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MATHEMATICS

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On generalized derivations of incline algebras

Şule Ayar Özbal *and Alev Firat †

Abstract

In this paper, as a generalization of derivation of an incline algebra, the notion of generalized derivation in an incline algebra is introduced and some of its properties are investigated in an incline and integral incline algebra.

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Keywords: (sub)incline, integral incline, ideal, derivation, generalized derivation.

1. Introduction

The notion of derivation on a ring has an important role for the characterization of rings. As generalizations of derivations, α-derivations, generalized derivations on prime and semi-prime rings are studied by a lot of researchers [10], [6]. In [11] the notion of derivation on a lattice was defined and some of its related properties were examined firstly by Szasz , G. In [5] the problems initiated by Szasz are pursued and completed by Luca Ferrari. Later, derivations, f-derivations, symmetric bi-derivations, symmetric f-biderivations on a lattice and some properties related with these derivations were discussed by Xin, X. L. , Li, T.Y. and Lu, J. H. , and Öztürk, M.A. and Çeven, Y.and A. Firat and S. Ayar Özbal in [12], [4],[4],[8] respectively.

In [2] the notion of inclines is introduced and their applications are studied by Cao, Z. Q. , Kim, K. H. , and Roush, F. W. . The notion of derivation for an incline algebra is introduced by Al-Shehri, N. O. in [1] and he discussed some of its properties. We introduced the notion of f-derivation in an incline algebra and studied it some properties in [9]. In this paper, as a generalization of derivation of an incline algebra, the notion of generalized derivation in an incline algebra is introduced and, some related properties are investigated.

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2. Preliminaries

2.1. Definition. [2] An incline algebra is a non-empty set R with binary operations denoted by $+$ and $*$ satisfying the following axioms for all $x, y, z \in R$:

(RI) $x + y = y + x$,

(RII) $x + (y + z) = (x + y) + z$,

(RIII) $x * (y * z) = (x * y) * z$,

(RIV) $x * (y + z) = (x * y) + (x * z)$,

(RV) $(y + z) * x = (y * x) + (z * x)$,

(RVI) $x + x = x$,

(RVII) $x + (x * y) = x$,

(RVIII) $y + (x * y) = y$.

Furthermore, an incline algebra R is said to be commutative if $x * y = y * x$ for all $x, y \in R$.

For convenience, we pronounce " $+$ " (resp." $*$ ") as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if $x * x = x$ for all $x \in R$ ([3, Proposition (1.1.1)]). A subincline of an incline R is a nonempty subset M of R which is closed under addition and multiplication. An ideal in an incline R is a subincline $M \subseteq R$ such that if $x \in M$ and $y \leq x$ then $y \in M$. An element 0 in an incline algebra R is a zero element if $x + 0 = x = 0 + x$ and $x * 0 = 0 * x = 0$ for any $x \in R$. An element 1 ($\neq$ zero element) in an incline algebra R is called a multiplicative identity if for any $x \in R$, $x * 1 = 1 * x = x$. A non-zero element a in an incline algebra R with zero element is said to be a left (resp. right) zero divisor if there exists a non-zero element $b \in R$ such that $a * b = 0 (resp.\ b * a = 0)$. A zero divisor is an element of R which is both a left zero divisor and a right zero divisor. An incline algebra R with multiplicative identity 1 and zero element 0 is called an integral incline if it has no zero divisors.

Note that $x \leq y$ if and only if $x + y = y$ for all $x, y \in R$. It is easy to see that $\leq$ is a partial order on R and that for any $x, y \in R$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$. It follows that

(1) $x * y \leq x$ and $y * x \leq x$ for all $x, y \in R$.

(2) $y \leq z$ implies $x * y \leq x$ and $y * x \leq z * x$ for any $x, y, z \in R$.

(3) If $x \leq y$, $a \leq b$, then $x + a \leq y + b$, $x * a \leq y * b$.

2.2. Definition. [1] Let R be an incline and $d : R \to R$ be a function. d is called as a derivation of R if it satisfies the following condition

$$d(x * y) \quad = \quad (d(x) * y) + (x * d(y))$$

for all $x, y \in R$.

