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CONTENTS

Mathematics

Octavian Agratini

Kantorovich-type operators preserving affine functions 1657

Natalia Chinchaladze

On one problem of a cusped elastic prismatic shells in case of the third model of Vekua's hierarchical model 1665

Sadulla Z. JAFAROV

On moduli of smoothness and approximation by trigonometric polynomials in weighted Lorentz spaces 1675

Junaid Alam Khan, Muhammad Ahsan Binyamin and Rabia

Subalgebra analogue to H-basis for ideals 1685

Saidakhmat LAKAEV and Ender ÖZDEMİR

The existence and location of eigenvalues of the one particle Hamiltonians on lattices 1693

Cui-Xia Li and Shi-Liang Wu

Eigenvalue distribution of relaxed mixed constraint preconditioner for saddle point problems 1705

Kai Liu and Tingbin Cao

Values sharing results on q-difference and derivative of meromorphic functions 1719

Paweł J. Szabłowski

Around Poisson–Mehler summation formula 1729

Birdal Şenoğlu and Özlem Türker Bayrak

Linear contrasts in one-way classification AR(1) model with gamma innovations 1743

Ş. Yılmaz and O. Kazancı

Approximations in a hyperlattice by using set-valued homomorphisms 1755

Statistics

Hamid Bidram, Rasool Roozegar and Vahid Nekoukhou

Exponentiated generalized geometric distribution: A new discrete distribution 1767

Gauss M. Cordeiro, Morad Alizadeh, Edwin M.M.Ortega and Luis H. Valdivieso Serrano

The Zografos-Balakrishnan odd log-logistic family of distributions: Properties and Applications 1781

Afshin Fallah and Zahra Goodarzi

Analysis of covariance by assuming a skew normal distribution for response variable 1805

Geeta Kalucha, Sat Gupta and Javid Shabbir

A two-step approach to ratio and regression estimation of finite population mean using optional randomized response models 1819

Akbar Ali Khan, Muhammad Aslam, Zawar Hussain and Muhammad Tahir

Comparison of loss functions for estimating the scale parameter of log-normal distribution using non-informative priors 1831

Fernando Nascimento, Marcelo Bourguignon and Jeremias Leão

Extended generalized extreme value distribution with applications in environmental data 1847

G. N. Singh, S. Maurya, M. Khetan and Cem Kadilar

Some imputation methods for missing data in sample surveys 1865

Ayfer Ezgi YILMAZ and Tulay SARACBASI

The effect of changing scores for multi-way tables with open-ended ordered categories 1881

Ehsan Zamanzade and Amer Ibrahim Al-Omari

New ranked set sampling for estimating the population mean and variance 1891

MATHEMATICS

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Kantorovich-type operators preserving affine functions

Octavian Agratini*

Abstract

Starting from positive linear operators which have the capability to reproduce affine functions, we design integral operators of Kantorovich-type which enjoy by the same property. We focus to show that the error of approximation can be smaller than in classical Kantorovich construction on some subintervals of its domain. Special cases are presented.

Keywords: Szász-Mirakjan operator, Baskakov operator, Stancu operator, Kantorovich operator, modulus of continuity.

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1. Introduction

In the field of Approximation Theory the study of positive and linear approximation processes holds an important place. In time numerous such sequences of discrete type operators have been investigated. In the sequel we generically denote a such sequence by $(L_n)_{n\geq 1}$. Among them, a special attention has been paid to operators which reproduce affine functions, property implied by the following two relations $L_n e_0 = e_0$ and $L_n e_1 = e_1$, $n \in \mathbb{N}$. Set e_j, $j \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the monomial of degree j. Since discrete operators are not suitable for approximating discontinuous functions, they were generalized into operators of integral type. One of the usual techniques is known as Kantorovich method which leads to an approximation process, say $(\widetilde{L}_n)_{n\geq 1}$, in spaces of integrable functions. Usually, the integral operators keep the property to reproduce constants, this means $\widetilde{L}_n e_0 = e_0$ but lose the property to reproduce affine functions, in other words $\widetilde{L}_n e_1 \neq e_1$.

The primary model for such construction is given by Bernstein operators defined as follows

$$(1.1)\qquad (B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),\ f \in \mathbb{R}^{[0,1]},\ x \in [0,1],$$

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and $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = \overline{0,n}$. In the above $\mathbb{R}^{[0,1]}$ represents the space of all real-valued functions defined on the compact interval $[0,1]$. Kantorovich extension has the form

$$(1.2)\qquad (\widetilde{B}_n f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt,\ \ x \in [0,1],$$

where $f \in L_1([0,1])$, the space of all Lebesgue integrable functions on $[0,1]$.

One has $B_n e_0 = e_0$, $B_n e_1 = e_1$, $\widetilde{B}_n e_0 = e_0$ and $\widetilde{B}_n e_1 \neq e_1$.

The purpose of this article is the following. Starting from a general discrete linear positive process reproducing polynomials of first degree, we indicate a technique to create an integral generalization in Kantorovich sense which will inherit the same property to reproduce affine functions. We study the error of approximation of the new sequence establishing the condition in which they are more useful than classical Kantorovich-type operators. Finally we present some particular examples.

2. The operators

Throughout the paper we consider an interval $J \subseteq \mathbb{R}$, which may be one of the types $J = [0,1]$ or $J = \mathbb{R}_+ = [0,\infty)$. The second variant will exhibit the problems caused by a finite endpoint and by the boundlessness of the interval. Let $(x_{n,k})_{k\in I_n}$ be a net on the interval J, where $I_n \subseteq \mathbb{N}$ is a set of indices. In what follows we consider that the net has equidistant nodes, meaning that for each $n \in \mathbb{N}$,

$$(2.1)\qquad x_{n,k+1} - x_{n,k} = p_n,\ \ k \in I_n,$$

where $\lim_n p_n = 0$. In fact, the overwhelming majority of discrete linear positive operators have this property. Most frequently encountered case is described by $x_{n,k} = k/n$, this implying $p_n = 1/n$.

We consider a sequence of linear positive operators of discrete type defined as follows

$$(2.2)\qquad (L_n f)(x) = \sum_{k\in I_n} \lambda_{n,k}(x) f(x_{n,k}),\ \ x \in J,$$

where $\lambda_{n,k} \in C(J)$, $\lambda_{n,k} \geq 0$ for each $(n,k) \in \mathbb{N}\times I_n$. For our purposes, if $Card(I_n)$ is finite, then $f \in C(J)$. If $Card(I_n)$ is non-finite, then

$$f \in \mathcal{F}(J) := \{g \in C(J) :\ \text{the series in (2.2) is absolutely convergent}\}.$$

Denoting by $C_B(J)$ the space of all real-valued continuous and bounded functions on J, we get $C_B(J) \subset \mathcal{F}(J)$. Anyway, we keep the assumption that $e_j \in \mathcal{F}(J)$, $j = 1$ and $j = 2$. As announced in Introduction, we consider that these operators reproduce affine functions, i.e.,

$$(2.3)\qquad \sum_{k\in I_n} \lambda_{n,k}(x) = 1$$

and

$$(2.4)\qquad \sum_{k\in I_n} \lambda_{n,k}(x) x_{n,k} = x,\ \ x \in J.$$

Set $p^* = \sup_{n\in\mathbb{N}} p_n$ where p_n is given at (2.1). If $J = \mathbb{R}_+$, then we consider $J^* = \left[\frac{p^*}{2}, \infty\right)$. If $J = [0,1]$, we take $J^* = \left[\frac{p^*}{2}, 1\right]$.

We define an integral generalization in Kantorovich sense of L_n, $n \in \mathbb{N}$, operators as follows

$$(2.5)\qquad (\widetilde{L}_n f)(x) = \frac{1}{p_n} \sum_{k \in I_n} \lambda_{n,k} \left(x - \frac{p_n}{2} \right) \int_{x_{n,k}}^{x_{n,k+1}} f(t)dt,\ x \in J^*.$$

If I_n is finite, the function f must be chosen integrable on J. Otherwise, f must be locally integrable function on J such that the antiderivative of f to belong to the space $\mathcal{F}(J)$. Also we mention that for a certain $k \in I_n$ such that $x_{n,k} \in J$ and $x_{n,k+1} \notin J$ we will replace $x_{n,k+1}$ with $x_{n,k}$. In other words, the integral will become null. This can happen if J is bounded.

Clearly, $\widetilde{L}_n$, $n \in \mathbb{N}$, are linear and positive operators.

The aim of this note is to show that our modified operators preserve affine functions.

In our opinion, the study of the convergence for $(\widetilde{L}_n)_{n\geq 1}$ sequence does not bring too much novelties. We will turn our attention to another direction. We determine in what circumstances this class of operators can offer a smaller approximation error than the classical Kantorovich operators. Some special cases are delivered.

3. The usefulness of $\widetilde{L}_n$ operator

At the beginning we calculate the first three moments of our integral operators.

Theorem 1. *Let $\widetilde{L}_n$, $n \in \mathbb{N}$, be defined by (2.5). For each $x \in J^*$ we have*

(i) $(\widetilde{L}_n e_0)(x) = 1$,

(ii) $(\widetilde{L}_n e_1)(x) = x$,

(iii) $(\widetilde{L}_n e_2)(x) = (L_n e_2)\left(x - \frac{p_n}{2} \right) + p_n x - \frac{p_n^2}{6}$,

where L_n and p_n are defined by (2.2) and (2.1), respectively.

Proof. The first statement is a direct consequence of (2.3). By using (2.1) and (2.4) we can write

$$\begin{aligned}(\widetilde{L}_n e_1)(x) &= \frac{1}{2p_n} \sum_{k \in I_n} \lambda_{n,k} \left(x - \frac{p_n}{2} \right) (2p_n x_{n,k} + p_n^2) \\ &= (L_n e_1)\left(x - \frac{p_n}{2} \right) + \frac{p_n}{2} (L_n e_0)\left(x - \frac{p_n}{2} \right) = x.\end{aligned}$$

Similarly, by using (2.1) and (2.5), we get

$$(\widetilde{L}_n e_2)(x) = \frac{1}{3p_n} \sum_{k \in I_n} \lambda_{n,k} \left(x - \frac{p_n}{2} \right) (3x_{n,k}^2 p_n + 3x_{n,k} p_n^2 + p_n^3)$$

which leads us to the last statement of Theorem 1. □

The first two identities of Theorem 1 guarantee that the operators $\widetilde{L}_n$, $n \in \mathbb{N}$, inherit property to reproduce affine functions.

At this point we introduce the second order central moment of the operator $\widetilde{L}_n$, that is

$$\mu_{n,2}(x) := (\widetilde{L}_n \varphi_x^2)(x),\ \text{where } \varphi_x(t) = t - x,\ (t,x) \in J \times J^*.$$

In view of Theorem 1, for any $x \in J^*$ we get

$$(3.1)\qquad \mu_{n,2}(x) = (L_n e_2)\left(x - \frac{p_n}{2} \right) + x(p_n - x) - \frac{p_n^2}{6}.$$

Starting from the same discrete operator L_n, $n \in \mathbb{N}$, it is well known that the classical Kantorovich generalization is designed as follows

$$(3.2)\qquad (L_n^* f)(x) = \frac{1}{p_n} \sum_{k \in I_n} \lambda_{n,k}(x) \int_{x_{n,k}}^{x_{n,k+1}} f(t)dt,\ x \in J.$$

Following a similar path set for $\widetilde{L}_n$, $n\in\mathbb{N}$, we easily deduce the formulas for each $x\in J$.

$$(L_n^* e_0)(x)=1,$$
$$(L_n^* e_1)(x)=x+\frac{p_n}{2},$$
$$(L_n^* e_2)(x)=(L_n e_2)(x)+p_n x+\frac{p_n^2}{3}.$$

The second order central moment is given by

$$\mu_{n,2}^*(x):=(L_n^*\varphi_x^2)(x)=(L_n e_2)(x)-x^2+\frac{p_n^2}{3}. \tag{3.3}$$

We turn our attention in comparing the approximation errors caused by the two classes of operators, $(\widetilde{L}_n)_{n\geq 1}$ and $(L_n^*)_{n\geq 1}$. To achieve it, we recall the notion of the first modulus of smoothness associated to a continuous function f on a compact interval $[a,b]$. It is denoted by $\omega(f;\cdot)_{[a,b]}$ and is defined as follows

$$\omega(f;\delta)_{[a,b]}=\sup\{|f(t)-f(x)|:\ |t-x|\leq\delta,\ t,x\in[a,b]\},\ \delta\geq 0.$$

Theorem 2. *(i) Let $J=[0,1]$ and $J^*=\left[\frac{p^*}{2},1\right]$. For any function $f\in C(J)$, the operators $\widetilde{L}_n$ and $\widetilde{L}_n^*$, $n\in\mathbb{N}$, satisfy*

$$|(L_n^* f)(x)-f(x)|\leq 2\omega_J\left(f;\sqrt{\mu_{n,2}^*(x)}\right)$$

and

$$|(\widetilde{L}_n f)(x)-f(x)|\leq 2\omega_{J^*}\left(f;\sqrt{\mu_{n,2}(x)}\right).$$

(ii) Let $J=[0,\infty)$ and $J^*=\left[\frac{p^*}{2},\infty\right)$. Let τ be fixed, $\tau>p^*/2$. For any function $f\in C_B(J)$, the operators $\widetilde{L}_n$ and L_n^*, $n\in\mathbb{N}$, satisfy

$$|(L_n^* f)(x)-f(x)|\leq 2\omega_{[0,\tau]}\left(f;\sqrt{\mu_{n,2}^*(x)}\right)$$

and

$$|(\widetilde{L}_n f)(x)-f(x)|\leq 2\omega_{[p^*/2,\tau]}\left(f;\sqrt{\mu_{n,2}(x)}\right).$$

Proof. These quantitative results given in terms of the modulus of smoothness are direct consequence of the following statement proved by Shisha and Mond [6]. If Λ is a linear positive operator defined on $C([a,b])$, then one has

$$|(\Lambda f)(x)-f(x)|\leq|f(x)|\ |(\Lambda e_0)(x)-1| + \left((\Lambda e_0)(x)+\frac{1}{\lambda}\sqrt{(\Lambda e_0)(x)(\Lambda\varphi_x^2)(x)}\right)\omega(f;\lambda),$$

for every $x\in[a,b]$ and $\lambda>0$. Taking in view that our operators L_n^* and $\widetilde{L}_n$ reproduce the constants and choosing $\lambda=\sqrt{(\Lambda\varphi_x^2)(x)}$ the conclusions of our theorem are taken place. □

Examining the upper bound of the approximation error, it is noticed that the operators $\widetilde{L}_n$, $n\in\mathbb{N}$, prove their usefulness if $\mu_{n,2}<\mu_{n,2}^*$ holds.

Theorem 3. *The operators $\widetilde{L}_n$, $n\in\mathbb{N}$, defined by (2.5) as compared to operators L_n^*, $n\in\mathbb{N}$, defined by (3.2) give a better error estimation for continuous and bounded functions if*

$$(L_n e_2)(x)\left(x-\frac{p_n}{2}\right)-(L_n e_2)(x)+p_n x<\frac{p_n^2}{2},\ x\geq\frac{p^*}{2}, \tag{3.4}$$

holds. The operators L_n, $n\in\mathbb{N}$, are defined by (2.2).

Proof. Imposing $\mu_{n,2} < \mu_{n,2}^*$ and using relations (3.1) and (3.3), the conclusion follows. □

4. Examples

The first two examples pertain to the case $J = \mathbb{R}_+$ and in the last example we consider $J = [0, 1]$.

1. *Modified Szász-Mirakjan-Kantorovich operators*
The classical Szász-Mirakjan operators are defined by

$$(L_n f)(x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \text{ where } s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \ x \geq 0.$$

The Szász-Kantorovich operators have been defined (see Butzer [1]) by

$$(L_n^* f)(x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_{k/n}^{(k+1)/n} f(t)dt, \ x \geq 0.$$

In this case, $p_n = 1/n$ and our operators $\widetilde{L}_n$, $n \in \mathbb{N}$, are given by

$$(\widetilde{L}_n f)(x) = ne^{\frac{1-2nx}{2}}\sum_{k=0}^{\infty}\frac{(2nx-1)^k}{2^k k!}\int_{k/n}^{(k+1)/n} f(t)dt, \ x \geq \frac{1}{2}.$$

These operators have been defined and studied by Duman, Özarslan and Della Vecchia [3]. The genuine Szász-Mirakjan operators satisfy

$$(L_n e_2)(x) = x^2 + \frac{x}{n}$$

and substituting in (3.4) we get $-3/(4n^2) < 0$. This fact guarantees that $\widetilde{L}_n$ operators generate a smaller approximation error.

2. *Modified Baskakov-Kantorovich operators*
The Baskakov operators are defined by

$$(L_n f)(x) = \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \ x \geq 0.$$

As mentioned in [2, *p. 115*], their Kantorovich extension has the form

$$(L_n^* f)(x) = n\sum_{k=0}^{\infty} v_{n,k}(x)\int_{k/n}^{(k+1)/n} f(t)dt.$$

We deduce $p_n = 1/n$ and $\widetilde{L}_n$ operators are defined by

$$(\widetilde{L}_n f)(x) = 2^n n^{n+1}\sum_{k=0}^{\infty}(2nx-1)^k(2nx+2n-1)^{-n-k}\int_{k/n}^{(k+1)/n} f(t)dt,$$

$x \geq \frac{1}{2}$. Since

$$(L_n e_2)(x) = x^2 + \frac{x(1+x)}{n},$$

relation (3.4) becomes

$$1 - 3n < 4nx, \ x \geq \frac{1}{2},$$

which implies a better error approximation by using $\widetilde{L}_n$, $n \in \mathbb{N}$, operators.

3. *Modified Stancu-Kantorovich operators*

Stancu polynomials [7] depend on parameter $\alpha \geq 0$ and are defined by

$$(L_n f)(x) = \sum_{k=0}^{n} \omega_{n,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{n}\right), \; x \in [0,1],$$

where

$$\omega_{n,k}^{\langle\alpha\rangle}(x) = \binom{n}{k} \frac{x^{(k,-\alpha)}(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}},$$

$$x^{(k,-\alpha)} := x(x+\alpha)\dots(x+(k-1)\alpha).$$

The following identities

$$L_n e_0 = e_0, \quad L_n e_1 = e_1, \quad L_n e_2 = e_2 + \frac{1+n\alpha}{n(1+\alpha)}(e_1 - e_2)$$

hold. The integral extension in Kantorovich sense of Stancu operators has been introduced and studied in [5]

$$(L_n^* f)(x) = (n+1) \sum_{k=0}^{n} \omega_{n,k}^{\langle\alpha\rangle}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt.$$

For operators generated by formula (2.5) we identify $p_n = 1/(n+1)$, $p^* = 1/2$ and $J^* = \left[\frac{1}{4}, 1\right]$. Relation (3.4) becomes

(4.1) $\quad -1 - \beta_n + 2(n+1)\beta_n(2x-1) < 0,$

where $\beta_n = \dfrac{1+n\alpha}{n(1+\alpha)}$.

For special case $\alpha = 0$, Stancu operators turn into Bernstein operators and L_n^* operators become the genuine Kantorovich operators [4], see (1.1) and (1.2). In this case $\beta_n = 1/n$ and for each $x < 3/4$ relation (4.1) hold. Consequently, at least on the interval $[1/4, 3/4]$ the operators $\widetilde{L}_n$, $n \in \mathbb{N}$, give a better error approximation.

For $\alpha > 0$, we set

$$\tau = \inf_{n \geq 1} \frac{2n\beta_n + 3\beta_n + 1}{4(n+1)\beta_n}.$$

We can easily verify that $1/2 < \tau < 1$.

In this case the interval I for which the operators $\widetilde{L}_n$, $n \in \mathbb{N}$, give a better error of approximation than the operators L_n^*, $n \in \mathbb{N}$, is the following $I = \left[\frac{1}{4}, \tau\right]$.

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On one problem of a cusped elastic prismatic shells in case of the third model of Vekua's hierarchical model

Natalia Chinchaladze*†

Abstract

In the present paper hierarchical model for cusped, in general, elastic prismatic shells is considered, when on the face surfaces a normal to the projection of the prismatic shell component of a traction vector and parallel to the projection of the prismatic shell components of a displacement vector are known.

Keywords: Cusped plates, cusped prismatic shells, degenerate elliptic systems, weighted spaces, Hardy's inequality, Korn's weighted inequality.

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1. Introduction

Investigations of cusped elastic prismatic shells actually takes its origin from the fifties of the last century, namely, in 1955 I.Vekua raised the problem of investigation of elastic cusped prismatic shells, whose thickness on the prismatic shell entire boundary or on its part vanishes (see [15], [16], [9] and references therein).

Let $Ox_1x_2x_3$ be an anticlockwise-oriented rectangular Cartesian frame of origin O. We conditionally assume the x_3-axis vertical. The elastic body is called a prismatic shell if it is bounded above and below by, respectively, the surfaces

$$x_3 = \overset{(+)}{h}(x_1, x_2) \text{ and } x_3 = \overset{(-)}{h}(x_1, x_2), \quad (x_1, x_2) \in \omega,$$

laterally by a cylindrical surface Γ of generatrix parallel to the x_3-axis and its vertical dimension is sufficiently small compared with other dimensions of the body. $\overline{\omega} := \omega \cup \partial\omega$ is the so-called projection of the prismatic shell on $x_3 = 0$.

The main difference between the prismatic shell of a constant thickness and the standard shell of a constant thickness is the following: the lateral boundary of the standard

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shell is orthogonal to the "middle surface" of the shell, while the lateral boundary of the prismatic shell is orthogonal to the prismatic shell's projection on $x_3 = 0$.

Let the thickness of the prismatic shell be

$$2h(x_1,x_2) := \overset{(+)}{h}(x_1,x_2) - \overset{(-)}{h}(x_1,x_2) \begin{cases} > 0 & \text{for } (x_1,x_2) \in \omega, \\ \geq 0 & \text{for } (x_1,x_2) \in \partial\omega. \end{cases}$$

If the thickness of the prismatic shell vanishes on $\gamma_0 \subset \partial\omega$, it is called cusped one.

Below we consider symmetric prismatic shell, i.e. the case when

$$\overset{(+)}{h}(x_1,x_2) = -\overset{(-)}{h}(x_1,x_2),$$

with the thickness as follows

$$2h := h_0 x_2^{\varkappa}, \ h_0, \varkappa = const, \ h_0, \ \varkappa > 0. \tag{1.1}$$

I. Vekua [15], [16] constructed hierarchical models for elastic prismatic shells, in particular, plates of variable thickness, when on the face surfaces either es (the first model) or displacements (the second model) are known. The updated survey of results concerning cusped elastic prismatic shells in the cases of the first and second models is given in [9] (see also [1], [5], [6], [10], [12], [14] and references therein). In the present paper the third hierarchical model for cusped elastic prismatic shells is analyzed. It means that on the face surfaces a normal to the projection of the prismatic shell component $Q_{\overset{(\pm)}{\nu}3}$ of a traction vector and parallel to the projection of the prismatic shell components $u_\alpha(x_1, x_2, \overset{(\pm)}{h}, t)$ of a displacement vector are known. The third model was first suggested in [8].

In what follows the usual notations are used: X_{ij} and e_{ij} are the stress and strain tensors, respectively, u_i are the displacements, F_i are the volume force components, ρ is the density, λ and μ are the Lamé constants, δ_{ij} is the Kronecker delta, subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables. Moreover, repeated indices imply summation (Greek letters run from 1 to 2 and Latin letters run from 1 to 3).

In the fifties of the twentieth century, I.Vekua ([9], [15], [16]) introduced a new mathematical model for elastic prismatic shells which was based on expansions of the three–dimensional displacement vector fields and the strain and stress tensors in linear elasticity into orthogonal Fourier-Legendre series with respect to the variable of plate thickness. By taking only the first $N + 1$ terms of the expansions, he introduced the so–called N–th approximation. Each of these approximations for $N = 0, 1, ...$ can be considered as an independent mathematical model of plates. In particular, in case of the first model the approximations for $N = 0$ and $N = 1$ correspond to the plane deformation and classical Kirchhoff-Love plate model, respectively (see [9]).

For the sake of simplicity we consider zero approximation of the hierarchical model. Basic equation system can be written as follows (see e.g. [8], [3])

$$\begin{aligned} &\mu(hv_{\alpha 0})_{,\beta\beta} + (\lambda+\mu)(hv_{\gamma 0})_{,\gamma\alpha} \\ &-(\ln h)_{,\beta}\left\{\lambda\delta_{\alpha\beta}(hv_{\gamma 0})_{,\gamma} + \mu\left[(hv_{\alpha 0})_{,\beta} + (hv_{\beta 0})_{,\alpha}\right]\right\} + \Phi_{\alpha 0} = \rho h \ddot{v}_{\alpha 0}, \end{aligned} \tag{1.2}$$

$$\mu(hv_{30,\beta})_{,\beta} + \Phi_{30} = \rho h \ddot{v}_{30}, \tag{1.3}$$

where

$$X_{\alpha\beta0}(x_1,x_2,t)=\lambda\delta_{\alpha\beta}\Big[(hv_{\gamma0}),_\gamma+\Psi_{\gamma\gamma}\Big]+\mu\Big[(hv_{\alpha0}),_\beta+(hv_{\beta0}),_\alpha+2\Psi_{\alpha\beta}\Big],$$
$$X_{3\beta0}(x_1,x_2,t)=\mu hv_{30,\beta},\quad X_{330}=\lambda\Big[(hv_{\gamma0}),_\gamma+\Psi_{\gamma\gamma}\Big],$$
$$e_{\alpha\beta0}=\frac{1}{2}\Big[(hv_{\alpha0}),_\beta+(hv_{\beta0}),_\alpha\Big]+\Psi_{\alpha\beta},\quad e_{3\beta0}=\frac{1}{2}hv_{30,\beta},\quad e_{330}=0,$$

$$\Phi_{\alpha0}:=2\mu\,\Psi_{\alpha\beta,\beta}+\lambda\,\Psi_{\gamma\gamma,\alpha}-(\ln h),_\beta[\lambda\delta_{\alpha\beta}\Psi_{\gamma\gamma}+2\mu\,\Psi_{\alpha\beta}]+F_{\alpha0},$$
$$\Phi_{30}:=Q_{\overset{(+)}{\nu}_3}\sqrt{(\overset{(+)}{h},_1)^2+(\overset{(+)}{h},_2)^2+1}+Q_{\overset{(-)}{\nu}_3}\sqrt{(\overset{(-)}{h},_1)^2+(\overset{(-)}{h},_2)^2+1}+F_{30},$$
$$\Psi_{\alpha\beta}:=\frac{1}{2}\Big[u_\beta(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h},_\alpha-u_\beta(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h},_\alpha$$
$$+u_\alpha(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h},_\beta-u_\alpha(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h},_\beta\Big],$$

X_{ij0}, e_{ij0}, u_{i0} and F_{i0} are the zeroth order moments of X_{ij}, e_{ij}, u_i and F_i, respectively; $v_{i0}:=h^{-1}u_{i0}$ are called weighted moments of the function u_i.

The case of cylindrical bending of the plates with the thickness (1.1) is considered in [8]. In this case the system (1.2)-(1.3) can be rewritten as follows

$$\mu(h(x_2)v_{10}(x_2)),_{22}-\mu(\ln h(x_2)),_2\,(h(x_2)v_{10}(x_2)),_2+\Phi_{10}(x_2)=0$$
$$(\lambda+2\mu)(h(x_2)v_{20}(x_2)),_{22}-(\lambda+2\mu)(\ln h(x_2)),_2\,(h(x_2)v_{20}(x_2)),_2+\Phi_{20}(x_2)=0,$$
$$\mu(h(x_2)v_{30,2}(x_2)),_2+\Phi_{30}(x_2)=0.$$

In [8] it is shown that $v_{\alpha0}$ can not be prescribed in cusped edge (i.e., Dirichlet problem are not satisfied) if $\varkappa>0$, and v_{30} can not be prescribed in cusped edge if $\varkappa\geq1$.

The weak setting of the homogeneous Dirichlet problem of the following system

$$\mu(hv_{\alpha0}),_{\beta\beta}+(\lambda+\mu)(hv_{\gamma0}),_{\gamma\alpha}$$
$$-(\ln h),_\beta\{\lambda\delta_{\alpha\beta}(hv_{\gamma0}),_\gamma+\mu[(hv_{\alpha0}),_\beta+(hv_{\beta0}),_\alpha]\}+\Phi_{\alpha0}=0,$$
$$\mu(hv_{30,\beta}),_\beta+\Phi_{30}=0,$$

is considered in [3].

2. Vibration problem

We will consider the case of harmonic vibration

$$v_{i0}(x,t):=e^{-\iota\vartheta t}\overset{0}{v}_{i0}(x),\quad \Phi_{i0}(x,t):=e^{-\iota\vartheta t}\overset{0}{\Phi}_{i0}(x),\quad \iota^2=-1,$$
$$\vartheta=\text{const}>0,\quad x:=(x_1,x_2)\in\omega,\quad i=1,2,3.$$

Taking into account of (1.1), (1.2), and (1.3) for $\overset{0}{v}_{i0}(x)$ we get the following system (the overscript index 0 is omitted below)

$$-\rho\vartheta^2hv_{10}-\mu\Delta_2(hv_{10})-(\lambda+\mu)\Big[(hv_{10}),_{11}+(hv_{20}),_{21}\Big]$$
$$+\mu(\ln h),_2[(hv_{10}),_2+(hv_{20}),_1]=\Phi_{10},$$

$$-\rho\vartheta^2hv_{20}-\mu\Delta_2(hv_{20})-(\lambda+\mu)\Big[(hv_{10}),_{12}+(hv_{20}),_{22}\Big]$$
$$+(\ln h),_2\{\lambda[(hv_{10}),_1+(hv_{20}),_2]+2\mu(hv_{20}),_2\}=\Phi_{20},$$

$$-\rho\vartheta^2hv_{30}-\mu[(hv_{30,1}),_1+(hv_{30,2}),_2]=\Phi_{30},$$

where Δ_2 is a two dimensional Laplace operator.

We can rewrite obtained system in the following vector form

$$\mathbf{A}v(x) = \Phi(x), \quad x \in \omega, \tag{2.1}$$

where

$$\mathbf{A} := \left\| \begin{array}{ccc} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{array} \right\|,$$

$$A_{11} := -\rho\vartheta^2 h - (\lambda + 2\mu)h\partial_{11} - \mu\left[h\partial_{22} + 2h,_2\partial_2 + h,_{22}\right] + \mu(\ln h),_2\left[h\partial_2 + h,_2\right],$$

$$A_{12} := -(\lambda+\mu)\left[h\partial_{12} + h,_2\partial_1\right] + \mu(\ln h),_2 h\partial_1,$$

$$A_{21} := -(\lambda+\mu)\left[h\partial_{12} + h,_2\partial_1\right] + \lambda(\ln h),_2 h\partial_1,$$

$$A_{22} := -\rho\vartheta^2 h - \mu h\partial_{11} - (\lambda+2\mu)\left[h\partial_{22} + 2h,_2\partial_2 + h,_{22}\right] + (\lambda+2\mu)(\ln h),_2\left[h\partial_1 + h,_2\right],$$

$$A_{13} = A_{23} = A_{31} = A_{32} = 0, \quad A_{33} := -\rho\vartheta^2 h - \mu h(\partial_{11} + \partial_{22}) + \mu h,_2\partial_2,$$

$$v := (v_{10}, v_{20}, v_{30})^\top, \quad \Phi := (\Phi_{10}, \Phi_{20}, \Phi_{30}),$$

the symbol $(\cdot)^\top$ means transposition.

Let

$$v, v^* \in C^2(\omega) \cap C^1(\overline{\omega}), \quad v^* := (v^*_{10}, v^*_{20}, v^*_{30})^\top,$$

where v and v^* are arbitrary vectors of the above class. We obtain the following Green's formula

$$\int_\omega \mathbf{A}v \cdot v^* d\omega = \mathbf{J}(v, v^*) - \int_{\partial\omega} X_n v \cdot v^* d\partial\omega = \int_\omega \Phi \cdot v^* d\omega. \tag{2.2}$$

Here $n := (n_1, n_2)$ is the inward normal to $\partial\omega$:

$$X_n := \{X_{n10}, X_{n20}, X_{n30}\},$$

with

$$X_{ni0} = X_{ij0} n_j,$$

$$\begin{aligned}
\mathbf{J}(v, v^*) := & \int_\omega -h\rho\vartheta^2 v_{i0} v^*_{i0} d\omega + \int_\omega \frac{\lambda}{h}\Big[(hv_{10}),_1 (hv^*_{10}),_1 + (hv_{20}),_2 (hv^*_{20}),_2 \\
& + (hv_{10}),_1 (hv^*_{20}),_2 + (hv_{20}),_2 (hv^*_{10}),_1\Big] d\omega + \int_\omega \frac{\mu}{h}\Big[2f(hv_{10}),_1 (hv^*_{10}),_1 \\
& + (hv_{10}),_2 (hv^*_{10}),_2 + (hv_{20}),_1 (hv^*_{10}),_2 + (hv_{20}),_1 (hv^*_{20}),_1 + (hv_{10}),_2 (hv^*_{20}),_1 \\
& + 2(hv_{20}),_2 (hv^*_{20}),_2 + hv_{30,1} hv^*_{30,1} + hv_{30,2} hv^*_{30,2}\Big] d\omega \\
= & \int_\omega -h\rho\vartheta^2 v_{i0} v^*_{i0} d\omega + \int_\omega \frac{\lambda}{h}(hv_{\alpha 0}),_\alpha (hv^*_{\beta 0}),_\beta d\omega \\
& + \int_\omega \frac{\mu}{h}\Big\{\left[(hv_{\alpha 0}),_\beta + (hv_{\beta 0}),_\alpha\right]\left[(hv^*_{\alpha 0}),_\beta + (hv^*_{\beta 0}),_\alpha\right] + (hv_{30,\alpha})(hv^*_{30,\alpha})\Big\} d\omega \\
= & \int_\omega a\left[-h^2\rho\vartheta^2 v_{i0} v^*_{i0} + \lambda e^1_{kk0}(v) e^1_{ii0}(v^*) + 2\mu e^1_{ij0}(v) e^1_{ij0}(v^*)\right] d\omega,
\end{aligned}$$

where

$$a := \frac{1}{h},$$

$$e^1_{ij0}(v) := \begin{cases} \dfrac{1}{2}\Big[(hv_{i0})_{,j} + (hv_{j0})_{,i}\Big], & i,j=1,2, \\ \dfrac{1}{2} hv_{i0,j}\,, & i=3,\ j=1,2, \\ 0, & i=j=3. \end{cases}$$

If we consider BVPs for system (2.1) with homogeneous boundary conditions for which the curvilinear integralalong $\partial\omega$ in (2.2) disappears, we arrive at the equation

$$\mathbf{J}(v,v^*) = \int_\omega \Phi \cdot v^* d\omega.$$

Denote by $\mathcal{D}(\omega)$ a space of infinity differentiable functions with compact support in ω and introduce the linear form $[\mathcal{D}(\omega)]^3$ by the formula:

$$(v,v^*)_{X^\varkappa_\vartheta} = \int_\omega \Big[h^2\rho\vartheta^2 v_{i0}v^*_{i0} + e^1_{ij0}(v)e^1_{ij0}(v^*)\Big]\frac{1}{h}d\omega,$$

$$\|v\|^2_{X^\varkappa_\vartheta} = \int_\omega \Big[h\rho\vartheta^2 v_{i0}v_{i0} + \frac{1}{4h}\Big(4[(hv_{10})_{,1}]^2$$

$$+4[(hv_{20})_{,2}]^2 + 2((hv_{10})_{,2} + (hv_{20})_{,1})^2 + 2(hv_{30,1})^2 + 2(hv_{30,2})^2\Big)\Big]d\omega.$$

$X^\varkappa_\vartheta$ is a Hilbert space.

The classical and weak setting of the homogeneous Dirichlet problem can be formulated as follows:

2.1. Problem. Find a 3-dimensional vector v in ω satisfying the system of differential equations (2.1) in ω and the homogeneous Dirichlet boundary condition

(2.3) $\qquad [v(x)]^+ = 0, \quad x \in \partial\omega.$

2.2. Problem. Find a vector $v \in X^\varkappa_\vartheta$ satisfying the equality

(2.4) $\qquad \mathbf{J}(v,v^*) = \langle \Phi, v^* \rangle \quad \text{for all} \quad v^* \in X^\varkappa_\vartheta,$

here, the vector Φ belongs to the adjoint space $[X^\varkappa_\vartheta]^*$, and $\langle\cdot,\cdot\rangle$ denotes duality brackets between the spaces $[X^\varkappa_\vartheta]^*$ and $X^\varkappa_\vartheta$.

Further, we construct the vectors in $\Omega := \{(x;x_3) : x\in\omega, -h(x) < x_3 < h(x)\}$:

$$w_i(x,x_3) = \frac{1}{2}v_{i0}(x), \quad i=1,2,3,$$

$$w^*_i(x,x_3) = \frac{1}{2}v^*_{i0}(x), \quad i=1,2,3.$$

It can be shown that

(2.5) $\qquad J(w,w^*) := \displaystyle\int_\Omega [-\rho\vartheta^2 w_i w^*_i + \sigma_{ij}(w)e_{ij}(w^*)]d\Omega = \mathbf{J}(v,v^*),$

where $w(x,x_3) := (w_1,w_2,w_3)$ and $w^*(x,x_3) := (w^*_1,w^*_2,w^*_3)$ are vectors and $J(w,w^*)$ is the bilinear form corresponding to the three-dimensional potential energy for the displacement vector w.

In view of the homogeneous Dirichlet boundary condition (2.3), if $\varkappa > 1$, the following Hardy inequality holds (see [13], p. 69; [11])

(2.6) $\qquad \displaystyle\int_\varepsilon^l x_2^{\varkappa-2} v^2_{\alpha 0} dx_2 \le \frac{4}{(\varkappa-1)^2}\int_\varepsilon^l x_2^\varkappa (v_{\alpha 0,2})^2 dx_2, \quad \varkappa > 1.$

Replacing in (2.6) $\varkappa$ by $\varkappa+2$, we obtain

$$(2.7)\qquad \int_\varepsilon^l x_2^\varkappa v_{\alpha 0}^2 dx_2 \le \frac{4}{(\varkappa+1)^2}\int_\varepsilon^l x_2^{\varkappa+2}(v_{\alpha 0,2})^2 dx_2, \ \text{ for any } \varkappa>0.$$

Now, considering the limit procedure as $\varepsilon\to 0+$, since the limits of the integrals in (2.7) exist for $v_{\alpha 0}\in X_\vartheta^\varkappa$, we immediately get the following

$$(2.8)\qquad \int_0^l x_2^\varkappa v_{\alpha 0}^2 dx_2 \le \frac{4}{(\varkappa+1)^2}\int_0^l x_2^{\varkappa+2}(v_{\alpha 0,2})^2 dx_2, \ \text{ for any } \varkappa>0.$$

Integrating by x_1 both side of (2.8) over $]x_1^0,x_1^1[$, we get

$$(2.9)\qquad \int_\omega x_2^\varkappa v_{\alpha 0}^2 d\omega \le \frac{4}{(\varkappa+1)^2}\int_\omega x_2^{\varkappa+2}(v_{\alpha 0,2})^2 d\omega, \ \text{ for any } \varkappa>0.$$

2.3. Lemma. *The bilinear form $\mathbf{J}(\cdot,\cdot)$ is bounded and strictly coercive in the space $X_\vartheta^\varkappa(\omega)$, i.e., there are positive constant C_0 and C_1 such that*

$$(2.10)\qquad |\mathbf{J}(v,v^*)|\le C_1\|v\|_{X_\vartheta^\varkappa}\|v^*\|_{X_\vartheta^\varkappa},$$

$$(2.11)\qquad \mathbf{J}(v,v)\ge C_0\|v\|^2_{X_\vartheta^\varkappa}$$

for all $v,v^*\in X_\vartheta^\varkappa$ and $\vartheta^2<\frac{\mu(\varkappa+1)^2}{16\rho l^2}$.

Proof. In view of (2.5) we have

$$\begin{aligned}
&|\mathbf{J}(v,v^*)|^2 = |J(w,w^*)|^2\\
&= \Big[\int_\Omega -\rho\vartheta^2 w_i w_i^* + (2\mu e_{ij}(w)+\lambda\delta_{ij}e_{kk}(w))e_{ij}(w^*)d\Omega\Big]^2\\
&\le \Big|\int_\Omega \rho\vartheta^2 w_i w_i^* d\Omega\Big|^2 + C_3\Big|\int_\Omega (2\mu e_{ij}(w)+\lambda\delta_{ij}e_{kk}(w))e_{ij}(w^*)d\Omega\Big|^2\\
&\le \Big|\int_\omega h\rho\vartheta^2 v_{i0}v_{i0}^* d\omega\Big|^2 + C_2\sum_{i,j=1}^3\int_\Omega e_{ij}^2(w)d\Omega\sum_{i,j=1}^3\int_\Omega e_{ij}^2(w^*)d\Omega\\
&\le \Big|\int_\omega h\rho\vartheta^2 v_{i0}v_{i0}^* d\omega\Big|^2 + C_2\int_\omega \frac12\sum_{i,j=1}^3 e_{ij0}^2(v)\frac{d\omega}{h}\int_\omega \frac12\sum_{i,j=1}^3 e_{ij0}^2(v^*)\frac{d\omega}{h}\\
&\le \int_\omega h\rho\vartheta^2\sum_{i=1}^3 v_{i0}^2 d\omega\int_\omega h\rho\vartheta^2\sum_{i=1}^3 {v^*}_{i0}^2 d\omega\\
&+C_2\int_\omega \frac12\sum_{i,j=1}^3 e_{ij0}^2(v)\frac{d\omega}{h}\int_\omega \frac12\sum_{i,j=1}^3 e_{ij0}^2(v^*)\frac{d\omega}{h}\\
&\le C_1\|v\|^2_{X_\vartheta^\varkappa}\|v^*\|^2_{X_\vartheta^\varkappa},
\end{aligned}$$

where

$$C_1 := \max\{1, C_2\}.$$

Whence (2.10) follows.

Further, taking into account of (2.9) and of the fact that $2\lambda+3\mu>0$, $\mu>0$ we get

$$
\begin{aligned}
\|v\|^2_{X^{\varkappa}_{\vartheta}} &\le \frac{\mathbf{J}(v,v)}{2\mu}+\frac{\vartheta^2\rho h_0}{\mu}\int_\omega x_2^{\varkappa}v_{i0}^2 d\omega \le \frac{\mathbf{J}(v,v)}{2\mu}+\frac{4\vartheta^2\rho h_0}{\mu(\varkappa+1)^2}\int_\omega x_2^{\varkappa+2}(v_{i0,2})^2 d\omega\\
&\le \frac{\mathbf{J}(v,v)}{2\mu}+\frac{4\vartheta^2\rho h_0 l^2}{\mu(\varkappa+1)^2}\int_\omega x_2^{\varkappa}(v_{i0,2})^2 d\omega \le \frac{\mathbf{J}(v,v)}{2\mu}+\frac{4\vartheta^2\rho l^2}{\mu(\varkappa+1)^2}\int_\omega \frac{(hv_{i0,2})^2}{h} d\omega\\
&= \frac{\mathbf{J}(v,v)}{2\mu}+\frac{4\vartheta^2\rho l^2}{\mu(\varkappa+1)^2}\int_\omega \Big[\frac{(hv_{10,2})^2}{h}+\frac{(hv_{20,2})^2}{h}+\frac{(hv_{30,2})^2}{h}\Big] d\omega\\
&\le \frac{\mathbf{J}(v,v)}{2\mu}+\frac{2\vartheta^2\rho l^2}{\mu(\varkappa+1)^2}\int_\omega \Big[\frac{2(hv_{10,2})^2}{h}+\frac{4(hv_{20,2})^2}{h}+\frac{2(hv_{30,2})^2}{h}\Big] d\omega\\
&\le \frac{\mathbf{J}(v,v)}{2\mu}+\frac{2\vartheta^2\rho l^2}{\mu(\varkappa+1)^2}\int_\omega \Big[\frac{2[(hv_{10})_{,2}]^2}{h}+\frac{4[(hv_{20})_{,2}]^2}{h}+\frac{2(hv_{30,2})^2}{h}\Big] d\omega\\
&\le \frac{\mathbf{J}(v,v)}{2\mu}+\frac{8\vartheta^2\rho l^2}{\mu(\varkappa+1)^2}\|v\|^2_{X^{\varkappa}_{\vartheta}},
\end{aligned}
$$

from here we have

$$
\mathbf{J}(v,v)\ge \Big(2\mu-\frac{16\vartheta^2\rho l^2}{(\varkappa+1)^2}\Big)\|v\|^2_{X^{\varkappa}_{\vartheta}}. \tag{2.12}
$$

If we assume $\vartheta^2<\frac{\mu(\varkappa+1)^2}{16\rho l^2}$ inequality (2.11) immediately follows from (2.12). ■

2.4. Remark. If $\mathbf{J}(v,v)=0$, then $v\equiv 0$ by (2.12).

2.5. Theorem. *Let $F\in[X^{\varkappa}_{\vartheta}]^*$. Then the variational problem (2.4) has a unique solution $v\in X^{\varkappa}_{\vartheta}$ for an arbitrary value of the parameter $\varkappa$ and* $\|v\|_{X^{\varkappa}_{\vartheta}}\le\frac{1}{C_0}\|F\|_{[X^{\varkappa}_{\vartheta}]^*}$.

Proof. The proof can be realized by means of Lax-Milgram theorem (see Appendix A.1). ■

It can be easily shown that if $\Phi\in[L(\omega)]^3$ and $\operatorname{supp}\Phi\cap\overline{\gamma}_0=\emptyset$, then $\Phi\in[X^{\varkappa}_{\vartheta}]^*$ and

$$
\langle\Phi\,,\,v^*\rangle=\int_\omega \Phi(x)\,v^*(x)\,d\omega,
$$

since $v^*\in[H^1(\omega_\varepsilon)]^3$, where ε is sufficiently small positive number such that $\operatorname{supp}\Phi\subset\omega_\varepsilon=\omega\cap\{x_2>\varepsilon\}$. Therefore,

$$
|\langle\Phi\,,\,v^*\rangle|=\Big|\int_\omega \Phi(x)\,v^*(x)\,d\omega\Big|\le ||\Phi||_{[L_2(\omega)]^3}\;||v^*||_{[L_2(\omega_\varepsilon)]^3}
$$

$$
\le ||\Phi||_{[L_2(\omega)]^3}\;||v^*||_{[H^1(\omega_\varepsilon)]^3}\le C_\varepsilon\,||\Phi||_{[L_2(\omega)]^3}\;||v^*||_{X^{\varkappa}_{\vartheta}}\,.
$$

In this case, we obtain the estimate

$$
||v||_{X^{\varkappa}_{\vartheta}}\le\frac{C_\varepsilon}{C_0}\,||\Phi||_{[L_2(\omega)]^3}\,.
$$

For establishing a representation of the space $X^{\varkappa}_0$ as a weighted Sobolev space, we introduce the following space:

$$
Y^{\varkappa}_0:=\Big[\overset{0}{W}{}^1_{2\,,\varkappa}(\omega)\Big]^2,
$$

where $\overset{0}{W}{}^1_{2\,,\varkappa}(\omega)$ is a completion $\mathcal{D}(\omega)$ by means of the norm

$$
\|f\|^2_{\overset{0}{W}{}^1_{2\,,\varkappa}(\omega)}:=\int_\omega x_2^{\varkappa}\Big(|\nabla f|^2\Big)d\omega,\quad \nabla f=(f_{,1}\,,f_{,2}\,).
$$

The norm in the space $Y_0^{\varkappa}$ for a vector (v_{10}, v_{20}, v_{30}) reads as

$$\|v\|^2_{Y_0^{\varkappa}} := \int_\omega x_2^{\varkappa}\Big(\sum_{\alpha=1}^{2} |\nabla v_{\alpha 0}|^2\Big) d\omega.$$

Using Korn's and Hardy's inequalities (see Appendix) the following theorem can be proved (similarly, to the Theorem 5.1 of [4])

2.6. Theorem. *The linear spaces $X_0^{\varkappa}$ and $Y_0^{\varkappa}$ as sets of vector functions coincide and the norms $\|\cdot\|_{X_0^{\varkappa}}$, $\|\cdot\|_{Y_0^{\varkappa}}$ are equivalent if $\varkappa = 0$ and $\vartheta^2 < \min\{\frac{\mu(\varkappa+1)^2}{16\rho l^2}, \frac{2}{h_0 \rho l^2}\}$.*

2.7. Remark. Note that if $v \in X_\vartheta^{\varkappa}$, then all the components of v posses the zero traces on part γ_1 of the boundary $\partial\omega$ for arbitrary $\varkappa$ due to the well-known trace theorem in the Sobolev space W^1. This follows, on the one hand, from the fact that the elliptic system under consideration is non-degenerated at the curve γ_1 and, on the other hand, from the construction of the space $X_\vartheta^{\varkappa}$.

3. Appendix

A.1. The Lax-Milgram theorem. Let V be a real Hilbert space and let $J(w, v)$ be a bilinear form defined on $V \times V$. Let this form be continuous, i.e., let there exist a constant $K > 0$ such that

$$|J(w, v)| \le K \|w\|_V \|v\|_V$$

holds $\forall w, v \in V$ and V-elliptic, i.e., let there exist a constant $\alpha > 0$ such that

$$J(w, w) \ge \alpha \|w\|_V^2$$

holds $\forall w \in V$. Further let F be a bounded linear functional from V^* dual of V. Then there exists one and only one element $z \in V$ such that

$$J(z, v) = \langle F, v\rangle \equiv Fv \quad \forall v \in V$$

and

$$\|z\|_V \le \alpha^{-1} \|F\|_{V^*}.$$

Let ω be as in Section 1 and let $\mathcal{D}(\omega)$ be a space of infinitely differentiable functions with compact support in ω.

A.2. Hardy's inequality. For every $f \in \mathcal{D}(\omega)$ and $\nu \neq 1$ there holds the inequality

$$\text{(A.1)} \qquad \int_\omega x_2^{\nu-2} f^2(x)\, d\omega \le C_\nu \int_\omega x_2^{\nu} |\nabla f(x)|^2\, d\omega,$$

where the positive constant C_ν is independent of f.

By completion of $\mathcal{D}(\omega)$ with the norm

$$||f||^2_{\overset{\circ}{W}{}^1_{2,\nu}(\omega)} := \int_\omega x_2^{\nu} |\nabla f(x)|^2\, d\omega,$$

we conclude that the inequality (A.1) holds for arbitrary $f \in \overset{\circ}{W}{}^1_{2,\nu}(\omega)$.

For proof see [7].

A.3. Korn's weighted inequality. Let $\varphi = (\varphi_1, \varphi_2) \in [\overset{\circ}{W}{}^1_{2,\nu}(\omega)]^2$ and $\nu \neq 1$. Then

$$\int_\omega x_2^{\nu} [\,|\nabla \varphi_1(x)|^2 + |\nabla \varphi_2(x)|^2\,]\, d\omega$$

$$\leq C_\nu \int_\omega x_2^\nu [\varphi_{1,1}^2(x) + \varphi_{2,2}^2(x) + (\varphi_{1,2}(x) + \varphi_{2,1}(x))^2] \, d\omega,$$

where the positive constant C_ν is independent of φ.

The proof can be found in [7], [17].

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On moduli of smoothness and approximation by trigonometric polynomials in weighted Lorentz spaces

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Abstract

We investigate the approximation properties of the functions by trigonometric polynomials in weighted Lorentz spaces with weights satisfying so called Muckenhoupt's A_p condition. Relations between moduli of smoothness of the derivatives of the functions and those of the functions itself are studied. In weighted Lorentz spaces we also prove a theorem on the relationship between the derivatives of a polynomial of best approximation and the best approximation of the function. Moreover, we study relationship between modulus of smoothness of the function and its de la Vallée-Poussin sums in these spaces.

Keywords: moduli of smoothness, weighted Lorentz spaces, Muckenhoupt weight, trigonometric approximation, best approximation.

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1. Introduction and the main results

Let $\mathbb{T} = [-\pi, \pi]$. A function $\omega : \mathbb{T} \to [0, \infty]$ will be called a *weight function* if ω is locally integrable and almost everywhere (a.e.) positive. The function ω generates the Borel measure

$$\omega(E) = \int_E \omega(x)dx.$$

By

$$f^*_\omega(t) = \inf\{\nu \geq 0 : \ \omega(\{x \in \mathbb{T}: |f(x)| > \nu\}) \leq t\}$$

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we denote the nondecreasing rearrangement of a function $f : \mathbb{T} \to [0, \infty]$. We denote also

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*_\omega(u) du.$$

Let $0 < p < \infty$, $0 < q < \infty$. A measurable and a.e. finite function f on $\mathbb{T}$ belongs to the Lorentz space $L^{pq}_\omega(\mathbb{T})$ if

$$\|f\|_{L^{pq}_\omega} := \left(\int_{\mathbb{T}} (t^{\frac{1}{p}} f^{**}(t))^q \frac{dt}{t} \right)^{1/q} < \infty.$$

Note that Lorentz spaces, introduced by G. Lorentz in the 1950 s. [24], [25]. As seen the weighted Lorentz spaces $L^{pq}_\omega(\mathbb{T})$ is expressed not only in terms of the parameter p, but also in terms of the second parameter q. If $p = q$, then $L^{pq}_\omega(\mathbb{T})$ is the weighted Lebesgue space $L^p_\omega(\mathbb{T})$ [10, p. 20]. İf $q < r$, then the space $L^{pq}_\omega(\mathbb{T})$ is contained in $L^{pr}_\omega(\mathbb{T})$. Detailed information about properties of the Lorentz spaces can be found in [12], [20], [26] and [31].

Let $1 < p < \infty$, $p' = \frac{p}{p-1}$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy *Muckenhoupt's A_p-condition* on $\mathbb{T}$ if

$$\sup_J \left(\frac{1}{|J|} \int_J \omega\,(t)\, dt \right) \left(\frac{1}{|J|} \int_J \omega^{1-p'}\,(t)\, dt \right)^{p-1} < \infty\ ,$$

where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length. Note that the weight functions belonging to the A_p- class, introduced by Muckenhoupt [27], play a very important role in different fields of mathematical analysis.

We use $c,\ c_1,\ c_2,\ ...$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the questions of interest. We shall also employ the symbol $A \asymp B$, denoting that $cA \leq B \leq C$, where $c,\ C$ are constants.

Let $\alpha \in \mathbb{Z}^+$ and $f \in L^1(\mathbb{T})$. Suppose that $x,\ h$ are real, and let us take into

$$\Delta^\alpha_t f(x) := \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j}\ f\ (x + (\alpha - j)t),$$

where $\binom{\alpha}{j} := \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-j+1)}{j!}$, $j > 1$ is the Binomial coefficients and $\binom{\alpha}{0} : = 1, \binom{\alpha}{1} : = \alpha$.

Let $1 < p, q < \infty$, $\omega \in A_p(\mathbb{T})$, $f \in L^{pq}_\omega(\mathbb{T})$. We put

$$\sigma^\alpha_\delta f(x) := \frac{1}{\delta} \int_0^\delta |\Delta^\alpha_t f(x)|\, dt.$$

If $f \in L^{pq}_\omega(\mathbb{T})$, $\omega \in A_p(\mathbb{T})$ according to [6] the Hardy-Littlewood Maximal function is bounded in $L^{pq}_\omega(\mathbb{T})$, $\omega \in A_p(\mathbb{T})$. Then we have

$$\|\sigma^\alpha_\delta f\|_{L^{pq}_\omega} \leq c_1 \|f\|_{L^{pq}_\omega.} < \infty.$$

For $1 < p, q < \infty$, $\omega \in A_p(\mathbb{T})$, $f \in L^{pq}_\omega(\mathbb{T})$, $\alpha \in \mathbb{Z}^+$ we define the *$\alpha - th$ mean modulus of smoothness* $\omega_\alpha(f,\ .)_{L^{pq}_\omega}$ by

$$\omega_\alpha(f,\ h)_{L^{pq}_\omega} := \sup_{|\delta| \leq h} \|\sigma^\alpha_\delta f(x\|_{L^{pq}_\omega}$$

Let $f\in L_{\omega}^{pq}(\mathbb{T})$, $\alpha\in\mathbb{Z}^{+}$ the modulus of smoothness $\omega_{\alpha}(f,\ .)_{L_{\omega}^{pq}}$ is a nondecreasing, nonnegative, function and

$$\begin{aligned}\omega_{\alpha}^{p}(f_1+f_2,\ .)_{\ L_{\omega}^{pq}} &\le \omega_{\alpha}^{p}(f_1,\ .)_{\ L_{\omega}^{pq}}+\omega_{\alpha}^{p}(f_2,\ .)_{\ L_{\omega}^{pq}},\\ \lim_{\delta\to 0^+}\omega_{\alpha}(f,\ \delta)_{L_{\omega}^{pq}} &= 0.\end{aligned}$$

For $f\in L_{\omega}^{pq}(\mathbb{T})$, we define the $\alpha-th$ derivative of f as function $g\in L_{\omega}^{pq}(\mathbb{T})$ satisfying

$$\lim_{h\to 0^+}\left\|\frac{\Delta_h^{\alpha}(f)}{h^{\alpha}}-g\right\|_{L_{\omega}^{pq}}=0, \tag{1.1}$$

in which case we write $g=f^{\ (\alpha)}$.

Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}A_k(f,x),\ \ A_k(f,x):=a_k(f)\cos kx+b_k(f)\sin kx \tag{1.2}$$

be the Fourier series of thefunction $L^1(\mathbb{T})$. The *nth partial sums*, and *de la Vallée-Poussin sum* of the series (1.2) are defined, respectively, as

$$S_n(f)=\frac{a_0}{2}+\sum_{k=1}^{n}A_k(f,x),$$

$$V_n(f)=\frac{1}{n}\sum_{\nu=1}^{2n-1}S_{\nu}(f).$$

We denote by $E_n(f)_{L_{\omega}^{pq}}$ $(n=0,1,2,...)$ the best approximation of $f\in L_{\omega}^{pq}(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n, i. e.,

$$E_n(f)_{L_{\omega}^{pq}}:=\inf\left\{\|f-T_n\|_{L_{\omega}^{pq}}:T_n\in\Pi_n\right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

Let $W_{pq,\omega}^{\alpha}(\mathbb{T})$ $(r=1,2,...)$ be the linear space of functions $f\in L_{\omega}^{pq}(\mathbb{T})$, $1<p,q<\infty$, $\omega\in A_p(\mathbb{T})$, such that $f^{\ (\alpha)}\in L_{\omega}^{pq}(\mathbb{T})$. It becomes a Banach space with the norm

$$\|f\|_{W_{pq,\omega}^{\alpha}(\mathbb{T})}:=\|f\|_{L_{\omega}^{pq}}+\left\|f^{(\alpha)}\right\|_{L_{\omega}^{pq}}.$$

The problems of approximation theory in the weighted and nonweighted Lorentz space have been investigated in [1], [21], [35] and [37]. The approximation problems by trigonometric polynomials in different spaces have been investigated by several authors (see, for example, [2-5], [7], [9], [11], [13-19], [22], [23], [28-30], [33] and [34]).

In this work we study the approximation problems of functions by trigonometric polynomials in the weighted Lorentz space $L_{\omega}^{pq}(\mathbb{T})$ with Muckenhoupt weights. Relations between moduli of smoothness of the derivatives of a function and those of the function itself are investigated. We also prove a theorem on the relationship between derivatives of a polynomial of best approximation and the best approximation of the function in the weighted Lorentz space $L_{\omega}^{pq}(\mathbb{T})$. In addition, in the weighted Lorentz space $L_{\omega}^{pq}(\mathbb{T})$ relationship between modulus of smoothness of the function and its de la Vallée-Poussin sums is studied. Similar problems in defferent spaces were investigated in [9], [30], [32].

Our main results are the following.

Theorem 1.1. *Let* $1<p,\ q<\infty$, $\omega\in A_p(\mathbb{T})$, $f\in L_{\omega}^{pq}(\mathbb{T})$ *and* T_n *a trigonometric polynomial of degree* n *satisfying the following conditions:*

$$\|f-T_n\|_{L_{\omega}^{pq}}=o\left(\frac{1}{n}\right)\ \ \textit{and}\ \ \left\|g-T_n'\right\|_{L_{\omega}^{pq}}=o(1),\quad n\to\infty.$$

Then we obtain $f' = g$, that is, the function g satisfies the condition (1.1).

Using the same method as in the proof of Theorem 1.1 we have the following Corollary.

Corollary1.1. *Let $1 < p,\ q < \infty$, $\omega \in A_p(\mathbb{T})$, $f, g_1, ..., g_k \in L^{pq}_{\omega}(\mathbb{T})$ and T_n be a trigonometric polynomial satisfying, for $i = 1, ..., k$, the conditions*

$$\|f - T_n\|_{L^{pq}_{\omega}} = o\left(\frac{1}{n^k}\right), \quad n \to \infty,$$

$$\left\|g_i - T_n^{(i)}\right\|_{L^{pq}_{\omega}} = o\left(\frac{1}{n^{k-i}}\right), \quad n \to \infty.$$

Then we obtain $g_i = g'_{i-1}$ $(f = g_0)$ in the sense of (1.1).

Theorem 1.2. *Let $1 < p < \infty$ and $1 < q \le 2$ or $p > 2$ and $q \ge 2$. Then, for a given $\omega \in A_p(T)$, $f \in L^{pq}_{\omega}(T)$ and integers α, r satisfying $\alpha > r$ we have*

$$\omega_{\alpha - r}\left(f^{(r)}, t\right)_{L^{pq}_{\omega}} \le c_2 \left\{ \int_0^t \frac{\omega_{\alpha}\left(f,\ u\right)^s_{L^{pq}_{\omega}}}{u^{sr+1}} du \right\}^{1/s},$$

where $s = \min\ (q, 2)$.

Theorem 1.3. *Let $1 < p,\ q < \infty$, $\omega \in A_p(T)$, $f \in L^{pq}_{\omega}(T)$, $\alpha, r \in \mathbb{Z}^+$ $(\alpha > r > 0)$ and let $T_n(f) \in \Pi_n$ be the polynomial of best approximation to f in $L^{pq}_{\omega}(T)$. In order that*

$$\left\|T_n^{(\alpha)}(f)\right\|_{L^{pq}_{\omega}} = O(n^{\alpha - r})$$

it is necessary and sufficient that

$$E_n(f)_{L^{pq}_{\omega}} = O(n^{-r}).$$

Theorem 1.4. *Let $1 < p,\ q < \infty$, $\omega \in A_p(T)$, $\alpha \in \mathbb{Z}^+$. If $f \in L^{pq}_{\omega}$, then*

1.

$$c_3 \omega_{\alpha}(f, \frac{1}{n})_{L^{pq}_{\omega}} \le \left(n^{-\alpha}\left\|V_n^{(\alpha)}(\ f)\right\|_{L^{pq}_{\omega}} + \|f(x) - V_n(\ f)\|_{L^{pq}_{\omega}}\right) \le c_4 \omega_{\alpha}(f, \frac{1}{n})_{L^{pq}_{\omega}} \tag{1.3}$$

where the constants c_4 and c_5 are dependent on α, p and q.

2.

$$c_5 \omega_{\alpha}(f, \frac{1}{n})_{L^{pq}_{\omega}} \le \left(n^{-\alpha}\left\|S_n^{(\alpha)}(f)\right\|_{L^{pq}_{\omega}} + \|f(x) - S_n(\ f)\|_{L^{pq}_{\omega}}\right) \le c_6 \omega_{\alpha}(f, \frac{1}{n})_{L^{pq}_{\omega}}, \tag{1.4}$$

where the constants $c_6$and c_7 are dependent on α, p and q.

2. Proofs of main results

We need the following results obtained in [35].

Lemma 2.1. *Let $\omega \in A_p(T)$, $1 < p, q < \infty$. If $f \in L^{pq}_{\omega}(T)$ and $\alpha = 1, 2, ...$, then there exists a constant $c_7 > 0$ depending α, p and q such that*

$$E_n(f)_{L^{pq}_{\omega}} \le c_7 \omega_{\alpha}(f, \frac{1}{n})_{L^{pq}_{\omega}}.$$

holds where $n = 0, 1, 2, ...$

Lemma 2.2. *Let $\omega \in A_p(T)$ and $\alpha \in Z^+$, $1 < p, q < \infty$. If $T_n \in \Pi_n$, $n \geq 1$, then there exists a constant $c_8 > 0$ depending only on α, p and q such that*

$$\omega_\alpha(T_n,\ h)_{L^{pq}_\omega} \leq c_8 h^\alpha \left\| T_n^{(\alpha)} \right\|_{L^{pq}_\omega},\ 0 < h \leq \pi$$

Lemma 2.3. *Let $\omega \in A_p(T)$, $1 < p, q < \infty$. If $T_n \in \Pi_n$, $n \geq 1$ and $\alpha \in Z^+$, then there exists a constant $c_9 > 0$ depending only on α, p and q such that*

$$\left\| T_n^{(\alpha)} \right\|_{L^{pq}_\omega} \leq c_9 n^\alpha \left\| T_n \right\|_{L^{pq}_\omega}.$$

Proof of Theorem 1.1. We take $\varepsilon > 0$. We choose a natural number $n_0 = n_0(\varepsilon)$ such that for $n \geq n_0$

$$\left\| f - T_n \right\|_{L^{pq}_\omega} \leq \varepsilon \frac{1}{n}, \qquad \left\| g - T_n' \right\|_{L^{pq}_\omega} \leq \varepsilon. \tag{2.1}$$

Taking account of (2.1) for h satisfying the condition $\frac{\sqrt{\varepsilon}}{n} \leq h \leq \frac{1}{n}$ we obtain

$$\left\| \frac{f(\cdot + h) - f(\cdot)}{h} - \frac{T(\cdot + h) - T_n(\cdot)}{h} \right\|^p_{L^{pq}_\omega} \leq 2^{\frac{p}{2}} \tag{2.2}$$

Considering [8] we have

$$\Delta_h^r T_n(x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i T_n\left(x + \left(\frac{r}{2} - i\right) h\right) =$$

$$= \sum_{j=r}^{\infty} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left(\frac{r}{2} - i\right)^j \frac{h^j}{j!} T_n^{(j)}(x) =$$

$$= h^r T_n^{(r)}(x) + \sum_{j=r+1}^{\infty} \eta(r,j)^{j-r} T_n^{(j)}(x), \tag{2.3}$$

where $-\frac{r}{2} < \eta(r,j) < \frac{r}{2}$ and $\eta(r,j) = 0$ if $j - r$ is odd. Then using (2.3) and Lemma 2.3 for $\frac{\sqrt{\varepsilon}}{n} \leq h < \frac{2\sqrt{\varepsilon}}{n}$ we find that

$$\left\| \frac{T_n(\cdot + h) - T_n(\cdot)}{h} - T_n'(\cdot) \right\|^p_{L^{pq}_\omega} \leq \sum_{m=2}^{\infty} \left(\frac{h^{m-1}}{m!} \right)^p \left\| T_n^{(m)} \right\|^p_{L^{pq}_\omega} \leq$$

$$\leq \sum_{m=2}^{\infty} (hn)^{(m-1)p} \left\| T_n \right\|^p_{L^{pq}_\omega} \leq 4 \frac{\varepsilon}{1 - 2^p \varepsilon^{p/2}} \left\| T_n \right\|^p_{L^{pq}_\omega} \leq c_{12} \varepsilon^p \left\| T_n \right\|^p_{L^{pq}_\omega}. \tag{2.4}$$

Using (2.2), (2.4) and (2.1) for $\frac{\sqrt{\varepsilon}}{n} \leq h < \frac{2\sqrt{\varepsilon}}{n}$ we reach

$$\left\| \frac{f(\cdot + h) - f(\cdot)}{h} - g \right\|^p_{L^{pq}_\omega} \leq \left\| \frac{f(\cdot + h) - f(\cdot)}{h} - \frac{T_n(\cdot + h) - T_n(\cdot)}{h} \right\|^p_{L^{pq}_\omega} +$$

$$+ \left\| \frac{T_n(\cdot + h) - T_n(\cdot)}{h} - T_n'(\cdot) \right\|^p_{L^{pq}_\omega} +$$

$$+ \left\| T_n' - g \right\|^p_{L^{pq}_\omega} \leq c_{10} \left(\varepsilon^{p/2} + \varepsilon^p \left\| f \right\|^p_{L^{pq}_\omega} + \varepsilon^p \right)$$

From the last inequality we have $g = f'$ in the sense of (1.2). Then the proof of Theorem 1.1 is completed.

Proof of Theorem 1. 2. The function $\omega_m(F,t)_{L^{pq}_\omega}$ non-decreasing and according to reference [34] the following inequality holds:

$$\omega_\alpha(F, 2t)_{L^{pq}_\omega} \leq c_{11} \omega_\alpha(F,\ t)_{L^{pq}_\omega} \tag{2.5}$$

It is sufficient to prove theorem for $t = 2^{-n}$. Then using of (2.5) we obtain

$$\left\{ \int_0^{2^{-n}} \frac{\omega_\alpha(f,\ u)^s_{L^{pq}_\omega}}{u^{sr+1}} du \right\}^{1/s} \asymp \left\{ \sum_{\nu=n}^{\infty} 2^{\nu sr} \omega_\alpha(f,\ 2^{-\nu})^s_{L^{pq}_\omega} \right\}^{1/s}.$$

Therefore for all n it is sufficient to prove the following inequality:

$$\omega_{\alpha-r}(f^{(r)}\ ,\ 2^{-n})_{L^{pq}_\omega} \le \left\{ \sum_{\nu=n}^{\infty} 2^{\nu sr} \omega_\alpha(f,\ 2^{-\nu})^s_{L^{pq}_\omega} \right\}^{1/s}. \tag{2.6}$$

By [34] for any trigonometric polynomial Q_n of degree cn and $F \in L^{pq}_\omega(\mathbb{T})$ we obtain

$$\omega_\alpha(F\ ,\ 1/n)_{L^{pq}_\omega} \le c_{12} \left(\|F - Q_n\|_{L^{pq}_\omega} + n^{-\alpha} \left\| Q_n^{(\alpha)} \right\|_{L^{pq}_\omega} \right). \tag{2.7}$$

Therefore we aim to find Q_{2^n} of degree $c2^n$ such that both $\left\| f^{(r)} - Q_{2^n} \right\|_{L^{pq}_\omega}$ and $2^{-n(\alpha-r)} \left\| Q_{2^n}^{(\alpha-r)} \right\|_{\|f\|_{L^{pq}_\omega}}$ are bounded by the right-hand side of inequality (2.6). Let $T_n \in \Pi_n$ $(n = 0, 1, 2, ...)$ be the polynomial of best approximation to f. It is known that [34] the set of trigonometric polynomials is dense in $L^{pq}_\omega(\mathbb{T})$. Then we have $\|f - T_{2^\nu}\|_{L^{pq}_\omega} \to 0$ as $\nu \to \infty$.

Let $f \in L^{pq}_\omega(\mathbb{T})$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(f).$$

We define trigonometric polynomial $\nu_N f$ as

$$\nu_N f = \sum_{k=0}^{\infty} \nu(\frac{k}{N}) A_k(f),$$

where $\nu \in C^\infty\,[0, \infty)$, $\nu(x) = 1$ for $x \le 1$ and $\nu(x) = 0$ for $x \ge 1$. Note that trigonometric polynomial $\nu_N f$ has the following properties:

I) $\nu_N f$ is a trigonometric polynomial of degree smaller than N;

II) If g is a trigonometric polynomial of degree $[N/2]$, then $\nu_N g = g$;

III) $\|\nu_N f\|_{L^{pq}_\omega} \le c \|f\|_{L^{pq}_\omega}$.

According to reference [34] we have

$$\|\nu_N f - f\|_{L^{pq}_\omega} \le c_{13} E_{N/2}(f)_{L^{pq}_\omega},$$

where $E_k(f)_{L^{pq}_\omega}$ is the best approximation of $f \in L^{pq}_\omega(\mathbb{T})$ trigonometric polynomials of degree not exceeding k. We now choose the Q_n of (2.7) for $F = f^{(r)}$ to be $(\nu_n f)^{(r)}$. It is cleary that $\|f - \nu_n f\|_{L^{pq}_\omega} = o(1)$ as $n \to \infty$.

The following identity holds:

$$\nu_{2^k} f - \nu_{2^n} f = \sum_{m=n}^{k-1} (\nu_{2^{m+1}} f - \nu_{2^m} f) \equiv \sum_{m=n}^{k-1} \gamma_m f.$$

Then

$$(\nu_{2^k} f)^{(r)} - (\nu_{2^n} f)^{(r)} = \sum_{m=1}^{k-1} (\gamma_m f)^{(r)}.$$

Using the Littlewood- Paley inequality for the weighted Lorentz spaces $L_\omega^{pq}(\mathbb{T})$ in [21] we have

$$\begin{aligned} & c_{14}\left\|(\nu_{2^k}f)^{(r)}-(\nu_{2^n}f)^{(r)}\right\|_{L_\omega^{pq}} \\ \leq\ & \left\|\left(\sum_{m=n}^{k-1}\left\{(\gamma_m f)^{(r)}\right\}^2\right)^{1/2}\right\|_{L_\omega^{pq}} \\ \leq\ & c_{15}\left\|(\nu_{2^k}f)^{(r)}-(\nu_{2^n}f)^{(r)}\right\|_{L_\omega^{pq}}. \end{aligned} \tag{2.8}$$

According to [21, Lemma 4.2 and 4.3] we get

$$\left\|\left(\sum_{m=n}^{k-1}\left\{(\gamma_m f)^{(r)}\right\}^2\right)^{1/2}\right\|_{L_\omega^{pq}} \leq \left(\sum_{m=n}^{k-1}\left\|(\gamma_m f)^{(r)}\right\|_{L_\omega^{pq}}^s\right)^{1/s}, \tag{2.9}$$

where $s=\min(q,2)$.

Note that $\nu_n f$ is the near best approximation to f in L_ω^{pq}. Then using [35] we reach the following equivalence

$$\omega_\alpha(f,1/n)\asymp\|f-\nu_n f\|_{L_\omega^{pq}}+n^{-\alpha}\left\|(\nu_n f)^{(\alpha)}\right\|_{L_\omega^{pq}}. \tag{2.10}$$

From (2.8)- (2.10) and Lemma 2.3 we conclude that

$$\begin{aligned} & \left\|(\nu_{2^k}f)^{(r)}-(\nu_{2^n}f)^{(r)}\right\|_{L_\omega^{pq}} \\ \leq\ & c_{16}\left(\sum_{m=n}^{k-1}2^{mrs}\left\|(\gamma_m f)\right\|_{L_\omega^{pq}}^s\right)^{1/s} \\ \leq\ & c_{17}\left(\sum_{m=n}^{k-1}2^{mrs}\omega_\alpha(f,2^{-m})_{L_\omega^{pq}}^s\right)^{1/s}, \end{aligned}$$

where c_1 independent of m,k and f.

Use of $Q_{2^n}=\nu_{2^n}f$ and (2.10) gives us

$$\begin{aligned} 2^{-n(\alpha-r)}\left\|((\nu_{2^n}f)^{(r)})^{(\alpha-r)}\right\|_{L_\omega^{pq}} &= 2^{-n(\alpha-r)}\left\|(\nu_{2^n}f)^{(\alpha)}\right\|_{L_\omega^{pq}} \\ &\leq 2^{nr}\omega_\alpha(f,2^{-n})_{L_\omega^{pq}}\leq c_{18}\left(\sum_{m=n}^{\infty}2^{mrs}\omega_\alpha(f,2^{-m})_{L_\omega^{pq}}^s\right)^{1/s}. \end{aligned}$$

The proof of Theorem 1.2 is completed.

Proof of Theorem 1. 3. We suppose that

$$E_n(f)_{L_\omega^{pq}}=\|f-T_n(f)\|_{L_\omega^{pq}}=O(n^{-r}),\qquad (r>0). \tag{2.11}$$

Taking into account Lemma 2.3 and the relations (2.11) we obtain

$$\left\|T_n^{(\alpha)}(f)\right\|_{L_\omega^{pq}}\leq c_{19}n^\alpha\|T_n(f)\|_{L_\omega^{pq}}\leq n^\alpha\|f-T_n(f)\|_{L_\omega^{pq}}+\|T_n(f)\|_{L_\omega^{pq}}\leq c_{20}n^{\alpha-r}.$$

Now we suppose that

$$\left\|T_n^{(\alpha)}(f)\right\|_{L_\omega^{pq}}=O(n^{\alpha-r}). \tag{2.12}$$

Using Lemma 2.1, Lemma 2.2 and (2.2) we get

$$\begin{aligned} \|T_{2n}(f)-T_n(T_{2n}(f))\|_{L^{pq}_\omega} &\leq E_n(T_{2n}(f))_{L^{pq}_\omega} \leq c_{21}\omega_\alpha(T_{2n},\frac{1}{n})_{L^{pq}_\omega}. \\ &\leq c_{22}n^{-\alpha}\left\|T^{(\alpha)}_{2n}\right\| \leq c_{23}n^{-\alpha}(n^{\alpha-r}) \leq c_{24}n^{-r}. \end{aligned} \tag{2.13}$$

On the other hand, since $T_n(T_{2n}(f))$ is a polynomial of order n the following inequality holds:

$$\begin{aligned} \|T_{2n}(f)-T_n(T_{2n}(f))\|_{L^{pq}_\omega} &= \|f-T_n(T_{2n}(f))-(f-T_{2n}(f))\|_{L^{pq}_\omega} \\ &\geq \|f-T_n(T_{2n}(f))\|_{L^{pq}_\omega}-\|f-T_{2n}(f))\|_{L^{pq}_\omega} \\ &\geq E_n(f)_{L^{pq}_\omega}-E_{2n}(f)_{L^{pq}_\omega} \geq 0. \end{aligned} \tag{2.14}$$

Use of (2.13) and (2.14) gives us

$$0 \leq E_n(f)_{L^{pq}_\omega}-E_{2n}(f)_{L^{pq}_\omega} \leq c_{25}n^{-r}. \tag{2.15}$$

Since $E_n(f)_{L^{pq}_\omega} \to 0$ from the inequality (2.15) we conclude that

$$\sum_{k=n_0}^{\infty}\left\{E_{2^k}(f)_{L^{pq}_\omega}-E_{2^{k+1}}(f)_{L^{pq}_\omega}\right\} \leq c_{26}\sum_{k=n_0}^{\infty}2^{-kr}.$$

Then from the last inequality we obtain

$$E_{2^{n_0}}(f)_{L^{pq}_\omega} \leq c_{27}2^{-n_0 r}. \tag{2.16}$$

It is clear that inequality (2.16) is equivalent to $E_n(f)_{L^{pq}_\omega} \leq c_{28}(n^{-r})$. This completes the proof.

Proof of Theorem 1. 4. In view of Lemma 2.2 the inequality

$$\omega_\alpha(T_n,\frac{1}{n})_{L^{pq}_\omega} \leq c_{29}n^{-\alpha}\left\|T^{(\alpha)}_n\right\|_{L^{pq}_\omega}, \tag{2.17}$$

holds, where T_n is a trigonometric polynomial of order n. Using the properties of smoothness $\omega_\alpha(f,.)_{L^{pq}_\omega}$ and (2.17), we reach

$$\begin{aligned} \omega_\alpha(f,\frac{1}{n})_{L^{pq}_\omega} &\leq \left(\omega_\alpha(f-T_n,\frac{1}{n})_{L^{pq}_\omega}+\omega_\alpha(T_n,\frac{1}{n})_{L^{pq}_\omega}\right) \\ &\leq c_{30}\left(\|f-T_n\|_{L^{pq}_\omega}+n^{-\alpha}\left\|T^{(\alpha)}_n\right\|_{L^{pq}_\omega}\right). \end{aligned} \tag{2.18}$$

Considering [34] there exists a constant $c>0$ depending only on α, p and q such that

$$n^{-\alpha}\left\|T^{(\alpha)}_n\right\|_{L^{pq}_\omega} \leq c_{31}\omega_\alpha(T_n,\frac{1}{n})_{L^{pq}_\omega}. \tag{2.19}$$

By virtue of Lemma 2.1

$$E_n(f)_{L^{pq}_\omega} \leq c_{32}\omega_\alpha(f,\frac{1}{n})_{L^{pq}_\omega}. \tag{2.20}$$

It is known that [34] for the de la Vallée-Poussin mean the inequality

$$\|f-V_n(f)\|_{L^{pq}_\omega} \leq c_{33}E_n(f)_{L^{pq}_\omega}. \tag{2.21}$$

holds. Use of (2.19)-(2. 21) gives us

$$\begin{aligned} & n^{-\alpha}\left\|V_n^{(\alpha)}(f)\right\|_{L_\omega^{pq}}+\|f-V_n(f)\|_{L_\omega^{pq}} \\ \leq\ & c_{34}\left(\omega_\alpha(V_n,\frac{1}{n})_{L_\omega^{pq}}+E_n(f)_{L_\omega^{pq}}\right) \\ \leq\ & c_{35}\left(\omega_\alpha(f,\frac{1}{n})_{L_\omega^{pq}}+\omega_\alpha(f-V_n,\frac{1}{n})_{L_\omega^{pq}}+E_n(f)_{L_\omega^{pq}}\right) \\ \leq\ & c_{36}\omega_\alpha(f,\frac{1}{n})_{L_\omega^{pq}}. \end{aligned}$$

The last inequality and (2.18) imply that (1.3).

According to [35] there exists a constant c_{25} such that

$$\|f-S_n(f)\|_{L_\omega^{pq}}\leq c_{37}E_n(f)_{L_\omega^{pq}}. \tag{2.22}$$

If the inequality (2.22) and the scheme of proof of the estimation (1.3) is used we obtain the estimation (1.4).

Theorem 1.4 is proved.

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Subalgebra analogue to H-basis for ideals

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Abstract

The H-basis concept allows an investigation of multivariate polynomial spaces degree by degree. In this paper we present the analogue of H-bases for subalgebras in polynomial rings, we call them "SH-bases". We present their connection to the Sagbi basis concept, characterize SH-basis and show how to construct them.

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1. Introduction

The concept of Gröbner bases, introduced by Buchberger [3] in 1965, has become an important ingredient for the treatment of various problems in computational algebra, (see [2] for an extensive survey). This concept has also been extended to more general situations, such as Gröbner bases of modules, for example, as in [9]. However, all approaches related to Gröbner bases are fundamentally tied to monomial orderings, which lead to asymmetry among the variables of interest. On the other hand, the concept of H-bases, introduced long ago by Macaulay [7], is based solely on homogeneous terms of a polynomial. In [12], an extension of Buchberger's algorithm is presented to construct H-bases algorithmically. Some applications of H-bases are given in [10], in addition, many of the problems in applications which can be solved by the Gröbner technique can also be treated successfully with H-bases.

The concept of Gröbner basis for ideals of a polynomial ring over a field K can be adopted in a natural way to K-subalgebras of a polynomial ring. In [11] Sagbi (Subalgebra Analogue to Gröbner Basis for Ideals) basis for the K-subalgebra of $K[x_1, \ldots, x_n]$ is

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defined; this concept was also independently developed in [6]. The properties and applications of Sagbi bases are typically similar to standard Gröbner basis results (see [1] and [4] for an overview of the standard theory). Like Gröbner bases, the concept of Sagbi basis is also tied to monomial orderings. Consequently, within the concept of H- bases for ideals, it is natural to probe the concept of subalgebra bases which may be based solely on homogenous terms of a polynomial. In this paper we will present the analogue to H-bases for ideals in polynomial rings, we call them "SH-bases". Unlike H-bases, SH-bases are not finite. This is not surprising because unlike ideals in polynomial rings, subalgebras in polynomial rings are not necessarily finitely generated. The subalgebras which are not finitely generated cannot have finite SH-basis. Moreover, a finitely generated subalgebra may have an infinite SH-basis (see Example 3.8).

The paper is organized as follows. In section 2, we briefly describe the underlying concept of grading which leads to Sagbi basis and SH-basis. Then, we give the notion of d-reduction, which is one of the key ingredients for the characterization and construction of SH-basis. After setting up the necessary notation, we present the d-reduction Algorithm (see Algorithm 1). Also, here we present some properties characterizing SH-basis (Theorem 2.4). In section 3, we present a criterion through which we can check that the given system of polynomials is an SH-basis of the subalgebra it generates (Theorem 3.4) and further on the basis of this theorem we present an algorithm for the construction of SH-basis (Algorithm 2).

2. SH-bases and Sagbi bases

Here and in the following sections we consider polynomials in n variables $x_1, \dots, x_n$ with coefficients from a field K. For short, we write

$$\mathcal{P} := K[x_1, \dots, x_n].$$

If G is a subset of $K[x_1, \dots, x_n]$ (not necessarily finite), then the subalgebra of $\mathcal{P}$ generated by G is $K[G]$. This notion is natural since the elements of $K[G]$ are precisely the polynomials in the set of formal variables G, viewed as elements of $K[G]$.

2.1. Definition. A G-monomial is a finite power product of the form $G^\alpha = g_1^{\alpha_1} \dots g_m^{\alpha_m}$ where $g_i \in G$ for $i = 1, \dots, m$, and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$.

Let Γ denote an ordered monoid, i.e., an abelian semigroup under an operation $+$, equipped with a total ordering $>$ such that, for all $\alpha, \beta, \gamma \in \Gamma$,

$$\alpha > \beta \Longrightarrow \alpha + \gamma > \beta + \gamma.$$

A direct sum

$$\mathcal{P} := \bigoplus_{\gamma \in \Gamma} \mathcal{P}_\gamma^{(\Gamma)}$$

is called grading (induced by Γ) or briefly a Γ-grading if for all $\alpha, \beta \in \Gamma$

$$f \in \mathcal{P}_\alpha^{(\Gamma)}, g \in \mathcal{P}_\beta^{(\Gamma)} \Longrightarrow f \cdot g \in \mathcal{P}_{\alpha+\beta}^{(\Gamma)}. \tag{2.1}$$

Since the decomposition above is a direct sum, each polynomial $f \neq 0$ has a unique representation

$$f = \sum_{i=1}^{s} f_{\gamma_i}, \quad 0 \neq f_{\gamma_i} \in \mathcal{P}_{\gamma_i}^{(\Gamma)}.$$

Assuming that $\gamma_1 > \gamma_2 > \ldots > \gamma_s$, the Γ-homogeneous term f_{γ_1} is called the maximal part of f, denoted by $M^{(\Gamma)}(f) := f_{\gamma_1}$, and $f - M^{(\Gamma)}(f)$ is called the d-reductum of f. For $G \subset \mathcal{P}$, $M^{(\Gamma)}(G) := \{\, M^{(\Gamma)}(g) \,|\, g \in G\,\}$.
There are two major examples of gradings. The first one is grading by degrees,

$$\mathcal{P}_d^{(\Gamma)} = \{p \in \mathcal{P} \,|\, p \text{ homogenous of degree } d\} \quad \forall\, d \in \mathbb{N}.$$

Here, $\Gamma = \mathbb{N}$ with the natural total ordering. This grading is called the H-grading because of the homogeneous polynomials. Therefore we also write H in place of this Γ. The space of all polynomials of degree at most d can now be written as

$$\mathcal{P}_d := \bigoplus_{k=0}^{d} \mathcal{P}_k^{(H)}.$$

The maximal part of a polynomial $f \neq 0$ is its homogeneous form of highest degree, $M^{(H)}(f)$. For simplicity, let $M^{(H)}(0) := 0$.

Now we introduce SH-bases and some of their properties. This concept is very similar to the concept of Sagbi bases. Therefore, we will briefly explain the underlying common structure.

2.2. Definition. A subset G of $\mathcal{P}$ is called SH- basis of the subalgebra $\mathcal{A}$ of $\mathcal{P}$ if, for all $0 \neq f \in \mathcal{A}$, there exist G-monomials G^{α_i} and $c_i \in K$, $i = 1, \ldots, p$ such that

$$f = \sum_{i=1}^{p} c_i G^{\alpha_i} \quad \text{and} \quad \max_{i=1}^{p}\{\deg(G^{\alpha_i})\} = \deg(f). \tag{2.2}$$

The representation for f in (2.2) is also called its SH-representation with respect to G.

Note that SH-basis of a subalgebra is also a generating set of it. To obtain more insight into SH-bases, we will give some equivalent definitions. First we need a more technical notion.

2.3. Definition. Let $f \in \mathcal{P}$ and $G \subset \mathcal{P}$. We say f d-reduces to $\widetilde{f}$ with respect to G if

$$\widetilde{f} = f - \sum_{i=1}^{m} c_i G^{\alpha_i}, \quad \deg(\widetilde{f}) < \deg(f),$$

holds with G-monomials G^{α_i} satisfying $\deg(G^{\alpha_1}) \leq \deg(f)$, $i = 1, \ldots, m$. In this case we write

$$f \rightarrow_G \widetilde{f}.$$

By $\rightarrow_{G,*}$ we denote the transitive closure of the binary relation $\rightarrow_G$[§].

The concept of d-reduction plays an important role in the characterization and construction of SH-basis. For $f \in P$ and $G \subset P$, the following algorithm computes h such that $f \rightarrow_{G,*} h$.

[§] $f \rightarrow_{G,*} h$ if we apply d-reduction iteratively such as $f \rightarrow_G h_1 \rightarrow_G h_2 \ldots \rightarrow_G h$, where h cannot be d-reduced any further with respect to G.

Algorithm 1

Input: Let G and f be subset and polynomial respectively in $\mathcal{P}$.
Output: $h \in \mathcal{P}$ such that $f \rightarrow_{G,*} h$.

1: $h := f$.
2: while ($h \neq 0$ and $G_h = \{\sum_i c_i G^{\alpha_i} \mid M^{(H)}(\sum_i c_i G^{\alpha_i}) = M^{(H)}(h)\} \neq \emptyset$)
3: (a) choose $\sum_i c_i G^{\alpha_i} \in G_h$.
4: (b) $h := h - \sum_i c_i G^{\alpha_i}$ and continue at 2.

We note that when step $2(b)$, has been performed, then $\deg(h)$ is strictly smaller than the $\deg(h - \sum_i c_i G^{\alpha_i})$ (by the choice of $\sum_i c_i G^{\alpha_i}$). This shows that the Algorithm 1 always terminate.

2.4. Theorem. *Let $G \subset \mathcal{P}$ and $\mathcal{A}$ be a subalgebra of $\mathcal{P}$. Then the following conditions are equivalent:*
i) G is an SH-basis of $\mathcal{A}$.
ii) $K[\{M^{(H)}(g) \mid g \in G\}] = K[\{M^{(H)}(f) \mid f \in \mathcal{A}\}]$.
iii) For all $f \in \mathcal{A}$, $f \rightarrow_{G,*} 0$.

Proof. $(i) \Rightarrow (ii)$ follows by

$$M^{(H)}(f) = \sum_{j \in J} c_j M^{(H)}(G^{\alpha_j}), \quad J := \{j \mid \deg(G^{\alpha_j}) = \deg(f)\}$$

for arbitrary $f \in \mathcal{A}$ with SH-representation $f = \sum c_i G^{\alpha_i}$.
$(ii) \Rightarrow (iii)$ If $0 \neq f \in \mathcal{A}$, then $M^{(H)}(f) = \sum_{j \in J} c_j M^{(H)}(G^{\alpha_j})$. Therefore, $\widetilde{f} = f - \sum_{j \in J} c_j G^{\alpha_j}$, where $\widetilde{f} \in \mathcal{A}$ and $\deg(\widetilde{f}) \leq \deg(f)$. Inductively, we get $f \rightarrow_{G,*} 0$.
$(iii) \Rightarrow (i)$ Let

$$g_0 = f \rightarrow_G g_1 \rightarrow_G \ldots \rightarrow_G g_d = 0$$

where $M^{(H)}(g_{i-1}) = M^{(H)}(G^{\alpha_i})$ and $\deg(G^{\alpha_{i+1}}) < \deg(G^{\alpha_i})$, $i = 1, \ldots, d$. Then

$$f = \sum_{i=1}^{d} c_i G^{\alpha_i} \quad \text{and} \quad \deg(f) = \deg(G^{\alpha_1}) = \max_{i=1}^{d}\{\deg(G^{\alpha_i})\}$$

i.e., f has an SH-representation with respect to G. ☐

The second major example of gradings leads to the Sagbi basis concept. Here, $\Gamma = \mathbb{N}^n$ with componentwise addition and equipped with a total ordering satisfying (2.1) and, in addition, $\gamma \geq 0 \; \forall \gamma \in \Gamma$. For arbitrary $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma$, the space $\mathcal{P}_\gamma^{(\Gamma)}$ is a vector space of dimension 1, namely,

$$\mathcal{P}_\gamma^{(\Gamma)} = \{\, c \cdot x^{\gamma_1} \ldots x^{\gamma_n} \mid c \in K \,\}.$$

The maximal part of a polynomial is now a product of a leading coefficient and a leading monomial, $M^{(\Gamma)}(f) = LC(f) \cdot LM(f)$, $LC(f) \in K, LM(f)$ a leading monomial. The s-reduction $f \rightarrow_G \widetilde{f}$ is defined if there exists a G-monomial G^α such that $LM(G^\alpha) = LM(f)$ and then we set $\widetilde{f} := f - cG^\alpha$, $c \in K$. The relation $\rightarrow_{G,*}$ is constructed as above.

A Sagbi basis G (with respect to a given monomial ordering and a given subalgebra $\mathcal{A}$) is a set of polynomials, generating the subalgebra $\mathcal{A}$ and satisfying one of the following equivalent conditions:
(i) Every $f \in \mathcal{A}$ has a representation $f = \sum_{i=1}^{s} c_i G^{\alpha_i}$, $LM(f) = \max_{i=1}^{s}\{LM(G^{\alpha_i})\}$.

(ii) $K[\{M^{(\Gamma)}(g) \mid g \in G\}] = K[\{M^{(\Gamma)}(f) \mid f \in \mathcal{A}\}]$.
(iii) Every $f \in \mathcal{A}$ s-reduces to 0 with respect to G.

The proof of this equivalence and many other equivalent conditions can be found in [11]. If a monomial ordering is compatible with the semi-ordering by degrees,

$$\deg(x^\gamma) > \deg(x^\beta) \Longrightarrow \gamma > \beta, \quad \gamma, \beta \in \mathbb{N}^n$$

then any Sagbi-representation as given in (i) is an SH-representation, in other words, a Sagbi basis with respect to a degree compatible ordering is an SH-basis as well. The converse is false, as the following example shows.

