_{n,k} \sum_j a_{kj}^{(n)} < \infty. \tag{10}$$

Then $\{T_j\}$ is an $A-$ *strong summation process* in $L_{p,q}^k(loc)$, *i.e.*,for any function $f \in L_{p,q}^k(loc)$ we have

$$\lim_k \sup_n \sum_j a_{kj}^{(n)} \|T_j(f;x) - f(x)\|_{p,q} = 0$$

if and only if

$$\lim_k \sup_n \sum_j a_{kj}^{(n)} \|T_j(f_i;x) - f_i(x)\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$.

Proof. We follow [6] up to a certain stage. If $f \in L_{p,q}^k(loc)$ then $f - k_f.q \in L_{p,q}^0(loc)$. So it is sufficient to prove the theorem for the function $f \in L_{p,q}^0(loc)$. For $\varepsilon > 0$, there exists a point x_0 such that the inequality

$$\left(\frac{1}{2h}\int_{x-h}^{x+h} |f(t)|^p\, dt\right)^{1/p} < \varepsilon q(x) \tag{11}$$

holds for all x, $|x| \geq x_0$. By the well known Lusin Theorem, there exists a continuous function φ on the finite interval $[-x_0-h, x_0+h]$ such that the inequality

$$\|f - \varphi; L_p(-x_0, x_0)\| < \varepsilon \tag{12}$$

is fulfilled. Setting

$$\delta < \min\left\{\frac{2h\varepsilon^p}{M^p(x_0)}, h\right\}, \tag{13}$$

where $M(x_0) = \max\left\{\max_{|x|\leq x_0+h} |\varphi(x)|, 1\right\}$, we can define a continuous function g by

$$g(x) = \begin{cases} \varphi(x) & , \ if\ \ |x| \leq x_0 + h \\ 0 & , \ if\ \ |x| \geq x_0 + h + \delta \\ linear & , \ otherwise. \end{cases}$$

Then by (11), (12), (13) and the Minkowski inequality, we obtain

$$\|f - g\|_{p,q} < \varepsilon \tag{14}$$

for any $\varepsilon > 0$ (see [6]).

Now we can find a point $x_1 > x_0$ such that

$$q(x_1) > \frac{M(x_0)}{\varepsilon} \text{ and } g(x) = 0 \text{ for } |x| > x_1, \tag{15}$$

where $M(x_0)$ is defined above. Then by (12), (13), (14) and the definiton of g and Lemma 2.1. we get

$$\begin{aligned}\sum_j a_{kj}^{(n)} \left\|T_j(f;x)-f(x)\right\|_{p,q} &\le \sum_j a_{kj}^{(n)} \left\|T_j(f-g)\right\|_{p,q} + \sum_j a_{kj}^{(n)} \left\|T_j g - g\right\|_{p,q} \\ &+ \sum_j a_{kj}^{(n)} \left\|f-g\right\|_{p,q} \\ &\le \varepsilon\left(\sum_j a_{kj}^{(n)} \left\|T_j\right\|_{p,q} + \sum_j a_{kj}^{(n)}\right) \\ &+ \sum_j a_{kj}^{(n)} \left\|T_j g - g; L_{p,q}(-x_1,x_1)\right\| \\ &+ \sum_j a_{kj}^{(n)} \left\|T_j g - g; L_{p,q}(|x|\ge x_1)\right\| \\ &\le \varepsilon\left(\sum_j a_{kj}^{(n)} \left\|T_j\right\|_{p,q} + \sum_j a_{kj}^{(n)} + 1\right) \\ &+ \sum_j a_{kj}^{(n)} \left\|T_j g; L_{p,q}(|x|\ge x_1)\right\|. \end{aligned} \tag{16}$$

Since $|g(x)| \le M(x_0)$ for all $x \in \mathbb{R}$, we can write

$$\begin{aligned}\sum_j a_{kj}^{(n)} \left\|T_j g; L_{p,q}(|x|\ge x_1)\right\|_{p,q} &\le M(x_o) \sum_j a_{kj}^{(n)} \left\|T_j 1; L_{p,q}(|x|\ge x_1)\right\| \\ &\le M(x_o) \sum_j a_{kj}^{(n)} \left\|T_j 1 - 1; L_{p,q}(|x|\ge x_1)\right\| \\ &+ M(x_o) \sum_j a_{kj}^{(n)} \left\|1; L_{p,q}(|x|\ge x_1)\right\| \\ &\le M(x_o) \sum_j a_{kj}^{(n)} \left\|T_j 1 - 1\right\|_{p,q} \\ &+ \frac{M(x_o)}{q(x_1)} \sum_j a_{kj}^{(n)}. \end{aligned}$$

Considering hypothesis and (15) we get by (16) that

$$\lim_k \sup_n \sum_j a_{kj}^{(n)} \left\|T_j f - f\right\|_{p,q} = 0.$$

In the whole space $L_{p,q}(loc)$ we have the following**.**

2.5. Theorem. Let $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries for which (9) and (10) holds. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself satisfying

$$\lim_k \sup_n \sum_j a_{kj}^{(n)} \left\|T_j(f_i;x) - f_i(x)\right\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. Then for any functions $f \in L_{p,q}(loc)$ we have

$$\lim_k \sup_n \sum_j a_{kj}^{(n)} \left(\sup_{x\in\mathbb{R}} \frac{\left\|T_j f - f; L_p(x-h, x+h)\right\|_{p,q}}{q^*(x)}\right) = 0$$

where q^* is a weight function such that $\lim\limits_{|x|\to\infty} \frac{1+x^2}{q^*(x)} = 0$.

Proof. By hypothesis, given $\varepsilon > 0$, there exists x_0 such that for all x with $|x| \geq x_0$ we have

$$\frac{1+x^2}{q^*(x)} < \varepsilon. \tag{17}$$

Let $f \in L_{p,q}(loc)$. Then, for all n, k we get

$$\begin{aligned}
\gamma_n &:= \sum_j a_{kj}^{(n)} \left\| B_k^{(n)} f - f; L_p(|x| > x_0) \right\| \\
&\leq \sum_j a_{kj}^{(n)} \|T_j f\|_{p,q} + \sum_j a_{kj}^{(n)} \|f\|_{p,q} \\
&\leq \|f\|_{p,q} \left(\sum_j a_{kj}^{(n)} \|T_j\|_{p,q} + \sum_j a_{kj}^{(n)} \right) < N, \text{ say.}
\end{aligned}$$

Hence we have $\sup\limits_n \gamma_n < \infty$ is bounded. By Lusin's theorem we can find a continuous function φ on $[-x_0 - h, x_0 + h]$ such that

$$\|f - \varphi; L_p(-x_0 - h, x_0 + h)\| < \varepsilon. \tag{18}$$

Now we consider the following function G given in [6]

$$G(x) := \begin{cases} \varphi(-x_0 - h) & , \ x \leq -x_0 - h \\ \varphi(x_0) & , \ |x| < x_0 + h \\ \varphi(x_0 + h) & , \ x \geq x_0 + h. \end{cases}$$

We see that G is continuous and bounded on the whole real axis. Now let $f \in L_{p,q}(loc)$and we get for all n, k that

$$\begin{aligned}
\beta_n &:= \sum_j a_{kj}^{(n)} \|T_j f - f; L_{p,q}(-x_0, x_0)\| \\
&\leq \sum_j a_{kj}^{(n)} \|T_j(f - G); L_{p,q}(-x_0, x_0)\| + \sum_j a_{kj}^{(n)} \|T_j G - G; L_{p,q}(-x_0, x_0)\| \\
&+ \sum_j a_{kj}^{(n)} \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \\
&\leq \sum_j a_{kj}^{(n)} \|T_j\|_{p,q} \|(f - G); L_{p,q}(-x_0 - h, x_0 + h)\| \\
&+ \sum_j a_{kj}^{(n)} \|T_j G - G; L_{p,q}(-x_0, x_0)\| \\
&+ \sum_j a_{kj}^{(n)} \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \\
&\leq \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \left(\sum_j a_{kj}^{(n)} \|T_j\|_{p,q} + \sum_j a_{kj}^{(n)} \right) \\
&+ \sum_j a_{kj}^{(n)} \|T_j G - G; L_{p,q}(-x_0, x_0)\|.
\end{aligned}$$

Hence by the hypothesis and Lemma 2.1. we have

$$\lim_k \sup_n \beta_n = 0. \tag{19}$$

On the other hand, a simple calculation shows that

$$
\begin{aligned}
u_n &:= \sum_j a_{kj}^{(n)} \|T_j f - f\|_{p,q} \\
&< \sum_j a_{kj}^{(n)} \sup_{|x|<x_0} \frac{\left(\frac{1}{2h} \int\limits_{x-h}^{x+h} \left| \sum_j a_{kj}^{(n)} T_j f - f \right|^p dt \right)^{1/p}}{q^*(x)} \frac{q(x)}{q(x)} \\
&+ \sum_j a_{kj}^{(n)} \sup_{|x|\geq x_0} \frac{\left(\frac{1}{2h} \int\limits_{x-h}^{x+h} \left| \sum_j a_{kj}^{(n)} T_j f - f \right|^p dt \right)^{1/p}}{q^*(x)} \frac{q(x)}{q(x)} \\
&= \beta_n \sup_{|x|<x_0} \frac{q(x)}{q^*(x)} + \gamma_n \sup_{|x|\geq x_0} \frac{q(x)}{q^*(x)} \\
&< \beta_n q(x_0) + \varepsilon \gamma_n. \qquad (20)
\end{aligned}
$$

It follows from (17), (18), (19), (20) and Lemma 2.1. that

$$
\begin{aligned}
u_n &< q(x_0) \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \left(\sum_j a_{kj}^{(n)} \|T_j\|_{p,q} + \sum_j a_{kj}^{(n)} \right) \\
&+ q(x_0) \left\| \sum_j a_{kj}^{(n)} T_j G - G; L_{p,q}(-x_0, x_0) \right\| + \varepsilon N \\
&= K\varepsilon + q(x_0) \left\| \sum_j a_{kj}^{(n)} T_j G - G; L_{p,q}(-x_0, x_0) \right\|
\end{aligned}
$$

where $K := Mq(x_0) + N$ and $M := H + 1$. By Lemma 2.1. we get

$$
\lim_k \sup_n \sum_j a_{kj}^{(n)} \left(\sup_{x\in\mathbb{R}} \frac{\|T_j f - f; L_p(x-h, x+h)\|_{p,q}}{q^*(x)} \right) = 0.
$$

2.6. Remark. We now present an example of a sequence of positive linear operators which satisfies Theorem 2.5 but does not satisfy Theorem 2.1. Assume now that $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ is a sequence of infinite matrices defined by

$$
a_{kj}^{(n)} = \begin{cases} \frac{1}{k+1} & , \ n \leq j \leq n+k \\ 0 & , \quad \textit{otherwise}. \end{cases}
$$

In this case $\mathcal{A}-$summability method reduces to almost convergence,([8]).

Let $T_j : L_{p,q}(loc) \to L_{p,q}(loc)$ be given by

$$
T_j(f; x) = \begin{cases} \frac{x^2}{(x+h)^2} f(x+h) & , x \in [2(j-1)h, (2j+1)h] \\ f(x) & , \textit{otherwise} \end{cases}
$$

The sequence $\{T_j\}$ satisfies Theorem 2.1. (*see* [6]). It is shown that for all $j \in \mathbb{N}$, $\|T_j f\|_{p,q} \leq 4\|f\|_{p,q}$. Hence $\{T_j\}$ is an uniformly bounded sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Also

$$
\lim_k \sup_n \sum_j a_{kj}^{(n)} \|T_j(f_i; x) - f_i(x)\|_{p,q} = 0
$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. Now define $\{P_j\}$ by

$$
P_j(f; x) = (1 + u_j) T_j(f; x)
$$

where

$$u_j = \begin{cases} 1 & , \ j = 2^n, \ n \in \mathbb{N} \\ 0 & \quad d.d. \end{cases}$$

It is easy to see that $\{u_j\}$ almost convergent to zero. Therefore the sequence of positive linear operators $\{P_j\}$ satisfies Theorem 2.5. but does not satisfy Theorem 2.1.

3. Rates of Convergence For Strong $\mathcal{A}-$Summation Process in $L_{p,q}(loc)$

In this section, using the modulus of continuity, we study rates of convergence in $L_{p,q}(loc)$.
We now turn to introducing some notation and basic definitions to obtain the rate convergence of the operators given in Theorem 2.5.
Also, we consider the following modulus of continuity:

$$w(f,\delta) = \sup_{|x-y|\leq\delta} |f(y) - f(x)|,$$

where δ is a positive constant, $f \in L_{p,q}(loc)$. It is easy to see that, for any $c > 0$ and all $f \in L_{p,q}(loc)$,

$$w(f,\delta) \leq (1 + [c])\, w(f,\delta),$$

where $[c]$ is defined to be the greatest integer less than or equal to c, [3].
To obtain our main results we first need the following lemma.
3.1. Lemma. Let $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with non-negative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself.Then for each $j \in \mathbb{N}$ and $\delta > 0$, and for every function f that is continuous and bounded on the whole real axis, we have

$$\sum_j a_{kj}^{(n)} \|T_j f - f; L_{p,q}(a,b)\| \leq w(f;\delta) \sum_j a_{kj}^{(n)} \|T_j f_0 - f_0\|_{p,q} + 2w(f;\delta) \sum_j a_{kj}^{(n)} + C_1 \sum_j a_{kj}^{(n)} \|T_j f_0 - f_0\|_{p,q}$$

where $f_0(t) = 1, \varphi_x(t) := (t-x)^2, C_1 = \sup_{a<x<b} |f(x)|$ and $\delta := \alpha_j = \sqrt{\|T_j \varphi_x\|_{p,q}}$.

Proof. Let f be any continuous and bounded function on the real axis, and let $x \in [a,b]$ be fixed. Using linearity and monotonicity of T_j and for any $\delta > 0$, by modulus of continuity, we get

$$\begin{aligned} |T_j\ (f;x) - f(x)| &\leq T_j\ \left(w\left(f, \frac{|t-x|}{\delta}\delta\right), x\right) \\ &+ |f(x)|\, |T_j\ (f_0;x) - f_0(x)| \\ &\leq w_q(f,\delta)\, |T_j\ (f_0;x) - f_0(x)| + w_q(f,\delta) \\ &+ \frac{w_q(f,\delta)}{\delta^2} |T_j\ \varphi_x| + |f(x)|\, |T_j\ (f_0;x) - f_0(x)|. \end{aligned}$$

Let $\delta := \alpha_j = \sqrt{\|T_j\ \varphi_x\|_{p,q}}$. Then we have

$$\begin{aligned}\|T_j\ f - f; L_{p,q}(a,b)\| \le\ & w_q(f,\delta)\,\|T_j\ \ (f_0;x) - f_0(x)\|_{p,q} + \\ & + w_q(f,\delta) \\ & + \frac{w_q(f,\delta)}{\left(\sqrt{\|T_j\ \varphi_x\|_{p,q}}\right)^2}\|T_j\ \varphi_x\|_{p,q} \\ & + \|T_j\ \ (f_0;x) - f_0(x)\|_{p,q}\sup_{a<x<b}|f(x)|\end{aligned}$$

Now let $C_1 = \sup\limits_{a<x<b}|f(x)|$. Then we get

$$\begin{aligned}\sum_j a_{k,j}^{(n)}\|T_j\ f - f; L_{p,q}(a,b)\| \le\ & \sum_j a_{kj}^{(n)} w_q(f,\delta)\,\|T_j\ \ (f_0;x) - f_0(x)\|_{p,q} \\ & + 2\sum_j a_{kj}^{(n)} w_q(f,\delta) \\ & + C_1\sum_j a_{kj}^{(n)}\|T_j\ \ (f_0;x) - f_0(x)\|_{p,q}.\end{aligned}$$

3.2. Theorem. Let $\mathcal{A} : = \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries for which (10) holds. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Assume that for each continuous and bounded function f on the real line, the following conditions hold:

$$(i)\lim_k \sup_n \sum_j a_{kj}^{(n)}\|T_j\ \ (f_0;x) - f_0(x)\|_{p,q} = 0$$

$$(ii)\lim_k \sup_n \sum_j a_{kj}^{(n)} w_q(f,\delta) = 0$$

$$(iii)\lim_k \sup_n \sum_j a_{kj}^{(n)} w_q(f,\delta)\,\|T_j\ \ (f_0;x) - f_0(x)\|_{p,q} = 0$$

where $\delta = \alpha_j = \sqrt{\|T_j\ \varphi_x\|_{p,q}}$. Then we have

$$\lim_k \sup_n \sum_j a_{kj}^{(n)}\|T_j f - f; L_{p,q}(a,b)\| = 0.$$

Proof. Using Lemma 3.1. and considering (i), (ii), (iii) and (10) we have

$$\lim_k \sup_n \sum_j a_{kj}^{(n)}\|T_j f - f; L_{p,q}(a,b)\| = 0$$

for all continuous and bounded functions on the real axis.

3.3. Theorem. Let $\mathcal{A} : = \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries for which (9) and (10) holds. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Assume that

$$\lim_k \sup_n \sum_j a_{kj}^{(n)}\|T_j(f_i;x) - f_i(x)\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. If

$$(i) \lim_k \sup_n \sum_j a_{kj}^{(n)} \left\| T_j \ (f_0; x) - f_0(x) \right\|_{p,q} = 0$$

$$(ii) \lim_k \sup_n \sum_j a_{kj}^{(n)} w_q(G, \delta) = 0$$

$$(iii) \lim_k \sup_n \sum_j a_{kj}^{(n)} w_q(G, \delta) \left\| T_j \ (f_0; x) - f_0(x) \right\|_{p,q} = 0$$

where G is given as in the proof of Teorem 2.5. Then we have

$$\lim_k \sup_n \sum_j a_{kj}^{(n)} \left(\sup_{x \in \mathbb{R}} \frac{\| T_j f - f; L_p(x-h, x+h) \|}{q^*(x)} \right) = 0$$

where q^* is a weight function such that $\lim\limits_{|x| \to \infty} \frac{1+x^2}{q^*(x)} = 0$.

Proof. It is known from Theorem 2.5. that

$$\begin{aligned} u_n &< q(x_0) \| f - G; L_{p,q}(-x_0 - h, x_0 + h) \| \left(\sum_j a_{kj}^{(n)} \| T_j \|_{p,q} + \sum_j a_{kj}^{(n)} \right) \\ &+ q(x_0) \sum_j a_{kj}^{(n)} \| T_j G - G; L_{p,q}(-x_0, x_0) \| + \varepsilon N \\ &= K\varepsilon + q(x_0) \sum_j a_{kj}^{(n)} \| T_j G - G; L_{p,q}(-x_0, x_0) \| \end{aligned}$$

where $K := Mq(x_0) + N$ and $M := H + 1$. Then by Lemma 3.1. and Theorem 2.5. we get

$$\begin{aligned} u_k^{(n)} &\leq K\varepsilon + q(x_0) \sum_j a_{kj}^{(n)} w_q(G; \delta) \| T_j(f_0; x) - f_0(x) \|_{p,q} \\ &+ 2q(x_0) \sum_j a_{kj}^{(n)} w_q(G; \delta) \\ &+ q(x_0) C_1' \sum_j a_{kj}^{(n)} \| T_j(f_0; x) - f_0(x) \|_{p,q} \end{aligned}$$

where $C_1' := \sup\limits_{-x_0 < x < x_0} |G(x)|$ and the proof is completed.

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Bounds for the energy of graphs

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Abstract

The energy of a graph G, denoted by $E(G)$, is the sum of the absolute values of all eigenvalues of G. In this paper we present some lower and upper bounds for $E(G)$ in terms of number of vertices, number of edges, and determinant of the adjacency matrix. Our lower bound is better than the classical McClelland's lower bound. In addition, Nordhaus–Gaddum type results for $E(G)$ are established.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, $|E(G)| = m$. Let d_i be the degree of the vertex v_i for $i = 1, 2, \ldots, n$. The maximum and minimum vertex degrees are denoted by Δ and δ, respectively. If the vertices v_i and v_j are adjacent, we denote that by $v_iv_j \in E(G)$. The adjacency matrix $\mathbf{A} = \mathbf{A}(G)$ of G is defined by its entries as $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $\mathbf{A}(G)$. λ_1 is called the spectral radius of the graph G. Some well known properties of graph eigenvalues are:

$$\sum_{i=1}^{n} \lambda_i = 0 \quad , \quad \sum_{i=1}^{n} {\lambda_i}^2 = 2m \quad \text{and} \quad \det \mathbf{A} = \prod_{i=1}^{n} \lambda_i \, .$$

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A graph G is said to be singular if at least one of its eigenvalues is equal to zero. For singular graphs, evidently, $\det \mathbf{A} = 0$. A graph is nonsingular if all its eigenvalues are different from zero. Then, $\det \mathbf{A} \neq 0$.

The energy of the graph G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \tag{1.1}$$

where $\lambda_i,\ i = 1, 2, \ldots, n$, are the eigenvalues of graph G.

This spectrum-based graph invariant has been much studied in both chemical and mathematical literature. For details and an exhaustive list of references see the monograph [14]. What nowadays is referred to as graph energy, defined via Eq. (1.1), is closely related to the total π-electron energy calculated within the Hückel molecular orbital approximation; for details see in [8, 11, 18].

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we present a lower bound on the energy $E(G)$. In Section 4, we obtain an upper bound on $E(G)$. In Section 5, Nordhaus–Gaddum type results for $E(G)$ are established.

2. Preliminaries

In this section, we shall list some previously known results that will be needed in the next two sections.

2.1. Lemma. *(Cauchy interlace theorem)* [3, 17] *Let B be a $p \times p$ symmetric matrix and let B_k be its leading $k \times k$ submatrix; that is, B_k is a matrix obtained from B by deleting its last $p - k$ rows and columns. Then for $i = 1, 2, \ldots, k$,*

$$\rho_{p-i+1}(B) \leq \rho_{k-i+1}(B_k) \leq \rho_{k-i+1}(B) \tag{2.1}$$

where $\rho_i(B)$ is the i-th largest eigenvalue of B.

2.2. Lemma. [13] *Let $x_1, x_2, \ldots, x_N$ be non-negative numbers, and let*

$$\alpha = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad \text{and} \qquad \gamma = \left(\prod_{i=1}^{N} x_i\right)^{1/N}$$

be their arithmetic and geometric means. Then

$$\frac{1}{N(N-1)}\sum_{i<j}\left(\sqrt{x_i} - \sqrt{x_j}\right)^2 \leq \alpha - \gamma \leq \frac{1}{N}\sum_{i<j}\left(\sqrt{x_i} - \sqrt{x_j}\right)^2 .$$

Moreover, equality holds if and only if $x_1 = x_2 = \cdots = x_N$.

2.3. Lemma. [6] *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be non-negative real numbers. If $p > 1$, then*

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p\right)^{1/p} .$$

Moreover, the above equality holds if and only if the rows $\{a_i\}$ and $\{b_i\}$ are proportional.

2.4. Lemma. [1] *For a graph G,*

$$-\sqrt{\frac{2m(r-1)}{n(n-r+1)}} \le \lambda_r \le \sqrt{\frac{2m(n-r)}{nr}}\,, \quad 1 \le r \le n$$

2.5. Lemma. [2, 3] *Let G be a connected graph of order n. Then*

$$\lambda_1 \ge \frac{2m}{n}$$

with equality if and only if G is a regular graph.

3. Lower bound on graph energy

In this section we give a lower bound on energy $E(G)$ in terms of n, m and the determinant of the adjacency matrix.

First we mention some popular lower bounds on graph energy.

In the monograph [14] the following simple lower bound in terms of m is mentioned:

$$E(G) \ge 2\sqrt{m} \tag{3.1}$$

with equality holding if and only if G consists of a complete bipartite graph $K_{a,\,b}$ such that $a \cdot b = m$ and arbitrarily many isolated vertices.

McClelland [18] obtained the following lower bound in terms of n, m and the determinant of the adjacency matrix:

$$E(G) \ge \sqrt{2m + n(n-1)|\det \mathbf{A}|^{2/n}}\,. \tag{3.2}$$

Recently, Das et al. [5] have given the following lower bound, valid for non-singular graphs:

$$E(G) \ge \frac{2m}{n} + n - 1 + \ln\left(\frac{n\,|\det \mathbf{A}|}{2m}\right). \tag{3.3}$$

We now give an additional such lower bound, applicable for any graphs:

3.1. Theorem. *Let G be a simple graph of order $n > 2$ with m edges. Then*

$$E(G) \ge \sqrt{2m + n(n-1)|\det \mathbf{A}|^{2/n} + \frac{4}{(n+1)(n-2)}\left[\sqrt{\frac{2m}{n}} - \left(\frac{2m}{n}\right)^{1/4}\right]^2} \tag{3.4}$$

where equality holds if and only if $G \cong \frac{n}{2}\,K_2$ (n is even) or $G \cong \overline{K_n}$.

Proof. When $G \cong \overline{K_n}$, we have $m = 0$, $\det \mathbf{A} = 0$ and $E(G) = 0$. Hence the equality holds in (3.4). When $G \cong \frac{n}{2}\,K_2$ (n is even), we have $2m = n$, $\det \mathbf{A} = (-1)^{n/2}$ and $E(G) = n$. Hence the equality holds in (3.4). When $G \cong p\,K_2 \cup (n-2p)K_1$ ($\lceil \frac{n}{2} \rceil > p \ge 1$), we have $2m = 2p < n$, $\det \mathbf{A} = 0$ and $E(G) = 2p$. Hence the inequality in (3.4) is strict. Otherwise, G has at least one connected component with $m_1 \ge 2$ (m_1 is the number of edges in the connected component).

From Lemma 2.2, we get

$$\sum_{i=1}^{N} x_i \ge N\left(\prod_{i=1}^{N} x_i\right)^{1/N} + \frac{1}{(N-1)}\sum_{i<j}\left(\sqrt{x_i} - \sqrt{x_j}\right)^2. \tag{3.5}$$

Putting $N = \dfrac{n(n-1)}{2}$ and

$$\begin{aligned}(x_1,\, x_2, \dots,\, x_N) \quad &= \quad \Big(|\lambda_1||\lambda_2|,\, |\lambda_1||\lambda_3|, \dots,\, |\lambda_1||\lambda_n|,\, |\lambda_2||\lambda_3|,\\ &\dots,\quad |\lambda_2||\lambda_n|, \dots,\, |\lambda_{n-1}||\lambda_n|\Big)\end{aligned}$$

in (3.5), we get

$$\begin{aligned}\sum_{1\le i<j\le n} |\lambda_i||\lambda_j| \quad &\ge \quad \frac{n(n-1)}{2}\left(\prod_{i=1}^{n}|\lambda_i|\right)^{2/n}\\ &+ \quad \frac{2}{(n^2-n-2)}\sum_{i<j\le k<\ell}\left(\sqrt{|\lambda_i||\lambda_j|}-\sqrt{|\lambda_k||\lambda_\ell|}\right)^2\end{aligned}$$

that is,

$$\begin{aligned}2\sum_{1\le i<j\le n} |\lambda_i||\lambda_j| \quad &\ge \quad n(n-1)\ |\det \mathbf{A}|^{2/n}\\ &+ \quad \frac{4}{(n+1)(n-2)}\sum_{i<j\le k<\ell}\left(\sqrt{|\lambda_i||\lambda_j|}-\sqrt{|\lambda_k||\lambda_\ell|}\right)^2 .\end{aligned} \tag{3.6}$$

By Lemma 2.4,

$$\lambda_{n/2} \le \sqrt{\frac{2m}{n}} \qquad \text{and} \qquad \lambda_{(n+1)/2} \le \sqrt{\frac{2m(n-1)}{n(n+1)}} < \sqrt{\frac{2m}{n}}$$

for even and odd n, respectively.

From Lemma 2.5 and also from the above, we get for $n \ge 3$,

$$\lambda_1 \ge \frac{2m}{n} \qquad \text{and} \qquad \lambda_{\lceil \frac{n}{2}\rceil} \le \sqrt{\frac{2m}{n}}\ . \tag{3.7}$$

Since $m \ge 1$, by Lemma 2.1,

$$\lambda_n \le \lambda_2(\mathbf{A}_2) = -1\ .$$

From the above, we have that $|\lambda_n| \ge 1$. Since $n \ge 3$ and $m_1 \ge 2$, we further have

$$\begin{aligned}&\sum_{i<j\le k<\ell}\left(\sqrt{|\lambda_i||\lambda_j|}-\sqrt{|\lambda_k||\lambda_\ell|}\right)^2 \ge \left(\sqrt{|\lambda_1||\lambda_n|}-\sqrt{|\lambda_{\lceil \frac{n}{2}\rceil}||\lambda_n|}\right)^2\\ &+ \sum_{\substack{i<j\le k<\ell\\ (i,j)\ne(1,n),\\ (k,\ell)\ne(\lceil \frac{n}{2}\rceil,n)}}\left(\sqrt{|\lambda_i||\lambda_j|}-\sqrt{|\lambda_k||\lambda_\ell|}\right)^2 > |\lambda_n|\left(\sqrt{|\lambda_1|}-\sqrt{|\lambda_{\lceil \frac{n}{2}\rceil}|}\right)^2\\ &\ge \left[\sqrt{\frac{2m}{n}}-\left(\frac{2m}{n}\right)^{1/4}\right]^2 .\end{aligned}$$

Combining the above result with (3.6), we get

$$2\sum_{1\le i<j\le n}|\lambda_i||\lambda_j| > n(n-1)\ |\det \mathbf{A}|^{2/n} + \frac{4}{(n+1)(n-2)}\left[\sqrt{\frac{2m}{n}}-\left(\frac{2m}{n}\right)^{1/4}\right]^2 .$$

Adding to both sides $\sum_{i=1}^{n} \lambda_i^2$ $(= 2m)$, we get

$$E(G)^2 > 2m + n(n-1)\ |\det \mathbf{A}|^{2/n} + \frac{4}{(n+1)(n-2)} \left[\sqrt{\frac{2m}{n}} - \left(\frac{2m}{n}\right)^{1/4} \right]^2$$

which straightforwardly implies (3.4). □

Inequality (3.4), as well as (4.1), was mentioned in [9], but without details and without the characterization of the equality cases.

3.2. Remark. Our lower bound (3.4) is better than the lower bound (3.2).

3.3. Remark. In [5], it has been mentioned that sometimes the lower bound in (3.3) is better that the lower bounds in (3.1) and (3.2), but the lower bound in (3.3) is applicable for non-singular graphs.

4. Upper bound on graph energy

In this section we give an upper bound on energy $E(G)$ in terms of n, m, and $\det \mathbf{A}$. Other upper bounds on graph energy are discussed in the book [14] and the recent papers [4, 9, 19].

4.1. Theorem. *Let G be a connected non-singular graph of order n with m edges. Then*

$$E(G) \le 2m - \frac{2m}{n}\left(\frac{2m}{n} - 1\right) - \ln\left(\frac{n|\det \mathbf{A}|}{2m}\right) \tag{4.1}$$

where $\det \mathbf{A}$ $(\neq 0)$ *is the determinant of the adjacency matrix. Equality holds in* (4.1) *if and only if* $G \cong K_n$.

Proof. Since G is non-singular, we have $|\lambda_i| > 0$, $i = 1, 2, \ldots, n$. Thus

$$|\det \mathbf{A}| = \prod_{i=1}^{n} |\lambda_i| > 0\ .$$

Moreover, since G has no isolated vertices,

$$2m = \sum_{i=1}^{n} d_i \ge n \qquad \text{i.e.,} \qquad \frac{2m}{n} \ge 1\ .$$

Consider now the function

$$f(x) = x^2 - x - \ln x, \ \ x > 0$$

for which

$$f'(x) = 2x - 1 - \frac{1}{x}\ .$$

Thus $f(x)$ is an increasing function on $x \ge 1$ and a decreasing function on $0 < x \le 1$. Thus, $f(x) \ge f(1) = 0$ implying $x \le x^2 - \ln x$ for $x > 0$, with equality holding if and only if $x = 1$. Using this result, we get

$$\begin{aligned}
E(G) &= \lambda_1 + \sum_{i=2}^{n} |\lambda_i| \\
&\leq \lambda_1 + \sum_{i=2}^{n} \left(\lambda_i^2 - \ln|\lambda_i|\right) \qquad (4.2) \\
&= \lambda_1 + 2m - \lambda_1^2 - \ln \prod_{i=1}^{n} |\lambda_i| + \ln \lambda_1 \\
&= 2m + \lambda_1 - \lambda_1^2 - \ln|\det \mathbf{A}| + \ln \lambda_1 \ . \qquad (4.3)
\end{aligned}$$

From Lemma 2.5 we know that $\lambda_1 \geq 2m/n$. Since

$$g(x) = 2m + x - x^2 - \ln|\det \mathbf{A}| + \ln x$$

is an increasing function on $0 < x \leq 1$ and a decreasing function on $x \geq 1$, and since $x \geq \frac{2m}{n} \geq 1$, we have

$$g(x) \leq g\left(\frac{2m}{n}\right) = 2m + \frac{2m}{n} - \left(\frac{2m}{n}\right)^2 - \ln|\det \mathbf{A}| + \ln\left(\frac{2m}{n}\right) .$$

Combining this with (4.3), we arrive at (4.1). By this, the first part of the proof is done.

Suppose now that the equality holds in (4.1). Then all the inequalities in the above consideration must be equalities. From equality in (4.2), we get

$$|\lambda_2| = |\lambda_3| = \cdots = |\lambda_n| = 1 \ . \qquad (4.4)$$

Since G is connected, condition (4.4) is satisfied if and only if $G \cong K_n$ [3].

Conversely, one can see easily that the equality holds in (4.1) for K_n . □

Concluding this section, it should be mentioned that similar techniques (based on the inequalities stated in Section 2) have been used in estimating other spectrum–based graph indices, especially the Estrada index $EE(G)$ [7, 12, 15, 16, 21, 22]. Recall that this index is defined as

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$$

and that details of its theory can be found in the survey [10].

5. Nordhaus–Gaddum–type results for graph energy

Motivated by the seminal work of Noradhaus and Gaddum [20], we report here analogous results for graph energy. As usual, $\overline{G}$ denotes the complement of the graph G.

5.1. Theorem. *Let G and $\overline{G}$ be both connected non-singular graphs. If G has n vertices and m edges, then*

$$\begin{aligned}
3(n-1) + \ln\left(\frac{n^2\,|\det(A\,\overline{A})|}{2m(n(n-1)-2m)}\right) \leq E(G) + E(\overline{G}) &\leq 2(n-1) \\
+ \frac{4m(n(n-1)-2m)}{n^2} - \ln\left(\frac{n^2\,|\det(\mathbf{A}\,\overline{\mathbf{A}})|}{2m(n(n-1)-2m)}\right) & \qquad (5.1)
\end{aligned}$$

where $\det \mathbf{A}$ $(\neq 0)$ *and* $\det \overline{A}$ $(\neq 0)$ *are the determinants of the adjacency matrices of* G *and* $\overline{G}$*, respectively.*

Proof. By (3.3),

$$E(G) + E(\overline{G}) \geq \frac{2m + 2\overline{m}}{n} + 2(n-1) + \ln\left(\frac{n\,|\det \mathbf{A}|}{2m}\right) + \ln\left(\frac{n\,|\det \overline{A}|}{2\overline{m}}\right)$$

where $\overline{m}$ and $\overline{\mathbf{A}}$ are the number of edges and the adjacency matrix of $\overline{G}$.

Since $2m+2\overline{m} = n(n-1)$ and $\det \mathbf{A}\overline{A} = \det \mathbf{A}\ \det \overline{A}$, the lower bound in (5.1) follows.

By (4.1),

$$\begin{aligned} E(G) + E(\overline{G}) \quad &\leq \quad 2m + 2\overline{m} + \frac{2m + 2\overline{m}}{n} - \frac{4m^2 + 4\overline{m}^2}{n^2} \\ &- \quad \ln\left(\frac{n|\det \mathbf{A}|}{2m}\right) - \ln\left(\frac{n\,|\det \overline{\mathbf{A}}|}{2\overline{m}}\right). \end{aligned}$$

This straightforwardly leads to the upper bound in (5.1). □

5.2. Theorem. *Let* G *be a graph of order* n *with* m *edges. Then*

$$\begin{aligned} E(G) + E(\overline{G}) \quad &\leq \quad n + \Delta - \delta - 1 \\ &+ \quad \left[(n-1)\left(n - 1 + \frac{4m(n(n-1) - 2m)}{n^2}\right.\right. \\ &+ \quad \left.\left.\frac{2}{n^2}\sqrt{2m(2m+n)(n^2-2m)(n^2-2m-n)}\right)\right]^{1/2} \end{aligned} \tag{5.2}$$

where Δ *and* δ *are the maximum degree and minimum degree of* G*, respectively.*

Proof. By Lemma 2.3,

$$\left(\sum_{i=2}^{n}(|\lambda_i| + |\overline{\lambda}_i|)^2\right)^{1/2} \leq \left(\sum_{i=2}^{n}\lambda_i^2\right)^{1/2} + \left(\sum_{i=2}^{n}\overline{\lambda}_i^2\right)^{1/2}$$

where λ_i and $\overline{\lambda}_i$ are eigenvalues of G and $\overline{G}$, respectively. Since

$$\sum_{i=1}^{n}\lambda_i^2 = 2m \quad \text{and} \quad \sum_{i=1}^{n}\overline{\lambda}_1^2 = 2\overline{m}$$

we get

$$\sum_{i=2}^{n}(|\lambda_i|+|\overline{\lambda}_i|)^2 \le \sum_{i=2}^{n}\lambda_i^2+\sum_{i=2}^{n}\overline{\lambda}_i^2+2\sqrt{\sum_{i=2}^{n}\lambda_i^2\,\sum_{i=2}^{n}\overline{\lambda}_i^2}$$

$$= 2m-\lambda_1^2+2\overline{m}-\overline{\lambda}_1^2+2\sqrt{(2m-\lambda_1^2)\,(2\overline{m}-\overline{\lambda}_1^2)}$$

$$\le n(n-1)-\frac{4m^2+4\overline{m}^2}{n^2}+2\sqrt{\frac{4m\,\overline{m}}{n^4}\,(n^2-2m)\,(2m+n)}$$

$$= n-1+\frac{4m(n(n-1)-2m)}{n^2}$$

$$+ \frac{2}{n^2}\sqrt{2m(n^2-2m-n)(n^2-2m)(2m+n)}\,. \tag{5.3}$$

Since $\lambda_1 \le \Delta$, using the Cauchy–Schwarz inequality, we obtain

$$E(G)+E(\overline{G}) = |\lambda_1|+|\overline{\lambda}_1|+\sum_{i=2}^{n}(|\lambda_i|+|\overline{\lambda}_i|)$$

$$\le \Delta+n-\delta-1+\sqrt{(n-1)\sum_{i=2}^{n}(|\lambda_i|+|\overline{\lambda}_i|)^2}\,.$$

Together with (5.3) this yields (5.2). □

6. Concluding remarks

Studies of the structure–dependence of the total π-electron energy has a long history. Beginning with McClelland's seminal work [18] in the early 1970s, most of the researches along these lines were done by means of estimates (upper and lower bounds); for details see the surveys [8, 9]. Eventually, the concept of total π-electron energy was extended and redefined to the mathematically more general and more convenient concept of graph energy, Eq. (1.1), see [14].

In the present work we offer a few more estimates for graph energy, in terms of parameters that have direct and straightforward structural interpretation. By this, we deem to have somewhat improved the understanding of how graph energy (and thus total π-electron energy) are influenced by the respective structural features of the underlying graph.

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Simple groups with m-regular first prime graph component

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Abstract

Let G be a finite simple group and $\mathrm{GK}(G)$ be the prime graph of G. The connected component of $\mathrm{GK}(G)$ whose vertex set contains 2 is denoted by $\pi_1(G)$. In this paper, our purpose is to classify the finite simple groups G such that $\pi_1(G)$ is regular. We prove that $\pi_1(G)$ is regular if and only if all the connected components of $\mathrm{GK}(G)$ are cliques.

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1. Introduction

For a positive integer n, let $\pi(n)$ denote the set of all prime divisors of n. Given a finite group G, we set $\pi(G) = \pi(|G|)$. The *prime graph* (or *Gruenberg-Kegel graph*) $\mathrm{GK}(G)$ of G is a simple graph which is defined as follows. The vertex set of $\mathrm{GK}(G)$ is the set $\pi(G)$ and two distinct vertices p and q are adjacent (we write $(p, q) \in \mathrm{GK}(G)$) if G contains an element of order pq. If $2 \in \pi(G)$, then the connected component of $\mathrm{GK}(G)$ whose vertex set contains 2 is denoted by $\pi_1(G)$.

The concept of prime graph arose during the investigation of certain cohomological questions associated with integral representations of finite groups. And after that, prime graphs have received some attention in the theory of finite groups. For instance, it has been proved that some of finite simple groups can be characterized by their prime graphs (see [2, 3, 5, 17]). Moreover, some graph properties of this graph have been studied. It has been showed that for every finite group G, the number of connected components of $\mathrm{GK}(G)$ is at most 6 (see [6, 16]) and the diameter of $\mathrm{GK}(G)$ is at most 5 (see [8]). Also, in [9] the groups G such that $\mathrm{GK}(G)$ is a tree, have been investigated. Moreover, according to [12, 16], we know that if Δ is a connected component of $\mathrm{GK}(G)$ whose vertex set does

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not contain 2, then Δ is a clique. Note that a clique in a graph is a subset of its vertices such that every two vertices in the subset are connected by an edge. Motivated by this result, Lucido and Moghaddamfar in [10], described the finite nonabelian simple groups G such that $\pi_1(G)$ is a clique. Finally, in [4, 11, 18], the finite simple groups G such that $\pi_1(G)$ is m-regular, where $m \in \{0,1,2\}$, have been obtained. For a nonnegative integer m, a graph is called m-regular, when the degree of each vertex is m. Also, a graph is regular if the degrees of all vertices are the same. The aim of this paper is to extend the m-regularity results, for an arbitrary m. In fact, we prove the following main theorem:

Main theorem. *Let G be a finite nonabelian simple group and let m be a nonnegative integer. If $\pi_1(G)$ is m-regular, then $\pi_1(G)$ is a clique and one of the following statements holds:*

- $G = A_5, A_6, A_2(4), A_1(2^k)$, *where* $k > 1$, ${}^2B_2(2^{2k+1})$, *where* $k \geq 1$ *and* $m = 0$;
- $G = M_{11}, M_{22}, A_7$ *and* $m = 1$;
- $G = J_1, J_2, J_3, HiS, A_9, {}^3D_4(2), {}^2A_3(3), {}^2A_5(2), C_3(2), D_4(2)$ *and* $m = 2$;
- $G = A_{12}, A_{13}$ *and* $m = 3$;
- $G = A_1(q)$, *where* $q \equiv 1 \pmod 4$ *and* $m = |R_1(q)| - 1$;
- $G = A_1(q)$, *where* $q \equiv 3 \pmod 4$, $q > 3$, *and* $m = |R_2(q)| - 1$;
- $G = A_2(q)$, *where* $(q-1)_3 \neq 3$, $q + 1 = 2^k$, *and* $m = |R_1(q)| + 1$;
- $G = {}^2A_2(q)$, *where* $(q+1)_3 \neq 3$, $q - 1 = 2^k$, $C_2(q)$, *where* $q > 2$ *or* $G_2(3^k)$, *where* $k \geq 1$ *and* $m = |R_1(q)| + |R_2(q)|$.

It is worth remarking that by $R_k(q)$ we mean the set of all primitive prime divisors of $q^k - 1$.
As an immediate consequence of the main theorem, we have the following corollary:

Corollary. *Let G be a finite nonabelian simple group. Then $\pi_1(G)$ is regular if and only if all the connected components of the prime graph* $\mathrm{GK}(G)$ *are cliques.*

2. Notation and preliminary results

Throughout this paper, we use the following notation and definitions: By $\gcd(k,l)$ we denote the greatest common divisor of k and l. Let G be a finite group. For $p \in \pi(G)$, put $\deg(p) := |\{q \in \pi(G) |\ (p,q) \in \mathrm{GK}(G)\}|$.
The notation for groups of Lie type is according to [1] and sometimes for abbreviation, we write $A_n^\varepsilon(q)$ and $D_n^\varepsilon(q)$, where $\varepsilon \in \{+,-\}$, and $A_n^+(q) = A_n(q)$, $A_n^-(q) = {}^2A_n(q)$, $D_n^+(q) = D_n(q)$, $D_n^-(q) = {}^2D_n(q)$. Also, for an integer n, by $\eta(n)$, $\nu(n)$ and $\nu_\varepsilon(n)$ we denote the following functions:

$$\eta(n) = \begin{cases} n & \text{if } n \text{ is odd;} \\ n/2 & \text{otherwise.} \end{cases}$$

$$\nu(n) = \begin{cases} n & \text{if } n \equiv 0 \pmod 4; \\ n/2 & \text{if } n \equiv 2 \pmod 4; \\ 2n & \text{if } n \equiv 1 \pmod 2. \end{cases} \qquad \nu_\varepsilon(n) = \begin{cases} n & \text{if } \varepsilon = +; \\ \nu(n) & \text{if } \varepsilon = -. \end{cases}$$

All further unexplained group theory notation is standard and can be found in [1].

The following lemma describes the finite nonabelian simple groups G such that $\pi_1(G)$ is m-regular, where $m \in \{0,1,2\}$:

2.1. Lemma. [4, 11, 18] Let G be a finite nonabelian simple group.

1. If $\pi_1(G)$ is 0-regular, then $G = A_5, A_6; A_2(4), A_1(q)$, where q is a Fermat prime, a Mersenne prime or a prime power of 2; ${}^2B_2(q)$, where q is an odd prime power of 2.

2. If $\pi_1(G)$ is 1-regular, then $G = A_7; M_{11}, M_{22}; A_2(3), {}^2A_2(3), {}^2A_3(2), G_2(3); A_1(q)$, where q is a prime power such that $3 < q \equiv \varepsilon 1 \pmod 4$ and $|\pi(q - \varepsilon 1)| = 2$, for $\varepsilon \in \{+, -\}$.
3. If $\pi_1(G)$ is 2-regular, then $G = A_9; J_1, J_2, J_3, HiS; C_3(2), {}^2A_2(9), {}^2A_3(3), {}^3D_4(2), G_2(9), D_4(2); C_2(q)$, where $q = 4, 5, 7, 8, 9, 17$; $A_1(q)$, where q is a prime power such that $3 < q \equiv \varepsilon 1 \pmod 4$ and $|\pi(q - \varepsilon 1)| = 3$, for $\varepsilon \in \{+, -\}$.

The finite nonabelian simple groups G such that all the connected components of $\mathrm{GK}(G)$ are cliques, have been determined in [10]. Since this result plays a role in the proof of the main theorem, in the following, we state its revised version from [14]:

2.2. Lemma. Let G be a finite nonabelian simple group. Then all the connected components of $\mathrm{GK}(G)$ are cliques if and only if G is one of the following:

1. Sporadic groups $M_{11}, M_{22}, J_1, J_2, J_3, HiS$;
2. Alternating groups A_n, where $n = 5, 6, 7, 9, 12, 13$;
3. Groups of Lie type $A_1(q)$, where $q > 3$; $A_2(4)$; $A_2(q)$, where $(q-1)_3 \neq 3$, $q+1 = 2^k$; ${}^2A_3(3)$; ${}^2A_5(2)$; ${}^2A_2(q)$, where $(q+1)_3 \neq 3$, $q-1 = 2^k$; $C_3(2)$, $C_2(q)$, where $q > 2$; $D_4(2)$; ${}^3D_4(2)$; ${}^2B_2(q)$, where $q = 2^{2k+1}$; $G_2(q)$, where $q = 3^k$.

2.3. Remark. According to Table 1 in [7], we have $\pi_1({}^2A_5(2)) = \{2, 3, 5\}$ and $\pi_1({}^2A_2(17)) = \{2, 3, 17\}$. Moreover, by Lemma 2.2, the prime graph components of the groups ${}^2A_5(2)$ and ${}^2A_2(17)$ are cliques. Thus these mentioned groups should be added to the list of groups in Lemma 2.1(3).

3. Proof of the main theorem

If G is a finite nonabelian simple group, then by the classification of the finite simple groups, it follows that G is a sporadic simple group, an alternating group or a simple group of Lie type. We will consider each case separately.

According to [1], we can easily conclude the next statement for the sporadic simple groups:

3.1. Lemma. Let G be a sporadic simple group. If $\pi_1(G)$ is m-regular, then one of the following cases holds:

(1) $G = M_{11}, M_{22}$ and $m = 1$;
(2) $G = J_1, J_2, J_3, HiS$ and $m = 2$.

For considering the alternating groups, we need the following lemma:

3.2. Lemma. [7, Lemma 1] If $n \geq 19$ is a natural number, then there are at least three prime numbers q_i such that $(n+1)/2 < q_i < n$.

3.3. Lemma. Let $G = A_n$ be an alternating group of degree n. If $\pi_1(G)$ is m-regular, then $\pi_1(G)$ is a clique and one of the following cases holds:

(1) $G = A_5, A_6$ and $m = 0$;
(2) $G = A_7$ and $m = 1$;
(3) $G = A_9$ and $m = 2$;
(4) $G = A_{12}, A_{13}$ and $m = 3$.

Proof. According to Lemma 2.1, we can assume that $m \geq 3$. Note that for odd primes $r, s \in \pi(A_n)$, $(r, s) \notin \mathrm{GK}(A_n)$ if and only if $r + s > n$. Also, $(r, 2) \notin \mathrm{GK}(A_n)$ if and only if $r + 4 > n$ (see [14]). So, it easy to see that if $(s, r) \in \mathrm{GK}(A_n)$ and $(p, s) \neq (2, 3)$, where $2 \leq p < s < r$, then $(p, r), (p, s) \in \mathrm{GK}(A_n)$. Moreover, if $(p, r) \in \mathrm{GK}(A_n)$, then $(p, s) \in \mathrm{GK}(A_n)$.

If we denote the i-th prime number, by p_i, then since $\deg(2) = m$, according to $\pi(G)$, we see that $\{2, p_2, p_3, \cdots, p_m, p_{m+1}\} \subseteq \pi_1(G)$ and hence, $p_{m+1} \leq n$. We know that $p_{m+2} \leq n$, otherwise,

$$\pi(G) = \{2, p_2, p_3, \cdots, p_m, p_{m+1}\}$$

which implies that $\mathrm{GK}(G)$ is complete and this is impossible according to Lemma 2.2. Since $m \geq 3$, we have $p_{m+2} \geq 11$. Also, since $\pi_1(G)$ is m-regular, we conclude that $(3, p_{m+2}) \notin \mathrm{GK}(G)$, otherwise, $\deg(3) = m + 1$ which is a contradiction. Therefore, $n \leq 2 + p_{m+2}$. On the other hand, since $p_{m+1} \in \pi_1(G)$ and $\deg(p_{m+1}) = m$, we deduce that $(p_m, p_{m+1}) \in \mathrm{GK}(G)$ and hence, $p_m + p_{m+1} \leq n$. Thus $p_m + p_{m+1} \leq n \leq 2 + p_{m+2}$ which implies that

$$p_{m+2} - (p_m + p_{m+1}) \geq -2 \tag{3.1}$$

Now, if $p_{m+2} \geq 19$, then by Lemma 3.2 there exist at least three distinct primes q_i such that $(p_{m+2} - 1)/2 < q_i < p_{m+2}$. Thus we conclude that $(p_{m+2} - 1)/2 < p_{m-1} < p_m < p_{m+1} < p_{m+2}$ and hence,

$$1 + (p_{m+2} - 1)/2 \leq p_{m-1},$$

$$2 + (p_{m+2} - 1)/2 \leq p_m, \tag{3.2}$$

$$3 + (p_{m+2} - 1)/2 \leq p_{m+1}. \tag{3.3}$$

Summing 3.2 and 3.3, implies that $5 + 2 \times (p_{m+2} - 1)/2 \leq p_m + p_{m+1}$ and hence, $p_{m+2} - (p_m + p_{m+1}) \leq -4$, which contradicts 3.1. Thus $p_{m+2} \in \{11, 13, 17\}$. If $p_{m+2} = 11$, 13 or 17, then $m = 3$, 4 or 5 respectively. But according to 3.1, the last two cases cannot happen. So, $m = 3$ and by 3.1, we see that $n \in \{12, 13\}$, as desired. □

The rest of the paper will be devoted to the proof of the main theorem for the simple groups of Lie type. We will consider the classical and the exceptional groups of Lie type separately. For the classical simple groups, our method is based on the results of [14], concerning the arithmetic criterion of adjacency in their prime graphs.

Let s be a prime and let k be a natural number. The s-part of k which is denoted by k_s is equal to s^t if $s^t \mid k$ and $s^{t+1} \nmid k$. If q is a natural number, r is an odd prime and $\gcd(r, q) = 1$, then by $e(r, q)$ we denote the smallest natural number k such that $q^k \equiv 1 \pmod{r}$. If q is odd, we put $e(2, q) = 1$ whenever $q \equiv 1 \pmod 4$, and $e(2, q) = 2$ otherwise. The following lemma is considered as a corollary to Zsigmondy's theorem:

3.4. Lemma. [14, Lemma 1.4] Let q be a natural number greater than 1. For every natural number k, there exists a prime r with $e(r, q) = k$, but for the cases $q = 2$ and $k = 1$, $q = 3$ and $k = 1$, and $q = 2$ and $k = 6$.

A prime r with $e(r, q) = k$ is called a *primitive prime divisor* of $q^k - 1$. It is obvious that $q^k - 1$ can have more than one primitive prime divisor. We denote by $R_k(q)$ the set of all primitive prime divisors of $q^k - 1$ and by $r_k(q)$ any element of $R_k(q)$. When no confusion can arise, we will write r_k instead of $r_k(q)$ and R_k instead of $R_k(q)$.

3.5. Lemma. [14, Propositions 2.1-2.2],[15, Propositions 2.4-2.5] Let G be a finite simple group of Lie type over a field of order $q = p^\alpha$, for some prime p. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$.

(1) If $G = A^{\varepsilon}_{n-1}(q)$ and $2 \leq \nu_\varepsilon(k) \leq \nu_\varepsilon(l)$, then r and s are nonadjacent if and only if $\nu_\varepsilon(k) + \nu_\varepsilon(l) > n$ and $\nu_\varepsilon(k)$ does not divide $\nu_\varepsilon(l)$.
(2) If $G = B_n(q)$ or $C_n(q)$ and $1 \leq \eta(k) \leq \eta(l)$, then r and s are nonadjacent if and only if $\eta(k) + \eta(l) > n$ and l/k is not an odd natural number.

(3) If $G = D_n^{\varepsilon}(q)$ and $1 \leq \eta(k) \leq \eta(l)$, then r and s are nonadjacent if and only if $2\eta(k) + 2\eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$ and l/k is not an odd natural number. Moreover, if $\varepsilon = +$, then the chain of equalities $n = l = 2\eta(l) = 2\eta(k) = 2k$, is not true as well.

3.6. Lemma. [14, Proposition 3.1] Let G be a finite simple classical group of Lie type over a field of characteristic p, and let $r \in \pi(G)$ and $r \neq p$. Then r and p are nonadjacent if and only if one of the following holds:

(1) $G = A_{n-1}^{\varepsilon}(q)$, r is odd, and $\nu_{\varepsilon}(e(r, q)) > n - 2$;
(2) $G = C_n(q)$ or $G = B_n(q)$, $\eta(e(r, q)) > n - 1$;
(3) $G = D_n^{\varepsilon}(q)$, $\eta(e(r, q)) > n - 2$;
(4) $G = A_1(q), r = 2$;
(5) $G = A_2^{\varepsilon}(q), r = 3$ and $(q - \varepsilon 1)_3 = 3$.

3.7. Lemma. [14, Proposition 4.1-4.2] Let $G = A_{n-1}^{\varepsilon}(q)$ be a finite simple group of Lie type, r be a prime divisor of $q-\varepsilon 1$, and s be an odd prime distinct from the characteristic. Put $k = e(s, q)$. Then s and r are nonadjacent if and only if one of the following holds:

(1) $\nu_{\varepsilon}(k) = n$, $n_r \leq (q - \varepsilon 1)_r$, and if $n_r = (q - \varepsilon 1)_r$, then $2 < (q - \varepsilon 1)_r$;
(2) $\nu_{\varepsilon}(k) = n - 1$ and $(q - \varepsilon 1)_r \leq n_r$.

3.8. Lemma. [14, Propositions 4.3-4.4] Let G be a finite simple group of Lie type over a field of order $q = p^{\alpha}$, for some prime p. Let r be an odd prime divisor of $|G|$, $r \neq p$, and $k = e(r, q)$.

(1) If $G = B_n(q)$ or $C_n(q)$, then r and 2 are nonadjacent if and only if $\eta(k) = n$ and one of the following holds:
  (a) n is odd and $k = (3 - e(2, q))n$;
  (b) n is even and $k = 2n$.
(2) If $G = D_n^{\varepsilon}(q)$, then r and 2 are nonadjacent if and only if one of the following holds:
  (a) $\eta(k) = n$ and $\gcd(4, q^n - \varepsilon 1) = (q^n - \varepsilon 1)_2$;
  (b) $\eta(k) = k = n - 1$, n is even, $\varepsilon = +$, and $e(2, q) = 2$;
  (c) $\eta(k) = k/2 = n - 1$, $\varepsilon = +$, and $e(2, q) = 1$;
  (d) $\eta(k) = k/2 = n - 1$, n is odd, $\varepsilon = -$, and $e(2, q) = 2$.

3.9. Remark. Let G be a finite simple group over a field of order q, where $q = p^{\alpha}$ for an odd prime p. According to the above lemmas, it is evident that 2 and p are adjacent in all classical simple groups except $A_1(q)$. Moreover, for a fixed k, every two elements in $R_k(q)$ are adjacent in $\mathrm{GK}(G)$.

From now on, we assume that $q = p^{\alpha}$, where p is a prime number.

3.10. Lemma. Let G be a finite simple classical group of Lie type. If $\pi_1(G)$ is m-regular, then $\pi_1(G)$ is a clique and one of the following cases holds:

(1) $G = C_2(q)$, where $q > 2$, and $m = |R_1(q)| + |R_2(q)|$.
(2) $G = A_1(q)$, where $q > 3$. In this case, if $q \equiv 1 \pmod 4$, then $m = |R_1(q)| - 1$; and if $q \equiv 3 \pmod 4$, then $m = |R_2(q)| - 1$; also if q is even, then $m = 1$.
(3) $G = A_2(q)$, where $(q - 1)_3 \neq 3$, $q + 1 = 2^k$, and $m = 1 + |R_1(q)|$;
(4) $G = A_2(4)$, and $m = 0$;
(5) $G = {}^2A_3(3)$, ${}^2A_5(2)$, $C_3(2)$ or $D_4(2)$, and $m = 2$;
(6) $G = {}^2A_2(q)$, where $(q + 1)_3 \neq 3$, $q - 1 = 2^k$, and $m = |R_1(q)| + |R_2(q)|$;

Proof. According to the types of the classical groups, the proof will be divided into five parts.

Part A. $G = B_n(q)$ or $G = C_n(q)$, where $n \geq 2$ and $(n, q) \neq (2, 2)$:
If $(n, q) = (3, 2)$, then $B_n(q) \cong C_n(q)$ and according to Lemma 2.1(3), the result is obvious. Also, if $n = 2$, then $q > 2$ and $B_n(q) \cong C_n(q)$ and hence, Lemma 2.2 implies that $\pi_1(G)$ is a clique. Thus according to Remark 3.9, it is enough to calculate $\deg(p)$. Since $\pi(G) = \{p\} \cup R_1(q) \cup R_2(q) \cup R_4(q)$, Lemma 3.6(2) implies that $m = |R_1| + |R_2|$ as desired. So we may assume that $n \geq 3$ and $(n, q) \neq (3, 2)$.
Case 1. Let n be an odd number.

• If $2 \notin R_2(q)$, then $q \equiv 1 \pmod 4$ or $p = 2$. Since $n \geq 3$, we can see that $R_{2n} \cap R_2 = \emptyset$. Also, Lemma 3.4 and the fact that $(n, q) \neq (3, 2)$ imply that $R_{2n}(q)$ is nonempty. Moreover, since $n \geq 3$ is odd, according to Lemmas 3.8(1), 3.6(2) and 3.5(2), we have $(2, r_2), (r_2, r_{2n}) \in \mathrm{GK}(G)$. Thus $\{r_{2n}, r_2\} \subseteq \pi_1(G)$. Now, we claim that if $(r, r_{2n}) \in \mathrm{GK}(G)$, then $(r, r_2) \in \mathrm{GK}(G)$:

Since $n \geq 3$ is odd, by Lemmas 3.8(1) and 3.6(2), we see that $(r_2, 2), (r_2, p) \in \mathrm{GK}(G)$. Thus, we may assume that $r \in R_l(q) \setminus \{2\}$, where $l \in \mathbb{N}$. Considering Lemma 3.5(2) implies that $2n/l$ is an odd number, so is $l/2$. Lemma 3.5(2) now yields r and r_2 are adjacent in $\mathrm{GK}(G)$.

Moreover, since $n \geq 3$ and $2 \notin R_2(q)$, considering Lemma 3.5(2) implies that $(r_2, r_{2(n-1)}) \in \mathrm{GK}(G)$ but $(r_{2n}, r_{2(n-1)}) \notin \mathrm{GK}(G)$. Thus, $\deg(r_2) > \deg(r_{2n})$ and hence, in this case $\pi_1(G)$ cannot be m-regular.

• If $2 \in R_2(q)$, then $q \equiv -1 \pmod 4$ and hence, q is odd. In this case, according to Lemma 3.8(1), we have $(2, r) \in \mathrm{GK}(G)$ if and only if $r \notin R_n(q)$. Also, Lemma 3.5(2) implies that $(r_n, r_{2n}) \notin \mathrm{GK}(G)$. Thus if the vertex r is adjacent to r_{2n}, then r and 2 are adjacent as well. On the other hand, according to Lemma 3.5(2), we have r_{2n} and $r_{2(n-1)}$ are nonadjacent and hence $\deg(2) > \deg(r_{2n})$, which implies that $\pi_1(G)$ cannot be m-regular.
Case 2. If n is even, then $n \geq 4$. In this case, by Lemmas 3.6(2) and 3.8(1), we conclude that r and 2 are adjacent if and only if $r \notin R_{2n}$. Thus if $r_{2n} \notin \pi_1(G)$, then $\pi_1(G)$ is a clique, which is impossible according to Lemma 2.2. Thus $r_{2n} \in \pi_1(G)$ and $\mathrm{GK}(G)$ is connected. In this case, we have $n = 2^k \times m$, where $m \geq 1$ is odd and $k \geq 1$. If $m = 1$, then Lemmas 3.6(2), 3.5(2) and 3.8(1) imply that $R_{2n}(q)$ is an odd connected component of $\mathrm{GK}(G)$, which is a contradiction. Therefore, $m \geq 3$ and $n \neq 2^k$. Now we claim that, if $(r, r_{2n}) \in \mathrm{GK}(G)$, then $(r, r_{2^{k+1}}) \in \mathrm{GK}(G)$:

Since n is even, by Lemmas 3.6(2) and 3.8(1), we conclude that

$$(2, r_{2n}), (p, r_{2n}) \notin \mathrm{GK}(G).$$

If $(r, r_{2n}) \in \mathrm{GK}(G)$, then $r \in R_l(q) \setminus \{2\}$, where $l \in \mathbb{N}$ and according to Lemma 3.5(2), $l = 2^{k+1} \times j$, where $j \mid m$. Therefore, we can easily infer our assertion by using Lemma 3.5(2).

On the other hand, since $n \neq 2^k$, Lemma 3.8(1) implies that $(2, r_{2^{k+1}}) \in \mathrm{GK}(G)$. Thus $\deg(r_{2n}) < \deg(r_{2^{k+1}})$ and $\pi_1(G)$ cannot be m-regular.

Consequently, if $G = B_n(q)$ or $C_n(q)$, according to Cases 1 and 2, $\pi_1(G)$ is m-regular if and only if $(n, q) = (3, 2)$ or $n = 2$ and $q > 2$. Moreover, Lemma 2.2 implies that $\pi_1(G)$ is a clique.
Part B. $G = D_n(q)$, where $n \geq 4$:
Case 1. If $2 \in R_2(q)$, then $q \equiv -1 \pmod 4$ and hence, $p \neq 2$. In this case, if n is even, then according to Lemma 3.6(3) and $|G|$, we conclude that $(r, p) \in \mathrm{GK}(G)$ if and only if $r \notin R_{n-1} \cup R_{2(n-1)}$. Also, considering Lemma 3.8(2) and $|G|$ imply that $(r, 2) \in \mathrm{GK}(G)$ if and only if $r \notin R_{n-1}$. Thus $R_{2(n-1)} \subseteq \pi_1(G)$ and $\deg(2) > \deg(p)$. If n is odd, then by the same procedure, we can conclude that $(r, 2) \in \mathrm{GK}(G)$ if and only if $r \notin R_n$ and $(r, p) \in \mathrm{GK}(G)$ if and only if $r \notin R_n \cup R_{2(n-1)}$. Thus $\deg(2) > \deg(p)$ and $\pi_1(G)$ cannot be m-regular.

Case 2. If $2 \notin R_2(q)$, then $p = 2$ or $q \equiv 1 \pmod 4$. If $(n, q) = (4, 2)$, then according to Lemma 2.1(3), the result is obvious. Thus we may assume that $(n, q) \neq (4, 2)$ and hence, $R_2 \cap R_{2(n-1)} = \emptyset$. Lemma 3.4 now implies $R_{2(n-1)}$ is nonempty. Also, by Lemmas 3.6(3) and 3.8(2), we have $(p, r_2), (2, r_2) \in \mathrm{GK}(G)$. Moreover, by Lemma 3.5(3), we can easily see that if r is an odd number which is adjacent to $r_{2(n-1)}$, then r is adjacent to r_2 as well. On the other hand, if $n \geq 5$, then $(r_2, r_3) \in \mathrm{GK}(G)$, but $(r_3, r_{2(n-1)}) \notin \mathrm{GK}(G)$. Also, if $n = 4$, then $(r_2, r_4) \in \mathrm{GK}(G)$, but $(r_4, r_{2(n-1)}) \notin \mathrm{GK}(G)$. Therefore, $\deg(r_2) > \deg(r_{2(n-1)})$ and $\pi_1(G)$ cannot be m-regular.

Consequently, if $G = D_n(q)$, according to Cases 1 and 2, $\pi_1(G)$ is m-regular if and only if $(n, q) = (4, 2)$ and $m = 2$. Moreover, Lemma 2.2 implies that $\pi_1(G)$ is a clique.

Part C. $G = {}^2D_n(q)$, where $n \geq 4$:

Case 1. In this case, we assume that $2 \notin R_2(q)$ and hence, $q \equiv 1 \pmod 4$ or $p = 2$.

- If n is odd, then according to Lemmas 3.8(2), 3.6(3) and 3.5(3), we see that $(p, r_2), (2, r_2), (r_2, r_{2n}) \in \mathrm{GK}(G)$ and hence, $\{r_2, r_{2n}\} \subseteq \pi_1(G)$. Now, we claim that if $(r, r_{2n}) \in \mathrm{GK}(G)$, then $(r, r_2) \in \mathrm{GK}(G)$:

Since $(2, r_2), (p, r_2) \in \mathrm{GK}(G)$, it is sufficient to consider the case $r \in \pi_1(G) \setminus \{2, p\}$. Thus if $(r, r_{2n}) \in \mathrm{GK}(G)$, then there exists a natural number l, such that $r \in R_l(q)$. Applying Lemma 3.5(3) implies that $2n/l$ is an odd number, so is $l/2$. Thus by Lemma 3.5(3), we have $(r, r_2) \in \mathrm{GK}(G)$.

Moreover, Lemma 3.5(3) implies that $(r_2, r_4) \in \mathrm{GK}(G)$, but $(r_{2n}, r_4) \notin \mathrm{GK}(G)$. Thus $\deg(r_2) > \deg(r_{2n})$ and $\pi_1(G)$ cannot be m-regular.

- If $n \geq 4$ is even and $(n, q) \neq (4, 2)$, then $r_{2(n-1)} \in \pi(G)$ and it is enough to replace r_{2n} with $r_{2(n-1)}$ in the previous argument and conclude that $r_{2(n-1)} \in \pi_1(G)$ and $\deg(r_2) > \deg(r_{2(n-1)})$. If $G = {}^2D_4(2)$, then according to Lemma 3.6(3), we see that 2 is just adjacent to 3 and 5. Thus $\pi_1(G)$ should be 2-regular. But according to Lemma 3.5(3), 3 is adjacent to 2,5,7 and hence, $\pi_1(G)$ cannot be m-regular.

Case 2. If $2 \in R_2(q)$, then $q \equiv -1 \pmod 4$ and hence, q is odd and $q \equiv \varepsilon_0 \pmod 8$, where $\varepsilon_0 \in \{3, 7\}$.

- If n is even, then according to Lemma 3.8(2), we have $(2, r) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus R_{2n}$. Thus $(R_{2(n-1)} \cup R_{n-1}) \subseteq \pi_1(G)$. Also, Lemma 3.6(3) implies that $(r, p) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus (R_{2n} \cup R_{2(n-1)} \cup R_{n-1})$. Consequently, $\deg(2) > \deg(p)$ and Remark 3.9 implies that $\pi_1(G)$ cannot be m-regular.
- If n is odd and $q \equiv 7 \pmod 8$, then as in the even case we can see that $(2, r) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus R_{2(n-1)}$. Also, $(r, p) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus (R_{2n} \cup R_{2(n-1)})$. Thus similarly, we can conclude that $\pi_1(G)$ in not m-regular. Therefore, we may assume that n is odd and $q \equiv 3 \pmod 8$. Now, by Lemma 3.8(2), $(2, r) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus (R_{2n} \cup R_{2(n-1)})$. Thus $(r_3, 2), (2, r_{2(n-2)}) \in \mathrm{GK}(G)$. Also, Lemma 3.5(3) implies that $(r_3, r_{2(n-1)}), (r_3, r_{2n}), (r_3, r_{2(n-2)}) \notin \mathrm{GK}(G)$. Thus $\{r_3, r_{2(n-2)}\} \subseteq \pi_1(G)$ and the same argument in the above discussion conclude that $\deg(2) > \deg(r_3)$.

Consequently, if $G = {}^2D_n(q)$, according to Cases 1 and 2, $\pi_1(G)$ cannot be m-regular.

Part D. $G = A_{n-1}(q)$, where $n \geq 2$ and $(n, q) \neq (2, 2), (2, 3)$:

Case 1. In this case, we consider $n \in \{2, 3\}$. If $n = 2$, then Lemma 2.2 implies that the simple group G has complete prime graph components. Also, we can see that $\pi(G) = \{p\} \cup R_1 \cup R_2$. So, we can easily conclude the result by Lemmas 3.6(1,4) and 3.7. Thus it remains to consider the case $n = 3$. In this case, if $p \neq 2$, then $\pi(G) = \{p\} \cup R_i$, where $1 \leq i \leq 3$, and according to Remark 3.9 and Lemma 3.7, we infer that $(r, 2) \notin \mathrm{GK}(G)$ if and only if $r \in R_3(q)$. Thus since $2 \in R_1 \cup R_2$ and $(2, p) \in \mathrm{GK}(G)$, we conclude that $\deg(2) = |R_1| + |R_2|$. Lemma 3.6(1,5) now yields $(r, p) \in \mathrm{GK}(G)$ if and only if

$$r \in R_1(q) \cup \{2\}, \text{ where } (q-1)_3 \neq 3; \tag{3.4}$$

$$r \in (R_1(q) \cup \{2\}) \setminus \{3\}, \text{ where } (q-1)_3 = 3. \tag{3.5}$$

Now we find the possible cases which $\pi_1(A_2(q))$ is m-regular:

• If $\{2,3\} \cap R_1(q) = \emptyset$, then $2 \in R_2(q)$ and $(q-1)_3 \neq 3$. Thus according to 3.4 we have $\deg(p) = 1 + |R_1|$ and since $\pi_1(G)$ is m-regular and $\deg(2) = |R_1| + |R_2|$, we are supposed to have $|R_2(q)| = 1$. Therefore, $q + 1 = 2^k$ and by Lemma 2.2, the result is obtained.

• If $2 \in R_1$ and $3 \notin R_1$, then according to 3.4, $\deg(p) = |R_1|$ and hence $\deg(2) > \deg(p)$ which implies that $\pi_1(G)$ is not m-regular.

• If $2 \notin R_1$ and $3 \in R_1$, then since $p \neq 2$, we conclude that $2 \in R_2$. Also, by 3.4 and 3.5, we have $\deg(p) = 1+|R_1|$, where $(q-1)_3 > 3$ and $\deg(p) = |R_1|$, where $(q-1)_3 = 3$. Thus since $\pi_1(G)$ is m-regular, we infer that $\deg(2) = \deg(p)$. Since R_2 is nonempty, we conclude that $q + 1 = 2^k$. Therefore, the result is obvious by Lemma 2.2.

• If $\{2,3\} \subseteq R_1$. As in the previous case, we conclude that $\deg(p) = |R_1| - 1$, where $(q-1)_3 = 3$ and $\deg(p) = |R_1|$, where $(q-1)_3 > 3$. Thus both cases imply that $\deg(2) > \deg(p)$ and hence, in this case $\pi_1(G)$ is not m-regular.

Now it remains to consider the case $n = 3$ and $q = 2^\alpha \geq 4$:

• If $R_1 = \{3\}$, then according to Lemma 3.6(1,5), we conclude that 2 is a vertex with degree zero. On the other hand, since $3 \mid (2^\alpha - 1)$, we conclude that $\alpha = 2k$, where $k \in \mathbb{N}$. Thus $2^\alpha - 1 = (2^k - 1)(2^k + 1)$. But since $R_1 = \{3\}$, we have $2^k - 1 = 1$ and hence, $G \cong A_2(4)$ and $\pi_1(G)$ is 0-regular.

• If $R_1 \neq \{3\}$, then there is $r_1 \in R_1 \setminus \{3\}$. Now, according to Lemmas 3.6(5) and 3.7, we conclude that $(r, r_1) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus R_3$. Thus $\deg(r_1) = |R_1|+|R_2|$. Lemma 3.6(1,5) now yields $\deg(2) \leq |R_1|$ and hence, $\deg(2) < \deg(r_1)$ and in this case $\pi_1(G)$ cannot be m-regular.

Case 2. Let $n \geq 4$. If $2 \in R_2(q)$, then $q \equiv -1 \pmod 4$ and hence, $p \neq 2$. According to Lemma 3.6(1), $(p, r) \in \mathrm{GK}(G)$ if and only if $r \in \pi(G) \setminus (R_n \cup R_{n-1})$. On the other hand, since $4 \mid (q + 1)$, so $(q-1)_2 = 2$ and Lemma 3.7 implies that $(2, r) \in \mathrm{GK}(G)$ if and only if either $r \in \pi(G) \setminus R_n$ or $r \in \pi(G) \setminus R_{n-1}$. Thus $\deg(2) > \deg(p)$ and in this case $\pi_1(G)$ cannot be m-regular.

If $2 \notin R_2(q)$, then Lemmas 3.5(1), 3.6(1) and 3.7 imply that $(r_2, r_4), (r_2, 2) \in \mathrm{GK}(G)$ and hence, $\{r_2, r_4\} \subseteq \pi_1(G)$. Now, we claim that if $(r, r_4) \in \mathrm{GK}(G)$, then r and r_2 are adjacent as well:

Since $2 \notin R_2(q)$ and $n \geq 4$, according to Lemmas 3.6(1) and 3.7, we conclude that $(p, r_2), (2, r_2), (r_1, r_2) \in \mathrm{GK}(G)$. Thus if $(r, r_4) \in \mathrm{GK}(G)$, then it is enough to consider the case $r \in R_l(q)$, where $l \geq 2$. Since $(r_l, r_4) \in \mathrm{GK}(G)$, by Lemma 3.5(1), we have $l + 4 \leq n$ or $4 \mid l$ or $l \mid 4$. In each case, by using Lemma 3.5(1), we conclude that $(r_l, r_2) \in \mathrm{GK}(G)$.

If $n \geq 5$, then we can choose $l \in \{n-2, n-3\}$ as an odd integer greater than 1. Now by Lemma 3.5(1), we can easily check that $(r_l, r_4) \notin \mathrm{GK}(G)$, but $(r_l, r_2) \in \mathrm{GK}(G)$. Therefore, $\deg(r_2) > \deg(r_4)$. Now it remains to consider the case $n = 4$. In this case, we have $\pi(G) = \{p\} \cup R_i$, where $1 \leq i \leq 4$, and since $2 \notin R_2$, we have $p = 2$ or $4 \mid (q-1)$. Lemmas 3.7, 3.6(1) and 3.5(1) now yields $(r, r_4) \in \mathrm{GK}(A_3(q))$ if and only if $r \in R_4 \cup R_2$. Also, $(r, r_2) \in \mathrm{GK}(A_3(q))$ if and only if $r \in \pi(A_3(q)) \setminus R_3$. Thus $\deg(r_2) > \deg(r_4)$ and in this case $\pi_1(G)$ cannot be m-regular.

Consequently, according to Cases 1 and 2, we conclude that $\pi_1(A_{n-1}(q))$ is m-regular if and only if $\pi_1(A_{n-1}(q))$ is a clique.

Part E. $G = {}^2A_{n-1}(q)$, where $n \geq 3$ and $(n, q) \neq (3, 2)$:

Case 1. If $n = 3$, we consider the cases "q is even" and "q is odd", separately:

• If q is even, then $\pi(G) = \{2\} \cup R_1 \cup R_2 \cup R_6$. According to Lemma 3.6(1), we know that 2 is nonadjacent to r_1 and r_6. If $R_2 \neq \{3\}$, then Lemmas 3.7 and 3.6(1,5) imply that $(r_2, r) \in \mathrm{GK}(G)$, where $r_2 \in R_2 \setminus \{3\}$ and $r \in R_1 \cup R_2 \cup \{2\}$. Thus $\deg(2) \in \{|R_2|, |R_2|-1\}$ and $\deg(r_2) = |R_1|+|R_2|$ which imply that $\deg(2) < \deg(r_2)$. But this is a contradiction to the fact that $\pi_1(G)$ is m-regular and hence, $R_2 = \{3\}$. Now, since $q \neq 2$ is even we deduce that $(q+1) = 3^k$ and $(q+1)_3 \neq 3$. Thus by Lemmas 3.7, 3.5(1) and 3.6(1,5) we can see that $(2,r) \in \mathrm{GK}(G)$ if and only if $r = 3$ and also, $(3,r) \in \mathrm{GK}(G)$ if and only if $r \in \{2\} \cup R_1$. Therefore, $\deg(2) = 1 < \deg(3) = |R_1| + 1$ and in this case $\pi_1(G)$ cannot be m-regular.

• If q is odd, then by Lemma 3.7 and Remark 3.9, we can easily see that $\deg(2) = |R_1| + |R_2|$. Since $\pi_1(G)$ is m-regular, Remark 3.9 implies that $\deg(p) = \deg(2) = |R_1| + |R_2|$. On the other hand, according to Lemma 3.6(1,5) and Remark 3.9, we conclude that $(r,p) \in \mathrm{GK}(G)$ if and only if

$$r \in (\{2\} \cup R_2) \setminus \{3\} \text{ and } (q+1)_3 = 3 \tag{3.6}$$

or

$$r \in \{2\} \cup R_2 \text{ and } (q+1)_3 \neq 3 \tag{3.7}$$

Thus if 3.6 holds, then $\deg(p) = |R_2|$ or $\deg(p) = |R_2| - 1$, where $2 \in R_1$ or $2 \in R_2$ respectively. If 3.7 holds, then $\deg(p) = |R_2| + 1$ or $\deg(p) = |R_2|$, where $2 \in R_1$ or $2 \in R_2$ respectively.

Therefore, according to the above statements, we can easily conclude that $(q+1)_3 \neq 3$, $q - 1 = 2^k$ and $m = |R_1| + |R_2|$. Moreover, Lemma 2.2 implies that $\pi_1(G)$ is a clique.

Case 2. If $n \geq 4$, then we consider the following two subcases:

Subcase a. If $R_1(q) = \emptyset$, then $q \in \{2,3\}$. First we deal with the case $q = 2$. Since ${}^2A_3(2) \cong C_2(3)$ is 1-regular by Lemma 2.1(2), we can assume that $n \neq 4$. In this case, according to Lemma 3.6(1), $(r,2) \notin \mathrm{GK}(G)$ if and only if $r \in R_l$, where $\nu(l) \in \{n, n-1\}$. Since $(n,q) \neq (4,2)$, Lemma 3.7 implies that $\deg(2) < \deg(3)$. Similarly, if $n_3 > 3$, then we can conclude that $\deg(2) < \deg(3)$. Therefore, it remains to consider the case $n_3 = 3$. If $n = 6$, then by Lemmas 3.6(1), 3.7 and 3.5(1), we have $\pi_1(G) = \{2,3,5\}$ is 2-regular. Thus we may assume that $n \geq 12$ and in this case we know that $R_4(2) \cup R_8(2) = \{5,17\} \subseteq \pi(G)$. According to Lemmas 3.6(1) and 3.7, we have $(2,r_4),(3,r_4) \in \mathrm{GK}(G)$. Now we claim that if $(r,r_8) \in \mathrm{GK}(G)$, then $(r,r_4) \in \mathrm{GK}(G)$:

Since r_4 is adjacent to 2 and 3 and $R_2(2) = \{3\}$, it is enough to consider the case $r \in R_l(2)$, where $\nu(l) \geq 2$. Thus if $(r_l,r_8) \in \mathrm{GK}(G)$, then by Lemma 3.5(1), we can see that $\nu(l) + 8 \leq n$, $\nu(l) \mid 8$ or $8 \mid \nu(l)$ which imply that $\nu(l) + 4 \leq n$, $\nu(l) \in \{2,4,8\}$ or $8 \mid \nu(l)$, respectively and hence, $(r_l,r_4) \in \mathrm{GK}(G)$.

Set l be an integer, where $\nu(l) \in \{n-5, n-4\}$ and $\nu(l)$ is odd. Thus by Lemma 3.5(1), we can conclude that $(r_l,r_4) \in \mathrm{GK}(G)$, but $(r_l,r_8) \notin \mathrm{GK}(G)$. Therefore, $\deg(r_4(2)) > \deg(r_8(2))$. Thus if $n \geq 4$, then $\pi_1({}^2A_{n-1}(2))$ is m-regular if and only if $(n,m) = (4,1)$ or $(n,m) = (6,2)$. Moreover, according to Lemma 2.2, in both cases $\pi_1({}^2A_{n-1}(2))$ is a clique.

If $q = 3$, then according to Lemma 3.6(1), we have $(r,3) \notin \mathrm{GK}(G)$ if and only if $\nu(e(r,3)) \in \{n-1,n\}$. On the other hand, if $n_2 \neq 4$, then by Lemma 3.7 and as in the above discussion, we can see that $\deg(2) = \deg(r_2(3)) > \deg(3)$. Thus it is enough to consider the case $n_2 = 4$. Since according to Lemma 2.1(3), $\pi_1({}^2A_3(3))$ is 2-regular, we may assume that $n \geq 8$. Also, we know that $R_n(3) \subseteq \pi(G)$. Now, we claim that if $(r,r_n) \in \mathrm{GK}(G)$, then $(r,r_4) \in \mathrm{GK}(G)$:

Since $n \geq 8$, according to Lemmas 3.6(1) and 3.7, we can see that $(3,r_4),(2,r_4) \in \mathrm{GK}(G)$ and since $R_2(3) = \{2\}$ we may assume that $r \in R_l(3)$, where $\nu(l) \geq 2$. Now Lemma 3.5(1) implies that $\nu(l) \mid n$. If $\nu(l) = n$, then since $4 \mid n$, by Lemma 3.5(1), we conclude that $(r,r_4) \in \mathrm{GK}(G)$. If $\nu(l) \neq n$, then $\nu(l) \leq n/2$ and since $n \geq 8$, so we have $\nu(l) + 4 \leq /2 + 4 \leq n$. Now, using Lemma 3.5(1) completes the proof of our claim.

Since $(2,r_4) \in \mathrm{GK}(G)$ and $(2,r_n) \notin \mathrm{GK}(G)$, according to the above discussion, we conclude that $\deg(r_4) > \deg(r_n)$. As $n \geq 4$, we can see that $\pi_1({}^2A_{n-1}(3))$ is m-regular if and only if $(n,m)=(4,2)$. Moreover, according to Lemma 2.2, in this case $\pi_1({}^2A_{n-1}(3))$ is a clique.

Subcase b. If $R_1(q)\neq\emptyset$, then we have the following cases:

• If $2\in R_1$, then $q\equiv 1 \pmod 4$ and hence q is odd. According to Lemma 3.6(1) and Remark 3.9, we conclude that $(r,p)\notin \mathrm{GK}(G)$ if and only if $\nu(e(r,q))\in\{n-1,n\}$. Since $\pi_1(G)$ is m-regular, we should have $\deg(2)=\deg(p)$ and hence, Lemma 3.7 implies that $n_2=(q+1)_2>2$, which is impossible according to $q\equiv 1 \pmod 4$.

• If $2\notin R_1$ and $n\geq 5$, then by using Lemmas 3.6(1), 3.7 and 3.5(1), we can easily see that $(p,r_1),(2,r_1),(r_1,r_4)\in \mathrm{GK}(G)$ and each vertex which is adjacent to r_4 is adjacent to r_1, as well. On the other hand, according to Lemma 3.5(1), the vertex r_l, where $\nu(l)\in\{n-3,n-2\}$ is odd, is adjacent to r_1 but is nonadjacent to r_4. Thus $\deg(r_1)>\deg(r_4)$ and $\pi_1(G)$ cannot be m-regular. If $2\notin R_1$ and $n=4$, then $p=2$ or $q\equiv -1 \pmod 4$. Thus Lemmas 3.6(1) and 3.7 imply that $(2,r_4),(p,r_4)\notin \mathrm{GK}(G)$. Now, by Lemma 3.5(1), we conclude that $(r,r_4)\in \mathrm{GK}(G)$ if and only if $r\in R_1\cup R_4$. Also, we can see that $(r,r_1)\in \mathrm{GK}(G)$ if and only if $r\in\{p\}\cup R_1\cup R_2\cup R_4$. Thus $\deg(r_1)>\deg(r_4)$ and hence, $\pi_1(G)$ cannot be m-regular.

Consequently, according to Cases 1 and 2, we conclude that $\pi_1({}^2A_{n-1}(q))$ is m-regular if and only if $\pi_1({}^2A_{n-1}(q))$ is a clique. □

In order to complete the proof of the main theorem, we need the following lemmas for considering the Ree groups ${}^2G_2(3^{2n+1})$ and ${}^2F_4(2^{2n+1})$, where n is a natural number.

3.11. Lemma. [14, Lemma 1.5(2-3)] Let n be a natural number.

(1) Let $m_1(G,n)=3^{2n+1}-1$, $m_2(G,n)=3^{2n+1}+1$, $m_3(G,n)=3^{2n+1}-3^{n+1}+1$, $m_4(G,n)=3^{2n+1}+3^{n+1}+1$. Then $\gcd(m_1(G,n),m_2(G,n))=2$ and $\gcd(m_i(G,n),m_j(G,n))=1$ otherwise.

(2) Let $m_1(F,n)=2^{2n+1}-1$, $m_2(F,n)=2^{2n+1}+1$, $m_3(F,n)=2^{4n+2}+1$, $m_4(F,n)=2^{4n+2}-2^{2n+1}+1$, $m_5(F,n)=2^{4n+2}-2^{3n+2}+2^{2n+1}-2^{n+1}+1$, $m_6(F,n)=2^{4n+2}+2^{3n+2}+2^{2n+1}+2^{n+1}+1$.
Then $\gcd(m_2(F,n),m_4(F,n))=3$ and $\gcd(m_i(F,n),m_j(F,n))=1$ otherwise.

3.12. Lemma. [14, Propositions 3.3(2-3)] Let G be a finite simple Ree group over a field of characteristic p, let $r\in\pi(G)\setminus\{p\}$. Then r, p are nonadjacent if and only if one of the following holds:

(1) $G={}^2G_2(3^{2n+1})$, r divides $m_k(G,n)$ and $r\neq 2$.
(2) $G={}^2F_4(2^{2n+1})$, r divides $m_k(F,n)$, $r\neq 3$ and $k>2$.

3.13. Lemma. [14, Propositions 4.5(8)] If $G={}^2G_2(3^{2n+1})$ and $r\in\pi(G)\setminus\{2,3\}$, then r and 2 are nonadjecent if and only if r divides $m_3(G,n)$ or $m_4(G,n)$.

If $G={}^2F_4(2^{2n+1})$, then denote by $\mathcal{S}_i(G)$ the set $\pi(m_i(F,n))\setminus\{3\}$. Thus we have the following lemma:

3.14. Lemma. [15, Propositions 2.9(3)] Let $G={}^2F_4(2^{2n+1})$ and $r,s\in\pi(G)\setminus\{2\}$. Then r and s are nonadjecent if and only if either $r\in\mathcal{S}_k(G)$ and $s\in\mathcal{S}_l(G)$, where $l\neq k$, $\{k,l\}\neq\{1,2\},\{1,3\}$; or $r=3$ and $s\in\mathcal{S}_l(G)$, where $l\in\{3,5,6\}$.

3.15. Lemma. Let G be a finite simple exceptional group of Lie type. If $\pi_1(G)$ is m-regular, then $\pi_1(G)$ is a clique and one of the following cases holds:

(1) $G={}^2B_2(2^{2n+1})$ and $m=0$;
(2) $G={}^3D_4(2)$ and $m=2$;
(3) $G=G_2(3^n)$ and $m=|R_1(q)|+|R_2(q)|$.

Proof. According to the compact form of $\mathrm{GK}(G)$ in [15], where

$$G\in\{E_7(q),E_8(q),E_6(q),{}^2E_6(q),F_4(q)\},$$

we can easily find two vertices p and q in $\pi_1(G)$ which have the following properties:

(1) If $(p, r) \in \mathrm{GK}(G)$, then $(q, r) \in \mathrm{GK}(G)$;
(2) There exists a prime s in $\pi(G)$, where $(p, s) \notin \mathrm{GK}(G)$ but $(q, s) \in \mathrm{GK}(G)$.

Thus $\deg(q) > \deg(p)$ which implies that $\pi_1(G)$ cannot be m-regular. In the same manner we can see that $\pi_1({}^3D_4(q))$ is m-regular if and only if $q = 2$. We omit the details for convenience. Also, according to the compact form of $\mathrm{GK}(G_2(q))$, we can see that $\pi_1(G_2(q))$ is m-regular if and only if $q = 3^\alpha$. In this case, we have $m = |R_1| + |R_2|$. Moreover, by Lemma 2.1(1), we know that $\pi_1({}^2B_2(2^{2n+1}))$, where $n \in \mathbb{N}$, is 0-regular. Additionally, if $G = {}^2F_4(2)'$, then using [1] implies that $\deg(2) = 2$, $\deg(3) = 1$ and $(2, 3) \in \mathrm{GK}(G)$ and hence, $\pi_1({}^2F_4(2)')$ is not m-regular. Thus it remain to consider the simple groups, ${}^2G_2(3^{2n+1})$ and ${}^2F_4(2^{2n+1})$, where $n \in \mathbb{N}$. If $G = {}^2G_2(3^{2n+1})$, then Lemma 3.12(1) implies that $(3, r) \in \mathrm{GK}(G)$ if and only if $r = 2$. Also, according to Lemma 3.13, we can see that $(2, r) \in \mathrm{GK}(G)$ if and only if $r = 3$ or $r \mid m_1(G, n)$ or $r \mid m_2(G, n)$. Thus $\deg(2) > \deg(3)$ and $\pi_1({}^2G_2(3^{2n+1}))$ is not m-regular. Finally, if $G = {}^2F_4(2^{2n+1})$, then Lemma 3.12(2) implies that $(2, r) \in \mathrm{GK}(G)$ if and only if $r = 3$ or $r \mid m_1(F, n)$ or $r \mid m_2(F, n)$. Moreover, according to Lemma 3.14, we can see that $(3, r) \in \mathrm{GK}(G)$ if and only if $r = 2$ or $r \mid m_1(F, n)$ or $r \mid m_2(F, n)$ or $r \mid m_4(F, n)$. Thus $\deg(2) < \deg(3)$ and $\pi_1({}^2F_4(2^{2n+1}))$ is not m-regular. □

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Classical completely prime submodules

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Abstract

We define and characterize classical completely prime submodules which are a generalization of both completely prime ideals in rings and reduced modules (as defined by Lee and Zhou in [18]). A comparison of these submodules with other "prime" submodules in literature is done. If $\text{Rad}(M)$ is the Jacobson radical of M and $\beta^c_{cl}(M)$ the classical completely prime radical of M, we show that for modules over left Artinian rings R, $\text{Rad}(M) \subseteq \beta^c_{cl}(M)$ and $\text{Rad}(_RR) = \beta^c_{cl}(_RR)$.

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1. Introduction

All modules are left modules, the rings are associative but not necessarily unital. An ideal $\mathcal{P}$ of a ring R is completely prime (completely semiprime) if for any $a, b \in R$ ($a \in R$) such that $ab \in \mathcal{P}$ ($a^2 \in \mathcal{P}$) we have, $a \in \mathcal{P}$ or $b \in \mathcal{P}$ ($a \in \mathcal{P}$). A ring R is completely prime if the zero ideal is completely prime. A ring R is completely semiprime (or reduced) if and only if for all $a \in R$, $a^2 = 0 \Rightarrow a = 0$. An R-module M is reduced if for all $a \in R$ and every $m \in M$, $am = 0$ implies $\langle m \rangle \cap aM = 0$, where $\langle m \rangle = \mathbb{Z}m + Rm$ is the submodule of M generated by $m \in M$. It is worth noting that, if R is unital then $\langle m \rangle = Rm$, otherwise $Rm \subseteq \langle m \rangle$ but $\langle m \rangle \not\subseteq Rm$ in general. By $(P : N)$ (resp. $(P : m)$) where P, N are submodules of an R-module M and $m \in M$, we mean $\{r \in R \ : \ rN \subseteq P\}$ (resp. $\{r \in R \ : \ rm \in P\}$). If a is an element of a ring R, by $\langle a \rangle$ we denote the ideal of R generated by a. We write $N \leq M$ to mean N is a submodule of M. Our definition of a

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reduced module is a generalization of that in [18], where Rm is used in the place of $\langle m\rangle$. We state an equivalent but more handy definition for a reduced module.

1.1. Definition. An R-module M is reduced if for all $a \in R$ and every $m \in M$, $a^2m = 0$ implies $a\langle m\rangle = 0$.

This definition of a reduced (completely semiprime) module motivates the following two definitions:

1.2. Definition. A proper submodule P of an R-module M is completely semiprime if for all $a \in R$ and every $m \in M$, $a^2m \in P$ implies $a\langle m\rangle \subseteq P$.

1.3. Definition. A proper submodule P of an R-module M is classical completely prime if for all $a, b \in R$ and every $m \in M$, $abm \in P$ implies $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$.

An R-module M/P is a classical completely prime module if and only if P is a classical completely prime submodule of M. Thus, an R-module M is classical completely prime (completely semiprime) if and only if the zero submodule is a classical completely prime (completely semiprime) submodule of M. Although the phrase "completely prime" would seem suitable in the place of classical completely prime in Definition 1.3, we reserve it for a different meaning - one given by Tuganbaev in [24, p.1480] and discussed in [21] (in which it is most suitable).

1.1. Example. A free module M over a domain R is classical completely prime.

Proof. Suppose $abm = 0$ for some $a, b \in R$ and $m \in M$. Then

$$abm = ab\sum_{i=1}^{n}(r_im_i) = \sum_{i=1}^{n}(abr_i)m_i = 0$$

for some $r_i \in R$ and $m_i \in M$. Since M is free $abr_i = 0$ for all $i \in \{1, \cdots n\}$. For $m \neq 0$, there is atleast one $j \in \{1, \cdots n\}$ such that $r_j \neq 0$. Now $abr_j = 0$ implies $a = 0$ or $b = 0$ (since R is a domain) such that $a\langle m\rangle = 0$ or $b\langle m\rangle = 0$. □

1.2. Example. A torsionfree module M over a domain R is classical completely prime. It follows that flat modules over domains (and hence projective modules over domains) are classical completely prime modules.

Proof. Suppose for $a, b \in R$ and $m \in M$, $abm = 0$. If $m = 0$, $a\langle m\rangle = 0$ and $b\langle m\rangle = 0$. Let $m \neq 0$, then $ab = 0$ since M is torsionfree. Hence, $a = 0$ or $b = 0$ since R is a domain. Therefore, $a\langle m\rangle = 0$ or $b\langle m\rangle = 0$. The last part is due to the fact that flat modules are torsionfree, see [23, Example 1, p.15] and projective modules are flat modules. □

1.3. Example. Every submodule P of a module M over a division ring R is a classical completely prime submodule.

Proof. Suppose $a, b \in R$ and $m \in M$ such that $abm \in P$. If $ab = 0$, $a = 0$ or $b = 0$ such that $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$. Suppose $ab \neq 0$, then, $m \in (ab)^{-1}P \subseteq P$.[§] Thus, $a\langle m\rangle \subseteq P$ and $b\langle m\rangle \subseteq P$. □

1.4. Example. Any prime (sub)module over a commutative ring is a classical completely prime (sub)module.

Proposition 1.1 below and its corollaries provide more justification for our definition of classical completely prime submodules.

[§] $(ab)^{-1}$ is here used to mean the inverse of ab in R

1.1. Proposition. If $1 \in R$ and $P \lhd R$, then P is a completely prime ideal of R if and only if P is a classical completely prime submodule of ${}_RR$.

Proof. Suppose P is a completely prime ideal of R and for any $a, b \in R$ and $m \in_R R$, $abm \in P$. By definition of a completely prime ideal, $a \in P$ or $b \in P$ or $m \in P$. Thus, $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$. Conversely, suppose the ideal P of R is a classical completely prime submodule of ${}_RR$. Let for any $a, b \in R$, $ab \in P$. Since $1 \in R$ by definition of classical completely prime submodule, $ab.1 \in P$ implies $aR \in P$ or $bR \in P$ such that $a \in P$ or $b \in P$. □

1.1. Corollary. If $1 \in R$, then R is a domain if and only if ${}_RR$ is a classical completely prime module.

1.2. Corollary. If $1 \in R$ and $P \lhd R$, then P is a completely semiprime ideal of R if and only if it is a completely semiprime submodule of ${}_RR$.

1.3. Corollary. A unital ring R is reduced if and only if ${}_RR$ is a reduced module.

2. Investigation of properties

In this section, we investigate properties exhibited by classical completely prime (semiprime) submodules. First, we introduce notions of symmetric and IFP submodules that will prove useful later in the sequel. Lambek in [17, p.364] called a module M symmetric if $abm = 0$ implies $bam = 0$ for $a, b \in R$ and $m \in M$. We call a submodule P of an R-module M symmetric if $abm \in P$ implies $bam \in P$ for $a, b \in R$ and $m \in M$. So, a module M is symmetric if its zero submodule is symmetric. From [8], a right (or left) ideal I of a ring R is said to have the insertion-of-factor-property (IFP) if whenever $ab \in I$ for $a, b \in R$, we have $aRb \subseteq I$. We call a submodule N of an R-module M an IFP submodule if whenever $am \in N$ for $a \in R$ and $m \in M$, we have $aRm \subseteq N$. A module is IFP if its zero submodule is IFP.

2.1. Proposition. For any submodule P of an R-module M,

$$\text{completely semiprime} \Rightarrow \text{symmetric} \Rightarrow \text{IFP}.$$

Proof. Let $abm \in P$. $(bab)^2m \in P$ and P completely semiprime gives $bab\langle m\rangle \subseteq P$. Thus, $(ba)^2m = bab(am) \in bab\langle m\rangle \subseteq P$ and again P completely semiprime gives $bam \in ba\langle m\rangle \subseteq P$. For the second implication, let $am \in P$ for $a \in R$ and $m \in M$. Then $Ram \subseteq P$ and P symmetric implies $aRm \subseteq P$. □

2.1. Example. A module M over a left duo ring R (a ring whose all left ideals are two sided) is fully IFP (every submodule of M is IFP) but it need not be symmetric.

Proof. Let $P \leq M$, $a \in R$ and $m \in M$ such that $am \in P$, then $a \in (P : m)$. $(P : m)$ is a left ideal of R but since R is left duo, we have $(P : m) \lhd R$ and $aR \subseteq (P : m)$ such that $aRm \subseteq P$. Hence, P is IFP. $\mathbb{Z}_2$ is a left quasi duo ring (i.e., every maximal left ideal of $\mathbb{Z}_2$ is two sided). By [16, Prop. 2.1], any n-by-n upper triangular matrix ring R over $\mathbb{Z}_2$ is left quasi duo. Hence, every submodule of the module ${}_RR$ is IFP. We show that the zero submodule of ${}_RR$ is not symmetric. Take $m = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$, $a = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix}$, and $b = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix} \in R$; $abm = 0$ but $bam \neq 0$. □

2.2. Example. A submodule P of a module M over a commutative ring R is symmetric but it need not be completely semiprime.

2.1. Properties of underlying ring. We give information classical completely prime (completely semiprime) submodules of ${}_RM$ reveal about the underlying ring R. Propositions 2.2 and 2.3 indicate that there is a one to one correspondence between completely semiprime (classical completely prime) submodules P of the module ${}_RM$ and completely semiprime (completely prime) ideals of R of the form $(P : m)$ for all $m \in M \setminus P$.

2.2. Proposition. For $P \leq {}_RM$, the following statements are equivalent:

(1) P is a completely semiprime submodule of M;
(2) $(P : m) = (P : \langle m\rangle) = (\bar{0} : \bar{m})$ is a completely semiprime ideal of R for every $m \in M \setminus P$, where $\bar{m} = m + P$;
(3) for all $m \in M \setminus P$, $(P : m) \lhd R$ and for all $a \in R$ if $a^2m \in P$, then $am \in P$;
(4) for all $m \in M \setminus P$, $(P : m) \lhd R$ and for all $a \in R$ if $\langle a^2m\rangle \subseteq P$, then $\langle am\rangle \subseteq P$;
(5) for all $a \in R$ and every $m \in M$, if $\langle a^2m\rangle \subseteq P$, then $\langle a\langle m\rangle\rangle \subseteq P$.

Proof. (1) $\Rightarrow$ (2). Since $(P : m)$ is always a left ideal of R for all $m \in M \setminus P$, we show that if $a \in (P : m)$, then $aR \subseteq (P : m)$. Suppose $a \in (P : m)$, then $Ram \subseteq P$ and from Proposition 2.1, we have $aRm \subseteq P$ and therefore, $aR \subseteq (P : m)$ as required. Let $m \in M \setminus P$, $(P : m) = \{r \in R : rm \in P\} = \{r \in R \ : \ r\bar{m} = \bar{0}\} = (\bar{0} : \bar{m})$. The inclusion $(P : \langle m\rangle) \subseteq (P : m)$ is clear. Suppose $x \in (P : m)$, then $xR \subseteq (P : m)$. Hence, $x\langle m\rangle \subseteq P$ and we have $x \in (P : \langle m\rangle)$. Lastly, suppose $a^2 \in (P : m)$, i.e., $a^2m \in P$. Then, $am \in a\langle m\rangle \subseteq P$ since P is a completely semiprime submodule of M. Thus, $a \in (P : m)$.

(2) $\Rightarrow$ (1). Let for all $a \in R$ and $m \in M$, $a^2m \in P$. Then, $a^2 \in (P : m)$ which implies $a \in (P : m)$ by definition of a completely semiprime ideal of a ring R. Thus, $aR \subseteq (P : m)$ and $aRm \subseteq P$. Therefore, $a\langle m\rangle = \mathbb{Z}am + aRm \subseteq P$ and P is a completely semiprime submodule of M.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) and (5) $\Leftrightarrow$ (1) are trivial. □

2.1. Corollary. An R-module M is reduced if and only if for every $0 \neq m \in M$, $(0 : m)$ is a completely semiprime two sided-ideal of R.

2.3. Proposition. For a proper submodule P of an R-module M, the following statements are equivalent:

(1) P is a classical completely prime submodule of M;
(2) for every $m \in M \setminus P$, $(P : m) = (P : \langle m\rangle) = (\bar{0} : \bar{m})$ is a completely prime ideal of R;
(3) for all $m \in M \setminus P$, $(P : m) \lhd R$ and if $a, b \in R$ such that $abm \in P$, then $am \in P$ or $bm \in P$;
(4) for all $m \in M \setminus P$, $(P : m) \lhd R$ and if $a, b \in R$ such that $\langle abm\rangle \subseteq P$, then $\langle am\rangle \subseteq P$ or $\langle bm\rangle \subseteq P$;
(5) for all $a, b \in R$ and every $m \in M$, if $\langle abm\rangle \subseteq P$, then $\langle a\langle m\rangle\rangle \subseteq P$ or $\langle b\langle m\rangle\rangle \subseteq P$.

Proof. (1) $\Rightarrow$ (2). Every classical completely prime submodule of M is a completely semiprime submodule of M. We have seen in Proposition 2.2 that $(P : m)$ is an ideal of R and $(P : m) = (P : \langle m\rangle) = (\bar{0} : \bar{m})$. Let $a, b \in R$ and $0 \neq m \in M$ such that $ab \in (P : m)$, i.e., $abm \in P$. Now, P classical completely prime submodule gives $am \in a\langle m\rangle \subseteq P$ or $bm \in b\langle m\rangle \subseteq P$. Hence, $a \in (P : m)$ or $b \in (P : m)$.

(2) $\Rightarrow$ (1). Let for $a, b \in R$ and $0 \neq m \in M$, $abm \in P$, i.e., $ab \in (P : m)$. $(P : m)$ a completely prime ideal of R gives $a \in (P : m)$ or $b \in (P : m)$. Hence, ($am \in P$ and $aRm \subseteq P$) or ($bm \in P$ and $bRm \subseteq P$) such that $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) and (5) $\Leftrightarrow$ (1) are trivial. □

The zero divisor set of ${}_RM$ [3, p.316] is the set

$$Zd(M) := \{r \in R \ : \ \text{there exists } 0 \neq m \in M, \text{ with } rm = 0\}.$$

The following proposition provides us with two other ways of constructing completely prime ideals of a ring R from a submodule P of an R-module M.

2.4. Proposition. Let P be a classical completely prime submodule of an R-module M. Then,

(1) for any $m, n \in M \setminus P$ either $(P : n) \subseteq (P : m)$ or $(P : m) \subseteq (P : n)$;
(2) $Zd(M/P)$ is a completely prime ideal of R;
(3) for all submodules K and L of M not contained in P, $(P : L) \subseteq (P : K)$ or $(P : K) \subseteq (P : L)$;
(4) $(P : K)$ is a completely prime ideal of R for all submodules K of M such that $K \nsubseteq P$.

Proof. (1) Assume $n, m \in M \setminus P$. Then, $(P : n)(P : m) \subseteq (P : n) \cap (P : m) \subseteq (P : n + m)$. We know that, $(P : n + m)$ is a completely prime ideal of R and hence a prime ideal of R. So, we have $(P : n) \subseteq (P : n+m)$ or $(P : m) \subseteq (P : n+m)$. If $(P : n) \subseteq (P : n + m)$, then $(P : n) = (P : n) \cap (P : n + m) \subseteq (P : m)$. Similarly, if $(P : m) \subseteq (P : n + m)$, we get $(P : m) \subseteq (P : n)$.

(2) By definition, $Zd(M/P) = \bigcup_{m \in M \setminus P} (P : m)$. But $\{(P : m)\}_{m \in M \setminus P}$ form a chain of completely prime ideals of R. We see that $Zd(M/P)$ is the largest of all the $(P : m)$'s and hence a completely prime ideal of R.

(3) $(P : K)(P : L) \subseteq (P : K) \cap (P : L) \subseteq (P : K + L)$. Hence, $(P : K) \subseteq (P : K + L) \subseteq (P : L)$ or $(P : L) \subseteq (P : K + L) \subseteq (P : K)$.

(4) To show that $(P : K)$ is a completely prime ideal of R, it is enough to show that it is both prime and completely semiprime as an ideal of R. If P is classical completely prime, by Theorem 3.1 it is classical prime (see definition 3.2) and hence $(P : K)$ is a prime ideal of R for all $K \leq M$ such that $K \nsubseteq P$. Suppose $a^2 \in (P : K)$ for $a \in R$ and $K \leq M$ with $K \nsubseteq P$, then $a^2k \in P$ for all $k \in K$. By hypothesis, $a\langle k\rangle \subseteq P$ for all $k \in K$. Thus, $aK \subseteq P$ such that $a \in (P : K)$. □

2.2. Homomorphic images.

2.5. Proposition. Let M be an R-module, $N, P \leq M$ such that $N \subseteq P$. If $f : M \to M/N$ is a canonical epimorphism, then P is a classical completely prime submodule of M if and only if $f(P)$ is a classical completely prime submodule of M/N.

The proof is elementary, if $N \nsubseteq P$, P classical completely prime submodule of M does not in general imply $f(P)$ is a classical completely prime submodule of M/N (and hence classical completely prime is not in general closed under homomorphic images).

2.3. Example. The $\mathbb{Z}$-module $\mathbb{Z}$ is a classical completely prime module by Corollary 1.1 and $N = 8\mathbb{Z}$ is a submodule of $M = {}_{\mathbb{Z}}\mathbb{Z}$. By [1, Example 2.5], M/N is not a reduced module (i.e., not a completely semiprime module) and hence not a classical completely prime module.

2.6. Proposition. Let $f : R \to A$ be a ring epimorphism and M an A-module, then M is an R-module and ${}_AM$ is classical completely prime if and only if ${}_RM$ is classical completely prime.

Proof. Define a function from ${}_RM$ to ${}_AM$ by $rm = f(r)m$. This function turns M into an R-module whenever M is an ${}_AM$ module. Suppose ${}_AM$ is classical completely prime and for all $r, s \in R$ and $m \in M$, $rsm = 0$. Then, $0 = rsm = f(r)f(s)m$. Since ${}_AM$ is classical completely prime, $f(r)\langle m\rangle_A = 0$ or $f(s)\langle m\rangle_A = 0$. Then by structure of R-module, it follows that $r\langle m\rangle_R = 0$ or $s\langle m\rangle_R = 0$. Thus, ${}_RM$ is classical completely prime. Assume ${}_RM$ is classical completely prime and for all $a, b \in R$ and $m \in M$, $abm = 0$. Then since f is an epimorphism, there exists $r, s \in A$ such that $a = f(r)$ and $b = f(s)$, i.e., $f(r)f(s)m = rsm = 0$. By assumption, $r\langle m\rangle_R = 0$ or $s\langle m\rangle_R = 0$. If $r\langle m\rangle_R = 0$ (resp. $s\langle m\rangle_R = 0$), the fact that f is onto leads to $a\langle m\rangle_A = 0$ (resp. $b\langle m\rangle_A = 0$). Hence, ${}_AM$ is classical completely prime. □

2.3. Properties of submodules and direct summands.

2.7. Proposition. If M is a classical completely prime module, then any submodule N of M is also a classical completely prime module.

Proof. Elementary. □

2.8. Proposition. For an R-module M, the following statements are equivalent:

(1) M is a classical completely prime module,
(2) Each direct summand of M is a classical completely prime submodule of M.

Proof. (1) ⇒ (2). By Proposition 2.7 any submodule N of M is a classical completely prime module. If $M = K \oplus P$ where K and P are submodules, then M/K is isomorphic to P which is a classical completely prime module and so K is a classical completely prime submodule.

(2) ⇒ (1). If each direct summand of M is a classical completely prime submodule, then so is the zero submodule and hence M is a classical completely prime module. □

2.4. Classical multiplicative systems.

2.1. Definition. Let R be a ring and M an R-module. A nonempty set $S \subseteq M \setminus \{0\}$ is called a classical multiplicative system if, for all $a, b \in R$, $m \in M$ and for all submodules K of M, if $(K + a\langle m\rangle) \cap S \neq \emptyset$ and $(K + b\langle m\rangle) \cap S \neq \emptyset$, then $(K + abm) \cap S \neq \emptyset$.

2.9. Proposition. Let M be an R-module. A proper submodule P of M is classical completely prime if and only if its complement $M\backslash P$ is a classical multiplicative system.

Proof. Suppose $S := M\backslash P$. Let $a, b \in R$, $m \in M$ and K be a submodule of M such that $(K + a\langle m\rangle) \cap S \neq \emptyset$ and $(K + b\langle m\rangle) \cap S \neq \emptyset$. If $(K + \{abm\}) \cap S = \emptyset$, then $abm \in P$. Since P is classical completely prime , $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$. It follows that $(K + a\langle m\rangle) \cap S = \emptyset$ or $(K + b\langle m\rangle) \cap S = \emptyset$, a contradiction. Therefore, S is a classical multiplicative system in M. For the converse, let $S := M\backslash P$ be a classical multiplicative system in M. Suppose for $a, b \in R$ and $m \in M$, $abm \in P$. If $a\langle m\rangle \not\subseteq P$ and $b\langle m\rangle \not\subseteq P$, then $a\langle m\rangle \cap S \neq \emptyset$ and $b\langle m\rangle \cap S \neq \emptyset$. Thus, $abm \in S$, a contradiction. Therefore, P is a classical completely prime submodule of M. □

2.10. Proposition. Let M be an R-module, P be a proper submodule of M, and $S := M\backslash P$. Then, the following statements are equivalent:

(1) P is a classical completely prime submodule of M;
(2) S is a classical multiplicative system of M;
(3) for all $a, b \in R$ and $m \in M$, if $a\langle m\rangle \cap S \neq \emptyset$ and $b\langle m\rangle \cap S \neq \emptyset$, then $abm \in S$;
(4) for all $a, b \in R$ and $m \in M$, if $\langle a\langle m\rangle\rangle \cap S \neq \emptyset$ and $\langle b\langle m\rangle\rangle \cap S \neq \emptyset$, then $\langle abm\rangle \cap S \neq \emptyset$.

2.1. Lemma. Let M be an R-module, $S \subseteq M$ a classical multiplicative system of M and P a submodule of M maximal with respect to the property that $P \cap S = \emptyset$. Then, P is a classical completely prime submodule of M.

Proof. Suppose $a \in R$ and $m \in M$ such that $\langle abm \rangle \subseteq P$. If $\langle a\langle m\rangle\rangle \nsubseteq P$ and $\langle b\langle m\rangle\rangle \nsubseteq P$ then $(\langle a\langle m\rangle\rangle + P) \cap S \neq \emptyset$ and $(\langle b\langle m\rangle\rangle + P) \cap S \neq \emptyset$. By definition of a classical multiplicative system S of M, $(\langle abm\rangle + P) \cap S \neq \emptyset$. Since $\langle abm\rangle \subseteq P$, we have $P \cap S \neq \emptyset$, a contradiction. Hence, P must be a classical completely prime submodule. □

2.2. Definition. Let R be a ring and M an R-module. For $N \leq M$, if there is a classical completely prime submodule of M containing N, we define $\text{clc.}\sqrt{N} := \{m \in M$: every classical multiplicative system containing
m meets $N\}$. We write $\text{clc.}\sqrt{N} = M$ when there are no classical completely prime submodules of M containing N.

2.1. Theorem. Let M be an R-module and $N \leq M$. Then, either $\text{clc.}\sqrt{N} = M$ or $\text{clc.}\sqrt{N}$ equals the intersection of all classical completely prime submodules of M containing N, which is denoted by $\beta^c_{cl}(N)$.

Proof. Suppose $\text{clc.}\sqrt{N} \neq M$. Both $\text{clc.}\sqrt{N}$ and N are contained in the same classical completely prime submodules. By definition of $\text{clc.}\sqrt{N}$ it is clear that $N \subseteq \text{clc.}\sqrt{N}$. Hence, any classical completely prime submodule of M which contains $\text{clc.}\sqrt{N}$ must necessarily contain N. Suppose P is a classical completely prime submodule of M such that $N \subseteq P$, and let $t \in \text{clc.}\sqrt{N}$. If $t \notin P$, then the complement of P, $C(P)$ in M is a classical multiplicative system containing t and therefore we would have $C(P) \cap N \neq \emptyset$. However, since $N \subseteq P$, $C(P) \cap P = \emptyset$ and this contradiction shows that $t \in P$. Hence $\text{clc.}\sqrt{N} \subseteq P$ as we wished to show. Thus, $\text{clc.}\sqrt{N} \subseteq \beta_{co}(N)$. Conversely, assume $s \notin \text{clc.}\sqrt{N}$, then there exists a classical multiplicative system S such that $s \in S$ and $S \cap N = \emptyset$. From Zorn's Lemma, there exists a submodule $P \supseteq N$ which is maximal with respect to $P \cap S = \emptyset$. From Lemma 2.1, P is a classical completely prime submodule of M and $s \notin P$. □

2.5. Complete systems.

2.3. Definition. Let R be a ring and M an R-module. A nonempty set $T \subseteq M \setminus \{0\}$ is called a complete system if, for all $a \in R$, $m \in M$ and for all submodules K of M, if $(K + a\langle m\rangle) \cap T \neq \emptyset$, then $(K + \{a^2 m\}) \cap T \neq \emptyset$.

2.2. Corollary. Let M be an R-module. A proper submodule P of M is completely semiprime if and only if $M \backslash P$ is a complete system.

2.11. Proposition. Let M be an R-module, P be a proper submodule of M, and $T := M \backslash P$. Then, the following statements are equivalent:

(1) P is completely semiprime;
(2) T is a complete system;
(3) for all $a \in R$ and $m \in M$, if $a\langle m\rangle \cap T \neq \emptyset$, then $a^2 m \in T$;
(4) for all $a \in R$ and $m \in M$, if $\langle a\langle m\rangle\rangle \cap T \neq \emptyset$, then $\langle a^2 m\rangle \cap T \neq \emptyset$.

2.1. Remark. Every classical multiplicative system is a complete system but not conversely.

2.2. Question. Is every completely semiprime submodule of a module an intersection of classical completely prime submodules?

3. Comparison with "primes" in literature

In this section we compare classical completely prime (resp. completely semiprime) submodules with prime (resp. semiprime) and classical prime (resp. classical semiprime) submodules.

3.1. Definition. [2], [11] $P \leq {}_RM$ with $RM \not\subseteq P$ is prime if for any $N \leq {}_RM$ and any $\mathcal{A} \triangleleft R$ such that $\mathcal{A}N \subseteq P$, then $\mathcal{A}M \subseteq P$ or $N \subseteq P$. P is a semiprime submodule of M if for $a \in R$ and $m \in M$ such that $aRam \subseteq P$, then $am \in P$.

3.2. Definition. [4, p.338] $P \leq {}_RM$ with $RM \not\subseteq P$ is classical prime if for any $N \leq {}_RM$ and any $\mathcal{A}, \mathcal{B} \triangleleft R$ such that $\mathcal{A}\mathcal{B}N \subseteq P$, then $\mathcal{A}N \subseteq P$ or $\mathcal{B}N \subseteq P$. $P \leq_R M$ is classical semiprime if for every $\mathcal{A} \triangleleft R$, and $N \leq M$ such that $\mathcal{A}^2N \subseteq P$, then $\mathcal{A}N \subseteq P$.

Propositions 3.1 and 3.2 are modifications of [4, Proposition 1.1] and [4, Proposition 1.2] to suit a not necessarily unital module.

3.1. Proposition. Let $P \leq {}_RM$, the following statements are equivalent:
(1) P is a classical prime submodule of M;
(2) for all $a, b \in R$ and every $m \in M$, if $\langle a\rangle\langle b\rangle m \subseteq P$, then $\langle a\rangle m \subseteq P$ or $\langle b\rangle m \subseteq P$;
(3) for all $a, b \in R$ and every $m \in M$ such that $aRb\langle m\rangle \subseteq P$, then $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$.

3.2. Proposition. Let $P \leq {}_RM$, the following statements are equivalent:
(1) P is a classical semiprime submodule of M;
(2) for all $a \in R$ and every $m \in M$, if $\langle a\rangle^2 m \subseteq P$, then $\langle a\rangle m \subseteq P$;
(3) for all $a \in R$ and every $m \in M$, if $aRa\langle m\rangle \subseteq P$, then $a\langle m\rangle \subseteq P$.

3.1. Remark. In literature, classical prime is used interchangeably with weakly prime, cf., [3], [4], [5], [6]. We here use classical prime instead of weakly prime. In defense of our nomenclature, weakly prime modules exist in [13] when used in a totally different context - a context which generalizes the notion of weakly prime ideals for rings to modules. To the best of our knowledge, classical prime has never been used by other authors to mean something different. Our "classical semiprime" is what is called "semiprime" in [4], our nomenclature reflects that classical semiprime is derived from classical prime. Lastly, our "semiprime" is the semiprime in [11].

3.1. Theorem. For any submodule $P \leq {}_RM$, we have the following implications:

(i) in general			(ii)	P IFP	submodule
		prime	prime		
		$\Downarrow$	$\Downarrow$		
classical completely prime	$\Rightarrow$	classical prime	classical completely prime	$\Leftrightarrow$	classical prime

Proof. (i). By [22, Prop. 4.1.11], it is known that a prime submodule is classical prime. Now we show that a classical completely prime submodule is classical prime. Let $a, b \in R$ and $m \in M$ such that $\langle a\rangle\langle b\rangle m \subseteq P$. Then, $abm \in P$ and P classical completely prime in M implies $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$. Thus, ($am \in P$ and $aRm \subseteq P$) or ($bm \in P$ and $bRm \subseteq P$) so that $\langle a\rangle m = (\mathbb{Z}a + Ra + aR + RaR)m \subseteq P$ or $\langle b\rangle m = (\mathbb{Z}b + Rb + bR + RbR)m \subseteq P$. Hence, P is classical prime.