2.3. Proposition. [1] Let R be an incline with a zero element and d be a derivation of R. Then $d0 = 0$.

3. The Generalized Derivations of Incline Algebras

The following definition introduces the notion of a generalized derivation for an incline algebra.

3.1. Definition. Let R be an incline algebra. A function $D : R \to R$ is called a generalized derivation of R, if there exists a derivation d of R such that

$$D(x * y) \quad = \quad (D(x) * \ y) + (x * \ \ d(y))$$

for all $x, y \in R$.

3.1. Example. Let $R = \{0, a, b, c, d, 1\}$, and the sum " $+$ " and product " $*$ " be defined on R as follows:

+	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	1	1	1	1
b	b	1	b	1	b	1
c	c	1	1	c	1	1
d	d	1	b	c	d	1
1	1	1	1	1	1	1

*	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	0
b	0	0	d	d	d	d
c	0	0	d	d	d	d
d	0	0	d	d	d	d
1	0	0	d	d	d	d

Then $(R, +, *)$ is an incline but not a distributive lattice [7].
We define a function $D : R \to R$ for all x in R by

$$D(x) = \begin{cases} 0, & \text{x=0,c,1} \\ a, & \text{x=a} \\ d, & \text{b,d} \end{cases}$$

Then D is not a derivation of R since $D(1 * \ \ c) = D(d) = d$, but $(D(1) * c) + (1 * D(c)) = (0 * c) + (1 * 0) = 0 + 0 = 0$ and thus $D(1 * \quad c) \quad \neq (D(1) * \ \ c) + \ (1 * \ D(c))$.

But if we define d as a derivation of R as

$$d(x) = \begin{cases} 0, & \text{x=0,a} \\ d, & \text{otherwise} \end{cases}$$

then D satisfies the equation in Definition 3.1, i.e. D is a generalized derivation of R.

3.2. Proposition. Let R be an incline algebra and D be a generalized derivation of R associated with a derivation d of R. Then the followings hold for all x, y in R:

(i) $D(x * y) \leq D(x) + d(y)$,

(ii) If $x \leq y$ and then $D(x * y) \leq y$.

Proof:

(i) Let $x, y \in R$. We know that from (1) we have $D(x) * y \leq D(x)$ and $x * d(y) \leq d(y)$. Then by using (3) we get $D(x * y) = (D(x) * y) + (x * d(y)) \leq D(x) + d(y)$.
Hence we find $D(x * y) \leq D(x) + d(y)$.

(ii) Let $x \leq y$. Then by using (3) and (1) we get $x * d(y) \leq y * d(y) \leq y$. Similarly, we can get $D(x) * y \leq y$. Then we obtain, $D(x * y) = (D(x) * y) + (x * d(y)) \leq y + y = y$. Hence we have $D(x * y) \leq y$.

3.3. Proposition. Let R be an incline algebra with a zero element and D be a generalized derivation of R associated with a derivation d of R. Then $D(0) = 0$.

Proof: Since R is an incline algebra with a zero element we have $x * 0 = 0 * x = 0$ for all $x \in R$ then we can write
$D(0) = D(x*0) = (D(x)*0)+(x*d(0)) = x*d(0)$. By Proposition 2.3 we have $d(0) = 0$. Therefore $D(0) = 0$.

3.4. Proposition. Let R be an incline algebra with a multiplicative identity element and D be a generalized derivation of R associated with a derivation d of R. Then the followings hold for all $x \in R$:

(i) $x * d(1) \leq D(x)$,

(ii) If $d(1) = 1$, then $x \leq D(x)$.

Proof:

(i) Since R is an incline algebra with a multiplicative identity element we have $x * 1 = 1 * x = x$ for all $x \in R$, then we can write $D(x) = D(x * 1) = (D(x) * 1) + (x * d(1))$. Then we have $D(x) = D(x) + (x * d(1))$. Therefore we get, $x * d(1) \leq D(x)$.

(ii) It can be derived from (i).