2.5. *Example.* Let $f_1 = x^3 + x^2y$, $f_2 = y^3$, $f_3 = xy + y$ and $\mathcal{A} = K[f_1, f_2, f_3]$. Then f_1, f_2 and f_3 already constitute an SH-basis of $\mathcal{A}$. (This is consequence of Theorem 2.4). If we order the monomials by degree lexicographical ordering then

$$K[\{M^{(H)}(f) \mid f \in \mathcal{A}\}] = K[x^3, y^3, xy, x^2y^4].$$

Every Sagbi basis G with respect to this ordering contains at least four elements, for instance SINGULAR ([5]) computes $G = \{g_1, g_2, g_2, g_4\}$ with

$$g_1 = x^3 + x^2y = f_1$$

$$g_2 = y^3 = f_2$$

$$g_3 = xy + y = f_3$$

$$g_4 = x^2y^4 - 3x^2y^3 - 3xy^3$$

Obviously, this Sagbi basis is an SH-basis as well.

It is possible that a subalgebra has a finite SH-basis, but no finite Sagbi basis, as the following example shows.

2.6. *Example.* Let $G = \{f_1, f_2, f_3\} \subset K[x, y]$ where $f_1 = x + y$, $f_2 = xy$, $f_3 = xy^2$ and $\mathcal{A} = K[G]$. It is easy to see that G is an SH-basis of $\mathcal{A}$. However, the set $S = \{x + y, xy, xy^2, xy^3, xy^4, \ldots\} \subset \mathcal{A}$ is an infinite Sagbi basis for $\mathcal{A}$ with respect to a monomial ordering $x > y$. (see [11]).

3. Construction of SH-bases

In this section, we present an SH-basis criterion, through which we can construct SH-basis. For this purpose, we fix some notations which are necessary for this construction.

3.1. Definition. Let G be a set of polynomials in $\mathcal{P}$ and let $\mathcal{A} = K[G]$. We consider $f \in \mathcal{A}$ with the representation $f = \sum_{i=1}^{m} c_i G^{\alpha_i}$. Then the degree-height of f, written d-ht(f), with respect to this representation is $\max_{i=1}^{m}\{\deg(G^{\alpha_i})\}$.

Let $Y = \{y_1, \ldots, y_s\}$ and $K[Y]$ be a polynomial ring over a field K in variables $y_1, \ldots, y_s$. Let $P(Y) = P(y_1, \ldots, y_s) \in K[Y]$ and Y^α be a Y-monomial.

3.2. Definition. Let $G \subseteq \mathcal{P}$. A polynomial $P(Y) = \sum_{i=1}^{m} c_i Y^{\alpha_i} \in K[Y]$ (where $c_i \in K$) is called G-homogenous if $\deg(G^{\alpha_i})$ are same for $1 \le i \le m$.

3.3. Definition. Let $G = \{g_1, \ldots, g_s\}$ be a subset of $K[x_1, \ldots, x_n]$. We denote $AR((M^{(H)}(G))$, the ideal of algebraic relations between $M^{(H)}(g_i), i = 1, \ldots, s$ defined by:

$$AR((M^{(H)}(G)) = \{h \in K[y_1, \ldots, y_s] \mid h(M^{(H)}(g_1)), \ldots, M^{(H)}(g_s)) = 0\}$$

$AR((M^{(H)}(G))$ is an ideal in $K[y_1, \ldots, y_s]$.

In the case of Sagbi bases, there is as an algorithm for computing Sagbi bases by means of algebraic relations (see [8]) where algebraic relations and their connection to Sagbi bases are studied in detail.) The analogue for constructing SH-bases by means of algebraic relations is based on the following result.

3.4. Theorem. ***(SH-basis criterion)*** *Let $G = \{g_1, \dots, g_s\}$ be a subset of $K[x_1, \dots, x_n]$. Let $\mathcal{A} = K[G]$ and let $\{P_j(Y) \mid j \in J\}$ be a finite set of G-homogenous generators for $AR((M^{(H)}(G))$. Then the following conditions are equivalent:*
i) G is an SH-basis of $\mathcal{A}$.
ii) For each $j \in J$, $P_j(G) = P_j(g_1, \dots, g_s) \to_{G,*} 0$.

Proof. $(i) \Rightarrow (ii)$: This is trivial and follows from Theorem 2.4.
$(ii) \Rightarrow (i)$: For every $h \in K[G]$, we will show that

$$h = \sum_{i=1}^{m} c_i G^{\alpha_i} \text{ and } \deg(h) = \max_{i=1}^{m}\{\deg(G^{\alpha_i})\}.$$

Let $h \in K[G]$ and write $h = \sum_{i=1}^{m} c_i G^{\alpha_i}$; furthermore, assume that this representation has the smallest possible degree-height $t_0 = \max_{i=1}^{m}\{\deg(G^{\alpha_i})\}$ of all such representation. We know that $\deg(h) \le t_0$. Suppose that $\deg(h) < t_0$, without loss of generality, let the first N summands be the ones for which $\deg(M^{(H)}(G^{\alpha_i})) = t_0$. Then the cancelation of their maximal part must occur; i.e $\sum_{i=1}^{N} c_i M^{(H)}(G^{\alpha_i}) = 0$. From this condition, we obtain a polynomial $P(Y) = \sum_{i=1}^{N} c_i Y^{\alpha_i} \in AR((M^{(H)}(G))$. We can then write

$$\sum_{i=1}^{N} c_i Y^{\alpha_i} = P(Y) = \sum_{j=1}^{M} g_j(Y) P_j(Y) \tag{3.1}$$

where the polynomials $P_j(Y)$ are among the stated generators of $AR((M^{(H)}(G))$ and the polynomials $g_j(Y) \in K[y_1, \dots, y_s]$. Moreover, we may assume that each $g_j(Y)$ is G-homogenous (since $P(Y)$ and every $P_j(Y)$ are) and also that

$$\text{d-ht}(g_j(G)) + \text{d-ht}(P_j(G)) = \text{d-ht}(P(G)) = t_0 \quad \forall j. \tag{3.2}$$

We have assumed that each $P_j(G) \to_{G,*} 0$; therefore we have $P_j(G) = \sum_{k=1}^{n_j} c_{k_j} G^{\alpha_{k_j}}$ where $c_{k_j} \in K$. By definition, these sums must have degree heights $\max_k\{\deg(G^{\alpha_{k_j}})\} = \deg(P_j(G)) < \text{d-ht}(P_j(G))$ for each j, where the last inequality holds because $P_j(Y) \in AR((M^{(H)}(G))$. Then for each j, $1 \le j \le M$,

$$g_j(G) P_j(G) = \sum_{k=1}^{n_j} c_{kj} g_j(G) G^{\alpha_{kj}} \tag{3.3}$$

Note that

$$\deg(g_j(G)) P_j(G)) = \deg(g_j(G)) + \deg(P_j(G)) < \deg(g_j(G)) + \text{d-ht}(P_j(G)). \tag{3.4}$$

From our observation and using equation (3.2), we have

$$\deg(g_j(G)) + \text{d-ht}(P_j(G)) \le \text{d-ht}(g_j(G)) + \text{d-ht}(P_j(G)) = t_0 \tag{3.5}$$

Combining equations (3.4) and (3.5) we have

$$\deg(g_j(G) P_j(G)) < t_0 \tag{3.6}$$

Finally, equations (3.1) and (3.3) imply that

$$h = P(G) + \sum_{i=N+1}^{m} c_i G^{\alpha_i} = \underbrace{\sum_{j=1}^{M} (\sum_{k=1}^{n_j} c_{kj} g_j(G) G^{\alpha_{kj}})}_{} + \underbrace{\sum_{i=N+1}^{m} c_i G^{\alpha_i}}_{}. \tag{3.7}$$

Sum_1 Sum_2

If we examine the expression (3.7) closely, we see that:

- By (3.6), d-ht$(Sum_1) = \max_{j=1}^{M}\{\deg(g_j(G)Pj(G))\} < t_0$;
- By the choice of N, d-ht$(Sum_2) < t_0$;

But this contradicts our initial assumption that we have chosen a representation of h that has the smallest possible height. Thus, G is an SH-basis of $K[G]$. □

On the basis of Theorem 3.4, now we present an algorithm which computes SH-basis from a given set of generators. This algorithm is not necessarily terminating but does terminate, if and only if, the considered subalgebra has a finite SH-basis.

Algorithm 2

Input: A finite subset $G \subset \mathcal{P}$.
Output: SH-basis G.

1: Compute a generating set $\mathcal{S}$ for $AR(M^{(H)}(G))$.
2: For $P \in \mathcal{S}$
3: (a) $h \in \mathcal{P}$, such that $P(G) \to_{G,*} h$.
4: (b) If $h \neq 0$, set $G := G \cup \{h\}$ and continue at 1.

3.5. *Remark.* We have implemented SH-basis construction algorithm in the computer algebra system SINGULAR [5]. Code can be download from mathcity.org/junaid.

Now we present some examples which show the computation of SH-basis through Algorithm 2.

3.6. *Example.* The subalgebra $\mathcal{A} \subset \mathcal{P}$ of symmetric polynomials is well known to be finitely generated by a set S which is a set of elementary symmetric polynomials in $\mathcal{P}$. The set S is an SH-basis of $\mathcal{A}$ as $AR(M^{(H)}(S)) = \{0\}$ i.e, there is no polynomial $0 \neq P(Y) \in K[y_1, \ldots, y_n]$ such that $P(S) = 0$.

3.7. *Example.* Let $G = \{x + y + 1, x^2 + y^2 - x + 2, 2xy - y\}$ and $\mathcal{A} = \mathbb{Q}[G]$. The ideal $AR((M^{(H)}(G)) = AR(x + y, x^2 + y^2, xy)$ in $\mathbb{Q}[y_1, y_2, y_3]$ is generated by $P(Y) = y_1^2 - y_2 - y_3$. It is easy to see that the polynomial $P(G) = 3x + 3y - 1 \to_{G.*} 0$. This shows that G is an SH-basis of $\mathcal{A}$.

The next example shows that there are finitely generated algebras which do not admit a finite SH-basis.

3.8. *Example.* Let $G = \{g_1 = xz + y, g_2 = xyz, g_3 = xy^2z\}$ and $\mathcal{A} = \mathbb{Q}[G]$. Also we have $M^{(H)}(g_1) = xz, M^{(H)}(g_2) = xyz$ and $M^{(H)}(g_3) = xy^2z$.

In first step, $G = \{g_1 = xz + y, g_2 = xyz, g_3 = xy^2z\}$. It is evident that the ideal of relations $AR(M^{(H)}(G)) = AR(xz, xyz, xy^2z) \subset \mathbb{Q}[y_1, y_2, y_3]$ is generated by $P(Y) = y_1y_3 - y_2^2$. The polynomial $P(G) = (xz + y)(xy^2z) - (xyz)^2 = xy^3z \not\to_{G,*} 0$, so $G := G \cup \{g_4 = xy^3z\}$.

In second step, $G = \{g_1 = xz + y, g_2 = xyz, g_3 = xy^2z, g_4 = xy^3z\}$. The polynomial $P(Y) = y_1y_4 - y_2y_3$ is one the generators of the ideal of relations $AR(M^{(H)}(G)) = AR(xz, xyz, xy^2z, xy^3z) \subset \mathbb{Q}[y_1, y_2, y_3, y_4]$. Here we note that the polynomial $P(G) = (xz + y)(xy^3z) - (xyz)(xy^3z) = xy^4z \not\to_{G,*} 0$, therefore we have $G := G \cup \{xy^4z\} = \{g_1 = xz + y, g_2 = xyz, g_3 = xy^2z, g_4 = xy^3z, g_5 = xy^4z\}$.

By induction, we get $G = \{xz + y, xyz, xy^2z, xy^3z, xy^4z, xy^5z, \ldots\}$ which implies that $\mathcal{A}$ have an infinite SH-basis.

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The existence and location of eigenvalues of the one particle Hamiltonians on lattices

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Abstract

We consider a quantum particle moving in the one dimensional lattice $\mathbb{Z}$ and interacting with a indefinite sign external field $\hat{v}$. We prove that the associated Hamiltonian H can have one or two eigenvalues, situated as below the bottom of the essential spectrum, as well as above the its top. Moreover, we show that the operator H can have two eigenvalues outside of the essential spectrum and one of them is situated below the bottom of the essential spectrum, and other one above its top.

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1. Introduction

We consider the Hamiltonian H of a quantum particle moving in the one-dimensional lattice $\mathbb{Z}$ and interacting with a *indefinite sign* external field $\hat{v}$, i.e., the potential has positive and negative values.

In [9] of B.Simon the existence of eigenvalues of a family of continuous Schrödinger operators $H = -\Delta + \lambda V$, $\lambda > 0$ in one and two-dimensional cases have been considered. The result that H has bound state for all $\lambda > 0$ if only if $\int V(x)dx < 0$ is proven there for all $V(x)$ with $\int(1 + |x|^2)|V(x)|dx < +\infty$.

In [3] it is presented that under certain conditions on the potential a one-dimensional Schrödinger operator has a unique bound state in the limit of weak coupling while under other conditions no bound state in this limit. This question is studied for potentials obeying $\int(1 + |x|)|V(x)|dx < +\infty$.

The questions further discussed in R. Blankenbecker M.N. Goldberger and B.Simon [1].

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All these results require the use of the modified determinant. Throughout physics, stable composite objects are usually formed by the way of attractive forces, which allow the constituents to lower their energy by binding together. Repulsive forces separate particles in free space. However, in structured environment such as a periodic potential and in the absence of dissipation, stable composite objects can exist even for repulsive interactions [10].

The Bose-Hubbard model, which have been used to describe the repulsive pairs, is the theoretical basis for explanation of the experimental results obtained in [10].

Since the continuous Schrödinger operator has essential spectrum fulfilling semi-axis $[0,+\infty)$ and its eigenvalues appear below the bottom of the essential spectrum, it is a model, which well described the systems of two-particles with the attractive interaction.

Zero-range potentials are the mathematically correct tools for describing contact interactions. The latter reflects the fact that the zero-range potential is effective only in the s-wave [11].

The existence of eigenvalues of a family of Schrödinger operators $H=-\Delta-\mu V$, $\lambda>0$ with perturbation V of rank one in one and two-dimensional lattices have been considered in [7]. The result that H has a unique bound state for all $\mu>0$ is proven there and for the unique eigenvalue $e(\mu)$ lying below the bottom of the essential spectrum an asymptotic is found as $\mu\to 0$.

In [2] for the Hamiltonian H of two fermions with attractive interaction on a neighboring sites in the one-dimensional lattice $\mathbb{Z}$ has been considered and an asymptotics of the unique eigenvalue lying below the bottom of its essential spectrum has been proven.

For a family of the generalized Friedrichs models $H_\mu(p)$, $\mu>0, p\in\ T^2$ with the perturbation of rank one, associated to a system of two particles moving on the two-dimensional lattice $\mathbb{Z}$ has been considered in [6] and the existence or absence of a positive coupling constant threshold $\mu=\mu_0(p)>0$ depending on the parameters of the model has been proved.

In[5]a family $H_\mu(p)$, $\mu>0$, $p\in\ T$ of the generalized Friedrichs models with the perturbation of rank one, associated to a system of two particles, moving on the one-dimensional lattice $\mathbb{Z}$ is considered. The existence of a unique eigenvalue $E(\mu,p)$, of the operator $H_\mu(p)$ lying below the essential spectrum is proved. For any p from a neighborhood of the origin, the Puiseux series expansion for eigenvalue $E(\mu,p)$ at the point $\mu=\mu(p)\geq 0$ is found.

The main goal of this paper is to investigate the existence and location of eigenvalues of the one-particle Hamiltonian H with the zero-range interaction $\mu\neq 0$ and with interactions $\lambda\neq 0$ on a neighboring sites. We prove that the Hamiltonian H may have one or two eigenvalues, situating as below the bottom of the essential spectrum, as well as above its top. Moreover, the operator H can have two eigenvalues outside of the essential spectrum, where one of them is situated below the bottom of the essential spectrum and other one above its top.

This results are new and in accord with the known results of [9, 3, 1, 7, 6, 5].

2. The coordinate representation of the one particle Hamiltonian

Let $\mathbb{Z}$ be the one dimensional lattice(integer numbers) and $\ell^2(\mathbb{Z})$ be the Hilbert space of square summable functions on $\mathbb{Z}$ and $\ell^{2,e}(\mathbb{Z})\subset\ell^2(\mathbb{Z})$ be the subspace of functions(elements) $\hat f\in\ell^2(\mathbb{Z})$ satisfying the condition

$$\hat f(x)=\hat f(-x), x\in\ Z$$

The one particle operator $\hat{H}_{\mu\lambda}$ acting on $\ell^{2,e}(\mathbb{Z})$ is of the form

(2.1) $\hat{H}_{\mu\lambda} := \hat{H}_0 + \hat{V}_{\mu\lambda},$

where $\hat{H}_0$ is the Teoplitz type operator

(2.2) $(\hat{H}_0\hat{\varphi})(x) := \sum\limits_{s\in Z} \hat{\varepsilon}(s)\hat{\varphi}(x+s),\ \hat{\varphi} \in \ell^{2,e}(\mathbb{Z}),$

and

(2.3) $(\hat{V}_{\mu\lambda}\hat{\varphi})(x) := \hat{v}_{\mu\lambda}(x)\hat{\varphi}(x), \quad \hat{\varphi} \in \ell^{2,s}(\mathbb{Z}).$

The functions $\hat{\varepsilon}(s)$ and $\hat{v}_{\mu\lambda}(s)$ are defined on $\mathbb{Z}$ as follows

$$\hat{\varepsilon}(s) = \begin{cases} 1, & |s| = 0 \\ -\frac{1}{2}, & |s| = 1 \\ 0, & |s| > 1, \end{cases}$$

and

$$\hat{v}_{\mu\lambda}(s) = \begin{cases} \mu, & |s| = 0 \\ \frac{\lambda}{2}, & |s| = 1 \\ 0, & |s| > 1, \end{cases}$$

where μ, $\lambda \in R$ are real numbers.

We remark that $\hat{H}_{\mu\lambda}$ is a bounded self-adjoint operator on $\ell^{2,e}(\mathbb{Z})$.

3. The momentum representation of the Hamiltonian

Let $\mathbb{T} = (-\pi;\pi]$ be the one dimensional torus and $L^2(\mathbb{T}, d\nu)$ be the Hilbert space of integrable functions on $\mathbb{T}$, where $d\nu$ is the (normalized) Haar measure on $\mathbb{T}$, $d\nu(p) = \frac{dp}{2\pi}$.

Let $L^{2,e}(\mathbb{T}, d\nu) \subset L^2(\mathbb{T}, d\nu)$

be the subspace of elements $f \in L^2(\mathbb{T}, d\nu)$ satisfying the condition

$$f(p) = f(-p), \quad \text{a.e. } p \in T.$$

In the momentum representation the operator $H_{\mu\lambda}$ acts on $L^{2,e}(\mathbb{T}, d\nu)$ and is of the form

$$H_{\mu\lambda} = H_0 + V_{\mu\lambda},$$

where H_0 is the multiplication operator by function $\varepsilon(p) = 1 - \cos p$:

$$(H_0 f)(p) = \varepsilon(p) f(p), \quad f \in L^{2,e}(\mathbb{T}, d\nu),$$

and $V_{\mu\lambda}$ is the integral operator of rank 2

$$(V_{\mu\lambda} f)(p) = \int\limits_{\mathbb{T}} \Big(\mu + \lambda \cos p \cos t\Big) f(t) dt, \quad f \in L^{2,e}(\mathbb{T}, d\nu).$$

4. Spectral properties of the operators $H_{\mu 0}$ and $H_{0\lambda}$

Since the perturbation operator $V_{\mu 0}$ resp.$V_{0\lambda}$ is of rank 1, according the well known Weyl's theorem the essential spectrum $\sigma_{ess}(H_{\mu 0})$ resp.$\sigma_{ess}(H_{0\lambda})$ of $H_{\mu 0}$ resp. $H_{0\lambda}$ doesn't depend on $\mu \in R$ resp.$\lambda \in R$ and coincides to the spectrum $\sigma(H_0)$ of H_0 (see [8]), i.e.,

$$\sigma_{ess}(H_{\mu 0}) = \sigma_{ess}(H_{0\lambda}) = \sigma(H_0) = [\min_{p\in\mathbb{T}} \varepsilon(p), \max_{p\in\mathbb{T}} \varepsilon(p)] = [0, 2].$$

For any $\mu, \lambda \in R$ we introduce the Fredholm determinant $\Delta(\mu, \lambda; z)$, associating to the one particle Hamiltonian $H_{\mu,\lambda}$, as follows

(4.1) $\Delta(\mu, \lambda; z) = \big[1 - \mu a(z)\big]\big[1 - \lambda c(z)\big] - \mu\lambda b^2(z),$

where

$$a(z) := \int_{\mathbb{T}} \frac{d\nu}{z - \varepsilon(q)}, \tag{4.2}$$

$$b(z) := -\int_{\mathbb{T}} \frac{\cos q d\nu}{z - \varepsilon(q)}, \tag{4.3}$$

$$c(z) := \int_{\mathbb{T}} \frac{\cos^2 q d\nu}{z - \varepsilon(q)}. \tag{4.4}$$

are regular functions in $z \in \mathrm{C} \setminus [0, 2]$.

In the following theorem we have collected results on a unique eigenvalue of the operator $H_{\mu 0}$ resp. $H_{0\lambda}$ depending on the sign of $\mu \neq 0$ resp. $\lambda \neq 0$.

4.1. Theorem. *For any $0 \neq \mu \in \ R$ resp. $0 \neq \lambda \in \ R$ the operator $H_{\mu 0}$ resp. $H_{0\lambda}$ has a unique eigenvalue $\zeta(\mu)$ resp. $\zeta(\lambda)$ lying outside of the essential spectrum:*

(i) *If $\mu > 0$ resp. $\lambda > 0$, then the eigenvalue $\zeta(\mu)$ resp. $\zeta(\lambda)$ is lying in the interval $(2, +\infty)$.*
(ii) *If $\mu < 0$ resp.$\lambda < 0$, then the eigenvalue $\zeta(\mu)$ resp. $\zeta(\lambda)$ is lying in the interval $(-\infty, 0)$.*
(iii) *If $\mu > 0$ resp.$\lambda < 0$ then the eigenvalue $\zeta(\mu)$ resp. $\zeta(\lambda)$ is lying in the interval $(2, +\infty)$ resp. $(-\infty, 0)$.*
(iv) *If $\mu < 0$ resp.$\lambda > 0$ then the eigenvalue $\zeta(\mu)$ resp. $\zeta(\lambda)$ is lying in the interval $(-\infty, 0)$ resp. $(2, +\infty)$.*

The proof of Theorem 4.1 is a consequence of the formulated below Lemmas and corollaries, which can be deduced from the simple properties of determinant $\Delta(\mu, 0; z)$ resp.$\Delta(0, \mu; z)$.

4.2. Lemma. *The number $z \in \mathrm{C} \setminus [0, 2]$ is an eigenvalue of the operator $H_{\mu,0}$ resp. $H_{0,\lambda}$ if and only if $\Delta(\mu, 0; z) = 0$ resp. $\Delta(0, \lambda; z) = 0$.*

4.3. Lemma. *Let $\mu, \lambda \in \ R$.Then*

$$\lim_{z \to \pm\infty} \Delta(\mu, 0\,; z) = 1,$$
$$\lim_{z \to \pm\infty} \Delta(0, \lambda\,; z) = 1,$$
$$\lim_{z \to \pm\infty} \Delta(\mu, \lambda\,; z) = 1.$$

4.4. Lemma. *The functions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ are regular in the region $\mathrm{C} \setminus [0, 2]$, positive and monotone decreasing in the intervals $(-\infty, 0)$ and $(2, +\infty)$ and the following asymptotics are true:*

$$a(z) = C_1(z-2)^{-\frac{1}{2}} + O(z-2)^{\frac{1}{2}}, \textit{as } z \to 2+,$$
$$b(z) = C_1(z-2)^{-\frac{1}{2}} + 1 + O(z-2)^{\frac{1}{2}}, \textit{as } z \to 2+,$$
$$c(z) = C_1(z-2)^{-\frac{1}{2}} - 1 + O(z-2)^{\frac{1}{2}}, \textit{as } z \to 2+,$$

where $C_1 > 0$ and

$$a(z) = -C_0(-z)^{-\frac{1}{2}} + O(-z)^{\frac{1}{2}}, \textit{as } z \to 0-,$$
$$b(z) = -C_0(-z)^{-\frac{1}{2}} - 1 + O(-z)^{\frac{1}{2}}, \textit{as } z \to 0-,$$
$$c(z) = -C_0(-z)^{-\frac{1}{2}} - 1 + O(-z)^{\frac{1}{2}}, \textit{as } z \to 0-,$$

where $C_0 > 0$.

Proof. Since the functions under integral sign are positive the monotonicity of the Lebesgue integral gives that the functions $a(z)$ and $c(z)$ are positive. Now, we show that the function

$$b(z) := -\int\limits_{\mathbb{T}} \frac{\cos q d\nu}{z - \varepsilon(q)}$$

is positive. Representing $b(z)$ as

$$b(z) = -\int_{-\pi}^{0} \frac{\cos q d\nu}{z - \varepsilon(q)} - \int_{0}^{\pi} \frac{\cos q d\nu}{z - \varepsilon(q)}$$

and then changing of variables $q := q + \pi$ we have that

$$b(z) := \int\limits_{0}^{\pi} \frac{2\cos^2 q d\nu}{(z-1)^2 - \cos^2 q} > 0$$

The asymptotics of functions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ can be found in [2]. □

The Lemma 4.4 yields the following Corollary, which gives asymptotics for the functions $\Delta(\mu, 0; z)$ and $\Delta(0, \lambda; z)$.

4.5. Corollary. *The following asymptotics are true:*

(i) *If* $\mu,\ \lambda > 0$. *Then*

$$\lim_{z\to 2+} \Delta(\mu, 0; z) = -\infty,$$
$$\lim_{z\to 2+} \Delta(0, \lambda; z) = -\infty,$$

(ii) *If* $\mu,\ \lambda < 0$. *Then*

$$\lim_{z\to 2+} \Delta(\mu, 0; z) = +\infty,$$
$$\lim_{z\to 2+} \Delta(0, \lambda; z) = +\infty,$$

(iii) *If* $\mu, \lambda > 0$. *Then*

$$\lim_{z\to 0-} \Delta(\mu, 0; z) = +\infty,$$
$$\lim_{z\to 0-} \Delta(0, \lambda; z) = +\infty,$$

(iv) *If* $\mu, \lambda < 0$. *Then*

$$\lim_{z\to 0-} \Delta(\mu, 0; z) = -\infty,$$
$$\lim_{z\to 0-} \Delta(0, \lambda; z) = -\infty,$$

5. Spectral properties of the operator $H_{\mu\lambda}$

The perturbation operator $V_{\mu\lambda}$ is of rank 2 and hence by the well known Weyl's theorem the essential spectrum $\sigma_{ess}(H_{\mu\lambda})$ of $H_{\mu\lambda}$ doesn't depend on $\mu, \lambda \in R$ and coincides to the spectrum $\sigma(H_0)$ of H_0 (see [8]), i.e.,

$$\sigma_{ess}(H_{\mu\lambda}) = \sigma(H_0) = [\min_{p\in\mathbb{T}} \varepsilon(p),\ \max_{p\in\mathbb{T}} \varepsilon(p)] = [0, 2].$$

5.1. Remark. Note that since

$$(V_{\mu\lambda}f,f)=\mu|\int\limits_{\mathbb{T}} f(t)d\nu|^2+\lambda|\int\limits_{\mathbb{T}}\cos t f(t)d\nu|^2,\quad f\in L^{2,e}(\mathbb{T},d\nu)),$$

the operator $V_{\mu\lambda}$ is not only positive or only negative and hence the operator $H_{\mu\lambda}$ may have eigenvalues as below the bottom of the essential spectrum, as well as above the its top.

The following lemma describes the relations between the operator $H_{\mu,\lambda}$ and determinant $\Delta(\mu,\lambda;z)$ defined in (4.1).

5.2. Lemma. *The number $z\in \mathrm{C}\setminus[0,2]$ is an eigenvalue of the operator $H_{\mu,\lambda}$ if and only if $\Delta(\mu,\lambda;z)=0$.*

Proof. Let the operator $H_{\mu,\lambda}$ has an eigenvalue $z\in \mathrm{C}\setminus[0,2]$, i.e., the equation

$$(z-H_{\mu,\lambda})\psi(q)=(z-\varepsilon(q))\psi(q)-\mu\int\limits_T\psi(t)d\nu(t)-\lambda\cos p\int\limits_T\cos t\psi(t)d\nu(t)=0 \tag{5.1}$$

has a non-zero solution $\psi\in L^{2,e}(\,T,d\nu)$. We introduce the following linear continuous functionals defined on the Hilbert space $\psi\in L^{2,e}(\,T,d\nu)$

$$c_1:=c_1(\psi):=\int\limits_T\psi(t)d\nu(t) \tag{5.2}$$

$$c_2:=c_2(\psi):=\int\limits_T\cos(t)\psi(t) \tag{5.3}$$

Then we easily find that the solution of the equation (5.1) has form

$$\psi(q)=\mu\frac{c_1}{z-\varepsilon(q)}+\lambda\frac{c_2\cos(q)}{z-\varepsilon(q)}. \tag{5.4}$$

Putting the expression (5.6) for ψ to (5.2) and (4.7) we get the following homogeneous system of linear equations with respect to the functionals c_1 and c_2

$$\begin{cases} c_1=\mu c_1\int\limits_T\frac{d\nu}{z-\varepsilon(q)}+\lambda c_2\int\limits_T\frac{cos(q)d\nu}{z-\varepsilon(q)}\\ c_2=\mu c_1\int\limits_T\frac{\cos q d\nu}{z-\varepsilon(q)}+\lambda c_2\int\limits_T\frac{\cos^2 q d\nu}{z-\varepsilon(q)}\end{cases} \tag{5.5}$$

Hence, we can conclude that this homogenous system of linear equations has nontrivial solutions if and only if the associated determinant $\Delta(\mu,\lambda;z)$ has zero $z\in\mathbf{C}\setminus[0,2]$.

On the contrary, let a number $z\in\mathbf{C}\setminus[0,2]$ be a zero of determinant $\Delta(\mu,\lambda;z)$. Then it easily can be checked that z is eigenvalue of $H_{\mu,\lambda}$ and the function

$$\psi(q)=\mu\frac{c_1}{z-\varepsilon(q)}+\lambda\frac{c_2\cos q}{z-\varepsilon(q)}, \tag{5.6}$$

is the associated eigenfunction, where the vector (c_1,c_2) is a non-zero solution of the system (5.5). □

The following asymptotics for the determinant $\Delta(\mu,\lambda,z)$ can be received applying the asymptotics of the functions $a(\cdot),\ b(\cdot),\ c(\cdot)$ in Lemma 4.4.

5.3. Lemma.

(5.7) $\Delta(\mu,\lambda,z) = C^{+}_{-\frac{1}{2}}(\mu,\lambda)(z-2)^{-\frac{1}{2}} + C^{+}_{0}(\mu,\lambda) + O(z-2)^{\frac{1}{2}}$, *as* $z \to 2+$,

(5.8) $\Delta(\mu,\lambda,z) = C^{-}_{-\frac{1}{2}}(\mu,\lambda)(-z)^{-\frac{1}{2}} + C^{-}_{0}(\mu,\lambda) + O(-z)^{\frac{1}{2}}$, *as* $z \to 0-$,

where

(5.9) $C^{+}_{-\frac{1}{2}}(\mu,\lambda) = B_2(\mu\lambda - \mu - \lambda),\ B_2 > 0$

(5.10) $C^{+}_{0}(\mu,\lambda) = 1 + \lambda - \mu\lambda,$

(5.11) $C^{-}_{-\frac{1}{2}}(\mu,\lambda) = B_0(\mu\lambda + \mu + \lambda),\ B_0 > 0$

(5.12) $C^{-}_{0}(\mu,\lambda) = 1 - \lambda - \mu\lambda.$

The Lemma 5.3 yields the following results for the determinant $\Delta(\mu,\lambda;z)$.

5.4. Corollary. *For the determinant* $\Delta(\mu,\lambda;z)$ *the following results are true:*

(i) *Assume* $C^{+}_{-\frac{1}{2}}(\mu,\lambda) > 0$ *and* $C^{-}_{-\frac{1}{2}}(\mu,\lambda) > 0$. *Then*

$\lim_{z\to 2+} \Delta(\mu,\lambda;z) = +\infty.$

$\lim_{z\to 0-} \Delta(\mu,\lambda;z) = +\infty.$

(ii) *Assume* $C^{+}_{-\frac{1}{2}}(\mu,\lambda) = 0,\ \mu > 1$ *and* $C^{-}_{-\frac{1}{2}}(\mu,\lambda) = 0,\ \mu < -1$. *Then*

$\lim_{z\to 2+} \Delta(\mu,\lambda;z) < 0,$

$\lim_{z\to 0-} \Delta(\mu,\lambda;z) < 0.$

(iii) *Assume* $C^{+}_{-\frac{1}{2}}(\mu,\lambda) < 0$ *and* $C^{-}_{-\frac{1}{2}}(\mu,\lambda) < 0$.*Then*

$\lim_{z\to 2+} \Delta(\mu,\lambda;z) = -\infty,$

$\lim_{z\to 0-} \Delta(\mu,\lambda;z) = -\infty.$

(iv) *Assume* $C^{+}_{-\frac{1}{2}}(\mu,\lambda) = 0,\ \mu < 1$ *and* $C^{-}_{-\frac{1}{2}}(\mu,\lambda) = 0,\ \mu > -1$.*Then*

$\lim_{z\to 2+} \Delta(\mu,\lambda;z) > 0,$

$\lim_{z\to 0-} \Delta(\mu,\lambda;z) > 0.$

To formulate the main theorem we introduce the regions $\mathbb{G}^{+}_{02}, \mathbb{G}^{+}_{11}$ and $\mathbb{G}^{+}_{20}$ associated to the function $C^{+}_{-\frac{1}{2}}(\mu,\lambda)$ and also the regions $\mathbb{G}^{-}_{20}, \mathbb{G}^{-}_{11}$ and $\mathbb{G}^{-}_{02}$ associated to the function $C^{-}_{-\frac{1}{2}}(\mu,\lambda)$ as follows

(5.13) $\mathbb{G}_{2,+} = \{(\mu,\lambda) \in\ R^2 :\ C^{+}_{-\frac{1}{2}}(\mu,\lambda) > 0,\ \mu > 1\},$

(5.14) $\mathbb{G}_{1,+} = \{(\mu,\lambda) \in\ R^2 :\ C^{+}_{-\frac{1}{2}}(\mu,\lambda) = 0,\ \mu > 1 \text{ or } C^{+}_{-\frac{1}{2}}(\mu,\lambda) < 0\},$

(5.15) $\mathbb{G}_{0,+} = \{(\mu,\lambda) \in\ R^2\} : C^{+}_{-\frac{1}{2}}(\mu,\lambda) = 0,\ \mu < 1 \text{ or } C^{+}_{-\frac{1}{2}}(\mu,\lambda) > 0$

(5.16)

and

(5.17) $\mathbb{G}_{2,-} = \{(\mu,\lambda) \in R^2 : C^-_{-\frac{1}{2}}(\mu,\lambda) > 0,\ \mu < -1,\}$

(5.18) $\mathbb{G}_{1,-} = \{(\mu,\lambda) \in R^2 : C^-_{-\frac{1}{2}}(\mu,\lambda) = 0,\ \mu < -1 \text{ or } C^-_{-\frac{1}{2}}(\mu,\lambda) < 0\},$

(5.19) $\mathbb{G}_{0,-} = \{(\mu,\lambda) \in R^2\} : C^+_{-\frac{1}{2}}(\mu,\lambda) = 0,\ \mu > -1 \text{ or } C^-_{-\frac{1}{2}}(\mu,\lambda) > 0.$

(5.20)

The main results are given in the following theorem, where the existence and location of eigenvalues of the one-particle Hamiltonian H with indefinite sign interaction $v_{\mu\lambda}$ are stated.

The Hamiltonian $H_{\mu\lambda}$ can have one or two eigenvalues, situating as below the bottom of the essential spectrum, as well as above its top. Moreover, the operator $H_{\mu\lambda}$ has two eigenvalues outside of the essential spectrum, depending on $\mu \neq 0$ and $\lambda \neq 0$, where one of them is situated below the bottom of the essential spectrum and the other one above its top.

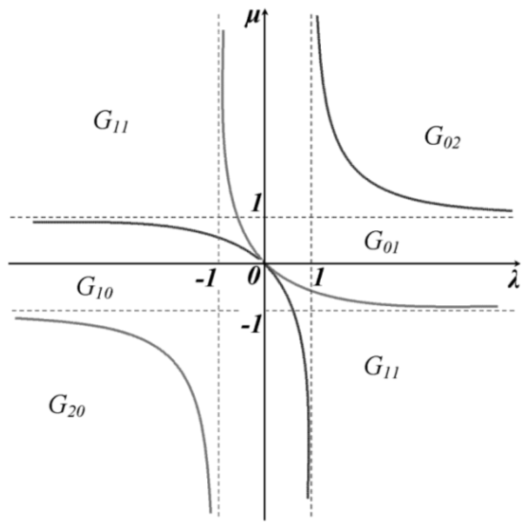

Figure 1.

5.5. Theorem. (i) *Assume* $(\mu,\lambda) \in \mathbb{G}_{0,-} \cap \mathbb{G}_{2,+}$. *Then the operator* $H_{\mu\lambda}$ *has no eigenvalue below the essential spectrum and it has two eigenvalues* $\zeta_1(\mu,\lambda)$ *and* $\zeta_2(\mu,\lambda)$ *satisfying the following relations*

$$2 < \zeta_1(\mu,\lambda) < \zeta_{\min}(\mu,\lambda) \leq \zeta_{\max}(\mu,\lambda) < \zeta_2(\mu,\lambda).$$

(ii) *Assume* $(\mu,\lambda) \in \mathbb{G}_{0,-} \cap \mathbb{G}_{1,+}$. *Then the operator* $H_{\mu\lambda}$ *has no eigenvalue below the essential spectrum and it has one eigenvalue* $\zeta_2(\mu,\lambda)$ *satisfying the following relation*
$\zeta_2(\mu,\lambda) > 2$.

(iii) *Let* $(\mu,\lambda) \in \mathbb{G}_{1,-} \cap \mathbb{G}_{1,+}$. *Then the operator* $H_{\mu\lambda}$ *has two eigenvalues* $\zeta_1(\mu,\lambda)$ *and* $\zeta_2(\mu,\lambda)$ *satisfying the following relations*

$$\zeta_1(\mu,\lambda) < 0 \text{ and } \zeta_2(\mu,\lambda) > 2.$$

(iv) *Assume* $(\mu,\lambda)\in\mathbb{G}_{1,-}\cap\mathbb{G}_{0,+}$. *Then the operator* $H_{\mu\lambda}$ *has one eigenvalue* $\zeta_1(\mu,\lambda)$ *satisfying the relation* $\zeta_1(\mu,\lambda)<0$ *it has no eigenvalue above the essential spectrum.*

(v) *Assume* $(\mu,\lambda)\in\mathbb{G}_{2,-}\cap\mathbb{G}_{0,+}$. *Then the operator* $H_{\mu\lambda}$ *has two eigenvalues* $\zeta_1(\mu,\lambda)$ *and* $\zeta_2(\mu,\lambda)$ *satisfying the following relations*

$$\zeta_1(\mu,\lambda)<\zeta_{\min}(\mu,\lambda)\le\zeta_{\max}(\mu,\lambda)<\zeta_2(\mu,\lambda)<0$$

and it has no eigenvalue above the essential spectrum.

5.6. Remark. The sets $\mathrm{G}_{02},\mathrm{G}_{01},\mathrm{G}_{11},\mathrm{G}_{10}$ and G_{20} which appears in Theorem 5.5 are shown in the figure 1.

Proof. (i) Assume $(\mu,\lambda)\in(\mu,\lambda)\in\mathbb{G}_{0,-}\cap\mathbb{G}_{2,+}$ and $z<0$. Then an application the Cauchy–Schwarz inequality for the functions $[\varepsilon(q)-z]^{-\frac{1}{2}}$ and $\cos q\,[\varepsilon(q)-z]^{-\frac{1}{2}}$ yields the inequality

$$\Delta(\mu,\lambda;z)=\big(1+\mu\int\limits_{\mathbb{T}}\frac{d\nu}{\varepsilon(q)-z}\big)+\big(1+\lambda\int\limits_{\mathbb{T}}\frac{\cos^2 q d\nu}{\varepsilon(q)-z}\big)$$

$$+\mu\lambda\Big[\int\limits_{\mathbb{T}}\frac{d\nu}{\varepsilon(q)-z}\int\limits_{\mathbb{T}}\frac{\cos^2 q d\nu}{\varepsilon(q)-z}-\big(\int\limits_{\mathbb{T}}\frac{\cos q d\nu}{\varepsilon(q)-z}\big)^2\Big]>0,$$

i.e., $\Delta(\mu,\lambda;z)$ has no zero in the interval $(-\infty,0)$. Lemma 5.2 gives that the operator $H_{\mu\lambda}$ has no eigenvalue below the bottom of the essential spectrum.

Let $(\mu,\lambda)\in(\mu,\lambda)\in\mathbb{G}_{0,-}\cap\mathbb{G}_{2,+}$ and $z>2$.

Since $\mu,\lambda>0$ the function $\Delta(\mu,0;\cdot)$ resp. $\Delta(0,\lambda;\cdot)$ is monotone increasing in $(1,+\infty)$. Applying Lemma 4.3 we have

$$\lim_{z\to+\infty}\Delta(\mu,0;z)=1 \text{ resp. } \lim_{z\to+\infty}\Delta(0,\lambda;z)=1.$$

Corollary 4.5 gives that

$$\lim_{z\to1+}\Delta(\mu,0;z)=-\infty, \text{ resp. } \lim_{z\to1+}\Delta(0,\lambda;z)=-\infty.$$

The continuous function $\Delta(\mu,0;\cdot)$ and $\Delta(0,\lambda;\cdot)$ has a zero $\zeta(\mu)$ resp. $\zeta(\lambda)$ in the interval $(1,+\infty)$. The representation (4.1) of the determinant $\Delta(\mu,\lambda\,;z)$ gives the inequality $\Delta(\mu,\lambda\,;\zeta(\mu))<0$ resp. $\Delta(\mu,\lambda\,;\zeta(\lambda))<0$. Denote by

$$\zeta_{\min}(\mu,\lambda)=\min\{\zeta(\mu),\zeta(\lambda)\}$$
$$\zeta_{\max}(\mu,\lambda)=\max\{\zeta(\mu),\zeta(\lambda)\}.$$

The representation (4.1) of determinant $\Delta(\mu,\lambda;z)$ gives the inequality $\Delta(\mu,\lambda;\zeta_{\min}(\mu,\lambda))<0$. Corollary 5.3 yields

$$\lim_{z\to1+}\Delta(\mu,\lambda;z)=+\infty$$

Hence there exist a number $z_1(\mu,\lambda)\in(1,\zeta_{\min}(\mu,\lambda))$ such that

$\Delta(\mu,\lambda;z_1(\mu,\lambda;0))=0.$

Lemma 5.2 gives the existence of the eigenvalue of the operator in the interval $(1,\zeta_{\min}(\mu,\lambda))$.

The monotonicity of function $\Delta(\mu,0;z)$ resp. $\Delta(\lambda,0;z)$ gives for $z>\zeta(\mu)$ resp. $z>\zeta(\lambda)$ the relation

$$\Delta(\mu,0;z)>\Delta(\mu,0;\zeta(\mu))=0, \text{ resp. } \Delta(\lambda,0;z)>\Delta(\lambda;\zeta(\lambda))=0.$$

Applying Lemma 4.4 we have in the interval $(2,+\infty)$ the inequality

$$\frac{\partial\Delta(\mu,\lambda;z)}{\partial z}=-\mu\Delta(0,\lambda;z)a'(z)-\lambda\Delta(\mu,0,;z)c'(z)-4\mu\lambda b(z)b'(z)>0,$$

i.e., the function $\Delta(\mu,\lambda;\cdot)$ is monotone increasing in the interval $(\zeta_{\max}(\mu,\lambda),+\infty)$. Lemma 4.3, i.e., the relation

$$\lim_{z\to+\infty}\Delta(\mu,\lambda;z)=1,$$

yields the existence a unique number $z_2(\mu,\lambda)\in(\zeta_{\max}(\mu,\lambda)$ such that

$$\Delta(\mu,\lambda;z_2(\mu,\lambda;0))=0.$$

Lemma 5.2 gives that the operator has two eigenvalues above the top of the essential spectrum. These eigenvalues obeys the relations (5.5).

(ii) Assume $(\mu,\lambda)\in\mathbb{G}_{0,-}\cap\mathbb{G}_{1,+}$ and $z<0$.

As in the case (i) we can show that operator $H_{\mu\lambda}$ has no eigenvalue below the essential spectrum.

It is easy to show that the operator $H_{\mu 0}$ has only one eigenvalue at the point $(\mu,0)\in\mathbb{G}_{0,-}\cap\mathbb{G}_{1,+},\mu>0$.

Lemma 4.3 and Corollary 5.4 give that

$$\lim_{z\to-\infty}\Delta(\mu,0;z)=1$$

and

$$\lim_{z\to 2+}\Delta(\mu,0;z)<0.$$

Hence, the continuous function $\Delta(\mu,0;\cdot)$ in $z\in(2,+\infty)$ has a unique zero $\zeta_1(\mu,0)\in(2,+\infty)$.

If $(\mu,\lambda)\in\mathbb{G}_{0,-}\cap\mathbb{G}_{1,+}$ is an other point belonging to the region, then there is a line
$\Gamma[(\mu,0),(\mu,\lambda)]\in\mathbb{G}_{0,-}\cap\mathbb{G}_{1,+}$, which connects the points $(\mu,0)$ and (μ,λ)(because this is a region).The compactness of $\Gamma[(\mu,0),(\mu,\lambda)]\in\mathbb{G}_{0,-}\cap\mathbb{G}_{1,+}$ yields that at the point (μ,λ) the function $\Delta(\mu,\lambda;z)$ has only one zero. Thus, Lemma 5.2 yields that the operator has only one eigenvalue above the top of the essential spectrum.

(iii) Assume $(\mu,\lambda)\in\mathbb{G}_{1,-}\cap\mathbb{G}_{1,+}$.

In this case applying Lemma 4.3 and Corollary 5.4 we have

$$\lim_{z\to 2+}\Delta(\mu,\lambda;z)=-\infty,$$

$$\lim_{z\to 0-}\Delta(\mu,\lambda;z)=-\infty,$$

and

$$\lim_{z\to\pm\infty}\Delta(\mu,\lambda;z)=1.$$

Hence, the continuous function $\Delta(\mu,\lambda;\cdot)$ in $z\in(-\infty,0)\cup(2,+\infty)$ has two zeros $\zeta_1(\mu,\lambda)$ in the interval $(-\infty,0)$ and $\zeta_2(\mu,\lambda)$ in the interval $(2,+\infty)$.

Thus, Lemma 5.2 yields that the operator has two eigenvalues: one of them lays below the bottom of the essential spectrum and other one lays above the top.

The other cases (iv) and (v) of Theorem 5.5 can be proven by the same way as the cases (i) and (ii).

□

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Eigenvalue distribution of relaxed mixed constraint preconditioner for saddle point problems

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Abstract

In this paper, the eigenvalue distribution of a family of relaxed mixed constraint preconditioner (RMCP) for the generalized saddle point problems is discussed in detail. Most of the bounds developed improve those appeared in previously published work.

Keywords: saddle point problems; inexact constraint preconditioner; eigenvalue

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1. Introduction

Consider the large, sparse and nonsingular linear system in saddle point form as

$$\mathcal{A}x \equiv \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite (SPD), $B \in \mathbb{R}^{m\times n}$ with $m \le n$ (possibly $m \ll n$) is of full rank and $C \in \mathbb{R}^{m\times m}$ is symmetric semi-positive definite. Systems of the form (1.1) arise in a variety of scientific and engineering applications, such as constrained optimization, least squares and Stokes problems. We refer the reader to [10] for a more detailed list of applications and numerical solution techniques of (1.1).

In recent years, considerable effort has been invested in developing efficient solvers for systems of form (1.1). Recent works on sparse direct methods for symmetric saddle point problems have been developed, such as direct solver package [18] and LDL^T-factorization technique [19]. In fact, the memory and the computational requirements for solving saddle point problems (1.1) may seriously challenge the most efficient direct solution method available today. In actual implements, many iterative methods have to be recommended to solve saddle point problems (1.1), such as generalized successive overrelaxation (GSOR) method [2], modified SSOR (MSSOR) method [33], Hermitian and skew-Hermitian splitting method [3–7, 11, 12] and so on. However, well established

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iterative methods such as Krylov subspace methods are very slow or even fail to converge if not conveniently preconditioned, it follows that preconditioning technique is a key ingredient for the success of Krylov subspace methods in applications. Most of the recent work on saddle point problems has focused on the development of preconditioners for Krylov subspace methods, especially block preconditioners and multilevel schemes. We refer the reader to [10] for a comprehensive survey of existing approaches for solving saddle point problems.

An important class of preconditioners is based on the block LU factorization of the coefficient matrix $\mathcal{A}$ [8, 9]. This class includes a variety of block diagonal and block triangular preconditioners [8, 9, 20–26, 28–30, 39–43]. Based on the Hermitian and skew-Hermitian splitting of the coefficient matrix $\mathcal{A}$, the HSS preconditioner is established [3–6, 12]. Based on the Dimensional Splitting (DS) of the coefficient matrix $\mathcal{A}$, a relaxed dimensional factorization preconditioner for Navier-Stokes equations is proposed [13, 14]. Based on the augmented Lagrangian (AL) reformulation of the saddle point problem, AL-type preconditioners appear to be remarkably robust for a broad range of problem parameters, and they are currently the focus of intense development in [15, 16].

As is known to all, the major issue of preconditioning technique is to find a good approximation of the inverse of the coefficient matrix $\mathcal{A}$. To accelerate Krylov solvers for saddle point problems, constraint preconditioner is another type of preconditioning techniques and has been first introduced in constrained optimization for $C = 0$ [31]. It has been proved [31] that the eigenvalues of the preconditioned matrix are all real and positive. The strategy of constraint preconditioner is that a suitable approximation of the (1,1) block A instead of the (1,1) block A leads to a good approximation of the inverse of the coefficient matrix $\mathcal{A}$. Dollar [32] has extended these results in [31] by allowing the (2,2) block to be symmetric and positive semidefinite. Further, the general symmetric (2,2) block has been discussed [1] and the nonsymmetric (1,1) block has been discussed [27]. Constraint preconditioner can be written as the inverse of a matrix whose non diagonal blocks are the same as those in $\mathcal{A}$, but their application may be very costly since it requires the solution of a linear system at each iteration with an appropriate Schur complement S as the coefficient matrix $\mathcal{A}$. A computationally efficient inexact constraint preconditioner (ICP) is represented by an approximation of S (or of S^{-1}) by means of an incomplete Cholesky factorization or a sparse approximate inverse. The application of ICP is cheaper with respect to the constraint preconditioner. An exhaustive analysis of spectral properties of ICP together with development of eigenvalue bounds are performed in [36]. ICP has been proved much more robust and performing than ILU preconditioners with variable fill-in, computed on the whole saddle point matrix from a number of realistic coupled consolidation problems [38].

Recently, drawing on the previous works: [34–36], Bergamaschi and Martínez [37] discussed a family of relaxed mixed constraint preconditioner (RMCP) as follows:

$$\mathcal{M}_\omega = \begin{bmatrix} I & 0 \\ BP_A^{-1} & I \end{bmatrix} \begin{bmatrix} P_A & 0 \\ 0 & -\omega P_S \end{bmatrix} \begin{bmatrix} I & P_A^{-1}B^T \\ 0 & I \end{bmatrix} = \begin{bmatrix} P_A & B^T \\ B & BP_A^{-1}B^T - \omega P_S \end{bmatrix},$$

where ω is a real acceleration parameter, P_A is a suitable approximation of the (1,1) block A and P_S is a suitable approximation of the Schur complement matrix $S = BP_A^{-1}B^T + C$. A detailed spectral analysis of RMCP was presented in [37]. In this paper, we focus on the relaxed mixed constraint preconditioner (RMCP) for symmetric saddle point problems (1.1). The spectral properties of the preconditioned matrix are given and some corresponding presented results in [36, 37, 44] are improved.

The paper is organized as follows. In Section 2, the spectral distribution of a class of the parameterized saddle point problems is characterized, which extends the corresponding theoretical results in [17, 36]. In Section 3, we discuss the eigenvalue distribution of

$\mathcal{M}_\omega^{-1}\mathcal{A}$ in detail and promote some corresponding presented results in [36, 37, 44]. The conclusions are drawn in Section 4.

2. Eigenvalues of $\mathcal{A}_\omega$

To make the spectral analysis of $\mathcal{M}_\omega^{-1}\mathcal{A}$ easily, the spectral distribution of a class of the parameterized saddle point matrix is characterized.

Given that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite (SPD), $B \in \mathbb{R}^{m\times n}$ $(m \le n)$ is of full rank and $C \in \mathbb{R}^{m\times m}$ is symmetric positive definite. For $\omega > 0$, we are interested in the eigenvalues of

$$\mathcal{A}_\omega u \equiv \begin{bmatrix} A & \frac{1}{\sqrt{\omega}}B^T \\ -\frac{1}{\sqrt{\omega}}B & \frac{1}{\omega}C \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv \lambda u, \tag{2.1}$$

or

$$Au_1 + \frac{1}{\sqrt{\omega}}B^T u_2 = \lambda u_1, \tag{2.2}$$

$$-\frac{1}{\sqrt{\omega}}Bu_1 + \frac{1}{\omega}Cu_2 = \lambda u_2. \tag{2.3}$$

For the purposes of our discussion, the following notation regarding the eigenvalues of SPD matrices A, BB^T and C are required:

$$\begin{aligned} &0 < \alpha_A = \lambda_{\min}(A), \quad &&\beta_A = \lambda_{\max}(A),\\ &0 \le \alpha_S = \lambda_{\min}(BB^T), &&\beta_S = \lambda_{\max}(BB^T),\\ &0 < \alpha_C = \lambda_{\min}(C), &&\beta_C = \lambda_{\max}(C). \end{aligned}$$

Obviously, matrix $\mathcal{A}_\omega$ has at most $n - m$ eigenvalues satisfying

$$\alpha_A \le \lambda \le \beta_A$$

with eigenvectors $u = (u_1^T, 0)^T$ and $Bu_1 = 0$. One can see for instance Proposition 2.2 in [17].

Throughout this section, we define, for some $s, u_2 \neq 0$,

$$\eta_A = \frac{s^T As}{s^T s} \in [\alpha_A, \beta_A], \eta_C = \frac{u_2^T Cu_2}{u_2^T u_2} \in [\alpha_C, \beta_C], \eta_S = \frac{u_2^T BB^T u_2}{u_2^T u_2} \in [\alpha_S, \beta_S].$$

The proof of Theorem 2.2 is based on the following Lemma 2.1, which is from [36].

2.1. Lemma. [36] *Let $\lambda \notin [\alpha_A, \beta_A]$. Then, for every $z \neq 0$, there exists a vector $s \neq 0$ such that*

$$\frac{z^T(A-\lambda I)^{-1}z}{z^T z} = (\frac{s^T As}{s^T s} - \lambda)^{-1} = (\eta_A - \lambda)^{-1}.$$

2.2. Theorem. *The real eigenvalues of Equation* (2.1) *not lying in* $[\alpha_A, \beta_A]$ *satisfy*

$$\frac{1}{\omega}\Big(\alpha_C + \frac{\alpha_S}{\beta_A}\Big) \le \frac{1}{\omega}\Big(\eta_C + \frac{\eta_S}{\eta_A}\Big) \le \lambda \le \max\Big\{\eta_A, \frac{1}{\omega}\eta_C\Big\} \le \max\Big\{\beta_A, \frac{1}{\omega}\beta_C\Big\}.$$

Proof. Let $\lambda \in \mathbb{R}$ with $\lambda \notin [\alpha_A, \beta_A]$ and let u such that $Bu_1 \neq 0$ and $B^T u_2 \neq 0$. Since $A - \lambda I$ is invertible, from (2.2) we have

$$u_1 = -\frac{1}{\sqrt{\omega}}(A-\lambda I)^{-1}B^T u_2. \tag{2.4}$$

Substituting (2.4) into (2.3) yields

$$\frac{1}{\omega}B(A-\lambda I)^{-1}B^T u_2 + \frac{1}{\omega}Cu_2 - \lambda u_2 = 0. \tag{2.5}$$

Premultiplying (2.5) by $\frac{u_2^T}{u_2^T u_2}$ leads to

$$\frac{1}{\omega}\frac{u_2^T B(A-\lambda I)^{-1}B^T u_2}{u_2^T u_2} + \frac{1}{\omega}\eta_C - \lambda = 0. \tag{2.6}$$

Based on Lemma 2.1, from (2.6) we have

$$\frac{1}{\omega}(\eta_A - \lambda)^{-1}\eta_S + \frac{1}{\omega}\eta_C - \lambda = 0.$$

or,

$$\omega\lambda^2 - (\eta_C + \omega\eta_A)\lambda + \eta_S + \eta_A\eta_C = 0. \tag{2.7}$$

The lager solution of (2.7) is

$$\begin{aligned}
\lambda_2 &= \frac{\eta_C + \omega\eta_A + \sqrt{(\eta_C + \omega\eta_A)^2 - 4\omega(\eta_S + \eta_A\eta_C)}}{2\omega} \\
&= \frac{\eta_C + \omega\eta_A + \sqrt{(\eta_C - \omega\eta_A)^2 - 4\omega\eta_S}}{2\omega} \\
&\le \max\{\eta_A, \frac{1}{\omega}\eta_C\}.
\end{aligned}$$

The smaller solution of (2.7) is

$$\begin{aligned}
\lambda_1 &= \frac{\eta_C + \omega\eta_A - \sqrt{(\eta_C + \omega\eta_A)^2 - 4\omega(\eta_S + \eta_A\eta_C)}}{2\omega} \\
&= \frac{2(\eta_S + \eta_A\eta_C)}{\eta_C + \omega\eta_A + \sqrt{(\eta_C - \omega\eta_A)^2 - 4\omega\eta_S}} \\
&\ge \frac{\eta_S + \eta_A\eta_C}{\max\{\omega\eta_A, \eta_C\}} = \frac{1}{\omega}(\eta_C + \frac{\eta_S}{\eta_A}).
\end{aligned}$$

The last equation follows from the inequality $\eta_C < \omega\eta_A$ (otherwise we would have $\lambda_1 > \eta_A > \alpha_A$ against the assumption). Hence,

$$\lambda_1 \ge \frac{1}{\omega}(\eta_C + \frac{\eta_S}{\eta_A}) \ge \frac{1}{\omega}(\alpha_C + \frac{\alpha_S}{\beta_A}).$$

2.1. Corollary. The real eigenvalues of Equation *(2.1)* satisfy

$$\min\left\{\alpha_A, \frac{1}{\omega}(\alpha_C + \frac{\alpha_S}{\beta_A})\right\} \le \lambda \le \max\left\{\beta_A, \frac{1}{\omega}\beta_C\right\}.$$

In the sequel, we will denote any complex eigenvalue as

$$\lambda = \lambda_R + i\lambda_I.$$

2.2. Corollary. The complex eigenvalues of Equation *(2.1)* satisfy

$$\frac{\omega\alpha_A + \alpha_C}{2\omega} \le \lambda_R \le \frac{\omega\beta_A + \beta_C}{2\omega},\ |\lambda_I| \le \sqrt{\frac{\beta_S}{\omega}}.$$

Proof. From (2.7), we have

$$\lambda_R = \frac{\eta_C + \omega\eta_A}{2\omega}, \tag{2.8}$$

$$\lambda_R^2 + \lambda_I^2 = \frac{\eta_S + \eta_A\eta_C}{\omega} \tag{2.9}$$

By simple computations, from (2.8) we have

$$\frac{\omega\alpha_A + \alpha_C}{2\omega} \le \lambda_R \le \frac{\omega\beta_A + \beta_C}{2\omega}.$$

Combining (2.8) and (2.9), we have

$$
\begin{aligned}
|\lambda_I| &= \sqrt{\frac{\eta_S+\eta_A\eta_C}{\omega}-\left(\frac{\eta_C+\omega\eta_A}{2\omega}\right)^2} \\
&= \sqrt{\frac{4\omega(\eta_S+\eta_A\eta_C)-(\eta_C+\omega\eta_A)^2}{4\omega^2}} \\
&= \sqrt{\frac{\eta_S}{\omega}-\frac{(\eta_C-\omega\eta_A)^2}{4\omega^2}} \le \sqrt{\frac{\eta_S}{\omega}} \le \sqrt{\frac{\beta_S}{\omega}}.
\end{aligned}
$$

Remark 2.1 When $\omega = 1$, Theorem 2.2 reduces to Theorem 1 [36], Corollary 2.1 reduces to Corollary 1 [36] and Corollary 2.2 reduces to Proposition 1 [36]. Specifically, this result in Corollary 2.1 with $\omega = 1$ improves that of Proposition 2.12 in [17], which provides a lower bound for $\lambda \ge \min\{\alpha_A, \alpha_C\}$.

Example 2.1

$$
\mathcal{A}_\omega = \begin{bmatrix} \beta_A & 0 & \frac{1}{\sqrt{\omega}}\times 1 \\ 0 & \alpha_A & \frac{1}{\sqrt{\omega}}\times 1 \\ -\frac{1}{\sqrt{\omega}}\times 1 & -\frac{1}{\sqrt{\omega}}\times 1 & \frac{1}{\omega}\times c \end{bmatrix}, \alpha_S = \beta_S = 2, \omega = 4.
$$

If $\beta_A = 3$, $\alpha_A = 2.9$ and $c = 1$, the eigenvalues of $\mathcal{A}_\omega$ are $\lambda(\mathcal{A}_\omega) = \{0.4501, 2.7372, 2.9627\}$. Obviously, $\alpha_C = \beta_C = 1$. From Corollary 2.1, we have

$$0.4167 < \lambda < 3.$$

If $\beta_A = 3$, $\alpha_A = 2$ and $c = 4$, the eigenvalues of $\mathcal{A}_\omega$ are $\lambda(\mathcal{A}_\omega) = \{2.8846, 1.5577 + 0.2949i, 1.5577 - 0.2949i\}$. Obviously, $\alpha_C = \beta_C = 4$. From Corollary 2.1, we have

$$\frac{7}{6} < \lambda < 3.$$

From Corollary 2.2, we have

$$\frac{3}{2} < \lambda_R < 2,\ |\lambda_I| \le \frac{\sqrt{2}}{2}.$$

Numerical results show that Corollary 2.1 provides some valid bounds for all the real eigenvalues of $\mathcal{A}_\omega$ and Corollary 2.2 provides some valid bounds for all the complex eigenvalues of $\mathcal{A}_\omega$.

3. Spectral analysis of $\mathcal{A}\mathcal{M}_\omega^{-1}$

It is not difficult to find that the spectral of $\mathcal{M}_\omega^{-1}\mathcal{A}$ is equivalent to the spectral of $\mathcal{A}\mathcal{M}_\omega^{-1}$. Here we focus on the bounds for the eigenvalues of $\mathcal{A}\mathcal{M}_\omega^{-1}$ to obtain the bounds for the eigenvalues of $\mathcal{M}_\omega^{-1}\mathcal{A}$. Making this strategy to discuss the bounds for the eigenvalues of the corresponding preconditioned matrix, one can see [24, 25, 36, 40, 41, 43] for more details.

In fact, $\mathcal{A}\mathcal{M}_\omega^{-1}z = \lambda z$ can be expressed as

$$\mathcal{A}\nu = \lambda\mathcal{M}_\omega\nu, \nu = \mathcal{M}_\omega^{-1}z.$$

To investigate the spectral properties of $\mathcal{M}_\omega^{-1}\mathcal{A}$, P_A and P_S, respectively, are SPD approximations of A and $S = BP_A^{-1}B^T + C$. P_A^{-1} and P_S^{-1} can also be viewed as preconditioners for the corresponding matrices, so that we can define the following SPD preconditioned matrices:

$$\mathcal{P} = \begin{bmatrix} P_A & 0 \\ 0 & P_S \end{bmatrix} \text{ and } S_P = P_S^{-1/2} S P_S^{-1/2}.$$

Since $\mathcal{P}$ is symmetric positive definite, the problem of finding the eigenvalues of $\mathcal{M}_\omega^{-1}\mathcal{A}$ with $u = \mathcal{P}_+^{\frac{1}{2}}\nu$ is equivalent to solving

$$\mathcal{P}^{-\frac{1}{2}}\mathcal{A}\mathcal{P}^{-\frac{1}{2}}u = \lambda\mathcal{P}^{-\frac{1}{2}}\mathcal{M}_\omega\mathcal{P}^{-\frac{1}{2}}u.$$

That is,

$$\begin{bmatrix} A_P & R^T \\ R & -\widehat{C} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda\begin{bmatrix} I & R^T \\ R & RR^T - \omega I \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{3.1}$$

where $R = P_S^{-1/2}BP_A^{-1/2}$, $A_P = P_A^{-1/2}AP_A^{-1/2}$ and $\widehat{C} = P_S^{-1/2}CP_S^{-1/2}$. Note that $RR^T = S_P - \widehat{C}$ and the inverse of the right side matrix product in (3.1) can be written as

$$\begin{aligned}\begin{bmatrix} I & R^T \\ R & RR^T - \omega I \end{bmatrix}^{-1} &= \left[\begin{bmatrix} I & 0 \\ R & -\sqrt{\omega}I \end{bmatrix}\begin{bmatrix} I & R^T \\ 0 & \sqrt{\omega}I \end{bmatrix}\right]^{-1} \\ &= \begin{bmatrix} I & R^T \\ 0 & \sqrt{\omega}I \end{bmatrix}^{-1}\begin{bmatrix} I & 0 \\ R & -\sqrt{\omega}I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I & -\frac{1}{\sqrt{\omega}}R^T \\ 0 & \frac{1}{\sqrt{\omega}}I \end{bmatrix}\begin{bmatrix} I & 0 \\ \frac{1}{\sqrt{\omega}}R & -\frac{1}{\sqrt{\omega}}I \end{bmatrix} \\ &\equiv \mathcal{U}\mathcal{L},\end{aligned}$$

so that the eigenvalues of (3.1) are the same as those of $\mathcal{L}\mathcal{P}^{-\frac{1}{2}}\mathcal{A}\mathcal{P}^{-\frac{1}{2}}\mathcal{U}x = \lambda x$ which reads:

$$\begin{bmatrix} A_P & \frac{1}{\sqrt{\omega}}(I - A_P)R^T \\ -\frac{1}{\sqrt{\omega}}R(I - A_P) & \frac{1}{\omega}(R(2I - A_P)R^T + \widehat{C}) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{3.2}$$

Let us assume that

$$\begin{aligned} &0 < \alpha_A = \lambda_{\min}(A_P) < 1 < \lambda_{\max}(A_P) = \beta_A, \\ &0 < \alpha_S = \lambda_{\min}(S_P) < 1 < \lambda_{\max}(S_P) = \beta_S, \\ &0 \le \alpha_C = \lambda_{\min}(\widehat{C}) < \lambda_{\max}(\widehat{C}) = \beta_C. \end{aligned}$$

Obviously, the eigenvalues of the projected matrix $A_R = (RR^T)^{-1}RA_PR^T$ is also important in the spectral analysis of the preconditioned matrices. In [36, 37, 44], it is shown that $[\alpha_A^R, \beta_A^R] \subset [\alpha_A, \beta_A]$, where $\alpha_A^R = \lambda_{\min}(A_R)$ and $\beta_A^R = \lambda_{\max}(A_R)$.

Throughout this section, we will use the following notation:

$$\theta_S = \frac{x_2^T S_P x_2}{x_2^T x_2}, \theta_A^R = \frac{x_2^T RA_PR^T x_2}{x_2^T RR^T x_2}, \theta_A = \frac{s^T A_P s}{s^T s}, \theta_C = \frac{x_2^T \widehat{C} x_2}{x_2^T x_2},$$

for some $s, x_2 \neq 0$. It follows that $\theta_A^R \in [\alpha_A^R, \beta_A^R]$ and $\frac{x_2^T RR^T x_2}{x_2^T x_2} = \theta_S - \theta_C \ge 0$.

To obtain the bounds for the eigenvalues of $\mathcal{M}_\omega^{-1}\mathcal{A}$, we need the following lemma.

3.1. Lemma. *Let $H = R(2I - A_P)R^T + \widehat{C}$, $P = R(I - A_P)^2R^T$ and $\beta_A^R < 2$. If $\alpha_A^R < 1$, then*

$$\begin{aligned} &\lambda(H) \in [\alpha_S(2 - \beta_A^R) + \alpha_C(\beta_A^R - 1), \beta_S(2 - \alpha_A^R) - \alpha_C(1 - \alpha_A^R)], \\ &\lambda(P) \le (\beta_S - \alpha_C)\max\left\{(1 - \alpha_A^R)^2, (\beta_A^R - 1)^2\right\}. \end{aligned}$$

If $\alpha_A^R \ge 1$, then

$$\begin{aligned} &\lambda(H) \in [\alpha_S(2 - \beta_A^R) + \alpha_C(\beta_A^R - 1), \beta_S(2 - \alpha_A^R) + \beta_C(\alpha_A^R - 1)], \\ &\lambda(P) \le (\beta_S - \alpha_C)\max\left\{(1 - \alpha_A^R)^2, (\beta_A^R - 1)^2\right\}. \end{aligned}$$

Proof. Based on the results in [36,44], here we only need prove that $\lambda(H) \le \beta_S(2-\alpha_A^R)+\beta_C(\alpha_A^R-1)$ for $\alpha_A^R \ge 1$.

In fact, $\lambda(H) \in [\min q(x_2,H), \max q(x_2,H)]$, where

$$\begin{aligned} q(x_2,H) &= \frac{x_2^T(R(2I-A_P)R^T+\widehat{C})x_2}{x_2^Tx_2} \\ &= (\theta_S-\theta_C)(2-\theta_A^R)+\theta_C. \end{aligned}$$

Because the function on the right hand side is decreasing in θ_A^R, then

$$\begin{aligned} \max q(x_2,H) &\le (\theta_S-\theta_C)(2-\alpha_A^R)+\theta_C \\ &= \theta_S(2-\alpha_A^R)+\theta_C(\alpha_A^R-1) \\ &\le \beta_S(2-\alpha_A^R)+\beta_C(\alpha_A^R-1). \end{aligned}$$

The proof is completed.

Investigating the results in Lemma 2 [36,44], the bounds for the eigenvalues of $R(2I-A_P)R^T+\widehat{C}$ and $R(I-A_P)^2R^T$ are provided just when $\beta_A^R<2$ and $\alpha_A^R<1$. In this case, it is easy to see that the results in Lemma 3.1 perfect the corresponding theoretical results in Lemma 2 [36, 44]. Based on Lemma 3.1, it is easy to obtain the following results.

3.1. Corollary. Let $H=\frac{1}{\omega}(R(2I-A_P)R^T+\widehat{C})$, $P=\frac{1}{\omega}R(I-A_P)^2R^T$ and $\beta_A^R<2$. If $\alpha_A^R<1$, then

$$\begin{aligned} \lambda(H) &\in \Big[\frac{\alpha_S(2-\beta_A^R)+\alpha_C(\beta_A^R-1)}{\omega}, \frac{\beta_S(2-\alpha_A^R)-\alpha_C(1-\alpha_A^R)}{\omega}\Big], \\ \lambda(P) &\le \frac{(\beta_S-\alpha_C)}{\omega}\max\big\{(1-\alpha_A^R)^2,(\beta_A^R-1)^2\big\}. \end{aligned}$$

If $\alpha_A^R \ge 1$, then

$$\begin{aligned} \lambda(H) &\in \Big[\frac{\alpha_S(2-\beta_A^R)+\alpha_C(\beta_A^R-1)}{\omega}, \frac{\beta_S(2-\alpha_A^R)+\beta_C(\alpha_A^R-1)}{\omega}\Big], \\ \lambda(P) &\le \frac{(\beta_S-\alpha_C)}{\omega}\max\big\{(1-\alpha_A^R)^2,(\beta_A^R-1)^2\big\}. \end{aligned}$$

Obviously, Corollary 3.1 is a generalization of Lemma 3.1. When $\omega=1$, Corollary 3.1 reduces to Lemma 3.1.

Based on Theorem 3 in [36,44] and Corollary 3.1, we have the following results.

3.2. Theorem. *Let $\beta_A<2$.*
For $\alpha_A^R<1$, the real eigenvalues of (3.2) satisfy

$$\min\left\{\alpha_A, \frac{\alpha_S+\alpha_C(\beta_A-1)}{\omega\beta_A}\right\} \le \lambda \le \max\left\{\beta_A, \frac{\beta_S(2-\alpha_A^R)-\alpha_C(1-\alpha_A^R)}{\omega}\right\}. \tag{3.3}$$

And if $\lambda_I \ne 0$, then the complex eigenvalues of (3.2) satisfy

$$\frac{\omega\alpha_A+\alpha_S(2-\beta_A^R)+\alpha_C(\beta_A^R-1)}{2\omega} \le \lambda_R \le \frac{\omega\beta_A+\beta_S(2-\alpha_A^R)-\alpha_C(1-\alpha_A^R)}{2\omega}, \tag{3.4}$$

$$|\lambda_I| \le \sqrt{\frac{\beta_S-\alpha_C}{\omega}}\max\{1-\alpha_A^R, |\beta_A^R-1|\}. \tag{3.5}$$

For $\alpha_A^R \ge 1$, the real eigenvalues of (3.2) satisfy

$$\min\left\{\alpha_A, \frac{\alpha_S+\alpha_C(\beta_A-1)}{\omega\beta_A}\right\} \le \lambda \le \max\left\{\beta_A, \frac{\beta_S(2-\alpha_A^R)+\beta_C(\alpha_A^R-1)}{\omega}\right\}. \tag{3.6}$$

And if $\lambda_I \neq 0$, then the complex eigenvalues of (3.2) *satisfy*

$$\frac{\omega\alpha_A + \alpha_S(2-\beta_A^R) + \alpha_C(\beta_A^R - 1)}{2\omega} \leq \lambda_R \leq \frac{\omega\beta_A + \beta_S(2-\alpha_A^R) + \beta_C(\alpha_A^R - 1)}{2\omega}, \tag{3.7}$$

$$|\lambda_I| \leq \sqrt{\frac{\beta_S - \alpha_C}{\omega}} \max\{\alpha_A^R - 1, |\beta_A^R - 1|\}. \tag{3.8}$$

Proof. The proof is similar to the proof of Theorem 3 in [36]. One can see [36] for more details.

Obviously, when $\omega = 1$, the following results are obtained.

3.2. Corollary. Let $\beta_A < 2$.
For $\alpha_A^R < 1$, the real eigenvalues of *(3.2)* satisfy

$$\min\left\{\alpha_A, \frac{\alpha_S}{\beta_A} + \frac{\alpha_C(\beta_A - 1)}{\beta_A}\right\} \leq \lambda \leq \max\left\{\beta_A, \beta_S(2-\alpha_A^R) - \alpha_C(1-\alpha_A^R)\right\}.$$

And if $\lambda_I \neq 0$, then the complex eigenvalues of *(3.2)* satisfy

$$\frac{\alpha_A + \alpha_S(2-\beta_A^R) + \alpha_C(\beta_A^R - 1)}{2} \leq \lambda_R \leq \frac{\beta_A + \beta_S(2-\alpha_A^R) - \alpha_C(1-\alpha_A^R)}{2},$$

$$|\lambda_I| \leq \sqrt{\beta_S - \alpha_C}(\beta_A^R - 1).$$

For $\alpha_A^R \geq 1$, the real eigenvalues of *(3.2)* satisfy

$$\min\left\{\alpha_A, \frac{\alpha_S}{\beta_A} + \frac{\alpha_C(\beta_A - 1)}{\beta_A}\right\} \leq \lambda \leq \max\left\{\beta_A, \beta_S(2-\alpha_A^R) + \beta_C(\alpha_A^R - 1)\right\}.$$

And if $\lambda_I \neq 0$, then the complex eigenvalues of *(3.2)* satisfy

$$\frac{\alpha_A + \alpha_S(2-\beta_A^R) + \alpha_C(\beta_A^R - 1)}{2} \leq \lambda_R \leq \frac{\beta_A + \beta_S(2-\alpha_A^R) + \beta_C(\alpha_A^R - 1)}{2},$$

$$|\lambda_I| \leq \sqrt{\beta_S - \alpha_C}(\beta_A^R - 1).$$

Remark 3.1 From Corollary 3.2, we know that for $\alpha_A^R < 1$ and $\lambda_I \neq 0$, the upper bound of λ_R is sharper than the upper bound of λ_R in [36, 44]. In fact, one can easily see the following result, that is,

$$0 < \frac{\beta_A + \beta_S(2-\alpha_A^R) - \alpha_C(1-\alpha_A^R)}{2} \leq \frac{\beta_A + \beta_S(2-\alpha_A^R) + \alpha_C(1-\alpha_A^R)}{2}.$$

If $C \equiv 0$, then $\widehat{C} = P_S^{-1/2} C P_S^{-1/2} = 0$. It follows that $\alpha_C = \beta_C = 0$. Then the bounds of Theorem 3.2 simplify is stated in the following.

3.3. Corollary. Let $\beta_A < 2$ and $C = 0$. Then the real eigenvalues of *(3.2)* satisfy

$$\min\left\{\alpha_A, \frac{\alpha_S}{\omega\beta_A}\right\} \leq \lambda \leq \max\left\{\beta_A, \frac{\beta_S(2-\alpha_A^R)}{\omega}\right\}.$$

And if $\lambda_I \neq 0$, then the complex eigenvalues of *(3.2)* satisfy

$$\frac{\omega\alpha_A + \alpha_S(2-\beta_A^R)}{2\omega} \leq \lambda_R \leq \frac{\omega\beta_A + \beta_S(2-\alpha_A^R)}{2\omega}, |\lambda_I| \leq \sqrt{\frac{\beta_S}{\omega}} \max\{1-\alpha_A^R, |\beta_A^R - 1|\}.$$

To develop eigenvalue bounds for RMCP we will use Theorem 3.2, and particularly the results regarding the real eigenvalues of $\mathcal{M}_\omega^{-1}\mathcal{A}$. The following theorem gives very simple estimates of the eigenvalues of the RMCP preconditioned matrix in terms of ω.

3.3. Theorem. *Let* $1 \le \beta_A^R \le \beta_A < 2$.
For $\alpha_A^R < 1$, *any real eigenvalue* λ *of* $\mathcal{M}_\omega^{-1}\mathcal{A}$ *satisfies*

$$\min\Big\{\alpha_A, \frac{\alpha_S}{2\omega}\Big\} \le \lambda \le \max\Big\{\beta_A, \frac{2\beta_S}{\omega}\Big\}.$$

Moreover, the complex eigenvalues λ *of* $\mathcal{M}_\omega^{-1}\mathcal{A}$ *satisfy*

$$\frac{\alpha_A}{2} \le \lambda_R \le \frac{\beta_A}{2} + \frac{2\beta_S - \alpha_C}{2\omega},\ \ |\lambda_I| \le \sqrt{\frac{\beta_S}{\omega}}.$$

For $\alpha_A^R \ge 1$, *any real eigenvalue* λ *of* $\mathcal{M}_\omega^{-1}\mathcal{A}$ *satisfies*

$$\min\Big\{\alpha_A, \frac{\alpha_S}{2\omega}\Big\} \le \max\Big\{\beta_A, \frac{2(\beta_S+\beta_C)}{\omega}\Big\}.$$

Moreover, the complex eigenvalues λ *of* $\mathcal{M}_\omega^{-1}\mathcal{A}$ *satisfy*

$$\frac{\alpha_A}{2} \le \lambda_R \le \frac{\beta_A}{2} + \frac{\beta_S+\beta_C}{\omega},\ \ |\lambda_I| \le \sqrt{\frac{\beta_S}{\omega}}(\beta_A^R - 1).$$

Proof. For $\alpha_A^R < 1$, from (3.3) we have

$$\min\left\{\alpha_A, \frac{\alpha_S + \alpha_C(\beta_A - 1)}{\omega\beta_A}\right\} \le \lambda \le \max\left\{\beta_A, \frac{\beta_S(2-\alpha_A^R) - \alpha_C(1-\alpha_A^R)}{\omega}\right\}.$$

Using $\alpha_C \ge 0$, $1 < \beta_A < 2$ and $\alpha_A^R < 1$, we have

$$\min\Big\{\alpha_A, \frac{\alpha_S}{2\omega}\Big\} \le \lambda \le \max\Big\{\beta_A, \frac{2\beta_S}{\omega}\Big\}.$$

Using $1 < \beta_A^R < 2$ and $\alpha_A^R < 1$, from (3.4) and (3.5) we have

$$\begin{aligned}\frac{\alpha_A}{2} \le \lambda_R &\le \frac{\omega\beta_A + \beta_S(2-\alpha_A^R) - \alpha_C(1-\alpha_A^R)}{2\omega}\\ &= \frac{\beta_A}{2} + \frac{\beta_S + (\beta_S - \alpha_C)(1-\alpha_A^R)}{2\omega}\\ &\le \frac{\beta_A}{2} + \frac{2\beta_S - \alpha_C}{2\omega}\end{aligned}$$

and

$$|\lambda_I| \le \sqrt{\frac{\beta_S}{\omega}}(\beta_A^R - 1).$$

For $\alpha_A^R \ge 1$, from (3.6) we have

$$\min\left\{\alpha_A, \frac{\alpha_S + \alpha_C(\beta_A - 1)}{\omega\beta_A}\right\} \le \lambda \le \max\left\{\beta_A, \frac{\beta_S(2-\alpha_A^R) + \beta_C(\alpha_A^R - 1)}{\omega}\right\}.$$

Using $\alpha_C \ge 0$, $1 < \beta_A < 2$ and $\alpha_A^R \ge 1$, we have

$$\min\Big\{\alpha_A, \frac{\alpha_S}{2\omega}\Big\} \le \lambda \le \max\Big\{\beta_A, \frac{2\beta_S + \beta_C\alpha_A^R}{\omega}\Big\} \le \max\Big\{\beta_A, \frac{2(\beta_S+\beta_C)}{\omega}\Big\}.$$

Using $1 < \beta_A^R < 2$ and $\alpha_A^R \ge 1$, from (3.7) and (3.8) we have

$$\frac{\alpha_A}{2} \le \lambda_R \le \frac{\beta_A}{2} + \frac{2\beta_S + \alpha_A^R\beta_C}{2\omega} \le \frac{\beta_A}{2} + \frac{\beta_S+\beta_C}{\omega},$$

$$|\lambda_I| \le \sqrt{\frac{\beta_S}{\omega}}(\beta_A^R - 1).$$

Remark 3.2 Theorem 2 in [37] also gives very simple estimates of the eigenvalues of the RMCP preconditioned matrix in terms of ω, but this result in Theorem 2 is not generally true. In fact, by investigating the proof of Theorem 2, the bound of the eigenvalue λ of the preconditioned matrix $\mathcal{M}_\omega^{-1}\mathcal{A}$ not only depends on whether α_A^R is smaller or larger than 1, but depends on whether β_A^R is smaller or larger than 1. However in [37] it is not specify whether $\alpha_A^R < 1$ or $\alpha_A^R > 1$. It is only stated that $0 < \alpha_A \leq \alpha_A^R$ from $[\alpha_A^R, \beta_A^R] \subset [\alpha_A, \beta_A]$. Similarly, the conditions in Theorem 2 [37] do not also specify whether $\beta_A^R < 1$ or $\beta_A^R > 1$. It is only stated that $\beta_A^R \leq \beta_A$ from $[\alpha_A^R, \beta_A^R] \subset [\alpha_A, \beta_A]$. In this case, Theorem 3.3 perfects the results in Theorem 2 in [37].

Example 3.1 Let

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.6 \end{bmatrix}.$$

For convenience, we can choose $P_A = I$ and $P_S = 2I$. Then $A_P = P_A^{-1/2} A P_A^{-1/2} = A$,

$$S = BP_A^{-1}B^T + C = \begin{bmatrix} 2.5 & 0 \\ 0 & 1.6 \end{bmatrix}, S_P = P_S^{-1/2} S P_S^{-1/2} = \begin{bmatrix} 1.25 & 0 \\ 0 & 0.8 \end{bmatrix}$$

and

$$\widehat{C} = P_S^{-1/2} C P_S^{-1/2} = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

Therefore, $\alpha_A = \lambda_{\min}(A_P) = 0.5$, $\beta_A = \lambda_{\max}(A_P) = 1.5$, $\alpha_S = \lambda_{\min}(S_P) = 0.8$, $\beta_S = \lambda_{\max}(S_P) = 1.25$, $\alpha_C = \lambda_{\min}(\widehat{C}) = 0.25$ and $\beta_C = \lambda_{\max}(\widehat{C}) = 0.3$.

Since $R = P_S^{-1/2} B P_A^{-1/2} = P_S^{-1/2} B$,

$$A_R = (RR^T)^{-1} R A_P R^T = \begin{bmatrix} 0.75 & 0 \\ 0 & 1.5 \end{bmatrix},$$

This shows that $\alpha_A^R = \lambda_{\min}(A_R) = 0.75$, $\beta_A^R = \lambda_{\max}(A_R) = 1.5 < 2$.

If $\omega = 2$, all the eigenvalues of $\mathcal{M}_\omega^{-1}\mathcal{A}$ are $\lambda(\mathcal{M}_\omega^{-1}\mathcal{A}) = \{0.3283, 1, 1.4467, 0.6250 \pm 0.2165i\}$. From Theorem 3.3, any real eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.2 < \lambda < 1.5,$$

and the complex eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.25 < \lambda_R < 1.3125, |\lambda_I| < 0.7906.$$

Obviously, 0.3283,1 and 1.4467 lie in (0.2, 1.5), $0.6250 \in (0.25, 1.3125)$ and $|\pm 0.2165| < 0.7906$.

Based on Theorem 2 in [37], any real eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.5 < \lambda < 6,$$

and the complex eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.25 < \lambda_R < 2.75, |\lambda_I| < 1.5811.$$

Obviously, $0.3283 \notin (0.5, 6)$, $0.6250 \in (0.25, 2.75)$ and $|\pm 0.2165| < 1.5811$.

If $\omega = \frac{1}{2}$, all the eigenvalues of $\mathcal{M}_\omega^{-1}\mathcal{A}$ are $\lambda(\mathcal{M}_\omega^{-1}\mathcal{A}) = \{0.6044, 1, 2.8956, 1.3 \pm 0.4583i\}$. From Theorem 3.3, any real eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.5 < \lambda < 5,$$

and the complex eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.25 < \lambda_R < 3, |\lambda_I| < 1.5811.$$

Obviously, 0.6044 ,1, 2.8956 lie in (0.5, 5), $1.3 \in (0.25, 3)$ and $|\pm 0.4583| < 1.5811$.

Based on Theorem 2 in [37], any real eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.2 < \lambda < 1.5,$$

and the complex eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.25 < \lambda_R < 1.8125, |\lambda_I| < 0.7906.$$

Obviously, $2.8956 \notin (0.2, 1.5)$, $1.3 \in (0.25, 1.8125)$ and $|\pm 0.4583| < 0.7906$.

Example 3.2 Let

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

For convenience, we can choose $P_A = I$ and $P_S = \frac{5}{3}I$. Then $A_P = P_A^{-1/2} A P_A^{-1/2} = A$,

$$S = BP_A^{-1}B^T + C = \begin{bmatrix} 2 & 0 \\ 0 & 1.2 \end{bmatrix}, S_P = P_S^{-1/2} S P_S^{-1/2} = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.72 \end{bmatrix}$$

and

$$\widehat{C} = P_S^{-1/2} C P_S^{-1/2} = \begin{bmatrix} 0 & 0 \\ 0 & 0.12 \end{bmatrix}.$$

Therefore, $\alpha_A = \lambda_{\min}(A_P) = 0.25$, $\beta_A = \lambda_{\max}(A_P) = 1.5$, $\alpha_S = \lambda_{\min}(S_P) = 0.72$, $\beta_S = \lambda_{\max}(S_P) = 1.2$, $\alpha_C = \lambda_{\min}(\widehat{C}) = 0$ and $\beta_C = \lambda_{\max}(\widehat{C}) = 0.12$.

Since $R = P_S^{-1/2} B P_A^{-1/2} = P_S^{-1/2} B$,

$$A_R = (RR^T)^{-1} R A_P R^T = \begin{bmatrix} 0.375 & 0 \\ 0 & 1.5 \end{bmatrix},$$

This shows that $\alpha_A^R = \lambda_{\min}(A_R) = 0.375$, $\beta_A^R = \lambda_{\max}(A_R) = 1.5 < 2$.

If $\omega = 3$, all the eigenvalues of $\mathcal{M}_\omega^{-1}\mathcal{A}$ are $\lambda(\mathcal{M}_\omega^{-1}\mathcal{A}) = \{0.2147, 0.3927, 1.4533, 0.5687 \pm 0.3675i\}$. From Theorem 3.3, any real eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.12 < \lambda < 1.5,$$

and the complex eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.125 < \lambda_R < 1.15, |\lambda_I| < 0.6325.$$

Obviously, 0.2147, 0.3927 and 1.4533 lie in (0.12, 1.5), $0.5687 \in (0.125, 1.15)$ and $|\pm 0.3675| < 0.6325$.

Based on Theorem 2 in [37], any real eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.25 < \lambda < 7.2,$$

and the complex eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.125 < \lambda_R < 3.3, |\lambda_I| < 1.8974.$$

Obviously, $0.2147 \notin (0.25, 7.2)$, $0.5687 \in (0.125, 3.3)$ and $|\pm 0.3675| < 1.8974$.

If $\omega = \frac{1}{3}$, all the eigenvalues of $\mathcal{M}_\omega^{-1}\mathcal{A}$ are $\lambda(\mathcal{M}_\omega^{-1}\mathcal{A}) = \{0.3649, 0.6642, 5.571, 1.38 \pm 0.66i\}$. From Theorem 3.3, any real eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.25 < \lambda < 7.2,$$

and the complex eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.125 < \lambda_R < 4.35, |\lambda_I| < 1.8974.$$

Obviously, 0.3649, 0.6642 and 5.571 lie in (0.25, 7.2), $1.38 \in (0.125, 4.35)$ and $|\pm 0.66| < 1.8974$.

Based on Theorem 2 in [37], any real eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.12 < \lambda < 1.5,$$

and the complex eigenvalue λ of $\mathcal{M}_\omega^{-1}\mathcal{A}$ satisfies

$$0.125 < \lambda_R < 1.7, |\lambda_I| < 0.6325.$$

Obviously, $5.571 \notin (0.12, 1.5)$, $1.38 \in (0.125, 1.7)$ and $|\pm 0.66| \not< 0.6325$.

Numerical results of Examples 3.1 and 3.2 show that the eigenvalue distribution of the preconditioned matrix $\mathcal{M}_\omega^{-1}\mathcal{A}$ in Theorem 3.3 is more tighter than that of Theorem 2 in [37]. This shows that Theorem 3.3 provide valid bounds for all the real eigenvalues of the preconditioned matrix $\mathcal{M}_\omega^{-1}\mathcal{A}$ and also provide valid bounds for the real and imaginary parts of all the complex eigenvalues of the preconditioned matrix $\mathcal{M}_\omega^{-1}\mathcal{A}$.

4. Conclusion

In this paper, our goal is to discuss the eigenvalue distribution of a family of relaxed mixed constraint preconditioner (RMCP) for saddle point problems. Some valid bounds for all the eigenvalues of the corresponding preconditioned matrix are obtained and some corresponding theoretical results in [36, 37, 44] have been improved. With regard to the application of RMCP, one can see [34, 36, 37] for more details.

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Values sharing results on q-difference and derivative of meromorphic functions

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Abstract

In this paper, we mainly deal with the problem that $f(qz)$ and $f'(z)$ share common values. One of the purpose is to explore whether the classical uniqueness results remain valid or not by considering some uniqueness theorems on $f(qz)$ and $f'(z)$ sharing common values. Some examples and remarks are given to show that our results are sharp in certain senses. We also consider the entire solutions of the equation $f'(z) = f(qz)$, which is important for the uniqueness results.

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1. Introduction

We use the standard symbols and fundamental results of Nevanlinna theory [7, 10, 18]. A meromorphic function $f(z)$ means meromorphic in the complex plane $\mathbb{C}$. If $f - a$ and $g - a$ have the same zeros, then we say that f and g share the value a IM (ignoring multiplicities). If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then f and g share the value a CM (counting multiplicities).

Recall a classical result given by Rubel and Yang [16] as follows.

Theorem A. Let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share two values $a, b \in \mathbb{C}$ CM, then $f'(z) = f(z)$.

Many improvements on Theorem A were investigated afterwards. For example, $f'(z)$ was improved to $f^{(k)}(z)$ or differential polynomials of $f(z)$, the condition CM was reduced

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to IM, an entire function $f(z)$ was extended to a meromorphic function, and so on. We only recall the following result given by Mues-Steinmetz [12].

Theorem B. Let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share two values $a, b \in \mathbb{C}$ IM, then $f'(z) = f(z)$.

Recently, Qi, Liu and Yang [14] considered the problem that $f(z)$ and $f(qz)$ share common values, where $f(z)$ is a zero-order meromorphic function and $|q| = 1$. One of the results can be stated as follows.

Theorem C. [14, Theorem 1.1] Let $f(z)$ be a zero-order meromorphic function and $a_1, a_2, a_3 \in \mathbb{C} \cup \{\infty\}$ be three distinct values. If $f(z)$ and $f(qz)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(qz)$.

Here, two remarks are given to show that the conditions of Theorem C are indispensable, which are not considered in [14].

1.1. Remark. Theorem C is not valid for meromorphic functions with finite order, which can be seen by the following two examples.

1.1. Example. If $f(z) = e^z$ and $q = -1$, then $f(z)$ and $f(qz)$ share $0, 1, \infty$ CM, but $f(z) \neq f(qz)$.

1.2. Example. If $f(z) = e^{z^2}$ and $q = i$, then $f(z)$ and $f(qz)$ share $0, 1, \infty$ CM, but $f(z) \neq f(qz)$.

1.2. Remark. The condition a_1, a_2 CM and a_3 IM can not be reduced to a_1, a_2 CM in Theorem C, which can be seen by the following example.

1.3. Example. If $f(z) = \frac{2z}{(z+1)^2}$ and $q = -1$, then $f(qz) = \frac{-2z}{(1-z)^2}$. We know that $f(z)$ and $f(qz)$ share $0, 1$ CM, but $f(z) \neq f(qz)$.

However, if f is an entire function with zero-order, then the conditions of Theorem C can be reduced as follows.

Theorem D. [14, Theorem 1.2] Let f be a zero-order entire function and $a_1, a_2 \in \mathbb{C}$ be two distinct values. If $f(z)$ and $f(qz)$ share a_1 and a_2 IM, then $f(z) = f(qz)$.