(ii). Suppose a classical prime submodule P is IFP, we show that P is classical completely prime. If $a, b \in R$ and $m \in M$ such that $abm \in P$, then $aRbm \subseteq P$ and $aRb(Rm) \subseteq P$ so that $aRb\langle m\rangle \subseteq P$. This implies, either $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$ by definition of classical prime submodule. So, P is classical completely prime. $\square$

3.2. Theorem. *P is a classical completely prime submodule of an R-module M if and only if P is both a classical prime and a completely semiprime submodule of M.*

Proof. Every classical completely prime submodule is completely semiprime. From Theorem 3.1, classical completely prime submodules are classical prime. For the converse, assume P is both a completely semiprime and a classical prime submodule of M. Now, let $a, b \in R$ and $m \in M$ such that $abm \in P$. By Proposition 2.1, P is IFP. Hence, $aRb\langle m\rangle \subseteq P$. P classical prime implies $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$. □

3.1. Example. Every maximal submodule P of an R-module M is a classical prime submodule but there exist modules with maximal submodules which are not classical completely prime. Let $\mathcal{A} \lhd R$ and $N \leq M$ such that $\mathcal{A}N \subseteq P$, where P is a maximal submodule of M. If $N \subseteq P$, we are through. Suppose $N \nsubseteq P$. Then, $M = N + P$ so that $\mathcal{A}M = \mathcal{A}N + \mathcal{A}P \subseteq P$. This shows P is a prime submodule and hence a classical prime submodule. We construct a maximal submodule which is not classical completely prime. Let $R = (M_2(\mathbb{Z}), +, .)$ be a ring of all 2-by-2 matrices with integral entries and $(M_2(\mathbb{Z}_2), +)$ be a group of all 2-by-2 matrices with entries from the ring $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. Then, $M_2(\mathbb{Z}_2)$ is an $M_2(\mathbb{Z})$-module and

$$P = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$$

is a maximal submodule of $M_2(\mathbb{Z}_2)$. Now, let $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $m = \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix}$. $abm = 0 \in P$ but $a\langle m\rangle \nsubseteq P$ and $b\langle m\rangle \nsubseteq P$ since $am \notin P$ and $bm \notin P$. Therefore, P is a maximal submodule of $M_2(\mathbb{Z}_2)$ but not a classical completely prime submodule of $M_2(\mathbb{Z})$-module $M_2(\mathbb{Z}_2)$.

In regard to Example 3.1, we point out that, although it is not true in general, we can find noncommutative rings for which every maximal submodule is classical completely prime. To illustrate this, we use left (quasi) duo rings. A ring R is called left (quasi) duo if every left (maximal left) ideal of R is two sided. A ring R is called left quasi-duo, if every maximal left ideal of R is two sided.

3.3. Proposition. [20, Proposition 3.6] *R is a left quasi-duo ring if and only if each simple R-module M is reduced.*

3.4. Proposition. *If R is a left quasi-duo ring, then each maximal submodule P of M is a classical completely prime submodule of M.*

Proof. Let P be a maximal submodule of M and R a left quasi-duo ring. M/P is simple and from Proposition 3.3, it is reduced. Hence, P is a completely semiprime submodule of M. Since every maximal submodule of M is classical prime, it follows from Theorem 3.2 that P is a classical completely prime submodule of M. □

3.2. Remark. It is not possible to get an example like Example 3.1 for a ring R which is a collection of all upper triangular matrices over $\mathbb{Z}$. This is because, upper triangular matrix rings are left quasi-duo and from Proposition 3.4, maximal submodules are always classical completely prime.

It is clear from Example 3.1 that simple modules are not always classical completely prime. We give another example to show that simple modules are not always classical completely prime. It makes use of Lemma 3.1.

3.1. Lemma. *For a simple and reduced module ${}_RM$, $am = 0$ implies $aM = 0$ for all $a \in R$ and $0 \neq m \in M$.*

Proof. Suppose $am = 0$. Since M is simple and reduced, we have $0 = aM \cap \langle m\rangle = aM \cap M = aM$. □

3.2. Example. Let $M = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ where entries of matrices in M are from $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and $R = M_2(\mathbb{Z})$. ${}_RM$ is a simple module which is not classical completely prime.

Proof. Let $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$,

$$rM = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} a & a \\ c & c \end{pmatrix}, \begin{pmatrix} b & b \\ d & d \end{pmatrix}, \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} \right\} \subseteq M$$

for any $a, b, c, d \in \mathbb{Z}$. The would be nontrivial proper submodules, namely; $N_1 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}$, $N_2 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ and
$N_3 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ are not closed under multiplication by R since, for a and c odd, $rN_1 \nsubseteq N_1$, for b and d odd, $rN_2 \nsubseteq N_2$ and for a odd but b, c, d even, $rN_3 \nsubseteq N_3$. Take $a = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \in R$ and $m = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \in M$, $am = 0$ but $aM \neq 0$ since $a = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \neq 0$. By Lemma 3.1, M is not reduced and hence not classical completely prime. □

3.3. Example. If P is a classical prime submodule of an R-module M, $(P : M)$ is a prime ideal of R which is not necessarily a completely prime ideal of R. On the other hand, if P is a classical completely prime submodule of an R-module M, then $(P : M)$ is a completely prime ideal of R. This shows that a classical prime submodule need not be classical completely prime.

Since over commutative rings classical completely prime submodules and classical submodules are indistinguishable, we have:

3.4. Example. [6, Example 1] Assume that R is a unital commutative domain and $\mathcal{P}$ is a non-zero prime ideal in R. $\mathcal{P} \oplus 0$ and $0 \oplus \mathcal{P}$ are classical completely prime submodules in the free module $M = R \oplus R$, but they are not prime submodules.

4. Comparison of "semiprimes"

4.1. Theorem. For any submodule P of an R-module M,

completely semiprime $\Rightarrow$ semiprime $\Rightarrow$ classical semiprime.

Proof. Suppose for $a \in R$ and $m \in M$, $aRam \subseteq P$, then $(a^2)^2 m \in P$ and P completely semiprime implies $a^2 m \in a^2\langle m\rangle \subseteq P$. Hence, $am \in a\langle m\rangle \subseteq P$ and P is semiprime. Now, suppose $aRa\langle m\rangle \subseteq P$ but $a\langle m\rangle \nsubseteq P$. Then, there exists $m_1 \in \langle m\rangle$ such that $am_1 \notin P$. By definition of semiprime submodules, $aRam_1 \nsubseteq P$ and so $aRa\langle m\rangle \nsubseteq P$ which is a contradiction. Therefore, whenever $aRa\langle m\rangle \subseteq P$, we have $a\langle m\rangle \subseteq P$ and semiprime $\Rightarrow$ classical semiprime. □

The reverse implications in Theorem 4.1 are not true in general. The simple module M constructed in Examples 3.2 is semiprime (because all simple modules are prime) but it is not completely semiprime. For the second implication, a counter example was constructed by Hongan in [15, p.119].

4.1. Corollary. If P is an IFP submodule of M, then

$$\text{completely semiprime} \;\Leftrightarrow\; \text{semiprime} \;\Leftrightarrow\; \text{classical semiprime}.$$

Proof. It is enough to show that classical semiprime $\Rightarrow$ completely semiprime, the rest follows from Theorem 4.1. Let $a^2m \in P$, where $a \in R$ and $m \in M$. For P IFP, $aRa\langle m\rangle \subseteq P$. By definition of classical semiprime, $a\langle m\rangle \subseteq P$ and P is completely semiprime. □

A ring R is left (right) permutable [10, p.258], if for all $a, b, c \in R$, $abc = bac$ ($abc = acb$). R is permutable if it is both left and right permutable. Commutative rings and nilpotent rings of index ≤ 3 are left (right) permutable. A ring R is medial [10], if for all $a, b, c, d \in R$, $abcd = acbd$. A left (right) permutable ring is medial but not conversely. A unital medial ring is indistinguishable from a commutative ring. A ring R is left self distributive, denoted by LSD (resp. right self distributive, denoted by RSD) if for all $a, b, c, d \in R$, the identity: $abc = abac$ (resp. $abc = acbc$) holds. LSD rings are left permutable, see [14, Corollary 2.2]. Left (right) permutable rings and medial rings exist in abundancy; according to Birkenmeier and Heatherly in [10, p.258], they are a special type of PI-rings and also exist as special subrings of every ring. Furthermore, if R is a noncommutative medial (left permutable, right permutable or permutable) ring, then the ring of polynomials (resp. formal power series or formal Laurent series) over R is a medial (left permutable, right permutable or permutable) ring which is not commutative, see [10, p.262-263].

4.2. Theorem. If P is a classical semiprime submodule of ${}_RM$ and R is a medial (left permutable, right permutable or LSD) ring then each of the following statements implies P is a completely semiprime submodule of ${}_RM$:

(1) M is finitely generated,
(2) M is free,
(3) M is cyclic.

Proof. We prove only the case for M cyclic, the proofs for other cases are similar. Suppose $a^2m \in P$ for $a \in R$ and $m \in M$, $R^2a^2m \subseteq P$. R medial implies $RaRam \subseteq P$. Since M is cyclic, $m = rm_0$ for some $r \in R$ and $m_0 \in M$. $RaRarm_0 \subseteq P$ and $R^2aRarm_0 \subseteq P$. Again, R medial leads to $RaRaRm \subseteq P$. It follows that $RaRa\langle m\rangle \subseteq P$. Since P is classical semiprime, $Ra\langle m\rangle \subseteq P$, i.e., $Ra \subseteq (P : \langle m\rangle)$. P classical semiprime implies $(P : \langle m\rangle)$ is a semiprime ideal of R and hence $a \in (P : \langle m\rangle)$, i.e., $a\langle m\rangle \subseteq P$. □

4.2. Corollary. If P is a prime (semiprime, classical prime) submodule of ${}_RM$ with R medial (left permutable, right permutable or LSD), then each of the following statements implies P is completely prime and hence classical completely prime.

(1) M is finitely generated,
(2) M is free,
(3) M is cyclic.

5. The radicals $\beta^c_{cl}(M)$ and $\beta_{co}(R)$

Let $\mathcal{M}^c$ be the class of all completely prime rings, i.e., rings which have no non-zero divisors. Then $\mathcal{M}^c_R$ is the class of all classical completely prime R-modules. We have $\mathcal{R}_c = \mathcal{N}_g$, the generalized nil radical which we shall call the completely prime radical of R (denoted by $\beta_{co}(R)$) with

$$\beta_{co}(R) := \cap\{I \triangleleft R \;:\; I \text{ is a completely prime ideal}\}.$$

The corresponding classical completely prime radical for the R-module M will be denoted by

$$\beta^c_{cl}(M) := \cap\{N \leq M : M/N \in \mathcal{M}^c_R\}.$$

Since each classical completely prime submodule of an R-module M is also classical prime submodule, we have $\beta_{cl}(M) \subseteq \beta^c_{cl}(M)$ where $\beta_{cl}(M)$ is the classical prime radical (the intersection of all classical prime submodules of M). If M is an R-module over a commutative ring, then the two radicals coincide.

5.1. Proposition. For any ring R, $\beta^c_{cl}({}_RR) \subseteq \beta_{co}(R)$.

Proof. Follows from [21, Lemma 4.1] and the fact that any completely prime module is classical completely prime. □

5.1. Lemma. For any R-module M, we have

$$\beta_{co}(R) \subseteq (\beta^c_{cl}(M) : M)_R.$$

Proof. We have $(\beta^c_{cl}(M) : M) = (\bigcap_{S \leq M} S : M)_R = \bigcap_{S \leq M} (S : M)_R$ where M/S is a classical completely prime module. Since $(S : M)_R$ is a completely prime ideal, we get $\beta_{co}(R) \subseteq (\beta^c_{cl}(M) : M)$. □

5.1. Remark. The containment in Lemma 5.1 is in general strict. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ for some prime number p. Now $\beta^c_{cl}(M) = \mathbb{Z}_{p^\infty}$ and $\beta_{co}(R) = 0$, i.e., $\beta_{co}(R)M = (0)$.

5.2. Lemma. For any ring R, we have $\beta_{co}(R) = (\beta^c_{cl}({}_RR) : R)_R$.

Proof. Follows from [12, Proposition 4.6]. □

Recall that for an R-module M, we have the Jacobson radical $\text{Rad}(M)$ of the module M defined as:

$$\text{Rad}(M) = \cap\{K \leq M \ : \ K \text{ is a maximal submodule of } M\}.$$

5.1. Theorem. Let M be a module over a left Artinian ring R. Then

$$\text{Rad}(M) \subseteq \beta^c_{cl}(M) \text{ and } \text{Rad}({}_RR) = \beta^c_{cl}({}_RR).$$

Proof. From [9, Cor. 4.3.17, p.178], $\text{Rad}(M) = \text{Jac}(R)M = \beta_{co}(R)M$ and from the fact that $\beta_{co}(R) \subseteq (\beta^c_{cl}(M) : M)_R$ we get $\text{Rad}(M) \subseteq \beta^c_{cl}(M)$. Again from [9, Cor. 4.3.17, p.178], and Lemma 5.2, $\text{Rad}({}_RR) = \text{Jac}(R)R = \beta_{co}(R)R = \beta^c_{cl}({}_RR)$. □

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Sharp Wilker and Huygens type inequalities for trigonometric and hyperbolic functions

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Abstract

In the article, some sharp Huygens and Wilker type inequalities involving trigonometric and hyperbolic functions are established.

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1. Introduction

The trigonometric and hyperbolic inequalities have been in recent years in the focus of many researchers. For many results and a long list of references we quote the papers [6, 10, 24], where many further references may be found. The following inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \quad 0 < x < \frac{\pi}{2} \tag{1.1}$$

is due to Wilker [13]. It has attracted attention of several researchers(see, e. g.,[4],[7], [8], [9],[14],[15],[21]). A hyperbolic counterpart of Wilker's inequality

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2. \tag{1.2}$$

$(x \neq 0)$ has been established by L. Zhu[16].

In [12], it was proved that

$$2 + \frac{8}{45}x^3 \tan x > \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x, \tag{1.3}$$

for $0 < x < \frac{\pi}{2}$. The constants $\frac{8}{45}$ and $\left(\frac{2}{\pi}\right)^4$ in the inequality (1.3) are the best possible.

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The famous Huygens inequality[11] for the sine and tangent functions states that for $x \in \left(0, \frac{\pi}{2}\right)$

$$2\sin x + \tan x > 3x. \tag{1.4}$$

The hyperbolic counterpart of (1.4) was established in [6] as follows: For $x > 0$

$$2\sinh x + \tanh x > 3x. \tag{1.5}$$

The inequalities (1.4) and (1.5) were respectively refined in [6, Theorem 2.6] as

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3, \tag{1.6}$$

and

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 2\frac{x}{\sinh x} + \frac{x}{\tanh x} > 3, \quad x \neq 0. \tag{1.7}$$

In the most recent paper [5], the inequalities (1.6) ,(1.7) and (1.1) were respectively further refined as

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > \frac{\sin x}{x} + 2\frac{\tan(x/2)}{x/2} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3. \tag{1.8}$$

and

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > \frac{\sinh x}{x} + 2\frac{\tanh(x/2)}{x/2} > 2\frac{x}{\sinh x} + \frac{x}{\tanh x} > 3. \tag{1.9}$$

and

$$\begin{aligned} \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} &> \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > \frac{\sin x}{x} + \left(\frac{\tan(x/2)}{x/2}\right)^2 \\ &> \frac{x}{\sin x} + \left(\frac{x/2}{\tan(x/2)}\right)^2 > 2. \end{aligned} \tag{1.10}$$

The hyperbolic counterparts of the last two inequalities in (1.10) were also given in [5] as follows:

$$\frac{\sinh x}{x} + \left[\frac{\tanh(x/2)}{x/2}\right]^2 > \frac{x}{\sinh x} + \left[\frac{x/2}{\tanh(x/2)}\right]^2 > 2. \tag{1.11}$$

Inspired by (1.3), Jiang et al. [19] first proved

$$3 + \frac{1}{60}x^3 \sin x < 2\frac{x}{\sin x} + \frac{x}{\tan x} < 3 + \frac{8\pi - 24}{\pi^3}x^3 \sin x. \tag{1.12}$$

and

$$2 + \frac{17}{720}x^3 \sin x < \frac{x}{\sin x} + \left(\frac{\frac{x}{2}}{\tan\frac{x}{2}}\right)^2 < 2 + \frac{\pi^2 + 8\pi - 32}{2\pi^3}x^3 \sin x. \tag{1.13}$$

holds for $0 < |x| < \frac{\pi}{2}$. Furthermore the constants $\frac{1}{60}$, $\frac{8\pi-24}{\pi^3}$ in (1.12) and the constants $\frac{17}{720}$, $\frac{\pi^2+8\pi-32}{2\pi^3}$ in (1.13) are the best possible.

Recently, Chen and Sándor [20] proved that

$$3 + \frac{3}{20}x^3 \tan x < 2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} < 3 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x.$$

for $0 < |x| < \frac{\pi}{2}$. The constants $\frac{3}{20}$ and $\left(\frac{2}{\pi}\right)^4$ are the best possible.

This paper is a continuation of our work [25] and is organized as follows. In Section 2, we give some lemmas and preliminary results. In Section 3, we prove some new sharp Wilker- and Huygens-type inequalities for trigonometric and hyperbolic functions.

2. some Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

2.1. Lemma. *The Bernoulli numbers B_{2n} for $n \in \mathbb{N}$ have the property*

$$(2.1)\qquad (-1)^{n-1}B_{2n} = |B_{2n}|,$$

where the Bernoulli numbers B_i for $i \geq 0$ are defined by

$$(2.2)\qquad \frac{x}{e^x - 1} = \sum_{i=0}^{\infty} \frac{B_i}{i!} x^i = 1 - \frac{x}{2} + \sum_{i=1}^{\infty} B_{2i} \frac{x^{2i}}{(2i)!}, \quad |x| < 2\pi.$$

Proof. In [2, p. 16 and p. 56], it is listed that for $q \geq 1$

$$(2.3)\qquad \zeta(2q) = (-1)^{q-1} \frac{(2\pi)^{2q}}{(2q)!} \frac{B_{2q}}{2},$$

where ζ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In [22, p.18, theorem 3.4], the following formula was given

$$(2.4)\qquad \sum_{n=1}^{\infty} \frac{1}{n^{2q}} = \frac{2^{2q-1}\pi^{2q}|B_{2q}|}{(2q)!}.$$

From (2.3) and (2.4), the formula (2.1) follows. □

2.2. Lemma. [17, 18] *Let B_{2n} be the even-indexed Bernoulli numbers. Then*

$$\frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{-2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{1-2n}}, n = 1, 2, 3, \cdots.$$

2.3. Lemma. *For $0 < |x| < \pi$, we have*

$$(2.5)\qquad \frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2\left(2^{2n-1} - 1\right)|B_{2n}|}{(2n)!} x^{2n}.$$

Proof. This is an easy consequence of combining the equality

$$(2.6)\qquad \frac{1}{\sin x} = \csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2\left(2^{2n-1} - 1\right)B_{2n}}{(2n)!} x^{2n-1},$$

see [1, p. 75, 4.3.68], with Lemma 2.1. □

2.4. Lemma ([1, p. 75, 4.3.70])**.** *For $0 < |x| < \pi$,*

$$(2.7)\qquad \cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} x^{2n-1}.$$

The following Lemma 2.5 and Lemma 2.6 can be found in [25].

2.5. Lemma. *For $0 < |x| < \pi$,*

$$(2.8)\qquad \frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!} x^{2(n-1)}.$$

2.6. Lemma. *For $0<|x|<\pi$,*

$$\frac{\cos x}{\sin^2 x}=\frac{1}{x^2}-\sum_{n=1}^{\infty}\frac{2(2n-1)\left(2^{2n-1}-1\right)|B_{2n}|}{(2n)!}x^{2(n-1)}. \tag{2.9}$$

2.7. Lemma. *For $0<|x|<\pi$,*

$$\begin{aligned}\frac{1}{\sin^3 x}=&\frac{1}{x^3}+\frac{1}{2}\sum_{n=2}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|(2n-1)(2n-2)x^{2n-3}\\ &+\frac{1}{2x}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n-1},\end{aligned} \tag{2.10}$$

and

$$\frac{\cos x}{\sin^3 x}=\frac{1}{x^3}-\sum_{n=2}^{\infty}\frac{(2n-1)(n-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n-3}. \tag{2.11}$$

Proof. Combining

$$\frac{1}{\sin^3 x}=\frac{1}{2\sin x}-\frac{1}{2}\left(\frac{\cos x}{\sin^2 x}\right)'$$

with Lemma 2.6, the identity (2.6), and Lemma 2.1 gives (2.10).

The equality (2.11) follows from combination of

$$\frac{\cos x}{\sin^3 x}=-\frac{1}{2}\left(\frac{1}{\sin^2 x}\right)'$$

with Lemma 2.5. □

2.8. Lemma. [23, 3, 15] *Let a_n and $b_n (n=0,1,2,\cdots)$ be real numbers, and let the power series $A(t)=\sum_{n=0}^{\infty}a_n t^n$ and $B(t)=\sum_{n=0}^{\infty}b_n t^n$ be convergent for $|t|<R$. If $b_n>0$ for $n=0,1,2,\cdots$, and if $\frac{a_n}{b_n}$ is strictly increasing (or decreasing) for $n=0,1,2,\cdots$, then the function $\frac{A(t)}{B(t)}$ is strictly increasing (or decreasing) on $(0,R)$.*

3. Main results

Now we are in a position to state and prove our main results.

3.1. Theorem. *For $0<|x|<\frac{\pi}{2}$, we have*

$$2+\frac{23}{720}x^3\sin x<\frac{\sin x}{x}+\left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)^2<2+\frac{128-16\pi^2+16\pi}{\pi^5}x^3\sin x. \tag{3.1}$$

The constants $\frac{23}{720}$ and $\frac{128-16\pi^2+16\pi}{\pi^5}$ in (3.1) are the best possible.

Proof. Let

$$\begin{aligned}f(x)=&\frac{\frac{\sin x}{x}+\left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)^2-2}{x^3\sin x}\\ =&\frac{x\sin^3 x-8\cos x-4\sin^2 x-2x^2\sin^2 x+8}{x^5\sin^3 x}\\ =&\frac{1}{x^5}\left(x+\frac{8}{\sin^3 x}-\frac{8\cos x}{\sin^3 x}-\frac{4}{\sin x}-\frac{2x^2}{\sin x}\right)\end{aligned}$$

for $x\in\left(0,\frac{\pi}{2}\right)$. By virtue of (2.10), (2.11), and (2.6), we have

$$f(x) = \frac{1}{x^5}\Bigg[x + \frac{8}{x^3} + \sum_{n=2}^{\infty} \frac{4(2n-1)(2n-2)(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n-3}$$
$$+ \frac{4}{x} + \sum_{n=1}^{\infty} \frac{4(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n-1}$$
$$- \frac{8}{x^3} + \sum_{n=2}^{\infty} \frac{8\cdot 2^{2n}(2n-1)(n-1)}{(2n)!}|B_{2n}|x^{2n-3}$$
$$- \frac{4}{x} - \sum_{n=1}^{\infty} \frac{4(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n-1}$$
$$- 2x - \sum_{n=1}^{\infty} \frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n+1}\Bigg]$$
$$= \frac{1}{x^5}\left[-x + \sum_{n=2}^{\infty} \frac{16(2n-1)(n-1)(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-3} - \sum_{n=1}^{\infty} \frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n+1}\right]$$
$$= \frac{1}{x^5}\left[\sum_{n=3}^{\infty} \frac{16(2n-1)(n-1)(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-3} - \sum_{n=1}^{\infty} \frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n+1}\right]$$
$$= \frac{1}{x^5}\left[\sum_{n=1}^{\infty} \frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!}|B_{2n+4}|x^{2n+1} - \sum_{n=1}^{\infty} \frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n+1}\right]$$
$$= \sum_{n=2}^{\infty}\left[\frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!}|B_{2n+4}| - \frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|\right]x^{2n-4}.$$

Let $a_n = \frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!}|B_{2n+4}| - \frac{2(2^{2n}-2)}{(2n)!}|B_{2n}|$ for $n \geq 2$.
By a simple computation, we have $a_2 = \frac{23}{720}$.
Furthermore, when $n \geq 3$, From Lemma 2.2 one can get

$$\begin{aligned} a_n =& \frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!}|B_{2n+4}| - \frac{2(2^{2n}-2)}{(2n)!}|B_{2n}| \\ >& \frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!} \cdot \frac{2(2n+4)!}{(2\pi)^{2n+4}} \frac{1}{1-2^{-2n-4}} \\ & - \frac{2(2^{2n}-2)}{(2n)!} \cdot \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1-2^{1-2n}} \\ =& \frac{4}{(\pi)^{2n}}\left[\frac{8(2n+3)(n+1)}{\pi^4} - 1\right] > 0. \end{aligned}$$

So the function $f(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. Moreover, it is easy to obtain

$$\lim_{x\to 0^+} f(x) = a_2 = \frac{23}{720} \quad \text{and} \quad \lim_{x\to(\pi/2)^-} f(x) = \frac{128 - 16\pi^2 + 16\pi}{\pi^5}.$$

The proof of Theorem 3.1 is complete. □

3.2. Remark. Since $f(x)$ is an even function we conclude that Theorem 3.1 holds for all x which satisfy $0 < |x| < \frac{\pi}{2}$.

3.3. Theorem. *For $x \neq 0$, we have*

$$3 + \frac{1}{40}x^3 \tanh x < \frac{\sinh x}{x} + 2\left(\frac{\tanh \frac{x}{2}}{\frac{x}{2}}\right) < 3 + \frac{1}{40}x^3 \sinh x. \tag{3.2}$$

The constant $\frac{1}{40}$ *is the best possible.*

Proof. Without loss of generality, we assume that $x > 0$.

We firstly prove the first inequality of (3.2).

Consider the function $F(x)$ defined by

$$F(x) = \frac{\frac{\sinh x}{x} + 2\frac{\tanh \frac{x}{2}}{\frac{x}{2}} - 3}{x^3 \tanh x} = \frac{\cosh 3x - 17\cosh x + 8\cosh 2x - 6x \sinh 2x + 8}{2x^4(\cosh 2x - 1)}.$$

and let

$$f(x) = \cosh 3x - 17\cosh x + 8\cosh 2x - 6x\sinh 2x + 8 \quad \text{and} \quad g(x) = 2x^4(\cosh 2x - 1).$$

From the power series expansions

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \tag{3.3}$$

it follows that

$$\begin{aligned} f(x) &= \cosh 3x - 17\cosh x + 8\cosh 2x - 6x\sinh 2x + 8 \\ &= \sum_{n=0}^{\infty} \frac{3^{2n}x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{17x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{2^{2n+3}x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6 \cdot 2^{2n+1}x^{2n+2}}{(2n+1)!} + 8 \\ &= \sum_{n=0}^{\infty} \frac{\left(3^{2n} + 2^{2n+3} - 17\right)x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6 \cdot 2^{2n+1}x^{2n+2}}{(2n+1)!} + 8 \\ &= \sum_{n=1}^{\infty} \frac{\left(3^{2n} + 2^{2n+3} - 17\right)x^{2n}}{(2n)!} - \sum_{n=1}^{\infty} \frac{6n2^{2n}x^{2n}}{(2n)!} \\ &= \sum_{n=3}^{\infty} \frac{3^{2n} + 2^{2n+3} - 17 - 6n2^{2n}}{(2n)!} x^{2n} \\ &\triangleq \sum_{n=3}^{\infty} a_n x^{2n} \end{aligned}$$

and

$$\begin{aligned} g(x) &= 2x^4(\cosh 2x - 1) \\ &= \sum_{n=1}^{\infty} \frac{2^{2n+1}x^{2n+4}}{(2n)!} \\ &= \sum_{n=3}^{\infty} \frac{4n(n-1)(2n-3)(2n-1)2^{2n-3}x^{2n}}{(2n)!} \\ &\triangleq \sum_{n=3}^{\infty} b_n x^{2n}. \end{aligned}$$

It is easy to see that the quotient

$$c_n = \frac{a_n}{b_n} = \frac{3^{2n} + 2^{2n+3} - 17 - 6n2^{2n}}{4n(n-1)(2n-3)(2n-1)2^{2n-3}}$$

satisfies $c_3 = \frac{1}{40}, c_4 = \frac{51}{1120}, c_5 = \frac{507}{8960}$ and

$$c_{n+1} - c_n = \frac{f_1 + f_2 + f_3}{2n(2n+3)\left(4n^2-1\right)\left(n^2-1\right)}, (n \geq 6),$$

where

$$f_1 = \left(\frac{9}{4}\right)^n \left(10n^2 - 57n + 23\right) = \left(\frac{9}{4}\right)^n (10n(n-6) + 3(n-6) + 41) > 0,$$

$$f_2 = \frac{1}{4^n}(102n^2 + 298n + 17) > 0,$$

$$f_3 = 144n^2 - 184n - 8 = 144n(n-6) + 680(n-6) + 4072 > 0.$$

for $n \geq 6$. This means that the sequence c_n is increasing. By Lemma 2.8, the function $F(x)$ is increasing on $(0, \infty)$. Moreover, it is not difficult to obtain $\lim_{x \to 0^+} F(x) = c_3 = \frac{1}{40}$. Therefore, the first inequality in (3.2) holds.

Finally, we prove the second inequality of (3.2).

Define a function $G(x)$ by

$$G(x) = \frac{\frac{\sinh x}{x} + 2\frac{\tanh \frac{x}{2}}{\frac{x}{2}} - 3}{x^3 \sinh x} = \frac{\cosh 2x + 8\cosh x - 6x \sinh x - 9}{x^4(\cosh 2x - 1)}.$$

and let

$$f(x) = \cosh 2x + 8\cosh x - 6x\sinh x - 9 \quad \text{and} \quad g(x) = x^4(\cosh 2x - 1).$$

By using (3.3), it follows that

$$\begin{aligned} f(x) &= \cosh 2x + 8\cosh x - 6x\sinh x - 9 \\ &= \sum_{n=0}^{\infty} \frac{2^{2n}x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{8x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6x^{2n+2}}{(2n+1)!} - 9 \\ &= \sum_{n=1}^{\infty} \frac{\left(2^{2n} + 8\right)x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6x^{2n+2}}{(2n+1)!} \\ &= \sum_{n=1}^{\infty} \frac{\left(2^{2n} + 8\right)x^{2n}}{(2n)!} - \sum_{n=1}^{\infty} \frac{12nx^{2n}}{(2n)!} \\ &= \sum_{n=3}^{\infty} \frac{\left(2^{2n} + 8 - 12n\right)x^{2n}}{(2n)!} \\ &\triangleq \sum_{n=3}^{\infty} a_n x^{2n} \end{aligned}$$

and

$$\begin{aligned} g(x) &= x^4(\cosh 2x - 1) \\ &= \sum_{n=1}^{\infty} \frac{2^{2n}x^{2n+4}}{(2n)!} \\ &= \sum_{n=3}^{\infty} \frac{4n(n-1)(2n-1)(2n-3)2^{2n-4}x^{2n}}{(2n)!} \\ &\triangleq \sum_{n=3}^{\infty} b_n x^{2n}. \end{aligned}$$

Let

$$c_n = \frac{a_n}{b_n} = \frac{2^{2n} - 12n + 8}{4n(n-1)(2n-1)(2n-3)2^{2n-4}}$$

satisfies $c_3 = \frac{1}{40}$. Furthermore, when $n \geq 3$, by a simple computation, we have

$$c_{n+1} - c_n = -4\frac{(8n-2)4^n - (18n^3+33n^2-16n-11)}{n(2n-3)\big(4n^2-1\big)(n^2-1)4^n},$$

for $n \geq 3$.

Since

$$\begin{aligned}
&(8n-2)4^n - (18n^3+33n^2-16n-11)\\
&> (8n-2)4n^2 - (18n^3+33n^2-16n-11)\\
&= 14n^3 - 41n^2 + 16n + 11\\
&= 14n(n-3)^2 + 43n(n-3) + 19(n-3) + 68 > 0.
\end{aligned}$$

This means that the sequence c_n is decreasing. By Lemma 2.8, the function $G(x)$ is decreasing on $(0, \infty)$. Moreover, it is not difficult to obtain $\lim_{x\to 0^+} G(x) = c_3 = \frac{1}{40}$.

This completes the proof of Theorem 3.3 .

□

3.4. Remark. Since $F(x)$ and $G(x)$ both are even functions, we conclude that Theorem 3.3 holds for all $x \neq 0$.

3.5. Theorem. *For $x \neq 0$,*

$$2 + \frac{23}{720}x^3 \tanh x < \frac{\sinh x}{x} + \left[\frac{\tanh(x/2)}{x/2}\right]^2 < 2 + \frac{23}{720}x^3 \sinh x. \tag{3.4}$$

The both constants $\frac{23}{720}$ in (3.4) are the best possible.

Proof. The left-hand side of inequality in (3.4) has been proved in [19], so we only need to prove the right-hand side of the inequality in (3.4).

Without loss of generality, we assume that $x > 0$.

Consider the function $H(x)$ defined by

$$\begin{aligned}
H(x) &= \frac{\frac{\sinh x}{x} + \left[\frac{\tanh(x/2)}{x/2}\right]^2 - 2}{x^3 \sinh x}\\
&= \frac{x\sinh x\cosh x + x\sinh x + 4\cosh x - 2x^2\cosh x - 2x^2 - 4}{x^5 \sinh x(1+\cosh x)}
\end{aligned}$$

and let

$$f(x) = x\sinh x\cosh x + x\sinh x + 4\cosh x - 2x^2\cosh x - 2x^2 - 4$$

and

$$g(x) = x^5 \sinh x(1+\cosh x).$$

By the power series expansions in (3.3), we obtain

$$
\begin{aligned}
f(x) &= x\sinh x\cosh x + x\sinh x + 4\cosh x - 2x^2\cosh x - 2x^2 - 4 \\
&= \sum_{n=0}^{\infty}\frac{2^{2n}}{(2n+1)!}x^{2n+2} + \sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n)!} + \sum_{n=0}^{\infty}\frac{4x^{2n}}{(2n)!} - \sum_{n=0}^{\infty}\frac{2x^{2n+2}}{(2n)!} - 2x^2 - 4 \\
&= \sum_{n=0}^{\infty}\frac{2^{2n}+1-2(2n+1)}{(2n+1)!}x^{2n+2} + \sum_{n=2}^{\infty}\frac{4}{(2n)!}x^{2n} \\
&= \sum_{n=1}^{\infty}\frac{2^{2n-2}+1-2(2n-1)}{(2n-1)!}x^{2n} + \sum_{n=2}^{\infty}\frac{4}{(2n)!}x^{2n} \\
&= \sum_{n=3}^{\infty}\frac{2n\left(2^{2n-2}-4n+3\right)+4}{(2n)!}x^{2n} \\
&\triangleq \sum_{n=3}^{\infty}a_n x^{2n}
\end{aligned}
$$

and

$$
\begin{aligned}
g(x) &= x^5\left[\frac{1}{2}\sinh(2x) + \sinh x\right] \\
&= \sum_{n=0}^{\infty}\frac{1+2^{2n}}{(2n+1)!}x^{2n+6} = \sum_{n=3}^{\infty}\frac{1+2^{2n-6}}{(2n-5)!}x^{2n} \\
&= \sum_{n=3}^{\infty}\frac{\left(1+2^{2n-6}\right)(2n-4)(2n-3)(2n-2)(2n-1)2n}{(2n)!}x^{2n} \\
&\triangleq \sum_{n=3}^{\infty}b_n x^{2n}.
\end{aligned}
$$

Let

$$
c_n = \frac{a_n}{b_n} = \frac{2n\left(2^{2n-2}-4n+3\right)+4}{\left(1+2^{2n-6}\right)(2n-4)(2n-3)(2n-2)(2n-1)2n}
$$

satisfies

$$
c_3 = \frac{23}{720} = 0.031\ldots, \qquad c_4 = \frac{17}{336} = 0.01226\ldots.
$$

Furthermore, when $n \geq 4$, by a simple computation, we have

$$
c_{n+1} - c_n = -4\frac{f_1(n)+f_2(n)+f_3(n)}{n(16+4^n)(64+4^n)(n-2)(2n-3)\left(4n^2-1\right)(n^2-1)},
$$

where

$$
\begin{aligned}
f_1(n) &= 16^n\left(8n^2+2n-6\right) \\
f_2(n) &= 4^n\left(-24n^4-138n^3+391n^2+153n-382\right) \\
f_3(n) &= -1536n^3-256n^2+2944n-256
\end{aligned}
$$

Since $n \geq 4$, one can easily check that $4^n \geq 16n^2$, this implies that

$$
\begin{aligned}
f_1(n)+f_2(n) &> 4^n 16n^2(8n^2+2n-6) + 4^n\left(-24n^4-138n^3+391n^2+153n-382\right) \\
&= 4^n(104n^4-106n^3+295n^2+153n-382)
\end{aligned}
$$

By a simple computation, one has

$$\begin{aligned}&104n^4 - 106n^3 + 295n^2 + 153n - 382\\&= 104n(n-4)^3 + 1142n(n-4)^2 + 4439n(n-4) + 6293(n-4) + 24790 > 0.\end{aligned}$$

On the other hand, when $n \geq 4$, one has $4^n > 16$, Hence

$$\begin{aligned}&f_1(n) + f_2(n) + f_3(n)\\&> 4^n(104n^4 - 106n^3 + 295n^2 + 153n - 382) - 1536n^3 - 256n^2 + 2944n - 256\\&> 16(104n^4 - 106n^3 + 295n^2 + 153n - 382) - 1536n^3 - 256n^2 + 2944n - 256\\&= 1664n^4 - 3232n^3 + 4464n^2 + 5392n - 6368\\&= 1664n(n-4)^3 + 16736n(n-4)^2 + 58480n(n-4) + 78032(n-4) + 305760 > 0.\end{aligned}$$

This means that the sequence c_n is decreasing. By Lemma 2.8, the function $H(x)$ is decreasing on $(0, \infty)$. Moreover, it is not difficult to obtain $\lim_{x\to 0^+} H(x) = c_3 = \frac{23}{720}$. □

3.6. Remark. Since $H(x)$ is an even function, we conclude that Theorem 3.5 holds for all $x \neq 0$.

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On some generalized soft mappings

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Abstract

In the present paper, we introduce and explore new form of continuity called soft pu-semi-continuity via soft semi-open set in soft topological spaces. Moreover we introduce the concepts of soft-pu-semi-open and soft pu-semi-closed functions and discuss many of their characterizations and properties. It is interesting to mention that the soft functions define and discuss here are the generalization of soft functions explored in [7][21].

Keywords: Soft topology, Soft semi-open(closed) sets, Soft semi-interior(closure), Soft semi-boundary, Soft pu-semi-continuous function, Soft pu-semi-open(closed) functions.

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1. Introduction

Researchers working in the fields of science including engineering physics, computer sciences, economics, social sciences and medical sciences usually deals with modelling of problems having uncertainty present in data and not clear objects. The difficulty arose, because of failure of classical methods to solve the problems having uncertainties and not enough information. Researches are going on for the development of new theories and ideas day by day and a lot of material is available in the literature.

In [15], Molodtsov introduced soft sets theory as a new general mathematical approach to deal with uncertain data and not clear objects. In soft systems, a very general frame work has been provided with the involvement of parameters. In [16], they applied successfully this approach for modelling the problems having uncertainties. In [13-14], Maji et. al explored the basics of soft set theory and presented its applications in decision making problems. Xiao et. al [20] and Pei et. al [18] discussed the relationship among soft sets and information systems. Using approach of soft sets, Kostek[11] introduced the criteria to measure sound quality. In [17], Mushrif et. al used the notions of soft set

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theory to develop the remarkable method for the classification of natural textures. Many researchers worked on the algebraic structures of soft set theory.
Shabir and Naz [19] explored and discussed the basics of soft topological spaces. After that Hussain [6-7], Hussain and Ahmad [8-9] [1], Aygunoglu et.al [2], Zorlutana et. al [21] continued studying the properties and introduced many interesting concepts in soft topological spaces. Bin Chen [3-4] presented and discussed soft semi-open sets and soft-semi-closed sets in soft topological spaces. S. Hussain [5] continued to add many notions and concepts toward soft semi-open sets and soft semi-closed sets in soft topological spaces.
Kharral and Ahmad[10] and then Zorlutana [21] discussed the mappings of soft classes and their properties in soft topological spaces. Recently, in 2015, S. Hussain[7], established fundamental and important characterizations of soft pu-continuous functions, soft pu-open functions and soft pu-closed functions via soft interior, soft closure, soft boundary and soft derived set.
In the present paper, we introduce and explore new form of continuity called soft pu-semi-continuity via soft semi-open set in soft topological spaces. Moreover we introduce the concepts of soft-pu-semi-open and soft pu-semi-closed functions and discuss many of their characterizations and properties. It is interesting to mention that the soft functions define and discuss here are the generalization of soft functions introduced in [7] [21].

2. Preliminaries

First we recall some definitions and results, which will use in the sequel.

2.1. Definition. [15] Let X be an initial universe and E be a set of parameters. Let $P(X)$ denotes the power set of X and A be a non-empty subset of E. A pair (F, A) is called a soft set over X , where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X . For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). Clearly, a soft set is not a set.

Here we consider only soft sets (F, A) over a universe X in which all the parameters of set A are same. We denote the family of these soft sets by $SS(X)_A$. For soft subsets, soft union , soft intersection, soft complement and their properties and relations to each other; the interested reader is refer to [13-16].

2.2. Definition. [21] The soft set $(F, A)\tilde{\in}SS(X)_A$ is called a soft point in X_A, denoted by e_F, if for the element $e \in A$, $F(e) \neq \phi$ and $F(e^c) = \phi$, for all $e^c \in A - \{e\}$.

2.3. Definition. [21] The soft point e_F is said to be in the soft set (G, A), denoted by $e_F\tilde{\in}(G, A)$, if for the element $e \in A$, $F(e) \subseteq G(e)$.

2.4. Definition. [21] A soft set (F, A) over X is said to be a null soft set, denoted by Φ_A , if for all $e \in A$, $F(e) = \phi$.

2.5. Definition. [21] A soft set (F, A) over X is said to be an absolute soft set, denoted by $\tilde{X}_A$, if for all $e \in A$, $F(e) = X$. Clearly, $\tilde{X}_A^c = \Phi_A$ and $\Phi_A^c = \tilde{X}_A$.

2.6. Definition. [19] Let τ be the collection of soft sets over X, then τ is said to be a soft topology on X, if
(1) Φ, $\tilde{X}$ belong to τ.
(2) the union of any number of soft sets in τ belongs to τ.
(3) the intersection of any two soft sets in τ belongs to τ.
The triplet (X, τ, E) is called a soft topological space over X.

2.7. Definition. [19][8] Let (X, τ, E) be a soft topological space over X and $A \subseteq E$ then
(1) soft interior of soft set (F, A) over X is denoted by $(F, A)^\circ$ and is defined as the union of all soft open sets contained in (F, A). Thus $(F, A)^\circ$ is the largest soft open set contained in (F, A). A soft set (F, A) over X is said to be a soft closed set in X, if its relative complement $(F, A)^c$ belongs to τ.
(2) soft closure of (F, A), denoted by $\overline{(F, A)}$ is the intersection of all soft closed super sets of (F, A). Clearly $\overline{(F, A)}$ is the smallest soft closed set over X which contains (F, A).
(3) soft boundary of soft set (F, A) over X is denoted by $\underline{(F, A)}$ and is defined as $\underline{(F, A)} = \overline{(F, A)} \cap \overline{((F, A)')}$. Obviously $\underline{(F, A)}$ is a smallest soft closed set over X containing (F, A).

For detailed properties of soft interior, soft closure and soft boundary, we refer to [8].

2.8. Definition. [3] Let (X, τ, E) be a soft topological space over X with $A \subseteq E$ and (F, A) be a soft set over X. Then (F, A) is called soft semi-open set if and only if there exists a soft open set (G, A) such that $(G, A) \tilde{\subseteq} (F, A) \tilde{\subseteq} \overline{(G, A)}$. The set of all soft semi-open sets is denoted by $S.S.O(X)$. Note that every soft open set is soft semi-open set. A sot set (F, A) is said to be soft semi-closed if its soft relative complement is soft semi-open . Equivalently there exists a soft closed set (G, A) such that $(G, A)^\circ \tilde{\subseteq} (F, A) \tilde{\subseteq} (G, A)$. Note that every soft closed set is soft semi-closed set.

2.9. Definition. [5] Let (X, τ, E) be a soft topological space over X and $A \subseteq E$. Then
(i) soft semi-interior of soft set (F, A) over X is denoted by $sint^s(F, A)$ and is defined as the union of all soft semi-open sets contained in (F, A).
(ii) soft semi-closure of (F, A) over X is denoted by $scl^s(F, A)$ is the intersection of all soft semi-closed super sets of (F, A).
(3) soft semi-exterior of soft set (F, A) over X is denoted by $sext^s(F, A)$ and is defined as $sext^s(F, A) = sint^s((F, A)^c)$.
(4) soft semi-boundary of soft set (F, A) over X is denoted by $sbd^s(F, A)$ and is defined as $sbd^s(F, A) = (sint^s(F, A) \cup sext^s(F, A))^c$.

For detailed properties of soft semi-interior, soft semi-exterior, soft semi-closure and soft semi-boundary, we refer to [5].

2.10. Definition. [10] Let $SS(X)_A$ and $SS(Y)_B$ be families of soft sets. $u : X \to Y$ and $p : A \to B$ be mappings. Then a function $f_{pu} : SS(X)_A \to SS(Y)_B$ defined as :
(1) Let (F, A) be a soft set in $SS(X)_A$. The image of (F, A) under f_{pu} , written as $f_{pu}(F, A) = (f_{pu}(F), p(A))$, is a soft set in $SS(Y)_B$ such that

$$f_{pu}(F)(y) = \begin{cases} \bigcup_{x \in p^{-1}(y) \cap A} u(F(x)), & p^{-1}(y) \cap A \neq \phi \\ \phi, & \text{otherwise} \end{cases},$$

for all $y \in B$.
(2) Let (G, B) be a soft set in $SS(Y)_B$. Then the inverse image of (G, B) under f_{pu} , written as $f_{pu}^{-1}(G, B) = (f_{pu}^{-1}(G), p^{-1}(B))$, is a soft set in $SS(X)_A$ such that

$$f_{pu}^{-1}(G)(x) = \begin{cases} u^{-1}(G(p(x))), & p(x) \in B \\ \phi, & \text{otherwise} \end{cases},$$

for all $x \in A$.

The soft function f_{pu} is called soft surjective, if p and u are surjective. The soft function f_{pu} is called soft injective, if p and u are injective.

For detailed properties of soft functions, we refer to [7][21].

3. Properties of soft pu-semi-continuous functions

3.1. Definition. Let (X, τ, A) and (Y, τ^*, B) be soft topological spaces over X and Y respectively and $u : X \to Y$ and $p : A \to B$ be mappings. Then the soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is soft pu-semi-continuous if and only if for any soft open set (G, B) in $SS(Y)_B$, $f_{pu}^{-1}(G, B)$ is a soft semi-open set in $SS(X)_A$.
Clearly it follows from the definition that the soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is soft pu-semi-continuous if and only if for any soft closed set (G, B) in $SS(Y)_B$, $f_{pu}^{-1}(G, B)$ is a soft semi-closed set in $SS(X)_A$.

3.2. Theorem. *Let (X, τ, A) and (Y, τ^*, B) be soft topological spaces over X and Y respectively and soft point $e_F \tilde{\in} \tilde{X}_A$. Then the soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is soft pu-semi-continuous if and only if for each soft open set (G, B) in $SS(Y)_B$ such that $f_{pu}(e_F) \tilde{\subseteq} (G, B)$, there exists soft semi-open set (F, A) in $SS(X)_A$ such that $e_F \tilde{\in} (F, A)$ and $f_{pu}(F, A) \tilde{\subseteq} (G, B)$.*

Proof. ($\Rightarrow$) Let f_{pu} is soft pu-semi-continuous function. Then for soft open set (G, B) in $SS(Y)_B$, $f_{pu}^{-1}(G, B)$ is soft semi-open in $SS(X)_A$. We show that for each soft open set (G, B) containing $f_{pu}(e_F)$, there exists soft semi-open set (F, A) in $SS(X)_A$ such that $e_F \tilde{\in} (F, A)$ and $f_{pu}(F, A) \tilde{\subseteq} (G, B)$. Let $e_F \tilde{\in} f_{pu}^{-1}(G, B)$, which is soft semi-open and $(F, A) \tilde{=} f_{pu}^{-1}(G, B)$. Then $e_F \tilde{\in} (F, A)$ and for soft open set (G, B),
$f_{pu}(F, A) \tilde{\subseteq} f_{pu}(f_{pu}^{-1}(G, B)) \tilde{\subseteq} (G, B)$, where (G, B) is soft open.
($\Leftarrow$) Suppose that (G, B) be a soft open set in $SS(Y)_B$. We prove that inverse image of soft open set in $SS(Y)_B$ is soft semi-open set in $SS(X)_A$. Let $e_F \tilde{\in} f_{pu}^{-1}(G, B)$. Then $f_{pu}(e_F) \tilde{\in} (G, B)$. Thus there exists soft semi-open set (F, A_{e_F}) such that $e_F \tilde{\in} (F, A_{e_F})$ and $f_{pu}(F, A_{e_F}) \tilde{\subseteq} (G, B)$. Then $e_F \tilde{\in} (F, A_{e_F})$
$\tilde{\subseteq} f_{pu}^{-1}(G, B)$ and $f_{pu}^{-1}(G, B) \tilde{=} \bigcup_{e_F \tilde{\in} f_{pu}^{-1}(G,B)} (F, A_{e_F})$. This follows that $f_{pu}^{-1}(G, B)$ is soft semi-open, by Theorem 3.2[3]. Hence f_{pu} is soft pu-semi-continuous. □

3.3. Theorem. *Suppose $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft function. Then the following statements are equivalent:*
(1) f_{pu} is soft pu-semi-continuous.
(2) For any soft subset (G, B) of $SS(Y)_B$, $sbd^s(f_{pu}^{-1}(G, B)) \tilde{\subseteq} f_{pu}^{-1}(\underline{(G, B)})$.
(3) For any soft subset (F, A) of $SS(X)_A$, $f_{pu}(scl^s(F, A)) \tilde{\subseteq} \overline{(f_{pu}(F, A))}$.

Proof. (1) $\Rightarrow$ (2) Let f_{pu} is soft pu-semi-continuous and (G, B) be a soft set in $SS(Y)_B$.
So, $sbd^s(f_{pu}^{-1}(G, B)) \tilde{=} scl^s(f_{pu}^{-1}(G, B)) \cap scl^s((f_{pu}^{-1}(G, B))^c)$
$\tilde{\subseteq} scl^s(f_{pu}^{-1}(\overline{(G, B)}) \tilde{\cap} scl^s(f_{pu}^{-1}(\overline{((G, B)^c)}) \tilde{=} f_{pu}^{-1}(\overline{(G, B)}) \tilde{\cap} f_{pu}^{-1}(\overline{(G, B)})$
$\tilde{=} f_{pu}^{-1}(\overline{(G, B)} \tilde{\cap} \overline{((G, B)^c)}) \tilde{=} f_{pu}^{-1}(\underline{(G, B)})$. Hence $sbd^s(f_{pu}^{-1}(G, B)) \tilde{\subseteq} f_{pu}^{-1}(\underline{(G, B)})$.
(2) $\Rightarrow$ (1) Suppose that (G, B) be a soft closed set in $SS(Y)_B$. We prove that $f_{pu}^{-1}(G, B)$ is soft semi-closed in $SS(X)_A$. As $sbd^s(f_{pu}^{-1}(G, B)) \tilde{\subseteq} f_{pu}^{-1}(\underline{(G, B)})$
$\tilde{\subseteq} f_{pu}^{-1}(G, B)$. This follows that $sbd^s(f_{pu}^{-1}(G, B)) \tilde{\subseteq} f_{pu}^{-1}(G, B)$. Therefore $f_{pu}^{-1}(G, B)$ is soft semi-closed in $SS(X)_A$. This gives that f_{pu} is soft pu-semi-continuous.
(1) $\Rightarrow$ (3) Suppose (F, A) be any soft set in $SS(X)_A$. As $\overline{(f_{pu}(F, A))}$ is soft closed in $SS(Y)_B$. Therefore, f_{pu} is soft pu-semi-continuous implies that $f_{pu}^{-1}(\overline{(f_{pu}(F, A))})$ is soft semi-closed in $SS(X)_A$ with $(F, A) \tilde{\subseteq} f_{pu}^{-1}(\overline{(f_{pu}(F, A))})$. This implies that
$scl^s(F, A) \tilde{\subseteq} scl^s(f_{pu}^{-1}(\overline{(f_{pu}(F, A))})) \tilde{=} f_{pu}^{-1}(\overline{(f_{pu}(F, A))})$. Which implies that
$f_{pu}(scl^s(F, A)) \tilde{\subseteq} f_{pu}(f_{pu}^{-1}(\overline{(f_{pu}(F, A))}))$. Hence $f_{pu}(scl^s(F, A)) \tilde{\subseteq} \overline{(f_{pu}(F, A))}$.
(3) $\Rightarrow$ (1) Suppose (G, B) be soft closed in $SS(Y)_B$. We prove that $f_{pu}^{-1}(G, B)$ is soft semi-closed. By (3), $f_{pu}(scl^s(f_{pu}^{-1}(G, B))) \tilde{\subseteq} \overline{(f_{pu}(f_{pu}^{-1}(G, B)))}$
$\tilde{\subseteq} \overline{(G, B)} \tilde{=} (G, B)$. This follows that $scl^s(f_{pu}^{-1}(G, B)) \tilde{\subseteq} f_{pu}^{-1} f_{pu}(scl^s(f_{pu}^{-1}(G, B))) \tilde{\subseteq}$

$f_{pu}^{-1}(G,B)$. Consequently $scl^s(f_{pu}^{-1}(G,B))\tilde{\subseteq}f_{pu}^{-1}(G,B)$. This shows that $f_{pu}^{-1}(G,B)$ is soft semi-closed in $SS(X)_A$. Thus f_{pu} is soft pu-semi-continuous. □

3.4. Theorem. *A soft function $f_{pu}: SS(X)_A \to SS(Y)_B$ is soft pu-semi-continuous if and only if $scl^s(f_{pu}^{-1}(G,B))\tilde{\subseteq}f_{pu}^{-1}(\overline{(G,B)})$, for any soft set (G,B) in $SS(Y)_B$.*

Proof. (⇒) Suppose that f_{pu} is soft pu-semi-continuous. Then by above Theorem 3.3, we get

$$f_{pu}(scl^s(F,A))\tilde{\subseteq}\overline{(f_{pu}(F,A))} \quad ...\ (A)$$

Suppose (G,B) be any soft set in $SS(Y)_B$. Take
$(F,A)\tilde{=}f_{pu}^{-1}(G,B)$ then $f_{pu}(scl^s(f_{pu}^{-1}(G,B)))\tilde{\subseteq}\overline{(f_{pu}f_{pu}^{-1}(G,B))}\tilde{\subseteq}\overline{(G,B)}$. This follows that $f_{pu}(scl^s(f_{pu}^{-1}(G,B)))\tilde{\subseteq}\overline{(G,B)}$.
(⇐) Suppose that $f_{pu}(scl^s(f_{pu}^{-1}(G,B)))\tilde{\subseteq}\overline{(G,B)}$, for any soft subset (G,B) in $SS(Y)_B$. Let $(G,B)\tilde{=}f_{pu}(F,A)$, for any soft set (F,A) in $SS(X)_A$. This gives
$scl^s(F,A)\tilde{\subseteq}scl^s(f_{pu}^{-1}(G,B))\tilde{\subseteq}f_{pu}^{-1}(\overline{(f_{pu}(F,A)})$. This follows that $f_{pu}(scl^s(F,A))\tilde{\subseteq}\overline{(f_{pu}(F,A))}$. Hence by above Theorem 3.3, f_{pu} is soft pu-semi-continuous. □

3.5. Lemma. [3][5] *The following properties of soft set (F,A) in $SS(X)_A$ are equivalent:*
(1) (F,A) is soft semi-closed.
(2) $(\overline{(F,A)})^\circ\tilde{\subseteq}(F,A)$.
(3) $(F,A)^c$ is soft semi-open.

3.6. Theorem. *Let $f_{pu}: SS(X)_A \to SS(Y)_B$ be a soft function. Then the following statements are equivalent:*
(1) f_{pu} is soft pu-semi-continuous.
(2) For any soft set (G,B) in $SS(Y)_B$, $\{\overline{(f_{pu}^{-1}(G,B))}\}^\circ\tilde{\subseteq}f_{pu}^{-1}(\overline{(G,B)})$.
(3) For any soft set (F,A) in $SS(X)_A$, $f_{pu}(\{\overline{(F,A)}\}^\circ)\tilde{\subseteq}\overline{(f_{pu}(F,A))}$.

Proof. (1) ⇒ (2) Suppose (G,B) be any soft set in $SS(Y)_B$. Then using soft pu-semi-continuity of f_{pu}, $f_{pu}^{-1}(\overline{(G,B)})$ is a soft semi-closed. Using Lemma 3.5 and $(G,B)\tilde{\subseteq}\overline{(G,B)}$, we have $f_{pu}^{-1}(\overline{(G,B)})\tilde{\supseteq}(\overline{(f_{pu}^{-1}(\overline{(G,B)}))})^\circ\tilde{\supseteq}(\overline{(f_{pu}^{-1}(G,B))})^\circ$. This follows that
$\{\overline{(f_{pu}^{-1}(G,B))}\}^\circ\tilde{\subseteq}f_{pu}^{-1}(\overline{(G,B)})$.
(2) ⇒ (3) Suppose that (F,A) be any soft set in $SS(X)_A$. Take $(G,B)\tilde{=}f_{pu}(F,A)$. Then $(F,A)\tilde{\subseteq}f_{pu}^{-1}(G,B)$. Using our supposition, we get $(\overline{(F,A)})^\circ\tilde{\subseteq}(\overline{(f_{pu}^{-1}(G,B))})^\circ$
$\tilde{\subseteq}f_{pu}^{-1}(\overline{(G,B)})$. This implies that $f_{pu}((\overline{(F,A)})^\circ)\tilde{\subseteq}f_{pu}f_{pu}^{-1}(\overline{(G,B)})\tilde{\subseteq}\overline{(G,B)}$
$\tilde{=}\overline{(f_{pu}(F,A))}$.
(3) ⇒ (1) Suppose that(G,B) be any soft closed set in $SS(Y)_B$. Take $(F,A)\tilde{=}$
$f_{pu}^{-1}(G,B)$, then $f_{pu}(F,A)\tilde{\subseteq}(G,B)$. Using our supposition, we get

$$f_{pu}((\overline{(F,A)})^\circ)\tilde{\subseteq}\overline{(f_{pu}(F,A))}\tilde{\subseteq}\overline{(G,B)}\tilde{=}(G,B)\ .....\ (B)$$

By (B), we have $(\overline{(F,A)})^\circ\tilde{\subseteq}f_{pu}^{-1}f_{pu}((\overline{(F,A)})^\circ)\tilde{\subseteq}f_{pu}^{-1}(\overline{(f_{pu}(F,A))})\tilde{\subseteq}f_{pu}^{-1}(\overline{(G,B)})\tilde{=}$
$f_{pu}^{-1}(G,B)$. This gives $(\overline{(F,A)})^\circ\tilde{\subseteq}f_{pu}^{-1}(G,B)\tilde{=}(F,A)$. Lemma 3.5 implies that $f_{pu}^{-1}(G,B)\tilde{=}(F,A)$ is a soft semi-closed set. Hence f_{pu} is soft pu-semi-continuous. □

3.7. Definition. Let (X,τ,A) be a soft topological spaces over X, (F,A) be a soft set in $SS(X)_A$ and soft point $e_F\tilde{\in}\tilde{X}_A$. Then e_F is called a soft semi-limit point of a soft set (F,A), if $(H,A)\tilde{\cap}((F,A)\tilde{-}\{e_F\})\tilde{\neq}\tilde{\phi}$, for any soft semi-open set (H,A) such that $e_F\tilde{\in}(H,A)$. The set of all soft semi-limit point of (F,A) is called as soft semi-derived set of (F,A) and is denoted by $sd^s(F,A)$.

Note that if $(F,A)\tilde{\subseteq}(H,A)$ then $sd^s(F,A)\tilde{\subseteq}sd^s(H,A)$..... (C)

3.8. Remark. Clearly e_F is a soft semi-limit point of (F,A) if and only if $e_F\tilde{\in}scl^s((F,A)\tilde{-}\{e_F\})$.

In the following theorem, we discuss the properties of soft semi-derived set "sd^s".

3.9. Theorem. *Let (X,τ,A) be a soft topological spaces over X and (F,A) be a soft set in $SS(X)_A$. Then*
(1) $scl^s(F,A)\tilde{=}(F,A)\tilde{\cup}sd^s(F,A)$.
(2) $sd^s((F,A)\tilde{\cup}(H,A))\tilde{=}sd^s(F,A)\tilde{\cup}sd^s(H,A)$.
In general,
(3) $\bigcup_i sd^s(F,A_i)\tilde{=}sd^s(\bigcup_i(F,A_i))$.
(4) $sd^s(sd^s(F,A))\tilde{\subseteq}sd^s(F,A)$.
(5) $scl^s(sd^s(F,A))\tilde{=}sd^s(F,A)$.

Proof. (1) Suppose $e_F\tilde{\in}scl^s(F,A)$. Then for any soft semi-closed set (K,A) such that $(F,A)\tilde{\subseteq}(K,A)$, we have $e_F\tilde{\in}(K,A)$. Now we consider two cases:
Case (i) If $e_F\tilde{\in}(F,A)$, then $e_F\tilde{\in}(F,A)\tilde{\cup}sd^s(F,A)$.
Case (ii) If $e_F\tilde{\notin}(F,A)$, then we prove that $e_F\tilde{\in}scl^s(F,A)$.
For this consider (L,A) is a soft semi-open set such that $e_F\tilde{\in}(L,A)$. Then $(L,A)\tilde{\cap}(F,A)\tilde{\neq}\tilde{\phi}$. If not, then , $(F,A)\tilde{\subseteq}(L,A)^c\tilde{=}(K,A)$, where (K,A) is a soft semi-closed soft superset of (F,A) such that $e_F\tilde{\notin}(F,A)$. Which is contradiction to the fact that e_F soft belongs to every soft semi-closed soft superset (K,A) of (F,A). This follows that $e_F\tilde{\in}sd^s(F,A)$. This implies that $e_F\tilde{\in}(F,A)\tilde{\cup}sd^s(F,A)$.
Conversely, suppose that $e_F\tilde{\in}(F,A)\tilde{\cup}sd^s(F,A)$, we prove that $e_F\tilde{\in}scl^s(F,A)$. If $e_F\tilde{\in}(F,A)$ then $e_F\tilde{\in}scl^s(F,A)$. If $e_F\tilde{\in}sd^s(F,A)$, then we show that e_F is in every soft semi-closed soft superset of (F,A). Contrarily suppose that there is a soft semi-closed soft superset (K,A) of (F,A) such that $e_F\tilde{\notin}(F,A)$. This follows that $e_F\tilde{\in}(K,A)^c\tilde{=}(L,A)$(say), which is soft semi-open and $(L,A)\tilde{\cap}(F,A)\tilde{=}\tilde{\phi}$. This gives $e_F\tilde{\notin}sd^s(F,A)$. This contradiction proves that $e_F\tilde{\in}scl^s(F,A)$. Hence $scl^s(F,A)\tilde{=}(F,A)\tilde{\cup}sd^s(F,A)$. This completes the proof of (1).
(2) First we prove that $sd^s((F,A)\tilde{\cup}(H,A))\tilde{\subseteq}sd^s(F,A)\tilde{\cup}sd^s(H,A)$.
Suppose $e_F\tilde{\in}sd^s((F,A)\tilde{\cup}(H,A))$. Then $e_F\tilde{\in}scl^s(((F,A)\tilde{\cup}(H,A))\tilde{-}\{e_F\})$
or $e_F\tilde{\in}scl^s(((F,A)\tilde{-}\{e_F\})\tilde{\cup}((H,A)\tilde{-}\{e_F\})$ implies $e_F\tilde{\in}scl^s((F,A)\tilde{-}\{e_F\})$
or $e_F\tilde{\in}scl^s((H,A)\tilde{-}\{e_F\})$. This gives $e_F\tilde{\in}sd^s(F,A)$ or $e_F\tilde{\in}sd^s(H,A)$. Therefore $e_F\tilde{\in}sd^s(F,A)\tilde{\cup}sd^s(H,A)$. This proves $sd^s((F,A)\tilde{\cup}(H,A))\tilde{\subseteq}sd^s(F,A)\tilde{\cup}sd^s(H,A)$. The reverse inclusion follows form property(C).
(3) This directly follows from property (C).
(4) Suppose that $e_F\tilde{\notin}sd^s(F,A)$. Then $e_F\tilde{\notin}scl^s((F,A)\tilde{-}\{e_F\})$. This follows that there is a soft semi-open set (L,A) such that $e_F\tilde{\in}(L,A)$ with $(L,A)\tilde{\cap}((F,A)\tilde{-}\{e_F\})$ $\tilde{=}\tilde{\phi}$. We show that $e_F\tilde{\notin}sd^s(sd^s(F,A))$. Contrarily suppose that $e_F\tilde{\in}sd^s(sd^s(F,A))$. This implies that $e_F\tilde{\in}scl^s(sd^s(F,A)\tilde{-}\{e_F\})$. $e_F\tilde{\in}(L,A)$ follows that
$(L,A)\tilde{\cap}(sd^s(F,A)\tilde{-}\{e_F\})\tilde{\neq}\tilde{\phi}$. Thus there exists a $q_F\tilde{\neq}e_F$ such that $q_F\tilde{\in}(L,A)$ $\tilde{\cap}(sd^s(F,A))$. This implies that $q_F\tilde{\in}((L,A)\tilde{-}\{e_F\})\tilde{\cap}(sd^s(F,A)\tilde{-}\{e_F\})$. Therefore
$((L,A)\tilde{-}\{e_F\})\tilde{\cap}(sd^s(F,A)\tilde{-}\{e_F\})\tilde{\neq}\tilde{\phi}$. This is contradiction to the fact that
$((L,A)\tilde{\cap}(sd^s(F,A)\tilde{-}\{e_F\})\tilde{=}\tilde{\phi}$. This follows that $e_F\tilde{\notin}sd^s(sd^s(F,A))$. Hence
$sd^s(sd^s(F,A))\tilde{\subseteq}sd^s(F,A)$. This proves (4).
(5) The proof follows form (1), (2) and (4). □

3.10. Theorem. *Let $f_{pu}:SS(X)_A\to SS(Y)_B$ be a soft function. Then the following statements are equivalent:*

(1) f_{pu} is soft pu-semi-continuous.
(2) For any soft set (F,A) in $SS(X)_A$, $f_{pu}(sd^s(F,A))\tilde{\subseteq}\overline{(f_{pu}(F,A))}$.

Proof. (1) $\Rightarrow$ (2) Let f_{pu} be soft pu-semi-continuous and (F,A) be any soft set in $SS(X)_A$. $\overline{(f_{pu}(F,A))}$ is soft closed implies that $f_{pu}^{-1}(\overline{f_{pu}(F,A)})$ is soft semi-closed in $SS(X)_A$. $(F,A)\tilde{\subseteq}f_{pu}^{-1}(f_{pu}(F,A))\tilde{\subseteq}f_{pu}^{-1}(\overline{(f_{pu}(F,A))})$. This follows that $scl^s(F,A)\tilde{\subseteq}scl^s(f_{pu}^{-1}(\overline{(f_{pu}(F,A))}))\tilde{=}f_{pu}^{-1}(\overline{(f_{pu}(F,A))})$. This implies that $f_{pu}(sd^s(F,A))\tilde{\subseteq}f_{pu}(scl^s(F,A))\tilde{\subseteq}f_{pu}f_{pu}^{-1}(\overline{(f_{pu}(F,A))})\tilde{\subseteq}\overline{(f_{pu}(F,A))}$. Therefore, $f_{pu}(sd^s(F,A))\tilde{\subseteq}\overline{(f_{pu}(F,A))}$.
(2) $\Rightarrow$ (1) Let $f_{pu}(sd^s(F,A))\tilde{\subseteq}\overline{(f_{pu}(F,A))}$, for any soft set (F,A) in $SS(X)_A$. Suppose that (G,B) be any soft closed subset in $SS(Y)_B$. We prove that $f_{pu}^{-1}(G,B)$ is soft semi-closed. By our supposition, $f_{pu}(sd^s(f_{pu}^{-1}(G,B)))\tilde{\subseteq}\overline{(f_{pu}(f_{pu}^{-1}(G,B)))}$ $\tilde{\subseteq}\overline{(G,B)}\tilde{=}(G,B)$. This follows that $f_{pu}(sd^s(f_{pu}^{-1}(G,B)))\tilde{\subseteq}(G,B)$. This implies that $sd^s(f_{pu}^{-1}(G,B))\tilde{\subseteq}f_{pu}^{-1}(G,B)$. This follows that $f_{pu}^{-1}(G,B)$ is soft semi-closed. Hence f_{pu} is soft pu-semi-continuous. □

3.11. Theorem. *Let $f_{pu}:SS(X)_A\to SS(Y)_B$ be a soft function. Then f_{pu} is soft pu-semi-continuous if and only if for any soft set (G,B) in $SS(Y)_B$, $f_{pu}^{-1}((G,B)^\circ)\tilde{\subseteq}sint^s$ $(f_{pu}^{-1}(G,B))$.*

Proof. ($\Rightarrow$) Since for any soft set $G,B)$ in $SS(Y)_B$, $(G,B)^\circ\tilde{=}(\overline{(G,B)^c})^c$ [5]. This follows that $f_{pu}^{-1}((G,B)^\circ)\tilde{=}f_{pu}^{-1}((\overline{(G,B)^c})^c)\tilde{=}(f_{pu}^{-1}(\overline{(G,B)^c}))^c$. Since f_{pu} is soft pu-semi-continuous, by Theorem 3.4, we get $scl^s(f_{pu}^{-1}((G,B)^c)\tilde{\subseteq}$ $f_{pu}^{-1}(\overline{(G,B)^c})$. Therefore $f_{pu}^{-1}((G,B)^\circ)\tilde{\subseteq}scl^s((f_{pu}^{-1}((G,B)^c))^c)$. This implies that $f_{pu}^{-1}((G,B)^\circ)\tilde{\subseteq}(scl^s((f_{pu}^{-1}(G,B))^c))^c$. Thus $f_{pu}^{-1}((G,B)^\circ)\tilde{\subseteq}X-(scl^s(f_{pu}^{-1}(G,B)))^c\tilde{=}$ $sint^s(f_{pu}^{-1}(G,B))$.
($\Leftarrow$) Suppose that (G,B) be any soft open set in $SS(Y)_B$. Then $(G,B)^\circ\tilde{=}(G,B)$. Using our supposition, we get $f_{pu}^{-1}(G,B)\tilde{=}f_{pu}^{-1}((G,B)^\circ)\tilde{\subseteq}sint^s(f_{pu}^{-1}(G,B))$. This follows that $f_{pu}^{-1}(G,B)\tilde{\subseteq}sint^s(f_{pu}^{-1}(G,B))$. But $sint^s(f_{pu}^{-1}(G,B))\tilde{\subseteq}f_{pu}^{-1}(G,B)$. Therefore, $f_{pu}^{-1}(G,B)\tilde{=}sint^s(f_{pu}^{-1}(G,B))$. This shows that $f_{pu}^{-1}(G,B)$ is soft semi-open. Hence f_{pu} is soft pu-semi-continuous. □

4. Properties of soft pu-semi-open functions

4.1. Definition. A soft set (F,A) in $SS(X)_A$ is said to be a soft semi-nbd of a soft point $e_F\tilde{\in}\tilde{X}_A$, if there exists a soft semi-open set (H,A) such that $e_F\tilde{\in}(H,A)\tilde{\subseteq}(F,A)$.

4.2. Definition. Let (X,τ,A) and (Y,τ^*,B) be soft topological spaces over X and Y respectively and $u:X\to Y$ and $p:A\to B$ be mappings. Then the soft function $f_{pu}:SS(X)_A\to SS(Y)_B$ is soft pu-semi-open if and only if for any soft open set (F,A) in $SS(X)_A$, $f_{pu}(F,A)$ is soft semi-open in $SS(Y)_B$.

4.3. Lemma. *If (F,A) is soft semi-open and (H,A) be any soft set such that $(F,A)\tilde{\subseteq}(H,A)$. Then $(F,A)\tilde{\subseteq}\overline{(H,A)^\circ}$.*

Proof. (F,A) is soft semi-open implies that $(F,A)\tilde{\subseteq}\overline{(F,A)^\circ}$[5]. Moreover, $(F,A)\tilde{\subseteq}(H,A)$ implies that $(F,A)^\circ\tilde{\subseteq}(H,A)^\circ$. Thus $\overline{(F,A)^\circ}\tilde{\subseteq}\overline{(H,A)^\circ}$ follows that $(F,A)\tilde{\subseteq}\overline{(H,A)^\circ}$. □

4.4. Theorem. *A soft function $f_{pu}:SS(X)_A\to SS(Y)_B$ is soft pu-semi-open if and only if for any soft subset (F,A) in $SS(X)_A$, $f_{pu}((F,A)^\circ)\tilde{\subseteq}\overline{(f_{pu}(F,A))^\circ}$.*

Proof. ($\Rightarrow$) Suppose that f_{pu} be soft pu-semi-open. Then $f_{pu}((F,A)^\circ)\tilde{\subseteq} f_{pu}(F,A))$ implies that $f_{pu}((F,A)^\circ)$ is soft semi-open. Thus by Lemma 4.3,
$f_{pu}((F,A)^\circ)\tilde{\subseteq}\overline{(f_{pu}(F,A))^\circ}$.
($\Leftarrow$) Suppose that (H,A) be any soft open set in $SS(X)_A$. Then
$(f_{pu}(H,A))^\circ\tilde{\subseteq} f_{pu}(H,A))\tilde{\subseteq} f_{pu}((H,A)^\circ)\tilde{\subseteq}\overline{(f_{pu}(H,A))^\circ}$. So $f_{pu}(H,A))$ is soft semi-open. Which implies that f_{pu} is soft pu-semi-open. Hence the proof. □

The following theorem can be proved in similar fashion.

4.5. Theorem. *A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is soft pu-semi-open if and only if for any soft subset (G,B) in $SS(Y)_B$, $(f_{pu}^{-1}(G,B))^\circ\tilde{\subseteq}\overline{f_{pu}^{-1}((G,B)^\circ)}$.*

4.6. Theorem. *If a soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ be soft pu-semi-continuous and soft pu-semi-open and (F,A) be soft semi-open set in $SS(X)_A$. Then $f_{pu}(F,A)$ is soft semi-open in $SS(Y)_B$.*

Proof. Since (F,A) is soft semi-open, then there exists soft open set (H,A) in $SS(X)_A$ such that $(H,A)\tilde{\subseteq}(F,A)\tilde{\subseteq}\overline{(H,A)}$. This implies that $f_{pu}(H,A)\tilde{\subseteq} f_{pu}(F,A)$
$\tilde{\subseteq} f_{pu}(\overline{(H,A)})\tilde{\subseteq}\overline{(f_{pu}(H,A))}$. Thus $f_{pu}(F,A)$ is soft semi-open in $SS(Y)_B$. This proves as required. □

The proof of the following lemma and proposition is easy and thus omitted.

4.7. Lemma. *Let (F,A) be any soft set and (H,A) be soft semi-closed set in $SS(X)_A$ such that $(F,A)\tilde{\subseteq}(H,A)$, then $sbd^s(F,A)\tilde{\subseteq}(H,A)$.*

4.8. Proposition. *If $(F,A)\tilde{\cap}(H,A)\tilde{=}\tilde{\phi}$ and (F,A) is soft open, then $(F,A)\tilde{\cap}\overline{(H,A)}\tilde{=}\tilde{\phi}$.*

The following lemma directly follows form Proposition 4.8.

4.9. Lemma. *If $(F,A)\tilde{\cap}(H,A)\tilde{=}\tilde{\phi}$ and (F,A) is soft open in $SS(X)_A$, then $(F,A)\tilde{\cap}\underline{(H,A)}\tilde{=}\tilde{\phi}$.*

4.10. Theorem. *If $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft bijective, soft function and (G,B) be any soft subset in $SS(Y)_B$. Then f_{pu} is soft pu-semi-open if and only if $f_{pu}^{-1}(sbd^s(G,B)\tilde{\subseteq}\underline{(f_{pu}^{-1}(G,B))}$.*

Proof. ($\Rightarrow$) Let soft function f_{pu} be soft pu-semi-open and (G,B) be soft open set in $SS(Y)_B$. Take

$$(F,A)\tilde{=}(\underline{f_{pu}^{-1}(G,B)})^c \qquad \ldots \text{(D)}$$

This follows that (F,A) is soft open. Therefore $f_{pu}(F,A)$ is soft semi-open in $SS(Y)_B$. This follows that $(f_{pu}(F,A))^c$ is soft semi-closed in $SS(Y)_B$. Therefore by equation (D) and soft bijectivity of soft function f_{pu}, we have $(G,B)\tilde{\subseteq}(f_{pu}(F,A))^c$. Using Lemma 4.7, we get $f_{pu}^{-1}(sbd^s(G,B))\tilde{\subseteq}$
$f_{pu}^{-1}((f_{pu}(F,A))^c)\tilde{\subseteq}(F,A)^c \tilde{=}(((\underline{f_{pu}^{-1}(G,B))})^c)^c\tilde{=}\underline{(f_{pu}^{-1}(G,B))}$. This gives
$f_{pu}^{-1}(sbd^s(G,B))\tilde{\subseteq}\underline{(f_{pu}^{-1}(G,B))}$.
($\Leftarrow$) Let (H,A) be soft open in $SS(X)_A$. Take $(G,B)\tilde{=}(f_{pu}(H,A))^c$. Clearly
$(G,B)\tilde{\cap} f_{pu}(H,A)\tilde{=}\tilde{\phi}$, follows $(H,A)\tilde{\cap} f_{pu}^{-1}(G,B)\tilde{=}\tilde{\phi}$. Lemma 4.9 implies that
$(H,A)\tilde{\cap}\underline{(f_{pu}^{-1}(G,B))}\tilde{=}\tilde{\phi}$. Therefore $f_{pu}^{-1}(sd^s(G,B))\tilde{\subseteq}\underline{(f_{pu}^{-1}(G,B))}$ implies $(H,A)\tilde{\cap}$
$f_{pu}^{-1}(sbd^s(G,B))\tilde{=}\tilde{\phi}$. Thus, $\tilde{\phi}\tilde{=} f_{pu}((H,A)\tilde{\cap} f_{pu}^{-1}(sbd^s(G,B)))\tilde{=} f_{pu}(H,A)\tilde{\cap} sbd^s(G,B)$ gives $sbd^s(G,B)\tilde{\subseteq}(f_{pu}(H,A))^c\tilde{=}(G,B)$. Therefore (G,B) is soft semi-closed. Hence $f_{pu}(H,A)$ is soft semi-open. Hence the proof. □

4.11. Theorem. *Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft function. Then the following statements are equivalent:*
(1) f_{pu} is soft pu-semi-open.
(2) For any soft set (F, A) in $SS(X)_A$, $f_{pu}((F, A)^{\circ})\tilde{\subseteq} sint^s(f_{pu}((F, A))$.
(3) For each $e_F \tilde{\in} \tilde{X}_A$ and each soft open-nbd (U, A) of soft point e_F, there exists a soft semi-open-nbd (V, A) of $f_{pu}(e_F)$ such that $(V, A)\tilde{\subseteq} f_{pu}(U, A)$.

Proof. (1) $\Rightarrow$ (2) Let f_{pu} be soft pu-semi-open and (F, A) be any soft set in $SS(X)_A$. Then $f_{pu}((F, A)^{\circ})$ is soft semi-open and $f_{pu}((F, A)^{\circ})\tilde{\subseteq} f_{pu}(F, A)$ implies that
$f_{pu}((F, A)^{\circ})\tilde{\subseteq} sint^s(f_{pu}((F, A))$.
(2) $\Rightarrow$ (3) Suppose that (U, A) be any soft open-nbd of soft point $e_F \tilde{\in} \tilde{X}_A$. Then there exists a soft open set (O, A) such that $e_F \tilde{\in} (O, A)\tilde{\subseteq}(U, A)$. Using our supposition, we get $f_{pu}(O, A)\tilde{=} f_{pu}((O, A)^{\circ})\tilde{\subseteq} sint^s(f_{pu}((O, A))$. This follows that
$f_{pu}(O, A)\tilde{\subseteq} sint^s(f_{pu}((O, A))$. Consequently, $f_{pu}(O, A)$ is soft semi-open-nbd in $SS(Y)_B$ such that $f_{pu}(e_F)\tilde{\in} f_{pu}(O, A)\tilde{\subseteq} f_{pu}(U, A)$.
(3) $\Rightarrow$ (1) Suppose that (U, A) be a soft open set in $SS(X)_A$. For any $q_F \tilde{\in} f_{pu}(U, A)$, by (3), there exists a soft semi-open-nbd (V_{q_F}, A) of $q_F \tilde{\in} \tilde{Y}_B$ such that $(V_{q_F}, A)\tilde{\subseteq}$
$f_{pu}(U, A)$. Since (V_{q_F}, A) is a soft semi-open-nbd of q_F. Then there exists a soft semi-open set (H_{q_F}, A) in $SS(Y)_B$ such that $q_F \tilde{\in} (H_{q_F}, A)\tilde{\subseteq}(V_{q_F}, A)$. This implies that $f_{pu}(U, A)\tilde{=}\tilde{\bigcup}\{(H_{q_F}, A) : q_F \tilde{\in} f_{pu}(U, A)\}$ is a soft semi-open in $SS(Y)_B$ [3]. Consequently, f_{pu} is a soft pu-semi-open function. □

4.12. Lemma. *Let (F, A) and (G, A) be soft sets in $SS(X)_A$. Then*
(1) $\overline{((F, A)\tilde{-}(H, A))}\tilde{\supseteq}\overline{(F, A)}\tilde{-}\overline{(H, A)}$.
(2) $((F, A)\tilde{-}(H, A))^{\circ}\tilde{\subseteq}(F, A)^{\circ}\tilde{-}(H, A)^{\circ}$.
(3) If (F, A) is soft open, then $(F, A)\tilde{\cap}\overline{(H, A)}\tilde{\subseteq}\overline{((F, A)\tilde{\cap}(H, A))}$.

Proof. (1). Suppose that $e_F \tilde{\in} \overline{(F, A)}\tilde{-}\overline{(H, A)}$. Then $e_F \tilde{\in} \overline{(F, A)}$ and $e_F \tilde{\notin} \overline{(H, A)}$. Thus there exists a soft open nbd (K, A) of e_F such that $(K, A)\tilde{\cap}(F, A)\tilde{\neq}\tilde{\phi}$ and $(K, A)\tilde{\cap}(H, A)\tilde{=}\tilde{\phi}$. This follows that $(K, A)\tilde{\cap}((F, A)\tilde{-}(H, A))\tilde{\neq}\tilde{\phi}$. Thus
$e_F \tilde{\in} \overline{((F, A)\tilde{-}(H, A))}$.
(2) This follows directly by (1) and using Demorgan's law.
(3) Given that (F, A) is soft open. Thus $(F, A)\tilde{=}(F, A)^{\circ}$. Thus $(F, A)\tilde{\cap}\overline{(H, A)}\tilde{=}$
$\overline{(H, A)}\tilde{\cap}(F, A)^{\circ}\tilde{=}\overline{(H, A)}\tilde{-}((F, A)^{\circ})^c\tilde{=}\overline{(H, A)}\tilde{-}\overline{((F, A)^c)}\tilde{\subseteq}\overline{((H, A)\tilde{-}(F, A)^c)}$
$\tilde{=}\overline{((H, A)\tilde{\cap}(F, A))}\tilde{=}\overline{((F, A)\tilde{\cap}(H, A))}$. Consequently, $(F, A)\tilde{\cap}\overline{(H, A)}\tilde{\subseteq}\overline{((F, A)\tilde{\cap}(H, A))}$. □

4.13. Theorem. *Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft bijective, soft function. Then f_{pu} is soft pu-semi-open if and only if for any soft set (G, B) in $SS(Y)_B$,*
$f^{-1}_{pu}(scl^s(G, B))\tilde{\subseteq}\overline{(f^{-1}_{pu}(G, B))}$.

Proof. ($\Rightarrow$) Suppose that (G, B) be any soft set in $SS(Y)_B$. Take

$$(H, A)\tilde{=}(\overline{(f^{-1}_{pu}(G, B))})^c \text{ ...... (E)}$$

This is clear that (H, A) is a soft open set in $SS(X)_A$. Then by our supposition, $f_{pu}(H, A)$ is a soft semi-open set in $SS(Y)_B$, or $(f_{pu}(H, A))^c$ is soft semi-closed set in $SS(Y)_B$. As f_{pu} is soft onto, from (E), it gives $(G, B)\tilde{\subseteq}(f_{pu}(H, A))^c$. Therefore, we get $scl^s(G, B)\tilde{\subseteq}(f_{pu}(H, A))^c$. f_{pu} is soft one-one, implies that
$f^{-1}_{pu}(scl^s(G, B))\tilde{\subseteq}(f^{-1}_{pu}((f_{pu}(H, A))))^c\tilde{=}(f^{-1}_{pu} f_{pu}(H, A))^c\tilde{\subseteq}(H, A)^c\tilde{=}\overline{(f^{-1}_{pu}(G, B))}$.
($\Leftarrow$) Suppose that (H, A) be any soft open set in $SS(X)_A$. Take $(G, B)\tilde{=}(f_{pu}(H, A))^c$. Since f_{pu} is soft bijective, then by our supposition, $f_{pu}(H, A)\tilde{\cap} scl^s(G, B)\tilde{=}$

$f_{pu}((H,A)\tilde{\cap} f_{pu}^{-1}(scl^s(G,B)))\tilde{\subseteq} f_{pu}((H,A)\tilde{\cap}\overline{(f_{pu}^{-1}(G,B))})$. Since (H,A) is soft open, therefore above Lemma 4.12(3) implies that $(H,A)\tilde{\cap}\overline{(f_{pu}^{-1}(G,B))}$ $\tilde{\subseteq}\overline{((H,A)\tilde{\cap} f_{pu}^{-1}(G,B))}$. Furthermore, it is obvious that $(H,A)\tilde{\cap} f_{pu}^{-1}(G,B)\tilde{=}\tilde{\phi}$. This implies that $f_{pu}(H,A)\tilde{\cap}$ $scl^s(G,B)\tilde{=}\tilde{\phi}$ and hence $scl^s(G,B)\tilde{\subseteq}(f_{pu}(H,A))^c\tilde{=}(G,B)$. This follows that (G,B) is a soft semi-closed and hence $f_{pu}(H,A)$ is a soft semi-open set in $SS(Y)_B$. This shows that f_{pu} is a soft pu-semi-open. □

4.14. Theorem. *Let $f_{pu}: SS(X)_A \to SS(Y)_B$ be soft bijective, soft function. Then f_{pu} is soft pu-semi-open if and only if for any soft subset (V,B) in $SS(Y)_B$ and for any soft closed set (F,A) in $SS(X)_A$ such that $f_{pu}^{-1}(V,B)\tilde{\subseteq}(F,A)$, there exists a soft semi-closed set (G,B) in $SS(Y)_B$ with $(V,B)\tilde{\subseteq}(G,B)$ such that $f_{pu}^{-1}(G,B)\tilde{\subseteq}(F,A)$.*

Proof. (⇒) Suppose that (V,B) be soft set in $SS(Y)_B$ and (F,A) be any soft closed set in $SS(X)_A$ such that $f_{pu}^{-1}(V,B)\tilde{\subseteq}(F,A)$. Take $(G,B)\tilde{=}(f_{pu}((F,A)^c))^c$. f_{pu} is soft pu-semi-open implies that (G,B) is soft semi-closed sets in $SS(Y)_B$. Since f_{pu} is bijective, it follows from $f_{pu}^{-1}(V,B)\tilde{\subseteq}(F,A)$ that $(V,B)\tilde{\subseteq}(G,B)$. By simple calculations, we have $f_{pu}^{-1}(G,B)\tilde{\subseteq}(F,A)$.
(⇐) Let (U,A) be soft open set in $SS(X)_A$. Take $(V,B)\tilde{=}(f_{pu}(U,A))^c$. Then $(U,A)^c$ is a soft closed set such that $f_{pu}^{-1}(V,B)\tilde{\subseteq}(U,A)^c$. By hypothesis, there exists a soft semi-closed set (G,B) in $SS(Y)_B$ such that $(V,B)\tilde{\subseteq}(G,B)$ and $f_{pu}^{-1}(G,B)\tilde{\subseteq}$ $(U,A)^c$. On the other hand, it follows from $(V,B)\tilde{\subseteq}(G,B)$ that $f_{pu}(U,A)\tilde{\subseteq}(G,B)^c$. Hence we get $f_{pu}(U,A)\tilde{=}(G,B)^c$, which is soft semi-open. This follows that soft function f_{pu} is soft pu-semi-open. □

5. Properties of soft pu-semi-closed functions

5.1. Definition. Let (X,τ,A) and (Y,τ^*,B) be soft topological spaces over X and Y respectively and $u: X\to Y$ and $p: A\to B$ are mappings. Then the soft function $f_{pu}: SS(X)_A \to SS(Y)_B$ is soft pu-semi-closed if and only if for any soft closed set (F,A) in $SS(X)_A$, $f_{pu}(F,A)$ is soft semi-closed in $SS(Y)_B$.

5.2. Theorem. *Let $f_{pu}: SS(X)_A \to SS(Y)_B$ be a soft function and (F,A) be soft set in $SS(X)_A$. Then f_{pu} is soft pu-semi-closed if and only if $f_{pu}(\overline{(F,A)})\tilde{\supseteq}\{\overline{(f_{pu}(F,A))}\}^\circ$.*

Proof. (⇒) Suppose that f_{pu} is a soft pu-semi-closed function and (F,A) be soft set in $SS(X)_A$. Then $f_{pu}(\overline{(F,A)})$ is soft semi-closed in $SS(Y)_B$. Then by Lemma 3.5, we get $f_{pu}(\overline{(F,A)})\tilde{\supseteq}\{\overline{f_{pu}(\overline{(F,A)})}\}^\circ\tilde{\supseteq}\{\overline{(f_{pu}(F,A))}\}^\circ$. This follows that $f_{pu}(\overline{(F,A)})\tilde{\supseteq}\{\overline{(f_{pu}(F,A))}\}^\circ$.
(⇐) Let (F,A) be a soft closed set in $SS(X)_A$. Then by hypothesis, we have $\{\overline{(f_{pu}(F,A))}\}^\circ\tilde{\subseteq} f_{pu}(\overline{(F,A)})\tilde{=} f_{pu}(F,A)$. By Lemma 3.5, $f_{pu}(F)$ is soft semi-closed in $SS(Y)_B$. This implies that f_{pu} is soft pu-semi-closed. □

5.3. Theorem. *Let $f_{pu}: SS(X)_A \to SS(Y)_B$ be a soft function and (F,A) be soft set in $SS(X)_A$. Then f_{pu} is soft semi-closed if and only if $scl^s(F,A)\tilde{\subseteq} f_{pu}(\overline{(F,A)})$.*

Proof. (⇒) Let $f_{pu}: SS(X)_A \to SS(Y)_B$ be a soft function and (F,A) be soft set in $SS(X)_A$. Then $f_{pu}(F,A)$ is soft semi-closed. Since $f_{pu}(F,A)\tilde{\subseteq} f_{pu}(\overline{(F,A)})$, then $scl^s(f_{pu}(F,A))\tilde{\subseteq} f_{pu}(\overline{(F,A)})$. Therefore $scl^s(f_{pu}(F,A))\tilde{\subseteq} f_{pu}(\overline{(F,A)})$.
(⇐) This follows from Theorem 5.2. □

5.4. Theorem. *Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft surjective, soft function. Then f_{pu} is soft pu-semi-closed if and only if for any soft subset (G,B) in $SS(Y)_B$ and any soft open set (F,A) in $SS(X)_A$ such that $f_{pu}^{-1}(G,B)\tilde{\subseteq}(F,A)$, there exists a soft semi-open set (V,B) in $SS(Y)_B$ with $(G,B)\tilde{\subseteq}(V,B)$ such that $f_{pu}^{-1}(V,B)\tilde{\subseteq}(F,A)$.*

Proof. $(\Rightarrow)$ Let (G,B) be any soft set in $SS(Y)_B$ and (F,A) be any soft open set in $SS(X)_A$ such that $f_{pu}^{-1}(G,B)\tilde{\subseteq}(F,A)$. Take

$$(V,B)\tilde{=}(f_{pu}((F,A)^c))^c \text{ ..... (F)}$$

Then (V,B) is soft semi-open set. Since $f_{pu}^{-1}(G,B)\tilde{\subseteq}(F,A)$. Simple calculations give $(G,B)\tilde{\subseteq}(V,B)$. Moreover, by (F), we have $f_{pu}^{-1}(V,B)\tilde{=}(f_{pu}^{-1}(f_{pu}((F,A)^c))^c$ $\tilde{\subseteq}((F,A)^c)^c\tilde{=}(F,A)$.

$(\Leftarrow)$ Let (F,A) be any soft closed set in $SS(X)_A$ and e_G be an arbitrary soft point in $(f_{pu}(F,A))^c$, then $f_{pu}^{-1}(e_G)\tilde{\subseteq}(f_{pu}^{-1}(f_{pu}(F,A)))^c\tilde{\subseteq}(F,A)^c$, and $(F,A)^c$ is soft open in $SS(X)_A$. Using our supposition, there exists a soft semi-open set (V_{e_G},B) containing e_G such that $f_{pu}^{-1}(V_{e_G},B)\tilde{\subseteq}(F,A)^c$. This follows $e_G\tilde{\in}(V_{e_G},B)\tilde{\subseteq}(f_{pu}(F,A))^c$. This implies that

$(f_{pu}(F,A))^c\tilde{=}\tilde{\bigcup}\{(V_{e_G},B) : e_G\tilde{\in}(f_{pu}(F,A))^c\}$ is soft semi-open in $SS(Y)_B$, since union of any collection of soft semi-open sets is soft semi-open[3]. Hence $f_{pu}(F,A)$ is soft pu-semi-closed. □

Conclusion: In recent years, many researchers worked on the findings of structures of soft sets theory initiated by Molodtsov and applied to many problems having uncertainties. In the present work, we introduced and explored new form of continuity called soft pu-semi-continuity via soft semi-open set in soft topological spaces. Moreover we also introduced the concepts of soft-pu-semi-open and soft pu-semi-closed functions and discussed many of their characterizations and properties. It is interesting to mention that the soft functions defined and discussed here are the generalization of soft functions introduced in [7]. This is need to continue further research in this direction to upgrade the general framework and to explore the practical life applications.

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Dedicated to the memory of
Prof. I. T. Mamedov (1955-2003)

Solvable time-delay differential operators for first order and their spectrums

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Abstract

In this work, firstly based on the M.I.Vishik's results and using methods of operator theory all solvable extensions of a minimal operator generated by linear delay differential-operator expression of first order in the Hilbert space of vector-functions in finite interval are described. Later on, sharp formulas for the spectrums of these solvable extensions have been found. Finally, the obtained results has been supported by applications.

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1. Introduction

Delay or time-delay differential equations or compound systems as generalization of an ordinary differential equations have been studied for at least 200 years. While some of the early investigation had its origins in certain types of geometric problems and number theory, much of the impetus for the development of the theory came from studies of viscoelasticity, population dynamics and control theory. More recent work has involved models from a wide variety of scientific fields, including nonlinear optics, economies, biology and as well population dynamics, engineering, ecology, chemistry, circadian rhythms, epidemiology, the respiratory system, tumor growth, neural networks.

Note that the fundamental theory of delay differential equations has been given in many of books. The detail analysis of this theory can be found in monographs of A. Ashyralyev and P.E. Sobolevskii [1], J.K. Hale and S.M.V. Lunel [2], O. Diekmann et al.[3], L. Edelstein-Keshet [4], L.E. El'sgol'ts and S.B. Norkin [5], T. Erneux [6], H. Smith [7] and etc.

One of the basis questions of this theory is to investigate the spectral properties of the corresponding problems.

The spectral analysis for the some delay differential equations with large delay first order with matrix coefficients has been investigated in work of M.Lichther, M.Wolfrum and S.Yanchuk [8]. Some aspects of the spectral theory have been investigated by A.Politi, G.Giacomelli, W.Huang, M.Lichther, M.Wolfrum and S.Yanchuk. In particular J.Mallet-Paret and R.D.Nussbaum [9] have studied in detail the appearance of periodic solutions for compound differential equation of first order with single delay in scalar and special cases.

Since analytical computation of solutions, eigenvalues and corresponding eigenfunctions problem is very theoretically and technically difficult, then here play significant role method of numerical analysis. Numerically computing of solutions, eigenvalues and corresponding eigenfunctions of the considered delay differential equations have been done, for example in works A. Ashyralyev with his group[10-12] and E. Jarlebring [13].

Recall that an operator $S : D(S) \subset H \to H$ in Hilbert space H is called solvable, if S is one-to-one, $SD(S) = H$ and $S^{-1} \in L(H)$.

In this work, by using methods of operator theory the all solvable extensions of minimal operator generated by delay differential operator expression for first order in the Hilbert space of vector functions at finite interval have been described in terms of boundary values. In addition, in section 3 sharp formula for the spectrum of these extensions has been given.Applications of obtained results to concrete models have been applied in section 4.

2. Description of Solvable Extensions

In the Hilbert space $L^2(H, (0, 1))$ of vector-functions consider a linear delay differential-operator expression for first order in the form

$$l(u) = u'(t) + A(t)u(t - \tau), \tag{2.1}$$

where:
(1) H is a separable Hilbert space with inner product $(\cdot\,,\,\cdot)_H$ and norm $\|\cdot\|_H$;
(2) operator-function $A(\cdot) : [0, 1] \to L(H)$ is continuous on the uniformly operator topology;
(3) 0< τ <1.

On the other hand here will be considered the following differential expression

$$m(u) = u'(t), \tag{2.2}$$

in the Hilbert space $L^2(H,(0,1))$ corresponding to (2.1). It is clear that formally adjoint expression of (2.2) is of the form

$$m^+(v) = -v'(t), \tag{2.3}$$

Now let us define operator M_0' on the dense in $L^2(H,(0,1))$ set of vector-functions D_0'

$$D_0' := \Big\{ u(t) \in L^2(H,(0,1)) : \; u(t) = \sum_{k=1}^{n} \varphi_k(t) f_k, \quad \varphi_k \in C_0^\infty(0,1), \; f_k \in H, k = 1, 2, \ldots, n, n \in \mathbb{N} \Big\}$$

as $M_0'u = m(u)$.

The closure of M_0' in $L^2(H,(0,1))$ is the minimal operator generated by differential-operator expression(2.2) and is denoted by M_0.

In a similar way the minimal operator M_0^+ in $L^2(H,(0,1))$ corresponding to differential expression (2.3) can be defined.

The adjoint operator of M_0^+ (M_0) in $L^2(H,(0,1))$ is called the maximal operator generated by (2.2)((2.3)) and it is denoted by $M(M^+)$. Now here define a operator S_τ, $0 < \tau < 1$ in $L^2(H,(0,1))$ in form

$$S_\tau u(t) := \begin{cases} u(t-\tau), & \text{if } \tau < t < 1, \\ 0, & \text{if } 0 < t < \tau. \end{cases}$$

From this it is obtained that

$$\begin{aligned} \|S_\tau u\|^2_{L^2(H,(0,1))} &= \int_\tau^1 (u(t-\tau), u(t-\tau))_H dt \\ &= \int_0^{1-\tau} (u(x), u(x))_H dx \\ &\le \int_0^1 \|u(x)\|_H^2 dx \\ &= \|u\|_{L^2(H,(0,1))} \end{aligned}$$

for all $u \in L^2(H,(0,1))$.

Then $\|S_\tau\| \le 1$, $0 < \tau < 1$. On the other words $S_\tau \in L(L^2(H,(0,1)))$ for any $\tau \in (0,1)$. In this situation the tensor product A with S_τ

$$A_\tau(t) = A(t) \otimes S_\tau, \; 0 < \tau < 1$$

is a linear bounded operator in $L^2(H,(0,1))$.

Along of this work the following defined operators

$$L_0 := M_0 + A_\tau(t),$$

$$L_0 : \overset{\circ}{W}{}_2^1 (H,(0,1)) \subset L^2(H,(0,1)) \to L^2(H,(0,1))$$

and

$$L := M + A_\tau(t),$$

$$L : W_2^1(H,(0,1)) \subset L^2(H,(0,1)) \to L^2(H,(0,1))$$

will be called the minimal and maximal operators corresponding to differential expression (2.1) in $L^2(H,(0,1))$ respectively.

Now let $U(t,s), t,s \in [0,1]$, be the family of evolution operators corresponding to the homogeneous differential equation

$$\begin{cases} U'_t(t,s)f + A_\tau(t)U(t,s)f = 0, t,s \in (0,1) \\ U(s,s)f = f, f \in H \end{cases}$$

The operator $U(t,s), t,s \in [0,1]$ is a linear continuous boundedly invertible in H and

$$U^{-1}(t,s) = U(s,t), s,t \in [0,1].$$

(for more detail analysis of this concept see [14]).

Let us introduce the operator

$$Uz(t) := U(t,0)z(t),\ U : L^2(H,(0,1)) \to L^2(H,(0,1)).$$

In this case it is easy to see that for the differentiable vector-function $z \in L^2(H,(0,1))$, $z : [0,1] \to H$, is valid the following relation:

$$l(Uz) = (Uz)'(t) + A(t)(Uz)(t-\tau) = U(z'(t)) + (U'_t + A_\tau(t)U)z(t) = Um(z)$$

From this $U^{-1}lU(z) = m(z)$. Hence it is clear that if the $\widetilde{L}$ is some extension of the minimal operator L_0, that is, $L_0 \subset \widetilde{L} \subset L$, then

$$U^{-1}L_0U = M_0,\ M_0 \subset U^{-1}\widetilde{L}U = \widetilde{M} \subset M,\ U^{-1}LU = M.$$

For example, can be easily to prove the validity of last relation. It is known that

$$D(M) = W_2^1(H,(0,1)),\ D(M_0) = \overset{o}{W}{}_2^1(H,(0,1)).$$

If $u \in D(M)$,then $l(Uz) = Um(z) \in L^2(H,(0,1))$, that is, $Uu \in D(L)$. From last relation $M \subset U^{-1}LU$. Contrary, if a vector-function $u \in D(L)$, then

$$m(U^{-1}v) = U^{-1}l(v) \in L^2(H,(0,1)),$$

that is, $U^{-1}v \in D(M)$. From last relation $U^{-1}L \subset MU$, that is $U^{-1}LU \subset M$. Hence $U^{-1}LU = M$.

The following assertions are true.

2.1. Theorem. *$KerL_0 = \{0\}$ and $\overline{R(L_0)} \neq L^2(H,(0,1))$.*

2.2. Theorem. *Each solvable extension $\widetilde{L}$ of the minimal operator L_0 in $L^2(H,(0,1))$ is generated by the differential-operator expression (2.1) and boundary condition*

$$(K+E)u(0) = KU(0,1)u(1), \tag{2.4}$$

where $K \in L(H)$ and E is a identity operator in H. The operator K is determined uniquely by the extension $\widetilde{L}$, i.e $\widetilde{L} = L_K$.

On the contrary, the restriction of the maximal operator L_0 to the manifold of vector-functions satisfy the condition (2.4) for some bounded operator $K \in L(H)$ is a solvable extension of the minimal operator L_0 in the $L^2(H,(0,1))$.

Proof. Firstly, it is described all solvable extensions $\widetilde{M}$ of the minimal operator M_0 in $L^2(H,(0,1))$ in terms of boundary values.

Consider the following so-called Cauchy extension M_c

$$M_cu = u'(t),\ M_c : D(M_c) = \{u \in W_2^1(H,(0,1)) : u(0) = 0\} \subset L^2(H,(0,1)) \to L^2(H,(0,1))$$

of the minimal operator M_0. It is clear that M_c is a solvable extension of M_0 and

$$M_c^{-1}f(t) = \int_0^t f(x)dx, f \in L^2(H,(0,1)),$$

$$M_c^{-1} : L^2(H,(0,1)) \to L^2(H,(0,1)).$$

Now assume that $\widetilde{M}$ is a solvable extension of the minimal operator M_0 in $L^2(H,(0,1))$. In this case it is known that domain of $\widetilde{M}$ can be written in direct sum in form

$$D(\widetilde{M}) = D(M_0) \oplus (M_c^{-1} + K)V,$$

where $V = KerM = H, K \in L(H)$ $(see[15])$. Therefore for each $u(t) \in D(\widetilde{M})$ it is true

$$u(t) = u_0(t) + M_c^{-1}f + Kf,\ u_0 \in D(M_0),\ f \in H.$$

That is,

$$u(t) = u_0(t) + tf + Kf,\ u_0 \in D(M_0),\ f \in H.$$

Hence

$$u(0) = Kf,\ u(1) = f + Kf = (K+E)f$$

and from these relations it is obtained that

$$(K+E)u(0) = Ku(1). \tag{2.5}$$

On the other hand uniqueness of operator $K \in L(H)$ it is clear from the work [15]. Therefore $\widetilde{M} = M_K$. This completes of necessary part of this assertion.

On the contrary, if M_K is a operator generated by differential expression (2.2) and boundary condition (2.5), then M_K is boundedly invertible and

$$M_K^{-1} : L^2(H,(0,1)) \to L^2(H,(0,1)),$$

$$M_K^{-1}f(t) = \int_0^t f(x)dx + K\int_0^1 f(x)dx,\ f \in L^2(H,(0,1)).$$

Consequently, all solvable extension of the minimal operator M_0 in $L^2(H,(0,1))$ is generated by differential expression (2.2) and boundary condition (2.5) with any linear bounded operator K.

Now consider the general case. For the this in the $L^2(H,(0,1))$ introduce a operator in form

$$U : L^2(H,(0,1)) \to L^2(H,(0,1)),\ (Uz)(t) := U(t,0)z(t), z \in L^2(H,(0,1)).$$

From the properties of family of evolution operators $U(t,s), t, s \in [0,1]$ imply that a operator U is a linear bounded, boundedly invertible and

$$(U^{-1}z)(t) = U(0,t)z(t).$$

On the other hand from the relations

$$U^{-1}L_0U = M_0, U^{-1}\widetilde{L}U = \widetilde{M}, U^{-1}LU = M$$

it is implies that a operator U is a one-to-one between of sets of solvable extensions of minimal operators L_0 and M_0 in $L^2(H,(0,1))$.

Extension $\widetilde{L}$ of the minimal operator L_0 is solvable in $L^2(H,(0,1))$ if and only if the operator $\widetilde{M} = U^{-1}\widetilde{L}U$ is a extension of the minimal M_0 in $L^2(H,(0,1))$. Then $u \in D(\widetilde{L})$ if and only if

$$(K+E)U(0,0)u(0) = KU(0,1)u(1),$$

that is,

$$(K+E)u(0) = KU(0,1)u(1).$$

This proves the validity of the claims in theorem. □

2.3. Remark. In general case $A(t)S_\tau \neq S_\tau A(t)$ in $L^2(H,(0,1))$. Indeed, if

$$(Af)(t) = tf(t),\ f \in L^2(H,(0,1)),\ A : L^2(0,1) \to L^2(0,1),$$

then for $0 < \tau < 1,\ \ f \in L^2(0,1)$ we have

$$(AS_\tau)f(t) = A(S_\tau f(t)) = A(f(t-\tau)) = tf(t-\tau), 0 < t < 1$$

and

$$(S_\tau A)f(t) = S_\tau(Af(t)) = S_\tau(tf(t)) = (t-\tau)f(t-\tau), 0 < t < 1.$$

2.4. Corollary. *Assume that $A(t) = A = const$ a.e. in $(0,1)$.*

In this case all solvable extensions of minimal operator L_0 are generated by delay differential expression

$$l(u) = u'(t) + Au(t-\tau), 0 < \tau < 1$$

and boundary condition

$$\begin{aligned}(K+E)u(0) &= K[u(1) - \frac{Au(1-\tau)}{1!} + \frac{A^2u(1-2\tau)}{2!} + ...] \\ &= K\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}A^n u(1-n\tau),\ K \in L(H)\end{aligned}$$

in the Hilbert $L^2(H,(0,1))$ and vice versa.

2.5. Remark. Since for any $0 < \tau < 1$ there exists $n_0 = n_0(\tau) \in \mathbb{N}$ such that

$$0 \le 1 - n_0\tau < 1 \ \text{ and } \ 1 - (n_0+1)\tau < 0.$$

Then

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}A^n u(1-n\tau) = \sum_{n=0}^{n_0}\frac{(-1)^n}{n!}A^n u(1-n\tau).$$

2.6. Remark. All solvable extensions of minimal operator are generated by delay differential expression

$$l(u) = u'(t) + u(t-\tau),\ 0 < \tau < 1$$

and boundary condition

$$\begin{aligned}(K+E)u(0) = K[u(1) - \frac{u(1-\tau)}{1!} + \frac{u(1-2\tau)}{1!} + ... \\ + \frac{(-1)^n u(1-n\tau)}{n!} + ...],\ K \in L(H),\end{aligned}$$

in the space $L^2(H,(0,1))$ and vice versa.

In addition note that following boundary value problem

$$u'(t) = -u(t-\tau),\ \tau < t < 1,\ \tau > 0,\ \ u(t) = 1,\ \tau < t < 0$$

by changing the function $u(t)$ with $y(t) = u(t) - 1,\ \tau < t < 1$ can be reduced to problem

$$y'(t) = -y(t-\tau) - 1,\ \ y(t) = 0,\ \tau < t < 0.$$

3. Spectrum of Solvable Extension

In this section will be investigated spectrum structure of solvable extensions of minimal operator L_0 in $L^2(H,(0,1))$.
Firstly, prove the following fact.

3.1. Theorem. *If $\widetilde{L}$ is a solvable extension of a minimal operator L_0 and $\widetilde{M} = U^{-1}\widetilde{L}U$ corresponding for the solvable extension of a minimal operator M_0, then for the spectrum of these extensions is true $\sigma(\widetilde{L})$=$\sigma(\widetilde{M})$.*

Proof. Let us consider a problem for the spectrum for a solvable extension L_K of a minimal operator L_0 generated by delay differential-operator expression (2.1), that is,

$$L_K u = \lambda u + f, \lambda \in \mathbb{C}, f \in L^2(H,(0,1)).$$

From this it is obtained that

$$(L_K - \lambda E)u = f \quad \text{or} \quad (UM_K U^{-1} - \lambda E)u = f$$

Hence

$$U(M_K - \lambda)(U^{-1}u) = f$$

the last equation explains the validity of the theorem. □

Now prove the following result for the spectrum of solvable extension.

3.2. Theorem. *If L_K a solvable extension of the minimal operator L_0 in the space $L^2(H,(0,1))$, then spectrum of L_K has the form:*

$$\sigma(L_K) = \{\lambda \in \mathbb{C} : \lambda = \ln|\frac{\mu+1}{\mu}| + i\arg(\frac{\mu+1}{\mu}) + 2n\pi i; \\ \mu \in \sigma(K) \setminus \{0,-1\}, n \in \mathbb{Z}\}.$$

Proof. Firstly, will be investigated the spectrum of the solvable extension $M_K = U^{-1}L_K U$ of the minimal operator M_0 in $L^2(H,(0,1))$. Consider the following problem for the spectrum, $M_K u = \lambda u + f, \lambda \in \mathbb{C},\ f \in L^2(H,(0,1))$. Then

$$u' = \lambda u + f,\ (K+E)u(0) = Ku(1), \lambda \in \mathbb{C}, f \in L^2(H,(0,1)), K \in L(H).$$

It is clear that a general solution of a above differential equation in $L^2(H,(0,1))$ has the form

$$u_\lambda(t) = e^{\lambda t} f_0 + \int_0^t e^{\lambda(t-s)} f(s)ds, f_0 \in H.$$

Therefore from the boundary condition $(K+E)u_\lambda(0) = Ku_\lambda(1)$ it is obtained that

$$(E + K(1-e^\lambda))f_0 = K\int_0^1 e^{\lambda(1-s)} f(s)ds.$$

For the $\lambda_m = 2m\pi i,\ m \in \mathbb{N}$ from the last relation it is established that

$$f_0^{(m)} = K\int_0^1 e^{\lambda_m(1-s)} f(s)ds, m \in \mathbb{N}.$$

Consequently, in this case the resolvent operator of M_K is in form

$$R_{\lambda_m}(M_K)f(t) = Ke^{\lambda_m t}\int_0^1 e^{\lambda_m(1-s)} f(s)ds + \int_0^t e^{\lambda_m(t-s)} f(s)ds,\ f \in L^2(H,(0,1)), m \in \mathbb{Z}.$$

On the other hand it is clear that $R_{\lambda_m}(M_K) \in L((L^2(H,(0,1)) \ , m \in \mathbb{Z}$.
If $\lambda \neq 2m\pi i, m \in \mathbb{Z}, \lambda \in \mathbb{C}$, then from boundary condition we have

$$(K - \frac{1}{e^{\lambda}-1}E)f_0 = \frac{1}{1-e^{\lambda}}K\int_0^1 e^{\lambda(1-s)}f(s)ds, f_0 \in H, \ f \in (L^2(H,(0,1)).$$

Therefore, for $\lambda \in \sigma(M_K)$ if and only if $\mu = \frac{1}{e^{\lambda}-1} \in \sigma(K)$.
In this case since $e^{\lambda} = \frac{\mu+1}{\mu}, \mu \in \sigma(K)$, then $\lambda_n = ln|\frac{\mu+1}{\mu}| + iarg(\frac{\mu+1}{\mu}) + 2n\pi i, n \in \mathbb{Z}$.
Later on, using the last relation and Theorem 3.1 it is proved the validity of claim in theorem. □

3.3. Corollary. *Let L_K be a solvable extension of minimal operator L_0 in $L^2(H,(0,1))$.*

(1) *If $\sigma(K) \subset \{0,1\}$, then $\sigma(L_K) = \emptyset$;*
(2) *If $\sigma(K) \setminus \{0,1\} \neq \emptyset$, then $\sigma(L_K)$ is infinite.*

Now will be proved one result on the asymptotically behaviour of eigenvalues of solvable extensions in special case.

3.4. Theorem. *If $K \in L(H), K \neq 0$, $\sigma(K) = \sigma_p(K)$, there exist $\alpha, \beta > 0$ such that for any $\mu \in \sigma_p(K)$ is true*

$$|\mu| \geq \alpha > 0 \quad and \quad |\mu + 1| \geq \beta > 0,$$

then $\lambda_n(M_K) \sim 2n\pi$, as $n \to \infty$.

Proof. In this case for $n \geq 1$

$$|\lambda_n(M_K)|^2 = ln^2|\frac{\mu+1}{\mu}| + |arg(\frac{\mu+1}{\mu}) + 2n\pi|^2.$$

Since for any $\mu \in \sigma_p(K)$

$$|\frac{\mu+1}{\mu}| \geq \frac{\beta}{|\mu|} \geq \frac{\beta}{\|K\|} > 0, \quad |\frac{\mu+1}{\mu}| \leq 1 + \frac{1}{|\mu|} \leq 1 + \frac{1}{\alpha},$$

then

$$ln\frac{\beta}{\|K\|} \leq ln|\frac{\mu+1}{\mu}| \leq ln(1+\frac{1}{\alpha}).$$

Therefore for any $\mu \in \sigma_p(K)$ is true

$$min\{|ln(\frac{\beta}{\|K\|})|, |ln(1+\frac{1}{\alpha})|\} \leq |ln|\frac{\mu+1}{\mu}|| \leq max\{|ln(\frac{\beta}{\|K\|})|, |ln(1+\frac{1}{\alpha})|\}.$$

On the other hand for any $n \in \mathbb{Z}$

$$(2n\pi)^2 \leq |arg(\frac{\mu+1}{\mu}) + 2n\pi|^2 \leq (2(n+1)\pi)^2.$$

Consequently, for any $n \in \mathbb{N}$

$$(2n\pi)^2\left(1 + \frac{1}{4n^2\pi^2}min^2\{|ln(\frac{\beta}{\|K\|})|, |ln(1+\frac{1}{\alpha})|\}\right)$$
$$\leq |\lambda_n(M_K)|^2 \leq (2n\pi)^2\left((\frac{2(n+1)\pi}{2n\pi})^2 + \frac{1}{(2n\pi)^2}max^2\{|ln(\frac{\beta}{\|K\|})|, |ln(1+\frac{1}{\alpha})|\}\right)$$

This means that $\lambda_n(M_K) \sim 2n\pi$, as $n \to \infty$. □

4. Applications

4.1. Example. Assume that

$$H = \mathbb{C}, (H, \|\cdot\|_H) = (\mathbb{C}, |\cdot|),\ A(\cdot) = a(\cdot) \in C(\mathbb{R})$$

and consider the following delay differential equation in from

$$u'(t) = a(t)u(t-\tau), 0 < \tau < 1$$

with history function $u(t) = 0, -\tau < t < 0$ in the Hilbert space $L^2(0,1)$.

Then the all solvable extension L_k of minimal operator L_0 is generated by delay differential expression

$$l(u) = u'(t) - a(t)u(t-\tau)$$

and boundary condition

$$(k+1)u(0) = kexp(\int_0^1 a(t)dt)u(1), k \in \mathbb{C}$$

in $L^2(0,1)$. In addition, spectrum of L_k is in form

$$\sigma(L_k) = \{\lambda \in \mathbb{C} : \lambda = \ln|\frac{k+1}{k}| + i\arg(\frac{k+1}{k}) + 2n\pi i, n \in \mathbb{Z}\}.$$

4.2. Example. Let us

$$(H, \|\cdot\|_H) = (\mathbb{C}, |\cdot|),\ a(\cdot), b(\cdot) \in C(\mathbb{R})$$

and consider the delay differential expression in form $l(u) = u'(t)+a(t)u(t)+b(t)u(t-\tau)$, $0 < t < 1$, $0 < \tau < 1$ with history function $u(t) = 0,\ -\tau < t < 0$. If change of function $u(\cdot)$ by $y(\cdot)$

$$y(t) = \lambda(t)u(t), \lambda(t) = exp(\int_0^t a(x)dx),$$

then

$$l(\lambda^{-1}y) = y'(t) + c(t)y(t-\tau),$$

where

$$c(t) = \frac{\lambda(t)b(t)}{\lambda(t-\tau)} = b(t)exp(\int_{t-\tau}^t a(x)dx).$$

In this case all solvable extension P_k of minimal operator P_0 is generated by delay differential expression

$$P(y) = y'(t) + c(t)y(t-\tau)$$

and boundary condition

$$(k+1)y(0) = kexp(-\int_0^1 c(t)dt)y(1), k \in \mathbb{C}$$

and vice versa.

Consequently, all solvable extension P_k of the minimal operator P_0 is generated by delay differential expression

$$l(u) = u'(t) + a(t)u(t) + b(t)u(t-\tau)$$

and boundary condition

$$(k+1)u(0) = kexp(-\int_0^1 b(t)exp(\int_{t-\tau}^{t} a(x)dx)dt)exp(\int_0^1 a(x)dx)u(1), k \in \mathbb{C}$$

and vice versa.
Moreover, spectrum of solvable extension L_k is in form

$$\sigma(L_k) = \{\lambda \in \mathbb{C} : \lambda = \ln|\frac{k+1}{k}| + i\arg(\frac{k+1}{k}) + 2n\pi i, n \in \mathbb{Z}\}, k \in \mathbb{C}.$$

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Strongly copure projective objects in triangulated categories

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Abstract

In this paper, we introduce and investigate the notions of ξ-strongly copure projective objects in a triangulated category. This extends Asadollahi's notion of ξ-Gorenstein projective objects. Then we study the ξ-strongly copure projective dimension and investigate the existence of ξ-strongly copure projective precover.

Keywords: strongly copure projective object; triangulated category; proper class of triangles.

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1. Introduction

Triangulated categories originated from algebraic geometry and algebraic topology and were introduced by Grothendieck and Verdier in the early sixties as the proper framework for doing homological algebra in an abelian category. By now triangulated categories have become indispensable in many different areas of mathematics, such as algebraic geometry, stable homotopy theory, and representation theory.

In [3], Beligiannis develops a classical homological algebra in a triangulated category $\mathcal{C} = (\mathcal{C}, \Sigma, \Delta)$. He introduced ξ-projective objects, ξ-projective resolution, ξ-projective dimension and their dualities. Based on the works of Auslander and Bridger [2], Enochs and Jenda [8] and Beligiannis [5], Asadollahi [3] introduced and studied ξ-Gorenstein projective objects and their dualities, which maked contributions to develop there relative homological algebra in a triangulated category.

At the other extreme, Mao [9] investigated strongly P-projective modules. M is called to be strongly P-projective if $\mathrm{Ext}^i_R(M,P) = 0$ for all projective left R-modules P, which is dual to strongly copure injective modules in Enochs and Jenda [6]. So we also call strongly P-projective modules as strongly copure projective modules in this paper. As we all known, strongly copure projective (resp.

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injective) modules are a generaliation of Gorenstein projective and projective (resp., Gorenstein injective and injective) modules in categories of modules.

Our aim in this paper is to introduce and study ξ-strongly copure projective (injective) objects in a triangulated category $\mathcal{C}$. This is denoted by ξ-$\mathcal{SC}$projective (ξ-$\mathcal{SC}$injective) objects for convenience. In Section 2, we introduce the notion of ξ-$\mathcal{SC}$projective objects and study some properties of ξ-$\mathcal{SC}$projective objects in $\mathcal{C}$. We also investigate ξ-$\mathcal{SC}$projective dimension. In Section 3, we introduce the concept of ξ-$\mathcal{SC}$projective precover and show the existence of ξ-$\mathcal{SC}$projective precover. We also prove that the equivalence between $\xi xt^{n+1}_{\mathcal{SCP}(\xi)}(A,-)=0$ and ξ-$\mathcal{SC}\mathrm{pd}A\leq n$ under some conditions.

Next we recall some known notions and facts of triangulated categories needed in the sequel. The basic reference for triangulated categories and derived categories is the original article of Verdier [15]. Also [3, 7, 11] give introduction to these concepts.