3.5. Proposition. Let R be an integral incline and D be a generalized derivation of R associated with a derivation d of R and a be an element of R. Then for all $x \in R$ we have:

(i) $a * D(x) = 0$ implies that $a = 0$ or $d = 0$,

(ii) $d(x) * a = 0$ and $D(x) * a = 0$ imply that $a = 0$ or $D = 0$.

Proof:

(i) Let $a * D(x) = 0$ for all $x \in R$. If we replace x by $x * y$ for $y \in R$ we get

$$\begin{aligned} 0 = a * D(x) = a * D(x * y) &= a * [(D(x) * y) + (x * d(y))] \\ &= (a * (D(x) * y)) + (a * (x * d(y))) \\ &= a * (x * d(y)) \end{aligned}$$

In this equation by taking $x = 1$ we get $a * d(y) = 0$. Since R is an integral incline we have $a = 0$ or $d = 0$.

(ii) Let $d(x) * a = 0$ and $D(x) * a = 0$ for all $x \in R$.If we replace x by $x * y$ for $y \in R$ in $D(x) * a = 0$ we get

$$\begin{aligned} 0 = D(x) * a = D(x * y) * a &= [(D(x) * y) + (x * d(y))] * a \\ &= ((D(x) * y) * a) + ((x * d(y)) * a) \\ &= (D(x) * y) * a \end{aligned}$$

In this equation by taking $y = 1$ we get $D(x) * a = 0$. Since R is an integral incline we have $a = 0$ or $D = 0$.

3.6. Theorem. Let M be a nonzero ideal of an integral incline R. If D is a nonzero generalized derivation of R associated with a nonzero derivation d of R, then D is nonzero on M.

Proof: Assume that D is a nonzero generalized derivation of R associated with a nonzero derivation d of R but D is zero generalized derivation on M $x \in M$. Then we have $D(x) = 0$. Let $y \in R$. By (1) $x * y \leq x$ and since M is an ideal of R, we have $D(x * y) = 0$, so we can write that

$0 = D(x * y) = (D(x) * y) + (x * d(y)) = x * d(y)$.

We know by our assumption that R has no zero divisors, so we have $x = 0$ for all $x \in M$ or $d(y) = 0$ for all $y \in R$. Since M is a nonzero ideal of R, we get that $d(y) = 0$ for all $y \in R$. This contradicts with our assumption that is d is a nonzero derivation on R. Hence, D is nonzero on M.

3.7. Theorem. Let D be a nonzero generalized derivation of an integral incline R associated with a nonzero derivation d on M. If M is a nonzero ideal of R, and $a \in R$ such that $a * D(M) = 0$, then $a = 0$.

Proof: By Theorem 3.6 we know that there is an element m in M such that $D(m) \neq 0$. Let M be a nonzero ideal of R, and $a \in R$ such that $a * d(M) = 0$. Then for $m, n \in M$ we can write

$$\begin{aligned} 0 = a * D(m * n) = a * (D(m) * n + m * d(n)) &= a * D(m) * n + a * m * d(n) \\ &= a * m * d(n). \end{aligned}$$

Since R is an integral incline, d is a nonzero derivation on M and M is a nonzero ideal we have $a = 0$.

3.8. Definition. Let D be a generalized derivation of an incline algebra R associated with a derivation of R. If $x \leq y$ implies that $D(x) \leq D(y)$ for all $x, y \in R$, then D is called an isotone generalized derivation.

3.9. Proposition. Let D be a generalized derivation of an incline algebra R associated with a derivation d of R. If for all $x, y \in R$ we have $D(x + y) = D(x) + D(y)$, then the followings hold for all $x, y \in R$:

(i) $D(x * y) \leq D(x)$,

(ii)$D(x * y) \leq D(y)$,

(iii) D is an isotone derivation.

Proof:

(i) Let $x, y \in R$. By using $(R7)$ we have $D(x) = D(x + x * y) = D(x) + D(x * y)$. Therefore we get that $D(x * y) \leq D(y)$.

(ii) Proof can be get similarly with the previous one.

(iii) Let $x \leq y$, then we have $x + y = y$, therefore we have $D(y) = D(x + y) = D(x) + D(y)$. Hence $D(x) \leq D(y)$.