Noticing the above four theorems, Theorem A and Theorem B are related to the value sharing problem on $f(z)$ and $f'(z)$, Theorem C and Theorem D are related to the value sharing problem on $f(z)$ and $f(qz)$. An interesting problem is what can we get if $f'(z)$ and $f(qz)$ share common values, where q is a non-zero constant. Some related results can be found in Section 3. Some results on the zeros distribution of q-difference differential polynomials of different types and uniqueness results can be seen in Section 4.

2. The entire solutions of $f'(z) = f(qz)$

As we all know that the differential equation $f'(z) = f(z)$ implies that $f(z) = Ae^z$, where A is a constant. Before considering the value sharing problem on $f(qz)$ and $f'(z)$, we should consider the solutions properties of the q-difference differential equation

$$f'(z) = f(qz), \tag{2.1}$$

where q is a non-zero constant. Obviously, the non-trivial entire solutions of (2.1) should be transcendental. Using the theory of series, we obtain the next result.

2.1. Theorem. *The non-trivial entire solutions of* (2.1) *must be have the form*

$$f(z) = \sum_{n=0}^{+\infty} \frac{a_0}{n!} q^{\frac{n(n-1)}{2}} z^n,$$

where a_0 is a free complex parameter.

Proof. Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$. Thus

$$f(qz) = a_0 + a_1 qz + a_2 (qz)^2 + \cdots + a_n (qz)^n + \cdots \tag{2.2}$$

and

$$f'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1} + \cdots . \tag{2.3}$$

By comparing with the coefficients of (2.2) and (2.3), we get

$$\begin{aligned} a_1 &= a_0 q^0, \\ a_2 &= \frac{a_0}{2} q^1, \\ a_3 &= \frac{a_0}{3\times 2} q^{1+2}, \\ a_4 &= \frac{a_0}{4\times 3\times 2} q^{1+2+3}, \\ a_5 &= \frac{a_0}{5\times 4\times 3\times 2} q^{1+2+3+4}, \\ a_6 &= \frac{a_0}{6\times 5\times 4\times 3\times 2} q^{1+2+3+4+5}, \ldots \end{aligned}$$

Using mathematical induction, we get $f(z)$ should have the form

$$f(z) = \sum_{n=0}^{+\infty} \frac{a_0}{n!} q^{\frac{n(n-1)}{2}} z^n.$$

□

2.1. Remark. As we all know that if $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function, the order's expression

$$\rho(g) = \overline{\lim}_{n\to\infty} \frac{n \log n}{\log \frac{1}{|a_n|}}.$$

Thus, we conclude that $\rho(f) = 0$ if $|q| \neq 1$ and $\rho(f) = 1$ if $|q| = 1$ in Theorem 2.1. Obviously, if $q = 1$, then $f(z) = \sum_{n=0}^{+\infty} \frac{a_0}{n!} z^n = a_0 e^z$.

3. Some results on $f(qz)$ and $f'(z)$ share common values

Let us recall the classical results in the uniqueness theory of meromorphic functions, the five-point, resp. four-point, theorems due to Nevanlinna [15].

The five-point theorem. If two meromorphic functions f, g share five distinct values in the extended complex plane IM, then $f \equiv g$.

The four-point theorem. If two meromorphic functions f, g share four distinct values in the extended complex plane CM, then $f \equiv T(g)$, where T is a Möbius transformation.

If the meromorphic function g has a special relationship with f, then the number five or four can be reduced. For example, considering the value sharing problem on $f(z)$ and $f(z+c)$ [8, Theorem 2] or $f(z)$ and $f(qz)$ [14, Theorem 1.1], the number is three. Before stating our results, we need the following lemma [18, Theorem 2.17].

3.1. Lemma. *Let f and g be non-constant meromorphic functions with the order less than one. If f and g share 0 and ∞ CM, then there exists a non-zero constant K satisfying $f = Kg$.*

Let $f_1 = \frac{f-a_1}{f-a_2}$ and $g_1 = \frac{g-a_1}{g-a_2}$. If f and g share a_1 and a_2 CM, then f_1 and g_1 share $0, \infty$ CM, thus we have $\frac{f-a_1}{f-a_2} = k\frac{g-a_1}{g-a_2}$.

3.2. Theorem. *Let f be a meromorphic function with order $\rho(f) < 1$ and let $a_1, a_2 \in \mathbb{C} \cup \{\infty\}$ and $a_3 \in \mathbb{C}$ be three distinct values. If $f(qz)$ and $f'(z)$ share a_1, a_2 CM and a_3 IM, then $f'(z) = f(qz)$.*

Proof. If $a_1, a_2, a_3 \in \mathbb{C}$. Let $F(z) = \frac{f'(z)-a_1}{f'(z)-a_2} \cdot \frac{a_3-a_2}{a_3-a_1}$ and $G(z) = \frac{f(qz)-a_1}{f(qz)-a_2} \cdot \frac{a_3-a_2}{a_3-a_1}$. Thus, we have $F(z)$ and $G(z)$ share $0, \infty$ CM and 1 IM. Since that $F(z)$ and $G(z)$ are meromorphic functions with $\rho(f) < 1$, then $F(z) = kG(z)$ follows from Lemma 3.1. If the value 1 is not the Picard exceptional value, then $k = 1$, thus $F(z) = G(z)$. If the value 1 is the Picard exceptional value, we have $\frac{F(z)-1}{G(z)-1}$ has no zeros and poles. Hence, we have $\frac{F(z)-1}{G(z)-1} = C$, which implies that $k = 1$, thus $F(z) = G(z)$. We conclude that $f'(z) = f(qz)$.

If one of a_1, a_2 is ∞, without loss of generality, we suppose that $a_1 = \infty$. Let $F(z) = f'(z) - a_2$ and $G(z) = f(qz) - a_2$. Thus $F(z)$ and $G(z)$ share $0, \infty$ CM. From Lemma 3.1, we have $F(z) = kG(z)$. Combining the above with the condition that a_3 is IM shared, then $k = 1$, thus $f'(z) = f(qz)$. □

3.1. Remark. Theorem 3.2 is not valid for meromorphic functions with $\rho(f) \geq 1$, which can be seen by taking $f(z) = e^z$ and $q = -1$. We see that $f(qz)$ and $f'(z)$ share $0, 1, -1$ CM, but $f'(z) \neq f(qz)$.

3.3. Theorem. *Let $f(z)$ be a non-constant entire function, q be a non-zero constant. If $f(qz)$ and $f'(z)$ share two distinct constants $a, b \in \mathbb{C}$ CM and one of a, b is the Picard exceptional value, then $f'(z) = f(qz)$ or $f(z) = e^{-Az+B}$, $-Ae^{2B} = b^2$ and $q = -1$.*

For the proof of Theorem 3.3, we need the following three lemmas.

3.4. Lemma. [18, Theorem 1.47] *Let $h(z)$ be a non-constant entire function and $f(z) = e^{h(z)}$. Then $T(r, h') = S(r, f)$.*

3.5. Lemma. [18, Theorem 1.56] *Let f_1, f_2, f_3 be meromorphic functions such that f_1 is not a constant. If $f_1 + f_2 + f_3 = 1$ and if*

$$\sum_{j=1}^{3} N(r, 1/f_j) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r),$$

where $\lambda < 1$ and $T(r) := \max_{1 \leq j \leq 3} T(r, f_j)$, then either $f_2 = 1$ or $f_3 = 1$.

3.6. Lemma. [7, Theorem 3.7] *Let $f(z)$ be an entire function. If $f(z)$ and $f^{(l)}(z)$ $(l \geq 2)$ have no zeros, then $f(z) = e^{Az+B}$, where A, B are constants.*

Proof. One of a, b is the Picard exceptional value, without loss of generality, we suppose that a is the Picard exceptional value. Thus

$$f(qz) - a = e^{\alpha(z)} \tag{3.1}$$

and

$$f'(z) - a = e^{\beta(z)}, \tag{3.2}$$

where $\alpha(z)$ and $\beta(z)$ are non-constant entire functions. From Lemma 3.4, we have $T(r, \alpha'(z)) = S(r, f(qz))$. Differentiating $f(qz)$, we have

$$f'(qz) = \frac{1}{q} e^{\alpha(z)} \alpha'(z) = a + e^{\beta(qz)}. \tag{3.3}$$

From (3.1) and (3.3), we get

$$T(r, e^{\alpha(z)}) \;=\; T(r, f(qz)) + O(1) \leq T(r, f'(qz)) + S(r, f(qz)), \tag{3.4}$$

$$T(r, e^{\beta(qz)}) = T(r, f'(qz)) + O(1). \tag{3.5}$$

If $a \neq 0$, from (3.3), we conclude that

$$\frac{1}{aq} e^{\alpha(z)} \alpha'(z) - \frac{e^{\beta(qz)}}{a} = 1. \tag{3.6}$$

Using the second main theorem for three small functions [7, Theorem 2.5], we have

$$\begin{aligned} T(r, e^{\alpha(z)}) &\leq \overline{N}\left(r, \frac{1}{e^{\alpha(z)} - \frac{aq}{\alpha'(z)}}\right) + S(r, e^{\alpha(z)}) \\ &= \overline{N}\left(r, \frac{1}{e^{\beta(qz)}}\right) + S(r, e^{\alpha(z)}) \\ &= S(r, e^{\alpha(z)}), \end{aligned} \tag{3.7}$$

which is a contradiction. Thus, $a = 0$. From (3.3), then

$$e^{\beta(qz) - \alpha(z)} = \frac{\alpha'(z)}{q}. \tag{3.8}$$

Since that $b \neq 0$ is CM shared by $f'(z)$ and $f(qz)$, then we get $\frac{f'(z)-b}{f(qz)-b} = e^{\gamma(z)}$, where $\gamma(z)$ is an entire function. Thus, combining the above with (3.1), (3.2), (3.8), we have

$$e^{\beta(z)} - b = f'(z) - b = e^{\gamma(z)}(f(qz) - b) = e^{\gamma(z)}(e^{\alpha(z)} - b).$$

Since $b \neq 0$, then

$$\frac{e^{\beta(z)}}{b} + e^{\gamma(z)} - \frac{e^{\gamma(z)+\alpha(z)}}{b} = 1.$$

From Lemma 3.5, if $e^{\gamma(z)} \equiv 1$, then $f'(z) = f(qz)$ follows. If $e^{\beta(z)} \equiv b$, which implies that $f'(z) \equiv b$, which is impossible. If $\frac{e^{\gamma(z)+\alpha(z)}}{-b} \equiv 1$, then we also have $e^{\beta(z)-\gamma(z)} \equiv -b$. Thus $e^{\alpha(z)+\beta(z)} = b^2$, which implies that $\alpha(z) + \beta(z) \equiv d$, where d is a constant. So $\beta(qz) \equiv -\alpha(qz) + d$. Combining the above with (3.8), we have

$$e^{-\alpha(qz)-\alpha(z)+d} = \frac{\alpha'(z)}{q}. \tag{3.9}$$

Remark that the left hand of (3.9) has no zeros, we have $\alpha'(z)$ has no zeros, thus either $\alpha(z) = Az + B$ or $\alpha(z)$ is a transcendental entire function. If $\alpha(z) = Az + B$, from (3.9), we have $q = -1$ and $e^{d-2B} = -A$. From (3.1), we have $f(z) = e^{-Az+B}$ and $-Ae^{2B} = b^2$. If $\alpha(z)$ is a transcendental entire function, since $\beta(z) \equiv -\alpha(z) + d$, then $\beta'(z)$ also has no zeros. Thus from $f''(z) = \beta'(z)e^{\beta(z)}$, then we have $f(z)$ and $f''(z)$ have no zeros, $f(z) = e^{az+b}$ follows by Lemma 3.6, which implies that $\alpha(z)$ is a polynomial, a contradiction. Thus, we have the proof of Theorem 3.3. □

3.2. Remark. If a, b are not Picard exceptional values, then Theorem 3.3 is not valid, which can be seen by the function $f(z) = 3a - \frac{a}{e^{2z}}$ and $q = -1$. Thus, $f'(z) = \frac{2a}{e^{2z}}$ and $f(qz) = 3a - ae^{2z}$ share a and $2a$ CM, but $f'(z) \neq f(qz)$.

In what follows, we will use the properties of the solutions of Fermat type equations to consider the problem that two functions share one common value. Recall the classical Fermat type equation

$$a(z)f(z)^n + b(z)g(z)^n = 1. \tag{3.10}$$

Yang [17, Theorem 1] obtained the following result.

Theorem E. Let $a(z), b(z), f(z), g(z)$ be meromorphic functions, m, n be positive integers. Then (3.10) can not hold, if $T(r, a(z)) = S(r, f)$ and $T(r, b(z)) = S(r, g)$, unless $m = n = 3$. If $f(z)$ and $g(z)$ are entire, then (3.10) can not hold, even if $m = n = 3$.

We get the following result, which is an improvement of [14, Corollary 1.4].

3.7. Theorem. *Let f be a zero-order non-constant entire function, and $q \neq 0$, $n \geq 2$ be an integer, and let $F = f^n$. If $F(z)$ and $F(qz)$ share a non-zero constant a CM, then $f(qz) = tf(z)$, where $t^n = 1$.*

Proof. Suppose that $F(z)$ and $F(qz)$ share a non-zero constant a CM, then we have $\frac{F(qz)-a}{F(z)-a} = C$. Thus, we have

(3.11) $$f(qz)^n - Cf(z)^n = a(1 - C).$$

If $C = 1$, then we have $f(qz) = tf(z)$, where $t^n = 1$. If $C \neq 1$, from Theorem E, we know that $n \leq 2$. From the condition $n \geq 2$, then $n = 2$. In this case $f(qz) - \sqrt{c}f(z)$ and $f(qz) + \sqrt{c}f(z)$ have no zeros. Since that $f(z)$ is zero-order entire function and combining the Hadamard factorization theorem, we obtain $f(z)$ should be a constant. □

3.3. Remark. (1) Theorem 3.7 is not valid for finite order entire function $f(z)$, which can be seen by taking $f(z) = e^z$, $q = -1$. Then $f(z)^n$ and $f(qz)^n$ share the value 1 CM, but $f(qz) \neq tf(z)$, where t is a constant.

(2) The condition of $a \neq 0$ can not be deleted, which can be seen by $f(z) = z^n$ and $f(qz) = q^n z^n$ and $q^n \neq 1$, thus $f(z)$ and $f(qz)$ share the value 0 CM, but $f(z) \neq f(qz)$.

(3) The condition $n \geq 2$ can not be improved to $n \geq 1$, which can be seen by $f(z) = z^n + a$ and $q^n = c$, thus $\frac{f(qz)-a}{f(z)-a} = c$. Here, $f(qz)$ and $f(z)$ share the value a CM, but $f(qz) \neq tf(z)$.

Brück conjecture is well-known as a classical problem in value sharing, which can be stated as follows.

Conjecture. Let $f(z)$ be a non-constant entire function, the hyper-order $\rho_2(f)$ is not a positive integer or infinite. If $f(z)$ and $f'(z)$ share a finite value b CM, then $\frac{f'-b}{f-b} = c$, where c is a non-zero constant.

The conjecture has been verified in special cases only: (1) f is of finite order, see [5]; (2) $b = 0$, see [3]; (3) $N(r, \frac{1}{f'}) = S(r, f)$, see [3]. we also want to summarize some results on q-difference analogue of Brück conjecture.

3.8. Theorem. *Let f be a non-constant entire function with $\rho(f) < 1$, and $q \neq 0$. If $f'(z)$ and $f(qz)$ share a constant a CM, then $\frac{f'(z)-a}{f(qz)-a} = c$.*

3.4. Remark. Theorem 3.8 is easily proved. Here, we state it to show a result similar as *Brück* conjecture. Theorem 3.8 is not valid for finite order entire functions, which can be seen by $f(z) = e^z$, $q = -1$, thus $\frac{f'(z)-1}{f(qz)-1} = -e^z$, where $f'(z)$ and $f(qz)$ share the value 1 CM.

4. Results on values shared by $f(qz)^n f'(z)$ and $g(qz)^n g'(z)$

Hayman conjecture [6] is an important problem in the theory of value distribution. It was also considered by some authors later, such as [2, 4, 13].

Theorem F.[4, Theorem 1] Let f be a transcendental meromorphic function. If $n \geq 1$ is a positive integer, then $f(z)^n f'(z) - 1$ has infinitely many zeros.

Recently, some authors investigated the zeros of $f(z)^n f(z+c) - a$, $f(z)^n f(qz) - a$ or their improvements, where a is a non-zero constant. Some related results can be found in [9, 11, 19]. The main aim of these results is to get the sharp value of n to ensure that the difference polynomials or q-difference polynomials admit infinitely many zeros. It is interesting to consider the value distribution of $f(qz)^n f'(z) - a(z)$, where $a(z)$ is a small function with respect to f. We obtain the following result.

4.1. Theorem. *Let $f(z)$ be a transcendental entire function with zero-order, $q \in \mathbb{C} \setminus \{0\}$ and $n \geq 1$. Then $f(qz)^n f'(z) - q(z)$ has infinitely many zeros, where $q(z)$ is a non-zero polynomial.*

4.2. Theorem. *Let $f(z)$ be a transcendental meromorphic function with zero-order, $q \in \mathbb{C} \setminus \{0\}$ and $n \geq 9$. Then $f(qz)^n f'(z) - a(z)$ has infinitely many zeros, where $a(z)$ is a non-zero small function with respect to $f(z)$.*

4.1. Remark. Theorem 4.1 is not valid for finite order entire functions, which can be seen by $f(z) = e^z$, $q = -\frac{1}{n}$, and $a(z)$ is a non-constant polynomial, thus $f(qz)^n f'(z) - a(z) = 1 - a(z)$ has finitely many zeros.

For the proofs of Theorems 4.1 and 4.2, we need the following results, which were firstly considered by Barnett et al.[1], Zhang and Korhonen [19] obtained the following version.

4.3. Lemma. [19, Theorem 1.1] *Let $f(z)$ be a non-constant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$T(r, f(qz)) = T(r, f) + S(r, f)$$

on a set of lower logarithmic density 1.

4.4. Lemma. [1, Theorem 1.1] *Let $f(z)$ be a non-constant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S(r, f)$$

on a set of logarithmic density 1.

4.5. Lemma. *Let $f(z)$ be a non-constant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$(n-2)T(r, f) \leq T(r, f(qz)^n f'(z)) + S(r, f) \leq (n+2)T(r, f) \tag{4.1}$$

on a set of lower logarithmic density 1. If $f(z)$ is a non-constant zero-order entire function, then

$$nT(r, f) \leq T(r, f(qz)^n f'(z)) + S(r, f) \leq (n+1)T(r, f). \tag{4.2}$$

Proof. From Lemma 4.3 and the fact that $T(r, f'(z)) \leq 2T(r, f) + S(r, f)$ when $f(z)$ is a meromorphic function, we get

$$T(r, f(qz)^n f'(z)) \leq (n+2)T(r, f) + S(r, f).$$

Hence, the right hand side of (4.1) is true. On the other hand

$$\begin{aligned}(n+1)T(r,f(z)) &= T(r,f(qz)^{n+1})+S(r,f)\\ &= T\left(r,\frac{f(qz)^{n+1}f'(z)}{f'(z)}\right)+S(r,f)\\ &\le T\left(r,\frac{f(qz)}{f'(z)}\right)+T\left(r,f(qz)^n f'(z)\right)+S(r,f)\\ &\le T\left(r,\frac{f'(z)}{f(qz)}\right)+T\left(r,f(qz)^n f'(z)\right)+S(r,f)\\ &\le N\left(r,\frac{1}{f(qz)}\right)+N(r,f'(z))+T\left(r,f(qz)^n f'(z)\right)+S(r,f)\\ &\le 3T(r,f)+T(r,f(qz)^n f'(z))+S(r,f) \qquad (4.3)\end{aligned}$$

on a set of lower logarithmic density 1. Thus, the left hand side of (4.1) is proved. If $f(z)$ is a transcendental zero-order entire function, then we have

$$\begin{aligned}(n+1)T(r,f(z) &= (n+1)m(r,f)\\ &\le m(r,f(qz)^{n+1})+S(r,f)\\ &\le m\left(r,\frac{f(qz)}{f'(z)}\right)+m\left(r,f(qz)^n f'(z)\right)+S(r,f)\\ &\le T\left(r,\frac{f'(z)}{f(qz)}\right)+T\left(r,f(qz)^n f'(z)\right)+S(r,f)\\ &\le T(r,f)+T\left(r,f(qz)^n f'(z)\right)+S(r,f) \qquad (4.4)\end{aligned}$$

on a set of lower logarithmic density 1. Combining Lemma 4.3 with the fact that $T(r,f'(z))\le T(r,f)+S(r,f)$ when $f(z)$ is an entire function, we get (4.2). □

Proofs of Theorems 4.1 and 4.2: Assume that $f(qz)^n f'(z)-q(z)$ has only finitely many zeros, if $f(z)$ is a transcendental zero-order entire function, from Hadmard factorization theorem, we have $f(qz)^n f'(z)-q(z)=p(z)$, where $p(z)$ is a non-zero polynomial. Thus, we have $nT(r,f)+S(r,f)\le T(r,f(qz)^n f'(z))=O(\log r)$, which is impossible.

If $f(z)$ is a transcendental zero-order meromorphic function, using the second main theorem, we have

$$\begin{aligned}(n-2)T(r,f(z)) &\le T(r,f(qz)^n f'(z))+S(r,f)\le \overline{N}(r,f(qz)^n f'(z))\\ &+\overline{N}\left(r,\frac{1}{f(qz)^n f'(z)}\right)+\overline{N}\left(r,\frac{1}{f(qz)^n f'(z)-a(z)}\right)+S(r,f)\\ &\le 6T(r,f)+\overline{N}\left(r,\frac{1}{f(qz)^n f'(z)-a(z)}\right)+S(r,f), \qquad (4.5)\end{aligned}$$

which is a contradiction with $n\ge 9$.

Let $P(z)=a_n z^n+a_{n-1}z^{n-1}+\cdots+a_1 z+a_0$ be a non-zero polynomial, where $a_0,a_1,\ldots,a_n(\ne 0)$ are complex constants and t_P is the number of the distinct zeros of $P(z)$. The following, we will consider the generally case of $P(f(qz))f'(z)-a(z)$, where $a(z)$ is a small function with respect to $f(z)$.

4.6. Theorem. *Let $f(z)$ be a transcendental entire function with zero-order, $q\in\mathbb{C}\setminus\{0\}$ and $n\ge 1$. Then $P(f(qz))f'(z)-q(z)$ has infinitely many zeros, where $q(z)$ is a non-zero polynomial.*

4.7. Theorem. *Let $f(z)$ be a transcendental meromorphic function with zero-order, $q\in\mathbb{C}\setminus\{0\}$ and $n\ge 2t_P+7$. Then $P(f(qz))f'(z)-a(z)$ has infinitely many zeros, where $a(z)$ is a non-zero small function with respect to $f(z)$.*

Using the similar method as the proof of Lemma 4.5, we have the following lemma, which is needed for the proofs of Theorems 4.6 and 4.7.

4.8. Lemma. *Let $f(z)$ be a non-constant zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$(n-2)T(r,f) \le T(r, P(f(qz))f'(z)) + S(r,f) \le (n+2)T(r,f) \tag{4.6}$$

on a set of lower logarithmic density 1. If $f(z)$ be a non-constant zero-order entire function,

$$nT(r,f) \le T(r, P(f(qz))f'(z)) + S(r,f) \le (n+1)T(r,f) \tag{4.7}$$

on a set of lower logarithmic density 1.

Finally, we consider the uniqueness of $f(qz)^n f'(z)$ and $g(qz)^n g'(z)$ sharing a non-zero polynomial and obtain the following result.

4.9. Theorem. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero-order, $q \in \mathbb{C} \setminus \{0\}$ and $n \ge 5$. If $f(qz)^n f'(z)$ and $g(qz)^n g'(z)$ share a non-zero polynomial $p(z)$ CM, then we have $f(qz)^n f'(z) = g(qz)^n g'(z)$.*

Proof. From the conditions, we get $\frac{f(qz)^n f'(z) - p(z)}{g(qz)^n g'(z) - p(z)} = c$. If $c = 1$, then $f(qz)^n f'(z) = g(qz)^n g'(z)$ follows. If $c \neq 1$, then we have

$$f(qz)^n f'(z) - cg(qz)^n g'(z) = p(z)(1-c). \tag{4.8}$$

Using the second main theorem, we get

$$\begin{aligned} T(r, f(qz)^n f'(z)) &\le \overline{N}(r, f(qz)^n f'(z)) + \overline{N}\left(r, \frac{1}{f(qz)^n f'(z)}\right) \\ &+ \overline{N}\left(r, \frac{1}{f(qz)^n f'(z) - (1-c)p(z)}\right) + S(r, f(qz)^n f'(z)) \\ &\le N\left(r, \frac{1}{f(qz)}\right) + N\left(r, \frac{1}{f'(z)}\right) + \overline{N}\left(r, \frac{1}{g(qz)^n g'(z)}\right) + S(r,f) \\ &\le 2T(r,f) + 2T(r,g) + S(r,f). \end{aligned} \tag{4.9}$$

Similar as the above, we also get

$$T(r, g(qz)^n g'(z)) \le 2T(r,f) + 2T(r,g) + S(r,g). \tag{4.10}$$

Combining (4.9), (4.10) with (4.2), we have

$$n[T(r,f) + T(r,g)] \le 4[T(r,f) + T(r,g)] + S(r,f) + S(r,g),$$

which is a contradiction with the condition $n \ge 5$. □

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Around Poisson–Mehler summation formula

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Abstract

We study polynomials in x and y of degree $n+m$: $\{Q_{m,n}(x,y|t,q)\}_{n,m\geq 0}$ that are related to the generalization of Poisson–Mehler formula i.e. to the expansion $\sum_{i\geq 0}\frac{t^{i}}{[i]_{q}!}H_{i+n}(x|q)H_{m+i}(y|q)$ $=Q_{n,m}(x,y|t,q)\sum_{i\geq 0}\frac{t^{i}}{[i]_{q}!}H_{i}(x|q)H_{m}(y|q)$, where $\{H_{n}(x|q)\}_{n\geq -1}$ are the so-called $q-$Hermite polynomials (qH). In particular we show that the spaces $span\{Q_{i,n-i}(x,y|t,q):i=0,\ldots,n\}_{n\geq 0}$ are orthogonal with respect to a certain measure (two-dimensional $(t,q)-$Normal distribution) on the square $\left\{(x,y):|x|,|y|\leq 2/\sqrt{1-q}\right\}$ being a generalization of two-dimensional Gaussian measure. We study structure of these polynomials showing in particular that they are rational functions of parameters t and q. We use them in various infinite expansions that can be viewed as simple generalization of the Poisson-Mehler summation formula. Further we use them in the expansion of the reciprocal of the right hand side of the Poisson-Mehler formula.

Keywords: $q-$Hermite, big $q-$Hermite, Al-Salam–Chihara, orthogonal polynomials, Poisson-Mehler summation formula. Orthogonal polynomials on the plane.

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1. Introduction and auxiliary results

1.1. Preface. We consider various generalizations of the celebrated Poisson–Mehler formula (see e.g. [10], (13.1.24) or [1], (10.11.17)):

$$\sum_{n\geq 0}\frac{\rho^{n}}{[n]_{q}!}H_{n}\left(x|q\right)H_{n}\left(y|q\right)=\frac{\left(\rho^{2}\right)_{\infty}}{\prod_{j=0}^{\infty}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{j}\right)}, \tag{1.1}$$

where $\{H_{n}\}_{n\geq 0}$ denote $q-$Hermite polynomials and $\omega\left(x,y|t\right)$ are certain polynomials symmetric in x and y of degree two. These polynomials as well as symbols $[n]_{q}!$ and $\left(\rho^{2}\right)_{\infty}$ are defined and explained in Sections 1.2 and 1.3. There exist many proofs of (1.1) (e.g. see [10], [1], [2], [21]). In [22] a certain generalization of (1.1) has been proved by the author. It was used in calculating moments of the so called Askey–Wilson distribution.

In the paper we consider functions

$$\gamma_{i,j}\left(x,y|\rho,q\right)=\sum_{n\geq 0}\frac{\rho^{n}}{[n]_{q}!}H_{n+i}\left(x|q\right)H_{n+j}\left(y|q\right). \tag{1.2}$$

for all $i,j\geq 0$. It was shown by the author in [21] (Lemma 3) that:

$$\gamma_{i,j}\left(x,y|\rho,q\right)=Q_{i,j}\left(x,y|\rho,q\right)\gamma_{0,0}\left(x,y|\rho,q\right), \tag{1.3}$$

where $Q_{i,j}\left(x,y|\rho,q\right)$ is a certain polynomial in x, y of degree $i+j$. Hence (1.3) can be viewed as a generalization of (1.1).

The main object of the paper is to study the properties and later the rôle of the polynomials $Q_{i,j}\left(x,y|\rho,q\right)$ in obtaining a family of two dimensional orthogonal polynomials as well as various expansions that can be viewed as either generalizations of (1.1) or expansions more or less directly related to this formula.

In particular we find generating function of these polynomials, we express them as linear combinations of polynomials belonging to families of polynomials of one variable.

We also analyze the measure (the so-called $(\rho,q)-2Normal$ measure) on the square $S\left(q\right)\times S\left(q\right)$ with the density defined by (2.5) below, that can be easily constructed from the densities of measures that make $q-$Hermite and the so-called Al-Salam–Chihara polynomials orthogonal and which can viewed as a generalization of bivariate Normal distribution. Interval $S\left(q\right)$ is defined by (1.5). The probabilistic aspects of this distribution were presented in [19]. We point out the rôle of the polynomials $Q_{n,m}$ in further analysis of this measure. In particular we introduce spaces of functions of two variables

$$\Lambda_{n}(x,y|\rho,q)=span\left\{Q_{i,n-i}(x,y,|\rho,q),i=0,\ldots,n\right\},n\geq 0 \tag{1.4}$$

and show that they are orthogonal with respect to $(\rho,q)-2Normal$ measure. Hence these spaces form the direct sum decomposition of the space of functions that are square integrable with respect to $(\rho,q)-2Normal$ measure.

Further we use these polynomials to obtain various infinite expansions. In particular we obtain an expansion of the reciprocal of the right hand side of (1.1) in an infinite series. In [21], (formula 5.3) one such expansion was presented. The expansions was non-symmetric in x and y (for each finite sum). This time the expansion is symmetric in x and y.

Among other possible views one can look at the results of paper as the generalization of the results of the two papers of Van der Jeugt et al. [12], [13]. The authors of these papers introduced convolutions of known families of classical orthogonal polynomials such as Hermite or Laguerre considered at two variables thus obtaining bivariate polynomials. They applied their results in Lie algebra and its generalizations.

Our "convolutions" concern generalizations of Hermite polynomials (q-Hermite, and Al-Salam–Chihara). As possible applications we mean the ones in analysis, two dimensional orthogonal polynomials theory or probability.

Since in our paper appear kernels built of mostly $q-$Hermite and Al-Salam–Chihara one should remark that some of the technics used in the proofs resemble those used in e.g. [8]. But by no means results are the same.

The paper is organized as follows. In the next two Subsections (i.e. 1.2 and 1.3) we provide simple introduction to $q-$series theory presenting typical notation used and presenting a few typical families of the so called basic orthogonal polynomials. The word basic comes from the base which is the parameter in most cases denoted by q. We do this since notation and terminology used in $q-$series theory is somewhat specific and not widely known to those not working within this field. We are also purposely not using notation based on hypergeometric series since it is mostly known to specialists of special functions theory. We believe that the results presented in the paper can be applied in various fields of traditional analysis like the theory of Fourier expansions, theory of reproducing kernels, orthogonal polynomials theory and last but not least probability theory. Then in Section 2 we present our main results, open questions and remarks are in Section 3 while laborious proofs are in Section 4.

1.2. Notation. We use notation traditionally used in the so called $q-$series theory. Since not all readers are familiar with it we will recall now this notation.

Throughout the paper, q is a parameter. We will assume that $-1 < q \leq 1$ unless otherwise stated. Let us define $[0]_q = 0$; $[n]_q = 1+q+\ldots+q^{n-1}$, $[n]_q! = \prod_{j=1}^{n}[j]_q$, with $[0]_q! = 1$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[n-k]_q![k]_q!} & , \quad n \geq k \geq 0 \\ 0 & , \quad otherwise \end{cases} .$$

It will be useful to use the so called $q-$Pochhammer symbol for $n \geq 1$:

$$(a;q)_n = \prod_{j=0}^{n-1}\left(1-aq^j\right),$$

$$(a_1,a_2,\ldots,a_k;q)_n = \prod_{j=1}^{k}(a_j;q)_n,$$

with $(a;q)_0 = 1$. Often $(a;q)_n$ as well as $(a_1,a_2,\ldots,a_k;q)_n$ will be abbreviated to $(a)_n$ and $(a_1,a_2,\ldots,a_k)_n$ respectively, if it will not cause misunderstanding.

It is easy to notice that for $|q|<1$ we have $(q)_n = (1-q)^n[n]_q!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_{n-k}(q)_k} & , \quad n \geq k \geq 0 \\ 0 & , \quad otherwise \end{cases}$.

Notice that $[n]_1 = n, [n]_1! = n!$, $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$, $(a;1)_n = (1-a)^n$ and $[n]_0 = \begin{cases} 1 & if \quad n \geq 1 \\ 0 & if \quad n = 0 \end{cases}$, $[n]_0! = 1$, $\begin{bmatrix} n \\ k \end{bmatrix}_0 = 1$, $(a;0)_n = \begin{cases} 1 & if \quad n = 0 \\ 1-a & if \quad n \geq 1 \end{cases}$.

In the sequel we shall also use the following useful notation:

$$S(q) = \begin{cases} [-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}] & if \quad |q|<1 \\ \mathbb{R} & if \quad q = 1 \end{cases}, \tag{1.5}$$

$$I_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}. \tag{1.6}$$

1.3. Polynomials.

1.3.1. *$q-$Hermite.* Let $\{H_n(x|q)\}_{n\geq 0}$ denote the family of the so called $q-$ Hermite (briefly qH) polynomials. That is the one parameter family of orthogonal polynomials satisfying the following three term recurrence:

$$H_{n+1}(x|q) = xH_n(x|q) - [n]_q H_{n-1}(x|q), \tag{1.7}$$

with $H_{-1}(x|q) = 0$ and $H_0(x|q) = 1$. In fact in the literature (see e.g. [1], [10], [15]) we encounter more often the re-scaled versions of these polynomials. Namely more often appear under the name of $q-$Hermite polynomials the following polynomials $\{h_n(x|q)\}_{n\geq 0}$ defined by their three term recurrence:

$$h_{n+1}(x|q) = 2xh_n(x|q) - (1-q^n)h_{n-1}(x|q), \tag{1.8}$$

with $h_{-1}(x|q) = 0$ and $h_0(x|q) = 1$. These polynomials are related to one another by the relationship $\forall n \geq -1$:

$$H_n(x|q) = \frac{h_n\left(x\sqrt{1-q}/2|q\right)}{(1-q)^{n/2}}, \tag{1.9}$$

for $|q| < 1$. For $q = 1$ we have $h_n(x|1) = 2^n x^n$ while $H_n(x|1) = H_n(x)$, where polynomials $H_n(x)$ are the so called 'probabilistic' Hermite polynomials i.e. classical, monic[†] polynomials orthogonal with respect to $\exp\left(-x^2/2\right)$. Observe further that $h_n(x|0) = U_n(x)$ and $H_n(x|0) = U_n(x/2)$, where U_n denotes the so called Chebyshev polynomial of the second kind (for details see e.g. [1]).

The polynomials H_n have nice probabilistic interpretation (see e.g. [22]) and besides they constitute the real generalization of the ordinary Hermite polynomials. That is why we will use them in this paper. The results presented here can be easily adopted and expressed in terms of polynomials h_n.

The generating function of these polynomials is given by the following formula that is in fact adapted to our setting formula (14.26.1) of [15]

$$\varphi_H(x|\rho,q) = \sum_{n\geq 0}\frac{\rho^n}{[n]_q!}H_n(x|q) = \frac{1}{\prod_{j=0}^{\infty} v\left(x\sqrt{1-q}/2|\rho q^j\sqrt{1-q}\right)}, \tag{1.10}$$

convergent for $|\rho(1-q)| < 1$, $x \in S(q)$, where we denoted

$$v(x|t) = 1 - 2xt + t^2. \tag{1.11}$$

Let us observe that $\forall x \in [-1,1]$, $t \in \mathbb{R} : v(x|t) \geq 0$.

Adapting formula (14.26.2) of [15] to our setting we have:

$$\int_{S(q)} H_n(x|q)\, H_m(x|q)\, f_N(x|q)\, dx = [n]_q!\delta_{mn}, \tag{1.12}$$

with

$$f_N(x|q) = \frac{\sqrt{(1-q)(4-(1-q)x^2)}\,(q)_\infty}{2\pi}\prod_{j=1}^{\infty} l\left(x\sqrt{1-q}/2|q^j\right), \tag{1.13}$$

for $x \in S(q)$, where

$$l(x|a) = (1+a)^2 - 4ax^2. \tag{1.14}$$

[†] i.e. polynomials with leading coefficient equal to 1.

Ismail et al. showed that (see [11])

$$\lim_{q\to 1^-} f_N(x|q) = \frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right), \tag{1.15}$$

$$\lim_{q\to 1^-} \varphi_H(x|\rho,q) = \exp\left(x\rho - x^2/2\right). \tag{1.16}$$

Apart from $q-$Hermite polynomials we will need the so called big $q-$Hermite (briefly bqH) polynomials $\{H_n(x|a,q)\}_{n\geq -1}$ with $a\in\mathbb{R}$. They are defined through their three term recurrence:

$$H_{n+1}(x|a,q) = (x-aq^n)H_n(x|a,q) - [n]_q H_{n-1}(x|a,q), \tag{1.17}$$

with $H_{-1}(x|a,q)=0$, $H_0(x|a,q)=1$. To support intuition let us remark that $H_n(x|a,1)$ $= H_n(x-a)$ and $H_n(x|a,0) = U_n(x/2) - aU_{n-1}(x/2)$.

One knows its relationship with the $q-$Hermite polynomials:

$$H_n(x|a,q) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q (-a)^k q^{\binom{k}{2}} H_{n-k}(x|q),$$

and that (see e.g. [15], (14.18.2) with an obvious modification for polynomials H_n):

$$\int_{S(q)} H_n(x|a,q)\,H_m(x|a,q)\,f_{bN}(x|a,q)\,dx = [n]_q!\delta_{mn},$$

$$\sum_{n\geq 0}\frac{t^n}{[n]_q!}H_n(x|a,q) = \varphi_H(x|t,q)\left((1-q)at\right)_\infty,$$

where

$$f_{bN}(x|a,q) = f_N(x|q)\,\varphi_H(x|a,q). \tag{1.18}$$

We will need the following Lemma concerning another relationship between polynomials $H_n(x|q)$ and $H_n(x|a,q)$.

1.1. Lemma. *Let us define for*
$\forall n\geq 0; x\in S(q); (1-q)t^2<1 : \eta_n(x|t,q) = \sum_{j\geq 0}\frac{t^j}{[j]_q!}H_{j+n}(x|q)$. *Then*

$$\eta_n(x|t,q) = H_n(x|t,q)\,\varphi_H(x|t,q),$$

where $H_n(x|t,q)$ is the bqH polynomial defined by (1.17).

Proof. In a version with continuous $q-$Hermite polynomials h defined by (1.8) and $h_n(x|t,q)$ are the big $q-$Hermite polynomials as defined in [15] (14.18.4) this formula has been proved as a particular case in [25] (2.1). We notice that $\eta_0(x|t,q) = \varphi_H(x|t,q)$. To switch to polynomials H_n using (1.9) is elementary. □

1.2. Remark. Let us remark that Carlitz in [7] considered similar shifted characteristic functions of the form $\sum_{j\geq 0}\frac{t^j}{(q)_j}w_{n+j}(x|q)$ with Rogers–Szegö polynomials w_n (see discussion below following formula (2.3)). From this result of Carlitz one can also deduce assertion of Lemma 1.1.

1.3.2. *Al-Salam–Chihara.* Next family of polynomials that we are going to consider depends on 2 (apart from q) parameters denoted by a and b, that satisfy the following three term recurrence (see e.g. [15],(14.8.4)):

$$A_{n+1}(x|a,b,q) = (2x-(a+b)q^n)A_n(x|a,b,q) - (1-abq^{n-1})(1-q^n)A_{n-1}(x|y,\rho,q), \tag{1.19}$$

with $A_{-1}(x|a,b,q)=0$, $A_0(x|a,b,q)=1$. These polynomials will be called Al-Salam–Chihara polynomials $\{A_n(x|a,b,q)\}_{n\geq -1}$ (briefly ASC). We will assume in the sequel that $|ab| < 1$. This assumption together with $|q| \leq 1$ guarantees that the measure that makes these polynomials orthogonal is positive. It follows directly follows from Favard's theorem since then $(1 - abq^{n-1})(1 - q^n) > 0$.

In the sequel in fact we will consider these polynomials with complex parameters forming a conjugate pair and also re-scaled. Namely we will take $a = \frac{\sqrt{1-q}}{2}\rho(y - i\sqrt{\frac{4}{1-q} - y^2})$, $b = \frac{\sqrt{1-q}}{2}\rho(y + i\sqrt{\frac{4}{1-q} - y^2})$, with $y \in S(q)$ and $|\rho| < 1$. More precisely we will consider polynomials $\{P_n(x|y,\rho,q)\}_{n\geq 0}$ defined by:

$$A_n\left(x\frac{\sqrt{1-q}}{2}|a,b,q\right)/(1-q)^{n/2} = P_n(x|y,\rho,q).$$

One can easily notice that $a + b = \rho y\sqrt{1-q}$, $ab = \rho^2$ and thus that the polynomials P_n satisfy the following three term recurrence:

$$P_{n+1}(x|y,\rho,q) = (x - \rho y q^n)P_n(x|y,\rho,q) - [n]_q(1 - \rho^2 q^{n-1})P_{n-1}(x|y,\rho,q), \tag{1.20}$$

with $P_{-1}(x|y,\rho,q) = 0$, $P_0(x|y,\rho,q) = 1$.

1.3. Remark. To support intuition let us remark (following e.g. [22]) that $P_n(x|y,\rho,1) = H_n\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)(1-\rho^2)^{n/2}$. On the other hand $P_n(x|y,\rho,0) = U_n(x/2) - \rho y U_{n-1}(x/2) + \rho^2 U_{n-2}(x/2)$, where $U_n(x)$ denotes Chebyshev polynomial of the second kind.

It is known see e.g. [15], (formula (14.8.13) adapted to our setting), [6], [22] that the polynomials P_n have the following generating function:

$$\varphi_P(x|y,\rho,t,q) = \sum_{n\geq 0}\frac{t^n}{[n]_q!}P_n(x|y,\rho,q) = \prod_{j=0}^{\infty}\frac{v\left(y\sqrt{1-q}/2|\rho t q^j\sqrt{1-q}\right)}{v\left(x\sqrt{1-q}/2|tq^j\sqrt{1-q}\right)},$$

convergent for $\left|t\sqrt{1-q}\right|, |\rho| < 1$, $x, y \in S(q)$.

We also have (see e.g. [22]) or :

$$\int_{S(q)} P_n(x|y,\rho,q)P_m(x|y,\rho,q)\, f_{CN}(x|y,\rho,q)\, dx = \delta_{nm}\, [n]_q!(\rho^2)_n, \tag{1.21}$$

where

$$f_{CN}(x|y,\rho,q) = f_N(x|q)\frac{(\rho^2)_\infty}{\prod_{j=0}^{\infty}\omega\left(x\sqrt{1-q}/2, y\sqrt{1-q}/2|\rho q^j\right)},$$

with

$$\omega(x,y|\rho) = \left(1-\rho^2\right)^2 - 4\rho(1+\rho^2)xy + 4\rho^2\left(x^2+y^2\right). \tag{1.22}$$

1.4. Remark. It was shown in [26](Lemma 1, (v)) that for $|q| < 1$ function $|f_{CN}(x|y,\rho,q)/f_N(x|q)|$ is bounded both from below and above hence square integrable on the square $S(q) \times S(q)$ with respect to the measure $f_N(x|q)\, f_{CN}(x|y,\rho,q)\, dxdy$. This will guarantee existence and convergence of some Fourier expansions considered in the next section.

We will call the densities f_N and f_{CN} respectively $q-$Normal and $(q,\rho)-$Conditional Normal. The names are justified by the nice probabilistic interpretations of these densities presented e.g. in [3], [4], [5], [6], [22] or [19]. Besides in [11] it was shown also that:

$$\lim_{q\to 1^-} f_{CN}(x|y,\rho,q) = \exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right)/\sqrt{2\pi(1-\rho^2)}. \tag{1.23}$$

1.5. Remark. Notice that convergence (1.15) and (1.23) in distribution of appropriate measures with these densities can be easily seen since we have $\lim_{q\longrightarrow 1^{-}} H_n(x|q) = H_n(x)$ and $\lim_{q\longrightarrow 1^{-}} P_n(x|y,\rho,q) = H_n\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)\left(1-\rho^2\right)^{n/2}$, hence we have convergence of appropriate moments. As stated above rigorous proofs of convergence of the densities can be found in [11].

We end up this section by recalling an auxiliary simple result that will be used in following sections many times. It has been formulated and proved in [25] Proposition 2.

1.6. Proposition. *Let* $\sigma_n(\rho|q) = \sum_{j\geq 0} \frac{\rho^j}{[j]_q!}\xi_{n+j}$ *for* $|\rho| < 1$, $-1 < q \leq 1$ *and certain sequence* $\{\xi_m\}_{m\geq 0}$ *such that* σ_n *exists for every* n. *Then*

$$(1.24)\qquad \sigma_n(\rho q^m|q) = \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\binom{k}{2}}(1-q)^k \rho^k \sigma_{n+k}(\rho|q).$$

1.7. Remark. Notice that this Proposition is trivially true for both $q = 0$ and $q = 1$.

2. Main Results

One of our main interests in this paper are the generalizations of the Poisson-Mehler formula (1.1).

It is well known that convergence in (1.1) takes place for $x, y \in S(q)$, $|\rho| < 1$ and for $|q| < 1$ is uniform. For $q = 1$ we have almost uniform convergence.

As a immediate corollary of Proposition 1.6 we have:

2.1. Corollary. *For* $|q| < 1$ *we have:*

$$(2.1)\qquad \gamma_{i,j}(x,y|\rho q^m,q) = \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\binom{k}{2}}(1-q)^k \rho^k \gamma_{i+k,j+k}(x,y|\rho,q),$$

$$(2.2)\qquad H_i(x|q)\,H_j(y|q) = \sum_{k\geq 0}(-1)^k q^{\binom{k}{2}}\frac{\rho^k}{(q)_k}\gamma_{i+k,j+k}(x,y|\rho,q),$$

where $\gamma_{i,j}(x,y|\rho,q)$ *is defined by (1.2). Formula (2.1) is also true trivially for* $q = 1$.

Proof. First assertion we get by applying directly (1.24) by setting $\sigma_{i,j} = \gamma_{i,j}$. Second assertion we get by passing in the first one with m to infinity and then noticing firstly that $\lim_{m\to\infty}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_q = \frac{1}{[k]_q!}$ and finally that $\frac{(1-q)^k}{[k]_q!} = \frac{1}{(q)_k}$. □

Now let us turn to polynomials $Q_{i,j}(x,y|\rho,q)$ defined by (1.3). It was shown in [22] that for all $-1 < q \leq 1$, $|\rho| < 1$, $x, y \in \mathbb{R}$:

$$(2.3)\qquad Q_{i,j}(x,y|\rho,q) = \sum_{s=0}^{j}(-1)^s q^{\binom{s}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q \rho^s H_{j-s}(y|q)\,P_{i+s}(x|y,\rho,q)\,/\left(\rho^2\right)_{i+s},$$

and $Q_{i,j}(x,y|\rho,q) = Q_{j,i}(y,x|\rho,q)$.

2.2. Remark. It has to be remarked that Carlitz in [7] considered the sum $\xi_{k,j}(x,y|\rho,q)$ $= \sum_{n\geq 0}\frac{\rho^n}{(q)_n}w_{n+k}(x|q)\,w_{n+j}(y|q)$, where $w_n(x|q)$ are the so called Rogers–Szegö polynomials related to polynomials $h_n(x|q)$ by the formula: $h_n(x|q) = e^{ni\theta}w_n(e^{-2i\theta}|q)$ with x $= \cos\theta$, $i-$imaginary unit. Indeed it turned out that functions $\xi_{k,j}$ also have the property that

$$\xi_{k,j}(x,y|\rho,q) = \nu_{k,j}(x,y|\rho,q)\xi_{0,0}(x,y|\rho,q),$$

where $\nu_{k,j}$ are polynomials of degree $k+j$ in x and y. However to show that $\nu_{k,j}(e^{-i\theta},e^{-i\eta}|\rho,q)$ can be expressed as $Q_{k,j}(\cos\theta,\cos\eta|\rho,q)$ is not an easy task. Discussion on this subject is in [23]. In particular see the proof of Proposition 5.

In particular we have

$$Q_{k,0}\left(x,y|\rho,q\right)=P_{k}\left(x|y,\rho,q\right)/\left(\rho^{2}\right)_{k}. \tag{2.4}$$

To analyze further properties of polynomials $Q_{k,j}$ let us introduce the following 2 dimensional density defined for $S^{2}\left(q\right)\overset{df}{=}S\left(q\right)\times S\left(q\right).$

$$f_{2D}\left(x,y|\rho,q\right)=f_{CN}\left(x|y,\rho,q\right)f_{N}\left(y|q\right). \tag{2.5}$$

Measure that has density f_{2D} will be called $(\rho,q)-$bivariate Normal (briefly $(\rho,q)-2N$). Obviously $f_{2D}\left(x,y|\rho,q\right)=\gamma_{0,0}\left(x,y|\rho,q\right)f_{N}\left(x|q\right)f_{N}\left(y|q\right).$ Its applications in theories of probability and Markov stochastic processes have been presented in [19] and [20].

Here below we give another interpretation of the polynomials $Q_{n,m}$ in particular its connection with the big $q-$Hermite polynomials.

2.3. Proposition. *For* $|q|<1,\ |\rho|<1,\ x,y\in\mathbb{R}$ *we have:*

i) $\forall i,j,m,k,i+j\neq m+k,$

$$\int_{S^{2}(q)}Q_{i,j}\left(x,y|\rho,q\right)Q_{m,k}\left(x,y|\rho,q\right)f_{2D}\left(x,y|\rho,q\right)dxdy=0,$$

ii) $\forall i,j,m,k,\ i+j=m+k,k>j:$

$$\int_{S^{2}(q)}Q_{n-j,j}\left(x,y|q\right)Q_{n-k,k}\left(x,y|\rho,q\right)f_{2D}\left(x,y|\rho,q\right)dxdy=$$
$$(-1)^{k-j}\frac{\rho^{k-j}q^{\binom{k-j}{2}}\left[j\right]_{q}!\left[n-j\right]_{q}!}{\left(\rho^{2}\right)_{n}}\sum_{s=0}^{j}q^{s(s-1)+ns}\begin{bmatrix}k\\k-j+s\end{bmatrix}_{q}$$
$$\times\begin{bmatrix}n-j+s\\s\end{bmatrix}_{q}\rho^{2s}\left(\rho^{2}q^{n-j+s}\right)_{j-s}.$$

iii)

$$\sum_{n,m\geq0}\frac{t^{n}s^{m}}{\left[n\right]_{q}!\left[m\right]_{q}!}Q_{n,m}\left(x,y|\rho,q\right)=\frac{f_{bN}\left(x|t,q\right)f_{bN}\left(y|s,q\right)}{f_{2D}(x,y|\rho,q)}\sum_{k\geq0}\frac{\rho^{k}}{\left[k\right]_{q}!}H_{k}\left(x|t,q\right)H_{k}\left(y|s,q\right),$$

where function f_{bN} *is defined by (1.18). The above mentioned formulae are also true for* $q=1$.

iv) $\forall m\geq0:$

$$Q_{i,j}\left(x,y|\rho q^{m},q\right)\prod_{i=0}^{m-1}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{i}\right)= \tag{2.6}$$
$$\left(\rho^{2}\right)_{2m}\sum_{k=0}^{m}(-1)^{k}\begin{bmatrix}m\\k\end{bmatrix}_{q}q^{\binom{k}{2}}\left(1-q\right)^{k}\rho^{k}Q_{i+k,j+k}\left(x,y|\rho,q\right),$$

where polynomial ω is defined by (1.22). In particular we have:

$$\prod_{j=0}^{n-1}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{j}\right)= \left(\rho^{2}\right)_{2n}\sum_{k=0}^{n}(-1)^{k}\begin{bmatrix}n\\k\end{bmatrix}_{q}q^{\binom{k}{2}}(1-q)^{k}\rho^{k}Q_{k,k}(x,y|\rho,q), \tag{2.7}$$

and

$$q^{\binom{n}{2}}\rho^{n}(1-q)^{n}Q_{n,n}(x,y|\rho,q)= \sum_{k=0}^{n}(-1)^{k}q^{\binom{n-k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{q}\frac{\prod_{j=0}^{k-1}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{j}\right)}{\left(\rho^{2}\right)_{2k}}, \tag{2.8}$$

with understanding that $\prod_{j=0}^{k-1}$ for $k=0$ is equal to 1.

Proof. Is shifted to section 4. □

Our main results follow in fact directly the results presented above.

2.4. Theorem. *Either for $|q|<1; x,y\in S(q); |\rho|<1$ we have:*
i)

$$H_{i}(x|q)H_{j}(y|q)\frac{\prod_{k=0}^{\infty}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{k}\right)}{\left(\rho^{2}\right)_{\infty}} =\sum_{k=0}^{\infty}(-1)^{k}q^{\binom{k}{2}}\frac{\rho^{k}}{[k]_{q}!}Q_{i+k,j+k}(x,y|\rho,q).$$

In particular we get:
ii)

$$1/\sum_{n\geq0}\frac{\rho^{n}}{[n]_{q}!}H_{n}(x|q)H_{n}(y|q)=\frac{\prod_{k=0}^{\infty}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{k}\right)}{\left(\rho^{2}\right)_{\infty}} =\sum_{k=0}^{\infty}(-1)^{k}q^{\binom{k}{2}}\frac{\rho^{k}}{[k]_{q}!}Q_{k,k}(x,y|\rho,q). \tag{2.9}$$

The last formula is valid also for $x,y\in\mathbb{R}$, $q=1$ and $|\rho|<1/2$.

Proof. To get i) we pass in (2.6) with m to infinity noting by (2.3) and (1.20) that $Q_{n,m}(x,y|0,q)=H_{n}(x|q)H_{m}(y|q)$. On the way we observe that $\lim_{m\to\infty}\begin{bmatrix}m\\k\end{bmatrix}_{q}=\frac{1}{(q)_{k}}=(1-q)^{-k}\frac{1}{[k]_{q}!}$. As far as the case $q=1$ is concerned denote by $g_{N}(x,y,\rho)$ density of the bivariate Normal density with parameters $\sigma_{1}=\sigma_{2}=1$, correlation coefficient ρ. Then notice that function $\exp(-\frac{1}{2}(x^{2}+y^{2}))/g_{N}(x,y,\rho)$ is square integrable on the plane with respect to $g_{N}(x,y,\rho)$ if $|\rho|<1/2$. □

3. Open problems and comments

3.1. Remark. The non-symmetric kernels constructed of bqH polynomials were given in [18]. Formula ii) of Proposition 2.3 gives its new interpretation. Besides, recall that these kernels were expressed using basic hypergeometric function ${}_3\phi_2$. Expansion on the left hand side of Proposition 2.3ii) gives new outlook on the properties of this function.

Notice also that for $q=1$ we have $\eta(x|t,1)=\exp(xt-\frac{t^2}{2})$, $H_n(x|t,1)=H_n(x-t)$ and

$$\sum_{n\geq 0}\frac{\rho^n}{n!}H_n(x)H_n(y)=\exp(\frac{x^2}{2}-\frac{(x-\rho y)^2}{2(1-\rho^2)}),$$

hence generating function of polynomials $Q_{i,j}$ can be calculated explicitly.

Similarly for $q=0$ we have $\eta(x|t,0)=\frac{1}{1-xt+t^2}$ (characteristic function of the Chebyshev polynomials) and $H_n(x|t,0)=U_n(x/2)-tU_{n-1}(x/2)$ (see e.g. [24]) hence also in this case we can get explicit form of the characteristic function of polynomials $Q_{i,j}$.

3.2. Remark. First of all notice that the left hand side of (2.9) is equal to $1/\gamma_{0,0}(x,y|\rho,q)$ $=f_N(x|q)/f_{CN}(x|y,\rho,q)$ and that it is a symmetric (with respect to x and y) function. In [21] there was presented (formula 5.3) an expansion of this function involving polynomials P_n and certain polynomials related to $q-$Hermite ones. The expansion was non-symmetric for every partial sum. Thus we get another expansion of known important special function.

3.3. Remark. Assertion i) of Proposition 2.3 states that polynomials $Q_{n,m}$ and $Q_{i,j}$ are orthogonal with respect to two dimensional measure μ_{2D} with the density given by (2.5) if only the $n+m\neq i+j$. Let us define space $\mathcal{L}=L_2\left(S^2(q),\mathcal{B},\mu_{2D}\right)$ of functions $f:S^2(q)\longrightarrow\mathbb{R}$ square integrable with respect to the measure μ_{2D}. Do polynomials $Q_{m,n}$ constitute a base of this space? It seems that yes, but not orthogonal. We can define subspaces of $\Lambda_m=span\{Q_{m,0},\ldots,Q_{0,m}\}$ of polynomials that are linear combinations of polynomials $Q_{i,j}$ such that $i+j=m$. Subspaces Λ_m are mutually orthogonal. Besides following argument that polynomials are dense in $\mathcal{L}$ we deduce that $\mathcal{L}=\bigoplus_{n=0}^{\infty}\Lambda_n$. What is the orthogonal base of $\mathcal{L}$? We have calculated covariances between polynomials $Q_{i,j}$ from Λ_m following (2.3) and (1.21). Thus we can follow Gram-Schmidt orthogonalization procedure within the spaces Λ_m. Is the union of orthogonal bases of Λ_m an orthogonal base of $\mathcal{L}$? Again it seems that yes. It would be interesting to find this base. Note that orthogonal polynomials on the plane are not an easy extension of the one-dimesional case. There are problems in defining them. For details see e.g. [14], [17], [16]. Recently in [9] there was defined a family of two dimensional polynomials that are two dimensional analogies of $q-$Hermite polynomials. Analogy is in the sense that many properties of the one-dimesional $q-$Hermite polynomials are retained in its two dimensional version.

3.4. Remark. In 2001 Wünsche in [27] considered Hermite and Laguerre polynomials on the plane. He has not however related his Hermite polynomials to any particular measure on the plane. In particular he defined Hermite polynomials depending on parameters forming a $2x2$ matrix. This matrix is however not connected in any way to the covariance matrix of the measure with respect to which these polynomials are supposed to be orthogonal.

On the other hand definition of polynomials $Q_{i,j}$ depends heavily on the measure with the density f_{2D}. For $q=1$ following (2.3), we have

$$Q_{i,j}(x,y|\rho,1)=\sum_{k=0}^{j}(-1)^k\binom{j}{k}H_{j-k}(y)H_{k+i}\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)/\left(\sqrt{1-\rho^2}\right)^{k+i}.$$

Hence polynomials $Q_{i,j}(x,y,\rho,1)$ are in fact another (different from that of Wünsche's) family of two dimensional generalization of Hermite polynomials.

4. Proofs

Proof of Proposition 2.3. i) We use (2.3), assume that $i>m$. We have:

$$\int_{S^2(q)} Q_{i,j}(x,y|q)\,Q_{m,k}(x,y|\rho,q)\,f_{2D}(x,y|\rho,q)\,dxdy=$$
$$\sum_{s=0}^{j}\sum_{t=0}^{k}(-1)^{s+t}q^{\binom{s}{2}}q^{\binom{t}{2}}\begin{bmatrix}j\\s\end{bmatrix}_q\begin{bmatrix}k\\t\end{bmatrix}_q\rho^{s+t}\frac{1}{(\rho^2)_{i+s}(\rho^2)_{m+t}}$$
$$\times\int_{S(q)} H_{j-s}(y|q)\,H_{k-t}(y|q)\,f_N(y|q)$$
$$\times\int_{S(q)} P_{i+s}(x|y,\rho,q)\,P_{m+t}(x|y,\rho,q)\,f_{CN}(x|y,\rho,q)\,dxdy=$$
$$(-1)^{i-m}\rho^{i-m}\sum_{s=0\vee m-i}^{j\wedge k+m-i}q^{\binom{s}{2}+\binom{i-m+s}{2}}\begin{bmatrix}j\\s\end{bmatrix}_q\begin{bmatrix}k\\i+s-m\end{bmatrix}_q\times$$
$$\rho^{2s}\frac{[i+s]_q!}{(\rho^2)_{i+s}}\int_{S(q)} H_{j-s}(y|q)\,H_{k+m-i-s}(y|q)\,f_N(y|q)\,dy=0,$$

if $j-s\neq k+m-i-s$ i.e. if $j+i\neq k+m$.

Now for $j+i=k+m$ and assuming that $k\geq j$ we get:

$$\int_{S^2(q)} Q_{n-j,j}(x,y|q)\,Q_{n-k,k}(x,y|\rho,q)\,f_{2D}(x,y|\rho,q)\,dxdy=$$
$$\sum_{s=0}^{j}\sum_{t=0}^{k}(-1)^{s+t}q^{\binom{s}{2}}q^{\binom{t}{2}}\begin{bmatrix}j\\s\end{bmatrix}_q\begin{bmatrix}k\\t\end{bmatrix}_q\rho^{s+t}\frac{1}{(\rho^2)_{n-j+s}(\rho^2)_{n-k+t}}$$
$$\times\int_{S(q)} H_{j-s}(y|q)\,H_{k-t}(y|q)\,f_N(y|q)$$
$$\times\int_{S(q)} P_{n-j+s}(x|y,\rho,q)\,P_{n-k+t}(x|y,\rho,q)\,f_{CN}(x|y,\rho,q)\,dxdy=$$
$$(-1)^{k-j}\rho^{k-j}\sum_{s=0}^{j}q^{\binom{s}{2}+\binom{k-j+s}{2}}\begin{bmatrix}j\\s\end{bmatrix}_q\begin{bmatrix}k\\k-j+s\end{bmatrix}_q\rho^{2s}\frac{[n-j+s]_q!}{(\rho^2)_{n-j+s}}$$
$$\times\int_{S(q)} H_{j-s}(y|q)\,H_{j-s}(y|q)\,f_N(y|q)\,dy$$

$$=(-1)^{k-j}\rho^{k-j}\sum_{s=0}^{j}q^{\binom{s}{2}+\binom{k-j+s}{2}}\begin{bmatrix}j\\s\end{bmatrix}_q\begin{bmatrix}k\\k-j+s\end{bmatrix}_q\rho^{2s}\frac{[n-j+s]_q!}{(\rho^2)_{n-j+s}}[j-s]_q!$$
$$=(-1)^{k-j}\frac{\rho^{k-j}q^{\binom{k-j}{2}}[j]_q!\,[n-j]_q!}{(\rho^2)_n}\sum_{s=0}^{j}q^{s(s-1)+ns}\begin{bmatrix}k\\k-j+s\end{bmatrix}_q$$
$$\times\begin{bmatrix}n-j+s\\s\end{bmatrix}_q\rho^{2s}\left(\rho^2q^{n-j+s}\right)_{j-s}$$

we use here $s(s-1)/2+(s+n)(s-1+n)/2-s(s-1)-n(n-1)/2=ns$

ii) We have

$$\sum_{i\geq 0,j\geq 0}\frac{s^it^j}{[i]_q![j]_q!}Q_{i,j}(x,y|\rho,q)=$$
$$\frac{1}{\gamma_{0,0}(x,y|\rho,q)}\sum_{i\geq 0,j\geq 0}\frac{t^is^j}{[i]_q![j]_q!}\sum_{n\geq 0}\frac{\rho^n}{[n]_q!}H_{i+n}(x|q)H_{n+j}(y|q)$$
$$=\sum_{n\geq 0}\frac{\rho^n}{[n]_q!}\sum_{j=0}^{\infty}\frac{s^j}{[j]_q!}H_{n+j}(y|q)\sum_{i\geq 0}\frac{t^i}{[i]_q!}H_{n+i}(x|q).$$

Now we use Lemma 1.1 twice and get

$$\sum_{i\geq 0,j\geq 0}\frac{s^it^j}{[i]_q![j]_q!}Q_{i,j}(x,y|\rho,q)=\frac{\varphi_H(x|t,q)\varphi_H(y|s,q)}{\gamma_{0,0}(x,y|\rho,q)}\sum_{n\geq 0}\frac{\rho^n}{[n]_q!}H_n(x|t,q)H_n(y|s,q)$$
$$=\frac{1}{(\rho^2)_\infty}\prod_{j=0}^{\infty}\frac{\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^j\right)}{v\left(x\sqrt{1-q}/2|t\sqrt{1-q}q^j\right)v\left(y\sqrt{1-q}/2|s\sqrt{1-q}q^j\right)}$$
$$\times\sum_{n\geq 0}\frac{\rho^n}{[n]_q!}H_n(x|t,q)H_n(y|s,q).$$

iii) First we notice that from (1.3) it follows that for $x,y\in S(q);\rho^2<1,-1<q\leq 1:$

$$\gamma_{i,j}(x,y|\rho q^m,q)=Q_{i,j}(x,y|\rho q^m,q)\frac{\left(\rho^2q^{2m}\right)_\infty}{\prod_{i=0}^{\infty}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{m+i}\right)}$$
$$=Q_{i,j}(x,y|\rho q^m,q)\frac{\prod_{i=0}^{m-1}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^i\right)}{(\rho^2)_{2m}}\gamma_{0,0}(x,y,\rho,q),$$

and also that $\gamma_{0,0}(x,y|\rho,q)=\frac{(\rho^2)_\infty}{\prod_{i=0}^{\infty}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^i\right)}$. Then we apply (2.1) to $\gamma_{i,j}$ above and then use (1.24) and cancel out $\gamma_{0,0}$ on both sides of (2.1). Finally we observe that on both sides we have polynomials hence one can extend the identity for all values of the variables. To get other formula of this assertion we argue by induction checking that the equality is true for $n=0$. Then we put (2.7) into (2.8) and get:

$$\sum_{k=0}^{n}(-1)^kq^{\binom{n-k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{\prod_{i=0}^{k-1}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^i\right)}{(\rho^2)_{2k}}=$$
$$\sum_{k=0}^{n}(-1)^kq^{\binom{n-k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q\sum_{j=0}^{k}(-1)^j\begin{bmatrix}k\\j\end{bmatrix}_q q^{\binom{j}{2}}\rho^jQ_{j,j}(x,y|\rho,q)=$$
$$\sum_{j=0}^{n}(-1)^jq^{\binom{j}{2}}\begin{bmatrix}n\\j\end{bmatrix}_q\rho^jQ_{j,j}(x,y|\rho,q)\sum_{k=j}^{n}(-1)^kq^{\binom{n-k}{2}}\begin{bmatrix}n-j\\k-j\end{bmatrix}_q=$$
$$\sum_{j=0}^{n}(-1)^jq^{\binom{j}{2}}\begin{bmatrix}n\\j\end{bmatrix}_q\rho^jQ_{j,j}(x,y|\rho,q)\sum_{m=0}^{n-j}(-1)^{m+j}q^{\binom{m}{2}}\begin{bmatrix}n-j\\m\end{bmatrix}_q=$$
$$q^{\binom{n}{2}}\rho^n(1-q)^nQ_{n,n}(x,y|\rho,q)$$

since $\forall n\geq 1:\sum_{i=0}^{n}(-1)^iq^{\binom{i}{2}}\begin{bmatrix}n\\i\end{bmatrix}_q=0.$ □

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Linear contrasts in one-way classification AR(1) model with gamma innovations

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Abstract

In this study, the explicit estimators of the model parameters in one-way classification AR(1) model with gamma innovations are derived by using modified maximum likelihood (MML) methodology. We also propose a new test statistic for testing linear contrasts. Monte Carlo simulation results show that the MML estimators have higher efficiencies than the traditional least squares (LS) estimators and the proposed test has much better power and robustness properties than the normal-theory test.

Keywords: Autoregressive model, linear contrasts, nonnormality, robustness, modified likelihood, gamma distribution.

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1. Introduction

Linear contrasts are widely used to make comparisons among the treatment means of interest. The usage of them require the independence assumption for the observations in each treatment. However, in numerous situations, the present state of a variable in each treatment is influenced by its past and this gives rise to autocorrelated time series structure. For instance in the agricultural and the biological sciences, the observations that are recorded over some time-space coordinate are extremely common, see, for example [7]. Some of the reasons for the lack of independence are (see [15]):

- Biased measurements,
- A poor allocation of treatments to experimental units,
- Adjacent experimental units or plots in a field.

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Another standard assumption is that the error terms are i.i.d (identically and independently distributed) as normal $N(\mu, \sigma^2)$. From the practical point of view, this assumption is also not realistic since nonnormal error distributions are more prevalent. There exists huge literature on the subject of nonnormal error distributions, see, for example, [8], [11], [5], [20], [31].

The normal theory test statistics for testing linear contrasts have low efficiencies when the normality assumption is not satisfied, see [18]. However, they can still be used for the situations where the normality assumption is violated to a slight or moderate degree. On the other hand, if the independence assumption is not met, traditional test statistics do not work well and give misleading results even if the observations exhibit low levels of correlation over time, see [16] and [12].

In recent years, the MML method has been applied to various time series models by Tiku and his colleagues. [21] developed a unit root test for the AR(1) model. The first order autoregressive model, AR(1), has been considered in [22] with asymmetric innovations of the gamma type. [24] extended the results of [22] to the symmetric non-normal innovations. [25] gave some engineering applications of the AR(1) models with nonnormal errors. [23] and [1] considered the simple regression model with first-order autoregressive errors when the error distribution is symmetric and asymmetric nonnormal, respectively. [26] and [3] extended this methodology to various independent sources of information and to multiple autoregressive model under non-normality; respectively. [31] extended the results of [23] to the generalized logistic distribution family representing very wide skew distributions ranging from highly right skewed to the highly left skewed.

Skew distributions are observed frequently in the context of experimental design; see for example, [18] and [17]. In their real life applications, they observed that the error terms are distributed as Generalized Logistic(b,σ) with shape parameters $b = 1, 2, 6$ and Weibull(p,σ) with shape parameter $p = 4$; respectively. Thus, positively skewed distributions fitted very well to the error terms. Therefore, different than the earlier studies, we assume that the error terms have Gamma which is another widely used and well known positive skewed distribution. Besides, we assume that the observations in each treatment are first order autocorrelated. This is the first study, dealing with both autocorrelation and non-normality in experimental design as far as we know. Thus, we aim to fill this gap in the literature.

We derive the estimators of the model parameters in this one-way classification model by using MML methodology. The methodology was first initiated by [19]. We also propose a new test statistic based on these MML estimators for testing linear contrasts and show that our solutions are much more efficient than the traditional normal-theory solutions.

The methodology developed in this paper can be extended to other designs, time series models (e.g. factorial designs AR(2) model) and any location-scale distribution (e.g., long-tailed symmetric and short-tailed symmetric distributions).

2. One-way classification AR(1) model

Consider the following one-way classification model with first-order autoregressive errors:

$$y_{i,j} - \phi y_{i,j-1} = \mu_i + e_{i,j}, \qquad -1 < \phi < 1;\ -\infty < \mu_i < \infty; \quad i = 1,\dots,a;\ j = 1,\dots,n \tag{2.1}$$

or alternatively reparametrized as

$$y_{i,j} - \phi y_{i,j-1} = \mu + \tau_i + e_{i,j}, \qquad -1 < \phi < 1; -\infty < \tau_i < \infty;$$
$$(2.2) \qquad -\infty < \mu < \infty; i = 1, \ldots, a; j = 1, \ldots, n$$

where $y_{i,j}$ is the jth observation in the ith treatment; μ is the constant representing the overall mean; μ_i is the mean of the ith treatment; τ_i is the ith treatment effect and $e_{i,j}$ is the error term.

Without loss of generality, we assume that $\sum \tau_i = 0$. Besides assume that $e_{i,j}$ are iid and have the gamma distribution

$$(2.3) \qquad f(e) = \frac{1}{\sigma^k \Gamma(k)} exp(-\frac{e}{\sigma}) e^{k-1}; \qquad 0 < e < \infty$$

where k is the shape parameter and is assumed to be known. Conditional on $y_{i,0}$, the likelihood function ignoring the constant term which has no effect on the estimators is

$$(2.4) \qquad L = \frac{1}{\sigma^n} e^{\sum_{i=1}^{a} \sum_{j=1}^{n} z_{i,j}} \prod_{i=1}^{a} \prod_{j=1}^{n} z_{i,j}^{k-1}$$

where $z_{i,j} = e_{i,j}/\sigma = (y_{i,j} - \phi y_{i,j-1} - \mu - \tau_i)/\sigma$.

The corresponding likelihood equations can be written as

$$\frac{\partial lnL}{\partial \mu} = \frac{N}{\sigma} - \frac{(k-1)}{\sigma} \sum_{i=1}^{a} \sum_{j=1}^{n} g(z_{i,j}) = 0$$

$$\frac{\partial lnL}{\partial \tau_i} = \frac{n}{\sigma} - \frac{(k-1)}{\sigma} \sum_{j=1}^{n} g(z_{i,j}) = 0$$

$$\frac{\partial lnL}{\partial \phi} = \frac{1}{\sigma} \sum_{i=1}^{a} \sum_{j=1}^{n} y_{i,j-1} - \frac{(k-1)}{\sigma} \sum_{i=1}^{a} \sum_{j=1}^{n} y_{i,j-1} g(z_{i,j}) = 0$$

$$(2.5) \qquad \frac{\partial lnL}{\partial \sigma} = -\frac{N}{\sigma} + \frac{1}{\sigma} \sum_{i=1}^{a} \sum_{j=1}^{n} z_{i,j} - \frac{(k-1)}{\sigma} \sum_{i=1}^{a} \sum_{j=1}^{n} z_{i,j} g(z_{i,j}) = 0$$

where $g(z) = 1/z$ and $N = an$: total number of observations.

These equations are in terms of $1/z_{i,j}$ and have no explicit solutions. Therefore they have to be solved by iteration which might be problematic especially when the data contains outliers, see, for example, [14], [27] and [28]. We, therefore, utilize the method of modified likelihood estimation which captures the beauty of maximum likelihood but alleviates its computational difficulties, see [20].

3. The MML estimators

The first step of obtaining the MML estimators is to express the likelihood equations (2.5) in terms of ordered $z_{i,(j)}$'s ($i = 1, \ldots, a;\ j = 1, \ldots, n$), since the complete sums are invariant to ordering. The second step is to linearize the term $g(z_{i,(j)}) = 1/z_{i,(j)}$ around $t_{(j)}$ by the use of the first two terms of a Taylor series expansion, since for large n, $z_{i,(j)}$ is close to its expected value $t_{(j)} = E(z_{i,(j)})$. Thus,

$$(3.1) \qquad g(z_{i,(j)}) \cong g(t_{(j)}) + (z_{i,(j)} - t_{(j)}) \left\{ \frac{\partial g(z)}{\partial z} \right\}_{z=t_{(j)}} = \alpha_j - \beta_j z_{i,(j)}$$

where $\alpha_j = 2/t_{(j)}$ and $\beta_j = 1/t_{(j)}^2$. Although the exact values of the $t_{(j)}$ are available, for convenience, we use their approximate values generated from the equation $\frac{1}{\Gamma(k)}\int_0^{t_{(j)}} e^{-z}z^{k-1}dz = \frac{j}{n+1}$, $1 \le j \le n$ for each treatment (i.e., for $i = 1, \ldots, a$).

Incorporating the linear approximation 3.1 into the likelihood equations 2.5 yields the modified likelihood equations. Then the MML estimators are obtained by solving these modified likelihood equations as:

$$\widehat{\mu} = \widehat{\mu}_{.[.]} + \frac{\Delta}{m}\widehat{\sigma}, \quad \widehat{\tau}_i = \widehat{\mu}_{i[.]} - \widehat{\mu}_{.[.]},$$

$$\widehat{\phi} = K + D\widehat{\sigma}, \quad \widehat{\sigma} = \frac{B + \sqrt{B^2 + 4NC}}{2\sqrt{N(N-a-1)}} \tag{3.2}$$

where

$$\widehat{\mu}_{i[.]} = \frac{\sum_{j=1}^{n} \beta_j (y_{i,[j]} - \phi y_{i,[j]-1})}{m}, \; \widehat{\mu}_{.[.]} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{n} \beta_j (y_{i,[j]} - \phi y_{i,[j]-1})}{am},$$

$$\Delta_j = \frac{1}{k-1} - \alpha_j, \quad \Delta = \sum_{j=1}^{n} \Delta_j, \quad m = \sum_{j=1}^{n} \beta_j,$$

$$K = \frac{\sum_{i=1}^{a}\sum_{j=1}^{n} \beta_j y_{i,[j]} y_{i,[j]-1} - \frac{1}{m}\sum_{i=1}^{a}(\sum_{j=1}^{n}\beta_j y_{i,[j]})(\sum_{j=1}^{n}\beta_j y_{i,[j]-1})}{\sum_{i=1}^{a}\sum_{j=1}^{n}\beta_j y_{i,[j]-1}^2 - \frac{1}{m}\sum_{i=1}^{a}(\sum_{j=1}^{n}\beta_j y_{i,[j]-1})^2},$$

$$D = \frac{\sum_{i=1}^{a}\sum_{j=1}^{n}(\Delta_j - \beta_j\frac{\Delta}{m}) y_{i,[j]-1}}{\sum_{i=1}^{a}\sum_{j=1}^{n}\beta_j y_{i,[j]-1}^2 - \frac{1}{m}\sum_{i=1}^{a}(\sum_{j=1}^{n}\beta_j y_{i,[j]-1})^2},$$

$$B = (k-1)\sum_{i=1}^{a}\sum_{j=1}^{n}(y_{i,[j]} - \phi y_{i,[j]-1} - \widehat{\mu}_{i[.]})\Delta_j, \; \textit{and}$$

$$C = (k-1)\sum_{i=1}^{a}\sum_{j=1}^{n}\beta_j (y_{i,[j]} - \phi y_{i,[j]-1} - \widehat{\mu}_{i[.]})^2. \tag{3.3}$$

It is clear that the MML estimators have closed forms. It should also be noted that they have exactly the same forms as other MML estimators irrespective of the underlying distribution besides having the invariance property, see [20]. The MML estimators are known to be asymptotically fully efficient, i.e. they are unbiased and minimum variance bounds (MVB) estimators, see [4] and [29]. For small sample sizes, they have very little or no bias and the true variances of the MML estimators are very close to minimum variance bounds, see [28].

For the computation of the MML estimators $\widehat{\mu}$, $\widehat{\tau}_i$, $\widehat{\phi}$ and $\widehat{\sigma}$, first the ordered variates of $z_{i,j} = e_{i,j}/\sigma = (y_{i,j} - \phi y_{i,j-1} - \mu - \tau_i)/\sigma$ $(i = 1, \ldots, a;\; j = 1, \ldots, n)$ has to be obtained. Since the ordering of $z_{i,j}$ only depends on ϕ (μ and τ_i are additive constants and σ is positive), it is done by using the LS estimate $\widehat{\phi}_{LS}$ of ϕ as an initial estimate. Then using the concomitants $(y_{i,[j]}, y_{i,[j]-1})$ corresponding to ordered variates $w_{i,(j)} = y_{i,[j]} - \widehat{\phi}_{LS} y_{i,[j]-1}$, the MML estimates $\widehat{\mu}$, $\widehat{\tau}_i$, $\widehat{\phi}$ and $\widehat{\sigma}$ are calculated from 3.2. A second iteration is carried out by replacing $\widehat{\phi}_{LS}$ with $\widehat{\phi}$ in the ordering of $w_{i,(j)}$ variates and new $\widehat{\mu}$, $\widehat{\tau}_i$, $\widehat{\phi}$ and $\widehat{\sigma}$ values are calculated. This is repeated till the estimates stabilize sufficiently enough. In our computations, two iterations were enough. Actually, in literature based on MML, it can be seen that at most three iterations are enough.

4. Efficiency of the MML estimators

In practice the LS estimators are widely used which will be shown that they are considerably less efficient than the MML estimators. Relative efficiencies (RE) of the LS estimators defined as

$$RE = 100 \times (variance\, of\, MMLE)/(variance\, of\, LSE) \tag{4.1}$$

are calculated by simulation based on $[100000/n]$ Monte Carlo runs. Although much other values are tried, the simulation results performed for sample sizes $n = 30$, 60 and 120 with the shape parameter taking the values $k = 2$, 3, 5 and 10 for $\phi = 0.0$, 0.5 and 0.9 are given in Table 1. It must be noted that the values for other ϕ values including negative ones yield the similar results so that they are not reported.

The model parameters μ_i, τ_i and σ are set as 0, 0 and 1 without loss of generality. Realize that for $\phi = 0.0$, the model 2.1 turns to be the usual one-way classification where the errors are distributed as gamma rather than normal. In fact, this is by its own a contribution since the model parameters in one-way classification model have not been estimated with gamma distributions so far.

The LS estimators of the model parameters are given by

$$\widetilde{\mu}_i = \frac{\sum_{j=1}^{n}(y_{i,j} - \phi y_{i,j-1})}{n} - k\widetilde{\sigma}, \quad \widetilde{\mu} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{n}(y_{i,j} - \phi y_{i,j-1}}{an} - k\widetilde{\sigma},$$

$$\widetilde{\tau}_i = \widetilde{\mu}_i - \widetilde{\mu}, \widetilde{\phi} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{n} y_{i,j}y_{i,j-1} - \frac{1}{n}\sum_{i=1}^{a}(\sum_{j=1}^{n} y_{i,j})(\sum_{j=1}^{n} y_{i,j-1})}{\sum_{i=1}^{a}\sum_{j=1}^{n} y_{i,j-1}^2 - \frac{1}{n}\sum_{i=1}^{a}(\sum_{j=1}^{n} y_{i,j-1})^2},$$

$$\widetilde{\sigma}^2 = \frac{\sum_{i=1}^{a}\sum_{j=1}^{n}((y_{i,j} - \phi y_{i,j-1}) - \widetilde{\mu}_i)^2}{(N-a-1)k}. \tag{4.2}$$

Note that the LS estimators $\widetilde{\mu}$ and $\widetilde{\sigma}^2$ are corrected for bias so that they become comparable with MML estimators. Besides, the initial values $y_{i,0}$ are taken as $e_{i,0}/\sqrt{1-\phi^2}$, which is, in fact, Model II of [30].

It can be seen from Table 1 that the MML estimators are more efficient than the LS estimators especially for the small values of the shape parameter k. It should be noted that the relative efficiency of the LS estimators decrease as the sample size n increase. This is another result of interest.

Table 1. Simulated means (1), $n\times$variances (2) and the relative efficiencies (RE) of the LS and MML estimators.

n		$\widetilde{\mu}_i$	$\widehat{\mu}_i$	RE	$\widetilde{\tau}_i$	$\widehat{\tau}_i$	RE	$\widetilde{\phi}$	$\widehat{\phi}$	RE	$\widetilde{\sigma}$	$\widehat{\sigma}$	RE
						$k = 2.0$, $\phi = 0.0$							
30	(1)	0.073	0.160	30	0.001	0.001	28	-0.031	-0.009	39	0.991	0.973	53
	(2)	0.125	0.038		1.444	0.404		0.309	0.121		0.426	0.227	
60	(1)	0.031	0.087	24	0.000	0.000	25	-0.015	-0.002	30	0.997	0.982	51
	(2)	0.060	0.015		1.346	0.334		0.317	0.096		0.408	0.210	
120	(1)	0.019	0.050	19	-0.001	-0.001	21	-0.008	0.000	21	1.000	0.988	45
	(2)	0.032	0.006		1.289	0.269		0.344	0.070		0.424	0.193	
						$k = 2.0$, $\phi = 0.5$							
30	(1)	0.242	0.229	32	0.008	0.004	27	0.441	0.478	38	0.993	0.972	52
	(2)	0.229	0.073		1.710	0.455		0.262	0.099		0.417	0.215	
60	(1)	0.109	0.109	26	0.004	0.002	22	0.473	0.493	30	0.999	0.983	50
	(2)	0.105	0.027		1.500	0.326		0.239	0.071		0.424	0.213	
120	(1)	0.058	0.063	20	-0.001	0.003	21	0.485	0.496	24	0.999	0.987	48
	(2)	0.054	0.011		1.513	0.317		0.255	0.060		0.405	0.196	
						$k = 2.0$, $\phi = 0.9$							
30	(1)	0.414	0.327	33	-0.008	-0.004	27	0.875	0.888	35	0.990	0.970	53
	(2)	0.413	0.136		1.974	0.529		0.039	0.014		0.410	0.218	
60	(1)	0.289	0.190	28	0.002	0.001	22	0.884	0.894	28	0.997	0.981	49
	(2)	0.218	0.061		1.933	0.415		0.036	0.010		0.415	0.204	
120	(1)	0.230	0.123	21	-0.003	0.000	19	0.888	0.896	22	0.999	0.987	48
	(2)	0.172	0.036		1.690	0.325		0.052	0.011		0.463	0.221	
						$k = 3.0$, $\phi = 0.0$							
30	(1)	0.115	0.183	48	-0.005	-0.001	47	-0.032	-0.013	56	0.993	0.978	60
	(2)	0.237	0.114		2.178	1.020		0.318	0.177		0.344	0.205	
60	(1)	0.055	0.093	43	-0.004	-0.003	42	-0.014	-0.004	51	0.998	0.986	57
	(2)	0.122	0.052		2.040	0.854		0.345	0.175		0.337	0.192	
120	(1)	0.031	0.055	39	0.006	0.002	40	-0.009	-0.002	45	1.000	0.991	54
	(2)	0.060	0.024		2.106	0.841		0.330	0.148		0.335	0.180	
						$k = 3.0$, $\phi = 0.5$							
30	(1)	0.377	0.333	49	0.003	0.003	43	0.440	0.468	53	0.991	0.978	57
	(2)	0.471	0.229		2.599	1.127		0.258	0.137		0.345	0.198	
60	(1)	0.178	0.153	41	-0.003	0.000	41	0.472	0.488	43	0.995	0.985	56
	(2)	0.231	0.095		2.230	0.911		0.266	0.116		0.331	0.186	
120	(1)	0.093	0.083	40	0.001	0.000	38	0.486	0.495	42	0.997	0.989	53
	(2)	0.107	0.043		2.001	0.767		0.261	0.109		0.363	0.192	
						$k = 3.0$, $\phi = 0.9$							
30	(1)	0.459	0.411	53	-0.006	0.000	44	0.882	0.889	53	0.992	0.977	60
	(2)	0.611	0.323		2.652	1.176		0.025	0.013		0.321	0.192	
60	(1)	0.359	0.269	43	0.000	-0.002	41	0.887	0.893	42	0.995	0.983	55
	(2)	0.395	0.169		2.463	1.004		0.027	0.012		0.317	0.173	
120	(1)	0.262	0.189	42	0.010	0.007	40	0.891	0.895	41	1.001	0.993	54
	(2)	0.257	0.109		2.431	0.980		0.034	0.014		0.336	0.180	
						$k = 5.0$, $\phi = 0.0$							
30	(1)	0.214	0.269	67	0.002	0.003	65	-0.033	-0.020	72	0.994	0.986	70
	(2)	0.582	0.388		3.685	2.393		0.333	0.241		0.273	0.192	
60	(1)	0.085	0.122	62	-0.001	-0.002	64	-0.013	-0.006	66	0.996	0.991	66
	(2)	0.291	0.181		3.401	2.175		0.337	0.221		0.263	0.173	
120	(1)	0.063	0.082	61	-0.004	-0.001	61	-0.011	-0.006	64	1.001	0.996	65
	(2)	0.136	0.083		3.192	1.952		0.302	0.193		0.262	0.171	
						$k = 5.0$, $\phi = 0.5$							
30	(1)	0.550	0.515	69	-0.009	-0.008	64	0.446	0.463	72	0.995	0.985	69
	(2)	1.157	0.795		4.091	2.618		0.253	0.182		0.274	0.190	
60	(1)	0.320	0.299	66	0.001	0.002	63	0.468	0.478	67	0.998	0.991	64
	(2)	0.569	0.373		3.690	2.340		0.250	0.167		0.295	0.188	
120	(1)	0.144	0.138	61	-0.004	-0.001	59	0.485	0.491	65	1.000	0.995	63
	(2)	0.262	0.160		3.417	2.029		0.232	0.152		0.263	0.165	
						$k = 5.0$, $\phi = 0.9$							
30	(1)	0.503	0.510	68	0.008	0.005	63	0.888	0.891	69	0.991	0.984	67
	(2)	1.058	0.722		3.973	2.518		0.015	0.011		0.268	0.181	
60	(1)	0.384	0.368	65	0.005	-0.002	62	0.892	0.894	67	0.999	0.991	63
	(2)	0.727	0.470		3.888	2.417		0.017	0.012		0.274	0.174	
120	(1)	0.267	0.255	62	-0.016	-0.007	60	0.894	0.896	65	1.001	0.996	62
	(2)	0.509	0.314		3.828	2.302		0.023	0.015		0.269	0.168	

Table 1.(cont.ed.)

		$\widetilde{\mu}_i$	$\widehat{\mu}_i$	RE	$\widetilde{\tau}_i$	$\widehat{\tau}_i$	RE	$\widetilde{\phi}$	$\widehat{\phi}$	RE	$\widetilde{\sigma}$	$\widehat{\sigma}$	RE
$k = 10.0$, $\phi = 0.0$													
30	(1)	0.420	0.469	82	0.002	-0.001	81	-0.036	-0.029	87	0.994	0.991	81
	(2)	2.082	1.711		6.969	5.624		0.326	0.284		0.223	0.181	
60	(1)	0.178	0.215	81	0.001	0.000	80	-0.015	-0.011	82	0.997	0.994	78
	(2)	0.903	0.732		7.191	5.752		0.320	0.263		0.223	0.173	
120	(1)	0.105	0.138	76	0.005	0.005	80	-0.008	-0.006	80	0.998	0.995	77
	(2)	0.491	0.375		6.897	5.514		0.318	0.254		0.221	0.170	
$k = 10.0$, $\phi = 0.5$													
30	(1)	1.095	1.069	84	0.011	0.013	81	0.448	0.456	86	0.995	0.991	82
	(2)	3.973	3.322		8.117	6.609		0.223	0.191		0.222	0.181	
60	(1)	0.527	0.522	80	-0.012	-0.012	80	0.474	0.479	83	0.999	0.995	81
	(2)	2.103	1.684		7.936	6.369		0.234	0.191		0.228	0.184	
120	(1)	0.292	0.287	79	-0.010	-0.005	78	0.486	0.489	81	0.997	0.995	75
	(2)	0.948	0.753		7.838	6.137		0.227	0.185		0.223	0.167	
$k = 10.0$, $\phi = 0.9$													
30	(1)	0.570	0.636	86	0.008	0.009	82	0.893	0.894	86	0.995	0.990	82
	(2)	2.334	2.005		7.448	6.068		0.008	0.007		0.210	0.172	
60	(1)	0.441	0.476	81	-0.001	0.001	80	0.895	0.896	82	0.995	0.992	78
	(2)	1.573	1.275		7.433	5.924		0.009	0.008		0.218	0.170	
120	(1)	0.368	0.385	75	0.012	0.012	79	0.896	0.896	78	1.000	0.997	76
	(2)	1.223	0.920		7.554	6.001		0.014	0.011		0.213	0.161	

5. Power and robustness properties of the proposed test

For testing the null hypothesis $H_0 : \sum_{i=1}^{a} l_i\tau_i = \sum_{i=1}^{a} l_i\mu_i = 0$ $(\mu_i = \mu+\tau_i)$; $\sum_{i=1}^{a} l_i = 0$, traditionally, where l_i $(1 \le i \le a)$ are constant coefficients of a linear contrast; we use the following test statistics based on the LS estimators given in 4.2

$$t = \frac{\sum_{i=1}^{a} l_i\widetilde{\mu}_i}{\sqrt{\sum_{i=1}^{a} l_i^2 \frac{\widetilde{\sigma}^2}{n}}}. \tag{5.1}$$

However, in this study, we propose the following test statistics based on MML estimators

$$t^* = \frac{\sum_{i=1}^{a} l_i\widehat{\mu}_i}{\sqrt{\sum_{i=1}^{a} l_i^2 \frac{\widehat{\sigma}^2}{m(k-1)}}}, \tag{5.2}$$

where the large values of t^* lead to the rejection of H_0. The null distribution of t^* is asymptotically normal N(0,1) due to the following lemmas:

5.1. Lemma. *For a given ϕ (σ known), the asymptotic distribution of $\widehat{\mu}_i(\phi,\sigma) = \widehat{\mu}_{i.} + (\Delta/m)\sigma$ which is the minimum variance bound estimator of $\mu_i = \mu + \tau_i$ $(1 \le i \le a)$ is normal with variance $V\{\widehat{\mu}_i(\phi,\sigma)\} \cong \sigma^2/m(k-1)$.*

Proof. Proof of the Lemma 5.1. The result follows from the fact that asymptotically $\partial lnL^*/\partial\mu_i$ is equivalent to $\partial lnL/\partial\mu_i$ [29] and assumes the form

$$\frac{\partial lnL^*}{\partial\mu_i} = \frac{m(k-1)}{\sigma^2}(\widehat{\mu}_i(\phi,\sigma) - \mu_i)$$

[10]. The normality follows from the fact that $E(\partial lnL^*/\partial\mu_i^r) = 0$ for all $r \ge 3$. □

5.2. Lemma. *For a given ϕ(μ known),the asymptotic distribution of $N\widehat{\sigma}^2(\phi,\mu)/\sigma^2$ is chi-square with $N = na$ degrees of freedom.*

Proof. Proof of the Lemma 5.2. Let

$$B_0 = (k-1)\sum_{i=1}^{a}\sum_{j=1}^{n}(y_{i,(j)} - \phi y_{i,(j-1)} - \mu_i)\Delta_j \quad and$$

$$C_0 = (k-1)\sum_{i=1}^{a}\sum_{j=1}^{n}\beta_j(y_{i,(j)} - \phi y_{i,(j-1)} - \mu_i)^2.$$

Since $B_0/\sqrt{nC_0} \cong 0$, α_j and β_j are bounded,

$$\begin{aligned}\frac{\partial lnL}{\partial\sigma} &\cong \frac{\partial lnL^*}{\partial\sigma} \\ &= -\frac{N}{\sigma^3}(\sigma - \frac{B_0+\sqrt{B_0^2+4NC_0}}{N})(\sigma - \frac{B_0-\sqrt{B_0^2+4NC_0}}{N}) \\ &\cong \frac{N}{\sigma^3}(\frac{C_0}{N} - \sigma^2).\end{aligned}$$

The result then follows from the values of $E(\partial^r lnL^*/\partial\sigma^r)$ as in [20]. □

5.3. Lemma. *Since $\widehat{\sigma}$ converges to σ as n tends to infinity, the asymptotic distribution of $\sqrt{n/v_{11}}(\widehat{\mu}(\phi,\widehat{\sigma}) - \mu)/\widehat{\sigma}$ is N(0,1) where v_{11} is the first element in the asymptotic covariance matrix.*

Proof. Proof of the Lemma 5.3. This follows from the well-known Slutsky's theorem. See [20]. □

Thus, when we have a linear contrast of 'a' MML estimators and $\widehat{\sigma}^2$ is the pooled MML estimator of σ^2, the [2] conditions are satisfied and $\sum_{i=1}^{a} l_i\mu_i$ and $\widehat{\sigma}^2$ are asymptotically independently distributed resulting the asymptotic distribution of $\sqrt{m(k-1)}\sum_{i=1}^{a} l_i\widehat{\mu}_i/(\widehat{\sigma}\sqrt{\sum_{i=1}^{a} l_i^2})$ being N(0,1).