Let $\mathcal{C}$ be an additive category and $\Sigma:\mathcal{C}\longrightarrow\mathcal{C}$ an additive functor. Let $\mathrm{Diag}(\mathcal{C},\Sigma)$ denotes the category whose objects are diagrams in $\mathcal{C}$ of the form $A\longrightarrow B\longrightarrow C\longrightarrow\Sigma A$, and morphisms between two objects $A_i\longrightarrow B_i\longrightarrow C_i\longrightarrow\Sigma A_i$, $i=1,2$, are triple of morphisms $\alpha:A_1\longrightarrow A_2$, $\beta:B_1\longrightarrow B_2$ and $\gamma:C_1\longrightarrow C_2$, such that the following diagram commutes:

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{h_1} & \Sigma A_1 \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\
A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{h_2} & \Sigma A_2
\end{array}$$

A triangle $(\mathcal{C},\Sigma,\Delta)$ is called a triangulated category, where $\mathcal{C}$ is an additive category. Σ is an autoequivalence of $\mathcal{C}$ and Δ is a full subcategory of $\mathrm{Diag}(\mathcal{C},\Sigma)$ which satisfies the following axioms. The elements of Δ are then called triangles.

(Tr1) Every diagram isomorphic to a triangle is a triangle. Every morphism $f:A\longrightarrow B$ in $\mathcal{C}$ can be embedded into a triangle $A\xrightarrow{f}B\xrightarrow{g}C\xrightarrow{h}\Sigma A$. For any object $A\in\mathcal{C}$, the diagram $0\longrightarrow A\xrightarrow{1_A}A\longrightarrow 0$ is a triangle, where 1_A denotes the identity morphism from A to A.

(Tr2) $A\xrightarrow{f}B\xrightarrow{g}C\xrightarrow{h}\Sigma A$ is a triangle if and only if $B\xrightarrow{g}C\xrightarrow{h}\Sigma A\xrightarrow{-\Sigma f}\Sigma B$ is so.

(Tr3) Given triangles $A_i\xrightarrow{f_i}B_i\xrightarrow{g_i}C_i\xrightarrow{h_i}\Sigma A_i$, $i=1,2$, and morphisms $\alpha:A_1\longrightarrow A_2$ and $\beta:B_1\longrightarrow B_2$ such that $\alpha f_2=f_1\beta$, there exists a morphism $\gamma:C_1\longrightarrow C_2$ such that (α,β,γ) is a morphism from the first triangle to the second.

(Tr4) (The Octahedral Axiom) Given triangles $A\xrightarrow{f}B\xrightarrow{i}C'\xrightarrow{i'}\Sigma A$, $B\xrightarrow{g}C\xrightarrow{j}A'\xrightarrow{f'}\Sigma B$, $A\xrightarrow{gf}C\xrightarrow{k}B'\xrightarrow{k'}\Sigma A$, there exist morphisms $f':C'\longrightarrow B'$ and $g':B'\longrightarrow A'$ such that the following diagram commutes and the third row is triangle:

$$\begin{array}{ccccccccc}
\Sigma^{-1}B' & \xrightarrow{\Sigma^{-1}k'} & A & \xrightarrow{1_A} & A & & & & \\
\downarrow{\scriptstyle\Sigma^{-1}g'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle gf} & & & & \\
\Sigma^{-1}A' & \xrightarrow{\Sigma^{-1}j'} & B & \xrightarrow{g} & C & \xrightarrow{j} & A' & \xrightarrow{j'} & \Sigma B \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle 1_{A'}} & & \downarrow{\scriptstyle\Sigma i} \\
 & & C' & \xrightarrow{f'} & B' & \xrightarrow{g'} & A' & \xrightarrow{\Sigma ij'} & \Sigma C' \\
 & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle k'} & & & & \\
 & & \Sigma A & \xrightarrow{1_{\Sigma A}} & \Sigma A & & & &
\end{array}$$

Throughout the paper, we fix a triangulated category $\mathcal{C}=(\mathcal{C},\Sigma,\Delta)$, Σ is the suspension functor and Δ is the triangulation.

1.1. Proposition. ([3, Proposition 2.1]) Let $\mathcal{C}$ be an additive category equipped with an autoequivalence $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ and a class of diagrams $\Delta \subseteq \mathrm{Diag}(\mathcal{C},\Sigma)$. Suppose that the triple $(\mathcal{C},\Sigma,\Delta)$, Σ satisfies all the axioms of a triangulated category except possibly of the Octahedral Axiom. Then the following are equivalent:

(a) **Base change**. For any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \Delta$ and morphism $\varepsilon : E \longrightarrow C$, there exists a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & M & = & M & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta} & & \downarrow \\
A & \xrightarrow{f'} & G & \xrightarrow{g'} & E & \xrightarrow{h'} & \Sigma A \\
\| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \varepsilon} & & \| \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \zeta} & & \downarrow \\
0 & \longrightarrow & \Sigma M & = & \Sigma M & \longrightarrow & 0
\end{array}$$

in which all horizontal and vertical diagrams are triangle in Δ.

(b) **Cobase change.** For any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \Delta$ and any morphism $\alpha : A \longrightarrow D$, there exists a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & = & N & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \delta} & & \downarrow \\
\Sigma^{-1}C & \xrightarrow{-\Sigma^{-1}(h)} & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \| \\
\Sigma^{-1}C & \xrightarrow{-\Sigma^{-1}(h')} & D & \xrightarrow{f'} & F & \xrightarrow{g'} & C \\
\downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \nu} & & \downarrow \\
0 & \longrightarrow & \Sigma N & = & \Sigma N & \longrightarrow & 0
\end{array}$$

in which all horizontal and vertical diagrams are triangles in Δ.

(c) **Octahedral Axiom** For any two morphisms $f_1 : A \longrightarrow B$, $f_2 : B \longrightarrow C$, there exists a commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{f_1} & B & \xrightarrow{g_1} & X & \xrightarrow{h_1} & \Sigma A \\
\| & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \alpha} & & \| \\
A & \xrightarrow{f_2 f_1} & C & \xrightarrow{g_3} & Y & \xrightarrow{h_3} & \Sigma A \\
\downarrow{\scriptstyle f_1} & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \Sigma f_1} \\
B & \xrightarrow{f_2} & C & \xrightarrow{g_2} & Z & \xrightarrow{h_2} & \Sigma B \\
\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \Sigma g_1 h_2} & & \downarrow \\
0 & \longrightarrow & \Sigma X & = & \Sigma X & \longrightarrow & 0
\end{array}$$

in which all horizontal and the third vertical diagrams are triangles in Δ.

A class of triangles ξ is closed under base change if for any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \xi$ and any morphism $\varepsilon : E \longrightarrow C$ as in Proposition 1.1(a), the triangle $A \xrightarrow{f'} G \xrightarrow{g'} E \xrightarrow{h'} \Sigma A$ belongs to ξ. Dually, a class of triangles is closed under cobase change if for any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \xi$ and any morphism $\alpha : A \longrightarrow D$ as in Proposition 1.1(b), the triangle $D \xrightarrow{f'} F \xrightarrow{g'} C \xrightarrow{h'} \Sigma D$ belongs to ξ. A class of triangles is closed under suspension if for any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \xi$ and any integer $i \in \mathbb{Z}$, the triangle

$$\Sigma^i A \xrightarrow{(-1)^i \Sigma^i f} \Sigma^i B \xrightarrow{(-1)^i \Sigma^i g} \Sigma^i C \xrightarrow{(-1)^i \Sigma^i h} \Sigma^{i+1} A$$

is in ξ. A class of triangles ξ is called saturated if in the situation of base change in Proposition 1.1, whenever the third vertical and the second horizontal triangle is in ξ, then the triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is in ξ.

1.2. Definition. ([3, Definition 2.2]) A full subcategory $\xi \subseteq \mathrm{Diag}(\mathcal{C}, \Sigma)$ is called a proper class of triangles if the following conditions hold:

(i) ξ is closed under isomorphisms, finite coproducts and $\Delta_0 \subseteq \xi \subseteq \Delta$, where Δ_0 denotes the full subcategory of split triangles.

(ii) ξ is closed under suspensions and is saturated.

(iii) ξ is closed under base and cobase change.

Throughout we fix a proper class of triangles ξ in the triangulated category $\mathcal{C}$.

2. Strongly copure projective objects

2.1. Definition. ([3, Definition 4.1]) An object $P \in \mathcal{C}$, (respectively $I \in \mathcal{C}$) is called ξ-projective (respectively ξ-injective) if for any triangle $A \to B \to C \to \Sigma A$ in ξ, the induced sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(P,A) \to \mathrm{Hom}_{\mathcal{C}}(P,B) \to \mathrm{Hom}_{\mathcal{C}}(P,C) \to 0$$

$$(\textit{respectively } \ 0 \to \mathrm{Hom}_{\mathcal{C}}(C,I) \to \mathrm{Hom}_{\mathcal{C}}(B,I) \to \mathrm{Hom}_{\mathcal{C}}(A,I) \to 0)$$

is exact in the category of abelian group $\mathcal{A}$b.

The symbol $\mathcal{P}(\xi)$ (res. $\mathcal{I}(\xi)$) will denote the full subcategory of ξ-projective (res. ξ-injective) objects of $\mathcal{C}$. It follows easily from the definition that the categories $\mathcal{P}(\xi)$ and $\mathcal{I}(\xi)$ are full, additive, closed under isomorphisms, direct summands and Σ-stable.

$\mathcal{C}$ is said to have enough ξ-projective objects if for any object $A \in \mathcal{C}$ there exists a triangle $K \to P \to A \to \Sigma K$ in ξ with $P \in \mathcal{P}(\xi)$. Dually one defines when $\mathcal{C}$ has enough ξ-injectives.

2.2. Lemma. ([3, Lemma 4.2]) Assume that $\mathcal{C}$ is a triangulated category with enough ξ-projective objects. Then $A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$ is in ξ if and only if for all $P \in \mathcal{P}(\xi)$ the induced sequence $0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P,A) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P,C) \longrightarrow 0$ is exact.

In [3], the ξ-projective dimension ξ-pd$A$ of an object $A \in \mathcal{C}$ is defined inductively.

2.3. Definition. ([3, Definition 4.7]) An ξ-exact complex $X_\bullet \to A$ over $A \in \mathcal{C}$ is a diagram $\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} A \to 0$ such that for each integer $n \geq 0$:

(i) There are triangles $K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$ in ξ, where $K_0 = A$.

(ii) The differential $d_n = g_{n-1} f_n$ for any $n \geq 1$ and $d_0 = f_0$.

An ξ-projective resolution of $A \in \mathcal{C}$ is an ξ-exact complex $P_\bullet \to A$ as above such that $P_n \in \mathcal{P}(\xi)$, $n \geq 0$.

2.4. Definition. ([2, Definition, 3.2]) A triangle $A \to B \to C \to \Sigma A$ in ξ is called $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact, if for any $Q \in \mathcal{P}(\xi)$, the induced complex

$$0 \to \mathrm{Hom}_{\mathcal{C}}(C,Q) \to \mathrm{Hom}_{\mathcal{C}}(B,Q) \to \mathrm{Hom}_{\mathcal{C}}(A,Q) \to 0$$

is exact in $\mathcal{A}$b.

2.5. Definition. An object A is said to be ξ-strongly copure projective object (ξ-$\mathcal{SC}$projective object) if there exists an ξ-projective resolution of $A: \cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \to \cdots \to P_1 \xrightarrow{d_0} P_0 \xrightarrow{d_0} A \to 0$ with $P_i \in \mathcal{P}(\xi)$ for all $i \geq 0$ such that $K_{n+1} \xrightarrow{g_n} P_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$ in ξ are $\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{P}(\xi))$-exact triangles for all integer n.

We denote $\mathcal{SCP}(\xi)$ the full subcategory of ξ-strongly copure projective objects of $\mathcal{C}$. It follows directly from the definition that the category $\mathcal{SCP}(\xi)$ is full, additive and closed under isomorphisms.

Remark. By [2, Definition 3.6], every ξ-Gorenstein projective object is ξ-strongly copure projective. In particular, there is an inclusion of categories $\mathcal{GP}(\xi) \subseteq \mathcal{SCP}(\xi)$, where $\mathcal{GP}(\xi)$ is the class of ξ-Gorenstein projective objects.

Let C be an object of $\mathcal{C}$. For any integer $n \geq 0$, the ξ-extension functor $\xi xt^n_\xi(-,C)$ is defined to be the nth right ξ-derived functor of the functor $\mathrm{Hom}_{\mathcal{C}}(-,C)$, that is $\xi xt^n_\xi(-,C) := R^n_\xi \mathrm{Hom}_{\mathcal{C}}(-,C)$.

2.6. Proposition. ([3, Corollary 4.12]) If $A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$ is a triangle in ξ, then for any $X \in \mathcal{C}$ we have a long exact sequence

$$0 \longrightarrow \xi xt^0_\xi(C,X) \longrightarrow \xi xt^0_\xi(B,X) \longrightarrow \xi xt^0_\xi(A,X) \longrightarrow \xi xt^1_\xi(C,X) \longrightarrow \cdots.$$

2.7. Lemma. Let A be a ξ-$\mathcal{SC}$projective object of $\mathcal{C}$. Then $\xi xt^0_\xi(A,Q) \cong \mathrm{Hom}_{\mathcal{C}}(A,Q)$ and $\xi xt^i_\xi(A,Q) = 0$ for any $Q \in \widetilde{\mathcal{P}}(\xi)$ and any $i > 0$, where $\widetilde{\mathcal{P}}(\xi)$ denote the full subcategory of $\mathcal{C}$ whose objects are of finite ξ-projective dimension.

Proof. Let ξ-pd$Q$=n for some nonnegative integer n and P_* a ξ-projective resolution of A. If $n = 0$, then Q is an ξ-projective object. Then $\mathrm{Hom}_{\mathcal{C}}(P_*,Q)$ is an exact sequence, and this implies that

$$\mathrm{Hom}_{\mathcal{C}}(A,Q) \cong H^0(0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P_0,Q) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P_1,Q) \longrightarrow \cdots) \cong \xi xt^0_\xi(A,Q).$$

Moreover, $\xi xt^i_\xi(A,Q) = 0$ for any $i > 0$. Inductively, suppose that the assertions follow for any object with ξ-projective dimension $n-1$. Consider the triangle $K \longrightarrow P \longrightarrow Q \longrightarrow \Sigma K$ in ξ , where $P \in \mathcal{P}(\xi)$ and ξ-pd$K = n-1$. For any $j \in \mathbb{Z}$, the triangle $\Sigma^j K \xrightarrow{(-1)^j\Sigma^j f} \Sigma^j P \xrightarrow{(-1)^j\Sigma^j g} \Sigma^j Q \xrightarrow{(-1)^j\Sigma^j h} \Sigma^{j+1} K$ is also in ξ. By Proposition 2.6, there is an exact sequence $0 \longrightarrow \xi xt^0_\xi(A,\Sigma^j K) \longrightarrow \xi xt^0_\xi(A,\Sigma^j P)$, and then $0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,\Sigma^j K) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,\Sigma^j P)$. This implies that $\mathrm{Hom}_{\mathcal{C}}(A,-)$ kills ξ-phantom map $(-1)^j\Sigma^j h$. Especially, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A,K) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A,P) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A,Q) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow & & \\
0 & \longrightarrow & \xi xt^0_\xi(A,K) & \longrightarrow & \xi xt^0_\xi(A,P) & \longrightarrow & \xi xt^0_\xi(A,Q) & \longrightarrow & \xi xt^1_\xi(A,K) = 0
\end{array},$$

where rows are exact. Hence $\xi xt^0_\xi(A,Q) \cong \mathrm{Hom}_{\mathcal{C}}(A,Q)$. Since

$$\xi xt^i_\xi(A,P) \longrightarrow \xi xt^i_\xi(A,Q) \longrightarrow \xi xt^{i+1}_\xi(A,K)$$

is exact by Proposition 2.6, where $\xi xt^i_\xi(A,P) = \xi xt^{i+1}_\xi(A,K) = 0$. Thus $\xi xt^i_\xi(A,Q) = 0$.

2.8. Proposition. Assume that $\mathcal{C}$ is a triangulated category with enough ξ-projective objects and X is an object in $\mathcal{P}(\xi)$. Then X is ξ-injective relative to $\mathcal{SCP}(\xi)$.

Proof. Let $A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$ be a triangle of $\mathcal{SCP}(\xi)$ in ξ. By Proposition 2.6, there is an exact sequence $0 \longrightarrow \xi xt^0_\xi(C,X) \longrightarrow \xi xt^0_\xi(B,X) \longrightarrow \xi xt^0_\xi(A,X) \longrightarrow \xi xt^1_\xi(C,X)$. Since $\xi xt^1_\xi(C,X) = 0$ by Lemma 2.7 and $\xi xt^0_\xi(G,X) \cong \mathrm{Hom}_{\mathcal{C}}(G,X)$ for any $G \in \mathcal{SCP}(\xi)$, there is an exact sequence $0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C,X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(B,X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,X) \longrightarrow 0$. So X is ξ-injective relative to $\mathcal{SCP}(\xi)$.

2.9. Theorem. Assume that $\mathcal{C}$ is a triangulated category with enough ξ-projective objects and $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$ is a triangle in ξ such that C is ξ-$\mathcal{SC}$projective. Then A is ξ-$\mathcal{SC}$projective if and only if B is ξ-$\mathcal{SC}$projective.

Proof. First assume that A is ξ-$\mathcal{SC}$projective. We will show that B is also such. Since A and C are ξ-$\mathcal{SC}$projective, there exist triangles $K_A \xrightarrow{g_A} P_A \xrightarrow{f_A} A \xrightarrow{h_A} \Sigma K_A$ and $K_C \xrightarrow{g_C} P_C \xrightarrow{f_C} C \xrightarrow{h_C} \Sigma K_C$ in ξ, where P_A and P_C are ξ-projective, K_A and K_C are ξ-$\mathcal{SC}$projective. By [3, Lemma 4.2], $\gamma f_C = 0$. Using that Σ is an automorphism and a result of Verdier [16], the commutative square on the top left corner below is embedded in a diagram

$$\begin{array}{ccccccc}
P_C & \xrightarrow{0} & \Sigma P_A & \xrightarrow{-\Sigma p} & \Sigma P_B & \xrightarrow{\Sigma q} & \Sigma P_C \\
\downarrow{\scriptstyle f_C} & & \downarrow{\scriptstyle -\Sigma f_A} & & \downarrow{\scriptstyle -\Sigma f_B} & & \downarrow{\scriptstyle \Sigma f_C} \\
C & \xrightarrow{\gamma} & \Sigma A & \xrightarrow{-\Sigma\alpha} & \Sigma B & \xrightarrow{-\Sigma\beta} & \Sigma C \\
\downarrow{\scriptstyle h_C} & & \downarrow{\scriptstyle -\Sigma h_A} & & \downarrow{\scriptstyle -\Sigma h_B} & & \downarrow{\scriptstyle \Sigma h_C} \\
\Sigma K_C & \xrightarrow{-\Sigma\Phi} & \Sigma^2 K_A & \xrightarrow{\Sigma^2\psi} & \Sigma^2 K_B & \xrightarrow{\Sigma^2\omega} & \Sigma^2 K_C \\
\downarrow{\scriptstyle -\Sigma g_C} & & \downarrow{\scriptstyle \Sigma^2 g_A} & & \downarrow{\scriptstyle \Sigma^2 g_B} & & \downarrow{\scriptstyle -\Sigma^2 g_C} \\
\Sigma P_C & \xrightarrow{0} & \Sigma^2 P_A & \xrightarrow{-\Sigma^2 p} & \Sigma^2 P_B & \xrightarrow{\Sigma^2 q} & \Sigma^2 P_C
\end{array},$$

which is commutative except the lower right square which anticommutes and where the rows and columns are triangles. But the above diagram is equivalent to the following commutative diagram:

$$\begin{array}{ccccccc}
K_A & \xrightarrow{-\psi} & K_B & \xrightarrow{\omega} & K_C & \xrightarrow{-\phi} & \Sigma K_A \\
\downarrow{\scriptstyle g_A} & & \downarrow{\scriptstyle g_B} & & \downarrow{\scriptstyle g_C} & & \downarrow{\scriptstyle \Sigma g_A} \\
P_A & \xrightarrow{p} & P_B & \xrightarrow{q} & P_C & \xrightarrow{0} & \Sigma P_A \\
\downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f_C} & & \downarrow{\scriptstyle \Sigma f_A} \\
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & \Sigma A \\
\downarrow{\scriptstyle h_A} & & \downarrow{\scriptstyle h_B} & & \downarrow{\scriptstyle h_C} & & \downarrow{\scriptstyle \Sigma h_A} \\
\Sigma K_A & \xrightarrow{-\Sigma\psi} & \Sigma K_B & \xrightarrow{\Sigma\omega} & \Sigma K_C & \xrightarrow{-\Sigma\phi} & \Sigma^2 K_A
\end{array}.$$

Since the second horizontal triangle is split and P_A, P_C are ξ-projective, P_B is ξ-projective. Applying to the above diagram the homological functor $\mathrm{Hom}_{\mathcal{C}}(P,-)$, $\forall P \in \mathcal{P}(\xi)$, a simple diagram chasing argument shows that $0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P,K_A^1) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P,K_B^1) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P,K_C^1) \longrightarrow 0$ is exact. By Lemma 2.2, the first horizontal triangle is in ξ. Similarly the sencond vertical triangle is in ξ. Since there is the commutative diagram for any $Q \in \mathcal{P}(\xi)$:

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathrm{Hom}_{\mathcal{C}}(C,Q) & \xrightarrow{f} & \mathrm{Hom}_{\mathcal{C}}(B,Q) & \xrightarrow{g} & \mathrm{Hom}_{\mathcal{C}}(A,Q) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(P_C,Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(P_B,Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(P_A,Q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Hom}_{\mathcal{C}}(K_C,Q) & \xrightarrow{h} & \mathrm{Hom}_{\mathcal{C}}(K_B,Q) & \xrightarrow{w} & \mathrm{Hom}_{\mathcal{C}}(K_A,Q) & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & & .
\end{array}$$

Obviously, f is monic and w is epic. Thus g is epic and h is monic. By $\mathrm{Hom}_{\mathcal{C}}(-,Q)$ is a cohomological functor and snake Lemma, the sequence $0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(B,Q) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P_B,Q) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(K_B,Q) \longrightarrow 0$ is exact. Proceeding the above procedure for the triangle $K_A \longrightarrow K_B \longrightarrow K_C \longrightarrow \Sigma K_A$, we get the ξ-projective resolution of B with appropriate properties. Hence B is ξ-$\mathcal{SC}$projective.

Assume that B is ξ-$\mathcal{SC}$projective. By base change, there is a commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}K_C^1 & = & \Sigma^{-1}K_C^1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}A & \longrightarrow & \Sigma^{-1}D & \longrightarrow & \Sigma^{-1}P_C^0 & \longrightarrow & A \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1}A & \longrightarrow & \Sigma^{-1}B & \longrightarrow & \Sigma^{-1}C & \longrightarrow & A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_C^1 & = & K_C^1 & \longrightarrow & 0.
\end{array}$$

Since $\Sigma^{-1}B$ and $\Sigma^{-1}K_C^1$ are ξ-$\mathcal{SC}$projective, we may use the previous case to deduce that $\Sigma^{-1}D$ is ξ-$\mathcal{SC}$projective. Then there exists an ξ-projective resolution of $\Sigma^{-1}D : \cdots \longrightarrow \Sigma^{-1}P_D^1 \longrightarrow \Sigma^{-1}P_D^0 \longrightarrow \Sigma^{-1}D$ satisfying the condition of definition. Since C is ξ-$\mathcal{SC}$projective, there exists a triangle $K_C^1 \xrightarrow{g_C} P_C^0 \xrightarrow{f_C} C \xrightarrow{h_C} \Sigma K_C$ in ξ with P_C^0 ξ-projective and K_C^1 ξ-$\mathcal{SC}$projective and $K_C^1 \longrightarrow P_C^0 \longrightarrow C \longrightarrow \Sigma K_C^1$ is $\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{P}(\xi))$ exact. For any $Q \in \mathcal{P}(\xi)$,there is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \xi xt_{\xi}^0(B,Q) & \longrightarrow & \xi xt_{\xi}^0(D,Q) & \longrightarrow & \xi xt_{\xi}^0(K_C^1,Q) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(B,Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(D,Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(K_C^1,Q) & \longrightarrow & 0
\end{array}$$

by Lemma 2.7. Moreover, there is the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \xi xt_{\xi}^{0}(C,Q) & \longrightarrow & \xi xt_{\xi}^{0}(B,Q) & \longrightarrow & \xi xt_{\xi}^{0}(A,Q) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\mathrm{Hom}_{\mathcal{C}}(\Sigma A,Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(C,Q) & \xrightarrow{\beta^*} & \mathrm{Hom}_{\mathcal{C}}(B,Q) & \xrightarrow{\alpha^*} & \mathrm{Hom}_{\mathcal{C}}(A,Q) & \xrightarrow{\gamma^*} & \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}C,Q)
\end{array}$$

with the below is exact. Since β^* is monic, γ^* is also so. So $0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C,Q) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(B,Q) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,Q) \longrightarrow 0$ is exact. Thus $\Sigma^{-1}A \longrightarrow \Sigma^{-1}D \longrightarrow \Sigma^{-1}P_C^0 \longrightarrow A$ is $\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{P}(\xi))$ exact. Now pasting $\cdots \longrightarrow \Sigma^{-1}P_D^1 \longrightarrow \Sigma^{-1}P_D^0 \longrightarrow \Sigma^{-1}D$ with $\Sigma^{-1}A \longrightarrow \Sigma^{-1}D \longrightarrow \Sigma^{-1}P_C^0 \longrightarrow A$, so A is ξ-$\mathcal{SC}$projective.

2.10. Proposition. Assume that $\mathcal{C}$ is a triangulated category with enough ξ-projective objects. If $X \in \mathcal{SCP}(\xi)$ is ξ-projective relative to $\mathcal{SCP}(\xi)$, then $X \in \mathcal{P}(\xi)$.

Proof. Since $\mathcal{C}$ has enough ξ-projectives, there exists a triangle $K \longrightarrow P \longrightarrow X \longrightarrow \Sigma K$ in ξ with $P \in \mathcal{P}(\xi)$. But X and P are ξ-$\mathcal{SC}$projective, then so is K by Theorem 2.9. Since X is ξ-projective relative to $\mathcal{SCP}(\xi)$, there exists an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,K) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,P) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,X) \longrightarrow 0.$$

So $K \longrightarrow P \longrightarrow X \longrightarrow \Sigma K$ is split. Then $P \cong K \oplus X$. Hence $X \in \mathcal{P}(\xi)$.

It is clear that $\mathcal{SCP}(\xi)$ is closed under countable direct sums. In the following, we use Eilenberg's trick to show that $\mathcal{SCP}(\xi)$ is closed under direct summands.

2.11. Corollary. $\mathcal{SCP}(\xi)$ is closed under direct summands.

Proof. Let A be an object of $\mathcal{SCP}(\xi)$ and B a direct summand of A. So $A = B \oplus B'$, for some object B' of $\mathcal{C}$. Set

$$K = B \oplus B' \oplus B \oplus B' \oplus \cdots .$$

Since $K = A \oplus A \oplus \cdots$ and $\mathcal{SCP}(\xi)$ is closed under countable direct sum, K belongs to $\mathcal{SCP}(\xi)$. We have $K \cong B \oplus K$ and so $B \oplus K$ also belongs to $\mathcal{SCP}(\xi)$. Now consider the split exact triangle

$$B \longrightarrow B \oplus K \longrightarrow K \xrightarrow{0} \Sigma B$$

in ξ to conclude, from the previous Theorem 2.9 , that B belongs to $\mathcal{SCP}(\xi)$.

Now we introduce a new invariant for an object A of $\mathcal{C}$, namely its ξ-$\mathcal{SC}$projective dimension, ξ-$\mathcal{SC}$pd$A$. It is defined inductively. When A=0, put ξ-$\mathcal{SC}$pd$A = -1$. If $A \in \mathcal{SCP}(\xi)$, then ξ-$\mathcal{SC}$pd$A = 0$. Next by induction, for an integer $n > 0$, put ξ-$\mathcal{SC}$pd$A \leq n$ if there exists a triangle $K \longrightarrow P \longrightarrow A \longrightarrow \Sigma K$ in $\mathcal{C}$ with $P \in \mathcal{SCP}(\xi)$ and ξ-$\mathcal{SC}$pd$K \leq n-1$.

We define ξ-$\mathcal{SC}$pd$A = n$ if ξ-$\mathcal{SC}$pd$A \leq n$ and ξ-$\mathcal{SC}$pd$A \nleq n-1$. If ξ-$\mathcal{SC}$pd$A \neq n$ for all $n \geq 0$, we set ξ-$\mathcal{SC}$pd$A = \infty$.

2.12. Theorem. Assume that $\mathcal{C}$ is a triangulated category with enough ξ-projective objects and $A \longrightarrow B \longrightarrow C \xrightarrow{\gamma} \Sigma A$ is a triangle in ξ such that $A \neq 0$ and C is ξ-$\mathcal{SC}$projective. Then ξ-$\mathcal{SC}$pd$A = \xi$-$\mathcal{SC}$pd$B$.

Proof. The result is clear from Theorem 2.9 if one of A or B is ξ-$\mathcal{SC}$projective. Let ξ-$\mathcal{SC}$pd$A = n > 0$. So there exists a triangle $K_A \longrightarrow P_A \longrightarrow A \longrightarrow \Sigma K_A$ in ξ where P_A is ξ-$\mathcal{SC}$projective and ξ-$\mathcal{SC}$pd$K_A = n-1$. Since C is ξ-$\mathcal{SC}$projective, there exists a triangle $K_C \longrightarrow P_C \longrightarrow C \longrightarrow \Sigma K_C$ in

ξ where P_C is ξ-projective and K_C is ξ-$\mathcal{SC}$projective. Then by the proof of Theorem 2.9, we have the following commutative diagram:

$$
\begin{array}{ccccccc}
P_C & \xrightarrow{0} & \Sigma P_A & \longrightarrow & \Sigma P_B & \longrightarrow & \Sigma P_C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C & \xrightarrow{\gamma} & \Sigma A & \longrightarrow & \Sigma B & \longrightarrow & \Sigma C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma K_C & \longrightarrow & \Sigma^2 K_A & \longrightarrow & \Sigma^2 K_B & \longrightarrow & \Sigma^2 K_C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma P_C & \longrightarrow & \Sigma^2 P_A & \longrightarrow & \Sigma^2 P_B & \longrightarrow & \Sigma^2 P_C,
\end{array}
$$

which is commutative except the lower right square which anticommutes and where the rows and columns are triangles. This is equivalent to the commutative diagram:

$$
\begin{array}{ccccccc}
K_A & \longrightarrow & K_B & \longrightarrow & K_C & \longrightarrow & \Sigma K_A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P_A & \longrightarrow & P_B & \longrightarrow & P_C & \xrightarrow{0} & \Sigma P_A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & B & \longrightarrow & C & \xrightarrow{\gamma} & \Sigma A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma K_A & \longrightarrow & \Sigma K_B & \longrightarrow & \Sigma K_C & \longrightarrow & \Sigma^2 K_A,
\end{array}
$$

in which the first three vertical and horizontal diagrams are triangles. The second horizontal triangle is split, and so belongs to ξ. Since P_A and P_C are both ξ-$\mathcal{SC}$projective, it follows from that P_B is also ξ-$\mathcal{SC}$projective. Applying $\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),-)$ to the above commutative diagram , by Lemma 2.2 and diagram chasing argument, the first horizontal and also second vertical triangles are $\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),-)$ exact and so belong to ξ. Now consider the triangle $K_A \longrightarrow K_B \longrightarrow K_C \longrightarrow \Sigma K_A$ in ξ, in which ξ-$\mathcal{SC}\mathrm{pd}K_A = n-1$ and use induction to deduce that ξ-$\mathcal{SC}\mathrm{pd}K_B = n-1$ and hence ξ-$\mathcal{SC}\mathrm{pd}B = n$.

Now suppose ξ-$\mathcal{SC}\mathrm{pd}B = n$. So there exists a triangle $K_B \longrightarrow P_B \longrightarrow B \longrightarrow \Sigma K_B$ in ξ, where P_B is ξ-$\mathcal{SC}$projective and ξ-$\mathcal{SC}\mathrm{pd}K_B = n-1$. Using (Tr2) and base change in Proposition 1.1, we get the following commutative diagram:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & K_B & = & K_B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1} C & \longrightarrow & P_A & \longrightarrow & P_B & \longrightarrow & C \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1} C & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K_B & = & \Sigma K_B & \longrightarrow & 0
\end{array}
$$

in which all horizontal and vertical diagrams are triangles. Since the third horizontal and third vertical triangles are in ξ, one can show the second horizontal and second vertical triangles are $\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),-)$ exact and so belong to ξ. Because both P_B and $\Sigma^{-1}C$ are ξ-$\mathcal{SC}$projective, by Theorem 2.9, so is P_A. So ξ-$\mathcal{SC}\mathrm{pd}A = n$.

2.13. Lemma. ([2, Proposition 3.15]) Let the following be a commutative diagram such that rows are triangles in ξ:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'. \end{array}$$

Then it may be completed to a morphism of triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

so that $X \longrightarrow X' \oplus Y \longrightarrow Y' \longrightarrow \Sigma X$ is a triangle in ξ.

2.14. Proposition. Assume that $\mathcal{C}$ is a triangulated category with enough ξ-projective objects and let A be an object of $\mathcal{C}$. Then the following are equivalent:

(i) ξ-$\mathcal{SC}\mathrm{pd}A \leq n$.

(ii) In any ξ-exact sequence $0 \longrightarrow B \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$, if P_i are ξ-$\mathcal{SC}$projective, then so is B.

Proof. (i) $\Rightarrow$ (ii). There exits a triangle $K \longrightarrow Q \longrightarrow A \longrightarrow \Sigma K$ in ξ where Q is ξ-$\mathcal{SC}$projective and ξ-$\mathcal{SC}\mathrm{pd}K \leq n-1$. Since $0 \longrightarrow B \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ is ξ-exact, by definition of ξ-exact sequence, there exists a triangle $K_1 \longrightarrow P_0 \longrightarrow A \longrightarrow \Sigma K$ in ξ. Since $\mathcal{C}$ have enough ξ-projectives, there exists a triangle $L \longrightarrow P \longrightarrow A \longrightarrow \Sigma L$ in ξ with P ξ-projective. So we can construct morphisms of triangles:

$$\begin{array}{ccccccc} L & \longrightarrow & P & \longrightarrow & A & \longrightarrow & \Sigma L \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & Q & \longrightarrow & A & \longrightarrow & \Sigma K, \end{array} \quad \begin{array}{ccccccc} L & \longrightarrow & P & \longrightarrow & A & \longrightarrow & \Sigma L \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & \Sigma K_1 \end{array}$$

Now consider the diagrams

$$\begin{array}{ccccccc} L & \longrightarrow & P & \longrightarrow & A & \longrightarrow & \Sigma L \\ \downarrow & & & & \downarrow & & \downarrow \\ K & \longrightarrow & Q & \longrightarrow & A & \longrightarrow & \Sigma K, \end{array} \quad \begin{array}{ccccccc} L & \longrightarrow & P & \longrightarrow & A & \longrightarrow & \Sigma L \\ \downarrow & & & & \downarrow & & \downarrow \\ K_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & \Sigma K_1 \end{array}$$

where the rows are triangles in ξ. By Lemma 2.13, we can complete them such that $L \longrightarrow K \oplus P \longrightarrow Q \longrightarrow \Sigma L$ and $L \longrightarrow K_1 \oplus P \longrightarrow P_0 \longrightarrow \Sigma L$ are both triangles in ξ. Since Q is ξ-$\mathcal{SC}$projective, by Theorem 2.12, ξ-$\mathcal{SC}\mathrm{pd}L = \xi$-$\mathcal{SC}\mathrm{pd}(K \oplus P)$. Since P_0 is ξ-$\mathcal{SC}$projective, by Theorem 2.12, ξ-$\mathcal{SC}\mathrm{pd}L = \xi$-$\mathcal{SC}\mathrm{pd}(K_1 \oplus P)$. Thus ξ-$\mathcal{SC}\mathrm{pd}(K \oplus P) = \xi$-$\mathcal{SC}\mathrm{pd}(K_1 \oplus P)$. But $K \longrightarrow K \oplus P \longrightarrow P \longrightarrow \Sigma K$ and $K_1 \longrightarrow K_1 \oplus P \longrightarrow P \longrightarrow \Sigma K_1$ are split triangles and so are in ξ, and P is ξ-projective, so is ξ-$\mathcal{SC}$projective. By Theorem 2.12 again, then ξ-$\mathcal{SC}\mathrm{pd}K = \xi$-$\mathcal{SC}\mathrm{pd}K_1$. The proof now can be completed by induction.

(ii) $\Rightarrow$ (i). Since $\mathcal{C}$ has enough ξ-projectives, there exists a ξ-exact complex

$$0 \longrightarrow B \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

where each P_i is ξ-projective. So by assumption B is ξ-$\mathcal{SC}$projective. This gives the result.

2.15. Proposition. Assume that $\mathcal{C}$ is a triangulated category with enough ξ-projective objects and let ξ-$\mathcal{SC}$pd$A \leq 1$ and $\xi xt^1_\xi(A,P) = 0$ for all $P \in \mathcal{P}(\xi)$. Then A is ξ-$\mathcal{SC}$projective.

Proof. Since $\mathcal{C}$ has enough ξ-projectives. We have a triangle $K \xrightarrow{f} P_0 \xrightarrow{g} A \xrightarrow{h} \Sigma K$ in ξ, where P_0 is ξ-projective. By proposition 2.14, K is ξ-$\mathcal{SC}$projective. Thus we have the following commutative diagram for any $P \in \mathcal{P}(\xi)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \xi xt^0_\xi(A,P) & \longrightarrow & \xi xt^0_\xi(P_0,P) & \longrightarrow & \xi xt^0_\xi(K,P) & \longrightarrow & \xi xt^1_\xi(A,P) = 0 \\
 & & \Big\downarrow \cong & & \alpha \Big\downarrow \cong & & \beta \Big\downarrow \cong & & \\
\mathrm{Hom}_{\mathcal{C}}(\Sigma K,P) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A,P) & \xrightarrow{g^*} & \mathrm{Hom}_{\mathcal{C}}(P_0,P) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathcal{C}}(K,P) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}A,P),
\end{array}$$

in which the rows are exact. Since the two isomorphisms α, β hold by Lemma 2.7, f^* is epic. So g^* is monic. Hence $K \xrightarrow{f} P_0 \xrightarrow{g} A \xrightarrow{h} \Sigma K$ is $\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{P}(\xi))$ exact. Since $\mathcal{C}$ has enough ξ-projectives, we have a triangle $K_1 \longrightarrow P_1 \longrightarrow K \longrightarrow \Sigma K_1$ in ξ with P_1 ξ-projective. Thus K_1 is ξ-$\mathcal{SC}$projective by Theorem 2.9. By Proposition 2.6 and Lemma 2.7, there is an exact sequence

$$0 \longrightarrow \xi xt^0_\xi(K,P) \longrightarrow \xi xt^0_\xi(P_1,P) \longrightarrow \xi xt^0_\xi(K_1,P) \longrightarrow 0.$$

This is equivalent to

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(K,P) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P_1,P) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(K_1,P) \longrightarrow 0$$

is exact. So $K_1 \longrightarrow P_1 \longrightarrow K \longrightarrow \Sigma K_1$ is also $\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{P}(\xi))$ exact. Proceeding this procedure, we get ξ-projective resolution of A satisfying the condition of definition of ξ-$\mathcal{SC}$projective object.

2.16. Theorem. Assume that $\mathcal{C}$ is a triangulated category with enough ξ-projective objects and let $A \in \mathcal{C}$ be of finite ξ-$\mathcal{SC}$projective dimension. Then ξ-$\mathcal{SC}$pd$A \leq n$ if and only if $\xi xt^i_\xi(A,Q) = 0$ for any $Q \in \widetilde{\mathcal{P}}(\xi)$ and $i > n$.

Proof. Let ξ-$\mathcal{SC}$pd$A \leq n$. So there exists a ξ-exact complex

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

with P_i ξ-$\mathcal{SC}$projective. But now in view of Lemma 2.7 and using the corresponding triangles, we see that $\xi xt^i_\xi(P_n,Q) \cong \xi xt^{n+i}_\xi(A,Q) = 0$ for all $i \geq 1$.

Let $0 \longrightarrow B \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ is ξ-exact sequence with P_i ξ-projective. Since ξ-$\mathcal{SC}$pd$A < \infty$, ξ-$\mathcal{SC}$pd$B < \infty$. Suppose ξ-$\mathcal{SC}$pd$B = m$. Then there exists an ξ-exact $\mathcal{SC}$projective resolution

$$0 \longrightarrow G_m \longrightarrow G_{m-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow B \longrightarrow 0,$$

with G_i ξ-$\mathcal{SC}$projective. Next we show that B is ξ-$\mathcal{SC}$projective. Consider a triangle $G_m \longrightarrow G_{m-1} \longrightarrow K_{m-1} \longrightarrow \Sigma G_m$ in ξ, where ξ-$\mathcal{SC}$pd$K_{m-1} \leq 1$. For any $Q \in \mathcal{P}(\xi)$, $\xi xt^1_\xi(K_{m-1},Q) \cong \xi xt^m_\xi(B,Q) \cong \xi xt^{m+n}_\xi(A,Q) = 0$. By Proposition 2.15, K_{m-1} is ξ-$\mathcal{SC}$projective. Proceeding this procedure, we get B is ξ-$\mathcal{SC}$projective. So ξ-$\mathcal{SC}$pd$A \leq n$.

3. ξ-$\mathcal{SC}$projective precover

3.1. Definition. Let A be an object of $\mathcal{C}$. A morphism $G \longrightarrow A$ where G is ξ-$\mathcal{SC}$projective is called a ξ-$\mathcal{SC}$projective precover of A if it can be completed to an $\mathrm{Hom}_{\mathcal{C}}(\mathcal{SCP}(\xi),-)$-exact triangle $K \longrightarrow G \longrightarrow A \longrightarrow \Sigma K$ in ξ.

The following proposition implies that the existence of ξ-$\mathcal{SC}$projective precover.

3.2. Theorem. Let A be an object of $\mathcal{C}$ of finite ξ-projective dimension. Then there exists an ξ-$\mathcal{SC}$projective precover.

Proof. By definition of ξ-projective dimension in [3], there exists a triangle $K \overset{f}{\longrightarrow} P \overset{g}{\longrightarrow} A \overset{h}{\longrightarrow} \Sigma K$ with P ξ-projective and ξ-pd$K < \infty$. For any ξ-$\mathcal{SC}$projective object Q, $\xi xt^1_\xi(Q,K) = 0$ by Lemma 2.7. But $\mathrm{Hom}_{\mathcal{C}}(Q,-)$ is a cohomological functor, then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \xi xt^0_\xi(Q,K) & \longrightarrow & \xi xt^0_\xi(Q,P) & \longrightarrow & \xi xt^0_\xi(Q,A) & \longrightarrow & \xi xt^1_\xi(Q,K)=0 \\
 & & \Big\downarrow \cong & & \alpha \Big\downarrow \cong & & \beta \Big\downarrow \cong & & \\
\mathrm{Hom}_{\mathcal{C}}(Q,\Sigma^{-1}A) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,K) & \overset{f_*}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(Q,P) & \overset{g_*}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(Q,A) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,\Sigma K),
\end{array}$$

in which the rows are exact. Since the two isomorphisms α, β hold by Lemma 2.7, g_* is epic. Thus f_* is monic. Hence $K \overset{f}{\longrightarrow} P \overset{g}{\longrightarrow} A \overset{h}{\longrightarrow} \Sigma K$ is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{SCP}(\xi),-)$-exact. Then $P \longrightarrow A$ is a ξ-$\mathcal{SC}$projective precover of A.

3.3. Proposition. Assume that $K_1 \overset{f_1}{\longrightarrow} P_1 \overset{g_1}{\longrightarrow} A \overset{h_1}{\longrightarrow} \Sigma K_1$ and $K_2 \overset{f_2}{\longrightarrow} P_2 \overset{g_2}{\longrightarrow} A \overset{h_2}{\longrightarrow} \Sigma K_2$ are triangles in ξ, where $P_1 \overset{g_1}{\longrightarrow} A$ and $P_2 \overset{g_1}{\longrightarrow} A$ are both ξ-$\mathcal{SC}$projective precovers of A. Then $K_1 \oplus P_2 \cong K_2 \oplus P_1$.

Proof. According to the base change in Proposition 1.1, we get the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_2 & \overset{=}{\longrightarrow} & K_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow \alpha & & \downarrow f_2 & & \downarrow \\
K_1 & \overset{f_1'}{\longrightarrow} & Y & \overset{g_1'}{\longrightarrow} & P_2 & \overset{h_1'}{\longrightarrow} & \Sigma K_1 \\
\| & & \downarrow \beta & & \downarrow g_2 & & \| \\
K_1 & \overset{f_1}{\longrightarrow} & P_1 & \overset{g_1}{\longrightarrow} & A & \overset{h_1}{\longrightarrow} & \Sigma K_1 \\
\downarrow & & \downarrow \gamma & & \downarrow h_2 & & \downarrow \\
0 & \longrightarrow & \Sigma K_2 & = & \Sigma K_2 & \longrightarrow & 0.
\end{array}$$

Since $P_2 \overset{g_2}{\longrightarrow} A$ is an ξ-$\mathcal{SC}$projective precover of A, there is an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(P_2,K_1) \overset{\alpha}{\longrightarrow} \mathrm{Hom}_{\mathcal{C}}(P_2,P_1) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P_2,A) \overset{(h_1)_*}{\longrightarrow} \mathrm{Hom}_{\mathcal{C}}(P_2,\Sigma K_1),$$

such that $(h_1)_* g_2 = 0$, i.e. $h_1 g_2 = 0$. Thus $h_1' = 0$. Then the second rows is split. Hence $Y \cong K_1 \oplus P_2$. Since $P_1 \overset{g_1}{\longrightarrow} A$ is an ξ-$\mathcal{SC}$projective precover of A, there is an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(P_1,K_2) \overset{\alpha}{\longrightarrow} \mathrm{Hom}_{\mathcal{C}}(P_1,P_2) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P_1,A) \overset{(h_2)_*}{\longrightarrow} \mathrm{Hom}_{\mathcal{C}}(P_1,\Sigma K_2),$$

such that $(h_2)_* g_1 = 0$, i.e. $h_2 g_1 = 0$. Thus $\gamma = 0$. Then the second column is split. So $Y \cong K_2 \oplus P_1$. Hence $K_2 \oplus P_1 \cong K_1 \oplus P_2$.

3.4. Definition. A ξ-$\mathcal{SC}$projective resolution of $A \in \mathcal{C}$ is a ξ-exact complex

$$\mathbf{P} := \cdots P_{n+1} \overset{d^{n+1}}{\longrightarrow} P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

such that $P_n \in \mathcal{SCP}(\xi)$ and for any $n \in \mathbb{N}_0$, in the relevant triangle $K_n \longrightarrow P_n \longrightarrow K_{n-1} \longrightarrow \Sigma K_n (n \geq 0) P_n \longrightarrow K_{n-1}$ is the ξ-$\mathcal{SC}$projective precover of K_{n-1}, in which $K_{-1} = A$. The resolution is said to be of length n if $P_n \neq 0$ and $P_i = 0$ for all $i > n$.

3.5. Definition. Let $\mathbf{P} := \cdots P_{n+1} \overset{d^{n+1}}{\longrightarrow} P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ be an ξ-$\mathcal{SC}$projective resolution of $A \in \mathcal{C}$. Then define $\xi xt^n_{\mathcal{SCP}(\xi)}(A,B)$ to be the nth-cohomology of the induced complex $\mathrm{Hom}_{\mathcal{C}}(\mathbf{P},B)$ for any $B \in \mathcal{C}$.

Remark. By the comparison theorem the above ξ-derived functors are well defined.

3.6. Corollary. Let $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A_1 \longrightarrow 0$ and $0 \longrightarrow P'_n \longrightarrow P'_{n-1} \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow A_2 \longrightarrow 0$ be ξ-$\mathcal{SC}$projective resolution of A_1 and A_2 respectively. If $A_1 \cong A_2$, then $P_0 \oplus P'_1 \oplus P_2 \oplus \cdots \cong P'_0 \oplus P_1 \oplus P'_2 \oplus \cdots$.

3.7. Proposition. Suppose $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$ in ξ such that $0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P,A) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P,C) \longrightarrow 0$ is exact for all $P \in \mathcal{SCP}(\xi)$. If $\cdots \longrightarrow P'_1 \longrightarrow P'_0 \xrightarrow{f'_0} A \longrightarrow 0$ and $\cdots \longrightarrow P''_1 \longrightarrow P''_0 \xrightarrow{f''_0} C \longrightarrow 0$ are ξ-$\mathcal{SC}$projective resolutions of A and C respectively, then there exists a ξ-$\mathcal{SC}$projective resolution of B.

Proof. Since $0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P''_0,A) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P''_0,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P''_0,C) \longrightarrow 0$ is exact with $P''_0 \in \mathcal{SCP}(\xi)$, $\gamma f''_0 = 0$. Using that Σ is an automorphism and a result of Verdier (see [16]), the commutative square on the top left corner below is embedded in a diagram

$$\begin{array}{ccccccc}
P''_0 & \xrightarrow{0} & \Sigma P'_0 & \xrightarrow{-\Sigma(p)} & \Sigma P_0 & \xrightarrow{\Sigma q} & \Sigma P''_0 \\
\Big\downarrow{\scriptstyle f''_0} & & \Big\downarrow{\scriptstyle -\Sigma f'_0} & & \Big\downarrow{\scriptstyle -\Sigma f_0} & & \Big\downarrow{\scriptstyle \Sigma f''_0} \\
C & \xrightarrow{\gamma} & \Sigma A & \xrightarrow{-\Sigma\alpha} & \Sigma B & \xrightarrow{-\Sigma\beta} & \Sigma C \\
\Big\downarrow{\scriptstyle h''_0} & & \Big\downarrow{\scriptstyle -\Sigma h'_0} & & \Big\downarrow{\scriptstyle -\Sigma h_0} & & \Big\downarrow{\scriptstyle \Sigma h''_0} \\
\Sigma K''_1 & \xrightarrow{-\Sigma\phi} & \Sigma^2 K'_1 & \xrightarrow{\Sigma^2\psi} & \Sigma^2 K_1 & \xrightarrow{\Sigma^2\omega} & \Sigma^2 K''_1 \\
\Big\downarrow{\scriptstyle -\Sigma g''_0} & & \Big\downarrow{\scriptstyle \Sigma^2 g'_0} & & \Big\downarrow{\scriptstyle \Sigma^2 g_0} & & \Big\downarrow{\scriptstyle -\Sigma^2 g''_0} \\
\Sigma P''_0 & \xrightarrow{0} & \Sigma^2 P'_0 & \xrightarrow{\Sigma^2 p} & \Sigma^2 P_0 & \xrightarrow{\Sigma^2 q} & \Sigma^2 P''_0
\end{array}$$

which is commutative except the lower right square which anticommutes and where the rows and columns are triangles. Then we have the following commutative diagram in which the first three vertical and horizontal diagrams are triangles:

$$\begin{array}{ccccccc}
K'_1 & \xrightarrow{\psi} & K_1 & \xrightarrow{\omega} & K''_1 & \xrightarrow{-\phi} & \Sigma K'_1 \\
\Big\downarrow{\scriptstyle g'_0} & & \Big\downarrow{\scriptstyle g_0} & & \Big\downarrow{\scriptstyle g''_0} & & \Big\downarrow{\scriptstyle \Sigma g'_0} \\
P'_0 & \xrightarrow{p} & P_0 & \xrightarrow{q} & P''_0 & \xrightarrow{0} & \Sigma P'_0 \\
\Big\downarrow{\scriptstyle f'_0} & & \Big\downarrow{\scriptstyle f_0} & & \Big\downarrow{\scriptstyle f''_0} & & \Big\downarrow{\scriptstyle \Sigma f'_0} \\
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & \Sigma A \\
\Big\downarrow{\scriptstyle h'_0} & & \Big\downarrow{\scriptstyle h_0} & & \Big\downarrow{\scriptstyle h''_0} & & \Big\downarrow{\scriptstyle -\Sigma h'_0} \\
\Sigma K'_1 & \xrightarrow{-\Sigma\psi} & \Sigma K_1 & \xrightarrow{\Sigma\omega} & \Sigma K''_1 & \xrightarrow{-\Sigma\phi} & \Sigma^2 K'_1.
\end{array}$$

Since the second horizontal triangle is split, we have $P_0 \in \mathcal{SCP}(\xi)$. Applying the cohomological $\mathrm{Hom}_{\mathcal{C}}(P,-)$ to the above diagram for any $P \in \mathcal{P}(\xi)$, a simple chasing argument shows that the first horizontal triangle and the second vertical triangle are both in ξ. Applying to the above diagram the cohomological $\mathrm{Hom}_{\mathcal{C}}(Q,-)$ for any $Q \in \mathcal{SCP}(\xi)$, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,K_1') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,K_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,K_1'') & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,P_0') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,P_0) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,P_0'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,A) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,B) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

Easily, we get dotted arrows. Then the second row and the second column are both exact. Inductively the above procedure completes the proof.

3.8. Definition. ([3, Definition 4.14]) Let $\mathcal{C}$ be a triangle category and $\mathcal{D}$ is a subcategory of $\mathcal{C}$. $\mathcal{D}$ is called generating subcategory if $\mathcal{D}$ is Σ-stable and $\mathrm{Hom}_{\mathcal{C}}(\mathcal{D},A) = 0 \Rightarrow A = 0$ for any $A \in \mathcal{C}$.

3.9. Theorem. If $\mathcal{SCP}(\xi)$ is a generating subcategory of a triangulated category $\mathcal{C}$, then the following two conditions are equivalent for any $A \in \mathcal{C}$ and $n \geq 0$:
(i) $\xi xt^{n+1}_{\mathcal{SCP}(\xi)}(A,B) = 0$ for any $B \in \mathcal{C}$;
(ii) there exists an ξ-$\mathcal{SC}$projective resolution $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$.

Proof. (ii) $\Rightarrow$ (i). It is obvious.

(i) $\Rightarrow$ (ii). Let $\cdots \longrightarrow P_{n+2} \xrightarrow{d_{n+2}} P_{n+1} \xrightarrow{d_{n+1}} P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ be an ξ-$\mathcal{SC}$projective resolution of A, where $d_n = g_{n-1}f_n$ and $P_n \xrightarrow{f_n} K_n$ is ξ-$\mathcal{SC}$projective precover of K_n, $\forall n \geq 0$. Since $\xi xt^{n+1}_{\mathcal{SCP}(\xi)}(A,B) = 0$ for any $B \in \mathcal{C}$, the complex

$$\mathrm{Hom}_{\mathcal{C}}(P_n,K_{n+1}) \xrightarrow{d^*_{n+1}} \mathrm{Hom}_{\mathcal{C}}(P_{n+1},K_{n+1}) \xrightarrow{d^*_{n+2}} \mathrm{Hom}_{\mathcal{C}}(P_{n+2},K_{n+1})$$

implies $\mathrm{Im}\, d^*_{n+1} = \mathrm{Ker}\, d^*_{n+2}$. But $f_{n+1}g_{n+1}f_{n+2} = 0$, then $f_{n+1}d_{n+2} = 0$. That is to say, $d^*_{n+2}f_{n+1} = 0$, i.e. $f_{n+1} \in \mathrm{Ker}\, d^*_{n+2}$. So there exists $\alpha : P_n \longrightarrow K_{n+1}$ such that $f_{n+1} = d_{n+1}\alpha$. Applying the functor $\mathrm{Hom}_{\mathcal{C}}(P,-)$, $\forall P \in \mathcal{SCP}(\xi)$, to the triangle $K_{n+1} \xrightarrow{g_n} P_n \xrightarrow{f_n} K_n \longrightarrow \Sigma K_{n+1}$, we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P,K_{n+1}) \xrightarrow{g_{n*}} \mathrm{Hom}_{\mathcal{C}}(P,P_n) \xrightarrow{f_{n*}} \mathrm{Hom}_{\mathcal{C}}(P,K_n) \longrightarrow 0.$$

Since $\alpha : P_n \longrightarrow K_{n+1}$ is ξ-$\mathcal{SC}$projective precover, $\mathrm{Hom}_{\mathcal{C}}(P,P_n) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathcal{C}}(P,K_{n+1})$ is epic. So $\alpha_* g_{n*} = 1_{\mathrm{Hom}_{\mathcal{C}}(P,K_{n+1})}$. Then the above exact sequence is split. So $\mathrm{Hom}_{\mathcal{C}}(P,P_n) \cong \mathrm{Hom}_{\mathcal{C}}(P,K_n \oplus K_{n+1})$. But $\mathcal{SCP}(\xi)$ be generating subcategory, then $P_n \cong K_n \oplus K_{n+1}$. Hence K_n is ξ-$\mathcal{SC}$projective. Thus the proof is completed.

Remark. Similar to the way that we define ξ-$\mathcal{SC}$projective objects, one can define ξ-$\mathcal{SC}$injective objects of triangulated category $\mathcal{C}$. The conclusions and their proofs in Sections 2 and 3 dualize perfectly, so all the resluts in these sections have valid analogs in terms of ξ-$\mathcal{SC}$injective objects.

4. Conclusions and a future work

In this paper, we generalize the notion of strongly copure projective modules in category of module to triangulated category, which is called to be ξ-strongly copure projective objects. This extends the notions of ξ-projective objects and ξ-Gorenstein projective objects in triangulated categories. We prove that $\mathcal{SCP}(\xi)$ has a resolving property in Theorem 2.9. We discuss the ξ-strongly copure projective dimension and show the relation between ξ-$\mathcal{SC}$pdA and $\xi xt^i_{\xi}(A,-)$ for any object A of $\mathcal{C}$ in Theorem 2.16. We also introduce the concept of ξ-$\mathcal{SC}$projective precover and investigate the existence in Theorem 3.2, and moreover, characterize the ξ-$\mathcal{SC}$projective resolution of object A by the functor $\xi xt^i_{\mathcal{SCP}(\xi)}(A,-)$ in Theorem 3.9.

Referee of this paper has suggested to study a relative quasi-Frobenius property of the category $\mathcal{C}$ in connection with the results obtained in [8] for module categories and in [14] for locally finitely presented Grothendieck categories. Following Referee's suggestion, we intend to study in future the following problem:

Problem 1. Assume that $\mathcal{C}$ is a triangulated category with enough ξ-projective objects as in Section 2. When are the following conditions equivalent?

(i) every ξ-$\mathcal{SC}$projective object in $\mathcal{C}$ is ξ-$\mathcal{SC}$injective;

(ii) every ξ-$\mathcal{SC}$injective object $\mathcal{C}$ is ξ-$\mathcal{SC}$projective;

(iii) every object in $\mathcal{C}$ is ξ-$\mathcal{SC}$projective or ξ-$\mathcal{SC}$injective (that is the global dimension of $\mathcal{C}$ is zero),

Let us recall that the equivalence of these three conditions are proved 40 years ago in [8] for the usual fp-purity in module categories and the equivalence is proved in [14] for the usual fp-purity in any locally finitely presented Grothendieck categories. Moreover, an analogous problem is also discussed in [3].

In the category R-Mod of unitary left modules over a ring R with an identity element, the classical equality is

$$\sup\{\mathrm{pd}_R A| \text{ for any left } R\text{-module } A\} = \sup\{\mathrm{id}_R A| \text{ for any left } R\text{-module } A\}$$

established in [13, Theorem 8.14] is extended by D. Bennis and N. Mahdou in [4] to the equality

$$\sup\{\mathrm{Gpd}_R A| \text{ for any left } R\text{-module } A\} = \sup\{\mathrm{Gid}_R A| \text{ for any left } R\text{-module } A\}$$

where $\mathrm{pd}_R A$(res. $\mathrm{id}_R A$) means the projective(res. injective) dimension of A, $\mathrm{Gpd}_R A$(res. $\mathrm{Gid}_R A$) means the Gorenstein projective(res. injective) dimension of A. Naturally, we also try to find some conditions such that the following conclusion holds in a triangulated category $\mathcal{C}$.

Problem 2. $\sup\{\xi\text{-}\mathcal{SC}\mathrm{pd}A| \text{ for any } A \in \mathcal{C}\} = \sup\{\xi\text{-}\mathcal{SC}\mathrm{id}A| \text{ for any } A \in \mathcal{C}\}$.

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Quantale algebras as a generalization of lattice-valued frames

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Abstract

Recently, I. Stubbe constructed an isomorphism between the categories of right Q-modules and cocomplete skeletal Q-categories for a given unital quantale Q. Employing his results, we obtain an isomorphism between the categories of Q-algebras and Q-quantales, where Q is additionally assumed to be commutative. As a consequence, we provide a common framework for two concepts of lattice-valued frame, which are currently available in the literature. Moreover, we obtain a convenient setting for lattice-valued extensions of the famous equivalence between the categories of sober topological spaces and spatial locales, as well as for answering the question on its relationships to the notion of stratification of lattice-valued topological spaces.

Keywords: (Cocomplete) (skeletal) Q-category, lattice-valued frame, lattice-valued partially ordered set, quantale, quantale algebra, quantale module, sober topological space, spatial locale, stratification degree, stratified topological space.

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1. Introduction

1.1. Lattice-valued frames. The well-known equivalence between the categories of sober topological spaces and spatial locales, initiated by D. Papert and S. Papert [42], and developed in a rigid way by J. R. Isbell [31] and P. T. Johnstone [32], opened an important relationship between general topology and universal algebra. In particular, it provided a convenient framework for the famous topological representation theorems of M. Stone for Boolean algebras [71] and distributive lattices [72], which in their turn (backed by the celebrated representation of distributive lattices of H. Priestley [45] and the plethora of its induced results) started the theory of natural dualities [8], presenting a general machinery (based in some elements of category theory, but for the most part in universal algebra) for obtaining topological representations of algebraic structures. The success of the evolving theory is mostly due to the fact that it translates algebraic problems, usually stated in an abstract symbolic language, into dual, topological problems, where geometric intuition comes to our help.

No wonder then that the beginning of the fuzzy era of L. A. Zadeh [82] and J. A. Goguen [17], together with the almost immediate fuzzification of the concept of topological space by C. L. Chang [7], R. Lowen [38] and S. E. Rodabaugh [51], turned the attention of the newly appearing fuzzy researchers to the fuzzification of the above-mentioned sobriety-spatiality equivalence. One of the first and the most successful attempt was made by S. E. Rodabaugh [52], who presented its both fixed- and variable-basis extensions, bringing the theory to its completion in [54, 56], thereby streamlining the initial machinery of P. T. Johnstone.

It soon appeared, however, that to develop properly lattice-valued pointless topology, one needs the corresponding lattice-valued generalization of locales, which should not be a direct fuzzification of the corresponding algebraic structure in the sense of fuzzy groups of A. Rosenfeld [59] and J. M. Anthony and H. Sherwood [4], or, more generally, lattice-valued algebras of A. Di Nola and G. Gerla [10], but should be capable of restoring the point-theoretic structure from a given extended locale of a lattice-valued topological space. One of the pioneering endeavors in this respect is due to D. Zhang and Y.-M. Liu [84], who introduced the concept of L-fuzzy locale as a frame homomorphism $L \xrightarrow{i_A} A$ and provided a lattice-valued sobriety-spatiality equivalence for the respective category of these structures (the comma category ($\mathbf{Loc} \downarrow L$)). A similar viewpoint was taken by W. Yao [79], who introduced L-frames through the notion of L-partially ordered set of L. Fan [12]. Moreover, in [78, 80], he developed the theory of lattice-valued domains, based in his newly established framework of L-order. Later on, W. Yao [81] constructed an isomorphism between his category of L-frames and the category of L-fuzzy frames of D. Zhang and Y.-M. Liu. On the other hand, there exists another and more sophisticated notion of lattice-valued frame, introduced by A. Pultr and S. E. Rodabaugh [48] and induced by the Lowen-Kubiák ι_L (fibre map) functor [37, 38], the latter providing a way of obtaining a crisp topological space from a lattice-valued one (it is important to notice that there exists another approach to the just mentioned fuzzy-crisp topological space passage, suggested by the notion of attachment of C. Guido [19] (see also [13, 14, 15, 20]), which extends the hypergraph functor of the fuzzy community [26]; whether the notion of attachment has its corresponding concept of lattice-valued frame is still an open and challenging question). The theory was given its maturity in [49, 50], which presented a new presheaf motivation for the concept as well as studied categorical properties of lattice-valued frames and deepened their relationships to lattice-valued topology.

1.2. Lattice-valued quantales. Motivated by the above-mentioned fuzzifications of the sobriety-spatiality equivalence, we extended the obtained theory in several ways [63, 65, 66, 67], thereby initiating categorically-algebraic topology [62], introduced as a common framework for the majority of modern approaches to lattice-valued topology, in order to provide convenient means of interaction between different theories. In particular, in [67] (see also [68]), we considered the notion of algebra over a given unital commutative quantale as a generalization of the concept of quantale module [36, 43, 61], whose theory has already been established as an important part of universal algebra, extending the classical theory of modules over a ring [3]. After a brief consideration, it became clear to us that the above-mentioned result of W. Yao on categorical equivalence between two concepts of lattice-valued frame is a direct consequence of a more general correspondence between quantale modules and lattice-valued $\bigvee$-semilattices, established recently by I. Stubbe [76], which, in its turn, extends the well-known isomorphism between the categories of **2**-modules and $\bigvee$-semilattices [61] (cf. the similar result for $\mathbb{Z}$-modules and abelian groups [3]). More precisely, having the just mentioned correspondence in hand, one easily obtains an isomorphism between the categories of quantale algebras and lattice-valued quantales, a particular instance of the latter providing the category of L-frames of W. Yao. Moreover, an analogue of the standard representation of unital algebras over a commutative ring with identity through central ring homomorphisms [18, 30] provides an isomorphism between the categories of L-frames of W. Yao and L-fuzzy frames of D. Zhang and Y.-M. Liu (obtained in a way different from [81]). The employed machinery clearly shows the strong dependance of this isomorphism on the existence of the unit in the considered algebras, the condition, which holds trivially in the frame case. In other words, the passage from frames to quantales makes the concepts of W. Yao as well as D. Zhang and Y.-M. Liu different. In view of the above-mentioned importance of lattice-valued frames in fuzzification of the sobriety-spatiality equivalence, as an additional consequence, quantale algebras give a convenient universally algebraic framework for developing lattice-valued analogues of the latter as well as for answering the long-standing question on its relationships to the notion of stratification of lattice-valued topology [58].

1.3. Skeletal Q-categories versus lattice-valued partial orders. The developments of this paper are highly dependant on the isomorphism between the categories $\mathbf{RMod}(Q)$ of right Q-modules and $\mathbf{CSCat}(Q)$ of cocomplete skeletal Q-categories, constructed by I. Stubbe [76] for every unital quantale Q (in fact, for a small quantaloid $\mathcal{Q}$). The result extends the classical representation of the category **Sup** of $\bigvee$-semilattices in terms of Eilenberg-Moore categories of two monads.

On the one hand, there exists the well-known powerset monad $\mathbb{P} = (\mathcal{P}, \eta, \mu)$ on the category **Set** of sets and maps, which is given by the following data:

(1) $\mathcal{P}(X_1 \xrightarrow{f} X_2) = \mathcal{P}X_1 \xrightarrow{\mathcal{P}f} \mathcal{P}X_2$, where $\mathcal{P}X_i = \{S \mid S \subseteq X_i\}$ and $\mathcal{P}f(S) = \{f(s) \mid s \in S\}$;
(2) $X \xrightarrow{\eta_X} \mathcal{P}X$ is defined by $\eta_X(x) = \{x\}$;
(3) $\mathcal{P}\mathcal{P}X \xrightarrow{\mu_X} \mathcal{P}X$ is defined by $\mu_X(\mathcal{S}) = \bigcup \mathcal{S}$.

The Eilenberg-Moore category $\mathbf{Set}^{\mathbb{P}}$ of the monad $\mathbb{P}$ is then precisely the above-mentioned category **Sup**.

On the other hand, there exists the down-set monad $\mathbb{D} = (\mathcal{D}, \zeta, \nu)$ on the category **Prost** of preordered sets (no anti-symmetry of partial order) and order-preserving maps, which is given by the following items:

(1) $\mathcal{D}(A_1 \xrightarrow{f} A_2) = \mathcal{D}A_1 \xrightarrow{\mathcal{D}f} \mathcal{D}A_2$, where $\mathcal{D}A_i = \{S \mid S \subseteq A_i \text{ and } S = \downarrow S\}$ and $\mathcal{D}f(S) = \downarrow \{f(s) \mid s \in S\}$;
(2) $A \xrightarrow{\zeta_A} \mathcal{D}A$ is defined by $\zeta_A(a) = \downarrow a$;

(3) $\mathcal{D}\mathcal{D}A \xrightarrow{\nu_A} \mathcal{D}A$ is defined by $\nu_A(\mathcal{S}) = \bigcup \mathcal{S}$.

The monad in question is easily seen to restrict to the full subcategory **Pos** of **Prost** of partially ordered sets (posets). The Eilenberg-Moore category $\mathbf{Pos}^{\mathbb{D}}$ of the monad $\mathbb{D}$ (whose objects have a simplified description due to the fact that the monad $\mathbb{D}$ is of Kock-Zöberlein type [35]) is again the category **Sup**. Moreover, the latter monad is induced by the reflective embedding $\mathbf{Sup} \overset{|-|}{\hookrightarrow} \mathbf{Pos}$ (which is precisely the forgetful functor), the left adjoint of which is given by the particular example of completion of posets, namely, by the above-mentioned functor $\mathcal{D}$, whose codomain is easily seen to be **Sup**, since the set $\mathcal{D}A$ is closed in $\mathcal{P}A$ under arbitrary set-theoretic unions (cf. Item (3) in the definition of the monad $\mathbb{D}$). Even more, since the forgetful functor $\mathbf{Pos} \xrightarrow{|-|} \mathbf{Set}$ (which is no more an embedding) has a left adjoint $\mathbf{Set} \xrightarrow{K} \mathbf{Pos}$, which is given by $K(X_1 \xrightarrow{f} X_2) = (X_1, =) \xrightarrow{f} (X_2, =)$, one easily gets that the composition of the just mentioned adjoint situations gives the one, which induces the powerset monad $\mathbb{P}$ on the category **Set**.

It is well-known that given a unital quantale Q, the Eilenberg-Moore category $\mathbf{Set}^{\mathbb{P}_Q}$ of the Q-powerset monad $\mathbb{P}_Q$ on the category **Set** provides the category $\mathbf{RMod}(Q)$ of right modules over Q, which essentially is a fuzzification of the above-mentioned isomorphism $\mathbf{Set}^{\mathbb{P}} \cong \mathbf{Sup}$, taking into consideration the simple fact that $\mathbf{Sup} \cong \mathbf{RMod}(\mathbf{2})$. Moreover, I. Stubbe [73] provided a lattice-valued analogue of both preordered and partially ordered set (Q-category and skeletal Q-category, respectively), the down-set monad $\mathbb{D}$ (the so-called contravariant presheaf monad on the category of (skeletal) Q-categories), and showed [76] that its Eilenberg-Moore category is precisely the category $\mathbf{CSCat}(Q)$ of cocomplete skeletal Q-categories, studying the properties of the latter structures in both stand-alone and category context. Additionally, he obtained that the category $\mathbf{CSCat}(Q)$ is isomorphic to the above category $\mathbf{RMod}(Q)$. Viewing the objects of the former category as a fuzzification of $\bigvee$-semilattices, we see that similar to the crisp case, where the categories $\mathbf{RMod}(\mathbf{2})$ and **Sup** are isomorphic, the categories $\mathbf{RMod}(Q)$ and $\mathbf{CSCat}(Q)$ are isomorphic as well.

The original results of I. Stubbe are more general than the above-mentioned ones, employing a (small) quantaloid $\mathcal{Q}$ instead of a quantale Q, and, therefore, using the language of enriched categories [34, 39]. As follows from the above discussion, however, their simplified Q-versions are closely related to lattice-valued mathematics. More precisely, Q-categories are nothing else than lattice-valued preorders of L. A. Zadeh [83] and S. V. Ovchinnikov [41] (see, e.g., [5] for a thorough discussion on the topic), whereas the assumption on being skeletal makes lattice-valued preorders into lattice-valued partial orders (see the above-mentioned references). Further, a contravariant Q-enriched presheaf is nothing else than a lattice-valued down-set (a covariant Q-enriched presheaf is then precisely a lattice-valued up-set), and the free cocompletion of a skeletal Q-category is a lattice-valued analogue of the above-mentioned completion of partially ordered sets (already studied elsewhere). Lastly, the assumption on cocompleteness of a skeletal Q-category provides the existence of a lattice-valued $\bigvee$-operation. As a consequence, one gets a convenient representation of lattice-valued $\bigvee$-semilattices through quantale modules (and vice versa), much relied upon in this paper.

When looking at the results of I. Stubbe though, ones notices that he neither uses the language of many-valued mathematics (even in the restricted Q-valued case), nor provides a proper (in fact, any, apart from [77], up to the knowledge of the author) placement of his achievements in that context. On the other hand, the theory of lattice-valued sets, going back up to 1965, can contribute a lot to the theory of Q-categories through the notion of lattice-valued preorder. More precisely, the theory of the latter structures is already well-developed, and, moreover, makes a significant part of lattice-valued mathematics. Since this paper targets the fuzzy community, we restate the above-mentioned isomorphism $\mathbf{RMod}(Q) \cong \mathbf{CSCat}(Q)$ of I. Stubbe in lattice-valued terms, and use it, later on, as an important tool in obtaining a characterization of lattice-valued frames. Our main point here is to contribute to the study of lattice-valued posets and not to the theory of Q-categories, the properties of which lie off the scope of this paper.

In the developments below, we rely on category theory and universal algebra. The necessary categorical background can be found in [2, 24, 39]. For algebraic notions we recommend [3, 9, 36, 43, 61]. Although we tried to make the paper as much self-contained as possible, it is expected from the reader to be acquainted with basic concepts of category theory, e.g., with that of category and functor.

2. Quantale modules and algebras

In this section, we briefly recall the notions of quantale module and algebra (notice that these structures are closely related to many-valued mathematics [67, 68]). Both concepts rely on the notion of quantale (introduced by C. J. Mulvey [40] as an attempt to provide a possible setting for constructive foundations of quantum mechanics, and to study the spectra of non-commutative C^*-algebras, which are locales in the commutative case), whose theory has found numerous applications in both universal algebra and category theory [36, 73, 74, 75, 76] as well as in lattice-valued mathematics [25, 27, 29, 57].

1. Definition. A $\bigvee$-*semilattice* is a partially ordered set (poset, for short), which has arbitrary joins (denoted $\bigvee$). A $\bigvee$-*semilattice homomorphism* $(A, \bigvee) \xrightarrow{\varphi} (B, \bigvee)$ is a $\bigvee$-preserving map $A \xrightarrow{\varphi} B$. **Sup** is the construct of $\bigvee$-semilattices and their homomorphisms. ∎

Notice that in this article, we use the term "$\bigvee$-semilattice" instead of the more usual term "sup-lattice" as in, e.g., [16, 36, 73, 76], or the term "join-semilattice" as in, e.g., [57]. Moreover, to be in line with the overall categorical notation of this paper, we use "**Sup**" instead of "$s\ell$" [33], or "$\mathcal{SL}$" [60], or "$\mathcal{S}up$" [36].

2. Definition. A *quantale* is a triple $(Q, \bigvee, \otimes)$ such that

(1) $(Q, \bigvee)$ is a $\bigvee$-semilattice;
(2) $(Q, \otimes)$ is a semigroup, i.e., $q_1 \otimes (q_2 \otimes q_3) = (q_1 \otimes q_2) \otimes q_3$ for every $q_1, q_2, q_3 \in Q$;
(3) $q \otimes (\bigvee S) = \bigvee_{s \in S}(q \otimes s)$ and $(\bigvee S) \otimes q = \bigvee_{s \in S}(s \otimes q)$ for every $q \in Q$ and every $S \subseteq Q$.

A *quantale homomorphism* $(P, \bigvee, \otimes) \xrightarrow{\varphi} (Q, \bigvee, \otimes)$ is a map $P \xrightarrow{\varphi} Q$, which preserves $\otimes$ and $\bigvee$. **Quant** is the category of quantales and their homomorphisms, concrete over the categories **Sup** of $\bigvee$-semilattices and **SGrp** of semigroups. ∎

Since the main results of the paper are much dependant on algebraic structures with ever growing signature (cf., e.g., the passage from $\bigvee$-semilattices to quantales), we will sometimes shorten the notion to just A (for $\bigvee$-semilattices) or Q (for quantales), making explicit just the algebraic structure which we need at the moment (cf., e.g., the notation for quantale modules of Definition 9).

The category **Quant** has been studied thoroughly in [36, 60], K. I. Rosenthal giving a coherent statement to the quantale theory. Throughout this paper, we will consider two specific types of quantales, which are mentioned below.

3. Definition. A quantale Q is said to be *unital* provided that there exists an element $1 \in Q$ such that $(Q, \otimes, 1)$ is a monoid. A *unital quantale homomorphism* should additionally preserve the unit. **UQuant** denotes the respective (non-full) subcategory of **Quant**. ∎

4. Definition. A quantale Q is said to be *commutative* provided that $q_1 \otimes q_2 = q_2 \otimes q_1$ for every q_1, $q_2 \in Q$. **CQuant** is the respective full subcategory of **Quant**. ∎

Every quantale, being a complete lattice, has the largest element $\top$ and the smallest element $\bot$. The following examples provide more intuition for the concept.

5. Example. Every *frame*, i.e., a complete lattice L such that $a \wedge (\bigvee S) = \bigvee_{s \in S}(a \wedge s)$ for every $a \in L$ and every $S \subseteq L$ [32], is a commutative unital quantale, where $\otimes = \wedge$ and $1 = \top$. In particular, the two-element chain $\mathbf{2} = \{\bot, \top\}$ is a commutative unital quantale. ∎

6. Example. Let $(A, \cdot)$ be a semigroup. The powerset $\mathcal{P}(A)$ is a quantale, where $\bigvee$ are unions and $S \otimes T = \{s \cdot t \,|\, s \in S,\ t \in T\}$. If $(A, \cdot, 1)$ is a monoid, then $\mathcal{P}(A)$ is unital, with the unit $\{1\}$. If $(A, \cdot)$ is commutative, then so is $\mathcal{P}(A)$. ∎

Example 6 provides the free quantale over a given semigroup [60] (the result is extended in [68]).

7. Example. Let X be a set and let $\mathcal{R}(X)$ be the set of all binary relations on X. $\mathcal{R}(X)$ is a quantale, where $\bigvee$ are unions and $\otimes$ is given by $S \otimes T = \{(x, y) \in X \times X \,|\, (x, z) \in T$ and $(z, y) \in S$ for some $z \in X\}$ (standard composition of relations). $\mathcal{R}(X)$ is unital, with the diagonal relation $\triangle = \{(x, x) \,|\, x \in X\}$ being the unit, but not commutative. ∎

It is shown in [6] that every unital quantale is isomorphic to a *relational quantale*, namely, a subset of $\mathcal{R}(X)$, which contains $\triangle$ and is closed under composition of relations, with $\bigvee$ being (in general) different from unions (see [22] for a more general result).

8. Example. Given a $\bigvee$-semilattice A, let $\mathcal{Q}(A)$ be the set $\mathbf{Sup}(A, A)$ of all $\bigvee$-preserving maps $A \xrightarrow{\varphi} A$. Equipped with the point-wise order, the set becomes a $\bigvee$-semilattice. Moreover, $\mathcal{Q}(A)$ is a unital quantale, where multiplication is given by the map composition and the unit is the identity map $A \xrightarrow{1_A} A$. ∎

It is shown in [44] that every quantale Q has a *faithful representation*, i.e., an embedding $Q \overset{\mu}{\hookrightarrow} \mathcal{Q}(A)$ for some $\bigvee$-semilattice A (which is actually Q itself).

On the next step, we recall the category $\mathbf{Mod}(Q)$ of unital left modules over a given unital quantale Q [36, 43, 61, 67, 68]. Its definition is very similar to the (well-known to algebraists) category $\mathbf{Mod}(R)$ of unital left modules over a unital ring R [3, 18, 30].

9. Definition. Given a unital quantale Q, a *unital left Q-module* is a pair $(A, *)$, where A is a $\bigvee$-semilattice and $Q \times A \xrightarrow{*} A$ is a map (the *action* of Q on A) such that

(1) $q * (\bigvee S) = \bigvee_{s \in S}(q * s)$ for every $q \in Q$ and every $S \subseteq A$;
(2) $(\bigvee T) * a = \bigvee_{t \in T}(t * a)$ for every $T \subseteq Q$ and every $a \in A$;
(3) $q_1 * (q_2 * a) = (q_1 \otimes q_2) * a$ for every q_1, $q_2 \in Q$ and every $a \in A$;
(4) $1_Q * a = a$ for every $a \in A$.

A *unital left Q-module homomorphism* $(A, *) \xrightarrow{\varphi} (B, *)$ is a map $A \xrightarrow{\varphi} B$, which preserves $\bigvee$ and satisfies the condition $\varphi(q * a) = q * \varphi(a)$ for every $a \in A$ and every $q \in Q$. $\mathbf{Mod}(Q)$ is the category of left unital Q-modules and their homomorphisms, concrete over the category **Sup**. ∎

Notice the possibility to define the category of modules over an arbitrary quantale, omitting Item (4) of Definition 9. Recently, however, we showed [69] that every category of modules over a non-unital quantale is equivalent to the category of unital modules over a unital extension of this quantale.

For the sake of shortness, from now on, “Q-module" means “unital left Q-module". It is easy to see that the category $\mathbf{Mod}(\mathbf{2})$ (recall the two-element quantale of Example 5) is isomorphic to the category **Sup** (cf. the well-known isomorphism between the categories of modules over the ring of integers $\mathbb{Z}$ and abelian groups [30]). Also notice that every unital quantale can be considered as a module over itself (with action given by quantale multiplication).

The concept of Q-module goes back to (at least) A. Joyal and M. Tierney [33]. More precisely, since **Sup** is a monoidal closed category (a convenient description of tensor products of $\bigvee$-semilattices is presented in [23]), unital (commutative) quantales are precisely the (commutative) monoids in **Sup**. Then Q-modules of Definition 9 are Q-modules in the sense of [33] (which essentially are just the Q-actions (in the sense of monoidal categories) on the objects of the monoidal category **Sup**, with Q-action morphisms (in the sense of monoidal categories again) serving as Q-module homomorphisms), provided that one notices that most of the results of [33], which deals with the commutative setting, are valid in the non-commutative case as well. Modules over a unital quantale form the central idea in the unified treatment of process semantics developed by S. Abramsky and S. Vickers in [1].

On the last step, we define the category $\mathbf{Alg}(Q)$ of algebras over a given unital commutative quantale Q. The definition was motivated by the category $\mathbf{Alg}(K)$ of algebras over a commutative ring K with identity [3, 18, 30]. Being started rather recently, the theory is less developed than that of quantale modules.

10. Definition. Given a unital commutative quantale Q, a Q-*algebra* is a triple $(A, \otimes, *)$ such that

(1) A is a $\bigvee$-semilattice;
(2) $(A, *)$ is a Q-module;
(3) $(A, \otimes)$ is a quantale;
(4) $q * (a_1 \otimes a_2) = (q * a_1) \otimes a_2 = a_1 \otimes (q * a_2)$ for every $a_1, a_2 \in A$, $q \in Q$.

A Q-*algebra homomorphism* $(A, \otimes, *) \xrightarrow{\varphi} (B, \otimes, *)$ is a map $A \xrightarrow{\varphi} B$, which is both a quantale homomorphism and a Q-module homomorphism. $\mathbf{Alg}(Q)$ is the category of Q-algebras and their homomorphisms, concrete over both the category $\mathbf{Mod}(Q)$ and the category **Quant**. ∎

It is not difficult to see that the category $\mathbf{Alg}(\mathbf{2})$ is isomorphic to the category **Quant** (cf. the isomorphism between the categories of algebras over the ring of integers $\mathbb{Z}$ and rings [30]). Notice as well that every unital commutative quantale is an algebra over itself (with action given by quantale multiplication).

Similar to the case of quantale modules, one can see that quantale algebras also go back to (at least) the already mentioned paper of A. Joyal and M. Tierney [33]. Given a unital commutative quantale Q, $\mathbf{Mod}(Q)$ is a monoidal closed category (see, e.g., [64] for the description of its monoidal structure, namely, tensor products of quantale modules). Then Q-algebras are precisely the monoids in $\mathbf{Mod}(Q)$.

It appears that the concept of quantale algebra provides a common framework for two concepts of lattice-valued frame, currently available in the fuzzy literature.

3. Quantale algebras as comma categories

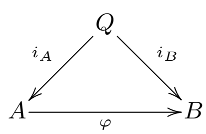
In [84] D. Zhang and Y.-M. Liu introduced the concept of L-fuzzy frame as an object of the comma category $(L \downarrow \mathbf{Frm})$, where **Frm** is the category of frames. This section extends the notion to quantales and shows its categorical equivalence to a particular instance of quantale algebras.

There exists the well-known representation of unital algebras over a commutative ring with identity through central ring homomorphisms [18, Proposition 1.1 of Chapter XIII], [30, Exercise 3 of Section IV.7]. In the following, we extend the result to quantale algebras. It should be noticed immediately that a similar achievement has been already attempted by W. Yao [81]. Due to its rather chaotic presentation, a significant flaw and the lack of proper universally algebraic background, we provide a more rigorous proof below. For convenience of the reader, we begin with certain algebraic and categorical preliminaries.

11. Definition. Given a unital commutative quantale Q, $\mathbf{UAlg}(Q)$ is the (non-full) subcategory of $\mathbf{Alg}(Q)$, whose objects additionally are unital quantales and whose morphisms additionally preserve the unit. ▪

12. Definition. The *center* of a Q-algebra A is the set $Z(A) = \{a \in A \mid a \otimes a' = a' \otimes a \text{ for every } a' \in A\}$. ▪

13. Definition. Given a unital commutative quantale Q, $(Q \downarrow \mathbf{UQuant})_z$ is the category, whose objects are the $\mathbf{UQuant}$-morphisms $Q \xrightarrow{i_A} A$ (i.e., from Q to any $\mathbf{UQuant}$-object) such that the image of i_A lies in the center of A. The morphisms of the category $(Q \xrightarrow{i_A} A) \xrightarrow{\varphi} (Q \xrightarrow{i_B} B)$ are the $\mathbf{UQuant}$-morphisms $A \xrightarrow{\varphi} B$, which make the triangle



commute. ▪

Notice that $(Q \downarrow \mathbf{UQuant})_z$ is a full subcategory of the comma category $(Q \downarrow \mathbf{UQuant})$, whose definition is written explicitly for convenience of the reader. Moreover, to be in line with the main goal of this article, we use the notation for comma categories of [84]. The preliminaries in hand, we proceed to the main result of this section, which makes use of the following two propositions.

14. Proposition. *There exists a functor* $\mathbf{UAlg}(Q) \xrightarrow{F} (Q \downarrow \mathbf{UQuant})_z$ *defined by* $F((A,*) \xrightarrow{\varphi} (B,*)) = (Q \xrightarrow{i_A} A) \xrightarrow{\varphi} (Q \xrightarrow{i_B} B)$, *where* $i_A(q) = q * 1_A$.

Proof. To show that the functor is correct on objects, we start by checking that the map $Q \xrightarrow{i_A} A$ is a unital quantale homomorphism. Given $S \subseteq Q$, $i_A(\bigvee S) = (\bigvee S) * 1_A = \bigvee_{s \in S}(s * 1_A) = \bigvee_{s \in S} i_A(s)$. Given q_1, $q_2 \in Q$, $i_A(q_1 \otimes q_2) = (q_1 \otimes q_2) * 1_A = q_1 * (q_2 * 1_A) = q_1 * (1_A \otimes (q_2 * 1_A)) = (q_1 * 1_A) \otimes (q_2 * 1_A) = i_A(q_1) \otimes i_A(q_2)$. Moreover, $i_A(1_Q) = 1_Q * 1_A = 1_A$.

To show that the image of i_A lies in the center of A, notice that given $q \in Q$ and $a \in A$, $i_A(q) \otimes a = (q * 1_A) \otimes a = q * (1_A \otimes a) = q * (a \otimes 1_A) = a \otimes (q * 1_A) = a \otimes i_A(q)$.

To verify that the functor is correct on morphisms, use the fact that given $q \in Q$, $\varphi \circ i_A(q) = \varphi(q * 1_A) = q * \varphi(1_A) = q * 1_B = i_B(q)$. □

An attentive reader will see that Proposition 14 makes no use of the centrality property of the objects of the category $(Q \downarrow \mathbf{UQuant})_z$. It is the functor in the opposite direction which employs the requirement.

15. Proposition. *There exists a functor* $(Q \downarrow \mathbf{UQuant})_z \xrightarrow{G} \mathbf{UAlg}(Q)$ *defined by* $G((Q \xrightarrow{i_A} A) \xrightarrow{\varphi} (Q \xrightarrow{i_B} B)) = (A,*) \xrightarrow{\varphi} (B,*)$, *where* $q * a = i_A(q) \otimes a$.

Proof. To check the correctness of the functor on objects, we show that $(A,*)$ is a unital Q-algebra. Given $q \in Q$ and $S \subseteq A$, $q * (\bigvee S) = i_A(q) \otimes (\bigvee S) = \bigvee_{s \in S}(i_A(q) \otimes s) = \bigvee_{s \in S}(q * s)$. Given $S \subseteq Q$ and $a \in A$, $(\bigvee S) * a = i_A(\bigvee S) \otimes a = (\bigvee_{s \in S} i_A(s)) \otimes a = \bigvee_{s \in S}(i_A(s) \otimes a) = \bigvee_{s \in S}(s * a)$. Given q_1, $q_2 \in Q$ and $a \in A$, $q_1 * (q_2 * a) = q_1 * (i_A(q_2) \otimes a) =$

$i_A(q_1)\otimes(i_A(q_2)\otimes a)=(i_A(q_1)\otimes i_A(q_2))\otimes a=i_A(q_1\otimes q_2)\otimes a=(q_1\otimes q_2)*a$. Given $a\in A$, $1_Q*a=i_A(1_Q)\otimes a=1_A\otimes a=a$. Lastly, given $q\in Q$ and $a_1,a_2\in A$, $q*(a_1\otimes a_2)=i_A(q)\otimes(a_1\otimes a_2)=(i_A(q)\otimes a_1)\otimes a_2=(q*a_1)\otimes a_2$. Moreover, the centrality property of $Q\xrightarrow{i_A}A$ gives $(i_A(q)\otimes a_1)\otimes a_2=(a_1\otimes i_A(q))\otimes a_2=a_1\otimes(i_A(q)\otimes a_2)=a_1\otimes(q*a_2)$.

For correctness of the functor on morphisms, use the fact that for $q\in Q$ and $a\in A$, $\varphi(q*a)=\varphi(i_A(q)\otimes a)=(\varphi\circ i_A(q))\otimes\varphi(a)=i_B(q)\otimes\varphi(a)=q*\varphi(a)$. □

It is important to underline that W. Yao [81] erroneously used the whole category $(Q\downarrow\mathbf{UQuant})$ as the domain of the functor G of Proposition 15.

16. Theorem. *$G\circ F=1_{\mathbf{UAlg}(Q)}$ and $F\circ G=(Q\downarrow\mathbf{UQuant})_z$, i.e., the two categories $\mathbf{UAlg}(Q)$ and $(Q\downarrow\mathbf{UQuant})_z$ are isomorphic.*

Proof. Given an $\mathbf{UAlg}(Q)$-object $(A,*)$, it follows that $G\circ F(A,*)=G(Q\xrightarrow{i_A}A)=(A,*')$, where $q*'a=i_A(q)\otimes a=(q*1_A)\otimes a=q*(1_A\otimes a)=q*a$. Given a $(Q\downarrow\mathbf{UQuant})_z$-object $Q\xrightarrow{i_A}A$, it follows that $F\circ G(Q\xrightarrow{i_A}A)=F(A,*)=Q\xrightarrow{i'_A}A$, where $i'_A(q)=q*1_A=i_A(q)\otimes 1_A=i_A(q)$. □

One should pay attention to the fact that the existence of the functor F of Proposition 14 depends on the availability of the unit in the objects of $\mathbf{UAlg}(Q)$.

4. Quantale algebras as lattice-valued quantales

In [79, 81] W. Yao developed the theory of lattice-valued frames, based in the concept of lattice-valued order of L. Fan [12]. In this section, we extend the notion to lattice-valued quantale and show its categorical equivalence to the concept of quantale algebra.

4.1. Quantale modules as lattice-valued $\bigvee$-semilattices. In view of the discussion in Subsection 1.3 on the isomorphism $\mathbf{RMod}(Q)\cong\mathbf{CSCat}(Q)$ of I. Stubbe [76], in this subsection, we restate his result in lattice-valued terms. More precisely, we consider the category $\mathbf{Sup}(Q)$ of Q-$\bigvee$-semilattices and show that it is isomorphic to the above-mentioned category $\mathbf{Mod}(Q)$. As a consequence, one obtains a particular (and very simple) case of [76, Corollary 4.13].

Before moving forward, we have to recall several basic properties of quantales and Q-modules. Given a quantale Q, there exist two residuations induced by its multiplication $\otimes$ and defined by $q_1\rightarrow_r q_2=\bigvee\{q\in Q\,|\,q_1\otimes q\leqslant q_2\}$ and $q_1\rightarrow_l q_2=\bigvee\{q\in Q\,|\,q\otimes q_1\leqslant q_2\}$, providing a single residuation $\cdot\rightarrow\cdot$ in case of Q being commutative. The operations enjoy the standard properties of Galois connections [11], i.e., $q_2\leqslant q_1\rightarrow_r q_3$ if and only if $q_1\otimes q_2\leqslant q_3$ iff $q_1\leqslant q_2\rightarrow_l q_3$. On the other hand, given a Q-module $(A,*)$, there exist residuations $a_1\twoheadrightarrow a_2=\bigvee\{q\in Q\,|\,q*a_1\leqslant a_2\}$ and $q\rightsquigarrow a=\bigvee\{a'\in A\,|\,q*a'\leqslant a\}$. The operations satisfy the Galois connection property $q\leqslant a_1\twoheadrightarrow a_2$ iff $q*a_1\leqslant a_2$ iff $a_1\leqslant q\rightsquigarrow a_2$. Moreover, the subsequent two lemmas recall a number of other standard properties of the above-mentioned residuations (for their simple proofs, the reader is referred to [36, 43, 60], or any other comprehensive reference on quantales).

17. Lemma. *Given a quantale Q, the following hold:*

(1) *$q_1\rightarrow_r(q_2\rightarrow_r q_3)=(q_2\otimes q_1)\rightarrow_r q_3$ and $q_1\rightarrow_l(q_2\rightarrow_l q_3)=(q_1\otimes q_2)\rightarrow_l q_3$ for every $q_1,q_2,q_3\in Q$;*
(2) *$q\rightarrow_r(\bigwedge S)=\bigwedge_{s\in S}(q\rightarrow_r s)$ and $q\rightarrow_l(\bigwedge S)=\bigwedge_{s\in S}(q\rightarrow_l s)$ for every $q\in Q$ and every $S\subseteq Q$;*
(3) *$(\bigvee S)\rightarrow_r q=\bigwedge_{s\in S}(s\rightarrow_r q)$ and $(\bigvee S)\rightarrow_l q=\bigwedge_{s\in S}(s\rightarrow_l q)$ for every $q\in Q$ and every $S\subseteq Q$.*

If Q is unital, then, additionally,

(4) $1_Q \rightarrow_r q = q$ *and* $1_Q \rightarrow_l q = q$ *for every* $q \in Q$.

18. Lemma. *Given a* Q*-module* $(A, *)$*, the following hold:*

(1) $q \rightarrow_l (a_1 \twoheadrightarrow a_2) = (q * a_1) \twoheadrightarrow a_2$ *for every* $q \in Q$ *and every* $a_1, a_2 \in A$*;*
(2) $a \twoheadrightarrow (\bigwedge S) = \bigwedge_{s \in S}(a \twoheadrightarrow s)$ *for every* $a \in A$ *and every* $S \subseteq A$*;*
(3) $(\bigvee S) \twoheadrightarrow a = \bigwedge_{s \in S}(s \twoheadrightarrow a)$ *for every* $a \in A$ *and every* $S \subseteq A$*;*
(4) $a_1 \twoheadrightarrow (q \rightsquigarrow a_2) = q \rightarrow_r (a_1 \twoheadrightarrow a_2)$ *for every* $q \in Q$ *and every* $a_1, a_2 \in A$.

Notice that the corresponding analogue for $\cdot \rightarrow_r \cdot$ in Item (1) of Lemma 18 requires commutativity of the quantale Q. All the properties mentioned in Lemmas 17, 18 will be heavily used throughout the paper, without mentioning them explicitly on each occasion.

Some results and notation from the theory of lattice-valued powerset operators will be also used throughout the paper (we notice that our employed powerset operators were first described by L. A. Zadeh in [82]; the arrow notation and the complete development is due to S. E. Rodabaugh [53, 55, 57]). Given a map $X \xrightarrow{f} Y$, there exist *forward* $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$ and *backward* $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$ *powerset operators*, defined by $f^{\rightarrow}(S) = \{f(x) \mid x \in S\}$ and $f^{\leftarrow}(T) = \{x \in X \mid f(x) \in T\}$ respectively. Given a $\bigvee$-semilattice L, the maps can be extended to *forward* $L^X \xrightarrow{f_L^{\rightarrow}} L^Y$ and *backward* $L^Y \xrightarrow{f_L^{\leftarrow}} L^X$ *L-powerset operators*, defined accordingly by $(f_L^{\rightarrow}(\alpha))(y) = \bigvee\{\alpha(x) \mid f(x) = y\}$ and $f_L^{\leftarrow}(\beta) = \beta \circ f$.

The necessary preliminaries in hand, we can proceed to the main definition of this subsection.

19. Definition. Let Q be a unital quantale. A *Q-partially ordered set* (*Q-poset*) is a pair (A, e), where A is a set, and $A \times A \xrightarrow{e} Q$ is a map (*Q-partial order* or *Q-order* on A) such that

(1) $1_Q \leqslant e(a, a)$ for every $a \in A$ (*Q-reflexivity*);
(2) $e(a_1, a_2) \otimes e(a_2, a_3) \leqslant e(a_1, a_3)$ for every $a_1, a_2, a_3 \in A$ (*Q-transitivity*);
(3) $1_Q \leqslant e(a_1, a_2)$ and $1_Q \leqslant e(a_2, a_1)$ imply $a_1 = a_2$, for every $a_1, a_2 \in A$ (*Q-antisymmetry*).

A *Q_r-$\bigvee$-semilattice* is a triple $(A, e, \sqcup)$, where (A, e) is a Q-poset, and $Q^A \xrightarrow{\sqcup} A$ is a map (*Q_r-join operation* on A) such that $e(\sqcup\alpha, a) = \bigwedge_{a' \in A}(\alpha(a') \rightarrow_r e(a', a))$ for every $\alpha \in Q^A$ and every $a \in A$. A *Q_r-$\bigvee$-semilattice homomorphism* $(A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)$ is a map $A \xrightarrow{\varphi} B$ such that $\varphi(\sqcup\alpha) = \sqcup\varphi_Q^{\rightarrow}(\alpha)$ for every $\alpha \in Q^A$ (*Q_r-join-preserving map*). $\mathbf{Sup}_r(Q)$ is the construct of Q_r-$\bigvee$-semilattices and their homomorphisms. ■

Replacing $\cdot \rightarrow_r \cdot$ with $\cdot \rightarrow_l \cdot$, one obtains the concept of *Q_l-$\bigvee$-semilattice*. Since they both share the same notion of lattice-valued order, we employ neither "r" nor "l" in the notation for this lattice-valued order. Moreover, the term "Q-$\bigvee$-semilattice" will suppose commutativity of the quantale Q. Given a Q-poset (A, e), there exists at most one Q-(r,l)-join operation $\sqcup$ on (A, e), since the condition $e(\sqcup\alpha, a) = \bigwedge_{a' \in A}(\alpha(a') \rightarrow_{r,l} e(a', a)) = e(\sqcup'\alpha, a)$ for every $a \in A$, implies $\sqcup\alpha = \sqcup'\alpha$ by Items (1) and (3) of Definition 19. One should also underline at once that Definition 19 uses the concepts of W. Yao [78, 79, 80, 81] developed for frames. An important difference though is the distinguishing between the two cases "r" and "l", the use of the unit 1_Q instead of the top element $\top$ and the inequality "$1_Q \leqslant \ldots$" instead of the equality "$1_Q = \ldots$". However, the case of Q being a frame, makes the two concepts coincide with that of W. Yao. To give the reader more intuition for the new notion, below, we provide its simple example based in non-commutative quantales.

20. Lemma. *Every unital quantale* Q *provides the* Q_r-$\bigvee$*-semilattice* $(Q, e, \sqcup)$*, where* $e(q_1, q_2) = q_1 \rightarrow_r q_2$ *and* $\sqcup\alpha = \bigvee_{q \in Q}(q \otimes \alpha(q))$.

Proof. To show that (Q,e) is a Q-poset, notice that given $q \in Q$, $q \otimes 1_Q \leqslant q$ provides $1_Q \leqslant q \to_r q = e(q,q)$. On the other hand, given $q_1, q_2, q_3 \in Q$, $e(q_1,q_2) \otimes e(q_2,q_3) = (q_1 \to_r q_2) \otimes (q_2 \to_r q_3) = \bigvee\{q \otimes q' \mid q_1 \otimes q \leqslant q_2 \text{ and } q_2 \otimes q' \leqslant q_3\}$ and then, $q_1 \otimes (q \otimes q') = (q_1 \otimes q) \otimes q' \leqslant q_2 \otimes q' \leqslant q_3$ gives $q \otimes q' \leqslant q_1 \to_r q_3 = e(q_1,q_3)$. As a result, $e(q_1,q_2) \otimes e(q_2,q_3) \leqslant e(q_1,q_3)$. Lastly, if $1_Q \leqslant e(q_1,q_2)$ and $1_Q \leqslant e(q_2,q_1)$, then $q_1 = q_1 \otimes 1_Q \leqslant q_2$ and $q_2 = q_2 \otimes 1_Q \leqslant q_1$ give $q_1 = q_2$.

To show that $\sqcup$ is the Q_r-join operation w.r.t. (Q,e), notice that for $\alpha \in Q^Q$ and $q \in Q$, it follows that

$$\bigwedge_{q' \in Q} (\alpha(q') \to_r e(q',q)) = \bigwedge_{q' \in Q} (\alpha(q') \to_r (q' \to_r q)) =$$
$$\bigwedge_{q' \in Q} ((q' \otimes \alpha(q')) \to_r q) = (\bigvee_{q' \in Q} (q' \otimes \alpha(q'))) \to_r q =$$
$$e(\bigvee_{q' \in Q} (q' \otimes \alpha(q')), q) = e(\sqcup\, \alpha, q),$$

which provides then the result in question. □

Notice that the machinery of Lemma 20 is not applicable to the residuation $\cdot \to_l \cdot$. Indeed, to show Item (2) of Definition 19, one starts with $e(q_1,q_2) \otimes e(q_2,q_3) = (q_1 \to_l q_2) \otimes (q_2 \to_l q_3) = \bigvee\{q \otimes q' \mid q \otimes q_1 \leqslant q_2 \text{ and } q' \otimes q_2 \leqslant q_3\}$ and has to show that $(q \otimes q') \otimes q_1 \leqslant q_3$, which is generally not true, unless Q is commutative. An analogue of this deficiency is the main reason for our using commutative quantales in the subsequent developments. We should notice, however, immediately that the above-mentioned machinery of Q-categories of I. Stubbe [76] does not depend on commutativity of its underlying quantale (indeed, its general version relies on a quantaloid $\mathcal{Q}$ instead of a quantale Q).

21. Proposition. *Given a unital commutative quantale Q, there exists a functor* $\mathbf{Mod}(Q) \xrightarrow{F} \mathbf{Sup}(Q)$ *defined by* $F((A,*) \xrightarrow{\varphi} (B,*)) = (A,e,\sqcup) \xrightarrow{\varphi} (B,e,\sqcup)$*, where* $e(a_1,a_2) = a_1 \twoheadrightarrow a_2$ *and* $\sqcup\, \alpha = \bigvee_{a \in A}(\alpha(a) * a)$.

Proof. To show that the functor is correct on objects, we begin by checking that (A,e) is a Q-poset. Given $a \in A$, $1_Q * a = a$ implies $1_Q \leqslant a \twoheadrightarrow a = e(a,a)$. Given $a_1, a_2, a_3 \in A$, $e(a_1,a_2) \otimes e(a_2,a_3) = (a_1 \twoheadrightarrow a_2) \otimes (a_2 \twoheadrightarrow a_3) = \bigvee\{q \otimes q' \mid q * a_1 \leqslant a_2 \text{ and } q' * a_2 \leqslant a_3\}$ and then, $(q \otimes q') * a_1 \overset{(\dagger)}{=} (q' \otimes q) * a_1 = q' * (q * a_1) \leqslant q' * a_2 \leqslant a_3$ provides $q \otimes q' \leqslant a_1 \twoheadrightarrow a_3 = e(a_1,a_3)$, where (†) uses commutativity of the quantale Q. As a result, $e(a_1,a_2) \otimes e(a_2,a_3) \leqslant e(a_1,a_3)$. Lastly, if $1_Q \leqslant e(a_1,a_2)$ and $1_Q \leqslant e(a_2,a_1)$, then $a_1 = 1_Q * a_1 \leqslant a_2$ and $a_2 = 1_Q * a_2 \leqslant a_1$ give $a_1 = a_2$.

To show that $\sqcup$ provides the Q-join operation w.r.t. (A,e), use the fact that given $\alpha \in Q^A$ and $a \in A$,

$$\bigwedge_{a' \in A} (\alpha(a') \to e(a',a)) = \bigwedge_{a' \in A} (\alpha(a') \to (a' \twoheadrightarrow a)) =$$
$$\bigwedge_{a' \in A} ((\alpha(a') * a') \twoheadrightarrow a) = (\bigvee_{a' \in A} (\alpha(a') * a')) \twoheadrightarrow a =$$
$$e(\bigvee_{a' \in A} (\alpha(a') * a'), a) = e(\sqcup\, \alpha, a).$$

To show that the functor F is correct on morphisms, notice that given $\alpha \in Q^A$ and $b \in B$,

$$e(\varphi(\sqcup \alpha), b) = \varphi(\sqcup \alpha) \twoheadrightarrow b = \varphi(\bigvee_{a\in A} (\alpha(a) * a)) \twoheadrightarrow b =$$
$$(\bigvee_{a\in A} (\alpha(a) * \varphi(a))) \twoheadrightarrow b = \bigwedge_{a\in A} ((\alpha(a) * \varphi(a)) \twoheadrightarrow b) =$$
$$\bigwedge_{a\in A} (\alpha(a) \to (\varphi(a) \twoheadrightarrow b)) = \bigwedge_{a\in A} (\alpha(a) \to e(\varphi(a), b)) =$$
$$\bigwedge_{b'\in B} \bigwedge_{\varphi(a)=b'} (\alpha(a) \to e(b', b)) = \bigwedge_{b'\in B} ((\bigvee_{\varphi(a)=b'} \alpha(a)) \to e(b', b)) =$$
$$\bigwedge_{b'\in B} ((\varphi_Q^{\to}(\alpha))(b') \to e(b', b)) = e(\sqcup \varphi_Q^{\to}(\alpha), b).$$

As a result, one obtains that $\varphi(\sqcup \alpha) = \sqcup \varphi_Q^{\to}(\alpha)$. □

The functor in the opposite direction requires the following specific notation. Given a $\bigvee$-semilattice L and a set X, for every $S \subseteq X$ and every $b \in L$, there exists a map $X \xrightarrow{\alpha_S^b} L$ defined by

$$\alpha_S^b(x) = \begin{cases} b, & x \in S \\ \bot, & \text{otherwise.} \end{cases}$$

In particular, if S is a singleton $\{s\}$, then we use the notation α_s^b. An important property of such maps is contained in the next "folklore" lemma.

22. Lemma. *Given a map $X \xrightarrow{f} Y$ and a $\bigvee$-semilattice L, for every map $\alpha_S^b \in L^X$, $f_L^{\to}(\alpha_S^b) = \alpha_{f^{\to}(S)}^b$.*

23. Proposition. *Given a unital commutative quantale Q, there exists a functor $\mathbf{Sup}(Q) \xrightarrow{G} \mathbf{Mod}(Q)$ defined by $G((A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)) = (A, \leqslant, \bigvee, *) \xrightarrow{\varphi} (B, \leqslant, \bigvee, *)$, where*

(1) *$a_1 \leqslant a_2$ iff $1_Q \leqslant e(a_1, a_2)$, for every $a_1, a_2 \in A$;*
(2) *$\bigvee S = \sqcup \alpha_S^{1_Q}$ for every $S \subseteq A$;*
(3) *$q * a = \sqcup \alpha_a^q$ for every $q \in Q$ and every $a \in A$.*

Proof. To check that G is well-defined on objects, we show that $(A, \leqslant, \bigvee, *)$ is a Q-module. The properties of Q-order of Definition 19 imply that $(A, \leqslant)$ is a poset (notice that reflexivity and antisymmetry can be obtained replacing 1_Q in Definition 19 by an arbitrary element of the quantale Q, whereas transitivity relies on the identity $1_Q = 1_Q \otimes 1_Q$).

To show that $\bigvee$ is the join operation on $(A, \leqslant)$, notice that given $S \subseteq A$, for every $s \in S$, it follows that

$$1_Q \leqslant e(\sqcup \alpha_S^{1_Q}, \sqcup \alpha_S^{1_Q}) = \bigwedge_{a\in A} (\alpha_S^{1_Q}(a) \to e(a, \sqcup \alpha_S^{1_Q})) =$$
$$\bigwedge_{s'\in S} (1_Q \to e(s', \sqcup \alpha_S^{1_Q})) = \bigwedge_{s'\in S} e(s', \sqcup \alpha_S^{1_Q}) \leqslant e(s, \sqcup \alpha_S^{1_Q})$$

and, therefore, $s \leqslant \sqcup \alpha_S^{1_Q}$. On the other hand, given $a \in A$ such that $s \leqslant a$ for every $s \in S$, it follows that

$$1_Q \leqslant \bigwedge_{s\in S} e(s, a) = \bigwedge_{a'\in A} (\alpha_S^{1_Q}(a') \to e(a', a)) = e(\sqcup \alpha_S^{1_Q}, a)$$

and, therefore, $\sqcup\alpha_S^{1_Q} \leqslant a$.

To show that $*$ is a module action on $(A, \bigvee)$, we verify the required conditions of Definition 9 in a row.

Item (1): For $q \in Q$ and $S \subseteq A$, it follows that $q * (\bigvee S) = \sqcup\alpha^q_{\bigvee S}$ and $\bigvee_{s\in S}(q * s) = \bigvee_{s\in S} \sqcup\alpha^q_s = \sqcup\alpha_T^{1_Q}$, where T is the shorthand for $\{\sqcup\alpha^q_s \mid s \in S\}$. To continue, we notice that

$$e(\sqcup\alpha^q_{\bigvee S}, \sqcup\alpha_T^{1_Q}) = \bigwedge_{a\in A} (\alpha^q_{\bigvee S}(a) \to e(a, \sqcup\alpha_T^{1_Q})) =$$
$$q \to e(\bigvee S, \sqcup\alpha_T^{1_Q}) = q \to e(\sqcup\alpha_S^{1_Q}, \sqcup\alpha_T^{1_Q}) =$$
$$q \to (\bigwedge_{a\in A} (\alpha_S^{1_Q}(a) \to e(a, \sqcup\alpha_T^{1_Q}))) =$$
$$q \to (\bigwedge_{s\in S} e(s, \sqcup\alpha_T^{1_Q})) = \bigwedge_{s\in S} (q \to e(s, \sqcup\alpha_T^{1_Q})).$$

For every $s \in S$, it follows that $\sqcup\alpha^q_s \leqslant \bigvee\{\sqcup\alpha^q_{s'} \mid s' \in S\} = \sqcup\alpha_T^{1_Q}$ and thus,

$$1_Q \leqslant e(\sqcup\alpha^q_s, \sqcup\alpha_T^{1_Q}) = \bigwedge_{a\in A} (\alpha^q_s(a) \to e(a, \sqcup\alpha_T^{1_Q})) = q \to e(s, \sqcup\alpha_T^{1_Q}).$$

As a consequence, one obtains that $1_Q \leqslant \bigwedge_{s\in S}(q \to e(s, \sqcup\alpha_T^{1_Q}))$ and, therefore, $\sqcup\alpha^q_{\bigvee S} \leqslant \sqcup\alpha_T^{1_Q}$.

For the converse inequality, one starts with the following:

$$e(\sqcup\alpha_T^{1_Q}, \sqcup\alpha^q_{\bigvee S}) = \bigwedge_{a\in A} (\alpha_T^{1_Q}(a) \to e(a, \sqcup\alpha^q_{\bigvee S})) =$$
$$\bigwedge_{s\in S} (1_Q \to e(\sqcup\alpha^q_s, \sqcup\alpha^q_{\bigvee S})) = \bigwedge_{s\in S} e(\sqcup\alpha^q_s, \sqcup\alpha^q_{\bigvee S}) =$$
$$\bigwedge_{s\in S} \bigwedge_{a\in A} (\alpha^q_s(a) \to e(a, \sqcup\alpha^q_{\bigvee S})) = \bigwedge_{s\in S} (q \to e(s, \sqcup\alpha^q_{\bigvee S})).$$

To continue, we notice that

$$1_Q \leqslant e(\sqcup\alpha^q_{\bigvee S}, \sqcup\alpha^q_{\bigvee S}) = \bigwedge_{a\in A} (\alpha^q_{\bigvee S}(a) \to e(a, \sqcup\alpha^q_{\bigvee S})) =$$
$$q \to e(\bigvee S, \sqcup\alpha^q_{\bigvee S})$$

and, therefore, $q \leqslant e(\bigvee S, \sqcup\alpha^q_{\bigvee S})$. For every $s \in S$, it follows that $q = 1_Q \otimes q \leqslant e(s, \bigvee S) \otimes e(\bigvee S, \sqcup\alpha^q_{\bigvee S}) \leqslant e(s, \sqcup\alpha^q_{\bigvee S})$ and, therefore, $1_Q \leqslant q \to e(s, \sqcup\alpha^q_{\bigvee S})$. As a consequence, one immediately obtains that $1_Q \leqslant \bigwedge_{s\in S}(q \to e(s, \sqcup\alpha^q_{\bigvee S}))$, which then yields the desired $\sqcup\alpha_T^{1_Q} \leqslant \sqcup\alpha^q_{\bigvee S}$.

Item (2): For $S \subseteq Q$ and $a \in A$, it follows that $(\bigvee S) * a = \sqcup\alpha_a^{\bigvee S}$ and $\bigvee_{s\in S}(s * a) = \bigvee_{s\in S} \sqcup\alpha^s_a = \sqcup\alpha_T^{1_Q}$, where T is a shorthand for $\{\sqcup\alpha^s_a \mid s \in S\}$. To continue, we notice that

$$e(\sqcup\alpha_a^{\bigvee S}, \sqcup\alpha_T^{1_Q}) = \bigwedge_{a'\in A} (\alpha_a^{\bigvee S}(a') \to e(a', \sqcup\alpha_T^{1_Q})) =$$
$$(\bigvee S) \to e(a, \sqcup\alpha_T^{1_Q}) = \bigwedge_{s\in S} (s \to e(a, \sqcup\alpha_T^{1_Q})).$$

For every $s \in S$, it follows that $\sqcup\alpha^s_a \leqslant \bigvee\{\sqcup\alpha^{s'}_a \mid s' \in S\} = \sqcup\alpha_T^{1_Q}$, which yields,

$$1_Q \leqslant e(\sqcup\alpha^s_a, \sqcup\alpha_T^{1_Q}) = \bigwedge_{a'\in A} (\alpha^s_a(a') \to e(a', \sqcup\alpha_T^{1_Q})) = s \to e(a, \sqcup\alpha_T^{1_Q}).$$

As a result, one gets, $1_Q \leqslant \bigwedge_{s\in S}(s \to e(a, \sqcup \alpha_T^{1_Q}))$ and, therefore, the desired $\sqcup \alpha_a^{\bigvee S} \leqslant \sqcup \alpha_T^{1_Q}$ follows.

For the converse inequality, use the fact that

$$e(\sqcup \alpha_T^{1_Q}, \sqcup \alpha_a^{\bigvee S}) = \bigwedge_{a'\in A} (\alpha_T^{1_Q}(a') \to e(a', \sqcup \alpha_a^{\bigvee S})) =$$
$$\bigwedge_{s\in S} (1_Q \to e(\sqcup \alpha_a^{s}, \sqcup \alpha_a^{\bigvee S})) = \bigwedge_{s\in S} e(\sqcup \alpha_a^{s}, \sqcup \alpha_a^{\bigvee S}) =$$
$$\bigwedge_{s\in S} \bigwedge_{a'\in A} (\alpha_a^{s}(a') \to e(a', \sqcup \alpha_a^{\bigvee S})) = \bigwedge_{s\in S} (s \to e(a, \sqcup \alpha_a^{\bigvee S})) =$$
$$(\bigvee S) \to e(a, \sqcup \alpha_a^{\bigvee S}) = \bigwedge_{a'\in A} (\alpha_a^{\bigvee S}(a') \to e(a', \sqcup \alpha_a^{\bigvee S})) =$$
$$e(\sqcup \alpha_a^{\bigvee S}, \sqcup \alpha_a^{\bigvee S}) \geqslant 1_Q.$$

Item (3): For $q_1, q_2 \in Q$ and $a \in A$, it follows that $q_1 * (q_2 * a) = q_1 * (\sqcup \alpha_a^{q_2}) = \sqcup \alpha_t^{q_1}$, where t is a shorthand for $\sqcup \alpha_a^{q_2}$, and $(q_1 \otimes q_2) * a = \sqcup \alpha_a^{q_1 \otimes q_2}$. To continue, we notice that

$$e(\sqcup \alpha_t^{q_1}, \sqcup \alpha_a^{q_1\otimes q_2}) = \bigwedge_{a'\in A} (\alpha_t^{q_1}(a') \to e(a', \sqcup \alpha_a^{q_1\otimes q_2})) =$$
$$q_1 \to e(\sqcup \alpha_a^{q_2}, \sqcup \alpha_a^{q_1\otimes q_2}) = q_1 \to (\bigwedge_{a'\in A} (\alpha_a^{q_2}(a') \to e(a', \sqcup \alpha_a^{q_1\otimes q_2}))) =$$
$$q_1 \to (q_2 \to e(a, \sqcup \alpha_a^{q_1\otimes q_2})) = (q_1 \otimes q_2) \to e(a, \sqcup \alpha_a^{q_1\otimes q_2}) =$$
$$\bigwedge_{a'\in A} (\alpha_a^{q_1\otimes q_2}(a') \to e(a', \sqcup \alpha_a^{q_1\otimes q_2})) = e(\sqcup \alpha_a^{q_1\otimes q_2}, \sqcup \alpha_a^{q_1\otimes q_2}) \geqslant 1_Q$$

and, therefore, $\sqcup \alpha_t^{q_1} \leqslant \sqcup \alpha_a^{q_1\otimes q_2}$.

For the converse inequality, we notice that

$$e(\sqcup \alpha_a^{q_1\otimes q_2}, \sqcup \alpha_t^{q_1}) = \bigwedge_{a'\in A} (\alpha_a^{q_1\otimes q_2}(a') \to e(a', \sqcup \alpha_t^{q_1})) =$$
$$(q_1 \otimes q_2) \to e(a, \sqcup \alpha_t^{q_1}) = q_1 \to (q_2 \to e(a, \sqcup \alpha_t^{q_1})) =$$
$$q_1 \to (\bigwedge_{a'\in A} (\alpha_a^{q_2}(a') \to e(a', \sqcup \alpha_t^{q_1}))) = q_1 \to e(\sqcup \alpha_a^{q_2}, \sqcup \alpha_t^{q_1}) =$$
$$\bigwedge_{a'\in A} (\alpha_t^{q_1}(a') \to e(a', \sqcup \alpha_t^{q_1})) = e(\sqcup \alpha_t^{q_1}, \sqcup \alpha_t^{q_1}) \geqslant 1_Q.$$

Item (4): Given $a \in A$, it follows that $1_Q * a = \sqcup \alpha_a^{1_Q} = \bigvee\{a\} = a$.

To show that the functor is correct on morphisms, notice that given $S \subseteq A$, we get, $\varphi(\bigvee S) = \varphi(\sqcup \alpha_S^{1_Q}) = \sqcup \varphi_Q^{\to}(\alpha_S^{1_Q}) \overset{(\dagger)}{=} \sqcup \alpha_{\varphi^{\to}(S)}^{1_Q} = \bigvee \varphi^{\to}(S)$, where (†) uses Lemma 22. Moreover, given $q \in Q$ and $a \in A$, it follows that $\varphi(q * a) = \varphi(\sqcup \alpha_a^{q}) = \sqcup \varphi_Q^{\to}(\alpha_a^{q}) \overset{(\dagger)}{=} \sqcup \alpha_{\varphi(a)}^{q} = q * \varphi(a)$, where (†) again relies on Lemma 22. □

Having constructed the two functors, we can prove the main result of this subsection and one of the main (and most interesting) results of this paper. More precisely, the following theorem provides a relation between lattice-valued $\bigvee$-semilattices of Definition 19, which are expressed through fuzzy concepts (e.g., fuzzy sets and fuzzy order) and quantale modules of Definition 9, which is a notion expressed in terms of universal algebra. As a consequence, one gets an additional tool for dealing with many-valued partial orders. In particular, the tool in question (i.e., the theory of quantale modules) is already rather well developed (see, e.g., [36, 43, 64]), which opens the possibility to bring an unsolved problem from the theory of lattice-valued partial orders to the theory of quantale modules, solve it in the new framework, and get the answer back to the initiating one. We obtain thus an analogue of the results of the theory of "natural dualities" [8], which allows an easy interchange between algebraic problems, usually stated in an abstract symbolic language, and their dual, topological problems, where geometric intuition comes to our help.