3.10. Theorem. Let D_1, D_2 be generalized derivations of an incline algebra R associated with derivations d_1, d_2 derivations of R respectively. Let $D_1(x+y) = D_1(x) + D_1(y)$ and $D_2(x+y) = D_2(x) + D_2(y)$ for all $x, y \in R$. Define $D_1D_2(x) = D_1(D_2(x))$ for all x in R. If $D_1D_2 = 0$ and $d_1d_2 = 0$ then D_2D_1 is a generalized derivation of R associated with a derivation d_2d_1 derivation of R.

Proof: Let $D_1D_2 = 0$. Then we have

$$\begin{aligned} D_1D_2(x * y) &= D_1(D_2(x) * y + x * d_2(y) \\ &= D_1(D_2(x)) * y + D_2(x) * d_1(y) + D_1(x) * d_2(y) + x * d_1d_2(y). \end{aligned}$$

Therefore we get $D_2(x) * d_1(y) + D_1(x) * d_2(y) = 0$.

$$\begin{aligned} D_2D_1(x * y) &= D_2(D_1(x) * y + x * d_1(y) \\ &= D_2(D_1(x)) * y + D_1(x) * d_2(y) + D_2(x) * d_1(y) + x * d_2d_1(y) \\ &= D_2(D_1(x)) * y + x * d_2d_1(y). \end{aligned}$$

Hence D_2D_1 is a generalized derivation of R associated with a derivation d_2d_1 derivation of R.

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Edelstein-type fixed point theorems in compact TVS-cone metric spaces

Thabet Abdeljawad *

Abstract

In this paper we prove two fixed point theorems in compact cone metric spaces over normal cones. The first theorem generalizes Edelstein theorem [8] and is different from the generalization obtained in [11]. The second theorem generalizes the main result in [10] and the first theorem. However, the two theorems fail in different categories. Moreover, different versions of the two theorems are proved in TVS-cone metric spaces by making use of the nonlinear scalarization function used very recently by Wei-Shih Du in [A note on cone metric fixed point theory and its equivalence, Nonlinear Analysis,72(5),2259-2261 (2010).] to prove the equivalence of the Banach contraction principle in cone metric spaces and usual metric spaces.

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Keywords: cone metric, normal cone, strictly normal cone, fixed point, contractive condition. TVS-cone metric space.

1. Introduction and Preliminaries

There are thousands of fixed points theorems in complete or compact metric spaces. However, these theorems can be categorized into the following four types. Let $T : (X, d) \to (X, d)$ be a mapping on a metric space not assumed to be convex or normed, then

(T1) Leader-type [1]: The mapping T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$. Such a mapping is called a Picard operator as in [2].

(T2) Unnamed-type: The mapping T has a unique fixed point and $\{T^n x\}$ does not necessary converge to the fixed point.

(T3) Subrahmanyam-type [3]: The mapping T may have more than one fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$. Such a mapping is called a weakly Picard operator as in [4] and [5].

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(T4) Caristi-type ([6], [7]): The mapping T may have more than one fixed point and $\{T^n x\}$ does not necessarily converge to a fixed point.

The following Edelstein fixed point theorem [8] belongs to (T1) because it can be proved using Meir-Keeler theorem [9].

1.1. Theorem. *Let (X, d) be a compact metric space and let T be a mapping on X. Assume $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.*

Motivated by the above, T. Suzuki in [10] generalized Theorem 1.1 by a theorem belongs to (T2). Indeed, he obtained the following theorem

1.2. Theorem. *Let (X, d) be a compact metric space and let T be a mapping on X. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y) \quad \text{for } x, y \in X. \tag{1.1}$$

Then T has a unique fixed point.

Recently, and in [11], the authors there rather implied that metric spaces do not provide enough space for fixed point theory. They introduced the notion of cone metric space, where they gave an example of a function which is contraction in the category of cone metric spaces but not contraction if considered over metric spaces and hence, by proving a fixed point theorem in cone metric spaces, ensured that this map must have a unique fixed point. They also there generalized Theorem 1.1 to compact cone metric spaces over regular cones. Namely, they proved

1.3. Theorem. *Let (X, d) be a compact cone metric space over a regular cone and let T be a mapping on X. Assume $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.*

After that series of articles about cone metric spaces started to appear. Some of those articles dealt with fixed point theorems in those spaces and some other with the structure of the spaces themselves. For example we name [12], [13], [14], [15], [16], [17], [18], [19] and [20]. For more recent articles about cone metric spaces we refer to [26, 27, 28, 29, 30].