Some of the simulated values of the probabilities $P(t^* \geq z_{0.05} = 1.645|H_0)$ for different sample sizes are given in Table 2.

Table 2. Values of the type I error of the t^* test; $\alpha = 0.050$.

	$k=2.0$	$k=3.0$	$k=5.0$	$k=10.0$	$k=15.0$
n			$\phi=0.0$		
50	0.030	0.042	0.046	0.046	0.048
100	0.032	0.053	0.052	0.055	0.051
150	0.032	0.042	0.054	0.054	0.053
200	0.032	0.048	0.050	0.056	0.046
			$\phi=0.4$		
50	0.034	0.047	0.054	0.058	0.048
100	0.030	0.045	0.047	0.053	0.040
150	0.030	0.041	0.053	0.051	0.056
200	0.036	0.046	0.054	0.052	0.040
			$\phi=0.8$		
50	0.044	0.057	0.053	0.059	0.052
100	0.033	0.050	0.054	0.052	0.043
150	0.039	0.048	0.047	0.047	0.053
200	0.030	0.046	0.044	0.056	0.048

It can be seen that the normal distribution provides satisfactory approximations to the percentage points. To have an idea about the power of the two tests given in 5.1 and 5.2, the simulated values for $n = 100$ where $l_1 = 1$, $l_2 = -2$ and $l_3 = 1$ for different k and ϕ values are reported in Table 3. We carried out simulations for several other k, n and l_i values but did not report since they give the similar results.

The values of power given in Table 3 are obtained by adding a constant d to the observations in the first and the third treatments and subtracting $2d$ from the observations

Table 3. Values of the power of the t^* and t tests; $n = 100$.

	$k = 2.0$		$k = 3.0$		$k = 5.0$		$k = 10.0$		$k = 15.0$	
d	t^*	t	t^*	t	t^*	t	t^*	t	t^*	t
					$\phi = 0.0$					
0.000	0.035	0.040	0.035	0.044	0.034	0.041	0.044	0.066	0.056	0.065
0.013	0.120	0.091	0.100	0.103	0.094	0.088	0.100	0.093	0.088	0.086
0.025	0.304	0.175	0.232	0.166	0.170	0.140	0.152	0.153	0.135	0.137
0.038	0.589	0.257	0.382	0.234	0.332	0.277	0.259	0.239	0.257	0.244
0.050	0.774	0.331	0.583	0.347	0.469	0.356	0.378	0.330	0.357	0.321
0.063	0.920	0.467	0.735	0.471	0.600	0.487	0.540	0.468	0.473	0.437
0.075	0.966	0.591	0.866	0.589	0.759	0.630	0.658	0.589	0.606	0.557
0.088	0.993	0.704	0.929	0.716	0.860	0.698	0.772	0.703	0.723	0.683
0.100	0.999	0.795	0.979	0.787	0.920	0.780	0.865	0.785	0.839	0.789
0.113	1.000	0.865	0.994	0.888	0.963	0.874	0.906	0.864	0.890	0.856
					$\phi = 0.4$					
0.000	0.026	0.052	0.035	0.048	0.045	0.059	0.040	0.056	0.048	0.057
0.013	0.082	0.090	0.110	0.113	0.084	0.077	0.067	0.067	0.085	0.078
0.025	0.225	0.130	0.189	0.139	0.160	0.130	0.166	0.164	0.159	0.154
0.038	0.383	0.188	0.322	0.220	0.276	0.223	0.258	0.237	0.264	0.245
0.050	0.623	0.267	0.487	0.322	0.404	0.306	0.372	0.332	0.363	0.338
0.063	0.816	0.368	0.613	0.373	0.549	0.427	0.522	0.441	0.498	0.458
0.075	0.888	0.474	0.758	0.483	0.685	0.520	0.605	0.541	0.609	0.584
0.088	0.960	0.543	0.869	0.594	0.802	0.611	0.752	0.681	0.708	0.676
0.100	0.981	0.607	0.935	0.701	0.874	0.739	0.833	0.768	0.816	0.771
0.113	0.996	0.726	0.967	0.769	0.927	0.781	0.906	0.857	0.894	0.866
					$\phi = 0.8$					
0.000	0.035	0.054	0.042	0.058	0.040	0.053	0.044	0.043	0.051	0.049
0.013	0.124	0.086	0.125	0.094	0.118	0.103	0.133	0.105	0.115	0.102
0.025	0.357	0.163	0.308	0.210	0.265	0.214	0.246	0.223	0.239	0.223
0.038	0.620	0.271	0.508	0.294	0.457	0.348	0.438	0.380	0.410	0.392
0.050	0.848	0.405	0.705	0.461	0.638	0.499	0.603	0.541	0.621	0.564
0.063	0.951	0.543	0.874	0.580	0.792	0.643	0.777	0.690	0.793	0.739
0.075	0.988	0.685	0.956	0.757	0.920	0.788	0.904	0.830	0.902	0.849
0.088	1.000	0.768	0.990	0.849	0.963	0.879	0.951	0.919	0.955	0.931
0.110	1.000	0.858	0.998	0.927	0.993	0.944	0.988	0.964	0.980	0.969
0.113	1.000	0.913	0.998	0.947	0.997	0.973	0.998	0.987	0.994	0.988

in the second treatment. The results show that t^* test is much more powerful than the classical t test.

In practice, we may be in error when we assume that our data follow a particular distribution, since the shape parameters might be misspecified or the data might contain outliers, or be contaminated. When these situations arise, the distribution of the test statistic may differ from that expected. Therefore, the accurate estimates of the probability of type I and type II errors (i.e. power of the test) will not be obtained. When the underlying assumptions are violated, robust test statistics are preferred to the traditional test statistics. A test is called robust if its type I error is never substantially higher than a pre-assigned value for plausible alternatives to an assumed model (Criterion Robustness) and if its power is high (Inference Robustness). It is clear that robustness is very desirable property for the hypothesis testing procedures. Table 4 summarizes the results of simulations for $k = 3$, $\phi = 0.4$ and $n = 100$ when we assume that the true model is Gamma(3, σ). For this simulation study, the plausible alternatives used are as follows:

(1) Gamma(2,σ),
(2) Gamma(4,σ),
(3) Outlier model: $(n - r)$ observations come from Gamma(3,σ) but r observation (we do not know which one) comes from Gamma(3,2σ); $r = [0.5 + 0.1n]$,
(4) Mixture model: 0.90Gamma(3,σ) + 0.10Gamma(3,2σ),
(5) Contamination model: 0.90Gamma(3,σ) + 0.10Gamma(5,σ)

Table 4. Power of the t^* and t tests for alternatives to Gamma(3,σ); $k = 3$, $n = 100$ and $\phi = 0.4$.

	Model (1)		Model (2)		Model (3)		Moel (4)		Model (5)	
d	t^*	t	t^*	t	t^*	t	t^*	t	t^*	t
0.00	0.042	0.058	0.042	0.052	0.030	0.015	0.046	0.048	0.043	0.051
0.02	0.127	0.090	0.163	0.122	0.111	0.026	0.156	0.093	0.131	0.100
0.04	0.335	0.213	0.339	0.232	0.314	0.082	0.368	0.212	0.303	0.163
0.06	0.608	0.341	0.619	0.381	0.569	0.161	0.668	0.334	0.544	0.239
0.08	0.811	0.539	0.809	0.519	0.808	0.303	0.872	0.490	0.739	0.350
0.10	0.946	0.704	0.936	0.685	0.947	0.479	0.970	0.648	0.899	0.489
0.12	0.987	0.819	0.985	0.823	0.981	0.628	0.992	0.786	0.964	0.603

The values are obtained by adding a constant d to the observations in the first and the third treatments and subtracting $2d$ from the observations in the second treatment as in efficiency analysis. From Table 4, we see that the power of the t^* test is higher than the t test for all sample models given above. For sample models, except Model (3), in fact, the t^* test has a double advantage: not only has it much smaller type I error but also has higher power. Similar results are obtained for other ϕ values.

6. Determination of the shape parameter

It is known that when location, scale and shape parameters are to be estimated, maximum likelihood method is doubtful unless large samples ($n > 250$ or so) are available; see [6]. Thus, one should consider estimating location, scale or location and shape parameters when the sample size is small which is the case for experimental design. Therefore, in this study, it is assumed that the shape parameter k in 2.3 is known. Actually, an assumption of known shape parameter is found to be quite reasonable for many real-life problems; see for example, [9]. See also [13] for a better understanding of the importance of a given shape parameter.

However, in practice, shape parameter is also unknown. A plausible value for it can be identified by using Q-Q plots, goodness-of-fit tests, or by matching (approximately) the sample skewness and kurtosis with the corresponding values of the distribution. Also it can be determined by trying a series of values of this parameter as in [24]. The one that maximizes the likelihood function is the required estimate. Due to the intrinsic robustness of MMLE shown in section 5, this value will yield essentially the same estimates and standard errors for plausible alternatives.

7. Conclusion

In this study, we proposed a new test statistic for testing the assumed values of linear contrasts in one-way classification AR(1) model. We believe that the results of this study will be very useful for researchers and practitioners. Since all the procedures related with linear contrasts are based on the assumption of normality, homogeneity of variances and independence of error terms. There is a huge literature about nonnormality and heterogeneity of variances. However, there is no too much work when the independence assumption of error terms is not satisfied. Dependency is tried to be prevented at the design stage by randomization and there is a gap about how to deal with it, if it exists. This paper fills this gap not only by dealing with dependency but also with non-normality. The proposed test directly use the original data rather than the transformed data and is straightforward both algebraically and computationally.

Besides it has nice properties like efficiency and being robust to plausible deviations from the assumed model, i.e. not much affected from the outliers, contamination or the misspesification of the shape parameter. The robustness of the test is due to the half-umbrella ordering of the β_j coefficients, i.e. they decrease in the direction of the long tail(s). Thus, the extreme observations in the direction of the long tail(s) automatically receive small weights. That is instrumental to achieve robustness; see [8] and [20].

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Approximations in a hyperlattice by using set-valued homomorphisms

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Abstract

In this paper, the concepts of set-valued homomorphism and strong set-valued homomorphism of a hyperlattice are introduced. The notions of generalized lower and upper approximation operators constructed by means of a set-valued mapping are provided. We also propose the notions of generalized lower and upper approximations with respect to a hyperideal of a hyperlattice which is an extended notion of rough hyperideal in a hyperlattice and discuss some significiant properties of them.

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1. Introduction

The theory of algebraic hyperstructures is a well-established branch of classical algebraic theory which were initiated by Marty [15]. In a classical algebraic structure, the composition of two elements is an element while in an algebraic hyperstructure the composition of two elements is a set. Hundreds of papers and several books have been written on hyperstructure theory, see for instance [5,6]. Hyperlattices were first studied by Konstantinidou and Mittas [18]. Since the concept of hyperlattice is a generalization of the concept of lattice, hyperlattice theory was studied by Konstantinidou [19-21], Ashrafi [3], Rahnamai-Barghi [29-30] Guo and Xin [14], Han and Zhao [12], Zhao and Han [37].

Rough set theory was proposed by Pawlak [26]; see also [27-28]. The theory of rough sets is an extension of set theory, in which a subset of a universe is described by a pair of

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ordinary sets called the lower and upper approximations. A key concept in Pawlak rough set model is the equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. However, the requirement of an equivalence relation in Pawlak rough set model seems to be a very restrictive condition that may limit the applications of rough set models. Thus, one of the main directions of research in rough set theory is naturally the generalization of Pawlak rough set approximations. For instance, the notion of approximations are extended to general binary relations, coverings, completely distributive lattices, fuzzy lattices and Boolean algebras. This research soon led to a natural question concerning the possible connection between rough sets and algebraic systems.

In [22], Kuroki introduced a rough ideal in a semigroup. Kuroki and Wang [23] presented some properties of the lower and upper approximations with respect to normal subgroups. Davvaz [8] investigated the relationship between rough sets and ring theory by considering a ring as a universal set and introducing the concepts of rough subrings and rough ideals with respect to an ideal of a ring. Kazancı and Davvaz [16] introduced the notions of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in a ring and presented some properties of such ideals. Rough semigroups, rough modules, rough lattices, rough MV-algebras, rough hemirings and rough γ semihyperrings have been investigated by many authors(see also [1,2,4,7,8,11,17,19,24,25,31,34]). Davvaz and Mahdavipour [10] presented a framework for generalizing the standard notion of rough set approximation space. They proposed new definitions of the lower and upper approximations which are basic concepts of rough set theory. In [9], Davvaz introduced the concept of set-valued homomorphism for groups which is a generalization of an ordinary homomorphism. The concepts of set-valued homomorphism and strong set-valued homomorphism of a ring were introduced by Yamak et al.[35] and Hooshmandasl et al. [13] .

The initiation and majority of studies on rough sets for algebraic structures have been concentrated on a congruence relation. The congruence relation, however, seems to restrict the application of the generalized rough set model for algebraic sets. This may be by reason of incomplete information about the objects under consideration. Sometimes due to imprecise human knowledge about the elements of the universe set, an equivalence relation among these elements is difficult to find. To overcome this problem, we require set-valued maps instead of equivalence relations in generalized rough sets. This technique is useful where it is not easy to find a equivalence relation among the objects of the universe set. This paper is structured as follows. After an introduction, in Section 2, we present some basic definitions and results about approximation operators. In Section 3, we restrict the universe of the approximation space to a hyperlattice and we introduce the axiomatic form of this concept. In Section 4, the concepts of generalized lower and upper approximation operators constructed by means of a set-valued homomorphism with respect to a hyperideal of a hyperlattice is presented and we examine some properties of these operators in a hyperlattice.

2. Preliminaries

In this section, we recall some notions and results (see [5,6,14,15,20]) which will be used throughout this article. Let L be a non-empty set and $P^*(L)$ be the set of all nonempty subsets of L. A hyperoperation on L is a map $\circ : L \times L \to P^*(L)$ which associates a nonempty subset $a \circ b$ with any pair (a, b) of elements of $L \times L$. The couple $(L, \circ)$ is called a hypergroupoid. If A and B are nonempty subsets of L, then for $a, b, x \in L$, we denote

(1) $x \circ A = \{x\} \circ A = \bigcup_{a \in A} x \circ a$, $A \circ x = A \circ \{x\} = \bigcup_{a \in A} a \circ x$. (2) $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$.

2.1. Definition. [14] Let L be a non-empty set endowed with two hyperoperations $\otimes$ and $\oplus$. The triple $(L, \otimes, \oplus)$ is called a hyperlattice if the following conditions hold for all $a, b, c \in L$:

(1) (idempotent laws) $a \in a \otimes a$, $a \in a \oplus a$,
(2) (commutative laws) $a \otimes b = b \otimes a$, $a \oplus b = b \oplus a$,
(3) (associative laws)$(a \otimes b) \otimes c = a \otimes (b \otimes c)$, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$,
(4) (absorption laws) $a \in a \otimes (a \oplus b)$, $a \in a \oplus (a \otimes b)$.

2.2. Definition. [14] Let $L = (L, \otimes, \oplus)$ be a hyperlattice and $S \in P^*(L)$. Then S is called a subhyperlattice of L if $a \otimes b$ and $a \oplus b \in P^*(S)$ for all $a, b \in S$. That is to say, S is subhyperlattice of L if and only if S is closed under the two hyperoperation $\otimes$ and $\oplus$ on L.

2.3. Example. Let $L = \{a, b, c, d\}$ be a set. Define the hyperoperations $''\otimes''$ and $''\oplus''$ on L with the following Cayley table :

$\otimes$	a	b	c	d
a	a	a	a	a
b	a	b	a	{a,b}
c	a	a	c	c
d	a	{a,b}	c	d

$\oplus$	a	b	c	d
a	a	b	{c,d}	d
b	b	b	d	d
c	{c,d}	d	{c,d}	d
d	d	d	d	d

It is easy to check that $(L, \otimes, \oplus)$ is a hyperlattice. Consider the subsets $S_1 = \{a, d\}$, $S_2 = \{c, d\}$. Then S_1 and S_2 are subhyperlattices of L. If we get $S_3 = \{a, c\}$, then S_3 is not a subhyperlattice of L. Because it isn't closed under the hyperoperation $\oplus$ on L.

2.4. Definition. [14] Let $L_1 = (L_1, \otimes_1, \oplus_1)$ and $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices. A map $\varphi : L_1 \to L_2$ is called a

(i) weak hyperlattice homomorphism if $\varphi(a \otimes_1 b) \subseteq \varphi(a) \otimes_2 \varphi(b)$ and $\varphi(a \oplus_1 b) \subseteq \varphi(a) \oplus_2 \varphi(b)$ for all $a, b \in L_1$,
(ii) strong hyperlattice homomorphism if $\varphi(a \otimes_1 b) = \varphi(a) \otimes_2 \varphi(b)$ and $\varphi(a \oplus_1 b) = \varphi(a) \oplus_2 \varphi(b)$ for all $a, b \in L_1$.

If such a homomorphism φ is surjective, injective or bijective, then φ is called an epimorphism, a monomorphism or an isomorphism from the hyperlattice $(L_1, \otimes_1, \oplus_1)$ to the hyperlattice $(L_2, \otimes_2, \oplus_2)$, respectively.

2.5. Definition. Let $L = (L, \otimes, \oplus)$ be a hyperlatice and $A \in P^*(L)$. Then A is called a hyperideal of L if and only if $a \otimes x \in P^*(A)$, $a \oplus x \in P^*(A)$ for all $a \in A, x \in L$.

Let $(L, \otimes, \oplus)$ be a hyperlattice. An equivalence relation θ is a reflexive, symmetric, and transitive binary relation on L. If θ is an equivalence relation on L, then the equivalence class of $a \in L$ is the set $\{y \in L \mid (a, y) \in \theta\}$. We write it as $[a]_\theta$.
Let θ be an equivalence relation on L. For any $A, B \in P^*(L)$, we write that $A\overline{\theta}B$ if the following two conditions are hold:

(1)$\forall a \in A, \exists b \in B$ such that $a\theta b$; (2) $\forall x \in B, \exists y \in A$ such that $x\theta y$.

We denote $A\overline{\overline{\theta}}B$ if for all $a \in A, b \in B$ we have $a\theta b$.

2.6. Definition. [32] An equivalence relation θ on a hyperlattice $L = (L, \otimes, \oplus)$ is called a regular (strongly regular) hypercongruence relation if for every $x \in L$, $(a, b) \in \theta$ implies

$$(a \otimes x)\overline{\theta}(b \otimes x) \quad \text{and} \quad (a \oplus x)\overline{\theta}(b \oplus x) \ ((a \otimes x)\overline{\overline{\theta}}(b \otimes y) \quad \text{and} \quad (a \oplus x)\overline{\overline{\theta}}(b \oplus y)).$$

Clearly, any strongly regular hypercongruence relation is a regular hypercongruence relation.

2.7. Example. Let $L = \{a, b, c, d\}$ and let the hyperoperations $''\otimes''$ and $''\oplus''$ on L be defined as follows:

$\otimes$	a	b	c	d
a	a	a	a	a
b	a	{a,b}	a	{a,b}
c	a	a	c	c
d	a	{a,b}	c	{c,d}

$\oplus$	a	b	c	d
a	{a,b}	b	{c,d}	d
b	b	b	d	d
c	{c,d}	d	{c,d}	d
d	d	d	d	d

Then $(L, \otimes, \oplus)$ is a hyperlattice [14]. Let θ be a hypercongruence relation on the hyperlattice L with the following equivalence classes: $[a]_\theta = [b]_\theta = \{a, b\}$, $[c]_\theta = [d]_\theta = \{c, d\}$. Then θ is a strongly regular hypercongruence relation on L.

2.8. Definition. Let $L = (L, \otimes, \oplus)$ be a hyperlattice and θ be a regular hypercongruence relation on L. Then θ is called a complete hypercongruence relation if $[a \otimes b]_\theta = \{x \otimes y \mid x \in [a]_\theta, y \in [b]_\theta\}$, and $[a \oplus b]_\theta = \{x \oplus y \mid x \in [a]_\theta, y \in [b]_\theta\}$ for all $a, b \in L$.

2.9. Example. Let $L = \{0, a, b, c, 1\}$ be a lattice $(L, \wedge, \vee)$, where the partial order relation on L is defined as shown in Figure 1. For all $x, y \in L$, $x \otimes y = \{x \wedge y\}$, $x \oplus y = \{x \vee y\}$, then $L = (L, \otimes, \oplus)$ is a hyperlattice.

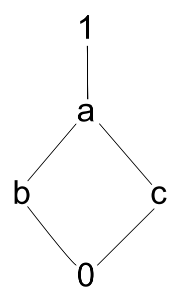

Figure 1. The lattice in Example 2.9.

(i) Let θ be a regular hypercongruence relation on the hyperlattice L with the following equivalence classes: $[1]_\theta = 1, [a]_\theta = [c]_\theta = \{a, c\}, [b]_\theta = [0]_\theta = \{b, 0\}$. Then θ is a complete hypercongruence relation.

(ii) Let θ be a regular hypercongruence relation on the hyperlattice L with the following equivalence classes: $[1]_\theta = [a]_\theta = \{1, a\}$, $[c]_\theta = \{c\}, [b]_\theta = \{b\}, [0]_\theta = \{0\}$. θ is not complete because $[c \oplus b]_\theta = \{1, a\}$, $[c]_\theta \oplus [b]_\theta = \{a\}$ and $[c \oplus b]_\theta \neq [c]_\theta \oplus [b]_\theta$.

2.10. Lemma. *Let $L = (L, \otimes, \oplus)$ be a hyperlattice and θ be a regular hypercongruence relation on L. Then for all $a, b, c, d \in L$,*

(i) *If $(a, b) \in \theta$ and $(c, d) \in \theta$, then $(a \otimes c)\overline{\theta}(b \otimes d)$ and $(a \oplus c)\overline{\theta}(b \oplus d)$,*

(ii) $\{x \otimes y \mid x \in [a]_\theta, y \in [b]_\theta\} \subseteq [a \otimes b]_\theta$,

(iii) $\{x \oplus y \mid x \in [a]_\theta, y \in [b]_\theta\} \subseteq [a \oplus b]_\theta$.

3. Rough subsets of a hyperlattice in the generalized approximation space

In this section, according to the notion of generalized approximation space presented in [9,35,36], we present some basic concepts about the generalized approximation space (U, W, T) and the associated lower and upper approximation operators. Let U and W be two non-empty universes. Let T be a set-valued mapping given by $T : U \to P(W)$. Then the triple (U, W, T) is referred to as a generalized approximation space. Any set-valued function from U to $P(W)$ defines a binary relation from U to W by setting $\rho_T = \{(x, y) \mid y \in T(x)\}$. Obviously, if ρ is an arbitrary relation from U to W, then it can be defined as a set-valued mapping $T_\rho : U \to P(W)$ by $T_\rho(x) = \{y \in W \mid (x, y) \in \rho\}$, where $x \in U$. For any set $X \subseteq W$, a pair of lower and upper approximations $\underline{T}(X)$ and $\overline{T}(X)$, are defined by
$\underline{T}(X) = \{x \in U \mid T(x) \subseteq X\}$ and $\overline{T}(X) = \{x \in U \mid T(x) \cap X \neq \emptyset\}$. The pair $(\underline{T}(X), \overline{T}(X))$ is referred to as a generalized rough set and $\underline{T}$ and $\overline{T}$ are referred to as lower and upper generalized approximation operators, respectively.

3.1. Definition. Let $L_1 = (L_1, \otimes_1, \oplus_1)$ and $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices. A mapping $T : L_1 \to P(L_2)$ is called a set-valued homomorphism if for all $a, b \in L_1$,

(i) $T(a) \otimes_2 T(b) \subseteq T(a \otimes_1 b)$,

(ii) $T(a) \oplus_2 T(b) \subseteq T(a \oplus_1 b)$.

3.2. Definition. Let $L_1 = (L_1, \otimes_1, \oplus_1)$ and $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices. A mapping $T : L_1 \to P(L_2)$ is called a strong set-valued homomorphism if for all $a, b \in L_1$,

(i) $T(a) \otimes_2 T(b) = T(a \otimes_1 b)$,

(ii) $T(a) \oplus_2 T(b) = T(a \oplus_1 b)$.

3.3. Example. Let $L_1 = (L_1, \otimes_1, \oplus_1)$ and $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices.

(i) The set-valued map $T : L_1 \to P(L_2)$ defined by $T(a) = L_2$ is a set-valued homomorphism.

(ii) If θ is a regular hypercongruence relation on a hyperlattice L_1 then $T_\theta : L_1 \to P(L_1)$ defined by $T_\theta(a) = [a]_\theta$ is a set-valued homomorphism. If θ is a complete regular hypercongruence then T_θ is a strong set-valued homomorphism.

(iii) If $\varphi : L_1 \to L_2$ is a strong hyperlattice homomorphism, then the set-valued map $T : L_1 \to P(L_2)$ defined by $T(a) = \{\varphi(a)\}$ is a strong set-valued homomorphism.

Note that Example 3.3. (ii) indicates that every regular hyper congruence relations may be considered as a set-valued homomorphism. On the other hand, hypercongruence relations are important in hyperalgebraic systems. So set-valued homomorphisms are interesting for pure algebraic systems.

3.4. Proposition. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$ and $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and $T : L_1 \to P(L_2)$ be a set valued homomorphism. If $X, Y \in P^*(L_2)$, then*

(i) $\overline{T}(X) \otimes_1 \overline{T}(Y) \subseteq \overline{T}(X \otimes_2 Y)$,
(ii) $\overline{T}(X) \oplus_1 \overline{T}(Y) \subseteq \overline{T}(X \oplus_2 Y)$.

Proof. (i) Assume that $x \in \overline{T}(X) \otimes_1 \overline{T}(Y)$. Then $x \in x_1 \otimes_1 x_2$ with $x_1 \in \overline{T}(X)$, $x_2 \in \overline{T}(Y)$. Hence $T(x_1) \cap X \neq \emptyset$ and $T(x_2) \cap Y \neq \emptyset$. Then there exist $a \in T(x_1) \cap X$ and $b \in T(x_2) \cap Y$ such that $a \in T(x_1), b \in T(x_2)$ and $a \in X, b \in Y$. Therefore $a \otimes_2 b \subseteq X \otimes_2 Y$. Since T is a set-valued homomorphism, we have $a \otimes_2 b \subseteq T(x_1) \otimes_2 T(x_2) \subseteq T(x_1 \otimes_1 x_2)$. Hence $T(x_1 \otimes_1 x_2) \cap (X \otimes_2 Y) \neq \emptyset$ which implies that $x \in \overline{T}(X \otimes_2 Y)$. So $\overline{T}(X) \otimes_1 \overline{T}(Y) \subseteq \overline{T}(X \otimes_2 Y)$.

(ii) The proof is similar to (i).

□

3.5. Corollary. *Let θ be a regular hypercongruence relation on a hyperlattice L and $X, Y \in P^*(L)$. Then*
(i) $\overline{T_\theta}(X) \otimes \overline{T_\theta}(Y) \subseteq \overline{T_\theta}(X \otimes Y)$,
(ii) $\overline{T_\theta}(X) \oplus \overline{T_\theta}(Y) \subseteq \overline{T_\theta}(X \oplus Y)$.

The following example shows that the inclusion symbol "$\subseteq$" in Propositions 3.4. may not be replaced by the equal sign.

3.6. Example. Consider the hyperlattice defined in Example 2.3. Let $T : L \to P(L)$ be a set-valued map defined as $T(x) = \{a\}$. Then it is easy to see that T is a set-valued homomorphism. If $X = \{b\}$ and $Y = \{d\}$, then $\overline{T}(X) \otimes \overline{T}(Y) = \emptyset$, $\overline{T}(X \otimes Y) = L$. Thus $\overline{T}(X) \otimes \overline{T}(Y) \neq \overline{T}(X \otimes Y)$. Further, if $T : L \to P(L)$ is a set-valued map defined as $T(x) = \{d\}$, then T is a set-valued homomorphism. If $X = Y = \{c\}$, then $\overline{T}(X) \oplus \overline{T}(Y) = \emptyset$, then $\overline{T}(X \oplus Y) = L$. Thus $\overline{T}(X) \oplus \overline{T}(Y) \neq \overline{T}(X \oplus Y)$.

3.7. Proposition. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and $T : L_1 \to P(L_2)$ be a strong set valued homomorphism. If $X, Y \in P^*(L_2)$, then*
(i) $\underline{T}(X) \otimes_1 \underline{T}(Y) \subseteq \underline{T}(X \otimes_2 Y)$,
(ii) $\underline{T}(X) \oplus_1 \underline{T}(Y) \subseteq \underline{T}(X \oplus_2 Y)$.

Proof. (i) Assume that $z \in \underline{T}(X) \otimes_1 \underline{T}(Y)$. Then $z \in x \otimes_1 y$ with $x \in \underline{T}(X)$, $y \in \underline{T}(Y)$. Hence $T(x) \subseteq X$ and $T(y) \subseteq Y$. Since T is a strong set-valued homomorphism, we have $T(x) \otimes_2 T(y) = T(x \otimes_1 x) \subseteq A \otimes_2 B$. Hence $z \in x \otimes_2 y \in \underline{T}(X \otimes_2 Y)$, that is $\underline{T}(X) \otimes_1 \underline{T}(Y) \subseteq \underline{T}(X \otimes_2 Y)$.

(ii) The proof is similar to (i).

□

3.8. Corollary. *Let θ be a regular hypercongruence relation on a hyperlattice L and $X, Y \in P^*(L)$. Then*
(i) $\underline{T_\theta}(X) \otimes \underline{T_\theta}(Y) \subseteq \underline{T_\theta}(X \otimes Y)$,
(ii) $\underline{T_\theta}(X) \oplus \underline{T_\theta}(Y) \subseteq \underline{T_\theta}(X \oplus Y)$.

The following example shows that the containment in the above proposition is proper.

3.9. Example. Consider the hyperlattice defined in Example 2.3. Let $T : L \to P(L)$ be a set-valued map defined as $T(x) = \{a\}$. Then it is easy to see that T is a set-valued homomorphism. If $X = \{d\}$, $Y = \{b\}$, then $\underline{T}(X) \otimes \underline{T}(Y) = \emptyset$, $\underline{T}(X \otimes Y) = L$.

Thus $\underline{T}(X) \otimes \underline{T}(Y) \neq \underline{T}(X \otimes Y)$. Further, if $T : L \to P(L)$ is a set-valued map defined as $T(x) = \{d\}$, then T is a set-valued homomorphism. If $X = Y = \{c\}$, then $\underline{T}(X) \oplus \underline{T}(Y) = \emptyset$, $\underline{T}(X \oplus Y) = L$. Thus $\underline{T}(X) \oplus \underline{T}(Y) \neq \underline{T}(X \oplus Y)$.

3.10. Proposition. *Let $T : L_1 \to P(L_2)$ be a (strong) set-valued homomorphism and $f : L_3 \to L_1$ be a weak (strong) hyperlattice homomorphism. Then $T \circ f$ is a (strong) set-valued homomorphism from $L_3 \to P(L_2)$ such that $\overline{T \circ f}(X) = f^{-1}(\overline{T}(X))$ and $\underline{T \circ f}(X) = f^{-1}(\underline{T}(X))$, for all $X \in P(L_2)$.*

Proof. The proof is straightforward.

□

3.11. Proposition. *Let $T : L_1 \to P(L_2)$ be a (strong) set-valued homomorphism and $f : L_2 \to L_3$ be a weak (strong) hyperlattice homomorphism. Then T_f is a (strong) set-valued homomorphism from $L_1 \to P(L_3)$ defined by $T_f(r) = f(T(r))$ such that $\underline{T_f}(X) = \underline{T}(f^{-1}(X))$ and $\overline{T_f}(X) = \overline{T}(f^{-1}(X))$, for all $X \in P(L_3)$.*

Proof. The proof is straightforward.

□

3.12. Definition. Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and let $T : L_1 \to P(L_2)$ be a set-valued mapping. If $\overline{T}(X)$ and $\underline{T}(X)$ are subhyperlattices (resp. hyperideals) of L_1, then $(\overline{T}(X), \underline{T}(X))$ is called a generalized rough subhyperlattice (resp. hyperideal).

3.13. Example. Let $L = (L, \otimes, \oplus)$ be a hyperlattice defined in Example 2.3. Let $T : L \to P(L)$ be a set-valued map defined as $T(x) = \{b\}$ and $X = \{a, b\}$. Then $\overline{T}(X)$ and $\underline{T}(X)$ are subhyperlattices (resp. hyperideals) of L. Hence $(\overline{T}(X), \underline{T}(X))$ is a generalized rough subhyperlattice (resp. hyperideal).

3.14. Theorem. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and $X \in P^*(L_2)$.*

(i) *If $T : L_1 \to P(L_2)$ is a set-valued homomorphism and X is a subhyperlattice of L_2, then $\overline{T}(X)$ is a subhyperlattice of L_1.*
(ii) *If $T : L_1 \to P(L_2)$ is a strong set-valued homomorphism and X is a subhyperlattice of L_2, then $\underline{T}(X)$ is, if it is non-empty, a subhyperlattice of L_1.*
(iii) *If $T : L_1 \to P^*(L_2)$ is a set-valued homomorphism and X is a hyperideal of L_2, then $\overline{T}(X)$ is a hyperideal of L_1.*
(iv) *If $T : L_1 \to P^*(L_2)$ is a strong set-valued homomorphism and X is a hyperideal of L_2, then $\underline{T}(X)$ is, if it is non-empty, a hyperideal of L_1.*

Proof. (i) Suppose that $x, y \in \overline{T}(X)$. Then $T(x) \cap X \neq \emptyset$ and $T(y) \cap X \neq \emptyset$. Hence there exist $a \in T(x) \cap X$ and $b \in T(y) \cap X$. Thus $a \otimes_2 b \subseteq T(x) \otimes_2 T(y) \subseteq T(x \otimes_1 y)$ and $a \oplus_2 b \subseteq T(x) \oplus_2 T(y) \subseteq T(x \oplus_1 y)$. Since X is a subhyperlattice of L_2, we have $a \otimes_2 b \subseteq X$ and $a \oplus_2 b \subseteq X$. So $T(x \otimes_1 y) \cap X \neq \emptyset$ and $T(x \oplus_1 y) \cap X \neq \emptyset$. Therefore $x \otimes_1 y, x \oplus_1 y \in \overline{T}(X)$. Consequently, $\overline{T}(X)$ is a subhyperlattice of L_1.

(ii) Suppose that $x, y \in \underline{T}(X)$. Then $T(x) \subseteq X$ and $T(y) \subseteq X$. Since X is a subhyperlattice of L_2 and T is a strong set-valued homomorphism, we have $T(x \otimes_1 y) =$

$T(x) \otimes_2 T(y) \subseteq X \otimes_2 X \subseteq X$ and $T(x \oplus_1 y) = T(x) \oplus_2 T(y) \subseteq X \oplus_2 X \subseteq X$. Thus $x \otimes_1 y, x \oplus_1 y \in \underline{T}(X)$. Therefore $\underline{T}(X)$ is a subhyperlattice of L_1.

(iii) By (i) $\overline{T}(X)$ is a subhyperlattice of L_1. Let $b \in L_1$. Since $T(b) \neq \emptyset$, there exist some $z \in L_2$ such that $z \in T(b)$. Let $x \in \overline{T}(X)$. Then $T(x) \cap X \neq \emptyset$ which implies that there exists $a \in T(x) \cap X$, that is $a \in T(x), a \in X$. Since X is a hyperideal of L_2 and T is a strong set-valued homomorphism, we have $a \otimes_2 z, a \oplus_2 z \subseteq X$ and $a \otimes_2 z \subseteq T(x) \otimes_2 T(b) = T(x \otimes_1 b)$, $a \oplus_2 z \subseteq T(x) \oplus_2 T(b) = T(x \oplus_1 b)$ which implies that $T(x \otimes_1 b) \cap X \neq \emptyset$ and $T(x \oplus_1 b) \cap X \neq \emptyset$. Thus $x \otimes_1 b, x \oplus_1 b \in \overline{T}(X)$. Therefore $\overline{T}(X)$ is a hyperideal of L_1.

(iv) Similarly, $\underline{T}(X)$ is a hyperideal of L_1.

□

The following example shows that the converse of the above theorem does not hold in general.

3.15. Example. Consider the hyperlattice defined Example 2.3. Let $T : L \to P(L)$ be a set-valued map defined as $T(x) = \{d\}$. Then it is easy to see that T is a set-valued homomorphism. If $X = \{b, d\}$, then X is not a subhyperlattice (hyperideal) of L. But $\overline{T}(X) = L$ is a subhyperlattice (hyperideal) of L.

3.16. Corollary. *Let θ be a regular hypercongruence relation on a hyperlattice $L = (L, \otimes, \oplus)$.*

(i) *If X is a hyperlattice of L, then $\overline{T_\theta}(X)$ is a subhyperlattice of L.*
(ii) *If θ is a complete regular hypercongruence relation and X is a subhyperlattice of L, then $\underline{T_\theta}(X)$ is, if it is non-empty, a subhyperlattice of L.*
(iii) *If X is a hyperideal of L, then $\overline{T_\theta}(X)$ is a hyperideal of L.*
(iv) *If θ is a complete regular hypercongruence relation and X is a hyperideal of L, then $\underline{T_\theta}(X)$ is, if it is non-empty, a hyperideal of L.*

Now we give a counterexample which shows that the condition that θ is a complete regular hypercongruence relation in Corollary 3.16. is necessary.

3.17. Example. Consider the hyperlattice L and the congruence relation on L defined in Example 2.9.(ii). If $X = \{a, b, c, 0\}$, then X is a subhyperlattice of L. But $\underline{T_\theta}(X) = \{b, c, 0\}$ is not a subhyperlattice of L.

4. Generalized lower and upper approximation operators with respect to a hyperideal of a hyperlattice

4.1. Definition. Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices, A be a hyperideal of L_2 and $T : L_1 \to P(L_2)$ be a set-valued mapping. Then we define $T_A : L_1 \to P(L_2)$ as $T_A(a) = T(a) \otimes_2 A$ for all $a \in L_1$. Then T_A is called the set-valued mapping with respect to a hyperideal A.

4.2. Definition. Let (L_1, L_2, T_A) be a generalized approximation space with respect to a hyperideal A and X be a non-empty subset of L_2. Then the sets
$\underline{T_A}(X) = \{a \in L_1 \mid T_A(a) \subseteq X\}$ and $\overline{T_A}(X) = \{a \in L_1 \mid T_A(a) \cap X \neq \emptyset\}$

are called generalized lower and upper approximations of X with respect to the hyperideal A, respectively.

4.3. Lemma. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and A, B be hyperideals of L_2. Let X be a subset of L_2 such that $A \subseteq B$. Then*

(i) $\overline{T_A}(X) \subseteq \overline{T_B}(X)$,
(ii) $\underline{T_B}(X) \subseteq \underline{T_A}(X)$.

Proof. (i) Suppose that $x \in \overline{T_A}(X)$. Then $(T(x) \otimes_2 A) \cap X \neq \emptyset$. So there exist $a \in (T(x) \otimes_2 A) \cap X$ such that $a \in (T(x) \otimes_2 A)$ and $a \in X$. Hence there exist $y \in T(x), z \in A$ such that $a = y \otimes_2 z$. Since $A \subseteq B$, we have $z \in B$. Thus $a = y \otimes_2 z \subseteq T(x) \otimes_2 B$ and $a \in X$. So $(T(x) \otimes_2 B) \cap X \neq \emptyset$. As a consequent, we obtain $\overline{T_A}(X) \subseteq \overline{T_B}(X)$.

(ii) The proof is similar to (i).

□

The following corollary follows from Lemma 4.3.

4.4. Corollary. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices and A, B be hyperideals of L_2. Let X be a subset of L_2 such that $A \subseteq B$. Then*

(i) $\overline{T_{A\cap B}}(X) \subseteq \overline{T_A}(X) \cap \overline{T_B}(X)$,
(ii) $\underline{T_A}(X) \cap \underline{T_B}(X) \subseteq \underline{T_{A\cap B}}(X)$.

4.5. Proposition. *Let (L_1, L_2, T_A) be a generalized approximation with respect to a hyperideal A and X, Y be a non-empty subsets of L_2.*

(i) *If $T : L_1 \to P(L_2)$ is a set-valued homomorphism, then $\overline{T_A}(X) \otimes_1 \overline{T_A}(Y) \subseteq \overline{T_A}(X \otimes_2 Y)$.*
(ii) *If $T : L_1 \to P(L_2)$ is a strong set-valued homomorphism, then $\underline{T_A}(X) \otimes_1 \underline{T_A}(Y) \subseteq \underline{T_A}(X \otimes_2 Y)$.*

Proof. (i) Suppose that $z \in \overline{T_A}(X) \otimes_1 \overline{T_A}(Y)$. Then there exist $x \in \overline{T_A}(X), y \in \overline{T_A}(Y)$ such that $z \in x \otimes_1 y$. Since $x \in \overline{T_A}(X)$ $y \in \overline{T_A}(Y)$ there exist $a \in T(x) \otimes_2 A$, $b \in T(y) \otimes_2 A$ such that $a \in T(x), b \in T(y), a \in X, b \in Y$. Since T is a set-valued homomorphism, we have $a \otimes_2 b \subseteq T(x) \otimes_2 T(y) \otimes_2 A \subseteq T(x \otimes_1 y) \otimes_2 A$ and $a \otimes_2 b \subseteq X \otimes_2 Y$. Hence $a \otimes_2 b \subseteq T(x \otimes_1 y) \otimes_2 A \cap (X \otimes_2 Y)$. So $z \in x \otimes_1 y \subseteq \overline{T_A}(X \otimes_2 Y)$.Therefore, we obtain $\overline{T_A}(X) \otimes_1 \overline{T_A}(Y) \subseteq \overline{T_A}(X \otimes_2 Y)$.

(ii) The proof is similar to (i).

□

4.6. Proposition. *Let $L_1 = (L_1, \otimes_1, \oplus_1)$, $L_2 = (L_2, \otimes_2, \oplus_2)$ be two hyperlattices, A, B be hyperideals of L_2 and X be a subhyperlatice of L_2.*

(i) *If $T : L_1 \to P(L_2)$ is a set-valued homomorphism, then $\overline{T_A}(X) \otimes_1 \overline{T_B}(X) \subseteq \overline{T_{A\otimes_2 B}}(X)$.*
(ii) *If $T : L_1 \to P(L_2)$ is a strong set-valued homomorphism, then $\underline{T_A}(X) \otimes_1 \underline{T_B}(X) = \underline{T_{A\otimes_2 B}}(X)$.*

Proof. The proof is straightforward.

□

4.7. Theorem. *Let (L_1, L_2, T_A) be a generalized approximation space with respect to a hyperideal A and X be a non-empty subset of L_2.*

(i) *If $T : L_1 \to P(L_2)$ is a set-valued homomorphism and X is a subhyperlattice of L_2, then $\overline{T_A}(X)$ is a subhyperlattice of L_1.*

(ii) *If* $T : L_1 \to P(L_2)$ *is a strong set-valued homomorphism and* X *is a subhyperlattice of* L_2*, then* $\underline{T_A}(X)$ *is, if it is non-empty, a subhyperlattice of* L_1*.*

(iii) *If* $T : L_1 \to P^*(L_2)$ *is a set-valued homomorphism and* X *is a hyperideal of* L_2*, then* $\overline{T_A}(X)$ *is a hyperideal of* L_1*.*

(iv) *If* $T : L_1 \to P^*(L_2)$ *be a strong set-valued homomorphism and* X *is a hyperideal of* L_2*, then* $\underline{T_A}(X)$ *is, if it is non-empty, a hyperideal of* L_1*.*

Proof. (i) Suppose that $x, y \in \overline{T_A}(X)$. Then, $(T(x) \otimes_2 A) \cap X \neq \emptyset$ and $(T(y) \otimes_2 A) \cap X \neq \emptyset$. Hence there exist $a \in (T(x) \otimes_2 A) \cap X$ and $b \in (T(y) \otimes_2 A) \cap X$. Since X is a subhyperlattice of L_2 , we have $a \otimes_2 b \subseteq X$ and $a \oplus_2 b \subseteq X$. On the other hand, $a \otimes_2 b \subseteq (T(x) \otimes_2 A) \otimes_2 (T(y) \otimes_2 A) \subseteq T(x) \otimes_2 T(y) \otimes_2 A \subseteq T(x \otimes_1 y) \otimes_2 A$ and $a \oplus_2 b \subseteq (T(x) \otimes_2 A) \oplus_2 (T(y) \otimes_2 A) \subseteq T(x) \oplus_2 T(y) \otimes_2 A \subseteq T(x \oplus_1 y) \otimes_2 A$. So $T(x \otimes_1 y) \otimes_2 A \cap X \neq \emptyset$ and $T(x \oplus_1 y) \otimes_2 A \cap X \neq \emptyset$. Thus $x \otimes_1 y, x \oplus_1 y \in \overline{T_A}(X)$. Therefore, $\overline{T_A}(X)$ is a subhyperlattice of L_1.

(ii) Similarly, $\underline{T_A}(X)$ is a subhyperlattice of L_1.

(iii) Using (i), $\overline{T_A}(X)$ is a subhyperlattice of L_1. Let $x \in \overline{T_A}(X)$ and $c \in L_1$. Then $(T(x) \otimes_2 A) \cap X \neq \emptyset$. So there exist $a \in (T(x) \otimes_2 A) \cap X$. Since $\overline{T_A}(X)$ is non-empty set, we can choose $z \in T(c)$. Since X is a hyperideal of L_2, we have $a \otimes_2 z, a \oplus_2 z \subseteq X$. On the other hand, $a \otimes_2 z \subseteq (T(x) \otimes_2 A) \otimes_2 T(c) \subseteq T(x \otimes_1 c) \otimes_2 A$, $a \oplus_2 z \subseteq (T(x) \otimes_2 A) \oplus_2 T(c) \subseteq T(x \oplus_1 c) \otimes_2 A$. So $(T(x \otimes_1 c) \otimes_2 A) \cap X \neq \emptyset$, $(T(x \oplus_1 c) \otimes_2 A) \cap X \neq \emptyset$ which implies $x \otimes_1 c, x \oplus_1 c \in \overline{T_A}(X)$. Therefore $\overline{T_A}(X)$ is a hyperideal of L_1.

(iv)The proof is straightforward.

□

The following example shows that the converse of the above theorem does not hold in general.

4.8. Example. Consider the hyperlattice defined in Example 2.9. Let $T : L \to P(L)$ be a set-valued map defined as $T(x) = \{d\}$. Then it is easy to see that T is a set-valued homomorphism. If $A = L$, $X = \{a, b, c\}$, then A is a hyperideal and X is not a subhyperlattice (hyperideal) of L. But $\overline{T_A}(X) = L$ is a subhyperlattice (hyperideal) of L.

5. Conclusion

The Pawlak rough sets on algebraic sets such as semigroups, groups, rings, modules and lattices were mainly studied by congruence relations. In this paper, a definition of set-valued homomorphism which was introduced for groups by Davvaz [9], for rings and modules by Yamak et al. [35-36], respectively, is considered as a regular hypercongruence relation for hyperlattices. We obtain some new properties of a set-valued homomorphism to provide opportunity for putting reasonable interpretations on the theory and applications of rough sets and adhering to the set-valued homomorphism and exploring the features of generalized rough approximations on hyperlattices. So, in this paper we propose a definition of set-valued homomorphism and explore the properties of generalized rough approximations on hyperlattices. Some new properties of set-valued homomorphisms which shall be very practical in the theory and applications of rough sets are obtained. Moreover, a new algebraic structure called generalized lower and upper approximations of a set with respect to a hyperideal is presented.

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Exponentiated generalized geometric distribution: A new discrete distribution

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Abstract

In this paper, a new three-parameter extension of the generalized geometric distribution of [6] is introduced. The new discrete distribution belongs to the resilience parameter family and handles a decreasing, increasing, upside-down and bathtub-shaped hazard rate function. The new distributions can also be considered as discrete analogs of some recent continuous distributions belonging to the known Marshall-Olkin family. Here, some basic statistical and mathematical properties of the new distribution are studied. In addition, estimation of the unknown parameters, a simulated example and an application of the new model are illustrated.

Keywords: Geometric distribution, Generalized geometric distribution, Generalized exponential geometric distribution, Marshall-Olkin family, Resilience parameter family.

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1. Introduction

Discretizing continuous distributions has recently received much attention in the literature. Let $F(x) = P(X \leq x)$ be the cumulative distribution function (cdf) of the absolutely continuous random variable X. The corresponding probability mass function (pmf) of X can be obtained by

$$P(X = x) = p_x = F(x+1) - F(x), \quad x \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \tag{1.1}$$

In recent years, several new discrete distributions have been appeared in the literature by Eq. (1.1). For example, we can address the works of [12], [15], [9] and [10] which are the discrete versions of Weibull, Rayleigh, half-normal and Burr distributions, respectively. A discrete version of Lindley distribution is introduced by [7] and [4]. [13] obtained a new discrete distribution by discretizing generalized exponential distribution of [8]. Discrete modified Weibull distributions, which are discrete versions of some known modified Weibull distributions, are introduced by [14] and [3]. In addition, [5] introduced the discrete additive Weibull distribution as a discrete version of the additive Weibull distribution of [19].

[11] introduced an extended family of distributions generated by the cdf

$$F(x; \alpha) = \frac{G(x)}{1 - \overline{\alpha}\overline{G}(x)}; \quad x \in \mathbb{R}_X, \quad \alpha > 0, \tag{1.2}$$

where $\overline{\alpha} = 1 - \alpha$ and $G(x)$ is the cdf of an absolutely continuous distribution. Several new continuous distributions have been obtained in the literature by inserting an arbitrary $G(x)$ into Eq. (1.2). For example, inserting the cdf of the exponential distribution into Eq. (1.2) yields a new distribution, called exponential Marshall-Olkin distribution, with cdf

$$F(x; \alpha, \beta) = \frac{1 - e^{-\beta x}}{1 - \overline{\alpha} e^{-\beta x}}; \quad x > 0, \quad \beta > 0, \alpha > 0, \tag{1.3}$$

(see [11]). For $0 < \alpha < 1$, (1.3) coincides with the cdf of the exponential-geometric (EG) distribution of [2].

[6] obtained the generalized geometric (GG) distribution by discretizing the exponential Marshall-Olkin distribution using Eq. (1.1). It is evident that when $0 < \alpha < 1$, the GG distribution corresponds to a discrete analogue of the EG distribution.

In this paper, we will introduce the exponentiated generalized geometric (EGG) distribution which is indeed an extension of the GG distribution. This new distribution can also be considered as a discrete version of the generalized exponential-geometric (GEG) distribution of [17].

The paper is organized as follows: In Section 2, we introduce the new distribution and investigate some of its statistical properties. We also derive expressions for the probability generating function, moment generating function and factorial moments. In Section 3, we will show that the proposed distributions are not infinitely divisible in general. The order statistics are discussed in Section 4. Estimation, Fisher information matrix and a kind of simulated example are discussed in Section 5. An application of the new model is illustrated in Section 6. Finally, Section 7 involves some concluding remarks.

2. Three-parameter EGG distribution

Consider the $GG(\alpha, \theta)$ distribution of [6] with the cdf

$$F_{GG}(x; \alpha, \theta) = \frac{1 - \theta^{x+1}}{1 - \overline{\alpha}\theta^{x+1}}, \quad x \geq 0, \tag{2.1}$$

where $\alpha > 0$ and $0 < \theta < 1$ are the model parameters. By inserting (2.1) into the *resilience parameter family* of distributions, the cdf of the resulting discrete distribution is given by

$$F(x;\alpha,\theta,\gamma) = [F_{GG}(x;\alpha,\theta)]^{\gamma} = \left[\frac{1-\theta^{x+1}}{1-\overline{\alpha}\theta^{x+1}}\right]^{\gamma}, \quad x \geq 0, \tag{2.2}$$

in which $\gamma > 0$ is the *resilience parameter.*

We call such a random variable X, with cdf (2.2), an *exponentiated generalized geometric distribution* with parameters $\alpha > 0$, $0 < \theta < 1$ and $\gamma > 0$ and denote it by $EGG(\alpha, \theta, \gamma)$.

It is evident that when $\gamma > 0$ is an integer value, the cdf given by (2.2) agrees with the cdf of the maximum of γ independent and identical $GG(\alpha, \theta)$ random variables.

The corresponding pmf of a random variable X following an $EGG(\alpha, \theta, \gamma)$ distribution for $x \in \mathbb{N}_0$ is given by

$$f(x;\alpha,\theta,\gamma) = P(X = x) = \left[\frac{1-\theta^{x+1}}{1-\overline{\alpha}\theta^{x+1}}\right]^{\gamma} - \left[\frac{1-\theta^{x}}{1-\overline{\alpha}\theta^{x}}\right]^{\gamma}. \tag{2.3}$$

[17] introduced the continuous generalized exponential-geometric (GEG) distribution with cdf

$$F(x;\alpha,\theta,\gamma) = \left[\frac{1-e^{-\beta x}}{1-\overline{\alpha}e^{-\beta x}}\right]^{\gamma}, \tag{2.4}$$

where $0 < \alpha < 1$, $\beta > 0$ and $\gamma > 0$ are the model parameters. The above cdf is indeed a kind of exponentiated distribution which contains the EG distribution of [2] as a special case, when $0 < \alpha < 1$. It is interesting to note that for $0 < \alpha < 1$ and $0 < e^{-\beta} = \theta < 1$, the EGG distribution can be viewed as a discrete version of the GEG distribution. In addition, the EGG distribution reduces to the GG distribution when $\gamma = 1$. Several properties of the $GG(\alpha, \theta)$ distribution are obtained for the case $0 < \alpha < 1$; see [6]. We will study several properties of the $EGG(\alpha, \theta, \gamma)$ distribution in this case. Figure 1 plots the pmfs of the $EGG(\alpha, \theta, \gamma)$ distribution for some parameters values.

The survival and hazard rate functions of the $EGG(\alpha, \theta, \gamma)$ distribution are given by

$$S(x;\alpha,\theta,\gamma) = 1 - \left[\frac{1-\theta^{x+1}}{1-\overline{\alpha}\theta^{x+1}}\right]^{\gamma}, \quad x \geq 0$$

and

$$h(x;\alpha,\theta,\gamma) = \frac{\left[\frac{1-\theta^{x+1}}{1-\overline{\alpha}\theta^{x+1}}\right]^{\gamma} - \left[\frac{1-\theta^{x}}{1-\overline{\alpha}\theta^{x}}\right]^{\gamma}}{1 - \left[\frac{1-\theta^{x+1}}{1-\overline{\alpha}\theta^{x+1}}\right]^{\gamma}}, \quad x \in \mathbb{N}_0,$$

respectively. As we see from Figure 2, the hazard rate function of the new distribution can be decreasing, increasing, upside-down bathtub and bathtub-shaped, depending on its parameters values, and hence presents a very flexible behavior.

Now, let $b > 1$ and $k > 0$ be real non-integers. If $|z| < 1$, we have the series representations

$$(1-z)^{b} = \sum_{i=0}^{\infty} \frac{\Gamma(b+1)}{\Gamma(b+1-i)i!}(-1)^{i} z^{i} \tag{2.5}$$

and

$$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)j!} z^{j}.$$

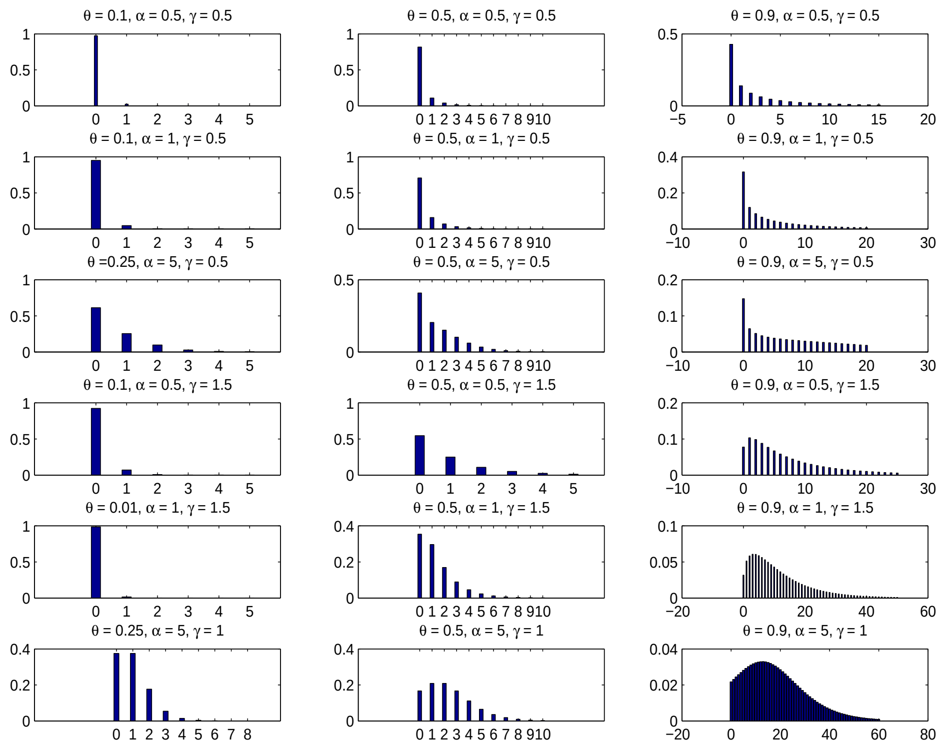

Figure 1. Pmfs of the $EGG(\alpha, \theta, \gamma)$ distribution for some parameter values.

The sum in Eq (2.5) stops at b for integer values of $b > 1$. Using the above series representations, Eq. (2.3), for $0 < \alpha < 1$, can be written as

$$(2.6)\quad f(x;\alpha,\theta,\gamma) = P(X = x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)(1-\theta^{i+j})\theta^{(i+j)x}, \quad x \in \mathbb{N}_0,$$

where

$$\omega_{i,j}(\alpha,\gamma) = \frac{\Gamma(\gamma+j)\gamma\overline{\alpha}^j(-1)^{i+1}}{i!j!\Gamma(\gamma+1-i)}.$$

It is clear that the pmf (2.6) is a linear combination of the geometric distributions

$$p_x = (1-\theta^{i+j})\theta^{(i+j)x}.$$

Hence, several properties of the $EGG(\alpha, \theta, \gamma)$ distribution can be obtained from those of the geometric distribution. For example, the moment and probability generating functions of the proposed distribution are given, respectively, by

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)\frac{1-\theta^{i+j}}{1-\theta^{i+j}e^t}$$

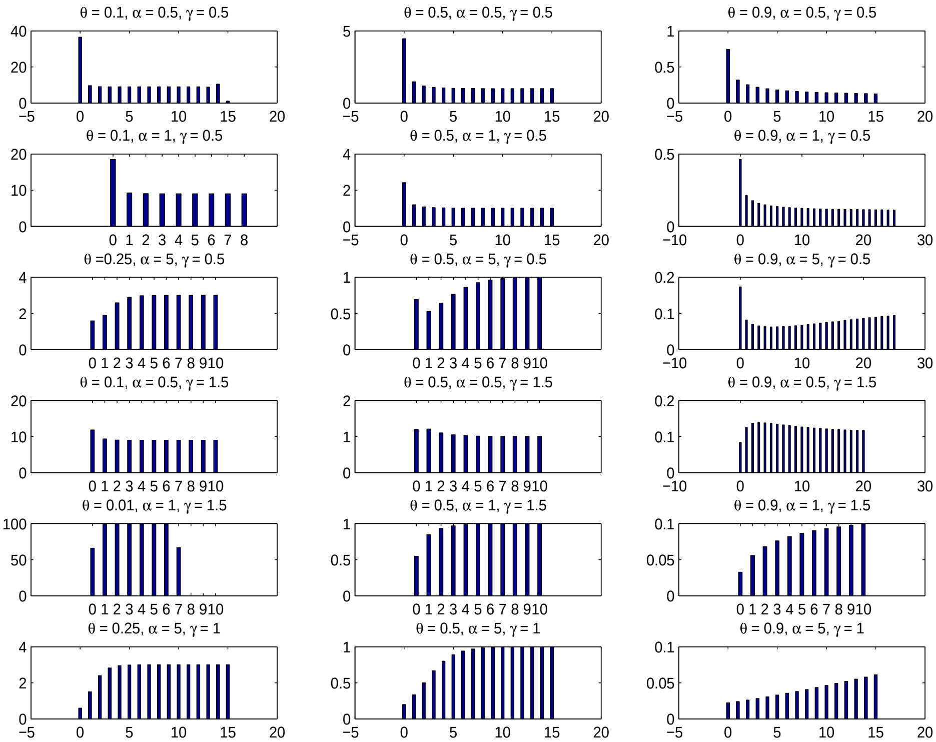

Figure 2. Hazard rate functions of the $EGG(\alpha, \theta, \gamma)$ distribution for some parameter values.

and

$$G_X(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)\frac{1-\theta^{i+j}}{1-\theta^{i+j}z}.$$

Moreover, the factorial moments are given by

$$E\left\{X(X-1)...(X-r+1)\right\} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)\left(\frac{\theta^{i+j}}{1-\theta^{i+j}}\right)^{r},$$

for $r = 1, 2, ...$. In particular, the mean and variance of the $EGG(\alpha, \theta, \gamma)$ distribution can be obtained by

$$E(X) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)\left(\frac{\theta^{i+j}}{1-\theta^{i+j}}\right) \tag{2.7}$$

and

$$\begin{aligned} Var(X) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)\left(\frac{\theta^{i+j}}{1-\theta^{i+j}}\right)\left(1+\frac{\theta^{i+j}}{1-\theta^{i+j}}\right) \\ &- \left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}(\alpha,\gamma)\left(\frac{\theta^{i+j}}{1-\theta^{i+j}}\right)\right\}^{2}. \end{aligned}$$

In addition, the median of the EGG model is given by

$$m = [\frac{1}{\log \theta} \left(\log \frac{1 - (1/2)^{1/\gamma}}{1 - \overline{\alpha}(1/2)^{1/\gamma}} \right) - 1],$$

where [.] denotes the integer part notation.

The mean and variance of the $EGG(\alpha, \theta, \gamma)$ distribution are calculated in Table 1 for different values of its parameters. It appears that the mean and variance increase, when α, θ, and γ increase. In addition, depending on the values of the parameters, the mean of the distribution can be smaller or greater than its variance.

Table 1 Mean (Variance) of $EGG(\alpha, \theta, \gamma)$ for different values of parameters.

γ	0.5					
α/θ	0.1		0.5		0.95	
	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)
0.1	0.0061 (0.0073)	0.0062 (0.0075)	0.0764 (0.1830)	0.0764 (0.1860)	2.4679 (47.6280)	2.4659 (48.6180)
0.5	0.0296 (0.0348)	0.0295 (0.0358)	0.3204 (0.7595)	0.3237 (0.7555)	7.6300 (169.2129)	7.6270 (170.2129)
1.0	0.0570 (0.0606)	0.0569 (0.0670)	0.5546 (1.2778)	0.5555 (1.3000)	11.5117 (268.8771)	11.5116 (269.8771)
2.0	0.1301 (0.1431)	0.1291 (0.1400)	1.0201 (2.2839)	1.0195 (2.29152)	16.0022 (403.8554)	16.6436 (413.8572)
γ	1.0					
α/θ	0.1		0.5		0.95	
	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)
0.1	0.0122 (0.0145)	0.0121 (0.0145)	0.1505 (0.3518)	0.1502 (0.3520)	4.5300 (84.5673)	4.5297 (85.5601)
0.5	0.0582 (0.0672)	0.0582 (0.0672)	0.6067 (1.3000)	0.6067 (1.2099)	13.0220 (259.8401)	13.0220 (259.8401)
1.0	0.1111 (0.1235)	0.1111 (0.1235)	1.0000 (2.0000)	1.0000 (2.0000)	19.0000 (380.0000)	19.0000 (380.0000)
2.0	0.2038 (0.1235)	0.2038 (0.1240)	1.5290 (2.8139)	1.5290 (2.8139)	26.5290 (519.9431)	26.5288 (519.9507)
γ	2.0					
α/θ	0.1		0.5		0.95	
	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)	Numeriacl	Eq. (2.7)
0.1	0.0255 (0.0285)	0.0241 (0.0285)	0.2909 (0.6513)	0.2909 (0.6513)	7.8655 (140.2428)	7.8655 (140.2428)
0.5	0.1136 (0.1255)	0.1136 (0.1255)	1.0761 (1.9730)	1.0761 (1.9730)	20.5445 (358.1039)	20.5445 (358.1039)
1.0	0.2121 (0.2163)	0.2121 (0.2163)	1.6667 (2.6667)	1.6667 (2.6667)	28.7436 (475.1874)	28.7436 (475.1874)
2.0	0.3742 (0.3313)	0.3742 (0.3313)	2.3854 (3.2973)	2.3854 (3.2973)	38.4915 (587.3795)	38.4907 (587.4363)

Remark 2.1 Remember that a random variable X with cdf G is stochastically smaller than Y with cdf F, denoted by $X \leq_{st} Y$, if for all x, $G(x) \geq F(x)$. This is the most basic and oldest stochastic order in Probability and Statistics. In this case, if G is simpler than F, $G(x)$ may provide a useful lower bound for $F(x)$ (see, e.g., [16] for more details). Now, let G and F denote the cdfs of the GG and EGG distributions which are defined via Eq.'s (2.1) and (2.2), respectively. It is obvious that for $\gamma > 1$, we have $X \leq_{st} Y$ because $[G(x)]^{\gamma} \leq G(x)$ and if $0 < \gamma < 1$, it follows that $X \geq_{st} Y$. Hence, For $\gamma \geq 1$ it follows that $E(X) \leq E(Y)$ and corresponding result holds if X is stochastically larger than Y. One can consider the results of Table 1 again.

3. Infinite divisibility

The researchers may also here make the following note in regards to the famous structural property of infinite divisibility of the distribution in question. Such a characteristic has a close relation to the Central Limit Theorem and waiting time distributions. Thus, it is a desirable question in modeling to know whether a given distribution is infinitely divisible or not. To settle this question, we recall that according to [18], (pp. 56), if p_x, $x \in \mathbb{N}_0$, is infinitely divisible, then $p_x \leq e^{-1}$ for all $x \in \mathbb{N}$. However, e.g., in an $EGG(0.65, 0.40, 3.80)$ distribution we see that $p_1 = 0.387 > e^{-1} = 0.367$. Therefore, in general, $EGG(\alpha, \theta, \gamma)$ distributions are not infinitely divisible. In addition, since the classes of self-decomposable and stable distributions, in their discrete concepts, are subclasses of infinitely divisible distributions, we conclude that an EDW distribution can be neither self-decomposable nor stable in general.

4. Order statistics

Let $F_i(x;\alpha,\theta,\gamma)$ be the cdf of the i-th order statistic of a random sample $X_1, X_2, ..., X_n$ from $EGG(\alpha,\theta,\gamma)$ distribution. Then, we have

$$F_i(x;\alpha,\theta,\gamma) = \sum_{k=i}^{n} \binom{n}{k} [F(x;\alpha,\theta,\gamma)]^k [1 - F(x;\alpha,\theta,\gamma)]^{n-k}.$$

Now, using the binomial expansion for $[1 - F(x;\alpha,\theta,\gamma)]^{n-k}$, the expression

$$\begin{aligned} F_i(x;\alpha,\theta,\gamma) &= \sum_{k=i}^{n}\sum_{j=0}^{n-k} \binom{n}{k}\binom{n-k}{j}(-1)^j [F(x;\alpha,\theta,\gamma)]^{k+j} \\ &= \sum_{k=i}^{n}\sum_{j=0}^{n-k} \binom{n}{k}\binom{n-k}{j}(-1)^j [G(x;\alpha,\theta)]^{\gamma(k+j)} \\ &= \sum_{k=i}^{n}\sum_{j=0}^{n-k} \binom{n}{k}\binom{n-k}{j}(-1)^j F_{EGG}(x;\alpha,\theta,(k+j)\gamma), \end{aligned}$$

is obtained for the cdf of the i-th order statistic. The corresponding pmf of the i-th order statistic, $f_i(x;\alpha,\theta,\gamma) = F_i(x;\alpha,\theta,\gamma) - F_i(x-1;\alpha,\theta,\gamma)$ for an integer value of x, then is given by

$$f_i(x;\alpha,\theta,\gamma) = \sum_{k=i}^{n}\sum_{j=0}^{n-k} \binom{n}{k}\binom{n-k}{j}(-1)^j f_{EGG}(x;\alpha,\theta,(k+j)\gamma),$$

where f_{EGG} denotes the pmf of an EGG distribution.

Remark 2.2 In view of the fact that $f_i(x;\alpha,\theta,\gamma)$ is a linear combination of a finite number of $EGG(\alpha,\theta,\gamma(k+j))$ distributions, we may obtain some properties of order statistics, such as their moments, from the corresponding EGG distribution (see [13]). For example, the mean of the i-th order statistic is given by

$$\mu_{i:n} = \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=i}^{n}\sum_{j=0}^{n-k}(-1)^j \binom{n}{k}\binom{n-k}{j}\omega_{i,l}(\alpha,(k+j)\gamma)\frac{\theta^{i+l}}{1-\theta^{i+l}}.$$

5. Estimation

Let $X_1, \ldots, X_n$ be a random sample of size n from the $EGG(\alpha,\theta,\gamma)$ distribution and $\mathbf{\Theta} = (\alpha,\theta,\gamma)$ be the unknown parameters vector. The log-likelihood function is given by

$$l(\mathbf{\Theta}) \quad = \quad \sum_{i=1}^{n} \log\left[\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\,\theta^{x_i+1}}\right)^{\gamma} - \left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\,\theta^{x_i}}\right)^{\gamma}\right].$$

The maximum likelihood estimation (MLE) of $\mathbf{\Theta}$ is obtained by solving the nonlinear equations, $U(\mathbf{\Theta}) = (U_\alpha(\mathbf{\Theta}), U_\theta(\mathbf{\Theta}), U_\gamma(\mathbf{\Theta}))^T = \mathbf{0}$, where

$$U_\alpha(\mathbf{\Theta}) = \frac{\partial l(\mathbf{\Theta})}{\partial \alpha} = \sum_{i=1}^{n} \frac{-\frac{\left(1-\theta^{x_i+1}\right)^{\gamma}\gamma\,\theta^{x_i+1}}{\left(1-(1-\alpha)\theta^{x_i+1}\right)^{\gamma-1}} + \frac{(1-\theta^{x_i})^{\gamma}\gamma\,\theta^{x_i}}{(1-(1-\alpha)\theta^{x_i})^{\gamma-1}}}{\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right)^{\gamma} - \left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)^{\gamma}},$$

$$U_\theta(\mathbf{\Theta}) = \frac{\partial l(\mathbf{\Theta})}{\partial \theta} = \sum_{i=1}^{n} \frac{-\alpha\,\gamma\,(x_i+1)\,\theta^{x_i}\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right)^{\gamma-1} + \alpha\, x_i \theta^{x_i-1}\gamma\left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)^{\gamma-1}}{\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right)^{\gamma} - \left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)^{\gamma}},$$

$$U_\gamma(\boldsymbol{\Theta}) = \frac{\partial l(\boldsymbol{\Theta})}{\partial \gamma} = \sum_{i=1}^{n} \frac{\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right)^\gamma \ln\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right) - \left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)^\gamma \ln\left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)}{\left(\frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}\right)^\gamma - \left(\frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}\right)^\gamma}.$$

We need the observed information matrix for interval estimation and hypotheses tests on the model parameters. The 3×3 Fisher information matrix, $J = J_n(\boldsymbol{\Theta})$, is given by

$$J = -\begin{bmatrix} J_{\alpha\alpha} & J_{\alpha\theta} & J_{\alpha\gamma} \\ J_{\theta\alpha} & J_{\theta\theta} & J_{\theta\gamma} \\ J_{\gamma\alpha} & J_{\gamma\theta} & J_{\gamma\gamma} \end{bmatrix}, \tag{5.1}$$

whose elements are given in Appendix.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, asymptotically

$$\sqrt{n}(\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}) \sim N_3(0, I(\boldsymbol{\Theta})^{-1}),$$

where $I(\boldsymbol{\Theta})$ is the expected information matrix. This asymptotic behavior is valid if $I(\boldsymbol{\Theta})$ replaced by $J_n(\hat{\boldsymbol{\Theta}})$, i.e., the observed information matrix evaluated at $\hat{\boldsymbol{\Theta}}$.

5.1. A simulated example. Let X be a random variable that follows a GEG distribution given by Eq. (2.4). Then, $[X]$ has an $EGG(\alpha, \theta, \gamma)$ distribution. Therefore, we can simulate an $EGG(\alpha, \theta, \gamma)$ random variable from the corresponding continuous GEG distribution. Table 2 below presents the maximum likelihood estimates of $\boldsymbol{\Theta} = (\alpha, \theta, \gamma)^T$ from an $EGG(\alpha, \theta, \gamma)$ distribution and also contains their standard errors for different values of n as a kind of simulated example. Standard errors are attained by means of the asymptotic covariance matrix of the MLEs of $EGG(\alpha, \theta, \gamma)$ parameters when the Newton-Raphson procedure converges in, e.g., MATLAB software.

Table 2 MLEs of the $EGG(\alpha,\theta,\gamma)$ parameters for different values of n.

	$\alpha=0.50$ $\theta=0.25$ $\gamma=0.75$			$\alpha=0.75$ $\theta=0.75$ $\gamma=0.50$		
n	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$
100	0.492(0.477)	0.213(0.527)	0.0.984(2.764)	0.812(0.396)	0.753(0.297)	0.442(0.393)
200	0.511(0.323)	0.288(0.410)	0.788(0.1.347)	0.730(0.268)	0.719(0.237)	0.551(0.379)
500	0.501(0.242)	0.217(0.264)	0.792(0.1.099)	0745(0.158)	0.751(0.129)	0.526(0.204)
1000	0.568(0.175)	0.257(0.185)	0.799(0.675)	0.743(0.108)	0.745(0.090)	0.534(0.144)
	$\alpha=2.0$ $\theta=0.5$ $\gamma=3.0$			$\alpha=3.0$ $\theta=0.9$ $\gamma=2.0$		
n	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$
100	2.077(0.951)	0.564(0.349)	2.656(2.652)	2.912(1.197)	0.897(0.156)	1.872(1.542)
200	1.904(0.663)	0.494(0.289)	2.941(2.352)	2.937(0.818)	0.888(0.113)	2.022(1.163)
500	1.915(0.465)	0.462(0.187)	3.290(1.880)	3.153(0.605)	0.914(0.065)	1.980(0.781)
1000	2.004(0.321)	0.511(0.124)	2.950(1.068)	2.918(0.306)	0.895(0.041)	1.981(0.427)
	$\alpha=1.00$ $\theta=0.50$ $\gamma=1.00$			$\alpha=1.50$ $\theta=0.95$ $\gamma=0.50$		
n	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$
100	1.278(0.723)	0.601(0.416)	0.864(1.005)	1.257(0.436)	0.867(0.150)	0.808(0.494)
200	0.933(0.363)	0.488(0.293)	0.974(0.850)	1.443(0.393)	0.947(0.060)	0.471(0.198)
500	0.982(0.230)	0.484(0.177)	1.043(0.553)	1.521(0.233)	0.957(0.023)	0.522(0.125)
1000	1.058(0.172)	0.542(0.122)	0.909(0.318)	1.507(0.177)	0.955(0.012)	0.481(0.087)

6. Application

In this section, the EGG model will be examined for a real data set. The data are integer parts of the lifetimes of fifty devices given by [1] and have also been analyzed by [14] and [3]. The data are: 0, 0, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

First, we obtain the MLE of the $EGG(\alpha,\theta,\gamma)$ parameters using the Newton-Raphson procedure. Then, we compare the EGG model with the discrete modified weibull (DMW) distribution of [14] as a rival model. In addition, a four-parameter discrete model, i.e., the discrete additive Weibull (DAddW) distribution of [5], is compared. A summery of computations which consists of the MLEs, Akaike information criterion (AIC) and the values of log-likelihood functions is given in Table 3.

Table 3 MLEs, maximized log-likelihoods and AIC values of the fitted models.

Model	Estimated parameters	$\ell(\hat{\boldsymbol{\theta}})$	AIC
EGG	$\hat{\alpha}$=36011.39, $\hat{\theta}$=0.8845, $\hat{\gamma}=0.1783$	-226.5	459.0
DMW	$\hat{q}$=0.9403, $\hat{c}$=1.0241, $\hat{\theta}$=0.3450	-229.1	464.2
DAddW	$\hat{q}$=0.9216, $\hat{b}$=0.000060091, $\hat{\theta}$=0.4541, $\hat{\gamma}=2.8387$	-228.2	464.4
GG	$\hat{\alpha}$=2.7934, $\hat{\theta}$=0.9674	-239.9	483.7

According to the results of Table 3, it seems that the EGG distribution gives a better fit than the GG (as a sub-model), the DMW (as a three-parameter rival model) and the four-parameter DAddW distributions.

7. Concluding remarks

We proposed the exponentiated generalized geometric (EGG) distribution belonging to the *resilience parameter family*. This new discrete distribution contains the generalized geometric (GG) distribution of [6] as a special case. Moreover, the EGG distribution coincides with the discrete counterpart of the generalized exponential-geometric distribution of [17]. We investigated the basic statistical and mathematical properties of the new model and illustrated that the hazard rate function of the new model can be increasing, decreasing, upside-down bathtub and bathtub-shaped. In addition, fitting the EGG model to a real data set indicated the capacity of the proposed distribution in data modeling.

Appendix

The elements of the 3×3 information matrix in Eq. (5.1) are given by

$$\begin{aligned} J_{\alpha\alpha} &= \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha^2} \\ &= \sum_{i=1}^{n} \frac{d_i^{\gamma}\left(\theta^{x_i+1}\right)^2\gamma\,(\gamma-1) - \omega_i^{\gamma}\theta^{2x_i}\gamma(\gamma-1)}{d_i^{\gamma} - {\omega_i}^{\gamma}} \\ &- \left[\frac{-d_i^{\gamma}\left(1-(1-\alpha)\theta^{x_i+1}\right)\gamma\,\theta^{x_i+1} + \omega_i^{\gamma}(1-(1-\alpha)\theta^{x_i})\gamma\theta^{x_i}}{d_i^{\gamma}-\omega_i^{\gamma}}\right]^2, \end{aligned}$$

$$\begin{aligned} J_{\alpha\theta} &= \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha\partial\theta} \\ &= \sum_{i=1}^{n}\left\{\frac{1}{d_i^{\gamma}-\omega_i^{\gamma}}\right. \\ &\times \left[d_i^{\gamma-1}\gamma\,\theta^{x}\,(x_i+1)\left(\gamma\,\theta^{x_i+1}-1+\theta^{x_i+1}\right) - d_i^{\gamma}\gamma\,\theta^{2\,x_i+1}\,(\gamma-1)\,(1-\alpha)\,(x_i+1)\right. \\ &- \left.\omega_i^{\gamma-1}\gamma\,\theta^{x_i-1}x_i\,(\gamma\,\theta^{x_i}-1+\theta^{x_i}) + \omega_i^{\gamma}\gamma\,\left(\theta\,(\gamma-1)\right)^{2\,x_i-1}(1-\alpha)\,x_i\right] \\ &+ \frac{d_i^{\gamma}(1-(1-\alpha)\theta^{x_i})\gamma\,\theta^{x_i+1} - \omega_i^{\gamma}\gamma\,\theta^{x_i}(1-(1-\alpha)\,\theta^{x_i})}{(d_i^{\gamma}-\omega_i^{\gamma})^2} \\ &\times \left[d_i^{\gamma-1}\gamma(1-(1-\alpha)\theta^{x_i+1})^{-1}\theta^{-1}(x_i+1)\theta^{x_i+1}\,(-1+d_i\,(1-\alpha))\right. \\ &- \left.\left.\omega_i^{\gamma-1}\gamma(1-(1-\alpha)\theta^{x_i})^{-1}\theta^{-1}x_i\theta^{x_i}\,(-1+\omega_i\,(1-\alpha))\right]\right\}, \end{aligned}$$

$$
\begin{aligned}
J_{\alpha\gamma} &= \frac{\partial^2 l(\mathbf{\Theta})}{\partial\alpha\partial\gamma}\\
&= \sum_{i=1}^{n}\Bigg\{\frac{1}{d_i^{\gamma}-\omega_i^{\gamma}}\\
&\times \left[d_i^{\gamma}\left(\theta\left(\gamma\,\ln\left(1-(1-\alpha)\,\theta^{x_i+1}\right)-\gamma\,\ln\left(1-\theta^{x_i+1}\right)-1\right)\right)^{x_i+1}\left(1-(1-\alpha)\,\theta^{x_i+1}\right)\right.\\
&+ \left.\omega_i^{\gamma}(1-\theta^{x_i})^{\gamma}\left(\theta\left(\ln\left(1-\theta^{x_i}\right)\gamma+1-\ln\left(1-(1-\alpha)\,\theta^{x_i}\right)\gamma\right)\right)^{x_i}\left(1-(1-\alpha)\,\theta^{x_i}\right)\right]\\
&+ \frac{\gamma\,\theta^{x_i}\left(\theta\,\left(1-\theta^{x_i+1}\right)^{\gamma}-\left(1-\theta^{x_i}\right)^{\gamma}\right)}{\left(1-(1-\alpha)\,\theta^{x_i+1}\right)^{\gamma-1}}\\
&\times \Big[\frac{d_i^{\gamma}\ln(d_i)-\omega_i^{\gamma}\ln(\omega_i)}{(d_i^{\gamma}-\omega_i^{\gamma})^2}\Big]\Bigg\},\\
J_{\theta\theta} &= \frac{\partial^2 l(\mathbf{\Theta})}{\partial\theta^2}\\
&= \sum_{i=1}^{n}\Bigg\{\frac{1}{d_i^{\gamma}-\omega_i^{\gamma}}\Big[d_i^{\gamma-2}\gamma^2\\
&\times \left(-\frac{\theta^{x_i+1}\,(x_i+1)}{\theta\,\left(1-(1-\alpha)\,\theta^{x_i+1}\right)}+d_i(1-\alpha)\,\theta^{x_i+1}\,(x_i+1)\left(1-(1-\alpha)\,\theta^{x_i+1}\right)^{-1}\theta^{-1}\right)^2\\
&+ {d_i}^{\gamma-1}\gamma\Big[\frac{\theta^{x_i+1}\,(x_i+1)\,(-x_i)}{\theta^2\,\left(1-(1-\alpha)\,\theta^{x_i+1}\right)}\\
&+ 2\,d_i\left(1-\theta^{x_i+1}\right)(1-\alpha)^2\left(\theta^{x_i+1}\right)^2(x_i+1)^2\left(1-(1-\alpha)\,\theta^{x_i+1}\right)^{-2}\theta^{-2}\\
&+ \frac{(-1+\alpha)\,\theta^{x_i+1}\,(x_i+1)\left(3\,\theta^{x_i+1}x+2\,\theta^{x_i+1}-x\right)}{\theta^2\,\left(1-(1-\alpha)\,\theta^{x_i+1}\right)^2}\Big]\\
&- \alpha\,(x_i+1)^2\left(\theta^{x_i+1}\right)^2\gamma\left(-\frac{-1+\theta^{x_i+1}}{\theta^{x_i+1}\alpha-\theta^{x_i+1}+1}\right)^{\gamma}\\
&\times \left(-1+\theta^{x_i+1}\right)^{-2}\theta^{-2}\left(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1\right)^{-1}\\
&+ \left(-\frac{-1+\theta^{x_i+1}}{\theta^{x_i+1}\alpha-\theta^{x_i+1}+1}\right)^{\gamma}\gamma\left(\theta^{x_i+1}\right)^2(x_i+1)^2\\
&\times \alpha\,(-1+\alpha)\left(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1\right)^{-2}\theta^{-2}\\
&\times \left(-1+\theta^{x_i+1}\right)^{-1}-\alpha^2{x_i}^2\left(\theta^{x_i}\right)^2\gamma^2\left(-\frac{-1+\theta^{x_i}}{\theta^{x_i}\alpha-\theta^{x_i}+1}\right)^{\gamma}\\
&\times (-1+\theta^{x_i})^{-2}\theta^{-2}\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^{-2}-\omega_i^{\gamma-1}\gamma\Big[-\frac{\theta^{x_i}x_i\,(x_i-1)}{\theta^2\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)}\\
&+ 2\,\frac{\left(1-\theta^{x_i}\right)(1-\alpha)^2\left(\theta^{x_i}\right)^2{x_i}^2}{\left(1-(1-\alpha)\,\theta^{x_i}\right)^3\theta^2}+\frac{\theta^{x_i-2}x_i\,(-1+\alpha)\left(3\,\theta^{x_i}x_i-\theta^{x_i}-x_i+1\right)}{\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^2}\Big]\\
&+ \omega_i^{\gamma-1}(1-\theta^{x_i})\Big[\frac{\gamma\,\theta^{x_i}{x_i}^2\alpha\,\theta^{x_i-2}}{\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^2}\Big]+\frac{\left(1-\theta^{x_i}\right)^{\gamma-1}\gamma\,\theta^{2\,x_i-2}{x_i}^2\alpha\,(-1+\alpha)}{\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^{\gamma-2}}\Big]\\
&- \frac{1}{(d_i^{\gamma}-\omega_i^{\gamma})^2}\\
&\times \Big[d_i^{\gamma-1}\gamma\Big(-\frac{\theta^{x_i+1}\,(x_i+1)\,\alpha}{\left(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1\right)^2\theta}\Big)+\frac{\omega_i^{\gamma-1}\gamma\,\theta^{x_i-1}x\alpha}{\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^2}\Big]^2\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
J_{\theta\gamma} &= \frac{\partial^2 l(\mathbf{\Theta})}{\partial\theta\partial\gamma} \\
&= \sum_{i=1}^{n}\left\{\frac{1}{d_i^\gamma-\omega_i^\gamma}\left[d_i^{\gamma-1}\ln(d_i)\gamma(-\frac{\theta^{x_i}\,(x_i+1)\,\alpha}{\left(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1\right)^2})\right.\right. \\
&- d_i^{\gamma-1}\left(\frac{\theta^{x_i}\,(x_i+1)\,\alpha}{\left(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1\right)^2}\right) \\
&- \left.\omega_i^{\gamma-1}\ln(\omega_i)\gamma\left(-\frac{\theta^{x_i-1}x_i\alpha}{\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^2}\right)+\omega_i^{\gamma-1}\left(\frac{\theta^{x_i}x_i\alpha}{\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^2\theta}\right)\right] \\
&- \frac{1}{\left(d_i^\gamma-\omega_i^\gamma\right)^2}\{\left[d_i^{\gamma-1}\left(-\frac{\gamma\,\theta^{x_i}\,(x+1)\,\alpha}{\left(\theta^{x_i+1}\alpha-\theta^{x_i+1}+1\right)^2}\right)+\omega_i^{\gamma-1}\left(\frac{\gamma\,\theta^{x_i-1}x_i\alpha}{\left(\theta^{x_i}\alpha-\theta^{x_i}+1\right)^2}\right)\right] \\
&\times \left.\left[d_i^\gamma\ln(d_i)-\omega_i^\gamma\ln(\omega_i)\right]\}\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{\gamma\gamma} &= \frac{\partial^2 l(\mathbf{\Theta})}{\partial\gamma^2} \\
&= \sum_{i=1}^{n}\{\frac{d_i^\gamma\left(\ln(d_i)\right)^2-\omega_i^\gamma\left(\ln(\omega_i)\right)^2}{d_i^\gamma-\omega_i^\gamma}-\frac{\left(d_i^\gamma\ln(d_i)-\omega_i^\gamma\ln(\omega_i)\right)^2}{\left(d_i^\gamma-\omega_i^\gamma\right)^2}\},
\end{aligned}
$$

where $d_i = \frac{1-\theta^{x_i+1}}{1-(1-\alpha)\theta^{x_i+1}}$ and $\omega_i = \frac{1-\theta^{x_i}}{1-(1-\alpha)\theta^{x_i}}$.

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The Zografos-Balakrishnan odd log-logistic family of distributions: Properties and Applications

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Abstract

We study some mathematical properties of a new generator of continuous distributions with two additional shape parameters called the *Zografos-Balakrishnan odd log-logistic* family. We present some special models and investigate the asymptotes and shapes. The density function of the new family can be expressed as a mixture of exponentiated densities based on the same baseline distribution. We derive a power series for its quantile function. Explicit expressions for the ordinary and incomplete moments, quantile and generating functions, Shannon and Rényi entropies and order statistics, which hold for any baseline model, are determined. We estimate the model parameters by maximum likelihood. Two real data sets are used to illustrate the potentiality of the proposed family.

Keywords: Estimation, Gamma distribution, Generated family, Maximum likelihood, Mean deviation, Moment, Quantile function.

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1. Introduction

The statistics literature is filled with hundreds of continuous univariate distributions: see Johnson *et al.* (1994, 1995). Recent developments have been focused to define new families by adding shape parameters to control skewness, kurtosis and tail weights thus providing great flexibility in modeling skewed data in practice, including the two-piece approach introduced by Hansen (1994) and the generators pioneered by Eugene *et al.* (2002), Cordeiro and de Castro (2011), Alexander *et al.* (2012) and Cordeiro *et al.* (2013). Many subsequent articles apply these techniques to induce skewness into well-known symmetric distributions such as the symmetric Student t. For a review, see Aas and Haff (2006).

We study several mathematical properties of a new family of distributions called the *Zografos-Balakrishnan odd log-logistic-G* ("ZBOLL-G" for short) family with two additional shape parameters. These parameters can provide great flexibility to model the skewness and kurtosis of the generated distribution. Indeed, for any baseline G distribution, the new family can extend several common models such as the normal, Weibull and Gumbel distributions by adding these parameters to a parent G. The proposed family is an extension of that one introduced recently by Zografos and Balakrishnan ("ZB") (2009) and Ristic and Balakrishnan (2012), although both are based on the same gamma generator.

Let W be any continuous distribution defined on a finite or an infinite interval. The ZB family is defined from the cumulative distribution function (cdf) (for $\beta > 0$)

$$F(x) \quad = \quad \frac{\gamma\left(\beta, -\log[1 - W(x)]\right)}{\Gamma(\beta)}, \quad x \in \mathbb{R}, \tag{1.1}$$

where $\Gamma(\alpha) = \int_0^\infty t^{a-1}\, \mathrm{e}^{-t}\mathrm{dt}$ and $\gamma(\beta, z) = \int_0^z t^{\beta-1}\, \mathrm{e}^{-t} dt$ are the gamma function and lower incomplete gamma function, respectively.

Further, we define $W(x)$ from any baseline cdf $G(x; \boldsymbol{\tau})$ ($x \in \mathbb{R}$), where $\boldsymbol{\tau}$ denotes the parameters in the parent G, as

$$W(x) = \frac{G^{\alpha}(x; \boldsymbol{\tau})}{G^{\alpha}(x; \boldsymbol{\tau}) + \bar{G}^{\alpha}(x; \boldsymbol{\tau})}, \tag{1.2}$$

where $\alpha > 0$ and $\bar{G}(x; \boldsymbol{\tau}) = 1 - G(x; \boldsymbol{\tau})$ is the baseline survival function. According to Marshall and Olkin (2007, equation (21)), the function $W(x)$ in (1.2) is the odd log-logistic-G (OLL-G) cdf. By inserting (1.2) in equation (1.1), we have

$$F(x) = \frac{1}{\Gamma(\beta)} \gamma \left\{ \beta, -\log\left[1 - \frac{G^{\alpha}(x; \boldsymbol{\tau})}{G^{\alpha}(x; \boldsymbol{\tau}) + \bar{G}^{\alpha}(x; \boldsymbol{\tau})}\right] \right\}. \tag{1.3}$$

The model (1.3) is called the ZBOLL-G distribution with parameters α and β. Let $g(x; \boldsymbol{\tau}) = dG(x; \boldsymbol{\tau})/dx$ be the baseline probability density function (pdf). The density function corresponding to (1.3) is given by

$$f(x) \quad = \quad \frac{\alpha\, g(x; \boldsymbol{\tau})\, G^{\alpha-1}(x; \boldsymbol{\tau})\, \bar{G}^{\alpha-1}(x; \boldsymbol{\tau})}{\Gamma(\beta) [G^{\alpha}(x; \boldsymbol{\tau}) + \bar{G}^{\alpha}(x; \boldsymbol{\tau})]^2} \left\{ -\log\left[\frac{\bar{G}^{\alpha}(x; \boldsymbol{\tau})}{G^{\alpha}(x; \boldsymbol{\tau}) + \bar{G}^{\alpha}(x; \boldsymbol{\tau})} \right] \right\}^{\beta-1}. \tag{1.4}$$

Henceforth, a random variable X with density function (1.4) is denoted by $X \sim$ZBOLL-G$(\alpha, \beta, \boldsymbol{\tau})$. The ZBOLL-G family has the same parameters of the parent G plus the parameters α and β. For $\alpha = \beta = 1$, it reduces to the baseline G distribution. For $\alpha = 1$, we obtain the gamma-G (G-G) family and, for $\beta = 1$, we have the OLL-G family. The hazard rate function (hrf) of X is given by

$$h(x) = \frac{\alpha\, g(x;\boldsymbol{\tau})\, G^{\alpha-1}(x;\boldsymbol{\tau})\, \bar{G}^{\alpha-1}(x;\boldsymbol{\tau})}{[G^{\alpha}(x;\boldsymbol{\tau})+\bar{G}^{\alpha}(x;\boldsymbol{\tau})]^2} \times \frac{\left\{-\log\left[\frac{\bar{G}^{\alpha}(x;\boldsymbol{\tau})}{G^{\alpha}(x;\boldsymbol{\tau})+\bar{G}^{\alpha}(x;\boldsymbol{\tau})}\right]\right\}^{\beta-1}}{\Gamma(\beta)-\gamma\left\{\beta,-\log\left[1-\frac{G^{\alpha}(x;\boldsymbol{\tau})}{G^{\alpha}(x;\boldsymbol{\tau})+\bar{G}^{\alpha}(x;\boldsymbol{\tau})}\right]\right\}}. \tag{1.5}$$

Each new ZBOLL-G distribution can be defined from a specified G distribution. The ZBOLL family is easily simulated by inverting (1.3) as follows: if V has the $\gamma(\beta,1)$ distribution, then the solution of the nonlinear equation

$$X = G^{-1}\left\{\frac{\left(1-\mathrm{e}^{-V}\right)^{\frac{1}{\alpha}}}{(1-\mathrm{e}^{-V})^{\frac{1}{\alpha}}+\mathrm{e}^{\frac{-V}{\alpha}}}\right\} \tag{1.6}$$

has density (1.4).