24. Theorem. *Given a unital commutative quantale Q, $G \circ F = 1_{\mathbf{Mod}(Q)}$ and $F \circ G = 1_{\mathbf{Sup}(Q)}$, i.e., the two categories $\mathbf{Mod}(Q)$ and $\mathbf{Sup}(Q)$ are isomorphic.*

Proof. Given a Q-module $(A, *)$, $G \circ F(A, *) = G(A, e, \sqcup) = (A, \leqslant', \bigvee', *')$. On the other hand, given $a_1, a_2 \in A$, $a_1 \leqslant' a_2$ iff $1_Q \leqslant e(a_1, a_2) = a_1 \twoheadrightarrow a_2$ iff $a_1 = 1_Q * a_1 \leqslant a_2$. Then $\bigvee = \bigvee'$, which can be verified directly, since given $S \subseteq A$, $\bigvee' S = \sqcup\, \alpha_S^{1_Q} = \bigvee_{a \in A}(\alpha_S^{1_Q}(a) * a) = \bigvee_{s \in S}(1_Q * s) = \bigvee S$. Moreover, given $q \in Q$ and $a \in A$, $q *' a = \sqcup\, \alpha_a^q = \bigvee_{a' \in A}(\alpha_a^q(a') * a') = q * a$. Altogether, it follows that $(A, \leqslant', \bigvee', *') = (A, \leqslant, \bigvee, *)$.

Given a Q-$\bigvee$-semilattice $(A, e, \sqcup)$, $F \circ G(A, e, \sqcup) = F(A, \leqslant, \bigvee, *) = (A, e', \sqcup')$. On the other hand, given $a_1, a_2 \in A$, it follows that

$$e'(a_1, a_2) = a_1 \twoheadrightarrow a_2 = \bigvee\{q \in Q \mid q * a_1 \leqslant a_2\} =$$
$$\bigvee\{q \in Q \mid 1_Q \leqslant e(\sqcup\, \alpha_{a_1}^q, a_2)\} = (\dagger).$$

Since $e(\sqcup\, \alpha_{a_1}^q, a_2) = \bigwedge_{a \in A}(\alpha_{a_1}^q(a) \to e(a, a_2)) = q \to e(a_1, a_2)$, we get that

$$(\dagger) = \bigvee\{q \in Q \mid 1_Q \leqslant q \to e(a_1, a_2)\} =$$
$$\bigvee\{q \in Q \mid q \leqslant e(a_1, a_2)\} = e(a_1, a_2)$$

and, therefore, $e'(a_1, a_2) = e(a_1, a_2)$. Given $\alpha \in Q^A$, $\sqcup' \alpha = \bigvee_{a \in A}(\alpha(a) * a) = \bigvee_{a \in A} \sqcup\, \alpha_a^{\alpha(a)}$ $= \sqcup\, \alpha_T^{1_Q}$, where T is a shorthand for $\{\sqcup\, \alpha_a^{\alpha(a)} \mid a \in A\}$. Given $a' \in A$,

$$e(\sqcup\, \alpha_T^{1_Q}, a') = \bigwedge_{a'' \in A}(\alpha_T^{1_Q}(a'') \to e(a'', a')) = \bigwedge_{a \in A}(1_Q \to e(\sqcup\, \alpha_a^{\alpha(a)}, a')) =$$
$$\bigwedge_{a \in A} e(\sqcup\, \alpha_a^{\alpha(a)}, a') = \bigwedge_{a \in A} \bigwedge_{a'' \in A}(\alpha_a^{\alpha(a)}(a'') \to e(a'', a')) =$$
$$\bigwedge_{a \in A}(\alpha(a) \to e(a, a')) = e(\sqcup\, \alpha, a')$$

and thus, $\sqcup' \alpha = \sqcup\, \alpha$. Taken together, it follows that $(A, e', \sqcup') = (A, e, \sqcup)$. □

Notice that Theorem 24 essentially provides two descriptions of the same concept. In the current paper, we are inclined to favor the category $\mathbf{Mod}(Q)$, whose many properties are already known, and (which is more important) whose definition enjoys an easy and straightforward universally algebraic presentation. The subsequent results of this paper will provide additional reasons for our viewpoint.

4.2. Some properties of lattice-valued $\bigvee$-semilattices. Looking closely at the category $\mathbf{Sup}(Q)$ of lattice-valued $\bigvee$-semilattices from the previous subsection, an experienced reader could ask whether its properties resemble those of the well-known and much studied category $\mathbf{Sup}$. A more general question on the overall fruitfulness of such an extension is ultimately looming in the background. It is the main purpose of this subsection, to remove the possible doubts of that kind through considering several simple (but important) properties of the category $\mathbf{Sup}(Q)$. More precisely, we restate several of the properties of skeletal Q-categories (already obtained by, e.g., I. Stubbe [73, 74, 75, 76]) in lattice-valued terms (cf., e.g., Lemma 30 and Proposition 31). Such a restatement is required for a better development of the theory of lattice-valued partial orders, whose tools are different from the already mentioned theory of skeletal Q-categories, based in the technique of enriched categories.

The first feature we extend is the trivial fact that every $\bigvee$-preserving map is automatically monotone. Our intuition suggests that the statement should be valid in the framework of the category $\mathbf{Sup}(Q)$ as well. Strikingly enough, however, the papers of W. Yao [78, 79, 80, 81] keep silence on the topic, strictly distinguishing between lattice-valued monotonicity and preservation of lattice-valued $\bigvee$. With the help of Theorem 24 from the previous subsection, we can clarify the matter. We begin with the extension of crisp monotonicity, modifying the respective many-valued concept of W. Yao [78, 79, 80, 81] developed for frames.

25. Definition. Given two Q-ordered sets (A, e) and (B, e), a map $A \xrightarrow{f} B$ is said to be *Q-monotone* provided that $e(a_1, a_2) \leqslant e(f(a_1), f(a_2))$ for every $a_1, a_2 \in A$. ∎

Notice that we do not require the quantale Q to be commutative. On the other hand, if this is really the case, one easily obtains the following result.

26. Proposition. *Given a unital commutative quantale Q, every $\mathbf{Sup}(Q)$-morphism is Q-monotone.*

Proof. Given a $\mathbf{Sup}(Q)$-morphism $(A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)$, there exists a $\mathbf{Mod}(Q)$-morphism $(A, *) \xrightarrow{\varphi} (B, *)$ such that $F((A, *) \xrightarrow{\varphi} (B, *)) = (A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)$ and, therefore, we can assume that the maps e, $\sqcup$ are induced by the action $*$. Given $a_1, a_2 \in A$, $q \leqslant e(a_1, a_2) = a_1 \twoheadrightarrow a_2$ implies $q * a_1 \leqslant a_2$ implies $q * \varphi(a_1) \leqslant \varphi(a_2)$ implies $q \leqslant \varphi(a_1) \twoheadrightarrow \varphi(a_2) = e(\varphi(a_1), \varphi(a_2))$. Altogether, $e(a_1, a_2) \leqslant e(\varphi(a_1), \varphi(a_2))$. □

Proposition 26 illustrates the technique, which will be used throughout this subsection, i.e., replacing the abstract maps e and $\sqcup$ of a Q-$\bigvee$-semilattice with their concrete realizations through a module action. Simple as it looks, the machinery is capable of providing several useful results.

Our next property extends another well-known result that every $\bigvee$-semilattice is actually a complete lattice, i.e., has additionally a $\bigwedge$-operation. This fact was heavily employed in the definition of Q-$\bigvee$-semilattices in the previous subsection and also in the most important results of the latter and, therefore, the simple property should be most welcome in the extended framework. In the following, we show that this really is the case. Start with the extension of the crisp $\bigwedge$-operation to our new framework (notice that we still follow the frame path of W. Yao [78, 79, 80, 81]).

27. Definition. Given a Q-poset (A, e), the map $Q^A \xrightarrow{\sqcap} A$ is called a *Q_r-meet operation* on A provided that $e(a, \sqcap \alpha) = \bigwedge_{a' \in A}(\alpha(a') \rightarrow_r e(a, a'))$ for every $\alpha \in Q^A$ and every $a \in A$. ∎

Replacing $\cdot \rightarrow_r \cdot$ with $\cdot \rightarrow_l \cdot$, one obtains the concept of *Q_l-meet operation.* The case of a commutative quantale Q provides a nice property of these notions.

28. Proposition. *Given a unital commutative quantale Q, every $\mathbf{Sup}(Q)$-object has Q-meets.*

Proof. Given a Q-$\bigvee$-semilattice $(A, e, \sqcup)$, we know that both e and $\sqcup$ are induced by a module action $*$ on A. Define a map $Q^A \xrightarrow{\sqcap} A$ by $\sqcap \alpha = \bigwedge_{a \in A}(\alpha(a) \rightsquigarrow a)$ (recall the notation, stated before Lemma 17). To show that the map is the desired Q-meet operation on A, notice that given $\alpha \in Q^A$ and $a \in A$, it follows that $e(a, \sqcap \alpha) = a \twoheadrightarrow \sqcap \alpha = a \twoheadrightarrow (\bigwedge_{a' \in A}(\alpha(a') \rightsquigarrow a')) = \bigwedge_{a' \in A}(a \twoheadrightarrow (\alpha(a') \rightsquigarrow a')) = \bigwedge_{a' \in A}(\alpha(a') \rightarrow (a \twoheadrightarrow a')) = \bigwedge_{a' \in A}(\alpha(a') \rightarrow e(a, a'))$. □

It should be underlined that in case of lattice-valued frames, W. Yao [78, 79] provides a stronger result, namely, that the conditions of the existence of L-join- or L-meet operation for a frame L are equivalent. We will not pursue, however, the topic any further, which would lead us off the goal of the paper.

The last property concerns the concept of Galois connection on $\bigvee$-semilattices. The standard result (see, e.g., [11] or [16, Section 0-3]) says that every **Sup**-morphism $(A,\bigvee) \xrightarrow{\varphi} (B,\bigvee)$ has an upper adjoint map $B \xrightarrow{\psi} A$ characterized uniquely by the condition $\varphi(a) \leqslant b$ iff $a \leqslant \psi(b)$, for every $a \in A$ and every $b \in B$. The explicit formula for the map is then given by $\psi(b) = \bigvee\{a \in A \mid \varphi(a) \leqslant b\} = \bigvee \varphi^{\leftarrow}(\downarrow b)$, where $\downarrow b = \{b' \in B \mid b' \leqslant b\}$. Moreover, one can show that ψ is $\bigwedge$-preserving. Since the above machinery was much used in the previous subsection, its analogue in the extended setting seems to be highly desirable. In the following, we provide its generalization, employing the frame notions of W. Yao [78, 79, 81].

29. Definition. Given Q-posets (A,e) and (B,e), a pair (g,f) of maps $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ is a *Q-Galois connection* or a *Q-adjunction* between (A,e) and (B,e) provided that $e(f(a),b) = e(a,g(b))$ for every $a \in A$ and every $b \in B$. The map f (resp. g) is called *Q-lower* (resp. *Q-upper*) *adjoint*. ∎

The following lemma provides the extension of two well-known properties of Galois connections.

30. Lemma. *Given a Q-Galois connection (g,f) between (A,e) and (B,e), the following hold:*

(1) *both g and f are Q-monotone;*
(2) *g (resp. f) preserves the existing Q-(r,l)-$\bigwedge$ (resp. Q-(r,l)-$\bigvee$).*

Proof. To show Item (1), notice that given $a_1, a_2 \in A$, $e(f(a_1), f(a_2)) = e(a_1, g\circ f(a_2)) \geqslant e(a_1,a_2) \otimes e(a_2, g\circ f(a_2)) = e(a_1,a_2) \otimes e(f(a_2), f(a_2)) \geqslant e(a_1,a_2) \otimes 1_Q = e(a_1,a_2)$. On the other hand, given $b_1, b_2 \in B$, it follows that $e(g(b_1), g(b_2)) = e(f\circ g(b_1), b_2) \geqslant e(f\circ g(b_1), b_1) \otimes e(b_1,b_2) = e(g(b_1), g(b_1)) \otimes e(b_1,b_2) \geqslant 1_Q \otimes e(b_1,b_2) = e(b_1,b_2)$.

For Item (2), use the fact that given $\alpha \in Q^A$ such that $\sqcap\,\alpha$ exists and $a \in A$,

$$\begin{aligned} e(a, g(\sqcap\,\alpha)) = e(f(a), \sqcap\,\alpha) = \bigwedge_{b\in B} (\alpha(b) \rightarrow_{r,l} e(f(a),b)) = \\ \bigwedge_{b\in B} (\alpha(b) \rightarrow_{r,l} e(a,g(b))) = \bigwedge_{a'\in A} \bigwedge_{g(b)=a'} (\alpha(b) \rightarrow_{r,l} e(a,g(b))) = \\ \bigwedge_{a'\in A} \bigwedge_{g(b)=a'} (\alpha(b) \rightarrow_{r,l} e(a,a')) = \bigwedge_{a'\in A} ((\bigvee_{g(b)=a'} \alpha(b)) \rightarrow_{r,l} e(a,a')) = \\ \bigwedge_{a'\in A} ((g_Q^{\rightarrow}(\alpha))(a') \rightarrow_{r,l} e(a,a')). \end{aligned}$$

It follows that $\sqcap\, g_Q^{\rightarrow}(\alpha)$ exists and equals $g(\sqcap\,\alpha)$.

Given $\alpha \in Q^A$ such that $\sqcup \alpha$ exists and $b \in B$,

$$e(f(\sqcup \alpha), b) = e(\sqcup \alpha, g(b)) = \bigwedge_{a \in A} (\alpha(a) \rightarrow_{r,l} e(a, g(b))) =$$
$$\bigwedge_{a \in A} (\alpha(a) \rightarrow_{r,l} e(f(a), b)) = \bigwedge_{b' \in B} \bigwedge_{f(a)=b'} (\alpha(a) \rightarrow_{r,l} e(f(a), b)) =$$
$$\bigwedge_{b' \in B} \bigwedge_{f(a)=b'} (\alpha(a) \rightarrow_{r,l} e(b', b)) = \bigwedge_{b' \in B} ((\bigvee_{f(a)=b'} \alpha(a)) \rightarrow_{r,l} e(b', b)) =$$
$$\bigwedge_{b' \in B} ((f_Q^{\rightarrow}(\alpha))(b') \rightarrow_{r,l} e(b', b)).$$

It follows that $\sqcup f_Q^{\rightarrow}(\alpha)$ exists and equals $f(\sqcup \alpha)$. □

Notice that in order to illustrate the extension of the classical duality machinery to the fuzzy setting, Lemma 30 provides the proofs, which usually are replaced with something like "follows through duality".

Turning back to quantale modules, to employ the standard machinery, we introduce a simple notation. Given a Q-poset (A, e), every $a \in A$ provides a map $A \xrightarrow{\downarrow_e a} Q$ defined by $(\downarrow_e a)(b) = e(b, a)$ (notice the fuzzification of the above-mentioned lower set $\downarrow a$).

31. Proposition. *Given a unital commutative quantale Q, every* $\mathbf{Sup}(Q)$*-morphism* $(A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)$ *has a Q-upper adjoint.*

Proof. We again assume that the maps e, $\sqcup$ are induced by their respective module actions on A. Define a map $B \xrightarrow{\psi} A$ by $\psi(b) = \sqcup \varphi_Q^{\leftarrow}(\downarrow_e b)$. To check the adjunction property, notice that given $a \in A$ and $b \in B$, $e(\varphi(a), b) = \varphi(a) \twoheadrightarrow b$, whereas $e(a, \psi(b)) = a \twoheadrightarrow \psi(b)$, where

$$\psi(b) = \sqcup \varphi_Q^{\leftarrow}(\downarrow_e b) = \bigvee_{a' \in A} ((\varphi_Q^{\leftarrow}(\downarrow_e b))(a') * a') = \bigvee_{a' \in A} ((\downarrow_e b)(\varphi(a')) * a') =$$
$$\bigvee_{a' \in A} (e(\varphi(a'), b) * a') = \bigvee_{a' \in A} ((\varphi(a') \twoheadrightarrow b) * a') = \bigvee_{a' \in A} ((\bigvee_{q * \varphi(a') \leqslant b} q) * a') =$$
$$\bigvee_{a' \in A} \bigvee_{q * \varphi(a') \leqslant b} (q * a') = \bigvee_{q * \varphi(a') \leqslant b} (q * a')$$

and, therefore, $e(a, \psi(b)) = a \twoheadrightarrow (\bigvee_{q * \varphi(a') \leqslant b} (q * a')) = a \twoheadrightarrow (\bigvee S)$. Given $q \in Q$, $q \leqslant \varphi(a) \twoheadrightarrow b$ implies $q * \varphi(a) \leqslant b$ implies $q * a \in S$ implies $q * a \leqslant \bigvee S$ implies $q \leqslant a \twoheadrightarrow (\bigvee S)$. On the other hand, $q \leqslant a \twoheadrightarrow (\bigvee S)$ implies $q * a \leqslant \bigvee S$ implies $\varphi(q * a) \leqslant \varphi(\bigvee S)$ implies $q * \varphi(a) \leqslant \bigvee_{q * \varphi(a') \leqslant b} (q * \varphi(a')) \leqslant b$ implies $q \leqslant \varphi(a) \twoheadrightarrow b$. Altogether, one obtains, $e(\varphi(a), b) = e(a, \psi(b))$. □

The challenging task of generalizing other important results to the new setting will be left to the subsequent developments of the topic, whereas here, we will extend Q-$\bigvee$-semilattices to lattice-valued quantales.

4.3. Quantale algebras as lattice-valued quantales. This subsection provides the main result of the section, namely, a representation of quantale algebras as lattice-valued quantales. With the concept of lattice-valued frame of W. Yao [79, 81] in mind, we introduce the latter notion in the following way (cf. the crisp case of Definition 2).

32. Definition. Given a unital quantale Q, a Q_r*-quantale* is a tuple $(A, e, \sqcup, \otimes)$, where $(A, e, \sqcup)$ is a Q_r-$\bigvee$-semilattice and $A \times A \xrightarrow{\otimes} A$ is a map (Q*-multiplication* on A) such that

(1) $(A, \otimes)$ is a semigroup;
(2) $a \otimes (\sqcup \alpha) = \sqcup (a \otimes \cdot)^{\rightarrow}_{Q}(\alpha)$ and $(\sqcup \alpha) \otimes a = \sqcup (\cdot \otimes a)^{\rightarrow}_{Q}(\alpha)$ for every $a \in A$ and every $\alpha \in Q^A$.

A Q_r*-quantale homomorphism* $(A, e, \sqcup, \otimes) \xrightarrow{\varphi} (B, e, \sqcup, \otimes)$ is a map $A \xrightarrow{\varphi} B$, which is a Q_r-$\bigvee$-semilattice homomorphism $(A, e, \sqcup) \xrightarrow{\varphi} (B, e, \sqcup)$ such that $\varphi(a_1 \otimes a_2) = \varphi(a_1) \otimes \varphi(a_2)$ for every $a_1, a_2 \in A$. $\mathbf{Quant}_r(Q)$ is the category of Q_r-quantales and their homomorphisms, concrete over both $\mathbf{Sup}_r(Q)$ and $\mathbf{SGrp}$. ∎

Similarly, one gets the category $\mathbf{Quant}_l(Q)$. Below, we generalize the fact mentioned after Definition 10 that the category $\mathbf{Alg}(\mathbf{2})$ is isomorphic to the category $\mathbf{Quant}$ (cf. the isomorphism between the category $\mathbf{Alg}(\mathbb{Z})$ of algebras over the ring of integers $\mathbb{Z}$ and the category $\mathbf{Rng}$ of rings [30]), namely, we show that given a unital commutative quantale Q, the categories $\mathbf{Alg}(Q)$ and $\mathbf{Quant}(Q)$ are isomorphic (notice that $\mathbf{Quant}_l(Q) = \mathbf{Quant}_r(Q) = \mathbf{Quant}(Q)$ for a commutative quantale Q). The underlying machinery will rely on the isomorphism between the categories $\mathbf{Mod}(Q)$ and $\mathbf{Sup}(Q)$ of Theorem 24.

33. Proposition. *Given a unital commutative quantale Q, there exists a functor $\mathbf{Alg}(Q) \xrightarrow{F} \mathbf{Quant}(Q)$ defined by $F((A, \otimes, *) \xrightarrow{\varphi} (B, \otimes, *)) = (A, e, \sqcup, \otimes) \xrightarrow{\varphi} (B, e, \sqcup, \otimes)$, where the maps e and $\sqcup$ are obtained as in Proposition 21.*

Proof. In view of Proposition 21, it will be enough to check the correctness of the functor on objects and that will follow from verification of Item (2) of Definition 32. Given $a \in A$ and $\alpha \in Q^A$, for every $\bar{a} \in A$,

$$e(\sqcup (a \otimes \cdot)^{\rightarrow}_{Q}(\alpha), \bar{a}) = \bigwedge_{a' \in A} (((a \otimes \cdot)^{\rightarrow}_{Q}(\alpha))(a') \rightarrow e(a', \bar{a})) =$$
$$\bigwedge_{a' \in A} ((\bigvee_{a \otimes a'' = a'} \alpha(a'')) \rightarrow (a' \twoheadrightarrow \bar{a})) = \bigwedge_{a' \in A} \bigwedge_{a \otimes a'' = a'} (\alpha(a'') \rightarrow (a' \twoheadrightarrow \bar{a})) =$$
$$\bigwedge_{a' \in A} \bigwedge_{a \otimes a'' = a'} (\alpha(a'') \rightarrow ((a \otimes a'') \twoheadrightarrow \bar{a})) = \bigwedge_{a'' \in A} (\alpha(a'') \rightarrow ((a \otimes a'') \twoheadrightarrow \bar{a})) =$$
$$\bigwedge_{a'' \in A} ((\alpha(a'') * (a \otimes a'')) \twoheadrightarrow \bar{a}) = \bigwedge_{a'' \in A} ((a \otimes (\alpha(a'') * a'')) \twoheadrightarrow \bar{a}) =$$
$$(\bigvee_{a'' \in A} (a \otimes (\alpha(a'') * a''))) \twoheadrightarrow \bar{a} = (a \otimes (\bigvee_{a'' \in A} (\alpha(a'') * a''))) \twoheadrightarrow \bar{a} =$$
$$(a \otimes (\sqcup \alpha)) \twoheadrightarrow \bar{a} = e(a \otimes (\sqcup \alpha), \bar{a}).$$

As a result, one obtains that $a \otimes (\sqcup \alpha) = \sqcup (a \otimes \cdot)^{\rightarrow}_{Q}(\alpha)$.

On the other hand,

$$e(\sqcup (\cdot \otimes a)^{\rightarrow}_{Q}(\alpha), \bar{a}) = \bigwedge_{a' \in A} (((\cdot \otimes a)^{\rightarrow}_{Q}(\alpha))(a') \rightarrow e(a', \bar{a})) =$$
$$\bigwedge_{a' \in A} ((\bigvee_{a'' \otimes a = a'} \alpha(a'')) \rightarrow (a' \twoheadrightarrow \bar{a})) = \bigwedge_{a' \in A} \bigwedge_{a'' \otimes a = a'} (\alpha(a'') \rightarrow (a' \twoheadrightarrow \bar{a})) =$$
$$\bigwedge_{a' \in A} \bigwedge_{a'' \otimes a = a'} (\alpha(a'') \rightarrow ((a'' \otimes a) \twoheadrightarrow \bar{a})) = \bigwedge_{a'' \in A} (\alpha(a'') \rightarrow ((a'' \otimes a) \twoheadrightarrow \bar{a})) =$$
$$\bigwedge_{a'' \in A} ((\alpha(a'') * (a'' \otimes a)) \twoheadrightarrow \bar{a}) = \bigwedge_{a'' \in A} (((\alpha(a'') * a'') \otimes a) \twoheadrightarrow \bar{a}) =$$
$$(\bigvee_{a'' \in A} ((\alpha(a'') * a'') \otimes a)) \twoheadrightarrow \bar{a} = ((\bigvee_{a'' \in A} (\alpha(a'') * a'')) \otimes a) \twoheadrightarrow \bar{a} =$$
$$((\sqcup \alpha) \otimes a) \twoheadrightarrow \bar{a} = e((\sqcup \alpha) \otimes a, \bar{a}).$$

As a result, we get that $(\sqcup \alpha) \otimes a = \sqcup(\cdot \otimes a)^{\rightarrow}_{Q}(\alpha)$. $\Box$

Notice that to illustrate the use of the properties of quantale algebras, we provided the full proof for both the right and the left distributivity laws.

34. Proposition. *Given a unital commutative quantale Q, there exists a functor* $\mathbf{Quant}(Q) \xrightarrow{G} \mathbf{Alg}(Q)$, $G((A,e,\sqcup,\otimes) \xrightarrow{\varphi} (B,e,\sqcup,\otimes)) = (A,\leqslant,\bigvee,*,\otimes) \xrightarrow{\varphi} (B,\leqslant,\bigvee,*,\otimes)$, *where $\leqslant$, $\bigvee$ and $*$ are obtained as in Proposition 23.*

Proof. In view of Proposition 23, it will be enough to show that the functor is correct on objects and that will follow from verification of Item (3) of Definition 2 and Item (3) of Definition 10.

For the first item, notice that given $S \subseteq A$ and $a \in A$, $a \otimes (\bigvee S) = a \otimes (\sqcup \alpha_S^{1_Q}) = \sqcup(a \otimes \cdot)^{\rightarrow}_{Q}(\alpha_S^{1_Q})$ and $\bigvee_{s\in S}(a\otimes s) = \sqcup\, \alpha_T^{1_Q}$, where T is a shorthand for $\{a\otimes s \mid s \in S\}$. For every $\bar{a} \in A$, it follows that

$$e(\sqcup(a\otimes\cdot)^{\rightarrow}_{Q}(\alpha_S^{1_Q}),\bar{a}) = \bigwedge_{a'\in A}(((a\otimes\cdot)^{\rightarrow}_{Q}(\alpha_S^{1_Q}))(a') \rightarrow e(a',\bar{a})) =$$
$$\bigwedge_{a'\in A}((\bigvee_{a\otimes a''=a'}\alpha_S^{1_Q}(a'')) \rightarrow e(a',\bar{a})) = \bigwedge_{a'\in A}\bigwedge_{a\otimes a''=a'}(\alpha_S^{1_Q}(a'') \rightarrow e(a',\bar{a})) =$$
$$\bigwedge_{s\in S}(1_Q \rightarrow e(a\otimes s,\bar{a})) = \bigwedge_{s\in S} e(a\otimes s,\bar{a}).$$

On the other hand,

$$e(\sqcup\,\alpha_T^{1_Q},\bar{a}) = \bigwedge_{a'\in A}(\alpha_T^{1_Q}(a') \rightarrow e(a',\bar{a})) = \bigwedge_{s\in S}(1_Q \rightarrow e(a\otimes s,\bar{a})) =$$
$$\bigwedge_{s\in S} e(a\otimes s,\bar{a}).$$

Altogether, $e(\sqcup(a\otimes\cdot)^{\rightarrow}_{Q}(\alpha_S^{1_Q}),\bar{a}) = e(\sqcup\,\alpha_T^{1_Q},\bar{a})$ and, therefore, $\sqcup(a\otimes\cdot)^{\rightarrow}_{Q}(\alpha_S^{1_Q}) = \sqcup\,\alpha_T^{1_Q}$, which yields then the desired $a\otimes(\bigvee S) = \bigvee_{s\in S}(a\otimes s)$.

To show the second distributivity law, notice that $(\bigvee S)\otimes a = (\sqcup\,\alpha_S^{1_Q})\otimes a = \sqcup(\cdot\otimes a)^{\rightarrow}_{Q}(\alpha_S^{1_Q})$ and $\bigvee_{s\in S}(s\otimes a) = \sqcup\,\alpha_T^{1_Q}$, where T is a shorthand for $\{s\otimes a \mid s\in S\}$. For every $\bar{a}\in A$, it follows that

$$e(\sqcup(\cdot\otimes a)^{\rightarrow}_{Q}(\alpha_S^{1_Q}),\bar{a}) = \bigwedge_{a'\in A}(((\cdot\otimes a)^{\rightarrow}_{Q}(\alpha_S^{1_Q}))(a') \rightarrow e(a',\bar{a})) =$$
$$\bigwedge_{a'\in A}((\bigvee_{a''\otimes a=a'}\alpha_S^{1_Q}(a'')) \rightarrow e(a',\bar{a})) = \bigwedge_{a'\in A}\bigwedge_{a''\otimes a=a'}(\alpha_S^{1_Q}(a'') \rightarrow e(a',\bar{a})) =$$
$$\bigwedge_{s\in S}(1_Q \rightarrow e(s\otimes a,\bar{a})) = \bigwedge_{s\in S} e(s\otimes a,\bar{a}).$$

Moreover,

$$e(\sqcup\,\alpha_T^{1_Q},\bar{a}) = \bigwedge_{a'\in A}(\alpha_T^{1_Q}(a') \rightarrow e(a',\bar{a})) = \bigwedge_{s\in S}(1_Q \rightarrow e(s\otimes a,\bar{a})) =$$
$$\bigwedge_{s\in S} e(s\otimes a,\bar{a}).$$

As a result, we obtain that $e(\sqcup(\cdot\otimes a)^{\rightarrow}_{Q}(\alpha_S^{1_Q}),\bar{a}) = e(\sqcup\,\alpha_T^{1_Q},\bar{a})$, namely, $\sqcup(\cdot\otimes a)^{\rightarrow}_{Q}(\alpha_S^{1_Q}) = \sqcup\,\alpha_T^{1_Q}$, which provides then the desired $(\bigvee S)\otimes s = \bigvee_{s\in S}(s\otimes a)$.

For the second item, notice that given $q\in Q$ and $a_1,a_2\in A$, $q*(a_1\otimes a_2) = \sqcup\,\alpha^q_{a_1\otimes a_2}$, $(q*a_1)\otimes a_2 = (\sqcup\,\alpha^q_{a_1})\otimes a_2 = \sqcup(\cdot\otimes a_2)^{\rightarrow}_{Q}(\alpha^q_{a_1})$ and $a_1\otimes(q*a_2) = a_1\otimes(\sqcup\,\alpha^q_{a_2}) =$

$\sqcup(a_1 \otimes \cdot)^{\rightarrow}_{Q}(\alpha^q_{a_2})$. For every $\bar{a} \in A$, it follows that

$$e(\sqcup(\cdot \otimes a_2)^{\rightarrow}_{Q}(\alpha^q_{a_1}), \bar{a}) = \bigwedge_{a \in A} (((\cdot \otimes a_2)^{\rightarrow}_{Q}(\alpha^q_{a_1}))(a) \rightarrow e(a, \bar{a})) =$$

$$\bigwedge_{a \in A} ((\bigvee_{a' \otimes a_2 = a} \alpha^q_{a_1}(a')) \rightarrow e(a, \bar{a})) = \bigwedge_{a \in A} \bigwedge_{a' \otimes a_2 = a} (\alpha^q_{a_1}(a') \rightarrow e(a, \bar{a})) =$$

$$q \rightarrow e(a_1 \otimes a_2, \bar{a})$$

as well as

$$e(\sqcup(a_1 \otimes \cdot)^{\rightarrow}_{Q}(\alpha^q_{a_2}), \bar{a}) = \bigwedge_{a \in A} (((a_1 \otimes \cdot)^{\rightarrow}_{Q}(\alpha^q_{a_2}))(a) \rightarrow e(a, \bar{a})) =$$

$$\bigwedge_{a \in A} ((\bigvee_{a_1 \otimes a' = a} \alpha^q_{a_2}(a')) \rightarrow e(a, \bar{a})) = \bigwedge_{a \in A} \bigwedge_{a_1 \otimes a' = a} (\alpha^q_{a_2}(a') \rightarrow e(a, \bar{a})) =$$

$$q \rightarrow e(a_1 \otimes a_2, \bar{a}).$$

On the other hand, we obtain that

$$e(\sqcup \alpha^q_{a_1 \otimes a_2}, \bar{a}) = \bigwedge_{a \in A} (\alpha^q_{a_1 \otimes a_2}(a) \rightarrow e(a, \bar{a})) = q \rightarrow e(a_1 \otimes a_2, \bar{a}).$$

As a consequence, one gets that

$$e(\sqcup(\cdot \otimes a_2)^{\rightarrow}_{Q}(\alpha^q_{a_1}), \bar{a}) = e(\sqcup \alpha^q_{a_1 \otimes a_2}, \bar{a}) = e(\sqcup(a_1 \otimes \cdot)^{\rightarrow}_{Q}(\alpha^q_{a_2}), \bar{a}).$$

It immediately follows that $\sqcup(\cdot \otimes a_2)^{\rightarrow}_{Q}(\alpha^q_{a_1}) = \sqcup \alpha^q_{a_1 \otimes a_2} = \sqcup(a_1 \otimes \cdot)^{\rightarrow}_{Q}(\alpha^q_{a_2})$, which then gives the desired equality $(q * a_1) \otimes a_2 = q * (a_1 \otimes a_2) = a_1 \otimes (q * a_2)$. □

The two propositions in hand, we can prove the main result of this section.

35. Theorem. *Given a unital commutative quantale Q, $G \circ F = 1_{\mathbf{Alg}(Q)}$ and $F \circ G = 1_{\mathbf{Quant}(Q)}$, i.e., the two categories $\mathbf{Alg}(Q)$ and $\mathbf{Quant}(Q)$ are isomorphic.*

Proof. Follows from Theorem 24, in view of Propositions 33, 34. □

Similar to the case of Theorem 24, Theorem 35 provides two descriptions of the same concept. It is our opinion that $\mathbf{Alg}(Q)$ is better suited for applications due to its compact universally algebraic definition and a certain knowledge on its properties. The next section will give more reasons for such an opinion.

4.4. Quantale algebras as lattice-valued frames. The main results of the previous two subsections can be summarized as follows (we notice that the prefix "**U**" in the notations for categories in Theorem 36 stands for "unital", which in case of, e.g., the category $\mathbf{UQuant}(Q)$ means that Item (1) of Definition 32 employs a monoid $(A, \otimes, \imath)$).

36. Theorem. *Given a unital commutative quantale Q, the categories $(Q \downarrow \mathbf{UQuant})_z$, $\mathbf{UAlg}(Q)$ and $\mathbf{UQuant}(Q)$ are isomorphic.*

Proof. Follows from Theorems 16, 35 and the construction of functors of Propositions 33, 34. □

The isomorphism between $(Q \downarrow \mathbf{UQuant})_z$ and $\mathbf{UAlg}(Q)$ is more demanding, since it requires the existence of the unit in the underlying quantales of Q-algebras, and the isomorphism between $\mathbf{UAlg}(Q)$ and $\mathbf{UQuant}(Q)$ is the restriction of a more general one between $\mathbf{Alg}(Q)$ and $\mathbf{Quant}(Q)$. In other words, one easily gets the following result (see the construction of the functors of Propositions 33, 34).

37. Corollary. *For every unital commutative quantale Q, the category $(Q \downarrow \mathbf{UQuant})_z$ is isomorphic to a non-full subcategory of the category $\mathbf{Quant}(Q)$.*

Corollary 37 acquires more importance when one considers the concepts of lattice-valued frame of D. Zhang and Y.-M. Liu [84] as well as W. Yao [79]. To make the handling of the corresponding situation easier, below we introduce two additional categories.

38. Definition. For a frame L, $\mathbf{UAlg}_{Frm}(L)$ is the full subcategory of $\mathbf{UAlg}(L)$, whose objects have frames as their underlying quantales. $\mathbf{Frm}(L)$ is the image of the subcategory $\mathbf{UAlg}_{Frm}(L)$ under the isomorphism F of Proposition 33. ▪

It is easy to check that the category $\mathbf{Frm}(L)$ is isomorphic to the category L-$\mathbf{Frm}_Y$ of L-frames of W. Yao [81]. Also notice the double simplification of the category $\mathbf{Alg}(Q)$, not only taking frames as the underlying algebraic structures of quantale algebras, but also replacing the quantale Q with a frame L. Such a reduced case makes Corollary 37 stronger and, possibly, more interesting.

39. Corollary. *Given a frame L, the categories $(L \downarrow \mathbf{Frm})$ and $\mathbf{Frm}(L)$ are isomorphic.*

Proof. By Theorem 16, $(L \downarrow \mathbf{Frm})$ is isomorphic to the category $\mathbf{UAlg}_{Frm}(L)$. □

Since the category $(L \downarrow \mathbf{Frm})$ provides the concept of lattice-valued frame of D. Zhang and Y.-M. Liu [84], Corollary 39 says that the notions of D. Zhang and Y.-M. Liu as well as of W. Yao are categorically equivalent. It should be noticed immediately that W. Yao [81] obtained the same result. Corollary 39, however, provides a more general viewpoint on this relation and employs completely different machinery. In particular, Corollary 37 shows that the passage from frames to quantales makes the setting of D. Zhang and Y.-M. Liu different from that of W. Yao. Moreover, since both concepts of lattice-valued frame are instances of quantale algebras, by our opinion, they both are categorically redundant in mathematics. The next section elaborates our opinion in full extent.

5. Applications to lattice-valued topology

When looking closely into the papers, which introduce the concepts of lattice-valued frames, considered in this article, one sees immediately that both of them are motivated by the wish of their authors to extend the well-known equivalence between sober topological spaces and spatial locales [32] to the setting of lattice-valued topology. The crucial point here is the following. Since locales come essentially from the crisp world, e.g., are nicely and conveniently related to crisp topology, they can easily loose their efficiency in the lattice-valued framework. Indeed, one is confronted with the use of one and the same algebraic structure to encode the information on both crisp and lattice-valued topological spaces. While the passage from spaces to locales causes no difficulty, the converse transformation is liable to miss some information on its way. Despite the fact that S. E. Rodabaugh [52] successfully extended the crisp localic machinery to the lattice-valued case, later on, he himself cast certain doubts on its fruitfulness and introduced a fuzzification on the localic side as well, considering lattice-valued locales [49, 50]. The previous section gave another framework for dealing with the notion. It is the main purpose of this section to show its fruitfulness in this respect.

To begin with, we recall the concept of stratified topological space [47, 58]. Notice that given a set X and a $\bigvee$-semilattice L, the L-powerset L^A is a $\bigvee$-semilattice with the pointwise algebraic structure. The result is easily extendable to other algebraic structures, e.g., unital Q-algebras. Moreover, for every $a \in L$, we denote by $\underline{a}$ the constant map $X \xrightarrow{\underline{a}} L$ with the value a.

40. Definition. Given a unital quantale Q, a *Q-topological space* or *Q-space* is a pair (X, τ), where X is a set and τ is a unital subquantale of Q^X. Given Q-spaces (X, τ) and (Y, σ), a map $X \xrightarrow{f} Y$ is said to be *Q-continuous* provided that $(f_Q^{\leftarrow})^{\rightarrow}(\sigma) \subseteq \tau$.

$\mathbf{Top}(Q)$ is the category of Q-topological spaces and Q-continuous maps, concrete over the category **Set**. ▪

41. Definition. Given a unital quantale Q, a Q-space (X, τ) is called *stratified* provided that $\{\underline{q} \mid q \in Q\} \subseteq \tau$. $\mathbf{STop}(Q)$ is the full subcategory of $\mathbf{Top}(Q)$ consisting of stratified Q-spaces. ▪

42. Definition. Given a unital quantale Q and a unital subquantale D of Q, a Q-space (X, τ) is called *stratified to degree* D provided that $\{\underline{q} \mid q \in D\} \subseteq \tau$. $\mathbf{STop}_D(Q)$ is the full subcategory of $\mathbf{Top}(Q)$ consisting of Q-spaces, which are stratified to degree D. ▪

Notice that the stratification idea is due to R. Lowen [38], the term itself first occurring in [46]. Stratification degree was first encountered by the author in [47]. It appears that there exists a nice relation between quantale algebras and (stratified) lattice-valued topological spaces. Start with one preliminary notion.

43. Definition. A unital Q-algebra A is said to be $*$-*divisible w.r.t.* 1_A (*divisible*, for short) provided that for every $a \in A$, there exists $q \in Q$ such that $a = q * 1_A$. ▪

Every unital quantale Q provides a unital divisible Q-algebra, since given $q \in Q$, $q = q \otimes 1_Q = q * 1_Q$. In particular, every frame L is a unital divisible L-algebra.

44. Proposition. *Let A be a unital Q-algebra and let (X, τ) be an A-space. If (X, τ) is stratified, then τ is a unital sub(Q-)algebra of A^X. If A is divisible and τ is a unital sub(Q-)algebra of A^X, then τ is stratified.*

Proof. For the first statement, notice that it is enough to check the closure of τ under the module action. Given $\alpha \in \tau$ and $q \in Q$, $(q * \alpha)(x) = q * \alpha(x) = q * (1_A \otimes \alpha(x)) = (q * 1_A) \otimes \alpha(x) = \underline{q * 1_A}(x) \otimes \alpha(x) = (\underline{q * 1_A} \otimes \alpha)(x)$ for every $x \in X$. As a result, $q * \alpha = \underline{q * 1_A} \otimes \alpha \in \tau$, since $\underline{q * 1_A} \in \tau$ by stratification.

For the second statement, notice that given $a \in A$, by the condition of the proposition, there exists some $q \in Q$ such that $a = q * 1_A$. Since $\underline{1_A} \in \tau$ and τ is a Q-module, $\underline{a} = \underline{q * 1_A} = q * \underline{1_A} \in \tau$. □

With Proposition 44 in hand, one obtains the following result.

45. Theorem. *Given a unital commutative quantale Q and a Q-algebra A, there is a functor $\mathbf{STop}(A) \xrightarrow{\Omega_A} \mathbf{UAlg}(Q)$ defined by $\Omega_A((X, \tau) \xrightarrow{f} (Y, \sigma)) = \tau \xrightarrow{(f_A^{\leftarrow})^{op}} \sigma$.*

The real power of the above result can be exploited in the framework of variety-based topology [63, 65]. In particular, one easily obtains the functor in the opposite direction as well as the related concepts of sobriety and spatiality, providing an equivalence between sober topological spaces and spatial Q-algebras (see [65], where the case $A = Q$ is considered). The resulting issue here is as follows. Since the concept of Q-algebra incorporates the above-mentioned two notions of lattice-valued frames, the respective extensions of the sobriety-spatiality equivalence of D. Zhang and Y.-M. Liu [84] and W. Yao [79] are particular instances of that for Q-algebras and, therefore, are categorically redundant in lattice-valued mathematics. Based in this observation, we strongly believe in the desirability to shift from lattice-valued frames to quantale algebras.

As a final remark, we notice that the passage from unital Q-algebras to stratified topologies in Proposition 44 requires divisibility of the respective Q-algebra A. Since, in general, the property rarely holds, it is time for stratification degree to come in play. Recall from Proposition 14 that every unital Q-algebra A provides a map $Q \xrightarrow{i_A} A$ defined by $i_A(q) = q * 1_A$ and denote by D_A the image of i_A.

46. Proposition. *Given a unital Q-algebra A, every A-space is stratified to degree D_A.*

Proof. Given an A-space (X, τ), and $a \in D_A$, there exists some $q \in Q$ such that $a = q * 1_A$. Similar to the proof of the second part of Proposition 44, one obtains that $\underline{a} \in \tau$. □

As a consequence, it follows that every category $\mathbf{Top}(A)$ over a unital Q-algebra A is essentially the category $\mathbf{STop}_{D_A}(A)$ of A-spaces stratified to degree D_A. The observation provides a convenient framework for studying the concept of stratification in lattice-valued topology.

6. Conclusion: open problems

Employing the isomorphism between the categories of right Q-modules and cocomplete skeletal Q-categories, obtained by I. Stubbe [76] for every unital quantale Q (in fact, a small quantaloid $\mathcal{Q}$), in this paper, we showed that the concept of quantale algebra, introduced recently [67] as a generalization of the well-known notion of algebra over a commutative ring with identity, has a significant merit of providing a common framework for (at least) two notions of lattice-valued frames available in the literature, namely, L-fuzzy frames of D. Zhang and Y.-M. Liu [84] and L-frames of W. Yao [79]. The obtained results suggest categorical redundancy of these concepts in mathematics in (at least) two respects. Firstly, both of them are isomorphic to particular subcategories of the category of quantale algebras and, moreover, are categorically equivalent to each other (as already observed by W. Yao [81]). Secondly, their motivating extensions of the classical equivalence of the categories of sober topological spaces and spatial locales to the lattice-valued world can be done much easier and more straightforward in the setting of quantale algebras. The quantale algebra extension in its turn follows from the results obtained in the realm of variety-based topology, providing another fruitful example of its usefulness as well as making its current generalization to categorically-algebraic (catalg) topology [62, 70] most desirable. Moreover, the isomorphism between the categories of quantale algebras and lattice-valued quantales of Theorem 35, suggest categorical redundancy of lattice-valued quantales (and, in particular, lattice-valued frames) in fuzzy mathematics. On the other hand, the results of Subsection 4.2 make the development of non-categorical properties of lattice-valued quantales highly desirable, in order to streamline and study deeper the classical properties of crisp quantales. It will be the topic of our forthcoming papers to investigate this issue in its full generality.

As it happens with every new theory, certain open problems arise in its development, some of which are worth (by our opinion) to be presented to the reader.

6.1. From lattice-valued frames to lattice-valued quantales. In Corollary 39, we showed categorical equivalence between the concepts of lattice-valued frame of D. Zhang, Y.-M. Liu [84] and W. Yao [79]. On the other hand, Corollary 37 shows that the frameworks are different in case of arbitrary quantales. In particular, it suggests that the setting of D. Zhang and Y.-M. Liu can be partly incorporated into that of W. Yao. The obtained relationships, however, are by no means complete, requiring further studies on the topic. At the moment, one can pose the following open problems.

47. Problem. Does the category $(Q \downarrow \mathbf{UQuant})_z$ provide a (co)reflective subcategory of $\mathbf{Quant}(Q)$? ▪

48. Problem. Is the category $\mathbf{Quant}(Q)$ isomorphic to a subcategory of $(Q \downarrow \mathbf{Quant})$? ▪

49. Problem. To what extent is it possible to lift the isomorphism of Corollary 39 to quantale setting? ▪

The first problem deals with a generalization of the issue of adding a unit to a non-unital quantale considered in [69]. The last problem is ultimately the most important and, probably, the most difficult one.

6.2. Lattice-valued frames of A. Pultr and S. E. Rodabaugh. Having incorporated two concepts of lattice-valued frame in the setting of quantale algebras, we have exhausted the topic by no means. In particular, there exists another famous instance of the notion, introduced by A. Pultr and S. E. Rodabaugh [48] and studied by them further in [49, 50]. As has been mentioned in Introduction, its motivation came from the Lowen-Kubiák ι_L (fibre map) functor [37, 38]. As a result, the ultimate definition is more complicated than the respective concepts of this paper.

Start with a preliminary notation, namely, given a $\bigwedge$-semilattice L, let $L_\top$ denote the set $L\backslash\{\top\}$.

50. Definition. Given a chain L, an *L-frame* is a system of frame homomorphisms $A = (A^{\mathfrak{u}} \xrightarrow{\varphi_t^A} A^{\mathfrak{l}})_{t\in L_\top}$ such that

(1) $\varphi^A_{\bigwedge S} = \bigvee_{s\in S} \varphi^A_s$ for every non-empty $S \subseteq L_\top$;
(2) A is an extremal epi-sink;
(3) A is a mono-source. ∎

The condition of L being a chain deals mostly with the meet-irreducibles of L (as was pointed out by U. Höhle) and, therefore, its various modifications has already been considered by U. Höhle and S. E. Rodabaugh [28] as well as J. Gutiérrez García, U. Höhle and M. A. de Prada Vicente [21]. Despite these changes, the notion is still considerably out of the scope of the classical definitions of lattice-valued frames. In view of the results of this paper, the next problem springs into mind immediately.

51. Problem. Does there exist any connection between quantale algebras and lattice-valued frames of A. Pultr and S. E. Rodabaugh? ∎

Notice that while the concept of quantale algebra essentially provides an extension of partially ordered sets, employing generalization of partial order in the sense of Principle of Fuzzification of J. A. Goguen [17], the just mentioned notion of lattice-valued frame seems to be more sophisticated, the first of its conditions stemming from the realm of sheaves [49]. As a result, a quick look at Problem 51 inspired the author with nothing more than the following observations.

Every Q-algebra $(A, *)$ provides two families of maps: $\mathcal{A}_1 = (A \xrightarrow{q*\cdot} A)_{q\in Q}$ and $\mathcal{A}_2 = (Q \xrightarrow{\cdot * a} A)_{a\in A}$. Moreover, the Q-action on A can be restored from each of them. The next lemma shows several simple (but important) properties of these families.

52. Lemma. *Given a Q-algebra A, the following hold:*

(1) *every element of $\mathcal{A}_1$, $\mathcal{A}_2$ is a $\bigvee$-semilattice homomorphism;*
(2) *if every element of Q (resp. A) is idempotent w.r.t. the multiplication, then every element of $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) is a quantale homomorphism;*
(3) *$\mathcal{A}_1$ is both a mono-source and an epi-sink, whereas $\mathcal{A}_2$ is an epi-sink; both are extremal epi-sinks in the category* **Sup***;*
(4) *if $Q = \mathbf{2}$, then $\mathcal{A}_1 = (A \xrightarrow{\perp} A, A \xrightarrow{1_A} A)$, whereas $\mathcal{A}_2 = (\mathbf{2} \xrightarrow{\cdot * a} A)_{a\in A}$ with* $q * a = \begin{cases} a, & q = 1_{\mathbf{2}} \\ \perp, & \textit{otherwise}; \end{cases}$
(5) *if $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) satisfies Item (1) of Definition 50, then $(\bigwedge S) * a = (\bigvee S) * a$ for every $a \in A$ and every non-empty $S \subseteq Q_\top$ (resp. A has no more than two elements).*

Proof. Item (1) follows from the properties of Q-algebras (Definition 10).

To show Item (2), notice that given $q \in Q$ and $a_1, a_2 \in A$, it follows that $q*(a_1 \otimes a_2) \overset{(\dagger)}{=} (q \otimes q) * (a_1 \otimes a_2) = q * (q * (a_1 \otimes a_2)) = q * (a_1 \otimes (q * a_2)) = (q * a_1) \otimes (q * a_2)$, where (†) uses the idempotency of Q. On the other hand, given $a \in A$ and $q_1, q_2 \in Q$, $(q_1 \otimes q_2) * a \overset{(\dagger)}{=} (q_1 \otimes q_2) * (a \otimes a) = q_1 * (q_2 * (a \otimes a)) = q_1 * (a \otimes (q_2 * a)) = (q_1 * a) \otimes (q_2 * a)$, where (†) uses the idempotency of A.

The first part of Item (3) follows from the fact that $1_Q * \cdot$ is the identity map on A. For the second part, notice that given $a \in A$, $1_Q * a = a$ and, therefore, $\bigcup_{a \in A} (\cdot * a)^{\rightarrow}(Q) = A$. For the last part, use the fact that both sinks are jointly surjective (cf. [2, Examples 10.65(1)]).

Item (4) is straightforward.

To verify Item (5), notice that in case of $\mathcal{A}_1$, the requirement provides $(\bigwedge S) * a = \bigvee_{s \in S} (s * a)$ for every $a \in A$ and every non-empty $S \subseteq Q_\top$. With Definition 10 in mind, one obtains, $(\bigwedge S) * a = (\bigvee S) * a$ for every $a \in A$ and every non-empty $S \subseteq Q_\top$.

The case of $\mathcal{A}_2$ gives $q * (\bigwedge S) = \bigvee_{s \in S} (q * s)$ for every $q \in Q$ and every non-empty $S \subseteq A_\top$. By Definition 10, substituting 1_Q for q, we get, $\bigwedge S = \bigvee S$ for every non-empty $S \subseteq A_\top$. Now, given $a_1, a_2 \in A_\top$, $a_1 \leqslant a_1 \vee a_2 = a_1 \wedge a_2 \leqslant a_2$ and, similarly, $a_2 \leqslant a_1$, resulting in $a_1 = a_2$. □

Taking into consideration the properties of frames (e.g., idempotency of the meet operation), Lemma 52 provides a point in favor of the above-mentioned representations of Q-algebras. However, its Item (5) eliminates the use of the representation $\mathcal{A}_2$ (also suggested by the second part of Item (4) of Lemma 52). It will be the topic of our further research to study the issue in full detail.

The above open problems will be addressed in our forthcoming papers.

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Chen inequalities for submanifolds of generalized space forms with a semi-symmetric metric connection

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Abstract

We investigate sharp inequalities for submanifolds in both generalized complex space forms and generalized Sasakian space forms with a semi-symmetric metric connection.

Keywords: Chen inequality, generalized complex space form, generalized Sasakian space form, semi-symmetric metric connection.

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1. Introduction

A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric linear connection on a Riemannian manifold in [10]. Later, H. A. Hayden [11] gave the definition of a semi-symmetric metric connection. In 1970, K. Yano [19] studied semi-symmetric metric connection and proved that a Rimannian manifold admits a semi-symmetric metric connection with vanishing curvature tensor if and only if the manifold is conformally flat. Then, in [12], [13] and [16] T. Imai and Z. Nakao considered some properties of a Riemannian manifold admitting a semi-symmetric metric connection and they studied submanifolds of a Riemannian manifold with a semi-symmetric metric connection.

On the other hand, B. Y. Chen introduced *Chen inequality* and he gave the definition of new types of curvature invariants (called extrinsic and intrinsic invariants) in [6]. Then,

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in [7], [8] and [9], he established sharp inequalities for different submanifolds in various ambient spaces.

In [3] and [4], K. Arslan, R. Ezentaş, I. Mihai, C. Murathan and C. Özgür studied Chen inequalities for submanifolds in locally conformal almost cosymplectic manifolds and (κ, μ)-contact space forms, respectively. Later, P. Alegre, A. Carriazo, Y. H. Kim and D. W. Yoon considered same inequalities for submanifolds of generalized space forms in [2].

Recently, in [14], A. Mihai and C. Özgür proved Chen inequalities for submanifolds of real space forms admitting a semi-symmetric metric connection. They also studied same problems for submanifolds of complex space forms and Sasakian space forms with a semi-symmetric metric connection in [15]. As a generalization of the results of [15], in tis study, we prove similar inequalities for submanifolds of generalized complex space forms and generalized Sasakian space forms with respect to a semi-symmetric metric connection.

2. Preliminaries

Let N be an $(n+p)$-dimensional Riemannian manifold with a Riemannian metric g. A linear connection $\widetilde{\nabla}$ on a Riemannian manifold N is called a *semi-symmetric connection* if the torsion tensor $\widetilde{T}$ of the connection $\widetilde{\nabla}$

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}] \tag{2.1}$$

satisfies

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = w(\widetilde{Y})\widetilde{X} - w(\widetilde{X})\widetilde{Y}, \tag{2.2}$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N, where w is a 1-form associated with the vector field U on N defined by

$$w(\widetilde{X}) = g(\widetilde{X}, U). \tag{2.3}$$

$\widetilde{\nabla}$ is called a *semi-symmetric metric connection* if

$$\widetilde{\nabla} g = 0.$$

If $\overset{\circ}{\widetilde{\nabla}}$ is the Levi-Civita connection of a Riemannian manifold N, the semi-symmetric metric connection $\widetilde{\nabla}$ is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + w(\widetilde{Y})\widetilde{X} - g(\widetilde{X},\widetilde{Y})U, \tag{2.4}$$

(see [19]).

Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N. We will consider the induced semi-symmetric metric connection by ∇ and the induced Levi-Civita connection by $\overset{\circ}{\nabla}$ on the submanifold M.

Let $\widetilde{R}$ and $\overset{\circ}{\widetilde{R}}$ be curvature tensors of $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{\nabla}}$ of a Riemannian manifold N, respectively. We also denote by R the curvature tensor of M with respect to ∇ and $\overset{\circ}{R}$ the curvature tensor of M with respect to $\overset{\circ}{\nabla}$. Then the Gauss formulas with a semi-symmetric metric connection ∇ and the Levi-Civita connection $\overset{\circ}{\nabla}$, respectively, are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$$

and

$$\overset{\circ}{\widetilde{\nabla}}_X Y = \overset{\circ}{\nabla}_X Y + \overset{\circ}{\sigma}(X, Y),$$

for any vector fields X, Y tangent to M, where $\overset{\circ}{\sigma}$ is the second fundamental form of M in N and σ is a $(0, 2)$-tensor on M. Also, the mean curvature vector of M in N is denoted by $\overset{\circ}{H}$.

The equation of Gauss for an n-dimensional submanifold M in an $(n+p)$-dimensional Riemannian manifold N is given by

$$\overset{\circ}{\widetilde{R}}(X, Y, Z, W) = \overset{\circ}{R}(X, Y, Z, W) + g(\overset{\circ}{\sigma}(X, Z), \overset{\circ}{\sigma}(Y, W)) - g(\overset{\circ}{\sigma}(Y, Z), \overset{\circ}{\sigma}(X, W)) \tag{2.5}$$

Then, $\widetilde{R}$ and $\overset{\circ}{\widetilde{R}}$ are related by

$$\begin{aligned} \widetilde{R}(X, Y, Z, W) \quad = \quad & \overset{\circ}{\widetilde{R}}(X, Y, Z, W) - \alpha(Y, Z)g(X, W) + \alpha(X, Z)g(Y, W) - \\ & -\alpha(X, W)g(Y, Z) + \alpha(Y, W)g(X, Z), \end{aligned} \tag{2.6}$$

for any vector fields X, Y, Z, W on N [19], where $(0, 2)$-tensor field α is given by

$$\alpha(X, Y) = \left(\overset{\circ}{\widetilde{\nabla}} w\right) Y - w(X)w(Y) + \frac{1}{2} w(U)g(X, Y),$$

for $X, Y \in \chi(M)$, where the trace of α is denoted by

$$trace\alpha = \lambda.$$

Denote by $K(\pi)$ or $K(u, v)$ the sectional curvature of M associated with a 2-plane section $\pi \subset T_x M$ with respect to the induced semi-symmetric non-metric connection ∇, where $\{u, v\}$ is an orthonormal basis of π. The scalar curvature τ at $x \in M$ is denoted by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j),$$

where $\{e_1, ..., e_n\}$ is any orthonormal basis of $T_x M$ [8].

We will need the following Chen's lemma for later use:

2.1. Lemma. [6] Let $n \ge 2$ and $a_1, a_2, ..., a_n, b$ be real numbers such that

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right). \tag{2.7}$$

Then $2a_1 a_2 \ge b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = ... = a_n.$$

Let M be an n-dimensional Riemannian manifold, L a k-plane section of T_xM, $x \in M$ and X a unit vector in L.

For an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = X$, the *Ricci curvature* (or k-Ricci curvature) of L at X is defined by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k},$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i and e_j. For any integer k, $2 \leq k \leq n$, the Riemannian invariant Θ_k of M is denoted by

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad x \in M,$$

where L runs over all k-plane sections in T_xM and X runs over all unit vectors in L.

3. Chen inequality for submanifolds of generalized complex space forms

We consider as an ambient space a generalized complex space form with a semi-symmetric metric connection.

A $2m$-dimensional almost Hermitian manifold (N, J, g) is said to be a *generalized complex space form* (see [17] and [18]) if there exist two functions F_1 and F_2 on N such that

$$\overset{\circ}{\widetilde{R}}(X,Y,Z,W) = F_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + \tag{3.1}$$
$$+F_2[g(X,JZ)g(JY,W) - g(Y,JZ)g(JX,W) + 2g(X,JY)g(JZ,W)],$$

for any vector fields X, Y, Z, W on N, where $\overset{\circ}{\widetilde{R}}$ is the curvature tensor of N with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$. In such a case, we will write $N(F_1, F_2)$.

If $N(F_1, F_2)$ is a generalized complex space form with a semi-symmetric metric connection $\widetilde{\nabla}$, then by the use of (2.6) and (3.1), the curvature tensor $\widetilde{R}$ of $N(F_1, F_2)$ can be written as

$$\widetilde{R}(X,Y,Z,W) = F_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + \tag{3.2}$$
$$+F_2[g(X,JZ)g(JY,W) - g(Y,JZ)g(JX,W) + 2g(X,JY)g(JZ,W)] -$$
$$-\alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z).$$

Let M be an n-dimensional, $n \geq 3$, submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$. We put

$$JX = PX + FX,$$

for any vector field X tangent to M, where PX and FX are tangential and normal components of JX, respectively.

We also set

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j).$$

On the other hand, $\Theta^2(\pi)$ is denoted by $\Theta^2(\pi) = g^2(Pe_1, e_2) = g^2(Je_1, e_2)$ in [2], where $\{e_1, e_2\}$ is an orthonormal basis of a 2-plane section π. $\Theta^2(\pi)$ is a real number in $[0, 1]$, independent of the choice of e_1 and e_2.

For submanifolds of generalized complex space forms with respect to the semi-symmetric metric connection we establish the following sharp inequality:

3.1. Theorem. Let M, $n \geq 3$, be an n-dimensional submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Then we have:

$$\begin{aligned} \tau(x) - K(\pi) \quad \leq \quad & \frac{n-2}{2}\left[\frac{n^2}{n-1}\|H\|^2 + (n+1)F_1 - 2\lambda\right] - \\ & - \left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{F_2}{2} - trace(\alpha_{|\pi^\perp}), \end{aligned} \tag{3.3}$$

where π is a 2-plane section of T_xM, $x \in M$.

Proof. Let $\{e_1, e_2, ...e_n\}$ be an orthonormal basis of T_xM and $\{e_{n+1}, ..., e_{2m}\}$ be an orthonormal basis of $T_x^\perp M$, $x \in M$, where e_{n+1} is parallel to the mean curvature vector H.

Taking $X = W = e_i$ and $Y = Z = e_j$ such that $i \neq j$ and by the use of (3.2), we get

$$\widetilde{R}(e_i, e_j, e_j, e_i) = F_1 + 3F_2 g^2(Je_i, e_j) - \alpha(e_i, e_i) - \alpha(e_j, e_j). \tag{3.4}$$

From [16], the Gauss equation with respect to the semi-symmetric metric connection can be written as

$$\widetilde{R}(e_i, e_j, e_j, e_i) = R(e_i, e_j, e_j, e_i) + g(\sigma(e_i, e_j), \sigma(e_i, e_j)) - g(\sigma(e_i, e_i), \sigma(e_j, e_j)). \tag{3.5}$$

Comparing the right hand sides of the equations (3.4) and (3.5), we obtain

$$\begin{aligned} & F_1 + 3F_2 g^2(Je_i, e_j) - \alpha(e_i, e_i) - \alpha(e_j, e_j) \\ = \quad & R(e_i, e_j, e_j, e_i) + g(\sigma(e_i, e_j), \sigma(e_i, e_j)) - g(\sigma(e_i, e_i), \sigma(e_j, e_j)). \end{aligned}$$

Then, by summation over $1 \leq i, j \leq n$, the above equation turns into

$$\begin{aligned} & 2\tau + \|\sigma\|^2 - n^2\|H\|^2 \\ = \quad & n(n-1)F_1 + 3F_2\sum_{i,j=1}^{n} g^2(Je_i, e_j) - 2(n-1)\lambda, \end{aligned} \tag{3.6}$$

where

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j))$$

and

$$H = \frac{1}{n} trace\sigma.$$

We set

$$\delta = 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 + 2(n-1)\lambda - n(n-1)F_1 - 3F_2\|P\|^2. \tag{3.7}$$

Then, the equation (3.6) can be written as follows

(3.8) $$n^2\left\|H\right\|^2=(n-1)\left(\left\|\sigma\right\|^2+\delta\right).$$

For a chosen orthonormal basis, the relation (3.8) takes the following form

$$\left(\sum_{i=1}^{n}\sigma_{ii}^{n+1}\right)^2=(n-1)\left[\sum_{i=1}^{n}(\sigma_{ii}^{n+1})^2+\sum_{i\neq j}(\sigma_{ij}^{n+1})^2+\sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(\sigma_{ij}^{r})^2+\delta\right].$$

So, by the use of Chen's Lemma, we have

$$2\sigma_{11}^{n+1}\sigma_{22}^{n+1}=\sum_{1\le i\neq j\le n}(\sigma_{ij}^{n+1})^2+\sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(\sigma_{ij}^{r})^2+\delta.$$

Let π be a 2-plane section of T_xM at a point x, where $\pi=sp\{e_1,e_2\}$. Then, the Gauss equation for $X=Z=e_1$ and $Y=W=e_2$ gives us

$$\begin{aligned}
K(\pi)&=F_1+3F_2g^2(Je_1,e_2)-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\sum_{r=n+1}^{2m}[\sigma_{11}^{r}\sigma_{22}^{r}-(\sigma_{12}^{r})^2]\ge\\
&\ge F_1+3F_2g^2(Je_1,e_2)-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\\
&+\frac{1}{2}\left(\sum_{1\le i\neq j\le n}(\sigma_{ij}^{n+1})^2+\sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(\sigma_{ij}^{r})^2+\delta\right)+\sum_{r=n+2}^{2m}\sigma_{11}^{r}\sigma_{22}^{r}-\sum_{r=n+1}^{2m}(\sigma_{12}^{r})^2\\
&=F_1+3F_2g^2(Je_1,e_2)-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\\
&+\frac{1}{2}\sum_{1\le i\neq j\le n}(\sigma_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{2m}\sum_{i,j>2}(\sigma_{ij}^{r})^2+\\
&+\frac{1}{2}\sum_{r=n+2}^{2m}(\sigma_{11}^{r}+\sigma_{22}^{r})^2+\sum_{j>2}[(\sigma_{1j}^{n+1})^2+(\sigma_{2j}^{n+1})^2]+\frac{1}{2}\delta\ge\\
&\ge F_1+3F_2g^2(Je_1,e_2)-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\frac{1}{2}\delta
\end{aligned}$$

which implies

$$K(\pi)\ge F_1+3F_2g^2(Je_1,e_2)-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\frac{1}{2}\delta.$$

From (3.7), it is easy to see that

$$\begin{aligned}
K(\pi)\quad\ge\quad&\tau-\frac{n-2}{2}\left[\frac{n^2}{n-1}\left\|H\right\|^2+(n+1)F_1-2\lambda\right]+\\
&+\left[6\Theta^2(\pi)-3\left\|P\right\|^2\right]\frac{F_2}{2}+trace(\alpha_{|\pi^\perp}),
\end{aligned}$$

where $trace(\alpha_{|\pi^\perp})$ is denoted by

$$\alpha(e_1,e_1)+\alpha(e_2,e_2)=\lambda-trace(\alpha_{|\pi^\perp})$$

(see [15]). Hence, we finish the proof of the theorem. ∎

3.2. Proposition. The mean curvature H of M admitting semi-symmetric metric connection coincides with the mean curvature $\overset{\circ}{H}$ of M admitting Levi-Civita connection if and only if the vector field U is tangent to M.

As a consequence of Proposition 3.2 we can give the following result:

3.3. Theorem. If the vector field U is tangent to M, then the equality case of (3.3) holds at a point $x \in M$ if and only if there exist an orthonormal basis $\{e_1, e_2, ...e_n\}$ of T_xM and an orthonormal basis $\{e_{n+1}, ..., e_{2m}\}$ of $T_x^{\perp}M$ such that the shape operators of M in $N(F_1, F_2)$ at x have the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu$$

and

$$A_{e_r} = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 & \cdots & 0 \\ \sigma_{12}^r & -\sigma_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad n+2 \leq i \leq 2m,$$

where we denote by $\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n+2 \leq r \leq 2m$.

Proof. Equality case holds at a point $x \in M$ if and only if the equality holds in each of the previous inequalities and hence the Lemma yields equality.

$$\sigma_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$\sigma_{ij}^r = 0, \quad \forall i \neq j, i, j > 2, r = n+1, ..., 2m,$$

$$\sigma_{11}^r + \sigma_{22}^r = 0, \quad \forall r = n+2, ..., 2m,$$

$$\sigma_{1j}^{n+1} = \sigma_{2j}^{n+1} = 0, \quad \forall j > 2,$$

$$\sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} = ... = \sigma_{nn}^{n+1}.$$

If we choose $\{e_1, e_2\}$ such that $\sigma_{12}^{n+1} = 0$ and denote by $a = \sigma_{11}^r$, $b = \sigma_{22}^r$, $\mu = \sigma_{33}^{n+1} = ... = \sigma_{nn}^{n+1}$, then the shape operators take the desired forms. ∎

4. Ricci curvature for submanifolds of generalized complex space forms

In this section we establish relationship between the Ricci curvature of a submanifold M in a generalized complex space form $N(F_1, F_2)$ with a semi-symmetric metric connection, and the squared mean curvature $\|H\|^2$.

Now, let begin with the following theorem:

4.1. Theorem. Let M, $n \geq 3$, be an n-dimensional submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Then we have:

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} + \frac{2}{n}\lambda - F_1 - \frac{3F_2}{n(n-1)} \|P\|^2 . \tag{4.1}$$

Proof. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of T_xM and $\{e_{n+1}, ..., e_{2m}\}$ be an orthonormal basis of $T_x^{\perp}M$ at $x \in M$, where e_{n+1} is parallel to the mean curvature vector H.

Then, the equation (3.7) can be written as follows

$$n^2 \|H\|^2 = 2\tau + \|\sigma\|^2 + 2(n-1)\lambda - n(n-1)F_1 - 3F_2 \|P\|^2 . \tag{4.2}$$

For a choosen orthonormal basis, let $e_1, e_2, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then, the shape operators take the forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

and

$$A_{e_r} = (\sigma_{ij}^r), \quad i, j = 1, ..., n; \quad r = n+2, ..., 2m, \quad traceA_{e_r} = 0.$$

By the use of (4.2), we obtain

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2 + \\ &\quad +2(n-1)\lambda - n(n-1)F_1 - 3F_2 \|P\|^2 . \end{aligned} \tag{4.3}$$

On the other hand, since

$$0 \le \sum_{i<j} (a_i - a_j)^2 = (n-1)\sum_{i=1}^{n} a_i^2 - 2\sum_{i<j} a_i a_j,$$

we get

$$n^2 \|H\|^2 = \left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \le n\sum_{i=1}^{n} a_i^2,$$

which means

$$\sum_{i=1}^{n} a_i^2 \ge n \|H\|^2 . \tag{4.4}$$

Thus, in view of (4.4) in (4.3) we get (4.1), which completes the proof of the theorem. ∎

In view of Theorem 4.1, we can give the following theorem:

4.2. Theorem. Let M, $n \ge 3$, be an n-dimensional submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field U is tangent to M. Then, for any integer k, $2 \le k \le n$ and for any point $x \in M$, we have:

$$\|H\|^2 (x) \ge \Theta_k(\pi) + \frac{2}{n}\lambda - F_1 - \frac{3F_2}{n(n-1)} \|P\|^2 . \tag{4.5}$$

Proof. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of T_xM at $x \in M$. The k-plane section spanned by $e_{i_1}, ..., e_{i_k}$ is denoted by $L_{i_1 \dots i_k}$. Then, by the definitions, we can write

$$\tau(L_{i_1 \dots i_k}) = \frac{1}{2} \sum_{i \in \{i_1 \dots i_k\}} Ric_{L_{i_1 \dots i_k}}(e_i) \tag{4.6}$$

and

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \le i_1 \le \dots \le i_k \le n} \tau(L_{i_1 \dots i_k}). \tag{4.7}$$

By making use of (4.6) and (4.7) in (4.1), we obtain

$$\tau(x) \ge \frac{n(n-1)}{2} \Theta_k(\pi),$$

which gives us (4.5). ∎

5. Chen inequality for submanifolds of generalized Sasakian space forms

Let N be a $(2m+1)$-dimensional *almost contact metric manifold* [5] with an almost contact metric structure (φ, ξ, η, g) consisting of a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g on N satisfying

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X),$$

for all vector fields X, Y on N. Such a manifold is said to be a *contact metric manifold* if $d\eta = \Phi$, where $\Phi(X,Y) = g(X, \varphi Y)$ is called the *fundamental* 2*-form* of N [5].

On the other hand, the almost contact metric structure of N is said to be *normal* if

$$[\varphi, \varphi](X,Y) = -2d\eta(X,Y)\xi,$$

for any vector fields X, Y on N, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi, \varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal contact metric manifold is called a *Sasakian manifold* [5].

Given an almost contact metric manifold N with an almost contact metric structure (φ, ξ, η, g), N is called a *generalized Sasakian space form* [1] if there exist three functions f_1, f_2 and f_3 on N such that

$$\begin{aligned} \overset{\circ}{\tilde{R}}(X,Y,Z,W) = {} & f_1\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} + \\ & + f_2\{g(X,\varphi Z)g(\varphi Y, W) - g(Y, \varphi Z)g(\varphi X, W) + 2g(X, \varphi Y)g(\varphi Z, W)\} + \\ & + f_3\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z)\}, \end{aligned} \tag{5.1}$$

for any vector fields X, Y, Z, W on N, where $\overset{\circ}{\tilde{R}}$ denotes the curvature tensor of N with respect to the Levi-Civita connection $\overset{\circ}{\tilde{\nabla}}$. In such a case, we will write $N(f_1, f_2, f_3)$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, then N is a Sasakian space form.

If $N(f_1, f_2, f_3)$ is a $(2m+1)$-dimensional generalized Sasakian space form with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Then, from (2.6) and (5.1) the curvature tensor $\widetilde{R}$ of $N(f_1, f_2, f_3)$ can be written as follows

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= f_1\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}+ \\ &+f_2\{g(X,\varphi Z)g(\varphi Y,W) - g(Y,\varphi Z)g(\varphi X,W) + 2g(X,\varphi Y)g(\varphi Z,W)\}+ \\ &+f_3\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z)\}- \\ &-\alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z).\end{aligned} \tag{5.2}$$

Let M, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional generalized Sasakian space form. We put

$$\varphi X = PX + FX,$$

for any vector field X tangent to M, where PX and FX are tangential and normal components of φX, respectively.

We also set

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(\varphi e_i, e_j).$$

Decompose

$$\xi = \xi^{\top} + \xi^{\perp},$$

where $\xi^{\top}$ and $\xi^{\perp}$ denote the tangential and normal components of ξ.

From [2], recall $\Theta^2(\pi) = g^2(Pe_1, e_2) = g^2(\varphi e_1, e_2)$, where $\{e_1, e_2\}$ is an orthonormal basis of a 2-plane section π, is a real number in $[0,1]$, independent of the choice of e_1 and e_2.

Now, let begin with the following theorem which gives us a sharp inequality for submanifolds of generalized Sasakian space forms with respect to the semi-symmetric metric connection:

5.1. Theorem. Let M, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional generalized Sasakian space form $N(f_1, f_2, f_3)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Then we have:

$$\begin{aligned}\tau(x) - K(\pi) \quad &\leq \quad (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{f_1}{2} - \lambda\right] - \\ &- \left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{f_2}{2} + [\|\xi_\pi\|^2 - (n-1)\left\|\xi^{\top}\right\|^2]f_3 - \\ &- trace(\alpha_{|\pi^{\perp}}),\end{aligned} \tag{5.3}$$

where π is a 2-plane section of T_xM, $x \in M$.

Proof. Let $\{e_1, e_2, ...e_n\}$ be an orthonormal basis of T_xM and $\{e_{n+1}, ..., e_{2m+1}\}$ be an orthonormal basis of $T_x^{\perp}M$, $x \in M$, where e_{n+1} is parallel to the mean curvature vector H.

For $X = W = e_i$ and $Y = Z = e_j$ such that $i \neq j$, the equation (5.2) can be written as

$$\widetilde{R}(e_i, e_j, e_j, e_i) = f_1 + 3f_2 g^2(\varphi e_i, e_j) - f_3[\eta(e_i)^2 + \eta(e_j)^2] - \alpha(e_1, e_1) - \alpha(e_2, e_2). \tag{5.4}$$

Comparing the right hand sides of the equations (3.5) and (5.4) we can write

$$\begin{aligned} & f_1 + 3f_2 g^2(\varphi e_i, e_j) - f_3[\eta(e_i)^2 + \eta(e_j)^2] - \alpha(e_1, e_1) - \alpha(e_2, e_2) \\ = \; & R(e_i, e_j, e_j, e_i) + g(\sigma(e_i, e_j), \sigma(e_i, e_j)) - g(\sigma(e_i, e_i), \sigma(e_j, e_j)). \end{aligned}$$

Then, by summation over $1 \leq i, j \leq n$, the above relation reduces to

$$2\tau + \|\sigma\|^2 - n^2 \|H\|^2 = n(n-1)f_1 + 3f_2 \|P\|^2 - 2(n-1)f_3 \left\|\xi^\top\right\|^2 - 2(n-1)\lambda. \tag{5.5}$$

If we put

$$\delta = 2\tau - \frac{n^2(n-2)}{n-1} \|H\|^2 + 2(n-1)\lambda - n(n-1)f_1 - 3f_2 \|P\|^2 + 2(n-1)f_3 \left\|\xi^\top\right\|^2, \tag{5.6}$$

the equation (5.5) turns into

$$n^2 \|H\|^2 = (n-1)\left(\|\sigma\|^2 + \delta\right). \tag{5.7}$$

For a chosen orthonormal basis, the relation (5.7) takes the following form

$$\left(\sum_{i=1}^{n} \sigma_{ii}^{n+1}\right)^2 = (n-1)\left[\sum_{i=1}^{n} (\sigma_{ii}^{n+1})^2 + \sum_{i \neq j} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2 + \delta\right].$$

So, by the use of Chen's Lemma, we have

$$2\sigma_{11}^{n+1}\sigma_{22}^{n+1} = \sum_{1 \leq i \neq j \leq n} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2 + \delta.$$

Let π be a 2-plane section of T_xM at a point x, where $\pi = sp\{e_1, e_2\}$. We need to denote $\xi_\pi = pr_\pi \xi$ for the later use as follows

$$\|\xi_\pi\|^2 = \eta(e_1)^2 + \eta(e_2)^2.$$

Then, from the Gauss equation for $X = Z = e_1$ and $Y = W = e_2$ we get

$$\begin{aligned} K(\pi) = {} & f_1 + 3f_2 g^2(Pe_1, e_2) - f_3 \|\xi_\pi\|^2 - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \sum_{r=n+1}^{2m+1} [\sigma_{11}^r \sigma_{22}^r - (\sigma_{12}^r)^2] \geq \\ & \geq f_1 + 3f_2 g^2(Pe_1, e_2) - f_3 \|\xi_\pi\|^2 - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \\ & + \frac{1}{2}\left(\sum_{1 \leq i \neq j \leq n} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2 + \delta\right) + \sum_{r=n+2}^{2m+1} \sigma_{11}^r \sigma_{22}^r - \sum_{r=n+1}^{2m+1} (\sigma_{12}^r)^2 \\ & = f_1 + 3f_2 g^2(Pe_1, e_2) - f_3 \|\xi_\pi\|^2 - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \\ & + \frac{1}{2} \sum_{1 \leq i \neq j \leq n} (\sigma_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m+1} \sum_{i,j>2}^{n} (\sigma_{ij}^r)^2 + \\ & + \frac{1}{2} \sum_{r=n+2}^{2m+1} (\sigma_{11}^r + \sigma_{22}^r)^2 + \sum_{j>2} [(\sigma_{1j}^{n+1})^2 + (\sigma_{2j}^{n+1})^2] + \frac{1}{2}\delta \geq \\ & \geq f_1 + 3f_2 g^2(Pe_1, e_2) - f_3 \|\xi_\pi\|^2 - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{1}{2}\delta, \end{aligned}$$

which implies

$$K(\pi) \geq f_1 + 3f_2 g^2(Pe_1, e_2) - f_3 \|\xi_\pi\|^2 - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{1}{2}\delta.$$

From (5.6), it easy to see that

$$\begin{aligned} K(\pi) \quad &\geq \quad \tau - (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{f_1}{2} - \lambda\right] - \\ &- \left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{f_2}{2} - [\|\xi_\pi\|^2 - (n-1)\left\|\xi^\top\right\|^2]f_3 + \\ &+ trace(\alpha_{|\pi^\perp}), \end{aligned}$$

which gives us (5.3). Hence, we complete the proof of the theorem. ∎

5.2. Corollary. Let M, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional generalized Sasakian space form $N(f_1, f_2, f_3)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$.

If the structure vector field ξ is tangent to M, we have

$$\begin{aligned} \tau(x) - K(\pi) \quad &\leq \quad (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{f_1}{2} - \lambda\right] - \\ &- \left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{f_2}{2} + [\|\xi_\pi\|^2 - (n-1)]f_3 - \\ &- trace(\alpha_{|\pi^\perp}). \end{aligned} \tag{5.8}$$

If the structure vector field ξ is normal to M, we have

$$\begin{aligned} \tau(x) - K(\pi) \quad &\leq \quad (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{f_1}{2} - \lambda\right] - \\ &- \left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{f_2}{2} - trace(\alpha_{|\pi^\perp}). \end{aligned} \tag{5.9}$$

As a consequence of Proposition 3.2, for both submanifolds of generalized Sasakian space forms, we can give the following corollary:

5.3. Corollary. Under the same assumptions as in the Theorem 5.1, if the vector field U is tangent to M, then we have:

$$\begin{aligned} \tau(x) - K(\pi) \quad &\leq \quad (n-2)\left[\frac{n^2}{2(n-1)}\left\|\overset{\circ}{H}\right\|^2 + (n+1)\frac{f_1}{2} - \lambda\right] - \\ &- \left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{f_2}{2} + [\|\xi_\pi\|^2 - (n-1)]f_3 - \\ &- trace(\alpha_{|\pi^\perp}). \end{aligned}$$

5.4. Theorem. The equality case of (5.3) holds at a point $x \in M$ if and only if there exist an orthonormal basis $\{e_1, e_2, ...e_n\}$ of T_xM and an orthonormal basis $\{e_{n+1}, ..., e_{2m+1}\}$ of $T_x^\perp M$ such that the shape operators of M in $N(f_1, f_2, f_3)$ at x have the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu$$

and

$$A_{e_r} = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 & \cdots & 0 \\ \sigma_{12}^r & -\sigma_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad n+2 \le i \le 2m+1,$$

where we denote by $\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r)$, $1 \le i, j \le n$ and $n+2 \le r \le 2m+1$.

Proof. Equality case holds at a point $x \in M$ if and only if the equality holds in each of the previous inequalities and hence the Lemma yields equality.

$$\sigma_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$\sigma_{ij}^r = 0, \quad \forall i \neq j, i, j > 2, r = n+1, ..., 2m+1,$$

$$\sigma_{11}^r + \sigma_{22}^r = 0, \quad \forall r = n+2, ..., 2m+1,$$

$$\sigma_{1j}^{n+1} = \sigma_{2j}^{n+1} = 0, \quad \forall j > 2,$$

$$\sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} = ... = \sigma_{nn}^{n+1}.$$

If we choose $\{e_1, e_2\}$ such that $\sigma_{12}^{n+1} = 0$ and denote by $a = \sigma_{11}^r$, $b = \sigma_{22}^r$, $\mu = \sigma_{33}^{n+1} = ... = \sigma_{nn}^{n+1}$, then the shape operators take the mentioned forms. ∎

6. Ricci curvature for submanifolds of generalized Sasakian space forms

In this section we establish relationship between the Ricci curvature of a submanifold M of a generalized Sasakian space form $N(f_1, f_2, f_3)$ with a semi-symmetric metric connection and the squared mean curvature $\|H\|^2$.

Now, let begin with the following theorem:

6.1. Theorem. Let M, $n \ge 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional generalized Sasakian space form $N(f_1, f_2, f_3)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Then we have:

$$\begin{aligned} \|H\|^2 \quad \ge \quad & \frac{2\tau}{n(n-1)} + \frac{2}{n}\lambda - f_1 - \frac{3f_2}{n(n-1)} \|P\|^2 + \\ & + \frac{2}{n} f_3 \left\| \xi^\top \right\|^2 . \end{aligned} \tag{6.1}$$

Proof. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of $T_x M$ and $\{e_{n+1}, ..., e_{2m+1}\}$ be an orthonormal basis of $T_x^\perp M$, $x \in M$, where e_{n+1} is parallel to the mean curvature vector H . Then, the equation (5.5) can be written as follows

$$\begin{aligned} n^2 \|H\|^2 \quad = \quad & 2\tau + \|\sigma\|^2 + 2(n-1)\lambda - n(n-1) f_1 \\ & -3 f_2 \|P\|^2 + 2(n-1) f_3 . \end{aligned} \tag{6.2}$$

For a choosen orthonormal basis, let $e_1, e_2, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then, the shape operators take the forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

and

$$A_{e_r} = (\sigma_{ij}^r), \quad i,j = 1,...,n; \quad r = n+2,...,2m+1, \quad traceA_{e_r} = 0.$$

By the use of (6.2), we obtain

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2 + \\ &\quad +2(n-1)\lambda - n(n-1)f_1 - 3f_2 \|P\|^2 + 2(n-1)f_3. \end{aligned} \tag{6.3}$$

On the other hand, we know that

$$\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2 . \tag{6.4}$$

Hence, by the use of (6.4) in (6.3), we obtain (6.1). ∎

In view of Theorem 6.1, we can give the following theorem:

6.2. Theorem. Let M, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional generalized Sasakian space form $N(f_1, f_2, f_3)$ with respect to the semi-symmetric metric connection $\widetilde{\widetilde{\nabla}}$ such that the vector field U is tangent to M. Then, for any integer k, $2 \leq k \leq n$ and for any point $x \in M$, we have:

$$\|H\|^2 (x) \geq \Theta_k(\pi) + \frac{2}{n}\lambda - f_1 - \frac{3f_2}{n(n-1)} \|P\|^2 + \frac{2}{n} f_3 \left\|\xi^\top\right\|^2 . \tag{6.5}$$

Proof. Similar to the proof of the Theorem 4.2, we easily get (6.5). ∎

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Some extended trapezoid-type inequalities and applications

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Abstract

In this paper, we shall establish some extended trapezoid-type

$$\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \quad (a\le x\le c\le y\le b)$$

inequalities for differentiable convex functions and differentiable concave functions which are connected with Hermite-Hadamard inequality. Some error estimates for the midpoint, trapezoidal and Ostrowski formulae are also given.

Keywords: Hermite-Hadamard Inequality, Midpoint Inequality, Trapezoid Inequality, Ostrowski Inequality, Convex Function, Concave Functions, Special Means, Quadrature Rules.

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1. Introduction

Throughout in this paper, let $a<b$ in $\mathbb{R}$.

The inequality

$$f\left(\frac{a+b}{2}\right)\le(\ge)\,\frac{1}{b-a}\int_a^b f(t)\,dt\le(\ge)\,\frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex (concave) functions $f:[a,b]\to\mathbb{R}$, is known in the literature as Hermite-Hadamard inequality [7].

For some results which generalize, improve, and extend the inequality (1.1), see [1]-[6] and [8]-[15].

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In [11], Tseng *et al.* established the following Hermite-Hadamard-type inequality which refines the first inequality of (1.1).

A. Theorem. *Suppose that* $f:[a,b]\to\mathbb{R}$ *is a convex function on* $[a,b]$. *Then we have the inequality*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\le\frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.2}$$

Using the similar proof of Theorem A, we also note that the inequalities in (1.2) are reversed when f is concave on $[a,b]$.

In [4], Dragomir and Agarwal established the following results connected with the second inequality in the inequality (1.1).

B. Theorem. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function on* (a,b) *with* $a<b$. *If* $|f'|$ *is convex on* $[a,b]$, *then we have*

$$\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{b-a}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \tag{1.3}$$

which is the trapezoid inequality provided $|f'|$ *is convex on* $[a,b]$.

C. Theorem. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function on* (a,b) *with* $a<b$ *and let* $p>1$. *If* $|f'|^{p/(p-1)}$ *is convex on* $[a,b]$, *then we have*

$$\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}} \tag{1.4}$$

which is the trapezoid inequality provided $|f'|^{p/(p-1)}$ *is convex on* $[a,b]$.

In [10], Pearce and Pečarić established the following theorems that improve Theorem C, generalize Theorem D and give similar results of Theorems B-C with a concavity property instead of convexity.

D. Theorem. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function on* (a,b) *with* $a<b$ *and* $q\ge 1$. *If* $|f'|^q$ *is convex on* $[a,b]$, *then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}} \tag{1.5}$$

which is the trapezoid inequality provided $|f'|^q$ *is convex on* $[a,b]$.

E. Theorem. *Under the assumptions of Theorem D. Then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}} \tag{1.6}$$

which is the midpoint inequality provided $|f'|^q$ *is convex on* $[a,b]$.

F. Theorem. *Under the assumptions of Theorem D and* $|f'|^q$ $(q\ge 1)$ *is concave on* $[a,b]$. *Then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right| \tag{1.7}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right| \tag{1.8}$$

which are the trapezoid inequality and the midpoint inequality provided $|f'|^q$ *is concave on* $[a,b]$*, respectively.*

In [1], Alomari and Darus established the following Ostrowski-type inequalities.

G. Theorem. *Under the assumptions of Theorem B. Then, for all* $x\in[a,b]$*, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le(b-a)\left[\left(\frac{1}{6}+\frac{1}{3}\left(\frac{b-x}{b-a}\right)^3\right)|f'(a)|+\left(\frac{1}{6}+\frac{1}{3}\left(\frac{x-a}{b-a}\right)^3\right)|f'(b)|\right]. \tag{1.9}$$

H. Theorem. *Under the assumptions of Theorem D. Then, for all* $x\in[a,b]$*, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{b-a}{2^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^2\left|f'\left(\frac{a+2x}{3}\right)\right|+\left(\frac{b-x}{b-a}\right)^2\left|f'\left(\frac{2x+b}{3}\right)\right|\right]. \tag{1.10}$$

From the above results, it is natural to consider the extended trapezoid-type formulae in the following lemma.

1.1. Lemma. *Let* $a\le x\le c\le y\le b$*. Then we have the extended trapezoid-type formula*

$$\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

as follows:

(1) *The trapezoid-type formula*

$$\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| = \left|\frac{f((1-\alpha)a+\alpha b)+f(\alpha a+(1-\alpha)b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

as $0\le\alpha\le\frac{1}{2}, x=(1-\alpha)a+\alpha b, c=\frac{a+b}{2}$ *and* $y=\alpha a+(1-\alpha)b$*.*

(2) *The trapezoid formula*

$$\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| = \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

as $x=a, c=\frac{a+b}{2}$ *and* $y=b$*.*

(3) *The midpoint formula*

$$\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| = \left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

as $x=c=y=\frac{a+b}{2}$*.*

(4) *The Ostrowski formula*

$$\begin{aligned} &\left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ = &\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \end{aligned}$$

as $x=c=y$.

In this paper, we establish some extended trapezoid-type inequalities which reduce the trapezoid-type, midpoint-type, Ostrowski-type inequalities, and improve Theorems B and D-H. Some applications to special means of real numbers are given. Finally, the approximations for quadrature formulae are also given.

2. Extended trapezoid-type Inequality

Throughout in this section, let $0 \le \alpha \le \gamma \le 1-\beta \le 1, a \le x \le c \le y \le b$ and let $P_1, P_2, I_i\ (i=1,\cdots,4), h(t), h_1(t)\ (t \in [a,b])$ be defined as follows.

$$\begin{aligned} P_1 = &\frac{1}{3(b-a)^3}\left[(x-a)^2\left(\frac{3b-a}{2}-x\right)+(c-x)^2\left(\frac{3b-c}{2}-x\right)\right. \\ &\left.+(y-c)^2\left(\frac{3b-c}{2}-y\right)+(b-y)^3\right]. \end{aligned} \tag{2.1}$$

$$\begin{aligned} P_2 = &\frac{1}{3(b-a)^3}\left[(x-a)^3+(c-x)^2\left(\frac{c-3a}{2}+x\right)\right. \\ &\left.+(y-c)^2\left(\frac{c-3a}{2}+y\right)+(b-y)^2\left(\frac{b-3a}{2}+y\right)\right]. \end{aligned} \tag{2.2}$$

$$I_1 = \frac{1}{3(b-a)^2(c-a)}\left[(x-a)^2\left(\frac{3c-a}{2}-x\right)+(c-x)^3\right], \tag{2.3}$$

$$I_2 = \frac{1}{3(b-a)^2(c-a)}\left[(c-x)^2\left(\frac{c-3a}{2}+x\right)+(x-a)^3\right], \tag{2.4}$$

$$I_3 = \frac{1}{3(b-a)^2(b-c)}\left[(y-c)^2\left(\frac{3b-c}{2}-y\right)+(b-y)^3\right], \tag{2.5}$$

and

$$I_4 = \frac{1}{3(b-a)^2(b-c)}\left[(b-y)^2\left(\frac{b-3c}{2}+y\right)+(y-c)^3\right] \tag{2.6}$$

where $a < c < b$.

$$h(t) = \begin{cases} t-a, & a \le t < x \\ t-c & x \le t < y \\ t-b & y \le t \le b \end{cases} \quad \text{and } h_1(t) = \begin{cases} t-a, & a \le t < x \\ c-t, & x \le t < c \\ t-c, & c \le t < y \\ b-t, & y \le t \le b \end{cases}.$$

In order to prove our main results, we need the following lemma and remark whose proof can be obtained by simple computations and $r^2+s^2=(r+s)^2-2rs, r^2+s^2+t^2+u^2=(r+s+t+u)^2-[2(r+s)(t+u)+2rs+2tu]$ where $r,s,t,u \in \mathbb{R}$.

2.1. Lemma. *Let* $a, b, x, c, y, P_1, P_2, I_i \ (i = 1, \cdots, 4), h(t), h_1(t) \ (t \in [a, b])$ *be defined as above. Then we have*

$$|h(t)| = h_1(t) \, (t \in [a, b]),$$

$$P_1 = \frac{1}{(b-a)^3} \int_a^b (b-t) \, h_1(t) \, dt \text{ and } P_2 = \frac{1}{(b-a)^3} \int_a^b (t-a) \, h_1(t) \, dt,$$

As $a < c < b$,

$$\begin{aligned} I_1 &= \frac{1}{(b-a)^2 (c-a)} \int_a^c (c-t) \, h_1(t) \, dt, \\ I_2 &= \frac{1}{(b-a)^2 (c-a)} \int_a^c (t-a) \, h_1(t) \, dt, \\ I_3 &= \frac{1}{(b-a)^2 (b-c)} \int_c^b (b-t) \, h_1(t) \, dt, \\ I_4 &= \frac{1}{(b-a)^2 (b-c)} \int_c^b (t-c) \, h_1(t) \, dt, \end{aligned}$$

(2.7) $$I_1 + \frac{b-c}{b-a} (I_2 + I_3) = P_1,$$

(2.8) $$I_4 + \frac{c-a}{b-a} (I_2 + I_3) = P_2,$$

(2.9) $$\begin{aligned} I_1 + I_2 &= \frac{1}{(b-a)^2} \int_a^c h_1(t) \, dt = \frac{1}{2(b-a)^2} \left[(x-a)^2 + (c-x)^2 \right] \\ &= \frac{1}{2} \left(\frac{c-a}{b-a} \right)^2 - \frac{(x-a)(c-x)}{(b-a)^2}, \end{aligned}$$

(2.10) $$\begin{aligned} I_3 + I_4 &= \frac{1}{(b-a)^2} \int_c^b h_1(t) \, dt = \frac{1}{2(b-a)^2} \left[(y-c)^2 + (b-y)^2 \right] \\ &= \frac{1}{2} \left(\frac{b-c}{b-a} \right)^2 - \frac{(y-c)(b-y)}{(b-a)^2}, \end{aligned}$$

(2.11) $$\begin{aligned} & I_1 + I_2 + I_3 + I_4 = P_1 + P_2 = \frac{1}{(b-a)^2} \int_a^b h_1(t) \, dt \\ &= \frac{1}{2(b-a)^2} \left[(x-a)^2 + (c-x)^2 + (y-c)^2 + (b-y)^2 \right] \\ &= \frac{1}{2} - \left[\frac{(c-a)(b-c)}{(b-a)^2} + \frac{(x-a)(c-x)}{(b-a)^2} + \frac{(y-c)(b-y)}{(b-a)^2} \right], \end{aligned}$$

$$\begin{aligned} & aI_1 + cI_2 \\ &= \frac{1}{(b-a)^2} \int_a^c h_1(t) \, t dt \\ &= \frac{1}{6} \left(\frac{x-a}{b-a} \right)^2 (2x+a) + \frac{1}{6} \left(\frac{c-x}{b-a} \right)^2 (2x+c), \end{aligned}$$

$$\begin{aligned} & cI_3 + bI_4 \\ = & \frac{1}{(b-a)^2}\int_c^b h_1(t)\,tdt \\ = & \frac{1}{6}\left(\frac{y-c}{b-a}\right)^2(2y+c)+\frac{1}{6}\left(\frac{b-y}{b-a}\right)^2(2y+b) \end{aligned}$$

and

$$0 < P_1, P_2, I_i \leq I_1 + I_2 + I_3 + I_4 \leq \frac{1}{2} \ \ (i = 1 \cdots 4).$$

2.2. Remark. Let $\alpha \in [0,1]$, $x = (1-\alpha)a + \alpha b, c = \frac{a+b}{2}$ and $y = \alpha a + (1-\alpha)b$ in the identities $(2.1) - (2.11)$. Then we have the identities

$$I_1 = I_4 = \frac{1}{3}\left[\alpha^2\left(\frac{3}{2}\gamma - 2\alpha\right) + 2\left(\frac{1}{2}-\alpha\right)^3\right] \quad \text{as} \quad 0 < \gamma \leq 1, \tag{2.12}$$

$$I_2 = I_3 = \frac{1}{3}\left[\left(\frac{1}{2}-\alpha\right)^2\left(\frac{1}{2}+2\alpha\right) + 2\alpha^3\right] \quad \text{as} \quad 0 \leq \gamma < 1,$$

$$P_1 = P_2 = I_1 + I_2 = I_3 + I_4 = \frac{1}{8} - \alpha\left(\frac{1}{2}-\alpha\right), \tag{2.13}$$

and

$$I_1 + I_2 + I_3 + I_4 = P_1 + P_2 = \frac{1}{4} - \alpha(1-2\alpha). \tag{2.14}$$

2.3. Remark. In Theorem 2.4, Let $x = c = y$ in the identities $(2.1) - (2.6)$ and (2.11). Then we have the identities

$$P_1 = \frac{(x-a)^2(3b-a-2x)}{6(b-a)^3} + \frac{1}{3}\left(\frac{b-x}{b-a}\right)^3,$$

$$P_2 = \frac{(b-x)^2(b-3a+2x)}{6(b-a)^3} + \frac{1}{3}\left(\frac{x-a}{b-a}\right)^3,$$

$$I_1 = \frac{1}{2}I_2 = \frac{1}{6}\left(\frac{x-a}{b-a}\right)^2, \qquad I_4 = \frac{1}{2}I_3 = \frac{1}{6}\left(\frac{b-x}{b-a}\right)^2,$$

$$I_1 + I_2 + I_3 + I_4 = P_1 + P_2 = \frac{1}{2} - \frac{(x-a)(b-x)}{(b-a)^2}$$

Now, we are ready to state and prove the main results.

2.4. Theorem. *Let* $a, b, x, c, y, P_1, P_2, I_i \ (i = 1, \cdots, 4), h(t), h_1(t) \ (t \in [a,b])$ *be defined as above and let* q, f *be defined as in Theorem D. Then we have the following extended trapezoid-type inequalities.*

(1) The following inequality holds:

$$\begin{aligned} & \left|\frac{c-a}{b-a}f(x) + \frac{b-c}{b-a}f(y) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \leq \ & (P_1+P_2)(b-a)\left(\frac{P_1|f'(a)|^q + P_2|f'(b)|^q}{P_1+P_2}\right)^{\frac{1}{q}}. \end{aligned} \tag{2.15}$$

(2) As $a < c < b$, we have the inequality

$$\begin{aligned}
(2.16)\qquad & \left|\frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq\ & (I_1+I_2+I_3+I_4)(b-a)\left(\frac{I_1|f'(a)|^q+(I_2+I_3)|f'(c)|^q+I_4|f'(b)|^q}{I_1+I_2+I_3+I_4}\right)^{\frac{1}{q}} \\
\leq\ & (P_1+P_2)(b-a)\left(\frac{P_1|f'(a)|^q+P_2|f'(b)|^q}{P_1+P_2}\right)^{\frac{1}{q}}
\end{aligned}$$

which refines the inequality (2.15).

Proof. Using the integration by parts and simple computation, we have the following identity:

$$\begin{aligned}
(2.17)\qquad & \frac{1}{b-a}\int_a^b h(t)\,f'(t)\,dt \\
=\ & \frac{c-a}{b-a}f(x)+\frac{b-c}{b-a}f(y)-\frac{1}{b-a}\int_a^b f(t)\,dt.
\end{aligned}$$

(1) Now, using Hölder's inequality, the convexity of $|f'|^q$ and Lemma 2.1, we have the inequality

$$\begin{aligned}
(2.18)\qquad & \left|\frac{1}{b-a}\int_a^b h(t)\,f'(t)\,dt\right| \\
\leq\ & \frac{1}{b-a}\int_a^b |h(t)|\left|f'(t)\right|dt \\
=\ & \frac{1}{b-a}\int_a^b h_1(t)\left|f'(t)\right|dt \\
\leq\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\int_a^b h_1(t)\left|f'(t)\right|^q dt\right)^{\frac{1}{q}} \\
=\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\int_a^b h_1(t)\left|f'\left(\frac{b-t}{b-a}\cdot a+\frac{t-a}{b-a}\cdot b\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left[\int_a^b h_1(t)\frac{b-t}{b-a}\left|f'(a)\right|^q+h_1(t)\frac{t-a}{b-a}\left|f'(b)\right|^q dt\right]^{\frac{1}{q}} \\
=\ & \left(\frac{1}{(b-a)^2}\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\frac{1}{(b-a)^3}\int_a^b h_1(t)(b-t)\,dt\cdot\left|f'(a)\right|^q\right. \\
& \left.+\frac{1}{(b-a)^3}\int_a^b h_1(t)(t-a)\,dt\cdot\left|f'(b)\right|^q\right)^{\frac{1}{q}}\cdot(b-a) \\
=\ & (P_1+P_2)^{1-\frac{1}{q}}\left(P_1\left|f'(a)\right|^q+P_2\left|f'(b)\right|^q\right)^{\frac{1}{q}}(b-a) \\
=\ & (P_1+P_2)(b-a)\left(\frac{P_1|f'(a)|^q+P_2|f'(b)|^q}{P_1+P_2}\right)^{\frac{1}{q}}.
\end{aligned}$$

The inequality (2.15) follows from the identity (2.17) and the inequality (2.18).

(2) Let $a < c < b$. Using the inequality (2.18), the convexity of $|f'|^q$ and Lemma 2.1, we have the inequalities

$$\begin{aligned}
(2.19)\quad & \left|\frac{1}{b-a}\int_a^b h(t)\,f'(t)\,dt\right| \\
\leq\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\int_a^b h_1(t)\,|f'(t)|^q\,dt\right)^{\frac{1}{q}} \\
=\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\int_a^c h_1(t)\,|f'(t)|^q\,dt+\int_a^c h_1(t)\,|f'(t)|^q\,dt\right)^{\frac{1}{q}} \\
=\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left[\int_a^c h_1(t)\left|f'\left(\frac{c-t}{c-a}\cdot a+\frac{t-a}{c-a}\cdot c\right)\right|^q dt\right. \\
& \left.+\int_c^b h_1(t)\left|f'\left(\frac{b-t}{b-c}\cdot c+\frac{t-c}{b-c}\cdot b\right)\right|^q\right]^{\frac{1}{q}} \\
\leq\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left[\int_a^c h_1(t)\left(\frac{c-t}{c-a}\,|f'(a)|^q+\frac{t-a}{c-a}\,|f'(c)|^q\right)dt\right. \\
& \left.+\int_c^b h_1(t)\left(\frac{b-t}{b-c}\,|f'(c)|^q+\frac{t-c}{b-c}\,|f'(b)|^q\right)\right]^{\frac{1}{q}} \\
=\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left[\int_a^c h_1(t)\left(\frac{c-t}{c-a}\,|f'(a)|^q+\frac{t-a}{c-a}\,|f'(c)|^q\right)dt\right. \\
& \left.+\int_c^b h_1(t)\left(\frac{b-t}{b-c}\,|f'(c)|^q+\frac{t-c}{b-c}\,|f'(b)|^q\right)\right]^{\frac{1}{q}} \\
=\ & \left(\frac{1}{(b-a)^2}\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\frac{|f'(a)|^q}{(b-a)^2(c-a)}\int_a^c (c-t)\,h_1(t)\,dt\right. \\
& +\frac{|f'(c)|^q}{(b-a)^2(c-a)}\int_a^c (t-a)\,h_1(t)\,dt+\frac{|f'(c)|^q}{(b-a)^2(b-c)}\int_c^b (b-t)\,h_1(t)\,dt \\
& \left.+\frac{|f'(b)|^q}{(b-a)^2(b-c)}\int_c^b h_1(t)\,(t-c)\,dt\right)^{\frac{1}{q}}\cdot(b-a) \\
=\ & (I_1+I_2+I_3+I_4)^{1-\frac{1}{q}}\left(I_1\left|f'(a)\right|^q+(I_2+I_3)\left|f'(c)\right|^q+I_4\left|f'(b)\right|^q\right)^{\frac{1}{q}}(b-a) \\
=\ & (I_1+I_2+I_3+I_4)\,(b-a)\left(\frac{I_1\,|f'(a)|^q+(I_2+I_3)\,|f'(c)|^q+I_4\,|f'(b)|^q}{I_1+I_2+I_3+I_4}\right)^{\frac{1}{q}}
\end{aligned}$$

and

$$\begin{aligned}
(2.20)\quad & \frac{I_1\,|f'(a)|^q+(I_2+I_3)\,|f'(c)|^q+I_4\,|f'(b)|^q}{I_1+I_2+I_3+I_4} \\
=\ & \frac{I_1\,|f'(a)|^q+I_4\,|f'(b)|^q}{P_1+P_2}+\frac{I_2+I_3}{P_1+P_2}\left|f'\left(\frac{b-c}{b-a}c+\frac{c-a}{b-a}b\right)\right|^q \\
\leq\ & \frac{\left[I_1+\frac{b-c}{b-a}(I_2+I_3)\right]|f'(a)|^q+\left[I_4+\frac{c-a}{b-a}(I_2+I_3)\right]|f'(b)|^q}{P_1+P_2} \\
=\ & \frac{P_1\,|f'(a)|^q+P_2\,|f'(b)|^q}{P_1+P_2}.
\end{aligned}$$

The inequality (2.16) follows from the identities (2.11), (2.17) and the inequalities (2.19)$-$(2.20). This completes the proof. ∎

Under the conditions of Theorem 2.4, Remark 2.2 and the identities (2.11), (2.1) $-$ (2.6), we have the following corollaries and remarks.

2.5. Corollary. *Let* $0 \leq \alpha \leq 1, x = (1-\alpha)a + \alpha b$, $c = \frac{a+b}{2}$ *and* $y = \alpha a + (1-\alpha)b$. *Then, using Theorem 2.4 and Remark 2.2, we have the trapezoid-type inequality*

$$\begin{aligned} &\left| \frac{f((1-\alpha)a+\alpha b) + f(\alpha a + (1-\alpha)b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq\ & \left[\frac{1}{4} - \alpha(1-2\alpha)\right](b-a)\left(\frac{I_1|f'(a)|^q + 2I_2\left|f'\left(\frac{a+b}{2}\right)\right|^q + I_1|f'(b)|^q}{2(I_1+I_2)}\right)^{\frac{1}{q}} \\ \leq\ & \left[\frac{1}{4} - \alpha(1-2\alpha)\right](b-a)\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

which is provided $|f'|^q$ *is convex on* $[a,b]$.

2.6. Remark. Let $\alpha = 0$. Then, using Corollary 2.5 and Remark 2.2, we have $I_1 = \frac{1}{12}$, $I_2 = \frac{1}{24}$ and the trapezoid inequality

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq\ & \frac{b-a}{4}\left(\frac{|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q}{3}\right)^{\frac{1}{q}} \\ \leq\ & \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

which improves Theorems B and D.

2.7. Remark. Let $\alpha = \frac{1}{2}$. Then, using Corollary2.5 and Remark 2.2, we have $I_1 = \frac{1}{24}$, $I_2 = \frac{1}{12}$ and the midpoint inequality

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq\ & \frac{b-a}{4}\left(\frac{|f'(a)|^q + 4\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q}{6}\right)^{\frac{1}{q}} \\ \leq\ & \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

which improves Theorem E.

2.8. Remark. Let $\alpha = \frac{1}{4}$. Then, using Corollary 2.5 and Remark 2.2, we have $I_1 = I_2 = \frac{1}{32}$ and the inequality

$$\begin{aligned} &\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq\ & \frac{b-a}{8}\left(\frac{|f'(a)|^q + 2\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}} \\ \leq\ & \frac{b-a}{8}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

which is the second inequality in (1.2) provided $|f'|^q$ is convex on $[a,b]$.

2.9. Corollary. *Let P_1, P_2, I_i $(i = 1, \cdots, 4)$ be defined as in Remark 2.3. Then, using Theorem 2.4 and Remark 2.3, we have the following Ostrowski-type inequalities which are provided $|f'|^q$ is convex on $[a,b]$.*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ \leq\ & (P_1 + P_2)(b-a) \left(\frac{P_1 |f'(a)|^q + P_2 |f'(b)|^q}{P_1 + P_2} \right)^{\frac{1}{q}} \end{aligned}$$

as $x = c = y$ and $a \leq x \leq b$.

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ \leq\ & 3(I_1 + I_4)(b-a) \left[\frac{I_1 |f'(a)|^q + 2(I_1 + I_4)|f'(x)|^q + I_4 |f'(b)|^q}{3(I_1 + I_4)} \right]^{\frac{1}{q}} \\ \leq\ & (P_1 + P_2)(b-a) \left(\frac{P_1 |f'(a)|^q + P_2 |f'(b)|^q}{P_1 + P_2} \right)^{\frac{1}{q}} \end{aligned}$$

as $x = c = y$ and $a < x < b$.

2.10. Remark. Let $k_1(t) = (t-a)^2 (3b - a - 2t)$ and $k_2(t) = (b-t)^2 (b - 3a + 2t)$ be defined on $[a,b]$. By simple computations,we obtain that k_1 is strictly increasing on $[a,b]$, k_2 is strictly decreasingon $[a,b]$ and $k_1(t),\ k_2(t) \leq (b-a)^3$ $(t \in [a,b])$. The , using the above inequalities, Corollary 2.9 improves Theorem G as $q = 1$.

2.11. Theorem. *Let $a, b, x, c, y, P_1, P_2, I_i$ $(i = 1, \cdots, 4), h(t), h_1(t)$ $(t \in [a,b])$ be defined as above and let q, f be defined as in Theorem F. Then we have the following extended trapezoid-type inequalities.*

(1) *The following inequality holds:*

$$\begin{aligned} & \left| \frac{c-a}{b-a} f(x) + \frac{b-c}{b-a} f(y) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ \leq\ & (P_1 + P_2)(b-a) \left| f'\left(\frac{P_1 a + P_2 b}{P_1 + P_2} \right) \right|. \end{aligned} \tag{2.21}$$

(2) *As $a < c < b$, we have the inequality*

$$\begin{aligned} & \left| \frac{c-a}{b-a} f(x) + \frac{b-c}{b-a} f(y) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ \leq\ & (b-a) \left[(I_1 + I_2) \left| f'\left(\frac{I_1 a + I_2 c}{I_1 + I_2} \right) \right| + (I_3 + I_4) \left| f'\left(\frac{I_3 c + I_4 b}{I_3 + I_4} \right) \right| \right] \\ \leq\ & (P_1 + P_2)(b-a) \left| f'\left(\frac{P_1 a + P_2 b}{P_1 + P_2} \right) \right| \end{aligned} \tag{2.22}$$

which refines the inequality (2.21).

Proof. We observe that $|f'|^q$ is concave on $[a,b]$ implies $|f'| = \left(|f'|^q\right)^{\frac{1}{q}}$ is also concave on $[a,b]$. Using the inequality (2.18), the Jensen's integral inequality and Lemma 2.1, we have the following inequalities:

$$
\begin{aligned}
(2.23)\qquad & \left|\frac{1}{b-a}\int_a^b h(t)\, f'(t)\, dt\right| \\
\leq\ & \frac{1}{b-a}\int_a^b h_1(t)\left|f'(t)\right| dt \\
\leq\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\, dt\right)\left|f'\left(\frac{\int_a^b h_1(t)\, tdt}{\int_a^b h_1(t)\, dt}\right)\right| \\
=\ & (b-a)\left(\frac{1}{(b-a)^2}\int_a^b h_1(t)\, dt\right)\left|f'\left(\frac{\frac{1}{(b-a)^2}\int_a^b h_1(t)\, tdt}{\frac{1}{(b-a)^2}\int_a^b h_1(t)\, dt}\right)\right| \\
=\ & (P_1+P_2)(b-a)\left|f'\left(\frac{P_1 a+P_2 b}{P_1+P_2}\right)\right|.
\end{aligned}
$$

The inequality (2.21) follows from the identity (2.17) and the inequality (2.23). Now, let $a<c<b$. Then we have the inequality

$$
\begin{aligned}
(2.24)\qquad & \left|\frac{1}{b-a}\int_a^b h(t)\, f'(t)\, dt\right| \\
\leq\ & \frac{1}{b-a}\int_a^b h_1(t)\left|f'(t)\right| dt \\
=\ & \frac{1}{b-a}\left(\int_a^c h_1(t)\left|f'(t)\right| dt+\int_c^b h_1(t)\left|f'(t)\right| dt\right) \\
\leq\ & \frac{1}{b-a}\left[\int_a^c h_1(t)\, dt\left|f'\left(\frac{\frac{1}{(b-a)^2}\int_a^c h_1(t)\, tdt}{\frac{1}{(b-a)^2}\int_a^c h_1(t)\, dt}\right)\right|\right. \\
& \left.\qquad+\int_c^b h_1(t)\, dt\left|f'\left(\frac{\int_c^b h_1(t)\, tdt}{\int_c^b h_1(t)\, dt}\right)\right|\right] \\
=\ & (b-a)\left[\frac{1}{(b-a)^2}\int_a^c h_1(t)\, dt\left|f'\left(\frac{\frac{1}{(b-a)^2}\int_a^c h_1(t)\, tdt}{\frac{1}{(b-a)^2}\int_a^c h_1(t)\, dt}\right)\right|\right. \\
& \left.+\frac{1}{(b-a)^2}\int_c^b h_1(t)\, dt\left|f'\left(\frac{\frac{1}{(b-a)^2}\int_c^b h_1(t)\, tdt}{\frac{1}{(b-a)^2}\int_c^b h_1(t)\, dt}\right)\right|\right] \\
=\ & (b-a)\left[(I_1+I_2)\left|f'\left(\frac{I_1 a+I_2 c}{I_1+I_2}\right)\right|+(I_3+I_4)\left|f'\left(\frac{I_3 c+I_4 b}{I_3+I_4}\right)\right|\right].
\end{aligned}
$$

Using the inequality (2.18), the convcavity of $|f'|^q$ and Lemma 2.1, we have the inequalitiy

$$
\begin{aligned}
(2.25) \qquad & (I_1+I_2)\left|f'\left(\frac{I_1a+I_2c}{I_1+I_2}\right)\right|+(I_3+I_4)\left|f'\left(\frac{I_3c+I_4b}{I_3+I_4}\right)\right| \\
= \; & (I_1+I_2+I_3+I_4)\left[\frac{I_1+I_2}{I_1+I_2+I_3+I_4}\left|f'\left(\frac{I_1a+I_2c}{I_1+I_2}\right)\right|\right. \\
& \left.+\frac{I_3+I_4}{I_1+I_2+I_3+I_4}\left|f'\left(\frac{I_3c+I_4b}{I_3+I_4}\right)\right|\right] \\
\leq \; & (I_1+I_2+I_3+I_4)\left|f'\left(\frac{I_1a+(I_2+I_3)c+I_4b}{I_1+I_2+I_3+I_4}\right)\right| \\
= \; & (I_1+I_2+I_3+I_4)\left|f'\left(\frac{I_1a+(I_2+I_3)\left(\frac{b-c}{b-a}a+\frac{c-a}{b-a}b\right)+I_4b}{I_1+I_2+I_3+I_4}\right)\right| \\
= \; & (I_1+I_2+I_3+I_4)\left|f'\left(\frac{\left(I_1+(I_2+I_3)\frac{b-c}{b-a}\right)a+\left(I_4+(I_2+I_3)\frac{c-a}{b-a}\right)b}{I_1+I_2+I_3+I_4}\right)\right| \\
= \; & (P_1+P_2)\left|f'\left(\frac{P_1a+P_2b}{P_1+P_2}\right)\right|.
\end{aligned}
$$

The inequality (2.22) follows from the identity (2.17) and the inequalities $(2.24)-(2.25)$. This completes the proof. ∎

Under the conditions of Theorem 2.11 nad Remark 2.2, we have the following corollaries and remarks.

2.12. Corollary. *Let $0\leq\alpha\leq 1, x=(1-\alpha)a+\alpha b$, $c=\frac{a+b}{2}$ and $y=\alpha a+(1-\alpha)b$. Then, using Theorem 2.11 and Remark 2.2, we have the trapezoid-type inequality*

$$
\begin{aligned}
& \left|\frac{f((1-\alpha)a+\alpha b)+f(\alpha a+(1-\alpha)b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq \; & \left[\frac{1}{8}-\alpha\left(\frac{1}{2}-\alpha\right)\right](b-a)\left(\left|f'\left(\frac{I_1a+I_2\frac{a+b}{2}}{I_1+I_2}\right)\right|+\left|f'\left(\frac{I_2\frac{a+b}{2}+I_1b}{I_1+I_2}\right)\right|\right) \\
\leq \; & \left[\frac{1}{4}-\alpha(1-2\alpha)\right](b-a)\left|f'\left(\frac{a+b}{2}\right)\right|
\end{aligned}
$$

which is provided $|f'|^q$ is convex on $[a,b]$.

2.13. Remark. Let $\alpha=0$. Then, using Corollary 2.12 and Remark 2.2, we have $I_1=\frac{1}{12}$, $I_2=\frac{1}{24}$ and the trapezoid inequality

$$
\begin{aligned}
& \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq \; & \frac{b-a}{8}\left(\left|f'\left(\frac{5a+b}{6}\right)\right|+\left|f'\left(\frac{a+5b}{6}\right)\right|\right) \\
\leq \; & \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|
\end{aligned}
$$

which refines the inequality (1.7).

2.14. Remark. Let $\alpha = \frac{1}{2}$. Then, using Corollary2.12 and Remark 2.2, we have $I_1 = \frac{1}{24}$, $I_2 = \frac{1}{12}$ and the midpoint inequality

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq & \frac{b-a}{8}\left(\left| f'\left(\frac{2a+b}{3}\right)\right| + \left| f'\left(\frac{a+2b}{3}\right)\right| \right) \\ \leq & \frac{b-a}{4}\left| f'\left(\frac{a+b}{2}\right)\right| \end{aligned}$$

which refines the inequality (1.8).

2.15. Remark. Let $\alpha = \frac{1}{4}$. Then, using Corollary 2.12 and Remark 2.2, we have $I_1 = I_2 = \frac{1}{32}$ and the inequality

$$\begin{aligned} & \left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq & \frac{b-a}{16}\left(\left| f'\left(\frac{3a+b}{4}\right)\right| + \left| f'\left(\frac{a+3b}{4}\right)\right| \right) \\ \leq & \frac{b-a}{8}\left| f'\left(\frac{a+b}{2}\right)\right| \end{aligned}$$

which is the second inequality in (1.2) provided $|f'|^q$ is concave on $[a,b]$.

2.16. Corollary. *Let P_1, P_2, I_i $(i = 1, \cdots, 4)$ be defined as in Remark 2.3. Then, using Theorem 2.11 and Remark 2.3, we have the following Ostrowski-type inequalities which are provided $|f'|^q$ is convex on $[a,b]$.*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq & (P_1+P_2)(b-a)\left| f'\left(\frac{P_1 a + P_2 b}{P_1+P_2}\right)\right| \end{aligned}$$

as $x = c = y$ and $a \leq x \leq b$.

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq & \frac{b-a}{2}\left[\left(\frac{x-a}{b-a}\right)^2 \left| f'\left(\frac{a+2x}{3}\right)\right| + \left(\frac{b-x}{b-a}\right)^2 \left| f'\left(\frac{2x+b}{3}\right)\right| \right] \\ \leq & (P_1+P_2)(b-a)\left| f'\left(\frac{P_1 a + P_2 b}{P_1+P_2}\right)\right| \end{aligned}$$

as $x = c = y$ anda $< x < b$.

2.17. Remark. Using the fact that $2 \geq 2^{\frac{1}{q}}$ as $q \geq 1$, Corollary 3.3 improves Theorem H.

3. Applications for Special Means

In the literature, let us recall the following special means:

(1) The weighted arithmetic mean

$$A_\alpha(u,v) = \alpha u + (1-\alpha)v,\ 0 \leq \alpha \leq 1,\ u,v \in \mathbb{R}.$$

(2) The unweighted arithmetic mean

$$A(u,v)=\frac{u+v}{2},\ u,v\in\mathbb{R}.$$

(3) The harmonic mean

$$H(u,v)=\frac{2}{\frac{1}{u}+\frac{1}{v}},\ u,v>0.$$

(4) The identric mean

$$I(u,v)=\begin{cases}\frac{1}{e}\left(\frac{v^v}{u^u}\right)^{\frac{1}{v-u}} & \text{if } u\neq v\\ u & \text{if } u=v\end{cases},\ u,v>0.$$

(5) The logarithmic mean

$$L(u,v)=\begin{cases}\frac{v-u}{\ln v-\ln u} & \text{if } u\neq v\\ u & \text{if } u=v\end{cases},\ u,v>0.$$

(6) The p-logarithmic mean

$$L_p(u,v)=\begin{cases}\left[\frac{v^{p+1}-u^{p+1}}{(p+1)(v-u)}\right]^{\frac{1}{p}} & \text{if } u\neq v\\ u & \text{if } u=v\end{cases},\ u,v>0,\ p\in\mathbb{R}\backslash\{-1,0\}.$$

(7) The p-power mean

$$M_p(u,v)=\left(\frac{u^p+v^p}{2}\right)^{\frac{1}{p}},\ u,v>0,p\in\mathbb{R}\backslash\{0\}.$$

(8) The weighted p-power mean

$$M_p\begin{pmatrix}\alpha_1,\alpha_2,\cdots,\alpha_n\\ u_1,u_2,\cdots,u_n\end{pmatrix}=\left(\sum_{i=1}^{n}\alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0\leq\alpha_i\leq 1$, $u_i>0$ $(i=1,2,\cdots,n)$ with $\sum_{i=1}^{n}\alpha_i=1$.