In this article, we shall generalize Theorem 1.2 to compact cone metric spaces over normal cones. Also, recalling that there exist normal cones with constant 1 which are not regular, we shall prove another generalization to Theorem 1.3. Moreover, by the help of the nonlinear scalarization, we prove other different versions of Edelstein fixed point theorem in TVS-cone metric spaces without any normality or regularity assumptions.

Throughout this paper, (E, S) stands for real Hausdorff locally convex topological vector space (t.v.s.) with S its generating system of seminorms. A non-empty subset P of E is called cone if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \{0\}$. The cone P will be assumed to be closed and has nonempty interior as well. For a given cone P, one can define a partial ordering (denoted by $\leq$: or $\leq_P$) with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x \neq y$ while $x << y$ will show $y - x \in int(P)$, where $int(P)$ denotes the interior of P. Continuity of the algebric operations in a topological vector space and the properties of the cone imply the relations:

$$int(P) + int(P) \subseteq int(P) \text{ and } \lambda int(P) \subseteq int(P) \ (\lambda > 0).$$

We appeal to these operations in the following.

1.4. Definition. [23] A cone P of a topological vector space (X, τ) is said to be normal whenever τ has a base of zero consisting of $P-$ full sets. Where a subset of A of an order vector space via a cone P is said to be $P-$full if for each $x, y \in A$ we have $\{a \in E : x \leq a \leq y\} \subset A$.

1.5. Theorem. [23] *(a) A cone P of a topological vector space (X,τ) is normal if and only if whenever $\{x_\alpha\}$ and $\{y_\alpha\}$, $\alpha \in \Delta$ are two nets in X with $0 \leq x_\alpha \leq y_\alpha$ for each $\alpha \in \Delta$ and $y_\alpha \to 0$, then $x_\alpha \to 0$.*

(b) The cone of an ordered locally convex space (X,τ) is normal if and only if τ is generated by a family of monotone $\tau-$ continuous seminorms. Where a seminorm q on X is called monotone if $q(x) \leq q(y)$ for all $x, y \in X$ with $0 \leq x \leq y$.

1.6. Definition. A cone P of a locally convex topological vector space (E,S) is said to be (sequentially) regular if every sequence in E which is increasing and bounded above must be convergent in (E,S). That is, if $\{a_n\}$ is a sequence in E such that $a_1 \leq a_2 \leq ... \leq a_n \leq ... \leq b$ for some $b \in E$, then there is $a \in E$ such that $q(a_n) \to 0$ as $n \to \infty$, for all $q \in S$. Equivalently, if every sequence in E which is decreasing and bounded below is convergent.

It is well-known that if E is a real Banach space then every (sequentially) regular cone is normal.

In particular, if P is a cone of a real Banach space E, then it is called *normal* if there is a number $K \geq 1$ such that for all $x, y \in E$: $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$. The least positive integer K, satisfying this inequality, is called the normal constant of P. Also, P is said to be *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. For more details about cones in locally convex topological vector spaces we may refer the reader to [23].

It is worthwhile to mention here, that there are certain real Banach spaces whose positive cones have empty interior such as the sequences spaces l^p, $1 \leq p < \infty$ and the Lebesgue integrable spaces L_p, $1 \leq p < \infty$ [21]. On the other hand the positive cone in the Euclidean space $\mathbb{R}^n$ does not have empty interior. For example in $\mathbb{R}^2$ the interior of the positive cone $P = \{(x,y) : x \geq 0, y \geq 0\}$ is $\{(x,y) : x > 0, y > 0\}$ which is nonempty. For the infinite dimensional case the positive cones of $AM-spaces$ can have nonempty interiors [22]. For the purposes in defining convergence [11] and other topological concepts in cone metric spaces [15], the cones under consideration are always assumed to have nonempty interiors.