The parameters α and β have a clear interpretation. Following the key idea of Zografos and Balakrishnan (2009) and Ristic and Balakrishnan (2012), we can also interpret (1.4) in this way: if $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ are upper record values from a sequence of independent random variables with common pdf

$$w(x) = W'(x) = \frac{\alpha\, g(x;\boldsymbol{\tau})\{G(x;\boldsymbol{\tau})[1-G(x;\boldsymbol{\tau})]\}^{\alpha-1}}{\{G^{\alpha}(x;\boldsymbol{\tau})+\bar{G}^{\alpha}(x;\boldsymbol{\tau})\}^2},$$

then the pdf of the nth upper record value has the pdf (1.4).

It is important to mention that the results presented in this paper follow similar lines of those developed by Nadarajah et al. (2015), although their model is completely different from that one discussed in this paper.

The rest of the paper is organized as follows. In Section 2, we present some new distributions. In Section 3, we introduce the asymptotic properties of equations (1.3), (1.4) and (1.5). Section 4 deals with two useful representations for (1.3) and (1.4). In Section 5, we derive a power series for the quantile function (qf) of X. In Sections 6 and 7, we obtain the entropies and order statistics. Estimation of the model parameters by maximum likelihood and the observed information matrix are presented in Section 8. Two applications to real data prove empirically the importance of the new family in Section 9. Finally, some conclusions and future work are noted in Section 10.

2. Special ZOBLL-G distributions

The ZOBLL-G family of density functions (1.4) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. In this section, we present and study some special cases of this family because it extends several widely-known distributions in the literature. The density function (1.4) will be most tractable when $G(x;\boldsymbol{\tau})$ and $g(x;\boldsymbol{\tau})$ have simple analytic expressions.

2.1. Zografos-Balakrishnan odd log-logistic Weibull (ZBOLL-W) model. If $G(x;\boldsymbol{\tau})$ is the Weibull cdf with scale parameter $\kappa > 0$ and shape parameter $\lambda > 0$, where $\boldsymbol{\tau} = (\lambda,\kappa)^T$, say $G(x;\boldsymbol{\tau}) = 1-\exp\{-(x/\lambda)^{\kappa}\}$, the ZOBBLL-W density function (for $x > 0$) is given by

$$f_{\text{ZBOLL-W}}(x) = \frac{\alpha\,\kappa\,\lambda^{-\kappa}x^{\kappa-1}\exp[-(x/\lambda)^{\kappa}]\{1-\exp[-(x/\lambda)^{\kappa}]\}^{\alpha-1}}{\Gamma(\beta)\{\{1-\exp[-(x/\lambda)^{\kappa}]\}^{\alpha}+\exp[-\alpha(x/\lambda)^{\kappa}]\}^{2}}$$
$$\times \quad \exp[-(\alpha-1)(x/\lambda)^{\kappa}]\left\{-\log\left[\frac{\exp[-\alpha(x/\lambda)^{\kappa}]}{\{1-\exp[-(x/\lambda)^{\kappa}]\}^{\alpha}+\exp[-\alpha(x/\lambda)^{\kappa}]}\right]\right\}^{\beta-1}. \tag{2.1}$$

Figure 1 displays some possible shapes of the ZBOLL-W density function.

2.2. Zografos-Balakrishnan odd log-logistic normal (ZBOLL-N) model. The ZBOLL-N distribution is defined from (1.4) by taking $G(x;\boldsymbol{\tau}) = \Phi(\frac{x-\mu}{\sigma})$ and $g(x;\boldsymbol{\tau}) = \sigma^{-1}\,\phi(\frac{x-\mu}{\sigma})$ to be the cdf and pdf of the normal $N(\mu,\sigma^2)$ distribution, where $\boldsymbol{\tau} = (\mu,\sigma)^T$. Its density function is given by

$$f_{\text{ZBOLL-N}}(x) = \frac{\alpha\,\phi(z)\Phi^{\alpha-1}(z)[1-\Phi(z)]^{\alpha-1}}{\sigma\Gamma(\beta)\{\Phi^{\alpha}(z)+[1-\Phi(z)]^{\alpha}\}^{2}} \times \left\{-\log\left[\frac{[1-\Phi(z)]^{\alpha}}{\Phi^{\alpha}(z)+[1-\Phi(z)]^{\alpha}}\right]\right\}^{\beta-1}, \tag{2.2}$$

where $z = (x-\mu)/\sigma$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter, α and β are shape and scale parameters, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. For $\mu = 0$ and $\sigma = 1$, we obtain the ZBOLL-standard normal (ZBOLL-SN) distribution. Plots of the ZBOLL-N density function for selected parameter values are displayed in Figure 2.

2.3. Zografos-Balakrishnan odd log-logistic Gumbel (ZBOLL-Gu) model. Consider the Gumbel distribution with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$, $\boldsymbol{\tau} = (\mu,\sigma)^T$, and the pdf and cdf (for $x \in \mathbb{R}$) given by

$$g(x;\boldsymbol{\tau}) = \frac{1}{\sigma}\exp\left[\left(\frac{x-\mu}{\sigma}\right) - \exp\left(\frac{x-\mu}{\sigma}\right)\right]$$

and

$$G(x;\boldsymbol{\tau}) = 1 - \exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right],$$

respectively. The mean and variance are equal to $\mu - \gamma\sigma$ and $\pi^2\sigma^2/6$, respectively, where γ is the Euler's constant ($\gamma \approx 0.57722$). Inserting these expressions in (1.4) gives the ZBOLL-Gu density function

$$f_{\text{ZBOLL}-\text{Gu}}(x) = \frac{\alpha\,\exp[z-\exp(z)]\{1-\exp[-\exp(z)]\}^{\alpha-1}\exp[-(\alpha-1)\exp(z)]}{\sigma\Gamma(\beta)\{\{1-\exp[-\exp(z)]\}^{\alpha}+\exp[-\alpha\exp(z)]\}^{2}} \times \left\{-\log\left(\frac{\exp[-\alpha(\exp(z))]}{\{1-\exp[-\exp(z)]\}^{\alpha}+\exp[-\alpha\exp(z)]}\right)\right\}^{\beta-1}, \tag{2.3}$$

where $z = (x-\mu)/\sigma$,$x, \mu \in \mathbb{R}$ and $\alpha, \beta, \sigma > 0$. Plots of (2.3) for selected parameter values are displayed in Figure 3.

(a) (b) (c)

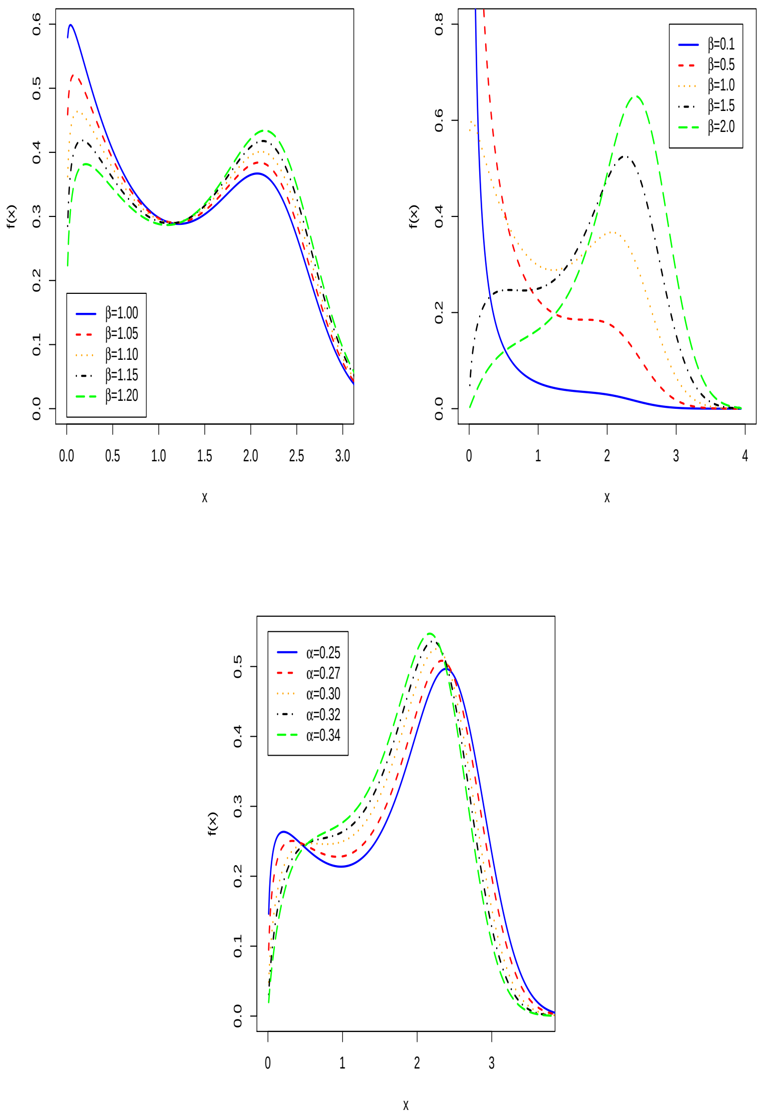

Figure 1. Plots of the ZBOLL-W density function for some parameter values. (a) For different values of β, with $\alpha = 0.3$, $\kappa = 3.5$ and $\lambda = 1.4$. (b) For different values of β with $\alpha = 0.3$, $\kappa = 3.5$ and $\lambda = 1.4$. (c) For different values of α with $\beta = 1.5$, $\kappa = 3.5$ and $\lambda = 1.4$.

3. Asymptotics

Let $c = \inf\{x|G(x) > 0\}$, then the asymptotics of equations (1.3), (1.4) and (1.5) when $x \to c$ are given by

$$F(x) \sim \frac{G(x)^{\alpha\,\beta}}{\Gamma(\beta+1)} \quad \text{as x} \to \text{c},$$

$$f(x) \sim \frac{\alpha\, g(x)\, G(x)^{\alpha\,\beta-1}}{\Gamma(\beta)} \quad \text{as x} \to \text{c},$$

$$h(x) \sim \frac{\alpha\, g(x)\, G(x)^{\alpha\,\beta-1}}{\Gamma(\beta)} \quad \text{as x} \to \text{c}.$$

(a) (b) (c)

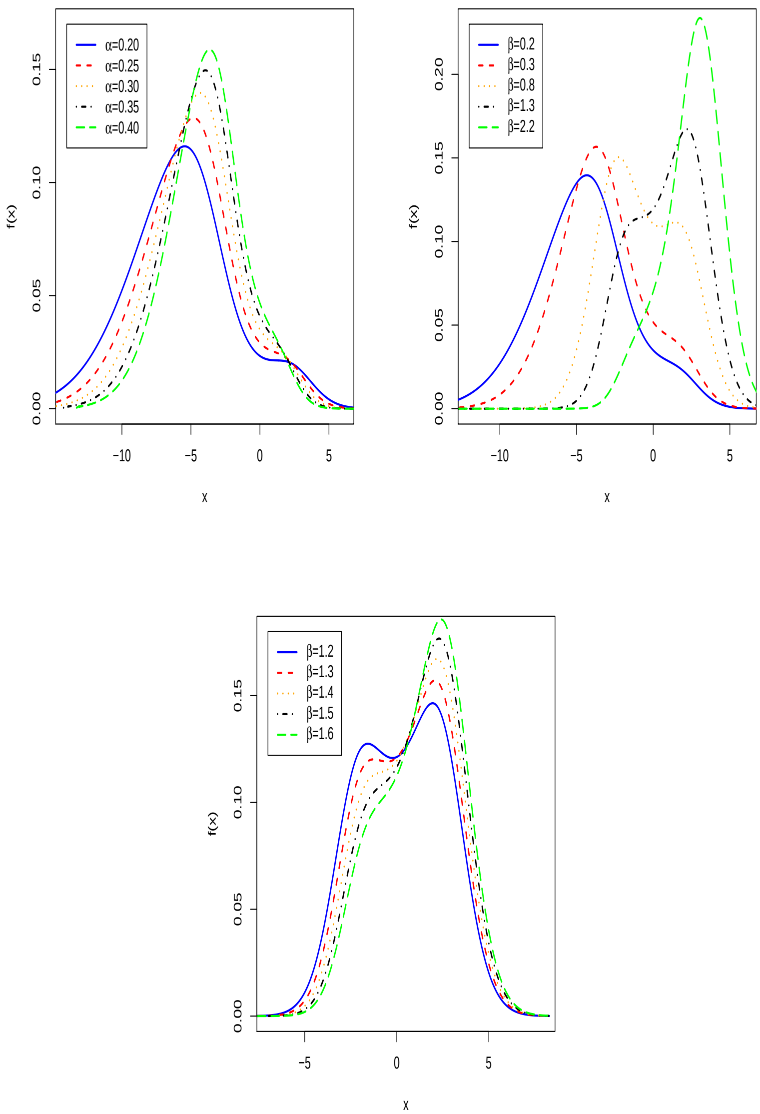

Figure 2. Plots of the ZBOLL-N density function for some parameter values. (a) For different values of α with $\beta = 0.2$, $\mu = 0$ and $\sigma = 1$. (b) For different values of β with $\alpha = 0.3$, $\mu = 0$ and $\sigma = 1.0$. (c) For different values of β with $\alpha = 0.3$, $\mu = 0$ and $\sigma = 0.1$.

The asymptotics of equations (1.3), (1.4) and (1.5) when $x \to \infty$ are given by

$$1 - F(x) \sim \frac{1}{\Gamma(\beta)} \left\{-\alpha \log\left[\bar{G}(x)\right]\right\}^{\beta-1} \bar{G}(x)^{\alpha} \quad \text{as x} \to \infty,$$

$$f(x) \sim \frac{\alpha\, g(x)\, \bar{G}(x)^{\alpha-1} \left\{-\alpha \log\left[\bar{G}(x)\right]\right\}^{\beta-1}}{\Gamma(\beta)} \quad \text{as x} \to \infty,$$

$$h(x) \sim \frac{\alpha\, g(x)}{\bar{G}(x)} \quad \text{as x} \to \infty.$$

(a) (b) (c)

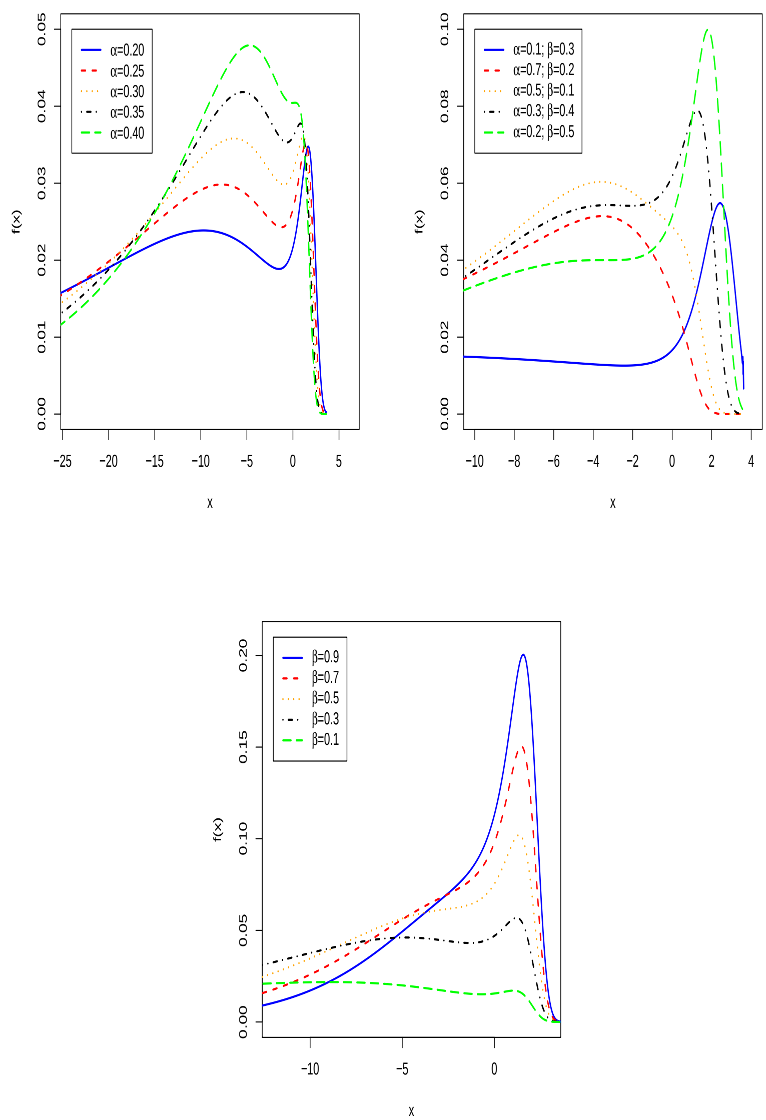

Figure 3. Plots of the ZBOLL-Gu density function for some parameter values. (a) For different values of α with $\beta = 0.2$, $\mu = 0$ $\sigma = 1$. (b) For different values of α and β with $\mu = 0$ and $\sigma = 1.0$. (c) For different values of β with $\alpha = 0.3$, $\mu = 0$ and $\sigma = 0$.

4. Two useful representations

Two useful linear representations for (1.3) and (1.4) can be derived using the concept of exponentiated distributions. For an arbitrary baseline cdf $G(x)$, a random variable is said to have the exponentiated-G (exp-G) distribution with power parameter $a > 0$, say $Z \sim$exp-$G(a)$, if its pdf and cdf are $h_a(x) = aG^{a-1}(x)g(x)$ and $H_a(x) = G^a(x)$, respectively. The properties of exponentiated distributions have been studied by many authors in recent years, see Mudholkar and Srivastava (1993) for exponentiated Weibull, Gupta *et al.* (1998) for exponentiated Pareto, Gupta and Kundu (1999) for exponentiated exponential, Nadarajah (2005) for exponentiated Gumbel, Kakde and Shirke (2006) for exponentiated lognormal, and Nadarajah and Gupta (2007) for exponentiated gamma.

The generalized binomial coefficient for real arguments is given by $\binom{x}{y} = \Gamma(x+1)/[\Gamma(y+1)\Gamma(x-y+1)]$. By using the incomplete gamma function expansion, we can write

$$F(x) = \frac{1}{\Gamma(\beta)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(\beta+i)} \left\{-\log\left[1 - \frac{G(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha}\right]\right\}^{\beta+i}.$$

For any real positive power parameter, the formula below holds (http://functions.wolfram.com/ElementaryFunctions/Log/06/01/04/03/)

$$\begin{aligned}\left\{-\log\left[1 - \frac{G(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha}\right]\right\}^{\beta+i} &= (\beta+i) \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j+k} \binom{k-\beta-i}{k} \binom{k}{j}}{(\beta+i-j)} \\ &\times \; p_{j,k} \left[\frac{G(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha}\right]^{\beta+i+k},\end{aligned} \tag{4.1}$$

where the constants $p_{j,k}$ can be determined recursively by

$$p_{j,k} = k^{-1} \sum_{m=1}^{k} [k - m(j+1)]\, c_m\, p_{j,k-m} \tag{4.2}$$

for $k = 1, 2, \ldots$, $c_k = (-1)^{k+1}/(k+1)$ and $p_{j,0} = 1$.

Further,

$$\left[\frac{G(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha}\right]^{\beta+i+k} = \frac{\sum_{r=0}^{\infty} \lambda_r\, G(x)^r}{\sum_{r=0}^{\infty} \rho_r\, G(x)^r} = \sum_{r=0}^{\infty} a_r\, G(x)^r,$$

where

$$\lambda_r = \sum_{l=r}^{\infty} (-1)^{l+r} \binom{\alpha(\beta+i+k)}{l} \binom{l}{r}, \quad \rho_r = h_r(\alpha, \beta+i+k),$$

and (for $r \geq 1$)

$$a_r = a_r(\alpha, \beta, i, k) = \frac{1}{\rho_0} \left(\rho_r - \frac{1}{\rho_0} \sum_{s=1}^{r} \rho_s\, a_{r-s}\right),$$

$a_0 = \lambda_0/\rho_0$ and $h_r(\alpha, \beta+i+k)$ is defined in Appendix A.

Then, equation (1.3) can be expressed as

$$F(x) = \sum_{r=0}^{\infty} b_r\, H_r(x), \tag{4.3}$$

where

$$b_r = \frac{1}{\Gamma(\beta)} \sum_{i,k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{i+j+k}\, p_{j,k}\, a_r(\alpha, \beta, i, k)}{(\beta+i-j)\, i!} \binom{k-\beta-i}{k} \binom{k}{j},$$

and $H_r(x)$ denotes the cdf of the exp-$G(r)$ distribution. The pdf (1.4) reduces to

$$f(x) = \sum_{r=0}^{\infty} b_{r+1}\, h_{r+1}(x), \tag{4.4}$$

where $h_{r+1}(x)$ denotes the pdf of the exp-$G(r+1)$ distribution. So, several mathematical properties of the proposed family can be obtained by knowing those of the exp-G distribution, see, for example, Mudholkar *et al.* (1996), Gupta and Kundu (2001) and Nadarajah and Kotz (2006), among others.

5. Quantile function

The gamma regularized function is defined by $Q(\beta, z) = \int_z^\infty x^{\beta-1}\,\mathrm{e}^{-x}/\Gamma(\beta)$. The inverse gamma regularized function $Q^{-1}(\beta, u)$ admits a power series expansion given by (http://functions.wolfram.com/GammaBetaErf/InverseGammaRegulari zed/06/01/03/)

$$Q^{-1}(\beta, u) = u\ \sum_{i=0}^{\infty} m_i\, u^i,$$

where $w = [\Gamma(\beta+1)\,(1-u)]^{1/\beta}$, $m_0 = 1$, $m_1 = 1/(\beta+1)$, $m_2 = (3\beta+5)/[2(\beta+1)^2\,(\beta+2)]$, $m_3 = [\beta(8\beta+3)+31]/[2(\beta+1)^3\,(\beta+2)(\beta+3)]$, etc.

First,

$$B = \frac{(1-\mathrm{e}^{-\mathrm{v}})^{\frac{1}{\beta}}}{(1-\mathrm{e}^{-\mathrm{v}})^{\frac{1}{\beta}} + \mathrm{e}^{\frac{-\mathrm{v}}{\beta}}} = \frac{1}{1+\mathrm{e}^{\frac{-\mathrm{v}}{\beta}}\,(1-\mathrm{e}^{-\mathrm{v}})^{\frac{-1}{\beta}}}$$

By using Taylor expansion and generalized binomial expansion, we have obtain

$$\mathrm{e}^{\frac{-\mathrm{v}}{\beta}}\,(1-\mathrm{e}^{-\mathrm{v}})^{\frac{-1}{\beta}} = \sum_{\mathrm{k}=0}^{\infty} \mathrm{b}_{\mathrm{k}}^*\,\mathrm{v}^{\mathrm{k}},$$

where $b_0^* = 1$ and, for $k \geq 1$, $b_k^* = \frac{(-1)^{j+k}\,(j+\beta^{-1})^k}{k!}\binom{-1/\beta}{j}$.

Then,

$$B = \frac{1}{\sum_{k=0}^{\infty} b_k^*\,v^k} = \sum_{k=0}^{\infty} c_k^*\,v^k$$

where $c_0^* = 1/b_0^*$ and c_k^* (for $k \geq 1$) is obtained from the last equation as

$$c_k^* = -\frac{1}{b_0^*}\sum_{r=1}^{k} b_r^*\,c_{k-r}^*.$$

Further, we can write

$$\begin{aligned} A &= \frac{(1-\mathrm{e}^{-\mathrm{Q}^{-1}(\beta,\mathrm{u})})^{\frac{1}{\beta}}}{(1-\mathrm{e}^{-\mathrm{Q}^{-1}(\beta,\mathrm{u})})^{\frac{1}{\beta}} + \mathrm{e}^{\frac{-\mathrm{Q}^{-1}(\beta,\mathrm{u})}{\beta}}} = \sum_{k=0}^{\infty} c_k^*\left[Q^{-1}(\beta,u)\right]^k \\ &= \sum_{k=0}^{\infty} c_k^*\left(u\ \sum_{i=0}^{\infty} m_i\,u^i\right)^k. \end{aligned} \tag{5.1}$$

We use an equation by Gradshteyn and Ryzhik (2000, Section 0.314) for a power series raised to a positive integer j

$$\left(\sum_{i=0}^{\infty} a_i\,u^i\right)^j = \sum_{i=0}^{\infty} c_{j,i}\,u^i. \tag{5.2}$$

Here, for $j \geq 0$, $c_{j,0} = a_0^j$, and the coefficients $c_{j,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation

$$c_{j,i} = (i\, a_0)^{-1} \sum_{p=1}^{i} [p\,(j+1) - i]\, a_p\, c_{j,i-p}, \tag{5.3}$$

So, the coefficient $c_{j,i}$ follows from $c_{j,0}, \ldots, c_{j,i-1}$ and then from $a_0, \ldots, a_i$.

Based on equations (5.2) and (5.3), we can rewrite (5.1) as

$$A = \sum_{i,k=0}^{\infty} c_k^*\, v_{k,i}\, u^{i+k} = \sum_{l=0}^{\infty} d_l^*\, u^l,$$

where, for $k \geq 0$, the coefficients $v_{k,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation

$$v_{k,i} = (i\, m_0)^{-1} \sum_{p=1}^{i} [p\,(j+1) - i]\, m_p\, v_{k,i-p},$$

with $v_{k,0} = m_0^k$ and $d_l^* = \sum_{(i,k)\in I_l} c_k^*\, v_{k,i}$ and $I_l = \{(i,k) | i + k = l; i, k = 0, 1, 2, \ldots\}$.

Then, the qf of X reduces to

$$Q(u) = Q_G\left(\sum_{l=0}^{\infty} d_l^*\, u^l\right). \tag{5.4}$$

In general, even when $Q_G(u)$ does not have a closed-form expression, this function can usually be expressed in terms of a power series

$$Q_G(u) = \sum_{i=0}^{\infty} s_i\, u^i, \tag{5.5}$$

where the coefficients s_i's are suitably chosen real numbers. For several important distributions, such as the normal, Student t, gamma and beta distributions, $Q_G(u)$ does not have a closed-form expression but it can be expanded as in equation (5.5).

By combining (5.4) and (5.5) and using again (5.2) and (5.3), we obtain

$$Q(u) = \sum_{l=0}^{\infty} h_l\, u^l, \tag{5.6}$$

where $h_l = \sum_{i=0}^{\infty} s_i\, h_{i,l}$ (for $i \geq 0$ and $l \geq 0$), $h_{i,l} = (l\, d_0^*)^{-1} \sum_{p=1}^{l} [p\,(i+1) - l]\, d_p^*\, h_{i,l-p}$, for $l \geq 1$, and $h_{l,0} = d_0^*$.

Hence, equation (5.6) reveals that the qf of the ZBOLL-G distribution can be expressed as a power series. For practical purposes, we can adopt ten terms in this power series.

Let $W(\cdot)$ be any integrable function in the positive real line. We can write

$$\int_{-\infty}^{\infty} W(x)\, f(x) dx = \int_0^1 W\left(\sum_{l=0}^{\infty} h_l\, u^l\right) du. \tag{5.7}$$

Equations (5.6) and (5.7) are the main results of this section. We can obtain from them various ZBOLL-G mathematical properties using integrals over $(0, 1)$, which are usually more simple than if they are based on the left integral. For example, an alternative formula for the nth ordinary moment of X follows from (5.7) combined with (5.2) and (5.3) as

$$\mu_n' = \int_0^1 \left(\sum_{l=0}^{\infty} h_l\, u^l\right)^n du = \sum_{l=0}^{\infty} \frac{f_{n,l}}{(l+1)},$$

where (for $n \geq 0$) $f_{n,0} = h_0^n$ and, for $n \geq 1$,

$f_{n,l} = (l\, h_0)^{-1} \sum_{r=1}^{l} [r\,(n+1) - l]\, h_r\, f_{n,l-r}$.

6. Entropies

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi and Shannon entropies. The Rényi entropy of a random variable with pdf $f(\cdot)$ is defined by

$$I_R(\gamma) = \frac{1}{1-\gamma} \log\left(\int_0^\infty f^\gamma(x)\right) dx,$$

for $\gamma > 0$ and $\gamma \neq 1$. The Shannon entropy of a random variable X is defined by $E\{-\log[f(X)]\}$. It is a special case of the Rényi entropy when $\gamma \uparrow 1$.

Here, we derive expressions for the Rényi and Shannon entropies of the ZBOLL-G family. By using (4.1), we can write

$$\left\{-\log[1 - \frac{G(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha}]\right\}^{\gamma\beta-\gamma} = (\gamma\beta-\gamma) \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{j+k}\binom{k-\gamma\beta+\gamma}{k}\binom{k}{j} p_{j,k}}{[\gamma(\beta-1)-j]} \times \left[\frac{G(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha}\right]^{[\gamma(\beta-1)+k]}.$$

Hence,

$$\left\{-\log[1 - \frac{G(x)^\alpha}{G(x)^\alpha + \bar{G}(x)^\alpha}]\right\}^{\gamma\beta-\gamma} \left[\frac{\alpha\, g(x)\, G(x)^{\alpha-1}\bar{G}(x)^{\alpha-1}}{\Gamma(\beta)\,\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^2}\right]^\gamma =$$

$$\alpha^\gamma(\gamma\beta-\gamma) \sum_{k,s=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{j+k}\binom{k-\gamma\beta+\gamma}{k}\binom{k}{j}\binom{\gamma(\alpha-1)}{s} p_{j,k}}{\Gamma(\beta)^{-\gamma}\,[\gamma(\beta-1)-j]} \times \frac{G(x)^{\alpha\gamma(\beta-1)+k\,\alpha+\gamma(\alpha-1)+s}}{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^{\gamma(\beta-1)+k+2\gamma}}.$$

Further,

$$\frac{G(x)^{\alpha\gamma(\beta-1)+k\,\alpha+\gamma(\alpha-1)+s}}{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^{\gamma(\beta-1)+k+2\gamma}} = \frac{\sum_{r=0}^{\infty}\lambda'_r\, G(x)^r}{\sum_{r=0}^{\infty}\rho'_r\, G(x)^r} = \sum_{r=0}^{\infty} a'_r\, G(x)^r,$$

where

$$\lambda'_r = \sum_{l=r}^{\infty}(-1)^{l+r}\binom{\alpha\gamma(\beta-1)+k\,\alpha+\gamma(\alpha-1)+s}{l}\binom{l}{r}$$

$$\rho'_r = h_r(\alpha, \gamma(\beta-1)+k+2\gamma)$$

$$a'_r = a'_r(\alpha,\beta,i,k) = \frac{1}{\rho'_0}\left(\rho'_r - \frac{1}{\rho'_0}\sum_{s=1}^{r}\rho'_s\, a'_{r-s}\right), \text{ for } r \geq 1,$$

$a_0 = \lambda'_0/\rho'_0$ and $h_r(\alpha, \beta+i+k)$ is defined in the Appendix. Then,

$$\int_0^\infty g^\gamma(x)dx = \frac{1}{\Gamma(\beta)^\gamma}\int_0^\infty \left\{-\log\left[1 - \frac{G(x)^\alpha}{G(x)^\alpha + \bar{G}(x;)^\alpha}\right]\right\}^{\gamma a-\gamma} \times \left\{\frac{\alpha\, g(x)\, G(x)^{\alpha-1}\bar{G}(x)^{\alpha-1}}{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^2}\right\}^\gamma dx$$

$$= \frac{\alpha^\gamma(\gamma\beta-\gamma)}{\Gamma(\beta)^\gamma} \sum_{k,r,s=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{j+k}\binom{k-\gamma\beta+\gamma}{k}\binom{k}{j}\binom{\gamma(\alpha-1)}{s} p_{j,k} a'_r}{[\gamma(\beta-1)-j]} S_r,$$

where S_r can be evaluated from the baseline distribution as

$$S_r = \int_0^\infty G(x)^r \, g^\gamma(x) dx.$$

Hence, the Rényi entropy of X is given by

$$\begin{aligned} I_R(\gamma) &= \frac{\gamma}{1-\gamma}\log(\alpha) - \frac{\gamma}{1-\gamma}\log[\Gamma(\beta)] + \frac{1}{1-\gamma}\log(\gamma\beta-\gamma) \\ &+ \frac{1}{1-\gamma}\log\left\{\sum_{k,r,s=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k-\gamma\beta+\gamma}{k}\binom{k}{j}\binom{\gamma(\alpha-1)}{s}p_{j,k}\,a'_r}{[\gamma(\beta-1)-j]}\,I_r\right\}. \end{aligned}$$

The Shannon entropy can be obtained by limiting $\gamma \uparrow 1$ in $I_R(\gamma)$. However, it is easier to derive an expression for it from first principles. Using the power series for $\log(1-z)$, we can write

$$\begin{aligned} E\{-\log[f(X)]\} &= -\log(\alpha) + \log[\Gamma(\beta)] - E\{\log[g(X)]\} + (1-\alpha)E\{\log[G(X)]\} \\ &\quad +(1-\alpha)E\{\log[\bar{G}(X)]\} + 2E\{\log[G^\alpha(X) + \bar{G}^\alpha(X)]\} \\ &\quad +(1-\beta)E\left\{-\log\left[1-\frac{G^\alpha(X)}{G^\alpha(X)+\bar{G}^\alpha(X)}\right]\right\}. \end{aligned}$$

First, we define and compute

$$A(a_1,a_2,a_3,a_4;\alpha) = \int_0^1 \frac{u^{a_1}(1-u)^{a_2}}{[u^\alpha+(1-u)^\alpha]^{a_3}}\left\{-\log\left[1-\frac{u^\alpha}{u^\alpha+(1-u)^\alpha}\right]\right\}^{a_4} du.$$

Along the same lines of the derivation of the Rényi entropy, we obtain

$$\begin{aligned} A(a_1,a_2,a_3,a_4;\alpha) &= a_4 \sum_{k,s=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k+s}\binom{k-a_4}{k}\binom{k}{j}\binom{a_2}{s}p_{j,k}}{a_4-j} \\ &\quad \times \int_0^1 \frac{u^{\alpha(a_4+k)+a_1+s}}{[u^\alpha+(1-u)^\alpha]^{a_4+k+a_3}}du. \end{aligned}$$

Also,

$$\frac{u^{\alpha(a_4+k)+a_1+s}}{[u^\alpha+(1-u)^\alpha]^{a_4+k+a_3}} = \frac{\sum_{r=0}^{\infty}\lambda''_r\,u^r}{\sum_{r=0}^{\infty}\rho''_r\,u^r} = \sum_{r=0}^{\infty}a''_r\,u^r,$$

where (for $r \geq 1$)

$$\lambda''_r = \sum_{l=r}^{\infty}(-1)^{l+r}\binom{\alpha(a_4+k)+a_1+s}{l}\binom{l}{r},$$

$$\rho''_r = h_r(\alpha, a_4+k+a_3),$$

$$a''_r = a''_r(\alpha,\beta,i,k) = \frac{1}{\rho''_0}\left(\rho'_r - \frac{1}{\rho''_0}\sum_{s=1}^{r}\rho''_s\,a''_{r-s}\right),$$

$a''_0 = \lambda''_0/\rho''_0$ and $h_r(\alpha, a_4+k+a_3)$ is defined in the Appendix. Then,

$$A(a_1,a_2,a_3,a_4;\alpha) =$$

$$a_4\sum_{k,s,r=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k+s}\binom{k-a_4}{k}\binom{k}{j}\binom{\alpha-1}{s}}{(a_4-j)(r+1)}\,p_{j,k}a''_r(\alpha,\beta,i,k).$$

Hence,

$$\mathrm{E}\{\log[G(X)]\} = \frac{\alpha}{\Gamma(\beta)}\frac{\partial}{\partial t}A(\alpha+t-1,\alpha-1,2,\beta-1;\alpha)\Big|_{t=0},$$

$$\mathrm{E}\left\{\log\left[\bar{G}(X)\right]\right\} = \frac{\alpha}{\Gamma(\beta)}\frac{\partial}{\partial t}A(\alpha-1,\alpha+t-1,2,\beta-1;\alpha)\Big|_{t=0},$$

$$\mathrm{E}\left\{\log\left[G(X)^{\alpha}+\bar{G}(X)^{\alpha}\right]\right\} = \frac{\alpha}{\Gamma(\beta)}\frac{\partial}{\partial t}A(\alpha-1,\alpha-1,2-t,\beta-1;\alpha)\Big|_{t=0}$$

and

$$E\left\{-\log\left[1-\frac{G^{\alpha}(X)}{G^{\alpha}(X)+\bar{G}^{\alpha}(X)}\right]\right\} = \frac{\alpha}{\Gamma(\beta)}\frac{\partial}{\partial t}A(\alpha-1,\alpha-1,2,\beta+t-1;\alpha)\Big|_{t=0}.$$

The simplest formula for the Shannon entropy of X is given by

$$\begin{aligned}\mathrm{E}\{-\log[f(X)]\} &= -\log(\alpha)+\log[\Gamma(\beta)]-\mathrm{E}\{\log[g(X;\boldsymbol{\tau})]\}\\ &\quad+\frac{\alpha(1-\alpha)}{\Gamma(\beta)}\frac{\partial}{\partial t}A(\alpha+t-1,\alpha-1,2,\beta-1;\alpha)\Big|_{t=0}\\ &\quad+\frac{\alpha(1-\alpha)}{\Gamma(\beta)}\frac{\partial}{\partial t}A(\alpha-1,\alpha+t-1,2,\beta-1;\alpha)\Big|_{t=0}\\ &\quad+\frac{2\alpha}{\Gamma(\beta)}\frac{\partial}{\partial t}A(\alpha-1,\alpha-1,2-t,\beta-1;\alpha)\Big|_{t=0}\\ &\quad+\frac{\alpha(1-\beta)}{\Gamma(\beta)}\frac{\partial}{\partial t}A(\alpha-1,\alpha-1,2,\beta+t-1;\alpha)\Big|_{t=0}.\end{aligned}$$

We provide in Figures 4a-b a numerical investigation to identify how the parameter values change the shapes of the Rényi entropy of X for some parameter ranges. To evaluate the values of $I_R(\gamma)$ we consider the random variable X having the ZBOLL-W distribution given in equation (2.1).

7. Order statistics

Suppose $X_1,\ldots,X_n$ is a random sample from the ZBOLL-G family. Denote the random variables in ascending order as $X_{1:n}\leq\ldots\leq X_{n:n}$. The pdf of $X_{i:n}$ is given by (David and Nagarajah, 2003)

$$\begin{aligned}f_{i:n}(x) &= K\,f(x)\,F^{i-1}(x)\,\{1-F(x)\}^{n-i} = K\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}f(x)\,F(x)^{j+i-1}\\ &= \sum_{r,k=0}^{\infty}\sum_{j=0}^{n-i}m_{j,r,k}\,h_{r+k+1}(x), \qquad (7.1)\end{aligned}$$

where $K=n!/[(i-1)!\,(n-i)!]$, $h_{r+k+1}(x)$ denotes the exp-G density function with power parameter $r+k+1$ and

$$m_{j,r,k}=\frac{(-1)^j\,n!}{(i-1)!\,(n-i-j)!\,j!}\,\frac{(r+1)\,b_{r+1}\,f_{j+i-1,k}}{(r+k+1)},$$

where b_k is defined by (4.3). Here, the quantities $f_{j+i-1,k}$ are obtained recursively by $f_{j+i-1,0}=b_0^{j+i-1}$ and (for $k\geq 1$)

$$f_{j+i-1,k}=(k\,b_0)^{-1}\sum_{m=1}^{k}[m(j+i)-k]\,b_m\,f_{j+i-1,k-m}.$$

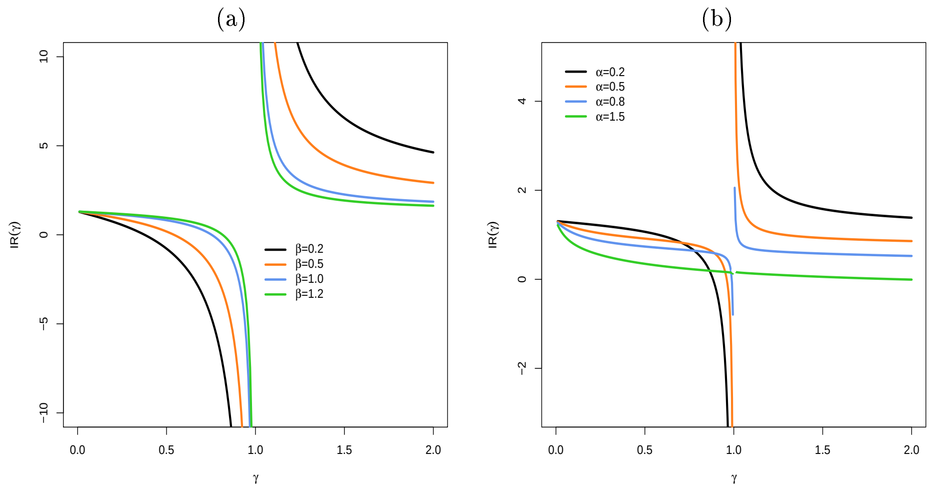

Figure 4. The Rényi entropy of X as function of γ for $\lambda = 1.5$, $\kappa = 3.5$ and: (a) $\alpha = 0.2$ for some values of β; (b) $\beta = 1.5$ for some values of α.

Thus, one can easily obtain ordinary and incomplete moments and generating function of ZBOLL-G order statistics from (7.1) for any parent G.

8. Maximum likelihood estimation

In this section, we determine the maximum likelihood estimates (MLEs) of the model parameters of the new family from complete samples only. Let $x_1, \ldots, x_n$ be observed values from the ZBOLL-G family with parameters α, β and $\boldsymbol{\tau}$. Let $\boldsymbol{\theta} = (\alpha, \beta, \boldsymbol{\tau}^\top)^\top$ be the $r \times 1$ parameter vector. The total log-likelihood function for $\boldsymbol{\theta}$ is given by

$$\begin{aligned} \ell_n(\boldsymbol{\theta}) = \ell_n &= n\log(\alpha) - n\log[\Gamma(\beta)] + \sum_{i=1}^{n}\log[g(x_i;\boldsymbol{\tau})] \\ &+ (\alpha-1)\sum_{i=1}^{n}\log[G(x_i;\boldsymbol{\tau})] + +(\alpha-1)\sum_{i=1}^{n}\log[1-G(x_i;\boldsymbol{\tau})] \\ &- 2\sum_{i=1}^{n}\log\{G^{\alpha}(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^{\alpha}\} \\ &+ (\beta-1)\sum_{i=1}^{n}\log\left\{-\log\left[\frac{[1-G(x_i;\boldsymbol{\tau})]^{\alpha}}{G^{\alpha}(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^{\alpha}}\right]\right\}. \end{aligned} \quad (8.1)$$

The log-likelihood function can be maximized either directly by using the SAS (PROC NLMIXED) or by solving the nonlinear likelihood equations obtained by differentiating (8.1). The components of the score function

$U_n(\Theta) = (\partial\ell_n/\partial\alpha, \partial\ell_n/\partial\beta, \partial\ell_n/\partial\boldsymbol{\tau})^\top$ are given by

$$\frac{\partial\ell_n}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log[G(x_i;\boldsymbol{\tau})] + \sum_{i=1}^{n}\log[1-G(x_i;\boldsymbol{\tau})]$$
$$- 2\sum_{i=1}^{n}\frac{G^\alpha(x_i;\boldsymbol{\tau})\log[G(x_i;\boldsymbol{\tau})] + [1-G(x_i;\boldsymbol{\tau})]^\alpha\log[1-G(x_i;\boldsymbol{\tau})]}{G^\alpha(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^\alpha}$$
$$+ (\beta-1)\sum_{i=1}^{n}\frac{G^\alpha(x_i;\boldsymbol{\tau})\ \log\left\{\frac{[1-G(x_i;\boldsymbol{\tau})]}{G(x_i;\boldsymbol{\tau})}\right\}}{[G^\alpha(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^\alpha]\log\left\{\frac{[1-G(x_i;\boldsymbol{\tau})]^\alpha}{G^\alpha(x_i;\boldsymbol{\tau})+[1-G(x_i;\boldsymbol{\tau})]^\alpha}\right\}},$$
$$\frac{\partial\ell_n}{\partial\beta} = -n\psi(\beta) + \sum_{i=1}^{n}\log\left\{-\log\left[\frac{[1-G(x_i;\boldsymbol{\tau})]^\alpha}{G^\alpha(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^\alpha}\right]\right\},$$
$$\frac{\partial\ell_n}{\partial\boldsymbol{\tau}} = \sum_{i=1}^{n}\frac{[\dot{g}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}}{g(x_i;\boldsymbol{\tau})} - (\alpha-1)\sum_{i=1}^{n}\frac{[\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}}{G(x_i;\boldsymbol{\tau})}$$
$$- 2\alpha\sum_{i=1}^{n}\frac{[\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}\{G^\alpha(x_i;\boldsymbol{\tau}) - [1-G(x_i;\boldsymbol{\tau})]^{\alpha-1}\}}{G^\alpha(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^\alpha}$$
$$+ \alpha(\beta-1)\sum_{i=1}^{n}\frac{[\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}\ [G^\alpha(x_i;\boldsymbol{\tau}) + G^{\alpha-1}(x_i;\boldsymbol{\tau})]}{[G^\alpha(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^\alpha]\log\left\{\frac{[1-G(x_i;\boldsymbol{\tau})]^\alpha}{G^\alpha(x_i;\boldsymbol{\tau})+[1-G(x_i;\boldsymbol{\tau})]^\alpha}\right\}},$$

where

$$[\dot{g}(x_i;\boldsymbol{\tau})]_\alpha = \frac{dg(x_i;\boldsymbol{\tau})}{d\alpha}, \qquad [\dot{G}(x_i;\boldsymbol{\tau})]_\alpha = \frac{dG(x_i;\boldsymbol{\tau})}{d\alpha},$$
$$[\dot{g}(x_i;\boldsymbol{\tau})]_\beta = \frac{dg(x_i;\boldsymbol{\tau})}{d\beta}, \qquad [\dot{G}(x_i;\boldsymbol{\tau})]_\beta = \frac{dG(x_i;\boldsymbol{\tau})}{d\beta},$$
$$[\dot{g}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}} = \frac{dg(x_i;\boldsymbol{\tau})}{d\boldsymbol{\tau}}, \qquad [\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}} = \frac{dG(x_i;\boldsymbol{\tau})}{d\boldsymbol{\tau}},$$

and the functions $g(\cdot)$ and $G(\cdot)$ are defined in Section 1 and $\psi(\cdot)$ is the digamma function.

The MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is obtained by solving the nonlinear likelihood equations $U_\alpha(\boldsymbol{\theta}) = 0$, $U_\beta(\boldsymbol{\theta}) = 0$ and $U_{\boldsymbol{\tau}}(\boldsymbol{\theta}) = \mathbf{0}$. These equations cannot be solved analytically and statistical software can be used to solve them numerically. We can use iterative techniques such as a Newton-Raphson type algorithm to obtain the estimate $\widehat{\boldsymbol{\theta}}$. We employ the numerical procedure NLMixed in SAS.

For interval estimation of $(\alpha, \beta, \boldsymbol{\tau})$ and hypothesis tests on these parameters, we obtain the observed information matrix since the expected information matrix is very complicated and requires numerical integration. The $(p+2)\times(p+2)$ observed information matrix $J(\boldsymbol{\theta})$, where p is the dimension of the vector $\boldsymbol{\tau}$, becomes

$$J(\boldsymbol{\theta}) = -\begin{pmatrix} \mathbf{L}_{\alpha\alpha} & \mathbf{L}_{\alpha\beta} & \mathbf{L}_{\alpha\boldsymbol{\tau}} \\ . & \mathbf{L}_{\beta\beta} & \mathbf{L}_{\beta\boldsymbol{\tau}} \\ . & . & \mathbf{L}_{\boldsymbol{\tau}\boldsymbol{\tau}} \end{pmatrix},$$

whose elements are given in Appendix B.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_{p+2}(\mathbf{0}, I(\boldsymbol{\theta})^{-1})$, where $I(\boldsymbol{\theta})$ is the expected information matrix. The multivariate normal $N_{p+2}(\mathbf{0}, J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution, where $I(\boldsymbol{\theta})$ is replaced by $J(\widehat{\boldsymbol{\theta}})$, i.e., the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$, can be used to construct approximate confidence intervals for the individual parameters.

We can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain likelihood ratio (LR) statistics for testing some special models of the proposed family. Tests of the hypotheses of the type $H_0 : \boldsymbol{\psi} = \boldsymbol{\psi}_0$ *versus* $H : \boldsymbol{\psi} \neq \boldsymbol{\psi}_0$, where $\boldsymbol{\psi}$ is a subset of parameters of $\boldsymbol{\theta}$, can be performed through LR statistics in the usual way.

9. Applications

In this section, we use two real data sets to compare the fits of the ZBOLL-G family with others commonly used lifetime models. In each case, the parameters are estimated by maximum likelihood (Section 8) using the subroutine NLMixed in SAS. First, we describe the data sets and give the MLEs (the corresponding standard errors and 95% confidence intervals) of the model parameters and the values of the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) and Kolmogorov-Smirnov (K-S) statistics. The lower the values of these criteria, the better the fit. Note that over-parametrization is penalized in these criteria, so that the two additional parameters in the proposed family do not necessarily lead to smaller values of these statistics. Next, we perform LR tests for testing some special models. Finally, we provide the histograms of the data sets to have a visual comparison of the fitted density functions.

9.1. Application 1: Zootechnics data. The data come from the zootechnics records of a Brazilian company engaged in raising beef cattle, where the farms stocked with the Nelore breed are located in the States of Bahia and São Paulo. In the analysis, only data on females born in 2000 were used and the age at first calving was the reproductive characteristic analyzed. In this case, the response variable is the logarithm of the age of the cows at first calving (measured in days). The first calving age is an important characteristic for beef cattle breeders because the faster cows reach reproductive maturity, the more calves they will produce during their breeding cycle and the greater the breeder's return on investment will be. Further, this trait is easy and inexpensive to measure. The sample size in this study is $n = 897$.

First, we describe the descriptive statistics of the data in Table 1. They suggest negatively skewed distributions with different degrees of variability, skewness and kurtosis. Then, we report the MLEs (and the corresponding standard errors in parentheses) of the

Table 1. Descriptive statistics.

Mean	Median	Mode	Variance	Skewness	Kurtosis	Min.	Max.	n
1004.32	1053.0	1074.0	13838.9	-0.405	-0.139	722	1453	897

parameters in Table 2. Additionally, we compare the models using the AIC, CAIC, BIC and K-S statistics (see Table 3). The figures in this table indicate that the ZBOLL-W model gives the best fit among the fitted models.

A comparison of the proposed distribution with some of its sub-models using LR statistics is performed in Table 4. The figures in this table, specially the p-values, reveal that the ZBOLL-W model gives a better fit to these data than the other three sub-models.

More information is provided by a visual comparison of the histogram of the data with the fitted density functions. The plots of the fitted ZBOLL-W, OLL-W, gamma-W and Weibull density functions are displayed in Figure 5. We also conclude that the ZBOLL-W distribution provides an adequate fit to these data.

Table 2. Estimates of the parameters, standard errors in [·] and 95% confidence intervals in (·) for the zootechnics data.

Model	α	β	κ	λ
ZBOLL-W	86.8186	0.1612	0.4420	2582.45
	[4.4833]	[0.0136]	[0.035]	[168.38]
	(78.0196, 95.6175)	(0.1346, 0.1879)	(0.3735, 0.5106)	(2251.98, 2912.91)
OLL-W	1.3982	1	7.5136	1068.10
	[0.1074]	-	[0.4975]	[5.1663]
	(1.1874, 1.6089)	-	(6.5372, 8.4899)	(1057.97, 1078.24)
Gamma-W	1	2.0976	6.4239	924.76
	-	[0.3808]	[0.6520]	[40.0309]
	-	(1.3503, 2.8449)	(5.1443, 7.7035)	(846.20, 1003.33)
Weibull	1	1	9.4418	1054.36 3
	-	-	[0.2198]	[3.9260]
	-	-	(9.0104, 9.8732)	(1046.07, 1062.07)

Table 3. The AIC, CAIC, BIC and K-S statistics for the zootechnics data.

Model	AIC	CAIC	BIC	K-S	p-value
ZBOLL-W	10838	10839	10857	0.1381	<0.001
OLL-W	11081	11082	11095	0.1519	<0.001
Gamma-W	11078	11079	11092	0.1717	<0.001
Weibull	11103	11104	11113	0.1595	<0.001

Table 4. LR statistics for the zootechnics data.

Model	Hypotheses	Statistic w	p-value
ZBOLL-W vs OLL-W	$H_0 : \beta = 1$ vs $H_1 : H_0$ is false	244.0	<0.00001
ZBOLL-W vs Gamma-W	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	241.0	<0.00001
ZBOLL-W vs Weibull	$H_0 : \alpha = \beta = 1$ vs $H_1 : H_0$ is false	268.0	<0.00001

9.2. Application 2: Temperature data. The variable temperature (^{o}C) corresponding to daily data for the period from January 1 to December 31, 2011, obtained from the weather station of the Department of Biosystem Engineering of the Luiz de Queiroz School of Agriculture (ESALQ) of the University of São Paulo (USP), located in the City of Piracicaba, at latitude 22"42"30"S, longitude 47"38"30"W and altitude of 546 meters. First, we describe the data set in Table 5.

Table 5. Descriptive statistics.

Mean	Median	Mode	Variance	Skewness	Kurtosis	Min.	Max.	n
22.32	22.90	19.25	8.71	-0.50	-0.73	14.68	27.25	365

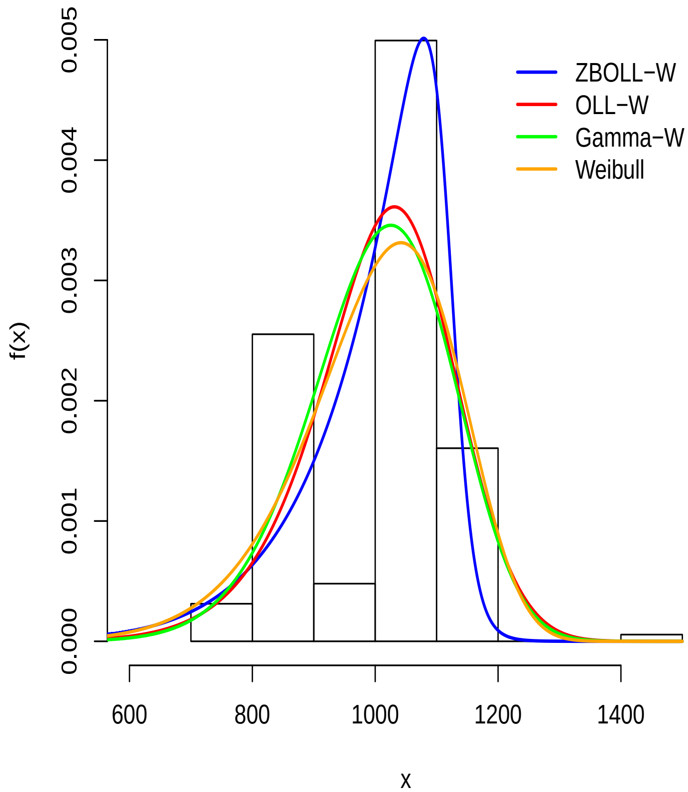

Figure 5. (a) Fitted ZBOLL-W, OLL-W, gamma-W and Weibull densities for the zootechnics data.

For these data, we compare the fitted ZBOLL-N, OLL-N, gamma-N and normal distributions. The MLEs of μ and σ for the normal distribution are taking as starting values for the iterative procedure to fit the ZBOLL-N, OLL-N and gamma-N models. The MLEs of the parameters, standard errors and 95% confidence intervals for the parameters are given in Table 6. Additionally, we compare the models using the AIC, CAIC, BIC and K-S statistics (see Table 7). Since the values of these statistics are smaller for the ZBOLL-N distribution compared to those values of the other models (see Table 6), the new distribution produces a fit to the current data quite better than its special models.

A comparison of the proposed distribution with some of its sub-models using LR statistics is performed in Table 8. The figures in this table, specially the p-values, indicate that the ZBOLL-N model gives a better fit to these data than the other three sub-models.

Table 6. Estimates of the parameters, standard errors in [·] and 95% confidence intervals in (·) for the ZBOLL-N model and its special models and three criteria for the temperature data.

Model	α	β	μ	σ
ZBOLL-N	0.1783	1.3744	21.0200	0.9293
	[0.0262]	[0.1355]	[0.3640]	[0.0729]
	(0.1183, 0.2382)	(1.1579, 1.5907)	(20.4648, 21.5757)	(0.7422, 1.1163)
OLL-N	0.1861	1	21.9071	0.8915
	[0.0448]	-	[0.1220]	[0.1281]
	(0.0958, 0.2763)	-	(21.6668, 22.1474)	(0.6332, 1.1498)
Gamma-N	1	0.1246	26.698	1.4409
	-	[0.0069]	[0.1910]	[0.0319]
	-	(0.1109, 0.1383)	(26.1933, 26.9446)	(1.3782, 1.5037)
Normal	1	1	22.3271	2.9463
	-	-	[0.1542]	[0.1090]
	-	-	()	()

Table 7. AIC, CAIC, BIC and K-S statistics for the temperature data.

Model	AIC	CAIC	BIC	K-S	p-values
ZBOLL-N	1777.9	1778.1	1793.5	0.0617	0.0731
OLL-N	1790.4	1791.4	1802.1	0.1108	0.0002
Gamma-N	1797.6	1797.7	1809.3	0.0818	0.0151
Normal	1828.7	1829.7	1836.5	0.1029	0.0005

Table 8. LR statistics for the temperature data.

Model	Hypotheses	Statistic w	p-value
ZBOLL-N vs OLL-N	$H_0 : \beta = 1$ vs $H_1 : H_0$ is false	14.5	0.00014
ZBOLL-N vs Gamma-N	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	22.0	<0.00001
ZBOLL-N vs Normal	$H_0 : \alpha = \beta = 1$ vs $H_1 : H_0$ is false	54.8	<0.00001

More information is provided by a visual comparison of the histogram of the data and the fitted density functions. The plots of the fitted ZBOLL-N, OLL-N, gamma-N and normal densities are displayed in Figure 6. We conclude that the ZBOLL-N distribution provides the best fit to these data.

10. Conclusions

In this paper, we propose a new family of distributions with two extra generator parameters, which includes as special cases all classical continuous distributions. For any parent continuous distribution G, we define the so-called *Zografos-Balakrishnan odd log-logistic-G* family with two extra positive parameters. The new family extends several widely known distributions and some of its special models are discussed. We demonstrate that the new family density function is a linear mixture of exponentiated-G densities.

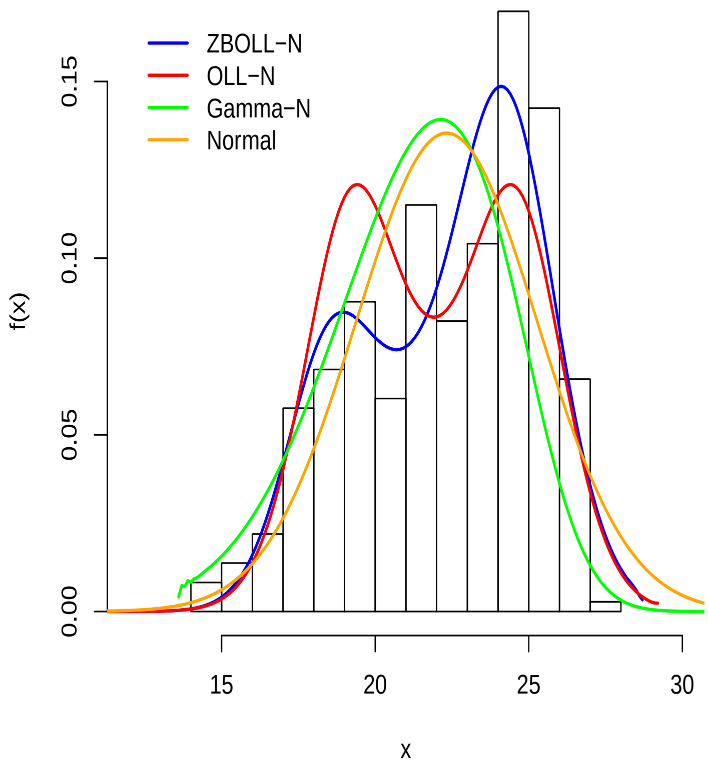

Figure 6. (a) Fitted ZBOLL-N, OLL-N, gamma-N and normal densities for the zootechnics data.

We obtain some of its mathematical properties, which include ordinary and incomplete moments, generating and quantile functions, mean deviations, Bonferroni and Lorenz curves, two types of entropies and order statistics. The application of the new family is straightforward. The model parameters are estimated by maximum likelihood. Two real examples are used for illustration, where the new family does fit well both data sets.

Appendix A: Three useful power series

We present three power series required for the algebraic developments in Section 3 and 6. First, for $b > 0$ real non-integer and $-1 < u < 1$, we have the binomial expansion

$$(1-u)^a = \sum_{j=0}^{\infty} (-1)^j \binom{a}{j} u^j, \tag{10.1}$$

where the binomial coefficient is defined for any real.

Second, expanding z^λ in Taylor series, we can write

$$(10.2)\qquad z^\lambda = \sum_{k=0}^{\infty} (\lambda)_k\,(z-1)^k/k! = \sum_{i=0}^{\infty} f_i\, z^i$$

where

$$(10.3)\qquad f_i = f_i(\lambda) = \sum_{k=i}^{\infty} \frac{(-1)^{k-i}}{k!}\binom{k}{i}(\lambda)_k$$

and $(\lambda)_k = \lambda(\lambda-1)\ldots(\lambda-k+1)$ denotes the descending factorial.

Third, we obtain an expansion for $[G(x)^a + \bar{G}(x)^a]^c$. We can write from equation (10.2) and (10.1)

$$(10.4)\qquad [G(x)^a + \bar{G}(x)^a] = \sum_{j=0}^{\infty} t_j\, G(x)^j,$$

where $t_j = t_j(a) = a_j(a) + (-1)^j\binom{a}{j}$ and $a_j(a)$ is defined by (10.2). Then, using (10.2), we have

$$[G(x)^a + \bar{G}(x)^a]^c = \sum_{i=0}^{\infty} f_i\left(\sum_{j=0}^{\infty} t_j\, G(x)^j\right)^i,$$

where $f_i = f_i(c)$.

Finally, using again equations (10.3) and (10.4), we have

$$(10.5)\qquad [G(x)^a + \bar{G}(x)^a]^c = \sum_{j=0}^{\infty} h_j(a,c)\, G(x)^j,$$

where $h_j(a,c) = \sum_{i=0}^{\infty} f_i\, m_{i,j}$ and (for $i \geq 0$) $m_{i,j} = (j\,t_0)^{-1}\sum_{m=1}^{j}[m(j+1)-j]\,t_m\, m_{i,j-m}$ (for $j \geq 1$) and $m_{i,0} = t_0^i$.

Appendix B

The elements of the observed information matrix $J(\boldsymbol{\theta})$ for the parameters $(\alpha, \beta, \boldsymbol{\tau})$ are given by

$$\begin{aligned} J_{\alpha\alpha} &= \frac{-n}{\alpha^2} \\ &- 2\sum_{i=1}^{n} \frac{G^\alpha(x_i;\boldsymbol{\tau})[1-G(x_i;\boldsymbol{\tau})]^\alpha \left\{\log[G(x_i;\boldsymbol{\tau})]\log[\frac{G(x_i;\boldsymbol{\tau})}{1-G(x_i;\boldsymbol{\tau})}]\right\}}{[G^\alpha(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^\alpha]^2} \\ &- 2\sum_{i=1}^{n} \frac{G^\alpha(x_i;\boldsymbol{\tau})[1-G(x_i;\boldsymbol{\tau})]^\alpha \left\{\log[1-G(x_i;\boldsymbol{\tau}]\log[\frac{1-G(x_i;\boldsymbol{\tau})}{G(x_i;\boldsymbol{\tau})}]\right\}}{[G^\alpha(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^\alpha]^2}, \end{aligned}$$

$$J_{\alpha\,\beta} = -\sum_{i=1}^{n} \frac{G^\alpha(x_i;\boldsymbol{\tau})\,\log\left[\frac{G(x_i;\boldsymbol{\tau})}{\bar{G}(x_i;\boldsymbol{\tau})}\right]}{[G^\alpha(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^\alpha]\log\left[1-\frac{G^\alpha(x_i;\boldsymbol{\tau})}{G^\alpha(x_i;\boldsymbol{\tau})+[1-G(x_i;\boldsymbol{\tau})]^\alpha}\right]},$$

$$J_{\alpha\boldsymbol{\tau}} = \sum_{i=1}^{n} \frac{[\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}}{G(x_i;\boldsymbol{\tau})} - \sum_{i=1}^{n} \frac{[\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}}{1-G(x_i;\boldsymbol{\tau})}$$
$$- \quad 2\sum_{i=1}^{n} \frac{[\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}} \left[G^{\alpha-1}(x_i;\boldsymbol{\tau}) - [1-G(x_i;\boldsymbol{\tau})]^{\alpha-1}\right]}{G^{\alpha}(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^{\alpha}}$$
$$- \quad 2\alpha\sum_{i=1}^{n} \frac{[\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}\, G^{\alpha}(x_i;\boldsymbol{\tau})\, [1-G(x_i;\boldsymbol{\tau})]^{\alpha} \log\left[\frac{G(x_i;\boldsymbol{\tau})}{1-G(x_i;\boldsymbol{\tau})}\right]}{\left[G^{\alpha}(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^{\alpha}\right]^2}$$

$$J_{\beta\beta} = -n\,\psi'(\beta)$$

$$J_{\beta\boldsymbol{\tau}} \quad = \quad -\alpha\sum_{i=1}^{n} \frac{[\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}\; G^{\alpha-1}(x_i;\boldsymbol{\tau})}{[1-G(x_i;\boldsymbol{\tau})]\left[G^{\alpha}(x_i;\boldsymbol{\tau}) + [1-G(x_i;\boldsymbol{\tau})]^{\alpha}\right] \log\left[1 - \frac{G^{\alpha}(x_i;\boldsymbol{\tau})}{G^{\alpha}(x_i;\boldsymbol{\tau})+[1-G(x_i;\boldsymbol{\tau})]^{\alpha}}\right]}$$

$$J_{\boldsymbol{\tau}\boldsymbol{\tau}} \quad = \quad (\alpha-1)\sum_{i=1}^{n}\left\{\frac{[\ddot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}\boldsymbol{\tau}}}{G(x_i;\boldsymbol{\tau})} - \frac{[\dot{G}(x_i;\boldsymbol{\tau})]^2_{\boldsymbol{\tau}}}{[G(x_i;\boldsymbol{\tau})]^2}\right\}$$
$$+(\alpha-1)\sum_{i=1}^{n}\left\{\frac{[\ddot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}\boldsymbol{\tau}}}{[1-G(x_i;\boldsymbol{\tau})]} + \frac{[\dot{G}(x_i;\boldsymbol{\tau})]^2_{\boldsymbol{\tau}}}{[1-G(x_i;\boldsymbol{\tau})]^2}\right\}$$
$$-\beta\sum_{i=1}^{n}\left\{\frac{[\ddot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}\boldsymbol{\tau}}}{[1-G(x_i;\boldsymbol{\tau})]^2} + \frac{2[\dot{G}(x_i;\boldsymbol{\tau})]^2_{\boldsymbol{\tau}}}{[1-G(x_i;\boldsymbol{\tau})]^3}\right\} + \sum_{i=1}^{n}\left\{\frac{[\ddot{g}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}\boldsymbol{\tau}}}{g(x_i;\boldsymbol{\tau})} - \frac{[\dot{g}(x_i;\boldsymbol{\tau})]^2_{\boldsymbol{\tau}}}{[g(x_i;\boldsymbol{\tau})]^2}\right\},$$

where

$$[\dot{g}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}} = \frac{dg(x_i;\boldsymbol{\tau})}{d\boldsymbol{\tau}},\; [\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}} = \frac{dG(x_i;\boldsymbol{\tau})}{d\boldsymbol{\tau}},\; [G(x_i;\boldsymbol{\tau})]^2 = \frac{dG(x_i;\boldsymbol{\tau})}{d\boldsymbol{\tau}}\left(\frac{dG(x_i;\boldsymbol{\tau})}{d\boldsymbol{\tau}}\right)^T,$$

$$[\ddot{g}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}\boldsymbol{\tau}} = \frac{d^2g(x_i;\boldsymbol{\tau})}{d\boldsymbol{\tau}\boldsymbol{\tau}^T},\; [\ddot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}\boldsymbol{\tau}} = \frac{d^2G(x_i;\boldsymbol{\tau})}{d\boldsymbol{\tau}\boldsymbol{\tau}^2},$$

and $g(\cdot)$ and $G(\cdot)$ are defined in Section 1.

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Analysis of covariance by assuming a skew normal distribution for response variable

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Abstract

The traditional theory of analysis of covariance (ANCOVA) is based on normality assumption, while in many real world applications the data violate normality and this theory is not adequate. In this paper, we expand a model for analysis of covariance with a skew normal response variable. The maximum likelihood estimates of the model parameters are provided via an EM algorithm. We also developed asymptotic confidence intervals for parameters. A simulation study is performed to assess the performance of the proposed model. The methodology is illustrated using a real data set.

Keywords: Analysis of covariance, skew normal, maximum likelihood, EM algorithm.

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1. Introduction

Analysis of covariance is a widely used technique for exploring possible relation between a usually continuous response variable and a set of covariates and treatments. This methodology is a combination of regression and analysis of variance (ANOVA) that profits the benefits of both of these two efficient modeling methods. The ANCOVA can be employed for a wide range of different purposes. It can be used to filter out error variance, to explore pre-test vs. post-test effects, to control the variables, to finding significant difference between groups by reducing the within-groups variations etc. It also provides a useful approach to treat the potentially confounding variables. In many practical situations, one cannot provide the ideal homogenous experimental units for all treatments,

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even after blocking, which is an essential requirement for comparative experiments analyzed by ANOVA. Thus one has to appeal to ANCOVA. In this context, adjusting treatment effects for nuisance covariates effects on the response variable is of paramount importance for the researcher. This practice allows finding the net effect of treatments under specified collection of covariates and provides a clear guidance for users of the results. This concept is foreign to proper regression analysis which does not discriminate between treatment and covariate. ANCOVA was firstly motivated by Fisher [14]. During the years many researchers have investigated different theoretical and applied aspects of ANCOVA in different sciences. Cochran [10] and Cox and McCullagh [11] and references therein are good sources for more information about ANCOVA. As it is pointed out by [19] the traditional theory of normal ANCOVA is not adequate when the data violate the normality assumption. This creates a strong motivation for considering ANCOVA under other distributions that are more flexible than normal distribution. Many researches have been recently focused to develop suitable methods for dealing with non-normality. These considerations are not limited to ANCOVA and other modeling techniques such as regression, ANOVA, discriminant analysis etc. have investigated repeatedly for use in situations that the normality assumption does not hold. In particular, the skew normal family of distributions as a generalization of the normal family has attracted considerable attentions in literature. Though the earlier appearance of skew normal distribution returns to Roberts [21] and O'Hagan and Leonard [20] and Aigner *et al.* [1], but the first formal definition of this family of distributions was provided by Azzalini [3]. The multivariate form of the skew normal distribution is expanded in [4] and [5]. During the three past decades many skew normal distribution have been introduced and discussed in literature. References [22, 15, 16, 4] are excellent sources of information about the skew normal family of distributions and their properties. Different modeling approaches such as regression analysis (Sahu *et al.* [22], Ferreira and Steel [13] and Cancho [9]), Bayesian nonlinear regression (De la Cruz and Branco [12]), linear mixed models (Arellano-Valle [2]) and analyzing longitudinal data (Baghfalaki *et al.* [7] and Lin and Lee [18]) have been developed under the assumption of skew normal distribution. This paper investigates ANCOVA under the assumption of skew normal distribution for response variable. We show that the skew normal ANCOVA model leads to the more efficient estimations of the model parameters than the traditional models.

The rest of paper is structured as follows. In section 2, we give some brief preliminaries and necessary background about the concept of ANCOVA and its formulation. The skew normal ANCOVA model is developed in section 3. We provide the ML estimates of the model parameters and their adjusted counterparts via EM algorithm. In section 4, we construct the asymptotic confidence intervals for the model parameters. A simulation study is performed to assess the performance of the proposed model, in section 5. In section 6, a real data set is analyzed to explain the proposed methodology.

2. Preliminaries and Notations

The aim of ANCOVA is to explore possible relation between a response variable and a set of treatments and covariates. Consider a balanced complete randomized design with t treatments and r replications. We treat the balanced design to avoid cluttered notations, but the problem can be cast in general unbalanced design, as it is explained by Meshkani *et. al.* [19]. In the simplest case, an ANCOVA model with a covariate and a two-level factor is given by

$$\boldsymbol{E}[Y_{ij}|\boldsymbol{x},\boldsymbol{z}] = \beta_0 + \beta_i + \gamma(z_{ij} - \bar{z}_i) \qquad i = 1, \ldots, t, j = 1, \ldots, r, \tag{2.1}$$

where β_0 shows the intercept term and $\beta_i, i = 1, \ldots, t$ denote the factor effects which satisfy the constraint $\sum_{i=1}^{t} \beta_i = 0$. The vector of model parameters is

$$\boldsymbol{\theta} = (\boldsymbol{\beta}', \boldsymbol{\gamma}')' = (\beta_0, \beta_1, \ldots, \beta_{t-1}, \gamma)'.$$

In model (2.1) the regression equation of Y on Z has a fixed slope γ for all treatments. If the slopes of the regression model for different treatments is not the same, the ANCOVA model would be of the form

$$\boldsymbol{E}[Y_{ij}|\boldsymbol{x}, \boldsymbol{z}] = \beta_0 + \beta_i + \gamma_i(z_{ij} - \bar{z}_i) \quad i = 1, \ldots, t, j = 1, \ldots, r,$$

therefore there is a vector of slopes $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_t)'$. In general case, an ANCOVA model can be written as

$$\boldsymbol{E}(Y_{11}, \ldots, Y_{tr}) = \boldsymbol{X\beta} + \boldsymbol{Z\gamma} = \boldsymbol{W\theta}, \tag{2.2}$$

where $\boldsymbol{X}$ denotes the design matrix, $\boldsymbol{Z}$ includes the observed covariates, $\boldsymbol{W} = [\boldsymbol{X}, \boldsymbol{Z}]$ and $\boldsymbol{\theta} = (\boldsymbol{\beta}', \boldsymbol{\gamma}')'$ with $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_{t-1})'$, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_q)'$ and $p = t + q$. For example, in model (2.1) we have

$$\boldsymbol{W} = \left[\begin{array}{ccccc|c} \mathbf{1}_r & \mathbf{1}_r & \mathbf{0}_r & \ldots & \mathbf{0}_r & \tilde{\boldsymbol{z}}_1 \\ \mathbf{1}_r & \mathbf{0}_r & \mathbf{0}_r & \ldots & \mathbf{0}_r & \tilde{\boldsymbol{z}}_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{1}_r & \mathbf{0}_r & \mathbf{0}_r & \ldots & \mathbf{1}_r & \tilde{\boldsymbol{z}}_{t-1} \\ \mathbf{1}_r & -\mathbf{1}_r & -\mathbf{1}_r & \ldots & -\mathbf{1}_r & \tilde{\boldsymbol{z}}_t \end{array}\right] = [\boldsymbol{X}|\boldsymbol{Z}] \tag{2.3}$$

where $\mathbf{1}_r = (1, \ldots, 1)'$, $\mathbf{0}_r = (0, \ldots, 0)'$ and $\tilde{\boldsymbol{z}}_i = ((z_{i1} - \bar{z}_i), \ldots, (z_{ij} - \bar{z}_i), \ldots, (z_{ir} - \bar{z}_i))$. As it can be clearly seen, in an ANCOVA model the relationship between the mean of a response variable and treatments and covariates is determined by the structure of design matrix $\boldsymbol{X}$ and covariate matrix $\boldsymbol{Z}$. For model (2.2) the design matrix, $\boldsymbol{X}$, and the vector of treatments effects, $\boldsymbol{\beta}$, are the same as model (2.1), but the matrix of observed covariates is given by

$$\boldsymbol{Z} = \left[\begin{array}{ccccc} \tilde{\boldsymbol{z}}_1 & \mathbf{0}_r & \ldots & \mathbf{0}_r & \mathbf{0}_r \\ \mathbf{0}_r & \tilde{\boldsymbol{z}}_2 & \ldots & \mathbf{0}_r & \mathbf{0}_r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_r & \mathbf{0}_r & \ldots & \tilde{\boldsymbol{z}}_{t-1} & \mathbf{0}_r \\ \mathbf{0}_r & \mathbf{0}_r & \ldots & \mathbf{0}_r & \tilde{\boldsymbol{z}}_t \end{array}\right].$$

The unbalanced form of ANCOVA models can also be represented by the general form given in equation (2.2). Considering the constraint $\sum_{i=1}^{t} r_i\beta_i = 0$, it would suffice to replace the $-\frac{1}{r_t}(r_1, \ldots, r_{t-1})$ for $-\mathbf{1}_r$ in the last row of the matrix $\boldsymbol{W}$ where $r_i, , i = 1, \ldots, t$, denotes the number of replications for i-th treatment. For Other common designs such as split-plot, Latin squares, Greco-Latin etc., the modeling method is similar, i.e., the design matrix and the covariate matrix can be written in the general form of $\boldsymbol{W} = [\boldsymbol{X}, \boldsymbol{Z}]$. It should be noted that the constraint $\sum_{i=1}^{t} \beta_i = 0$ has been absorbed into the design matrix $\boldsymbol{W}$. More details and examples about other common designs can be found in [19]

Considering the general formulation of an ANCOVA model given in equation (2.2), the main goal of ANOCVA is to estimate the vector of parameters $\boldsymbol{\theta}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ using the vector of responses $\boldsymbol{y} = (y_{11}, \ldots, y_{tr})$ and the matrix of observations $\boldsymbol{W}$. In what follows we follow the notations of [19].

3. The model and parameter estimation

In this section, we develop an ANCOVA model under the assumption of skew normal distribution for response variable. Consider the general form of an ANCOVA model given in equation (2.2). Let the skew normal ANCOVA model be

$$Y_{ij}|\boldsymbol{w}_{ij} \sim SSN(\boldsymbol{w}_{ij}\boldsymbol{\theta} - \sqrt{\frac{2}{\pi}}\lambda, \sigma^2, \lambda) \quad i = 1, ..., s;\ j = 1, ..., r, \tag{3.1}$$

where s is the number of treatments, r is the number of replications, $\boldsymbol{w}_{ij} = (\boldsymbol{x}_i, z_{ij})$ denotes the ij-th row of matrix $\boldsymbol{W}$ and $SSN(\mu, \sigma^2, \lambda)$ denotes the Sahu skew normal distribution with location parameter μ, scale parameter σ^2 and skewness parameter λ, given by

$$f_{Y_{ij}}(y|\mu, \sigma^2, \lambda) = 2\phi(y; \mu - \sqrt{\frac{2}{\pi}}\lambda, \sigma^2 + \lambda^2)\Phi(\frac{\lambda}{\sigma}\frac{(y-\mu)}{(\sigma^2+\lambda^2)^{\frac{1}{2}}}), \tag{3.2}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote, respectively, the density and cumulative distribution function of the normal distribution. The likelihood function of the model (3.1) is

$$\begin{aligned} L(\boldsymbol{\theta}, \lambda, \sigma^2|\boldsymbol{y}, \boldsymbol{W}) &= \prod_{i=1}^{s}\prod_{j=1}^{r} f_{Y_{ij}}(y_{ij}|\boldsymbol{\theta}, \lambda, \sigma^2) \\ &= \prod_{i=1}^{s}\prod_{j=1}^{r} 2\phi(y_{ij}; \boldsymbol{w}_{ij}\boldsymbol{\theta} - \sqrt{\frac{2}{\pi}}\lambda, \sigma^2 + \lambda^2)\Phi(\frac{\lambda}{\sigma}\frac{(y_{ij} - \boldsymbol{w}_{ij}\boldsymbol{\theta})}{(\sigma^2+\lambda^2)^{\frac{1}{2}}}). \end{aligned} \tag{3.3}$$

Due to the complexity of likelihood function (3.3) there are no explicit form for the ML estimators of the model parameters. Therefore, we provide an EM algorithm to compute the numerical values of the ML estimates. For this, it is necessary to formulate the problem in terms of a missing data problem. The skew normal ANCOVA model (3.1) can be written in a hierarchical structure as a mixture of normal and halfnormal distributions given by

$$\begin{cases} Y_{ij}|T_{ij} = t_{ij} \sim N(\boldsymbol{w}_{ij}\boldsymbol{\theta} + \lambda(t_{ij} - \sqrt{\frac{2}{\pi}}), \sigma^2) \\ T_{ij} \sim HN(0,1) \qquad\qquad\qquad\qquad i = 1, ..., s;\ j = 1, ..., r. \end{cases}$$

Therefore, considering $\{T_{ij}; i = 1, \ldots, s; j = 1, \ldots, r\}$ and $\{y_{ij}; i = 1, .., s; j = 1, ..., r\}$, respectively, as missing and incomplete data, the joint density of the complete data (y_{ij}, T_{ij}) is given by

$$\begin{aligned} f_{(Y_{ij},T_{ij})}(y_{ij}, t_{ij}) &= f_{Y_{ij}|T_{ij}=t_{ij}}(y_{ij}) \times g_{T_{ij}}(t_{ij}) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2}(y_{ij} - \boldsymbol{w}_{ij}\boldsymbol{\theta} - \lambda(t_{ij} - \sqrt{\frac{2}{\pi}}))^2 \right\} \\ &\times \sqrt{\frac{2}{\pi}} \exp\left\{ -\frac{t_{ij}^2}{2} \right\}. \end{aligned}$$

Hence, the complete data likelihood and log-likelihood functions are obtained to be

$$\begin{aligned} L_c(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W}, \boldsymbol{t}) &= \prod_{i=1}^{s}\prod_{j=1}^{r} f_{(Y_{ij},T_{ij})}(y_{ij}, t_{ij}) \\ &= \exp\Bigg\{ -\frac{1}{2\sigma^2}\sum_{i=1}^{s}\sum_{j=1}^{r}\Bigg[(y_{ij} - \boldsymbol{w}_{ij}\boldsymbol{\theta})^2 - 2\lambda(y_{ij} - \boldsymbol{w}_{ij}\boldsymbol{\theta} \\ &+ \lambda\sqrt{\frac{2}{\pi}})t_{ij} + (\lambda^2 + \sigma^2)t_{ij}^2 + 2\lambda\sqrt{\frac{2}{\pi}}(y_{ij} - \boldsymbol{w}_{ij}\boldsymbol{\theta}) + \lambda^2\frac{2}{\pi}\Bigg]\Bigg\} \end{aligned}$$

$$\times \quad (\pi)^{-sr}(\sigma^2)^{-\frac{sr}{2}}$$

and

$$\begin{aligned}\ell_c(\boldsymbol{\theta},\sigma^2,\lambda|\boldsymbol{y},\boldsymbol{W},\boldsymbol{t}) &= -\frac{1}{2\sigma^2}\Big[(\boldsymbol{y}-\boldsymbol{W\theta})'(\boldsymbol{y}-\boldsymbol{W\theta}) - 2\lambda(\boldsymbol{y}-\boldsymbol{W\theta}+\lambda\sqrt{\frac{2}{\pi}}\mathbf{1})'\boldsymbol{t} \\ &+ (\lambda^2+\sigma^2)(\boldsymbol{t}^2)'\mathbf{1} + 2\lambda\sqrt{\frac{2}{\pi}}(\boldsymbol{Y}-\boldsymbol{W\theta})'\mathbf{1} + \frac{2sr\lambda^2}{\pi}\Big] \\ &- sr\log\pi - \frac{sr}{2}\log\sigma^2,\end{aligned}$$

respectively, where $\boldsymbol{t} = (t_{11},...,t_{sr})'$, $\boldsymbol{t}^2 = (t_{11}^2,...,t_{sr}^2)$ and $\mathbf{1}_{sr}$ denotes a $sr\times 1$ unit vector. To proceeds the EM algorithm, the conditional expectation of the complete data log-likelihood given incomplete data is obtained to be

$$\begin{aligned}E(\ell_c(\boldsymbol{\theta},\sigma^2,\lambda|\boldsymbol{y},\boldsymbol{W})) &= -sr\log\pi - \frac{sr}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\Big[(\boldsymbol{y}-\boldsymbol{W\theta})'(\boldsymbol{y}-\boldsymbol{W\theta}) \\ &- 2\lambda(\boldsymbol{y}-\boldsymbol{W\theta}+\lambda\sqrt{\frac{2}{\pi}}\mathbf{1}_{sr})'\hat{\boldsymbol{t}} + (\lambda^2+\sigma^2)(\hat{\boldsymbol{t}^2})'\mathbf{1}_{sr} \\ &+ 2\lambda\sqrt{\frac{2}{\pi}}(\boldsymbol{Y}-\boldsymbol{W\theta})'\mathbf{1}_{sr} + \frac{2sr\lambda^2}{\pi}\Big],\end{aligned} \tag{3.4}$$

where $\hat{t}$ and $\hat{t^2}$ denote the first and second order conditional moments of random variable $T_{ij}|y_{ij}$, respectively. Using the equations of the truncated normal moments (see for example, Barr *et al.* 1999), these moments are given by

$$\begin{aligned}\hat{t}_{ij} &= E(t_{ij}|\hat{\boldsymbol{\theta}},y_{ij}) = \eta_{ij}+\tau\delta_{ij}, \\ \hat{t^2}_{ij} &= E(t_{ij}^2|\hat{\boldsymbol{\theta}},y_{ij}) = \eta_{ij}^2+\tau^2+\tau\delta_{ij}\eta_{ij},\end{aligned}$$

where $\eta_{ij} = \frac{\lambda}{\sigma^2+\lambda^2}(y_{ij}-\boldsymbol{w}_{ij}\boldsymbol{\theta}+\sqrt{\frac{2}{\pi}}\lambda)$, $\tau^2 = \frac{\sigma^2}{\sigma^2+\lambda^2}$ and $\delta_{ij} = \frac{\phi(\frac{\hat{\eta}_{ij}}{\hat{\tau}})}{\Phi(\frac{\hat{\eta}_{ij}}{\hat{\tau}})}$. The M-step of EM algorithm searches the parameter space to maximize the conditional expectation (3.4). Given the values of the parameters in k-th iteration of algorithm, the ML estimates of the parameters in (k+1)-th iteration are obtained as,

$$\begin{aligned}\hat{\boldsymbol{\theta}}^{(k+1)} &= (\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'\Big(\boldsymbol{y}+\hat{\lambda}^{(k)}(\sqrt{\frac{2}{\pi}}\mathbf{1}_{sr}-\hat{\boldsymbol{t}}^{(k)})\Big), \\ \hat{\sigma^2}^{(k+1)} &= \frac{1}{sr}\Big[(\boldsymbol{y}-\boldsymbol{W}\hat{\boldsymbol{\theta}}^{(k)})'(\boldsymbol{y}-\boldsymbol{W}\hat{\boldsymbol{\theta}}^{(k)}) \\ &- 2\hat{\lambda}^{(k)}(\boldsymbol{y}-\boldsymbol{W}\hat{\boldsymbol{\theta}}^{(k)}+\hat{\lambda}^{(k)}\sqrt{\frac{2}{\pi}}\mathbf{1}_{sr})'\hat{\boldsymbol{t}}^{(k)} + \hat{\lambda^2}^{(k)}(\boldsymbol{t}^2)'\mathbf{1} \\ &+ 2\lambda\sqrt{\frac{2}{\pi}}(\boldsymbol{Y}-\boldsymbol{W\theta})'\mathbf{1} + \frac{2sr\lambda^2}{\pi}\Big], \\ \hat{\lambda}^{(k+1)} &= \Big(\sqrt{\frac{2}{\pi}}\mathbf{1}_{sr}'(\boldsymbol{y}-\boldsymbol{W}\hat{\boldsymbol{\theta}}^{(k)}) - (\boldsymbol{y}-\boldsymbol{W}\hat{\boldsymbol{\theta}}^{(k)})'\hat{\boldsymbol{t}}^{(k)}\Big) \\ &\times \Big(\mathbf{1}_{sr}'(2\sqrt{\frac{2}{\pi}}\hat{\boldsymbol{t}}^{(k)} - \hat{\boldsymbol{t}^2}^{(k)}) - sr\frac{2}{\pi}\Big)^{-1}.\end{aligned}$$

The E and M steps are repeated alternately until a convergence rule holds.

3.1. Adjusted effects. As it can be clearly seen, from equation (2.2), there are two types of parameters in an ANCOVA model. The first type, denoted by $\boldsymbol{\beta}$, corresponds to the treatment effects. Whereas the second type, denoted by $\boldsymbol{\gamma}$, corresponds to covariate effects. The EM algorithm expounded in previous section, provides the ML estimator for the vector of parameters, $\boldsymbol{\theta}$, without separating these two types of parameters. We may be interested in estimating either the effects of treatments adjusted for the effects of covariates or the effects of covariates adjusted for the effects of treatments. In this section, we provide the adjusted estimators for both covariates and treatments effects. For this purpose, we rewrite the equation (3.4) by using the equality of $\boldsymbol{w}_{ij}\boldsymbol{\theta} = \boldsymbol{x}_i\boldsymbol{\beta} + \boldsymbol{z}_{ij}\boldsymbol{\gamma}$, as:

$$\begin{aligned} E(\ell_c(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{\beta}, \boldsymbol{\gamma})) \quad &\propto \quad \sum_{i=1}^{s}\sum_{j=1}^{r}\Big(y_{ij} - (\boldsymbol{x}_i\boldsymbol{\beta} + \boldsymbol{z}_{ij}\boldsymbol{\gamma})\Big)^2 \\ &- \quad 2\lambda\sum_{i=1}^{s}\sum_{j=1}^{r}\Big(y_{ij} - \boldsymbol{x}_i\boldsymbol{\beta} - \boldsymbol{z}_{ij}\boldsymbol{\gamma}\Big)\hat{t}_{ij} - 2\lambda^2\sqrt{\frac{2}{\pi}}\sum_{i=1}^{s}\sum_{j=1}^{r}\hat{t}_{ij}, \\ &+ \quad \lambda^2\sum_{i=1}^{s}\sum_{j=1}^{r}\hat{t^2}_{ij} + 2\lambda\sqrt{\frac{2}{\pi}}\sum_{i=1}^{s}\sum_{j=1}^{r}\Big(y_{ij} - \boldsymbol{x}_i\boldsymbol{\beta} - \boldsymbol{z}_{ij}\boldsymbol{\gamma}\Big) + sr\lambda^2\frac{2}{\pi} \\ &\propto \quad \boldsymbol{y}'\boldsymbol{y} - 2\boldsymbol{y}'(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{\gamma}) + (\boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{X}\boldsymbol{\beta}) + 2(\boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{Z}\boldsymbol{\gamma}) \\ &- \quad 2\lambda^2\sqrt{\frac{2}{\pi}}\boldsymbol{1}'_{sr}\hat{\boldsymbol{t}} + \lambda^2\boldsymbol{1}'_{sr}\widehat{\boldsymbol{t}^2} + 2\lambda\sqrt{\frac{2}{\pi}}\boldsymbol{1}'_{sr}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{Z}\boldsymbol{\gamma}) \\ &+ \quad (\boldsymbol{Z}\boldsymbol{\gamma})'(\boldsymbol{Z}\boldsymbol{\gamma}) + sr\lambda^2\frac{2}{\pi}. \end{aligned} \tag{3.5}$$

Equating the first order derivations of (3.5) with respect to the model parameters $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ to zero, leads to the following system of equations:

$$\begin{cases} \dfrac{\partial(E(\ell_c(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{\beta}, \boldsymbol{\gamma}))}{\partial\boldsymbol{\beta}} = -2\boldsymbol{X}'\boldsymbol{y} + 2\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} + 2\boldsymbol{X}'\boldsymbol{Z}\boldsymbol{\gamma} - 2\lambda\sqrt{\tfrac{2}{\pi}}\boldsymbol{X}'\boldsymbol{1}_{sr} = \boldsymbol{0} \\ \dfrac{\partial(E(\ell_c(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{\beta}, \boldsymbol{\gamma}))}{\partial\gamma} = -2\boldsymbol{Z}'\boldsymbol{y} + 2\boldsymbol{Z}'\boldsymbol{X}\boldsymbol{\beta} + 2\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\gamma} - 2\lambda\sqrt{\tfrac{2}{\pi}}\boldsymbol{Z}'\boldsymbol{1}_{sr} = \boldsymbol{0}. \end{cases}$$

Therefore, the adjusted ML estimators of treatments and covariates effects in $k-th$ iteration of the EM algorithm are obtained to be:

$$\hat{\boldsymbol{\beta}}^{(k+1)}_{ML.z} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'(\boldsymbol{y} - \boldsymbol{Z}\boldsymbol{\gamma}^{(k)}_{ML} + \lambda^{(k)}\sqrt{\frac{2}{\pi}}\boldsymbol{1}_{sr}),$$

$$\hat{\boldsymbol{\gamma}}^{(k+1)}_{ML.x} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}^{(\hat{k})}_{ML} + \lambda^{(k)}\sqrt{\frac{2}{\pi}}\boldsymbol{1}_{sr}).$$

Note that, as it is well known, in the proper regression analysis each regression coefficient shows the effect of corresponding explanatory variable on the response variable, given all other explanatory variables (qualitative and quantitative) are kept fixed. But in ANCOVA, one needs the treatment effects for the situation that only the whole set of quantitative variables, i.e., covariates are kept fixed. Moreover, the partition used in ANCOVA is dictated by the context of each special experiment. For example, some experiments may have no covariate thus no partition is considered and some may have one or more covariates whose effects should be removed from the treatment effects. Thus, there is a natural partition of treatments and covariates correspond to each problem.