Using the above results, we have the following propositions, corollaries and remarks about the above special means:

3.1. Proposition. *In Corollary 2.5 and Corollary 2.9, let* $s\in(-\infty,1]\cup\left[1+\frac{1}{q},\infty\right)\setminus\{-1,0\}$, $q\geq 1, 0<a<b$ *and let* $f(t)=t^s$ *on* $[a,b]$. *Then we have the following trapezoid-type and Ostrowski-type inequalities.*

$$\begin{aligned}(3.1)\qquad &\left|A\left(A_\alpha^s(b,a),A_\alpha^s(a,b)\right)-L_s^s(a,b)\right|\\ \leq\ &\left[\frac{1}{4}-\alpha(1-2\alpha)\right]|s|(b-a)M_q\begin{pmatrix}\frac{I_1}{2(I_1+I_2)},\frac{I_2}{I_1+I_2},\frac{I_1}{2(I_1+I_2)}\\ a^{s-1},\left(\frac{a+b}{2}\right)^{s-1},b^{s-1}\end{pmatrix}\\ \leq\ &\left[\frac{1}{4}-\alpha(1-2\alpha)\right]|s|(b-a)M_q\left(a^{s-1},b^{s-1}\right)\end{aligned}$$

as $\alpha\in[0,1]$, $x=(1-\alpha)a+\alpha b$, $c=\frac{a+b}{2}$ *and* $y=\alpha a+(1-\alpha)b$.

$$\begin{aligned}&\left|x^s-L_s^s(a,b)\right|\\ \leq\ &(P_1+P_2)|s|(b-a)M_q\begin{pmatrix}\frac{P_1}{P_1+P_2},\frac{P_2}{P_1+P_2}\\ a^{s-1},b^{s-1}\end{pmatrix}\end{aligned}$$

as $x = c = y$ *and* $a \leq x \leq b$.

$$\begin{aligned} & \left|x^{s} - L_{s}^{s}(a,b)\right| \\ \leq \quad & 3\left(I_{1}+I_{4}\right)|s|(b-a) M_{q}\begin{pmatrix} \frac{I_{1}}{3(I_{1}+I_{4})}, \frac{2}{3}, \frac{I_{4}}{3(I_{1}+I_{4})} \\ a^{s-1}, x^{s-1}, b^{s-1} \end{pmatrix} \\ \leq \quad & \left(P_{1}+P_{2}\right)|s|(b-a) M_{q}\begin{pmatrix} \frac{P_{1}}{P_{1}+P_{2}}, \frac{P_{2}}{P_{1}+P_{2}} \\ a^{s-1}, b^{s-1} \end{pmatrix} \end{aligned}$$

as $x = c = y$ *and* $a < x < b$.

3.2. Corollary. *Let* $\alpha = 0$ *and in the inequality* (3.1). *Then, using the Hermite-Hadamard inequality* (1.1), *we have the following Hermite-Hadamard-type inequalities.*

$$\begin{aligned} 0 \quad \leq \quad & A\left(a^{s}, b^{s}\right) - L_{s}^{s}(a,b) \\ \leq \quad & \frac{|s|(b-a)}{4} M_{q}\begin{pmatrix} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ a^{s-1}, \left(\frac{a+b}{2}\right)^{s-1}, b^{s-1} \end{pmatrix} \\ \leq \quad & \frac{|s|(b-a)}{4} M_{q}\left(a^{s-1}, b^{s-1}\right) \end{aligned}$$

as $s \in (-\infty, 0) \cup \left[1+\frac{1}{q}, \infty\right) \setminus \{-1\}$.

$$\begin{aligned} 0 \quad \leq \quad & L_{s}^{s}(a,b) - A\left(a^{s}, b^{s}\right) \\ \leq \quad & \frac{s(b-a)}{4} M_{q}\begin{pmatrix} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ a^{s-1}, \left(\frac{a+b}{2}\right)^{s-1}, b^{s-1} \end{pmatrix} \\ \leq \quad & \frac{s(b-a)}{4} M_{q}\left(a^{s-1}, b^{s-1}\right) \end{aligned}$$

as $s \in (0,1]$.

3.3. Corollary. *Let* $\alpha = \frac{1}{2}$ *and in the inequality* (3.1). *Then, using the Hermite-Hadamard inequality* (1.1), *we have the following Hermite-Hadamard-type inequalities.*

$$\begin{aligned} 0 \quad \leq \quad & L_{s}^{s}(a,b) - A^{s}(a,b) \\ \leq \quad & \frac{|s|(b-a)}{4} M_{q}\begin{pmatrix} \frac{1}{6}, \frac{2}{3}, \frac{1}{6} \\ a^{s-1}, \left(\frac{a+b}{2}\right)^{s-1}, b^{s-1} \end{pmatrix} \\ \leq \quad & \frac{|s|(b-a)}{4} M_{q}\left(a^{s-1}, b^{s-1}\right) \end{aligned}$$

as $s \in (-\infty, 0) \cup \left[1+\frac{1}{q}, \infty\right) \setminus \{-1\}$.

$$\begin{aligned} 0 \quad \leq \quad & A^{s}(a,b) - L_{s}^{s}(a,b) \\ \leq \quad & \frac{s(b-a)}{4} M_{q}\begin{pmatrix} \frac{1}{6}, \frac{2}{3}, \frac{1}{6} \\ a^{s-1}, \left(\frac{a+b}{2}\right)^{s-1}, b^{s-1} \end{pmatrix} \\ \leq \quad & \frac{s(b-a)}{4} M_{q}\left(a^{s-1}, b^{s-1}\right) \end{aligned}$$

as $s \in (0,1]$.

3.4. Corollary. *Let $\alpha = \frac{1}{4}$ in the inequality* (3.1)*. Then, using the inequality* (1.2)*, we have the Hermite-Hadamard-type inequalities.*

$$\begin{aligned} 0 &\leq L_s^s(a,b) - A\left(A_{\frac{1}{4}}^s(b,a), A_{\frac{1}{4}}^s(a,b)\right) \\ &\leq \frac{|s|(b-a)}{8} M_q \left(\begin{array}{c} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ a^{s-1}, \left(\frac{a+b}{2}\right)^{s-1}, b^{s-1} \end{array} \right) \\ &\leq \frac{|s|(b-a)}{8} M_q\left(a^{s-1}, b^{s-1}\right) \end{aligned}$$

as $s \in (-\infty, 0) \cup \left[1 + \frac{1}{q}, \infty\right) \setminus \{-1\}$.

$$\begin{aligned} 0 &\leq A\left(A_{\frac{1}{4}}^s(b,a), A_{\frac{1}{4}}^s(a,b)\right) - L_s^s(a,b) \\ &\leq \frac{s(b-a)}{8} M_q \left(\begin{array}{c} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ a^{s-1}, \left(\frac{a+b}{2}\right)^{s-1}, b^{s-1} \end{array} \right) \\ &\leq \frac{s(b-a)}{8} M_q\left(a^{s-1}, b^{s-1}\right) \end{aligned}$$

as $s \in (0, 1]$.

3.5. Proposition. *In Corollary 2.12 and Corollary 2.16, let* $s \in \left[1, 1 + \frac{1}{q}\right]$, $q \geq 1$, $0 \leq a \leq x \leq c \leq y \leq b$ *and let* $f(t) = t^s$ *on* $[a, b]$*. Then we have the following trapezoid-type and Ostrowski-type inequalities.*

$$\begin{aligned} &|A(A_\alpha^s(b,a), A_\alpha^s(a,b)) - L_s^s(a,b)| \\ \leq\ & \left[\frac{1}{8} - \alpha\left(\frac{1}{2} - \alpha\right)\right] s(b-a)\left[A^{s-1}_{\frac{I_1}{I_1+I_2}}\left(a, \frac{a+b}{2}\right) + A^{s-1}_{\frac{I_1}{I_1+I_2}}\left(b, \frac{a+b}{2}\right)\right] \\ \leq\ & \left[\frac{1}{4} - \alpha(1 - 2\alpha)\right] s(b-a) A^{s-1}(a,b). \end{aligned} \tag{3.2}$$

As $x = c = y$,

$$\begin{aligned} &|x - L_s^s(a,b)| \\ \leq\ & (P_1 + P_2) s(b-a) A^{s-1}_{\frac{P_1}{P_1+P_2}}(a,b) \end{aligned}$$

As $x = c = y$ *and* $a < x < b$,

$$\begin{aligned} &|x - L_s^s(a,b)| \\ \leq\ & 3(I_1 + I_4) s(b-a)\left[A_{\frac{1}{3}}^{s-1}(a,x) + A_{\frac{1}{3}}^{s-1}(b,x)\right] \\ \leq\ & (P_1 + P_2) s(b-a) A^{s-1}_{\frac{P_1}{P_1+P_2}}(a,b). \end{aligned}$$

3.6. Corollary. *Let $\alpha = 0$ and in the inequality* (3.2)*. Then, using the Hermite-Hadamard inequality* (1.1)*, we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq L_s^s(a,b) - A(a^s, b^s) \leq \frac{s(b-a)}{8}\left[A_{\frac{1}{6}}^{s-1}(a,b) + A_{\frac{1}{6}}^{s-1}(b,a)\right] \\ &\leq \frac{s(b-a)}{4} A^{s-1}(a,b). \end{aligned}$$

3.7. Corollary. *Let $\alpha = \frac{1}{2}$ in the inequality* (3.2)*. Then, using the Hermite-Hadamard inequality* (1.1)*, we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq A^s(a,b) - L_s^s(a,b) \leq \frac{s(b-a)}{8}\left[A_{\frac{1}{3}}^{s-1}(a,b) + A_{\frac{1}{3}}^{s-1}(b,a)\right] \\ &\leq \frac{s(b-a)}{4} A^{s-1}(a,b). \end{aligned}$$

3.8. Corollary. *Let $\alpha = \frac{1}{4}$ in the inequality* (3.2)*. Then, using the inequality* (1.2)*, we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq A\left(A_{\frac{1}{4}}^s(b,a), A_{\frac{1}{4}}^s(a,b)\right) - L_s^s(a,b) \\ &\leq \frac{s(b-a)}{16}\left[A_{\frac{1}{4}}^{s-1}(a,b) + A_{\frac{1}{4}}^{s-1}(b,a)\right] \\ &\leq \frac{s(b-a)}{8} A^{s-1}(a,b). \end{aligned}$$

3.9. Proposition. *In Corollary 2.5 and Corollary 2.9, let $q \geq 1, 0 < a \leq x \leq c \leq y \leq b$ and let $f(t) = \frac{1}{t}$ on $[a,b]$. Then we have the following trapezoid-type and Ostrowski-type inequalities.*

$$\begin{aligned} &\left|H^{-1}(A_\alpha(b,a), A_\alpha(a,b)) - L^{-1}(a,b)\right| \\ &\leq \left[\frac{1}{4} - \alpha(1-2\alpha)\right](b-a) M_q\begin{pmatrix} \frac{I_1}{2(I_1+I_2)}, \frac{I_2}{I_1+I_2}, \frac{I_1}{2(I_1+I_2)} \\ a^{-2}, \left(\frac{a+b}{2}\right)^{-2}, b^{-2} \end{pmatrix} \\ &\leq \left[\frac{1}{4} - \alpha(1-2\alpha)\right](b-a) M_q\left(a^{-2}, b^{-2}\right). \end{aligned} \tag{3.3}$$

As $x = c = y$,

$$\begin{aligned} &\left|\frac{1}{x} - L^{-1}(a,b)\right| \\ &\leq (P_1+P_2)(b-a) M_q\begin{pmatrix} \frac{P_1}{P_1+P_2}, \frac{P_2}{P_1+P_2} \\ a^{-2}, b^{-2} \end{pmatrix}. \end{aligned}$$

As $x = c = y$ and $a < x < b$,

$$\begin{aligned} &\left|\frac{1}{x} - L^{-1}(a,b)\right| \\ &\leq 3(I_1+I_4)(b-a) M_q\begin{pmatrix} \frac{I_1}{3(I_1+I_4)}, \frac{2}{3}, \frac{I_4}{3(I_1+I_4)} \\ a^{-2}, x^{-2}, b^{-2} \end{pmatrix} \\ &\leq (P_1+P_2)(b-a) M_q\begin{pmatrix} \frac{P_1}{P_1+P_2}, \frac{P_2}{P_1+P_2} \\ a^{-2}, b^{-2} \end{pmatrix}. \end{aligned}$$

3.10. Corollary. *Let $\alpha = 0$ and in the inequality* (3.3)*. Then, using the Hermite-Hadamard inequality* (1.1)*, we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq H^{-1}(a,b) - L^{-1}(a,b) \leq \frac{b-a}{4} M_q\begin{pmatrix} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ a^{-2}, \left(\frac{a+b}{2}\right)^{-2}, b^{-2} \end{pmatrix} \\ &\leq \frac{b-a}{4} M_q\left(a^{-2}, b^{-2}\right). \end{aligned}$$

3.11. Corollary. *Let* $\alpha = \frac{1}{2}$ *in the inequality* (3.3). *Then, using the Hermite-Hadamard inequality* (1.1), *we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq L^{-1}(a,b) - A^{-1}(a,b) \leq \frac{b-a}{4} M_q \begin{pmatrix} \frac{1}{6}, \frac{2}{3}, \frac{1}{6} \\ a^{-2}, \left(\frac{a+b}{2}\right)^{-2}, b^{-2} \end{pmatrix} \\ &\leq \frac{b-a}{4} M_q \left(a^{-2}, b^{-2}\right). \end{aligned}$$

3.12. Corollary. *Let* $\alpha = \frac{1}{4}$ *in the inequality* (3.3). *Then, using the inequality* (1.2), *we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq L^{-1}(a,b) - H^{-1}\left(A_{\frac{1}{4}}(b,a), A_{\frac{1}{4}}(a,b)\right) \\ &\leq \frac{b-a}{8} M_q \begin{pmatrix} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ a^{-2}, \left(\frac{a+b}{2}\right)^{-2}, b^{-2} \end{pmatrix} \\ &\leq \frac{b-a}{8} M_q \left(a^{-2}, b^{-2}\right). \end{aligned}$$

3.13. Proposition. *In Corollary 2.5 and Corollary 2.9, let* $q \geq 1, 0 < a \leq x \leq c \leq y \leq b$ *and let* $f(t) = \ln t$ *on* $[a,b]$. *Then we have the following trapezoid-type and Ostrowski-type inequalities.*

$$\begin{aligned} & \left|A\left(\ln A_\alpha(a,b), \ln A_\alpha(b,a)\right) - \ln I(a,b)\right| \\ \leq & \left[\frac{1}{4} - \alpha(1-2\alpha)\right](b-a) M_q \begin{pmatrix} \frac{I_1}{2(I_1+I_2)}, \frac{I_2}{I_1+I_2}, \frac{I_1}{2(I_1+I_2)} \\ a^{-1}, \left(\frac{a+b}{2}\right)^{-1}, b^{-1} \end{pmatrix} \\ \leq & \left[\frac{1}{4} - \alpha(1-2\alpha)\right](b-a) M_q \left(a^{-1}, b^{-1}\right). \end{aligned} \tag{3.4}$$

As $x = c = y$,

$$\begin{aligned} & \left|\ln x - \ln I(a,b)\right| \\ \leq & \left(P_1 + P_2\right)(b-a) M_q \begin{pmatrix} \frac{P_1}{P_1+P_2}, \frac{P_2}{P_1+P_2} \\ a^{-1}, b^{-1} \end{pmatrix}. \end{aligned}$$

As $x = c = y$ *and* $a < x < b$,

$$\begin{aligned} & \left|\ln x - \ln I(a,b)\right| \\ \leq & 3\left(I_1 + I_4\right)(b-a) M_q \begin{pmatrix} \frac{I_1}{3(I_1+I_4)}, \frac{2}{3}, \frac{I_4}{3(I_1+I_4)} \\ a^{-1}, x^{-1}, b^{-1} \end{pmatrix} \\ \leq & \left(P_1 + P_2\right)(b-a) M_q \begin{pmatrix} \frac{P_1}{P_1+P_2}, \frac{P_2}{P_1+P_2} \\ a^{-1}, b^{-1} \end{pmatrix} \end{aligned}$$

3.14. Corollary. *Let* $\alpha = 0$ *and in the inequality* (3.4). *Then, using the Hermite-Hadamard inequality* (1.1), *we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq \ln I(a,b) - A(\ln a, \ln b) \leq \frac{b-a}{4} M_q \begin{pmatrix} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ a^{-1}, \left(\frac{a+b}{2}\right)^{-1}, b^{-1} \end{pmatrix} \\ &\leq \frac{b-a}{4} M_q \left(a^{-1}, b^{-1}\right). \end{aligned}$$

3.15. Corollary. *Let* $\alpha = \frac{1}{2}$ *in the inequality* (3.3). *Then, using the Hermite-Hadamard inequality* (1.1), *we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq \ln A(a,b) - \ln I(a,b) \leq \frac{b-a}{4} M_q \begin{pmatrix} \frac{1}{6}, \frac{2}{3}, \frac{1}{6} \\ a^{-1}, \left(\frac{a+b}{2}\right)^{-1}, b^{-1} \end{pmatrix} \\ &\leq \frac{b-a}{4} M_q \left(a^{-1}, b^{-1}\right). \end{aligned}$$

3.16. Corollary. *Let $\alpha = \frac{1}{4}$ in the inequality* (3.3)*. Then, using the inequality* (1.2)*, we have the Hermite-Hadamard-type inequality*

$$\begin{aligned} 0 &\leq A\left(\ln A_{\frac{1}{4}}(a,b), \ln A_{\frac{1}{4}}(b,a)\right) - \ln I(a,b) \\ &\leq \frac{b-a}{8} M_q \left(\begin{array}{c} \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \\ a^{-1}, \left(\frac{a+b}{2}\right)^{-1}, b^{-1} \end{array} \right) \\ &\leq \frac{b-a}{8} M_q\left(a^{-1}, b^{-1}\right). \end{aligned}$$

4. Applications for the extended Trapezoid Quadrature Formula

Throughout in this section, let $0 \leq \alpha \leq 1, L_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a partition of the interval $[a,b]$, $\xi_i \leq x_i \leq \zeta_i$ in $[t_i, t_{i+1}]$, $l_i = t_{i+1} - t_i$, $(i = 0, 1, \cdots, n-1)$ let $P_1(i), P_2, I_j(i)$ $(j = 1, \cdots, 4; i = 1, \cdots, n)$ be defined as follows.

$$\begin{aligned} P_1(i) &= \frac{1}{3l_i^3}\left[(x_i - t_i)^2\left(\frac{3t_{i+1} - t_i}{2} - \xi_i\right) + (x_i - \xi_i)^2\left(\frac{3t_{i+1} - x_i}{2} - \xi_i\right)\right. \\ &\quad \left. + (\zeta_i - x_i)^2\left(\frac{3t_{i+1} - x_i}{2} - \zeta_i\right) + (t_{i+1} - \zeta_i)^3\right] \end{aligned}$$

and

$$\begin{aligned} P_2(i) &= \frac{1}{3l_i^3}\left[(\xi_i - t_i)^3 + (x_i - \xi_i)^2\left(\frac{x_i - 3t_i}{2} + \xi_i\right)\right. \\ &\quad \left. + (\zeta_i - x_i)^2\left(\frac{x_i - 3t_i}{2} + \zeta_i\right) + (t_{i+1} - \zeta_i)^2\left(\frac{t_{i+1} - 3t_i}{2} + \zeta_i\right)\right]. \end{aligned}$$

As $t_i < x_i < t_{i+1}$,

$$I_1(i) = \frac{1}{3l_i^2(x_i - t_i)}\left[(\xi_i - t_i)^2\left(\frac{3x_i - t_i}{2} - \xi_i\right) + (x_i - \xi_i)^3\right],$$

$$I_2(i) = \frac{1}{3l_i^2(x_i - t_i)}\left[(x_i - \xi_i)^2\left(\frac{x_i - 3t_i}{2} + \xi_i\right) + (\xi_i - t_i)^3\right],$$

$$I_3(i) = \frac{1}{3l_i^2(t_{i+1} - x_i)}\left[(\zeta_i - x_i)^2\left(\frac{3t_{i+1} - x_i}{2} - \zeta_i\right) + (t_{i+1} - \zeta_i)^3\right]$$

and

$$I_4(i) = \frac{1}{3l_i^2(t_{i+1} - x_i)}\left[(t_{i+1} - \zeta_i)^2\left(\frac{t_{i+1} - 3x_i}{2} + \zeta_i\right) + (\zeta_i - x_i)^3\right].$$

Define the extended Trapezoid quadrature formula

$$\int_a^b f(t)dt = T(f, L_n, \xi, \zeta) + R(f, L_n, \xi, \zeta) \tag{4.1}$$

where

$$T(f, L_n, \xi, \zeta) = \sum_{i=0}^{n-1} \frac{x_i - t_i}{t_{i+1} - t_i} f(\xi_i) + \frac{t_{i+1} - x_i}{t_{i+1} - t_i} f(\zeta_i) \tag{4.2}$$

and the remainder term $R(f, L_n, \xi_i, \zeta_i)$ denotes the associated approximation error of $\int_a^b f(t)dt$ by $T(f, L_n, \xi, \zeta)$.

Now, we have the following special formulae.

(1) The trapezoid formula

$$T(f,L_n,\xi,\zeta)=T_0(f,L_n,\xi,\zeta)=\sum_{i=0}^{n-1}\frac{f(t_i)+f(t_{i+1})}{2}l_i \tag{4.3}$$

where $\xi_i=t_i$ and $\zeta_i=t_{i+1}$ $(i=0,1,\cdots,n-1)$.

(2) The midpoint formula

$$T(f,L_n,\xi,\zeta)=M(f,L_n,\xi,\zeta)=\sum_{i=0}^{n-1}f\left(\frac{t_i+t_{i+1}}{2}\right)l_i \tag{4.4}$$

where $\xi_i=\zeta_i=\frac{t_i+t_{i+1}}{2}$ $(i=0,1,\cdots,n-1)$.

(3) The Ostrowski formula

$$T(f,L_n,\xi,\zeta)=O(f,L_n,\xi,\zeta)=\sum_{i=0}^{n-1}f(x_i)l_i \tag{4.5}$$

where $\xi_i=\zeta_i=x_i$ $(i=0,1,\cdots,n-1)$.

4.1. Theorem. *Let f be defined as in Theorem 2.4 and let $\int_a^b f(t)dt, T(f,L_n,\xi,\zeta)$ and $R(f,L_n,\xi,\zeta)$ be defined as in the identity (4.1). Then, the remainder term $R(f,L_n,\xi,\zeta)$ satisfies the following estimates.*

(1) We have the inequality

$$\begin{aligned}(4.6)\qquad &|R(f,L_n,\xi,\zeta)|\\ &\le \sum_{i=0}^{n-1}(P_1(i)+P_2(i))l_i^2\left(\frac{P_1(i)|f'(\xi_i)|^q+P_2(i)|f'(\zeta_i)|^q}{P_1(i)+P_2(i)}\right)^{\frac{1}{q}}\\ &\le \max\left\{|f'(a)|,|f'(b)|\right\}\sum_{i=0}^{n-1}(P_1(i)+P_2(i))l_i^2.\end{aligned}$$

(2) Let $t_i<x_i<t_{i+1}$ $(i=0,1,\cdots,n-1)$. Then we have the inequality

$$\begin{aligned}(4.7)\qquad &|R(f,L_n,\xi,\zeta)|\\ \le\ &\sum_{i=0}^{n-1}\left\{\left(\sum_{j=1}^{4}I_j(i)\right)l_i^2\right.\\ &\left.\times\left(\frac{I_1(i)|f'(t_i)|^q+(I_2(i)+I_3(i))|f'(x_i)|^q+I_4(i)|f'(t_{i+1})|^q}{\sum_{j=1}^{4}I_j(i)}\right)^{\frac{1}{q}}\right\}\\ \le\ &\sum_{i=0}^{n-1}(P_1(i)+P_2(i))l_i^2\left(\frac{P_1(i)|f'(t_i)|^q+P_2(i)|f'(t_{i+1})|^q}{P_1(i)+P_2(i)}\right)^{\frac{1}{q}}\\ \le\ &\max\left\{|f'(a)|,|f'(b)|\right\}\sum_{i=0}^{n-1}(P_1(i)+P_2(i))l_i^2.\end{aligned}$$

Proof. Apply Theorem 2.4 on the intervals $[t_i,t_{i+1}]$ $(i=0,1,\cdots,n-1)$ to get the following inequalities.

(1) For all $i=0,1,\cdots,n-1$, we have the inequality

$$\left|\frac{f(\xi_i)+f(\zeta_i)}{2}l_i-\int_{t_i}^{t_{i+1}}f(s)ds\right| \tag{4.8}$$

$$\leq \quad (P_1(i)+P_2(i))\,l_i^2\left(\frac{P_1(i)\left|f'(t_i)\right|^q+P_2(i)\left|f'(t_{i+1})\right|^q}{P_1(i)+P_2(i)}\right)^{\frac{1}{q}}.$$

(2) Let $t_i<x_i<t_{i+1}$ $(i=0,1,\cdots,n-1)$. Then we have the inequality

$$\left|\frac{f(\xi_i)+f(\zeta_i)}{2}l_i-\int_{t_i}^{t_{i+1}}f(s)ds\right| \tag{4.9}$$

$$\leq \quad \left(\sum_{j=1}^{4}I_j(i)\right)l_i^2$$

$$\times\left(\frac{I_1(i)\left|f'(t_i)\right|^q+(I_2(i)+I_3(i))\left|f'(x_i)\right|^q+I_4(i)\left|f'(t_{i+1})\right|^q}{\sum_{j=1}^{4}I_j(i)}\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq \quad (P_1(i)+P_2(i))\,l_i^2\left(\frac{P_1(i)\left|f'(t_i)\right|^q+P_2(i)\left|f'(t_{i+1})\right|^q}{P_1(i)+P_2(i)}\right)^{\frac{1}{q}}.$$

Using the convexity of $|f'|^q$, we have the inequality

$$\left(\frac{P_1(i)\left|f'(t_i)\right|^q+P_2(i)\left|f'(t_{i+1})\right|^q}{P_1(i)+P_2(i)}\right)^{\frac{1}{q}} \tag{4.10}$$

$$\leq \quad \left[\frac{P_1(i)}{P_1(i)+P_2(i)}\left(\frac{b-t_i}{b-a}\left|f'(a)\right|^q+\frac{t_i-a}{b-a}\left|f'(b)\right|^q\right)\right.$$

$$\left.+\frac{P_2(i)}{P_1(i)+P_2(i)}\left(\frac{b-t_{i+1}}{b-a}\left|f'(a)\right|^q+\frac{t_{i+1}-a}{b-a}\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}$$

$$\leq \quad \left(\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}=\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}$$

for all $i=0,1,\cdots,n-1$.

The inequalities (4.6) and (4.7) follow from the inequalities (4.10) – (4.10) and the generalized triangle inequality.

This completes the proof. ∎

4.2. Corollary. *In Theorem 4.1, let* $\xi_i=t_i,\zeta_i=t_{i+1}$ *and* $x_i=\frac{t_i+t_2}{2}$ $(i=0,1,\cdots,n-1)$. *Then* $P_1(i)=P_2(i)=\frac{1}{8},I_1(i)=I_4(i)=\frac{1}{12},\ I_2(i)=I_3(i)=\frac{1}{24}$ $(i=0,1,\cdots,n-1)$ *and the trapzoid-type error satisfies*

$$\left|R(f,L_n,\xi,\zeta)\right|$$

$$\leq \quad \sum_{i=0}^{n-1}\frac{l_i^2}{4}\left[\frac{\left|f'(t_i)\right|^q+\left|f'\left(\frac{t_i+t_{i+1}}{2}\right)\right|^q+\left|f'(t_{i+1})\right|^q}{3}\right]^{\frac{1}{q}}$$

$$\leq \quad \sum_{i=0}^{n-1}\frac{l_i^2}{4}\left[\frac{\left|f'(t_i)\right|^q+\left|f'(t_{i+1})\right|^q}{2}\right]^{\frac{1}{q}}$$

$$\leq \quad \max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\sum_{i=0}^{n-1}\frac{l_i^2}{4}$$

which improves Proposition 3 *in* [11].

4.3. Corollary. *In Theorem 4.1, let* $\xi_i = \zeta_i = x_i = \frac{t_i+t_{i+1}}{2} = (i = 0, 1, \cdots, n-1)$*. Then* $P_1(i) = P_2(i) = \frac{1}{8}, I_1(i) = I_4(i) = \frac{1}{24},\ I_2(i) = I_3(i) = \frac{1}{12}\ (i = 0, 1, \cdots, n-1)$ *and the midpoint-type error satisfies*

$$\begin{aligned}
&|R(f, L_n, \xi, \zeta)| \\
\leq\ & \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left[\frac{|f'(t_i)|^q + 4\left|f'\left(\frac{t_i+t_{i+1}}{2}\right)\right|^q + |f'(t_{i+1})|^q}{6} \right]^{\frac{1}{q}} \\
\leq\ & \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left[\frac{|f'(t_i)|^q + |f'(t_{i+1})|^q}{2} \right]^{\frac{1}{q}} \\
\leq\ & \max\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\} \sum_{i=0}^{n-1} \frac{l_i^2}{4}.
\end{aligned}$$

4.4. Corollary. *In Theorem 4.1, let* $\xi_i = \zeta_i = x_i \in (t_i, t_{i+1})\ (i = 0, 1, \cdots, n-1)$*. Then*

$$\begin{aligned}
P_1(i) &= \frac{(x_i - t_i)^2 (3t_{i+1} - t_i - 2x)}{6(t_{i+1} - t_i)^3} + \frac{1}{3}\left(\frac{t_{i+1} - x}{t_{i+1} - t_i}\right)^3, \\
P_2(i) &= \frac{(t_{i+1} - x)^2 (t_{i+1} - 3t_i + 2x)}{6(t_{i+1} - t_i)^3} + \frac{1}{3}\left(\frac{x_i - t_i}{t_{i+1} - t_i}\right)^3, \\
I_1(i) &= \frac{1}{2} I_2(i) = \frac{1}{6}\left(\frac{x_i - t_i}{t_{i+1} - t_i}\right)^2, \\
I_4(i) &= \frac{1}{2} I_3(i) = \frac{1}{6}\left(\frac{t_{i+1} - x_i}{t_{i+1} - t_i}\right)^2,
\end{aligned}$$

$$\begin{aligned}
\sum_{j=0}^{4} I_j(i) &= P_1(i) + P_2(i) \\
&= 3(I_1(i) + I_4(i)) = \frac{1}{2} - \frac{(x_i - t_i)(t_{i+1} - x_i)}{(t_{i+1} - t_i)^2}
\end{aligned}$$

and the Ostrowski-type error satisfies

$$\begin{aligned}
&|R(f, L_n, \xi, \zeta)| \\
\leq\ & \sum_{i=0}^{n-1} \Bigg\{ 3(I_1(i) + I_4(i))\, l_i^2 \\
& \times \left[\frac{I_1(i)|f'(t_i)|^q + 2(I_1(i) + I_4(i))\left|f'\left(\frac{t_i+t_{i+1}}{2}\right)\right|^q + I_4(i)|f'(t_{i+1})|^q}{3(I_1(i) + I_4(i))} \right]^{\frac{1}{q}} \Bigg\} \\
\leq\ & \sum_{i=0}^{n-1} (P_1(i) + P_2(i))\, l_i^2 \left[\frac{P_1(i)|f'(t_i)|^q + P_2(i)|f'(t_{i+1})|^q}{P_1(i) + P_2(i)} \right]^{\frac{1}{q}} \\
\leq\ & \max\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\} \sum_{i=0}^{n-1} (P_1(i) + P_2(i))\, l_i^2.
\end{aligned}$$

Similarly, using Theorem 2.11 we can prove the following theorem.

4.5. Theorem. *Let* f *be defined as in Theorem 2.11 and let* $\int_a^b f(t)dt, T(f, L_n, \xi, \zeta)$ *and* $R(f, L_n, \xi, \zeta)$ *be defined as in the identity* (4.1)*. Then, the remainder term* $R(f, L_n, \xi, \zeta)$ *satisfies the following estimates.*

(1) We have the inequality

$$|R(f,I_n,\xi,\zeta)| \leq \sum_{i=0}^{n-1} (P_1(i)+P_2(i))\, l_i^2 \left| f'\left(\frac{P_1(i)\, t_i + P_2(i)\, t_{i+1}}{P_1(i)+P_2(i)} \right) \right|$$

for all $i = 0, 1, ..., n-1$.

(2) Let $t_i < x_i < t_{i+1}$ $(i = 0, 1, \cdots, n-1)$. Then we have the inequality

$$\begin{aligned} & |R(f,L_n,\xi,\zeta)| \\ \leq\ & \sum_{i=0}^{n-1} l_i^2 \left[(I_1(i)+I_2(i)) \left| f'\left(\frac{I_1(i)\, t_i + I_2(i)\, x_i}{I_1(i)+I_2(i)} \right) \right| \right. \\ & \left. + (I_3(i)+I_4(i)) \left| f'\left(\frac{I_3(i)\, x_i + I_4(i)\, t_{i+1}}{I_3(i)+I_4(i)} \right) \right| \right] \\ \leq\ & \sum_{i=0}^{n-1} (P_1(i)+P_2(i))\, l_i^2 \left| f'\left(\frac{P_1(i)\, t_i + P_2(i)\, t_{i+1}}{P_1(i)+P_2(i)} \right) \right|. \end{aligned}$$

4.6. Corollary. *In Theorem 4.5, let* $\xi_i = t_i, \zeta_i = t_{i+1}$ *and* $x_i = \frac{t_i+t_2}{2}$ $(i = 0, 1, \cdots, n-1)$. *Then* $P_1(i) = P_2(i) = \frac{1}{8}, I_1(i) = I_4(i) = \frac{1}{12}$, $I_2(i) = I_3(i) = \frac{1}{24}$ $(i = 0, 1, \cdots, n-1)$ *and the trapzoid-type error satisfies*

$$\begin{aligned} & |R(f,L_n,\xi,\zeta)| \\ \leq\ & \sum_{i=0}^{n-1} \frac{l_i^2}{8} \left(\left| f'\left(\frac{5t_i + t_{i+1}}{6} \right) \right| + \left| f'\left(\frac{t_i + 5t_{i+1}}{6} \right) \right| \right) \\ \leq\ & \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left| f'\left(\frac{t_i + t_{i+1}}{2} \right) \right| \end{aligned}$$

which improves Proposition 4 *in* [11].

4.7. Corollary. *In Theorem 4.5, let* $\xi_i = \zeta_i = x_i = \frac{t_i+t_{i+1}}{2} = (i = 0, 1, \cdots, n-1)$. *Then* $P_1(i) = P_2(i) = \frac{1}{8}, I_1(i) = I_4(i) = \frac{1}{24}$, $I_2(i) = I_3(i) = \frac{1}{12}$ $(i = 0, 1, \cdots, n-1)$ *and the midpoint-type error satisfies*

$$\begin{aligned} & |R(f,L_n,\xi,\zeta)| \\ \leq\ & \sum_{i=0}^{n-1} \frac{l_i^2}{8} \left(\left| f'\left(\frac{2t_i + t_{i+1}}{3} \right) \right| + \left| f'\left(\frac{t_i + 2t_{i+1}}{3} \right) \right| \right) \\ \leq\ & \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left| f'\left(\frac{t_i + t_{i+1}}{2} \right) \right| \end{aligned}$$

4.8. Corollary. *In Theorem 4.5, let* $\xi_i = \zeta_i = x_i \in (t_i, t_{i+1})$ $(i = 0, 1, \cdots, n-1)$. *Then the Ostrowski-type error satisfies*

$$\begin{aligned} & |R(f,L_n,\xi,\zeta)| \\ \leq\ & \sum_{i=0}^{n-1} \frac{l_i^2}{2} \left[\left(\frac{x_i - t_i}{t_{i+1} - t_i} \right)^2 \left| f'\left(\frac{t_i + 2x_i}{3} \right) \right| + \left(\frac{t_{i+1} - x_i}{t_{i+1} - t_i} \right)^2 \left| f'\left(\frac{2x_i + t_{i+1}}{3} \right) \right| \right] \\ \leq\ & \sum_{i=0}^{n-1} (P_1(i)+P_2(i))\, l_i^2 \left| f'\left(\frac{P_1(i)\, t_i + P_2(i)\, t_{i+1}}{P_1(i)+P_2(i)} \right) \right|, \end{aligned}$$

where $P_1(i), P_2(i)$ $(i = 0, 1, \cdots, n-1)$ *is defined as in Corollary 4.7 and* $t_i < x_i < t_{i+1}$ $(i = 0, 1, \cdots, n-1)$.

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The (P-A-L) extended Weibull distribution: A new generalization of the Weibull distribution

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Abstract

Recently, some attempts have been made to construct new families of models to extend well-known distributions and at the same time provide great flexibility in modeling data in practice. So, several classes by adding shape parameters to generate new models have been explored in the statistical literature. We propose a new generalization of the three-parameter extended Weibull distribution pioneered by Pappas et al. (2012) by using the generator by Marshall and Olkin (1997). The new model is called the (P-A-L) extended Weibull, where (P-A-L) denote the first letters of the scientists Pappas, Adamidis and Loukas.

Keywords: Extended Weibull distribution, Hazard rate function, Maximum likelihood estimation, Moment, Order statistic, Weibull distribution.

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1. Introduction

For more than half a century the Weibull distribution has attracted the attention of statisticians working on theory and methods in various fields of applied researchers. Thousands of papers have been written on this distribution. It is of most interest to the theory because of its great number of special features and to practitioners because of its ability to fit to real data from various fields, ranging from life data to weather data or observations made in economics and business administration, hydrology, biology and engineering sciences. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, this distribution does not provide a reasonable parametric fit for some practical applications where the underlying hazard rates may be bathtub or unimodal shapes. [13] introduced a new generalization of any distribution, which is derived by using the generator by [10]. In the literature, several generalizations of the Weibull distribution have been proposed such as those studied by [3], [19], [14] and [20].

The *extended Weibull* (EW) distribution with parameters $\alpha > 0$, $\beta > 0$ and $\nu > 0$ has probability density function (pdf) given by

$$g(t) = \frac{\frac{\nu\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{\left[1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{2}}. \tag{1.1}$$

The reliability function corresponding to (1.1) becomes

$$\overline{G}(t) = \frac{\nu\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}. \tag{1.2}$$

Let $G(t)$ be a baseline cumulative distribution function (cdf) with corresponding survival function $\overline{G}(t) = 1 - G(t)$, pdf $g(t) = dG(t)/dt$ and hazard rate function (hrf) $\lambda(t)$. [13] proposed the *(P-A-L) extended family* with the additional parameter $p > 1$, where the survival function $\overline{F}(t)$, cdf $F(t)$ and pdf $f(t)$ are given by (for $t > 0$)

$$\overline{F}(t) = \frac{\log\left[1-(1-p)\,\overline{G}(t)\right]}{\log(p)}, \tag{1.3}$$

$$F(t) = 1 - \frac{\log\left[1-(1-p)\,\overline{G}(t)\right]}{\log(p)} \tag{1.4}$$

and

$$f(t) = \frac{(p-1)\;g(t)}{\left[1-(1-p)\,\overline{G}(t)\right]\;\log(p)},$$

respectively.

Further, [13] studied the *(P-A-L) extended modified Weibull* distribution. In this paper, we take the EW distribution given by (1.1) as the baseline model to define a new four-parameter *(P-A-L) extended Weibull*, say the (P-A-L)EW distribution.

The rest of the paper is organized as follows. In Section 2, we provide the pdf and cdf of the new distribution and present some special models. In Section 3, we study some of its structural properties including moments, moment generating function (mgf), quantile and residual life functions. The mean deviations and two types of entropies are determined in Sections 4 and 5, respectively. Section 6 is devoted to the reliability function. In Section 7, we present the reliability function, hrf, cumulative hazard rate function (chrf) and mean residual lifetime function (mrlf). The order statistics and the minimum and maximum order statistics are investigated in Section 8. In Section 9, we obtain the maximum likelihood estimates (MLEs) of the model parameters. In Section 10, we apply a particle swarm optimization (PSO) method to estimate the parameters. In Section 11, we provide one application to real data in order to illustrate the potentiality of the new model. Concluding remarks are addressed in Section 12.

2. The (P-A-L) Extended Weibull Distribution

Combining (1.2) and (1.4), the cdf of the (P-A-L)EW distribution follows as

$$F(t) = 1 - \frac{1}{\log(p)} \log \left\{ \frac{1-(1-p\,\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}} \right\}. \tag{2.1}$$

By differentiating (2.1), the corresponding pdf reduces to

$$f(t) = \frac{1}{\log(p)} \left\{ \frac{(p-1)\,\frac{\nu\beta}{\alpha}\,\left(\frac{t}{\alpha}\right)^{\beta-1}\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{\left[1-(1-p\,\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]\left[1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]} \right\}. \tag{2.2}$$

Henceforth, we denote by $T\ \sim$(P-A-L)EW(α, β, ν, p) a random variable having pdf (2.2). It is clear that the new distribution is very flexible (as it can seen from Table 1). In fact, several distributions can be obtained as special cases of the new model for selected parameter values. These special cases include at least eleven distributions displayed in Figure 1: the (P-A-L) extended Rayleigh (P-A-L)ER, (P-A-L) extended Exponential (P-A-L)EE, (P-A-L) Weibull (P-A-L)W, (P-A-L) Rayleigh (P-A-L)R, (P-A-L) exponential (P-A-L)E, extended Weibull (EW) (Marshall and Olkin, 1997), extended Rayleigh (ER), extended exponential (EW), Weibull (W), Rayleigh (R) and exponential (E) distributions.

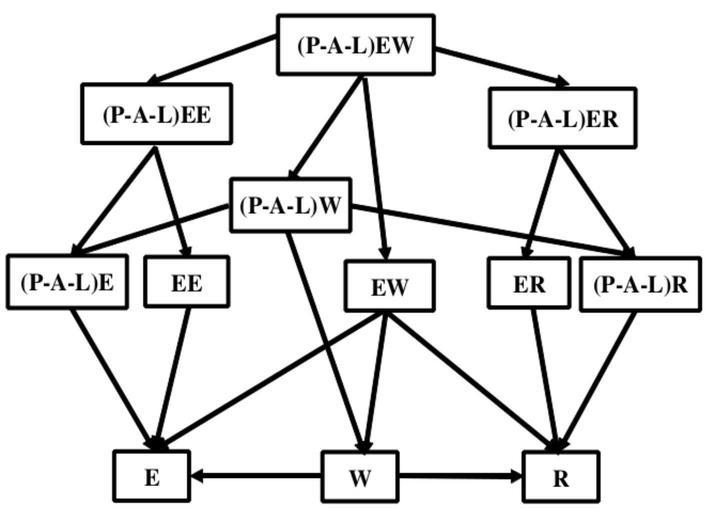
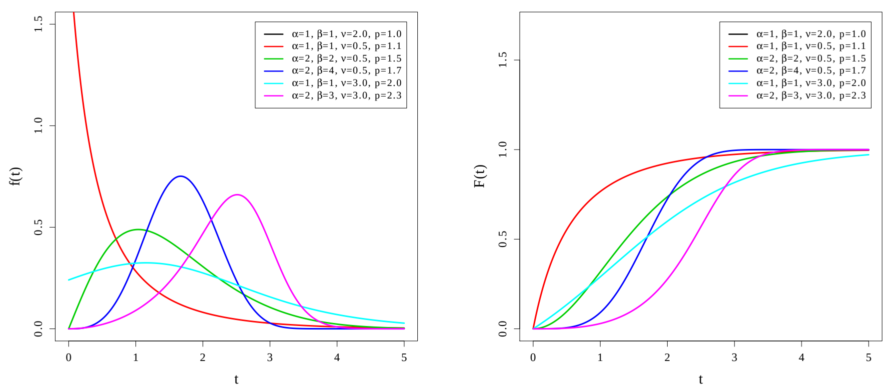


Figure 1. Sub-models of the (P-A-L)EW distribution.

Figures 2(a) and 2(b) display some of the possible shapes of the pdf and cdf of the new distribution, respectively, for different values of the parameters α, β, $\nu > 0$ and $p > 1$.

Table 1. Special cases of the (P-A-L)EW distribution.

Sub-Models	Parameters of (P-A-L)EWD				Cumulative distribution function
	α	β	ν	p	
(P-A-L)ER	-	2	-	-	$1-\frac{1}{\log p}\log\left\{\frac{1-(1-p\,\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^2}}{1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^2}}\right\}$
(P-A-L)EE	-	1	-	-	$1-\frac{1}{\log p}\log\left\{\frac{1-(1-p\,\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)}}{1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)}}\right\}$
(P-A-L)W	-	-	1	-	$1-\frac{1}{\log p}\log\left\{1-(1-p)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right\}$
(P-A-L)R	-	2	1	-	$1-\frac{1}{\log p}\log\left\{1-(1-p)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{2}}\right\}$
(P-A-L)E	-	1	1	-	$1-\frac{1}{\log p}\log\left\{1-(1-p)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)}\right\}$
W	-	-	1	$p\to 1$	$F(t)=1-\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}$
R	-	2	1	$p\to 1$	$F(t)=1-\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{2}}$
ED	-	1	1	$p\to 1$	$F(t)=1-\mathrm{e}^{-\left(\frac{t}{\alpha}\right)}$



(a) (b)

Figure 2. The (P-A-L)EW for (a) densities and for (b) distributions.

3. Mathematical Properties

In this section, we derive some mathematical properties of the (P-A-L)EW distribution such as the quantile, median, random number generator, central and non-central moments and mgf.

3.1. Quantile Function. The quantile function (qf) is used to obtain the quantiles of a probability distribution. Consider $F_X : \mathbb{R} \to [0,1]$ a distribution function of the continuous random variable X. The pth quantile of $F(x)$ is given by the value of x such that

$$Q(u) = \inf\{\mathrm{x} \in \mathbb{R} : \mathrm{u} \leq \mathrm{F(x)}\},$$

where $u \in (0,1)$. The qf $Q(u) = F^{-1}(u)$ of T comes by inverting (2.1) as

$$t = Q(u) = \left[\alpha^{\beta}\log\left(-\frac{\nu p^{u+1} - p^{u} - \nu p + p}{p^{u} - p}\right)\right]^{1/\beta}. \tag{3.1}$$

3.2. Central and Non-Central Moments. The rth non-central moment of T can be expressed as

$$E(T^r) = \mu_r = \frac{\alpha^r \nu(p-1)}{\log(p)} \Gamma\left(\frac{r}{\beta}+1\right) \sum_{i,j=0}^{\infty} \frac{(1-p\nu)^i \, (1-\nu)^j}{(1+i+j)^{\frac{r}{\beta}+1}}. \tag{3.2}$$

The nth central moment of T, say m_n, can be easily obtained from the non-central moments by (for $n \geq 1$)

$$m_n = E(T-\mu)^n = \sum_{r=0}^{n} \binom{n}{r} (-\mu)^{n-r} \, E(T^r).$$

Let $\alpha = 1.5$, $\beta = 1.3$, $\nu = 1.2$ and $p = 1.2$. We can easily check that equation (3.2) holds. The following script written in the `Julia` language implements equation (3.2) with $r = 2$. The `Julia` language can be obtained from http://julialang.org/downloads/ (see [4]). So, we provide a numerical check for $i = 0, \ldots, 5000$ and $j = 0, \ldots, 5000$. The code follows below.

```
alpha = 1.5
beta = 1.3
nu = 1.2
p = 1.2
I = 5000
J = 5000
r=2 # Moment of order 2.
constant = alpha^r*nu*(p-1)/log(p)*gamma(r/beta+1)
sum_I = zeros(Float64,I+1,1)
sum_J = zeros(Float64,J+1,1)
for i = 0:I
    for j = 0:J
        numerator = (1-p*nu)^i * (1-nu)^j
        denominator = (1+i+j)^(r/beta+1)
        sum_J[j+1] = numerador/denominador
   end
   sum_I[i+1] = sum(sum_J)
end
constant*sum(sum_I) # The result is 3.66655262332183.
```

Thus, for the fixed parameters, $r = 2$ and using the above code, we obtain $E(T^2) = 3.6665526$. The same result follows by numerical integration of (2.2). Established algebraic expansions to determine the moments of T can be more efficient than computing these moments directly by using this numerical integration, which can be prone to rounding off errors among others.

3.3. Residual life function. Given that a component survives up to time $x \geq 0$, the residual life is the period beyond x until the time of failure and it is defined by the expectation of the conditional random variable $T|T > x$. In reliability, it is well-known that the mrlf and the ratio of two consecutive moments of residual life determine the distribution uniquely (see [9]). Therefore, we obtain the rth order moment of the residual life by

$$m_r(x) = E[(T-x)^r \mid T > x] = \frac{1}{\overline{F}(x)} \int_x^{\infty} (t-x)^r \, f(t) dt. \tag{3.3}$$

Applying the binomial expansion for $(T-x)^r$ and substituting $\overline{F}(x)$ given by (1.3) into equation (3.3), the rth moment of the residual life of T is

$$m_r(x) = \frac{\nu(p-1)}{\overline{F}(x)\,\log(p)} \sum_{i=0}^{r} \sum_{j,k=0}^{\infty} \binom{r}{i} \frac{(1-p\nu)^j\,(1-\nu)^k \alpha^i\,(-x)^{r-i}}{(j+k+1)^{\frac{i}{\beta}+1}} \times\Gamma\left(\frac{i}{\beta}+1,(j+k+1)\left(\frac{x}{\alpha}\right)^{\beta}\right), \tag{3.4}$$

where $\Gamma(a,y)=\int_y^\infty x^{a-1}\,\mathrm{e}^{-\mathrm{x}}\mathrm{dx}$ is the upper incomplete gamma function.

Another important characteristic for T is the mrlf obtained by setting $r=1$ in equation (3.4). It represents the mean lifetime left for an item of age x. Whereas the hrf at x provides information on a random variable T about a small interval after x, the mrlf at x considers information about the whole remaining interval (x,∞).

We obtain the mrlf of T as

$$m(x) = -x + \frac{\nu\alpha(p-1)}{\overline{F}(x)\,\log(p)} \sum_{j,k=0}^{\infty} \frac{(1-p\nu)^j\,(1-\nu)^k}{(j+k+1)^{\frac{1}{\beta}+1}} \times \Gamma\left(\frac{1}{\beta}+1,(j+k+1)\left(\frac{x}{\alpha}\right)^{\beta}\right).$$

3.4. Reversed residual life function. The waiting time since failure is the waiting time elapsed since the failure of an item on condition that this failure had occurred in $[0,x]$. The rth order moment of the reversed residual life function (rrlf) is given by $M_r(x)=E[(x-T)^r \mid T<x]=\frac{1}{F(x)}\int_0^x (x-t)^r\,f(t)dt$.

Following similar algebra as before, we obtain

$$M_r(x) = \frac{\nu(p-1)}{F(x)\,\log(p)} \sum_{i=0}^{r} \sum_{j,k=0}^{\infty} \frac{(-1)^i\,\alpha^i\,\binom{r}{i}\,(1-p\nu)^j\,(1-\nu)^k\,x^{r-i}}{(j+k+1)^{\frac{i}{\beta}+1}} \times\gamma\left(\frac{i}{\beta}+1,(j+k+1)\left(\frac{x}{\alpha}\right)^{\beta}\right),$$

where $\gamma(a,y)=\int_0^y x^{a-1}\,\mathrm{e}^{-\mathrm{x}}\mathrm{dx}$ is the lower incomplete gamma function.

Then, the mean reversed residual life of T becomes

$$M(x) = x - \frac{\nu\alpha(p-1)}{F(x)\log(p)} \sum_{j,k=0}^{\infty} \frac{(1-p\nu)^j\,(1-\nu)^k}{(j+k+1)^{\frac{i}{\beta}+1}} \times \gamma\left(\frac{i}{\beta}+1,(j+k+1)\left[\frac{x}{\alpha}\right]^{\beta}\right), \tag{3.5}$$

where $M(x)$ represents the mean time elapsed since the failure of T given that it fails at or before x.

4. Mean Deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviations about the mean and the median – defined by $D_1(T)=\int_0^\infty |t-\mu|\,f(t)dt$ and $D_2(T)=\int_0^\infty |t-M|\,f(t)dt$, respectively, where $\mu=E(T)$ is the mean and $M=Q(0.5)$ is the median given by (3.1).

The measures $D_1(T)$ and $D_2(T)$ can be expressed as $D_1(T) = 2\mu F(\mu) - 2Z(\mu)$ and $D_2(T) = \mu - 2Z(M)$, where $Z(x) = \int_0^x t\, f(t)dt$ is the first incomplete mean of T. This integral can be determined from (2.2) by

$$Z(x) = \frac{\nu\alpha(p-1)}{\log(p)} \sum_{i,j=0}^{\infty} \frac{(1-p\nu)^i(1-\nu)^j}{(i+j+1)^{\frac{1}{\beta}+1}} \Gamma\left(\frac{1}{\beta}+1, (i+j+1)\left(\frac{x}{\alpha}\right)^{\beta}\right). \tag{4.1}$$

Thus, the mean deviations $D_1(T)$ and $D_2(T)$ can be obtained from (4.1).

Important applications of (4.1) refer to the Bonferroni and Lorenz curves to study income and poverty, but also in other fields such as reliability, demography, medicine and insurance. For given probability p, they are given by $B(p) = Z(q)/(p\mu)$ and $L(p) = Z(q)/\mu$, respectively, where $q = Q(p)$ comes directly from (3.1).

5. Rényi and Shannon Entropies

The entropy of a random variable T with density function $f(t)$ is a measure of variation of the uncertainty. One of the popular entropy measure is the Rényi entropy given by

$$I_R(\eta) = \frac{1}{1-\eta} \log[\int_{\Re} f^{\eta}(t)dt],$$

where $\eta > 0,\ \eta \neq 1$. The quantity $f^{\eta}(t)$ for T reduces to

$$f^{\eta}(t) = \frac{1}{[\log(p)]^{\eta}} \left\{ \frac{(p-1)^{\eta}\ \left(\frac{\nu\beta}{\alpha}\right)^{\eta}\ \left(\frac{t}{\alpha}\right)^{\eta\beta-\eta}\ \mathrm{e}^{-\eta\left(\frac{t}{\alpha}\right)^{\beta}}}{\left[1-(1-p\ \nu)\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{\eta} \left[1-(1-\nu)\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{\eta}} \right\}. \tag{5.1}$$

Using the power series in equation (5.1), we can write

$$\begin{aligned} f^{\eta}(t) &= \frac{(p-1)^{\eta}}{[\log(p)]^{\eta}} \left(\frac{\nu\beta}{\alpha}\right)^{\eta} \sum_{i,j=0}^{\infty} \frac{\Gamma(\eta+i)\Gamma(\eta+j)}{[\Gamma(\eta)]^2\, j!} (1-p\ \nu)^i (1-\nu)^j \left(\frac{t}{\alpha}\right)^{\eta\beta-\eta} \\ &\times\ \mathrm{e}^{-(\mathrm{i}+\mathrm{j}+\eta)\left(\frac{t}{\alpha}\right)^{\beta}}. \end{aligned}$$

Then, after some calculations, $I_R(\eta)$ reduces to

$$\begin{aligned} I_R(\eta) &= \frac{1}{1-\eta} \log\left[\frac{(p-1)^{\eta}}{[\log(p)]^{\eta}} \nu^{\eta} \left(\frac{\beta}{\alpha}\right)^{\eta-1}\right] \\ &+ \frac{1}{1-\eta} \log\left[\sum_{i,j=0}^{\infty} \frac{\Gamma(\eta+i)\Gamma(\eta+j)}{[\Gamma(\eta)]^2\, j!} \frac{(1-p\ \nu)^i(1-\nu)^j}{(i+j+\eta)^{\eta-\frac{\eta-1}{\beta}}} \Gamma\left(\eta - \frac{\eta-1}{\beta}\right)\right]. \end{aligned}$$

The Shannon entropy, which is defined by $E\{-\log[f(T)]\}$, can be derived numerically from $\lim_{\eta\to 1} I_R(\eta)$.

6. Reliability Function

In the context of reliability, the stress-strength model describes the life of a component which has a random strength T_1 that is subjected to a random stress T_2. The component fails at the instant when the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $T_1 > T_2$. Hence, $R = \Pr(T_2 < T_1)$ is a measure of component reliability. It has many applications especially in engineering concepts such as strength failure and system collapse. Now, we obtain the reliability R when T_1 and T_2 have independent (P-A-L)EW$(\alpha, \beta, \nu_1, p_1)$ and (P-A-L)EW$(\alpha, \beta, \nu_2, p_2)$ distributions with the same shape parameter β and scale parameter α. The reliability R is defined by $R = \int_0^{\infty} f_1(t)\, F_2(t)dx$.

By using the power series $\log(1+x)=\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\,x^i$, $F_2(t)$ can be written as

$$F_2(t)=1-\frac{1}{\log(p_2)}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\,\frac{\nu_2^i(p_2-1)^i\,\mathrm{e}^{-\mathrm{i}\left(\frac{t}{\alpha}\right)^{\beta}}}{\left[1-(1-\nu_2)\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{i}}.$$

By expanding $f_1(t)$ and $F_2(t)$, we can write $f_1(t)\,F_2(t)$ as

$$\begin{aligned} f_1(t)F_2(t) &= f_1(t)-\frac{1}{\log\left(p_1+p_2\right)}\sum_{i=1}^{\infty}\sum_{j,k,l=0}^{\infty}\frac{(-1)^{i+1}\Gamma(i+l)}{\Gamma(i+1)\,l!}\,\frac{\nu_1\beta,\nu_2^i}{\alpha}(p_1-1)\\ &\quad\times(p_2-1)^i\,\left(1-p_1\,\nu_1\right)^j(1-\nu_1)^k(1-\nu_1)^l\,\left(\frac{t}{\alpha}\right)^{\beta-1}\,\mathrm{e}^{-(\mathrm{i+j+k+l})\left(\frac{t}{\alpha}\right)^{\beta}}. \end{aligned}\tag{6.1}$$

Inserting (6.1) into the general expression for R and, after some algebra, we obtain

$$\begin{aligned} R &= 1-\frac{1}{\log\left(p_1+p_2\right)}\sum_{i=1}^{\infty}\sum_{j,k,l=0}^{\infty}\frac{(-1)^{i+1}\Gamma(i+l)\,\nu_1\,(\nu_2)^i}{\Gamma(i+1)\,l!\,(i+j+k+l)}\\ &\quad\times(p_1-1)\,(p_2-1)^i\,(1-p_1\nu_1)^j\,(1-\nu_1)^k(1-\nu_1)^l. \end{aligned}$$

7. Reliability Analysis

Here, we present the reliability function, hrf, chrf and mrlf of T.

7.1. Survival function. The (P-A-L)EW distribution can be a useful characterization of lifetime data analysis for a given system. Its survival function is

$$\overline{F}(t)=\frac{1}{\log(p)}\log\left\{\frac{1-(1-p\,\nu)\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{1-(1-\nu)\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}\right\}.$$

Figure 3 illustrates the survival behavior of the new distribution for some parameter values.

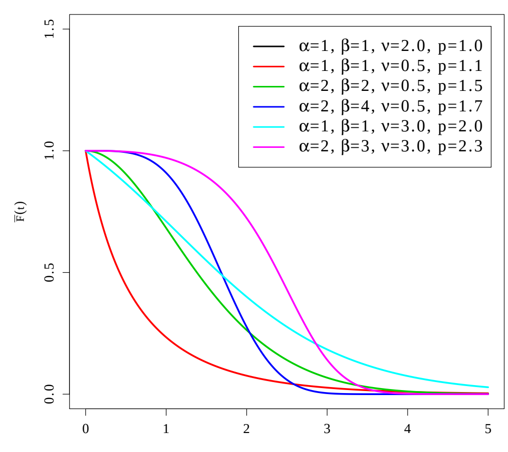


Figure 3. The survival function of the (P-A-L)EW distribution.

7.2. Hazard rate function. The hrf of T is given by

$$h(t) = \frac{(p-1)\frac{\nu\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}} \left[1-(1-\mathrm{p}\nu)\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{-1}}{\left[1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right] \log\left\{\frac{1-(1-p\,\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}\right\}}.$$

We note that $h(t)$ can be constant, increasing, or decreasing depending on the parameter values. For example, if $p \to 1$, $\nu = 1$ and $\beta = 1$, then $h(t) = \frac{1}{\alpha}$ is constant, whereas if $p \to 1$ and $\nu = 1$, then $h(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}$, which is increasing for $\beta > 1$ and decreasing for $\beta < 1$. Figure 4 displays some plots of the hrf of T.

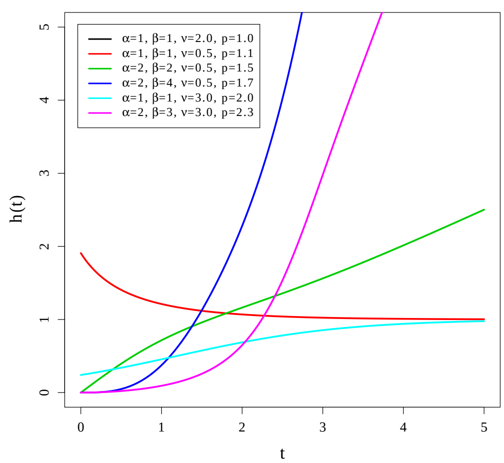


Figure 4. The hrf of the (P-A-L)EW distribution

7.3. Cumulative hazard rate function. Many generalized Weibull models have been proposed in reliability literature through the relationship between the reliability function $R(t)$ and the chrf $H(t)$, which is a non–decreasing function of t, given by $H(t) = -\log[R(t)]$. The chrf of T becomes

$$H(t) = \int_0^t h(u)du = \log(\log p) - \log\left\{\log\left[\frac{1-(1-p\,\nu)\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{1-(1-\nu)\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}\right]\right\},$$

where $H(t)$ is the total number of failures or deaths over an interval of time. Figure 5 illustrates the behavior of the chrf of T for some parameter values.

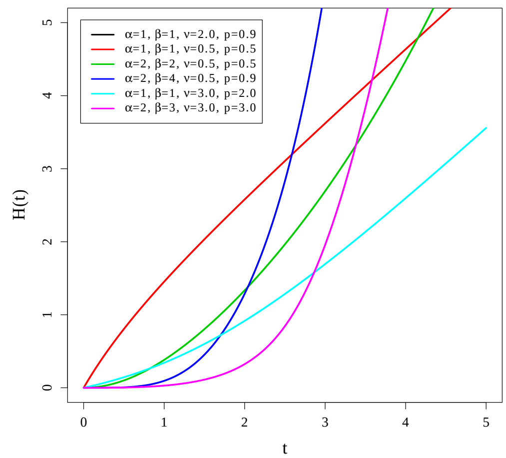


Figure 5. Cumulative Hazard Rate Function.

7.4. Mean residual lifetime function. The additional life time given that the component has survived up to time t is the rlf of the component. Then, the expectation of the random variable T_t represents the remaining lifetime reduces to

$$m(t) = E(T_t) = E(T - t \mid T > t) = \frac{\int_t^\infty R(u)du}{R(t)} .$$

The mrlf and the hrf are important since they characterize uniquely the corresponding lifetime distribution. We obtain

$$\begin{aligned} m(t) \quad &= \quad -t + \frac{\alpha}{\log\left[\frac{1-(1-p\,\nu)\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^\beta}}{1-(1-\nu)\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^\beta}}\right]} \sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{1}{\beta}+1,(j+1)\left(\frac{t}{\alpha}\right)^\beta\right)}{(i+1)^{\frac{1}{\beta}+1}} \\ &\quad \times \left\{(1-\nu)^{i+1} - (1-p\nu)^{i-1}\right\}. \end{aligned}$$

7.5. Order Statistics. Let $T_1, \ldots, T_n$ denote n independent random variables from a distribution function $F(t)$ with pdf $f(t)$, and $T_{(1)}, \ldots, T_{(n)}$ denote the order sample arrangement. So, the pdf of $T_{(j)}$ is given by

$$f_{T_{(j)}}(t) = \frac{n!}{(j-1)!(n-j)!}\, f(t)\, F(t)^{j-1} \left[1 - F(t)\right]^{n-j} \quad \text{for} \;\; j = 1, \ldots, n.$$

Using equations (2.1) and (2.2), the pdf of $T_{(j)}$ becomes

$$\begin{aligned} f_{T_{(j)}}(t) &= \frac{n!}{(j-1)!(n-j)!} \times \left\{1 - \frac{1}{\log p} \log\left[\frac{1-(1-p\,\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}\right]\right\}^{j-1} \\ &\times \frac{(p-1)\frac{\nu\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{[\log(p)]^{n-j+1}\left[1-(1-p\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]\left[1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]} \\ &\times \left\{\log\left[\frac{1-(1-p\,\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}\right]\right\}^{n-j}. \end{aligned}$$

Therefore, the pdf's of the smallest order statistic $T_{(1)}$ and of the largest order statistic $T_{(n)}$ are easily obtained from the last equation with $i = 1$ and $i = n$, respectively. Then, the minimum and maximum order statistics can be derived for some special models of the new distribution. For example, for the (P-A-L)ER ($\beta = 2$), (P-A-L)EE model ($\beta = 1$), (P-A-L)W ($\nu = 1$), (P-A-L)R ($\nu = 1$ and $\beta = 2$), (P-A-L)E ($\nu = 1$ and $\beta = 1$) and EW ($p \to 1$) distributions, among others.

The pdf's of the $(k+1)$th and kth ordered statistics from the (P-A-L)EW model obey the relationship

$$f_{T_{(k+1)}}(t) = \left(\frac{n-k}{k}\right) \frac{1 - \frac{1}{\log(p)} \log\left[\frac{1-(1-p\ \nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}\right]}{\frac{1}{\log(p)} \log\left[\frac{1-(1-p\ \nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}{1-(1-\nu)\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^{\beta}}}\right]}\, f_{T_{(k)}}(t).$$

8. Estimation of the Parameters

Inference can be carried out in three different ways: point estimation, interval estimation and hypothesis testing. Several approaches for parameter point estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimates (MLEs) enjoy desirable properties and can be used in constructing confidence intervals and also in test-statistics. Large sample theory for these estimates delivers simple approximations that work well in finite samples. Statisticians often seek to approximate quantities such as the density of a test-statistic that depend on the sample size in order to obtain better approximate distributions. The resulting approximation for the MLEs in distribution theory is easily handled either analytically or numerically. In this section, we use the method of likelihood to estimate the model parameters and use them to obtain confidence intervals for the unknown parameters.

8.1. Maximum Likelihood Estimation. Let $t_1, \ldots, t_n$ be a sample of size n from the (P-A-L)EW distribution. Let $\theta = (\alpha, \beta, \nu, p)^T$ be the parameter vector. Then, the log-likelihood function $\ell = \ell(\theta)$ is given by

$$\begin{aligned} \ell &= n\log[\log(p)] + n\log(p-1) - n\log\left(\frac{\nu\beta}{\alpha}\right) + (\beta-1)\sum_{i=1}^{n}\log\left(\frac{t_i}{\alpha}\right) \\ &- \sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta} + \sum_{i=1}^{n}\log\left[1-(1-p\nu)\mathrm{e}^{-\left(\frac{t_i}{\alpha}\right)^{\beta}}\right] - \sum_{i=1}^{n}\log\left[1-(1-\nu)\mathrm{e}^{-\left(\frac{t_i}{\alpha}\right)^{\beta}}\right]. \end{aligned}$$

Then, the MLE of θ can be derived from the derivatives of ℓ. They should satisfy the following equations:

$$
\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= -\frac{n\beta}{\alpha} + \frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta} + \frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta} \mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}} \\
&\quad \times \left[\frac{1-p\nu}{\left[1-(1-p\nu)\mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}}\right]} + \frac{(1-\nu)}{\left[1-(1-\nu)\mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}}\right]}\right] = 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n}\left[1-\left(\frac{t_i}{\alpha}\right)^{\beta}\right]\log\left(\frac{t_i}{\alpha}\right)^{\beta} - \sum_{i=1}^{n}\mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}}\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}\log\left(\frac{\mathrm{t_i}}{\alpha}\right) \\
&\quad \times \left[\frac{(1-p\nu)}{\left[1-(1-p\nu)\mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}}\right]} + \frac{(1-\nu)}{\left[1-(1-\nu)\mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}}\right]}\right] = 0,
\end{aligned}
$$

$$
\frac{\partial \ell}{\partial \nu} = \frac{n}{\nu} - p\sum_{i=1}^{n}\frac{\mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}}}{\left[1-(1-p\nu)\mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}}\right]} - p\sum_{i=1}^{n}\frac{\mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}}}{\left[1-(1-\nu)\mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}}\right]} = 0
$$

and

$$
\frac{\partial \ell}{\partial p} = \frac{-n}{p\log p} + \frac{-n}{p-1} - \nu\sum_{i=1}^{n}\frac{\mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}}}{\left[1-(1-p\nu)\mathrm{e}^{-\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}}\right]} = 0.
$$

These equations cannot be solved analytically, and statistical softwares are required to solve them numerically. To solve these equations, it is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function. For interval estimation of the parameters, we obtain the 3×3 observed information matrix $J(\theta) = \{\frac{\partial^2 \ell}{\partial r\, s}\}$ (for $r, s = \alpha, \beta, \nu, p$), whose elements can be computed numerically.

Under standard regularity conditions when $n \to \infty$, the distribution of the MLE can be approximated by a multivariate normal $N_4(0, J(\widehat{\theta})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\theta})$ is the total observed information matrix evaluated at $\widehat{\theta}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained through the bootstrap percentile method.

9. Particle Swarm Optimization

In computer science, the particle swarm optimization (PSO) is a computational method for optimization of parametric and multiparametric functions. The PSO algorithm is a meta-heuristic which has been providing good solutions for problems of optimization global functions with box-constrained. The use of meta-heuristic methods such as PSO has proved to be useful for maximizing complicated log-likelihood functions without the need for early kick functions as the BFGS, L-BFGS-B, Nelder-Mead and simulated annealing methods. As in most heuristic methods that are inspired by biological phenomena, the PSO is inspired by the behavior of flying birds. The philosophical idea of the PSO algorithm is based on the collective behavior of birds (particle) in search of food (point of global optimal). The PSO technique was first defined by [6] in a paper published in the Proceedings of the IEEE International Conference on Neural Networks IV.

A modification of the PSO algorithm was proposed by [16] published in the Proceedings of IEEE International Conference on Evolutionary Computation. Further details on the philosophy of the PSO method are given in the book Swarm Intelligence (see [8]).

The PSO optimizes a problem by having a population of candidate solutions and moving these particles around in the search-space according to simple mathematical formulae over the particle's position and velocity. The movement of the particles in the search space is randomized. Each iteration of the PSO algorithm, there is a leader particle, which is the particle that minimizes the objective function in the respective iteration. The remaining particles arranged in the search region will follow the leader particle randomly and sweep the area around this leading particle. In this local search process, another particle may become the new leader particle and the other particles will follow the new leader randomly. Each particle arranged in the search region has a velocity vector and position vector and its movement in the search region is given by changes in these vectors. The PSO algorithm is presented below, where $f : \mathbb{R}^n \mapsto \mathbb{R}$ is the objective function to be minimized, S is the number of particles in the swarm (set of feasible points, i.e. search region), each having particle a vector position $x_i \in \mathbb{R}^n$ in the search-space and a vector velocity defined by $v_i \in \mathbb{R}^n$. Let p_i be the best known position of particle i and g the best position of all particles.

(1) For each particle $i = 1, \ldots, S$ do:
- Initialize the particle's position with a uniformly distributed random vector: $x_i \sim U(b_{lo}, b_{up})$, where b_{lo} and b_{up} are the lower and upper boundaries of the search-space.
- Initialize the particle's best known position to its initial position: $p_i \hookleftarrow x_i$.
- If $f(p_i) < f(g)$ update the swarm's best known position: $g \hookleftarrow p_i$.
- Initialize the particle's velocity: $v_i \sim U(-|b_{up} - b_{lo}|, |b_{up} - b_{lo}|)$.

(2) Until a termination criterion is met (e.g. number of iterations performed, or a solution with adequate objective function value is found), repeat:
- For each particle $i = 1, \ldots, S$ do:
  - Pick random numbers: $r_p, r_g \sim U(0, 1)$.
  - For each dimension $d = 1, \ldots, n$ do:
    - Update the particle's velocity: $v_{i,d} \hookleftarrow \omega\, v_{i,d} + \varphi_p r_p (p_{i,d} - x_{i,d}) + \varphi_g r_g (g_d - x_{i,d})$.
  - Update the particle's position: $x_i \hookleftarrow x_i + v_i$
  - If $f(x_i) < f(p_i)$ do:
    - Update the particle's best known position: $p_i \hookleftarrow x_i$
    - If $f(p_i) < f(g)$ update the swarm's best known position: $g \hookleftarrow p_i$.

(3) Now g holds the best found solution.

The parameter ω is called inertia coefficient and as the name implies controls the inertia of each particle arranged in the search region. The quantities ω_p and ω_g control the acceleration of each particle and are called acceleration coefficients.

10. Application

We consider an application using the (P-A-L)EW distribution. We use the `AdequacyModel` script version 1.0.8 available for the programming language `R`. The script is currently maintained by one of the authors of this paper and more information can be obtained f rom http://cran.rstudio.com/web/packages/AdequacyModel/index.html. The package is distributed under the terms of the licenses GNU General Public License (GPL-2 or GPL-3).

The application take into account the data relating to the percentage of body fat determined by underwater weighing and various body circumference measurements for 250 men. For details about the data set, see http://lib.stat.cmu.edu/datasets/.

Table 2. Descriptive statistics.

Statistics	Real data sets
	Body Fat (%)
Mean	19.3012
Median	19.2500
Mode	22.5000
Variance	67.7355
Skewness	0.1953
Kurtosis	-0.3815
Maximum	47.5000
Minimum	3.0000
n	250

In order to determine the shape of the most appropriate hazard function for modeling, graphical analysis data may be used. In this context, the total time in test (TTT) plot proposed by [1] is very useful. Let T be a random variable with non-negative values which represents the survival time. The TTT curve is obtained by constructing the plot of $G(r/n) = [(\sum_{i=1}^{r} T_{i:n}) + (n-r)T_{r:n}]/(\sum_{i=1}^{n} T_{i:n})$ versus r/n, for $r = 1, \ldots, n$ and $T_{i:n}$ $(i = 1, \ldots, n)$ are the order statistics of the sample (see [11]). The plots can be easily obtained using the function `TTT` of the script `AdequacyModel`. For more details on this function, see `help(TTT)`. The TTT plot for the current data is displayed in Figure 6, which is concave and according to [1] provides evidence that the monotonic hrf is adequate.

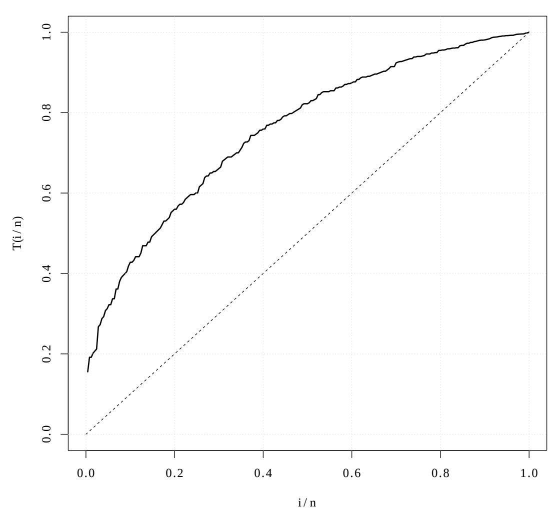


Figure 6. The TTT plot for percentage of body fat.

Figure 7 displays the estimated density to the data obtained in a nonparametric manner using kernel density estimation with the Gaussian filter. Let $X_1, \ldots, X_n$ be a random vector of independent and identically distributed random variables, when each random variable follows an unknown pdf f. The kernel density estimator is given by

$$\hat{f}_h(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - x_i) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right), \tag{10.1}$$

where $K(\cdot)$ is the kernel function usually symmetrical and $\int_{-\infty}^{\infty} K(x)dx = 1$. Here, $h > 0$ is a smoothing parameter known in literature as bandwidth. Numerous kernel functions are adopted in the literature. The normal standard distribution is the most widely used because it has convenient mathematical properties. [17] demonstrated that for the K standard normal, the bandwidth ideal is $h = \left(\frac{4\hat{\sigma}^5}{3n}\right)^{\frac{1}{5}} \approx 1.06\,\hat{\sigma}\,n^{-1/5}$, where $\hat{\sigma}$ is the standard deviation of the sample.

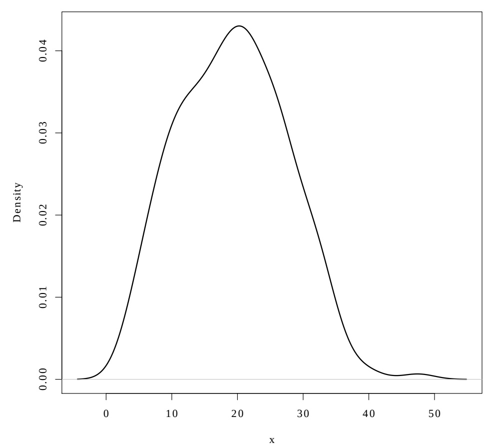


Figure 7. Gaussian kernel density estimation for percentage of body fat.

In order to verify which distribution fits better these data, we consider the Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics described by [5]. Chen and Balakrishnan (see [5]) constructed the Cramér-von Mises and Anderson-Darling corrected statistics based on the suggestions from [18]. We use these statistics, where we have a random sample $(x_1, \ldots, x_n)$ with empirical distribution function $F_n(x)$ and we want to test if the sample comes from a special distribution. The Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics are, respectively, given by

$$\begin{aligned}
W^* &= \left\{n\int_{-\infty}^{+\infty}\{F_n(x) - F(x;\widehat{\theta}_n)\}^2 dF(x;\widehat{\theta}_n)\right\}\left(1 + \frac{0.5}{n}\right) = W^2\left(1 + \frac{0.5}{n}\right),\\
A^* &= \left\{n\int_{-\infty}^{+\infty}\frac{\{F_n(x) - F(x;\widehat{\theta}_n)\}^2}{\{F(x;\widehat{\theta})(1 - F(x;\widehat{\theta}_n))\}}dF(x;\widehat{\theta}_n)\right\}\left(1 + \frac{0.75}{n} + \frac{2.25}{n^2}\right)\\
&= A^2\left(1 + \frac{0.75}{n} + \frac{2.25}{n^2}\right),
\end{aligned}$$

where $F_n(x)$ is the empirical distribution function, $F(x;\hat{\theta}_n)$ is the postulated distribution function evaluated at the MLE $\hat{\theta}_n$ of θ. Note that the statistics W^* and A^* are given by the differences of $F_n(x)$ and $F(x;\hat{\theta}_n)$. Thus, the lower are the statistics W^* and A^* more evidence we have that $F(x;\hat{\theta}_n)$ generates the sample. The details to compute the statistics W^* and A^* are given by Chen and Balakrishnan.

The `goodness.fit` function provides various adequacy of fit statistics, among them, the Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics proposed by [5], Consistent Akaike Information Criterion (CAIC) defined by [2], Bayesian Information Criterion (BIC) defined by [15] and Hannan-Quinn Information Criterion (HQIC) given by [7]. These statistics are used to assess the adequacy of the fit of the distributions considered in the two real data sets.

The PSO methodology was used for the improvement of the MLEs. Initially, we use the Nelder-Mead method to maximize the log-likelihood function of the models under study using the `goodness.fit` function of the script `AdequacyModel`. After obtaining convergence using the Nelder-Mead method (see [12]), we use the PSO method as an attempt to obtain best candidates for global maximums of their log-likelihood functions for the compared models. We consider $S = 550$ (550 particles) and 500 iterations as stopping criterion. We choose as optimal candidates for the estimates, those MLEs calculated by the PSO method when ℓ (the maximized log-likelihood function for the current model) is higher than the log-likelihood function evaluated at the estimates computed by the Nelder-Mead method. Figure 8 displays the fitted densities to the current data. The MLEs used in Figure 8 are highlighted in Table 3. It is noted in Table 4 that the proposed distribution provides the best fit to the data.

Table 3. MLEs obtained by Nelder-Mead and PSO methods.

Distributions		Estimates				ℓ
(P-A-L)EW	PSO	1.8571	0.7700	63.4424	37.5844	871.0364
	Nelder-Mead	**19.6993**	**2.5831**	**0.2865**	**28.0810**	**874.7802**
Kw-W	PSO	**71.3501**	**77.4079**	**0.1635**	**25.1942**	**888.7122**
	Nelder-Mead	0.6960	2.0492	3.3057	0.0314	875.8679
Exp-W	PSO	45.30062	69.5828	55.3411	-	875.8749
	Nelder-Mead	**0.0418**	**3.0356**	**0.7436**	**-**	**870.6432**
Weibull	PSO	**21.7567**	**2.5373**	**-**	**-**	**876.4216**
	Nelder-Mead	21.7552	2.5371	-	-	876.1854
Gamma	PSO	**5.8060**	**0.3036**	**-**	**-**	**888.5930**
	Nelder-Mead	4.6090	0.2388	-	-	884.6877

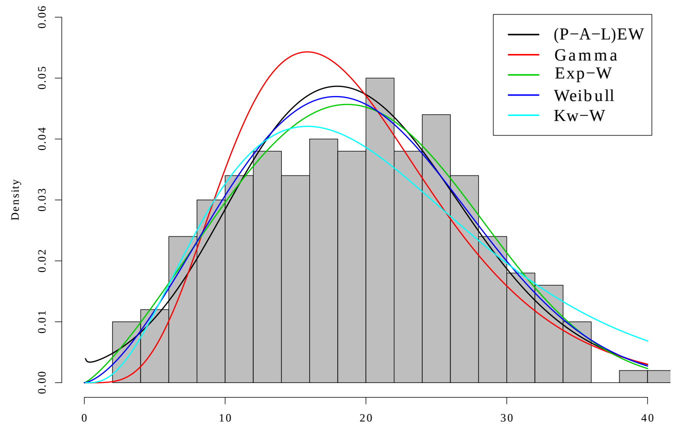


Figure 8. Fitted densities to the percentage of body fat data.

Table 4. Statistics of adequacy to adjust.

Distributions	AIC	CAIC	BIC	HQIC	A^*	W^*
(P-A-L)EW	1757.560	1757.724	1771.646	1763.230	**0.1192**	**0.0144**
Kw-W	1785.429	1785.592	1799.515	1791.098	1.8205	0.3005
Exp-W	1757.750	1757.847	1768.314	1762.002	0.2477	0.0334
Weibull	1756.843	1756.892	1763.886	1759.678	0.4357	0.0668
Gamma	1781.186	1781.235	1788.229	1784.021	1.9548	0.3233

11. Concluding Remarks

The idea of generating new extended models from classic ones has been of great interest among researchers in the past decade. A new four-parameter generalization of the Weibull model, called the (P-A-L) extended Weibull, (P-A-L)EW for short, distribution is defined and some of its mathematical properties studied. They include moments, generating, quantile, reliability and residual life functions, mean deviations and two types of entropies. Many well-known distributions emerge as special cases of the proposed distribution by using special parameter values. We use maximum likelihood and a particle swarm optimization method to estimate the model parameters. By means of a real data set, we prove that this model has the capability to provide consistent estimates from the considered estimation methods.

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Family of generalized gamma distributions: Properties and applications

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Abstract

In this paper, a family of generalized gamma distributions, T-gamma family, has been proposed using the T-$R\{Y\}$ framework. The family of distributions is generated using the quantile functions of uniform, exponential, log-logistic, logistic and extreme value distributions. Several general properties of the T-gamma family are studied in details including moments, mean deviations, mode and Shannon's entropy. Three new generalizations of the gamma distribution which are members of the T-gamma family are developed and studied. The distributions in the T-gamma family are very flexible due to their various shapes. The distributions can be symmetric, skewed to the right, skewed to the left, or bimodal. Four data sets with various shapes are fitted by using members of the T-gamma family of distributions.

Keywords: T-$R\{Y\}$ framework, quantile function, Shannon's entropy, bimodality.

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1. Introduction

The origin of gamma distribution, from the book by Johnson et al. (1994, p. 343), can be attributed to Laplace (1836) who obtained a gamma distribution as the distribution of a “precision constant". The gamma distribution has been used to model waiting times. For example in life testing, the waiting time until “death" is a random variable that has a gamma distribution (Hogg et al. 2013, p. 156). The gamma distribution is used in Bayesian statistics, where it is used as a conjugate prior distribution for various types of scale parameters such as the parameter θ in an exponential distribution or a normal distribution with a known mean. Other applications include the size of insurance claims (Boland, 2007), hydrology (Aksoy, 2000), and bacterial gene expression (Friedman et al. 2006). For other types of applications, see for example the works of Costantino and Desharnais (1981), Dennis and Patil (1984), and Johnson et al. (1994, Chapter 17) and the references therein.

The early generalization of gamma distribution can be traced back to Amoroso (1925) who discussed a generalized gamma distribution and applied it to fit income rates. Johnson et al. (1994, Chapter 8) gave a four parameter generalized gamma distribution which reduces to the generalized gamma distribution defined by Stacy (1962) when the location parameter is set to zero. Mudholkar and Srivastava (1993) introduced the exponentiated method to derive a distribution. The generalized gamma defined by Stacy (1962) is a three-parameter exponentiated gamma distribution. Agarwal and Al-Saleh (2001) applied generalized gamma to study hazard rates. Balakrishnan and Peng (2006) applied this distribution to develop generalized gamma frailty model. Cordeiro et al. (2012) derived another generalization of Stacy’s generalized gamma distribution using exponentiated method, and applied it to life time and survival analysis. Nadarajah and Gupta (2007) proposed another type of generalized gamma distribution with application to fitting drought data.

Eugene et al. (2002) introduced the beta-generated family of distributions and since then, many variants of this family have been studied. Based on the beta-generated family and its variants, more generalized gamma distributions have been defined and studied. Some examples are the beta-gamma distribution by Kong et al. (2007), the Kumaraswamy-gamma distribution by Cordeiro and de Castro (2011), the Kumaraswamy-generalized gamma distribution by de Pascoa et al. (2011), and the beta generalized gamma distribution by Cordeiro et al. (2013).

The beta-generated family was extended by Alzaatreh et al. (2013) to the T-$R(W)$ family. The cumulative distribution function (CDF) of the T-$R(W)$ distribution is $G(x) = \int_a^{W(F(x))} r(t)dt$, where $r(t)$ is the probability density function (PDF) of a random variable T with support (a, b) for $-\infty \leq a < b \leq \infty$. The function $W(F(x))$ of the CDF $F(x)$ is monotonic and absolutely continuous. Aljarrah et al. (2014) considered the function $W(F(x))$ to be the quantile function of a random variable Y and defined the T-$R\{Y\}$ family. This framework can be applied to derive generalized families of any existing distribution.

Some generalizations of the gamma distribution that fall into the T-$R\{Y\}$ framework include the family of generalized gamma-generated distributions by Zografos and Balakrishnan (2009), the gamma-Pareto distribution by Alzaatreh et al. (2012) and the gamma-normal distribution by Alzaatreh et al. (2014a). These distributions belong to the gamma-R\{exponential\} family. Various applications to biological data, lifetime data, hydrological data and others were provided in these literatures. For a review of methods for generating continuous distributions, one may refer to Lee et al. (2013).

Various distributions in the T-$R\{Y\}$ family have been studied in the literature. The distributions, in general, have more parameters which add more flexibility to their usefulness. These distributions have shown their usefulness in many fields. They have been applied in many areas and found to provide better fit to complex real life situations. Examples include the following: the beta-normal (Eugene et al., 2002) was applied to bimodal data; the Kumaraswamy-Weibull (Cordeiro et al., 2010) was applied to model failure time data; the beta-Weibull (Famoye et al., 2005), the beta Pareto (Akinsete et al., 2008) and the beta generalized Pareto (Mahmoudi, 2011) were applied to model flood data.

This article focuses on the generalization of the gamma distribution using the T-gamma$\{Y\}$ framework and studies some new distributions in this family and their applications. Section 2 gives a brief review of the T-$R\{Y\}$ framework, defines several new generalized gamma sub-families. Section 3 gives some general properties of the T-gamma$\{Y\}$ distributions. Section 4 develops several new T-gamma$\{Y\}$ distributions and derives some properties. Section 5 gives some applications. Summary and conclusions are given in section 6.

2. The T-gamma$\{Y\}$ family of distributions

The T-$R\{Y\}$ framework defined in Aljarrah et al. (2014) (see also Alzaatreh et al., 2014b) is briefly described in the following. Let T, R and Y be random variables with CDF $F_T(x) = P(T \leq x)$, $F_R(x) = P(R \leq x)$, $F_Y(x) = P(Y \leq x)$ and corresponding quantile functions $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$, where the quantile function is defined as $Q_Z(p) = \inf\{z : F_Z(z) \geq p\}$, $0 < p < 1$. If densities exist, we denote them by $f_T(x)$, $f_R(x)$ and $f_Y(x)$. Now assume the random variables T, $Y \in (a,\ b)$ for $-\infty \leq a < b \leq \infty$. The random variable X in T-$R\{Y\}$ family of distributions is defined as

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = F_T(Q_Y(F_R(x))). \tag{2.1}$$

The corresponding PDF associated with (2.1) is

$$f_X(x) = f_T(Q_Y(F_R(x))) \times Q'_Y(F_R(x)) \times f_R(x). \tag{2.2}$$

Alternatively, (2.2) can be written as

$$f_X(x) = f_R(x) \times \frac{f_T(Q_Y(F_R(x)))}{f_Y(Q_Y(F_R(x)))}. \tag{2.3}$$

The hazard function of the random variable X can be written as

$$h_X(x) = h_R(x) \times \frac{h_T(Q_Y(F_R(x)))}{h_Y(Q_Y(F_R(x)))}. \tag{2.4}$$

Alzaatreh et al. (2013) studied the T-$R\{$exponential$\}$ distributions. Aljarrah et al. (2014) studied the general framework and some properties of T-$R\{Y\}$.

Let R be a gamma random variable with PDF $f_R(x) = \beta^{-\alpha}(\Gamma(\alpha))^{-1}x^{\alpha-1}e^{-x/\beta}$, $x > 0$ and CDF $F_R(x) = \beta^{-\alpha}(\Gamma(\alpha))^{-1}\int_0^x t^{\alpha-1}e^{-t/\beta}dt$, then (2.2) reduces to

$$\begin{aligned} f_X(x) &= \frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta} \times \frac{f_T(Q_Y(F_R(x)))}{f_Y(Q_Y(F_R(x)))} \\ &= \text{gamma}(\alpha, \beta) \times \frac{f_T(Q_Y(F_R(x)))}{f_Y(Q_Y(F_R(x)))}. \end{aligned} \tag{2.5}$$

Gamma(α, β) is the PDF of gamma random variable. Hereafter, the family of distributions in (2.5) will be called the T-gamma$\{Y\}$ family and it will be denoted by T-$G\{Y\}$.

It is clear that the PDF in (2.5) is a generalization of gamma distribution. For consistency, the notation $f_G(x)$ and $F_G(x)$ will respectively be used in place of $f_R(x)$ and $F_R(x)$ for the gamma random variable in the remaining sections. From (2.1), if $T \stackrel{d}{=} Y$, then $X \stackrel{d}{=}$ gamma(α β). Also, if $Y \stackrel{d}{=}$ gamma(α β), then $X \stackrel{d}{=} T$.

Various existing generalizations of the gamma distributions can be seen as members of T-$G\{Y\}$ family. When $T \sim$ beta(a, b) and $Y \sim$ uniform(0, 1), the T-$G\{Y\}$ reduces to the beta-gamma distribution (Kong et al., 2007). When $T \sim$ Power(a) and $Y \sim$ uniform(0, 1), the T-$G\{Y\}$ reduces to the exponentiated-gamma distribution (Nadarajah and Kotz, 2006) and when $T \sim$ Kumaraswamy(a, b) and $Y \sim$ uniform(0, 1), the T-$G\{Y\}$ reduces to the Kumaraswamy-gamma distribution (Cordeiro and de Castro, 2011). Table 1 gives five quantile functions of known distributions which will be applied to generate T-$G\{Y\}$ sub-families in the following subsections.

Table 1. Quantile functions for different Y distributions

Y	$Q_Y(p)$
(a) Uniform	p
(b) Exponential	$-b\log(1-p),\ \ b>0$
(c) Log-logistic	$a(p/(1-p))^{1/b},\ \ a,\, b>0$
(d) Logistic	$a+b\log[p/(1-p)],\ \ b>0$
(e) Extreme value	$a+b\log[-\log(1-p)],\ \ b>0$

2.1. T-gamma{uniform} family of distributions (T-G{uniform}). By using the quantile function of the uniform distribution in Table 1, the corresponding CDF to (2.1) is

$$F_X(x) = F_T\left\{F_G(x)\right\}, \tag{2.6}$$

and the corresponding PDF to (2.6) is

$$\begin{aligned} f_X(x) &= \frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta} \times f_T\left(F_G(x)\right) \\ &= \text{gamma}(\alpha,\beta) \times f_T\left(F_G(x)\right), \quad x>0. \end{aligned} \tag{2.7}$$

2.2. T-gamma{exponential} family of distributions (T-G{exponential}). By using the quantile function of the exponential distribution in Table 1, the corresponding CDF to (2.1) is

$$F_X(x) = F_T\left\{-b\log(1-F_G(x))\right\}, \tag{2.8}$$

and the corresponding PDF to (2.8) is

$$\begin{aligned} f_X(x) &= \frac{b}{\beta^\alpha\Gamma(\alpha)(1-F_G(x))}x^{\alpha-1}e^{-x/\beta} \times f_T\left(-b\log\left(1-F_G(x)\right)\right) \\ &= \text{gamma}(\alpha,\beta) \times \frac{b}{(1-F_G(x))} \times f_T\left(-b\log\left(1-F_G(x)\right)\right), \quad x>0. \end{aligned} \tag{2.9}$$

Note that the CDF and the PDF in (2.8) and (2.9) can be written as $F_X(x) = F_T\left(-bH_G(x)\right)$ and $f_X(x) = bh_G(x)f_T\left(-bH_G(x)\right)$ where $h_G(x)$ and $H_G(x)$ are the hazard and cumulative hazard functions for the gamma distribution, respectively. Therefore, the T-G{exponential} family of distributions arises from the 'hazard function of the gamma distribution'.

2.3. T-gamma{log-logistic} family of distributions (T-G{log-logistic}). By using the quantile function of the log-logistic distribution in Table 1, the corresponding CDF to (2.1) is

$$F_X(x) = F_T\left\{a(F_G(x)/[1-F_G(x)])^{1/b}\right\}, \tag{2.10}$$

and the corresponding PDF is

$$\begin{aligned} f_X(x) &= \frac{a}{b\beta^\alpha\Gamma(\alpha)}\frac{x^{\alpha-1}e^{-x/\beta}}{(1-F_G(x))^2}\left(\frac{F_G(x)}{1-F_G(x)}\right)^{1/b-1} f_T\left(a\left(\frac{F_G(x)}{1-F_G(x)}\right)^{1/b}\right) \\ &= \frac{a\cdot \text{gamma}(\alpha,\beta)}{b(1-F_G(x))^2}\left[\frac{F_G(x)}{1-F_G(x)}\right]^{1/b-1} f_T\left\{a\left[\frac{F_G(x)}{1-F_G(x)}\right]^{1/b}\right\},\ x>0. \quad (2.11)\end{aligned}$$

Note that if $a = b = 1$, (2.11) reduces to

$$f_X(x) = \frac{gamma(\alpha,\beta)}{(1-F_G(x))^2} \times f_T\left(F_G(x)/[1-F_G(x)]\right),\ x>0,$$

which is a family of generalized gamma distributions arising from the 'odds' of the gamma distribution.

2.4. T-gamma{logistic} family of distributions (T-G{logistic}). By using the quantile function of the logistic distribution in Table 1, the corresponding CDF to (2.1) is

$$F_X(x) = F_T\left\{a + b\log\left(F_G(x)/[1-F_G(x)]\right)\right\}, \tag{2.12}$$

and the corresponding PDF is

$$f_X(x) = \frac{bx^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)F_G(x)[1-F_G(x)]} f_T\left(a + b\log\left(\frac{F_G(x)}{1-F_G(x)}\right)\right),\ x>0. \tag{2.13}$$

Note that if $a = 0$ and $b = 1$, (2.13) reduces to

$$f_X(x) = \frac{h_G(x)}{F_G(x)} \times f_T\left(\log\left(\frac{F_G(x)}{1-F_G(x)}\right)\right), \quad x>0,$$

which is a family of generalized gamma distributions arising from the 'logit function' of the gamma distribution.

2.5. T-gamma{extreme value} family of distributions (T-G{extreme value}). By using the quantile function of the extreme value distribution in Table 1, the corresponding CDF to (2.1) is

$$F_X(x) = F_T\left\{a + b\log(-\log[1-F_G(x)])\right\}, \tag{2.14}$$

and the corresponding PDF is

$$f_X(x) = \frac{bx^{\alpha-1}e^{-x/\beta} f_T\left\{a + b\log[-\log(1-F_G(x))]\right\}}{\beta^\alpha\Gamma(\alpha)[F_G(x)-1]\log(1-F_G(x))},\ x>0. \tag{2.15}$$

The CDF in (2.14) and the PDF in (2.15) can be written as

$$F_X(x) = F_T\left(a + b\log H_G(x)\right)$$

and

$$f_X(x) = b\left\{h_G(x)/H_G(x)\right\} f_T\left(a + b\log H_G(x)\right)$$

respectively.

3. Some properties of the T-$G\{Y\}$ family of distributions

In this section, we discuss some general properties of the T-gamma family of distributions in detail. We omit the proof for some straightforward results.

3.1. Lemma. *Let T be a random variable with PDF $f_T(x)$, then the random variable $X = Q_G(F_Y(T))$, where $Q_G(.)$ is the quantile function of gamma(α, β), follows the T-gamma$\{Y\}$ distribution.*

3.2. Corollary. *Based on Lemma 3.1, we have*

(i) $X = Q_G(T)$ *follows the distribution of T-$G\{$uniform$\}$ family.*
(ii) $X = Q_G(1 - e^{-T/b})$ *follows the distribution of T-$G\{$exponential$\}$ family.*
(iii) $X = Q_G\left(\left[1 + (T/a)^{-b}\right]^{-1}\right)$ *follows the distribution of T-$G\{$log-logistic$\}$ family.*
(iv) $X = Q_G\left(\left[1 + e^{-(T-a)/b}\right]^{-1}\right)$ *follows the distribution of T-$G\{$logistic$\}$ family.*
(v) $X = Q_G\left(1 - e^{-e^{(T-a)/b}}\right)$ *follows the distribution of T-$G\{$extreme value$\}$ family.*

3.3. Lemma. *The quantile functions for T-gamma$\{Y\}$ family is given by $Q_X(p) = Q_G(F_Y(Q_T(p)))$.*

3.4. Corollary. *Based on Lemma 3.3, the quantile function for the (i) T-$G\{$uniform$\}$, (ii) T-$G\{$exponential$\}$, (iii) T-$G\{$log-logistic$\}$, (iv) T-$G\{$logistic$\}$ and (v) T-$G\{$extreme value$\}$, are respectively,*

(i) $Q_X(p) = Q_G\left(Q_T(p)\right)$,
(ii) $Q_X(p) = Q_G\left(1 - e^{-b^{-1}Q_T(p)}\right)$,
(iii) $Q_X(p) = Q_G\left(\left[1 + (Q_T(p)/a)^{-b}\right]^{-1}\right)$,
(iv) $Q_X(p) = Q_G\left(\left[1 + e^{-(Q_T(p)-a)/b}\right]^{-1}\right)$,
(v) $Q_X(p) = Q_G\left(1 - e^{-e^{(Q_T(p)-a)/b}}\right)$.

3.5. Proposition. *The mode(s) of the T-gamma$\{Y\}$ family are the solutions of the equation*

$$x = \frac{\alpha - 1}{\beta^{-1} - \Psi\{f_T(Q_Y(F_G(x)))\} - \Psi\{Q'_Y(F_G(x))\}}, \tag{3.1}$$

where $\Psi(f) = f'/f$.

Proof. For gamma distribution,

$$f_G(x) = \beta^{-\alpha}(\Gamma(\alpha))^{-1}x^{\alpha-1}e^{-x/\beta},$$

we have $f'_G(x) = [(\alpha - 1)/x - \beta^{-1}]f_G(x)$. Using this fact; one can show the result in (3.1) by equating the derivative of the equation (2.5) to zero and then solving for x. □

The entropy of a random variable X is a measure of variation of uncertainty (Rényi, 1961). Shannon's entropy has been used in many fields such as engineering and information theory. Shannon's entropy (Shannon, 1948) for a random variable X with PDF $f(x)$ is defined as $\eta_X = -E\left\{\log\left(f(X)\right)\right\}$.

3.6. Proposition. *The Shannon's entropy for the T-$G\{Y\}$ family (2.1) is given by*

$$\eta_X = \eta_T + E\left(\log f_Y(T)\right) - E\left\{\log f_G(Q_G(F_Y(T)))\right\}. \tag{3.2}$$

Proof. See Theorem 2 of Aljarrah et al. (2014). □

3.7. Corollary. *The Shannon's entropy for the T-G{Y} family can be written as*

$$\eta_X = \eta_T + E\left(\log f_Y(T)\right) + \log\Gamma(\alpha) + \alpha\log(\beta) + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X.$$

Proof. For the T-$G\{Y\}$ family, we have $\log(f_G(x)) = -\log(\Gamma(\alpha)) - \alpha\log(\beta) + (\alpha-1)\log(x) - x/\beta$. The result follows from Proposition 3.6. □

3.8. Corollary. *Based on Corollary 3.7, the Shannon's entropies for the (i) T-G{uniform}, (ii) T-G{exponential}, (iii) T-G{log-logistic}, (iv) T-G{logistic} and (v) T-G{extreme value}, distributions, respectively, are given by*

(i) $\eta_X = C_1 + \eta_T + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X$,
(ii) $\eta_X = C_2 + \eta_T - b^{-1}\mu_T + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X$,
(iii) $\eta_X = C_3 + \eta_T + (b-1)E(\log T) - 2E(\log(1+(T/a)^b)) + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X$,
(iv) $\eta_X = C_4 + \eta_T - b^{-1}\mu_T - 2E(\log(1+e^{-(T-a)/b})) + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X$,
(v) $\eta_X = C_5 + \eta_T + b^{-1}\mu_T - E(e^{(T-a)/b}) + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X$,

where $C_1 = \log\Gamma(\alpha) + \alpha\log(\beta)$, $C_2 = -\log b + \log\Gamma(\alpha) + \alpha\log(\beta)$, $C_3 = \log b - b\log a + \log\Gamma(\alpha) + \alpha\log(\beta)$, $C_4 = -\log b + ab^{-1} + \log\Gamma(\alpha) + \alpha\log(\beta)$ *and* $C_5 = -\log b - ab^{-1} + \log\Gamma(\alpha) + \alpha\log(\beta)$.

3.9. Proposition. *The rth moment for the T-gamma{Y} family of distributions is given by*

$$E(X^r) = \beta^r \sum_{k=0}^{\infty} c_k E[F_Y(T)]^{k+r}, \tag{3.3}$$

where $c_0 = 1$, $c_m = m^{-1}\sum_{k=1}^{m}(kr - m + k)g_{k+1}c_{m-k}$, $m \geq 1$ *and* g_k *satisfies the following:*

$$\begin{aligned} g_1 = 1, n(n+\alpha)g_{n+1} \quad &= \quad \sum_{i=1}^{n}\sum_{j=1}^{n-i+1} g_i g_j g_{n-i-j+2}\, j(n-i-j+2) \\ &- \quad \Delta(n)\sum_{i=2}^{n} g_i g_{n-i+2}\, i[i-\alpha-(1-\alpha)(n+2-i)], \\ \textit{and} \quad &\Delta(n) = \begin{cases} 0, & n < 2 \\ 1, & n \geq 2. \end{cases} \end{aligned}$$

Proof. From Lemma 3.1, the rth moment for the T-$G\{Y\}$ family can be written as $E(X^r) = E(Q_G(F_Y(T)))^r$, where $Q_G(p)$ is the quantile function of gamma distribution with parameters α and β. Steinbrecher and Shaw (2008) showed that a power series expansion of $Q_G(p)$ is possible and can be written as $Q_G(p) = \beta\sum_{n=1}^{\infty} g_n\, p^n$ where g_n can be obtained from the recurrence relation defined in the statement of Proposition 3.9. For example, the first three terms of g_n are 1, $(\alpha+1)^{-1}$ and $(3\alpha+5)/[2(\alpha+1)^2(\alpha+2)]$. Other terms can be similarly obtained. Therefore, $(Q_G(p))^r = \beta^r\sum_{k=0}^{\infty} c_k\, p^{k+r}$ (see Gradshteyn and Ryzhik, 2007), where c_k can be obtained from the recurrence relation defined in Proposition 3.9. □

3.10. Corollary. *Based on Proposition 3.9, the rth moments for the (i) T-G{uniform}, (ii) T-G{exponential}, (iii) T-G{log-logistic}, (iv) T-G{logistic} and (v) T-G{extreme value} distributions, respectively, are given by*

(i) $E(X^r) = \beta^r\sum_{k=0}^{\infty} c_k E(T^{k+r})$,
(ii) $E(X^r) = \beta^r\sum_{k=0}^{\infty}\sum_{j=0}^{k+r}(-1)^j \binom{k+r}{j} c_k M_T(-j/b)$,
(iii) $E(X^r) = \beta^r\sum_{k=0}^{\infty} c_k E\left(1+(T/a)^{-b}\right)^{-k-r}$,
(iv) $E(X^r) = \beta^r\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j c_k M_{T-a}(-j/b)$,
(v) $E(X^r) = \beta^r\sum_{k=0}^{\infty}\sum_{j=0}^{k+r}\sum_{i=0}^{\infty}(-1)^{i+j}\frac{(j)^i}{i!}\binom{k+r}{j} c_k M_{T-a}(i/b)$,

where $M_X(t) = E(e^{tX})$.

3.11. Proposition. *The mean deviations from the mean and the median for the T-gamma{Y} family, respectively, are given by*

$$D(\mu) = 2\mu F_T(Q_Y(F_G(\mu))) - 2\Pi_\mu \text{ and } D(M) = \mu - 2\Pi_M, \tag{3.4}$$

where μ *and* M *are the mean and median for* X, *and*

$$\Pi_c = \beta \sum\nolimits_{k=1}^{\infty} g_k \int_{-\infty}^{Q_Y(F_G(c))} f_T(u)(F_Y(u))^k du.$$

Proof. For a nonnegative random variable X, we have $D(\mu) = 2\mu F_X(\mu) - 2\Pi_\mu$ and $D(M) = \mu - 2\Pi_M$, where $\Pi_c = \int_0^c x f_X(x)dx$. From (2.5) and Lemma 3.1, one can easily see that $\Pi_c = \beta \int_{-\infty}^{Q_Y(F_G(c))} f_T(u) Q_G(F_Y(u))du$. The results in (3.4) can be obtained using the series expansion of $Q_G(.)$ in Proposition 3.9. □

3.12. Corollary. *Based on Proposition 3.11, the* Π_c*'s for (i) T-G{uniform}, (ii) T-G{exponential}, (iii) T-G{log-logistic}, (iv) T-G{logistic} and (v) T-G{extreme value} distributions, are respectively given by*

(i)

$$\Pi_c = \beta \sum_{k=1}^{\infty} g_k S_u(c, 0, k), \tag{3.5}$$

where $S_\xi(c, a, k) = \int_a^{Q_Y(F_G(c))} \xi^k f_T(u)du$ *and* $Q_Y(F_G(c)) = F_G(c)$ *for uniform distribution.*

(ii)

$$\Pi_c = \beta \sum_{k=1}^{\infty} \sum_{j=0}^{k} g_k \binom{k}{j} (-1)^j S_{e^{u/b}}(c, 0, -j), \tag{3.6}$$

where $Q_Y(F_G(c)) = -b \log(1 - F_G(c))$ *for exponential distribution.*

(iii)

$$\Pi_c = \beta \sum_{k=1}^{\infty} \sum_{j=0}^{k} g_k \binom{k}{j} (-1)^j S_{1+(u/a)^b}(c, 0, -j), \tag{3.7}$$

where $Q_Y(F_G(c)) = a[F_G(c)/(1 - F_G(c))]^{1/b}$ *for log-logistic distribution.*

(iv)

$$\Pi_c = \beta \sum_{k=1}^{\infty} g_k S_{1+e^{-(u-a)/b}}(c, -\infty, -j), \tag{3.8}$$

where $Q_Y(F_G(c)) = a + b \log\{F_G(c)/(1 - F_G(c))\}$ *for logistic distribution.*

(v)

$$\Pi_c = \beta \sum_{k=1}^{\infty} \sum_{j=0}^{k} (-1)^j \binom{k}{j} g_k S_{e^{e^{(u-a)/b}}}(c, -\infty, -j), \tag{3.9}$$

where $Q_Y(F_G(c)) = a + b \log\{-\log(1 - F_G(c))\}$ *for extreme value distribution.*

Proposition 3.11 and Corollary 3.12 can be used to obtain the mean deviations for T-G{uniform}, T-G{exponential}, T-G{log-logistic}, T-G{logistic} and T-G{extreme value} distributions.

3.13. Proposition. *Let X be a random variable that follows the T-gamma$\{Y\}$ family in (2.5). Assume that $E(X^n) < \infty$ for all n, then $E(X^n) \leq [\beta^n\Gamma(\alpha+n)/\Gamma(\alpha)] \times E\left(1/[1-F_Y(T)]\right)$.*

Proof. If the random variable R is nonnegative and X follows the T-$R\{Y\}$ family in (2.1) with $E(X^n) < \infty$, one can show that $E(X^n) \leq E(R^n)E[1/(1-F_Y(T))]$ (see Theorem 1 in Aljarrah et al., 2014). The result follows by using the fact that R follows a gamma distribution with parameters α and β, and $E(R^n) = \beta^n\Gamma(\alpha+n)/\Gamma(\alpha)$. □

3.14. Corollary. *If $E(X^n) < \infty$ and by using Proposition 3.13, we have the following results:*

- *(i) If X follows T-G\{uniform\}, then* $E(X^n) \leq \left[\frac{\beta^n\Gamma(\alpha+n)}{\Gamma(\alpha)}\right] E\left((1-T)^{-1}\right)$.
- *(ii) If X follows T-G\{exponential\}, then* $E(X^n) \leq \left[\frac{\beta^n\Gamma(\alpha+n)}{\Gamma(\alpha)}\right] M_T(1/b)$.
- *(iii) If X follows T-G\{log-logistic\}, then* $E(X^n) \leq \left[\frac{\beta^n\Gamma(\alpha+n)}{\Gamma(\alpha)}\right] [1+E(T/a)^b]$.
- *(iv) If X follows T-G\{logistic\}, then* $E(X^n) \leq \left[\frac{\beta^n\Gamma(\alpha+n)}{\Gamma(\alpha)}\right] [1+M_{T-a}(1/b)]$.
- *(v) If X follows T-G\{extreme value\}, then* $E(X^n) \leq \left[\frac{\beta^n\Gamma(\alpha+n)}{\Gamma(\alpha)}\right] E(e^{e^{(T-a)/b}})$.

4. Some examples of T-$G\{Y\}$ family of distributions

In this section, we present some members of the T-$G\{Y\}$ family, namely, Weibull-G\{exponential\}, Weibull-G\{log-logistic\} and Cauchy-G\{logistic\}. For simplicity, we only use the standard form (i.e. no parameters in the distribution of Y) of the quantile functions in Table 1.

4.1. The Weibull-G\{exponential\} distribution. If a random variable T follows the Weibull distribution with parameters c and γ, then

$$f_T(t) = c\gamma^{-1}(t/\gamma)^{c-1}e^{-(t/\gamma)^c},\ c,\gamma > 0.$$

From (2.9), the PDF of the Weibull-G\{exponential\} is given by

$$\text{(4.1)} \qquad f_X(x) = \frac{c}{\gamma^c\beta^\alpha\Gamma(\alpha)}\frac{x^{\alpha-1}e^{-x/\beta}}{1-F_G(x)}(-\log(1-F_G(x)))^{c-1} \times \exp\left\{-\gamma^{-c}(-\log(1-F_G(x)))^c\right\}, \quad x > 0.$$

When $c = 1$, (4.1) reduces to the exponential-G\{exponential\}. When $c = \gamma = 1$, equation (4.1) reduces to the gamma distribution. From (2.8), the CDF of the Weibull-G\{exponential\} is given by

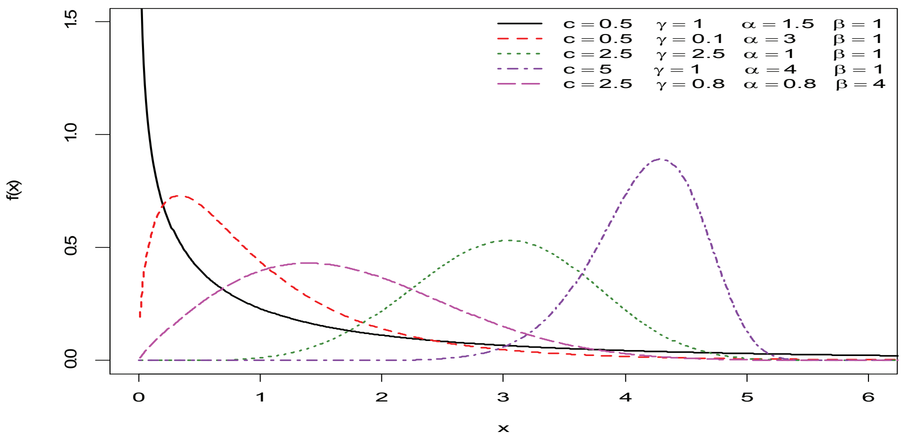
$$F_X(x) = 1 - \exp\left\{-\gamma^{-c}(-\log(1-F_G(x)))^c\right\}, \quad x > 0.$$

In Figure 1, various graphs of Weibull-G\{exponential\} PDF for different parameter values are provided. These plots show that the PDF can be left skewed, right skewed, approximately symmetric or have a reversed J-shape.
Some properties of the Weibull-G\{exponential\} are obtained in the following by using the general properties discussed in section 3.

(1) Quantile function: By using Lemma 3.3, the quantile function of the Weibull-G\{exponential\} distribution is given by

$$Q_X(p) = Q_G\left\{1 - e^{-\gamma(-\log(1-p))^{1/c}}\right\}.$$

Figure 1. The PDFs of Weibull-G\{exponential\} for various parameter values



(2) Mode: By using Proposition 3.5, the mode of Weibull-G\{exponential\} distribution can be obtained by solving the following equation numerically

$$x = (\alpha - 1)\left(\beta^{-1} - h_G(x)\left\{\frac{c-1}{(1-F_G(x))H_G(x)} + \gamma^{-c+1}(H_G(x))^{c-1}\right\}\right)^{-1}.$$

(3) Shannon entropy: By using Corollary 3.8 and the fact that $\mu_T = \gamma\Gamma(1+1/c)$ and $\eta_T = 1+\xi(1-1/c)+\log(\gamma/c)$, the Shannon's entropy of Weibull-G\{exponential\} distribution is

$$\eta_X = C + (1-\alpha)E(\log X) + \beta^{-1}\mu_X,$$

where $C = \log\Gamma(\alpha) + \alpha\log(\beta) + \xi(1-1/c) + \log(\gamma/c) - \gamma\Gamma(1+1/c) + 1$ and $\xi \approx 0.5772$ is the Euler's constant.

(4) Moments: By using Corollary 3.10, the rth moment of the Weibull-G\{exponential\} distribution can be written as

$$E(X^r) = \beta^r \sum\nolimits_{k=0}^{\infty} \sum\nolimits_{j=0}^{k+r} \sum\nolimits_{i=0}^{\infty} \frac{(-1)^{j+i} j^i \gamma^i}{i!} \binom{k+r}{j} c_k \Gamma(1+i/c).$$

(5) Mean deviations: By using Corollary 3.12, the mean deviation from the mean and the mean deviation from the median of Weibull-G\{exponential\} distribution can be obtained from (3.4) where

$$\Pi_c = \beta \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{\infty} \frac{(-1)^j k^i \gamma^i}{i!} \binom{k}{j} g_k \Gamma\left[1+i/c, (Q_Y(F_G(c))/\gamma)^c\right],$$

$Q_Y(F_G(c)) = -\log(1-F_G(c))$ and $\Gamma(\alpha, x) = \int_0^x u^{\alpha-1}e^{-u}du$ is the incomplete gamma function.

(6) Moments upper bound: By Corollary 3.14, $E(X^n) \leq [\beta^n\Gamma(\alpha+n)/\Gamma(\alpha)] \times M_T(1)$, where T follows Weibull(c, γ). If $c = 1$ and $\gamma < 1$, one can show that $E(X^n) \leq \frac{\beta^n\Gamma(\alpha+n)}{(1-\gamma)\Gamma(\alpha)}$.

4.2. The Weibull-G\{log-logistic\} distribution. If a random variable T follows the Weibull distribution with parameters c and γ, then

$$f_T(t) = c\gamma^{-1}(t/\gamma)^{c-1}e^{-(t/\gamma)^c}, \; c, \gamma > 0.$$

From (2.11), the PDF of the Weibull-G\{log-logistic\} is given by

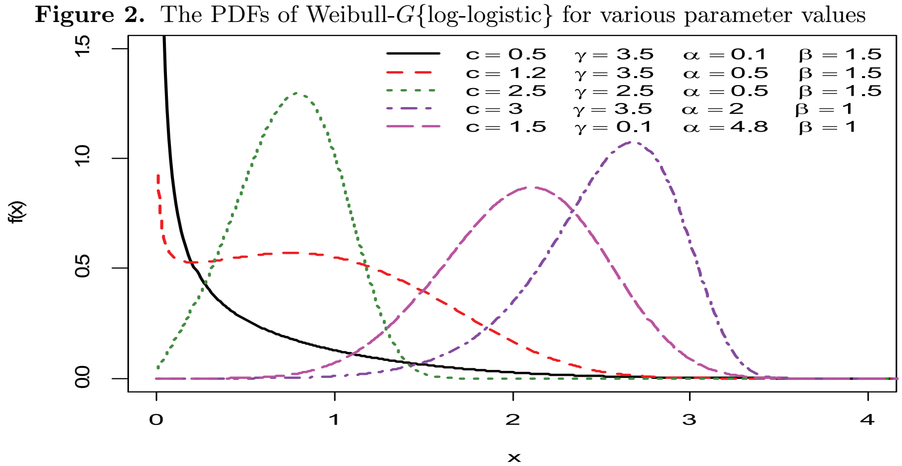
$$f_X(x) = \frac{c}{\gamma^c\beta^\alpha\Gamma(\alpha)}\frac{x^{\alpha-1}e^{-x/\beta}}{(1-F_G(x))^2}\left(\frac{F_G(x)}{1-F_G(x)}\right)^{c-1} \times \exp\left\{-\left(\frac{F_G(x)}{\gamma(1-F_G(x))}\right)^c\right\}, \quad x>0. \tag{4.2}$$

When $c = 1$, the Weibull-G\{log-logistic\} reduces to the exponential-G\{log-logistic\}. From (2.10), the CDF of the Weibull-G\{log-logistic\} is given by

$$F_X(x) = 1 - \exp\left\{-\left(\frac{F_G(x)}{\gamma(1-F_G(x))}\right)^c\right\},\ x>0.$$

Various graphs of Weibull-G\{log-logistic\} PDF for different parameter values are provided in Figures 2 and 3. These plots show the PDF has great shape flexibility. It can be left skewed, right skewed, approximately symmetric or have a reversed J-shape. Also, the distribution can be unimodal or bimodal.

Figure 2. The PDFs of Weibull-G\{log-logistic\} for various parameter values



4.3. The Cauchy-G\{logistic\} distribution. If a random variable T follows the Cauchy distribution with parameters c and γ, then

$$f_T(t) = \pi^{-1}\{1+[(t-c)/\gamma]^2\}^{-1},\ \gamma>0,\ c\in R.$$

From (2.13), the PDF of the Cauchy-G\{logistic\} is defined as

$$f_X(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\pi\gamma\beta^\alpha\Gamma(\alpha)F_G(x)(1-F_G(x))} \times\left[1+\gamma^{-2}(\log(F_G(x)/(1-F_G(x))-c)^2\right]^{-1}, \quad x>0. \tag{4.3}$$

In Figure 4, various graphs of the Cauchy-G\{logistic\} distribution for various parameter values are provided. These graphs indicate that the Cauchy-G\{logistic\} distribution can be right skewed, approximately symmetric or have a reversed J-shape.

Figure 3. Some bimodal PDFs of Weibull-G\{log-logistic\} for various parameter values

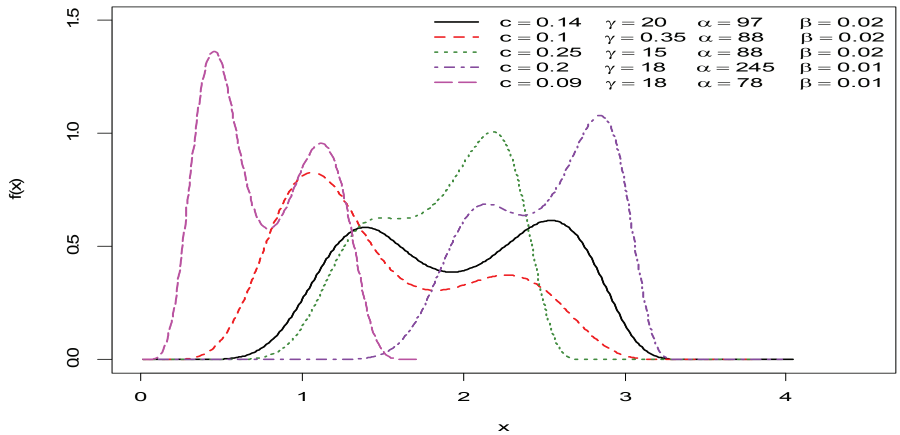
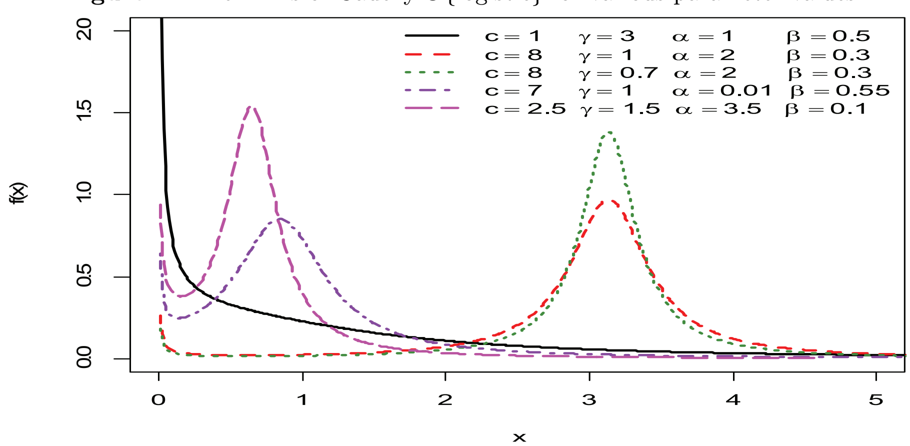


Figure 4. The PDFs of Cauchy-G\{logistic\} for various parameter values



5. Applications

In this section, the applications of the T-gamma distribution are illustrated by fitting some members of the family to different data sets including unimodal and bimodal data sets.