The cone $[0,\infty)$ in $(\mathbb{R}, |.|)$ and the cone $P = \{(x,y) : x \geq 0, y \geq 0\}$ in $\mathbb{R}^2$ are normal cones with constant $K = 1$. However, there are examples of non-normal cones.

1.7. Example. [24] Let $E = C^1[0,1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, and consider the cone $P = \{f \in E : f \geq 0\}$.

For each $k \geq 1$, put $f(x) = x$ and $g(x) = x^{2k}$. Then, $0 \leq g \leq f$, $\|f\| = 2$ and $\|g\| = 2k + 1$. Since $k\|f\| < \|g\|$, k is not normal constant of P and hence P is a non-normal cone.

Normal cones are very important when some fixed point theorems are generalized to cone metric spaces. Most of normal cones have normal constant 1. Hence, it is still of great interest when some fixed point theorems are generalized to cone metric spaces over normal cones with normal constant 1. Here are some examples of normal cones with normal constant 1.

1.8. Example. Let $E = \mathbb{R}^n$, $n \geq 1$ and $P = \{(x_1, x_2, ..., x_n) : x_i \geq 0, i = 1, 2, ..., n\}$. Then, P is a normal cone with normal constant $K = 1$.

1.9. Example. Let $E = C[0,1]$ with the supremum norm and $P = \{f \in E : f \geq 0\}$. Then P is a cone with normal constant $K = 1$ which is not regular. This is clear, since the sequence x^n is monotonically decreasing, but not uniformly convergent to 0.

A cone P of real Banach space E is called strictly normal with normal constant K, if K is the least number such that

$$0 \leq a < b \text{ implies } \|a\| < K\|b\|, \text{ for all } a, b \in E.$$

The cone given in Example 1.8 is clearly strictly normal. Also, the positive cones of the L_p-spaces with $1 \leq p < \infty$ are strictly normal with $K = 1$. Actually, the norms of those spaces are strictly monotone ([22]). However, the cone given in Example 1.9 is not strictly normal.

1.10. Definition. (See [25], [30]) For $e \in int(P)$, the nonlinear scalarization function $\xi_e : E \to \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{t \in \mathbb{R} : y \in te - P\}, \text{ for all } y \in E.$$

1.11. Lemma. *(See* [25],[30]*) For each $t \in \mathbb{R}$ and $y \in E$, the following are satisfied:*

- (i) $\xi_e(y) \leq t \Leftrightarrow y \in te - P$,
- (ii) $\xi_e(y) > t \Leftrightarrow y \notin te - P$,
- (iii) $\xi_e(y) \geq t \Leftrightarrow y \notin te - int(P)$,
- (iv) $\xi_e(y) < t \Leftrightarrow y \in te - int(P)$,
- (v) $\xi_e(y)$ *is positively homogeneous and continuous on E,*
- (vi) *if $y_1 \in y_2 + P$, then $\xi_e(y_2) \leq \xi_e(y_1)$,*
- (vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$*, for all $y_1, y_2 \in E$,*
- $(viii)$ *if $y_1 \in y_2 + int(P)$, then $\xi_e(y_2) < \xi_e(y_1)$.*

Lemma 1.11 (vi) and Lemma 1.11 (viii) mean that the nonlinear scalarization function ξ_e is monotone and strictly monotone. However, it is not strongly monotone (see [30]).

1.12. Definition. Let X be a non-empty set. and E as usual a Hausdorff locally convex topological space. Suppose a vector-valued function $p : X \times X \to E$ satisfies:

- $(M1)$ $0 \leq p(x, y)$ for all $x, y \in X$,
- $(M2)$ $p(x, y) = 0$ if and only if $x = y$,
- $(M3)$ $p(x, y) = p(y, x)$ for all $x, y \in X$
- $(M4)$ $p(x, y) \leq p(x, z) + p(z, y)$, for all $x, y, z \in X$.

Then, p is called TVS-cone metric on X, and the pair (X, p) is called a TVS-cone metric space (in short, TVS-CMS).