4. Asymptotic Confidence Intervals

To construct exact confidence intervals for the model parameters requires exact knowledge of the sampling distribution of the ML estimators. Due to the complexity of these estimators, derivation of their exact distributions is a challenging problem, if it be feasible at all. Therefore, in this section we use the asymptotic distributions of these estimators to construct asymptotic confidence intervals for the model parameters. The results of this section are valid when r or equivalently $n(= tr)$ goes to infinity.
Consider the skew normal ANCOVA model (3.1). Let

$$\ell_{ij}(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W}) = \log 2 - \frac{1}{2}\log 2\pi - \frac{1}{2}\log a - \frac{b_{ij}}{2} + \log\Phi(k_{ij}),$$

with $a = \sigma^2 + \lambda^2$ and

$$\begin{aligned} b_{ij} &= \frac{\left(y_{ij} - \boldsymbol{w}_{ij}\boldsymbol{\theta} + \sqrt{\frac{2}{\pi}}\lambda\right)^2}{\sigma^2 + \lambda^2}, \\ k_{ij} &= \frac{\lambda}{\sigma^{\frac{1}{2}}}\left(y_{ij} - \boldsymbol{w}_{ij}\boldsymbol{\theta} + \sqrt{\frac{2}{\pi}}\lambda\right). \end{aligned}$$

Then, the log-likelihood function of the model is given by

$$\ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W}) = \sum_{i=1}^{s}\sum_{j=1}^{r} \ell_{ij}(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W}).$$

The first order partial derivations of the log-likelihood function with respect to the model parameters $\boldsymbol{\theta}$, σ^2 and λ are given by

$$\begin{aligned} \frac{\partial \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial \boldsymbol{\theta}} &= \sum_{i=1}^{s}\sum_{j=1}^{r} \frac{\partial \ell_{ij}(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial \boldsymbol{\theta}}, \\ \frac{\partial \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial \sigma^2} &= \sum_{i=1}^{s}\sum_{j=1}^{r} \frac{\partial \ell_{ij}(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial \sigma^2}, \\ \frac{\partial \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial \lambda} &= \sum_{i=1}^{s}\sum_{j=1}^{r} \frac{\partial \ell_{ij}(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial \lambda}, \end{aligned}$$

respectively, where

$$\begin{aligned} \frac{\partial \ell_{ij}(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial \boldsymbol{\theta}} &= \frac{y_{ij} - \boldsymbol{w}_{ij}\boldsymbol{\theta} + \sqrt{\frac{2}{\pi}}}{a}\boldsymbol{w}'_{ij} - \delta_{\Phi(k_{ij})}\frac{\lambda}{\sigma\sqrt{a}}\boldsymbol{w}'_{ij}, \\ \frac{\partial \ell_{ij}(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial \sigma^2} &= -\frac{1}{2a} + \frac{b_{ij}}{2a} - \delta_{\Phi(k_{ij})}\frac{k_{ij}(2\sigma^2 + \lambda^2)}{\sigma^2 a}, \\ \frac{\partial \ell_{ij}(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial \lambda} &= -\frac{\lambda}{a} - \frac{1}{a^2}\{a\sqrt{\frac{2}{\pi}}(y_{ij} - \boldsymbol{w}_{ij}\boldsymbol{\theta} + \sqrt{\frac{2}{\pi}}\lambda) - \lambda b_{ij}\} \\ &+ \delta_{\Phi(k_{ij})}k_{ij}(1 - \frac{\lambda^2}{a}) + \frac{\lambda}{\sigma\sqrt{a}}\sqrt{\frac{2}{\pi}}, \end{aligned}$$

and $\delta_{\Phi(u)} = \dfrac{\phi(u)}{\Phi(u)}$. The second order derivations of the log-likelihood function with respect to the parameters are similarly given by

$$\frac{\partial^2 \ell_{ij}}{\partial \boldsymbol{\nu}\partial \boldsymbol{\xi}'} = -\frac{1}{2}\frac{\partial^2 \log a}{\partial \boldsymbol{\nu}\partial \boldsymbol{\xi}'} - \frac{1}{2}\frac{\partial^2 b_{ij}}{\partial \boldsymbol{\nu}\partial \boldsymbol{\xi}'} + \frac{\partial^2 \log \Phi(k_{ij})}{\partial \boldsymbol{\nu}\partial \boldsymbol{\xi}'},$$

where $\boldsymbol{\nu}$ represents the parameters $\boldsymbol{\theta}, \sigma^2$ or λ and

$$\begin{aligned} \frac{\partial^2 \log \Phi(k_{ij})}{\partial \boldsymbol{\nu}\partial \boldsymbol{\xi}'} &= \delta_{\Phi}(k_{ij})\Big(\frac{\partial^2 k_{ij}}{\partial \boldsymbol{\nu}\partial \boldsymbol{\xi}'}\Big) + \Delta_{\Phi}(k_{ij})\Big(\frac{\partial k_{ij}}{\partial \boldsymbol{\nu}}\Big)\Big(\frac{\partial k_{ij}}{\partial \boldsymbol{\xi}}\Big)' \\ \frac{\partial^2 \log a}{\partial \lambda^2} &= 2\frac{(a - 2\lambda^2)}{a^2}, \quad \frac{\partial^2 b_{ij}}{\partial \lambda^2} = \frac{4}{\pi a} - 8\frac{\sigma k_{ij}}{\lambda a^{\frac{3}{2}}} + 2\frac{b_{ij}}{a^2}(4\lambda - a), \\ \frac{\partial^2 b_{ij}}{\partial \lambda \partial \sigma^2} &= 2\sqrt{\frac{2}{\pi}}\frac{\sigma k_{ij}}{\lambda a^{\frac{3}{2}}} + 4\frac{\lambda}{a^2}b_{ij}, \quad \frac{\partial^2 b_{ij}}{\partial {\sigma^2}^2} = -2\frac{b_{ij}}{a^2}, \\ \frac{\partial^2 b_{ij}}{\partial \lambda \partial \boldsymbol{\theta}} &= 2(\frac{k_{ij}}{a^{\frac{3}{2}}} - \sqrt{\frac{2}{\pi}}\frac{1}{a})\boldsymbol{w}'_{ij}, \quad \frac{\partial^2 b_{ij}}{\partial \sigma^2 \partial \boldsymbol{\theta}} = \frac{\sigma k_{ij}}{\lambda a^{\frac{3}{2}}}\boldsymbol{w}'_{ij}, \\ \frac{\partial^2 b_{ij}}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}'} &= \frac{2}{a}\boldsymbol{w}'_{ij}(\boldsymbol{w}_{ij}), \quad \frac{\partial^2 k_{ij}}{\partial \lambda \partial \boldsymbol{\theta}} = -\frac{\sigma}{a^{\frac{3}{2}}}\boldsymbol{w}'_{ij}, \\ \frac{\partial^2 k_{ij}}{\partial \lambda^2} &= 2\sqrt{\frac{2}{\pi}}\frac{\sigma}{a^{\frac{3}{2}}} - 3\frac{\sigma^2}{a^2}k_{ij} \end{aligned}$$

$$\frac{\partial^2 k_{ij}}{\partial\lambda\partial\sigma^2} = \frac{(2\sigma^2+\lambda^2)}{2\sigma a^{\frac{3}{2}}}(\frac{k_{ij}}{\lambda a^{\frac{1}{2}}} - \sqrt{\frac{2}{\pi}}\lambda) + \frac{\lambda k_{ij}}{a^2}(\frac{3}{2} - \frac{a}{\sigma^2}), \quad \frac{\partial^2 k_{ij}}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'} = 0.$$

with $\Delta_\phi(u) = \delta_{\Phi(u)}(u + \delta_{\Phi(u)})$. Therefore the Hessian matrix of model is obtained as:

$$\boldsymbol{H}(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W}) = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix},$$

where

$$\begin{aligned} h_{11} &= \frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}, \quad h_{22} = \frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial\lambda^2}, \quad h_{33} = \frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial(\sigma^2)^2}, \\ h_{12} &= h_{21} = \frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial\boldsymbol{\theta}\partial\lambda}, \quad h_{13} = h_{31} = \frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial\boldsymbol{\theta}\partial\sigma^2}, \\ h_{23} &= h_{32} = \frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial\sigma^2\partial\lambda}. \end{aligned}$$

Consequently, the Fisher information matrix of model is given by

$$\boldsymbol{I}(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W}) = -H(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W}) = \begin{pmatrix} \boldsymbol{I}(\boldsymbol{\theta}) & \boldsymbol{I}(\boldsymbol{\theta}.\lambda) & \boldsymbol{I}(\sigma^2, \boldsymbol{\theta}) \\ \boldsymbol{I}(\boldsymbol{\theta}, \lambda) & \boldsymbol{I}(\lambda) & \boldsymbol{I}(\sigma^2, \lambda) \\ \boldsymbol{I}(\sigma^2, \boldsymbol{\theta}) & \boldsymbol{I}(\sigma^2, \lambda) & \boldsymbol{I}(\sigma^2) \end{pmatrix},$$

where

$$\begin{aligned} \boldsymbol{I}(\boldsymbol{\theta}) &= -\boldsymbol{E}(\frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}) \\ &= -\frac{1}{a}\boldsymbol{W}'\boldsymbol{W} + \boldsymbol{E}\left(\frac{\phi(k_{ij})}{\Phi(\boldsymbol{k})}(k_{ij} + \frac{\phi(k_{ij})}{\Phi(k_{ij})})\right) - \frac{\lambda}{\sigma\sigma^2 a}\boldsymbol{W}'\boldsymbol{W}, \\ \boldsymbol{I}(\lambda) &= -\boldsymbol{E}(\frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial\lambda^2}) = -\frac{a - 2\lambda^2}{a^2} \\ &- \left(\frac{2}{\pi a} - \frac{4}{a^2}\sqrt{\frac{2}{\pi}}\lambda + \frac{(4-a)}{a^2}\frac{1}{\sigma^2+\lambda^2}(\sigma^2 + \frac{2\lambda^2}{\pi})\right) \\ &+ \boldsymbol{E}\Big(\frac{\phi(k_{ij})}{\Phi(k_{ij})}(2\sqrt{\frac{2}{\pi}}\frac{\sigma}{a^{\frac{3}{2}}}) - 3\frac{\sigma^2}{a^2}k_{ij}\frac{\phi(k_{ij})}{\Phi(k_{ij})}(k_{ij} + \frac{\phi(k_{ij})}{\Phi(k_{ij})})k_{ij}\Big(1 - \frac{\lambda^2}{a}\Big) \\ &+ \frac{\lambda}{\sigma a^{\frac{1}{2}}}\sqrt{\frac{2}{\pi}}\Big), \\ \boldsymbol{I}(\sigma^2) &= -\boldsymbol{E}(\frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial\sigma^2}), \\ \boldsymbol{I}(\sigma^2, \lambda) &= -\boldsymbol{E}(\frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial\sigma^2\partial\lambda}), \\ \boldsymbol{I}(\sigma^2, \boldsymbol{\theta}) &= -\boldsymbol{E}(\frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial\sigma^2\partial\boldsymbol{\theta}}), \\ \boldsymbol{I}(\boldsymbol{\theta}, \lambda) &= -\boldsymbol{E}(\frac{\partial^2 \ell(\boldsymbol{\theta}, \sigma^2, \lambda|\boldsymbol{y}, \boldsymbol{W})}{\partial\boldsymbol{\theta}\partial\lambda}). \end{aligned}$$

Now, one can use the inverse of expected Fisher information matrix to approximate the variance of ML estimators. Thus, the asymptotic distributions of ML estimators and asymptotic confidence intervals for the model parameters are given by

$$\begin{aligned} \hat{\boldsymbol{\theta}}_{ML} &\sim AN(\boldsymbol{\theta}, \boldsymbol{I}^{-1}(\boldsymbol{\theta})), \\ \hat{\sigma^2}_{ML} &\sim AN(\sigma^2, \boldsymbol{I}^{-1}(\sigma^2)), \\ \hat{\lambda}_{ML} &\sim AN(\lambda, \boldsymbol{I}^{-1}(\lambda)), \end{aligned}$$

and

$$(\hat{\boldsymbol{\theta}}_{ML} - Z_{1-\frac{\alpha}{2}}\sqrt{\boldsymbol{I}^{-1}(\hat{\boldsymbol{\theta}}_{ML})} \quad , \quad \hat{\boldsymbol{\theta}}_{ML} + Z_{1-\frac{\alpha}{2}}\sqrt{\boldsymbol{I}^{-1}(\hat{\boldsymbol{\theta}}_{ML})}),$$

$$(\hat{\sigma^2}_{ML} - Z_{1-\frac{\alpha}{2}}\sqrt{\boldsymbol{I}^{-1}(\hat{\sigma^2}_{ML})} \quad , \quad \hat{\sigma^2}_{ML} + Z_{1-\frac{\alpha}{2}}\sqrt{\boldsymbol{I}^{-1}(\hat{\sigma^2}_{ML})}),$$

$$(\hat{\lambda}_{ML} - Z_{1-\frac{\alpha}{2}}\sqrt{\boldsymbol{I}^{-1}(\hat{\lambda}_{ML})} \quad , \quad \hat{\sigma^2}_{ML} + Z_{1-\frac{\alpha}{2}}\sqrt{\boldsymbol{I}^{-1}(\hat{\lambda}_{ML})}), \tag{4.1}$$

respectively. Notice that in (4.1) we substituted the ML estimates of parameters in Fisher information matrix to estimate it. According to the large sample theory results, if $\boldsymbol{I}(\xi)$ is a continuous function of ξ, as it is typically the case, then $\boldsymbol{I}(\hat{\theta}_{ML})$ is a consistent estimator of $\boldsymbol{I}(\theta_{ML})$. See, for example, Lehmann [17], p.p. 525, for more details.

5. Simulation Study

In this section, we perform a simulation study to assess the efficiency of the ML estimators of the model parameters for the proposed model. We consider an ANCOVA model with a covariate and a two-level treatment of the form

$$\mu_{ij} = E(y_{ij}|\boldsymbol{X}, \boldsymbol{Z}) = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_i + \boldsymbol{\gamma}(\boldsymbol{z}_{ij} - \bar{\boldsymbol{z}}_i), \tag{5.1}$$

with $i = 1, 2$ and $j = 1, \ldots, r$. The values of the model parameters are set to be $\beta_0 = 2, \beta_1 = 5$ and $\gamma = 1$. The covariate values are simulated from normal distribution. Then, the values of $\mu_{ij}, i = 1, 2, j = 1, \ldots, r$ are computed using the equality of $\mu_{ij} = \boldsymbol{w}_{ij}\boldsymbol{\theta}$. Finally, the response variable observations $\{y_{ij}, i = 1, 2; j = 1, ..., r\}$ are simulated from $SSN(\boldsymbol{w}_{ij}\boldsymbol{\theta} - \sqrt{\frac{2}{\pi}}, \sigma^2, \lambda)$. In order to evaluate the effect of sample size, $n = sr$, on efficiency of the ML estimators, we consider the number of replications, r, to be $\{10, 25, 50, 100\}$.

Also, to assess the ability of the proposed model for modeling observations with both symmetric and asymmetric structures, we consider different values for the skewness parameter as $\{-2, -1, 0, 1, 2\}$. Taking these considerations into account, the values of the root mean square error (RMSE) for the ML estimators of the model parameters are computed and presented in Table 1. We also provide the corresponding values of the ML estimators under normal distribution as the usual traditional ANCOVA model in order to compare and evaluate the robustness of different models against violation from normality. The number of repetitions in simulations fixed to be 5000 in order to take into account the uncertainty in random number generating procedure. As it is expected, for $\lambda = 0$, there are no significant differences between the values of RMSE for normal and skew normal ANCOVA models. This is because for $\lambda = 0$ the skew normal distribution reduces to normal distribution. For positive and negative values of the skewness parameter, which respectively correspond to the right-skewed and left-skewed data, the skew normal model provides more efficiency (in terms of smaller RMSE) than the normal model because it truly takes into account the skewed structure of data. Obviously, due to the asymptotic optimality of ML estimators, the efficiency of estimators for both normal and skew normal models increase when the sample size increases.

6. Real Example

To illustrate the proposed methodology and to evaluate its applicability, we provide a real example in this section. Table 2 shows the salary data for 58 employees in a company in Iran by the level of proficiency and working experience. Our aim is to find the possible relation between salary as the response variable and working experience as a covariate for different levels of the proficiency factor. Therefore, we consider an ANCOVA model with a covariate and a two-level treatment as

$$\mu_{ij} = E(y_{ij}|\boldsymbol{X}, \boldsymbol{Z}) = \beta_0 + \beta_i + \gamma(\boldsymbol{z}_{ij} - \bar{\boldsymbol{z}}_i),$$

Table 1. The values of RMSE for ML estimators of the model parameters.

		ANCOVA Model						
		Skew Normal				Normal		
λ	Sample Size	λ	β_0	β_1	γ	β_0	β_1	γ
-2	20	4.5209	0.9088	4.7756	27.5127	1.0760	5.0431	27.6104
	50	4.6625	0.9106	4.6961	22.2627	1.0288	5.0194	22.9811
	100	4.4520	0.9126	4.7334	12.2603	1.0186	5.0084	12.6771
	200	3.9090	0.9206	4.8485	10.1249	1.0077	5.0029	9.6656
-1	20	3.5050	0.6819	4.7952	18.3095	1.0264	5.0178	18.3095
	50	1.1328	0.6630	4.9245	10.9095	1.0105	5.0063	11.4319
	100	3.7464	0.6995	4.7370	8.2709	1.0068	5.0068	8.6453
	200	3.4097	0.7170	4.7982	5.7749	1.0017	5.0040	6.1442
0	20	$< 1 \times 10^{-13}$	1.0098	5.0072	10.5319	1.0098	5.0072	10.5319
	50	$< 1 \times 10^{-13}$	1.0055	5.0018	7.5567	1.0055	5.0018	7.5567
	100	$< 1 \times 10^{-13}$	1.0036	5.0013	5.4093	1.0036	5.0013	5.4093
	200	$< 1 \times 10^{-13}$	1.0019	4.9981	3.6826	1.0019	4.9981	3.6826
1	20	1.4593	0.6506	4.8993	18.1243	1.0292	5.0137	19.0754
	50	1.1328	0.6630	4.9245	10.9095	1.0105	5.0063	11.4319
	100	0.7788	0.6711	4.9595	7.4625	1.0067	5.0032	7.8143
	200	0.9599	0.9989	4.9984	6.1875	1.0037	5.0002	6.2877
2	20	2.4142	0.6868	4.6382	29.2781	1.0835	5.0349	36.4821
	50	2.9663	0.6086	4.3624	15.1047	1.0414	4.9991	19.1748
	100	3.3649	0.5308	4.1750	9.6994	1.0131	5.0090	12.9511
	200	3.8389	0.4401	3.8845	7.6109	1.0100	5.0012	10.2676

with $i = 1, 2$ and $j = 1, \ldots, 29$. The histogram and box plot of the response variable observations presented in Figure 1, indicate unimodality and right-skewed structure of data. The result of goodness-of-fit tests indicate that skew normal, lognormal and inverse gaussian distributions could be fitted to the response observations at the 5% significance level . The ML estimates of model parameters and their corresponding 95% asymptotic confidence intervals are presented in Table 3. We also provide the corresponding values for the ML estimators of the parameters for inverse Gaussian ANCOVA model, developed by [19], and lognormal distribution as other possible candidates for modeling a positively skewed data. The normal model is also considered to assess the effect of ignoring the skewness in modeling process. As it can be clearly seen, in skew normal, lognormal and inverse Gaussian ANCOVA models both the effects of covariate and proficiency factor are significant. While the normal model incorrectly indicates that the working experience is not a significant covariate. The negative log-likelihood values along with AIC and BIC criteria for different models are provided in Table 4. Notice that as it is pointed out by [19] the regression coefficients and factor effects are not directly comparable for different models due to their different link functions. Therefore the predicted mean, $\hat{\boldsymbol{\mu}}$, under different models of interest can be compared via the root mean square error of prediction (RMSEP) criterion, presented in Table 4.

Table 2. Salary data for 58 employees in a company in Iran by the level of proficiency and working experience.

	Proficiency			
	Level I		Level II	
No.	Salary (1000,000 Rial)	Working Experience (Year)	Salary (1000,000 Rial)	Working Experience (Year)
1	13.21067	13	22.37932	16
2	17.06074	22	18.49741	11
3	15.59944	17	15.24213	10
4	14.71969	15	15.09468	11
5	15.36749	18	15.91385	10
6	16.06207	20	18.95504	15
7	19.89032	11	15.32940	9
8	13.40146	13	15.11344	8
9	11.32948	13	14.69974	10
10	16.11468	16	17.81728	11
11	12.02035	16	19.44098	17
12	11.43113	11	18.08340	13
13	15.06282	13	12.64286	4
14	14.66817	17	14.48483	8
15	13.40854	19	20.44149	14
16	14.63256	13	18.36701	12
17	16.38201	18	25.21042	22
18	13.41755	10	15.72097	10
19	13.49431	16	20.43385	14
20	19.67476	16	17.28177	11
21	12.79719	15	16.72111	11
22	13.80254	18	19.04832	12
23	17.15738	23	14.69331	8
24	16.31889	16	14.76271	10
25	12.91568	14	17.64608	12
26	12.68716	11	17.11542	10
27	11.24524	10	19.80369	17
28	14.62928	18	16.30396	7
29	17.96321	26	18.22075	6

Table 3. The ML estimates and 95% asymptotic confidence intervals of parameters for the skew normal ANCOVA model along with corresponding values for normal, lognormal and inverse Gaussian models.

	Parameters		
ANCOVA Model	β_0	β_1	γ
Normal	7.67	4.70	0.44
	(5.59,9.76)	(3.20,6.21)	(-0.07,0.96)
Lognormal	2.25	0.29	0.03
	(2.12,2.39)	(0.22,0.36)	(0.02,0.03)
Inverse Gaussian	0.063	0.005	-0.001
	(0.061,0.065)	(0.003,0.007)	(-0.002,-0.001)
Skew Normal	14.65	2.95	0.46
	(13.65,15.65)	(1.93,3.97)	(0.28,0.65)

It is seen that the skew normal model has better fit to the data than those of the other models. Moreover, the values of the RMSEP for different models indicate that the skew normal model leads to a model with higher predictive power than other models. Of course, it is clear that the main advantage of the skew normal model to lognormal and inverse Gaussian models is its applicability for symmetric, right-skewed and left-skewed

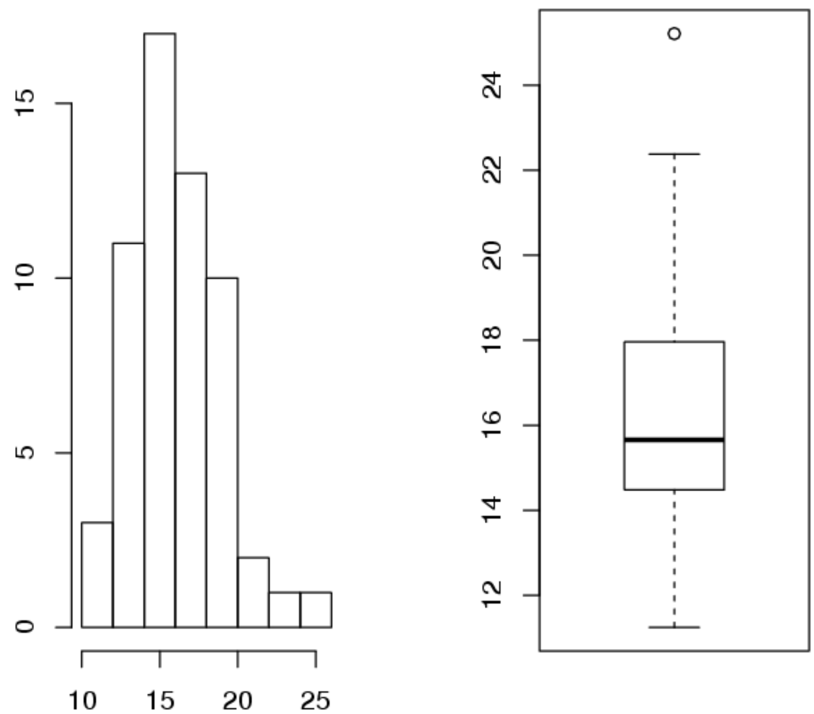

Figure 1. The histogram and box plot of the response variable observations.

Table 4. The values of negative log-likelihood, AIC and BIC and RMSEP criteria for skew normal ANCOVA model along with corresponding values for normal, lognormal and inverse Gaussian mode.

	Goodness-of-fit Criteria			
ANCOVA Model	-loglike	AIC	BIC	RMSEP
Normal	408.65	821.31	825.43	6.5931
Lognormal	291.40	588.80	594.98	5.2246
Inverse Gaussian	156.59	319.19	325.37	2.7994
Skew Normal	140.34	286.68	292.86	1.8062

data, whereas lognormal and inverse Gaussian models can be used only for analyzing right-skewed data.

7. Conclusions

In many real world applications the normality assumption does not hold. Too many researches have been recently focused to develop suitable methods for dealing with non-normality. Particularly, in many real world applications the response variable reflects a unimodal skewed structure. In these cases, the skew normal family of distributions due to its flexibility can be used for data analysis. The results show that in this situations the skew normal ANCOVA model leads to the more efficient estimations of the model parameters than the normal model. Moreover it is a considerably good rival for other traditional models such as lognormal and inverse Gaussian for analyzing skewed data. It is obvious that, the proposed ANCOVA model can be used when the data are symmetric, because the skew normal family of distribution includes the normal distribution as a special case. In this paper, we employed Sahu [21] skew normal distribution among other families of skew normal distributions due to its interesting distributional properties such as simple implementing of the EM algorithm. But other families of the skew normal distribution can be employed in a similar manner.

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A two-step approach to ratio and regression estimation of finite population mean using optional randomized response models

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Abstract

We propose a modified two-step approach for estimating the mean of a sensitive variable using an additive optional RRT model which allows respondents the option of answering a quantitative sensitive question directly without using the additive scrambling if they find the question non-sensitive. This situation has been handled before in Gupta et al. (2010) using the split sample approach. In this work we avoid the split sample approach which requires larger total sample size. Instead, we estimate the finite population mean by using an Optional Additive Scrambling RRT Model but the corresponding sensitivity level is estimated from the same sample by using the traditional Binary Unrelated Question RRT Model of Greenberg et al. (1969). The initial mean estimation is further improved by utilizing information from a non-sensitive auxiliary variable by way of ratio and regression estimators. Expressions for the *Bias* and *MSE* of the proposed estimators (correct up to first order approximation) are derived. We compare the results of this new model with those of the split-sample based Optional Additive RRT Model of Kalucha et al. (2015), Gupta et al. (2015) and the simple optional additive RRT Model of Gupta et al. (2010). We see that the regression estimator for the new model has the smallest *MSE* among all of the estimators considered here when they have the same sample size.

Keywords: Auxiliary Information, Mean square error, Optional randomized response technique, Ratio estimator, Regression estimator, Unrelated Question RRT Model

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1. Introduction

The randomized response technique of reducing respondent bias in obtaining answers to sensitive questions developed by Warner (1965) has been extended from the situation where response is categorical to that in which the response is quantitative. Choice of scrambling mechanism plays an important role in quantitative response models. Eichhron and Hayre (1983), Gupta and Shabbir (2004), Gupta et al. (2002, 2010), Wu et al. (2008) and many others have estimated the mean of a sensitive variable when the study variable is sensitive and no auxiliary information is available. While Eichhron and Hayre (1983) have used multiplicative scrambling, Gupta et al. (2010) have used additive scrambling in the context of optional randomized response models where a respondent provides a true response if he/she considers the question non-sensitive, and provides a scrambled response if the question is deemed sensitive. The researcher will not know which type of response has been provided. Sousa et al. (2010) and Gupta et al. (2012) suggested mean estimators based on full additive RRT models using an auxiliary variable. Kalucha et al. (2015) and Gupta et al. (2015) improved the mean estimators further by using optional additive RRT models which apart from estimating μ_Y (the mean of sensitive variable Y) also estimated W (the sensitivity level of the research question) using a split-sample approach. Recently Singh and Tarray (2014) have studied optional randomized response model in the stratified sampling setting.

The main motivation for the proposed model is to avoid the split sample approach which requires unnecessarily larger total sample sizes. We estimate the mean of the sensitive characteristic by using an Additive Optional RRT model but the corresponding sensitivity level is estimated from the same sample by using the Greenberg et al. (1969) model. This eliminates the need for split-sample approach that requires a larger total sample size.

Let μ_Y and σ_Y^2 be the unknown mean and variance of the sensitive variable Y, μ_X and σ_X^2 be the known mean and variance of the auxiliary variable X. Let W be the unknown sensitivity level of the survey question in the population.

2. The Split-Sample Model – Gupta et al. (2010)

Here the sample of size n is split into two sub-samples of sizes n_1 and n_2 $(n_1+n_2=n)$. Let S_1, S_2 be scrambling variables used in the two sub-samples. Let the mean and variance respectively of S_i $(i=1,2)$ be θ_i and $\sigma_{S_i}^2$. We assume that Y, X and S_i $(i=1,2)$ are mutually independent. For the i^{th} population unit $(i=1,2,\ldots,N)$, let y_i and x_i respectively be the values of the study variable Y and the auxiliary variable X. Moreover let $\bar{y}=\frac{\sum_1^n y_i}{n}$, $\bar{x}=\frac{\sum_1^n x_i}{n}$, $\bar{z}=\frac{\sum_1^n z_i}{n}$ be the sample means, and $\mu_Y=E(Y)$, $\mu_X=E(X)$ and $\mu_Z=E(Z)$ be the corresponding population means for Y, X and the scrambled response Z respectively. We assume that μ_X is known. In each sub sample, we will observe X directly but will only have an additively scrambled version of Y. According to this model, the reported response Z_i in the i^{th} sub-sample is given by

$$Z_i=\left\{\begin{array}{ll} Y & \text{with probability } (1-W) \\ (Y+S_i) & \text{with probability } W \end{array}\right\} \quad i=1,2$$

The mean and variance respectively for Z_i $(i=1,2)$ are given by

$$(2.1) \qquad E(Z_i)=\mu_Y+\theta_i W \quad \text{where } E(S_i)=\theta_i \ (i=1,2),$$

and

$$\sigma^2_{Z_i} = \sigma^2_Y + \sigma^2_{S_i} W + \theta_i^2 W(1-W) \tag{2.2}$$

It follows easily from (2.1) that for $\theta_1 \neq \theta_2$,

$$\mu_Y = \frac{\theta_2 E(Z_1) - \theta_1 E(Z_2)}{\theta_2 - \theta_1} \quad \text{and} \quad W = \frac{E(Z_2) - E(Z_1)}{(\theta_2 - \theta_1)}\,. \tag{2.3}$$

Hence if information on X is ignored, expressions in (2.3) lead to the following unbiased estimators of μ_Y and W:

$$\hat{\mu}_Y = \frac{\theta_2 \bar{z}_1 - \theta_1 \bar{z}_2}{\theta_2 - \theta_1},\ \theta_1 \neq \theta_2 \quad \text{and} \quad \hat{W} = \frac{\bar{z}_2 - \bar{z}_1}{(\theta_2 - \theta_1)},\ \theta_1 \neq \theta_2, \tag{2.4}$$

where $\bar{z}_1$, $\bar{z}_2$ respectively are the sample mean of reported responses in the two subsamples.

It can be verified that $\hat{\mu}_Y$ and $\hat{W}$ are unbiased estimators of the population mean μ_Y and the sensitivity level W. Variances of these estimators are given by

$$\text{Var}(\hat{\mu}_Y) = \frac{1}{(\theta_2 - \theta_1)^2}\left[\theta_2^2\left(\frac{1-f_1}{n_1}\right)\sigma^2_{Z_1} + \theta_1^2\left(\frac{1-f_2}{n_2}\right)\sigma^2_{Z_2}\right] \tag{2.5}$$

and

$$\text{Var}(\hat{W}) = \frac{1}{(\theta_2 - \theta_1)^2}\left[\left(\frac{1-f_1}{n_1}\right)\sigma^2_{Z_1} + \left(\frac{1-f_2}{n_2}\right)\sigma^2_{Z_2}\right],$$

where $\theta_1 \neq \theta_2$, $f_1 = \frac{n_1}{N}$, $f_2 = \frac{n_2}{N}$, $f = \frac{n}{N} = f_1 + f_2$,

$$\sigma^2_{Z_1} = \frac{1}{N-1}\sum_{i=1}^{N}(Z_{1_i} - \mu_Z)^2 \quad \text{and} \quad \sigma^2_{Z_2} = \frac{1}{N-1}\sum_{i=1}^{N}(Z_{2_i} - \mu_Z)^2\,.$$

3. The Proposed Model

In the proposed model, the underlying sensitivity level W and its variance are estimated by using the Greenberg et al. (1969) model. Here the sensitive question is "Whether or not you consider the underlying main research question sensitive for a face-to-face survey". Let π_b be the known probability of the binary innocuous unrelated question and p_b be the known probability of the respondent selecting the sensitivity question. We consider a finite population $U = \{1, 2, \ldots, N\}$ of size N and a random sample of size n be drawn without replacement. When estimating the mean, let S be the scrambling variable used to additively scramble the responses in the sample with mean $E(S) = \theta$. We assume that Y, X and S are mutually independent.

3.1. Estimation of Sensitivity Level (W)**.** The probability of "yes response" to the sensitivity question is given by

$$P_y = p_b W + (1-p_b)\pi_b \tag{3.1}$$

Solving for W, we have

$$W = \frac{P_y - (1-p_b)\pi_b}{p_b} \tag{3.2}$$

Thus the estimate of W, as per the Greenberg et al. (1969) model, is given by

$$\hat{W} = \frac{\hat{P}_y - (1-p_b)\pi_b}{p_b}, \tag{3.3}$$

where $\hat{P}_y$ is the proportion of yes response in the sample.

We know that $\hat{W}$ is an unbiased estimator and its variance is given by

$$\text{Var}(\hat{W}) = \left(\frac{1-f}{n}\right)\frac{P_y(1-P_y)}{p_b^2} \tag{3.4}$$

An unbiased estimator of this variance is given by

$$\hat{\text{Var}}(\hat{W}) = \left(\frac{1-f}{n-1}\right)\frac{\hat{P}_y(1-\hat{P}_y)}{p_b^2} \tag{3.5}$$

3.2. Estimation of Mean. The reported quantitative response Z to the main research question according to optional additive RRT model can be expressed as

$$Z = \left\{\begin{array}{ll} Y+S & \text{with probability } W \\ Y & \text{with probability } 1-W \end{array}\right\}$$

The mean and variance respectively of Z are given by

$$\begin{aligned} E(Z) &= WE(Y+S) + (1-W)E(Y) \\ &= E(Y) + WE(S) \\ &= \mu_Y + W\theta, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \text{Var}(Z) &= WE(Y+S)^2 + (1-W)E(Y^2) - \mu_Z^2 \\ &= \sigma_Y^2 + W\sigma_S^2 + \theta^2 W(1-W) \end{aligned} \tag{3.7}$$

From equation (3.6) we have

$$\mu_Y = \mu_Z - W\theta$$

This leads to an estimator for μ_Y given by

$$\hat{\mu}_{YW^*} = \hat{\mu}_Z - \hat{W}\theta, \tag{3.8}$$

where $\hat{\mu}_Z = \bar{z}$ is the sample mean of reported responses and $\hat{W}$ is given by equation (3.3).

We note that $\hat{\mu}_{YW^*}$ is an unbiased estimator of μ_Y and its variance is given by

$$\begin{aligned} \text{Var}(\hat{\mu}_{YW^*}) &= \text{Var}(\bar{z} - \hat{W}\theta) \\ &= \text{Var}(\bar{z}) + \theta^2\,\text{Var}(\hat{W}) \\ &= \left(\frac{1-f}{n}\right)(\sigma_Z^2) + \theta^2\left(\frac{1-f}{n}\right)\frac{P_y(1-P_y)}{p_b^2} \end{aligned} \tag{3.9}$$

The variance of the estimator in (3.9) can be conveniently estimated by

$$\hat{\text{Var}}(\hat{\mu}_{YW^*}) = \left(\frac{1-f}{n}\right)(s_z^2) + \theta^2\hat{\text{Var}}(\hat{W}) \tag{3.10}$$

where s_z^2 is the sample variance of reported responses given by $s_z^2 = (n-1)^{-1}\sum_{i=1}^{n}(z_i - \bar{z})^2$ and $\hat{\text{Var}}(\hat{W})$ is as given in (3.5) above.

We further modify the proposed mean estimator $\hat{\mu}_{YW^*}$ in the presence of an auxiliary variable by proposing ratio ($\hat{\mu}_{RW^*}$) and regression ($\hat{\mu}_{RegW^*}$) estimators and compare it with the estimators proposed in Kalucha et al. (2015) and Gupta et al. (2015), both based on split-sample approach.

4. Ratio Estimator

4.1. Kalucha et al. (2015) – Split-Sample Based Ratio Estimator. Kalucha et al. (2015) proposed the following additive ratio estimator for the mean of Y:

$$\hat{\mu}_{AR} = \left(\frac{\theta_2\bar{z}_1 - \theta_1\bar{z}_2}{\theta_2 - \theta_1}\right)\left(\frac{\mu_X}{\bar{x}_1} + \frac{\mu_X}{\bar{x}_2}\right)\left(\frac{1}{2}\right), \quad \theta_1 \neq \theta_2. \tag{4.1}$$

where $\left(\frac{\theta_2\bar{z}_1-\theta_1\bar{z}_2}{\theta_2-\theta_1}\right)$ is the unbiased estimator of μ_Y given by Gupta et al. (2010), and $\bar{x}_1$ and $\bar{x}_2$ are the respective sub-sample means for X. It was shown that this estimator performs better than the ratio estimator proposed by Sousa et al. (2010) utilizing a non-optional additive RRT model.

Bias and *MSE* of $\hat{\mu}_{AR}$, correct up to first order of approximation, are given by

$$\begin{aligned} Bias(\hat{\mu}_{AR}) &= \left(\frac{1-f_1}{n_1}\right)\left[\frac{\mu_Y}{2}C_x^2 - \left(\frac{\theta_2}{\theta_2-\theta_1}\right)\frac{\rho_{yx}\sigma_Y C_x}{2}\right] \\ &\quad + \left(\frac{1-f_2}{n_2}\right)\left[\frac{\mu_Y}{2}C_x^2 + (\frac{\theta_1}{\theta_2-\theta_1})\frac{\rho_{yx}\sigma_Y C_x}{2}\right] \\ &= C_x^2\mu_Y\left[\alpha - \rho_{yx}\frac{\beta}{2}\right] \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} MSE(\hat{\mu}_{AR}) &= \frac{1}{(\theta_2-\theta_1)^2}\left[\theta_2^2\left(\frac{1-f_1}{n_1}\right)\sigma_{Z_1}^2 + \theta_1^2\left(\frac{1-f_2}{n_2}\right)\sigma_{Z_2}^2\right] \\ &\quad + \frac{\mu_Y^2 C_x^2}{4}\alpha - \mu_Y\rho_{yx}\sigma_Y C_x\beta \end{aligned} \tag{4.3}$$

where $\alpha = \left(\frac{1-f_1}{n_1}\right) + \left(\frac{1-f_2}{n_2}\right)$, $\beta = \left(\frac{1-f_1}{n_1}\right)\left(\frac{\theta_2}{\theta_2-\theta_1}\right) - \left(\frac{1-f_2}{n_2}\right)\left(\frac{\theta_1}{\theta_2-\theta_1}\right)$, and C_x is the coefficient of variation for X.

4.2. Proposed Ratio Estimator-New Approach. In this section we propose a ratio estimator where the RRT estimator of the mean of Y given by (3.8) above is further improved by using information on an auxiliary variable X. We define $\delta_z = (\bar{z} - \mu_Z)/\mu_Z$, $\delta_x = (\bar{x} - \mu_X)/\mu_X$. Note that $E(\delta_i) = 0$ for $i = z, x$.

The proposed estimator is given by

$$\hat{\mu}_{RW^*} = (\bar{z} - \hat{W}\theta)\left(\frac{\mu_X}{\bar{x}}\right) = (\mu_Z(1+\delta_z) - \hat{W}\theta)(1+\delta x)^{-1} \tag{4.4}$$

Using Taylor's approximation and retaining terms of order up to 2, (4.4) can be rewritten as

$$\hat{\mu}_{RW^*} - \mu_Z \cong \mu_Z(\delta_z - \delta_x - \delta_z\delta_x + \delta_x^2) - \hat{W}\theta(1 - \delta_x + \delta_x^2) \tag{4.5}$$

Substituting the value of μ_Z from (3.6) in (4.5), we have

$$\hat{\mu}_{RW^*} - \mu_Y \cong \mu_Y(\delta_z - \delta_x - \delta_z\delta_x + \delta_x^2) + (W - \hat{W})\theta(1 - \delta_x + \delta_x^2) + W\theta(\delta_z - \delta_z\delta_x) \tag{4.6}$$

Under the assumption of bivariate normality (see Sukhatme and Sukhatme, 1970), we have

$$E(\delta_z^2) = \frac{1-f}{n}C_z^2,\ E(\delta_x^2) = \frac{1-f}{n}C_x^2,\ E(\delta_z\delta_x) = \frac{1-f}{n}C_{zx}$$

where $C_{zx} = \rho_{zx}C_zC_x$, C_z and C_x are the coefficients of variation of Z and X, respectively.

Also, we have:

$$(4.7)\qquad C_z^2 = \frac{\sigma_y^2 + W\sigma_S^2 + \theta^2 W(1-W)}{(\bar{Z})^2} \quad \text{and} \quad \rho_{zx} = \frac{\rho_{yx}}{\sqrt{1 + W\frac{\sigma_S^2}{\sigma_y^2} + \frac{\theta^2 W(1-W)}{\sigma_y^2}}}$$

From equation (4.6), we can get expression for the *Bias* of $\hat{\mu}_{RW^*}$, correct up to first order of approximation, as given by

$$(4.8)\qquad Bias(\hat{\mu}_{RW^*}) \cong \mu_Y \left(\frac{1-f}{n}\right)(C_x^2 - \rho_{zx}C_zC_x) - W\theta\left(\frac{1-f}{n}\right)\rho_{zx}C_zC_x$$

Similarly from (4.6), *MSE* of $\hat{\mu}_{RW^*}$, correct to first order of approximation, is given by

$$\begin{aligned} MSE(\hat{\mu}_{RW^*}) &= E(\hat{\mu}_{RW^*} - \mu_Y)^2 \\ &\cong \mu_Y^2 E(\delta_z^2 + \delta_x^2 - 2\delta_z\delta_x) + \theta^2 E(W - \hat{W})^2 E(1 - 2\delta_x + 3\delta_x^2) \\ &\quad + W^2\theta^2 E(\delta_z^2) + 2\mu_Y W\theta E(\delta_z^2 - \delta_z\delta_x) \end{aligned}$$

or

$$\begin{aligned} MSE(\hat{\mu}_{RW^*}) &\cong \left(\frac{1-f}{n}\right)\mu_Y^2(C_z^2 + C_x^2 - 2\rho_{zx}C_zC_x) \\ &\quad + \theta^2\,\mathrm{Var}(\hat{W})\left(1 + 3\left(\frac{1-f}{n}\right)C_x^2\right) + W^2\theta^2\left(\frac{1-f}{n}\right)C_z^2 \\ &\quad + 2\mu_Y W\theta\left(\frac{1-f}{n}\right)(C_z^2 - \rho_{zx}C_zC_x) \end{aligned} \qquad (4.9)$$

where $\mathrm{Var}(\hat{W})$ is given by (3.4) above.

4.3. Mean and Variance of the Proposed Ratio Estimator. The proposed ratio estimator can be rewritten as

$$(4.10)\qquad \hat{\mu}_{RW^*} = \left(\frac{\bar{y}}{\bar{x}}\right)\mu_X, \quad \text{where } \bar{y} = \bar{z} - \hat{W}\theta$$

Hence

$$(4.11)\qquad E(\hat{\mu}_{RW^*}) = \mu_X E\left\{\frac{\bar{y}}{\bar{x}}\right\}$$

Using a Taylor series expansion of $\frac{\bar{y}}{\bar{x}}$ around (μ_Y,μ_X):

$$\begin{aligned} \frac{\bar{y}}{\bar{x}} &\cong \frac{\vec{y}}{\bar{x}}\Big|_{(\mu_Y,\mu_X)} + (\bar{y}-\mu_Y)\frac{\partial}{\partial\bar{y}}\left(\frac{\bar{y}}{\bar{x}}\right)\Big|_{(\mu_Y,\mu_X)} + (\bar{x}-\mu_X)\frac{\partial}{\partial\bar{x}}\left(\frac{\bar{y}}{\bar{x}}\right)\Big|_{(\mu_Y,\mu_X)} \\ &\quad + \frac{1}{2}(\bar{y}-\mu_Y)^2\frac{\partial^2}{\partial\bar{y}^2}\left(\frac{\bar{y}}{\bar{x}}\right)\Big|_{(\mu_Y,\mu_X)} + \frac{1}{2}(\bar{x}-\mu_X)^2\frac{\partial^2}{\partial\bar{x}^2}\left(\frac{\bar{y}}{\bar{x}}\right)\Big|_{(\mu_Y,\mu_X)} \\ &\quad + (\bar{y}-\mu_Y)(\bar{x}-\mu_X)\frac{\partial^2}{\partial\bar{y}\partial\bar{x}}\left(\frac{\bar{y}}{\bar{x}}\right)\Big|_{(\mu_Y,\mu_X)} \\ &\quad + O\left(\left((\bar{y}-\mu_Y)\frac{\partial}{\partial\bar{y}} + (\bar{x}-\mu_X)\frac{\partial}{\partial\bar{x}}\right)^3\left(\frac{\bar{y}}{\bar{x}}\right)\right) \end{aligned}$$

The mean of $\frac{\bar{y}}{\bar{x}}$ can now be found by taking expected value, ignoring all terms higher than 2.

$$\begin{aligned} E\left\{\frac{\bar{y}}{\bar{x}}\right\} &\cong \frac{\mu_Y}{\mu_X} + \mathrm{Var}(\bar{x})\frac{\mu_Y}{\mu_X^3} - \frac{\mathrm{Cov}(\bar{y},\bar{x})}{\mu_X^2} \\ &\cong \frac{\mu_Y}{\mu_X} + \frac{(1-f)}{n}\left(\mathrm{Var}(x)\frac{\mu_Y}{\mu_X^3} - \frac{\mathrm{Cov}(y,x)}{\mu_X^2}\right) \end{aligned} \qquad (4.12)$$

Substituting (4.12) in (4.11), we get

$$E(\hat{\mu}_{RW^*}) \cong \mu_Y + \frac{(1-f)}{n}\left(\mathrm{Var}(x)\frac{\mu_Y}{\mu_X^2} - \frac{\mathrm{Cov}(y,x)}{\mu_X}\right) \tag{4.13}$$

It is clear from the above expression that $\hat{\mu}_{RW^*}$ is asymptotically unbiased. Now

$$\mathrm{Var}(\hat{\mu}_{RW^*}) = \mu_X^2 \,\mathrm{Var}\left(\frac{\bar{y}}{\bar{x}}\right) \tag{4.14}$$

An approximation of the variance of $\frac{\bar{y}}{\bar{x}}$ is obtained by using the first order terms of Taylor series expansion:

$$\begin{aligned}
\mathrm{Var}\left(\frac{\bar{y}}{\bar{x}}\right) &= E\left\{\left(\frac{\bar{y}}{\bar{x}} - E\left\{\frac{\bar{y}}{\bar{x}}\right\}\right)^2\right\} \\
&\cong E\left\{\left(\frac{\bar{y}}{\bar{x}} - \frac{\mu_Y}{\mu_X}\right)^2\right\} \\
&\cong \frac{\mathrm{Var}(\bar{y})}{\mu_X^2} + \frac{\mu_Y^2\,\mathrm{Var}(\bar{x})}{\mu_X^4} - \frac{2\mu_Y\,\mathrm{Cov}(\bar{y},\bar{x})}{\mu_X^3} \\
&\cong \frac{(1-f)}{n}\left(\frac{\mathrm{Var}(y)}{\mu_X^2} + \frac{\mu_Y^2\,\mathrm{Var}(x)}{\mu_X^4} - \frac{2\mu_Y\,\mathrm{Cov}(y,x)}{\mu_X^3}\right) \qquad (4.15)
\end{aligned}$$

Substituting (4.15) in (4.14), we have

$$\mathrm{Var}(\hat{\mu}_{RW^*}) \cong \frac{(1-f)}{n}\left(\mathrm{Var}(y) + \frac{\mu_Y^2\,\mathrm{Var}(x)}{\mu_X^2} - \frac{2\mu_Y\,\mathrm{Cov}(y,x)}{\mu_X}\right) \tag{4.16}$$

Substituting for $\mathrm{Var}(y)$ and using the fact that $\mathrm{Cov}(y,x) = \mathrm{Cov}(z,x)$ in (4.16), we get

$$\begin{aligned}
\mathrm{Var}(\hat{\mu}_{RW^*}) \cong \frac{(1-f)}{n}\Bigg(&\mathrm{Var}(z) - W\,\mathrm{Var}(S) - \theta^2 W(1-W) \\
&+ \frac{\mu_Y^2\,\mathrm{Var}(x)}{\mu_X^2} - \frac{2\mu_Y\,\mathrm{Cov}(z,x)}{\mu_X}\Bigg)
\end{aligned} \tag{4.17}$$

The above variance can be estimated by using:

$$\hat{\mathrm{Var}}(z) = s_z^2, \quad \hat{W} = \frac{\hat{P}_y - (1-p_b)\pi_b}{p_b}, \text{ and } \hat{\mathrm{Cov}}(z,x) = s_{zx},$$

where sample covariance $s_{zx} = (n-1)^{-1}\sum_{i=1}^{n}(z_i - \bar{z})(x_i - \bar{x})$.

5. Regression Estimator

5.1. Gupta et al. (2015) – Split-Sample Based Regression Estimator. Gupta et al. (2015) suggested a regression estimator of the mean using split-sample approach, as given by:

$$\hat{\mu}_{Areg} = \left(\frac{\theta_2\bar{z}_1 - \theta_1\bar{z}_2}{\theta_2 - \theta_1}\right) + \left\{\hat{\beta}_{Z_1X_1}(\mu_X - \bar{x}_1) + \hat{\beta}_{Z_2X_2}(\mu_X - \bar{x}_2)\right\}\left(\frac{1}{2}\right), \tag{5.1}$$

where $\hat{\beta}_{Z_iX_i}(i = 1,2)$ are the sample regression coefficients between Z_i and X_i respectively, and $\bar{z}_i$, $\bar{x}_i$ $(i = 1,2)$ are the two sub-sample means. It was shown that this estimator performs better than the regression estimator proposed by Gupta et al. (2012) utilizing a non-optional additive RRT model. *Bias* and *MSE* of $\hat{\mu}_{Areg}$, correct up to first order of approximation, are given by

$$Bias(\hat{\mu}_{Areg}) \cong \left[-\frac{1}{2}\beta_{Z_1X}\left(\frac{1-f_1}{n_1}\right) - \frac{1}{2}\beta_{Z_2X}\left(\frac{1-f_2}{n_2}\right)\right]\left\{\frac{\mu_{12}}{\mu_{11}} - \frac{\mu_{03}}{\mu_{02}}\right\} \tag{5.2}$$

and

$$(5.3)\qquad MSE^{(1)}(\hat{\mu}_{Areg}) = \frac{1}{(\theta_2-\theta_1)^2}\left[\theta_2^2\left(\frac{1-f_1}{n_1}\right)\sigma_{Z_1}^2+\theta_1^2\left(\frac{1-f_2}{n_2}\right)\sigma_{Z_2}^2\right] + \frac{\rho_{yx}^2\sigma_Y^2}{4}\alpha - \rho_{yx}^2\sigma_Y^2\beta,$$

where $\theta_2 \neq \theta_1$; α and β are defined earlier and $\mu_{rs} = \frac{1}{N-1}\sum_{i=1}^{N}(z_i-\bar{Z})^r(x_i-\bar{X})^s$.

5.2. Proposed Regression Estimator-New Approach. We modify the mean estimator in (3.8) above by using the regression estimation approach and propose the following estimator for the population mean of Y:

$$(5.4)\qquad \hat{\mu}_{RegW^*} = (\bar{z}-\hat{W}\theta)+\hat{\beta}_{zx}(\mu_X-\bar{x})$$

We obtain the expressions for the bias and the mean square error for the proposed regression estimator $\hat{\mu}_{RegW^*}$. If $e_0 = (\bar{z}-\mu_Z)/\mu_Z$, $e_1 = (\bar{x}-\mu_X)/\mu_X$, $e_2 = (\sigma_x^2-\sigma_X^2)/\sigma_X^2$ and $e_3 = (\sigma_{zx}-\sigma_{ZX})/\sigma_{ZX}$, then we have $E(e_i) = 0$, $i = 0, 1, 2, 3$.

Using Taylor's approximation and retaining terms of order up to 2, (5.4) can be rewritten as

$$(5.5)\qquad \hat{\mu}_{RegW^*} - \mu_Z \cong \mu_Z e_0 - \hat{W}\theta - \beta_{zx}\mu_X[e_1+e_1e_3-e_1e_2]$$

Substituting for μ_Z, (5.5) can be written as

$$(5.6)\qquad \hat{\mu}_{RegW} - \mu_Y \cong \mu_Z e_0 - \beta_{zx}\mu_X[e_1+e_1e_3-e_1e_2]+(W-\hat{W})\theta$$

From Mukhopadhyay (1998, p. 123), we have $E(e_1^2) = \frac{1-f}{n}C_x^2$, $E(e_0^2) = \frac{1-f}{n}C_z^2$, $E(e_1e_2) = \frac{1-f}{n}\frac{1}{\bar{X}}\frac{\mu_{03}}{\mu_{02}}$, $E(e_1e_3) = \frac{1-f}{n}\frac{1}{\bar{X}}\frac{\mu_{12}}{\mu_{11}}$, where $\mu_{rs} = \frac{1}{N-1}\sum_{i=1}^{N}(z_i-\bar{Z})^r(x_i-\bar{X})^s$ and C_x, C_z are the coefficients of variation of x and z, respectively. Also, we have:

$$(5.7)\qquad \beta_{zx} = \frac{\sigma_{zx}}{\sigma_x^2} = \frac{\sigma_{yx}}{\sigma_x^2} = \rho_{yx}\frac{\sigma_y}{\sigma_x} = \beta_{yx}$$

where ρ_{yx} and ρ_{zx} are the coefficients of correlation between y and x, and between z and x, respectively.

Using this in (5.6), the *Bias* of $\hat{\mu}_{RegW}$, to first order of approximation, is given by

$$(5.8)\qquad Bias(\hat{\mu}_{RegW^*}) \cong -\beta_{zx}\left(\frac{1-f}{n}\right)\left\{\frac{\mu_{12}}{\mu_{11}}-\frac{\mu_{03}}{\mu_{02}}\right\}$$

The expression for *MSE* of $\hat{\mu}_{RegW^*}$ to first order of approximation, is given by

$$(5.9)\qquad \begin{aligned} MSE(\hat{\mu}_{RegW^*}) &\cong \left(\frac{1-f}{n}\right)\left[\sigma_z^2-\frac{\sigma_{yx}^2}{\sigma_x^2}\right]+\theta^2\,\mathrm{Var}(\hat{W}) \\ &= \left(\frac{1-f}{n}\right)\sigma_y^2\left\{\left(1+\frac{W\sigma_S^2+\theta^2W(1-W)}{\sigma_y^2}\right)-\rho_{yx}^2\right\}+\theta^2\,\mathrm{Var}(\hat{W}) \end{aligned}$$

where $\mathrm{Var}(\hat{W})$ is given by (3.4) above.

We note that $\hat{\mu}_{RegW^*}$ is an unbiased estimator and hence

$$(5.10)\qquad \begin{aligned} \mathrm{Var}(\hat{\mu}_{RegW^*}) &= MSE(\hat{\mu}_{RegW^*}) \\ &\cong \left(\frac{1-f}{n}\right)\left[\sigma_z^2-\frac{\sigma_{yx}^2}{\sigma_x^2}\right]+\theta^2\,\mathrm{Var}(\hat{W}) \end{aligned}$$

The above variance can be estimated by using:

$$\hat{\sigma}_z^2 = s_z^2,\ \hat{\sigma}_{yx}^2 = \hat{\sigma}_{zx}^2 = s_{zx}^2 \quad \text{and} \quad \hat{\mathrm{Var}}(\hat{W}) = \frac{(1-f)}{(n-1)}\frac{\hat{P}_y(1-\hat{P}_y)}{p_b^2}.$$

6. Efficiency comparisons

6.1. Efficiency Comparison of $\hat{\mu}_{RW^*}$ and $\hat{\mu}_{YW^*}$. We have from equations (3.9) and (4.9), $MSE(\hat{\mu}_{RW^*}) < MSE(\hat{\mu}_{YW^*})$ if

$$(6.1) \qquad 1 + \frac{3\theta^2 \operatorname{Var}(\hat{W})}{\mu_Y^2} < 2\rho_{yx}\frac{C_y}{C_x}$$

Since $\frac{3\theta^2 \operatorname{Var}(\hat{W})}{\mu_Y^2}$ approaches 0 because $\operatorname{Var}(\hat{W})$ approaches 0 as the sample becomes larger, (6.1) will generally hold if

$$(6.2) \qquad 1 < 2\rho_{yx}\frac{C_y}{C_x} \text{ or } \rho_{yx} > \frac{1}{2}\frac{C_x}{C_y}$$

If we assume $(C_x \approx C_y)$, we can conclude from (6.2) that

$$(6.3) \qquad MSE(\hat{\mu}_{RW^*}) < MSE(\hat{\mu}_{YW^*}) \text{ if } \rho_{yx} > \frac{1}{2}.$$

Hence the proposed ratio estimator $(\hat{\mu}_{RW^*})$ is more efficient than the proposed ordinary mean estimator $(\hat{\mu}_{YW^*})$ when the correlation between the study variable and the auxiliary variable is high $\left(\rho_{yx} > \frac{1}{2}\right)$.

6.2. Efficiency Comparison of $\hat{\mu}_{RegW^*}$ with $\hat{\mu}_{RW^*}$ and $\hat{\mu}_{YW^*}$.

(i) It can be verified from (3.9) and (5.9) that according to first order approximation $MSE(\hat{\mu}_{RegW^*}) < MSE(\hat{\mu}_{YW^*})$ if

$$(6.4) \qquad \left(\frac{1-f}{n}\right)\frac{\sigma_{yx}^2}{\sigma_x^2} > 0$$

(ii) It can be verified from (4.9) and (5.9) that up to first order approximation $MSE(\hat{\mu}_{RegW^*}) < MSE(\hat{\mu}_{RW^*})$ if

$$(6.5) \qquad 1 - 2\rho_{yx}\frac{C_y}{C_x} + \rho_{yx}^2\frac{C_y^2}{C_x^2} + \frac{3\theta^2 \operatorname{Var}(\hat{W})}{\mu_Y^2} > 0$$

With $(C_x \cong C_y)$, (6.5) can be rewritten as

$$(6.6) \qquad (1-\rho_{yx})^2 + \frac{3\theta^2 \operatorname{Var}(\hat{W})}{\mu_Y^2} > 0$$

Since the conditions (6.4) and (6.6) will always hold true, up to first order of approximation, the regression estimator $\hat{\mu}_{RegW^*}$ performs better than the ordinary mean estimator $\hat{\mu}_{YW^*}$ and the ratio estimator $\hat{\mu}_{RW^*}$.

7. Simulation Study

7.1. Comparison of the Proposed Model with the Split-Sample Model in the Presence of Auxiliary Information. The tables below provide a comparison between the proposed model and the split-sample additive scrambling models of Kalucha et al. (2015) and Gupta et al. (2015) in the presence of non-sensitive auxiliary information. We choose the parameters as per the observation A1 (given below) that was obtained in Gupta et al. (2015) under which the regression estimator $\hat{\mu}_{Areg}$ is more efficient than both additive ratio estimator $\hat{\mu}_{AR}$ and the ordinary mean estimator $\hat{\mu}_Y$ under the split sample approach:

A1. We choose our scrambling variables S_1 and S_1 in such a way that their means θ_1 and θ_2 are opposite in signs and associate the one with the smaller magnitude to the larger sub-sample and vice-versa. Also if one of the chosen means is zero then we associate it to the larger split sample.

In the simulation study, we consider a finite population of size $N = 5000$ generated from a bivariate normal distribution. The simulated bivariate normal population has theoretical mean of $[Y, X]$ as $\mu = [6, 4]$. The covariance matrix $(\sum)$ is as given below:

$$\sum = \begin{bmatrix} 9 & 4.8 \\ 4.8 & 4 \end{bmatrix}, \quad \rho_{YX} = 0.7996$$

We estimate the empirical MSE using 5000 samples of various sizes selected from this population. The scrambling variables S_1 and S_2 are taken to be normal variates with $\sigma^2_{S_1} = 2$ and $\sigma^2_{S_2} = 1$. The scrambling variable means are chosen as per **A1 (given above).** The selected means are $\theta_1 = 5$, $\theta_2 = -0.5$ and $n_2 > n_1$. For the population we consider two sample sizes: $n = 500, 1000$ for different values of the sensitivity level $W = 0.3, 0.7, 0.9$.

For the proposed model we choose $\theta = \theta_2 = -0.5$ with $\pi_b = 0.25$ and $p_b = 0.7$.

Table 1. Theoretical **(bold)** and empirical MSE comparisons of the mean estimator $(\hat{\mu}_{YW^*})$, the ratio estimator $(\hat{\mu}_{RW^*})$ and the regression estimator $(\hat{\mu}_{RegW^*})$ of the proposed model with the mean estimator $(\hat{\mu}_Y)$, the additive ratio estimator $(\hat{\mu}_{AR})$ and the regression estimator $(\hat{\mu}_{Areg})$ of the split-sample model with $\rho_{YX} = 0.7996$.

n	W	MSE Estimation									
		Proposed Model				Split-Sample Model					
		Var$(\hat{W})$	$MSE(\hat{\mu}_{YW^*})$	$MSE(\hat{\mu}_{RW^*})$	$MSE(\hat{\mu}_{RegW^*})$	n_1	n_2	Var$(\hat{W})$	$MSE(\hat{\mu}_Y)$	$MSE(\hat{\mu}_{AR})$	$MSE(\hat{\mu}_{Areg})$
500	0.3	**0.000749**	**0.017141**	**0.007283**	**0.006706**	200	300	**0.003511**	**0.024982**	**0.019001**	**0.017437**
		0.000821	0.016916	0.007221	0.006638			0.004487	0.023217	0.018106	0.01665
	0.7	**0.000903**	**0.0179**	**0.008041**	**0.007465**	200	300	**0.003688**	**0.02605**	**0.020069**	**0.018505**
		0.000999	0.017614	0.008264	0.007608			0.004821	0.025906	0.020948	0.019584
	0.9	**0.000764**	**0.018171**	**0.008313**	**0.007736**	200	300	**0.003277**	**0.026387**	**0.020406**	**0.018842**
		0.000853	0.018221	0.008534	0.008002			0.002443	0.029625	0.023441	0.022628
1000	0.3	**0.000333**	**0.007618**	**0.003237**	**0.002981**	450	550	**0.001665**	**0.012846**	**0.009044**	**0.008528**
		0.000416	0.00738	0.003224	0.002915			0.003114	0.011986	0.009241	0.008602
	0.7	**0.000401**	**0.007956**	**0.003574**	**0.003318**	450	550	**0.001748**	**0.013394**	**0.009593**	**0.009076**
		0.000497	0.007744	0.003589	0.003319			0.002965	0.012035	0.009007	0.008506
	0.9	**0.000340**	**0.008076**	**0.003694**	**0.003438**	450	550	**0.001568**	**0.013578**	**0.009777**	**0.009260**
		0.000423	0.008367	0.003914	0.003693			0.001270	0.012051	0.008914	0.008395

We note from the table that consistently the regression estimator $(\hat{\mu}_{RegW^*})$ is more efficient than the ratio $(\hat{\mu}_{RW^*})$ and the mean estimator $(\hat{\mu}_{YW^*})$ of the proposed model for all values of W. Also as the sensitivity W increases, the MSE's increase, highlighting the usefulness of an Optional RRT model since W is highest (equal to 1) for non-optional model. While comparing the proposed model with the split-sample model, we note that MSE's of the proposed model estimators $(\hat{\mu}_{YW^*}, \mu_{RW^*} \mu_{RegW^*})$ are consistently smaller as compared to $(\hat{\mu}_Y, \hat{\mu}_{AR}, \hat{\mu}_{Areg})$ estimators. We observe that for a fixed sample size the MSE's for the proposed model are reduced by more than two and a half times as compared to the split-sample based model.

7.2. Comparison of the Point Estimates of Proposed Model with the Split-Sample Model in the Presence of Auxiliary Information.

Table 2. Empirical values of the estimators $\hat{W}$, the mean estimator ($\hat{\mu}_{YW^*}$), the ratio estimator ($\hat{\mu}_{RW^*}$) and the regression estimator ($\hat{\mu}_{RegW^*}$) of the proposed model and the corresponding split sample model for $W = 0.3, 0.7, 0.9$ and the population mean $\mu_Y = 6$.

n	W	Point Estimates							
		Proposed Model				Split-sample Model			
		$\hat{W}$	$\hat{\mu}_{YW^*}$	$\hat{\mu}_{RW^*}$	$\hat{\mu}_{RegW^*}$	$\hat{W}$	$\hat{\mu}_{Y}$	$\hat{\mu}_{AR}$	$\hat{\mu}_{Areg}$
500	0.3	0.30049	5.91234	5.90924	5.90958	0.34439	5.90478	5.90812	5.90471
	0.7	0.69978	5.90947	5.91254	5.91158	0.6523	5.86084	5.86545	5.86143
	0.9	0.89957	5.91218	5.90169	5.91065	0.90461	5.83561	5.83925	5.83557
1000	0.3	0.30052	5.91076	5.912	5.91161	0.34351	5.92844	5.93066	5.92885
	0.7	0.69979	5.9116	5.91047	5.91053	0.65809	5.89841	5.90048	5.8986
	0.9	0.89997	5.91107	5.91144	5.91125	0.90812	5.89409	5.89618	5.89436

We note that both methods produce nearly unbiased estimators of the population mean. However, the proposed model produces better estimates of the sensitivity level.

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Comparison of loss functions for estimating the scale parameter of log-normal distribution using non-informative priors

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Abstract

The estimation of parameters of distributions is a core topic in the literature on Statistical methodology. Many Bayesian and classical approaches have been derived for estimating parameters. In this study, Bayesian estimation technique is adopted for the comparison of two non-informative priors and six loss functions to estimate the scale parameter of Log-Normal distribution assuming fixed values of location parameter. The main purpose of this study is to search for a suitable prior when no prior information is available and to look for an appropriate loss function for estimation of the scale parameter of Log-Normal distribution. Through simulation study, comparisons are made on the basis of the posterior variances, coefficients of skewness, ex-kurtosis and Bayes risks. The simulation results are verified through a real data set of lung cancer patients.

Keywords: Prior distribution, Posterior distribution, Log-Normal distribution, Inverted Gamma distribution.

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1. Introduction

Suppose a random variable Y is normally distributed with mean θ and variance ϕ then $X = exp(Y)$ is distributed Log-Normally with location and scale parameters θ and ϕ, respectively. The probability density function (pdf) of Log-Normal random variable X is:

$$f(x;\theta,\phi) = \frac{1}{x\sqrt{2\pi\phi}}\, e^{-\frac{1}{2\phi}(\ln x-\theta)^2},\ x>0,\ -\infty<\theta<\infty,\ \phi>0 \tag{1.1}$$

where, θ is location and ϕ is scale parameter.

The cumulative distribution function of this distribution is given by

$$F(x) = \frac{1}{2} + \frac{1}{2}Erf\left(\frac{\ln x-\theta}{\sqrt{2\phi}}\right) \tag{1.2}$$

The Log-normal distribution, defined in equation (1.1), has become a convenient model for different biological, social and life testing phenomena. This distribution has wide applications in business and economics such as modeling of firm sizes, incomes, stock prices, lengths of service in labor turnover contexts and many other fields. Finney [2] obtained formulae for efficient estimation of the mean and variance of a population using sample information from the Log-Normal distribution. Tiku [13] found the estimators of parameters of Log-Normal distribution using type-II censored sample data. He obtained the asymptotic variances and covariances of the estimators. Zellner [14] used Bayesian and non-Bayesian methods for estimating parameters of the log-normal distribution and of log-normal regression processes. He derived posterior distributions for parameters of interest and described their statistical properties. Stedinger [12] evaluated efficiency of different methods for fitting two-parameter and three-parameter log-normal distributions. He made the comparison using mean square error of estimators. Alternatively, Shen [10] combined the orthogonal transformation and the Rao-Blackwell Theorem for deriving uniform minimum variance unbiased estimators (UMVUEs) for the parameters of the Log-Normal distribution. Limpert et al. [5] discussed the use of Log-Normal distribution in different fields of science, specially in biological sciences. For the two-parameter log-normal distribution, Khan et al. [4] derived the prediction of future responses assuming a non-informative prior and an informative prior for the parameters under type-II censored sampling and type-II median censored sampling. Mehta et al. [7] proposed a simple and novel method to approximate the sum of several log-normal random variables with a single log-normal random variable. Martín and Pérez [6] presented generalized form of the log-normal distribution and analyzed it through Bayesian tools. Saleem and Aslam [9] used Bayesian tools of inference to estimate the parameters of two-component mixture of Rayleigh distributions assuming the uniform and the Jeffreys priors. Rupasov et al. [8] showed that trial to trial neuronal variability of electromyographic (EMG) signals can be well described by the Log-Normal distribution. They also found that the variability of temporal parameters of handwriting duration and response time can also be well described by the Log-Normal distribution. Sindhu and Aslam [11] estimated the parameters of Inverse Weibul distribution under different loss functions and different priors.

Loss function plays a vital rule in Bayesian estimation problems. Loss function is the penalty for not getting the actual value. Zellner [15] discussed estimation of parameters of different models including Log-Normal under Varian's asymmetric LINEX loss function. Fabrizi and Trivisano [1] proposed a generalized inverse Guassian prior for the variance parameter of Log-Normal distribution and discussed the estimation of its mean under Quadratic Loss Function (QLF).

Estimation of scale parameter of Log-Normal distribution is not yet been considered under different loss functions using non-informative priors. Keeping in mind the above discussion and motivated by importance of the Log-Normal distribution in different fields, this study is made to look for best loss function and appropriate non-informative prior. In this paper, the posterior distributions for the scale parameter ϕ are derived under Uniform and Jeffreys priors. Also, Bayes estimators (BEs) and Bayes risks (BRs) are obtained using Squared Error Loss Function (SELF), Quadratic Loss Function (QLF), Weighted Loss Function (WLF), Precautionary Loss Function (PLF), Simple Asymmetric Precautionary Loss Function (SAPLF) and DeGroot Loss Function (DLF).

Comparisons of the priors and the loss functions are made on the basis of posterior variances, coefficients of skewness, ex-kurtosis and Bayes risks. For these comparisons, different sample sizes, different loss functions and different choices of location parameter θ have been considered. The rest of the paper is designed as follows.

In section 2, the posterior distributions of the scale parameter have been derived under Uniform and Jeffreys priors. A simulation study is presented in section 3 to compare the performance of the two priors on the basis of posterior variance, skewness and ex-kurtosis. Bayes estimators (BEs) and Bayes risks (BRs) under the considered loss functions are given in section 4. To look for best non-informative prior and loss function for the estimation of the scale parameter, a simulation study is carried out in section 5. A real data set of lung cancer patients is used in section 6 to draw graphs of the posterior distributions for different values of the location parameter and to verify the simulation results discussed in section 5.

2. The Posterior Distributions of the Scale Parameter under Non-Informative Priors

The Likelihood function of the Log-normal distribution can be written as under.

$$L(\phi) = \frac{(2\pi\phi)^{-\frac{n}{2}}}{\prod_{i=1}^{n} x_i}\, e^{-\frac{1}{2\phi}\sum_{i=1}^{n}(\ln x_i-\theta)^2} \tag{2.1}$$

We assume the improper Uniform prior ($U(0,\infty)$) for ϕ which can be written as

$$\Phi_U(\phi) \propto 1,\ \phi > 0. \tag{2.2}$$

The Posterior Distribution of ϕ given the data, under the above prior is given by

$$p(\phi|\mathbf{x}) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}\phi^{-(\alpha_1+1)}e^{-\frac{\beta_1}{\phi}},\ \ \phi > 0 \tag{2.3}$$

where $\alpha_1 = \frac{n}{2} - 1$ and $\beta_1 = \frac{1}{2}\sum_{i=1}^{n}(\ln x_i - \theta)^2$.

The expression in 2.3 can be identified as Inverted Gamma distribution.

The Jeffreys prior for ϕ is

$$\Phi_J(\phi) \propto \frac{1}{\phi},\ \phi > 0. \tag{2.4}$$

The posterior distribution of ϕ given the data, using Jeffreys prior is given by

$$p(\phi|\mathbf{x}) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}\phi^{-(\alpha_2+1)}e^{-\frac{\beta_2}{\phi}},\ \phi > 0 \tag{2.5}$$

where $\alpha_2 = \frac{n}{2}$ and $\beta_2 = \frac{1}{2}\sum_{i=1}^{n}(\ln x_i - \theta)^2$.

The expression in 2.5 can be identified as Inverted Gamma distribution.

3. Simulation Study for Comparison of Priors on the basis of Posterior Variances, Co-Efficient of Skewness and Ex-Kurtosis

Consider the generation of random samples of sizes $n = 30, 50, 100, 200$ and 500 from the Log-Normal distribution assuming the location parameter $\theta = 1, 2, 3$ and the scale parameter $\phi = 1, 4, 7$. The simulation process is repeated $10,000$ times and the results have then been averaged.

The results of posterior variance, skewmess and ex-kurtosis are showcased in the following tables.

Table 1. Posterior Variance for Different Values of θ and ϕ

n	Prior	$\theta = 1$			$\theta = 2$			$\theta = 3$		
		$\phi = 1$	$\phi = 4$	$\phi = 7$	$\phi = 1$	$\phi = 4$	$\phi = 7$	$\phi = 1$	$\phi = 4$	$\phi = 7$
30	Uniform	0.11885	30.4636	283.5813	0.11764	30.2204	281.9675	0.11712	30.2193	284.025
50		0.05619	14.2796	133.6523	0.05604	14.3247	134.2029	0.05569	14.2144	134.045
100		0.02360	6.01455	56.68350	0.02352	6.03511	56.61045	0.02347	5.99707	56.4623
200		0.01085	2.77936	26.00710	0.01087	2.77255	26.01307	0.01083	2.76955	25.9318
500		0.00414	1.05660	9.922510	0.00413	1.05823	9.913580	0.00413	1.05553	9.91950
30	Jeffreys	0.09363	24.2955	226.2839	0.09394	24.1826	225.9903	0.09475	24.0629	225.914
50		0.04923	12.5365	118.0433	0.04864	12.5922	117.5760	0.04932	12.5574	118.546
100		0.02210	5.66527	53.18104	0.02198	5.67742	52.97626	0.02220	5.68842	53.2441
200		0.01054	2.69131	25.18842	0.01052	2.69231	25.19030	0.01051	2.69701	25.3189
500		0.00409	1.04692	9.805566	0.00408	1.04212	9.804080	0.00408	1.04248	9.79537

Table 2. Posterior Skewness for Different Values of θ and ϕ

n	Prior	$\theta = 1$			$\theta = 2$			$\theta = 3$		
		$\phi = 1$	$\phi = 4$	$\phi = 7$	$\phi = 1$	$\phi = 4$	$\phi = 7$	$\phi = 1$	$\phi = 4$	$\phi = 7$
30	Uniform	0.31492	0.31492	0.31492	0.31492	0.31492	0.31492	0.31492	0.31492	0.31492
50		0.22335	0.22335	0.22335	0.22335	0.22335	0.22335	0.22335	0.22335	0.22335
100		0.14904	0.14904	0.14904	0.14904	0.14904	0.14904	0.14904	0.14904	0.14904
200		0.10259	0.10259	0.10259	0.10259	0.10259	0.10259	0.10259	0.10259	0.10259
500		0.06389	0.06389	0.06389	0.06389	0.06389	0.06389	0.06389	0.06389	0.06389
30	Jeffreys	0.30046	0.30046	0.30046	0.30046	0.30046	0.30046	0.30046	0.30046	0.30046
50		0.21799	0.21799	0.21799	0.21799	0.21799	0.21799	0.21799	0.21799	0.21799
100		0.14741	0.14741	0.14741	0.14741	0.14741	0.14741	0.14741	0.14741	0.14741
200		0.10206	0.10206	0.10206	0.10206	0.10206	0.10206	0.10206	0.10206	0.10206
500		0.06376	0.06376	0.06376	0.06376	0.06376	0.06376	0.06376	0.06376	0.06376

Table 3. Posterior Ex-Kurtosis for Different Values of θ and ϕ

n	Prior	$\theta=1$			$\theta=2$			$\theta=3$		
		$\phi=1$	$\phi=4$	$\phi=7$	$\phi=1$	$\phi=4$	$\phi=7$	$\phi=1$	$\phi=4$	$\phi=7$
30	Uniform	3.21818	3.21818	3.21818	3.21818	3.21818	3.21818	3.21818	3.21818	3.21818
50		1.55714	1.55714	1.55714	1.55714	1.55714	1.55714	1.55714	1.55714	1.55714
100		0.67826	0.67826	0.67826	0.67826	0.67826	0.67826	0.67826	0.67826	0.67826
200		0.31842	0.31842	0.31842	0.31842	0.31842	0.31842	0.31842	0.31842	0.31842
500		0.12285	0.12285	0.12285	0.12285	0.12285	0.12285	0.12285	0.12285	0.12285
30	Jeffreys	2.90909	2.90909	2.90909	2.90909	2.90909	2.90909	2.90909	2.90909	2.90909
50		1.48052	1.48052	1.48052	1.48052	1.48052	1.48052	1.48052	1.48052	1.48052
100		0.66327	0.66327	0.66327	0.66327	0.66327	0.66327	0.66327	0.66327	0.66327
200		0.31508	0.31508	0.31508	0.31508	0.31508	0.31508	0.31508	0.31508	0.31508
500		0.12235	0.12235	0.12235	0.12235	0.12235	0.12235	0.12235	0.12235	0.12235

To search for a suitable prior for the scale parameter ϕ of the Log-Normal distribution, different properties of the posterior distributions have been checked under the two assumed priors and for different values of the location parameter θ.

It is clear from Tables 1, 2 and 3 that as the sample size increases, posterior variances decrease. From Table 1, it can be seen that the posterior variances for Jeffreys prior are smaller for all the values of θ, considered in the simulation study. Specifically, for $\theta = 3$, the posterior variances are minimum. Tables 2 and 3 show that both the skewness and ex-kurtosis are positive. Therefore, both the posteriors are positively skewed and are lepto-kurtic. Skewness and ex-kurtosis decrease with the increase in sample size. The choices of location parameter θ and the scale parameter ϕ put no effect on these two quantities. Both co-efficients of skewness and ex-kurtosis for posterior distribution obtained under Jeffreys prior are minimum.

It can be concluded, when no prior information is in hand, that Jeffreys prior performs better than the Uniform prior for estimating the scale parameter ϕ of the Log-Normal model.

4. Bayes Estimators and Bayes Risks under Different Loss Functions

In this section, Bayes estimators (BE) and Bayes risks (BR) are derived for different loss functions under the considered priors.

4.1. BE and BR under Squared Error Loss Function (SELF). For an estimator ϕ^* of ϕ, the SELF is defined as follows.

$$L(\phi, \phi^*) = (\phi - \phi^*)^2 \tag{4.1}$$

The BE under this loss function is:

$$\phi^* = E_{\phi|\mathbf{x}}(\phi) \tag{4.2}$$

where $E_{\phi|\mathbf{x}}$ is the Expectation over the posterior distribution.

The BR under SELF is given by:

$$\rho(\phi^*) = E_{\phi|\mathbf{x}}(\phi^2) - (E_{\phi|\mathbf{x}}(\phi))^2 \tag{4.3}$$

The BEs and BRs under SELF are given in the following table.

Table 4. BEs and BRs under SELF

Prior	$BE=\frac{\beta}{\alpha-1}$	$BR=\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n-4}$	$\frac{2\left\{\sum_{i=1}^{n}(\ln x_i-\theta)^2\right\}^2}{(n-4)^2(n-6)}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n-2}$	$\frac{2\left\{\sum_{i=1}^{n}(\ln x_i-\theta)^2\right\}^2}{(n-2)^2(n-4)}$

4.2. BE and BR under Quadratic Loss Function (QLF). The QLF, for an estimator ϕ^* of the parameter ϕ, is defined as:

$$L(\phi,\phi^*)=\frac{(\phi-\phi^*)^2}{\phi^2} \tag{4.4}$$

The BE under the above loss function is given below.

$$\phi^*=\frac{E_{\phi|\mathbf{x}}\left(\phi^{-1}\right)}{E_{\phi|\mathbf{x}}\left(\phi^{-2}\right)} \tag{4.5}$$

The BR under QLF is of the following form.

$$\rho(\phi^*)=1-\frac{\left(E_{\phi|\mathbf{x}}\left(\phi\right)^{-1}\right)^2}{E_{\phi|\mathbf{x}}\left(\phi\right)^{-2}} \tag{4.6}$$

The BEs and BRs, using the two priors, under QLF are given in the following table.

Table 5. BEs and BRs under QLF

Prior	$BE=\frac{\beta}{\alpha+1}$	$BR=\frac{1}{\alpha+1}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n}$	$\frac{2}{n}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n+2}$	$\frac{2}{n+2}$

4.3. BE and BR under Weighted Loss Function (WLF). The mathematical form of this loss function, for an estimator ϕ^* of the parameter ϕ, is as under.

$$L(\phi,\phi^*)=\frac{(\phi-\phi^*)^2}{\phi} \tag{4.7}$$

The BE under WLF is of the following form.

$$\phi^*=\left\{E_{\phi|x}\left(\frac{1}{\phi}\right)\right\}^{-1} \tag{4.8}$$

The BR under WLF is written as under.

$$\rho(\phi^*) = E_{\phi|\mathbf{x}}(\phi) - \left\{E_{\phi|\mathbf{x}}\left(\frac{1}{\phi}\right)\right\}^{-1} \tag{4.9}$$

The following table contains BEs and BRs, using the two priors, under WLF.

Table 6. BEs and BRs under WLF

Prior	$BE = \frac{\beta}{\alpha}$	$BR = \frac{\beta}{\alpha(\alpha-1)}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i - \theta)^2}{n-2}$	$\frac{2\sum_{i=1}^{n}(\ln x_i - \theta)^2}{(n-2)(n-4)}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i - \theta)^2}{n}$	$\frac{2\sum_{i=1}^{n}(\ln x_i - \theta)^2}{n(n-2)}$

4.4. DeGroot Loss Function (DLF). For an estimator ϕ^* of the parameter ϕ, the DLF is written mathematically as follows.

$$L(\phi, \phi^*) = \frac{(\phi - \phi^*)^2}{{\phi^*}^2} \tag{4.10}$$

The BE under this loss function is as under.

$$\phi^* = \frac{E_{\phi|\mathbf{x}}(\phi^2)}{E_{\phi|\mathbf{x}}(\phi)} \tag{4.11}$$

Under DLF, the BR is of the form.

$$\rho(\phi^*) = 1 - \frac{\left\{E_{\phi|\mathbf{x}}(\phi)\right\}^2}{E_{\phi|\mathbf{x}}(\phi)^2} \tag{4.12}$$

The BEs and BRs, using the two priors, under DLF are contained in the following table.

Table 7. BEs and BRs under DLF

Prior	$BE = \frac{\beta}{\alpha-2}$	$BR = \frac{1}{\alpha-1}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i - \theta)^2}{n-6}$	$\frac{2}{n-4}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i - \theta)^2}{n-4}$	$\frac{2}{n-2}$

4.5. BE and BR under Precautionary Loss Function (PLF). Let ϕ^* be an estimator of a parameter ϕ, then PLF can be defined through the following equation.

$$L(\phi,\phi^*) = \frac{(\phi-\phi^*)^2}{\phi^*} \tag{4.13}$$

The BE under PLF is given below.

$$\phi^* = \sqrt{E_{\phi|\mathbf{x}}(\phi^2)} \tag{4.14}$$

The BR under PLF is as under.

$$\rho(\phi^*) = 2\left\{\sqrt{E_{\phi|\mathbf{x}}(\phi^2)} - E_{\phi|\mathbf{x}}(\phi)\right\} \tag{4.15}$$

The BEs and BRs, using the two priors, under PLF are presented in the following table.

Table 8. BEs and BRs under PLF

Prior	$BE = \frac{\beta}{\sqrt{(\alpha-1)(\alpha-2)}}$	$BR = 2\left\{\frac{\beta}{\sqrt{(\alpha-1)(\alpha-2)}} - \frac{\beta}{\alpha-1}\right\}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{(n-4)(n-6)}}$	$\frac{2\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{(n-4)(n-6)}} - \frac{2\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n-4}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{(n-2)(n-4)}}$	$\frac{2\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{(n-2)(n-4)}} - \frac{2\sum_{i=1}^{n}(\ln x_i-\theta)^2}{n-2}$

4.6. BE and BR under Simple Asymmetric Precautionary Loss Function (SAPLF). The SAPLF, for an estimator ϕ^* of a parameter ϕ^*, is defined as follows

$$L(\phi,\phi^*) = \frac{(\phi-\phi^*)^2}{\phi\phi^*} \tag{4.16}$$

The BE under SAPLF is as under.

$$\phi^* = \sqrt{\frac{E_{\phi|\mathbf{x}}(\phi)}{E_{\phi|\mathbf{x}}(\phi^{-1})}} \tag{4.17}$$

The BR under SAPLF is as follows.

$$\rho(\phi^*) = 2\left\{\sqrt{E_{\phi|\mathbf{x}}(\phi)\,E_{\phi|\mathbf{x}}(\phi^{-1})} - 1\right\} \tag{4.18}$$

The BEs and BRs, using the two priors, under SAPLF are shown in the following table.

Table 9. BEs and BRs under SAPLF

Prior	$BE = \frac{\beta}{\sqrt{\alpha(\alpha-1)}}$	$BR = 2\left\{\sqrt{\frac{\alpha}{\alpha-1}} - 1\right\}$
Uniform	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{(n-2)(n-4)}}$	$2\left\{\sqrt{\frac{n-2}{n-4}} - 1\right\}$
Jeffreys	$\frac{\sum_{i=1}^{n}(\ln x_i-\theta)^2}{\sqrt{n(n-2)}}$	$2\left\{\sqrt{\frac{n}{n-2}} - 1\right\}$

It can easily be depicted from the expressions of BRs in Tables 4−9 that Jeffreys prior requires less number of observations than the Uniform prior.

5. Simulation Study for Bayes Estimators and Bayes Risks under Different Loss Functions

A simulation study is carried out to obtain the BEs and BRs under different loss functions using different priors. The simulation process is repeated 10,000 times considering generation of random samples of sizes 30, 50, 100, 200 and 500 from Log-Normal distribution assuming $\phi = 1, 4, 7$ and $\theta = 1, 2, 3$, and the results have then been averaged. These results are presented in the following tables.