5.1. Unimodal data sets. In this subsection, we fit the Weibull-G\{exponential\}, Weibull-G\{log-logistic\} and Cauchy-G\{logistic\} in equations 4.1, 4.2 and 4.3, respectively, to three data sets with various shapes that are approximately symmetric or left skewed or right skewed. The maximum likelihood method is used to estimate the model parameters. The initial values for the parameters α and β are obtained by assuming the random sample $x_i,\ i = 1, 2, \dots, n$ is from the gamma distribution with parameters α and β. The moment estimates from the gamma distribution are used as the initial values, which are $\alpha_0 = \bar{x}^2/s^2$ and $\beta_0 = s^2/\bar{x}$. Now, by Lemma 3.1, $t_i = Q_Y(F_G(x_i)),\ i = 1, 2, \dots, n$ follows the T distribution with parameters c and γ in all the examples in section 4. The moment estimates or the maximum likelihood estimates of the T-distribution can be used as the initial values for c and γ.

The first data set ($n = 80$) in Table 2 represents the annual maximum temperatures at Oxford and Worthing in England for the period of 1901-1980. Chandler and Bate (2007) used the generalized extreme value distribution to model the annual maximum temperatures in Table 2. The summary statistics from the first data set are: $\bar{x} = 85.3250$, $s = 4.2658$, $\gamma_1 = -0.0162$ and $\gamma_2 = 2.7309$, where γ_1 and γ_2 are the sample skewness and kurtosis respectively. The second data set ($n = 202$) in Table 3 is from Weisberg (2005) and it represents the sum of skin folds in 202 athletes collected at the Australian Institute of Sports. The summary statistics from the second data set are: $\bar{x} = 69.0218$, $s = 32.5653$, $\gamma_1 = 1.1660$ and $\gamma_2 = 4.3220$. The third data set ($n = 40$) in Table 4 is from Xu et al. (2003) and it represents the time to failure ($10^3\,h$) of turbocharger of one type of engine. The summary statistics from the third data set are: $\bar{x} = 6.2525$, $s = 1.9555$, $\gamma_1 = -0.6542$ and $\gamma_2 = 2.5750$.

The data sets are fitted to the Weibull-G{exponential}, Weibull-G{log-logistic} and Cauchy-G{logistic} distributions. The maximum likelihood estimates, the log-likelihood value, the Akaike Information Criterion (AIC), the Kolmogorov-Smirnov (K-S) test statistic, and the p-value for the K-S statistic for the fitted distributions to the three data sets are reported in Table 5. The results in Table 5 show that all the generalized gamma distributions provide adequate fit to the data set in Table 2. For the data set in Table 3, the Weibull-G{exponential} provides the best fit followed by the Weibull-G{log-logistic}, while the Cauchy-G{logistic} does not provide an adequate fit. For the data set in Table 4, all the three generalized gamma distributions provide an adequate fit.

On examining the summary statistics of the data sets, it is noticed that the data set in Table 2 is approximately symmetric, the data set in Table 3 is right skewed and the data set in Table 4 is left skewed. This shows the flexibility of these generalized gamma distributions in fitting various data sets with different distribution shapes. We also fit the three data sets to the gamma distribution. The resulting K-S statistics p-values are less than 0.0001 for all data sets. Figure 5 displays the histogram and the fitted density functions for the three data sets, which support the results in Table 5.

Table 2. The annual maximum temperatures data ($n = 80$)

75	92	87	86	85	95	84	87	86	82	77
89	79	83	79	85	89	84	84	82	86	81
84	84	87	89	80	86	85	84	89	80	87
84	85	82	86	87	86	89	90	90	91	81
85	79	83	93	87	83	88	90	83	82	80
81	95	89	86	89	87	92	89	87	87	83
89	88	84	84	77	85	77	91	94	80	80
85	83	88								

5.2. Bimodal data. In this subsection, we fit the Weibull-G{log-logistic} to a bimodal data set obtained from Emlet et al. (1987) on the asteroid and echinoid egg size. The data consists of 88 asteroids species divided into three types; 35 planktotrophic larvae, 36 lecithotrophic larvae, and 17 brooding larvae. The logarithm of the egg diameters of the asteroids data has a bimodal shape. We fit the logarithm of the egg diameters of the asteroids data and compared it with the beta-normal distribution (Famoye et al., 2004) and logistic-normal{logistic} distribution (Alzaatreh et al., 2014b). The results of the

Table 3. The sum of skin folds data ($n = 202$)

28.0	98	89.0	68.9	69.9	109.0	52.3	52.8	46.7	82.7	42.3
109.1	96.8	98.3	103.6	110.2	98.1	57.0	43.1	71.1	29.7	96.3
102.8	80.3	122.1	71.3	200.8	80.6	65.3	78.0	65.9	38.9	56.5
104.6	74.9	90.4	54.6	131.9	68.3	52.0	40.8	34.3	44.8	105.7
126.4	83.0	106.9	88.2	33.8	47.6	42.7	41.5	34.6	30.9	100.7
80.3	91.0	156.6	95.4	43.5	61.9	35.2	50.9	31.8	44.0	56.8
75.2	76.2	101.1	47.5	46.2	38.2	49.2	49.6	34.5	37.5	75.9
87.2	52.6	126.4	55.6	73.9	43.5	61.8	88.9	31.0	37.6	52.8
97.9	111.1	114.0	62.9	36.8	56.8	46.5	48.3	32.6	31.7	47.8
75.1	110.7	70.0	52.5	67	41.6	34.8	61.8	31.5	36.6	76.0
65.1	74.7	77.0	62.6	41.1	58.9	60.2	43.0	32.6	48	61.2
171.1	113.5	148.9	49.9	59.4	44.5	48.1	61.1	31.0	41.9	75.6
76.8	99.8	80.1	57.9	48.4	41.8	44.5	43.8	33.7	30.9	43.3
117.8	80.3	156.6	109.6	50.0	33.7	54.0	54.2	30.3	52.8	49.5
90.2	109.5	115.9	98.5	54.6	50.9	44.7	41.8	38.0	43.2	70.0
97.2	123.6	181.7	136.3	42.3	40.5	64.9	34.1	55.7	113.5	75.7
99.9	91.2	71.6	103.6	46.1	51.2	43.8	30.5	37.5	96.9	57.7
125.9	49.0	143.5	102.8	46.3	54.4	58.3	34.0	112.5	49.3	67.2
56.5	47.6	60.4	34.9							

Table 4. The time to failure of turbocharger data ($n = 40$)

1.6	3.5	4.8	5.4	6.0	6.5	7.0	7.3	7.7	8.0	8.4
2.0	3.9	5.0	5.6	6.1	6.5	7.1	7.3	7.8	8.1	8.4
2.6	4.5	5.1	5.8	6.3	6.7	7.3	7.7	7.9	8.3	8.5
3.0	4.6	5.3	6.0	8.7	8.8	9.0				

maximum likelihood estimates, the log-likelihood value, the AIC, the K-S test statistic, and the p-value for the K-S statistic for the fitted distributions are reported in Table 6. The results in Table 6 show that all distributions provide an adequate fit to the data set. Figure 6 displays the histogram and the fitted density functions for the data.

6. Summary and Conclusions

The gamma distribution is a commonly used distribution for fitting lifetime data, survival data, hydrological data, and others. The generalization of the gamma distribution provides more flexible distributions for these different applications. This article applies the T-$R\{Y\}$ framework proposed by Aljarrah et al. (2014) to define T-gamma$\{Y\}$ family by using the gamma random variable. Some general properties of the family are studied. Five types of generalized gamma sub-families are defined by using five different quantile functions for uniform, exponential, log-logistic, logistic, and extreme value distributions. Various properties for each of these sub-families are studied including moments, modes, entropy, deviation from the mean and deviation from the median. Three generalized gamma distributions, namely, Weibull-$G\{$exponential$\}$, Weibull-$G\{$log-logistic$\}$ and Cauchy-$G\{$logistic$\}$ are defined and some of their properties investigated. It is noticed that the shapes of T-$G\{Y\}$ distributions can be symmetric, skewed to the right, skewed to the left or bimodal. This shows that the new generalized gamma distributions are very flexible in fitting real world data. For future research, many other types of generalizations of gamma distribution can be derived using the methodology described in this paper.

Table 5. Parameter estimates for the three data sets in Tables 2, 3, and 4

Parameter estimates for the annual maximum temperatures data in Table 2								
Distribution	$\hat{c}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\ell}$	AIC	K-S	K-S p-value
Weibull-G\{E\}	1.4579 (0.7993)*	2.8324 (4.0113)	392.4465 (169.4705)	0.2048 (0.0901)	–228.9830	465.9661	0.0638	0.9006
Weibull-G\{LL\}	0.4753 (0.1934)	0.0481 (0.1168)	423.0032 (169.6370)	0.2232 (0.0904)	–229.0198	466.0396	0.0635	0.9041
Cauchy-G\{L\}	0.1349 (0.8799)	1.1913 (0.2113)	534.8853 (90.7804)	0.1591 (0.0269)	–239.0094	486.0187	0.0727	0.7922
Parameter estimates for the sum of skin folds data in Table 3								
Distribution	$\hat{c}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\ell}$	AIC	K-S	K-S p-value
Weibull-G\{E\}	0.7291 (0.0404)	3.9319 (0.4010)	17.3862 (0.0025)	2.6521 (0.0025)	–953.5709	1915.1420	0.0634	0.3921
Weibull-G\{LL\}	0.3184 (0.0518)	0.0219 (0.0131)	13.4018 (2.7954)	10.4219 (2.8058)	–962.2296	1932.4590	0.0793	0.1578
Cauchy-G\{L\}	-0.9076 (0.3571)	3.2642 (0.3125)	29.4223 (0.0203)	2.1914 (0.0236)	–977.9650	1963.9300	0.1174	0.0076
Parameter estimates for the time to failure of turbocharger data in Table 4								
Distribution	$\hat{c}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\ell}$	AIC	K-S	K-S p-value
Weibull-G\{E\}	8.3877 (1.8836)	4.2085 (0.9729)	0.1116 (0.0714)	4.9569 (1.6596)	–81.3549	170.7098	0.1114	0.7039
Weibull-G\{LL\}	0.6094 (0.2749)	28.3338 (43.2957)	7.5745 (5.5233)	0.5396 (0.3396)	–78.9643	165.9286	0.0820	0.9507
Cauchy-G\{L\}	1.6246 (1.4562)	3.3557 (1.1973)	57.8285 (21.4393)	0.1015 (0.0354)	–85.9245	179.8491	0.1442	0.3766

*standard error

Table 6. Parameter estimates for the asteroids data

Distribution	Weibull-G\{LL\}	Beta-normal*	Logistic-N\{L\}
Parameter Estimates	$\hat{\alpha} = 410.7779(16.1145)$ $\hat{\beta} = 0.0151(0.0196)$ $\hat{c} = 0.1390(0.0865)$ $\hat{\gamma} = 3.6233(0.4476)$	$\hat{\alpha} = 0.0129$ $\hat{\beta} = 0.0070$ $\hat{\mu} = 5.7466$ $\hat{\sigma} = 0.0675$	$\hat{\lambda} = 0.1498(0.0185)$ $\hat{\mu} = 6.0348(0.0685)$ $\hat{\sigma} = 0.2604(0.0100)$
Log-likelihood	-111.2091	-109.4800	-111.4287
AIC	230.4182	226.9600	228.4974
K-S statistic	0.1088	0.1233	0.0988
p-value	0.2486	0.1377	0.3572

*From Famoye et al. (2004) and the MLE standard errors were not provided

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Figure 5. PDF for the fitted distributions for the three data sets

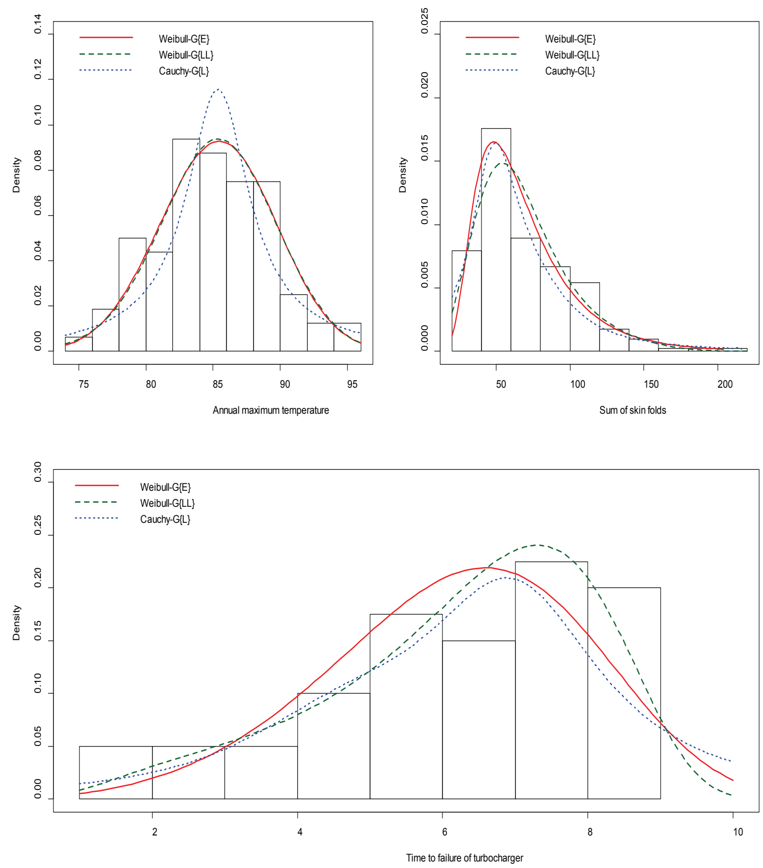

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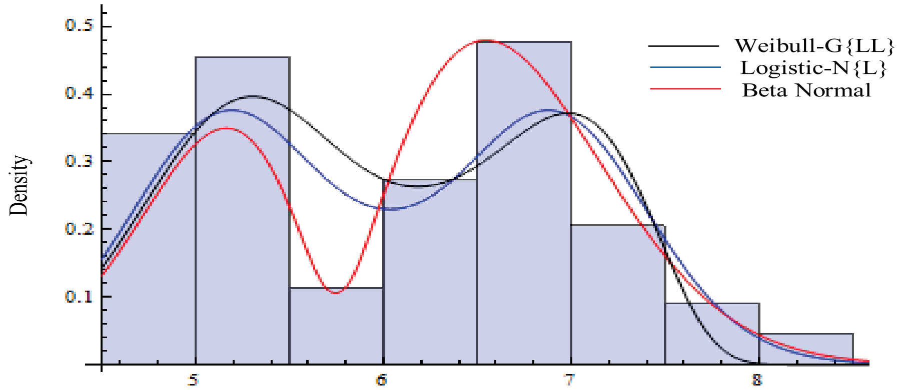
Figure 6. PDF for the fitted distributions for the asteroids data



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Q-Q plots with confidence for testing Weibull and exponential distributions

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Abstract

One of the basic graphical methods for assessing the validity of a distributional assumption is the Q-Q plot which compares quantiles of a sample against the quantiles of the distribution. In this paper, we focus on how a Q-Q plot can be augmented by intervals for all the points so that, if the population distribution is Weibull or exponential then all the points should fall inside the corresponding intervals simultaneously with probability $1-\alpha$. These simultaneous $1-\alpha$ probability intervals provide therefore an objective mean to judge whether the plotted points fall close to the straight line: the plotted points fall close to the straight line if and only if all the points fall within the corresponding intervals. The powers of five Q-Q plot based graphical tests and the most popular non-graphical Anderson-Darling and Cramér-von-Mises tests are compared by simulation. Based on this power study, the tests that have better powers are identified and recommendations are given on which graphical tests should be used in what circumstances. Examples are provided to illustrate the methods.

Keywords: Exponential distribution, Graphical methods, Hypotheses testing, Power; Q-Q plot, Simultaneous inference, Statistical simulation, Weibull distribution.

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1. Introduction

When a simple random sample $Y_1, \cdots, Y_n$ is drawn from a population, one important question is whether the population has a distribution of the form $F_0((y-\mu)/\sigma)$, where $F_0(\cdot)$ is a given cumulative distribution function (cdf), and $-\infty < \mu < \infty$ and $\sigma > 0$ are two unknown parameters. Note that μ is not necessarily the mean and σ is not necessarily the standard deviation of Y. One widely used graphical technique for dealing with this question is the Q-Q plot. In order to provide an objective judgement on whether the points $(z_k, Y_{[k]})$ fall close to a straight line and building on the work of Michael (1983). Chantarangsi et al. (2015) consider augmenting the normal probability plot by providing an interval for each $Y_{[k]}$ $(k = 1, \ldots, n)$ so that, if the population is normally distributed then all the $Y_{[k]}$ $(k = 1, \cdots, n)$ will fall into the corresponding intervals simultaneously with probability $1 - \alpha$. In this paper, the authors use the idea of Chantarangsi et al. (2015) on Q-Q plots to judge whether a sample is drawn from the Weibull or exponential distributions.

The exponential distribution $Exp(\mu, \sigma)$ is a location-scale family, but the Weibull distribution is not. Therefore, log-transformation is applied to the Weibull distribution to obtain the smallest extreme value distribution $SEV(\mu, \sigma)$, which is a location-scale family. A Q-Q plot consists of the n points $(q_k, Y_{[k]})$, $k = 1, \cdots, n$, where $Y_{[1]} \leq \cdots \leq Y_{[n]}$ are the ordered Y_k's and $q_1 < \ldots < q_n$ are a set of n reference values which represent the ordered values of a typical sample of size n from the distribution $F_0(y)$. There are several ways to choose the reference values $q_k = F_0^{-1}(p_k)$ where $F_0^{-1}(\cdot)$ is the inverse function of $F_0(\cdot)$. Various slightly different forms of p_k have been suggested in the statistical literature. See, e.g., Weibull (1939) [23], Blom (1958) [2] and Filliben (1975) [7]. Throughout this paper, we use $p_k = (k - 0.5)/n$ $(k = 1, \ldots, n)$, which are firstly given in Hazen (1914) [8] and used in the software packages R (when $n > 10$) and Matlab. Note that the choices of the p_k's do not affect the tests discussed in this paper.

If $Y_1, \cdots, Y_n$ have the distribution $F_0((y-\mu)/\sigma)$, then the n points $(q_k, Y_{[k]})$ should fall close to a straight line. In order to provide an objective judgement on whether the points $(q_k, Y_{[k]})$ fall close to a straight line, one can augment the Q-Q plot by providing an interval for each $Y_{[k]}$ $(k = 1, \cdots, n)$ so that, if the population follows the distribution $F_0((y-\mu)/\sigma)$, then all the $Y_{[k]}$ $(k = 1, \cdots, n)$ will fall inside the corresponding intervals simultaneously with probability $1 - \alpha$. Each of these n intervals can be depicted in the Q-Q plot as a vertical interval at the corresponding q_k. Therefore, if at least one point $(q_k, Y_{[k]})$ $(1 \leq k \leq n)$ does not fall within the corresponding interval then one can claim, with $1-\alpha$ confidence, that the population does not follow the distribution $F_0((y-\mu)/\sigma)$. This is in effect a size α test for the null hypothesis H_0: the population distribution is $F_0((y-\mu)/\sigma)$ for some $-\infty < \mu < \infty$ and $\sigma > 0$ against the alternative hypothesis H_a: H_0 is not true, but with a clear graphical interpretation on the Q-Q plot.

One way to construct the intervals is to use the Kolmogorov-Smirnov statistic

$$(1.1) \qquad D = \max_{1 \leq k \leq n} \left| F_0\left((Y_{[k]} - \hat{\mu})/\hat{\sigma}\right) - (k - 0.5)/n \right|$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the estimates of μ and σ, respectively. Note that D is sometimes also referred to as Lilliefors' (1967) statistic [11] when F_0 is the cdf of the standard normal distribution $\Phi(\cdot)$. Let c_D be a critical constant so that $P\{D \leq c_D\} = 1 - \alpha$ under H_0. This probability statement can be rewritten as

$$(1.2) \qquad P\left\{ \; Y_{[k]} \in \hat{\mu} + \hat{\sigma} F_0^{-1}\left((k - 0.5)/n \pm c_D\right), \quad k = 1, \cdots, n \; \right\} = 1 - \alpha.$$

Hence, under H_0, each $Y_{[k]}$ should fall in the corresponding interval
$\hat{\mu} + \hat{\sigma} F_0^{-1}\left((k - 0.5)/n \pm c_D\right)$ simultaneously for $k = 1, \cdots, n$ with probability $1 - \alpha$.

The second set of intervals is due to Michael (1983) [16] and based on the statistic

$$D_m = \max_{1\le k\le n}\left|(2/\pi)\arcsin\sqrt{F_0\left((Y_{[k]}-\hat{\mu})/\hat{\sigma}\right)} - (2/\pi)\arcsin\sqrt{(k-0.5)/n}\right|. \tag{1.3}$$

Let c_{D_m} be a critical constant so that $P\{D_m \le c_{D_m}\} = 1-\alpha$ under H_0. This probability statement can be rewritten as

$$P\Big\{Y_{[k]} \in \hat{\mu}+\hat{\sigma}F_0^{-1}\big(\sin^2[\arcsin\sqrt{(k-0.5)/n}\pm\frac{\pi}{2}c_{D_m}]\big) \text{ for } k=1,\dots,n\Big\} = 1-\alpha. \tag{1.4}$$

The purpose of this paper is to propose three new graphical tests and to compare the powers of these graphical tests in order to identify the one having larger overall power.

The layout of the paper is as follows. Section 2 presents the methods of parameter estimation for Weibull and exponential distributions. Section 3 then constructs graphical tests for testing Weibull and exponential distributions based on the tests proposed in Chantarangsi et al. (2015) [5]. The powers of these graphical and two non-graphical tests are then compared in a simulation study in order to identify the tests that have overall good power in Section 4. An illustrative example is presented in Section 5.

2. Distribution function and Parameter estimation

2.1. Weibull distribution. A random variable X is said to have the Weibull distribution, $Wbl(a,b,c)$, if its cdf is given by

$$F(x|a,b,c) = 1-\exp\Big\{-\Big[\frac{x-a}{b}\Big]^c\Big\}, \quad x>a,\ b>0,\ c>0 \tag{2.1}$$

where a is called the location parameter, b the scale parameter and c the shape parameter. In this paper, it is assumed a is known and so $Y=\ln(X-a)$ has the so-called **smallest extreme value (SEV) distribution**. The cdf of Y is given by

$$F(y|\mu,\sigma) = 1-\exp\Big(-\exp\Big(\frac{y-\mu}{\sigma}\Big)\Big), \quad -\infty<y<\infty \tag{2.2}$$

where $-\infty<\mu=\ln b<\infty$ is the location parameter and $\sigma=1/c>0$ is the scale parameter. In short, $Y\sim SEV(\mu,\sigma)$. The original null hypothesis $H_0: X_1,\dots,X_n$ come from $Wbl(a,b,c)$, where a is known, is therefore the same as $H_0: Y_1=\ln(X_1-a),\dots,Y_n=\ln(X_n-a)$ are from $SEV(\mu,\sigma)$ for some unknown parameters μ and σ.

Note that the $\boldsymbol{p^{th}}$ **quantile** of the distribution $SEV(0,1)$ is given by $F^{-1}(p) = \ln(-\ln(1-p))$. Hence a Q-Q plot contains the n points $(\ln(-\ln(1-p_k)), Y_{[k]})$, $k=1,\dots,n$ where $p_k=\frac{k-0.5}{n}$.

Since both the location and scale parameters of $SEV(\mu,\sigma)$ are unknown, they have to be estimated. We consider three popular estimators proposed in the statistical literature: the maximum likelihood estimators (MLE), the best linear unbiased estimators (BLUE) and the best linear invariant estimators (BLIE). They are studied to see which one gives better power. The MLEs are given (cf. Krishnamoorthy (2006) [9]) by

$$\tilde{\mu} = \tilde{\sigma}\ln\Big(\frac{1}{n}\sum_{k=1}^n\exp\Big(\frac{Y_k}{\tilde{\sigma}}\Big)\Big), \tag{2.3}$$

$$\tilde{\sigma} = -\bar{Y}+\frac{\sum_{k=1}^n Y_k\exp\Big(\frac{Y_k}{\tilde{\sigma}}\Big)}{\sum_{k=1}^n\exp\Big(\frac{Y_k}{\tilde{\sigma}}\Big)}. \tag{2.4}$$

Pirouzi–Fard and Holmquist (2013) [19] considered the statistic D_m in which the BLUEs of μ and σ in $SEV(\mu,\sigma)$ are obtained by the generalised least squares (GLS) method. Let $Z_{[1]} \leq \ldots \leq Z_{[n]}$ be the ordered values of a sample of size n from $SEV(0,1)$ with

$$\mu_k = \mathrm{E}(Z_{[k]}),\ k = 1, \ldots, n \tag{2.5}$$

$$\sigma_k^2 = \mathrm{Var}(Z_{[k]}),\ k = 1, \ldots, n. \tag{2.6}$$

Pirouzi–Fard and Holmquist (2007) [17] propose the approximations

$$\mu_k \approx \begin{cases} -\ln(n) - \gamma, & \text{for } k = 1, \\ \ln(-\ln(1 - [\frac{k-0.4866}{n+0.1840}])), & \text{for } k = 2, \ldots, n \end{cases} \tag{2.7}$$

where $\gamma \approx 0.577215665$ is Euler's constant. Pirouzi–Fard and Holmquist (2008) [18] propose the approximations

$$\sigma_{rk}^2 \approx \begin{cases} \frac{\pi^2}{6}, & \text{for } r = k = 1, \\ \frac{(k-0.469)([n+0.831-k][n+0.073])^{-1}}{\ln(\frac{n+0.831-k}{n+0.356})\ln(\frac{n+0.779-k}{n+0.356})}, & \text{for } 1 \leq r \leq k \leq n \end{cases} \tag{2.8}$$

where $\sigma_{rk}^2 = \sigma_{kr}^2$ is the covariance of the $Z_{[r]}$ and $Z_{[k]}$ and so, if $r = k$, $\sigma_{rk}^2 = \sigma_k^2$. Let $\boldsymbol{\mu} = [\mu_1 \ldots \mu_n]'$, $V = (\sigma_{rk}^2)_{n\times n}$ and $\mathbf{Y} = [Y_{[1]} \ldots Y_{[n]}]'$. Then $Y_{[k]} = \mu + \sigma Z_{[k]}$ and $E(Y_{[k]}) = \mu + \sigma E(Z_{[k]}) = \mu + \sigma\mu_k$.

Consider the regression model

$$Y_{[k]} = \mu + \sigma\mu_k + \varepsilon_k,\ k = 1, \ldots, n, \tag{2.9}$$

with $\mathrm{Cov}(Y_{[r]}, Y_{[k]}) = \sigma^2\mathrm{Cov}(Z_{[r]}, Z_{[k]}) = \sigma^2\sigma_{rk}^2$. Since the $Y_{[k]}$'s are heteroscedastic and autocorrelated, the unknown $\boldsymbol{\beta} = [\mu, \sigma]'$ in (2.9) can be estimated by using the GLS method, which result in the BLUEs $\dot{\boldsymbol{\beta}} = (\mathbf{X}'V^{-1}\mathbf{X})^{-1}\mathbf{X}'V^{-1}\mathbf{Y}$ where $\mathbf{X} = [\mathbf{1}, \boldsymbol{\mu}]$ and $V = (\sigma_{rk}^2)_{n\times n}$. Lloyd (1952) [14] is the first to apply the GLS method for estimating the parameters of a location–scale distribution.

Although BLUEs have some very nice properties, they often have larger mean square errors than some other linear estimators. The BLIEs are given in Mann (1969) [15] by

$$\ddot{\mu} = \dot{\mu} - \dot{\sigma}\Big(\frac{E_{12}}{1 + E_{22}}\Big),\ \ddot{\sigma} = \frac{\dot{\sigma}}{1 + E_{22}} \tag{2.10}$$

where $\dot{\mu}$ and $\dot{\sigma}$ are the BLUEs of μ and σ and $\begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix} = \mathbf{X}'V^{-1}\mathbf{X}$.

2.2. Exponential distribution. The cdf of the two-parameter exponential distribution with the location parameter μ and the scale parameter σ is given by

$$F(y|\mu, \sigma) = 1 - \exp\Big(-\frac{y - \mu}{\sigma}\Big), \quad y > \mu,\ \sigma > 0. \tag{2.11}$$

Specifically, the $\boldsymbol{p^{th}}$ **quantile** of a random variable $Y \sim Exp(\mu, \sigma)$ is given by

$$F^{-1}(p) = \mu - \sigma\ln(1 - p). \tag{2.12}$$

In particular, the p^{th} quantile of the random variable $\frac{Y-\mu}{\sigma} \sim Exp(0,1)$ is $-\ln(1-p)$.

2.2.1. *Parameter estimation.* Again the three popular estimators MLE, BLUE and BLIE are investigated in order to find the estimator that gives good overall powers.

The MLEs of μ and σ are given (cf. Krishnamoorthy, 2006 [9]) by

$$(\tilde{\mu}, \tilde{\sigma}) = \Big(Y_{[1]}, \frac{1}{n}\sum_{k=1}^{n}(Y_k - Y_{[1]})\Big) = \big(Y_{[1]}, \bar{Y} - Y_{[1]}\big). \tag{2.13}$$

Let $Z_{[1]} \leq \ldots \leq Z_{[n]}$ be the ordered sample from $Exp(0,1)$. Then we have (cf. Ahsanullah and Hamedani, 2010 [1])

$$\mu_k = \mathrm{E}(Z_{[k]}) = \sum_{i=1}^{k} \frac{1}{n-i+1}, \; k = 1, \ldots, n \tag{2.14}$$

$$\sigma_k^2 = \mathrm{Var}(Z_{[k]}) = \sum_{i=1}^{k} \frac{1}{(n-i+1)^2}, \; k = 1, \ldots, n \tag{2.15}$$

$$\sigma_{rk}^2 = \mathrm{Cov}(Z_{[r]}, Z_{[k]}) = \sum_{i=1}^{k} \frac{1}{(n-i+1)^2}, \; 1 \leq r \leq k \leq n. \tag{2.16}$$

where $\sigma_{rk}^2 = \sigma_{kr}^2 = \mathrm{Cov}(Z_{[r]}, Z_{[k]})$. Similar to the case of Weibull distribution, the BLUEs of (μ, σ) can be obtained by the generalised least squares method and are given by

$$\dot{\mu} = \frac{nY_{[1]} - \bar{Y}}{n-1}, \tag{2.17}$$

$$\dot{\sigma} = \frac{n(Y_{[1]} - \bar{Y})}{n-1}. \tag{2.18}$$

See, e.g., Ahsanullah and Hamedani (2010) [1] for details.

The BLIEs are given in Mann (1969)[15] by

$$\ddot{\mu} = (1 + \frac{1}{n})Y_{[1]} - \frac{\bar{Y}}{n}, \tag{2.19}$$

$$\ddot{\sigma} = \bar{Y} - Y_{[1]}. \tag{2.20}$$

3. The tests

The five graphical tests considered in this paper include the two existing tests D, D_m mentioned in the introduction and the three new tests D_e, D_{be} and D_{bi} based on those in Chantarangsi et al.(2015) [5] for testing normality. The $(\hat{\mu}, \hat{\sigma})$ in each test can therefore be substituted by $(\tilde{\mu}, \tilde{\sigma})$, $(\dot{\mu}, \dot{\sigma})$ or $(\ddot{\mu}, \ddot{\sigma})$. In this section, we assume H_0 is true and provide all the tests of size α. The D and D_m tests using $(\tilde{\mu}, \tilde{\sigma})$ have been considered by Kimber (1985) [10]. The D and D_m tests using $(\tilde{\mu}, \tilde{\sigma})$ have been studied in Coles (1989) [3], which shows that the $(\dot{\mu}, \dot{\sigma})$ gives better powers than the $(\tilde{\mu}, \tilde{\sigma})$.

Recall that $Z_1, ..., Z_n$ denote a simple random sample drawn from $SEV(0,1)$ or $Exp(0,1)$ and $Z_{[1]} \leq ... \leq Z_{[n]}$ be the ordered values. The expected values and variances of $Z_{[k]}$ for $k = 1, \ldots, n$ are given by

$$\mu_k = \mathrm{E}(Z_{[k]}), \tag{3.1}$$

$$\sigma_k^2 = \mathrm{Var}(Z_{[k]}) = \mathrm{E}(Z_{[k]}^2) - \mu_k^2 \tag{3.2}$$

where $f_k(z)$ is the probability density function of $Z_{[k]}$ and is defined by

$$f_k(z) = \frac{n!}{(k-1)!(n-k)!} \left(F_Z(z)\right)^{k-1} \left(1 - F_Z(z)\right)^{n-k} f_Z(z) \;, \; -\infty \leq z \leq \infty.$$

First, we consider testing the Weibull distribution. Recall that $Z_1, ..., Z_n$ denote a simple random sample from $SEV(0,1)$, $Z_{[1]} \leq ... \leq Z_{[n]}$ are the ordered values, and $\mu_k = \mathrm{E}(Z_{[k]})$, $\sigma_k^2 = \mathrm{Var}(Z_{[k]})$. It is clear that $(Y_{[1]}, ..., Y_{[n]})$ have the same joint distribution as $(\mu + \sigma Z_{[1]}, ..., \mu + \sigma Z_{[n]})$. In particular, we have $\mathrm{E}(Y_{[k]}) = \mu + \sigma\mu_k$ and $\mathrm{Var}(Y_{[k]}) = \sigma^2\sigma_k^2$.

The test D_e uses the test statistic

$$D_e = \max_{1 \leq k \leq n} \left| \frac{Y_{[k]} - (\hat{\mu} + \hat{\sigma}\mu_k)}{\hat{\sigma}\sigma_k} \right|, \tag{3.3}$$

where $(\hat{\mu}, \hat{\sigma})$ is the estimator of (μ, σ) and can be any one of the three estimators MLE $(\tilde{\mu}, \tilde{\sigma})$, BLUE $(\dot{\mu}, \dot{\sigma})$ and BLIE $(\ddot{\mu}, \ddot{\sigma})$ considered in Section 2.

It is clear from expression (3.3) that the distribution of D_e does not depends on the unknown parameters μ and σ^2. The critical constant c_e, which satisfies $P\{D_e \leq c_e\} = 1-\alpha$ under H_0, can easily be computed accurately by using a large number of simulations, as in Chantarangsi et al. (2015) [5]. See Edwards and Berry (1987) [6] and Liu et al. (2005) [13] for ways to assess the accuracy of this approach. It is noteworthy that simulation methods are also used to compute the critical constants of the D and D_m tests; see, e.g., Michael (1983) [16] and Scott and Stewart (2011) [20].

The probability statement $P\{D_e \leq C_e\} = 1-\alpha$ produces the following simultaneously probability intervals for $Y_{[1]}, ..., Y_{[n]}$:

$$P\left\{\ Y_{[k]} \in [\hat{\mu} + \hat{\sigma}\mu_k \pm c_e\hat{\sigma}\sigma_k] \ \text{ for } \ k = 1, \cdots, n\ \right\} = 1-\alpha. \tag{3.4}$$

The D_{be} test is constructed in the following steps. Let $F_0(\cdot)$ denote the cdf of $SEV(0,1)$. Note that, under H_0, $U_k = F_0\Big(\dfrac{Y_k - \mu}{\sigma}\Big)$, $k = 1, \ldots, n$ has a uniform distribution on the interval $(0,1)$ and the order statistic $U_k = F_0\Big(\dfrac{Y_k - \mu}{\sigma}\Big)$ has the beta distribution with parameters k and $n-k+1$.

- **Step 1**. Construct p^* level highest-density probability interval $[L(p^*, k, n), U(p^*, k, n)]$ for $U_{[k]}$, which is the shortest probability interval for $U_{[k]}$ among all the p^* level probability intervals for $U_{[k]}$.
- **Step 2**. Find p^* so that

$$K(p^*) \equiv P\Big\{F_0^{-1}(L(p^*, k, n)) \leq \frac{Y_{[k]} - \hat{\mu}}{\hat{\sigma}} \leq F_0^{-1}(U(p^*, k, n)) \ \text{ for } k = 1, ..., n\Big\} \quad = \quad 1-\alpha.$$

  Such a p^* can be found by simulation and a standard numerical searching algorithm in a similar way as in Chantarangsi et al. (2015) [5].
- **Step 3**. Under H_0, the simultaneous $1-\alpha$ probability intervals for $Y_{[1]} \leq ... \leq Y_{[n]}$ are therefore given by

$$\hat{\mu} + \hat{\sigma}F_0^{-1}(L(p^*, k, n)) \leq Y_{[k]} \leq \hat{\mu} + \hat{\sigma}F_0^{-1}(U(p^*, k, n)), \quad k = 1, \ldots, n.$$

Hence test D_{be} rejects H_0 if and only if at least one $Y_{[k]}$ is not included in its corresponding interval $[\hat{\mu} + \hat{\sigma}F_0^{-1}(L(p^*, k, n)), \hat{\mu} + \hat{\sigma}F_0^{-1}(U(p^*, k, n))]$.

The D_{bi} test uses statistic

$$D_{bi} \;=\; \max_{1\leq k\leq n} \frac{\left|F_0\big((Y_{[k]} - \hat{\mu})/\hat{\sigma}\big) - (k-0.5)/n\right|}{\sqrt{(k-0.5)(n-k+0.5)/n^3}}.$$

Let c_{bi} be a critical constant so that $P\Big\{D_{bi} < c_{bi}\Big\} = 1-\alpha$, under H_0, which can be determined by using simulation as before. The simultaneous $1-\alpha$ probability intervals for $Y_{[1]} \leq ... \leq Y_{[n]}$ are therefore given by

$$Y_{[k]} \in \hat{\mu} + \hat{\sigma}F_0^{-1}\Big(\frac{k-0.5}{n} \pm c_{bi}\sqrt{\frac{(k-0.5)(n-k+0.5)}{n^3}}\Big) \text{ for } k = 1, \ldots, n. \tag{3.5}$$

The test D and D_m are specified in (1.1), (1.2) and (1.3), (1.4), respectively, but with $F_0(\cdot)$ being the cdf of $SEV(0,1)$.

The non-graphical Anderson-Darling (AD) test rejects H_0 if and only if AD $> c$ where

$$\text{AD} = -\sum_{k=1}^{n}\Big[\frac{(2k-1)\{\ln(F_0(Y_{[k]})) + \ln(1 - F_0(Y_{[n+1-k]}\}}{n}\Big] - n. \tag{3.6}$$

The critical constant c, which satisfies $P\{\text{AD} < c\} = 1-\alpha$ under H_0, can be determined by simulation as before.

The non-graphical Cramér-von Mises (CvM) test rejects H_0 if and only if $\text{CvM} > c$ where

$$\text{CvM} = \sum_{k=1}^{n}\left[F_0(Y_{[k]}) - \frac{2k-1}{2n}\right]^2 + \frac{1}{12n} \tag{3.7}$$

The critical constant c, which satisfies $P\{\text{CvM} < c\} = 1-\alpha$ under H_0, can again be determined by simulation.

For testing the Exponential distribution $Exp(\mu, \sigma)$, the five graphical and two non-graphical tests for testing the Weibull distribution given above are easily modified by simply assuming that $F_0(\cdot)$ is the cdf of $Exp(0,1)$ and that $Z_1, \ldots, Z_n$ are a simple random sample from $Exp(0,1)$ to give the five graphical and two non-graphical tests also denoted as D, D_m, D_e, D_{be}, AD and CvM.

Our focus is on the five graphical tests D, D_m, D_e, D_{bi} and D_{be}, each providing a set of simultaneous $1-\alpha$ probability intervals for the $Y_{[k]}$'s. These intervals can be used in the Q-Q plot to objectively judge whether the n points $(q_k, Y_{[k]})$ fall close to a straight line. We also want to compare the powers of the five graphical and the two non-graphical tests.

From many simulation studies on power comparison published in statistical literature (cf. Littell et al. (1979) [12] and Sürücü (2008) [21]), the AD and CvM tests usually have larger power than other tests, for testing Weibull or Exponential distributions. This is the reason why AD and CvM tests are included in our power comparison study.

4. Power comparisons

The power of a test is evaluated by simulation as the proportion of times the null hypothesis H_0 is rejected by the test for a given alternative distribution. In our simulation study, each critical constant c is based on 30,000 simulations and each power value is based on 10,000 simulations. The powers of the seven tests are computed for all possible combinations of $\alpha = 0.01, 0.05, 0.1$, the three estimators (MLE, BLUE, BLIE), sample size n from a set of values, and the alternative distribution from a set of distributions. The set of alternative distributions includes many of the distributions used in several published studies on power comparison of tests for Weibull or Exponential distributions (cf. Littell, et al. (1979) [12], Kimber (1985) [10], Coles (1989) [3], Tiku and Singh (1981) [22], Castro-Kusiss (2011) [4], Pirouzi-Fard and Holmquist et al. (2013) [19]).

4.1. For Weibull distribution. The alternatives are divided into the following three groups. The **first** group of seven distributions are asymmetric on the support $(0,\infty)$ and includes $\chi^2(1)$, $\chi^2(3)$, $\chi^2(4)$, $\chi^2(6)$, $\chi^2(10)$, $LogN(0,1)$ and Half-normal(0,1) ($HN(0,1)$). The **second** group of seven distributions are on the interval $(0,1)$ and includes $U(0,1)$, $beta(2,2)$, $beta(2,5)$, $beta(5,1.5)$, $beta(0.5,0.5)$, $beta(0.5,3)$ and $beta(1,2)$. The **third** group of seven distributions are symmetric on the support $(-\infty,\infty)$ and include $Laplace(0,1)$, $logistic(0,1)$, $N(0,1)$, $t(1)$, $t(3)$, $t(4)$ and $t(6)$.

Sample sizes $n =$ 10, 25, 40 ,100, 150, 200, 250, 300, 350, 400 and 500 are used for the alternative distributions from Group I and Group II. For the alternative distributions from Group III, the considered sample sizes are $n = 5(5)30$, 40, 50, 100, 150, and 200 since the powers are very close to 100% already at sample size $n = 200$.

From the results of our study, which one of the three estimators is used has little effect on the powers of the seven tests. Hence any one of the three estimators can be used with any one of the seven tests. Tables 1-3 give the powers of the tests when BLUE is used.

From the power results in Table 1 for **the first group** of alternative distributions, the following observations can be made. The D_{be} test has good power, even relative to the non-graphical tests AD and CvM, against the alternatives in Group I except $\chi^2(1)$ and $HN(0,1)$. D_m and D_{be} have similar powers. Overall D and D_e tend to be less powerful than the other tests. While D_{bi} has better power than D and D_e on many cases, it is less powerful than D_{be} and D_m overall.

From the power results in Table 2 for **the second group** of alternative distributions, the D_{bi} test often has the best powers and is more powerful than the non-graphical AD and CvM tests on most occasions whereas the D and D_e tests generally have least powers. However, when $n \leqslant 40$, D_e seems to have greater powers than all the other tests. Additionally, the powers of D_m and D_{be} are close to each other. All tests have little power in detecting the departure from the Weibull distribution of $beta(2,5)$. Also, the D_{bi} test is more powerful than the non-graphical AD and CvM tests on most occasions.

From the power results in Table 3 for **the third group** of alternative distributions, the AD and CvM tests are overall more powerful than the other tests. Nevertheless, for $N(0,1)$, the D_{bi} test is more powerful than the AD and CvM tests. The D, D_e and D_{bi} tests have low power over the distributions in Group III and the powers of D_{bi} are less than those of D and D_e for larger sample sizes. Among the graphical tests, the D_m and D_{be} tests are more powerful overall.

4.2. For exponential distribution. The **first** group of nine distributions are asymmetric on the support $(0,\infty)$ includes $\chi^2(1)$, $\chi^2(3)$, $\chi^2(4)$, $\chi^2(6)$, $\chi^2(10)$, $LogN(0,1)$, $HN(0,1)$, $Wbl(0,0.5,0.5)$ and $Wbl(0,2,2)$. The **second** and the **third** groups of the distributions are the same as the second and third groups, respectively, given in Section 4.1

From our simulation study, the BLIE often gives the best power, even though the power differences between BLIE and BLUE are often small. Hence BLIE is recommended for testing Exponential distribution.

From the power results given in Table 4 for **the first group** of alternative distributions, the following observations can be made. The two non–graphical AD and CvM tests are the most powerful against all alternative distributions exception $LogN(0,1)$. Interestingly, the powers of the D test are as good as those of the others. Moreover, the D_e test is the best choice against $LogN(0,1)$. On the other hand, it has low powers in comparison with the other tests in this group. Also, the D_{bi} test is the best choice against $HN(0,1)$; however, it has the least power among $\chi^2(1)$, $LogN(0,1)$ and $Wbl(0,0.5,0.5)$. For the other alternative distributions, powers of the D_{bi} test is slightly better than those of D_m and D_{be}.

From the power results given in Table 5 for **the second group** of the alternative distributions, we can observe that the D_{bi} test shows good power, even relative to the non-graphical AD and CvM tests, except for $beta(0.5,3)$. The D_e test has the worst power among all the tests except for $beta(0.5,3)$. The powers of the D_m and D_{be} tests are not as high as those of the D_{bi}, AD and CvM tests in many cases, but they perform quite well overall the alternative distributions generally.

From the power results given in Tables 6 for **the third group** of the alternative distributions, the powers of all tests are very similar. Nevertheless, the CvM test is slightly more powerful than the other tests.

The overall conclusions from this power study for both Weibull and Exponential distributions are as follows. Although not completely dominated, the D and D_e are less powerful than the other three graphical tests in most scenarios and so not recommended. Therefore, the graphical tests D_m, D_{be} and D_{bi} are recommended for use with Q-Q plot.

Table 1. Powers (in %) for testing Weibull distribution with BLUE and $\alpha = 0.05$ against the alternative distributions from Group I

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$\chi^2(1)$	10	5.88	5.27	**8.26**	5.35	3.72	5.96	6.01
	25	7.92	5.28	**12.20**	4.69	3.62	8.90	7.60
	40	8.59	6.08	**15.10**	5.50	4.14	10.54	9.45
	100	15.12	11.76	**21.55**	11.83	8.97	20.02	17.65
	150	18.26	17.09	28.31	18.26	13.23	**30.18**	22.92
	200	24.79	23.03	32.11	24.50	15.90	**36.20**	30.45
	250	28.68	29.80	36.00	31.73	20.39	**44.26**	36.02
	300	33.80	35.66	40.64	39.01	23.22	**52.24**	43.40
	350	40.00	42.05	44.83	44.21	27.60	**59.38**	50.78
	400	44.94	48.46	49.11	49.36	31.80	**64.56**	56.43
	500	53.12	57.82	54.76	59.67	37.64	**75.64**	66.21
$\chi^2(3)$	10	4.95	5.56	4.26	5.30	**6.35**	5.22	5.36
	25	5.19	6.95	3.49	6.86	**7.90**	4.90	5.45
	40	5.69	8.05	3.41	7.80	**8.61**	5.22	5.92
	100	6.42	10.25	4.15	**10.42**	10.25	7.72	7.23
	150	7.60	11.63	5.42	**12.32**	10.91	10.08	9.00
	200	8.90	13.49	6.17	**14.26**	12.48	11.82	10.62
	250	9.53	15.12	6.08	**15.46**	13.48	12.28	11.15
	300	10.46	16.67	6.61	**17.40**	13.29	15.54	12.76
	350	11.89	18.54	7.58	**18.94**	14.85	17.26	14.84
	400	12.59	19.61	8.46	**20.36**	15.34	18.60	15.64
	500	14.85	23.04	9.59	**23.24**	16.16	22.08	18.56
$\chi^2(4)$	10	5.38	6	3.61	5.94	**7.15**	5.10	5.78
	25	5.86	8.51	3.28	7.82	**9.94**	5.76	6.65
	40	6.52	10.8	3.69	10.56	**12.73**	7.30	7.25
	100	9.46	16.89	5.73	**16.46**	16.36	12.88	11.57
	150	11.94	**21.11**	8.48	21.04	19.32	18.50	15.73
	200	15.81	26.15	10.32	**26.42**	21.74	22.88	19.73
	250	18.3	31.00	11.59	**32.24**	25.62	26.9	23.20
	300	20.29	35.16	14.34	**36.96**	26.09	34.22	27.74
	350	23.49	39.30	15.93	**40.20**	29.91	38.24	31.85
	400	25.85	42.15	18.41	**45.44**	30.54	42.80	34.45
	500	31.83	49.39	21.69	**52.28**	35.57	51.98	42.92
$\chi^2(6)$	10	5.27	6.31	3.25	7.16	**8.16**	5.64	6.11
	25	6.89	11.60	3.55	10.16	**13.83**	8.02	8.22
	40	8.68	14.66	4.67	14.64	**16.47**	10.06	10.47
	100	14.87	**28.75**	9.88	27.98	27.93	22.66	19.81
	150	20.91	38.06	15.08	**39.04**	33.85	34.52	28
	200	27.7	47.71	19.75	**48.26**	40.11	43.62	37.24
	250	33.44	56.15	23.16	**56.50**	45.65	52.74	45.45
	300	38.48	62.91	28.89	**65.20**	49.38	64.00	53.22
	350	44.89	69.20	33.12	**71.16**	55.36	69.82	60.17
	400	49.49	74.85	39.38	75.80	59.42	**76.5**	65.55
	500	59.44	82.92	46.73	84.48	66.89	**85.54**	76.69
$\chi^2(10)$	10	6.05	7.42	3.25	7.62	**9.90**	6.58	6.94
	25	8.45	15.36	4.53	15.18	**18.09**	10.70	10.44
	40	12.10	20.96	6.79	20.46	**23.40**	15.80	14.34
	100	22.95	**44.68**	16.41	43.98	42.34	37.64	31.43
	150	32.45	58.86	25.65	**60.14**	52.66	54.08	45.41
	200	43.97	70.82	32.42	**72.00**	62.19	68.92	58.92
	250	52.61	80.28	40.01	**81.68**	70.69	78.58	69.77
	300	60.13	86.40	49.10	**87.44**	75.20	86.82	77.76
	350	69.08	91.21	57.34	91.68	81.42	**91.76**	84.74
	400	74.35	94.05	65.87	**94.66**	85.55	94.38	88.95
	500	84.01	97.62	76.72	98.10	91.39	**98.30**	95.04
$LogN(0,1)$	10	9.34	12.40	3.97	12.43	**16.88**	10.36	11.25
	25	18.62	34.49	11.94	33.89	**40.13**	28.40	25.46
	40	28.12	52.20	20.64	51.58	**56.83**	43.92	38.57
	100	63.39	**91.66**	51.29	91.59	90.10	86.94	79.81
	150	81.86	98.38	73.18	**98.48**	97.35	97.18	94.17
	200	93.03	99.80	87.30	**99.80**	99.33	99.56	98.54
	250	97.35	**100**	94.65	99.98	99.89	99.90	99.74
	300	98.93	**99.98**	98.58	**99.98**	99.95	99.98	99.94
	350	99.71	99.99	99.60	99.99	**100**	99.98	99.97
	400	99.88	**100**	99.88	**100**	**100**	**100**	**100**
	500	99.99	**100**	99.99	**100**	**100**	**100**	**100**
$HN(0,1)$	10	5.93	5.43	**8.56**	5.25	4.00	6.62	5.99
	25	7.15	5.28	**12.18**	4.51	3.05	8.64	7.45
	40	8.75	5.91	**14.58**	5.38	4.06	10.40	9.28
	100	14.39	11.21	**21.20**	10.78	9.02	19.24	16.33
	150	18.87	17.01	27.84	17.83	13.15	**28.12**	23.71
	200	25.35	23.58	32.23	25.03	16.05	**36.26**	31.01
	250	30.30	29.43	35.25	30.53	20.49	**43.74**	37.25
	300	34.27	35.62	40.75	38.35	23.16	**52.94**	44.61
	350	40.56	41.92	45.11	44.39	27.38	**59.48**	51.30
	400	44.67	47.59	48.83	49.75	30.69	**65.56**	56.10
	500	53.89	58.04	55.33	59.65	37.40	**75.70**	66.90

The bolded number is the highest power among the seven tests for each sample size.

Table 2. Powers (in %) for testing Weibull distribution with BLUE and $\alpha = 0.05$ against the alternative distributions from Group II

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$U(0,1)$	10	14.21	12.00	**20.92**	9.97	5.44	19.68	15.94
	25	32.05	31.54	42.77	25.61	36.30	**48.62**	39.43
	40	49.17	66.52	57.26	60.33	**77.42**	72.12	60.46
	100	91.16	99.99	92.79	99.93	**100**	99.32	97.21
	150	98.66	100	99.86	**100**	**100**	**100**	99.89
	200	99.88	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	10	6.59	5.50	**9.59**	4.80	3.14	7.52	6.88
	25	9.56	6.47	**14.47**	5.53	4.15	13.38	10.62
	40	12.84	9.65	18.01	8.36	10.64	**18.16**	15.15
	100	28.22	46.57	29.62	39.38	**56.22**	49.62	38.56
	150	41.99	76.58	39.77	70.98	**84.94**	70.94	56.61
	200	56.74	92.79	47.98	90.94	**96.39**	83.74	71.93
	250	68.90	98.58	57.59	97.30	**99.50**	92.04	82.95
	300	75.78	99.76	70.69	99.55	**99.92**	97.06	90.43
	350	84.35	99.98	81.58	99.89	**99.99**	98.90	94.79
	400	89.27	**100**	89.49	99.99	**100**	99.46	97.29
	500	94.82	**100**	97.48	**100**	**100**	99.96	99.25
$beta(2,5)$	10	4.62	4.41	4.78	4.28	4.39	**4.86**	4.85
	25	**5.20**	4.62	5.16	3.64	4.27	4.34	5.10
	40	**4.57**	3.98	4.56	3.96	4.30	4.54	4.65
	100	5.37	4.85	3.94	4.23	**6.14**	5.50	5.64
	150	5.67	5.27	4.07	5.30	6.67	**7.26**	6.04
	200	6.24	6.88	4.16	6.22	**8.42**	7.28	7.01
	250	6.71	8.16	3.29	6.39	**10.59**	8.06	7.99
	300	6.57	8.53	3.49	8.11	**11.78**	9.66	8.18
	350	7.58	10.32	3.54	9.15	**14.76**	10.70	9.57
	400	8.96	12.58	3.90	10.45	**16.46**	11.28	10.85
	500	9.14	15.62	3.29	12.60	**20.97**	14.32	11.99
$beta(5,1.5)$	10	8.01	6.43	**11.80**	5.54	3.14	9.92	8.34
	25	15.01	11.07	**22.14**	8.52	8.50	21.30	17.66
	40	20.65	21.21	29.10	17.81	27.54	**35.22**	26.04
	100	52.08	87.37	52.56	82.54	**92.98**	79.70	66.78
	150	72.15	99.11	70.78	98.45	**99.75**	94.98	86.69
	200	86.56	99.98	86.87	99.96	**100**	99.22	95.78
	250	93.99	**100**	95.62	**100**	**100**	99.78	98.62
	300	97.42	**100**	98.47	**100**	**100**	**100**	99.68
	350	99.16	**100**	99.93	**100**	**100**	**100**	99.93
	400	99.59	**100**	**100**	**100**	**100**	**100**	99.98
	500	99.96	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	10	33.39	30.47	42.47	25.68	19.10	**46.44**	38.96
	25	73.77	85.45	79.02	80.91	**92.48**	90.24	82.42
	40	92.10	99.60	93.11	99.37	**99.84**	99.30	97.09
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	10	7.64	6.09	**10.92**	5.59	3.40	8.78	7.67
	25	11.56	8.25	**18.65**	6.62	4.76	17.96	12.59
	40	15.57	12.17	**24.85**	10.12	10.80	21.82	18.44
	100	34.31	44.73	43.08	40.37	46.97	**53.34**	44.78
	150	48.78	68.59	55.84	67.44	72.02	**74.86**	62.26
	200	62.84	86.65	65.45	85.04	**88.69**	86.86	76.82
	250	74.08	95.30	73.12	93.69	**96.18**	93.42	87.03
	300	81.81	98.42	81.59	97.86	**98.85**	97.84	93.06
	350	88.79	99.57	88.39	99.41	**99.71**	98.88	96.52
	400	92.53	99.81	92.62	99.82	**99.94**	99.74	98.00
	500	97.01	100	97.31	99.97	**100**	99.98	99.68
$beta(1,2)$	10	6.89	5.53	**10.12**	4.95	3.19	9.56	6.88
	25	11.17	8.10	**17.98**	6.36	5.07	15.70	12.68
	40	15.92	12.97	**23.00**	10.83	13.73	22.46	18.94
	100	35.17	56.25	39.37	50.40	**65.34**	57.34	46.89
	150	50.68	84.91	53.03	81.43	**91.37**	79.70	66.85
	200	67.18	97.10	64.43	95.51	**98.58**	90.42	81.10
	250	77.27	99.60	72.79	99.35	**99.83**	96.64	89.98
	300	86.15	99.93	84.35	99.88	**99.96**	98.92	95.14
	350	90.96	**100**	92.33	99.99	**100**	99.66	97.74
	400	94.92	**100**	97.08	**100**	**100**	99.96	99.07
	500	98.28	**100**	99.71	**100**	**100**	99.98	99.89

The bolded number is the highest power among the seven tests for each sample size.

Table 3. Powers (in %) for testing Weibull distribution with BLUE and $\alpha = 0.05$ against the alternative distributions from Group III

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$Laplace(0,1)$	5	9.73	9.71	8.24	10.60	**11.44**	8.42	9.82
	10	19.56	24.02	12.10	24.88	**27.00**	23.03	24.12
	15	28.93	34.3	21.59	35.23	34.12	35.57	**35.92**
	20	37.33	43.57	32.12	46.18	41.05	**47.03**	46.43
	25	46.92	52.22	40.26	54.51	46.24	**57.06**	56.08
	30	55.42	58.61	48.28	60.53	51.40	**65.73**	65.04
	40	67.43	68.97	61.76	70.24	57.72	**78.08**	76.93
	50	76.62	75.98	71.28	77.85	62.10	**85.56**	84.58
	100	96.73	95.10	94.00	95.95	79.88	**98.80**	98.60
	150	99.68	98.97	98.98	99.27	89.72	**99.93**	99.91
	200	99.91	99.84	99.84	99.91	95.01	**99.98**	99.98
$logistic(0,1)$	5	6.92	6.96	6.11	7.49	**8.39**	6.00	6.92
	10	12.14	16.66	6.23	16.95	**21.01**	15.21	15.6
	15	17.08	24.35	11.14	24.59	**28.35**	22.95	22.02
	20	21.99	32.40	17.64	33.89	**36.11**	31.65	29.29
	25	27.67	40.51	23.56	**41.59**	41.50	39.05	36.56
	30	33.29	46.99	29.12	**47.51**	46.99	47.12	43.69
	40	41.67	57.45	39.37	57.19	55.02	**58.72**	54.4
	50	49.23	64.32	47.57	65.94	60.65	**67.92**	62.79
	100	80.52	89.51	77.33	90.48	81.39	**93.46**	90.59
	150	93.54	96.73	92.68	97.40	91.26	**98.8**	98.01
	200	98.02	99.20	97.08	99.43	95.98	**99.8**	99.58
$N(0,1)$	5	6.12	6.14	5.35	6.35	**7.10**	5.19	6.33
	10	9.34	12.40	3.97	12.31	**16.88**	10.66	11.25
	15	11.99	19.50	6.00	19.18	**25.58**	15.97	15.54
	20	15.08	26.60	9.04	26.15	**33.13**	22.62	20.79
	25	18.62	34.49	11.94	34.17	**40.13**	28.44	25.46
	30	21.20	40.80	14.26	40.64	**46.48**	33.81	29.35
	40	28.12	52.20	20.64	51.58	**56.83**	44.20	38.57
	50	34.00	63.23	25.66	62.38	**66.57**	54.84	47.11
	100	63.39	91.66	51.29	**91.65**	90.10	87.06	79.81
	150	81.86	98.38	73.18	**98.48**	97.35	97.11	94.17
	200	93.03	99.80	87.30	**99.80**	99.33	99.59	98.54
$t(1)$	5	29.51	29.54	27.90	30.42	**31.14**	29.09	30.53
	10	57.46	59.16	52.82	60.89	58.12	60.83	**61.63**
	15	74.7	75.48	71.96	76.67	72.05	78.63	**78.87**
	20	84.57	84.78	82.93	86.41	79.99	**88.29**	88.27
	25	90.87	90.96	89.64	91.88	85.82	**93.61**	93.57
	30	94.48	94.24	94.23	**98.23**	89.47	96.59	96.57
	40	98.48	98.15	98.16	98.04	95.13	99.24	**99.25**
	50	99.55	99.39	99.17	99.46	97.38	**99.82**	99.81
	100	99.99	99.99	99.99	99.99	99.95	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	10.22	10.22	9.11	11.07	**11.78**	9.47	10.68
	10	20.44	24.62	14.82	25.56	**26.67**	24.35	24.79
	15	29.91	35.99	25.70	36.78	**37.79**	37.32	36.57
	20	38.28	46.30	36.23	47.67	44.65	**48.31**	46.71
	25	44.87	52.07	44.71	54.12	48.93	**55.70**	53.75
	30	53.21	59.70	51.88	61.79	55.00	**64.49**	62.31
	40	65.06	70.69	65.83	71.58	61.89	**76.47**	74.48
	50	73.21	77.30	73.89	78.96	67.17	**84.37**	82.35
	100	94.82	94.91	95.01	95.87	84.33	**98.24**	97.79
	150	99.17	99.05	99.08	99.25	92.34	**99.81**	99.78
	200	99.90	99.82	99.79	99.91	96.78	**99.99**	**99.99**
$t(4)$	5	8.42	8.34	7.19	9.38	**10.01**	7.86	9.03
	10	16.23	20.52	10.06	21.19	**24.02**	19.26	19.71
	15	24.30	31.57	18.27	31.69	**34.57**	31.35	30.48
	20	29.66	39.30	**27.64**	**41.26**	39.82	40.30	38.41
	25	36.57	46.94	34.70	48.43	44.96	**48.65**	46.28
	30	43.92	54.04	42.72	55.34	50.89	**57.18**	54.20
	40	54.16	63.65	55.10	65.51	57.76	**68.83**	65.64
	50	63.98	71.89	64.72	73.25	63.75	**78.45**	75.41
	100	89.52	92.18	90.06	93.15	80.28	**96.31**	95.22
	150	97.67	98.05	97.77	98.56	89.32	**99.62**	99.50
	200	99.71	99.67	99.44	99.69	95.12	**99.98**	99.94
$t(6)$	5	7.36	7.44	6.39	8.06	**8.94**	6.60	8.00
	10	13.12	17.30	7.71	17.84	**21.37**	15.56	16.08
	15	19.79	27.33	13.72	27.05	**31.09**	25.84	25.03
	20	24.37	34.85	20.20	35.55	**37.02**	34.02	32.06
	25	28.99	41.57	25.69	**42.88**	42.01	41.23	38.45
	30	35.57	48.13	32.68	**49.40**	47.53	48.97	45.51
	40	45.09	59.16	44.56	60.42	55.11	**61.44**	57.07
	50	52.67	66.73	53.43	67.79	60.94	**70.89**	66.24
	100	82.40	89.59	81.38	90.57	80.02	**94.11**	91.83
	150	94.14	96.94	94.13	97.61	89.50	**98.94**	98.26
	200	98.59	99.17	98.32	99.32	94.79	**99.84**	99.74

The bolded number is the highest power among the seven tests for each sample size.

Table 4. Powers (in %) for testing exponential distribution with BLIE and $\alpha = 0.05$ against the alternative distributions from Group I

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$\chi^2(1)$	5	3.88	3.93	6.43	4.22	3.92	**6.91**	4.38
	10	10.66	10.95	11.78	10.29	5.98	**17.95**	12.18
	15	19.85	21.14	15.45	19.33	8.04	**31.62**	22.77
	20	29.58	31.39	19.63	29.14	10.39	**44.49**	34.35
	25	39.12	42.05	23.29	41.13	13.56	**55.70**	44.23
	30	48.89	52.44	28.02	52.94	16.41	**66.29**	55.15
	40	65.77	70.48	38.14	68.94	25.37	**81.21**	71.65
	50	77.49	81.98	47.76	81.63	32.17	**90.29**	82.33
	100	98.44	99.19	91.11	99.25	77.25	**99.76**	99.16
	150	99.8	99.95	99.04	99.98	95.65	99.97	99.95
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	5	6.71	6.68	6.03	6.5	6.72	6.55	**7.13**
	10	9.14	9.38	8.09	9.31	**9.99**	8.35	9.85
	15	11.41	12.33	10.13	12	**13.27**	11.11	12.82
	20	12.79	13.49	11.58	14.08	**15.26**	12.9	14.53
	25	14.86	16.54	14.33	17.05	**18.98**	15.67	17.36
	30	17.48	20.55	18.15	20.5	**23.20**	19.07	20.93
	40	22.36	27.02	23.75	26.2	**29.30**	26.17	27.44
	50	28.63	33.3	29.57	33.12	32.83	32.79	**33.31**
	100	55.11	62.31	57.49	62.72	55.78	**66.06**	64.9
	150	75.82	81.42	77.27	80.94	73.85	**85.92**	84.15
	200	88.34	91.63	88.78	91.78	84.15	**94.96**	93.84
$\chi^2(4)$	5	7.53	7.52	6.83	**8.40**	7.55	7.31	8.38
	10	12.49	12.71	10.84	12.52	13.65	12.47	**14.34**
	15	17.87	18.81	15.87	19.37	20.18	18.21	**20.34**
	20	22.79	24.45	21.38	25.45	26.9	24.29	**27.25**
	25	28.83	31.47	28.2	31.62	**34.64**	32.04	34.48
	30	35.45	38.87	35	39.77	41.44	39.55	**42.07**
	40	47.26	52.1	48.1	52	53.31	54.19	**55.67**
	50	59.32	64.21	59.55	63.68	62.12	67.96	**68.35**
	100	91.96	93.2	90.83	93.39	89.79	**96.13**	95.92
	150	98.9	99.05	98.33	98.97	97.78	**99.76**	99.67
	200	99.86	99.85	99.71	99.88	99.48	**99.97**	99.94
$\chi^2(6)$	5	9.01	8.76	7.76	9.82	9.09	8.82	**9.91**
	10	18.46	18.81	15.91	18.81	20.24	18.83	**21.69**
	15	28.86	29.97	25.94	30.47	31.66	30.47	**33.50**
	20	39.04	40.25	36.76	40.41	42.5	43.67	**46.71**
	25	49.3	52.05	47.95	51.09	54.19	55.2	**57.70**
	30	59.48	62.14	58.25	62.56	63.71	66.24	**68.38**
	40	75.66	78.24	74.9	76.43	78	82.77	**83.69**
	50	86.15	86.82	83.99	86.31	85.44	91.07	**91.42**
	100	99.54	99.47	99.07	99.45	98.92	99.8	**99.81**
	150	99.99	99.99	99.99	**100**	99.96	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(10)$	5	11.25	10.8	9.24	11.59	11.31	10.15	11.63
	10	25.03	25.23	21.74	25.71	26.69	26.66	**29.69**
	15	40.73	41.76	37.59	42.23	42.87	45.36	**48.66**
	20	55.33	55.86	51.79	55.47	57.69	61.25	**63.83**
	25	67.44	68.15	64.6	67.48	69.71	73.69	**75.90**
	30	77.25	77.87	74.37	78.26	78.83	83.11	**84.53**
	40	89.81	89.72	87.86	89.06	89.67	93.82	**94.33**
	50	95.93	96.11	94.85	95.74	95.32	98.07	**98.18**
	100	99.97	99.96	99.91	99.98	99.89	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	4.94	4.86	5.46	5.07	4.92	**5.60**	4.98
	10	6.6	6.45	**10.54**	6.48	6.05	8.55	7.11
	15	8.41	8.25	**13.68**	8.24	7.28	10.84	9.22
	20	9.98	9.73	**17.84**	9.63	8.53	12.6	11.09
	25	11.5	10.76	**20.89**	10.67	8.9	13.91	12.43
	30	13.09	12.49	**23.86**	12.69	10.04	16.17	14.93
	40	16.56	15.92	**29.88**	15.67	11.7	19.52	18.62
	50	19.22	18.29	**34.85**	18.23	12.62	22.86	21.12
	100	34.56	36.09	**59.85**	38.55	23.17	41.63	40.03
	150	48.77	55.58	**76.67**	57.09	37.67	62.01	57.11
	200	62.53	71.6	**87.66**	74.74	50.32	77.89	71.49
$HN(0,1)$	5	7.71	7.56	6.47	**8.05**	7.73	7.14	8.01
	10	11.95	11.51	9.81	11.89	11.87	11.56	**13.63**
	15	15.4	14.68	12.49	14.9	15.01	15.28	**17.56**
	20	18.66	17.42	15.07	17.37	19.53	19.5	**21.97**
	25	22.44	20.66	17.79	20.32	23.72	23.84	**26.89**
	30	26.36	24.34	20.92	24.39	28.17	28.53	**31.49**
	40	33.68	31.05	25.74	29.35	37.97	38.52	**42.03**
	50	41.27	38.31	29.78	35.99	44.46	47.83	**50.45**
	100	71.14	70.53	51.8	68.05	74.89	80.67	**82.66**
	150	87.36	87.97	68.69	85.55	90.08	94.22	**94.95**
	200	95.33	95.82	80.58	95.21	95.53	98.39	**98.51**
$Wbl(0,2,2)$	5	12.13	11.97	10.16	13.08	12.18	11.35	**13.10**
	10	26.38	26.68	23.32	27.15	27.95	28.51	**31.66**
	15	42	41.66	37.49	42.47	42.88	46.8	**49.91**
	20	56.21	55.18	51.39	56.12	58.68	63.61	**66.42**
	25	69.1	67.8	63.63	68.26	71.4	76.72	**78.60**
	30	79.46	78.94	75.08	78.38	81.81	86.19	**87.49**
	40	90.65	90.41	86.89	89.52	92.1	95.26	**95.75**
	50	96.17	96.11	93.6	95.57	96.25	98.61	**98.75**
	100	99.99	**100**	99.95	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Wbl(0,0.5,0.5)$	5	8.09	8.08	14.47	8.58	8.09	**15.83**	9.42
	10	34.64	35.48	33.86	33.55	23.84	**48.41**	38.89
	15	58.36	59.64	48.38	57.67	39.07	**72.17**	64.11
	20	75.35	76.39	59.78	75.66	51.32	**86.42**	80
	25	85.75	86.82	70.47	86.84	62.04	**93.62**	89.27
	30	92.51	93.55	79.13	93.13	72.78	**97.25**	94.6
	40	97.97	98.55	91.08	98.43	86.55	**99.45**	98.89
	50	99.52	99.64	97.03	99.67	93.63	**99.93**	99.71
	100	**100**	**100**	**100**	**100**	99.96	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bolded number is the highest power among the seven tests for each sample size.

Table 5. Powers (in %) for testing exponential distribution with BLIE and $\alpha = 0.05$ against the alternative distributions from Group II

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$U(0,1)$	5	15.67	15.55	14.28	16.24	15.71	16.46	**17.66**
	10	31.33	29.25	27.39	28.75	27.13	37.18	**38.87**
	15	45.01	40.55	37.26	39.61	47.03	55.29	**56.27**
	20	56.22	52.96	46.61	50.21	**75.36**	69.99	70.11
	25	66.49	72.22	55.16	64.58	**91.51**	81.01	80.71
	30	76.06	87.6	63.65	80.84	**97.90**	88.46	88.06
	40	86.99	98.51	76.27	96.24	**99.93**	96.62	96.14
	50	93.95	99.94	83.99	99.64	**100**	99.18	98.96
	100	99.95	**100**	99.03	**100**	**100**	**100**	**100**
	150	**100**	**100**	99.97	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	5	17.46	17.34	15.33	17.35	17.48	17.36	**19.58**
	10	39.35	37.79	34.76	38.38	37.23	45.71	**48.30**
	15	59.77	56.97	53.34	56.85	59.87	69.63	**71.17**
	20	74.61	72.36	68.37	71.85	82.53	85.9	**86.51**
	25	85.39	85.76	79.74	83.82	93.97	93.73	**94.01**
	30	92.35	94.41	87.41	92.36	**98.52**	97.28	97.43
	40	97.95	99.47	95.49	98.83	**99.92**	99.64	99.57
	50	99.66	99.96	98.94	99.96	**99.99**	99.96	99.96
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,5)$	5	11.54	11.21	9.58	11.68	11.55	10.35	12.07
	10	22.41	22.2	19.41	22.78	23.13	23.4	**26.49**
	15	34.2	33.89	29.72	35.67	34.7	38.87	**41.73**
	20	46.85	46.9	42.85	46.37	50.73	55.33	**58.30**
	25	59.29	59.51	54.71	58.35	65.09	69.04	**70.72**
	30	69.49	69.56	64.45	68.86	74.73	78.33	**79.83**
	40	83.18	84.2	78.88	82.76	88.58	91.46	**92.18**
	50	91.68	92.59	87.17	91.88	94.65	**96.61**	96.51
	100	99.89	99.95	99.62	99.97	99.97	**99.99**	**99.99**
	150	**100**	**100**	99.99	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(5,1.5)$	5	29.41	29.25	26.77	30.17	29.43	32.5	**34.46**
	10	67.79	65.79	63.03	66.24	63.23	77.22	**78.49**
	15	88.88	87.03	84.94	86.69	91.03	94.62	**94.72**
	20	96.36	96.32	94.22	95.58	**99.06**	98.85	98.88
	25	99.04	99.53	97.9	99.24	**99.93**	99.83	99.83
	30	99.77	99.94	99.41	99.92	**100**	99.96	99.97
	40	99.99	**100**	99.99	**100**	**100**	**100**	**100**
	50	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	5	15.37	17.09	17.51	15.55	15.02	**19.08**	18.55
	10	25.68	23.86	24.3	23.7	17.77	**33.07**	30.86
	15	34.68	30.67	29.02	29.62	39.82	**46.82**	42.86
	20	42.05	43.44	33.96	37.41	**67.97**	58.65	52.89
	25	51.63	64.34	39.1	53.45	**85.32**	69.81	63.54
	30	60.5	82.04	46.19	72.82	**94.24**	79.64	73.03
	40	73.47	96.83	56.8	93.4	**99.34**	91.34	85.51
	50	84.76	99.67	66.83	99.02	**99.97**	97.12	93.44
	100	99.58	**100**	95.72	**100**	**100**	**100**	99.94
	150	**100**	**100**	99.8	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	4.2	4.29	5.74	4.68	4.22	**6.20**	4.43
	10	6.89	7.12	5.54	6.57	3.41	**11.14**	7.24
	15	11.39	12.59	6.15	11.45	3.69	**19.21**	12.77
	20	16.4	18.64	5.8	18.03	4.06	**27.30**	18.89
	25	21.82	25.79	6.1	25.21	5.23	**35.77**	25.36
	30	29.42	35.96	7.52	34.63	7.15	**46.15**	33.48
	40	41.8	50.45	10.36	49.5	10.79	**61.25**	47.45
	50	53.19	63.21	14.96	62.88	13.39	**73.41**	58.37
	100	88.7	95.48	62	95.76	46.13	**97.09**	91.98
	150	98.05	99.38	90.74	99.59	79.03	**99.79**	98.87
	200	99.75	99.94	98.74	**99.99**	94.15	**99.99**	99.85
$beta(1,2)$	5	9.27	9	7.91	9.37	9.29	8.25	**9.51**
	10	13.61	13.05	11.31	13.28	12.79	13.94	**16.16**
	15	18.9	17.48	15.04	17.5	17.18	19.96	**21.98**
	20	22.78	20.16	18.07	20.65	24.65	26.83	**28.78**
	25	27.42	24.72	20.68	24.74	34.97	33.24	**35.41**
	30	33.8	31.55	25.3	28.85	**46.92**	39.23	41.61
	40	42.05	45	30.49	37.17	**68.24**	51.83	53.67
	50	50.05	59.11	33.95	51.04	**82.03**	61.94	62.77
	100	82.37	98.78	58.94	96.71	**99.93**	92.94	92.8
	150	95.21	**100**	77.37	99.97	**100**	99.26	99.11
	200	99.06	**100**	89.27	**100**	**100**	99.94	99.93

The bolded number is the highest power among the seven tests for each sample size.

Table 6. Powers (in %) for testing exponential distribution with BLIE and $\alpha = 0.05$ against the alternative distributions from Group III

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$Laplace(0,1)$	5	25.61	24.7	21.48	26.16	25.67	25.04	**27.65**
	10	60.42	60.3	56.88	61.58	60.62	62.09	**65.41**
	15	83.94	83.34	81.32	82.93	82.41	85.7	**87.23**
	20	93.38	92.83	91.86	92.78	92.59	94.33	**95.17**
	25	97.35	97.23	96.71	97.19	96.87	98.02	**98.31**
	30	98.95	98.74	98.56	99.07	98.51	99.26	**99.39**
	40	99.92	99.89	99.85	99.83	99.75	99.95	**99.96**
	50	99.99	99.97	99.97	99.98	99.98	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$logistic(0,1)$	5	21.7	20.94	18.16	22.39	21.76	21.18	23.72
	10	53	52.19	48.92	53.43	52.87	56.92	**60.18**
	15	77.37	75.92	72.94	75.67	76.04	81.49	**83.42**
	20	89.05	88.33	86.86	88.07	88.63	92.34	**93.20**
	25	94.98	94.46	93.29	94.59	94.89	96.95	**97.39**
	30	97.64	97.47	**99.51**	97.89	97.6	98.76	99
	40	99.75	99.65	**99.92**	99.57	99.62	99.88	99.92
	50	99.97	99.96	**100**	99.95	99.92	99.98	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Cauchy(0,1)$	5	36.91	36.15	34.7	37.87	36.95	37.42	**38.16**
	10	71.92	72.42	71.82	72.93	73.18	72.74	**73.67**
	15	89.01	90.12	89.82	89.63	90.32	90.3	**90.66**
	20	95.56	96.1	96.2	96	96.2	96.62	**96.64**
	25	98.37	98.53	98.63	98.44	98.43	**98.96**	98.84
	30	99.26	99.48	99.53	99.57	99.33	**99.61**	99.59
	40	99.92	99.93	99.93	99.91	99.9	**99.97**	**99.97**
	50	99.98	99.96	99.98	99.98	99.95	**99.99**	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	19.96	19.38	16.72	20.56	20	19.36	**21.68**
	10	48.77	48.2	44.51	48.66	47.92	53.85	**56.93**
	15	72.26	70.56	67.53	70.91	70.84	78.64	**80.52**
	20	86.74	85.46	83.21	85.28	86.97	91.58	**92.48**
	25	93.69	93.11	91.48	92.8	94.22	96.61	**97.10**
	30	97.4	96.97	95.94	96.93	97.79	98.89	**98.97**
	40	99.53	99.6	99.36	99.54	99.7	99.87	**99.87**
	50	99.9	99.93	99.74	99.9	99.97	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(1)$	5	36.95	36.49	35.15	37.38	36.99	37.63	38.10
	10	71.96	72.62	71.82	73.13	73.33	72.87	**73.95**
	15	88.84	89.8	89.55	88.73	90.01	90.29	**90.42**
	20	95.55	95.99	95.99	95.8	95.96	96.28	**96.40**
	25	98.1	98.47	98.38	98.38	98.5	98.73	**98.75**
	30	99.44	99.57	99.54	99.55	99.46	**99.70**	99.64
	40	99.94	99.94	99.94	99.91	99.89	**99.96**	**99.96**
	50	**100**	**100**	**100**	99.99	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	24.99	24.26	21.54	25.92	25.03	24.09	**26.51**
	10	57.86	57.82	54.6	59.09	58.19	60.54	**63.44**
	15	79.54	79.53	77.17	79.71	79.48	81.94	**83.82**
	20	91.11	90.99	89.6	90.34	90.63	92.85	**93.56**
	25	96.16	96.11	95.66	96.26	95.85	97.23	**97.55**
	30	98.54	98.27	98.01	98.24	97.99	99.02	**99.15**
	40	99.78	99.71	99.69	99.74	99.66	99.84	**99.87**
	50	99.92	99.95	99.86	99.94	99.88	**99.96**	**99.96**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(4)$	5	23.11	22.34	19.63	24.39	23.22	22.4	24.80
	10	56.85	56.46	53.18	56.42	56.33	59.36	**62.35**
	15	77.96	77.74	75.06	78.02	77.67	80.79	**82.63**
	20	89.87	89.29	88.05	89.12	89.5	92.19	**92.92**
	25	96.04	95.82	95.24	95.62	95.75	97.18	**97.59**
	30	98.27	98	97.55	98.3	97.75	98.93	**99.11**
	40	99.67	99.62	99.54	99.61	99.54	99.88	**99.91**
	50	**100**	99.98	99.95	99.94	99.94	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(6)$	5	22.14	21.71	18.71	22.41	22.19	21.57	**24.02**
	10	54.25	53.75	50.25	54.45	53.79	57.52	**60.85**
	15	76.8	76.15	73.28	75.82	75.88	80.59	**82.37**
	20	89.28	88.48	86.95	88.44	88.94	91.94	**92.83**
	25	95.62	95.19	94.29	95.04	95.05	97.1	**97.37**
	30	98.15	98.08	97.46	97.75	97.92	98.83	**99.06**
	40	99.72	99.64	99.52	99.72	99.6	99.83	**99.87**
	50	99.98	99.98	99.95	99.94	99.94	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bolded number is the highest power among the seven tests for each sample size.

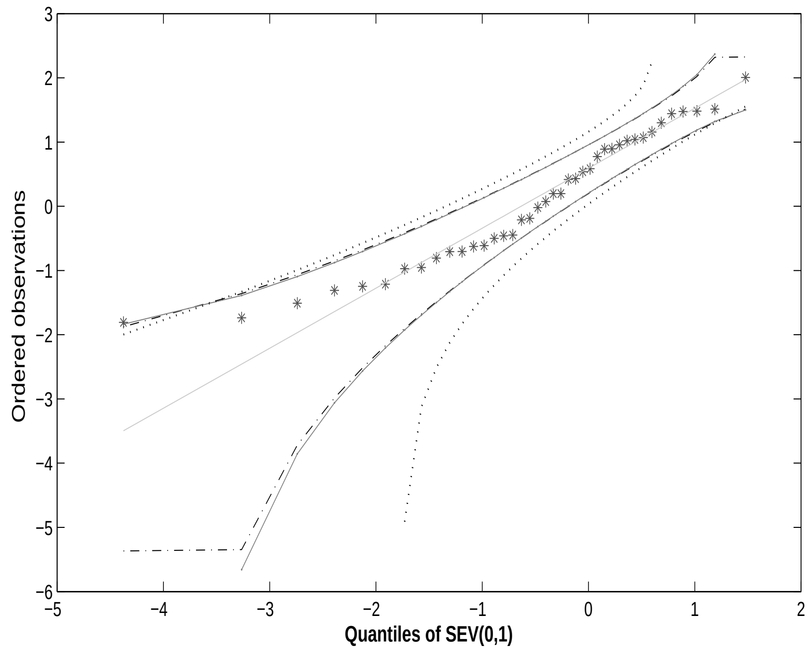

Figure 1. The Q-Q plot and simultaneous probability intervals of D_m (dot dashed lines), D_{be} (solid lines) and D_{bi} (dotted lines) for testing the $SEV(\mu, \sigma)$ distribution by using BLUE at $\alpha = 0.05$

5. Illustrative examples

Example 1 The following sample of $n = 40$ observations is available: 0.1638, 0.176, 0.2208, 0.2697, 0.2872, 0.2976, 0.3782, 0.3851, 0.4464, 0.4934, 0.4946, 0.5341, 0.5413, 0.6063, 0.631, 0.6395, 0.8083, 0.829, 0.9798, 1.0765, 1.2162, 1.2174, 1.5189, 1.539, 1.7137, 1.7962, 2.1652, 2.4304, 2.445, 2.6073, 2.772, 2.8333, 2.9133, 3.1765, 3.6735, 4.2328, 4.3731, 4.4028, 4.5422 and 7.4225. We want to test whether the population from which the sample is taken has the distribution $SEV(\mu, \sigma)$ for some unknown μ and $\sigma > 0$. Following the recommendations in the last section, we can apply any one of the D_m, D_{be} and D_{bi} tests at $\alpha = 0.05$. The Q-Q plot together with the corresponding intervals for the $Y_{[k]}$'s of D_m, D_{be} and D_{bi} are given in Figure 1. Since $Y_{[1]}$ is outside the corresponding interval of each of D_m, D_{be} and D_{bi} tests, H_0 is rejected by each of D_m, D_{be} and D_{bi} tests, i.e. we can claim that the sample does not follow a $SEV(\mu, \sigma)$ distribution.

For the non-graphical tests AD and CvM, the tests statistics of AD and CvM are 0.5111 and 0.0783, respectively. Also, the corresponding critical values of AD and CvM at $\alpha = 0.05$ are 0.7529 and 0.1240, respectively. Hence the hypothesis H_0 is not rejected by either AD or CvM in this case.

Example 2 The following sample of $n = 40$ observations is available: 3.4966, 3.6591, 4.2103, 4.7391, 4.9138, 5.0151, 5.7313, 5.7879, 6.264, 6.6019, 6.6103, 6.8772, 6.9248, 7.3341, 7.4817, 7.5318, 8.4437, 8.5465, 9.2456, 9.6554, 10.2049, 10.2092, 11.2582, 11.3228, 11.8595, 12.0993, 13.0847, 13.72, 13.7534, 14.1161, 14.4676, 14.5948, 14.7573, 15.2703, 16.1591, 17.0582, 17.2696, 17.3136, 17.5179 and 20.9488. We want to test that H_0 the population from which the sample is taken has the distribution $Exp(\mu, \sigma)$ for some unknown μ and $\sigma > 0$.

The usual Q-Q plot with the corresponding intervals for the $Y_{[k]}$'s of D_{bi} with $\alpha = 0.05$ are given in Figure 2. Since several points are outside the corresponding intervals of D_{bi}, e.g., $Y_{[3]}$, $Y_{[4]}$, $Y_{[39]}$ and $Y_{[40]}$, the null hypothesis H_0 is rejected by D_{bi}.

For the non-graphical tests, the test statistics AD and CvM are 1.5627 and 0.2773, respectively. Also, the critical values at $\alpha = 0.05$ and $n = 40$ are 1.1755 and 0.2107, respectively. Hence the null hypothesis H_0 is also rejected by AD or CvM.

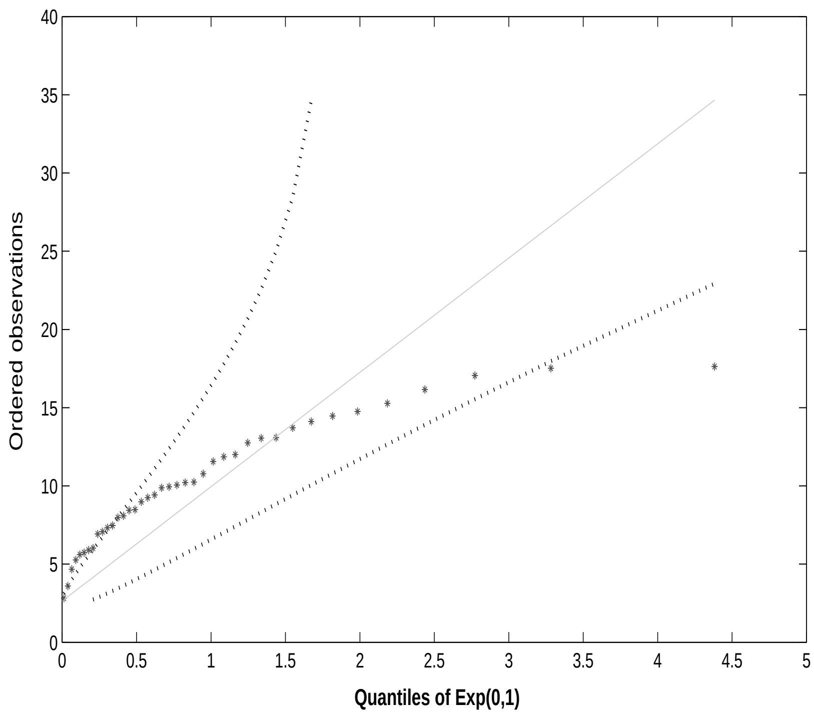

Figure 2. The Q-Q plot and simultaneous probability intervals of D_{bi} (dotted lines) for testing $Exp(\mu, \sigma)$ distribution by using BLIE at $\alpha = 0.05$

6. Conclusions

Generally, the Kolmogorov-Smirnov test (D test) has a very low power. Although the Anderson-Darling and Craḿer-von-Mises tests are non-graphical, they may not be more powerful than the graphical tests. According to Wanpen et al.(2015), the D_{bi} and D_e tests should be used for testing normality based on a simple random sample. For testing the Weibull and exponential distributions, the D_{be}, D_{bi} and D_{sp} tests should be used. Although the D_e test is one of the graphical tests recommended for testing normality when a simple random sample is considered, it is a bad choice for testing Weibull and exponential distributions.

Specifically, we obtain the simultaneous $1 - \alpha$ probability intervals suitable for Q-Q plots on testing the Weibull and exponential distributions. They become the objective judgement on Q-Q plots for practitioners who want to use the graphical test.

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Dividend moments for two classes of risk processes with phase-type interclaim times

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Abstract

In this paper, we consider the distribution of discounted dividend payments until ruin under a risk model with two independent classes of claims in which both of the two interclaim times have phase-type distributions and a constant dividend barrier. We obtain the integro-differential equations with boundary conditions for the moment-generating function of the sum of the discounted dividend payments until ruin. Explicit expressions for arbitrary moments of the discounted dividend payments are derived if the distribution of the two classes claim amount both belong to the rational family. Finally, numerical illustrations are presented to show how the results are applied.

Keywords: Two classes of risk processes, Dividend payments, Moment-generating function, Dividend barrier, Phase-type distribution.

2000 AMS Classification: 62P05, 91B30.

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1. Introduction

The ruin problems for a risk model involving two independent classes of risks have been considered by many researchers, see, for example, [9], [10], [15], and among others. As an extension of these papers, [5] investigated the risk model with two classes of renewal risk processes by assuming that both of the two claim number processes have phase-type interclaim times. The topics of these literatures are concentrated on the Gerber-Shiu discounted penalty function, which is an important tool to quantify the riskiness of the risk model.

In recent years, particular attention has been devoted to the risk models with dividend strategies. We refer the readers to, e.g. [6], [12], [13], [14] for details. The distribution of

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the discounted sum of dividend payments until ruin which is an important quantity in assessing the quality of dividend strategies has been studied by [7], [11], and the references therein. In particular, [1] presented some results on the distribution of dividend payments until ruin in a Sparre Andersen risk model with generalized Erlang(n)-distributed interclaim times and a constant dividend barrier which complemented the results of [8]. [16] considered dividend payments with a threshold strategy in the compound Poisson risk model perturbed by diffusion. [4] extended the results of [16] via assuming that the interclaim times follow a generalized Erlang(n) distribution. As a more general framework, [3] considered surplus processes of which the claim number is a Markovian arrival process perturbed by diffusion with dividend barrier strategies.

The main purpose of the current paper is to investigate the distribution of the discounted sum of dividend payments until ruin for two classes of risk processes in the presence of a constant dividend barrier, where both of the two claim number processes have phase-type interclaim. This paper is a natural extension of [1] and enriches the results for two classes of renewal risk processes. The rest of the paper is structured as follows. Section 2 describes the risk model. In Section 3, we derive systems of integro-differential equations for the moment-generating function of the sum of discounted dividend payments until ruin. Section 4 presents the results for arbitrary moments of the discounted dividend payments and derives explicit expressions when the two classes claim amount distributions both belong to the rational family. In Section 5, a numerical example is given.

2. Model setup

The surplus process $R(t)$ of an insurance portfolio is given by

$$R(t) = u + ct - S(t), \quad t \geq 0, \tag{2.1}$$

where $u \geq 0$ is the initial surplus, c denotes the insurer's premium income per unit time, and the aggregate-claim process $\{S(t) : t \geq 0\}$ is defined by

$$S(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i, \quad t \geq 0,$$

where $\{X_1, X_2, \cdots\}$ and $\{Y_1, Y_2, \cdots\}$ are independent and identically distributed (i.i.d.) positive random variables representing the successive individual claim amounts from the first and the second class, respectively. The random variables $\{X_1, X_2, \cdots\}$ are assumed to have common cumulative distribution function $F(x) = 1 - \bar{F}(x), x \geq 0$, with probability density function $f(x) = F'(x)$, of which the Laplace transform is $\tilde{f}(s) = \int_0^\infty e^{-sx} f(x)dx$, $s \in \mathbb{C}$, $\mathbb{C}$ denotes the complex space. Similarly, common cumulative distribution function, density function and the Laplace transform of the density function of $\{Y_1, Y_2, \cdots\}$ are given by $G(x) = 1 - \bar{G}(x), x \geq 0$, $g(x) = G'(x)$ and $\tilde{g}(s) = \int_0^\infty e^{-sx} g(x)dx$. The renewal processes $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ denote the number of claims up to time t caused by the first and the second class of claim respectively, and are defined as follows:

$$N_1(t) = \sup\{n : T_1 + T_2 + \cdots + T_n \leq t\},$$

$$N_2(t) = \sup\{n : V_1 + V_2 + \cdots + V_n \leq t\},$$

where the i.i.d. interclaim times $\{T_1, T_2, \cdots\}$ have common cumulative distribution function $K_1(t), t \geq 0$ and density function $k_1(x) = K_1'(x)$, and $\{V_1, V_2, \cdots\}$ have common cumulative distribution function $K_2(t), t \geq 0$ and density function $k_2(x) = K_2'(x)$.

In addition, we assume that $\{X_1, X_2, \cdots\}$, $\{Y_1, Y_2, \cdots\}$, $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are mutually independent. The net profit condition is given by $c > E(X_1)/E(T_1) + E(Y_1)/E(V_1)$.

In the present paper, we consider the risk model (2.1) with a constant dividend barrier $d(\geq 0)$. For such a dividend strategy, it is assumed that whenever the surplus process reaches the level d, the premium income is paid out as dividends to policyholders; otherwise, no dividend is paid. Let $R_d(t)$ be the surplus of an insurance company at time t under a constant dividend barrier d, then

$$dR_d(t) = \begin{cases} cdt - dS(t), & R_d(t) < b, \\ -dS(t), & R_d(t) \geq b. \end{cases}$$

The time of (ultimate) ruin is $T = \inf\{t | R(t) < 0\}$, where $T = \infty$ if $R(t) \geq 0$ for all $t \geq 0$. The probability of ruin is $\psi(u) = Pr(T < \infty)$.

Denote by $D(t)$ the cumulative amount of dividends paid out up to time t and $\delta > 0$ the force of interest, then $\mathbb{D} = \int_0^T e^{-\delta t} dD(t)$ is the present value of all dividends until ruin time T. In the following text, we turn to the moment generating function of $\mathbb{D}$,

$$M(u, y, d) = E[e^{y\mathbb{D}} | R(0) = u]$$

(for those values of y where it exists) and the rth moment

$$W(u, r, d) = E[\mathbb{D}^r | R(0) = u], \quad r \in \mathbb{N}.$$

Note that $W(u, 0, d) \equiv 1$. We will always assume that $0 \leq u \leq d$ (otherwise the overflow is immediately paid out as dividends) and that $M(u, y, d)$ and $W(u, r, d)$ are sufficiently smooth functions in u and y, respectively.

Throughout the text of the paper, all bold-faced letters represent either vectors or matrices and all vectors are column vectors. We assume that the distribution $K_1(t)$ of the interclaim time random variable T_1 is phase-type with representation $(\boldsymbol{\alpha}^\top, \mathbf{A}, \mathbf{a})$, where $\boldsymbol{\alpha}^\top = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, with $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$, $\mathbf{A} = (a_{ij})_{i,j=1}^n$ is an $n \times n$ matrix with $a_{ii} < 0, a_{ij} \geq 0$, for $i \neq j$, $\sum_{j=1}^n a_{ij} \leq 0$, for any $i = 1, 2, \cdots, n$, and $\mathbf{a} = (a_1, a_2, \cdots, a_n)^\top$ with $\mathbf{a} = -\mathbf{A}\mathbf{e}_n$, where $\mathbf{x}^\top$ denotes the transpose of $\mathbf{x}$ and $\mathbf{e}_n$ denotes a n-dimensional column vector with all elements being one. Following [2], we have

$$K_1(t) = 1 - \boldsymbol{\alpha}^\top e^{\mathbf{A}t}\mathbf{e}_n, \quad k_1(t) = \boldsymbol{\alpha}^\top e^{\mathbf{A}t}\mathbf{a}, \quad t \geq 0,$$

and

$$\tilde{k}_1(s) = \int_0^\infty e^{-st} k_1(t) dt = \boldsymbol{\alpha}^\top (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}. \tag{2.2}$$

By the definition of phase-type distributions, each of the interclaim times $T_i, i = 1, 2, \cdots$, corresponds to the time to absorption in a terminating continuous-time Markov Chain, say, $I_t^{(i)}$ with n transient states $\{E_1, E_2, \cdots, E_n\}$ and one absorbing state E_0.

Correspondingly, the distribution $K_2(t)$ of the interclaim time random variable V_1 is phase-type with representation $(\boldsymbol{\beta}^\top, \mathbf{B}, \mathbf{b})$, where $\boldsymbol{\beta}^\top = (\beta_1, \beta_2, \cdots, \beta_m)$, $\mathbf{B} = (b_{ij})_{i,j=1}^m$ is an $m \times m$ matrix, $\mathbf{b} = (b_1, b_2, \cdots, b_m)^\top$ with $\mathbf{b} = -\mathbf{B}\mathbf{e}_m$. Then we have

$$K_2(t) = 1 - \boldsymbol{\beta}^\top e^{\mathbf{B}t}\mathbf{e}_m, k_2(t) = \boldsymbol{\beta}^\top e^{\mathbf{B}t}\mathbf{b}, t \geq 0,$$

and

$$\tilde{k}_2(s) = \int_0^\infty e^{-st} k_2(t) dt = \boldsymbol{\beta}^\top (s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}. \tag{2.3}$$

Similarly, $J_t^{(i)}$ denotes the terminating continuous-time Markov Chain of $V_i, i = 1, 2, \cdots$, with m transient states $\{F_1, F_2, \cdots, F_m\}$ and one absorbing state F_0.

Now, we construct a two-dimensional Markov process $\{(I(t),J(t));t\geq 0\}$ by piecing the $\{I_t^{(i)};i=1,2,\cdots\}$ and $\{J_t^{(i)};i=1,2,\cdots\}$ together,

$$I(t)=\{I_t^{(1)}\},0\leq t<T_1,\quad I(t)=\{I_{t-T_1}^{(2)}\},T_1\leq t<T_1+T_2,\cdots,$$

$$J(t)=\{J_t^{(1)}\},0\leq t<V_1,\quad J(t)=\{J_{t-V_1}^{(2)}\},V_1\leq t<V_1+V_2,\cdots.$$

So $\{(I(t),J(t));t\geq 0\}$ is the underlying state process with states $\{(E_1,F_1),(E_2,F_1),\cdots,(E_n,F_1),(E_1,F_2),(E_2,F_2),\cdots,(E_n,F_2),\cdots,(E_1,F_m),(E_2,F_m),\cdots,(E_n,F_m)\}$, initial distribution $\boldsymbol{\gamma}=\boldsymbol{\beta}\otimes\boldsymbol{\alpha}$, where $\otimes$ denotes the Kronecker product of two matrices.

For $k=1,2;i=1,2,\cdots,n;j=1,2,\cdots,m$, let $M^{(k)}(u,y,d)$ denote the moment generating function of $\mathbb{D}$ if the ruin is caused by a claim from class k and $R(0)=u$. $M_{ij}^{(k)}(u,y,d)$ denotes the moment generating function of $\mathbb{D}$ when the ruin is caused by a claim from class k and initial state $(I_0^{(1)},J_0^{(1)})=(E_i,F_j)$, then the moment generating function can be written as

$$M^{(k)}(u,y,d)=\boldsymbol{\gamma}^\top\mathbf{M}^{(k)}(u,y,d), \tag{2.4}$$

where $\mathbf{M}^{(k)}(u,y,d)\equiv\Big(M_{11}^{(k)}(u,y,d),M_{21}^{(k)}(u,y,d),\cdots,M_{n1}^{(k)}(u,y,d),\ M_{12}^{(k)}(u,y,d),$ $M_{22}^{(k)}(u,y,d),\cdots,M_{n2}^{(k)}(u,y,d),\cdots,\ M_{1m}^{(k)}(u,y,d),M_{2m}^{(k)}(u,y,d),\cdots,M_{nm}^{(k)}(u,y,d)\Big)^\top$. Thus

$$M(u,y,d)=\boldsymbol{\gamma}^\top\mathbf{M}(u,y,d)=\boldsymbol{\gamma}^\top[\mathbf{M}^{(1)}(u,y,d)+\mathbf{M}^{(2)}(u,y,d)]. \tag{2.5}$$

Let $W_{ij}(u,r,d)$ denote the rth moment of $\mathbb{D}$ if $(I_0^{(1)},J_0^{(1)})=(E_i,F_j)$. Then the moment can be computed by

$$W(u,r,d)=\boldsymbol{\gamma}^\top\mathbf{W}(u,r,d), \tag{2.6}$$

where $\mathbf{W}(u,r,d)\equiv(W_{11}(u,r,d),W_{21}(u,r,d),\cdots,W_{n1}(u,r,d),\ W_{12}(u,r,d),W_{22}(u,r,d),\cdots,W_{n2}(u,r,d),\cdots,\ W_{1m}(u,r,d),\ W_{2m}(u,r,d),\cdots,\ W_{nm}(u,r,d))^\top$.

3. Integro-differential system for $\mathbf{M}^{(k)}(u,y,d)$

Let $\frac{\partial\cdot}{\partial u}$ and $\frac{\partial\cdot}{\partial y}$ denote the differentiation operators with respect to u and y, respectively.

3.1. Theorem. *The vectors $\mathbf{M}^{(k)}(u,y,d),0\leq u\leq d,k=1,2$ satisfy the following partial integro-differential system, respectively,*

$$\begin{aligned}&\Big(c\frac{\partial}{\partial u}-y\delta\frac{\partial}{\partial y}\Big)\mathbf{M}^{(1)}(u,y,d)+I_{m\times m}\otimes\mathbf{A}\mathbf{M}^{(1)}(u,y,d)+\\&\mathbf{B}\otimes I_{n\times n}\mathbf{M}^{(1)}(u,y,d)+I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\int_0^u\mathbf{M}^{(1)}(u-x,y,d)f(x)dx+\\&(\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\int_0^u\mathbf{M}^{(1)}(u-x,y,d)g(x)dx+(\mathbf{e}_m\otimes\mathbf{a})\bar{F}(u)=\mathbf{0},\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&\Big(c\frac{\partial}{\partial u}-y\delta\frac{\partial}{\partial y}\Big)\mathbf{M}^{(2)}(u,y,d)+I_{m\times m}\otimes\mathbf{A}\mathbf{M}^{(2)}(u,y,d)+\\&\mathbf{B}\otimes I_{n\times n}\mathbf{M}^{(2)}(u,y,d)+I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\int_0^u\mathbf{M}^{(2)}(u-x,y,d)f(x)dx+\\&(\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\int_0^u\mathbf{M}^{(2)}(u-x,y,d)g(x)dx+(\mathbf{b}\otimes\mathbf{e}_n)\bar{G}(u)=\mathbf{0},\end{aligned} \tag{3.2}$$

with boundary conditions

$$\frac{\partial\mathbf{M}^{(k)}(u,y,d)}{\partial u}\bigg|_{u=d}=y\mathbf{M}^{(k)}(d,y,d),\quad k=1,2, \tag{3.3}$$

where $I_{n\times n}$ denotes the $n\times n$ identity matrix, $\mathbf{0}$ denotes a column vector of length mn with all elements being 0.

Proof. Considering an infinitesimal time interval $(0, dt)$ for $0 \le u \le d$, there are four possible events regarding to the occurrence of the claim and change of the environment: (i) no claim arrival and no change of state; (ii) a claim arrival but no change of state; (iii) a change of state but no claim arrival; (iv) two or more events occur. Taking into account the above four events in $(0, dt)$ and using the total expectation formula, it follows that

$$\begin{aligned}
&M_{ij}^{(1)}(u,y,d) \\
&= (1+a_{ii}dt)(1+b_{jj}dt)M_{ij}^{(1)}(u+cdt, ye^{-\delta dt}, d) \\
&+(1+b_{jj}dt)\sum_{k=1,k\neq i}^{n}(a_{ik}dt)M_{kj}^{(1)}(u+cdt, ye^{-\delta dt}, d) \\
&+(1+a_{ii}dt)\sum_{h=1,h\neq j}^{m}(b_{jh}dt)M_{ih}^{(1)}(u+cdt, ye^{-\delta dt}, d) \\
&+(1+b_{jj}dt)(a_i dt)\left[\sum_{s=1}^{n}\alpha_s\int_0^{u+cdt}M_{sj}^{(1)}(u+cdt-x, ye^{-\delta dt}, d)f(x)dx\right. \\
&\left.+\int_{u+cdt}^{\infty}f(x)dx\right] \\
&+(1+a_{ii}dt)(b_j dt)\sum_{r=1}^{m}\beta_r\int_0^{u+cdt}M_{ir}^{(1)}(u+cdt-x, ye^{-\delta dt}, d)g(x)dx + o(dt).
\end{aligned} \tag{3.4}$$

By Taylor expansion,

$$\begin{aligned}
&M_{ij}^{(1)}(u+cdt, ye^{-\delta dt}, d) \\
= \quad &M_{ij}^{(1)}(u,y,d) + cdt\frac{\partial M_{ij}^{(1)}(u,y,d)}{\partial u} + y(e^{-\delta dt}-1)\frac{\partial M_{ij}^{(1)}(u,y,d)}{\partial y} + o(dt).
\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4), dividing by dt and then letting $dt \to 0$, it yields that

$$\begin{aligned}
&c\frac{\partial M_{ij}^{(1)}(u,y,d)}{\partial u} - y\delta\frac{\partial M_{ij}^{(1)}(u,y,d)}{\partial y} + \sum_{k=1}^{n}a_{ik}M_{kj}^{(1)}(u,y,d) + \sum_{h=1}^{m}b_{jh}M_{ih}^{(1)}(u,y,d) \\
&+a_i\left(\sum_{s=1}^{n}\alpha_s\int_0^u M_{sj}^{(1)}(u-x,y,d)f(x)dx + \int_u^{\infty}f(x)dx\right) \\
&+b_j\sum_{r=1}^{m}\beta_r\int_0^u M_{ir}^{(1)}(u-x,y,d)g(x)dx = 0.
\end{aligned} \tag{3.6}$$

Rewriting (3.6) in matrix form and rearranging it, we have (3.1). By similar derivation to (3.4)-(3.6), we get (3.2).

When $u = d$, we have

$$\begin{aligned}
&M_{ij}^{(1)}(d,y,d) \\
&= (1+a_{ii}dt)(1+b_{jj}dt)e^{ycdt}M_{ij}^{(1)}(d, ye^{-\delta dt}, d) \\
&+(1+b_{jj}dt)e^{ycdt}\sum_{k=1,k\neq i}^{n}(a_{ik}dt)M_{kj}^{(1)}(d, ye^{-\delta dt}, d) \\
&+(1+a_{ii}dt)e^{ycdt}\sum_{h=1,h\neq j}^{m}(b_{jh}dt)M_{ih}^{(1)}(d, ye^{-\delta dt}, d) \\
&+(1+b_{jj}dt)(a_i dt)e^{ycdt}\left[\sum_{s=1}^{n}\alpha_s\int_0^{d}M_{sj}^{(1)}(d-x, ye^{-\delta dt}, d)f(x)dx\right. \\
&\left.+\int_{d}^{\infty}f(x)dx\right] \\
&+(1+a_{ii}dt)(b_j dt)e^{ycdt}\sum_{r=1}^{m}\beta_r\int_0^{d}M_{ir}^{(1)}(d-x, ye^{-\delta dt}, d)g(x)dx + o(dt).
\end{aligned} \tag{3.7}$$

It follows from Taylor expansion that

$$
\begin{aligned}
(3.8)\qquad & ycM_{ij}^{(1)}(d,y,d) - y\delta\frac{\partial M_{ij}^{(1)}(d,y,d)}{\partial y} + \sum_{k=1}^{n} a_{ik}M_{kj}^{(1)}(d,y,d) + \sum_{h=1}^{m} b_{jh}M_{ih}^{(1)}(d,y,d) \\
& + a_i\left(\sum_{s=1}^{n}\alpha_s\int_0^d M_{sj}^{(1)}(d-x,y,d)f(x)dx + \int_d^\infty f(x)dx\right) \\
& + b_j\sum_{r=1}^{m}\beta_r\int_0^d M_{ir}^{(1)}(d-x,y,d)g(x)dx = 0.
\end{aligned}
$$

Comparing the above equations with the corresponding equations in (3.6) and utilizing the continuity of $M_{ij}^{(1)}(u,y,d)$ at $u=d$, then

$$\left.\frac{\partial \mathbf{M}^{(1)}(u,y,d)}{\partial u}\right|_{u=d} = y\mathbf{M}^{(1)}(d,y,d).$$

By the same approach, we can obtain the boundary conditions (3.3) for $k=2$. □

3.2. Remark. When $m=1$ and $G(0)=1$, from Eq.(3.1), we have

$$
\begin{aligned}
(3.9)\qquad & \left(c\frac{\partial}{\partial u} - y\delta\frac{\partial}{\partial y}\right)\mathbf{M}^{(1)}(u,y,d) + \mathbf{A}\mathbf{M}^{(1)}(u,y,d) \\
& + \left[\int_0^u \boldsymbol{\alpha}^\top\mathbf{M}^{(1)}(u-x,y,d)f(x)dx + \bar{F}(u)\right]\mathbf{a} = \mathbf{0}.
\end{aligned}
$$

In this case, $\mathbf{M}^{(2)}(u,y,d)$ need not be considered. Specially, when the distribution $K_1(t)$ of the interclaim time is a generalized Erlang(n) distribution, i.e.,

$$\boldsymbol{\alpha}^\top = (1,0,\ldots,0), \mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & -\lambda_n \end{pmatrix}, \mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \lambda_n \end{pmatrix}.$$

Then (3.9) can be expressed as

$$\left(\prod_{i=1}^{n}\frac{y\delta\frac{\partial}{\partial y} - c\frac{\partial}{\partial u} + \lambda_i}{\lambda_i}\right)M^{(1)}(u,y,d) - \int_0^u M^{(1)}(u,y,d)f(x)dx - \bar{F}(u) = 0,$$

which is identical to (2) in [1].

4. The moments of the discounted dividend payments

4.1. Integro-differential system. Adding (3.1) to (3.2), by virtue of (2.5) leads to

$$
\begin{aligned}
(4.1)\qquad & \left(c\frac{\partial}{\partial u} - y\delta\frac{\partial}{\partial y}\right)\mathbf{M}(u,y,d) + I_{m\times m}\otimes\mathbf{A}\mathbf{M}(u,y,d) + \\
& \mathbf{B}\otimes I_{n\times n}\mathbf{M}(u,y,d) + I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\int_0^u\mathbf{M}(u-x,y,d)f(x)dx + \\
& (\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\int_0^u\mathbf{M}(u-x,y,d)g(x)dx + (\mathbf{e}_m\otimes\mathbf{a})\bar{F}(u) \\
& +(\mathbf{b}\otimes\mathbf{e}_n)\bar{G}(u) = \mathbf{0}.
\end{aligned}
$$

Note that $W(u,r,d) = E[\mathbb{D}^r|R(0)=u]$. With the help of the representation

$$\mathbf{M}(u,y,d) = \mathbf{e}_{mn} + \sum_{r=1}^{\infty}\frac{y^r}{r!}\mathbf{W}(u,r,d),$$

by equating the coefficients of $y^r (r\in\mathbb{N})$ in (4.1), using $\mathbf{a} = -\mathbf{A}\mathbf{e}_n$, $\mathbf{b} = -\mathbf{B}\mathbf{e}_m$, $I_{m\times m}\otimes\mathbf{A}\mathbf{e}_{mn} = -I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\mathbf{e}_{mn} = -\mathbf{e}_m\otimes\mathbf{a}$ and $\mathbf{B}\otimes I_{n\times n}\mathbf{e}_{mn} = -(\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\mathbf{e}_{mn} = -\mathbf{b}\otimes\mathbf{e}_n$, we have the following result.

4.1. Theorem. *The vector* $\mathbf{W}(u,r,d), 0 \le u \le d$, *satisfies the following integro-differential system,*

$$(4.2)\qquad \begin{array}{l} c\frac{d\mathbf{W}(u,r,d)}{du} - r\delta\mathbf{W}(u,r,d) + I_{m\times m}\otimes\mathbf{A}\mathbf{W}(u,r,d)+ \\ \mathbf{B}\otimes I_{n\times n}\mathbf{W}(u,r,d) + I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\int_0^u \mathbf{W}(u-x,r,d)f(x)dx+ \\ (\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\int_0^u \mathbf{W}(u-x,r,d)g(x)dx = \mathbf{0}, \end{array}$$

with boundary conditions

$$(4.3)\qquad \frac{\partial \mathbf{W}(u,r,d)}{\partial u}\bigg|_{u=d} = r\mathbf{W}(d,r-1,d).$$

4.2. Remark. When $m=1$ and $G(0)=1$, from Eq.(4.2), we get

$$(4.4)\qquad c\frac{d\mathbf{W}(u,r,d)}{du} - r\delta\mathbf{W}(u,r,d) + \mathbf{A}\mathbf{W}(u,r,d) + (\mathbf{a}\boldsymbol{\alpha}^\top)\int_0^u \mathbf{W}(u-x,r,d)f(x)dx = \mathbf{0}.$$

Furthermore, when the distribution $K_1(t)$ of the interclaim time is a generalized Erlang(n) distribution, see Remark 3.1 for the representation $(\boldsymbol{\alpha}^\top, \mathbf{A}, \mathbf{a})$. Under this scenario, we recover (9) in [1] from (4.4) as follows:

$$\left(\prod_{i=1}^{n}\frac{r\delta - c\frac{\partial}{\partial u} + \lambda_i}{\lambda_i}\right) W(u,r,d) - \int_0^u W(u-x,r,d)f(x)dx = 0.$$

4.2. Explicit results for claim-size with rational family distributions. Now define the Laplace transforms $\tilde{\mathbf{W}}(s,r,d) = \int_0^\infty e^{-su}\mathbf{W}(u,r,d)du$ by ignoring for a moment that $\mathbf{W}(u,r,d)$ is only defined for $0 \le u \le d$.

Taking Laplace transforms on both sides of (4.2) yields

$$(4.5)\qquad \begin{array}{l} \Big[(cs - r\delta)\, I_{mn\times mn} + I_{m\times m}\otimes\mathbf{A} + \mathbf{B}\otimes I_{n\times n} + I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\tilde{f}(s) \\ +\ (\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\tilde{g}(s)\Big]\,\tilde{\mathbf{W}}(s,r,d) = c\mathbf{W}(0,r,d). \end{array}$$

Let $\mathbf{L}(s) = (cs - r\delta)\, I_{mn\times mn} + I_{m\times m}\otimes\mathbf{A} + \mathbf{B}\otimes I_{n\times n} + I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\tilde{f}(s) + (\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\tilde{g}(s)$, and $\mathbf{L}^*(s)$ be the adjoint of matrix $\mathbf{L}(s)$. In the following, we assume $det[\mathbf{L}(s)] \ne 0$. So, from (4.5), it holds that

$$(4.6)\qquad \tilde{\mathbf{W}}(s,r,d) = \frac{\mathbf{L}^*(s)}{det[\mathbf{L}(s)]}c\mathbf{W}(0,r,d).$$

Thanks to [5], the generalized Lundberg's equation $det[\mathbf{L}(s)] = 0$ has exactly mn roots in the right half of the complex plane when $\delta > 0$. We denote them by $\rho_1, \rho_2, \cdots, \rho_{mn}$ respectively, and for simplicity, we assume that they are different from each other.

Next, we present some explicit results for the moments of the discounted dividend payments by assuming that the claim amount distributions F and G are both from the rational family distribution. That is, the Laplace transforms of the density functions are of the forms

$$\tilde{f}(s) = \frac{p_{m_1-1}(s)}{p_{m_1}(s)}, \quad \tilde{g}(s) = \frac{q_{m_2-1}(s)}{q_{m_2}(s)}, \quad m_1, m_2 \in \mathbb{N}^+,$$

where $p_{m_1-1}(s), q_{m_2-1}(s)$ are polynomials of degree m_1-1 and m_2-1 or less, respectively, while $p_{m_1}(s)$ and $q_{m_2}(s)$ are polynomials of degree m_1 and m_2 with only negative roots, and satisfy $p_{m_1-1}(0) = p_{m_1}(0)$, $q_{m_2-1}(0) = q_{m_2}(0)$. Without loss of generality, we assume that $p_{m_1}(s)$ and $q_{m_2}(s)$ have leading coefficient 1. This wide class of distributions includes the phase-type distributions, and in particular, it includes the Erlang, Coxian and exponential distribution and all the mixtures of them.

In what follows, let $h(s) = [p_{m_1}(s)q_{m_2}(s)]^{mn}$. Multiplying both numerator and denominator of (4.6) by $h(s)$ results in

$$\tilde{\mathbf{W}}(s,r,d) = \frac{h(s)\mathbf{L}^*(s)}{h(s)det[\mathbf{L}(s)]} c\mathbf{W}(0,r,d). \tag{4.7}$$

Obviously, the factor $h(s)det[\mathbf{L}(s)]$ of the denominator is a polynomial of degree $mn(m_1+m_2+1)$ with leading coefficient c^{mn}. Therefore, the equation $h(s)det[\mathbf{L}(s)] = 0$ has $mn(m_1+m_2+1)$ roots on the complex plane. We can factorize $h(s)det[\mathbf{L}(s)]$ as follows

$$h(s)det[\mathbf{L}(s)] = c^{mn}\prod_{i=1}^{mn}(s-\rho_i)\prod_{j=1}^{(m_1+m_2)mn}(s+R_j), \tag{4.8}$$

where R_j for each j has positive real part and we assume that all of them are distinct from each other.

Since the numerator $h(s)\mathbf{L}^*(s)$ in (4.7) is a polynomial with degree less than $mn(m_1+m_2+1)$. By the partial fraction decomposition, it follows that

$$\tilde{\mathbf{W}}(s,r,d) = \sum_{j=1}^{mn}\frac{\mathbf{\Gamma}_j(d)}{s-\rho_j} + \sum_{j=1}^{(m_1+m_2)mn}\frac{\mathbf{\Lambda}_j(d)}{s+R_j}, \tag{4.9}$$

where $\mathbf{\Gamma}_j(d)$, for $j = 1, 2, \cdots, mn$, and $\mathbf{\Lambda}_j(d)$, for $j = 1, 2, \cdots, (m_1+m_2)mn$, are the coefficient matrices defined respectively by

$$\mathbf{\Gamma}_j(d) = -\frac{h(\rho_j)\mathbf{L}^*(\rho_j)\mathbf{W}(0,r,d)}{c^{mn-1}\left[\prod_{k=1}^{(m_1+m_2)mn}(R_k+\rho_j)\right]\left[\prod_{i=1,i\neq j}^{mn}(\rho_i-\rho_j)\right]}, \tag{4.10}$$

and

$$\mathbf{\Lambda}_j(d) = \frac{h(-R_j)\mathbf{L}^*(-R_j)\mathbf{W}(0,r,d)}{c^{mn-1}\left[\prod_{k=1}^{mn}(\rho_k+R_j)\right]\left[\prod_{i=1,i\neq j}^{(m_1+m_2)mn}(R_i-R_j)\right]}. \tag{4.11}$$

Obviously, $\mathbf{\Gamma}_j(d)$ and $\mathbf{\Lambda}_j(d)$ depend on dividend barrier d. Inverting (4.9) leads to

$$\mathbf{W}(u,r,d) = \sum_{j=1}^{mn}\mathbf{\Gamma}_j(d)e^{\rho_j u} + \sum_{j=1}^{(m_1+m_2)mn}\mathbf{\Lambda}_j(d)e^{-R_j u}. \tag{4.12}$$

Since we don't need to distinguish $\mathbf{\Gamma}_j(d)$ and $\mathbf{\Lambda}_j(d)$, for notational convenience, (4.12) can be reexpressed as

$$\mathbf{W}(u,r,d) = \sum_{j=1}^{mn(m_1+m_2+1)}\mathbf{\Upsilon}_j(d)e^{\kappa_j u}, \tag{4.13}$$

where $\kappa_j, j = 1, \ldots, mn(m_1+m_2+1)$ denote $mn(m_1+m_2+1)$ roots of $h(s)det[\mathbf{L}(s)] = 0$.

Now we announce that the explicit forms for arbitrary moments of the discounted dividend payments can be obtained from (4.13) if the two classes claim amount distributions both belong to the rational family. The coefficients $\mathbf{\Upsilon}_j(d)$ can be determined by boundary conditions (4.3), and we can obtain the other demand equations for determining these coefficients by substituting (4.13) into (4.2), and equating coefficients of the resulting exponential terms. At the same time, the asymptotic behavior $\lim_{d\to\infty}\mathbf{W}(u,r,d) = \mathbf{0}$ holds.

5. Numerical illustrations

In this section, we will illustrate numerically an application of the main results in this paper. We assume that the claim amounts from class 1 and class 2 both follow exponentially distributions with density functions, respectively,

$$f(x) = \mu_1 e^{-\mu_1 x}, \quad \mu_1 > 0, x > 0, \quad g(y) = \mu_2 e^{-\mu_2 y}, \quad \mu_2 > 0, y > 0.$$

We also assume $\mu_1 \neq \mu_2$ for simplicity. Thus, the Laplace transforms $\tilde{f}(s) = \frac{\mu_1}{s+\mu_1}$, $\tilde{g}(s) = \frac{\mu_2}{s+\mu_2}$. At the same time, we suppose that the interclaim times from class 1 occur following a Poisson process with parameter λ and interclaim times from class 2 occur following a phase-type distribution with the following parameters: $\boldsymbol{\beta}^\top = (1/2, 1/2), \mathbf{B} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$. So, we also have $\boldsymbol{\alpha} = (1), \mathbf{A} = (-\lambda), \mathbf{a} = (\lambda)$, and

$$\mathbf{L}(s) = \begin{pmatrix} cs - r\delta - \lambda - \lambda_1 + \frac{\lambda\mu_1}{s+\mu_1} + \frac{\lambda_1\mu_2}{2(s+\mu_2)} & \frac{\lambda_1\mu_2}{2(s+\mu_2)} \\ \frac{\lambda_2\mu_2}{2(s+\mu_2)} & cs - r\delta - \lambda - \lambda_2 + \frac{\lambda\mu_1}{s+\mu_1} + \frac{\lambda_2\mu_2}{2(s+\mu_2)} \end{pmatrix}.$$

From (4.2), we have

$$\begin{aligned} &c\tfrac{d\mathbf{W}(u,r,d)}{du} - r\delta\mathbf{W}(u,r,d) + \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \mathbf{W}(u,r,d) + \\ &\begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \mathbf{W}(u,r,d) + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \int_0^u \mathbf{W}(u-x,r,d) f(x)dx + \\ &\begin{pmatrix} \frac{\lambda_1}{2} & \frac{\lambda_1}{2} \\ \frac{\lambda_2}{2} & \frac{\lambda_2}{2} \end{pmatrix} \int_0^u \mathbf{W}(u-x,r,d) g(x)dx = \mathbf{0}. \end{aligned} \tag{5.1}$$

Using (4.13), we obtain the representation

$$\mathbf{W}(u,r,d) = \sum_{j=1}^{6} \boldsymbol{\Upsilon}_j(d) e^{\kappa_j u}. \tag{5.2}$$

Obviously, $s = -\mu_2$ is one of the roots of $h(s)det[\mathbf{L}(s)] = 0$. Hence, (5.2) can be rewritten as

$$\mathbf{W}(u,r,d) = \sum_{j=1}^{5} \boldsymbol{\Upsilon}_j(d) e^{\kappa_j u} + \boldsymbol{\Upsilon}_6(d) e^{-\mu_2 u}. \tag{5.3}$$

Substituting (5.3) into (5.1) results in

(5.4)

$$\begin{aligned} &\sum_{j=1}^{5} \mathbf{L}(\kappa_j) \boldsymbol{\Upsilon}_j(d) e^{\kappa_j u} = \\ &\left[\sum_{j=1}^{5} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \boldsymbol{\Upsilon}_j(d) \tfrac{\mu_1}{\kappa_j+\mu_1} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \boldsymbol{\Upsilon}_6(d) \tfrac{\mu_1}{-\mu_2+\mu_1} \right] e^{-\mu_1 u} + \\ &\left\{ \begin{pmatrix} c\mu_2 + r\delta + \lambda + \lambda_1 - \frac{\lambda\mu_1}{-\mu_2+\mu_1} & 0 \\ 0 & c\mu_2 + r\delta + \lambda + \lambda_2 - \frac{\lambda\mu_1}{-\mu_2+\mu_1} \end{pmatrix} \boldsymbol{\Upsilon}_6(d) + \right. \\ &\left. \sum_{j=1}^{5} \begin{pmatrix} \frac{\lambda_1}{2} & \frac{\lambda_1}{2} \\ \frac{\lambda_2}{2} & \frac{\lambda_2}{2} \end{pmatrix} \boldsymbol{\Upsilon}_j(d) \tfrac{\mu_2}{\kappa_j+\mu_2} - \begin{pmatrix} \frac{\lambda_1}{2} & \frac{\lambda_1}{2} \\ \frac{\lambda_2}{2} & \frac{\lambda_2}{2} \end{pmatrix} \boldsymbol{\Upsilon}_6(d) \mu_2 u \right\} e^{-\mu_2 u}, \end{aligned}$$

from which we have the following conditions

$$\sum_{j=1}^{5} \mathbf{L}(\kappa_j) \boldsymbol{\Upsilon}_j(d) = \mathbf{0}, \tag{5.5}$$

$$\sum_{j=1}^{5}\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\boldsymbol{\Upsilon}_j(d)\frac{\mu_1}{\kappa_j+\mu_1}+\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\boldsymbol{\Upsilon}_6(d)\frac{\mu_1}{-\mu_2+\mu_1}=\mathbf{0}, \tag{5.6}$$

$$\begin{aligned}&\begin{pmatrix}c\mu_2+r\delta+\lambda+\lambda_1-\frac{\lambda\mu_1}{-\mu_2+\mu_1} & 0\\ 0 & c\mu_2+r\delta+\lambda+\lambda_2-\frac{\lambda\mu_1}{-\mu_2+\mu_1}\end{pmatrix}\boldsymbol{\Upsilon}_6(d)+\\ &\sum_{j=1}^{5}\begin{pmatrix}\frac{\lambda_1}{2} & \frac{\lambda_1}{2}\\ \frac{\lambda_2}{2} & \frac{\lambda_2}{2}\end{pmatrix}\boldsymbol{\Upsilon}_j(d)\frac{\mu_2}{\kappa_j+\mu_2}=\mathbf{0},\end{aligned} \tag{5.7}$$

and

$$\begin{pmatrix}\frac{\lambda_1}{2} & \frac{\lambda_1}{2}\\ \frac{\lambda_2}{2} & \frac{\lambda_2}{2}\end{pmatrix}\boldsymbol{\Upsilon}_6(d)=\mathbf{0}. \tag{5.8}$$

For $r=1$ we have from (4.3) $\left.\frac{\partial \mathbf{W}(u,1,d)}{\partial u}\right|_{u=d}=\mathbf{e}_{mn}$, which yields

$$\sum_{j=1}^{5}\boldsymbol{\Upsilon}_j(d)\kappa_j e^{\kappa_j d}-\boldsymbol{\Upsilon}_6(d)\mu_2 e^{-\mu_2 d}=\mathbf{e}_2. \tag{5.9}$$

By virtue of the asymptotic behavior $\lim_{d\to\infty}\mathbf{W}(u,r,d)=\mathbf{0}$, we have

$$\lim_{d\to\infty}\left[\sum_{j=1}^{5}\boldsymbol{\Upsilon}_j(d)e^{\kappa_j u}+\boldsymbol{\Upsilon}_6(d)e^{-\mu_2 u}\right]=\mathbf{0}. \tag{5.10}$$

Thus the coefficients $\boldsymbol{\Upsilon}_j(d), j=1,\ldots,6$, can be determined from Eqs. (5.5)-(5.10), then we obtain $\mathbf{W}(u,1,d)$. By the same arguments and in view of the boundary conditions (4.3), we can derive $\mathbf{W}(u,r,d)$ for $r=2,3,\ldots$.

For illustration purpose, we set $c=2.5, \delta=0.01, \lambda=1, \lambda_1=1, \lambda_2=2, \mu_1=1, \mu_2=2$. It is easy to check that the net profit condition holds. Now, we consider the expectation of discounted dividend payments, namely, $r=1$. In this case the solutions of $h(s)det[\mathbf{L}(s)]=0$ are $\kappa_1=0.8082, \kappa_2=0.0118, \kappa_3=-0.4017, \kappa_4=-0.7713, \kappa_5=-1.6390, \kappa_6=-\mu_2=-2.000$. In the following, Table 1 gives some numerical values of $W(u,1,d)=\boldsymbol{\gamma}^{\top}\mathbf{W}(u,1,d)$.

Table 1. Exact values for $W(u,1,d)$.

d\u	0	1	2	3	4	5	6	7	8
0	1.0372								
1	0.7157	1.4795							
2	0.3980	0.7842	1.4831						
3	0.2003	0.3858	0.7050	1.3935					
4	0.0621	0.1376	0.2784	0.5856	1.2718				
5	0.0449	0.0847	0.1498	0.2878	0.5942	1.2803			
6	0.0206	0.0388	0.0680	0.1296	0.2663	0.5722	1.2582		
7	0.0094	0.0176	0.0307	0.0582	0.1191	0.2555	0.5613	1.2472	
8	0.0042	0.0079	0.0138	0.0261	0.0532	0.1140	0.2503	0.5560	1.2419

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A class of Hartley-Ross type unbiased estimators for population mean using ranked set sampling

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Abstract

In this paper, we propose a class of Hartley-Ross type unbiased estimators for estimating the finite population mean of the study variable under ranked set sampling (RSS), when population mean of the auxiliary variable is known. The variances of the proposed class of unbiased estimators are obtained to first degree of approximation. Both theoretically and numerically, the proposed estimators are compared with some competitor estimators, using three different data sets. It is identified through numerical study that the proposed estimators are more efficient as compared to all other competitor estimators.

Keywords: Ranked set sampling, auxiliary variable, variance.

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1. Introduction

In applications, there might be a situation when the variable of interest cannot be easily measured or is very expensive to do so, but it can be ranked easily at no cost or at very little cost. In view of this, Mclntyre [5] was the first who proposed the concept of ranked set sampling (RSS) in the context of obtaining reliable farm yield estimates based on sampling of pastures and crop yield. He provided a clear and insightful introduction to the basic framework of RSS and laid out the rationale for how it can be lead to improved estimation relative to simple random sampling (SRS). Takahasi and Wakimoto [11] have provided the necessary mathematical theory of RSS and showed that the sample mean under RSS is an unbiased estimator of the finite population mean and more precise than the sample mean estimator under SRS.

The auxiliary information plays an important role in increasing efficiency of the estimator. Samawi and Muttlak [6] have suggested an estimator for population ratio in RSS and showed that it has less variance as compared to usual ratio estimator in SRS.

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In RSS, perfect ranking of elements was considered by Mclntyre [5] and Takahasi and Wakimoto [11] for estimation of population mean. In some situations, ranking may not be perfect. Dell and Clutter [2] have studied the case in which there are errors in ranking. They pointed out that a loss in efficiency would be caused by the errors in ranking. The sample mean in RSS is an unbiased estimator of the population mean regardless of errors in ranking of the elements. To reduce the error in ranking, several modifications of the RSS method had been suggested. Stokes [10] has proposed use of the concomitant variable to aid in the ranking process to obtain ranked set data. She has also studied the ranked set sample approach for making inferences about the population variance and correlation coefficient. Here, the ranking of elements was done on basis of the auxiliary variable instead of judgment. Singh et al. [7] have proposed an estimator for population mean and ranking of the elements is observed on basis of the auxiliary variable. Singh et al.[9] have also proposed the ratio and the product type estimators for population mean under stratified ranked set sampling (SRSS).

Hartley and Ross [3] proposed an unbiased ratio estimators for finite population mean in SRS. Motivated by Singh et al. [8], we suggest a class of Hartley-Ross type unbiased estimators based on RSS for population mean, using some known population parameters of the auxiliary variable. It is shown that the proposed estimators outperform as compared to some existing estimators in RSS.

2. RSS procedure

To create ranked sets, we must partition the selected first phase sample into sets of equal size. In order to plan RSS design, we must therefore choose a set of size m that is typically small, around three or four, to minimize ranking error. Here m is the number of sample units allocated to each set. The RSS procedure can be summarized as follows:

- **Step 1:**Randomly select m^2 bivariate sample units from the population.
- **Step 2:**Allocate m^2 selected units randomly as possible into m sets, each of size m.
- **Step 3:**Each sample is ranked with respect to one of the variables Y or X. Here, we assume that the perfect ranking is done on basis of the auxiliary variable X while the ranking of Y is with error.
- **Step 4:**An actual measurement from the first sample is then taken of the unit with the smallest rank of X, together with variable Y associated with smallest rank of X. From second sample of size m, the variable Y associated with the second smallest rank of X is measured. The process is continued until from the mth sample, the Y associated with the highest rank of X is measured.
- **Step 5:**Repeat Steps 1 through 4 for r cycles until the desired sample size $n = mr$, is obtained for analysis.

As an illustration, we select a sample of size 36 from a population by simple random sampling with replacement (SRSWR). These data are grouped into 3 sets each of size 3 and we repeat this process 4 times. According to RSS methodology, we order the X values from smaller to larger and assume that there is no judgment error in this ordering. Then, the smallest unit is selected from the first ordered set, the second smallest unit is selected from the second ordered set and so on. Similarly from the third ordered set, the third smallest unit is selected. By this way, we select $n = mr = 12$ observations. A ranked set sample design with set size $m = 3$ and number of sampling cycles $r = 4$ is illustrated in Figure 1. Although 36 sample units have been selected from the population, only the 12 circled units are actually included in the final sample for quantitative analysis.

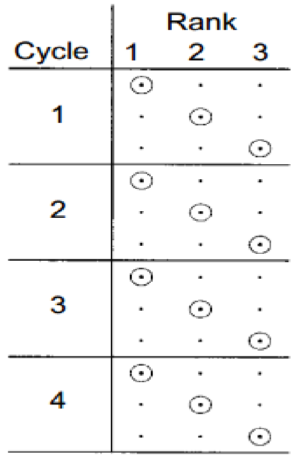

Figure 1. Illustration of ranked set sampling.

3. Symbols and Notations

We consider a situation when rank the elements on the auxiliary variable. Let $(y_{[i]j}, x_{(i)j})$ be the ith judgment ordering in the ith set for the study variable Y based on the ith order of the ith set of the auxiliary variable X in the jth cycle. To obtain bias and variance of the estimators, we define:
$\bar{y}_{[i]} = \bar{Y}(1+e_0),\ \bar{x}_{(i)} = \bar{X}(1+e_1),\ \bar{r}_{(i)} = \bar{R}(1+e_2),\ \bar{x}^*_{(i)} = \bar{X}^*(1+e^*_1),$
$\bar{r}^*_{(i)} = \bar{R}^*(1+e^*_2)$, such that
$E(e_i) = 0,$ i=0,1,2. $E(e^*_i) = 0,$ i=1,2.
and
$E(e_0^2) = \gamma C_y^2 - W^2_{y[i]},\quad E(e_1^2) = \gamma C_x^2 - W^2_{x(i)},\quad E(e_0e_1) = \gamma C_{yx} - W_{yx(i)},$
$E(e_1^{*2}) = \gamma C^2_{x^*} - W^2_{x^*(i)},\quad E(e_0e^*_1) = \gamma C_{yx^*} - W_{yx^*(i)},\ E(e^*_1e^*_2) = \gamma C_{r^*x^*} - W_{r^*x^*(i)},$
where

$$W_{yx(i)} = \frac{1}{m^2r\bar{X}\bar{Y}}\sum_{i=1}^{m}\tau_{yx(i)},\ W^2_{x(i)} = \frac{1}{m^2r\bar{X}^2}\sum_{i=1}^{m}\tau^2_{x(i)},\ W^2_{y[i]} = \frac{1}{m^2r\bar{Y}^2}\sum_{i=1}^{m}\tau^2_{y[i]},$$

$W_{yx^*(i)} = \frac{1}{m^2r\bar{X}^*\bar{Y}}\sum_{i=1}^{m}\tau_{yx^*(i)},\ W^2_{x^*(i)} = \frac{1}{m^2r\bar{X}^{*2}}\sum_{i=1}^{m}\tau^2_{x^*(i)},$
$W_{r^*x^*(i)} = \frac{1}{m^2r\bar{X}^*\bar{R}^*}\sum_{i=1}^{m}\tau_{r^*x^*(i)},$

$$\tau_{x(i)} = (\mu_{x(i)} - \bar{X}),\ \tau_{y[i]} = (\mu_{y([i]} - \bar{Y}),\ \tau_{yx(i)} = (\mu_{y[i]} - \bar{Y})(\mu_{x(i)} - \bar{X}),$$

$\tau_{x^*(i)} = (\mu_{x^*(i)} - \bar{X}^*), \tau_{yx^*(i)} = (\mu_{y[i]} - \bar{Y})(\mu_{x^*(i)} - \bar{X}^*), \tau_{r^*x^*(i)} = (\mu_{r^*(i)} - \bar{R}^*)(\mu_{x^*(i)} - \bar{X}^*).$
Here $\gamma = (\frac{1}{mr})$ and $C_{yx} = \rho C_y C_x$, where C_y and C_x are the coefficients of variation of Y and X respectively. Also $\bar{Y}$ and $\bar{X}$ are the population means of Y and X respectively. The values of $\mu_{y[i]}$ and $\mu_{x(i)}$ depend on order statistics from some specific distributions (see Arnold et al.[1]).
The following notations will be used through out this paper.

$\bar{y}_{[i]} = (1/n)\sum_{j=1}^{n} y_{[i]j}$, $\bar{x}_{(i)} = (1/n)\sum_{j=1}^{n} x_{(i)j}$, $\bar{r}_{(i)} = \frac{\sum_{j=1}^{n} r_{(i)j}}{n}$, $r_{(i)j} = \frac{y_{[i]j}}{x_{(i)j}}$,
$\bar{R} = E(\bar{r}_{(i)})$, $\bar{r}^*_{(i)} = \frac{\sum_{j=1}^{n} r^*_{(i)j}}{n}$, $r^*_{(i)j} = \frac{y_{[i]j}}{x^*_{(i)j}}$, $x^*_{(i)j} = (ax_{(i)j} + b)$,
$\bar{x}^*_{(i)} = (a\bar{x}_{(i)} + b)$, $\bar{X}^* = (a\bar{X} + b)$, $\bar{R}^* = E(\bar{r}^*_{(i)})$, $\bar{r}^{'}_{(i)} = \frac{\sum_{j=1}^{n} r^{'}_{(i)j}}{n}$,
$r^{'}_{(i)j} = \frac{y_{[i]j}}{x^{'}_{(i)j}}$, $x^{'}_{(i)j} = (x_{(i)j}C_x + \rho)$, $\bar{x}^{'}_{(i)} = (\bar{x}_{(i)}C_x + \rho)$, $\bar{X}^{'} = (\bar{X}C_x + \rho)$,
$\bar{R}^{'} = E(\bar{r}^{'}_{(i)})$, $\bar{r}^{''}_{(i)} = \frac{\sum_{j=1}^{n} r^{''}_{(i)j}}{n}$, $r^{''}_{(i)j} = \frac{y_{[i]j}}{x^{''}_{(i)j}}$, $x^{''}_{(i)j} = (x_{(i)j}\beta_2(x) + C_x)$,
$\bar{x}^{''}_{(i)} = (\bar{x}_{(i)}\beta_2(x) + C_x)$, $\bar{X}^{''} = (\bar{X}\beta_2(x) + C_x)$, $\bar{R}^{''} = E(\bar{r}^{''}_{(i)})$, $\bar{r}^{'''}_{(i)} = \frac{\sum_{j=1}^{n} r^{'''}_{(i)j}}{n}$,
$r^{'''}_{(i)j} = \frac{y_{[i]j}}{x^{'''}_{(i)j}}$, $x^{'''}_{(i)j} = (x_{(i)j}C_x + \beta_2(x))$, $\bar{x}^{'''}_{(i)} = (\bar{x}_{(i)}C_x + \beta_2(x))$,
$\bar{X}^{'''} = (\bar{X}C_x + \beta_2(x))$ and $\bar{R}^{'''} = E(\bar{r}^{'''}_{(i)})$,
where a and b are known population parameters, which can be coefficient of variation, coefficient of skewness and coefficient of kurtosis and the coefficient of correlation of the auxiliary variable.

Following Singh [8], the variance of the Hartley-Ross type unbiased estimator based on Upadhyaya and Singh [12] estimator in SRS, is given by

$$(3.1)\qquad V(\bar{y}^{(u)}_{US2(SRS)}) \cong \gamma\left(\bar{Y}^2C_y^2 + \bar{X}^{'''2}\bar{R}^{'''2}C^2_{x^{'''}} - 2\bar{R}^{'''}\bar{Y}\bar{X}^{'''}C_{yx^{'''}}\right).$$

Under RSS scheme, the variance of $\bar{y}_{RSS} = \bar{y}_{[i]} = (1/n)\sum_{j=1}^{n} y_{[i]j}$, is given by

$$(3.2)\qquad V(\bar{y}_{RSS}) = \bar{Y}^2\left(\gamma C_y^2 - W^2_{y[i]}\right).$$

4. Proposed Hartley-Ross unbiased estimator in RSS

Following Singh et al. [8], we consider the following ratio estimator:

$$(4.1)\qquad \bar{y}_{H(RSS)} = \bar{r}_{(i)}\bar{X}.$$

The bias of $\bar{y}_{H(RSS)}$, is given by

$$B(\bar{y}_{H(RSS)}) = -\frac{(N-1)}{N}S_{r_{(i)}x_{(i)}},$$

where $S_{r_{(i)}x_{(i)}} = \frac{1}{N}\sum_{j=1}^{N}(r_{(i)j} - \bar{R})(x_{(i)j} - \bar{X})$ and an unbiased estimator of $S_{r_{(i)}x_{(i)}}$ is given by

$$s_{r_{(i)}x_{(i)}} = \frac{1}{n-1}\sum_{j=1}^{n}(r_{(i)j} - \bar{r}_{(i)})(x_{(i)j} - \bar{x}_{(i)}) = \frac{n}{n-1}(\bar{y}_{[i]} - \bar{r}_{(i)}\bar{x}_{(i)}).$$

So bias of $\bar{y}_{H(RSS)}$ becomes

$$(4.2)\qquad B(\bar{y}_{H(RSS)}) = -\frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}_{(i)}\bar{x}_{(i)}).$$

Thus an unbiased Hartley-Ross type estimator of population mean based on RSS is given by

$$(4.3)\qquad \bar{y}^{(u)}_{H(RSS)} = \bar{r}_{(i)}\bar{X} + \frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}_{(i)}\bar{x}_{(i)}).$$

In terms of $e's$, we have

$$\bar{y}^{(u)}_{H(RSS)} = \bar{X}\bar{R}(1+e_2) + \frac{n(N-1)}{N(n-1)}\left[\bar{Y}(1+e_0) - \bar{X}\bar{R}(1+e_1)(1+e_2)\right].$$

Under the assumption $\frac{n(N-1)}{N(n-1)} \cong 1$, we can write

$$(\bar{y}^{(u)}_{H(RSS)} - \bar{Y}) \cong (\bar{Y}e_0 - \bar{X}\bar{R}e_1).$$

Taking square and then expectation, the variance of $\bar{y}^{(u)}_{H(RSS)}$, is given by

$$V(\bar{y}^{(u)}_{H(RSS)}) \cong \bar{Y}^2(\gamma C_y^2 - W_{y[i]}^2) + \bar{X}^2\bar{R}^2(\gamma C_x^2 - W_{x(i)}^2) - 2\bar{R}\bar{Y}\bar{X}(\gamma C_{yx} - W_{yx(i)}). \quad (4.4)$$

5. Proposed class of Hartley-Ross type unbiased estimators in RSS

Consider the following ratio estimator:

$$\bar{y}_{P(RSS)} = \bar{r}^*_{(i)}\bar{X}^*. \quad (5.1)$$

The bias of $\bar{y}_{P(RSS)}$, is given by

$$B(\bar{y}_{P(RSS)}) = -\frac{(N-1)}{N} S_{r^*_{(i)}x^*_{(i)}},$$

where $S_{r^*_{(i)}x^*_{(i)}} = \frac{1}{N}\sum_{j=1}^{N}(r^*_{(i)j} - \bar{R}^*)(x^*_{(i)j} - \bar{X}^*)$ and an unbiased estimator of $S_{r^*_{(i)}x^*_{(i)}}$ is given by

$$\begin{aligned} s_{r^*_{(i)}x^*_{(i)}} &= \frac{1}{n-1}\sum_{j=1}^{n}(r^*_{(i)j} - \bar{r}^*_{(i)})(x^*_{(i)j} - \bar{x}^*_{(i)}) \\ &= \frac{n}{n-1}(\bar{y}_{[i]} - \bar{r}^*_{(i)}\bar{x}^*_{(i)}). \end{aligned}$$

We give the following theorem.

5.1. Theorem. *An unbiased estimator of* $S_{r^*_{(i)}x^*_{(i)}} = \frac{1}{N}\sum_{j=1}^{N}(r^*_{(i)j} - \bar{R}^*)(x^*_{(i)j} - \bar{X}^*)$ *is given by*
$s_{r^*_{(i)}x^*_{(i)}} = \frac{1}{n-1}\sum_{j=1}^{n}(r^*_{(i)j} - \bar{r}^*_{(i)})(x^*_{(i)j} - \bar{x}^*_{(i)})$.

Proof. We have to prove that $E(s_{r^*_{(i)}x^*_{(i)}}) = S_{r^*_{(i)}x^*_{(i)}}$. Here for fixed i, $j = 1, 2, ..., n$, $r^*_{(i)j}$ and $x^*_{(i)j}$ are simple random samples of size n.

$$\begin{aligned} E(s_{r^*_{(i)}x^*_{(i)}}) &= E\left[\frac{1}{n-1}\sum_{j=1}^{n}(r^*_{(i)j} - \bar{r}^*_{(i)})(x^*_{(i)j} - \bar{x}^*_{(i)})\right], \\ &= \frac{1}{n-1}E\left[\sum_{j=1}^{n} r^*_{(i)j}x^*_{(i)j} - n\bar{r}^*_{(i)}\bar{x}^*_{(i)}\right], \\ &= \frac{1}{n-1}\left[\sum_{j=1}^{n} E(r^*_{(i)j}x^*_{(i)j}) - nE(\bar{r}^*_{(i)}\bar{x}^*_{(i)})\right], \\ &= \frac{1}{n-1}\left[\frac{n}{N}\sum_{j=1}^{N} r^*_{(i)j}x^*_{(i)j} - n\left(Cov(\bar{r}^*_{(i)}, \bar{x}^*_{(i)}) + \bar{R}^*\bar{X}^*\right)\right], \\ &= \frac{n}{n-1}\left[\frac{1}{N}\sum_{j=1}^{N} r^*_{(i)j}x^*_{(i)j} - \bar{R}^*\bar{X}^* - \frac{S_{r^*_{(i)}x^*_{(i)}}}{n}\right], \\ &= \frac{n}{n-1}\left(S_{r^*_{(i)}x^*_{(i)}} - \frac{S_{r^*_{(i)}x^*_{(i)}}}{n}\right), \\ &= S_{r^*_{(i)}x^*_{(i)}}. \end{aligned}$$

□

So bias of $\bar{y}_{P(RSS)}$ becomes

$$B(\bar{y}_{KP(RSS)}) = -\frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}^*_{(i)}\bar{x}^*_{(i)}). \tag{5.2}$$

Thus an unbiased class of Hartley-Ross type estimators of population mean based on RSS is given by

$$\bar{y}^{(u)}_{P(RSS)} = \bar{r}^*_{(i)}\bar{X}^* + \frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}^*_{(i)}\bar{x}^*_{(i)}). \tag{5.3}$$

In terms of $e's$, we have

$$\bar{y}^{(u)}_{P(RSS)} = \bar{X}^*\bar{R}^*(1+e_2^*) + \frac{n(N-1)}{N(n-1)}\left(\bar{Y}(1+e_0) - \bar{X}^*\bar{R}^*(1+e_1^*)(1+e_2^*)\right).$$

Under the assumption $\frac{n(N-1)}{N(n-1)} \cong 1$, we have

$$(\bar{y}^{(u)}_{P(RSS)} - \bar{Y}) \cong (\bar{Y}e_0 - \bar{X}^*\bar{R}^*e_1^*).$$

Taking square and then expectation, the variance of $\bar{y}^{(u)}_{P(RSS)}$, is given by

$$\begin{aligned} V(\bar{y}^{(u)}_{P(RSS)}) \cong & \bar{Y}^2(\gamma C_y^2 - W^2_{y[i]}) + \bar{X}^{*2}\bar{R}^{*2}(\gamma C^2_{x^*} - W^2_{x^*(i)}) \\ & - 2\bar{R}^*\bar{Y}\bar{X}^*(\gamma C_{yx^*} - W_{yx^*(i)}). \end{aligned} \tag{5.4}$$

Note: (i). If $a = C_x$ and $b = \rho$, then from Equation (5.3), we get the Hartley-Ross type unbiased estimator based on Kadilar and Cingi [4] estimator $\bar{y}^{(u)}_{KC(RSS)}$, as:

$$\bar{y}^{(u)}_{KC(RSS)} = \bar{r}'_{(i)}\bar{X}' + \frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}'_{(i)}\bar{x}'_{(i)}). \tag{5.5}$$

The variance of $\bar{y}_{KC(RSS)}$, is given by

$$\begin{aligned} V(\bar{y}^{(u)}_{KC(RSS)}) \cong & \bar{Y}^2(\gamma C_y^2 - W^2_{y[i]}) + \bar{X}'^2\bar{R}'^2(\gamma C^2_{x'} - W^2_{x'(i)}) \\ & - 2\bar{R}'\bar{Y}\bar{X}'(\gamma C_{yx'} - W_{yx'(i)}). \end{aligned} \tag{5.6}$$

(ii). If $a = \beta_2(x)$ and $b = C_x$, then Equation (5.3) becomes the Hartley-Ross type unbiased estimator based on Upadhyaya and Singh [12] estimator $\bar{y}^{(u)}_{US1(RSS)}$ and is given by

$$\bar{y}^{(u)}_{US1(RSS)} = \bar{r}''_{(i)}\bar{X}'' + \frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}''_{(i)}\bar{x}''_{(i)}). \tag{5.7}$$

The variance of $\bar{y}_{US1(RSS)}$, is given by

$$\begin{aligned} V(\bar{y}^{(u)}_{US1(RSS)}) \cong & \bar{Y}^2(\gamma C_y^2 - W^2_{y[i]}) + \bar{X}''^2\bar{R}''^2(\gamma C^2_{x''} - W^2_{x''(i)}) \\ & - 2\bar{R}''\bar{Y}\bar{X}''(\gamma C_{yx''} - W_{yx''(i)}). \end{aligned} \tag{5.8}$$

(iii). If $a = C_x$ and $b = \beta_2(x)$, then Equation (5.3) becomes the Hartley-Ross type unbiased estimator based on Upadhyaya and Singh [12] estimator $\bar{y}^{(u)}_{US2(RSS)}$ and is given by

$$\bar{y}^{(u)}_{US2(RSS)} = \bar{r}'''_{(i)}\bar{X}''' + \frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}'''_{(i)}\bar{x}'''_{(i)}). \tag{5.9}$$

The variance of $\bar{y}_{US2(RSS)}$, is given by

$$\begin{aligned} V(\bar{y}^{(u)}_{US2(RSS)}) \cong & \bar{Y}^2(\gamma C_y^2 - W^2_{y[i]}) + \bar{X}'''^2\bar{R}'''^2(\gamma C^2_{x'''} - W^2_{x'''(i)}) \\ & - 2\bar{R}'''\bar{Y}\bar{X}'''(\gamma C_{yx'''} - W_{yx'''(i)}). \end{aligned} \tag{5.10}$$

6. Efficiency comparison

The proposed estimator $\bar{y}^{(u)}_{US2(RSS)}$ is more efficient than $\bar{y}^{(u)}_{US2(SRS)}$, $\bar{y}_{(RSS)}$, $\bar{y}^{(u)}_{H(RSS)}$, $\bar{y}^{(u)}_{KC(RSS)}$ and $\bar{y}^{(u)}_{US1(RSS)}$ respectively if the following conditions hold:

(i). $-(\bar{Y}W_{y[i]} - \bar{X}^{'''}\bar{R}^{'''}W_{x'''(i)})^2 < 0$

(ii). $\bar{X}^{'''}\bar{R}^{'''}(\gamma C^2_{x'''} - W^2_{x'''(i)}) - 2\bar{Y}(\gamma C_{yx'''} - W_{yx'''(i)}) < 0$

(iii). $\bar{X}^{'''2}\bar{R}^{'''2}(\gamma C^2_{x'''} - W^2_{x'''(i)}) - 2\bar{X}^{'''}\bar{R}^{'''}\bar{Y}(\gamma C_{yx'''} - W_{yx'''(i)}) - \bar{X}^2\bar{R}^2(\gamma C^2_{x} - W^2_{x(i)}) + 2\bar{R}\bar{X}\bar{Y}(\gamma C_{yx} - W_{yx(i)}) < 0$

(iv). $\bar{X}^{'''2}\bar{R}^{'''2}(\gamma C^2_{x'''} - W^2_{x'''(i)}) - 2\bar{X}^{'''}\bar{R}^{'''}\bar{Y}(\gamma C_{yx'''} - W_{yx'''(i)}) - \bar{X}^{'2}\bar{R}^{'2}(\gamma C^2_{x'} - W^2_{x'(i)}) + 2\bar{R}^{'}\bar{X}^{'}\bar{Y}(\gamma C_{yx'} - W_{yx'(i)}) < 0.$

(v). $\bar{X}^{'''2}\bar{R}^{'''2}(\gamma C^2_{x'''} - W^2_{x'''(i)}) - 2\bar{X}^{'''}\bar{R}^{'''}\bar{Y}(\gamma C_{yx'''} - W_{yx'''(i)}) - \bar{X}^{''2}\bar{R}^{''2}(\gamma C^2_{x''} - W^2_{x''(i)}) + 2\bar{R}^{''}\bar{X}^{''}\bar{Y}(\gamma C_{yx''} - W_{yx''(i)}) < 0.$

7. Numerical Illustration

To observe performances of the estimators, we use the following three data sets. The descriptions of these populations are given below.

Population I [source: Valliant et al.[13]]
The summary statistics are:
y : Breast cancer mortality in 1950-1969,
x : Adult female population in 1960.

$N = 301$,	$n = 12$,	$m = 3$,	$r = 4$,
$\bar{X} = 11288.1800$,	$\bar{Y} = 39.8500$,	$\rho = 0.9671$,	$\beta_2(x) = 10.79$,
$\bar{R} = 0.0039$,	$\bar{R}^{'} = 0.0032$,	$\bar{R}^{''} = 0.00036$,	$\bar{R}^{'''} = 0.0032$,
$\bar{X}^{'} = 13780.84$,	$\bar{X}^{''} = 121852.40$,	$\bar{X}^{'''} = 13290.67$,	$C_y = 1.2794$,
$C_x = 1.2207$,	$C_{x'} = 1.2206$,	$C_{x''} = 1.2207$,	$C_{x'''} = 1.2198$,
$C_{yx} = 1.5105$,	$C_{yx'} = 1.5104$,	$C_{yx''} = 1.5104$,	$C_{yx'''} = 1.5093$,
$W^2_{y[i]} = 0.014502$,	$W^2_{x(i)} = 0.002234$,	$W_{yx(i)} = 0.022478$,	$W^2_{x'(i)} = 0.002234$,
$W_{yx'(i)} = 0.022476$,	$W^2_{x''(i)} = 0.002234$,	$W_{yx''(i)} = 0.022478$,	$W^2_{x'''(i)} = 0.002231$,
$W_{yx'''(i)} = 0.022461$.			

Population II [source: Valliant et al. [13]]
The summary statistics are:
y : Number of patients discharged,
x : Number of beds.

Population III [source: Valliant et al. [13]]
The summary statistics are:
y : Population, excluding residents of group quarters in 1960,

$N = 393$, $n = 15$, $m = 3$, $r = 5$,
$\bar{X} = 274.70$, $\bar{Y} = 814.65$, $\rho = 0.9105$, $\beta_2(x) = 3.5670$,
$\bar{R} = 3.1548$, $\bar{R}^{'} = 3.6842$, $\bar{R}^{''} = 0.9286$, $\bar{R}^{'''} = 3.5520$,
$\bar{X}^{'} = 214.13$, $\bar{X}^{''} = 980.63$, $\bar{X}^{'''} = 216.78$, $C_y = 0.7239$,
$C_x = 0.7762$, $C_{x'} = 0.7729$, $C_{x''} = 0.7756$, $C_{x'''} = 0.7634$,
$C_{yx} = 0.5116$, $C_{yx'} = 0.5094$, $C_{yx''} = 0.5112$, $C_{yx'''} = 0.5031$,
$W^2_{y[i]} = .016234$, $W^2_{x(i)} = 0.003354$, $W_{yx(i)} = 0.041280$, $W^2_{x'(i)} = 0.003353$,
$W_{yx'(i)} = 0.041277$, $W^2_{x''(i)} = 0.003354$, $W_{yx''(i)} = 0.041279$, $W^2_{x'''(i)} = 0.003348$,
$W_{yx'''(i)} = 0.04148$.

x : Number of households in 1960.

$N = 304$, $n = 12$, $m = 3$, $r = 4$,
$\bar{X} = 8931.17$, $\bar{Y} = 32916.19$, $\rho = 0.9979$, $\beta_2(x) = 14.6079$,
$\bar{R} = 3.7993$, $\bar{R}^{'} = 2.9703$, $\bar{R}^{''} = 0.2589$, $\bar{R}^{'''} = 2.9580$,
$\bar{X}^{'} = 11627.52$, $\bar{X}^{''} = 130466.90$, $\bar{X}^{'''} = 11641.13$, $C_y = 1.2390$,
$C_x = 1.3018$, $C_{x'} = 1.3017$, $C_{x''} = 1.3018$, $C_{x'''} = 1.3002.98$,
$C_{yx} = 1.6096$, $C_{yx'} = 1.6094$, $C_{yx''} = 1.6095$, $C_{yx'''} = 1.6075$,
$W^2_{y[i]} = .006744$, $W^2_{x(i)} = 0.005193$, $W_{yx(i)} = 0.023651$, $W^2_{x'(i)} = 0.005192$,
$W_{yx'(i)} = 0.023649$, $W^2_{x''(i)} = 0.005193$, $W_{yx''(i)} = 0.023652$, $W^2_{x'''(i)} = 0.005179$,
$W_{yx'''(i)} = 0.023622$.

Table 1. Comparison values

Population	$V(\bar{y}^{(u)}_{US2(RSS)}) < V(\bar{y}^{(u)}_{US2(SRS)})$	$V(\bar{y}^{(u)}_{US2(RSS)}) < V(\bar{y}_{RSS})$	$V(\bar{y}^{(u)}_{US2(RSS)}) < V(\bar{y}^{(u)}_{H(RSS)})$	$V(\bar{y}^{(u)}_{US2(RSS)}) < V(\bar{y}^{(u)}_{KC(RSS)})$	$V(\bar{y}^{(u)}_{US2(RSS)}) < V(\bar{y}^{(u)}_{US1(RSS)})$
I	$-7.7800 < 0$	$-3.0556 < 0$	$-3.5006 < 0$	$-4.2868 < 0$	$-3.4670 < 0$
II	$-3509.73 < 0$	$-0.55052 < 0$	$-0.43858 < 0$	$-0.39893 < 0$	$-0.43292 < 0$
III	$-50649.03 < 0$	$-2773.45 < 0$	$-199146.50 < 0$	$-185356.01 < 0$	$-197541.90 < 0$

We investigate the percent relative efficiency (PRE) of Hartley-Ross unbiased estimator $\bar{y}^{(u)}_{H(RSS)} = \hat{\theta}_1$ (say), Hartley-Ross type unbiased estimator based on Kadilar and Cingi [4] estimator $\bar{y}^{(u)}_{KC(RSS)} = \hat{\theta}_2$, Hartley-Ross type unbiased estimator based on Upadhyaya and Singh [12] estimator $\bar{y}^{(u)}_{US1(RSS)} = \hat{\theta}_3$ and $\bar{y}^{(u)}_{US2(RSS)} = \hat{\theta}_4$ with respect to conventional estimator $\bar{y}_{RSS} = \hat{\theta}_0$ (say) .

The PRE of proposed estimators $\hat{\theta}_j, \quad j = 1, 2, 3, 4,$ with respect to conventional estimator $\bar{y}_{RSS} = \hat{\theta}_0$, is defined as:

$$PRE(\hat{\theta}_0, \hat{\theta}_j) = \frac{V(\hat{\theta}_0)}{V(\hat{\theta}_j)} \times 100, \quad j = 1, 2, 3, 4. \tag{7.1}$$

The $PRE's$ of our proposed estimators and other existing estimators for Populations I, II and III are given in Tables 2, 3 and 4 respectively.

Table 2. $PRE's$ of various estimators for Population I.

m	r	n	$\bar{y}_{RSS}$	$\bar{y}^{(u)}_{H(RSS)}$	$\bar{y}^{(u)}_{KC(RSS)}$	$\bar{y}^{(u)}_{US1(RSS)}$	$\bar{y}^{(u)}_{US2(RSS)}$
3	3	9	100	178.37	178.40	178.38	**178.74**
	4	12	100	354.74	354.77	354.75	**354.92**
	5	15	100	326.14	326.23	326.22	**326.96**
4	3	12	100	397.23	397.30	397.25	**397.80**
	4	16	100	119.37	119.40	119.38	**119.56**
	5	20	100	114.70	114.75	114.74	**114.86**
5	3	15	100	217.06	217.10	217.09	**217.30**
	4	20	100	108.68	108.71	108.70	**108.85**
	5	25	100	177.16	177.20	177.18	**177.50**
	10	50	100	355.90	355.98	355.94	**356.77**

Table 3. $PRE's$ of various estimators for Population II.

m	r	n	$\bar{y}_{RSS}$	$\bar{y}^{(u)}_{H(RSS)}$	$\bar{y}^{(u)}_{KC(RSS)}$	$\bar{y}^{(u)}_{US1(RSS)}$	$\bar{y}^{(u)}_{US2(RSS)}$
3	3	9	100	199.15	201.21	199.54	**207.09**
	4	12	100	147.75	149.47	148.08	**154.18**
	5	15	100	119.02	119.06	119.04	**119.45**
4	3	12	100	259.28	259.33	259.29	**259.86**
	4	16	100	177.56	177.60	177.57	**177.96**
	5	20	100	141.97	142.01	141.98	**142.78**
5	3	15	100	111.53	111.56	111.54	**112.72**
	4	20	100	138.47	138.50	138.48	**139.75**
	5	25	100	167.65	167.69	167.67	**168.08**
	10	50	100	260.15	260.20	260.17	**260.66**

Table 4. $PRE's$ of various estimators for Population III.

m	r	n	$\bar{y}_{RSS}$	$\bar{y}^{(u)}_{H(RSS)}$	$\bar{y}^{(u)}_{KC(RSS)}$	$\bar{y}^{(u)}_{US1(RSS)}$	$\bar{y}^{(u)}_{US2(RSS)}$
3	3	9	100	158.03	158.08	158.04	**158.66**
	4	12	100	330.60	330.71	330.61	**332.27**
	5	15	100	288.63	288.68	288.64	**289.37**
4	3	12	100	194.50	194.56	194.51	**195.34**
	4	16	100	116.76	116.82	116.78	**117.51**
	5	20	100	322.73	322.84	322.75	**324.23**
5	3	15	100	146.69	146.73	146.70	**147.21**
	4	20	100	122.19	122.23	122.20	**122.71**
	5	25	100	124.24	124.28	124.26	**124.76**
	10	50	100	215.39	215.46	215.40	**216.32**

From Tables 2, 3 and 4, we see that the proposed Hartley-Ross type unbiased estimators are more efficient than usual conventional estimator in RSS. Thus, if population coefficient of variation, population coefficient of kurtosis and population correlation coefficient are known in advance, then our proposed estimators can be used in practice.

8. Conclusion

Table 1 has established the conditions obtained in Section 6 numerically. It is shown that all conditions are satisfied for all considered populations. On the basis of results given in Tables 2, 3 and 4, we conclude that the proposed class of Hartley-Ross type unbiased estimators are preferable over its competitive estimators under RSS. It is also observed that the proposed unbiased estimator $\bar{y}^{(u)}_{US2(RSS)}$ has highest PRE in comparison to all other considered estimators in all three populations.

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A multi-item EPQ model with imperfect production process for time varying demand with shortages

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Abstract

In this paper, economic production quantity(EPQ)models for breakable or deteriorating items are developed with time dependent linear variable demands. Here rate of production and holding cost are time dependent and unit production cost is a function of both production reliability indicator and production rate. Set-up cost is also partially production rate dependent. Here two models are developed in optimal control framework considering the effect of time value of money and inflation. Shortages are allowed for both the models. The problems are solved using Eulers-Lagrangian function based on variational calculus and applying generalized reduced gradient method using LINGO 13.0 software to determine the optimal reliability indicator (r) and then corresponding production rates and total profits. Numerical experiments are performed for both the models to illustrate the models both numerically and graphically.

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1. Introduction

In real life, we are accustomed with two categories of items mainly-damageable and non-damageable items. Again damageable items can be divided into two sub-categories namely-breakable item and deteriorating items. Deteriorating items deteriorate with time like seasonal fruits, different vegetable items etc. Since the items are deteriorated with time ,as a result the holding cost of the items is increased. For example, the fruits like grapes are available in the market from march to July in every year. Therefore the business time of that type of fruit is finite. Naturally, demand of the grapes increases with time and it exist in the market for a short period of time i.e. the business time horizon is finite. Also, fruits like mango, apple,vegetable like ladies finger, cabbage, beet and carrot are available in the market for a finite period and their demands increase with time. Some research works already have been investigated so far by several researchers on EOQ and EPQ/EMQ models with time dependent demand (cf. Dave and Patel[15], Dutta and Pal[14],Cheng[9],Lee and Hsu[25],Sana[47],Sarkar et al.[48], Maihami and Kamalabadi[32], Guchhait et al.[20]).

On the other hand, items made of glass, clay, ceramic, etc. belong to breakable category. Mainly fashionable/decorating items are made of glass, ceramic, etc., and demand of these types of items exists over finite time only. As sale of these fashionable products increases with the exhibition of stock, manufacturers of these items face a conflicting situation in their business. To stimulate the demand, they are tempted to go for huge number of production to have a large display and in this process, invites more damage to his units, as breakability increases with the increase of piled stock and the duration of stress due to the stock. So, breakability depend on huge stock and duration of accumulated stress due to stock. In the literature, there are only very few articles with this type of items(cf. Maiti and Maiti[[35],[36],[37]], Mandal[[38],[39]],Lee[[26]]). Still there is a scope to develop/modify some inventory models in this area considering time dependent breakability specially in different environments.

In real life, basically in metropolitan cities, holding cost increases with time due to non availability of space, bank interest etc. Also set-up is cost partially dependent on production rate. The researchers gave the less attention for research in this area. A notable remarks have been highlighted in inventory control problems with variable holding cost (cf. Alfares[1], Urban[56]) and Set-up cost (cf. Matsuyama[31], Darwish[13]). As per our knowledge, no one has formulated an inventory model for breakable/demegable items with the assumption of variable set-up cost or time dependent holding cost.

In recent times, the economy of developing countries like India, Bangladesh, Nepal, Bhutan, Pakistan etc. changes rigorously due to high inflation. The effect of inflation and time value of money are also well established in inventory problems. Initially, Buzacott[4] used the inflation subject to different types of pricing policies. Then consequently in the subsequent years, Mishra[33] , Padmanavan and Vrat[42] , Hariga and Ben-daya[21] , Bierman and Thomas[6] , Chen[8] , Moon and Lee[34] , Dey et al.[16] , Shah [50] etc. have worked in this area. Liao et al.[28] investigated the model of Aggarwal and Jaggi[2] with the assumption of inflation and stock dependent demand rate. Chen and Kang[7] presented integrated models with permissible delay in payments and variant pricing energy.

In most of the previous production inventory models, the researchers considered that all the produced items are of perfect quality. But, in real life, due to complex design of mechinaries items, it is not possible to produce all the items of perfect quality and is directly affected by the reliability of the production process.Recently some research works have been done in an imperfect production process like as Bazan[5] , Paul[43], Dey[17], Sarkar[[51],[52]], Mohammadia[40], Haidar[22] etc. In the literature there are few research publications in the two ware-house inventory model with defective items like as Rad[45], Pal[44] etc . In the literature there are some notable works in the area of rework of the imperfect product such as Cardenas-Barron [[10][11][12]], Taleizadeh [[54],[55]], Sarkar[53], Wee[58] etc. In imperfect production-inventory models, reliability of the production process is considered in different ways. Firstly a fraction r of produced units are considered as good product and remaining (1-r) defective units. Some authors consider r as crisp (cf. Cheng[9], Maiti and Maiti[29]) and others consider r as uncertain (cf. Yoo et al.[57],Liao and Sheu[28]) and they tried to determine optimal r so as to optimize cost or profit.In reality if r is maximum, the manufactures are highly satisfied. Considering this fact some research works have been done in this area (cf. Sana[47] and Sarkar et al.[48] ,Guchhait[20]) . In this research work, we consider this approach.

Nearly all inventory models are formulated with constant holding cost (cf. Sana[47], Sarkar et al.[48], Maiti,[30]). In reality, due to rental charges, inflation, preservation cost, bank interest, etc., holding cost increases with time. Thus some factors contributing to the holding cost change with time (cf. Giri et al.[18]) . Also set-up cost depends on production rate as high production rate require sophisticated modern mechanism. In this paper set-up and holding costs are considered as functions of production rate and time respectively.

Variational principle is a straightforward process for the analysis of optimal control problems. Few researchers have formulated the production- inventory models as optimal control problems and solved using this method (cf. Sana[47],Sarkar et al.[48] and Guchhait[20]). But all the researchers formulated their models with single item.To the best of our knowledge, none has considered the multi-item with shortages via variational principle. The present problem has been solved under the assumption of multi-item with shortages using variational principle.

Most of the EPQ models, unit production cost is taken as a constant. But in reality production cost varies with production rate, raw material cost, labour charge, wear and tear cost and reliability of the production process (cf. Khouja[24]). In this study, unit production cost is dependent on production rate, reliability indicator, raw material, labour charge and wear-and-tear costs.

Thus, major contributions of the present investigation is as follows:

• A notable remark has been put in the area of production-inventory research where the models are developed with the assumption of infinite time horizon (cf. Porteous[41], Cheng[9],Maiti and Maiti[29], Yoo et al. [57]etc). According to their assumptions, demand of an item remains unchanged for interminably. But, in real life, Gurnani[19] pointed out that rapid development of technology leads to the change in product specification with latest feature which in turn, motivates the customers to go for buy new products. For this reason, many researchers have investigated and analysised the inventory models with finite time horizon (cf. Khanra and Choudhuri[23], Maiti[30] etc). But in the existing literature of inventory model with demagable/deteriorating items,they overlooked this phenomenon (cf. Maiti and Maiti[29], Guchhait et al.[20]). For this reason, here a finite time horizon multi-item production manufacturing model of a damageable items with shortages has been formulated and solved.

• In this paper, due to this reasons mentioned above holding and set-up costs are considered as functions of time and production rate for both the item respectively.

• In imperfect production inventory control problem,reliability factors play an important role in manufacturing process. But in the competitive market, due to existence in the market, managers of the production firms are highly satisfied if r i.e. reliability (also called process reliability)reaches its maximum levels and they can not allow the reliability to fall below a minimum level. Following, this approach, recently some works have been done by Sana[47],Sarker et al.[48], Sarkar[46] and Guchait et al.[20]. In the present

investigation, the authors have considered this approach for both the items.
• But in the common business practices that customers are allured with displayed stock and for that, demand is considered as stock-dependent. Some works have been done by Levin et al.[27], Baker and Urban[3], Alfares[1], Stavrulaki[49], etc. In recent market policy of the big departmental stores like Big Bazar, Metro Bazar, Bazar Kolkata, Wall Mart, TESCO, Carrefour etc.,where the items are displayed in huge stock and for the breakable items, huge stocks invites more breakability / damageability along with more sales. Hence, a balanced is to be maintained between increased breakability and sale for maximum profit. Till now, inventory practitioners have been paid a little attention in this area of inventory problem with damageable items (cf. Maiti and Maiti[29]). In this present investigation, optimum reliability indicator and the inventory level of breakable items made a balance between the process reliability and increased sale so as to maximize the profit.
• Due to simplicity and effectiveness of the variational principle as mentioned above , the present models are solved using Variational principle method by considering the augmented profit function.
• Thus, here an attempt has made to formulate and solve multi-item EPQ models incorporating all the features. As per the above arguments,in this present investigation, unit production cost taken depends on production rate, reliability indicator, raw material cost, etc.
•Till now, none has considered all the above features into account in a single model.

In this paper, a multi-item production-inventory model with imperfect production process is formulated for a breakable or deteriorating items over a finite time horizon. Here we formulate two models with shortages. First model is for two items with shortages and the second model is for single item with shortages. The unit production cost is a function of production rate, raw material cost, labour charge, wear and tear cost and product reliability indicator. The first model is formulated as optimal control problems for the maximization of total profits over the planning horizon with budget constraint and optimum profit with profits along with optimum reliability indicator(r) are obtained using Euler-Lagrange equation based on variational principle. The second model is of single item also solved under the same assumptions and technique. Both problems have been solved using a non-linear optimization technique -GRG (LINGO-13.0) and illustrated with some numerical data. Several particular cases are derived and the results are presented in both tabular and graphical forms. Finally, some sensitivity analyses can be made with respect to different parameters.

The rest of the research paper is structured as follows. Some notations and assumptions are given by section 2. Section 3 is followed by the mathematical development and description of the proposed model with shortages through optimal control framework. Here three lemmas are proposed and proved. Also, the mathematical development and description of the model with single item are proposed in section 4. Section 5 proposed the solution procedure. Section 6 represents the numerical data and results of different models and pictographic representation of the effect of different parameters. Discussion and managerial insights are discussed in section 7. After that a summarization of this study is included in section 8 by naming it as conclusion and future research work. At last, the list references that are used to make this study possible.

2. Notations and assumptions for the proposed model

2.1. Notations:

(i) $q_1(t)$ and $q_2(t)$ be the inventory at any time t of item-1 and -2 respectively.

(ii) $\dot{q_1}(t)$ and $\dot{q_2}(t)$ are the derivative of $q_1(t)$ and $q_2(t)$ with respect to time t respectively.
(iii) $B_1(q_1,t)$ and $B_2(q_2,t)$ be the breakability or damageability function of item-1 and item- 2 respectively.
(iv) $P_1(t)$ and $P_2(t)$ are the production rate of item-1 and item-2 respectively at any time t.
(v) r_1 and r_2 are the production reliability indicator for item-1 and item-2 respectively,
$0 \leq r_1, r_2 \leq 1$.
(vi) r_{1min}, r_{2min}. and r_{1max}, r_{2max} are the minimum and maximum value of r_1 and r_2 respectively,
$0 \leq r_{1min}, r_{2min} \leq 1$,$0 \leq r_{1max}, r_{2max} \leq 1$.
(vii) λ_1 and λ_2 are the variation constant of tool or die costs for item-1 and-2 respectively,$\lambda_1 > 0$, $\lambda_2 > 0$.
(viii) $\chi(r_1)$ and $\chi(r_2)$ are the development cost of item-1 and-2 respectively.
(ix) C_{p_1} and C_{p_2} are the unit production cost of item-1 and item-2 respectively.
(x) C_{d_1} and C_{d_2} are the rework cost per defective item-1 and item-2 respectively.
(xi) $C_{h1}(t)$ and $C_{h2}(t)$ are the unit holding cost of item-1 and item-2 respectively.
(xii) C_3 and C_4 are the setup cost of item-1 and item-2 respectively.
(xiii) S_{p_1} and S_{p_2} is the unit selling price for the item-1 and item-2 respectively,$S_{p_1} > C_{p_1}$,$S_{p_2} > C_{p_2}$.
(xiv) S_{h_1} and S_{h_2} is the unit shortages cost for the item-1 and item-2 respectively.

2.2. Assumptions:

(i) The imperfect production-inventory system involves single and multi-item and which are to be sold .
(ii) The planning horizon for both the models are limited i.e. T is finite.
(iii) Here, it is assume that the inventory levels at $t = 0$ is $-S_1$ for item-1 and $-S_2$ for item-2 and both the inventory reaches to 0 at $t = T$.
(iv) In the show-rooms, the items made of China-clay, mud, glass,ceramic, etc., are kept in a heaped stocks. Due to this reason, the items at the bottom are under stress due to weight and for a long time, items are get damaged and break. Therefore, the breakability or damageability rate depends upon the stock of item and as well as how many times is under stress. Therefore the breakability rate of item-1 can be expressed as a function of stock levels and time and is of the form: $B_1(q_1,t) = b_{10}q_1 + b_{11}t$ for $q_1 > 0$ where b_{10} and b_{11} are the parameters can be chosen for best fit for the reliability function. Similarly, $B_2(q_2,t) = b_{20}q_2 + b_{21}t$ for $q_2 > 0$ where b_{20} and b_{21} are the parameters can be chosen for best fit for the reliability function.
(v) For the seasonal fruits like mango, apple etc., theirs demand is increases with time though their business period is limited and finite. Here demand rate is linear time-dependent for both the item.
(vi) Production rate for both items increases with time.
(vii) r_1 and r_2 indicates the defective rate of the production. Therefore, $r_1P_1(t)$, $r_2P_2(t)$ are the rate of producing defective item-1 and -2 respectively.
(viii) λ_1 and λ_2 are the variation constant of tool or die costs for item-1 and-2 respectively.
(ix) $\chi(r_1)$ and $\chi(r_2)$ depends upon the production reliability indicator, r_1 and r_2 respectively and are represented as $\chi(r_1) = N_1 + N_2 e^{C_A(r_{1max}-r_1)/(r_1-r_{1min})}$ and $\chi(r_2) = N_3 + N_4$

$e^{C_A(r_{2max}-r_2)/(r_2-r_{2min})}$ where N_1 and N_3 are the fixed cost like labour, energy, etc., and is independent of r_1 and r_2. N_2 and N_4 are the cost of modern technology, resource and design complexity for production when $r_1 = r_{1max}$, $r_2 = r_{2max}$,. Also, C_A represents the difficulties in increasing reliability, which depends on the design complexity, technology and resource limitations, etc for both the items.

(x) Unit production cost,C_{p_1} and C_{p_2} are the function of production rate $P_1(t)$,$P_2(t)$ respectively and production reliability and can be expressed in the form $C_{p_1}(r_1,t) = C_{r_1} + \frac{\chi(r_1)}{P_1(t)} + \lambda_1 P_1(t)$ for item-1 and $C_{p_2}(r_2,t) = C_{r_2} + \frac{\chi(r_2)}{P_2(t)} + \lambda_2 P_2(t)$ for item-2, where C_{r_1} and C_{r_2} are the fixed material cost for item-1 and 2 respectively. Second term is the development cost which is equally distributed over the production $P_1(t)$,$P_2(t)$ at any time t. Also, the third term $\lambda_1 P_1(t)$ and $\lambda_2 P_2(t)$ are the tool/ die cost which is proportional to the production rate respectively for both the item.

(xi) Now-a-days, due to inflation, bank interest, hiring charge, etc., holding cost increases with time. For this reason the holding cost changes with time and other factors remain constant. Hence the holding cost $C_{h1}(t)$ and $C_{h2}(t)$ can be expressed as $C_{h1}(t) = C_{10} + C_{11}t$ and $C_{h2}(t) = C_{20} + C_{21}t$ respectively for item-1 and item-2, where C_{10}, C_{11}, C_{20} and C_{21} are constants.

(xii) Set-up cost, C_3 and C_4, are normally constant with time for both the items. But, , if dynamic production rate is considered, some machineries, etc., are to be set-up and maintained in such a way that the production system can stand with the pressure of increasing demand. Thus, a part of C_3,C_4 are linearly proportional to production rate and hence C_3, C_4 are of the form: $C_3(P_1(t)) = C_{30} + C_{31}P_1(t)$ and $C_4(P_2(t)) = C_{40} + C_{41}P_2(t)$, where C_{30}, C_{31}, C_{40} andC_{41} are the constants.

(xiii) In the developing countries, inflation is predominant and interest rate depends on the inflation value. Thus $\mu = R - i$, where R and i are the interest and inflation per unit currency, respectively,$\mu > 0$.

(xiv) All inventory costs are positive.

3. Mathematical formulation of the proposed multi-item model:

3.1. Model-1: Model with stock and time dependent breakable items: In real life, a production company not only produce one item but produce different types of item i.e. multi-item. Due to continuous long operation of machinery units and over duty of the workers , the production firm produces good quality item as well as imperfect quality items.These defective or imperfect quality items are instantly reworked at a per unit cost to make the product as new as perfect one to maintain the brand image of the manufacturer. The production of the defective items increases with time and reliability parameter of the produced item. The parameters r_1 and r_2 are the reliability indicator of the item-1 and -2 respectively. The production system became more stable and reliable , if r_1 and r_2 decreases i.e. smaller value of r_1 and r_2 provides the better quality product and produced smaller imperfect quality unites.

The inventory levels decreases due to demand and breakability/deterioration. Thus, the rate of change of inventory level at any time t for the item -1 can be represented by the following differential equation:

$$\begin{aligned} \frac{dq_1(t)}{dt} &= P_1 - D_1 - B_1(q_1,t) \\ \text{i.e } P_1(t) &= \dot{q_1} + D_1 + B_1(q_1,t), \text{with } q_1(0) = -S_1 \text{ and } q_1(T) = 0\,, \end{aligned} \tag{3.1}$$

where $D_1 \equiv D_1(t)$

Thus, the rate of change of inventory level at any time t for the item -2 can be represented by the following differential equation:

$$\begin{aligned}
\frac{dq_2(t)}{dt} &= P_2 - D_2 - B_2(q_2,t) \\
\text{i.e. } P_2(t) &= \dot{q_2} + D_2 + B_2(q_2,t) \text{with} \quad q_2(0) = -S_2 \text{ and } q_2(T) = 0, \qquad (3.2)\\
where\, D_2 &\equiv D_2(t)
\end{aligned}$$

where D_1 and D_2 are the demand function of time t and is of the form $D_1(t) = a_1 + b_1 t$ and $D_2(t) = a_2 + b_2 t$ for item-1 & 2 respectively.
The end condition $q_1(0) = -S_1$, $q_2(0) = -S_2$ and $q_1(T) = 0$ and $q_2(T) = 0$ indicate that at time $t = 0$ the maximum shortages is $-S_1$ for item-1 and $-S_2$ for item-2 i.e. the inventory starts with shortages at time $t = 0$. As P_1 and D_1 are the function of time t and combined effect of theses two the shortages reaches to zero and the inventory build-up as $P_1(t) > D_1 + B_1(q_1,t)$ in the first part of the cycle. After some time, as demand is a function of time t, D_1 is more than the combined effect of $D_1 + B_1(q_1,t)$ i.e. the accumulated stock decreases as $P_1(t) < D_1 + B_1(q_1,t)$ and ultimately the stock reaches to zero. Similar process is also followed for the item-2.
Since the production firm manufacturers two different types of items, then a budget constraint is imposed for procurement of the raw materials cost. Here C_{r_1} and C_{r_2} are the fixed material cost for item-1 and -2 respectively and if M be the maximum available budget for both the items, then the budget constraint can be expressed as

$$C_{r_1}q_1 + C_{r_2}q_2 \leq M \qquad (3.3)$$

The corresponding profit function for both the items, incorporation the inflation and time value of money during the time duration $[0,T]$ is given by

$$Z_p = \int_0^T \Bigg\{ e^{-\mu t}\Bigg[S_{p_1}D_1 - C_{p_1}(r_1,t)P_1(t) - C_{d_1}r_1P_1(t) - C_{h_1}(t)q_1 - \frac{C_3(P_1(t))}{T} - S_{h_1}S_1\Bigg] + e^{-\mu t}\Bigg[S_{p_2}D_2 - C_{p_2}(r_2,t)P_2(t) - C_{d_2}r_2P_2(t) - C_{h_2}(t)q_2 - \frac{C_4(P_2(t))}{T} - S_{h_2}S_2\Bigg]\Bigg\} dt$$

$$= \int_0^T e^{-\mu t}\Bigg[S_{p_1}D_1 + S_{p_2}D_2 - (C_{r_1} + C_{d_1}r_1)(\dot{q_1} + D_1 + B_1) - (C_{r_2} + C_{d_2}r_2)(\dot{q_2} + D_2 + B_2) - \chi(r_1) - \chi(r_2) - \lambda_1(\dot{q_1} + D_1 + B_1)^2 - \lambda_2(\dot{q_2} + D_2 + B_2)^2 - (C_{10} + C_{11}t)q_1 - (C_{20} + C_{21}t)q_2$$

$$-\{C_{30} + C_{31}(\dot{q_1} + D_1 + B_1])\}/T - \{C_{40} + C_{41}(\dot{q_2} + D_2 + B_2)\}/T - S_{h_1}S_1 - S_{h_2}S_2\Bigg] dt$$

$$= \int_0^T f(q_1, q_2, \dot{q}_1, \dot{q}_2, t)dt$$

$$\text{where } f(q_1, q_2, \dot{q}_1, \dot{q}_2, t) = e^{-\mu t}\Bigg[S_{p_1}D_1 + S_{p_2}D_2 - (C_{r_1} + C_{d_1}r_1)(\dot{q_1} + D_1 + B_1)$$

$$-(C_{r_2} + C_{d_2}r_2)(\dot{q_2} + D_2 + B_2) - \chi(r_1) - \chi(r_2) - \lambda_1(\dot{q_1} + D_1 + B_1)^2 - \lambda_2(\dot{q_2} + D_2 + B_2)^2$$

$$-(C_{10}+C_{11}t)q_1 \quad - \quad (C_{20}+C_{21}t)q_2 - \{C_{30}+C_{31}(\dot{q_1}+D_1+B_1])\}/T - \{C_{40}+C_{41}$$

$$(\dot{q_2}+D_2+B_2)\}/T \quad - \quad S_{h_1}S_1 - S_{h_2}S_2\Big] \tag{3.4}$$

Now our problem is to find the path of $q_1(t)$, $q_2(t)$, $P_1(t)$ and $P_2(t)$ such that Z_p is maximum with respect to the budget constraint. Since the problem is involved with a constraint, then to find the optimal solution of the optimal control problem, we construct the augmented profit functional as

$$Z_T = \int_0^T \left[f(q_1,q_2,\dot{q}_1,\dot{q}_2,t) + \lambda e^{-\mu t}(C_{r_1}q_1 + C_{r_2}q_2 - M) \right] dt \tag{3.5}$$

where,$F(q_1,q_2,\dot{q}_1,\dot{q}_2,t) = f(q_1,q_2,\dot{q}_1,\dot{q}_2,t) + \lambda e^{-\mu t}(C_{r_1}q_1 + C_{r_2}q_2 - M)$ and λ is the Lagrange multiplier having any real value.

3.1. Lemma. *Z_T has a maximum value for a path $q_1 = q_1(t)$ and $q_2 = q_2(t)$in the interval $[0,T]$*

Proof. Proof of the Lemma 3.1 . we consider a path(curve)$q_1 = q_1(t)$ and $q_2 = q_2(t)$ such that the functional Z_T is maximum in that path in the interval $[0,T]$ i.e. $t=0$ and $t=T$. Let us consider a path q_0 which is given by the path $q=q_0$ for which Z_T has a maximum value.We consider a class of neighboring curves p_ρ which is given by $q_1 = q_{1\rho}(t) = q_0(t) + \rho_1\eta_1(t)$ and $q_2 = q_{2\rho}(t) = q_0(t) + \rho_2\eta_2(t)$, where ρ_1 and ρ_2 is a very small constant and $\eta_1(t)$ and $\eta_2(t)$ (>0, for all values of t) is any two differential functions of t.Therefore, the value of Z_T for the path p_ρ is given by the relation $Z_T(\rho) = \int_0^T F_{\rho_1\rho_2}$ dt, where $F_{\rho_1\rho_2} = F(q_0(t)+\rho_1\eta_1(t), \dot{q_0}(t)+\rho_1\dot{\eta_1}(t), q_0(t)+\rho_2\eta_2(t), \dot{q_0}(t)+\rho_2\dot{\eta_2}(t), t)$
For maximum value of Z_T, we must have $\frac{\partial}{\partial\rho_1}(Z_T(\rho_1,\rho_2)) \mid_{\rho_1=0} = 0$
and $\frac{\partial}{\partial\rho_2}(Z_T(\rho_1,\rho_2)) \mid_{\rho_2=0} = 0$ and $\left[\frac{\partial^2 Z_T}{\partial\rho_1^2}\ \frac{\partial^2 Z_T}{\partial\rho_2^2}\ -\frac{\partial^2 Z_T}{\partial\rho_1\partial\rho_2}\right] > 0$ and $\frac{\partial^2}{\partial\rho_1^2}(Z_T(\rho_1,\rho_2)) < 0$
Now,

$$\begin{aligned}
\frac{\partial}{\partial\rho_1}(Z_T(\rho_1,\rho_2)) &= \int_0^T \Big\{\eta_1(t)\frac{\partial F_\rho}{\partial q_1} + \dot{\eta_1}(t)\frac{\partial F_\rho}{\partial \dot{q_1}}\Big\}dt \\
&= \int_0^T \Big[\eta_1(t)\frac{\partial F_\rho}{\partial q_1}\Big]dt + \Big[\eta_1(t)\frac{\partial F_\rho}{\partial \dot{q_1}}\Big]_0^T - \int_0^T \eta_1(t)\frac{d}{dt}\Big(\frac{\partial F_\rho}{\partial \dot{q_1}}\Big)dt \\
&= \int_0^T \eta_1(t)\Big\{\frac{\partial F_\rho}{\partial q_1} - \frac{d}{dt}\Big(\frac{\partial F_\rho}{\partial \dot{q_1}}\Big)\Big\}dt
\end{aligned}$$

As $q_1(t)$ is fixed at the end points $t=0$ and $t=T$, so, $\eta_1(0) = \eta_1(T) = 0$. Therefore, $\frac{d}{d\rho_1}(Z_T(\rho_1,\rho_2)) \mid_{\rho_1=0} = 0$ gives

$$\frac{\partial F_\rho}{\partial q_1} - \frac{d}{dt}(\frac{\partial F_\rho}{\partial \dot{q_1}}) = 0 \tag{3.6}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial\rho_2}(Z_T(\rho_1,\rho_2)) &= \int_0^T \Big\{\eta_2(t)\frac{\partial F_\rho}{\partial q_2} + \dot{\eta_2}(t)\frac{\partial F_\rho}{\partial \dot{q_2}}\Big\}dt \\
&= \int_0^T \Big[\eta_2(t)\frac{\partial F_\rho}{\partial q_2}\Big]dt + \Big[\eta_2(t)\frac{\partial F_\rho}{\partial \dot{q_2}}\Big]_0^T - \int_0^T \eta_2(t)\frac{d}{dt}\Big(\frac{\partial F_\rho}{\partial \dot{q_2}}\Big)dt \\
&= \int_0^T \eta_2(t)\Big\{\frac{\partial F_\rho}{\partial q_2} - \frac{d}{dt}\Big(\frac{\partial F_\rho}{\partial \dot{q_2}}\Big)\Big\}dt \qquad (3.7)
\end{aligned}$$

As $q_2(t)$ is fixed at the end points $t = 0$ and $t = T$, so, $\eta_2(0) = \eta_2(T) = 0$. Therefore, $\frac{\partial}{\partial \rho_2}(Z_T(\rho_1, \rho_2)) \mid_{\rho_2=0}= 0$ gives

$$\frac{\partial F_\rho}{\partial q_2} - \frac{d}{dt}(\frac{\partial F_\rho}{\partial \dot{q_2}}) = 0 \tag{3.8}$$

Equations (3.6) and (3.8) are the necessary conditions for extreme value of P_T.
Again, to find the maximum value of Z_T we must have, $\left[\frac{\partial^2 Z_T}{\partial \rho_1^2}\ \frac{\partial^2 Z_T}{\partial \rho_2^2}\ -\frac{\partial^2 Z_T}{\partial \rho_1 \partial \rho_2}\right] > 0$ and $\frac{\partial^2 Z_T}{\partial \rho_1^2} < 0$
Now,

$$\begin{aligned}
\frac{\partial^2 Z_T}{\partial \rho_1^2} &= \int_0^T \Big\{\eta_1^2 \frac{\partial^2 Z_p}{\partial q_1^2} + 2\eta_1 \dot{\eta_1} \frac{\partial^2 Z_p}{\partial q_1 \partial \dot{q_1}} + \dot{\eta_1}^2 \frac{\partial^2 Z_p}{\partial \dot{q_1}^2}\Big) dt \\
&= -2\lambda_1 e^{-\mu t}\Big\{\eta_1^2 b_{10}^2 + 2\eta_1 \dot{\eta}_1 b_{10} + \dot{\eta_1}^2\Big\} < 0 \text{ as } 2\lambda_1 e^{-\mu t} > 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 Z_T}{\partial \rho_2^2} &= \int_0^T \Big\{\eta_2^2 \frac{\partial^2 Z_p}{\partial q_2^2} + 2\eta_2 \dot{\eta_2} \frac{\partial^2 Z_p}{\partial q_2 \partial \dot{q_2}} + \dot{\eta_2}^2 \frac{\partial^2 Z_p}{\partial \dot{q_2}^2}\Big) dt \\
&= -2\lambda_2 e^{-\mu t}\Big\{\eta_2^2 b_{20}^2 + 2\eta_2 \dot{\eta}_2 b_{20} + \dot{\eta_2}^2\Big\} < 0 \text{ as } 2\lambda_2 e^{-\mu t} > 0 \\
\text{Finally,} \frac{\partial^2 Z_T}{\partial \rho_1 \partial \rho_2} &= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left[\frac{\partial^2 Z_T}{\partial \rho_1^2} \frac{\partial^2 Z_T}{\partial \rho_2^2} - \frac{\partial^2 Z_T}{\partial \rho_1 \partial \rho_2}\right] &= 2\lambda_1 e^{-\mu t}\Big\{\eta_1^2 b_{10}^2 + 2\eta_1 \dot{\eta}_1 b_{10} + \\
&\quad \dot{\eta_1}^2\Big\} 2\lambda_2 e^{-\mu t}\Big\{\eta_2^2 b_{20}^2 + 2\eta_2 \dot{\eta}_2 b_{20} + \dot{\eta_2}^2\Big\} > 0 \\
\text{and} \frac{\partial^2 Z_T}{\partial \rho_1^2} &= -\lambda_1 e^{-\mu t}\Big\{\eta_1^2 b_{10}^2 + 2\eta_1 \dot{\eta}_1 b_{10} + \dot{\eta_1}^2\Big\} < 0 \text{ as } \lambda_1 e^{-\mu t} > 0
\end{aligned}$$

Hence the sufficient condition, $\left[\frac{\partial^2 Z_T}{\partial \rho_1^2}\ \frac{\partial^2 Z_T}{\partial \rho_2^2}\ -\frac{\partial^2 Z_T}{\partial \rho_1 \partial \rho_2}\right] > 0$ and $\frac{\partial^2 Z_T}{\partial \rho_1^2} < 0$ shows that Z_T has a maximum in $[0, T]$. □

3.2. Lemma. *$\frac{\partial Z_T(r_1, r_2)}{\partial r_1} = 0$ must have at least one solution in $[r_{1min}, r_{1max}]$, if $\frac{\partial Z_T(r_1, r_2)}{\partial r_1} < 0$, provided $\frac{\partial Z_T(r_1, r_2)}{\partial r_1} \rightarrow \propto$ at $r_1 = r_{1min}$ for all r_2, otherwise $\frac{\partial Z_T(r_1, r_2)}{\partial r_1} = 0$ may have or may not have a solution in $[r_{1min}, r_{1max}]$. The solution gives a maximum value of Z_T, if $\frac{\partial^2 Z_T}{\partial r_1^2} < 0$ and $\frac{\partial^2 Z_T}{\partial r_1^2} \frac{\partial^2 Z_T}{\partial r_2^2} - \Big(\frac{\partial^2 Z_T}{\partial r_1 \partial r_2}\Big)^2 > 0$ in the rectangle $[r_{1min}, r_{1max} : r_{2min}, r_{2max}]$*

Proof. Proof of the Lemma 3.2 .For maximization of the associate profit for both the items, $Z_T(r_1, r_2)$, differentiating $Z_T(r_1, r_2)$ with respect to r_1, we have

$$\frac{\partial Z_T}{\partial r_1} = N_2 e^{C_A (r_{1max} - r_1)/(r_1 - r_{1min})} C_A \frac{r_{1min} - r_{1max}}{(r_1 - r_{1min})^2} \frac{e^{-\mu T} - 1}{\mu}$$

As $r_1 \to r_{1min}$, then $\dfrac{\partial Z_T}{\partial r_1} \to\propto$
Again,

$$\frac{\partial^2 Z_T}{\partial r_1^2} = \frac{N_2 C_A e^{-\mu T-1}}{\mu}\Big[e^{C_A(r_{1max}-r_1)/(r_1-r_{1min})}\frac{r_{1min}-r_{1max}}{(r_1-r_{1min})^4} + e^{C_A(r_{1max}-r_1)/(r_1-r_{1min})} \frac{r_{1min}-r_{1max}}{(r_1-r_{1min})^3}\Big]$$

As $r_1 \to r_{1min}$ then $\dfrac{\partial Z_T}{\partial r_1} \to\propto$, therefore $\dfrac{\partial Z_T}{\partial r_1}$ has at least one solution if $\dfrac{\partial Z_T}{\partial r_1} \to\propto$ holds; otherwise $\dfrac{\partial Z_T(r_1,r_2)}{\partial r_1} = 0$ may have or may not have a solution in $[r_{1min}, r_{1max}]$. If $\dfrac{\partial Z_T}{\partial r_1}\,|_{r_1=r_1^*}= 0$ for $r_1^* \in [r_{1min}, r_{1max}]$ and $\dfrac{\partial^2 Z_T}{\partial r_1^2} < 0$ and $\dfrac{\partial^2 Z_T}{\partial r_1^2}\dfrac{\partial^2 Z_T}{\partial r_2^2} - (\dfrac{\partial^2 Z_T}{\partial r_1 \partial r_2})^2 > 0$, then $P_T(r_1^*)$ is maximum.
Similarly,**Lemma 3.3** can be written as, □

3.3. Lemma. *$\dfrac{\partial Z_T(r_1,r_2)}{\partial r_2} = 0$ must have at least one solution in $[r_{2min}, r_{2max}]$, if $\dfrac{\partial Z_T(r_1,r_2)}{\partial r_2} < 0$, provided $\dfrac{\partial Z_T(r_1,r_2)}{\partial r_2} \to\propto$ at $r_2 = r_{2min}$ for all r_1 otherwise $\dfrac{\partial Z_T(r_1,r_2)}{\partial r_2} = 0$ may have or may not have a solution in $[r_{2min}, r_{2max}]$. The solution gives a maximum value of Z_T, if $\dfrac{\partial^2 Z_T}{\partial r_2^2} < 0$ and $\dfrac{\partial^2 Z_T}{\partial r_1^2}\dfrac{\partial^2 Z_T}{\partial r_2^2} - (\dfrac{\partial^2 Z_T}{\partial r_1 \partial r_2})^2 > 0$ in the rectangle $[r_{1min}, r_{1max} : r_{2min}, r_{2max}]$*

Proof. Proof of the Lemma 3.3.we can proof the Lemma 3.3 following the same of Lemma 3.2. □

Now, for find the optimal path, we have from the Euler-Lagranges equation for the maximum value of $F(q_1, q_2, \dot{q}_1, \dot{q}_2, t)$ is

$$\frac{\partial F}{\partial q_1} - \frac{d}{dt}(\frac{\partial F}{\partial \dot{q}_1}) = 0 \tag{3.9}$$

$$\frac{\partial F}{\partial q_2} - \frac{d}{dt}(\frac{\partial F}{\partial \dot{q}_2}) = 0 \tag{3.10}$$

Firstly, we consider the first Euler-Lagrangian equation (3.9) and the boundary condition (3.1), we have

$$\ddot{q_1} - \mu\dot{q_1} - (b_{10}+\mu)b_{10}q_1 = H_1(t) \tag{3.11}$$

where

$$\begin{aligned} H_1(t) &= (\mu + b_{10})D_1 - b_1 - b_{11}(b_{10}t - 1 + \mu t) \\ &+ \frac{(C_{r_1} + r_1 C_{d_1} + C_{31}/T)(\mu + b_{10}) + (C_{10} + C_{11})t}{2\lambda_1} - \frac{\lambda}{2\lambda_1}C_{r_1} \\ &= K_1 + K_2 t + K_1^{'} \end{aligned}$$

where $K_1 = a_1(\mu + b_{10}) - b_1 - b_{11} + \dfrac{(C_{r_1} + r_1 C_{d_1} + C_{31}/T)(\mu + b_{10}) + C_{10}}{2\lambda_1}$

$$K_2 = \big[b_1(\mu + b_{10}) + b_{11}(b_{10} + \mu) + \frac{C_{11}}{2\lambda_1}], K_1^{'} = -\frac{\lambda}{2\lambda_1}C_{r_1} \tag{3.12}$$

The complementary function of the Eq. (3.11) is $C_1e^{(b_{10}+\mu)t}+C_2e^{-b_{10}t}$, where C_1 and C_2 are arbitrary constants and the particular integral is given by the $\frac{1}{D^2-\mu D-(\mu+b_{10})b_{10}}\Big\{K_1+K_2t+K_1^{'}\Big\}$. Here $D(\equiv \frac{d}{dt})$ represents the differential operator. Therefore, the complete solution of the Eq.(3.11) can be represented as

$$q_1(t) = C_1e^{(b_{10}+\mu)t}+C_2e^{-b_{10}t}-\frac{1}{K_3^2}\big[K_1K_3+K_2\big(K_3t-\mu\big)\big]-\frac{K_1^{'}}{K_3} \tag{3.13}$$

$$P_1(t) = K_4e^{(b_{10}+\mu)t}+K_5t+K_6+K_7, \tag{3.14}$$

$$\text{where}, K_4 = C_1(2b_{10}+\mu), K_3=b_{10}(b_{10}+\mu), K_5=(b_1+b_{11}-\frac{b_{10}K_2}{K_3}), K_6=-\frac{K_1^{'}}{K_3}b_{10}$$

$$K_7 = \frac{1}{K_3^2}\left[a_1K_3^2-K_2K_3-b_{10}(K_1K_3-K_2\mu),\right]$$

$$C_2=\frac{1}{\left[e^{(b_{10}+\mu)T}-e^{-b_{10}T}\right]}\Big(\frac{1}{K_3^2}[(K_1K_3-K_2\mu)e^{(b_{10}+\mu)T}-(K_1K_3+K_2(K_3T-\mu))]$$
$$-\frac{K_1^{'}}{K_3}(e^{(b_{10}+\mu)T}-1)-S_1e^{(b_{10}+\mu)T}\Big)$$
$$C_1=-S_1-C_2+\frac{1}{K_3^2}(K_1K_3-K_2\mu)+\frac{K_1^{'}}{K_3}$$

Substituting the value of $q_1(t)$ and $P_1(t)$ in the expression of (3.4), the corresponding profit function for the item-1 can be expressed as

$$Z_{p_1}=\int_0^T e^{-\mu t}\big[S_{p_1}D_1-(C_{r_1}+C_{d_1}r_1)(\dot{q_1}+D_1+B_1)-\chi(r_1)-\lambda_1(\dot{q_1}+D_1+B_1)^2$$
$$-(C_{10}+C_{11}t)q_1-\{C_{30}+C_{31}(\dot{q_1}+D_1+B_1)\}/T-S_{h_1}S_1\big]dt$$

$$\begin{aligned}
&= S_{p_1}\big[-a_1\frac{(e^{-\mu T}-1)}{\mu}-b_1\frac{(e^{-\mu T}-1+\mu Te^{-\mu T})}{\mu^2}\big]-(C_{r_1}+r_1C_{d_1})\big[\frac{K_4(e^{b_{10}T}-1)}{b_{10}}\\
&+ K_5(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{K_6}{\mu}(1-e^{-\mu T})+\frac{K_7}{\mu}(1-e^{-\mu T})\big]+\big[N_1+N_2\\
&\quad e^{C_A(r_{1max}-r_1)/(r_1-r_{1min})}\big](\frac{e^{-\mu T}-1}{\mu})+\lambda_1\big[(K_4e^{(b_{10}+\mu)T}+K_5T+K_6+K_7)^2\frac{e^{-\mu T}}{\mu}\\
&- \frac{(K_4+K_6+K_7)^2}{\mu}+\frac{2}{\mu^2}(\frac{K_4^2(b_{10}+\mu)}{(2b_{10}+\mu)}(e^{(2b_{10}+\mu)T}-1)+\frac{K_4K_5}{b_{10}}(e^{b_{10}T}-1)+\\
&\quad \frac{K_4K_5(b_{10}+\mu)}{b_{10}}\{\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2}\}+\frac{(K_7+K_6)K_4(b_{10}+\mu)}{b_{10}}(e^{b_{10}T}-1)\\
&- K_5^2(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{(K_6+K_7)K_5}{\mu}(1-e^{-\mu T}))\big]-C_{10}\big[\frac{C_1}{b_{10}}(e^{b_{10}T}-1)\\
&- \frac{C_2}{(b_{10}+\mu)}(e^{-(b_{10}+\mu)T}-1)+\frac{K_1^{'}}{K_3\mu}(e^{-\mu T}-1)+\frac{1}{K_3^2}(K_1K_3+K_2(K_3T-\mu))\frac{e^{-\mu T}}{\mu}\\
&+ \frac{K_2}{K_3\mu^2}(e^{-\mu T}-1)-\frac{1}{K_3^2\mu}(K_1K_3-K_2\mu)\big]-C_{11}\big[C_1(\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2})+
\end{aligned}$$

$$C_2(\frac{-Te^{-(b_{10}+\mu)T}}{(b_{10}+\mu)} - \frac{e^{-(b_{10}+\mu)T}}{(b_{10}+\mu)^2} + \frac{1}{(b_{10}+\mu)^2}) - \frac{K_1}{K_3}(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2})$$
$$- \frac{K_2\mu}{K_3^2}(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}) + \frac{K_2}{K_3}(\frac{-T^2e^{-\mu T}}{\mu} - \frac{2Te^{-\mu T}}{\mu^2} - \frac{2}{\mu^3}e^{-\mu T} + \frac{2}{\mu^3})$$
$$- \frac{K_1^{'}}{K_3}(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2})] - \frac{1}{T}[\frac{C_{30}}{\mu}(1-e^{-\mu T}) + C_{31}(\frac{K_4}{b_{10}}(e^{b_{10}T}-1)$$

$$(3.15) + \quad K_5(-\frac{Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}) + \frac{(K_6+K_7)}{\mu}(1-e^{-\mu T}))] - S_{h_1}S_1(\frac{1-e^{-\mu T}}{\mu})$$

From the second Euler-Lagranges Equation and using the boundary condition, we have

$$(3.16) \quad \ddot{q_2} - \mu\dot{q_2} - (b_{20}+\mu)b_{20}q_2 = H_2(t)$$

where

$$\begin{aligned} H_2(t) &= (\mu+b_{20})D_2 - b_2 - b_{21}(b_{20}t - 1 + \mu t) \\ &+ \frac{(C_{r_2} + r_2C_{d_2} + C_{41}/T)(\mu+b_{20}) + (C_{20}+C_{21})t}{2\lambda_2} - \frac{\lambda}{2\lambda_2}C_{r_2} \\ &= K_{11} + K_{22}t + K_{11}^{'} \\ \text{where}, K_{11} &= a_2(\mu+b_{20}) - b_2 - b_{21} + \frac{(C_{r_2} + r_2C_{d_2} + C_{41}/T)(\mu+b_{20}) + C_{20}}{2\lambda_2} \\ K_{22} &= \big[b_2(\mu+b_{20}) + b_{21}(b_{20}+\mu) + \frac{C_{21}}{2\lambda_2}, \ \ K_{11}^{'} = -\frac{\lambda}{2\lambda_2}C_{r_2} \end{aligned}$$

and $K_{33} = b_{20}(b_{20}+\mu)$

The complementary function of the Eq. (3.16) is $C_3e^{(b_{20}+\mu)t} + C_4e^{-b_{20}t}$, where C_3 and C_4 are arbitrary constants and the particular integral is given by the $\frac{1}{D^2-\mu D-(\mu+b_{20})b_{20}}\left\{K_{11}+K_{22}t+K_{11}^{'}\right\}$. Here $D(\equiv \frac{d}{dt})$ represents the differential operator. Therefore, the complete solution of the Eq.(16) can be represented as

$$(3.17) \qquad q_2(t) = C_3e^{(b_{20}+\mu)t} + C_4e^{-b_{20}t} - \frac{1}{K_{33}^2}\big[K_{11}K_{33} + K_{22}\big(K_{33}t-\mu\big)\big] - \frac{K_{11}^{'}}{K_{33}}$$

$$\begin{aligned} (3.18) \text{ and } \ P_2(t) &= K_8e^{(b_{20}+\mu)t} + K_9t + K_{10} + K_{12}, \\ \text{where, } K_8 &= C_3(2b_{20}+\mu), K_9 = (b_2 + b_{21} - b_{20}\frac{K_{22}}{K_{33}}), K_{10} = -\frac{K_{11}^{'}b_{20}}{K_{33}}, \\ K_{12} &= \frac{1}{K_{33}^2}[a_2{K_{33}}^2 - K_{22}K_{33} - b_{20}(K_{11}K_{33} - K_{22}\mu)] \end{aligned}$$

$$C_4 = \frac{1}{[e^{(b_{20}+\mu)T} - e^{-b_{20}T}]}(\frac{1}{K_{33}^2}[(K_{11}K_{33} - K_{22}\mu)e^{(b_{20}+\mu)T} - (K_{11}K_{33} + K_{22}(K_{33}T-\mu))]$$
$$-\frac{K_{11}^{'}}{K_{33}}(e^{(b_{20}+\mu)T} - 1) - S_2e^{(b_{20}+\mu)T})$$
$$C_3 = -S_2 - C_4 + \frac{1}{K_{33}^2}(K_{11}K_{33} - K_{22}\mu) + \frac{K_{11}^{'}}{K_{33}}$$

Substituting the value of $q_2(t)$ and $P_2(t)$ in the expression of (3.4), the corresponding profit function for the item-2 can be expressed as

$$\begin{aligned}
Z_{p_2} &= \int_0^T e^{-\mu t}\big[S_{p_2}D_2 - (C_{r_2} + C_{d_2}r_2)(\dot{q_2} + D_2 + B_2) - \chi(r_2) - \lambda_2(\dot{q_2} + D_2 + B_2)^2 \\
&- (C_{20} + C_{21}t)q_2 - \{C_{40} + C_{41}(\dot{q_2} + D_2 + B_2])\}/T - S_{h_2}S_2\big]dt
\end{aligned}$$

$$\begin{aligned}
&= S_{p_2}\big[-a_2\frac{(e^{-\mu T}-1)}{\mu} - b_2\frac{(e^{-\mu T}-1+\mu Te^{-\mu T})}{\mu^2}\big] - (C_{r_2} + r_2C_{d_2})\big[\frac{K_8(e^{b_{20}T}-1)}{b_{20}} \\
&+ K_9\big(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}\big) + \frac{K_{10}}{\mu}(1-e^{-\mu T}) + \frac{K_{12}}{\mu}(1-e^{-\mu T})\big] + \big[N_3 \\
&+ N_4e^{C_A(r_{2max}-r_2)/(r_2-r_{2min})}\big]\frac{e^{-\mu T}-1}{\mu} + \lambda_1\big[(K_8e^{(b_{20}+\mu)T} + K_{10}T + K_{10} + K_{12})^2 \\
&\quad \frac{e^{-\mu T}}{\mu} - \frac{(K_8+K_{10}+K_{12})^2}{\mu} + \frac{2}{\mu^2}\big(\frac{K_8^2(b_{20}+\mu)}{(2b_{20}+\mu)}(e^{(2b_{20}+\mu)T}-1) + \frac{K_8K_9}{b_{20}}(e^{b_{20}T} \\
&- 1) + \frac{K_8K_9(b_{20}+\mu)}{b_{20}}\{\frac{Te^{b_{20}T}}{b_{20}} - \frac{e^{b_{20}T}}{b_{20}^2} + \frac{1}{b_{20}^2}\} + \frac{(K_{12}+K_{10})K_8(b_{20}+\mu)}{b_{20}}(e^{b_{20}T} - \\
&\quad 1) - K_9^2(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}) + \frac{(K_{10}+K_{12})K_9}{\mu}(1-e^{-\mu T}))\big] - C_{20}\big[\frac{C_3}{b_{20}}(e^{b_{20}T} \\
&- 1) - \frac{C_4}{(b_{20}+\mu)}(e^{-(b_{20}+\mu)T}-1) + \frac{K_{11}'}{K_{33}\mu}(e^{-\mu T}-1) + \frac{1}{K_{33}^2}(K_{11}K_{33} + K_{22}(K_{33}T \\
&\quad -\mu))\frac{e^{-\mu T}}{\mu} + \frac{K_{22}}{K_{33}\mu^2}(e^{-\mu T}-1) - \frac{1}{K_{33}^2\mu}(K_{11}K_{33} - K_{22}\mu)\big] - C_{21}\big[C_3(\frac{Te^{b_{20}T}}{b_{20}} - \\
&\quad \frac{e^{b_{20}T}}{b_{20}^2} + \frac{1}{b_{20}^2}) + C_4(\frac{-Te^{-(b_{20}+\mu)T}}{(b_{20}+\mu)} - \frac{e^{-(b_{20}+\mu)T}}{(b_{20}+\mu)^2} + \frac{1}{(b_{20}+\mu)^2}) - \frac{K_{11}}{K_{33}}(\frac{-Te^{-\mu T}}{\mu} \\
&- \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}) - \frac{K_{22}\mu}{K_{33}^2}(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}) + \frac{K_{22}}{K_{33}}(\frac{-T^2e^{-\mu T}}{\mu} - \frac{2Te^{-\mu T}}{\mu^2} - \frac{2}{\mu^3} \\
&\quad e^{-\mu T} + \frac{2}{\mu^3}) - \frac{K_{11}'}{K_{33}}(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2})\big] - \frac{1}{T}\big[\frac{C_{40}}{\mu}(1-e^{-\mu T}) + C_{41}(\frac{K_8}{b_{20}}(e^{b_{20}T} - \\
&\quad 1) + K_9(-\frac{Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}) + \frac{(K_{10}+K_{12})}{\mu}(1-e^{-\mu T}))\big] - S_{h_2}S_2(\frac{1-e^{-\mu T}}{\mu})
\end{aligned} \tag{3.19}$$

Therefore total profit for item-1 and -2 can be expressed as $Z_p = Z_{p_1} + Z_{p_2}$, where Z_{p_1} and Z_{p_2} are given by (3.15)&(3.19) respectively, Therefore,

$$\begin{aligned}
Z_p = \int_0^T e^{-\mu t}\Bigg[&S_{p_1}D_1 + S_{p_2}D_2 - (C_{r_1} + C_{d_1}r_1)(\dot{q_1} + D_1 + B_1) - (C_{r_2} + C_{d_2}r_2)(\dot{q_2} + D_2 \\
&+B_2) - \chi(r_1) - \chi(r_2) - \lambda_1(\dot{q_1} + D_1 + B_1)^2 - \lambda_2(\dot{q_2} + D_2 + B_2)^2 - (C_{10} + C_{11}t)q_1 - (C_{20} \\
&+C_{21}t)q_2 - \{C_{30} + C_{31}(\dot{q_1} + D_1 + B_1)\}/T - \{C_{40} + C_{41}(\dot{q_2} + D_2 + B_2)\}/T - S_{h_1}S_1 \\
&-S_{h_2}S_2 + \lambda(C_{r_1}q_1 + C_{r_2}q_2 - M)\Bigg]dt
\end{aligned}$$

$$
\begin{aligned}
= \quad & S_{p_1}\big[-a_1\frac{(e^{-\mu T}-1)}{\mu}-b_1\frac{(e^{-\mu T}-1+\mu Te^{-\mu T})}{\mu^2}\big]-(C_{r_1}+r_1C_{d_1})\big[\frac{K_4(e^{b_{10}T}-1)}{b_{10}} \\
+ \quad & K_5\big(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}\big)+\frac{K_6}{\mu}(1-e^{-\mu T})+\frac{K_7}{\mu}(1-e^{-\mu T})\big]+\big[N_1+N_2 \\
& e^{C_A(r_{1max}-r_1)/(r_1-r_{1min})}\big]\big(\frac{e^{-\mu T}-1}{\mu}\big)+\lambda_1\big[(K_4e^{(b_{10}+\mu)T}+K_5T+K_6+K_7)^2\frac{e^{-\mu T}}{\mu} \\
- \quad & \frac{(K_4+K_6+K_7)^2}{\mu}+\frac{2}{\mu^2}\big(\frac{K_4^2(b_{10}+\mu)}{(2b_{10}+\mu)}(e^{(2b_{10}+\mu)T}-1)+\frac{K_4K_5}{b_{10}}(e^{b_{10}T}-1) \\
+ \quad & \frac{K_4K_5(b_{10}+\mu)}{b_{10}}\{\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2}\}+\frac{(K_7+K_6)K_4(b_{10}+\mu)}{b_{10}}(e^{b_{10}T}-1) \\
- \quad & K_5^2(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{(K_6+K_7)K_5}{\mu}(1-e^{-\mu T}))]-C_{10}\big[\frac{C_1}{b_{10}}(e^{b_{10}T}-1)- \\
& \frac{C_2}{(b_{10}+\mu)}(e^{-(b_{10}+\mu)T}-1)+\frac{K_1^{'}}{K_3\mu}(e^{-\mu T}-1)+\frac{1}{K_3^2}(K_1K_3+K_2(K_3T-\mu))\frac{e^{-\mu T}}{\mu} \\
+ \quad & \frac{K_2}{K_3\mu^2}(e^{-\mu T}-1)-\frac{1}{K_3^2\mu}(K_1K_3-K_2\mu)\Big]-C_{11}\big[C_1(\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2}) \\
+ \quad & C_2(\frac{-Te^{-(b_{10}+\mu)T}}{(b_{10}+\mu)}-\frac{e^{-(b_{10}+\mu)T}}{(b_{10}+\mu)^2}+\frac{1}{(b_{10}+\mu)^2})-\frac{K_1}{K_3}(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}) \\
- \quad & \frac{K_2\mu}{K_3^2}(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{K_2}{K_3}(\frac{-T^2e^{-\mu T}}{\mu}-\frac{2Te^{-\mu T}}{\mu^2}-\frac{2}{\mu^3}e^{-\mu T}+\frac{2}{\mu^3})- \\
& \frac{K_1^{'}}{K_3}(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})\big]-\frac{1}{T}\big[\frac{C_{30}}{\mu}(1-e^{-\mu T})+C_{31}(\frac{K_4}{b_{10}}(e^{b_{10}T}-1)+ \\
& K_5(-\frac{Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{(K_6+K_7)}{\mu}(1-e^{-\mu T}))]-S_{h_1}S_1(\frac{1-e^{-\mu T}}{\mu})
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & S_{p_2}\big[-a_2\frac{(e^{-\mu T}-1)}{\mu}-b_2\frac{(e^{-\mu T}-1+\mu Te^{-\mu T})}{\mu^2}\big]-(C_{r_2}+r_2C_{d_2})\big[\frac{K_8(e^{b_{20}T}-1)}{b_{20}} \\
+ \quad & K_9\big(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}\big)+\frac{K_{10}}{\mu}(1-e^{-\mu T})+\frac{K_{12}}{\mu}(1-e^{-\mu T})\big]+\big[N_3+N_4 \\
& e^{C_A(r_{2max}-r_2)/(r_2-r_{2min})}\big]\frac{e^{-\mu T}-1}{\mu}+\lambda_2\big[(K_8e^{(b_{20}+\mu)T}+K_{10}T+K_{10}+K_{12})^2 \\
& \frac{e^{-\mu T}}{\mu}-\frac{(K_8+K_{10}+K_{12})^2}{\mu}+\frac{2}{\mu^2}\big(\frac{K_8^2(b_{20}+\mu)}{(2b_{20}+\mu)}(e^{(2b_{20}+\mu)T}-1)+\frac{K_8K_9}{b_{20}} \\
& (e^{b_{20}T}-1)+\frac{K_8K_9(b_{20}+\mu)}{b_{20}}\{\frac{Te^{b_{20}T}}{b_{20}}-\frac{e^{b_{20}T}}{b_{20}^2}+\frac{1}{b_{20}^2}\}+\frac{(K_{12}+K_{10})K_8(b_{20}+\mu)}{b_{20}}- \\
& (e^{b_{20}T}-1)K_9^2(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2})+\frac{(K_{10}+K_{12})K_9}{\mu}(1-e^{-\mu T}))]-C_{20}\big[\frac{C_3}{b_{20}} \\
& (e^{b_{20}T}-1)-\frac{C_4}{(b_{20}+\mu)}(e^{-(b_{20}+\mu)T}-1)+\frac{K_{11}^{'}}{K_{33}\mu}(e^{-\mu T}-1)+\frac{1}{K_{33}^2}(K_{11}K_{33}+K_{22} \\
& (K_{33}T-\mu))\frac{e^{-\mu T}}{\mu}+\frac{K_{22}}{K_{33}\mu^2}(e^{-\mu T}-1)-\frac{1}{K_{33}^2\mu}(K_{11}K_{33}-K_{22}\mu)\Big]-C_{21}\big[C_3(\frac{Te^{b_{20}T}}{b_{20}} \\
- \quad & \frac{e^{b_{20}T}}{b_{20}^2}+\frac{1}{b_{20}^2})
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & C_4(\frac{-Te^{-(b_{20}+\mu)T}}{(b_{20}+\mu)} - \frac{e^{-(b_{20}+\mu)T}}{(b_{20}+\mu)^2} + \frac{1}{(b_{20}+\mu)^2}) - \frac{K_{11}}{K_{33}}(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}) \\
- \quad & \frac{K_{22}\mu}{K_{33}^2}(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}) + \frac{K_{22}}{K_{33}}(\frac{-T^2e^{-\mu T}}{\mu} - \frac{2Te^{-\mu T}}{\mu^2} - \frac{2}{\mu^3}e^{-\mu T} + \frac{2}{\mu^3}) - \\
& \frac{K_{11}^{'}}{K_{33}}(\frac{-Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2})] - \frac{1}{T}[\frac{C_{40}}{\mu}(1-e^{-\mu T}) + C_{41}(\frac{K_8}{b_{20}}(e^{b_{20}T}-1) \\
+ \quad & K_9(-\frac{Te^{-\mu T}}{\mu} - \frac{e^{-\mu T}}{\mu^2} + \frac{1}{\mu^2}) + \frac{(K_{10}+K_{12})}{\mu}(1-e^{-\mu T}))] - S_{h_2}S_2\frac{1-e^{-\mu T}}{\mu} + \lambda \\
& \Bigg[C_{r_1}\Big\{\frac{C_1}{b_{10}}(e^{b_{10}T}-1) - \frac{C_2}{(b_{10}+\mu)}(e^{-(b_{10}+\mu)T}-1) + \frac{1}{K_3^2\mu}(K_1K_3 + K_2(K_3T-\mu)) \\
& e^{-\mu T} + \frac{K_2}{K_3\mu^2}(e^{-\mu T}-1) - \frac{1}{K_3^2\mu}(K_1K_3 - K_2\mu) + \frac{K_1^{'}}{K_3\mu}(e^{-\mu T}-1)\Big\} + C_{r_2}\Big\{\frac{C_3}{b_{20}} \\
& (e^{b_{20}T}-1) - \frac{C_4}{(b_{20}+\mu)}(e^{-(b_{20}+\mu)T}-1) + \frac{1}{K_{33}^2\mu}(K_{11}K_{33} + K_{22}(K_{33}T-\mu))e^{-\mu T} \\
+ \quad & \frac{K_{22}}{K_{33}\mu^2}(e^{-\mu T}-1) - \frac{1}{K_{33}^2\mu}(K_{11}K_{33} - K_{22}\mu) + \frac{K_{11}^{'}}{K_{33}\mu}(e^{-\mu T}-1)\Big\} \\
& \qquad\qquad + \quad \frac{M}{\mu}(e^{-\mu T}-1)\Bigg] \qquad (3.20)
\end{aligned}
$$

3.2. Model-1a: Model with two stock-dependent breakable items. In the above Model-1, if we take the the parametric values of breakability/deterioration which are directly related to the time equal to zero i.e. $b_{11} = 0$ and $b_{21} = 0$, then we get another Model-1a. Therefore, the Model-1 reduces to a production-inventory model for deteriorating items with stock dependent breakability/deterioration. So, the total profit can be obtain by optimizing the Eq. (3.20) with $b_{11} = 0$ and $b_{21} = 0$

3.3. Model-1b: Model with two items without breakability. In the above Model-1, if we take the parametric value of deterioration which is directly related to stock and time is equal to zero i.e. $b_{10} = 0$, $b_{11} = 0$,$b_{20} = 0$, $b_{21} = 0$, then we get a another Model-1b. Therefore, the Model-1 reduces to a production-inventory model with out deteriorating item. As b_{10},b_{11}, b_{20},b_{21} appears in the denominator of the expression of (3.20) So, the total profit can not obtain by optimizing the Eq. (3.20) by directly putting with $b_{10} = 0$,$b_{11} = 0$, $b_{20} = 0$, $b_{21} = 0$. Thus, for the total profit of Model-1b can be obtain by omitting the breakability term from the expression of 1 and 2 and processing the same way as before in Model-1.

3.4. Model-1c: Model with two breakable items with constant demand. In the above Model-1, if we take the the parametric value of demand which is directly related to the time is equal to zero i.e. $b_1 = 0$,$b_2 = 0$, then we get a another Model-1c. Therefore, the Model-1 reduces to a production-inventory model for breakable item with constant demand . So, the total profit can be obtain by optimizing the Eq. (3.20) with $b_1 = 0$ and $b_2 = 0$.

3.5. Model-1d: Model two breakable items with constant holding cost. In the above Model-1, if we take the the parametric value of holding cost which is directly related to the time is equal to zero i.e. $C_{11} = 0$,$C_{21} = 0$, then we get a another Model-1d. Therefore, the Model-1 reduces to a production-inventory model for breakable item with constant holding cost. So, the total profit can be obtain by optimizing the Eq. (3.20) with $C_{11} = 0$ and $C_{21} = 0$.

3.6. Model-1e: Model with two breakable items with constant set-up cost. In the above Model-1, if we take the the parametric value of setup cost which is directly related to the production rate is equal to zero i.e. $C_{31} = 0$,$C_{41} = 0$, then we get a another Model-1e. Therefore, the Model-1 reduces to a production-inventory model for breakable item with constant set up cost. So, the total profit can be obtain by optimizing the Eq. (3.20) with $C_{31} = 0$ and $C_{41} = 0$.

4. Mathematical formulation of the proposed model with single item:

4.1. Model-2: Model with single item. In real life, the manager of a production firm always wants to produce more quantity through a long-run process by imposing over-time to its labour as well as machinery items. As a result, there may aries different types of difficulties in the production process which results the production of perfect quality item as well as defective item. These defective items are reworked instantly at a per unit cost to make the product as new as perfect one to maintain the brand image of the manufacturer. The production of the defective items increases with time and the reliability parameter of the produced item. The parameter r_1 is the reliability indicator of the item-1. The production system became more stable and reliable , if r_1 decreases i.e. smaller value of r_1 provides the better quality product and produced smaller imperfect quality unites.

The inventory levels decreases due to demand and deterioration. Thus, the change of inventory level at any time t can be represented by the following differential equation:

$$\frac{dq_1(t)}{dt} = P_1 - D_1 - B_1(q_1, t)$$

$$\text{i.e. } P_1(t) = \dot{q_1} + D_1 + B_1(q_1, t)$$

$$\text{with } q_1(0) = -S_1 \text{ and } q_1(T) = 0 \ \ , where\ D_1 \equiv D_1(t) \tag{4.1}$$

where D_1 is the demand function of time t and is of the form $D_1(t) = a_1 + b_1 t$.

The end condition $q_1(0) = -S_1$ and $q_1(T) = 0$ indicate that at time $t = 0$ the maximum shortages is $-S_1$ i.e. the inventory starts with shortages at time $t = 0$. As P_1 and D_1 are the function of time t and combined effect of theses two the shortages reaches to zero and the inventory build-up as $P_1(t) > D_1 + B_1(q_1, t)$ in the first part of the cycle. After some time, as demand is a function of time t, D_1 is more than the combined effect of $D_1 + B_1(q_1, t)$ i.e. the accumulated stock decreases as $P_1(t) < D_1 + B_1(q_1, t)$ and ultimately the stock reaches to zero.

The corresponding profit function, incorporation the inflation and time value of money during the time duration $[0, T]$ is given by

$$Z_p = \int_0^T e^{-\mu t}\big[S_{p_1} D_1 - C_{p_1}(r_1, t) P_1(t) - C_{d_1} r_1 P_1(t) - C_{h_1}(t) q_1 - C_3(P_1(t))/T - S_{h_1} S_1\big] dt$$

$$= \int_0^T e^{-\mu t}\big[S_{p_1} D_1 - (C_{r_1} + C_{d_1} r_1)(\dot{q_1} + D_1 + B_1) - \chi(r_1) - \lambda_1(\dot{q_1} + D_1 + B_1)^2 - (C_{10}$$

$$+ C_{11} t) q_1 - \{C_{30} + C_{31}(\dot{q_1} + D_1 + B_1)\}/T - S_{h_1} S_1\big] dt \tag{4.2}$$

$$= \int_0^T F(q_1, \dot{q}_1, t)dt$$

where $F(q_1, \dot{q}_1, t) = e^{-\mu t}\big[S_{p1}D_1 - (C_{r1} + C_{d1}r_1)(\dot{q}_1 + D_1 + B_1) - \chi(r_1) - \lambda_1$
$(\dot{q} + D_1 + B_1)^2 - (C_{10} + C_{11}t)q - \{C_{30} + C_{31}(\dot{q}_1 + D_1 + B_1)\}/T - S_{h_1}S_1\big]$

Now our problem is to find the path of $q_1(t)$ and $P_1(t)$ such that $F(q_1, \dot{q}_1, t)$ is to be maximized. Now, for find the optimal path, we have from the Euler-Lagranges equation for the maximum value of $F(q_1, \dot{q}_1, t)$ is

$$\frac{\partial F}{\partial q_1} - \frac{d}{dt}(\frac{\partial F}{\partial \dot{q}_1}) = 0 \tag{4.3}$$

using (4.2), we have,

$$\ddot{q}_1 - \mu\dot{q}_1 - (b_{10} + \mu)b_{10}q_1 = H_1(t) \tag{4.4}$$

where

$$\begin{aligned} H_1(t) &= a_1(\mu + b_{10}) - b_1 - b_{11} + \frac{(C_{r_1} + r_1C_{d_1} + C_{31}/T)(\mu + b_{10}) + C_{10}}{2\lambda_1} + t \\ &\quad \big[b_1(\mu + b_{10}) + b_{11}(b_{10} + \mu) + \frac{C_{11}}{2\lambda_1}\big] \\ &= K_1 + K_2t \\ \text{where } K_1 &= a_1(\mu + b_{10}) - b_1 - b_{11} + \frac{(C_{r_1} + r_1C_{d_1} + C_{31}/T)(\mu + b_{10})}{2\lambda_1} \\ K_2 &= \big[b_1(\mu + b_{10}) + b_{11}(b_{10} + \mu) + \frac{C_{11}}{2\lambda_1}\big] \end{aligned}$$

The complementary function of the Eq. (4.4) is $C_1^{'}e^{(b_{10}+\mu)t} + C_2^{'}e^{-b_{10}t}$, where $C_1^{'}$ and $C_2^{'}$ are arbitrary constants and the particular integral is given by the $\frac{1}{D^2-\mu D-(\mu+b_{10})b_{10}}$ $H_1(t)$. Here $D(\equiv \frac{d}{dt})$ represents the differential operator. Therefore, the complete solution of the Eq.(4.4) can be represented as

$$q_1(t) = C_1^{'}e^{(b_{10}+\mu)t} + C_2^{'}e^{-b_{10}t} - \frac{1}{K_3^2}\big[K_1K_3 + K_2\big(K_3t - \mu\big)\big]$$

and the corresponding rate is

$$P_1(t) = K_4e^{(b_{10}+\mu)t} + K_5t + K_7, \tag{4.5}$$

$$\text{where, } K_3 = b_{10}(b_{10} + \mu), K_4 = C_1^{'}(2b_{10} + \mu), K_5 = (b_1 + b_{11} - \frac{b_{10}K_2}{K_3}),$$

$$\text{and } \quad K_7 = \frac{1}{K_3^2}(a_1K_3^2 + b_{10}K_2\mu - b_{10}K_1K_3 - K_2K_3)$$

Using the boundary conditions given with $q_1(0) = -S_1$ and $q_1(T) = 0$ in the expression of $q_1(t)$,we can get the value of $C_1^{'}$ and $C_2^{'}$. Substituting the value of $q_1(t)$ and $P_1(t)$ in the expression of (4.2), the corresponding profit function can be expressed as

$$\begin{aligned} Z_p = \int_0^T e^{-\mu t}\big[&S_{p_1}D_1 - (C_{r_1} + C_{d_1}r_1)(\dot{q}_1 + D_1 + B_1) - \chi(r_1) - \lambda_1(\dot{q}_1 + D_1 + B_1)^2 \\ &-(C_{10} + C_{11}t)q_1 - \{C_{30} + C_{31}(\dot{q}_1 + D_1 + B_1)\}/T - S_{h_1}S_1\big]dt \end{aligned}$$

$$
\begin{aligned}
= \quad & S_{p_1}\big[-a_1\frac{(e^{-\mu T}-1)}{\mu}-b_1\frac{(e^{-\mu T}-1+\mu T e^{-\mu T})}{\mu^2}\big]-(C_{r_1}+r_1 C_{d_1})\big[\frac{K_4(e^{b_{10}T}-1)}{b_{10}} \\
+ \quad & K_5\big(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}\big)+\frac{K_7}{\mu}(1-e^{-\mu T})\big]+\big[N_1+N_2 \\
& e^{C_A(r_{1max}-r_1)/(r_1-r_{1min})}\big]\big(\frac{e^{-\mu T}-1}{\mu}\big)+\lambda_1\big[(K_4 e^{(b_{10}+\mu)T}+K_5 T+K_7)^2\frac{e^{-\mu T}}{\mu}- \\
& \frac{(K_4+K_7)^2}{\mu}+\frac{2}{\mu^2}\{\frac{K_4^2(b_{10}+\mu)}{(2b_{10}+\mu)}(e^{(2b_{10}+\mu)T}-1)+\frac{K_4K_5}{b_{10}}(e^{b_{10}T}-1)+\frac{K_4K_5}{b_{10}} \\
& (b_{10}+\mu)\{\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2}\}+\frac{K_4K_7(b_{10}+\mu)}{b_{10}}(e^{b_{10}T}-1)-\frac{K_5K_7}{\mu}(e^{-\mu T}-1)\}\big] \\
& -C_{10}\big[\frac{C_1^{'}}{b_{10}}(e^{b_{10}T}-1)-\frac{C_2^{'}}{(b_{10}+\mu)}(e^{-(b_{10}+\mu)T}-1)+\frac{K_1}{K_3\mu}(e^{-\mu T}-1)+\frac{1}{K_3^2}(K_1K_3 \\
+ \quad & K_2(K_3T-\mu))\frac{e^{-\mu T}}{\mu}+\frac{K_2}{K_3\mu^2}(e^{-\mu T}-1)-\frac{1}{\mu {K_3}^2}(K_1K_3-K_2\mu)\big] \\
- \quad & C_{11}\big[C_1^{'}\big(\frac{Te^{b_{10}T}}{b_{10}}-\frac{e^{b_{10}T}}{b_{10}^2}+\frac{1}{b_{10}^2}\big)+C_2^{'}\big(\frac{-Te^{-(b_{10}+\mu)T}}{(b_{10}+\mu)}-\frac{e^{-(b_{10}+\mu)T}}{(b_{10}+\mu)^2}+\frac{1}{(b_{10}+\mu)^2}\big) \\
- \quad & \frac{K_1}{K_3}\big(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}\big)-\frac{K_2\mu}{K_3^2}\big(\frac{-Te^{-\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}\big)+\frac{K_2}{K_3}\big(\frac{-T^2e^{-\mu T}}{\mu} \\
- \quad & \frac{2Te^{-\mu T}}{\mu^2}-\frac{2}{\mu^3}e^{-\mu T}+\frac{2}{\mu^3}\big)\big]-\frac{1}{T}\big[\frac{C_{30}}{\mu}(1-e^{-\mu T})+C_{31}\big(\frac{K_4}{b_{10}}(e^{b_{10}T}-1) \\
+ \quad & K_5\big(-\frac{-Te^{\mu T}}{\mu}-\frac{e^{-\mu T}}{\mu^2}+\frac{1}{\mu^2}\big)+\frac{K_7}{\mu}(1-e^{-\mu T})\big)\big]-S_{h_1}S_1\big(\frac{1-e^{-\mu T}}{\mu}\big)
\end{aligned}
\tag{4.6}
$$

4.2. Model-2a: Model with single stock-dependent breakable item. In the above Model-2, if we take the the parametric values of breakability/deterioration which are directly related to the time equal to zero i.e. $b_{11}=0$, then we get another Model-2a. Therefore, the Model-2 reduces to a production-inventory model for deteriorating items with stock dependent breakability/deterioration. So, the total profit can be obtained by optimizing the Eq. (4.6) with $b_{11}=0$

4.3. Model-2b: Model with single non-breakable item. In the above Model-2, if we take the the parametric value of deterioration which is directly related to stock and time is equal to zero i.e. $b_{10}=0$, $b_{11}=0$, then we get a another model-2b. Therefore, the Model-2 reduces to a production-inventory model with out deteriorating item. As b_{10}, b_{11} appears in the denominator of the expression of (4.6), So the total profit can not obtain by optimizing the Eq. (4.6) by directly putting with $b_{10}=0$, $b_{11}=0$. Thus, for the total profit of Model-2b can be obtain by omitting the reliability term from the expression of (4.6) and processing the same way as before in Model-2.

4.4. Model-2c: Model with single breakable item with constant demand. In the above Model-2, if we take the the parametric value of demand which is directly related to the time is equal to zero i.e. $b_1=0$, then we get a another Model-2c. Therefore, the Model-2 reduces to a production-inventory model for breakable item with constant demand . So, the total profit can be obtain by optimizing the Eq. (4.6) with $b_1=0$.

4.5. Model-2d: Model with single breakable item with constant holding cost. In the above Model-2, if we take the the parametric value of holding cost which is directly related to the time is equal to zero i.e. $C_{11} = 0$, then we get a another Model-2d. Therefore, the Model-2 reduces to a production-inventory model for breakable item with constant holding cost. So, the total profit can be obtain by optimizing the Eq. (4.6) with $C_{11} = 0$.

4.6. Model-2e: Model with single breakable item with constant set-up cost. In the above Model-2, if we take the the parametric value of setup cost which is directly related to the production rate is equal to zero i.e. $C_{31} = 0$, then we get a another Model-2e. Therefore, the Model-2 reduces to a production-inventory model for breakable item with constant set up cost. So, the total profit can be obtain by optimizing the Eq. (4.6) with $C_{31} = 0$.

5. Solution procedure:

In section 3.2, we already prove that there exists a path $q = q_1(t)$ and $q = q_2(t)$ lying between the interval $[0, T]$ for which Z_p is maximum. In this problem, only the reliability indicator is the decision variable and others parameters are known, so the profit function Z_p given by (3.20) and (4.6) are the function of a two variable r_1 and r_2 for Model-1 and single variable r_1 for Model-2 respectively. So, there are two method for finding the optimal value of r_1 and r_2. First we discussed the analytical method for finding the optimal value of r_1 and r_2. To find the optimal value of r_1 and r_2, the first order partial derivative of the profit function with respect to r_1 and r_2 are made equal to zero. Thus for the Model-1, we get two different transcendental equation on r_1 and r_2 and for Model-2, we get one transcendental equation on r_1 and solve using Newton-Raphson method. Now to find the second order derivative of Z_p with respect to r_1 and r_2 are calculate separately for both the models. Both the value of second order derivative with to the calculated r_1 and r_2 value are less than zero i.e. $\frac{\partial^2 Z_T}{\partial r_1^2} < 0$, $\frac{\partial^2 Z_T}{\partial r_2^2} < 0$, $\frac{\partial^2 Z_T}{\partial r_1^2}\frac{\partial^2 Z_T}{\partial r_2^2} - \left(\frac{\partial^2 Z_T}{\partial r_1 \partial r_2}\right)^2 > 0$ for model-1 and $\frac{d^2 Z_T}{dr_1^2} < 0$ for Model-2. So for both the model, we conclude that both the profit function are maximized and the corresponding profit can be calculated by putting the value of r_1 and r_2 respectively for both the models. Also the profit functions are optimized using LINGO-13 software and the result obtained are same as those obtained by analytical method. Therefore, we conclude that the result obtained by the above mentioned procedure is a global optimal solution for different models.

6. Numerical Experiment:

Model-1: The following parametric value have been used to validate the model: $a_1 = 60$; $b_1 = 50$; $\lambda_1 = 0.05$; $C_{r_1} = 4$; $C_{d_1} = 4$; $C_{10} = 1$, $C_{11} = 0.02$; $C_{30} = 10$; $C_{31} = 0.02$; $C_A = 0.002$; $S_{p_1} = 75$; $b_{10} = 0.05$; $b_{11} = 1.5$; $r_{1max} = 0.9$; $r_{1min} = 0.1$ $N_1 = 200$; $N_2 = 30$; $T = 12$; $S_{h1} = 1.02$; $S_1 = 10$;$S_2 = 20$; $\mu = 0.03$;$M = 2000$; $a_2 = 65$; $b_2 = 55$; $\lambda_2 = 0.06$; $C_{r_2} = 5$; $C_{d_2} = 5$; $C_{20} = 2$, $C_{21} = 0.03$; $C_{40} = 11$; $C_{41} = 0.03$; $S_{p_2} = 76$; $b_{20} = 0.06$; $b_{21} = 1.6$; $r_{2max} = 0.9$; $r_{2min} = 0.1$, $S_{h2} = 1.03$; $N_3 = 205$; $N_4 = 32$;

Model-2: In this model i.e.,only one item is considered. In this case, we consider the inputs of 1st item and all the parameters are same as Model 1.