Table 10. BE and BR under SELF for different values of θ and ϕ

n	Prior	$\theta = 1$						$\theta = 2$						$\theta = 3$					
		$\phi = 1$		$\phi = 4$		$\phi = 7$		$\phi = 1$		$\phi = 4$		$\phi = 7$		$\phi = 1$		$\phi = 4$		$\phi = 7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30	Uniform	1.15610	0.11885	18.52246	30.5066	56.6850	285.3746	1.15067	0.11764	18.4261	30.18859	56.4148	283.1549	1.14740	0.11712	18.3917	30.07997	56.6257	284.8302
50		1.08966	0.05619	17.37730	14.2768	53.2779	134.0584	1.08900	0.05604	17.3905	14.30565	53.2108	133.8097	1.08531	0.05569	17.4230	14.3490	53.3653	134.5817
100		1.04295	0.02360	16.70100	6.05753	51.0567	56.58021	1.04104	0.02352	16.7038	6.053948	50.9718	56.40364	1.03978	0.02347	16.7008	6.05122	51.0835	56.62360
200		1.02092	0.01085	16.2807	2.75962	49.9615	25.98787	1.02177	0.01087	16.3434	2.781177	49.9828	26.01366	1.02010	0.01080	16.3055	2.76811	49.9114	25.94280
500		1.00884	0.00414	16.1477	1.05986	49.3953	9.918169	1.00857	0.00413	16.1233	1.056755	49.3473	9.898489	1.00763	0.00413	16.1270	1.05722	49.3409	9.89536
30	Jeffreys	1.06848	0.09363	17.0768	23.9193	52.3748	225.4040	1.06970	0.09394	17.1377	24.0990	52.4906	225.9511	1.07458	0.09475	17.1855	24.2322	52.5921	226.905
50		1.04362	0.04923	16.6159	12.4797	50.9967	117.5658	1.03705	0.04864	16.6749	12.5786	51.0691	117.9625	1.04420	0.04932	16.6611	12.5524	50.9845	117.615
100		1.01979	0.02210	16.3062	5.65119	49.9702	53.07748	1.01701	0.02198	16.3365	5.67291	50.0336	53.21180	1.02176	0.02220	16.2968	5.64316	50.0850	53.2689
200		1.01145	0.01054	16.1601	2.69165	49.5063	25.2586	1.01029	0.01052	16.1555	2.68966	49.4593	25.21420	1.00971	0.01051	16.1558	2.6904	49.5561	25.3112
500		1.00457	0.00409	16.0601	1.04411	49.1995	9.80001	1.00423	0.00408	16.0619	1.04442	49.1835	9.792590	1.00390	0.00408	16.0679	1.04520	49.2420	9.81679

Table 11. BE and BR under QLF for different values of θ and ϕ

n	Prior	$\theta = 1$						$\theta = 2$						$\theta = 3$					
		$\phi = 1$		$\phi = 4$		$\phi = 7$		$\phi = 1$		$\phi = 4$		$\phi = 7$		$\phi = 1$		$\phi = 4$		$\phi = 7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30	Uniform	0.99683	0.06667	15.9892	0.06667	49.1442	0.06667	1.00001	0.06667	16.0197	0.06667	49.0209	0.06667	0.99872	0.06667	15.9924	0.06667	48.9732	0.06667
50		1.00010	0.04000	15.9874	0.04000	49.0865	0.04000	0.99895	0.04000	15.9871	0.04000	49.0378	0.04000	0.99732	0.04000	16.0079	0.04000	49.0887	0.04000
100		0.99876	0.02000	15.9954	0.02000	49.0019	0.02000	1.00020	0.02000	15.9781	0.02000	49.1506	0.02000	1.00026	0.02000	16.0037	0.02000	48.9685	0.02000
200		1.00048	0.01000	15.9864	0.01000	49.0171	0.01000	0.99951	0.01000	16.0114	0.01000	48.9663	0.01000	1.00032	0.01000	16.0211	0.01000	49.0125	0.01000
500		0.99990	0.00400	16.0153	0.00400	49.0476	0.00400	0.99996	0.00400	15.9822	0.00400	49.0226	0.00400	1.00059	0.00400	16.0141	0.00400	49.0054	0.00400
30	Jeffreys	0.93610	0.06250	14.9223	0.06250	45.8692	0.06250	0.93545	0.06250	15.0120	0.06250	45.8977	0.06250	0.93735	0.06250	14.9988	0.06250	45.9757	0.06250
50		0.96133	0.03846	15.3730	0.03846	47.0391	0.03846	0.96016	0.03846	15.3760	0.03846	47.3039	0.03846	0.96199	0.03846	15.3717	0.03846	46.9715	0.03846
100		0.98191	0.01961	15.6737	0.01961	47.9365	0.01961	0.97958	0.01961	15.6617	0.01961	48.0052	0.01961	0.98258	0.01961	15.6731	0.01961	48.0314	0.01961
200		0.99101	0.00990	15.8256	0.00990	48.5146	0.00990	0.99037	0.00990	15.8430	0.00990	48.5811	0.00990	0.98921	0.00990	15.8360	0.00990	48.5113	0.00990
500		0.99697	0.00398	15.9414	0.00398	48.7507	0.00398	0.99711	0.00398	15.9281	0.00398	48.7708	0.00398	0.99693	0.00398	15.9442	0.00398	48.7972	0.00398

Table 12. BE and BR under DLF for different values of θ and ϕ

n	Prior	$\theta = 1$						$\theta = 2$						$\theta = 3$					
		$\phi = 1$		$\phi = 4$		$\phi = 7$		$\phi = 1$		$\phi = 4$		$\phi = 7$		$\phi = 1$		$\phi = 4$		$\phi = 7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30	Uniform	1.24656	0.07692	19.9602	0.076923	61.1146	0.076923	1.24301	0.07692	19.9337	0.076923	61.3169	0.076923	1.24796	0.07692	19.9892	0.076923	61.1388	0.076923
50		1.13850	0.04348	18.1615	0.043478	55.5427	0.043478	1.13462	0.04348	18.1893	0.043478	55.6496	0.043478	1.13137	0.04348	18.1539	0.043478	55.5593	0.043478
100		1.06318	0.02083	17.0610	0.020833	52.2324	0.020833	1.06191	0.02083	17.0426	0.020833	52.1436	0.020833	1.06028	0.02083	17.0164	0.020833	52.1689	0.020833
200		1.03232	0.01020	16.4914	0.010204	50.5055	0.010204	1.03061	0.01020	16.4988	0.010204	50.5139	0.010204	1.03112	0.01020	16.4932	0.010204	50.5463	0.010204
500		1.01192	0.00403	16.1892	0.004032	49.5339	0.004032	1.01224	0.00403	16.2136	0.004032	49.6621	0.004032	1.01274	0.00403	16.1880	0.004032	49.6351	0.004032
30	Jeffreys	1.15722	0.07143	18.4422	0.071429	56.6918	0.071429	1.15580	0.07143	18.4283	0.071429	56.3957	0.071429	1.14590	0.07143	18.4731	0.071429	56.6648	0.071429
50		1.08962	0.04167	17.3342	0.041667	53.2016	0.041667	1.08643	0.04167	17.3816	0.041667	53.4161	0.041667	1.09065	0.04167	17.3178	0.041667	53.2236	0.041667
100		1.04305	0.02041	16.6493	0.020408	50.8967	0.020408	1.04361	0.02041	16.6594	0.020408	51.0393	0.020408	1.04216	0.02041	16.6128	0.020408	51.0043	0.020408
200		1.02001	0.01010	16.3188	0.010101	49.9748	0.010101	1.02134	0.01010	16.3336	0.010101	49.9855	0.010101	1.01959	0.01010	16.3251	0.010101	50.0521	0.010101
500		1.00817	0.00402	16.1329	0.004016	49.4114	0.004016	1.00777	0.00402	16.1279	0.004016	49.4766	0.004016	1.00817	0.00402	16.1195	0.004016	49.4526	0.004016

Table 13. BE and BR under PLF for different values of θ and ϕ

n	Prior	$\theta=1$						$\theta=2$						$\theta=3$					
		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30	Uniform	1.19900	0.09412	19.2689	1.50411	58.9176	4.62280	1.20447	0.09406	19.1396	1.50174	58.8260	4.61562	1.20300	0.09400	19.1579	1.50317	58.7515	4.60977
50		1.10650	0.04890	17.8331	0.78000	54.1936	2.38243	1.11411	0.04889	17.8105	0.78298	54.4042	2.39169	1.11085	0.04892	17.7623	0.78086	54.2870	2.38653
100		1.04918	0.02208	16.8483	0.35191	51.4916	1.07839	1.05408	0.02203	16.8532	0.35296	51.7212	1.08320	1.05465	0.02206	16.8587	0.35307	51.5888	1.08042
200		1.02585	0.01051	16.4034	0.16813	50.3456	0.51505	1.02526	0.01049	16.4234	0.16802	50.2900	0.51448	1.02659	0.01050	16.4062	0.16784	50.2758	0.51433
500		1.01034	0.00408	16.1673	0.06524	49.4900	0.19976	1.01074	0.00408	16.1734	0.06528	49.5453	0.19998	1.01002	0.00408	16.1553	0.06521	49.4755	0.19970
30	Jeffreys	1.10422	0.08083	17.7628	1.29227	54.4858	3.96394	1.11366	0.08065	17.8432	1.29813	54.7219	3.98112	1.10893	0.08077	17.7381	1.29048	54.7425	3.98262
50		1.06769	0.04483	17.0319	0.71721	52.1369	2.19548	1.06427	0.04496	17.0239	0.71688	52.1163	2.19462	1.06592	0.04502	17.0836	0.71939	52.0445	2.19159
100		1.03147	0.02115	16.5329	0.33915	50.4415	1.03473	1.03123	0.02111	16.4857	0.33818	50.5669	1.03730	1.03206	0.02114	16.5087	0.33865	50.5803	1.03757
200		1.01443	0.01029	16.2330	0.16439	49.6779	0.50307	1.01520	0.01028	16.2528	0.16459	49.7859	0.50417	1.01576	0.01029	16.2391	0.16445	49.8353	0.50466
500		1.00522	0.00404	16.0991	0.06472	49.3332	0.19832	1.00423	0.00405	16.0972	0.06471	49.3168	0.19826	1.00528	0.00405	16.0805	0.06465	49.2540	0.19801

Table 14. BE and BR under WLF for different values of θ

n	Prior	$\theta=1$						$\theta=2$						$\theta=3$					
		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30	Uniform	1.07093	0.08238	17.0964	1.31511	52.3845	4.02958	1.07238	0.08249	17.1583	1.31987	52.2952	4.02271	1.06978	0.08229	17.1576	1.31982	52.7499	4.05769
50		1.04137	0.04528	16.7085	0.72646	51.1505	2.22394	1.04207	0.04531	16.6322	0.72314	51.1362	2.22332	1.04296	0.04535	16.6528	0.72404	51.1519	2.22400
100		1.02233	0.02130	16.3509	0.34065	49.9914	1.04149	1.02088	0.02127	16.3070	0.33973	49.8856	1.03928	1.01842	0.02122	16.3514	0.34066	49.9854	1.04136
200		1.01161	0.01032	16.1406	0.16470	49.4889	0.50499	1.01038	0.01031	16.1580	0.16488	49.4440	0.50453	1.01029	0.01031	16.1529	0.16483	49.5732	0.50585
500		1.01161	0.01032	16.0685	0.06479	49.1605	0.19823	1.01038	0.01031	16.0550	0.06474	49.1999	0.19839	1.01029	0.01031	16.0480	0.06471	49.2116	0.19843
30	Jeffreys	0.99817	0.07130	16.0086	1.14347	48.8909	3.49221	1.00151	0.07154	15.9995	1.14283	48.9981	3.49986	0.99775	0.07127	15.9993	1.14281	49.0448	3.50320
50		1.00190	0.04175	15.9760	0.66567	49.1387	2.04745	1.00016	0.04167	15.9847	0.66603	49.0197	2.04249	0.99927	0.04164	16.0619	0.66925	48.8018	2.03341
100		0.99962	0.02040	16.0068	0.32667	48.9493	0.99897	1.00114	0.02043	16.0117	0.32677	49.1360	1.00278	1.00192	0.02045	15.9928	0.32638	49.0456	1.00093
200		1.00433	0.00405	16.0147	0.16177	48.9206	0.49415	1.00342	0.00405	16.0418	0.16204	49.0315	0.49527	1.00444	0.00405	16.0020	0.16164	48.9709	0.49466
500		1.00061	0.00402	15.9865	0.06420	48.9984	0.19678	1.00041	0.00402	15.9710	0.06414	48.9624	0.19664	0.99911	0.00401	16.0096	0.06430	49.0144	0.19685

Table 15. BE and BR under SAPLF for different values of θ

n	Prior	$\theta=1$						$\theta=2$						$\theta=3$					
		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$		$\phi=1$		$\phi=4$		$\phi=7$	
		BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR	BE	BR
30	Uniform	1.10882	0.07550	17.8537	0.07550	54.2717	0.07550	1.10565	0.07550	17.7617	0.07550	54.1956	0.07550	1.11006	0.07550	17.8116	0.07550	54.4152	0.07550
50		1.06607	0.04302	17.0353	0.04302	52.1882	0.04302	1.06244	0.04302	17.0284	0.04302	52.1682	0.04302	1.05939	0.04302	17.0585	0.04302	52.2884	0.04302
100		1.03035	0.02073	16.5174	0.02073	50.6166	0.02073	1.02912	0.02073	16.4974	0.02073	50.4981	0.02073	1.02754	0.02073	16.5164	0.02073	50.5237	0.02073
200		1.01661	0.01018	16.2418	0.01018	49.7278	0.01018	1.01493	0.01018	16.2476	0.01018	49.7359	0.01018	1.01543	0.01018	16.2452	0.01018	49.8430	0.01018
500		1.00585	0.00403	16.1019	0.00403	49.2450	0.00403	1.00647	0.00403	16.1005	0.00403	49.2936	0.00403	1.00464	0.00403	16.0890	0.00403	49.3088	0.00403
30	Jeffreys	1.03686	0.07020	16.5563	0.07020	50.5246	0.07020	1.02791	0.07020	16.5673	0.07020	50.6038	0.07020	1.03678	0.07020	16.6341	0.07020	50.6590	0.07020
50		1.02010	0.04124	16.2593	0.04124	50.1087	0.04124	1.02411	0.04124	16.3179	0.04124	49.8674	0.04124	1.02081	0.04124	16.3662	0.04124	50.0359	0.04124
100		1.01204	0.02031	16.1777	0.02031	49.4667	0.02031	1.01064	0.02031	16.1720	0.02031	49.4841	0.02031	1.01039	0.02031	16.1187	0.02031	49.4309	0.02031
200		1.00596	0.01008	16.0850	0.01008	49.1865	0.01008	1.00423	0.01008	16.0760	0.01008	49.2391	0.01008	1.00501	0.01008	16.0741	0.01008	49.3851	0.01008
500		1.00243	0.00401	16.0383	0.00401	49.0779	0.00401	1.00275	0.00401	16.0324	0.00401	49.1231	0.00401	1.00125	0.00401	16.0218	0.00401	49.0598	0.00401

From Tables 10− 15, it is clear that Bayes estimates approach to the true value and Bayes risks approach to zero with increase in sample size. Jeffreys prior requires less number of observations than the Uniform prior for efficient estimation. Also, Bayes risks under SELF, QLF, DLF, PLF, WLF and SAPLF using Jeffreys prior are minimum. The results under QLF are much better than the results of the remaining assumed loss functions. Also, the results show that the parameter is over and under estimated and the degree of over and under estimation is minimum under Jeffreys and QLF. For large scale, the degree of over estimation is significant and SELF gives much poor estimates under both the priors.

6. Application

To illustrate the proposed methodology, we used a published data on 137 inoperable lung cancer patients. The data were published in a book titled "The Statistical Analysis of Failure Time Data", by Kalbfleisch and Prentice [3]. For our purpose, obviously these are anonymous data since we don't know the patients' identifications. Also, no identification of the patients are given by the authors in their book. In our study, these data are analyzed anonymously. There is no ethics committee/institutional review board (or data production agency/commissioner) that approved this retrospective study.

The BEs and BRs under the assumed loss functions, for the real data set, are showcased in the tables given below.

Table 16. BEs and BRs under SELF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.65964	2.075529	6.316937	0.6092167	3.034387	0.1405726
Jeffreys	11.48690	1.984194	6.223353	0.582408	2.989433	0.1343866

Table 17. BEs and BRs under QLF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.31921	0.014598	6.132501	0.0145985	2.945792	0.014598
Jeffreys	11.15634	0.014389	6.0442630	0.014389	2.903407	0.014388

Table 18. BEs and BRs under DLF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.83765	0.015037	6.413379	0.015037	3.080714	0.015037
Jeffreys	11.65964	0.014815	6.316937	0.014815	3.034387	0.014815

Table 19. BEs and BRs under PLF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.748300	25.77324	6.364975	13.36979	3.057463	6.258806
Jeffreys	11.572950	25.31613	6.269970	13.15072	3.011826	6.16111

Table 20. BEs and BRs under WLF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.48690	0.1727354	6.223353	0.0935842	2.989433	0.0449539
Jeffreys	11.31921	0.167692	6.1325010	0.090852	2.94579	0.043641

Table 21. BEs and BRs under SAPLF using Real Data Set

Prior	θ=1		θ=2		θ=3	
	BE	BR	BE	BR	BE	BR
Uniform	11.572950	0.0149815	6.269970	0.0149815	3.011826	0.0149815
Jeffreys	11.402750	0.0147603	6.1777600	0.0147603	2.967532	0.0147603

After examining the results presented in tables 16−21, it can easily be concluded that the performance of Jeffreys prior is better. Also, QLF performs much better than the rest of the assumed loss functions for estimating the scale parameter of Log-Normal distribution.

6.1. Graphical Presentation. The graphs of the posterior distributions, under the two priors, are sketched for different values of the location parameter using the real data set.

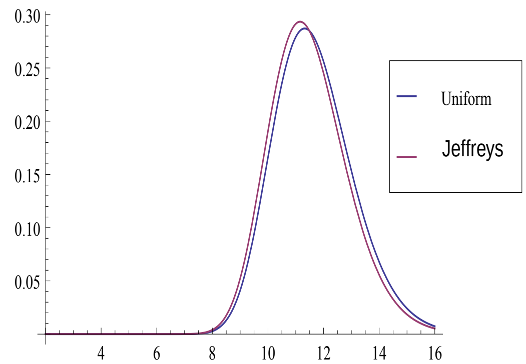

Figure 1. Graph of the Posterior Distributions under Uniform and Jeffreys Priors for $\theta = 1$

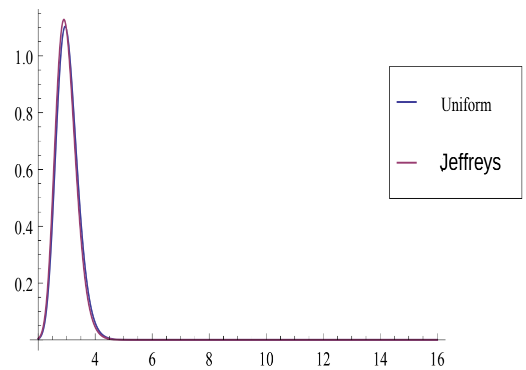

Figure 2. Graph of the Posterior Distributions under Uniform and Jeffreys Priors for $\theta = 2$

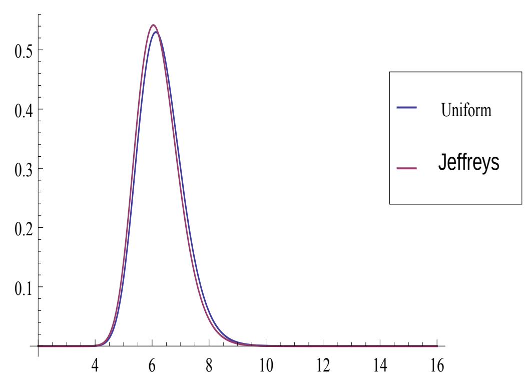

Figure 3. Graph of the Posterior Distributions under Uniform and Jeffreys Priors for $\theta = 3$

The figures 1, 2 and 3 clearly favor Jeffreys as the best non-informative prior for the scale parameter, as the curves of the posterior distribution under this prior are less skewed. For $\theta = 3$, the graph has least posterior skewness.

Conclusions

In this paper, comparisons of non-informative priors for estimating the scale parameter of log-normal distribution have been presented. The two non-informative priors, Uniform and Jeffreys, have been considered and then their posterior distributions were derived. Comparisons have been made on the basis of posterior variance, coefficients of skewness, ex-kurtosis and Bayes risks. In these comparisons, we have observed that Jeffreys prior gives the less posterior variances, less posterior skewness, less ex-kurtosis and less Bayes risks. The simulation results for large value of ϕ are not much convincing. The degree of over estimation significantly increases for large values of scale. SELF performs poorly for large values of ϕ. Also, the results under Quadratic Loss Function (QLF) are efficient with minimum risks. A real data set of lung cancer patients had also been analyzed which verified the simulation results.

Therefor, it is concluded that Jeffreys prior is the appropriate prior when no prior information is available. Also, the Quadratic Loss Function (QLF) is recommended to be used for the estimation of the scale parameter of Log-Normal distribution.

This work can further be extended by considering different (weakly) informative priors and different loss functions. The location parameter can also be estimated for known scale parameter in future research work. Both the parameters can be assumed unknown and can be estimated simultaneously.

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Extended generalized extreme value distribution with applications in environmental data

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Abstract

In probability theory and statistics, the generalized extreme value (GEV) distribution is a family of continuous probability distributions developed within extreme value theory, which has wide applicability in several areas including hydrology, engineering, science, ecology and finance. In this paper, we propose three extensions of the GEV distribution that incorporate an additional parameter. These extensions are more flexible than the GEV distribution, i.e., the additional parameter introduces skewness and to vary tail weight. In these three cases, the GEV distribution is a particular case. The parameter estimation of these new distributions is done under the Bayesian paradigm, considering vague priors for the parameters. Simulation studies show the efficiency of the proposed models. Applications to river quotas and rainfall show that the generalizations can produce more efficient results than is the standard case with GEV distribution.

Keywords. **Keywords:** Extreme value theory; Generalized extreme value distribution, Generalized classes of distributions, Environmental and Economic data.

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1. Introduction

In many areas of knowledge, the study of the behavior of a variable is related to the tail of the distribution. These observations, although they can occur with lower frequency than the central part of the distribution, can be of the greatest interest to the researcher; in some situations, an occurrence in the tail can cause a great impact on society, such as an index of very high rainfall or a high level of river flow, among other variables such as temperature. These changes cause more impact than in the mean of the index, according to [16] and [18]. One of the major challenges of analyzing extremes is proposing a model and estimating its parameters with little information due to the scant data available. Often the tail of a statistical distribution is commonly seen as normal, or exponential tails may not be the most suitable for this type of data. Another challenge in analyzing this type of data is estimating with a high level of precision the probability of the occurrence of events that not have been observed. [2] shows in detail the difficulties in estimating extreme events. To answer these questions, the extreme value theory has been developed to analyze these types of occurrences, proposing specific distributions for this type of observation. One approach is to analyze this type of data and group data in maxima every n observations, and then to model the block maxima. A pioneering work in this area was done by [1], who presented an asymptotic result for distribution of maximum block n. There was not much progress in the area until the 1950s, when works such as those by [21] and [8] showed that the only non-trivial limiting distribution of affinely normalised maximum is the generalized extreme value (GEV) distribution.

A random variable X follows the GEV distribution if its cumulative distribution function (cdf) is given by

$$G(x;\mu,\sigma,\xi) = \begin{cases} \exp\left\{-[1+\xi(x-\mu)/\sigma]^{-1/\xi}\right\}, & \xi \neq 0, \\ \exp\left\{-\exp[-(x-\mu)/\sigma]\right\}, & \xi \to 0, \end{cases} \tag{1.1}$$

and is defined in the set $\{x : 1+\xi(x-\mu)/\sigma > 0\}$, where $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter and $\xi \in \mathbb{R}$ is a shape parameter. Thus, for $\xi > 0$, the expression just given for the cumulative distribution function is valid for $x > \mu - \sigma/\xi$, while for $\xi < 0$ it is valid for $x < \mu + \sigma/(-\xi)$. In the first case, at the lower end-point it equals 0; in the second case, at the upper end-point, it equals 1. For $\xi = 0$ the expression in (1.1) is interpreted by taking the limit as $\xi \to 0$. The probability density function (pdf) corresponding to (1.1) is given by

$$g(x;\mu,\sigma,\xi) = \begin{cases} \sigma^{-1}[1+\xi(x-\mu)/\sigma]^{-(1/\xi)-1}\exp\left\{-[1+\xi(x-\mu)/\sigma]^{-1/\xi}\right\}, & \xi \neq 0, \\ \sigma^{-1}\exp[-(x-\mu)/\sigma]\exp\left\{-\exp[-(x-\mu)/\sigma]\right\}, & \xi \to 0. \end{cases}$$

Estimates of extreme quantiles z_u of the annual maximum distribution are then obtained by inverting Equation (1.1)

$$z_u = \begin{cases} \mu + \frac{\sigma}{\xi}\left\{[-\log(u)]^{-\xi} - 1\right\}, & \xi \neq 0, \\ \mu - \sigma\log\left[-\log(u)\right], & \xi \to 0, \end{cases}$$

where $u \in [0,1]$.

In GEV distribution, if x^* is the upper limit of distribution G, according to [5] and [4] the shape parameter ξ satisfies

$$\lim_{x\to x^*} \frac{1-G(x;\mu,\sigma,\xi)}{xg(x;\mu,\sigma,\xi)} = \xi, \tag{1.2}$$

if $\xi > 0$ and $x^* = \infty$, and

$$\lim_{x \to x^*} \frac{1 - G(x; \mu, \sigma, \xi)}{(x - x^*)\, g(x; \mu, \sigma, \xi)} = \xi, \tag{1.3}$$

if $\xi < 0$ and $x^* < \infty$.

There has been an increased interest in defining new classes of univariate continuous distributions introducing additional shape parameters to the baseline model. In many applied areas such as lifetime analysis [7], environmental [17], medical [14], economy [11], there is a clear need for extended forms of the classical distributions, that is, new distributions which are more flexible to model real data in these areas since the data can present a high degree of skewness and kurtosis. In the context of extreme values, Papastathopoulos and Tawn (2013) studied three extensions of the generalised Pareto distribution. The extended distributions have attracted several statisticians to develop new models because the computational and analytical facilities available in standard softwares can easily tackle the problems involved in computing special functions in these extended distributions.

In recent years, several common distributions have been generalized via exponentiation. Let $G(x)$ be the cdf of any continuous baseline distribution. The cdf of the exponentiated-G distribution is defined by elevating $G(x)$ to the power α, say $F(x) = G(x)^\alpha$, where $\alpha > 0$ denotes an extra shape parameter. The baseline distribution is obtained as a special case when $\alpha = 1$. The pdf corresponding can be written as

$$f(x) = \alpha\, g(x)\, G(x)^{\alpha - 1}, \; x \in \mathbb{R}. \tag{1.4}$$

where $g(x)$ is the pdf of baseline distribution. Following this idea, [6] introduced the exponentiated exponential distribution as a generalization of the exponential distribution. In the same way, [13] proposed four more exponentiated distributions which generalize the gamma, Weibull, Gumbel and Fréchet distributions and provided some mathematical properties for each distribution. Several other authors have considered exponentiated distributions, for example, [12], [7], [20], [9] and [10]. Recently, [17] studied a broad family of univariate distributions through a particular case of Stacy's generalized gamma distribution. This new family stems from the general class: if $G(x)$ denotes the baseline cdf of a random variable, then a generalized class of distributions can be defined by

$$F(x) = 1 - \gamma\{\delta, -\log[G(x)]\}, \quad x \in \mathbb{R}, \; \delta > 0, \tag{1.5}$$

where

$$\gamma(\delta, z) = \frac{1}{\Gamma(\delta)} \int_0^z t^{\delta - 1} \mathrm{e}^{-t} \mathrm{d}t,$$

denotes the incomplete gamma function and $\Gamma(\cdot)$ is the gamma function. This family of distributions has pdf given by

$$f(x) = \frac{1}{\Gamma(\delta)} \{-\log[G(x)]\}^{\delta - 1}\, g(x).$$

[19] proposed a class of generalized distributions based on the transmutation map approach. Let F_1 and F_2 be the cdf's of two distributions with a common sample space. The general rank transmutation as given in [19] is defined as $G_{R_{12}}(u) = F_2(F_1^{-1}(u))$ and $G_{R_{21}}(u) = F_1(F_2^{-1}(u))$. Notice that the inverse cdf also known as quantile function is defined as $F^{-1}(y) = \inf_{x \in \mathbb{R}}\{F(x) \geq y\}$, for $y \in [0, 1]$. The functions $G_{R_{12}}(u)$ and $G_{R_{21}}(u)$ are both mapped in the unit interval $I = [0, 1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $G_{R_{ij}}(0) = 0$ and $G_{R_{ij}}(1) = 1$, for $i = 1, 2$. A quadratic rank transmutation map is defined as $G_{R_{12}}(u) = u + \lambda\, u(1 - u), \; |\lambda| \leq 1$ from which follows that the cdf satisfies the relationship

$$F_2(x) = (1 + \lambda) F_1(x) - \lambda [F_1(x)]^2, \tag{1.6}$$

which on differentiation yields $f_2(x) = f_1(x)[1 + \lambda - 2\,\lambda\, F_1(x)]$, where $f_1(x)$ and $f_2(x)$ are the corresponding pdfs associated with cdf $F_1(x)$ and $F_2(x)$ respectively.

The aim of this paper is to propose new modifications to GEV models that incorporate an additional parameter, with the hope that it will yield "better" results in certain practical situations. We create three new modifications for the GEV distribution: dual gamma GEV distribution, exponentiated GEV distribution and transmuted GEV distribution. The major benefit of these models is their ability to fit the skewed data better than GEV distribution.

The article is organized as follows. In Section 2, we define the dual gamma generalized extreme value (GGEV), exponentiated generalized extreme value (EGEV) and transmuted generalized extreme value (TGEV) distributions, derive the quantile functions of models and provide plots of such functions for selected parameter values. In Section 3, inference procedure is carried out under the Bayesian paradigm, with prior information playing an important role in the estimation procedures. Section 4 illustrates the method with a few simulated examples. Section 5 presents two applications to extreme data analysis. Concluding remarks are addressed in Section 6.

2. Construction of extreme value models

In this section, we present three new probability density functions that are generalizations of the GEV density. We illustrate the flexibility of these distributions and provide plots of the density function for selected parameter values.

2.1. The dual gamma generalized extreme value distribution (GGEV). Taking the GEV distribution as the baseline model in Equation (1.5), we have

$$(2.1)\qquad F(x;\mu,\sigma,\xi,\delta) = \begin{cases} 1-\gamma(\delta,[1+\xi(x-\mu)/\sigma]^{-1/\xi}), & \xi\neq 0,\\ 1-\gamma(\delta,\exp[-(x-\mu)/\sigma]), & \xi\to 0,\end{cases}$$

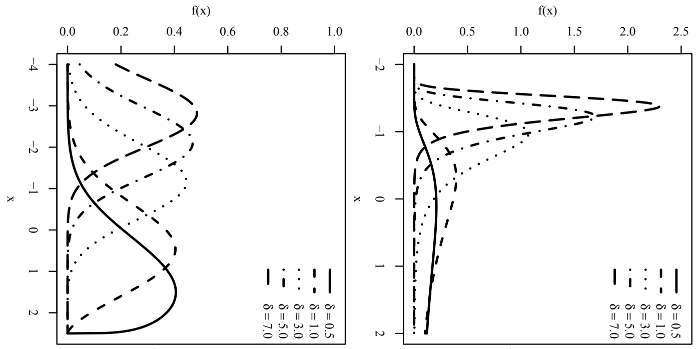
where $\delta>0$. The corresponding pdf has a very simple form

$$f(x;\mu,\sigma,\xi,\delta) = \begin{cases} \frac{\sigma^{-1}}{\Gamma(\delta)}[1+\xi(x-\mu)/\sigma]^{-(\delta/\xi)-1}\exp\left\{-[1+\xi(x-\mu)/\sigma]^{-1/\xi}\right\}, & \xi\neq 0,\\ \frac{\sigma^{-1}}{\Gamma(\delta)}\exp\left\{-\delta[(x-\mu)/\sigma]\right\}\exp\left\{-\exp[-(x-\mu)/\sigma]\right\}, & \xi\to 0.\end{cases}$$

The quantile function of GGEV distribution is given by

$$z_u = \begin{cases} \mu+\frac{\sigma}{\xi}\left\{\left[Q^{-1}(\delta,(1-u))\right]^{-\xi}-1\right\}, & \xi\neq 0,\\ \mu-\sigma\log\left[Q^{-1}(\delta,(1-u))\right], & \xi\to 0,\end{cases}$$

where $u\in[0,1]$ and $Q^{-1}(\delta,u)$ is the inverse function of $Q(\delta,x)=\gamma(\delta,x)$. Some plots of the GGEV density functions are displayed in Figure 1. The case where $\delta=1$ is the particular case of standard GEV distribution. As this distribution has not the form of a GEV distribution, it not has some properties as max-stability. However, applications results shown that the flexibility of this class of distribution allow some predictive advantages compared with standard GEV.



(a) $\xi = -0.4$ (b) $\xi = 0.4$

Figure 1. Plot for the GGEV density for some parameter values; $\mu = 0$ and $\sigma = 1$.

In the Proposition 2.1, we provide some useful properties of the GGEV distribution.

2.1. Proposition. *Let $X \sim (\mu, \sigma, \xi, \delta)$. Then, first moment, variance, skewness and kurtosis are given by*

(a) $\mathbb{E}(X) = \dfrac{\mu\xi\Gamma(\delta) + \sigma[\Gamma(\delta - \xi) - \Gamma(\delta)]}{\xi\Gamma(\delta)}$, $\xi \neq 0$ *and* $\xi < \delta$. *When* $\xi = 0$, *we have* $\mathbb{E}(X) = \mu - \sigma\psi(\delta)$, *where* $\psi(\cdot)$ *is derivative of the logarithm of the gamma function.*

(b) $Var(X) = \dfrac{\sigma^2[\Gamma(\delta)\Gamma(\delta - 2\xi) - \Gamma^2(\delta - \xi)]}{\xi^2\Gamma^2(\delta)}$, $\xi \neq 0$ *and* $\xi < \delta/2$. *When* $\xi = 0$, $Var(X) = \sigma^2\psi(1, \delta)$.

(c) $$\gamma_1 = \begin{cases} \dfrac{\Gamma^2(\delta)\Gamma(\delta-3\xi) - 3\Gamma(\delta)\Gamma(\delta-\xi)\Gamma(\delta-2\xi) + 2\Gamma^3(\delta-\xi)}{[\Gamma(\delta)\Gamma(\delta-2\xi) - \Gamma^2(\delta-\xi)]^{3/2}}, & \text{if } \xi > 0 \text{ and } \xi < \delta/3, \\ -\dfrac{\Gamma^2(\delta)\Gamma(\delta-3\xi) - 3\Gamma(\delta)\Gamma(\delta-\xi)\Gamma(\delta-2\xi) + 2\Gamma^3(\delta-\xi)}{[\Gamma(\delta)\Gamma(\delta-2\xi) - \Gamma^2(\delta-\xi)]^{3/2}}, & \text{if } \xi < 0, \\ -\dfrac{\psi(2,\delta)}{[\psi(1,\delta)]^{3/2}}, & \text{if } \xi = 0. \end{cases}$$

(d) $$\gamma_2 = \begin{cases} \dfrac{\Gamma^3(\delta)\Gamma(\delta-4\xi) - 4\Gamma^2(\delta)\Gamma(\delta-3\xi)\Gamma(\delta-\xi) + 6\Gamma(\delta)\Gamma(\delta-2\xi)\Gamma^2(\delta-\xi) - 3\Gamma^4(\delta-\xi)}{[\Gamma(\delta)\Gamma(\delta-2\xi) - \Gamma^2(\delta-\xi)]^2} - 3, & \text{if } \xi \neq 0 \text{ and } \xi < \frac{\delta}{4}, \\ \dfrac{3[\psi(1,\delta)]^2 + \psi(3,\delta)}{[\psi(1,\delta)]^2} - 3, & \text{if } \xi = 0, \\ \infty, & \text{if } \xi \geq \delta/4. \end{cases}$$

These results (a), (b), (c) and (d) are directly obtained from the definition of each measure.

2.1. Remark. The density function of X (GGEV distribution) can be expressed as

$$f(x; \mu, \sigma, \xi, \delta) = \frac{[1 + \xi(x - \mu)/\sigma]^{\frac{-(\delta-1)}{\xi}}}{\Gamma(\delta)} \cdot g(x; \mu, \sigma, \xi),$$

where $g(x; \mu, \sigma, \xi)$ is the pdf of the GEV distribution. The multiplying quantity $\frac{[1+\xi(x-\mu)/\sigma]^{\frac{-(\delta-1)}{\xi}}}{\Gamma(\delta)}$ works as a corrected factor for the pdf of the GEV distribution.

2.2. The exponentiated generalized extreme value distribution (EGEV). Now inserting (1.1) into (1.4) we obtain the pdf of exponentiated generalized extreme value (EGEV) distribution

$$f(x; \mu, \sigma, \xi, \alpha) = \begin{cases} \alpha\,\sigma^{-1}[1 + \xi(x - \mu)/\sigma]^{-(1/\xi)-1} \exp\left\{-\alpha[1 + \xi(x - \mu)/\sigma]^{-1/\xi}\right\}, & \xi \neq 0, \\ \alpha\,\sigma^{-1} \exp[-(x - \mu)/\sigma] \exp\{-\alpha \exp[-(x - \mu)/\sigma]\}, & \xi \to 0. \end{cases} \quad (2.2)$$

The EGEV cdf can be expressed as

$$F(x;\mu,\sigma,\xi,\alpha) = \begin{cases} \exp\left\{-\alpha[1+\xi(x-\mu)/\sigma]^{-1/\xi}\right\}, & \xi \neq 0, \\ \exp\left\{-\alpha \exp[-(x-\mu)/\sigma]\right\}, & \xi \to 0, \end{cases} \tag{2.3}$$

where $\alpha > 0$. The quantile function corresponding to Equation (2.3) is

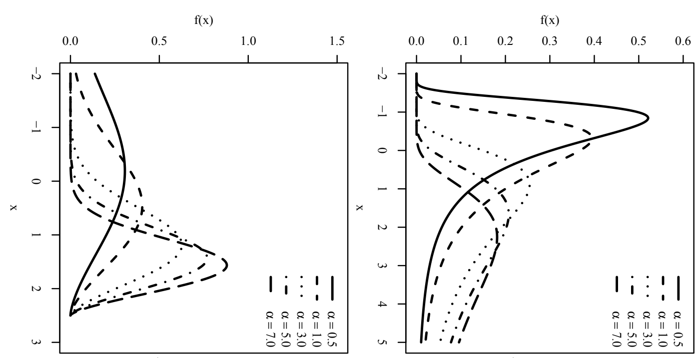
$$z_u = \begin{cases} \mu + \frac{\sigma}{\xi}\left\{\left[-\frac{1}{\alpha}\log(u)\right]^{-\xi} - 1\right\}, & \xi \neq 0, \\ \mu - \sigma \log\left[-\frac{1}{\alpha}\log(u)\right], & \xi \to 0, \end{cases}$$

where $u \in [0, 1]$.

2.2. Proposition. *The EGEV distribution is a particular case of GEV distribution. The Proof is shown in appendix.*

The result of this proposition hold important properties for EGEV distribution as for example the max-stability, and the shape parameter form obtained from Equations (1.2) and (2.5) is ξ.

Figure 2 displays some plots of the density function (2.2) for some parameter values. The case where $\alpha = 1$ is the particular case of standard GEV distribution.



(a) $\xi = -0.4$ (b) $\xi = 0.4$

Figure 2. Plot for the EGEV density for some parameter values; $\mu = 0$ and $\sigma = 1$.

In the Proposition 2.3, we provide some useful properties of the EGEV distribution.

2.3. Proposition. *Let* $X \sim (\mu, \sigma, \xi, \alpha)$. *Then, first moment and variance are given by*

(a) $\mathbb{E}(X) = \dfrac{\mu\xi + \sigma[\alpha^{\xi}\Gamma(1-\xi) - 1]}{\xi}$, $\xi \neq 0$ *and* $\xi < 1$. *When* $\xi = 0$, *we have* $\mathbb{E}(X) = \mu + \sigma[\zeta + \ln\alpha]$, *where* $\zeta = 0.577215$ *is the Euler's constant.*

(b) *Var*$(X) = \dfrac{\sigma^2\alpha^{2\xi}[\Gamma(1-2\xi) - \Gamma^2(1-\xi)]}{\xi^2}$, $\xi \neq 0$ *and* $\xi < 1/2$. *When* $\xi = 0$, *the variance is Var*$(X) = \dfrac{\pi^2\sigma^2}{6}$.

These results (a) and (b) are directly obtained from the definition of each measure.

2.3. The transmuted generalized extreme value distribution (TGEV). If $G(x)$ is the GEV cumulative distribution in (1.1), then, applying it in the function (1.6), the transmuted generalized extreme value (TGEV) cumulative distribution is given by

$$F(x;\mu,\sigma,\xi,\lambda) = \begin{cases} (1+\lambda)\exp\left\{-[1+z\,\xi]^{-1/\xi}\right\} - \lambda\exp\left\{-2[1+z\,\xi]^{-1/\xi}\right\}, \xi \neq 0, \\ (1+\lambda)\exp\left\{-\exp[-z]\right\} - \lambda\exp\left\{-2\exp[-z]\right\},\ \xi \to 0. \end{cases} \tag{2.4}$$

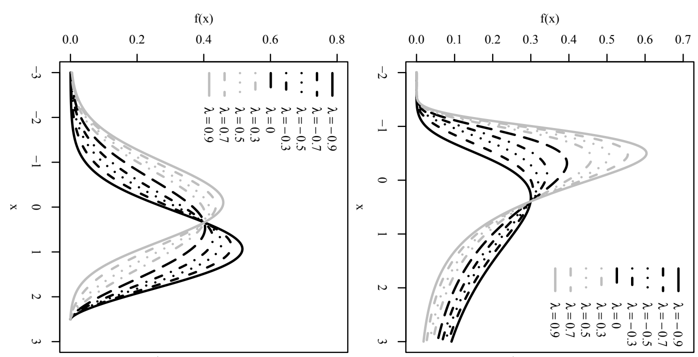
The corresponding pdf is

$$f(x;\mu,\sigma,\xi,\lambda) = \begin{cases} \dfrac{\exp\left\{-[1+z\,\xi]^{-1/\xi}\right\}[1+\lambda-2\lambda\exp\left\{-[1+z\,\xi]^{-1/\xi}\right\}]}{\sigma[1+z\,\xi]^{1+(1/\xi)}}, \xi \neq 0, \\ \dfrac{\exp[-z]\exp\left\{-\exp[-z]\right\}[1+\lambda-2\lambda\exp\left\{-\exp[-z]\right\}]}{\sigma}, \xi \to 0, \end{cases}$$

where $\lambda \in [-1,1]$, $z = (x-\mu)/\sigma$. The quantile function of TGEV distribution, say z_u, is given by

$$z_u = \begin{cases} \mu + \frac{\sigma}{\xi}\left\{\left[-\log\left[\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\,\lambda\,u}}{2\,\lambda}\right]\right]^{-\xi} - 1\right\}, & \xi \neq 0, \\ \mu - \sigma\log\left\{-\log\left[\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\,\lambda\,u}}{2\,\lambda}\right]\right\}, & \xi \to 0, \end{cases}$$

for $\lambda \neq 0$ and $u \in [0,1]$. In Figure 3, we plot the density of the TGEV distribution for selected parameter values. The case where $\lambda = 0$ is the particular case of standard GEV distribution.



(a) $\xi = -0.4$ (b) $\xi = 0.4$

Figure 3. Plot for the TGEV density for some parameter values; $\mu = 0$ and $\sigma = 1$.

In the Proposition 2.4, we provide some useful properties of the TGEV distribution.

2.4. Proposition. *Let $X \sim (\mu,\sigma,\xi,\alpha)$. Then, first moment and variance are given by*

(a) $\mathbb{E}(X) = \dfrac{\mu\xi + \sigma[(1+\lambda-2^{\xi}\lambda)\Gamma(1-\xi)-1]}{\xi}$, $\xi \neq 0$ *and* $\xi < 1$. *When* $\xi = 0$ *we have* $\mathbb{E}(X) = (\mu + \sigma\zeta) - \lambda\sigma\ln 2$.

(b) $\mathrm{Var}(X) = \dfrac{\sigma^2\{-[\lambda(2^{\xi}-1)-1]^2\Gamma^2(1-\xi) - [\lambda(4^{\xi}-1)-1]\Gamma(1-2\xi)\}}{\xi^2}$, $\xi \neq 0$ and $\xi < 1/2$. When $\xi = 0$ we have $\mathrm{Var}(X) = \sigma^2\left\{\dfrac{\pi^2}{6} - \lambda(1+\lambda)[\ln 2]^2\right\}$.

These results (a) and (b) are directly obtained from the definition of each measure.

2.5. Proposition. *The density function of X (TGEV) can be expressed as an finite linear combination of densities of GEV(μ, σ, ξ) and EGEV($\mu, \sigma, \xi, 2$) density functions, i.e.,*

$$f(x; \mu, \sigma, \xi, \lambda) = \beta \cdot g(x; \mu, \sigma, \xi) + (1 - \beta) \cdot z(x; \mu, \sigma, \xi, 2),$$

where $\beta = 1 + \lambda$ and $z(x)$ is pdf of EGEV distribution.

2.1. Corollary. If $\lambda = -1$, then $X \sim$ EGEV$(\mu, \sigma, \xi, 2)$.

2.6. Proposition. *If f is the density function of TGEV distribution, with cumulative function F, then*

$$\lim_{x \to x^*} \frac{1 - F(x; \mu, \sigma, \xi, \lambda)}{x f(x; \mu, \sigma, \xi, \lambda)} = \xi$$

if $\xi > 0$ and $x^* = \infty$, and

$$\lim_{x \to x^*} \frac{1 - F(x; \mu, \sigma, \xi, \lambda)}{(x - x^*)\, f(x; \mu, \sigma, \xi, \lambda)} = \xi, \tag{2.5}$$

if $\xi < 0$ and $x^* < \infty$.

The proof is shown in Appendix.

2.4. Return levels. In extreme values studies, is important to know with which probability a rare event can occur in the next periods of time, or every how many years is expected an event higher than r. For this, we can calculate the return level for every t periods of time. Specifically, the return level r_t is related to the quantile $1 - 1/t$ of the distribution of extreme values. Thus, for each of the three generalizations the return levels are given by $r_t = z_{1-1/t}$. In Bayesian estimation, as sampled points of the parameters for the respective posteriors, they sampled points with the return levels, obtaining a posterior distribution for r_t. We can verify some relationship between the standard GEV distribution and its generalizations.

2.7. Proposition. *Let $r_{EGEV,t}$ the return level for EGEV distribution with parameters $(\mu, \sigma, \xi, \alpha)$ and let $r_{GEV,t}$ the return level for the GEV distribution with parameters (μ, σ, ξ). Then*

(1) *If $\alpha > 1$, then $r_{GEV,t} < r_{EGEV,t}$.*
(2) *If $\alpha < 1$, then $r_{GEV,t} > r_{EGEV,t}$.*

The proof is shown in the appendix

2.8. Proposition. *Let $r_{GGEV,t}$ the return level for GGEV distribution with parameters $(\mu, \sigma, \xi, \delta)$ and let $r_{GEV,t}$ the return level for the GEV distribution with parameters (μ, σ, ξ). Then*

(1) *If $\delta > 1$, then $r_{GEV,t} > r_{GGEV,t}$.*
(2) *If $\delta < 1$, then $r_{GEV,t} < r_{GGEV,t}$.*

The proof is shown in the appendix

2.9. Proposition. *Let $r_{TGEV,t}$ the return level for TGEV distribution with parameters $(\mu, \sigma, \xi, \lambda)$ and let $r_{GEV,t}$ the return level for the GEV distribution with parameters (μ, σ, ξ). Then*

(1) *If $\lambda > 0$, then $r_{GEV,t} > r_{TGEV,t}$.*
(2) *If $\lambda < 0$, then $r_{GEV,t} < r_{TGEV,t}$.*

The proof is shown in the appendix

3. Estimation and inference

3.1. Maximum likelihood estimation. In this section, we discuss maximum likelihood estimation for the new models. We present the log-likelihood function for all models considering the case $\xi \neq 0$. Thus, the log-likelihood are given by

$$\ell^{\text{GGEV}}(\boldsymbol{\theta}_1) = -n\,\log(\sigma\,\Gamma(\delta)) - (\delta/\xi+1)\sum_{i=1}^{n}\log[1+\xi(x_i-\mu)/\sigma] - \sum_{i=1}^{n}[1+\xi(x_i-\mu)/\sigma]^{-1/\xi},$$

$$\ell^{\text{EGEV}}(\boldsymbol{\theta}_2) = n\,\log(\alpha/\sigma) - (1/\xi+1)\sum_{i=1}^{n}\log[1+\xi(x_i-\mu)/\sigma] - \alpha\sum_{i=1}^{n}[1+\xi(x_i-\mu)/\sigma]^{-1/\xi},$$

$$\begin{aligned}\ell^{\text{TGEV}}(\boldsymbol{\theta}_3) \quad &= \quad -n\,\log(\sigma) - (1/\xi+1)\sum_{i=1}^{n}\log[1+\xi(x_i-\mu)/\sigma] - \sum_{i=1}^{n}[1+\xi(x_i-\mu)/\sigma]^{-1/\xi} \\ &+ \quad \sum_{i=1}^{n}\log\left(1+\lambda-2\,\lambda\,\exp\left\{-[1+\xi(x_i-\mu)/\sigma]^{-1/\xi}\right\}\right),\end{aligned}$$

where $\boldsymbol{\theta}_1 = (\mu,\sigma,\xi,\delta)$, $\boldsymbol{\theta}_2 = (\mu,\sigma,\xi,\alpha)$ and $\boldsymbol{\theta}_3 = (\mu,\sigma,\xi,\lambda)$, provided that

$$1+\xi(x_i-\mu)/\sigma > 0, \text{ for } i = 1,\ldots,n.$$

At parameter combinations for which the above result is violated, corresponding to a configuration for which at least one of the observed data falls beyond an end-point of the distribution, the likelihood is zero and the log-likelihood equals $-\infty$.

3.2. Bayesian analysis. In this work, we use the Bayesian paradigm to estimate the posterior parameters of these new class of distributions. We proposed vague prior distributions for the parameters, and perform the estimation combining the information of prior and the likelihood function to provide the posterior points. We have the posterior points by Markov chain Monte Carlo (MCMC) [3]. Base on the parametric space of the parameters, we proposed the following priors:

- $\mu \sim N(\mu_0, \sigma_0^2)$, μ_0 and σ_0^2 known;
- $\sigma \sim \Gamma(a_1, b_1)$, a_1 and b_1 known;
- $\xi \sim N(\mu_\xi, \sigma_\xi^2)$, μ_ξ and σ_ξ^2 known;
- $\delta \sim \Gamma(a_2, b_2)$, a_2 and b_2 known;
- $\alpha \sim \Gamma(a_3, b_3)$, a_3 and b_3 known;
- $\lambda \sim U(-1, 1)$.

Considering a case with a non-informative prior to the parameters, we consider $\mu_0 = \mu_\xi = 0$, $\sigma_0^2 = 1000$, σ_ξ^2, $a_i = 0.001$, $b_i = 0.001$, $i = 1, 2, 3$. Posterior points can be performed using MCMC algorithms. As we not have a closed form for the full conditional distributions for all the three cases, we use the Metropolis-Hastings algorithm technique of sampling.

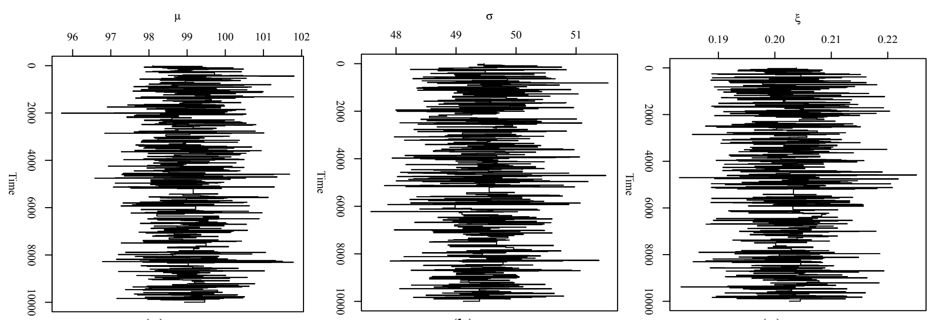
4. Simulation study

Simulations was performed in different configuration of the parameters, from the three extensions and the standard GEV distribution. The aim of this section if verify the ability of the estimation fits correctly the value of the parameters, in differents points of the generalization parameter. We performed all simulations with fixed (μ, σ, ξ) parameters at points $(100, 50, 0.2)$. For the δ of GGEV and α of EGEV, we used the values $(0.5, 2.0)$. For the λ of TGEV, we simulated points with $(-0.9, 0.9)$.

Table 1 shows the Bayes estimator with respect to quadratic loss (posterior mean), and credibility intervals of 95%. In all simulations, the posterior mean is near to the true value, and only for the parameter σ for the EGEV model, with $\alpha = 2.0$, the true value have been out of the credibility interval. The length of the intervals are similar between models, indicating that they have the same accuracy. For the GEV standard model, the credibility intervals was $(99.92; 101.97)$ for μ, $(49.78, 51.41)$ for σ and $(0.190, 0.218)$ for ξ. Figures 4-6 shows trace plots of the parameters for three simulations. In all cases, we observe that a stationary distribution whose central measure coincides with the true value.

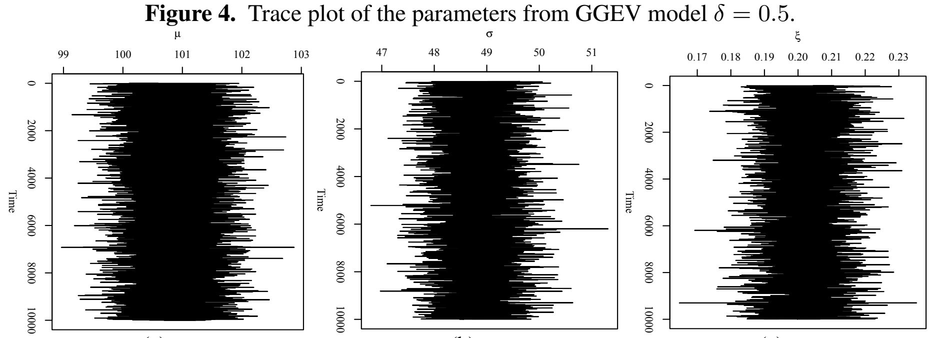
Table 1. Posterior means and 95% posterior credibility intervals for simulated data in the parameters models.

	EGEV, $\alpha = 0.5$		GGEV, $\delta = 0.5$		TGEV, $\lambda = -0.9$	
	Mean	95% C.I.	Mean	95% C.I.	Mean	95% C.I.
μ	99.4	(98.1; 100.7)	99.0	(97.6; 100.5)	100.4	(99.2; 101.4)
σ	49.7	(48.7; 50.7)	49.5	(48.4; 50.6)	50.6	(49.6; 51.7)
ξ	0.197	(0.182; 0.213)	0.203	(0.191; 0.214)	0.205	(0.189; 0.220)
	EGEV, $\alpha = 2.0$		GGEV, $\delta = 2.0$		TGEV, $\lambda = 0.9$	
	Mean	95% C.I.	Mean	95% C.I.	Mean	95% C.I.
μ	100.9	(99.9; 101.8)	100.3	(99.3; 101.3)	99.8	(99.0; 100.7)
σ	48.8	(47.8; 49.8)	50.1	(49.0; 51.2)	49.6	(48.8; 50.6)
ξ	0.202	(0.187; 0.218)	0.198	(0.176; 0.223)	0.181	(0.159; 0.203)



(a) (b) (c)

Figure 4. Trace plot of the parameters from GGEV model $\delta = 0.5$.



(a) (b) (c)

Figure 5. Trace plot of the parameters from EGEV model, $\alpha = 2.0$.

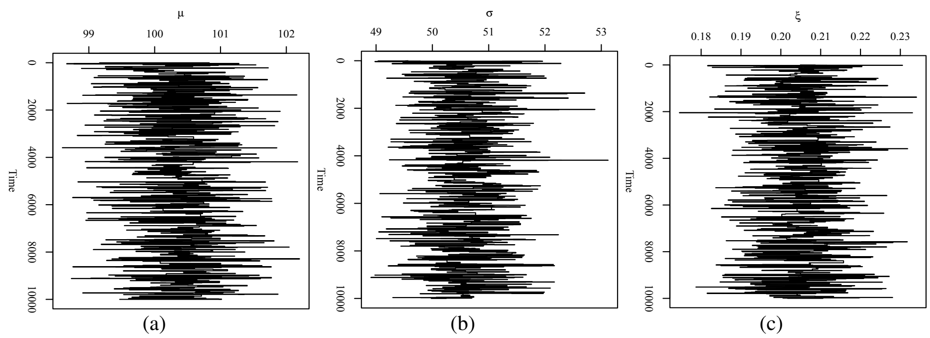


Figure 6. Trace plot of the parameters from TGEV model, $\lambda = 0.9$.

Figure 7 show the return level plot for standard GEV estimation. Altouthg this plot is common in literature of extremes, the objective of draw it in this work is to compare the GEV returns against it's generalizations.

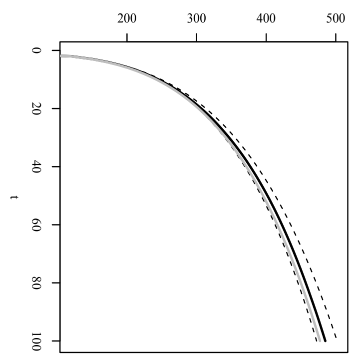
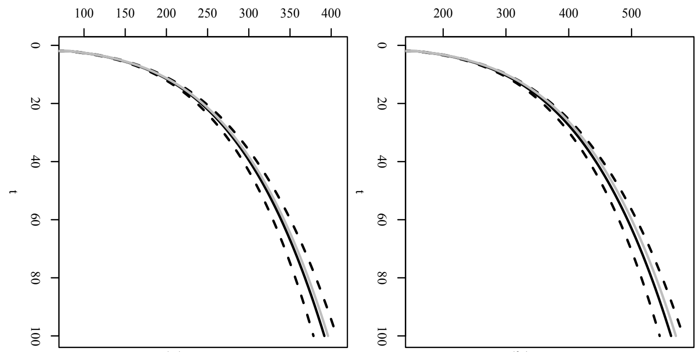


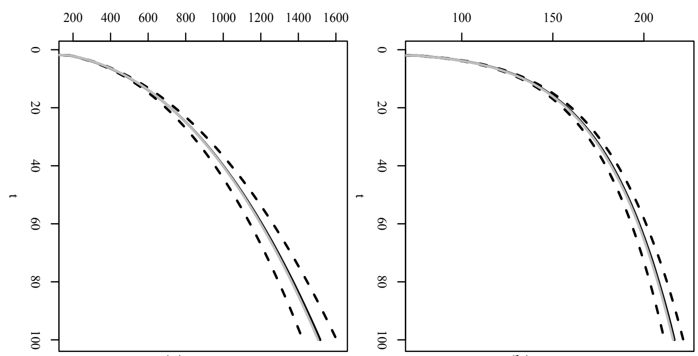
Figure 7. Posterior mean and 95% posterior credibility intervals of the return level plot for simulatated GEV model.

The Figures 8-10 show the posterior mean of the expected return levels, from $t = 2$ to 100, for each model simulated. From these figures, it is observed that for all simulations the true values of the returns are within the credibility interval, and the estimation is more accurate for GGEV model, where the line of the posterior mean returns is the nearest line of the true returns. The TGEV model is the less accurate model about the returns, which presents larger distance between the line of the mean and the true return, even so it is within the credibility interval. Comparing the three extensions proposed in this work with the returns of standard GEV, it is observed that increasing the parameter α in EGEV implies increasing values of returns, although even increasing $\alpha = 2.0$ we have similar results compared with standard GEV. About the GGEV returns, when δ decrease, we have a tail much more heavy than the standard GEV model. For TGEV model increase λ imply in lower returns values.



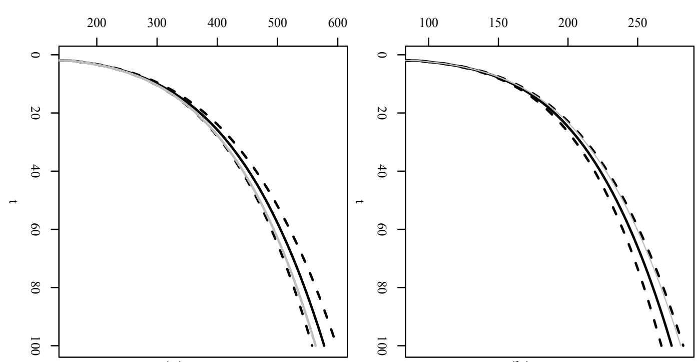
(a) (b)

Figure 8. Posterior mean and 95% posterior credibility intervals of the return level plot for simulatated EGEV model with $\alpha = 0.5$ (a) and $\alpha = 2.0$ (b). The grey line is the true return of the model.



(a) (b)

Figure 9. Posterior mean and 95% posterior credibility intervals of the return level plot for simulatated GGEV model with $\delta = 0.5$ (a) and $\delta = 2.0$ (b). The grey line is the true return of the model.



(a) (b)

Figure 10. Posterior mean and 95% posterior credibility intervals of the return level plot for simulatated TGEV model with $\lambda = -0.9$ (a) and $\lambda = 0.9$ (b). The grey line is the true return of the model.

5. Applications to real data

We conduct two applications with maxima are analyzed of the three extensions to real data for illustrative purpose. The first example is a data set that consists of monthly maxima quota of Gurgueia River, located in the State of Piauí, Brazil. A river quota is the height of the water in the section relative to a given reference. Conventionally the quotas are measured in centimeters (cm). Large quota values can cause floods in the regions close to the rivers. Daily data was collected from 1975 to 2012. We analyse the maximum for each every 30 days. The second application consist to analyse rainfall data in Barcelos Station, located in the North of Portugal. The daily data was collected daily from 1931 to 2008, and we analysed the maxima of each 30 days.

In both the modeling was done using the GEV and its three generalizations proposed in this work. About the additional parameter, identifiability problems were detected in the estimation of the parameters. In this case, we created a grid of possible values for the additional parameter to estimate the other parameters using the Bayesian approach for each point of the grid, and choose the one grid point that has the lowest $-2\ell(\boldsymbol{\theta})$, that is the primitive measure to calculate BIC and DIC. After choosing the best point for each of the three generalizations, they are compared with the standard GEV, to decide what the best model that fits each of the applications. In both application, the dual gamma extension showed be the best model. Figure 11 shows the measure for a grid of points for the applications. In the Gurgueia river quota, the best fit was when $\delta = 0.06$, while for the Barcelos rainfall data, the best fit measure was when $\delta = 0.26$. For the exponentiated and transmuted generalizations, the best additional parameter in the grid of points was points near the standard GEV case.

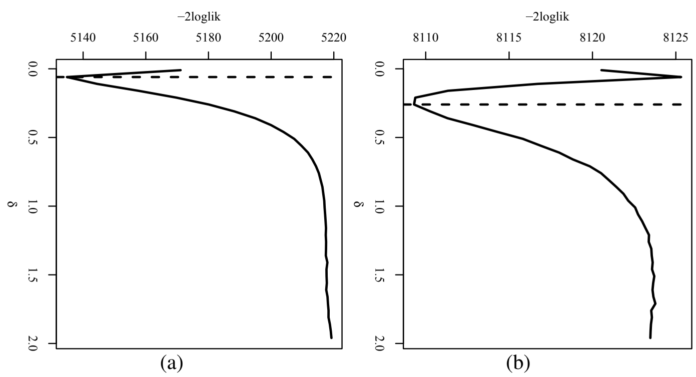


Figure 11. $-2\ell(\boldsymbol{\theta})$ for a grid of points to the δ of GGEV distribution. (a) Gurgueia river quota and (b) Barcelos rainfall.

Table 2 provides the BIC and DIC for the GEV and it's three generalizations proposed in this work. In general, the smaller the values of these statistics, the better the fit. We can note an advantage of the Dual-Gamma Generalization, followed by the standard GEV distribution. Table 3 shows de 95% credible interval for the parameters. We can verify a high value in location and scale parameter, and a negative value of the shape, indicating that the data has a lighted tail.

Figure 12 shows the return level plot for the applications. From this figure we can verify that, for gurgueia river quota, the returns of the model GGEV grow more slowly than the returns of the GEV, being more similar to the behavior of empirical returns. Based on the GGEV model, a return higher than 500cm once every $t = 20$ periods is expected. Each $t = 100$ periods of time, is waiting at least once a maximum higher than 604cm. For the Barcelos rainfall data, the return levels from the GGEV model is more similar than the return levels of the GEV model. Each $t = 20$ periods of time, is expected a return level higher than 76mm, while that for $t = 100$ periods of time, is expected that once the level would be equal or higher than 101mm. Figure 13 shows the predictive distribution for the applications based in GGEV model. We can verify a good fit for this model.

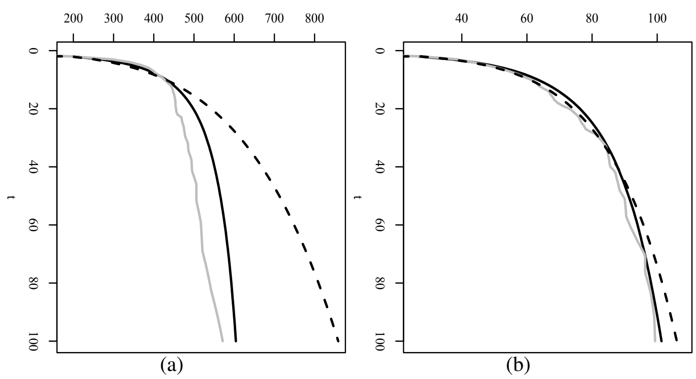


Figure 12. Return level plot for the applications. (a) Gurgueia river and (b) Barcelos Station, for the GGEV (full line), GEV (dotted Line:) and Empirical (grey line).

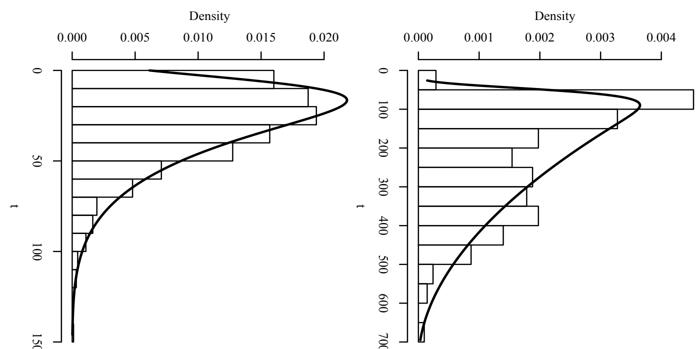
Table 2. BIC and DIC measures for applications

	River Quota at Gurgueia, Brazil				Rainfall at Barcelos, Portugal			
Model	GEV	EGEV	GGEV	TGEV	GEV	EGEV	GGEV	TGEV
DIC	5217	5215	5137	5218	8123	8122	8110	8123
BIC	5233	5239	5160	5241	8141	8148	8136	8149

Like the previous application, the best model pointed was the GGEV model, according to the Table 2. As the river data, in rainfall data, the model presents a lighted tail behavior, by the negative behavior of ξ (see Table 3).

Table 3. Mean ane 95% credibility intervals for parameters for the applications.

	River Quota at Gurgueia, Brazil		
Parameter	μ	σ	ξ
M (CI)	44.74 (38.35; 50.33)	14.37 (12.99; 16.08)	−0.020 (−0.023; −0.016)
	Rainfall at Barcelos, Portugal		
Parameter	μ	σ	ξ
M (CI)	5.21 (4.15; 6.20)	7.60 (7.13; 8.19)	−0.042 (−0.053; −0.031)



(a) (b)

Figure 13. Predictive density for Barcelos station and Gurgueia river, respectively.

6. Concluding remarks

In this paper, we proposed three extensions to the GEV distribution, with an additional parameter which modifies the behavior of the distribution, composing as a alternative models for single maxima events. In each generalization, the GEV distribution appears as a particular case. We performed the modelling under a Bayesian approach and the estimation of the parameters was proposed using the MCMC algorithm. The results of simulations show that the proposed method is efficient in recovering the true values of the parameters of generalizations, which credibility intervals were obtained with great accuracy in relation to the true parameter estimation. In fact, the three generalizations can be used to fit real data, where in both applications, the best model according to the fit measure was the generalization of GGEV model. These generalizations can be applied to any kinds of environmental data that involves the analysis of maxima.

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Appendix

MCMC Algorithm. For the additional parameters (α for the exponentiated, δ for the dual Gamma, and λ for the transmuted, we propose a Grid of points, and perform the Bayesian estimation via MCMC for each point of the Grid. The point of the grid with best Goodness of fit is the choosen point, denoted α^*, δ^* and λ^* for α and δ, the Grid is from 0.01 to 2, with intervals of 0.05 (the case 1.00 is the standard GEV). For λ, the Grid is from -0.9 to 0.9, with intervals of 0.1 (the case 0 is the standard GEV).

After choose the best point in grid for each case, the parameters (μ, σ, ξ) are sampled using the Metropolis-Hastings algorithm. Details of the MCMC sampling scheme are given below. At iteration s, parameters are updated as follows:

Sampling $\Theta = (\mu, \sigma, \xi)$: Propose new values for these parameters where $\mu^* \sim N(\mu^{(s)}, V_\mu)$, $\xi^* \sim N(\xi^{(s)}, V_\xi) I_{-0.5,\infty}(\xi)$ and $\sigma^* \sim \text{Gamma}(\sigma^{2(s)}/V_\sigma, \sigma^{(s)}/V_\sigma)$. Accept the new values $\Theta^{(s+1)} = \Theta^*$ with probability α_Θ, where

$$\alpha_\Theta = \min\left\{1, \frac{\pi(\Theta^*|\mathbf{x}) f_N(\mu^{(s)} \mid \mu^*, V_\mu) f_N(\xi^{(s)} \mid \xi^*, V_\xi) f_G(\sigma^{(s)} \mid \sigma^{*2}/V_\sigma, \sigma^*/V_\sigma)}{\pi(\Theta^{(s)}|\mathbf{x}) f_N(\mu^* \mid \mu^{(s)}, V_\mu) f_N(\xi^* \mid \xi^{(s)}, V_\xi) f_G(\sigma^* \mid \sigma^{(s)2}/V_\sigma, \sigma^{(s)}/V_\sigma)}\right\},$$

where $\pi(\Theta^*|\mathbf{x})$ is the posterior density given by the combination with the likelihood and the prior distribution given in Section 3.

Proof of the propsition 2.2. Let $X \sim EGEV(\mu, \sigma, \xi, \delta)$. Then, for $\xi \neq 0$

$$\begin{aligned} F(x;\mu,\sigma,\xi,\alpha) &= \exp\left\{-\alpha[1+\xi(x-\mu)/\sigma]^{-1/\xi}\right\} = \exp\left\{-[\alpha^{-\xi}+\alpha^{-\xi}\xi(x-\mu)/\sigma]^{-1/\xi}\right\} \\ &= \exp\left\{-\left[1+\frac{\alpha^{-\xi}\xi\left[x-\mu+\frac{\sigma(\alpha^{-\xi}-1)}{\xi\alpha^{-\xi}}\right]}{\sigma}\right]^{-1/\xi}\right\} \\ &= \exp\left\{-\left[1+\frac{\xi}{\sigma/\alpha^{-\xi}}\left(x-\left(\mu+\frac{\sigma}{\xi}(\alpha^{\xi}-1)\right)\right)\right]^{-1/\xi}\right\} \end{aligned}$$

which is the cumulative distribution function of $GEV(\mu', \sigma', \xi)$, where $\mu' = \left(\mu + \frac{\sigma}{\xi}(\alpha^{\xi}-1)\right)$ and $\sigma' = \sigma/\alpha^{-\xi}$.

The proof for the case $\xi = 0$ is similar, where $EGEV(\mu, \sigma)$ is a $GEV(\mu', \sigma)$, where $\mu' = \mu + \sigma\log(\alpha)$.

Proof of the proposition 2.6. If f and F are respectively the density and cumulative distribution of TGEV distribution, they can be written in function of a standard GEV distribution with density g and cumulative function G, weighted by a λ parameter, as written in (1.6). Then, the shape parameter of TGEV distribution, for the case where $\xi > 0$ is given by

$$\begin{aligned} \lim_{x\to x^*}\frac{1-F(x;\mu,\sigma,\xi,\lambda)}{xf(x;\mu,\sigma,\xi,\lambda)} &= \lim_{x\to x^*}\frac{[1-(1+\lambda)\,G(x;\mu,\sigma,\xi)+\lambda\,G(x;\mu,\sigma,\xi)^2]}{xg(x;\mu,\sigma,\xi)\,(1+\lambda-2\,\lambda\,G(x;\mu,\sigma,\xi))} \\ &= \lim_{x\to x^*}\frac{[1-G(x;\mu,\sigma,\xi)][1-\lambda G(x;\mu,\sigma,\xi)]}{xg(x;\mu,\sigma,\xi)\,[1+\lambda-2\,\lambda\,G(x;\mu,\sigma,\xi)]} \\ &= \lim_{x\to x^*}\frac{1-G(x;\mu,\sigma,\xi)}{xg(x;\mu,\sigma,\xi)}\lim_{x\to x^*}\frac{[1-\lambda\,G(x;\mu,\sigma,\xi)]}{[1+\lambda-2\,\lambda\,G(x;\mu,\sigma,\xi)]} \\ &= \xi\frac{\lim_{x\to x^*}[1-\lambda\,G(x;\mu,\sigma,\xi)]}{\lim_{x\to x^*}[1+\lambda-2\,\lambda\,G(x;\mu,\sigma,\xi)]} = \xi\frac{(1-\lambda)}{(1-\lambda)} = \xi. \end{aligned}$$

The proof for the case $\xi < 0$ is similar using (2.5).

Proof of the proposition 2.7. By simplicity of notation, consider $\Delta = \left[1+\frac{\xi(x-\mu)}{\sigma}\right]^{-1/\xi}$ for $\xi \neq 0$ and $\Delta = \exp\{-\frac{(x-\mu)}{\sigma}\}$ for $\xi = 0$. The cdf function of GEV distribution in (1.1) can be written as $G(x;\mu,\sigma,\xi) = \exp\{-\Delta\}$ and the cdf of EGEV in (2.3) can be written by $F(x;\mu,\sigma,\xi,\alpha) = \exp\{-\alpha\Delta\}$.

Then, for $\alpha > 1$, $G(x;\mu,\sigma,\xi) > F(x;\mu,\sigma,\xi,\alpha)$ and $\forall\ x$, $r_{GEV,t} < r_{EGEV,t}$. For $\alpha < 1$, $G(x;\mu,\sigma,\xi) < F(x;\mu,\sigma,\xi,\alpha)$ and $\forall\ x$, $r_{GEV,t} > r_{EGEV,t}$.

Proof of the proposition 2.8. The cdf of GGEV distribution in (2.1) can be rewritten as $F(x;\mu,\sigma,\xi,\delta) = 1 - F_G(\Delta;\delta,1)$, where F_G is the cdf of a Gamma distribution. Similarly, the GEV cdf in (1.1) can be rewritten as $G(x;\mu,\sigma,\xi) = 1 - F_G(\Delta;1,1)$.

For $\delta > 1$, $F_G(.;\delta,1) < F_G(.;1,1)$ and then $F(x;\mu,\sigma,\xi,\delta) > G(x;\mu,\sigma,\xi)$, which implies that $r_{GEV,t} > r_{GGEV,t}$. When $\delta < 1$, $F_G(.;\delta,1) > F_G(.;1,1)$ and then $F(x;\mu,\sigma,\xi,\delta) < G(x;\mu,\sigma,\xi)$, which implies that $r_{GEV,t} < r_{GGEV,t}$.

Proof of the proposition 2.9. Given the cdf $G(x;\mu,\sigma,\xi)$ of the GEV distribution in (1.1), the cdf of TGEV in (2.4) can be rewritten as

$$F(x;\mu,\sigma,\xi,\lambda) = G(x;\mu,\sigma,\xi) + \lambda\, G(x;\mu,\sigma,\xi)[1 - G(x;\mu,\sigma,\xi)].$$

Then, for $\lambda < 0$, $F(x;\mu,\sigma,\xi,\lambda) < G(x;\mu,\sigma,\xi)$ and then $r_{GEV,t} < r_{TGEV,t}$. When $\lambda > 0$, $F(x;\mu,\sigma,\xi,\lambda) > G(x;\mu,\sigma,\xi)$ and then $r_{GEV,t} > r_{TGEV,t}$.

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Some imputation methods for missing data in sample surveys

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Abstract

The present work suggests some imputation methods to deal with the problems of non-response in sample surveys. The imputation methods presented in this work lead to the precise estimation strategies of population mean. Empirical studies are carried out with the help of data borrowed from natural populations to show the superiorities of the suggested imputation methods over usual mean, ratio and regression methods of imputation in terms of the mean square error criterions. Suitable recommendations have been put forward for the survey practitioners.

Keywords: Imputation, non-response, auxiliary information, bias, mean square error.

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1. Introduction

The clinical or life savings drug testing experiments face the problems of missing data due to elimination of some of the experimental units during the course of experiments. Similarly in agricultural experiments, crops destroy due to some natural calamities or disease during the course of experiments. In demographic and socio-economic surveys, generally response from each unit in sample is not available due to various causes. Such incompleteness is known as non-response and if the appropriate information about the nature of non-response is not available, the conclusions concerning the population parameters may be spoiled.

In last couple of decades, significant advancements have been made to reduce the negative impact of non-response. Imputation is one which deals with the filling up

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method of incomplete data for adapting the standard analytic model in statistics. It is typically used when it is needed to substitute missing item values with certain fabricated values in a survey or census. To deal with the missing item values effectively [13], [14], [16] and [9] suggested imputation methods that make an incomplete data set structurally complete and its analysis simple. Imputation may also be carried out with the aid of an auxiliary variable if it is available. Some of the pioneer works which used information on an auxiliary variable under missing completely at random (MCAR) response mechanism were suggested by [10] ,[11] , [20],[22],[1],[4],[18],[21],[17],[19]and [2].

[15] advocated the use of multiple imputations to lessen the negative impact of missing data in more wise way. He showed multiple imputations provide a useful strategy for dealing with missing data by replacing each missing value with two or more acceptable fabricated values representing a distribution of possibilities. Motivated with this suggestion and in follow up we suggest some single and multiple imputations methods under MCAR response mechanism. The suggested imputation methods lead to some effective estimation procedures of population mean. Properties of the proposed imputation methods and subsequent estimation procedures have been examined and suitable recommendations are made.

2. Sample structure and notations

Consider $U = (U_1, U_2, U_3, ..., U_N)$ denote the finite population of size N and let y and x be the positively correlated study and auxiliary variables respectively. It is assumed that information on an auxiliary variable x is readily available for each unit of the population and we intend to estimate the population mean of the study variable y. Let a sample s of size n be drawn from the population under simple random sampling without replacement (SRSWOR) scheme and surveyed for study variable y but response from each sampled unit was not obtained which leads to the presence of non-response. Let r be the number of responding units out of sampled n units and the set of responding units is denoted by R and that of non-responding units by R^c . For sampled units $i \in R$, the values y_i are observed, while for the units $i \in R^c$,the y_i values are missing and respective imputed values are derived. We intend to develop some effective imputation methods with the aid of an auxiliary variable x, such that the value of x_i for unit U_i, is known and has positive value for each unit of the population. Hence onwards we use the following notations:
$\bar{Y}, \bar{X}$:The population means of the study and auxiliary variables y and x respectively.
S_y^2,S_x^2 :The population variances of the study and auxiliary variables y and x respectively.
C_y,C_x :The coefficients of variations of the study and auxiliary variables y and x respectively.
ρ_{yx}: The correlation coefficient between the study and auxiliary variables y and x.
$\bar{y}_r, \bar{x}_r$:The response means of the study and auxiliary variables y and x respectively.
$\bar{x}_n$:The sample mean of the auxiliary variable x based on the sample size n.

2.1. Proposed imputation methods and subsequent estimators. In this section, some more effective imputation methods and hence the corresponding estimators have been proposed under MCAR response mechanism. The derived resultant estimators have shown dominant performance over the existing methods of imputations and are more relevant for practical applications.

2.1.1. *Single imputation methods and subsequent estimators.* Following the MCAR response mechanism we suggest the following three single imputation methods for the missing values of the sample data.

(a) First method of imputation
The data after imputation takes the form,

$$(2.1)\qquad y_{.i}=\begin{cases} y_i exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right) & if \quad i\in R \\ \left(y_r+\hat{b}x_i-\hat{b}\bar{x}_i\right) exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right) & if \quad i\in R^c \end{cases}$$

where

$$\hat{b}=\frac{s_{yx}\left(r\right)}{s_x^2(r)}$$

Under the method of imputation discussed in equation (2.1), the point estimator of $\bar{Y}$ takes the following form

$$(2.2)\qquad \tau_1=\frac{1}{n}\sum_{i=1}^{n}y_{.i}=\frac{1}{n}\left[\sum_{i\in R}y_{.i}+\sum_{i\in R^c}y_{.i}\right]$$

which is simplified as

$$(2.3)\qquad \tau_1=\left[\bar{y}_r+\hat{b}\left(\bar{x}_n-\bar{x}_r\right)\right] exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)$$

(b) Second method of imputation
The data after imputation takes the form,

$$(2.4)\qquad y_{.i}=\begin{cases} y_i exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right) & if \quad i\in R \\ \left(\frac{\bar{y}_r}{\bar{x}_r}x_i\right) exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right) & if \quad i\in R^c \end{cases}$$

Under the method of imputation described in equation (2.4), the point estimator of $\bar{Y}$ takes the following form

$$(2.5)\qquad \tau_2=\frac{\bar{y}_r}{\bar{x}_r}\bar{x}_n exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)$$

(c) Third method of imputation
The data after imputation takes the form,

$$(2.6)\qquad y_{.i}=\begin{cases} y_i-\frac{n^2}{r^2}\bar{x}_n\hat{b} & if \quad i\in R \\ \left(\bar{y}_r+\frac{n}{n-r}\hat{b}\bar{x}_n exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)+\frac{n}{r}\hat{b}x_i\right) & if \quad i\in R^c \end{cases}$$

Under the method of imputation described in equation (2.6), the point estimator of $\bar{Y}$ takes the following form

$$(2.7)\qquad \tau_3=\bar{y}_r+\hat{b}\left[\left\{\bar{x}_n exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)\right\}-\bar{x}_r\right]$$

2.1.2. *Multiple imputations methods and resultant estimators.* In single imputation, the single value being imputed can reflect neither sampling variability about the actual value when one model for non-response is being considered nor additional uncertainty when more than one model is being entertained. Since, multiple imputations retain the virtues of single imputation and corrects its major flaws, therefore, we intend to use multiple imputations for each missing value in the sample of size n. The previously discussed methods of imputations have been considered to derive the imputed values for each missing value. After the generations of imputed values, complete data sets are produced and subsequently estimators based on sample of size n are reproduced. The final estimator of population mean $\bar{Y}$ is the average of estimates produced by imputation methods. Hence the final estimators of population mean $\bar{Y}$ based on the procedure of multiple imputations are considered as

$$\bar{y}_{MI_1}=\frac{1}{3}\left[\tau_1+\tau_2+\tau_3\right]$$

$$(2.8)\qquad \bar{y}_{MI_1}=\frac{1}{3}\left[\begin{array}{l}\left\{\bar{y}_r+\hat{b}\left(\bar{x}_n-\bar{x}_r\right)exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)\right\}\\+\left\{\frac{\bar{y}_r}{\bar{x}_r}\bar{x}_n exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)\right\}\\+\left\{\bar{y}_r+\hat{b}\left\{\left\{\bar{x}_n exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)\right\}-\bar{x}_r\right\}\right.\end{array}\right]$$

$$\bar{y}_{MI_2}=\frac{1}{2}\left[\tau_1+\tau_2\right]$$

$$(2.9)\qquad \bar{y}_{MI_2}=\frac{1}{2}\left[\begin{array}{l}\left\{\bar{y}_r+\hat{b}\left(\bar{x}_n-\bar{x}_r\right)exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)\right\}\\+\left\{\frac{\bar{y}_r}{\bar{x}_r}\bar{x}_n exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)\right\}\end{array}\right]$$

$$\bar{y}_{MI_3}=\frac{1}{2}\left[\tau_2+\tau_3\right]$$

$$(2.10)\qquad \bar{y}_{MI_3}=\frac{1}{2}\left[\begin{array}{l}\left\{\frac{\bar{y}_r}{\bar{x}_r}\bar{x}_n exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)\right\}\\+\left\{\bar{y}_r+\hat{b}\left\{\left\{\bar{x}_n exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)\right\}\right\}-\bar{x}_r\right\}\end{array}\right]$$

$$\bar{y}_{MI_4}=\frac{1}{2}\left[\tau_1+\tau_3\right]$$

$$(2.11)\qquad \bar{y}_{MI_4}=\frac{1}{2}\left[\begin{array}{l}\left\{\bar{y}_r+\hat{b}\left(\bar{x}_n-\bar{x}_r\right)exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)\right\}\\+\left\{\bar{y}_r+\hat{b}\left\{\left\{\bar{x}_n exp\left(\frac{\bar{X}-\bar{x}_r}{\bar{X}+\bar{x}_r}\right)\right\}\right\}-\bar{x}_r\right\}\end{array}\right]$$

3. Bias and mean square errors of the proposed estimators $\tau_1,\tau_2,\tau_3,\bar{y}_{MI_1},\bar{y}_{MI_2},\bar{y}_{MI_3}$ and $\bar{y}_{MI_4}$

Under the suggested method of imputation the estimators $\tau_1,\tau_2,\tau_3,\bar{y}_{MI_1},\bar{y}_{MI_2}$, $\bar{y}_{MI_3}$ and $\bar{y}_{MI_4}$defined in equations (2.3), (2.5), (2.7) and (2.8)-(2.11) are biased estimators of $\bar{Y}$. Since, we have considered the MCAR response mechanism, therefore, the bias and mean square errors of the proposed estimators are derived up to the first order of approximations using the following transformations:
$\bar{y}_r=\bar{Y}\left(1+e_1\right),\bar{x}_n=\bar{X}\left(1+e_2\right),\bar{x}_r=\bar{X}\left(1+e_3\right),s_{yx}(r)=S_{yx}\left(1+e_4\right)$,
$s_x^2(r)=S_x^2\left(1+e_5\right)$ such that $E(e_i)=0$ and $|e_i|<1$ for i=1,2,...,5.
Under the above transformation, the estimators τ_1,τ_2 and τ_3 take the following forms:

$$(3.1)\qquad \tau_1=\left[\begin{array}{l}\left\{\bar{Y}\left(1+e_1\right)+\beta_{yx}\bar{X}\left(1+e_4\right)\left(1+e_5\right)^{-1}\left(e_2-e_3\right)\right\}\\exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\end{array}\right]$$

$$(3.2)\qquad \tau_2=\left[\left\{\bar{Y}\left(1+e_1\right)\left(1+e_2\right)\left(1+e_3\right)^{-1}\right\}exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\right]$$

$$(3.3)\qquad \tau_3=\left[\begin{array}{l}\left\{\bar{Y}\left(1+e_1\right)+\beta_{yx}\bar{X}\left(1+e_4\right)\left(1+e_5\right)^{-1}\right\}\\\left\{\left\{\left(1+e_2\right)exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\right\}-\left(1+e_3\right)\right\}\end{array}\right]$$

The bias and the mean square errors up to the first order of approximations of the proposed estimators $\tau_1,\tau_2,\tau_3,\bar{y}_{MI_1},\bar{y}_{MI_2},\bar{y}_{MI_3}$ and $\bar{y}_{MI_4}$ are derived in the following theorems:

3.1. Theorem. *The bias of the estimators $\tau_1,\tau_2,\tau_3,\bar{y}_{MI_1},\bar{y}_{MI_2},\bar{y}_{MI_3}$ and $\bar{y}_{MI_4}$ are given by*

$$(3.4)\qquad B\left(\tau_1\right)=\left[\begin{array}{l}\bar{Y}\left\{\left(\frac{1}{r}-\frac{1}{N}\right)\frac{1}{2}\left(\frac{3}{4}\frac{\mu_{200}}{\bar{X}^2}-\frac{\mu_{110}}{\bar{X}\bar{Y}}\right)\right\}\\+\left\{\left(\frac{1}{r}-\frac{1}{n}\right)\beta_{yx}\left(\frac{1}{2}\frac{\mu_{200}}{\bar{X}}+\frac{\mu_{300}}{\mu_{200}}-\frac{\mu_{210}}{\mu_{110}}\right)\right\}\end{array}\right]$$

$$(3.5)\qquad B\left(\tau_2\right)=\bar{Y}\left[\begin{array}{l}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\left(\rho_{yx}C_yC_x-\frac{3}{2}C_x^2\right)\right\}\\+\left\{\left(\frac{1}{r}-\frac{1}{N}\right)\frac{1}{2}\left(\frac{15}{4}C_x^2-3\rho_{yx}C_yC_x\right)\right\}\end{array}\right]$$

$$B(\tau_3)=\beta_{yx}\begin{bmatrix}\left\{\left(\frac{1}{r}-\frac{1}{N}\right)\frac{3}{2}\left(\frac{1}{4}\frac{\mu_{200}}{\bar{X}}+\frac{\mu_{300}}{\mu_{200}}-\frac{\mu_{210}}{\mu_{110}}\right)\right\}\\+\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\left(\frac{\mu_{210}}{\mu_{110}}-\frac{\mu_{300}}{\mu_{200}}-\frac{1}{2}\frac{\mu_{200}}{\bar{X}}\right)\right\}\end{bmatrix} \tag{3.6}$$

$$B(\bar{y}_{MI_1})=\frac{1}{3}\left\{B(\tau_1)+B(\tau_2)+B(\tau_3)\right\} \tag{3.7}$$

$$B(\bar{y}_{MI_2})=\frac{1}{2}\left\{B(\tau_1)+B(\tau_2)\right\} \tag{3.8}$$

$$B(\bar{y}_{MI_3})=\frac{1}{2}\left\{B(\tau_2)+B(\tau_3)\right\} \tag{3.9}$$

$$B(\bar{y}_{MI_4})=\frac{1}{2}\left\{B(\tau_1)+B(\tau_3)\right\} \tag{3.10}$$

where $\mu_{rst}=E\left[\left(x_i-\bar{X}\right)^r\left(y_i-\bar{Y}\right)^s\left(z_i-\bar{Z}\right)^t\right];(r,s,t)\geq 0$ *are integers.*

$C_y^2=\frac{S_y^2}{\bar{Y}^2},C_x^2=\frac{S_x^2}{\bar{X}^2},\rho_{yx}=\frac{S_{yx}}{S_yS_x}$, S_y^2,S_x^2 *and* S_{yx} *have their usual meanings.*

Proof. The bias of the estimators τ_1,τ_2 and τ_3 are derived as

$B(\tau_1)=E\left[\tau_1-\bar{Y}\right]$

$$=E\left[\begin{bmatrix}\left\{\bar{Y}(1+e_1)+\beta_{yx}\bar{X}(1+e_4)(1+e_5)^{-1}(e_2-e_3)\right\}\\exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\end{bmatrix}-\bar{Y}\right] \tag{3.11}$$

$B(\tau_2)=E\left[\tau_2-\bar{Y}\right]$

$$=E\left[\left[\left\{\bar{Y}(1+e_1)(1+e_2)(1+e_3)^{-1}\right\}exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\right]-\bar{Y}\right] \tag{3.12}$$

$B(\tau_3)=E\left[\tau_3-\bar{Y}\right]$

$$=E\left[\begin{bmatrix}\left\{\bar{Y}(1+e_1)+\beta_{yx}\bar{X}(1+e_4)(1+e_5)^{-1}\right.\\\left.\left\{\left\{(1+e_2)exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\right\}-(1+e_3)\right\}\end{bmatrix}-\bar{Y}\right] \tag{3.13}$$

Now, expanding the right hand side of the equations (3.11) - (3.13) binomially and exponentially, taking expectations and retaining the terms up to first order of approximations, we get the expressions of the bias of the estimators τ_1,τ_2 and τ_3 as derived in equations (3.4) - (3.6).

The bias of the estimators $\bar{y}_{MI_1},\bar{y}_{MI_2},\bar{y}_{MI_3}$ and $\bar{y}_{MI_4}$ are derived as

$$B(\bar{y}_{MI_1})=E\left[\bar{y}_{MI_1}-\bar{Y}\right]$$

$$=E\left[\left\{\frac{1}{3}\{\tau_1+\tau_2+\tau_3\}\right\}-\bar{Y}\right]=\frac{1}{3}E\left[\left(\tau_1-\bar{Y}\right)+\left(\tau_2-\bar{Y}\right)+\left(\tau_3-\bar{Y}\right)\right]$$

$$=\frac{1}{3}\left[E\left(\tau_1-\bar{Y}\right)+E\left(\tau_2-\bar{Y}\right)+E\left(\tau_3-\bar{Y}\right)\right]$$

$$B(\bar{y}_{MI_1})=\frac{1}{3}\left\{B(\tau_1)+B(\tau_2)+B(\tau_3)\right\} \tag{3.14}$$

$$B(\bar{y}_{MI_2})=E\left[\bar{y}_{MI_2}-\bar{Y}\right]$$

$$=E\left[\left\{\frac{1}{2}\{\tau_1+\tau_2\}-\bar{Y}\right]=\frac{1}{2}E\left[\left(\tau_1-\bar{Y}\right)+\left(\tau_2-\bar{Y}\right)\right]$$

$$=\frac{1}{2}\left[E\left(\tau_1-\bar{Y}\right)+E\left(\tau_2-\bar{Y}\right)\right]$$

$$(3.15)\quad B\left(\bar{y}_{MI_2}\right)=\frac{1}{2}\left\{B(\tau_1)+B(\tau_2)\right\}$$

$$B\left(\bar{y}_{MI_3}\right)=E\left[\bar{y}_{MI_3}-\bar{Y}\right]$$

$$=E\left[\left\{\frac{1}{2}\left\{\tau_2+\tau_3\right\}\right\}-\bar{Y}\right]=\frac{1}{2}E\left[\left(\tau_2-\bar{Y}\right)+\left(\tau_3-\bar{Y}\right)\right]$$

$$=\frac{1}{2}\left[E\left(\tau_2-\bar{Y}\right)+E\left(\tau_3-\bar{Y}\right)\right]$$

$$(3.16)\quad B\left(\bar{y}_{MI_3}\right)=\frac{1}{2}\left\{B(\tau_2)+B(\tau_3)\right\}$$

$$B\left(\bar{y}_{MI_4}\right)=E\left[\bar{y}_{MI_4}-\bar{Y}\right]$$

$$=E\left[\left\{\frac{1}{2}\left\{\tau_1+\tau_3\right\}\right\}-\bar{Y}\right]=\frac{1}{2}E\left[\left(\tau_1-\bar{Y}\right)+\left(\tau_3-\bar{Y}\right)\right]$$

$$=\frac{1}{2}\left[E\left(\tau_1-\bar{Y}\right)+E\left(\tau_3-\bar{Y}\right)\right]$$

$$(3.17)\quad B\left(\bar{y}_{MI_4}\right)=\frac{1}{2}\left\{B(\tau_1)+B(\tau_3)\right\}$$

where $B\left(\tau_1\right)=E\left[\tau_1-\bar{Y}\right]$, B$(\tau_2)=E\left[\tau_2-\bar{Y}\right]$ and $B\left(\tau_3\right)=E\left[\tau_3-\bar{Y}\right]$

□

3.2. Theorem. *The mean square errors of the estimators* $\tau_1,\tau_2,\tau_3,\bar{y}_{MI_1},\bar{y}_{MI_2},\bar{y}_{MI_3}$ *and* $\bar{y}_{MI_4}$ *are given by*

$$(3.18)\quad M(\tau_1)=\bar{Y}^2\left[\begin{array}{l}\left(\frac{1}{r}-\frac{1}{N}\right)\left\{C_y^2+\frac{1}{4}C_x^2-\rho_{yx}C_yC_x\right\}\\+\left(\frac{1}{r}-\frac{1}{n}\right)\rho_{yx}C_yC_x\left\{C_x-\rho_{yx}C_y\right\}\end{array}\right]$$

$$(3.19)\quad M(\tau_2)=\bar{Y}^2\left[\begin{array}{l}\left(\frac{1}{r}-\frac{1}{N}\right)\left\{C_y^2+\frac{9}{4}C_x^2-3\rho_{yx}C_yC_x\right\}\\+2\left(\frac{1}{n}-\frac{1}{N}\right)\left\{\rho_{yx}C_yC_x-C_x^2\right\}\end{array}\right]$$

$$(3.20)\quad M(\tau_3)=\bar{Y}^2C_y^2\left(\frac{1}{r}-\frac{1}{N}\right)\left[1-\frac{3}{4}\rho_{yx}^2\right]$$

$$(3.21)\quad M(\bar{y}_{MI_1})=\left[\begin{array}{l}\frac{1}{9}\left[M(\tau_1)+M(\tau_2)+M(\tau_3)\right]\\+2\left\{C\left(\tau_1,\tau_2\right)\ +C\left(\tau_1,\tau_3\right)\ +C\left(\tau_2,\tau_3\right)\right\}\end{array}\right]$$

$$(3.22)\quad M(\bar{y}_{MI_2})=\frac{1}{4}\left[M(\tau_1)+M(\tau_2)+2C\left(\tau_1,\tau_2\right)\right]$$

$$(3.23)\quad M(\bar{y}_{MI_3})=\frac{1}{4}\left[M(\tau_2)+M(\tau_3)\ \ +2C\left(\tau_2,\tau_3\right)\right]$$

$$(3.24)\quad M(\bar{y}_{MI_4})=\frac{1}{4}\left[M(\tau_1)+M(\tau_3)\ \ +2C\left(\tau_1,\tau_3\right)\right]$$

where

$$(3.25)\quad C\left(\tau_1,\tau_2\right)=\bar{Y}^2\left[\begin{array}{l}\left(\frac{1}{r}-\frac{1}{N}\right)\left(C_y^2-\frac{1}{4}C_x^2-\rho_{yx}C_yC_x\right)\\+\left(\frac{1}{r}-\frac{1}{n}\right)\left(C_x^2-\rho_{yx}^2C_y^2\right)\end{array}\right]$$

$$(3.26)\quad C\left(\tau_1,\tau_3\right)=\bar{Y}^2\left[\begin{array}{l}\left(\frac{1}{r}-\frac{1}{N}\right)\left(C_y^2-\frac{1}{4}\rho_{yx}C_yC_x-\frac{1}{2}\rho_{yx}^2C_y^2\right)\\+\left(\frac{1}{r}-\frac{1}{n}\right)\frac{1}{2}\left(\rho_{yx}C_yC_x-\rho_{yx}^2C_y^2\right)\end{array}\right]$$

$$(3.27)\quad C\left(\tau_2,\tau_3\right)=\bar{Y}^2\left[\begin{array}{l}\left(\frac{1}{r}-\frac{1}{N}\right)\left(C_y^2-\frac{1}{4}\rho_{yx}C_yC_x-\frac{1}{2}\rho_{yx}^2C_y^2\right)\\+\left(\frac{1}{r}-\frac{1}{n}\right)\left(\rho_{yx}C_yC_x-\rho_{yx}^2C_y^2\right)\end{array}\right]$$

Proof. The mean square errors of the estimators τ_1, τ_2 and τ_3 are derived as
$M(\tau_1) = E\left[\tau_1 - \bar{Y}\right]^2$

$$= E\left[\left[\begin{array}{l} \left\{\bar{Y}(1+e_1) + \beta_{yx}\bar{X}(1+e_4)(1+e_5)^{-1}(e_2-e_3)\right\} \\ exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\} \end{array}\right] - \bar{Y}\right]^2 \tag{3.28}$$

$M(\tau_2) = E\left[\tau_2 - \bar{Y}\right]^2$

$$= E\left[\left[\left\{\bar{Y}(1+e_1)(1+e_2)(1+e_3)^{-1}\right\} exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\right] - \bar{Y}\right]^2 \tag{3.29}$$

$M(\tau_3) = E\left[\tau_3 - \bar{Y}\right]^2$

$$= E\left[\left[\begin{array}{l} \left\{\bar{Y}(1+e_1) + \beta_{yx}\bar{X}(1+e_4)(1+e_5)^{-1}\right. \\ \left.\left\{\left\{(1+e_2)\, exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\right\} - (1+e_3)\right\}\right. \end{array}\right] - \bar{Y}\right]^2 \tag{3.30}$$

Now, expanding the right hand side of the equations (3.28) - (3.30) binomially and exponentially, taking expectations and retaining the terms up to first order of approximations, we get the expressions of the mean square errors of the estimators τ_1, τ_2 and τ_3 as derived in equations (3.18) - (3.20).

The mean square errors of the estimators $\bar{y}_{MI_1}, \bar{y}_{MI_2}, \bar{y}_{MI_3}$ and $\bar{y}_{MI_4}$ are derived as

$$M(\bar{y}_{MI_1}) = E\left[\bar{y}_{MI_1} - \bar{Y}\right]^2$$

$$= E\left[\left\{\frac{1}{3}\{\tau_1 + \tau_2 + \tau_3\}\right\} - \bar{Y}\right]^2 = E\left[\frac{1}{3}\left(\tau_1 - \bar{Y}\right) + \frac{1}{3}\left(\tau_2 - \bar{Y}\right) + \frac{1}{3}\left(\tau_3 - \bar{Y}\right)\right]^2$$

$$M(\bar{y}_{MI_1}) = \left[\begin{array}{l} \frac{1}{9}\left\{E\left(\tau_1 - \bar{Y}\right)^2 + E\left(\tau_2 - \bar{Y}\right)^2 + E\left(\tau_3 - \bar{Y}\right)^2\right\} \\ \frac{2}{9}\left[E\left[\left(\tau_1 - \bar{Y}\right)\left(\tau_2 - \bar{Y}\right)\right] + E\left[\left(\tau_1 - \bar{Y}\right)\left(\tau_3 - \bar{Y}\right)\right]\right] \\ +\frac{2}{9}\left[E\left[\left(\tau_2 - \bar{Y}\right)\left(\tau_3 - \bar{Y}\right)\right]\right] \end{array}\right] \tag{3.31}$$

$$M(\bar{y}_{MI_2}) = E\left[\bar{y}_{MI_2} - \bar{Y}\right]^2$$

$$= E\left[\left\{\frac{1}{2}\{\tau_1 + \tau_2\}\right\} - \bar{Y}\right]^2 = E\left[\frac{1}{2}\left(\tau_1 - \bar{Y}\right) + \frac{1}{2}\left(\tau_2 - \bar{Y}\right)\right]^2$$

$$= \left[\frac{1}{4}E\left(\tau_1 - \bar{Y}\right)^2 + \frac{1}{4}E\left(\tau_2 - \bar{Y}\right)^2 + \frac{1}{2}E\left[\left(\tau_1 - \bar{Y}\right)\left(\tau_2 - \bar{Y}\right)\right]\right.$$

$$M(\bar{y}_{MI_2}) = \frac{1}{4}\left[M(\tau_1) + M(\tau_2) + 2C(\tau_1, \tau_2)\right] \tag{3.32}$$

$$M(\bar{y}_{MI_3}) = E\left[\bar{y}_{MI_3} - \bar{Y}\right]^2$$

$$= E\left[\left\{\frac{1}{2}\{\tau_2 + \tau_3\}\right\} - \bar{Y}\right]^2 = E\left[\frac{1}{2}\left(\tau_2 - \bar{Y}\right) + \frac{1}{2}\left(\tau_3 - \bar{Y}\right)\right]^2$$

$$= \left[\frac{1}{4}E\left(\tau_2 - \bar{Y}\right)^2 + \frac{1}{4}E\left(\tau_3 - \bar{Y}\right)^2 + \frac{1}{2}E\left[\left(\tau_2 - \bar{Y}\right)\left(\tau_3 - \bar{Y}\right)\right]\right.$$

$$M(\bar{y}_{MI_3}) = \frac{1}{4}\left[M(\tau_2) + M(\tau_3) \quad + 2C(\tau_2, \tau_3)\right] \tag{3.33}$$

$$M(\bar{y}_{MI_4}) = E\left[\bar{y}_{MI_4} - \bar{Y}\right]^2$$

$$= E\left[\left\{\frac{1}{2}\{\tau_1 + \tau_3\}\right\} - \bar{Y}\right]^2 = E\left[\frac{1}{2}\left(\tau_1 - \bar{Y}\right) + \frac{1}{2}\left(\tau_3 - \bar{Y}\right)\right]^2$$

$$= \left[\frac{1}{4}E\left(\tau_1 - \bar{Y}\right)^2 + \frac{1}{4}E\left(\tau_3 - \bar{Y}\right)^2 + \frac{1}{2}E\left[\left(\tau_1 - \bar{Y}\right)\left(\tau_3 - \bar{Y}\right)\right]\right.$$

$$(3.34)\quad M(\bar{y}_{MI_4}) = \frac{1}{4}\left[M(\tau_1) + M(\tau_3) \ \ + 2C\left(\tau_1, \tau_3\right)\right]$$

where $M\left(\tau_1\right) = E\left[\tau_1 - \bar{Y}\right]^2$, $\mathrm{M}(\tau_2) = E\left[\tau_2 - \bar{Y}\right]^2$, $\mathrm{M}(\tau_3) = E\left[\tau_3 - \bar{Y}\right]^2, C\left(\tau_1, \tau_2\right) = E\left[\left(\tau_1 - \bar{Y}\right)\left(\tau_2 - \bar{Y}\right)\right], C\left(\tau_1, \tau_3\right) = E\left[\left(\tau_1 - \bar{Y}\right)\left(\tau_3 - \bar{Y}\right)\right]$ and $C\left(\tau_2, \tau_3\right) = E\left[\left(\tau_2 - \bar{Y}\right)\left(\tau_3 - \bar{Y}\right)\right]$ The expressions of $C\left(\tau_1, \tau_2\right), C\left(\tau_1, \tau_3\right)$ and $C\left(\tau_2, \tau_3\right)$ are derived as

$$C\left(\tau_1, \tau_2\right) = E\left[\left(\tau_1 - \bar{Y}\right)\left(\tau_2 - \bar{Y}\right)\right]$$

$$(3.35)\quad = E\left[\begin{array}{l} \left[\left\{\left(\bar{Y}\left(1+e_1\right) + \beta_{yx}\bar{X}\left(1+e_4\right)\left(1+e_5\right)^{-1}\left(e_2 - e_3\right)\right)\right. \\ \left.\left(exp\left(-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right)\right)\right\} - \bar{Y}\right] \\ \left[\left[\left\{\bar{Y}\left(1+e_1\right)\left(1+e_2\right)\left(1+e_3\right)^{-1}\right\} exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\right] - \bar{Y}\right] \end{array}\right]$$

$$C\left(\tau_1, \tau_3\right) = E\left[\left(\tau_1 - \bar{Y}\right)\left(\tau_3 - \bar{Y}\right)\right]$$

$$(3.36)\quad = E\left[\begin{array}{l} \left[\left\{\left(\bar{Y}\left(1+e_1\right) + \beta_{yx}\bar{X}\left(1+e_4\right)\left(1+e_5\right)^{-1}\left(e_2 - e_3\right)\right)\right. \\ \left.\left(exp\left(-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right)\right)\right\} - \bar{Y}\right] \\ \left[\left\{\bar{Y}\left(1+e_1\right) + \beta_{yx}\bar{X}\left(1+e_4\right)\left(1+e_5\right)^{-1}\right.\right. \\ \left.\left.\left\{\left\{\left(1+e_2\right) exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\right\} - \left(1+e_3\right)\right\}\right\} - \bar{Y}\right] \end{array}\right]$$

$$C\left(\tau_2, \tau_3\right) = E\left[\left(\tau_2 - \bar{Y}\right)\left(\tau_3 - \bar{Y}\right)\right]$$

$$(3.37)\quad = E\left[\begin{array}{l} \left[\left[\left\{\bar{Y}\left(1+e_1\right)\left(1+e_2\right)\left(1+e_3\right)^{-1}\right\} exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\right] - \bar{Y}\right] \\ \left[\left\{\bar{Y}\left(1+e_1\right) + \beta_{yx}\bar{X}\left(1+e_4\right)\left(1+e_5\right)^{-1}\right.\right. \\ \left.\left.\left\{\left\{\left(1+e_2\right) exp\left\{-\frac{e_3}{2}\left(1+\frac{e_3}{2}\right)^{-1}\right\}\right\} - \left(1+e_3\right)\right\}\right\} - \bar{Y}\right] \end{array}\right]$$

Now, expanding the right hand side of the equations (3.35)− (3.37) binomially and exponentially, taking expectations and retaining the terms up to the first order of approximations, we get the expressions of the $C\left(\tau_1, \tau_2\right), C\left(\tau_1, \tau_3\right)$ and $C\left(\tau_2, \tau_3\right)$ as derived in equations (3.25) - (3.27). □

4. Some well-known methods of single imputation and resultant estimators

Following are the list of some existing methods of imputation and their resultant estimators which are often practiced in survey sampling.

4.1. Mean method of imputation. The data produced under mean method of imputation is described as

$$(4.1)\quad y_{.i} = \begin{cases} y_i & if \quad i \in R \\ y_r & if \quad i \in R^c \end{cases}$$

Under the method of imputation discussed in equation (4.1), the point estimator of the population mean $\bar{Y}$ is derived as

$$(4.2)\quad \bar{y}_M = \frac{1}{n}\sum_{i=1}^{n} y_{.i} = \frac{1}{n}\left[\sum_{i\in R} y_{.i} + \sum_{i\in R^c} y_{.i}\right] = \bar{y}_r$$

which is simplified as

The variance of the estimator $\bar{y}_M$ given in equation (4.2) is obtained under MCAR response mechanism and is given as

$$V\left(\bar{y}_M\right)=\left(\frac{1}{r}-\frac{1}{N}\right)\bar{Y}^2C_y^2$$

4.2. Ratio method of imputation. The ratio method of imputation is applied with the help of information obtained on an auxiliary variable x and consequently the data generated is described as

$$y_{.i}=\begin{cases} y_i & if \quad i\in R \\ \hat{b}_r x_i & if \quad i\in R^c \end{cases} \tag{4.3}$$

where $\hat{b}_r=\frac{\sum_{i\in R} y_i}{\sum_{i\in R} x_.}=\frac{\bar{y}_r}{\bar{x}_r}$

Under the method of imputation discussed in equation (4.3), the point estimator of population mean $\bar{Y}$ is derived as

$$\bar{y}_{RAT}=\frac{1}{n}\sum_{i=1}^{n}y_{.i}=\bar{y}_r\frac{\bar{x}_n}{\bar{x}_r} \tag{4.4}$$

The bias and mean square error of the estimator $\bar{y}_{RAT}$ are obtained under MCAR response mechanism up to first order of approximations and given as

$$B\left(\bar{y}_{RAT}\right)=\left(\frac{1}{r}-\frac{1}{n}\right)\bar{Y}\left(C_x^2-\rho_{yx}C_yC_x\right) \tag{4.5}$$

$$M\left(\bar{y}_{RAT}\right)=\bar{Y}^2\left[\left(\frac{1}{r}-\frac{1}{n}\right)C_y^2+\left(\frac{1}{r}-\frac{1}{n}\right)\left(C_x^2-\rho_{yx}C_yC_x\right)\right] \tag{4.6}$$

4.3. Regression method of imputation. The data generated by regression method of imputation is given as

$$y_{.i}=\begin{cases} y_i & if \quad i\in R \\ \hat{y}_i & if \quad i\in R^c \end{cases} \tag{4.7}$$

where

$$\hat{y}_i=\hat{a}+\hat{b}_{re}x_i, \hat{a}=\bar{y}_r-\hat{b}\bar{x}_r \text{ and } \hat{b}_{re}=\frac{S_{yx(r)}}{S_{x(r)}^2}$$

Under the method of imputation discussed in equation (4.5), the point estimator of population mean $\bar{Y}$ is derived as

$$\bar{y}_{REG}=\frac{1}{n}\sum_{i=1}^{n}y_{.i}=\bar{y}_r+\hat{b}_{re}\left(\bar{x}_n-\bar{x}_r\right) \tag{4.8}$$

The bias and mean square error of the estimator $\bar{y}_{REG}$ are obtained under MCAR response mechanism up to first order of approximations and given as

$$B\left(\bar{y}_{REG}\right)=\frac{\rho_{yx}C_y}{C_x\bar{X}}\left(\frac{1}{r}-\frac{1}{n}\right)\bar{Y}\left(\frac{\mu_{300}}{\mu_{200}}-\frac{\mu_{210}}{\mu_{110}}\right) \tag{4.9}$$

$$M\left(\bar{y}_{REG}\right)=\bar{Y}^2C_y^2\left[\left(\frac{1}{r}-\frac{1}{n}\right)-\left(\frac{1}{r}-\frac{1}{n}\right)\rho_{yx}^2\right] \tag{4.10}$$

5. Empirical study

In this section, we demonstrate the performances of the proposed imputation methods over mean, ratio and regression methods of imputation. To access the performances of the proposed methods, empirical studies are carried out on seventeen natural populations chosen from various survey literatures related to life sciences, agricultural and socio-economic characters. The details of the populations are provided in this section. The methodology of empirical study is as follows; from a finite population of size N a sample of size n is drawn under SRSWOR sampling scheme. The first m samples were selected from the all possible ${}^{N}C_n$ samples. First we drop (n-r) units randomly from each sample corresponding to the study variable y and imputed values are derived with six methods of imputations namely (i) Mean method of imputation (ii) Ratio method of imputation (iii) Regression method of imputation (iv) Suggested single imputations methods (v) Suggested multiple imputations methods

The percent relative efficiencies of the proposed single imputation methods with respect to the mean, ratio and regression methods of imputation are given as

$$PRE_1 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_M)_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\tau_1)_s - \bar{Y}\right]^2}\times 100, PRE_2 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_{RAT})_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\tau_1)_s - \bar{Y}\right]^2}\times 100,$$

$$PRE_3 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_{REG})_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\tau_1)_s - \bar{Y}\right]^2}\times 100, PRE_4 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_M)_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\tau_2)_s - \bar{Y}\right]^2}\times 100,$$

$$PRE_5 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_{RAT})_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\tau_2)_s - \bar{Y}\right]^2}\times 100, PRE_6 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_{REG})_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\tau_2)_s - \bar{Y}\right]^2}\times 100,$$

$$PRE_7 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_M)_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\tau_3)_s - \bar{Y}\right]^2}\times 100, PRE_8 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_{RAT})_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\tau_3)_s - \bar{Y}\right]^2}\times 100$$

and $PRE_9 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_{REG})_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\tau_3)_s - \bar{Y}\right]^2}\times 100$

The percent relative efficiencies of the proposed multiple imputations methods with respect to the mean, ratio, regression and proposed single imputation methods are given as

$$E_1 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_M)_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_1})_s - \bar{Y}\right]^2}\times 100, E_2 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_{RAT})_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_1})_s - \bar{Y}\right]^2}\times 100,$$

$$E_3 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_{REG})_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_1})_s - \bar{Y}\right]^2}\times 100, E_4 = \frac{\sum_{s=1}^{m}\left[(\tau_1)_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_1})_s - \bar{Y}\right]^2}\times 100,$$

$$E_5 = \frac{\sum_{s=1}^{m}\left[(\tau_2)_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_1})_s - \bar{Y}\right]^2}\times 100, E_6 = \frac{\sum_{s=1}^{m}\left[(\tau_3)_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_1})_s - \bar{Y}\right]^2}\times 100,$$

$$E_7 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_M)_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_2})_s - \bar{Y}\right]^2}\times 100, E_8 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_{RAT})_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_2})_s - \bar{Y}\right]^2}\times 100,$$

$$E_9 = \frac{\sum_{s=1}^{m}\left[(\bar{y}_{REG})_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_2})_s - \bar{Y}\right]^2}\times 100, E_{10} = \frac{\sum_{s=1}^{m}\left[(\tau_1)_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_2})_s - \bar{Y}\right]^2}\times 100,$$

$$E_{11} = \frac{\sum_{s=1}^{m}\left[(\tau_2)_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_2})_s - \bar{Y}\right]^2}\times 100, E_{12} = \frac{\sum_{s=1}^{m}\left[(\bar{y}_M)_s - \bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_3})_s - \bar{Y}\right]^2}\times 100,$$

$$E_{13}=\frac{\sum_{s=1}^{m}\left[(\bar{y}_{RAT})_s-\bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_3})_s-\bar{Y}\right]^2}\times 100, E_{14}=\frac{\sum_{s=1}^{m}\left[(\bar{y}_{REG})_s-\bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_3})_s-\bar{Y}\right]^2}\times 100,$$

$$E_{15}=\frac{\sum_{s=1}^{m}\left[(\tau_2)_s-\bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_3})_s-\bar{Y}\right]^2}\times 100, E_{16}=\frac{\sum_{s=1}^{m}\left[(\tau_3)_s-\bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_3})_s-\bar{Y}\right]^2}\times 100,$$

$$E_{17}=\frac{\sum_{s=1}^{m}\left[(\bar{y}_{M})_s-\bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_4})_s-\bar{Y}\right]^2}\times 100, E_{18}=\frac{\sum_{s=1}^{m}\left[(\bar{y}_{RAT})_s-\bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_4})_s-\bar{Y}\right]^2}\times 100,$$

$$E_{19}=\frac{\sum_{s=1}^{m}\left[(\bar{y}_{REG})_s-\bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_4})_s-\bar{Y}\right]^2}\times 100, E_{20}=\frac{\sum_{s=1}^{m}\left[(\tau_1)_s-\bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_4})_s-\bar{Y}\right]^2}\times 100,$$

$$\text{and } E_{21}=\frac{\sum_{s=1}^{m}\left[(\tau_2)_s-\bar{Y}\right]^2}{\sum_{s=1}^{m}\left[(\bar{y}_{MI_4})_s-\bar{Y}\right]^2}\times 100,$$

The percent relative efficiencies are computed for seventeen natural populations as described below and presented in Tables 1-7.

Population I [Source: [12]] (Page No. 399)
Y: Area under wheat in 1964
X: Area under wheat in 1963
N = 34, n = 7, r = 5, $\rho_{yx}=0.9800867$.

Population II [Source: [3]] (Page No. 58)
Y: Head length of second son.
X: Head length of first son.
$N=25, n=7, r=5, \rho_{yx}=0.7107518$.

Population III [Source: [5]] (Page No. 182)
Y: Number of placebo children.
X: Number of paralytic polio cases in the placebo group.
$N=34, n=7, r=5, \rho_{yx}=0.7328235$.

Population IV [Source: [8]] (Page No. 682)
Y: No. of hhś on ith block.
X: Eye estimate of no. of hhś on ith block
$N=20, n=7, r=5, \rho_{yx}=0.8662052$.

Population V [Source: [24]] (Page No. 349)
Y: Volume.
X: Diameter
$N=31, n=7, r=5, \rho_{yx}=0.9671194$.

Population VI [Source: [5]] (Page No. 182)
Y: Number of placebo children.
X: Number of paralytic polio cases in the not inoculated group.
$N=34, n=7, r=5, \rho_{yx}=0.6426412$.

Population VII [Source: [12]] (Page No. 399)
Y: Area under wheat in 1964
X: Cultivated area in 1961
$N=34, n=7, r=5, \rho_{yx}=0.9042627$.

Population VIII [Source: [3]] (Page No. 58)
Y: Head length of second son.
X: Head breadth of first son.
$N=34, n=7, r=5, \rho_{yx}=0.6931573$.

Population IX [Source: [5]] (Page No. 34)
Y: Food cost of family
X: Size of family

$N = 33, n = 7, r = 5, \rho_{yx} = 0.432738.$
Population X [Source: [7]] (Page No. 180)
Y: Sepal width of Iris setosa
X: Sepal length of Iris setosa
$N = 35, n = 7, r = 5, \rho_{yx} = 0.6315548.$
Population XI [Source: [6]] (Page No. 154)
Y: Average salary (in dollars) U. S.
X: Per pupil spending (in dollars) U. S.
$N = 26, n = 7, r = 5, \rho_{yx} = 0.8096703.$
Population XII [Source: [6]] (Page No. 274)
Y: Saving (in billions of dollars) U. S. (1970-1995).
X: Personal disposable income (in billions of dollars) U. S. (1970-1995).
$N = 26, n = 7, r = 5, \rho_{yx} = 0.8759079.$
Population XIII [Source: [6]] (Page No. 460)
Y: Index of real compensation per hour, business sector of U. S. (1959-1998).
X: Index of output per hour, business sector of U. S. (1959-1998).
$N = 30, n = 7, r = 5, \rho_{yx} = 0.9910549.$
Population XIV [Source: [6]] (Page No. 710)
Y: Investment in fixed plant and equipment in manufacturing (in billions of dollars) of U. S. (1970-1991).
X: Manufacturing sales (in billions of dollars) seasonally adjusted of U. S. (1970-1991).
$N = 22, n = 7, r = 5, \rho_{yx} = 0.9903192.$
Population XV [Source: [23]] (Page No. 166)
Y: Number of banana bunches.
X: Number of banana pits.
$N = 20, n = 7, r = 5, \rho_{yx} = 0.9800867.$
Population XVI [Source: [24]] (Page No. 349)
Y: Volume.
Z: Height
$N = 31, n = 7, r = 5, \rho_{yx} = 0.5982497.$
Population XVII [Source: [5]] (Page No. 32)
Y: Food cost of family
X: Income of family
$N = 33, n = 7, r = 5, \rho_{yx} = 0.2521603.$

Table 1: Percent relative efficiencies of the estimator τ_1 with respect to mean, ratio and regression method of imputation

Population Source	PRE_1	PRE_2	PRE_3
Population I	651.309	316.1384	323.7037
Population II	157.1894	126.3349	124.532
Population III	223.1392	162.9272	194.0364
Population IV	294.5976	188.9788	186.6463
Population V	164.7055	154.3641	158.9833
Population VI	200.7349	166.7052	181.3413
Population VII	284.5805	182.3409	178.0122
Population VIII	241.128	170.1591	155.7499
Population IX	146.6306	133.258	110.9385
Population X	100.5127	106.255	101.159
Population XI	182.2423	144.3668	142.4705
Population XII	264.9797	189.8048	184.0865
Population XIII	2139.517	735.6239	925.6935
Population XIV	287.5206	237.6237	237.7244
Population XV	236.8863	169.4697	172.0994

Table 2: Percent relative efficiencies of the estimator τ_2 with respect to mean, ratio and regression method of imputation

Population Source	PRE_1	PRE_2	PRE_3
Population I	609.8675	296.0231	303.1071
Population II	125.24594	100.6827	100.24594
Population III	177.9285	129.9161	154.7222
Population IV	248.101	147.7621	150.0241
Population V	301.875	282.9211	291.3873
Population VI	143.1064	118.8463	129.3873
Population VII	245.6476	157.3952	153.6587
Population VIII	181.8826	127.9263	117.0935
Population IX	116.9035	111.8727	106.1338
Population X	145.7711	115.4754	113.9586
Population XI	163.0738	142.7995	138.4974
Population XII	193.4761	198.1857	205.0121
Population XIII	3647.527	1254.118	1578.156
Population XIV	316.3238	261.4263	261.5392
Population XV	208.6929	149.2999	151.6167

Table 3: Percent relative efficiencies of the estimator τ_3 with respect to mean, ratio and regression method of imputation

Population Source	PRE_1	PRE_2	PRE_3
Population I	746.0278	362.1138	370.7794
Population II	148.3297	119.2142	117.5129
Population III	136.2724	100.50058	118.4991
Population IV	287.3633	184.3382	182.063
Population V	158.7588	148.7907	153.2432
Population VI	121.4111	100.8289	109.6812
Population VII	339.7371	217.6817	212.5141
Population VIII	261.7878	184.1273	168.5353
Population IX	149.2613	135.6488	112.9288
Population X	105.8197	111.8657	106.5001
Population XI	174.4859	138.224	136.4069
Population XII	241.7096	247.5934	256.1216
Population XIII	1264.535	434.813	547.1196
Population XVI	307.6482	254.2538	254.3616
Population XV	236.1414	168.9367	171.5582

Table 4: Percent relative efficiencies of the estimator $\bar{y}_{MI_1}$ with respect to mean, ratio, regression, τ_1, τ_2, and τ_3 method of imputation

Source	E_1	E_2	E_3	E_4	E_5	E_6
Population I	264.907	207.769	204.951	101.495	100.9956	100.4527
Population XVI	161.723	149.986	123.486	111.909	131.157	115.262
Population XVII	108.2	101.727	104.704	108.014	107.128	126.231

Table 5: Percent relative efficiencies of the estimator $\bar{y}_{MI_2}$ with respect to mean, ratio, regression, τ_1, and τ_2 method of imputation

Source	E_7	E_8	E_9	E_{10}	E_{11}
Population I	263.29	206.5075	203.701	100.8759	100.38554
Population II	204.77	134.7842	157.6394	105.1446	108.5116
Population VIII	192.0412	148.8212	173.3055	106.0514	111.1025
Population XI	142.7501	122.8817	144.1724	108.6922	101.2051
Population XV	239.2887	173.8419	171.1855	101.0125	101.3311
Population XVII	107.9156	101.4604	104.4289	107.7306	106.847

Table 6: Percent relative efficiencies of the estimator $\bar{y}_{MI_3}$ with respect to mean, ratio, regression, τ_2, and τ_3 method of imputation

Source	E_{12}	E_{13}	E_{14}	E_{15}	E_{16}
Population I	578.9268	251.5427	240.5896	102.2776	101.6885
Population II	189.3006	124.63	145.77	100.3426	138.7786
Population IV	280.08	169.36	166.809	102.6493	112.8907
Population VI	143.5103	131.2604	113.5755	110.1633	103.5136
Population VIII	120.7001	109.198	114.8592	101.4147	109.9245
Population XIV	314.2731	259.8436	259.7335	102.1552	100.3516
Population XVI	169.3458	157.0557	129.3067	137.3398	120.6951
Population XVII	109.7599	103.1943	106.2136	108.6731	128.0516

Table 7: Percent relative efficiencies of the estimator $\bar{y}_{MI_4}$ with respect to mean, ratio, regression, τ_1, and τ_3 method of imputation

Source	E_{17}	E_{18}	E_{19}	E_{20}	E_{21}
Population I	264.0247	207.0764	204.268	101.1567	100.1214
Population VII	118.5684	118.6753	128.109	100.5546	101.4809
Population X	152.753	127.9926	137.42	100.3505	101.3829
Population XVI	155.0982	143.8421	118.4277	107.3255	110.5407

6. Conclusions and recommendations

A close look on Tables 1-7 reveals that the proposed methods of imputations are rewarding in terms of percent relative efficiencies. These findings suggest that the proposed single and multiple methods of imputations described in this paper are highly beneficial in minimizing the negative impact of non-response to a greater extent as compared to the mean, ratio and regression methods of imputation. The survey statisticians may be encouraged for the practical applications of the suggested imputation methods, if non-response is unavoidable in the survey data.

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The effect of changing scores for multi-way tables with open-ended ordered categories

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Abstract

Log-linear models are used to analyze the contingency tables. If the variables are ordinal or interval, because the score values affect both the model significance and parameter estimates, selection of score values has importance. Sometimes an interval variable contains open-ended categories as the first or last category. While the variable has open-ended classes, estimates of the lowermost and/or uppermost values of distribution must be handled carefully. In that case, the unknown values of the first and last classes can be estimated firstly, and then the score values can be calculated. In the previous studies, the unknown boundaries were estimated by using interquartile range (IQR). In this study, we suggested interdecile range (IDR), interpercentile range (IPR), and mid-distance range (MDR) as alternative to IQR to detect the effects of score values on model parameters.

Keywords: Contingency tables, Log-linear models, Interval measurement, Open-ended categories, Scores.

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1. Introduction

Categorical variables, which have a measurement scale consisting of a set of categories, are of importance in many fields often in the medical, social, and behavioral sciences. The tables that represent these variables are called contingency tables. Log-linear model equations are applied to analyze these tables. Interaction, row effects, and association parameters are strictly important to interpret the tables.

In the presence of an ordinal variable, score values should be considered. As using row effects parameters for nominal–ordinal tables, association parameter is suggested for ordinal–ordinal tables. Score values are used to weight these parameters. In that case, selection of score values is important. For instance, taking the score values equal does not fit in many studies because these scores may not represent true intervals between categories. Choice of scores affects estimates of model parameters and results of goodness-of-fit test statistics.

To use quantitative data in contingency tables, the data need to be converted to qualitative form. If one category (class) of a variable has either no lower or upper limit, this category is called open-ended. Age, income, serum cholesterol levels, systolic blood pressure are some examples of variable which can have open-ended categories. Ku and Kullback [12] used a contingency table which one of its variable is systolic blood pressure with the levels: (1) "< 127", (2) 127-146, (3) 147-166, (4) "≥ 167". Lower bound of the first and upper bound of the fourth categories are unknown. Agresti [3] applied linear-by-linear association model to the data and accepted that the distance between (1–2), (2–3), and (3–4) categories are equal. If it is not allowed to get raw data, it is not possible to find minimum and maximum values. Therefore, it is impossible to find the boundaries of open-ended categories. In this situation, the boundaries need to be estimated first. Then the score values can be calculated.

Determining these boundaries and fitted score values have been discussed by authorities. The author who studied on score values initially was Birch [6]. Simon [14], Goodman [9], Agresti [3], Graubard and Korn [10] discussed the equally spaced score values in their studies. Inequally spaced scores were discussed in the studies of Bross [7] and Agresti [3]. Iki *et al.* [11] used ridit scores to analyze square contingency tables by using cumulative probabilities. More recently, Bagheban and Zayeri [5] proposed exponential score values as an alternative to equal spaced scores. Initially, Frigge *et al.* [8] proposed the interquartile range to illustrate the outlier, then Tibshirani and Hastie [15], and Liu and Wu [13] focused on the interquartile range (IQR) to detect genes with over-expressed outlier disease samples as we used on estimate of the open ended boundaries. Aktas and Saracbasi [4] used median and quartile ranges to calculate standardized score values on open-ended categories. We suggested three different methods as alternative to IQR for ordinal categories that are grouped from quantitative data.

In this paper, through an application with one open-ended variable, we discussed the effects of score values on model parameters. The proposed new methods used to determine the boundaries of open-ended classes. In section 2, the log-linear models were introduced. Section 3 outlined the score methods and suggested the methods to estimate the boundaries of open-ended categories were represented in Section 4. The log-linear models and the estimation methods were illustrated in Section 5 by an application.

2. Log-Linear Models

2.1. Models for Two-way Tables. Consider an $R \times C$ contingency table that the first variable is represented by X and the second variable is represented by Y. In this two-way table, cross-classifies constitute multinominal sample of n subjects on two categorical responses. Let n_{ij} denote the frequency of (i,j) cell and the cell probabilities are π_{ij} and the expected values m_{ij} where $i = 1, 2, \ldots, R$ and $j = 1, 2, \ldots, C$. The properties of independence [2], linear by linear association [9], and row effects [3] models for two-way contingency tables are given in Table 1.

Table 1. The properties of most used log-linear models for two-way contingency tables

Model	X	Y	Equation	*df*
Independence	N, O*	N,O	$log\ m_{ij} = \lambda + \lambda_i^X + \lambda_j^Y$	$(R-1)(C-1)$
Linear by Linear Association	O	O	$log\ m_{ij} = \lambda + \lambda_i^X + \lambda_j^Y + \beta u_i v_j$	$(R-1)(C-1)-1$
Row Effects	N	O	$log\ m_{ij} = \lambda + \lambda_i^X + \lambda_j^Y + \mu_i v_j$	$(R-1)(C-2)$

*:N: Nominal O: Ordinal

Here, in the equations λ is the overall effect parameter, λ_i^X is effect of variable X at i and λ_j^Y is effect of variable Y at j with constraints such as $\sum_{i=1}^{R} \lambda_i^X = \sum_{j=1}^{C} \lambda_j^Y = 0$. u_i and v_j in linear by linear association model are the the known scores where $u_1 \leq u_2 \leq \ldots \leq u_R$ are ordered row scores and $v_1 \leq v_2 \leq \ldots \leq v_C$ are column scores. β is the association parameter. Goodman [9] called the specifical case of model *uniform association model*, where $\{u_i = i\}$ and $\{v_j = j\}$. μ_i in row effect model is the row effect parameters where constraints are needed such as $\sum_{i=1}^{R} \mu_i = 0$.

The local log-odds ratios of linear by linear association, uniform association and row effects models are given in the Equations (2.1)-(2.3), respectively.

$$log\ \theta_{ij} = \beta(u_i - u_{i+1})(v_j - v_{j+1}), \tag{2.1}$$

$$log\ \theta_{ij} = \beta, \tag{2.2}$$

$$log\ \theta_{ij} = (\mu_{i+1} - \mu_i)(v_{j+1} - v_j). \tag{2.3}$$

2.2. Models for Multi-way Tables for Nominal × Ordinal × Ordinal Categorical Data. Let X be a nominal variable, Y and Z be ordinal variables and, u_j are score values for variable Y and v_k are score values for variable Z. Then the full model is:

$$log\ m_{ijk} = \lambda + \lambda_i^X + \lambda_j^Y + \lambda_k^Z + \mu_i^{XY} u_j + \mu_i^{XZ} v_k + \beta^{YZ} u_j v_k. \tag{2.4}$$

The constraints are $\sum_{i=1}^{R} \lambda_i^X = \sum_{j=1}^{C} \lambda_j^Y = \sum_{i=1}^{R} \mu_i^{XY} = \sum_{i=1}^{R} \mu_i^{XZ} = 0$. In this model, β^{YZ} represents the linear-by-linear association parameter, μ_i^{XY} and μ_i^{XZ} represent the row effects model parameters [2].

$log\ \theta_{ij(k)}$ is the conditional log-odds ratio between X and Y for fixed levels of Z, $log\ \theta_{i(j)k}$ is the conditional log-odds ratio between X and Z for fixed levels of Y and $log\ \theta_{(i)jk}$ is the conditional log-odds ratio between Y and Z for fixed levels of X can be calculated from Equation (2.5).

$$
(2.5)\qquad
\begin{aligned}
log\ \theta_{ij(k)} &= (\mu_{i+1}^{XY} - \mu_i^{XY})(u_{j+1} - u_j) \\
log\ \theta_{i(j)k} &= (\mu_{i+1}^{XZ} - \mu_i^{XZ})(v_{k+1} - v_k) \\
log\ \theta_{(i)jk} &= \beta^{YZ}(u_{j+1} - u_j)(v_{k+1} - v_k).
\end{aligned}
$$

2.3. Scoring Methods. For log-linear model studies, assignment of score values is important. Assuming all distance between adjacent categories equal is not always fit the data. In this situation, the way to assign the scores causes a problem. The score equality of best fitting model is chosen as the distance between adjacent categories. As $\pi_{.j}$, $j = 1, 2, \ldots, C$ are the marginal probabilities of the ordered variable Y, the properties of equal spaced, ridit [7, 11], and exponential [5] scores are summarized in Table 2.

Table 2. The recommended score equalities

Scores	**Variables**	u_i	v_j
Equal spaced	N, O	i	j
Ridit	O	-	$\sum_{k=1}^{j-1} \pi_{.k} + \frac{1}{2}\pi_{.j}$
Exponential	O	i^a	j^a

For application of equal spaced scores, all the intervals between adjacent categories are assumed as equal. The cumulative probabilities are used to calculate ridit scores. Sometimes, non-equality characteristic of scores are observed in the categories of variables. In this situation, the arithmetic progression between categories disappears. The exponential scores are used when the baseline characteristic of categories changing by a geometric progression. a in the exponential score equation is called the power parameter and the model gives the uniform association model with equal spaced score values for $a = 1$.

3. Suggested Methods to Estimate the Boundaries of Open-ended Categories

The most practical scoring method is the exponential scores because it permits different values of the power parameter. However, when working on the open-ended ordered categories, these methods are insufficient. Applying the same method both ordered and open-ended categories is only possible when ignoring the open-ended structure. It makes the minimum value (lower bound of the first category) and the maximum value (upper bound of the last category) unimportant. However, these unknown values are the proof of inequality of scores.

Instead of using equal or non-equal scoring method, the different methods need to be used. To avoid the outlier problem, the interquartile range was suggested as a measure of dispersion [13]. The first quartile of a raw data is defined as Q_1 and the third quartile is Q_3. Then, the interquartile range is $IQR = Q_3 - Q_1$. For a frequency table with k categories, the values which are less and greater than the limits in the Equation (3.1) were defined as outliers by Frigge *et al.* [8] under the normality assumption.

$$
(3.1)\qquad
\begin{aligned}
LowerBound(LB_1) &= Q_1 - 1.5 \times IQR \\
UpperBound(UB_k) &= Q_3 + 1.5 \times IQR.
\end{aligned}
$$

The definition of the quartiles can affect the number of observations which shown as outside. This estimation method is used with 25% trimmed range. Changes of trimmed range may have greater effects on the estimate of score values.

3.1. Interdecile and Interpercentile Ranges. In this study, *interdecile range (IDR)* and *interpercentile range (IPR)* were suggested as the alternatives of IQR, having 10% and 5% trimmed ranges, respectively. The calculations of IDR ($IDR = P_{90} - P_{10}$) and IPR ($IPR = P_{95} - P_5$) are similar with IQR.

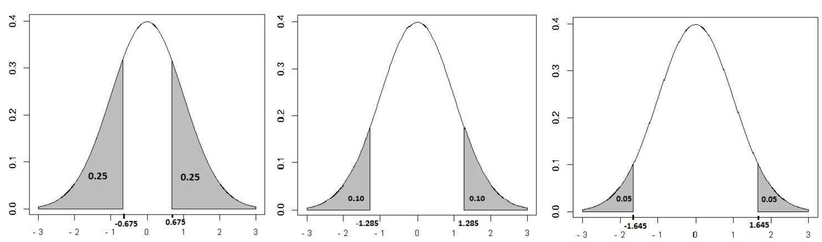
Under the normality assumption, the estimations of the boundaries with these methods can be limited as following equations, respectively.

$$(3.2) \qquad LB_1 = P_{10} - 0.78 \times IDR \quad \text{and} \quad UB_k = P_{90} + 0.78 \times IDR,$$

$$(3.3) \qquad LB_1 = P_5 - 0.61 \times IPR \quad \text{and} \quad UB_k = P_{95} + 0.61 \times IPR.$$

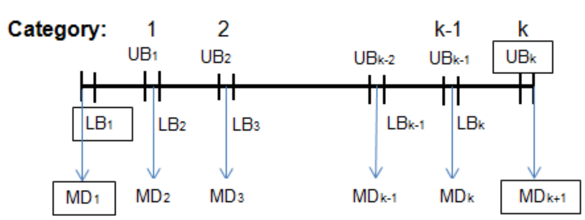
The standard normal distribution graphs and Z-values in order of IQR, IDR, and IPR are shown in Figure 1. Although the IPR seems to have wider range, this does not mean that it uses larger part of the distribution and it is better. The aim is to explain the data well and this depends on the distribution of frequencies.

Figure 1. The trimmed ranges for IQR, IDR, and IPR under the standard normal distribution



3.2. Mid-distance Range. Mid-distance range (MDR) was suggested to use as an alternative to IQR. The mid-distance ($MD_i = (LB_i + UB_{i-1})/2$) is the midpoint of i^{th} and $(i+1)^{th}$ categories where $i = 2, 3, \ldots, k$. The definition of MD is shown in Figure 2. In this figure, the first and last categories are open-ended and the values in the boxes are unknown. For a variable with k categories, the frequency table has $(k+1)$ MD. However, because of open-ended boundaries (LB_1 and UB_k), MD_1 and MD_{k+1} are not calculated.

Figure 2. The mid-distances of a k-categories frequency table



Under the normality assumption, the percentage of the first category is $p_1 = P(x < MD_2)$ and the k^{th} category is $p_k = P(x > MD_k)$. Then MDR is calculated from $MDR = MD_k - MD_2$. The distribution of frequencies is used to calculate MDR. Under the normality assumption, the boundaries are suggested,

$$LB_1 = MD_2 - [1/|Z_1|] \times MDR \quad \text{and} \quad UB_k = MD_k + [1/|Z_k|] \times MDR, \tag{3.4}$$

where $Z_1 = \Phi^{-1}(p_1)$ and $Z_k = \Phi^{-1}(p_k)$.

For Ku and Kullback [12] example, MD's of systolic blood pressure are calculated and shown in Table 3 [2].

Table 3. Mid-distances of systolic blood pressure

i	LB	UB	MD	
			–	MD_1
1	–	126		
			126.5	MD_2
2	127	146		
			146.5	MD_3
3	147	166		
			166.5	MD_4
4	167	–		
			–	MD_5

3.3. Standardized Score Values for Open-ended Categories. For an open-ended frequency table, because median is the appropriate measure of location and the quartile deviation is the appropriate measure of dispersion, Aktas and Saracbasi [4] suggested a score value that is calculated from quartile values. As s_i is the midpoint of i^{th} class, Q_2 is the median and Q_1, Q_3 are the first and third quartiles, respectively. The midpoint is,

$$s_i = \frac{LB_i + UB_i}{2}, \qquad i = 1, 2, \ldots, k. \tag{3.5}$$

Here, the estimated LB_1 and UB_k, which are defined in Equations (3.1)-(3.4), are used to calculate the midpoints. The standardized score values for row and column variables are

$$\begin{aligned} u_i &= \frac{s_i - Q_2}{(Q_3 - Q_1)/2}, \qquad i = 1, 2, \ldots, R \\ v_j &= \frac{s_j - Q_2}{(Q_3 - Q_1)/2}, \qquad j = 1, 2, \ldots, C. \end{aligned} \tag{3.6}$$

4. An Application

The $2 \times 4 \times 4$ contingency table, which is shown in Table 4, is taken from *General Social Survey, 1991, National Opinion Research Center.* It refers to the relationship between job satisfaction and income, stratified by gender, for 104 African–Americans [3].

The described models in Section 2 with equal spaced score values for $(nominal \times ordinal \times ordinal)$ structure were applied to the data in Table 4. Because the data set contains sampling zeros, a correction factor for zero of 6 cells ($n_{ij} = 0 + 0.5$) was used. Table 5 shows the value of likelihood ratio statistics (G^2) for testing the goodness-of-fit of each model. λ_i^G is the effect of gender at i, λ_j^I is the effect of income at j, and λ_k^S is the effect of job satisfaction at k. μ_i^{GI} and μ_i^{GS} are the row effects parameters between

Table 4. Job Satisfaction and income, controlling for gender

		Job Satisfaction			
Gender	**Income**	**Very Dissatisfied**	**A Little Satisfied**	**Moderately Satisfied**	**Very Satisfied**
Female	< 5000	1	3	11	2
	5000–15,000	2	3	17	3
	15,000–25,000	0	1	8	5
	$> 25,000$	0	2	4	2
Male	< 5000	1	1	2	1
	5000–15,000	0	3	5	1
	15,000–25,000	0	0	7	3
	$> 25,000$	0	1	9	6

gender–income and gender–job satisfaction, respectively. β^{IS} is the association parameter between income and job satisfaction. Then, Akaike Information Criteria (AIC) was used to select the best fitting model [1]. Regarding the presented results, all models were fit the data. Because the Model 6 that contains both association parameter between income–job satisfaction and the row effects parameter between gender–income had the smallest value of AIC, this model was chosen as the best fitting model.

Table 5. The results of goodness-of-fit test results for equal spaced score values

	Models	G^2	*df*	**P-Value**	**AIC**
1	$log\ m_{ijk} = \lambda + \lambda_i^G + \lambda_j^I + \lambda_k^S$	25.326	24	0.388	−22.674
2	$log\ m_{ijk} = \lambda + \lambda_i^G + \lambda_j^I + \lambda_k^S + \mu_i^{GI} u_j$	13.716	23	0.935	−32.284
3	$log\ m_{ijk} = \lambda + \lambda_i^G + \lambda_j^I + \lambda_k^S + \mu_i^{GS} v_k$	24.983	23	0.351	−21.017
4	$log\ m_{ijk} = \lambda + \lambda_i^G + \lambda_j^I + \lambda_k^S + \beta^{IS} u_j v_k$	20.794	23	0.594	−25.206
5	$log\ m_{ijk} = \lambda + \lambda_i^G + \lambda_j^I + \lambda_k^S + \mu_i^{GI} u_j + \mu_i^{GS} v_k$	13.373	22	0.922	−30.627
6	$log\ m_{ijk} = \lambda + \lambda_i^G + \lambda_j^I + \lambda_k^S + \mu_i^{GI} u_j + \beta^{IS} u_j v_k$	9.184	22	0.992	**−34.816**
7	$log\ m_{ijk} = \lambda + \lambda_i^G + \lambda_j^I + \lambda_k^S + \mu_i^{GS} v_k + \beta^{IS} u_j v_k$	20.451	22	0.555	−23.549
8	$log\ m_{ijk} = \lambda + \lambda_i^G + \lambda_j^I + \lambda_k^S + \mu_i^{GI} u_j + \mu_i^{GS} v_k + \beta^{IS} u_j v_k$	9.174	21	0.988	−32.826

Thereafter, the recommended score values were applied to Model 6 to choose the appropriate score values. Considering the open-ended structure, the standardized score values for income were calculated. Because gender is a nominal variable, score alternatives were not considered. For job satisfaction, equal spaced, exponential, and ridit scores were applied. The IQR, IDR, IPR, and MDR values for income were calculated as 17936.92, 25855.86, 30441.32, and 20000 respectively. To use mid-distance range, the percentages of first and fourth categories were calculated as $p_1 = 0.2056$ and $p_4 = 0.2337$. Then, LB_1 and UB_k from the methods, that were previously mentioned, were estimated. The estimated boundaries and range of income are shown in the Table 6. The estimated values of the lower bound are negative. This is reasonable when considering the people's loans. Between these methods, MDR has the largest value.

The score values in the first part of Table 7 were calculated for job satisfaction. In the second part of the table, the standardized score values in Equatin (3.6) were calculated for income. After analyzing the model with different power parameter values of exponential score, much appropriate a was found as 2. Because of the differences between estimated lowermost and uppermost values, the only alteration happens on the first and last classes.

Table 6. Estimated lower and upper boundaries of open-ended classes

Method	LB_1	UB_k	Range
IQR	–20,523	51,219	71,742
IDR	–15,304	50,887	67,191
IPR	–16,151	51,429	67,580
MDR	–19,330	52,510	71,840

Table 7. Estimated score values for income and job satisfaction

	Scores	v_1	v_2	v_3	v_4
Job Satisfaction	Equal Spaced	1	2	3	4
	Exponential	1	4	9	16
	Ridit	0.0304	0.1285	0.4906	0.8925
	Scores	u_1	u_2	u_3	u_4
Income	IQR	–2.457	–0.477	0.638	2.658
	IDR	–2.166	–0.477	0.638	2.639
	IPR	–2.213	–0.477	0.638	2.670
	MDR	–2.390	–0.477	0.638	2.730

Model 6 was analyzed with the score values in Table 7. The results with different score values for income and job satisfaction were shown in Table 8.

Table 8. The results of parameter estimates for different score values in Model 6

	Scores			$\hat{\beta}^{IS}$		$\hat{\mu}^{GI}$	
	Income–Job Satisfaction	G^2	P-value	Estimate	P-value	Estimate	P-value
1	IQR–Equal Sapced	10.063	0.986	0.146	0.057	–0.202	0.001
2	IQR–Exponential	9.584	0.990	0.028	0.043	–0.202	0.001
3	IQR–Ridit	9.687	0.989	0.458	0.045	–0.202	0.001
4	IDR–Equal Spaced	9.750	0.988	0.157	0.055	–0.215	0.001
5	IDR–Exponential	**9.273**	0.992	0.030	0.041	–0.215	0.001
6	IDR–Ridit	9.377	0.991	0.488	0.043	–0.215	0.001
7	IPR–Equal Spaced	9.794	0.988	0.154	0.056	–0.211	0.001
8	IPR–Exponential	9.321	0.991	0.030	0.042	–0.211	0.001
9	IPR–Ridit	9.426	0.991	0.480	0.044	–0.211	0.001
10	MDR–Equal Spaced	9.974	0.987	0.146	0.057	–0.202	0.001
11	MDR–Exponential	9.501	0.990	0.028	0.043	–0.202	0.001
12	MDR–Ridit	9.605	0.990	0.456	0.045	–0.202	0.001

Despite all the models in Table 8 fitted the data based on $df = 22$, the goodness-of-fit test statistics differed depending on the score alternatives. For these models, the best fitting one is Case 5 which has standardized scores for income with IDR method and exponential scores with $a = 2$ for job satisfaction. The 10% trimmed range was found as more appropriate. Besides the variation on G^2 statistics, estimated association parameter changed for different scores of income and job satisfaction. In general, the exponential score for job satisfaction had a decreasing effect on G^2 statistics for all the combinations.

The association between adjacent categories where the gender effect is constant could be explained by odds ratio that $\theta_{(i)jk} = exp\{\beta^{IS}(u_j - u_{j+1})(v_k - v_{k+1})\}$. The local odds ratios from the scores in Table 7 were estimated. The association between adjoint categories where job satisfaction effect was constant could be explained by odds ratio that $\theta_{ij(k)} = exp\{(\mu^{GI}_{i+1} - \mu^{GI}_i)(u_{j+1} - u_j)\}$. Table 9 and Table 10 show the odds ratios for different score values.

Table 9. $\theta_{(1)11}$ for income × job satisfaction for the fixed levels of gender

	Job Satisfaction		
Income	Equal Spaced	Exponential	Ridit
IQR	1.335	1.181	1.093
IDR	1.304	**1.164**	1.084
IPR	1.307	1.169	1.085
MDR	1.322	1.174	1.089

Table 10. $\theta_{11(1)}$ for gender × income for the fixed levels of job satisfaction

Scores for Income			
IQR	IDR	IPR	MDR
2.225	**2.067**	2.080	2.166

Regarding the presented results in Table 9, using different methods to estimate the lower and/or upper boundaries of open-ended categories was varying odds ratios. Using the estimation methods of IDR and IPR generated the odds ratios similar but different from the odds ratios estimated by using the IQR and MDR. Any category change on gender does not affect the odds ratio. The reason of this is the odds ratio depends on only changing scores of ordinal variable in row effects model. Regarding the presented results in Table 10, the odds ratios were varied between different scores of income.

By Case 5 in Table 8, the local odds ratios, which were calculated from parameter estimates, are shown in the following matrix.

$$\hat{\theta}_{(i)jk} = \begin{bmatrix} 1.164 & 1.288 & 1.426 \\ 1.105 & 1.182 & 1.264 \\ 1.197 & 1.350 & 1.522 \end{bmatrix}$$

$$\hat{\theta}_{ij(k)} = \begin{bmatrix} 2.067 & 1.615 & 2.364 \end{bmatrix}$$

The odds ratio that income was "$5000 - 15,000$" rather than "$15,000 - 25,000$" estimated to be 1.182 times higher than when the job satisfaction was "A little satisfied" rather than "Moderately satisfied". The odds ratio that males rather than female estimated to be 2.067 times higher than when the income was "< 5000" rather than "$5000 - 15,000$".

5. Conclusions

In this study, we focused on determining the model which explains the data well for open-ended categories. This determination depends on the changing score values. When working on the contingency tables, which contain open-ended ordered categories, the open-ended boundaries of the distribution is suggested to be estimated. In the previous studies, utilizing the interquartile range, which is calculated from the first and the third quartiles, the unknown boundaries were estimated. In this study, we suggested alternative methods of interquartile range. We estimated the unknown boundaries of the table with these methods.

The used method is important because different methods cause differences on the estimated boundaries and accordingly midpoints. Differences in midpoints cause differences in score values. The changing score values also influenced the model significance and model fit. Parameter estimates and odds ratios varied between the methods which we utilized.

The difference between these four methods is that the estimation methods of IQR, IDR, and IPR use the trimmed range, which is a constant value, and trimmed ranges from the both side of the frequency distribution is equal. However, to estimate the MDR, we used the trimmed range where the information comes from the distribution of open-ended variable itself. Therefore, the trimmed ranges are different between the left and the right sides of the distribution. This difference comes from the percentages of the first and last categories.

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New ranked set sampling for estimating the population mean and variance

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Abstract

The purpose of this study is to suggest a new modification of the usual ranked set sampling (RSS) method, namely; neoteric ranked set sampling (NRSS) for estimating the population mean and variance. The performances of the empirical mean and variance estimators based on NRSS are compared with their counterparts in ranked set sampling and simple random sampling (SRS) via Monte Carlo simulation. Simulation results indicate that the NRSS estimators perform much better than their counterparts using RSS and SRS designs when the ranking is perfect. When the ranking is imperfect, the NRSS estimators are still superior to their counterparts in ranked set sampling and simple random sampling methods. These findings show that the NRSS provides a uniform improvement over RSS without any additional costs. Finally, an illustrative example of a real data is provided to show the application of the new method in practice.

Keywords: Ranked set sampling, Imperfect rankings, Simple random sampling, Monte Carlo simulation.

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1. Introduction

The ranked set sampling (RSS) was first proposed by McIntyre [15] as an efficient sampling scheme for estimating the population mean of pasture and forage yields. This sampling scheme is suitable in situations where the ranking of observations can be easily done based on an auxiliary variable correlated with the variable of interest or any inexpensive method. The RSS has wide applications in many scientific problems, especially in environmental and ecological studies where the main focus is on economical and efficient sampling strategies. For example, assume that the Environmental Protection Agency wants to assure that the gasoline stations in metropolitan areas are distributing gasoline which complies with air clean regulations. However, the chemical parameters of gasoline can be easily ranked right after the collection at the gasoline pump by some crude field techniques which are cheap and easy. While bringing the sample units to the laboratory and use actual laboratory techniques to measure its chemical parameters is expensive. For similar applications of RSS in environmental studies, we refer to Cobby et al. [9], Halls and Dell [12], Martin et al. [14], and Ozturk et al. [17].
The standard ranked set sampling design can be described as follows:
I. Select a simple random sample of size k^2 units from the target population and divide them into k samples each of size k.
II. Rank the units within each sample in increasing magnitude by using personal judgment, eye inspection or based on a concomitant variable.
III. Select the ith ranked unit from the ith $(i = 1, \cdots, k)$ sample for actual quantification.
IV. The above Steps I through III can be repeated n times (cycles) if needed to obtain a ranked set sample of size $N = nk$.

Let $Y_1, \cdots, Y_k$ be a simple random sample of size k, then the measured ranked set sample units are denoted by $\left\{Y_{[i]j}, i = 1, \cdots, k, j = 1, \cdots, n\right\}$, where $Y_{[i]j}$ is the ith ranked unit from the jth cycle. It is of interest to note here that $Y_{[i]j}$ $(i = 1, \cdots, k)$ are independent random variables, and they follow the distribution of the ith order statistic of a sample of size k based on perfect ranking in the jth cycle, $j = 1, \cdots, n$. The cumulative distribution function (cdf) of $Y_{[i]}$ is given by $F_{[i]}(y) = i\binom{k}{i}\int_0^{F(y)} w^{i-1}(1-w)^{k-i}dw$, and its probability density function (pdf) is defined as $f_{[i]}(y) = i\binom{k}{i}\left[F(y)\right]^{i-1}\left[1 - F(y)\right]^{k-i}$. The mean and the variance of $Y_{[i]}$ are $\mu_{[i]} = \int_{-\infty}^{+\infty} y f_{[i]}(y)dy$ and $\sigma^2_{[i]} = \int_{-\infty}^{+\infty}(y-\mu_{[i]})^2 f_{[i]}(y)dy$, respectively.

Under imperfect ranking, the $Y_{[i]j}$'s follow the distribution of the ith judgment order statistic. McIntyre [15] used the empirical estimator of the mean based on RSS to estimate the population mean and deduced that his estimator is more efficient than its SRS counterpart via Monte Carlo simulation based on the same number of measured units. The RSS empirical mean estimator is defined as

$$(1.1)\qquad \bar{Y}_{RSS} = \frac{1}{nk}\sum_{j=1}^{n}\sum_{i=1}^{k} Y_{[i]j},$$

with variance

$$(1.2)\qquad Var\left(\bar{Y}_{RSS}\right) = \frac{\sigma^2}{nk} - \frac{1}{nk^2}\sum_{i=1}^{k}\left(\mu_{[i]} - \mu\right)^2.$$

Takahasi and Wakimoto [21] introduced the same method independently and was the first who proved mathematically that, $\bar{Y}_{RSS}$ is an unbiased estimator and has smaller variance than its counterpart in SRS regardless of the issue of ranking. They proved that

$$1 \le \frac{Var\left(\bar{Y}_{SRS}\right)}{Var\left(\bar{Y}_{RSS}\right)} \le \frac{k+1}{2},$$

where $\bar{Y}_{SRS} = \frac{1}{nk}\sum_{j=1}^{n}\sum_{i=1}^{k} Y_{ij}$ is the SRS estimator of the population mean with $Var\left(\bar{Y}_{SRS}\right) = \frac{\sigma^2}{nk}$.

The lower bound is attained if and only if the parent distribution is degenerate when the ranking is perfect, while the upper bound is attained if and only if the parent distribution is rectangular.

Bouza [7] and Al-Omari and Bouza [4] considered the problem of estimation of population mean in the RSS with missing values. Al-Saleh and Al-Omari [5], Al-Omari and Al-Saleh [3] and Al-Omari [1], [2] proposed some mean estimators in other variations of the RSS.

Stokes [20] suggested an estimator of the population variance based on RSS and showed that it is asymptotically ($n \to \infty$ or $k \to \infty$) unbiased of the population variance and has greater efficiency than the sample variance using SRS regardless of the issue of ranking. The variance estimator of Stokes [20] is given by

$$(1.3) \qquad S^2_{Stokes} = \frac{1}{nk-1}\sum_{j=1}^{n}\sum_{i=1}^{k}\left(Y_{[i]j} - \bar{Y}_{RSS}\right)^2.$$

Recently, an unbiased estimator of variance is proposed by MacEachern et al. [13] as

$$(1.4) \qquad S^2_M = \frac{1}{2n^2k^2}\sum_{i\neq j}^{k}\sum_{r=1}^{n}\sum_{s=1}^{n}\left(Y_{[i]r} - Y_{[j]s}\right)^2 + \frac{1}{2n(n-1)k^2}\sum_{i=1}^{k}\sum_{r=1}^{n}\sum_{s=1}^{n}\left(Y_{[i]r} - Y_{[i]s}\right)^2.$$

They showed that this estimator is more efficient than S^2_{Stokes} , especially when the ranking is perfect. However, S^2_M can be applied if the number of cycles is $n \geq 2$.

Perron and Sinha [18] demonstrated that S^2_M has the minimum variance among all unbiased estimators of the form $\sum_i\sum_j\sum_r\sum_s \gamma_{i,j,r,s}Y_{[i]r}Y_{[j]s}$, where the coefficients $\{\gamma_{i,j,r,s}\}$ satisfy $\gamma_{i,j,r,s} = \gamma_{j,i,r,s}$.

Another estimator of variance when the RSS is applied by measuring a concomitant variable is proposed by Zamanzade and Vock [22]. Their estimator was obtained by conditioning on observed concomitant values and using nonparametric kernel regression. Zamanzade and Vock [22]'s simulation results indicated that their proposed estimator considerably improves the estimation of variance when the rankings are fairly good. However, since our interest here is not about using values of concomitant variable, we do not consider their estimator for more investigations.

Biswas et al. [6] considered the problem of estimation of variance in finite population setting using jackknife method. Chen and Lim [8] considered the problem of estimation of variances of strata in a balanced ranked set sample. Sengupta and Mukhuti [19] proposed some unbiased variance estimators when the parent distribution is known to be simple exponential.

The rest of this paper is organized as follows: In Section 2, the suggested sampling scheme is explained and discussed for estimating the population mean and variance. In Section 3, we compare the performance of the mean and variance estimators using NRSS with their counterparts in RSS and SRS methods. In Section 4, a real data example is provided to show the application of the new sampling strategy in practice. Some concluding remarks are provided in Section 5.

2. Neoteric Ranked Set Sampling

Similar to the RSS, neoteric ranked set sampling (NRSS) is suggested to apply in situations where the ranking of the sample observations is much easier than obtaining

their precise values. The NRSS scheme can be described as follows:
I. Select a simple random sample of size k^2 units from the target population.
II. Rank the k^2 selected units in an increasing magnitude based on a concomitant variable, personal judgment or any inexpensive method.
III. If k is an odd, then select the $\left[\frac{k+1}{2}+(i-1)\,k\right]$th ranked unit for $i=1,\cdots,k$. But if k is an even, then select the $[l+(i-1)k]$th ranked unit, where $l=\frac{k}{2}$ if i is an even and $l=\frac{k+2}{2}$ if i is an odd for $i=1,\cdots,k$.
IV. Repeat Steps I through III n times (cycles) if needed to obtain a neoteric ranked set sample of size $N=nk$.
To illustrate the NRSS method, let us consider the following special case of univariate observations.

Let $Y_{ij}, Y_{2j}, \cdots, Y_{k^2j}$ be k^2 simple random units selected from the population of interest, and let $Y_{[i]j}, Y_{[2]j}, \cdots, Y_{[k^2]j}$ be the order statistics of $Y_{ij}, Y_{2j}, \cdots, Y_{k^2j}$ for $j=1,\cdots,n$.
1) Using NRSS
Assume that $k=3$ and $n=1$, then we have to select $k^2=9$ units as

$$Y_{11}, Y_{21}, Y_{31}, Y_{41}, Y_{51}, Y_{61}, Y_{71}, Y_{81}, Y_{91}.$$

Now, rank the units based on personal judgment or eye inspection to get

$$Y_{[1]1}, Y_{[2]1}, Y_{[3]1}, Y_{[4]1}, Y_{[5]1}, Y_{[6]1}, Y_{[7]1}, Y_{[8]1}, Y_{[9]1}.$$

Using NRSS method, we have to choose the units with the rank 2, 5, 8 for actual quantification as

$$\left\{Y_{[1]1}, \boxed{Y_{[2]1}}, Y_{[3]1}, Y_{[4]1}, \boxed{Y_{[5]1}}, Y_{[6]1}, Y_{[7]1}, \boxed{Y_{[8]1}}, Y_{[9]1}\right\}.$$

Then the measured NRSS units are $\left\{Y_{[2]1}, Y_{[5]1}, Y_{[8]1}\right\}$, where their mean and the variance are considered as estimators of the population mean and variance, respectively.
2) Using RSS

Now, using RSS method, we have to select 9 units:

$$\begin{bmatrix} Y_{11}, & Y_{12}, & Y_{13} \\ Y_{21}, & Y_{22}, & Y_{23} \\ Y_{31}, & Y_{32}, & Y_{33} \end{bmatrix}.$$

We then rank the units within each set with respect to a variable of interest and then select the ith ranked unit of the ith sample as:

$$\begin{bmatrix} \boxed{Y_{1[1]}}, & Y_{1[2]}, & Y_{1[3]} \\ Y_{2[1]}, & \boxed{Y_{2[2]}}, & Y_{2[3]} \\ Y_{3[1]}, & Y_{3[2]}, & \boxed{Y_{3[3]}} \end{bmatrix}.$$

The measured RSS units are $\left\{Y_{1[1]}, Y_{2[2]}, Y_{3[3]}\right\}$.
It is of interest to note here, that even if we select k^2 units in both methods RSS and NRSS, we only measure k units. Also, in RSS we rank k units in each of the k sets, while in the NRSS, we rank all the k^2 selected units at the same time.

In general, the resulting neoteric ranked set sample is denoted by $\left\{Y_{[(i-1)k+l]j}; i=1,\cdots,k, j=1,\cdots,n\right\}$, where $Y_{[(i-1)k+l]j}$ is the $[(i-1)k+l]$th measured unit from the jth cycle, and $l=\frac{k+1}{2}$ if k is odd, $l=\frac{k}{2}$ if k and i are both even and $l = \frac{k}{2} + 1$ if k is even but i is odd. Unlike RSS, NRSS measured units $\left\{Y_{[(i-1)k+l]j}; i = 1, \cdots, k\right\}$ are dependent, and they follow the distribution of $[(i-1)k+l]$th order statistics of a sample of size k^2 based on perfect ranking for $j = 1, \cdots, n$. In the case of imperfect rankings, the $\left\{Y_{[(i-1)k+l]j}; j = 1, \cdots, n\right\}$ follow distribution of judgment order statistics of a sample of size k^2.

To simplify the notations, if the sample size k is odd, then the measured units will be denoted by $Y_{[\frac{k+1}{2}]}, Y_{[\frac{3k+1}{2}]}, Y_{[\frac{5k+1}{2}]}, \cdots, Y_{[\frac{2k^2-k+1}{2}]}$. But if the sample size k is even, then the measured units are denoted by $Y_{[\frac{k+2}{2}]}, Y_{[\frac{3k}{2}]}, Y_{[\frac{5k+2}{2}]}, Y_{[\frac{7k}{2}]}, Y_{[\frac{9k+2}{2}]}, \cdots, Y_{[\frac{2k^2-k}{2}]}$.

The suggested estimator of the population mean using NRSS is defined by

$$\bar{Y}_{NRSS} = \frac{1}{nk} \sum_{j=1}^{n} \sum_{i=1}^{k} Y_{[(i-1)k+l]j}, \tag{2.1}$$

with variance

$$Var\left(\bar{Y}_{NRSS}\right) = \frac{1}{nk^2} \sum_{i=1}^{k} Var\left(Y_{[(i-1)k+l]1}\right) + \frac{2}{nk^2} \sum_{i<j}^{k} Cov\left(Y_{[(i-1)k+l]1}, Y_{[(j-1)k+l]1}\right). \tag{2.2}$$

In the following theorem, we prove that the proposed mean estimator is unbiased for symmetric distributions.

2.1. Theorem. *$\bar{Y}_{NRSS}$ is an unbiased estimator of population mean if the rankings are perfect and the parent distribution is symmetric.*

Proof. Without loss of generality, we may suppose that $n = 1$.
If k is odd, then the NRSS estimator of the population mean can be written as

$$\bar{Y}_{NRSS} = \frac{1}{k} \sum_{i=1}^{\frac{k-1}{2}} \left(Y_{[\frac{2ik-k+1}{2}]} + Y_{[\frac{2k^2-ik+1}{2}]} \right) + Y_{[\frac{k^2+1}{2}]}.$$

Take its expectation to have

$$E\left(\bar{Y}_{NRSS}\right) = E\left[\frac{1}{k} \sum_{i=1}^{\frac{k-1}{2}} \left(Y_{[\frac{2ik-k+1}{2}]} + Y_{[\frac{2k^2-ik+1}{2}]} \right) + Y_{[\frac{k^2+1}{2}]} \right]$$

$$= \frac{1}{k} \sum_{i=1}^{\frac{k-1}{2}} \left(E\left(Y_{[\frac{2ik-k+1}{2}]} \right) + E\left(Y_{[\frac{2k^2-ik+1}{2}]} \right) \right) + E\left(Y_{[\frac{k^2+1}{2}]} \right).$$

From symmetric assumption about μ , we have $Y_{[i]} - \mu \overset{d}{=} \mu - Y_{[i]}$, see for example David and Nagaraja [11]. Thus, $\mu - \mu_{[\frac{2ik-k+1}{2}]} = \mu_{\frac{2k^2-ik+1}{2}} - \mu$, and then $\mu_{[\frac{2ik-k+1}{2}]} + \mu_{\frac{2k^2-ik+1}{2}} = 2\mu$. Also, $E\left(Y_{[\frac{k^2+1}{2}]} \right) = \mu$ since it is the median of the chosen sample of size k^2. Therefore,

$$E\left(\bar{Y}_{NRSS}\right) = \frac{1}{k} \sum_{i=1}^{\frac{k-1}{2}} \left(\mu_{[\frac{2ik-k+1}{2}]} + \mu_{[\frac{2k^2-ik+1}{2}]} \right) + \mu_{[\frac{k^2+1}{2}]}$$

$$\frac{1}{k} \left[\frac{k-1}{2} (2\mu) + \mu \right] = \mu.$$

The case of the even sample size can be proved by rewriting $\bar{Y}_{NRSS}$ as:

$$\bar{Y}_{NRSS} = \frac{1}{k}\sum_{i=1}^{\frac{k}{4}}\left(Y_{\left[\frac{4ik-3k}{2}\right]} + Y_{\left[\frac{2k^2-4ik+3k+2}{2}\right]}\right) + \frac{1}{k}\sum_{i=1}^{\frac{k}{4}}\left(Y_{\left[\frac{4ik+k+2}{2}\right]} + Y_{\left[\frac{2k^2-4ik-k+4}{2}\right]}\right).$$

□

Let us consider the following two cases of symmetric and asymmetric distributions under perfect ranking.

1. Uniform distribution. Suppose that the random variable Y has a uniform $U(0,1)$ distribution. Therefore, the mean and variance of the ith ranked unit $Y_{[i]}$, respectively, are given by $E\left(Y_{[i]}\right) = \frac{i}{k+1}$ and $\text{Var}\left(Y_{[i]}\right) = \frac{i(k-i+1)}{(k+1)^2(k+2)}$.

For $k = 6$, we have to select 36 units from the population and then measure only 6 units of them to be a neoteric ranked set sample which are $Y_{[4]}, Y_{[9]}, Y_{[16]}, Y_{[21]}, Y_{[28]}, Y_{[33]}$. The NRSS mean estimator can be obtained as

$$\bar{Y}_{NRSS} = \frac{1}{6}\left[Y_{[4]} + Y_{[9]} + Y_{[16]} + Y_{[21]} + Y_{[28]} + Y_{[33]}\right].$$

The expectation of this estimator is

$$E\left(\bar{Y}_{NRSS}\right) = \frac{1}{6}\left[E\left(Y_{[4]}\right) + E\left(Y_{[9]}\right) + E\left(Y_{[16]}\right) + E\left(Y_{[21]}\right) + E\left(Y_{[28]}\right) + E\left(Y_{[33]}\right)\right]$$

$$= \frac{1}{6}\left(\frac{4}{37} + \frac{9}{37} + \frac{16}{37} + \frac{21}{37} + \frac{28}{37} + \frac{33}{37}\right) = \frac{1}{6}\left(\frac{111}{37}\right) = 0.5,$$

which is an unbiased estimator of the true population mean, $\mu = 0.5$. Recall that,

$$Var\left(\bar{Y}_{NRSS}\right) = \frac{1}{k^2}\sum_{i=1}^{k} Var\left(Y_{[(i-1)k+s]}\right) + \frac{2}{k^2}\sum_{i<j}^{k} Cov\left(Y_{[(i-1)k+s]}, Y_{[(j-1)k+s]}\right),$$

where for the uniform distribution

$$Cov\left(Y_{[j]}, Y_{[i]}\right) = E\left(Y_{[j]}.Y_{[i]}\right) - E\left(Y_{[j]}\right)E\left(Y_{[i]}\right) = \frac{j(k+1-i)}{(k+1)^2(k+2)}.$$

Therefore,

$$\begin{aligned} Var\left(\bar{Y}_{NRSS}\right) &= \tfrac{1}{36}\left(Var\left(Y_{[4]}\right) + Var\left(Y_{[9]}\right) + Var\left(Y_{[16]}\right) + Var\left(Y_{[21]}\right) + Var\left(Y_{[28]}\right) + Var\left(Y_{[33]}\right)\right) \\ &+ \tfrac{2}{36}\begin{pmatrix} Cov\left(Y_{[4]}, Y_{[9]}\right) + Cov\left(Y_{[4]}, Y_{[16]}\right) + Cov\left(Y_{[4]}, Y_{[21]}\right) + Cov\left(Y_{[4]}, Y_{[28]}\right) + Cov\left(Y_{[4]}, Y_{[33]}\right) + \\ Cov\left(Y_{[9]}, Y_{[16]}\right) + Cov\left(Y_{[9]}, Y_{[21]}\right) + Cov\left(Y_{[9]}, Y_{[28]}\right) + Cov\left(Y_{[9]}, Y_{[33]}\right) + Cov\left(Y_{[16]}, Y_{[21]}\right) + \\ Cov\left(Y_{[16]}, Y_{[28]}\right) + Cov\left(Y_{[16]}, Y_{[33]}\right) + Cov\left(Y_{[21]}, Y_{[28]}\right) + Cov\left(Y_{[21]}, Y_{[33]}\right) + Cov\left(Y_{28}, Y_{[33]}\right) \end{pmatrix} \\ &= \frac{1}{36}\left(\frac{66}{26011} + \frac{126}{26011} + \frac{168}{26011} + \frac{168}{26011} + \frac{126}{26011} + \frac{66}{26011}\right) \\ &+ \frac{2}{36}\left[\begin{array}{l} \left(\frac{56}{26011} + \frac{42}{26011} + \frac{32}{26011} + \frac{18}{26011} + \frac{8}{26011}\right) + \left(\frac{189}{52022} + \frac{72}{26011} + \frac{81}{52022} + \frac{18}{26011}\right) + \\ \left(\frac{128}{26011} + \frac{72}{26011} + \frac{32}{26011}\right) + \left(\frac{189}{52022} + \frac{42}{26011}\right) + \frac{56}{26011} \end{array}\right] \\ &= \frac{7}{2812}. \end{aligned}$$

Now, the variance of mean estimator based on a simple random sample of size $k = 6$ is $Var\left(\bar{Y}_{SRS}\right) = \frac{\sigma^2}{k} = \frac{1}{12(6)} = \frac{1}{72}$. Therefore, the relative efficiency (RE) of the NRSS estimator with respect to SRS estimator is $RE_1\left(\bar{Y}_{NRSS}, \bar{Y}_{SRS}\right) = \frac{MSE\left(\bar{Y}_{SRS}\right)}{MSE\left(\bar{Y}_{NRSS}\right)} = 5.5794$, and the RE of the RSS estimator with respect to its SRS counterpart is $RE_2\left(\bar{Y}_{RSS}, \bar{Y}_{SRS}\right) = \frac{MSE\left(\bar{Y}_{SRS}\right)}{MSE\left(\bar{Y}_{RSS}\right)} = 3.5$.

Exponential distribution. If Y has an exponential distribution with mean 1, then the mean and variance of the ith order statistic, $Y_{[i]}$ are given by

$E\left(Y_{[i]}\right) = \sum\limits_{w=k-i+1}^{k} \frac{1}{w}$, and $\operatorname{Var}\left(Y_{[i]}\right) = \sum\limits_{w=k-i+1}^{k} \frac{1}{w^2}$.

For $m = 5$, we have to select 25 units from the population and then measure only 5 units of them to be a neoteric ranked set sample, which are $Y_{[3]}, Y_{[8]}, Y_{[13]}, Y_{[18]}, Y_{[23]}$. Therefore, the NRSS mean estimator is

$$\bar{Y}_{NRSS} = \frac{1}{5}\left[Y_{[3]} + Y_{[8]} + Y_{[13]} + Y_{[18]} + Y_{[23]}\right].$$

The expectation of this estimator is

$$E\left(\bar{Y}_{NRSS}\right) = \frac{1}{5}\left[E\left(Y_{[3]}\right) + E\left(Y_{[8]}\right) + E\left(Y_{[13]}\right) + E\left(Y_{[18]}\right) + E\left(Y_{[23]}\right)\right]$$

$$= \frac{1}{5}\left(\frac{1727}{13800} + \frac{22798213}{60568200} + \frac{19081066231}{26771144400} + \frac{10914604807}{8923714800} + \frac{20666950267}{8923714800}\right)$$

$$= \frac{1}{5}\left(\frac{5090112581}{1070845776}\right) = 0.950671,$$

where $E\left(Y_{[3]}\right) = \sum\limits_{w=23}^{25} \frac{1}{w} = \frac{1727}{13800}$, $E\left(Y_{[8]}\right) = \sum\limits_{w=18}^{25} \frac{1}{w} = \frac{22798213}{60568200}$, $E\left(Y_{[13]}\right) = \sum\limits_{w=13}^{25} \frac{1}{w} = \frac{19081066231}{26771144400}$, $E\left(Y_{[18]}\right) = \sum\limits_{w=8}^{25} \frac{1}{w} = \frac{10914604807}{8923714800}$, $E\left(Y_{[23]}\right) = \sum\limits_{w=3}^{25} \frac{1}{w} = \frac{20666950267}{8923714800}$.

It can be seen that this estimator is biased with $Bias\left(\bar{Y}_{NRSS}\right) = -0.0493285$, which is very quite close to the bias value -0.05 obtained in Table 2, when $\rho = 1$.

The suggested NRSS estimator of the population variance is given by

$$S^2_{NRSS} = \frac{1}{nk-1}\sum_{j=1}^{n}\sum_{i=1}^{k}\left(Y_{[(i-1)k+l]j} - \bar{Y}_{NRSS}\right)^2 \tag{2.3}$$

It is of interest to note here that S^2_{NRSS} has a negligible bias of the population variance, which approaches to zero in most cases.

3. Monte Carlo Comparison

In this section, the performances of the proposed mean and variance estimators based on NRSS are compared with their counterparts using RSS and SRS methods. As we mentioned before, we only measure on $N = nk$ units using NRSS and RSS methods, to compare them with N units using SRS method.

For Monte Carlo simulation, we have used the model of imperfect ranking suggested by Dell and Clutter (1972), assuming (Z, X) follows a standard bivariate normal distribution with correlation coefficient ρ. Then, we take $Y = Z$, $\Phi(Z)$, $\log\left[\frac{\Phi(Z)}{1-\Phi(Z)}\right]$, $-\log\left[\Phi(Z)\right]$ and $\left[\Phi(Z)\right]^5$ as the variable of interest, where $\Phi(.)$ is the cdf of the standard normal distribution. Therefore, we allow the relation between the interest variable (Y) and the auxiliary variable (X) to be linear or non-linear, and the parent distributions to be Normal (0,1), Uniform (0,1), Logistic (0,1), Exponential (1) and Beta (0.2,1), respectively. Thus we have considered both symmetric and asymmetric distributions with bounded and unbounded supports in our simulation study.

The values of ρ are 0, 0.2, 0.4, 0.6, 0.8, 1. Without loss of generality, we assumed that the ranking is based on X. Therefore, as ρ gets large to 1, the ranking approaches to

Table 1. The relative efficiencies of NRSS mean estimator to SRS mean estimator (RE_1) and RSS mean estimator to SRS mean estimator (RE_2) for different values of (N, k).

Parent Distribution		Normal(0,1)		Uniform(0,1)		Logistic(0,1)		Exponential(1)		Beta(0.2,1)	
(N,k)	ρ	RE1	RE2	RE1	RE2	RE1	RE2	RE1	RE2	RE1	RE2
(10,5)	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0.2	1.04	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.01
	0.4	1.14	1.11	1.13	1.12	1.15	1.11	1.14	1.11	1.1	1.08
	0.6	1.40	1.28	1.34	1.30	1.42	1.29	1.38	1.24	1.29	1.21
	0.8	2.02	1.68	1.94	1.71	2.05	1.66	1.92	1.51	1.71	1.47
	1	4.75	2.78	4.68	3.00	4.88	2.56	4.37	2.16	4.05	2.14
(10,10)	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0.2	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.02
	0.4	1.17	1.14	1.16	1.14	1.18	1.14	1.14	1.11	1.12	1.10
	0.6	1.47	1.4	1.43	1.39	1.51	1.39	1.43	1.32	1.32	1.27
	0.8	2.37	2.03	2.25	2.03	2.38	1.99	2.19	1.77	1.94	1.71
	1	9.78	4.82	9.71	5.50	9.99	4.2	9.00	3.43	8.94	3.53
(20,5)	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0.2	1.03	1.02	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.01
	0.4	1.15	1.11	1.12	1.11	1.16	1.12	1.13	1.09	1.10	1.08
	0.6	1.41	1.31	1.35	1.27	1.42	1.29	1.37	1.23	1.29	1.21
	0.8	2.03	1.68	1.90	1.7	2.06	1.64	1.90	1.53	1.68	1.47
	1	4.74	2.78	4.66	3.00	4.89	2.58	3.99	2.19	3.93	2.12
(20,10)	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0.2	1.03	1.03	1.05	1.04	1.03	1.02	1.03	1.02	1.02	1.03
	0.4	1.16	1.13	1.15	1.14	1.17	1.13	1.15	1.12	1.11	1.09
	0.6	1.48	1.40	1.44	1.39	1.49	1.39	1.43	1.32	1.32	1.27
	0.8	2.35	2.02	2.24	2.03	2.37	1.96	2.16	1.79	1.93	1.70
	1	9.71	4.79	9.74	5.50	9.92	4.21	8.00	3.41	8.84	3.52

completely perfect. The relative efficiency (RE) of NRSS and RSS with respect to SRS is defined as

$$RE_1\left(\bar{Y}_{NRSS}, \bar{Y}_{SRS}\right) = \frac{MSE\left(\bar{Y}_{SRS}\right)}{MSE\left(\bar{Y}_{NRSS}\right)}, RE_2\left(\bar{Y}_{RSS}, \bar{Y}_{SRS}\right) = \frac{MSE\left(\bar{Y}_{SRS}\right)}{MSE\left(\bar{Y}_{RSS}\right)},$$

$$RE_3\left(S^2_{Stokes}, S^2_{SRS}\right) = \frac{MSE\left(S^2_{SRS}\right)}{MSE\left(S^2_{Stokes}\right)}, RE_4\left(S^2_{M}, S^2_{SRS}\right) = \frac{MSE\left(S^2_{SRS}\right)}{MSE\left(S^2_{M}\right)},$$

$$RE_5\left(S^2_{NRSS}, S^2_{SRS}\right) = \frac{MSE\left(S^2_{SRS}\right)}{MSE\left(S^2_{NRSS}\right)}.$$

where $MSE\left(\hat{\theta}\right) = Var\left(\hat{\theta}\right) + \left[Bias\left(\hat{\theta}\right)\right]^2$.

The values of (N, k) are selected to be $(10,5), (10,10), (20,5), (20,10)$. So, we can assess the effect of increasing total sample size for fixed k, and the effect of increasing k when the total sample size is fixed. The number of repetitions in the simulation study is set to be 100,000 for each sample size. The results are reported in Tables 1-4 for estimating the population mean and variance.

Table 2. Estimated biases of $\bar{Y}_{NRSS}$ for estimation of the population mean for different values of (N, k).

	(N,k)	(10,5)						(10,10)					
	ρ	0	0.2	0.4	0.6	0.8	1	0	0.2	0.4	0.6	0.8	1
Parent	Exponential (1)	0.00	0.00	-0.01	-0.02	-0.03	-0.05	0	0	-0.01	-0.01	-0.02	-0.04
Distribution	Beta(0.2,1)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(N,k)	(20,5)						(20,10)					
	ρ	0	0.2	0.4	0.6	0.8	1	0	0.2	0.4	0.6	0.8	1
Parent	Exponential (1)	0	0	-0.01	-0.02	-0.03	-0.05	0	0	0.01	-0.01	-0.02	-0.04
Distribution	Beta(0.2,1)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 3. The relative efficiencies of S^2_{NRSS} to S^2_{SRS} (RE_3), S^2_{Stokes} to S^2_{SRS} (RE_4) and S^2_M to S^2_{SRS} (RE_5) for different values of (N,k).

Parent Distribution		Normal(0,1)			Uniform(0,1)			Logistic(0,1)			Exponential(1)			Beta(0.2,1)		
(N,k)	ρ	RE3	RE4	RE5	RE3	RE4	RE5	RE3	RE4	RE5	RE3	RE4	RE5	RE3	RE4	RE5
	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0.2	1.02	1.00	1.00	1.00	1.00	1.00	1.01	1.00	1.00	1.01	1.00	1.00	1.01	1.00	1.00
(10,5)	0.4	1.04	1.01	1.01	1.01	1	1.02	1.09	1	1.03	1.15	1.05	1.07	1.05	1.01	1.04
	0.6	1.18	1.02	1.06	1.09	1.02	1.10	1.25	1.00	1.04	1.36	1.03	1.08	1.13	1.05	1.11
	0.8	1.56	1.03	1.17	1.33	1.08	1.26	1.77	1.01	1.10	2.10	1.03	1.11	1.35	1.11	1.23
	1	4.05	1.17	1.40	3.66	1.32	1.71	4.08	1.10	1.27	5.44	1.09	1.21	2.86	1.26	1.48
	0	1.00	1.00	-	1.00	1.00	-	1.01	1.00	-	1.00	1.00	-	1.00	1.00	-
	0.2	1.01	1.01	-	1.00	1.00	-	1.01	1.00	-	1.00	1.00	-	1.01	1.01	-
(10,10)	0.4	1.02	0.98	-	1.03	1.00	-	1.08	1.00	-	1.05	1.03	-	1.05	1.02	-
	0.6	1.16	1.00	-	1.10	1.03	-	1.23	1.00	-	1.32	1.05	-	1.14	1.07	-
	0.8	1.62	1.11	-	1.39	1.18	-	1.85	1.09	-	2.11	1.10	-	1.45	1.21	-
	1	9.14	1.55	-	5.68	1.9	-	8.52	1.35	-	9.97	1.20	-	6.35	1.69	-
	0	1.01	1.01	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0.2	1.02	1.01	1.02	1.00	1.00	1.00	1.03	1.00	1.00	1.03	1.00	1.01	1.01	1.00	1.00
(20,5)	0.4	1.05	1.02	1.01	1.02	1	1.02	1.08	1.01	1.01	1.14	1.02	1.03	1.06	1.03	1.04
	0.6	1.17	1.01	1.05	1.07	1.02	1.07	1.25	1.02	1.03	1.39	1.04	1.06	1.16	1.08	1.11
	0.8	1.48	1.07	1.14	1.31	1.10	1.2	1.65	1.05	1.11	1.96	1.05	1.09	1.36	1.16	1.22
	1	2.65	1.22	1.34	4.02	1.36	1.55	2.35	1.13	1.21	3.49	1.12	1.18	2.69	1.33	1.44
	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0.2	1.02	1.01	1.02	1.00	1.00	1.00	1.01	1.01	1.01	1.03	1.01	1.01	1.01	1.01	1.01
(20,10)	0.4	1.04	1.01	1.01	1.04	1.01	1.04	1.07	1.02	1.01	1.11	1.01	1.01	1.06	1.03	1.05
	0.6	1.17	1.02	1.07	1.10	1.04	1.10	1.24	1.03	1.07	1.36	1.05	1.08	1.17	1.11	1.15
	0.8	1.63	1.17	1.27	1.42	1.22	1.34	1.8	1.11	1.19	2.08	1.13	1.19	1.5	1.28	1.37
	1	5.98	1.65	1.85	7.66	2.07	2.49	4.42	1.42	1.55	5.86	1.26	1.35	6.6	1.83	2.03

Table 4. Estimated biases of variance estimators S^2_{NRSS} and S^2_{Stokes} for different values of (N, k).

Parent Distribution		Normal(0,1)		Uniform(0,1)		Logistic(0,1)		Exponential(1)		Beta(0.2,1)	
(N,k)	ρ	Bias of S^2_{NRSS}	Bias of S^2_{Stokes}	Bias of S^2_{NRSS}	Bias of S^2_{Stokes}	Bias of S^2_{NRSS}	Bias of S^2_{Stokes}	Bias of S^2_{NRSS}	Bias of S^2_{Stokes}	Bias of S^2_{NRSS}	Bias of S^2_{Stokes}
(10,5)	0	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00
	0.2	0.00	0.00	0.00	0.00	-0.02	0.01	0.00	0.00	0.00	0.00
	0.4	-0.02	0.01	0.00	0.00	-0.07	0.04	-0.03	0.01	0.00	0.00
	0.6	-0.04	0.02	0.00	0.00	-0.18	0.08	-0.07	0.02	0.00	0.00
	0.8	-0.07	0.04	0.00	0.00	-0.35	0.15	-0.14	0.04	0.00	0.00
	1	-0.10	0.07	0.00	0.00	-0.60	0.23	-0.24	0.06	0.00	0.00
(10,10)	0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.2	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00
	0.4	-0.01	0.02	0.00	0.00	-0.04	0.03	-0.02	0.01	0.00	0.00
	0.6	-0.02	0.03	0.00	0.00	-0.09	0.09	-0.05	0.02	0.00	0.00
	0.8	-0.03	0.06	0.00	0.00	-0.21	0.19	-0.09	0.05	0.00	0.00
	1	-0.04	0.09	0.00	0.00	-0.39	0.28	-0.17	0.08	0.00	0.00
(20,5)	0	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00
	0.2	-0.01	0.00	0.00	0.00	-0.02	0.01	-0.01	0.00	0.00	0.00
	0.4	-0.02	0.00	0.00	0.00	-0.10	0.02	-0.04	0.00	0.00	0.00
	0.6	-0.05	0.01	0.00	0.00	-0.23	0.05	-0.08	0.01	0.00	0.00
	0.8	-0.09	0.02	0.00	0.00	-0.43	0.06	-0.16	0.02	0.00	0.00
	1	-0.14	0.03	0.00	0.00	-0.70	0.11	-0.27	0.03	0.00	0.00
(20,10)	0	0.00	0.00	0.00	0.00	-0.01	0.01	0.00	0.00	0.00	0.00
	0.2	0.00	0.00	0.00	0.00	-0.02	0.00	0.00	0.00	0.00	0.00
	0.4	-0.01	0.01	0.00	0.00	-0.07	0.02	-0.02	0.01	0.00	0.00
	0.6	-0.03	0.01	0.00	0.00	-0.15	0.05	-0.06	0.01	0.00	0.00
	0.8	-0.05	0.03	0.00	0.00	-0.30	0.10	-0.12	0.02	0.00	0.00
	1	-0.09	0.04	0.00	0.00	-0.52	0.13	-0.21	0.04	0.00	0.00

Table 1 gives the relative efficiencies of the mean estimators based on NRSS and RSS schemes to SRS mean estimator for different distributions. We observe that when rankings are prefect ($\rho = 1$), the efficiency gain in using NRSS mean estimator is approximately two times higher than the mean estimator based on RSS scheme. Furthermore, the performance of NRSS mean estimator for the symmetric distributions is slightly better than the asymmetric distributions, and the best performance of NRSS mean estimator is for logistic distribution. It is clear from this table that the effect of the imperfect ranking on $\bar{Y}_{NRSS}$ is more than $\bar{Y}_{RSS}$, however, even in the case of imperfect ranking ($\rho \leq 0.8$), the $\bar{Y}_{NRSS}$ is still superior to $\bar{Y}_{RSS}$ for $\rho \geq 0.4$, and it is as efficient as $\bar{Y}_{RSS}$ for $\rho \leq 0.4$. When the rankings are completely random ($\rho = 0$), all estimators have the same performances. This can be justified by the fact that in the case of random rankings, RSS and NRSS schemes are intrinsically the same as SRS design. It is worth mentioning that in all considered cases, the relative efficiencies of $\bar{Y}_{NRSS}$ and $\bar{Y}_{RSS}$ increase as the set size (k) increases for fixed sample size (N).

Table 2 presents the estimated biases of the NRSS mean estimator for asymmetric distributions. We observe that the proposed mean estimator slightly underestimates the true population mean when the parent distribution is standard exponential and $\rho \geq 0.4$. Furthermore, the bias of $\bar{Y}_{NRSS}$ decreases in absolute value when set size (k) increases or the correlation of coefficient (ρ) decreases. In the case of the parent distribution being Beta(0.2,1), the NRSS mean estimator is almost unbiased.

The relative efficiencies of different variance estimators S^2_{NRSS} and S^2_{Stokes} to S^2_{SRS} are presented in Table 3. It is clear from this table that the performance of S^2_{NRSS} dominates all other estimators considered here when the rankings are perfect ($\rho = 1$), and S^2_{NRSS} performs at least twice as good as its competitors in RSS scheme. Although the imperfect ranking has more negative effect on S^2_{NRSS} than S^2_{Stokes} and S^2_M, S^2_{NRSS} is still superior to its RSS competitors for $\rho \leq 0.8$. Furthermore, we also observe that the relative efficiencies increase as the set size (k) increases for fixed sample size (N).

The estimated bias values of S^2_{NRSS} and S^2_{Stokes} are given in Table 4. We observe that for standard uniform and Beta(0.2,1) distributions, S^2_{NRSS} and S^2_{Stokes} are almost unbiased. However, for standard normal, standard exponential and standard logistic distributions, S^2_{Stokes} overestimates true population variance and S^2_{NRSS} underestimates σ^2. It is also evident that the bias of S^2_{NRSS} is larger than the bias of S^2_{Stokes} in absolute value. Furthermore, we observe that the biases of S^2_{Stokes} and S^2_{NRSS} decrease in absolute value as ρ decreases.

4. A real data set

In this section, a real data set is considered to illustrate the performance of NRSS method in estimating the population mean and variance. The data set consists of the percentage of body fat determined by underwater weighing and various body circumference measurements for 252 men. For more details about these data, see http://lib.stat.cmu.edu/datasets/bodyfat. We take the percentage of body fat as the interest variable (Y) and abdomen circumference as concomitant variable (X). Sampling with replacement is considered, so the assumption of independence is covered. The mean and variance of the target variable Y in the population are $\mu_Y = 19.15$ and $\sigma^2_Y = 70.03$, respectively, and the correlation of coefficient between the two variables is $\rho_{XY} = 0.81$. To select a sample of size 10, using using both RSS and NRSS designs, the following steps are carry out:

I. Select a bivariate simple random sample of size 25 of (X, Y).

II. On basis of NRSS, rank the X values and use their ordering for Y. Then, select the 3rd, 8th, 13th, 18th and 23rd judgment ranked values of Y for actual quantification to

Table 5. The values of the variable of interest Y using NRSS, RSS and SRS designs.

NRSS	25.5	5.3	19.7	27.2	27.0	15.1	5.7	22.9	26.0	32.3
RSS	27.3	18.5	19.7	27.0	18.5	31.6	10.6	15.2	10.6	15.2
SRS	0.7	29.6	26.7	11.5	19.2	27.3	17.5	16.5	3.0	20.5

constitute a neoteric ranked set sample of size 5.
III. For RSS, divide the 25 SRS observations into 5 sets each of size 5. Then, use the true ranked X values to rank the values of Y within each set of size 5 units. Finally, select the ith judgment ranked values of Y from the ith sample $(i = 1, \cdots, 5)$.
IV. Repeat Steps I to III two times to have a sample of size 10 from NRSS and RSS designs.
Also, a simple random sample of size 10 is selected from the same population. The results of measured values in NRSS, RSS and SRS designs are presented in Table 5.
The above results in Table 5 showed that

$$\bar{Y}_{NRSS} = 20.67,\ \bar{Y}_{RSS} = 19.42,\ \bar{Y}_{SRS} = 17.25,$$

$$S^2_{NRSS} = 85.22,,\ S^2_{Stokes} = 51.20,\ S^2_M = 49.89,\ S^2_{SRS} = 96.42.$$

Our results showed that the means of 100000 repeated values of the suggested estimators are all quite close to the real population parameters. For example, $Bias\left(\bar{Y}_{NRSS}\right) = 20.67 - 19.15 = 1.52$, and $Bias\left(S^2_{NRSS}\right) = 85.22 - 70.03 = 15.19$, which are more better than the SRS estimators. Also, the NRSS variance estimator is more efficient than its counterparts in Stokes [20], and MacEachern et al. [13].

5. Conclusion

In this paper, a new modification of the usual RSS is suggested for estimating the population mean and variance. The suggested estimators are compared with their competitors in SRS method. Our simulation results indicate that the suggested empirical mean and variance estimates are strongly better than their competitors in RSS and SRS designs for the same number of measured units with perfect ranking. In the case of imperfect rankings, the NRSS estimators are still superior to their counterparts in the RSS and SRS design and their superiority decrease as the quality of rankings decreases. We prove that the NRSS mean estimator is unbiased when the parent distribution is symmetric. For asymmetric distributions, the simulation results indicate that the NRSS mean estimator is slightly biased. Thus, based on the above observations, the NRSS can be recommended for estimating the population parameters due to its efficiency with respect to SRS and RSS methods.

In this paper, we consider the problem of estimation of mean and variance based on the NRSS. One can use the NRSS scheme for estimation of cumulative distribution function and population quantiles. It is also interesting to investigate the performance of goodness of fit tests based on empirical distribution function (e.g. Kolmogorov-Smirnov, Anderson–Darling, etc) NRSS and compare them with their counterparts in the RSS and SRS designs.

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