With the above input data, the optimum values of r_1 and r_2 and the corresponding value of profit function for both the models are obtained and presented in Tables-1 and 2.

Table-1: Optimum results of Models-1

	Model-1	Model-1a	Model-1b	Model-1c	Model-1d	Model-1e
r_1	0.1065489	0.1066123	0.107523	0.107528	0.101572368	0.124578
r_2	0.1064327	0.1066529	0.107439	0.157423	0.1792436	0.14132563
Z_p	609575.45	611359.89	702204.56	120451.97	617561.24	624578.57

Table-2: Optimum results of Models -2

	Model-2	Model-2a	Model-2b	Model-2c	Model-2d	Model-2e
r_1	0.1065357	0.1066117	0.1081474	0.0059423	0.1065271	0.1065355
r_2	$--$	$--$	$--$	$--$	$--$	$--$
Z_p	307580.8	307731.6	398616.8	50141.1	315486	307629

With the optimal values of r_1 and r_2, different pictorial representations of inventory, production and demand against time, profit and development cost against reliability indicator , unit production cost and set-up cost against time for Model-1 are depicted in Figs.1-6, respectively. Similar graphical representation for Model-2 are given in Figs.7-9.

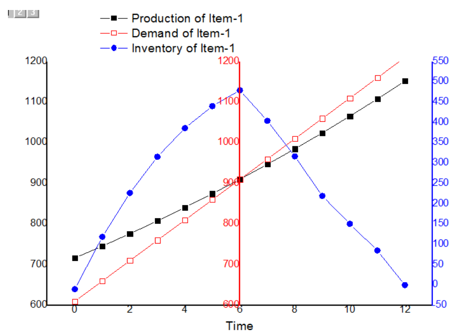

Figure 1. Time vs. production, demand and inventory of item-1 in Model-1

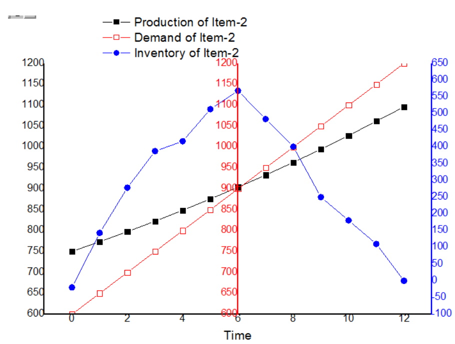

Figure 2. Time vs. production, demand and inventory of item-2 in Model-1

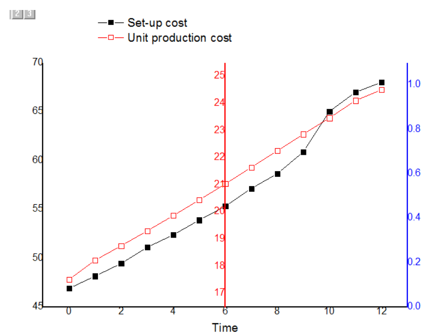

Figure 3. Time vs. unit production cost and set-up cost of item-1 in Model-1

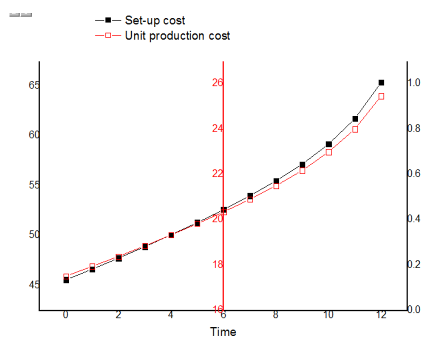

Figure 4. Time vs. unit production cost and set-up cost of item-2 in Model-1

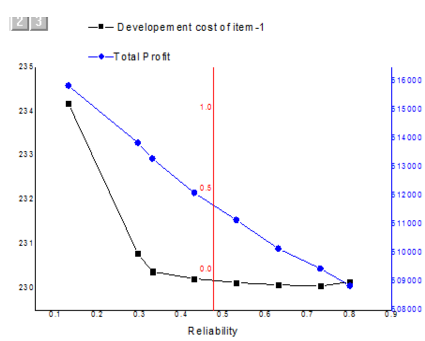

Figure 5. Reliability vs.developement cost and Profit of item-1 (Model-1)

7. Discussion:

For the presumed parametric values, it is very clear that profits for the models without damageability i.e. Model-1b and Model-2b gives the more profits than the corresponding models with damageability models such as Model-1, Model-1a, Model-2 and Model-2a. It is as per expectation of the real life phenomena. It occurs because profits decreases

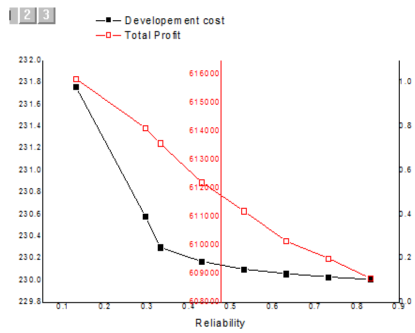

Figure 6. Reliability vs.developement cost and Profit of item-2 (Model-1)

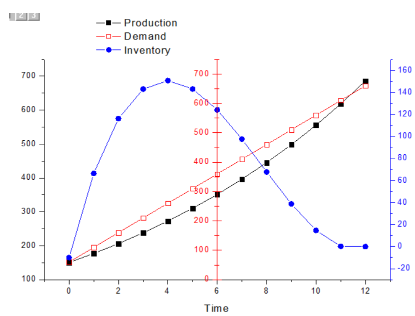

Figure 7. Time vs. production, demand and inventory for Model-2

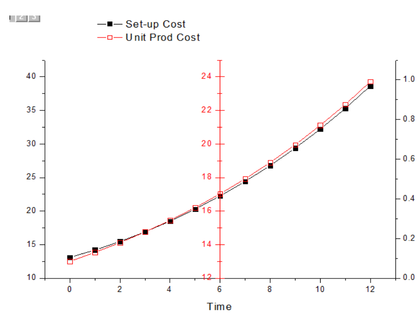

Figure 8. Time vs. set-up cost and unit production cost for Model-2

due to damageability of the units. Also from the Tables-1 and -2, it is observed that the models with stock dependent breakable items i.e. Model-1a and Model-2a give more profits than the corresponding models of breakable items i.e. Model-1 and Model-2. It is because the damageability rates for breakable items with both stock and time dependent breakability are higher than that of stock dependent breakability.

From Tables -1 and -2 it can be observe that the profit for the time dependent demand i.e Model -1 and -2 is greater than the constant demand i.e Model -1c and -2c, it can be explained from the real life situation that if demand increases with time then the profit will be more.It is also observed from Tables -1 and -2 that the profits for the constant

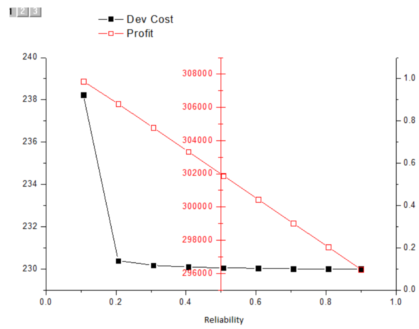

Figure 9. Reliability vs. profit and development cost for Model-2

holding cost and constant set up cost i.e profits for Model -1d,Model -2d,Model -1e and Model -2e are more than the profits for Model -1 and -2.It can be justified from the real fact that if unit holding cost and set up cost are constant than the retailer has to pay less amount for holding cost and set up cost and as a result the retailer gets more profit. Again process reliability indicators plays an important role for the profit making imperfect production system. Reliability indicator of a imperfect production process can be controlled using high quality machineries and skilled and efficient manpowers workers.
In our present investigation, demand of both models are time dependent. For both the models, production rate increases with time as demand increases with time. This phenomenon is justified by our pictorial representation i.e. Figs.-1,-2 and -7. It is observed from the Figs.1,2,7 that as the terminal conditions for stock are $q_1(0) = -10$, $q_1(T) = 0$, $q_2(0) = -20$, $q_2(T) = 0$ for Model-1 and $q_1(0) = -10$ and $q_1(T) = 0$ for Model-2, initially when time $t = 0$ shortages occurs at maximum level and as the demand and production dependent on time t and due to their combine effect, the shortages reach to zero after certain time. Due to this effect, the inventory is built-up as production is greater than the combine effect of demand and damageability. But after some time when considerable stock is built-up i.e., when the stock level becomes highest,production is discontinued. After this, to meet the demand, after allowing breakability, stock gradually reduces and ultimately becomes zero at $t = T$. Again from the Figs.-5,-6 and-9, it is observed that optimum profit Z_T is attained for a particular value of the process reliability r. Also it is noticed that the profit deceases with increasing process reliability,since reliability is defined as the ratio of number of damageable item with total items. Since the breakability increases with time i.e. damageability increases with time, so the profit is decreases with increasing reliability. This Phenomenon is also agree with the real life situation. Again from this figure, we observed that for some initial increasing value of r, the development cost sharply decreases and then become almost constant for higher values of r.Initially the profit become maximum and then decreases with increasing reliability. As set-up cost and production cost are partially production dependent, and production is time dependent, set-up cost and production cost increases with increasing time.These observation are found from the Figs.-3,-4 and-8.

8. Conclusions and Future Research work:

In this paper, for the first time, a multi-item production-inventory model with imperfect production process is considered for a breakable or deteriorating item over finite time horizon, where the process reliability indicator of the production process together with the production rate is controllable. For the present models, we observed that an optimum reliability indicator lures the maximum profit for an item having time dependent demand. Also it is found from our findings that minimum unit production cost for an item does not guaranty for giving maximum profit always. From the present models, it can be concluded that optimal control of production rate reduces holding cost as well as damageability which in turn increases profit separately for breakable/deteriorating items. The present investigation reveals that process reliability indicator is an important factor which determines the production rate and thus determining the optimal production path, unit production cost and optimal profit for the production-inventory managers. Here we formulate two types of models with shortages. First model is for two items with shortages and second model is for single item with shortages. The unit production cost is a function of production rate, raw material cost, labour charge, wear and tear cost and product reliability indicator. The first model is formulated as optimal control problems for the maximization of total profits over the planning horizon with budget constraint and optimum profit with profits along with optimum reliability indicator(r) are obtained using Euler-Lagrange equations based on variational principle.The second model is also solved under the same assumptions and using the same technique. Both the problems have been solved using a non-linear optimization technique -GRG (LINGO-13.0) and illustrated with some numerical data. Several particular cases are derived and the results are presented in both tabular and graphical forms. Finally, some sensitivity analyses can be made with respect to different parameters.The present models can be extended to fuzzy environment taking constant part of holding cost, set-up cost, etc as fuzzy in nature.Now a days due to inherent various and highly uncertinity of real life informations/data , impreciseness of fuzzy set i.e type-2 fuzzy sets in quite popular. Hence the present problem can be solved with type-2 fuzzy inventory cost,etc.This is a new area of research in which integrand of a finite integral is fuzzy or type-2 fuzzy and variational principle is applied.

More-over with the deterministic integrand and the limits of a finite integral as fuzzy, models can be formulated and solved using Fuzzy Riemann integral, not using variational principle.

The present model can also be extended to multi-period models where period starts with inventory and end with shortages or starts with inventory and end with inventory or different variations can be done with respect to shortages.

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Estimation of $P(Y < X)$ for the Lévy distribution

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Abstract

Three point estimators and two interval estimators of $P(Y < X)$ are derived when X and Y are independent Lévy random variables. Their performance with respect to relative biases, relative mean squared errors, coverage probabilities, and coverage lengths is assessed by simulation studies and a real data application.

Keywords: Bayesian estimator, Bootstrap confidence interval, Lévy distribution, Maximum likelihood estimator, Uniformly minimum variance unbiased estimator.

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1. Introduction

Let X be a Lévy random variable with scale parameter σ_x. Then the probability density function (pdf) and the cumulative distribution function (cdf) of X are:

$$f(x, \sigma_x) = \sqrt{\frac{\sigma_x}{2\pi}} x^{-\frac{3}{2}} \exp\left(-\frac{\sigma_x}{2x}\right)$$

and

$$F(x, \sigma_x) = 2\left[1 - \Phi\left(\sqrt{\frac{\sigma_x}{x}}\right)\right],$$

respectively, for $x > 0$ and $\sigma_x > 0$, where $\Phi(\cdot)$ denotes the standard normal cdf. According to O'Reilly and Rueda [28], $\frac{1}{X}$ is a gamma random variable with shape parameter $\frac{1}{2}$ and scale parameter $\frac{2}{\sigma}$. Lévy distribution has no moments.

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Lévy distribution and the more general alpha-stable distribution have received applications in many areas, including dispersive transport in disordered semiconductors, stock and stock-indexes returns, linear dynamical systems, income distribution, stochastic artificial neural networks, many-particle quantum systems, oil pricing time-series, distributions of stochastic payoff variations, real traffic flow, satellite magnetic field measurements, models for circular data, models of asset trading, directed percolation with incubation times, earthquake slip spatial distributions, models for financial markets with central regulation, long correlation times in supermarket sales, edge turbulence of fusion devices, network traffic behavior in switched Ethernet systems, modeling individual behavior in a large marine predator, evolutionary programming using mutations, distribution of marks in high school, fractal structures, models for fish locomotion, distribution of economical indices, south Spain seismic series, geophysical data analysis, supermarket sales, velocity difference in systems of vortex elements, currency exchange market, random field models for geological heterogeneity, structural reorganization in rice piles, and wave scattering from self-affine surfaces. Three of the most recent applications relate to daily price fluctuations in the Mexican financial market index (Alfonso *et al.* [1]), observations of anomalous diffusion (Sagi *et al.*, 2012), and bistable systems (Srokowski [33]).

In the stated areas, it is of interest to estimate the probability $R = P(Y < X)$ when X and Y are independent Lévy random variables. For example, X and Y could represent: stock returns for two different commodities; oil prices in two different countries; traffic at two different locations; earthquake magnitudes at two different locations; marks at two different high schools; and, so on.

Estimation of $P(Y < X)$ is widely known as stress-strength modeling: if X denotes the stress that a system is subjected to and Y the strength of the system then $P(Y < X)$ is the probability of the failure of the system. Many papers have investigated estimation of $P(Y < X)$ when X and Y arise from a specific distribution. For details, see Awad and Gharraf [4], Surles and Padgett [34] for the case X, Y are Burr distributed; Constantine *et al.* [11], Ismail *et al.* [20] for the case X, Y are gamma distributed; Obradovic *et al.* [27] for the case X, Y are geometric-Poisson distributed; Babayi *et al.* [5] for the case X, Y are generalized logistic distributed; Kundu and Raqab [22] for the case X, Y are generalized Rayleigh distributed; Saracoglu *et al.* [32] for the case X, Y are Gompertz distributed; Nadar *et al.* [25] for the case X, Y are Kumaraswamy distributed; Downtown [14], Reiser and Guttman [30] for the case X, Y are normal distributed; Genc [17] for the case X, Y are Topp-Leone distributed; McCool [24] for the case X, Y are Weibull distributed. There are also semiparametric and nonparametric methods for estimating $P(Y < X)$. Kotz *et al.* [21] provide an excellent review of known work.

There has not been much work on the estimation of $R = P(Y < X)$ when X and Y are independent Lévy random variables. The only paper we are aware of is Ali and Woo [3]. But the estimators given in Ali and Woo [3] are not those for R. A related paper by Ali *et al.* [2] studies the distribution of $X/(X + Y)$.

In this note, we provide point as well as interval estimators for $R = P(Y < X)$. The point estimators considered are: maximum likelihood estimator, uniformly minimum variance unbiased estimator (UMVUE) and Bayes estimator taken as the mean of the posterior distribution of R given suitable priors. The interval estimators considered are: asymptotic maximum likelihood estimator and bootstrap based percentile estimator. The performance of these estimators is assessed by simulation studies as well as by a real data application.

2. Point estimators of R

In this section, we give three point estimators for R. Their performances are compared by a simulation study in Section 5.1. Throughout, we suppose $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ are independent random samples from the Lévy distribution with scale parameters σ_x and σ_y, respectively.

2.1. Maximum likelihood estimator of R. The maximum likelihood estimators of σ_x and σ_y are:

$$\widehat{\sigma}_x = \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}}$$

and

$$\widehat{\sigma}_y = \frac{m}{\sum_{j=1}^{m} \frac{1}{Y_j}},$$

respectively. Ali and Woo [3] show that:

$$R = \frac{2}{\pi} \sin^{-1} \frac{1}{\sqrt{1 + \frac{\sigma_y}{\sigma_x}}}.$$

Thus, the maximum likelihood estimator of R follows by the invariance property:

$$\widehat{R} = \frac{2}{\pi} \sin^{-1} \frac{1}{\sqrt{1 + \frac{\widehat{\sigma}_y}{\widehat{\sigma}_x}}}. \tag{2.1}$$

2.2. UMVUE of R. To find the UMVUE of R, we use results in Ismail *et al.* [20]. It is easy to see that $\left(\sum_{i=1}^{n} 1/X_i, \sum_{j=1}^{m} 1/Y_j\right)$ is complete and sufficient for (σ_x, σ_y). Let

$$\Phi(X, Y) = \begin{cases} 1, & \text{if } \frac{1}{X} < \frac{1}{Y}, \\ 0, & \text{if } \frac{1}{X} > \frac{1}{Y}. \end{cases}$$

Then, one can see that $\Phi(X, Y)$ is an unbiased estimator of R. It follows by Lehmann-Scheffe theorem (see page 369 in Casella [8]) that:

$$\widetilde{R} = E\left(\Phi(X, Y) \middle| \sum_{i=1}^{n} \frac{1}{X_i}, \sum_{j=1}^{m} \frac{1}{Y_j}\right)$$

is an UMVUE. Since $\frac{1}{X}$, $\frac{1}{Y}$, $\sum_{i=1}^{n} \frac{1}{X_i}$, $\sum_{j=1}^{m} \frac{1}{Y_j}$ are gamma random variables, we have from Ismail *et al.* [20] that

$$\widetilde{R} = \begin{cases} \int_0^1 F_{W_2}\left(\frac{U}{V} w_1\right) f_{W_1}(w_1)\, dw_1, & \text{if } U \leq V, \\ \int_0^{\frac{V}{U}} F_{W_2}\left(\frac{U}{V} w_1\right) f_{W_1}(w_1)\, dw_1 + 1 - F_{W_1}\left(\frac{V}{U}\right), & \text{if } U > V, \end{cases} \tag{2.2}$$

where $W_1 \sim Beta\left(\frac{1}{2}, \frac{n-1}{2}\right)$ and $W_2 \sim Beta\left(\frac{1}{2}, \frac{m-1}{2}\right)$ are beta random variables, $U = \sum_{i=1}^{n} \frac{1}{X_i}$ and $V = \sum_{j=1}^{m} \frac{1}{Y_j}$. If $W \sim Beta(a,b)$ then its cdf is the incomplete beta function ratio defined by $I_w(a,b) = \int_0^w t^{a-1}(1-w)^{b-1} dt / B(a,b)$, where $B(a,b) = \int_0^1 t^{a-1}(1-w)^{b-1} dt$ denotes the beta function. So, (2.2) can be expressed as

$$
\widetilde{R} = \begin{cases} \frac{1}{B(1/2,(n-1)/2)} \int_0^1 I_{Uw/V}\left(\frac{1}{2}, \frac{m-1}{2}\right) w^{-1/2}(1-w)^{(n-3)/2} dw, \\ \quad \text{if } U \le V, \\ \frac{1}{B(1/2,(n-1)/2)} \int_0^{V/U} I_{Uw/V}\left(\frac{1}{2}, \frac{m-1}{2}\right) w^{-1/2}(1-w)^{(n-3)/2} dw \\ \quad +1 - I_{V/U}\left(\frac{1}{2}, \frac{n-1}{2}\right), \\ \quad \text{if } U > V. \end{cases} \tag{2.3}
$$

An alternative expression using the series expansion

$$
I_w(a,b) = \frac{w^a}{B(a,b)} \sum_{k=0}^{\infty} \frac{(1-b)_k w^k}{(a+k)k!},
$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial, is

$$
\widetilde{R} = \begin{cases} \frac{1}{B(1/2,(n-1)/2)\, B(1/2,(m-1)/2)} \\ \quad \cdot \sum_{k=0}^{\infty} \frac{((3-m)/2)_k}{(k+1/2)\,k!} B\left(k+1, \frac{n-1}{2}\right) \left(\frac{U}{V}\right)^{k+1/2}, \\ \quad \text{if } U \le V, \\ \frac{1}{B(1/2,(n-1)/2)\, B(1/2,(m-1)/2)} \\ \quad \cdot \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{((3-m)/2)_k\, ((3-n)/2)_\ell}{(k+1/2)\,(k+\ell+1)\,k!\ell!} \left(\frac{V}{U}\right)^{\ell} \\ \quad +1 - \frac{1}{B(1/2,(n-1)/2)} \sum_{k=0}^{\infty} \frac{((3-n)/2)_k}{(k+1/2)\,k!} \left(\frac{V}{U}\right)^{k+1/2}, \\ \quad \text{if } U > V. \end{cases} \tag{2.4}
$$

This expression can be used to compute measures like the variance, skewness and kurtosis of $\widetilde{R}$. For example, using equation (6.455.1) in Gradshteyn and Ryzhik [18], one can show that

$$
\begin{aligned}
E\left(\widetilde{R}^2\right) \quad = \quad & \frac{1}{B^2\left(1/2,(n-1)/2\right)B^2\left(1/2,(m-1)/2\right)} \\
& \cdot \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\left((3-m)/2\right)_k\left((3-m)/2\right)_\ell}{\left(k+1/2\right)k!\left(\ell+1/2\right)\ell!} \\
& \cdot B\left(k+1,\frac{n-1}{2}\right)B\left(\ell+1,\frac{n-1}{2}\right) \\
& \cdot I\left(k+\ell+1,-k-\ell-1\right) \\
& +\frac{1}{B^2\left(1/2,(n-1)/2\right)B^2\left(1/2,(m-1)/2\right)} \\
& \cdot \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\left((3-m)/2\right)_k}{\left(k+1/2\right)k!} \\
& \cdot \frac{\left((3-n)/2\right)_\ell\left((3-m)/2\right)_p\left((3-n)/2\right)_q}{\left(k+\ell+1\right)\ell!\left(p+1/2\right)\left(p+q+1\right)p!q!}J\left(\ell+q,-\ell-q\right) \\
& +1+\frac{1}{B^2\left(1/2,(n-1)/2\right)}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\left((3-n)/2\right)_k}{\left(k+1/2\right)k!} \\
& \cdot \frac{\left((3-n)/2\right)_\ell}{\left(\ell+1/2\right)\ell!}J\left(k+\ell+1,-k-\ell-1\right) \\
& +\frac{2}{B\left(1/2,(n-1)/2\right)B\left(1/2,(m-1)/2\right)} \\
& \cdot \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\left((3-m)/2\right)_k}{\left(k+1/2\right)k!} \\
& \cdot \frac{\left((3-n)/2\right)_\ell}{\left(k+\ell+1\right)\ell!}J\left(\ell,-\ell\right) \\
& -\frac{2}{B^2\left(1/2,(n-1)/2\right)B\left(1/2,(m-1)/2\right)} \\
& \cdot \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{p=0}^{\infty}\frac{\left((3-m)/2\right)_k}{\left(k+1/2)\right)k!} \\
& \cdot \frac{\left((3-n)/2\right)_\ell\left((3-n)/2\right)_p}{\left(k+\ell+1\right)\ell!\left(p+1/2\right)p!}J\left(\ell+p+\frac{1}{2},-\ell-p-\frac{1}{2}\right) \\
& -\frac{2}{B\left(1/2,(n-1)/2\right)}\sum_{k=0}^{\infty}\frac{\left((3-n)/2\right)_k}{\left(k+1/2\right)k!}J\left(k+\frac{1}{2},-k-\frac{1}{2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
I(\alpha,\beta) \quad = \quad & \frac{2^{\alpha+\beta}\sigma_x^{n/2}\sigma_y^{m/2}\Gamma\left(\alpha+\beta+(m+n)/2\right)}{\left(\sigma_x+\sigma_y\right)^{\alpha+\beta+(m+n)/2}\left(\alpha+n/2\right)\Gamma(m/2)\Gamma(n/2)} \\
& \cdot {}_2F_1\left(1,\alpha+\beta+\frac{m+n}{2};\alpha+\frac{n}{2}+1;\frac{\sigma_x}{\sigma_x+\sigma_y}\right)
\end{aligned}
$$

and

$$
J(\alpha,\beta)=\frac{2^{\alpha+\beta}\Gamma\left(\alpha+n/2\right)\Gamma\left(\beta+m/2\right)}{\sigma_x^{\alpha}\sigma_y^{\beta}\Gamma(m/2)\Gamma(n/2)}-I(\alpha,\beta),
$$

where

$$\Gamma(a) = \int_0^\infty t^{a-1}\exp(-t)dt$$

and

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}$$

denote the gamma and Gauss hypergeometric functions, respectively. So, the variance of $\widetilde{R}^2$ is $E\left(\widetilde{R}^2\right) - R^2$.

2.3. Bayes estimator of R. Suppose the scale parameters, σ_x and σ_y, have the following gamma priors:

$$\sigma_x \sim \Gamma\left(\frac{r_1}{2}, \lambda_1\right) \qquad \text{and} \qquad \sigma_y \sim \Gamma\left(\frac{r_2}{2}, \lambda_2\right).$$

There are several reasons why we have chosen gamma priors: i) the resulting posterior pdfs of σ_x and σ_y,

$$\sigma_x|x \sim \Gamma\left(\frac{n+r_1}{2}, \lambda_1 + \frac{1}{2}u\right) \qquad \text{and} \qquad \sigma_y|y \sim \Gamma\left(\frac{m+r_2}{2}, \lambda_1 + \frac{1}{2}v\right),$$

where $u = \sum_{i=1}^{n} \frac{1}{X_i}$ and $v = \sum_{i=1}^{m} \frac{1}{Y_i}$, belong to the same class; ii) According to Felsenstein [16], assuming a prior distribution "of rates such as a gamma distribution or lognormal distribution has deservedly been popular"; iii) According to Lambert *et al.* [23], gamma priors are "the most common used prior distribution for variance parameters, not least because it is used in many of the examples provided with the WinBUGS software"; iv) According to page 69 in Congdon [10], there has "been considerable debate about appropriate priors for variance and precision parameters ... the most common option is a gamma"; v) According to Dorfman and Karali [13], the gamma prior "on the error variance term is a standard one".

If we suppose σ_x and σ_y are independent then the joint posterior pdf of σ_x and σ_y is:

$$\begin{aligned} f(\sigma_x, \sigma_y|x,y) &= \sigma_x^{\frac{r_1+n}{2}-1} \frac{\left(\frac{1}{2}u+\lambda_1\right)^{\frac{r_1+n}{2}}}{\Gamma\left(\frac{r_1+n}{2}\right)} \exp\left(-\sigma_x\left[\frac{1}{2}u+\lambda_1\right]\right) \\ &\quad \cdot \exp\left(-\sigma_y\left[\frac{1}{2}v+\lambda_2\right]\right) \sigma_y^{\frac{r_2+m}{2}-1} \frac{\left(\frac{1}{2}v+\lambda_2\right)^{\frac{r_2+m}{2}}}{\Gamma\left(\frac{r_2+m}{2}\right)}. \end{aligned}$$

Thus, the posterior pdf of R is:

$$f_R(r|x,y) = C \frac{\cot\left(\frac{\pi}{2}r\right)^{r_2+m-1}\left[1+\cot\left(\frac{\pi}{2}r\right)^2\right]}{\left[\left(\frac{1}{2}u+\lambda_1\right)+\left(\frac{1}{2}v+\lambda_2\right)\cot\left(\frac{\pi}{2}r\right)^2\right]^{\frac{n+m+r_1+r_2}{2}}},$$

where

$$C = \pi \frac{\left(\frac{1}{2}u+\lambda_1\right)^{\frac{r_1+n}{2}}}{\Gamma\left(\frac{r_1+n}{2}\right)} \frac{\left(\frac{1}{2}v+\lambda_2\right)^{\frac{r_2+m}{2}}}{\Gamma\left(\frac{r_2+m}{2}\right)} \Gamma\left(\frac{r_1+r_2+m+n}{2}\right).$$

Under the mean squared error loss function, the Bayes estimator of R is:

$$\widehat{R}_{Bayes} = \int_0^1 r f_R(r|x,y)\, d$$

$$= C\int_0^1 r \frac{\cot\left(\frac{\pi}{2}r\right)^{r_2+m-1}\left[1+\cot\left(\frac{\pi}{2}r\right)^2\right]}{\left[\left(\frac{1}{2}u+\lambda_1\right)+\left(\frac{1}{2}v+\lambda_2\right)\cot\left(\frac{\pi}{2}r\right)^2\right]^{\frac{n+m+r_1+r_2}{2}}} dr. \tag{2.5}$$

Analytical solutions for the above integral are not available.

3. Interval estimators of R

In this section, we give two interval estimators for R. Their performances are compared by a simulation study in Section 5.2. Throughout, we suppose $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ are independent random samples from the Lévy distribution with scale parameters σ_x and σ_y, respectively.

3.1. Asymptotic confidence interval. For large sample sizes, a confidence interval for R can be obtained based on maximum likelihood estimation. For this purpose, we first obtain an asymptotic distribution of the maximum likelihood estimators, $\widehat{\sigma}_x$ and $\widehat{\sigma}_y$.

1. Theorem. If $n \to \infty$ and $m \to \infty$ such that $\frac{n}{m} \to p$ then

$$\left(\sqrt{n}\left(\widehat{\sigma}_x - \sigma_y\right), \sqrt{m}\left(\widehat{\sigma}_y - \sigma_y\right)\right) \to N\left(\mathbf{0}, \mathbf{\Sigma}\right),$$

where

$$\mathbf{\Sigma} = \begin{pmatrix} 2\sigma_x^2 & 0 \\ 0 & 2\sigma_y^2 \end{pmatrix}.$$

Proof. The proof is straightforward using asymptotic normality of $\widehat{\sigma}_x$ and $\widehat{\sigma}_y$. □

The asymptotic distribution of $\widehat{R}$ can now be easily deduced.

2. Theorem. If $n = m$ and $n \to \infty$ then

$$\sqrt{n}\left(\widehat{R} - R\right) \to N(0, D),$$

where

$$D = \frac{4\sigma_x\sigma_y}{\pi^2\left(\sigma_x+\sigma_y\right)^2}.$$

Hence, a 95 percent asymptotic confidence interval for R is

$$\left(\widehat{R} - 1.96\sqrt{\frac{D}{n}}, \widehat{R} + 1.96\sqrt{\frac{D}{n}}\right). \tag{3.1}$$

Proof. Follows by the delta method (see pages 33-35 of Davison [12]). □

3.2. Bootstrap confidence interval. Bootstrap confidence intervals are useful for small sample sizes. Here, we propose a percentile based bootstrap confidence interval due to Efron [15]. It can be constructed by the following scheme:

(1) From the samples $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$, compute the maximum likelihood estimates, $\widehat{\sigma}_x$ and $\widehat{\sigma}_y$;
(2) Using $\widehat{\sigma}_x$, generate a bootstrap sample $X_1^*, X_2^*, \ldots, X_n^*$ and similarly using $\widehat{\sigma}_y$ generate a bootstrap sample $Y_1^*, Y_2^*, \ldots, Y_m^*$. The inversion method was used to generate samples: this entails inverting the standard normal cdf and routines for this inversion are widely available even in pocket calculators. From the samples $X_1^*, X_2^*, \ldots, X_n^*$ and $Y_1^*, Y_2^*, \ldots, Y_m^*$, compute the maximum likelihood estimate of R, say $\widehat{R}^*$;
(3) Repeat step 2, B times, giving the estimates, say $\widehat{R}_1^*, \widehat{R}_2^*, \ldots, \widehat{R}_B^*$, of R;
(4) Compute the empirical cdf, say $\widehat{G}(\cdot)$, of $\widehat{R}_1^*, \widehat{R}_2^*, \ldots, \widehat{R}_B^*$. Then an approximate 95 percent confidence interval of R is

$$\left[\widehat{G}^{-1}(0.025), \widehat{G}^{-1}(0.975)\right], \tag{3.2}$$

where $\widehat{G}^{-1}(\cdot)$ denotes the inverse function of $\widehat{G}(\cdot)$.

Another bootstrap based interval is the bootstrap-t confidence interval for R. We shall not consider this here as it performed similarly to the percentile based bootstrap confidence interval.

4. A real data application

As mentioned in Section 1, one application of the Lévy distribution is to model stock index data. Here, we discuss such an application.

The data are S&P/IFC (Standard & Poor's / International Finance Corporation) global daily price indices in United States dollars for Egypt and South Africa, the two largest economies in Africa. The data cover the period from the 1st of January 1996 to the 31st of October 2008. The data were obtained from the database Datastream.

Following common practice, daily log returns were computed as first order differences of logarithms of daily price indices. Let X denote the daily log returns from South Africa and Y the daily log returns from Egypt. Some summary statistics for the data on X are: range = 0.078847, first quartile = 0.020640, median = 0.026720, and third quartile = 0.034270. Some summary statistics for the data on Y are: range = 0.086333, first quartile = 0.017030, median = 0.024120, and third quartile = 0.036920. The sample size for both data sets is 153.

The Lévy distribution was fitted to the data on X and Y by the method of maximum likelihood. We obtained the estimates $\widehat{\sigma_x} = 0.02392927$ and $\widehat{\sigma_y} = 0.01898238$. The chisquare and Kolmogorov-Smirnov tests for the fit to the log returns from South Africa gave the p-values 0.061 and 0.063. The chisquare and Kolmogorov-Smirnov tests for the fit to the log returns from Egypt gave the p-values 0.051 and 0.077. Since the Kolmogorov-Smirnov test assumes that the fitted distribution gives the "true" parameter values, the p-values were computed using Monte Carlo simulation.

Using the fitted estimates of σ_x and σ_y, we were able to compute $R = P(X < Y)$ using the three point estimation methods. For the maximum likelihood method, we obtained $\widehat{R} = 0.5367768$. For the UMVUE, we obtained $\widehat{R} = 0.5368976$. For the Bayes method, we obtained $\widehat{R} = 0.5367891$. It is remarkable that all three estimates are identical up to the first three decimal places. We took $\lambda_1 = \lambda_2 = 1$ and $r_1 = r_2 = 1$ for the Bayes method. Other choices gave similar results.

Using the fitted estimates of σ_x and σ_y, we were also able to compute $R = P(X < Y)$ using the two interval estimation methods. Using the asymptotic method, we obtained the 95 percent confidence interval (0.4866747, 0.5868788). Using the bootstrap method, we obtained the 95 percent confidence interval (0.4969806, 0.577812). The coverage length is smaller for the bootstrap method. We took $B = 500$ for the bootstrap method. Other choices gave similar results.

Both the confidence intervals contain $R = 0.5$ as a real value. Hence, there is no evidence that the daily log returns differ significantly between South Africa and Egypt. Further statistical analysis of the data set can be found in Nadarajah *et al.* [26].

5. Simulation studies

5.1. Simulation study for point estimators of R. Here, we perform a simulation study to compare the performances of the maximum likelihood estimator, the UMVUE and the Bayes estimator of R. The performance was assessed in terms of relative biases and relative mean squared errors. The following scheme was used:

(1) Generate ten thousand samples of $\{X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m\}$;

(2) Compute the estimators, (2.1), (2.2) and (2.5), for each of the ten thousand samples, say R_{1i}, R_{2i}, R_{3i} for $i = 1, 2, \ldots, 10000$. (2.2) and (2.5) were computed using the function integrate in R (R Development Core Team [29]);

(3) Compute the relative biases for the three estimators as

$$\text{Bias}_j = \frac{1}{10000} \sum_{i=1}^{10000} (R_{ji} - R)/R$$

for $j = 1, 2, 3$;

(4) Compute the relative mean squared errors for the three estimators as

$$\text{MSE}_j = \frac{1}{10000} \sum_{i=1}^{10000} (R_{ji} - R)^2/R$$

for $j = 1, 2, 3$.

We repeated this scheme for $m = n = 2, 3, \ldots, 100$ and $(\sigma_x, \sigma_y) = (1,1)$, $(1,2)$, $(1,5)$, $(2,2)$, $(2,5)$, $(5,5)$. For the Bayes estimator, we took $\lambda_1 = \lambda_2 = 1$ and $r_1 = r_2 = 1$, as in Section 4. Plots of the relative biases, bias_1, bias_2 and bias_3, versus n are shown in Figure 1. Plots of the relative mean squared errors, MSE_1, MSE_2 and MSE_3, versus n are shown in Figure 2. The red line in Figure 1 represents the relative biases being zero.

The following observations can be drawn from Figures 1 and 2:

(1) the magnitudes of the relative biases and relative mean squared errors generally decrease to zero with increasing n. Also the relative biases appear to take both positive and negative values when $\sigma_x = \sigma_y$;

(2) the relative biases for (2.1), (2.2) and (2.5) appear not too different when $\sigma_x = \sigma_y$;

(3) the relative biases for (2.1) and (2.5) appear generally positive when $\sigma_x < \sigma_y$;

(4) the relative biases for (2.1) and (2.2) appear smallest when $\sigma_x < \sigma_y$;

(5) the relative biases for (2.5) appear largest when $\sigma_x < \sigma_y$;

(6) the relative mean squared errors appear smallest, second smallest and largest for (2.5), (2.1) and (2.2), respectively, for small n;

(7) the relative biases and relative mean squared errors for all three estimators appear reasonable for all n and parameter values.

We have presented results for limited choices of (σ_x, σ_y) and for only one choice of $(\lambda_1, \lambda_2, r_1, r_2)$. But the results were the same for a wide range of other choices for

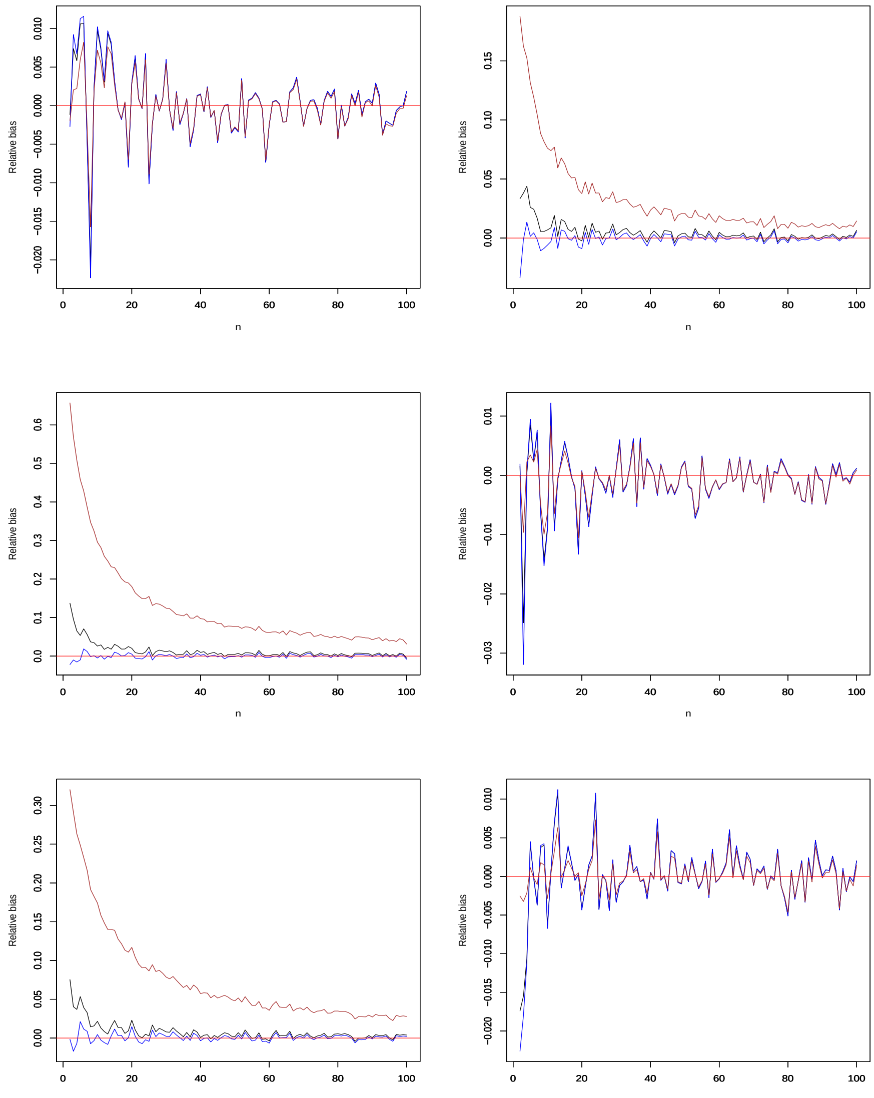

Figure 1. Relative biases of (2.1) in black, (2.2) in blue and (2.5) in brown. Top left is for $(\sigma_x, \sigma_y) = (1, 1)$, top right is for $(\sigma_x, \sigma_y) = (1, 2)$, middle left is for $(\sigma_x, \sigma_y) = (1, 5)$, middle right is for $(\sigma_x, \sigma_y) = (2, 2)$, bottom left is for $(\sigma_x, \sigma_y) = (2, 5)$, and bottom right is for $(\sigma_x, \sigma_y) = (5, 5)$.

(σ_x, σ_y) and $(\lambda_1, \lambda_2, r_1, r_2)$, including choices where $\lambda_1 \neq \lambda_2$ and $r_1 \neq r_2$. Similar results were also obtained when the gamma priors were replaced by non informative priors. In

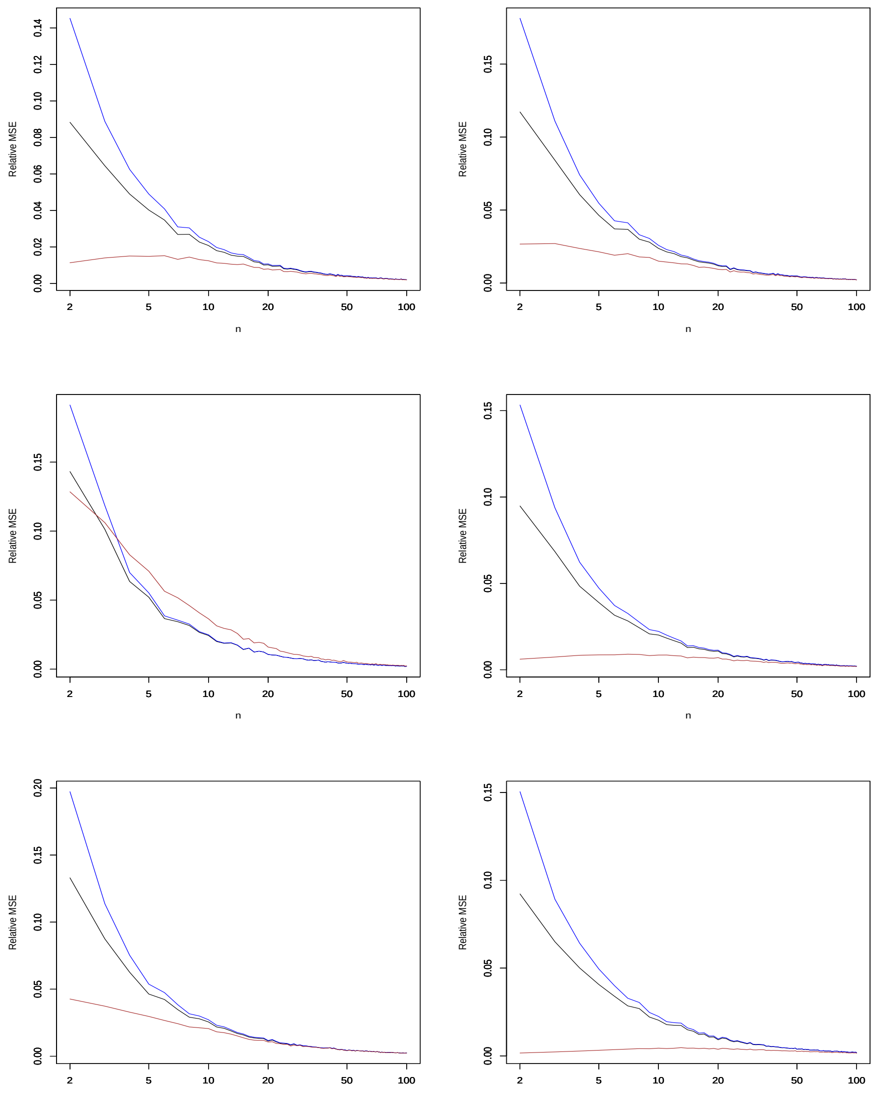

Figure 2. Relative mean squared errors of (2.1) in black, (2.2) in blue and (2.5) in brown. Top left is for $(\sigma_x, \sigma_y) = (1, 1)$, top right is for $(\sigma_x, \sigma_y) = (1, 2)$, middle left is for $(\sigma_x, \sigma_y) = (1, 5)$, middle right is for $(\sigma_x, \sigma_y) = (2, 2)$, bottom left is for $(\sigma_x, \sigma_y) = (2, 5)$, and bottom right is for $(\sigma_x, \sigma_y) = (5, 5)$.

particular, the magnitude of the relative biases generally decreased to zero with increasing n, the relative mean squared errors generally decreased to zero with increasing n, the relative biases for all three estimators appeared reasonable for all n, and the relative mean squared errors for all three estimators appeared reasonable for all n.

5.2. Simulation study for interval estimators of R. Here, we perform a simulation study to compare the performances of the asymptotic maximum likelihood and percentile based bootstrap confidence intervals for R. The performance was assessed in terms of coverage probabilities and coverage lengths. The following scheme was used:

(1) Generate ten thousand samples of $\{X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m\}$;
(2) Compute the confidence intervals, (3.1) and (3.2), for each of the ten thousand samples, say (L_{1i}, U_{1i}) and (L_{2i}, U_{2i}) for $i = 1, 2, \ldots, 10000$;
(3) Compute the coverage probabilities for the two intervals as

$$P_j = \frac{1}{10000} \sum_{i=1}^{10000} I \left\{ L_{ji} < \frac{2}{\pi} \sin^{-1} \frac{1}{\sqrt{1 + \dfrac{\sigma_y}{\sigma_x}}} < U_{ji} \right\}$$

for $j = 1, 2$;
(4) Compute the coverage lengths for the two intervals as

$$L_j = \frac{1}{10000} \sum_{i=1}^{10000} \left(U_{ji} - L_{ji} \right)$$

for $j = 1, 2$.

We repeated this scheme for $m = n = 1, 2, \ldots, 100$ and $(\sigma_x, \sigma_y) = (1, 1)$, $(1, 2)$, $(1, 5)$, $(2, 2)$, $(2, 5)$, $(5, 5)$. For the bootstrap confidence interval, we took $B = 500$, as in Section 4. Plots of the coverage probabilities, P_1 and P_2, versus n are shown in Figure 3. Plots of the coverage lengths, L_1 and L_2, versus n are shown in Figure 4. The red line in Figure 3 represents the 95 percent nominal level.

The following observations can be drawn from Figures 3 and 4:

(1) coverage probabilities generally approach the nominal level with increasing n and coverage lengths generally decrease with increasing n;
(2) coverage probabilities for (3.2) appear closer to the nominal level for all $n < 40$. Thereafter (3.1) and (3.2) appear to perform equally well.

We have presented results for limited choices of (σ_x, σ_y) and for only one choice of B. But the results were the same for a wide range of other choices for (σ_x, σ_y) and $B > 500$. In particular, the coverage probabilities generally approached the nominal level with increasing n and the coverage lengths generally decreased with increasing n.

6. Conclusions

In this note, we have studied estimation of $R = P(Y < X)$ when X and Y are independent Lévy random variables. We have considered three different point estimators for R: maximum likelihood estimator, UMVUE and Bayes estimator. We have considered two different interval estimators for R: asymptotic maximum likelihood estimator and bootstrap based percentile estimator.

Among the three point estimators, the Bayes estimator has the smallest relative mean squared errors but also the largest relative biases. The maximum likelihood estimator and the UMVUE have the smallest relative biases. But they do not have the smallest relative mean squared errors.

Among the two interval estimators, the bootstrap estimator has better coverage probabilities for small n. Both estimators perform equally well for all sufficiently large n.

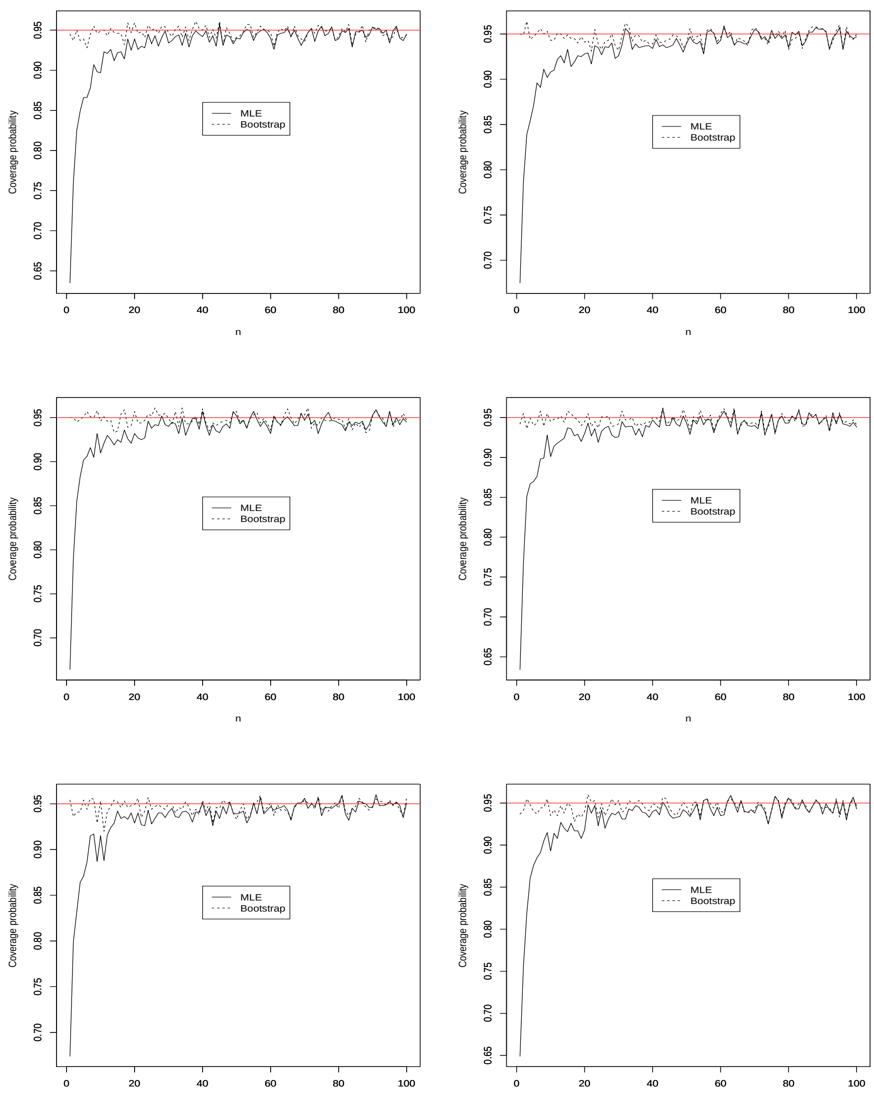

Figure 3. Coverage probabilities of (3.1) and (3.2). Top left is for $(\sigma_x, \sigma_y) = (1, 1)$, top right is for $(\sigma_x, \sigma_y) = (1, 2)$, middle left is for $(\sigma_x, \sigma_y) = (1, 5)$, middle right is for $(\sigma_x, \sigma_y) = (2, 2)$, bottom left is for $(\sigma_x, \sigma_y) = (2, 5)$, and bottom right is for $(\sigma_x, \sigma_y) = (5, 5)$.

In Sections 5.1 and 5.2, we have taken $m = n$ for simplicity. But the stated observations were the same when $m \neq n$.

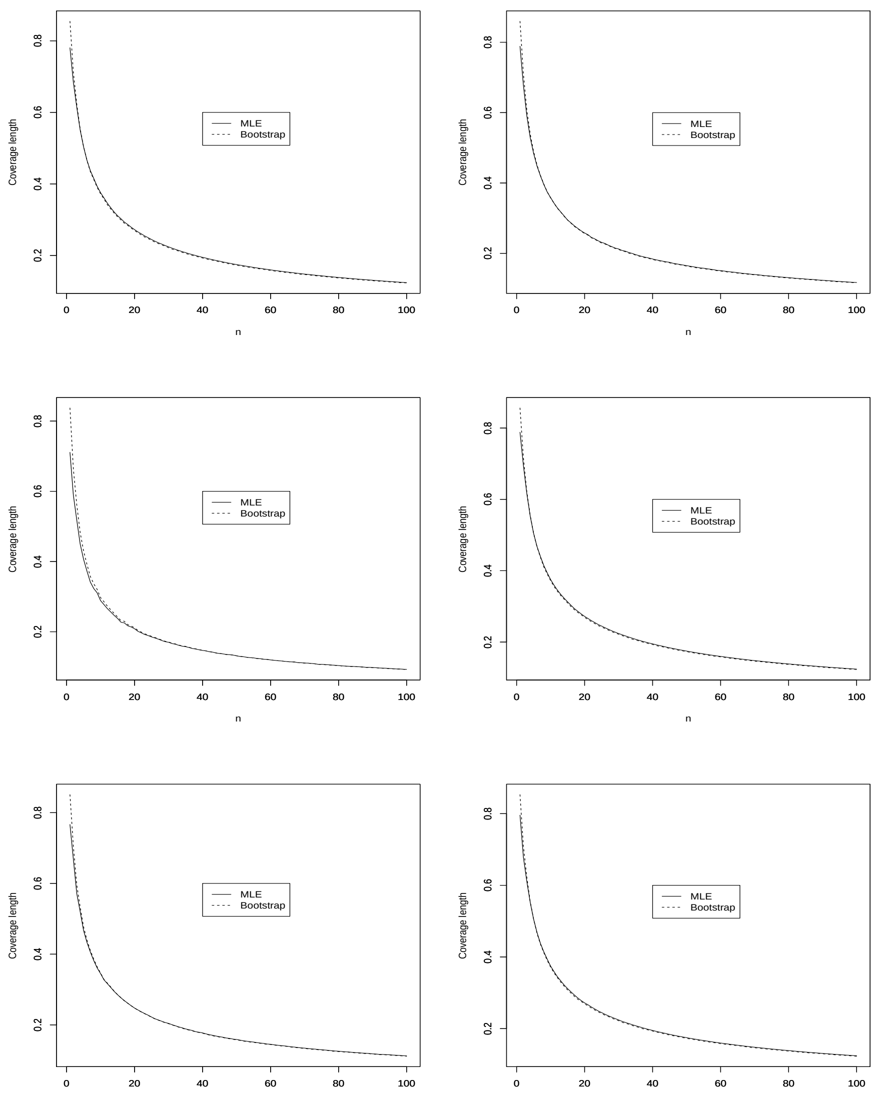

Figure 4. Coverage lengths of (3.1) and (3.2). Top left is for $(\sigma_x, \sigma_y) = (1, 1)$, top right is for $(\sigma_x, \sigma_y) = (1, 2)$, middle left is for $(\sigma_x, \sigma_y) = (1, 5)$, middle right is for $(\sigma_x, \sigma_y) = (2, 2)$, bottom left is for $(\sigma_x, \sigma_y) = (2, 5)$, and bottom right is for $(\sigma_x, \sigma_y) = (5, 5)$.

This is the first time estimation of $R = P(Y < X)$ for Lévy random variables has been studied in a comprehensive manner. Previously only maximum likelihood estimation of R has been considered for Lévy random variables.

A more comprehensive study of the estimation of $R = P(Y < X)$ for Lévy random variables could consider other point as well as interval estimators. These could include Bayesian highest posterior density intervals (Chen and Shao [9]), interval estimators based on the signed log-likelihood ratio due to Barndorff-Nielsen [6], interval estimators based on the modified signed log-likelihood ratio due to Barndorff-Nielsen [7], and robust estimators based on the theory of bounded influence M-estimators (Greco and Ventura [19]).

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The transmuted exponentiated Weibull geometric distribution: Theory and applications

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Abstract

A generalization of the exponentiated Weibull geometric model called the transmuted exponentiated Weibull geometric distribution is proposed and studied. It includes as special cases at least ten models. Some of its structural properties including order statistics, explicit expressions for the ordinary and incomplete moments and generating function are derived. The estimation of the model parameters is performed by the maximum likelihood method. The use of the new lifetime distribution is illustrated with an example. We hope that the proposed distribution will serve as a good alternative to other models available in the literature for modeling positive real data in several areas.

Keywords: Exponentiated Weibull geometric distribution, Goodness–of–fit statistic, Maximum likelihood estimation, Moment, Order statistic, Survival function, Transmutation map.

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1. Introduction

Several compounding distributions have been proposed in the literature to model lifetime data. Adamidis and Loukas [2] pioneered the two-parameter exponential-geometric (EG) distribution with decreasing failure rate. Kus [16] defined the exponential-Poisson distribution (following the same idea of the EG distribution) with decreasing failure rate and discussed various of its properties. Adamidis *et al.* [1] proposed the extended exponential-geometric (EEG) distribution which generalizes the EG distribution and discussed various of its structural properties along with its reliability features. The hazard rate function (hrf) of the EEG distribution can be monotone decreasing, increasing or constant. Lai *et al.* [17] introduced a modified Weibull distribution capable of modeling a bathtub-shaped hazard rate function (hrf). Mahmoudi and Shiran [19] proposed an exponentiated Weibull-geometric (EWG) distribution by compounding the EW and geometric distributions more flexible than the EW distribution and studied some of its properties. Wang and Elbatal [35] discussed a modified Weibull geometric distribution having monotonically increasing, decreasing, bathtub-shaped, and upside-down bathtub-shaped hazard rate functions. Finally, Saboor *et al.* [31] introduced a transmuted exponential Weibull distribution which have a bathtub-shaped and upside-down bathtub-shaped hazard rate functions.

The modeling of lifetime data by compounding a life model and a discrete distribution has been used to construct new lifetime models in the last few years. For some references, see Silva et al. [27]. *In practice, the exponential and Weibull are the most used baseline models. Suppose that a company has N systems functioning independently and producing a certain product at a given time, where N is a random variable, which is often determined by economy, customers demand, etc. The reason for considering N as a random variable comes from a practical viewpoint in which failure (of a device for example) often occurs due to the present of an unknown number of initial defects in the system. In this paper, we focus on the case in which N is a geometric random variable with probability mass function (pmf) $P(N = n) = (1-p)\,p^{n-1}$, for $0 < p < 1$ and $n = 1, 2, \cdots$. We can also consider that N follows other discrete distributions, such as the binomial, Poisson, etc, whereas they require to be truncated at zero since $N \geq 1$. Another reason by taking N to be a geometric random variable is that the "optimum" number can be interpreted as the "number to event", matching up with the definition of a geometric random variable as suggested by Nadarajah et al.* [22]. *Other motivations can also be found in Nadarajah et al.* [22]. Readers are referred to [34].

Suppose that $\{Z_i\}_{i=1}^N$ are independent and identically distributed (iid) random variables having the EW(α, β, θ) distribution with cumulative distribution function (cdf) given by

$$F(x; \alpha, \beta, \theta) = (1 - \mathrm{e}^{-(\alpha \mathrm{x})^\beta})^\theta,\ \mathrm{x} > 0,$$

and N a discrete random variable having a geometric distribution defined before. Let $Z_{(n)} = \max\{Z_i\}_{i=1}^N$. The cdf and probability density function (pdf) of $Z_{(n)}$ are given by

$$G(x; \alpha, \beta, \theta, p) = \frac{(1-p)\,(1 - \mathrm{e}^{-(\alpha \mathrm{x})^\beta})^\theta}{1 - p(1 - \mathrm{e}^{-(\alpha \mathrm{x})^\beta})^\theta} \tag{1.1}$$

and

$$g(x; \alpha, \beta, \theta, p) = (1-p)\theta\beta\alpha^\beta x^{\beta-1}\mathrm{e}^{-(\alpha \mathrm{x})^\beta}(1 - \mathrm{e}^{-(\alpha \mathrm{x})^\beta})^{\theta-1}\,[1 - \mathrm{p}(1 - \mathrm{e}^{-(\alpha \mathrm{x})^\beta})^\theta]^{-2}, \tag{1.2}$$

respectively, where α, β, $\theta > 0$ and $p \in [0, 1)$. The lifetime model defined by (1.1) and (1.2) is called the *exponentiated Weibull geometric* (EWG) distribution [19]. Hereafter, let Y be a random variable having the density (1.2) and write $Y \backsim$EWG$(\alpha, \beta, \theta, p)$.

In this paper, we define and study a new lifetime model called the *transmuted exponentiated Weibull-geometric* ("TEWG" for short) distribution. The main feature of this model is that a transmuted parameter is inserted in (1.2) to give greater flexibility in the form of the generated distribution. Using the quadratic rank transmutation map studied by [32], we construct the five-parameter TEWG model. We give a comprehensive description of some mathematical properties of the new distribution with the hope that it will attract wider applications in reliability, engineering and other areas of research. The concept of transmuted generator is explained below.

A Quadratic Rank Transmutation Map (QRTM) is defined by $G_{R12}(u) = u + \lambda u\,(1 - u), |\lambda| \leq 1$, from which the cdf's satisfy $F_2(x) = (1+\lambda)F_1(x) - \lambda F_1(x)^2$. By differentiating $F_2(x)$, we have

$$f_2(x) = f_1(x)\ [(1+\lambda) - 2\lambda F_1(x)]\,, \tag{1.3}$$

where $f_1(x)$ and $f_2(x)$ are the pdf's corresponding to the cdf's $F_1(x)$ and $F_2(x)$, respectively. For $\lambda = 0$, we have $f_2(x) = f_1(x)$.

1.1. Lemma. *The function $f_2(x)$ given by* (1.3) *is a well-defined density function.*

Proof. Rewriting $f_2(x)$ as $f_2(x) = f_1(x)[1 - \lambda\{2F_1(x) - 1\}]$, we note that $f_2(x)$ is nonnegative. We prove that the integration over its support is equal to one. Considering that the support of $f_1(x)$ is $(-\infty, \infty)$, we have

$$\int_{-\infty}^{\infty} f_2(x)dx = (1+\lambda)\int_{-\infty}^{\infty} f_1(x)\mathrm{d}x - 2\lambda\int_{-\infty}^{\infty} f_1(x)F_1(x)\mathrm{d}x = 1.$$

Similarly, for other cases, where the support of $f_1(x)$ is a part of the real line, the previous lemma holds. Hence, $f_2(x)$ is a well-defined pdf. We call $f_2(x)$ the transmuted pdf of a random variable with baseline density $f_1(x)$. This proves the current result.

Many authors constructed generalizations of some well-known distributions by using the *transmuted construction*. Aryal and Tsokos [4, 3] defined the transmuted generalized extreme value and transmuted Weibull distributions. Aryal [5] proposed and studied various structural properties of the transmuted log-logistic distribution, Shuaib and King [28] introduced the transmuted modified Weibull distribution, which extends the transmuted Weibull distribution by [3], and studied some of its mathematical properties and maximum likelihood estimation of the unknown parameters. Elbatal and Aryal [11] discussed the transmuted additive Weibull distribution. Elbatal [12, 13] presented the transmuted generalized inverted exponential and transmuted modified inverse Weibull distributions. Further, Merovci and Elbatal [20] proposed the transmuted Lindley-geometric distribution, Merovci *et al.* [20] defined the transmuted generalized inverse Weibull distribution and Elbatal *et al.* [10] studied the transmuted exponentiated Fréchet distribution.

The rest of the paper is organized as follows. In Section 2, we provide the pdf, cdf and survival function (sf) of the new distribution. Some special cases are given in Section 3. The density of the order statistics is given in Section 4. A mixture representation for the new pdf is derived in Section 5, where some of its structural properties can be easily obtained. Section 7 is related to the maximum likelihood estimates (MLEs) and the asymptotic confidence intervals for the unknown parameters. Finally, in Section 8, we present a real data analysis to illustrate the flexibility of the new lifetime model. Some conclusions are given in Section 9.

2. The TEWG Distribution

Let $\phi = (\alpha, \beta, \theta, p, \lambda)^T$. By inserting (1.1) and (1.2) in equation (1.3), the cdf and pdf of the TEWG distribution are given by

$$F_{TEWG}(x;\phi) = \frac{(1-p)\left(1-\mathrm{e}^{-(\alpha x)^\beta}\right)^\theta}{1-p\left(1-\mathrm{e}^{-(\alpha x)^\beta}\right)^\theta} \times \left\{1+\lambda-\lambda\left[\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha x)^\beta}\right)^\theta}{1-p\left(1-\mathrm{e}^{-(\alpha x)^\beta}\right)^\theta}\right]\right\} \tag{2.1}$$

and

$$f_{TEWG}(x;\phi) = \theta\beta\alpha^\beta(1-p)x^{\beta-1}\mathrm{e}^{-(\alpha x)^\beta} \times \left(1-\mathrm{e}^{-(\alpha x)^\beta}\right)^{\theta-1}\left[1-p\left(1-\mathrm{e}^{-(\alpha x)^\beta}\right)^\theta\right]^{-2} \times \left\{(1+\lambda)-2\lambda\left[\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha x)^\beta}\right)^\theta}{1-p\left(1-\mathrm{e}^{-(\alpha x)^\beta}\right)^\theta}\right]\right\}, \tag{2.2}$$

respectively, where $p \in [0,1), \alpha,\ \beta,\ \theta > 0$ and $|\lambda| \leq 1$. If X is a random variable with pdf (2.2), we use the notation $X \backsim$TEWG(ϕ).

We emphasize that the new model (2.2) is obtained by using the transmuted construction applied to a compounding life distribution from the exponentiated Weibull and geometric distributions.

The sf of X is given by $S_{TEWG}(x;\phi) = 1 - F_{TEWG}(x;\phi)$, whereas its hazard rate function (hrf) becomes $h_{TEWG}(x;\phi) = f_{TEWG}(x;\phi)/S_{TEWG}(x;\phi)$, which is an important quantity to characterize life phenomenon. The reversed hazard rate function (rhrf) of X is given by $\tau_{TEWG}(x;\phi) = f_{TEWG}(x;\phi)/F_{TEWG}(x;\phi)$.

2.1. Shapes of density and hazard function. The TEWG density (2.2) allows for greater flexibility of the tails. This function can exhibit different behavior depending on the parameter values as shown in Figures 1, 2 and 3. They display plots of the pdf of X for selected parameter values. Figure 1(a,b) and Figure 2(d) reveal that the mode of the pdf increases as λ, α and θ increases, respectively. Figure 2(c) and 3(e) indicate that the parameters β and p behave somewhat as scale parameters. Figure 3(f) and 4(g) display the increasing and bathtub-shaped of the hrf's, respectively.

3. Special Models

The TEWG distribution is a very flexible model that provides different distributions when its parameters are changed. It contains the following ten special models:

- For $\lambda = 0$, then (2.2) reduces to the EWG distribution pioneered by [19].
- The case $\theta = 1$ refers to the transmuted Weibull-geometric distribution.
- For $\lambda = 0$ and $\theta = 1$, we have the Weibull-geometric distribution given by [6].
- The transmuted generalized exponential geometric distribution arises as a special case of the TEWG distribution by taking $\theta = \beta = 1$.
- The case $\beta = 1$ refers to the transmuted exponentiated exponential geometric distribution.
- Setting $\lambda = 0$ and $\beta = 1$, we have the exponentiated exponential geometric distribution given by [18].

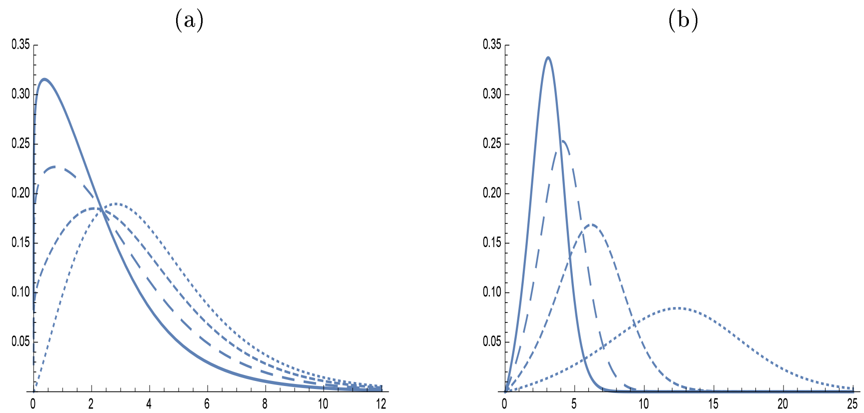

Figure 1. The TEWG density function: (a) $\alpha = 0.5$, $\theta = 1$, $\beta = 1.1$, $p = 0.5$ and $\lambda = -1$ (dotted line), $\lambda = -0.5$ (small dashed line), $\lambda = 0$ (long dashed line), $\lambda = 0.5$ (thick line). (b) $\lambda = -0.5$, $\theta = 2$, $\beta = 1.1$, $p = 0.5$ and $\alpha = 0.1$ (dotted line), $\alpha = 0.2$ (small dashed line), $\alpha = 0.3$ (long dashed line), $\alpha = 0.4$ (thick line).

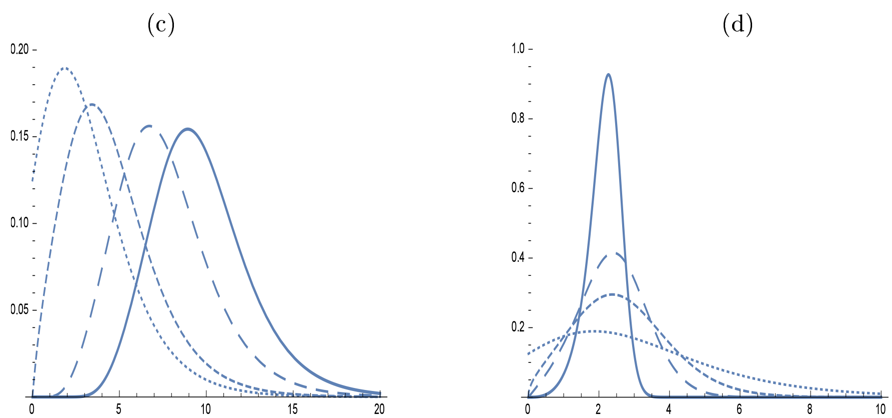

Figure 2. The TEWG density function: (c) $\alpha = 0.5$, $\theta = 1$, $p = 0.5$, $\lambda = -0.5$ and $\beta = 1$ (dotted line), $\beta = 2$ (small dashed line), $\beta = 10$ (long dashed line), $\beta = 30$ (thick line). (d) $\lambda = -0.5$, $\alpha = 0.5$, $\beta = 1$, $p = 0.5$ and $\theta = 1$ (dotted line), $\theta = 1.5$ (small dashed line), $\theta = 2$ (long dashed line), $\theta = 4$ (thick line).

- For $\theta = \beta = 1$, it follows the transmuted exponential geometric distribution.
- For $\lambda = 0$ and $\theta = \beta = 1$, we obtain the exponential geometric distribution given by [2].
- For $\beta = 2$, we have the transmuted generalized Rayleigh geometric distribution.
- The case $\beta = 2$ and $\theta = 1$ refers to the transmuted Rayleigh geometric distribution.

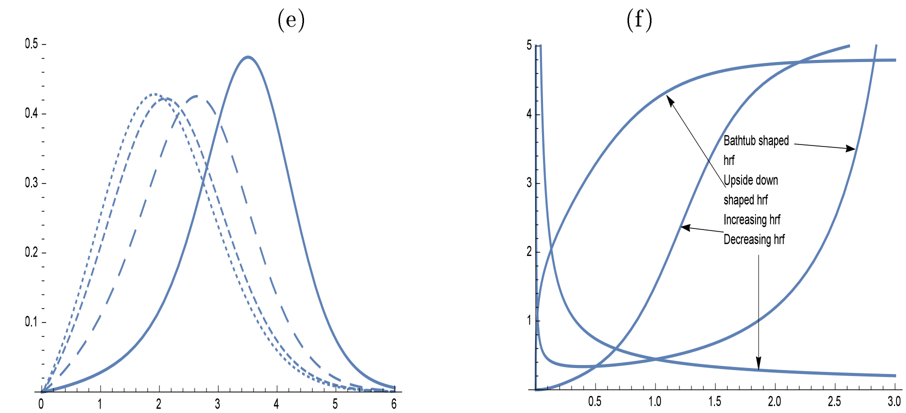

Figure 3. The TEWG density function: (e) $\alpha = 0.5$, $\theta = 2$, $\lambda = -0.5$, $\beta = 1.1$ and $p = 0$ (dotted line), $p = 0.3$ (small dashed line), $p = 0.6$ (long dashed line), $p = 0.9$ (thick line). The TEWG hazard rate function: (f) Increasing ($\alpha = 2.45, \beta = 1.2, \theta = 2.9, p = 0.9, \lambda = 0.15$), decreasing ($\alpha = 0.5, \beta = 0.4, \theta = 0.1, p = 0.2, \lambda = 0.1$), bathtub ($\alpha = 0.3, \beta = 3.3, \theta = 0.1, p = 0.8, \lambda = 1.2$) and upside-down bathtub ($\alpha = 2.4, \beta = 1, \theta = 1.3, p = 0.01, \lambda = 0.5$).

4. Order statistics

In this section, we derive closed-form expressions for the pdf of the rth order statistic of X. Let $X_1, \ldots, X_n$ be a simple random sample from the TEWG distribution with pdf and cdf given by (2.1) and (2.2), respectively. Let $X_{(1)} \leq X_{(2)} \leq, \ldots, \leq X_{(n)}$ denote the order statistics obtained from this sample. The pdf of $X_{i:n}$, say $f_{i:n}(x; \phi)$, is given by

$$f_{i:n}(x, \phi) = \frac{1}{B(i, n-i+1)} F(x; \phi)^{i-1} \left[1 - F(x; \phi)\right]^{n-i} f(x; \phi), \tag{4.1}$$

where $F(x; \phi)$ and $f(x; \phi)$ are the cdf and pdf of X given by (2.1) and (2.2), respectively, and $\mathrm{B}(\cdot, \cdot)$ is the beta function. We have

$$\begin{aligned} f_{i:n}(x; \phi) = &\frac{\theta\beta\alpha^{\beta}(1-p)}{B(i, n-i+1)} x^{\beta-1} \mathrm{e}^{-(\alpha \mathrm{x})^{\beta}} \mathrm{h}^{\theta-1} \\ &\times \left[1 - ph^{\theta}\right]^{-2} \left\{(1+\lambda) - 2\lambda \left[\frac{(1-p)h^{\theta}}{1-ph^{\theta}}\right]\right\} \\ &\times \left[\frac{(1-p)h^{\theta}}{1-ph^{\theta}} \left\{1 + \lambda - \lambda \left[\frac{(1-p)h^{\theta}}{1-ph^{\theta}}\right]\right\}\right]^{i-1} \\ &\times \left[1 - \frac{(1-p)h^{\theta}}{1-ph^{\theta}} \left\{1 + \lambda - \lambda \left[\frac{(1-p)h^{\theta}}{1-ph^{\theta}}\right]\right\}\right]^{n-i}, \end{aligned}$$

where $h = 1 - \mathrm{e}^{-(\alpha \mathrm{x})^{\beta}}$.

5. Mixture Representation

Based on equation (1.3), we can write

$$f(x)=\sum_{k=0}^{\infty}\left[w_{1k}\,h_{k+1}(x)+w_{2k}\,h_{k+2}(x)\right], \tag{5.1}$$

where $w_{1k}=(1+\lambda)(1-p)p^k$ and $w_{2k}=-(k+1)\lambda(1-p)^2p^k$. Equation (5.1) reveals that the density function of X is a mixture of EW densities.

5.1. Moments. Using the mixture representation, we obtain

$$\mu'_r=E(X^r)=\sum_{k=0}^{\infty}\left[w_{1k}\,E(Y_{k+1}^r)+w_{2k}\,E(Y_{k+2}^r)\right]. \tag{5.2}$$

We now provide two explicit expressions for $E(Y_{k+1}^r)$. First, Choudhury [7] derived the closed-form expression

$$E(Y_{k+1}^r)=\frac{(k+1)\theta}{\alpha^r}\,\Gamma\left(\frac{r}{\beta}+1\right)\left[1+\sum_{i=1}^{\infty}\frac{(-1)^i\,a_i((k+1)\theta)}{(i+1)^{r/\beta+1}}\right],$$

where $a_i=a_i(\gamma)=(-1)^i\,(\gamma-1)\cdots(\gamma-i)/i!$ for $i=1,2,\ldots$. The infinite series on the right hand side converges for all $\theta>0$.

Second, Nadarajah and Gubta [24] derived an infinite series representation applicable for any $r>-\beta$ real or integer given by

$$E(Y_k^r)=\frac{(k+1)\theta}{\alpha^r}\,\Gamma\left(\frac{r}{\beta}+1\right)\sum_{i=0}^{\infty}\frac{(1-(k+1)\theta)_i}{i!\,(i+1)^{(r+\beta)/\beta}}.$$

Inserting the last two expressions in (5.2) gives $E(X^r)$

5.2. Incomplete moments. The answers to many important questions in economics require more than just knowing the mean of the distribution, but its shape as well. This is obvious not only in the study of econometrics but in other areas as well.

For lifetime models, it is of interest to known the rth lower and upper incomplete moments of X defined by $m_r(x)=\int_0^x x^r\,f(x)dx$ and $v_r(x)=\int_x^{\infty} x^r f(x)dx$, respectively, for any real $r>0$. Clearly, these rth incomplete moments are related by $v_r(x)=\mu'_r-m_r(x)$.

Based on equation (5.1), we have

$$m_r(x)=\sum_{k=0}^{\infty}\left[w_{1k}\,m_r^{(k+1)}(x)+w_{2k}\,m_r^{(k+2)}(x)\right], \tag{5.3}$$

where $m_r^{(k+1)}(x)=\int_x^{\infty} x^r\,h_{k+1}(x)dx$ is the rth lower incomplete moment of Y_{k+1}.

Following a result of [23], we obtain

$$\begin{aligned} m_r^{(k+1)}(x)&=(k+1)\,\theta\,\alpha^{-r\,\beta}\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)^{(r+1)\beta}}\binom{(k+1)\theta-1}{j}\\ &\quad\times\gamma\left(\frac{r}{\beta}+1;(j+1)(\alpha x)^{\beta}\right), \end{aligned} \tag{5.4}$$

where $\gamma(s;t)=\int_0^t x^{s-1}\,\mathrm{e}^{-\mathrm{x}}\mathrm{dx}$ is the lower incomplete gamma function. Equation (5.4) gives $m_r(x)$ as a linear combination of incomplete gamma functions evaluated at different points.

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. For a given probability π, they are defined by $B(\pi) = m_1(q)/(\pi\, \mu_1')$ and $L(\pi) = m_1(q)/\mu_1'$, respectively, where $m_1(q)$ can be determined from (5.3) with $r = 1$ and $q = Q(\pi)$ is calculated by inverting numerically (2.1).

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean and median defined by $\delta_1 = \int_0^\infty |x - \mu_1'| f(x)dx$ and $\delta_2(x) = \int_0^\infty |x - M| f(x)dx$, respectively, where $\mu_1' = E(X)$ is the mean and $M = Q(0.5)$ is the median. These measures can be determined using the relationships $\delta_1 = 2\mu_1' F(\mu_1'; \phi)) - 2m_1(\mu_1')$ and $\delta_2 = \mu_1' - 2m_1(M)$, where $m_1(\mu_1')$ comes from (5.3) with $r = 1$.

5.3. Generating function. Let $M_{k+1}(t)$ be the moment generating function (mgf) of Y_{k+1}. We obtain the mgf of X, say $M(t)$, from equation (5.1) as

$$M(t) = \sum_{k=0}^{\infty} \left[w_{1k}\, M_{k+1}(t) + w_{2k}\, M_{k+2}(t) \right].$$

We provide an explicit expression for $M_{k+1}(t)$ when $\beta > 1$, which requires the complex parameter Wright generalized hypergeometric function with p numerator and q denominator parameters (Kilbas *et al.*, 2006, Equation (1.9)) defined by

$$(5.5) \qquad {}_p\Psi_q \left[\begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_p, A_p) \\ (\beta_1, B_1), \ldots, (\beta_q, B_q) \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{z^n}{n!}$$

for $z \in \mathbb{C}$, where $\alpha_j,\ \beta_k \in \mathbb{C},\ A_j,\ B_k \neq 0,\ j = \overline{1,p},\ k = \overline{1,q}$ and the series converges for $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$.

The mgf of Y_{k+1} (when $\beta > 1$) is given by

$$(5.6) \qquad M_{k+1}(t) = (k+1)\theta \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \binom{(k+1)\theta - 1}{j} {}_1\Psi_0 \left[\begin{array}{c} (1, \beta^{-1}) \\ - \end{array} ; \alpha\, t\, (j+1)^{-1/\beta} \right].$$

Generalized hypergeometric functions are included as in-built functions in most analytical softwares, so the special function in (5.5) and hence (5.6) can be evaluated by the softwares Maple, Matlab and Mathematica using known procedures.

6. Residual life and reversed failure rate functions

Given that a component survives up to time $t \geq 0$, the residual life is the period beyond t until the time of failure and defined by the conditional random variable $X - t | X > t$. In reliability, it is well-known that the mean residual life function and ratio of two consecutive moments of residual life determine the distribution uniquely [15]. Therefore, we obtain the rth order moment of the residual life using the general formula

$$\mu_r(t) = \frac{1}{\overline{F}(t)} \int_t^\infty (x-t)^r f(x; \varphi)\, dx,\ r \geq 1.$$

Applying the binomial expansion of $(x-t)^r$ and substituting $f(x;\varphi)$ given by (2.2) into the above formula and using the generalized binomial power series gives

$$\mu_r(t) = \frac{\theta\beta\alpha^\beta(1-p)}{\overline{F}(t)} \sum_{m=0}^{r}\sum_{k,j=0}^{\infty}(-1)^{m+k}\binom{r}{m}(j+1)p^j\Big\{(1+\lambda)\tbinom{(j+1)\theta-1}{k}\,t^m - \lambda(1-p)(j+2)\binom{(j+2)\theta-1}{k}\Big\}\int_t^\infty x^{r+\beta-m-1}\,\mathrm{e}^{-(\mathrm{k}+1)(\alpha \mathrm{x})^\beta}\mathrm{dx}$$

$$= \frac{\theta(1-p)}{\overline{F}(t)} \sum_{m=0}^{r}\sum_{k,j=0}^{\infty}(-1)^{m+k}\binom{r}{m}(j+1)\,p^j\,t^m\Big\{(1+\lambda)\binom{(j+1)\theta-1}{k} - \lambda(1-p)(j+2)\binom{(j+2)\theta-1}{k}\Big\}\left[\frac{\Gamma(\frac{r-m}{\beta}+1;(k+1)(\alpha t)^\beta)}{\alpha^{r-m}(k+1)^{\frac{r-m}{\beta}+1}}\right], \tag{6.1}$$

where $\Gamma(s;t)=\int_t^\infty x^{s-1}\,\mathrm{e}^{-\mathrm{x}}\mathrm{dx}$ is the upper incomplete gamma function.

Another important characteristic of the TEWG model is the mean residual life (MRL) function obtained by setting $r=1$ in equation (6.1). The importance of the MRL function is due to its uniquely determination of the lifetime distribution as well as the failure rate (FR) function. Lifetimes can exhibit IMRL (increasing MRL) or DMRL (decreasing MRL). The MRL function that first decreases (increases) and then increases (decreases) is usually called bathtub (upside-down bathtub) shaped, BMRL (UMRL). Ghitany [14], Mi [21], Park [30] and Tang *et al.* [33], among others, studied the relationship between the behaviors of the MLR and FR functions of a distribution.

7. Estimation and Inference

Here, we determine the maximum likelihood estimates (MLEs) of the parameters of the new distribution from complete samples only. Let $x_1,\dots,x_n$ be a random sample of size n from the TEWG$(x;\phi)$ model, where $\phi=(\alpha,\beta,\theta,p,\lambda)^T$. The log likelihood function for the vector of parameters ϕ can be expressed as

$$\ell(\phi) = n\log\theta + n\log\beta + n\beta\log\alpha + n\log(1-p) + (\beta-1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}(\alpha x_i)^\beta - 2\sum_{i=1}^{n}\log\left[1-p\left(1-\mathrm{e}^{-(\alpha \mathrm{x}_{(\mathrm{i})})^\beta}\right)^\theta\right] + (\theta-1)\sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-(\alpha \mathrm{x}_{(\mathrm{i})})^\beta}\right) + \sum_{i=1}^{n}\log\left\{(1+\lambda)-2\lambda\left(\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha \mathrm{x}_{(\mathrm{i})})^\beta}\right)^\theta}{1-p\left(1-\mathrm{e}^{-(\alpha \mathrm{x}_{(\mathrm{i})})^\beta}\right)^\theta}\right)\right\}.$$

The corresponding score function is given by

$$U_n(\varphi) = \left(\frac{\partial\ell(\phi)}{\partial\alpha},\frac{\partial\ell(\phi)}{\partial\beta},\frac{\partial\ell(\phi)}{\partial\theta},\frac{\partial\ell(\phi)}{\partial p},\frac{\partial\ell(\phi)}{\partial\lambda}\right)^T. \tag{7.1}$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained from (7.1), namely:

$$\begin{aligned}\frac{\partial\ell(\phi)}{\partial\alpha} &= \tfrac{n\beta}{\alpha}-\beta\alpha^{\beta-1}\sum_{i=1}^{n}(x_i)^{\beta}+(\theta-1)\sum_{i=1}^{n}\frac{\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\beta\alpha^{\beta-1}(\mathrm{x}_{\mathrm{i}})^{\beta}}{\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)}\\
&+2\theta p\sum_{i=1}^{n}\frac{e^{-(\alpha x_{(i)})^{\beta}}\beta\alpha^{\beta-1}(x_i)^{\beta}\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta-1}}{\left[1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\right]}\\
&-2\lambda\sum_{i=1}^{n}\frac{1}{\left\{(1+\lambda)-2\lambda\left(\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}{1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}\right)\right\}}\\
&\times\left[\frac{(1-p)\theta\beta\alpha^{\beta-1}\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}(\mathrm{x}_{\mathrm{i}})^{\beta}\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta-1}}{\left[1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\right]^{2}}\right]=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell(\phi)}{\partial\beta} &= \tfrac{n}{\beta}+n\log\alpha+\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}(\alpha x_i)^{\beta}\log(\alpha x_i)\\
&+2p\theta\sum_{i=1}^{n}\frac{\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta-1}\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}(\alpha\mathrm{x}_{\mathrm{i}})^{\beta}\log(\alpha\mathrm{x}_{\mathrm{i}})}{\left[1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\right]}\\
&+(\theta-1)\sum_{i=1}^{n}\frac{\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}(\alpha\mathrm{x}_{\mathrm{i}})^{\beta}\log(\alpha\mathrm{x}_{\mathrm{i}})}{\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)}\\
&-2\lambda\sum_{i=1}^{n}\frac{1}{\left\{(1+\lambda)-2\lambda\left(\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}{1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}\right)\right\}}\\
&\times\left[\frac{\theta(1-p)\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta-1}\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}(\alpha\mathrm{x}_{\mathrm{i}})^{\beta}\log(\alpha\mathrm{x}_{\mathrm{i}})}{\left[1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\right]^{2}}\right]=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell(\phi)}{\partial\theta} &= \tfrac{n}{\theta}+\sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)+2p\sum_{i=1}^{n}\frac{\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\log\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)}{1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}\\
&+\sum_{i=1}^{n}\frac{1}{\left\{(1+\lambda)-2\lambda\left(\frac{(1-p)\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}}{1-p\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}\right)\right\}}\\
&\times\left[\frac{(1-p)\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}\log\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)}{\left[1-p\left(1-\mathrm{e}^{-(\alpha\mathrm{x}_{(\mathrm{i})})^{\beta}}\right)^{\theta}\right]^{2}}\right]=0,\end{aligned}$$

$$\frac{\partial\ell(\phi)}{\partial p} = \frac{-n}{1-p} + 2\sum_{i=1}^{n} \frac{\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}{\left[1-p\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}\right]}$$

$$-2\lambda \sum_{i=1}^{n} \frac{1}{\left\{(1+\lambda)-2\lambda\left(\frac{(1-p)\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}{1-p\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}\right)\right\}}$$

$$\times \left[\frac{\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}\left(1-\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}\right)}{\left[1-p\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}\right]^{2}}\right] = 0,$$

and

$$\frac{\partial\ell(\phi)}{\partial\lambda} = \sum_{i=1}^{n} \frac{1-2\left(\frac{(1-p)\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}{1-p\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}\right)}{\left\{(1+\lambda)-2\lambda\left(\frac{(1-p)\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}{1-p\left(1-e^{-(\alpha x_{(i)})^{\beta}}\right)^{\theta}}\right)\right\}} = 0.$$

The above equations cannot be solved analytically but statistical software can be used to solve them numerically, for example, through the R-language or any iterative methods such as the NR (Newton-Raphson), BFGS (Broyden-Fletcher-Goldfarb-Shanno), BHHH (Berndt-Hall-Hall-Hausman), NM (Nelder-Mead), SANN (Simulated-Annealing) and L-BFGS-B (Limited-Memory Quasi-Newton code for Bound-Constrained Optimization).

The modified Anderson-Darling (A^*) and the modified Cramér-von Mises (W^*) statistics are widely used to determine how closely a specific cdf $F(\cdot)$ fits the empirical distribution for a given data set. The statistics A^* and W^* are given by

$$A^* = \left(\frac{2.25}{n^2} + \frac{0.75}{n} + 1\right)\left[-n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(z_i\left(1 - z_{n-i+1}\right)\right)\right],$$

and

$$W^* = \left(\frac{0.5}{n} + 1\right)\left[\sum_{i=1}^{n}\left(z_i - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}\right],$$

respectively, where $z_i = F(y_{(i)})$, and the $y_{(i)}$'s are the ordered observations.

The smaller these statistics are, the better the fit. Upper tail percentiles of the asymptotic distributions of them were tabulated by [25].

8. Application to the carbon fibres

We provide an application to a real data set to prove the flexibility of the TEWG distribution. We fit the gamma exponentiated exponential (GEE) [29], exponentiated Weibull–geometric (EWG) [19], extended Weibull (ExtW) [26], Kumaraswamy modified Weibull (KwMW) [9] and TEWG distributions to a real data on "carbon fibres" [25]. The parameters of the following distributions are estimated by maximizing the log-likelihood using the *NMaximize* procedure in the symbolic computational package *Mathematica*. The density functions (for $x > 0$) associated to these models are given by:

- The GEE density function,

$$f(x) = \frac{\lambda\,\alpha^{\delta}\,e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}\left(-\log\left(1-e^{-\lambda x}\right)\right)^{\delta-1}}{\Gamma(\delta)}, \quad \lambda,\alpha,\delta,x>0.$$

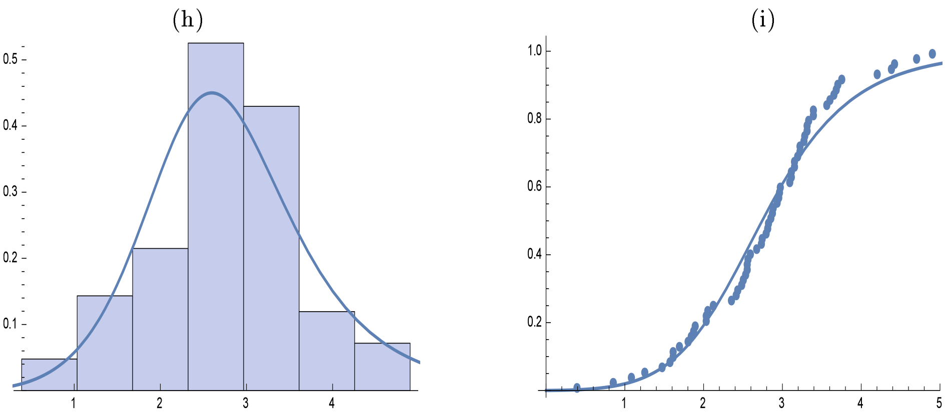

Figure 4. (a) The estimated TEWG density superimposed on the histogram for the carbon fibres. (b) The empirical cdf and the estimated TEWG cdf.

Table 1. MLEs of the parameters (standard errors in parentheses) for the carbon fibres

Distributions	Estimates				
GEE (λ, α, δ)	0.26555	10.0365	7.23658		
	(0.21621)	(2.59504)	(7.05288)		
EWG (α, θ, β, p)	520.24	0.35943	177.132	0.999778	
	(332.051)	(0.02509)	(207.54)	(0.00262)	
ExtW (a , b , c)	16.1979	0.001	8.05671		
	(25.7118)	(0.938764)	(1.65309)		
KwMW(α , γ , λ , a , b)	0.14981	1.7994	0.49987	0.64975	0.171114
	(0.326517)	(2.40813)	(0.616749)	(1.13328)	(0.529126)
TEWG (α, θ, β, p, λ)	59.2556	0.455874	1.42577	0.999917	-0.447535
	(27.5648)	(0.03366)	(1.60102)	(0.00937)	(0.49717)

- The ExtW density function,

$$f(x) = a\,(c + b\,x)\,x^{-2+b}\,\mathrm{e}^{-c/x - a x^{b} e^{-c/x}}\,, \quad a > 0\,,\, b > 0\,,\, c \geq 0\,,\, x > 0.$$

- The KwMW density function,

$$f(x) = a\,b\,\alpha\,x^{\gamma-1}(\gamma + \lambda\,x)\,\exp\left(\lambda\,x - \alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)\,\left[1 - \exp\left(-\alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)\right]^{a-1}$$
$$\times\,\left\{1 - \left[1 - \exp\left(-\alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)\right]^{a}\right\}^{b-1},$$

where $a > 0$, $b > 0$, $\alpha > 0$, $\gamma > 0$, $\lambda \geq 0$.

The estimated pdf and cdf of the TEWG distribution fitted to the uncensored breaking stress of carbon fibres (in Gba) reported by [8] are displayed in Figure 4. The estimates of the parameters and their standard errors (SEs) are listed in Table 1. The values of the statistics A^* and W^*, Akaike Information Criterion (AIC), Bayesian Information Criterioon (BIC), Hannan–Quinn Information Criterion (HQIC) and Consistent Akaike

Table 2. Goodness-of-fit statistics for the carbon fibres

Distributions	A^*	W^*	AIC	BIC	HQIC	CAIC
GEE $(\lambda, \alpha, \delta)$	1.43415	0.266823	189.787	196.356	192.383	190.175
EWG $(\alpha, \beta, \theta, p)$	0.789187	0.121661	118.164	127.922	122.625	119.82
ExtW (a, b, c)	2.26745	0.416152	207.471	214.04	210.067	207.858
KwMW$(\alpha\gamma, \lambda, a, b)$	1.28891	0.212227	180.676	191.624	185.002	181.676
TEWG $(\alpha, \beta, \theta, p, \lambda)$	0.77199	0.12016	117.586	128.534	121.912	118.586

Information Criterion (CAIC) are also given in Table 2. We note that the TEWG model provides the best fit among these models.

To compare the TEWG model with its EWG sub-model, the likelihood-ratio (LR) statistic is given by $w = 4.54198$ with p-value 0.033. The value if the LR statistic suggests that the TEWG model performs significantly better than its sub-model EWG.

9. Conclusions

We propose a new compounding lifetime model named the transmuted exponentiated Weibull geometric distribution, and study some of its general structural properties. The proposed model includes at least ten special lifetime models. A very useful mixture representation for its density function is derived. We provide explicit expressions for the moments and incomplete moments, generating and quantile functions, mean deviations and order statistics. These expressions are manageable using analytic and numerical computer resources, which may turn into adequate tools comprising the arsenal of applied statisticians. The model parameters are estimated by maximum likelihood. We prove that the proposed model can be superior to some models generated from other know families in terms of model fitting by means of an application to a real data set.

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On almost unbiased ridge logistic estimator for the logistic regression model

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Abstract

Schaefer et al. [15] proposed a ridge logistic estimator in logistic regression model. In this paper a new estimator based on the ridge logistic estimator is introduced in logistic regression model and we call it as almost unbiased ridge logistic estimator. The performance of the new estimator over the ridge logistic estimator and the maximum likelihood estimator in scalar mean squared error criterion is investigated. We also present a numerical example and a simulation study to illustrate the theoretical results.

Keywords: Almost unbiased ridge logistic estimator, Ridge logistic estimator, Logistic regression model, Scalar mean squared error

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1. Introduction

In this paper we consider the estimation of Euclidean parameters $\beta \in R^p$ in logistic regression model based on the dependent variable y_i is $Be(\pi_i)$. The parameters π_i relate to β and x_1, x_2,...,x_n with the following value:

$$\pi_i = \frac{exp(x_i\beta)}{1+exp(x_i\beta)},\ i=1,2,...,n \tag{1.1}$$

Usually the parameters of the model should be estimated using the maximum likelihood (ML) way by applying the following iterative weighted least square (IWLS) algorithm:

$$\hat{\beta}_{ML} = (X'\hat{W}X)^{-1}X'\hat{W}\hat{Z} \tag{1.2}$$

where $\hat{Z}$ is a vector with ith element equals $log(\hat{\pi}_i) + \frac{y_i-\hat{\pi}_i}{\hat{\pi}_i(1-\hat{\pi}_i)}$ and $W = diag(\frac{\hat{\pi}_i}{1-\hat{\pi}_i})$.

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Since ML estimation does not require any restriction on the characteristics of the independent variables, Maximum likelihood (ML) is the preferred estimation way in logistic regression. However, the ML estimator can be affected seriously by the presence of collinearity. It is known that ML parameter estimates have large variances in cases of multicollinearity. Many methods have proposed to combat this problem in linear regression model, such as the ridge estimator by Hoerl and Kennard [5], Liu estimator by Liu [10].

Schaefer et al. [15] use the ridge method to overcome the multicollinearity in logistic regression model and propose a ridge logistic estimator. Mansson and Shukur [13], Kibria et al. [12] proposed many methods to estimate the ridge parameter in ridge logistic estimator. Inan and Erdogan [9] proposed a Liu-type logistic estimator to overcome multicollinearity in logistic regression model.

Though the ridge logistic estimator proposed by Schaefer et al. [15] can overcome multicollinearity, however, this estimator has big bias. In this paper, we propose a new estimator which can be used not only overcome multicollinearity, but also can reduce the bias of the ridge estimator. We also discuss the statistical properties of the new estimator.

2. The almost unbiased ridge logistic estimator

The ridge logistic estimator (RLE) in the logistic regression model presented by Schaefer et al. [15] is denoted as follows:

$$\hat{\beta}_{RLE}(k) = (X'\hat{W}X + kI)^{-1}X'\hat{W}\hat{Z}, \ k > 0 \tag{2.1}$$

It is easy to obtain that:

$$\begin{aligned} Bias(\hat{\beta}_{RLE}(k)) &= E(\hat{\beta}_{RLE}(k)) - \beta \\ &= (X'\hat{W}X + kI)^{-1}X'\hat{W}X\beta - \beta \\ &= [(X'\hat{W}X + kI)^{-1}X'\hat{W}X - I]\beta \\ &= (X'\hat{W}X + kI)^{-1}[X'\hat{W}X - (X'\hat{W}X + kI)]\beta \\ &= -k(X'\hat{W}X + kI)^{-1}\beta \qquad (2.2) \end{aligned}$$

and

$$Cov(\hat{\beta}_{RLE}(k)) = (X'\hat{W}X + kI)^{-1}X'\hat{W}X(X'\hat{W}X + kI)^{-1} \tag{2.3}$$

In linear regression model, many authors have studied the almost unbiased estimator, such as Kadiyala [11], Akdeniz and Kaciranlar [1] and Xu and Yang [16, 17].

To obtain the almost unbiased ridge logistic estimator, we firstly list the following definitions.

Definition 2.1. [16, 17] Suppose $\hat{\beta}$ is a biased estimator of parameter vector β, and if the bias vector of $\hat{\beta}$ is given by $Bias(\hat{\beta}) = E(\hat{\beta}) - \beta = R\beta$, which shows that $E(\hat{\beta} - R\beta) = \beta$, then we call the estimator $\tilde{\beta} = \hat{\beta} - R\hat{\beta} = (I - R)\hat{\beta}$ is the almost unbiased estimator based on the biased estimator $\hat{\beta}$.

Now, we are ready to derive the almost unbiased ridge logistic estimator based on the RLE. Since: $Bias(\hat{\beta}_{RLE}(k)) = (X'\hat{W}X + kI)^{-1}X'\hat{W}X\beta - \beta$,we have

$$\begin{aligned}
\hat{\beta}_{AURLE}(k) &= [I - ((X'\hat{W}X + kI)^{-1}X'\hat{W}X - I)]\hat{\beta}_{RLE}(k) \\
&= [2I - (X'\hat{W}X + kI)^{-1}X'\hat{W}X]\hat{\beta}_{RLE}(k) \\
&= [2I - (X'\hat{W}X + kI)^{-1}X'\hat{W}X](X'\hat{W}X + kI)^{-1}X'\hat{W}\hat{Z} \\
&= [I + (X'\hat{W}X + kI)^{-1}X'\hat{W}X](X'\hat{W}X + kI)^{-1}X'\hat{W}X\hat{\beta}_{ML} \\
&= [I + k(X'\hat{W}X + kI)^{-1}][I - k(X'\hat{W}X + kI)^{-1}]\hat{\beta}_{ML} \\
&= [I - k^2(X'\hat{W}X + kI)^{-2}]\hat{\beta}_{ML} \qquad (2.4)
\end{aligned}$$

In the next section, we will discuss the properties of the new estimator.

For the convenience of the following discussions, let $\alpha = Q'\beta$, $\Lambda = diag(\lambda_1, ..., \lambda_p) = Q'(X'\hat{W}X)Q$, where $\lambda_1 \geq ... \geq \lambda_p > 0$ are the ordered eigenvalues of $X'\hat{W}X$.

3. The performance of the new estimator

The new estimator is proposed to reduce the bias of the ridge logistic estimator (RLE). So now we compare the new estimator with the RLE.

3.1. Theorem. *In logistic regression model we have* $\| Bias(\hat{\beta}_{AURLE}(k)) \|^2 < \| Bias(\hat{\beta}_{RLE}(k)) \|^2$ *for* $k > 0$.

Next we discuss the superiority of the new estimator in the scalar mean squared error (MSE) sense. Firstly we give its definition. Let $\hat{\beta}$ be an estimator of β, then the scalar mean squared error is defined as follows:

$$MSE(\hat{\beta}) = E(\hat{\beta} - \beta)'(\hat{\beta} - \beta) = tr\{Cov(\hat{\beta})\} + Bias(\hat{\beta})'Bias(\hat{\beta}) \qquad (3.1)$$

3.2. Theorem. *A sufficient of the new estimator superior to the RLE by the MSE criterion in logistic regression model is*

$$k > \frac{3 - \lambda_i\alpha_i^2 + \sqrt{(3 + \lambda_i\alpha_i^2)^2 + 4\lambda_i\alpha_i^2}}{4\alpha_i^2}$$

for all $i = 1, ..., p$.

3.3. Theorem. *The new estimator is superior to the maximum likelihood (ML) estimator in logistic regression model for* $k > 0$ *if* $1 - \lambda_i\alpha_i^2 > 0$ *for all* $i = 1, ..., p$ *and for* $k < \frac{2\lambda_i + \lambda_i\sqrt{2(1+\alpha_i^2\lambda_i)}}{\alpha_i^2\lambda_i - 1}$ *if* $1 - \lambda_i\alpha_i^2 < 0$ *for some* i.

4. The selection of ridge parameter k

In this section we consider that the ridge parameter which is obtained by using the ridge parameter introduced in the previous section and the ridge parameters proposed by Hoerl and Kennard [5], Hoerl et al. [6], Batah et al. [3], Lawless and Wang [7] and Khurana et al [4].

The ridge parameter corresponding to Eq. (7.2) is

$$k_{NEW} = \frac{p}{\sum_{i=1}^{p}[\alpha_i^2/[1 + (1 + \lambda_i * \alpha_i^2)^{1/2}]}$$

Second, the Hoerl and Kennard [5] ridge parameter is defined as

$$k_{HK} = \frac{\hat{\sigma}^2}{max\alpha_{iML}^2}$$

Table 1. Estimated quadratic bias with $\gamma = 0.9$

k	NEW	HK	HKB	LW	LS
RLE$\times 10^{-2}$	3.8401	0.5730	2.9811	0.0000	5.0532
AURLE $\times 10^{-2}$	0.0023	0.0000	0.0014	0.0000	0.0039

Table 2. Estimated quadratic bias with $\gamma = 0.95$

k	NEW	HK	HKB	LW	LS
RLE	0.1093	0.0020	0.0100	0.0000	0.1627
AURLE	0.0002	0.0006	0.0001	0.0000	0.0004

Table 3. Estimated quadratic bias with $\gamma = 0.99$

k	NEW	HK	HKB	LW	LS
RLE	0.9889	0.2645	0.9696	0.0000	1.3761
AURLE	0.1230	0.0090	0.0118	0.0000	0.2368

Third, the Hoerl et al. [6] ridge parameter is defined as

$$k_{HKB} = \frac{p\hat{\sigma}^2}{\hat{\beta}'_{ML}\hat{\beta}_{ML}}$$

Fourth, the Lawless and Wang [7] ridge parameter is defined as

$$k_{LW} = \frac{p\hat{\sigma}^2}{\hat{\beta}'_{ML}X'WX\hat{\beta}_{ML}}$$

Fifth, the Lindley and Smith [8] ridge parameter is defined as

$$k_{LS} = \frac{(n-p)(p+2)}{n+2}\,\frac{\hat{\sigma}^2}{\hat{\beta}'_{ML}\hat{\beta}_{ML}}$$

5. Monte Carlo simulation

The main purpose of this article is to compare the MSE properties and bias of the ML, RLE and AURLE when the regressors are highly intercorrelated. Hence, the core factor varied in the design of the experiment is the degree of correlation γ between the regressors. Therefore, the following formula which enables us to vary the strength of the correlation is used to generate the explanatory variables:

$$x_{ij} = (1-\gamma^2)^{1/2} z_{ij} + \gamma z_{ip},\ \ i = 1, ..., n,\ \ j = 1, ..., p \tag{5.1}$$

where z_{ij} are independent standard normal pseudo-random numbers,and γ is specified so that the correlation between any two explanatory variables is given by γ^2.

Four different values of γ corresponding to $0.9, 0.95, 0.99$ are considered and the sample size is equal to 50.

All simulation results are given in Tables 1-6.

From Tables 1-3, we can see that the new estimator has smaller quadratic bias than the RLE. When we see the estimated MSE of the new estimator and the RLE, we see that the new estimator is always superior to the RLE. The new estimator is superior to the RLE in the MSE criterion.

Table 4. Estimated MSE with $\gamma = 0.9$

k	0	NEW	HK	HKB	LW	LS
ML	0.0376	0.0376	0.0376	0.0376	0.0376	0.0376
RLE	0.0376	0.0709	0.0413	0.0629	0.0376	0.0823
AURLE	0.0376	0.0374	0.0376	0.0374	0.0376	0.0375

Table 5. Estimated MSE with $\gamma = 0.95$

k	0	NEW	HK	HKB	LW	LS
ML	0.0717	0.0717	0.0717	0.0717	0.0717	0.0717
RLE	0717	0.1645	0.0843	0.1558	0.0717	0.2146
AURLE	0.0717	0.0711	0.0714	0.0711	0.0717	0.0722

Table 6. Estimated MSE with $\gamma = 0.99$

k	0	NEW	HK	HKB	LW	LS
ML	0.3111	0.3111	0.3111	0.3111	0.3111	0.3111
RLE	0.3111	1.1128	0.4654	1.0948	0.3111	1.4769
AURLE	0.3111	0.2943	0.2955	0.3487	0.3111	0.4392

From the Tables, we also conclude that the new ridge parameter perform well.

6. Numerical example

In this section, we present a real data application in order to illustrate the benefits of the new estimator AURLE and satisfy the theoretical results. The data set is obtained from the official website of the Statistics Sweden (http://www.scb.se/) and it was also used in Asar and Genc [2] and a similar data set was used in Mansson et al. [4]. There are 271 observations which are the municipalities of Sweden in the data set. We fit a logistic regression model where the followings are the independent variables: the population (x1), the number of unemployed people (x2), the number of newly constructed buildings (x3) and the number of bankrupt firms (x4). We consider the net population change as the dependent variable such that it is coded as 1 if there is an increase in the population and 0 vice versa. We computed the bivariate correlations and observed that they are all greater than 0.90. The condition number being a measure of the degree of multicollinearity is computed as 38.3274 showing that there is severe multicollinearity problem with this data.

We provide the estimated theoretical MSE and coefficients of ML, RLE and AURLE for k_{NEW} k_{HK}, k_{HKB}, k_{LW} and k_{LS} in Table 7.

It is observed from Table 7 that MSE of ML is the largest among all possible situations. The new estimator NEW works well with the estimator AURLE such that AURLE has a less MSE than RLE when NEW is used. Moreover, AURLE has better performance when HKB and LS are used. In Figure 1, we plot the MSE values of RLE and AURLE for changing values of the parameter k between zero and 1. It is seen from Figure 1 that ARLE has less MSE values in this interval. According to Theorem 3.2, for $k > 4.0608$, AURLE should have a less MSE than that of RLE. This result can be seen from Figure 2.

Table 7. The estimated theoretical MSEs and coefficients of estimators for different estimators of k

	k	β_1	β_2	β_3	β_4	SMSE
RLE	kNEW	2.3757	0.2549	1.1752	-2.7641	973.7117
	kHK	18.1501	-11.8274	3.6423	-9.0784	1157.0656
	kHKB	4.1873	-0.9625	1.9584	-4.1488	904.7912
	KLW	0.1674	0.1576	0.1342	0.0622	1076.8163
	kLS	8.3886	-4.2012	2.8882	-6.0829	826.1210
AURLE	kNEW	3.9688	-0.6164	2.0167	-4.3217	931.4176
	kHK	23.2398	-15.7957	3.8558	-10.4687	1642.0958
	kHKB	7.1375	-3.1442	2.9676	-5.9501	865.1787
	KLW	0.2679	0.2492	0.2012	0.0581	1074.8677
	kLS	13.6711	-8.3300	3.6388	-8.0437	936.7677
ML		25.3151	-17.4071	3.8669	-10.9661	1894.3979

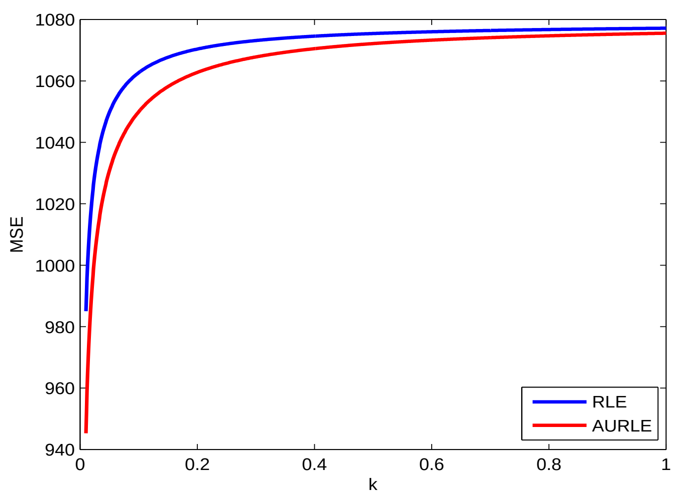

Figure 1. The estimated MSE of RLE and AURLE when $0 < k < 1$

Moreover, we plot the biases of the estimators to illustrate Theorem 3.1 in Figure 3. According to Figure 3, it is observed that the squared bias of AURLE is always less than that of RLE which coincides with Theorem 3.1.

Finally, Theorem 3.3 is also satisfied. Since $1 - \lambda_i \alpha_i^2 > 0$, AURLE has less MSE value than that of ML.

7. Conclusion

In this paper we propose a almost unbiased ridge logistic estimator based on the ridge logistic estimator and we also discuss the properties of the new estimator. The comparison results show that the new estimator has smaller quadratic bias the RLE, and under

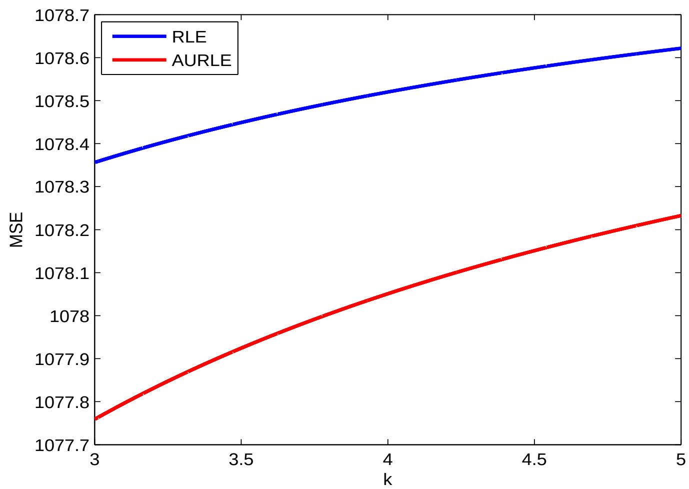

Figure 2. The estimated MSE of RLE and AURLE for satisfying Theorem 3.2

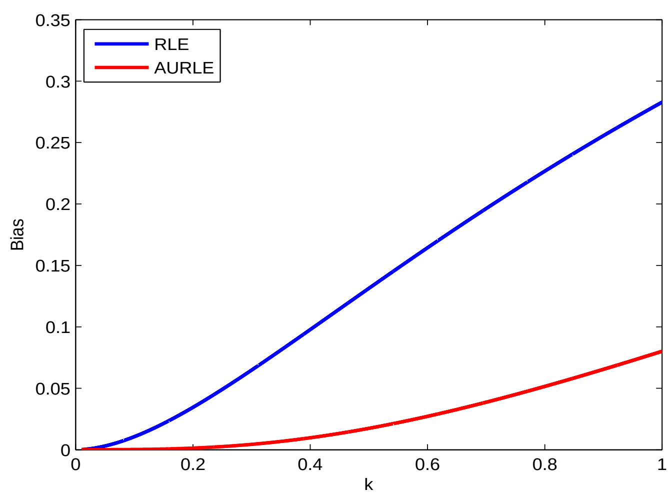

Figure 3. The biases of the estimator RLE and AURLE when $0 < k < 1$

certain conditions the new estimator is superior to the ML and RLE in the MSE sense.

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Appendix
3.1 **Theorem**

Proof. We have $Bias(\hat{\beta}_{RLE}(k)) = -k(X'\hat{W}X + kI)^{-1}\beta$ and

$$\begin{aligned} Bias(\hat{\beta}_{AURLE}(k)) &= [I - k^2(X'\hat{W}X + kI)^{-2}]\beta - \beta \\ &= -k^2(X'\hat{W}X + kI)^{-2}\beta \end{aligned} \tag{7.1}$$

Thus we have

$$\begin{aligned} & \|Bias(\hat{\beta}_{RLE}(k))\|^2 - \|Bias(\hat{\beta}_{AURLE}(k))\|^2 \\ = \ & \beta' k^2(X'\hat{W}X + kI)^{-2}\beta - \beta' k^4(X'\hat{W}X + kI)^{-4}\beta \\ = \ & \alpha' k^2(\Lambda + kI)^{-2}\alpha - \alpha' k^4(\Lambda + kI)^{-4}\alpha = \alpha' G\alpha \end{aligned} \tag{7.2}$$

where $G = k^2(\Lambda + kI)^{-2} - k^4(\Lambda + kI)^{-4} = diag(\frac{k^2\lambda_i(\lambda_i+2k)}{(\lambda_i+k)^4})$, thus for $k > 0$, $\alpha' G\alpha > 0$. The proof is completed. □

3.2 **Theorem**

Proof. By (2.2)-(2.3) and the definition of SMSE, we have

$$\begin{aligned} MSE(\hat{\beta}_{RLE}(k)) &= tr\{Cov\hat{\beta}_{RLE}(k)\} + Bias(\hat{\beta}_{RLE}(k))' Bias(\hat{\beta}_{RLE}(k)) \\ &= tr\{(X'\hat{W}X + kI)^{-1}X'\hat{W}X(X'\hat{W}X + kI)^{-1}\} \\ &\quad + \alpha' k^2(\Lambda + kI)^{-2}\alpha \\ &= \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + k)^2} + \sum_{i=1}^{p} \frac{k^2\alpha_i^2}{(\lambda_i + k)^2} \\ &= \sum_{i=1}^{p} \frac{\lambda_i + k^2\alpha_i^2}{(\lambda_i + k)^2} \end{aligned} \tag{7.3}$$

By (2.4), we can compute that:

$$\begin{aligned} & Cov\hat{\beta}_{AURLE}(k) \\ = \ & [I - k^2(X'\hat{W}X + kI)^{-2}](X'\hat{W}X)^{-1}[I - k^2(X'\hat{W}X + kI)^{-2}] \end{aligned} \tag{7.4}$$

Then we get

$$\begin{aligned} & MSE(\hat{\beta}_{AURLE}(k)) \\ = \ & tr\{Cov\hat{\beta}_{AURLE}(k)\} + Bias(\hat{\beta}_{AURLE}(k))' Bias(\hat{\beta}_{AURLE}(k)) \\ = \ & tr\{[I - k^2(X'\hat{W}X + kI)^{-2}](X'\hat{W}X)^{-1}[I - k^2(X'\hat{W}X + kI)^{-2}]\} \\ & + \alpha' k^4(\Lambda + kI)^{-4}\alpha \\ = \ & \sum_{i=1}^{p}(1 - \frac{k^2}{(\lambda_i + k)^2})^2 \frac{1}{\lambda_i} + \sum_{i=1}^{p} \frac{k^4\alpha_i^2}{(\lambda_i + k)^4} \\ = \ & \sum_{i=1}^{p} \frac{(\lambda_i + 2k)^2\lambda_i + k^4\alpha_i^2}{(\lambda_i + k)^4} \end{aligned} \tag{7.5}$$

Now we consider the difference:

$$\begin{aligned} \Delta_1 &= MSE(\hat{\beta}_{RLE}(k)) - MSE(\hat{\beta}_{AURLE}(k)) \\ &= \sum_{i=1}^{p} \frac{\lambda_i + k^2\alpha_i^2}{(\lambda_i + k)^2} - \sum_{i=1}^{p} \frac{(\lambda_i + 2k)^2\lambda_i + k^4\alpha_i^2}{(\lambda_i + k)^4} \\ &= \sum_{i=1}^{p} \frac{\lambda_i k[2k^2\alpha_i^2 + (\lambda_i\alpha_i^2 - 3)k - 2\lambda_i]}{(\lambda_i + k)^4} \end{aligned} \tag{7.6}$$

Δ_1 will be positive for $k > 0$ if and only if

$$2k^2\alpha_i^2 + (\lambda_i\alpha_i^2 - 3)k - 2\lambda_i > 0 \tag{7.7}$$

for all $i = 1, ..., p$. The expression in (7.7) is a quadratic function of k which has two distinct roots

$$k_{1,2} = \frac{3 - \lambda_i\alpha_i^2 \pm \sqrt{(3 + \lambda_i\alpha_i^2)^2 + 4\lambda_i\alpha_i^2}}{4\alpha_i^2} \tag{7.8}$$

Though the root $\frac{3-\lambda_i\alpha_i^2-\sqrt{(3+\lambda_i\alpha_i^2)^2+4\lambda_i\alpha_i^2}}{4\alpha_i^2}$ is negative. Thus when $k > 0$ and

$$k > \frac{3 - \lambda_i\alpha_i^2 + \sqrt{(3 + \lambda_i\alpha_i^2)^2 + 4\lambda_i\alpha_i^2}}{4\alpha_i^2}$$

for all $i = 1, ..., p$, the new estimator is superior to the RLE by the MSE criterion in logistic regression model. □

3.3 **Theorem**

Proof. It is easy to obtain that

$$MSE(\hat{\beta}_{ML}) = \sum_{i=1}^{p} \frac{1}{\lambda_i} \tag{7.9}$$

Now we study the following difference:

$$\begin{aligned}
\Delta_2 &= MSE(\hat{\beta}_{ML}) - MSE(\hat{\beta}_{AURLE}(k)) \\
&= \sum_{i=1}^{p} \frac{1}{\lambda_i} - \sum_{i=1}^{p} \frac{(\lambda_i + 2k)^2\lambda_i + k^4\alpha_i^2}{(\lambda_i + k)^4} \\
&= k^2 \sum_{i=1}^{p} \frac{(1 - \alpha_i^2\lambda_i)k^2 + 4\lambda_i k + 2\lambda_i^2}{\lambda_i(\lambda_i + k)^4}
\end{aligned} \tag{7.10}$$

Δ_2 will be positive if and only if

$$(1 - \alpha_i^2\lambda_i)k^2 + 4\lambda_i k + 2\lambda_i^2 > 0 \tag{7.11}$$

Now we discuss $(1 - \alpha_i^2\lambda_i)k^2 + 4\lambda_i k + 2\lambda_i^2$.
(1) If $1 - \lambda_i\alpha_i^2 > 0$ for all $i = 1, ..., p$, then $(1 - \alpha_i^2\lambda_i)k^2 + 4\lambda_i k + 2\lambda_i^2 > 0$.
(2) If $1 - \lambda_i\alpha_i^2 < 0$ for some $i = 1, ..., p$, then using the method in Theorem 3.2, we have if $k < \frac{2\lambda_i+\lambda_i\sqrt{2(1+\alpha_i^2\lambda_i)}}{\alpha_i^2\lambda_i-1}$, $(1 - \alpha_i^2\lambda_i)k^2 + 4\lambda_i k + 2\lambda_i^2 > 0$.
This completes the proof of Theorem. □

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