Note that in [11], the authors considered E as a real Banach space in the definition of TVS-CMS. Thus, a cone metric space (in short, CMS) in the sense of Huang and Zhang [11] is a special case of TVS-CMS.

1.13. Lemma. *(See* [25]*) Let (X, p) be a TVS-CMS. Then, $d_p : X \times X \to [0, \infty)$ defined by $d_p = \xi_e \circ p$ is a metric.*

1.14. Remark. Since a cone metric space (X, d) in the sense of Huang and Zhang [11], is a special case of TVS-CMS, then $d_p : X \times X \to [0, \infty)$ defined by $d_p = \xi_e \circ d$ is also a metric.

1.15. Definition. (See [25]) Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X.

- (i) $\{x_n\}_{n=1}^{\infty}$ TVS-cone converges to $x \in X$ whenever for every $0 << c \in E$, there is a natural number M such that $p(x_n, x) << c$ for all $n \geq M$ and denoted by $cone - \lim_{n\to\infty} x_n = x$ (or $x_n \overset{cone}{\to} x$ as $n \to \infty$),
- (ii) $\{x_n\}_{n=1}^{\infty}$ TVS-cone Cauchy sequence in (X, p) whenever for every $0 << c \in E$, there is a natural number M such that $p(x_n, x_m) << c$ for all $n, m \geq M$,

(*iii*) (X, p) is TVS-cone complete if every sequence TVS-cone Cauchy sequence in X is a TVS-cone convergent.

1.16. Lemma. *(See* [25]*) Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Set $d_p = \xi_e \circ p$. Then the following statements hold:*

(*i*) *If $\{x_n\}_{n=1}^{\infty}$ converges to x in TVS-CMS (X, p), then $d_p(x_n, x) \to 0$ as $n \to \infty$,*

(*ii*) *If $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in TVS-CMS (X, p), then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence (in usual sense) in (X, d_p),*

(*iii*) *If (X, p) is complete TVS-CMS, then (X, d_p) is a complete metric space.*

1.17. Remark. (see also [30, 31]) Note that the implications in (i) and (ii) of Lemma 1.16 are also conversely true. Regarding (i) we prove that if $x_n \to x$ in (X, d_p) then $x_n \to x$ in (X, p). To this end, let $c >> 0$ be given, then find $q \in S$ and $\delta > 0$ such that $q(b) < \delta$ implies that $b << c$. Since $\frac{e}{n} \to 0$ in (E, S) find $\epsilon = \frac{1}{n_0}$ such that $\epsilon q(e) = q(\epsilon e) < \delta$ and hence $\epsilon e << c$. Now, find n_0 such that $d_p(x_n, x) = \xi_e \circ p(x_n, x) < \epsilon$ for all $n \geq n_0$. Hence, by Lemma 1.11 (iv) $p(x_n, x) << \epsilon e << c$ for all $n \geq n_0$. The proof of the converse of implication (ii) is similar. Now it is possible to say that (iii) of Lemma 1.16 is immediate from (i) and (ii). This remark clearly is also valid for any topological vector space since neighborhood elements are symmetric.

The following lemma characterizes convergence and Cauchyness in cone metric spaces by means of the scalar norm of the Banach space E under the normality assumption for the cone P.

1.18. Lemma. [11] *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then,*

(i) $\{x_n\}$ converges to x if and only if $lim_{n\to\infty} d_s(x_n, x) = 0$,

(ii) $\{x_n\}$ is Cauchy if and only if $lim_{m,n\to\infty} d_s(x_n, x_m) = 0$,

where $d_s(x, y) = \|d(x, y)\|$, for $x, y \in X$.

Regarding the above Lemma, It is always worthwhile to mention that normality is only used in proving the necessity (for example see [24]).

1.19. Lemma. [11] *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $lim_{n\to\infty} x_n = x$ and $lim_{n\to\infty} y_n = y$. Then*

$$lim_{n\to\infty} \|d(x_n, y_n) - d(x, y)\| = 0$$

In case $Int(P) \neq \emptyset$ it was proved in [15] that cone metric spaces are topological spaces and compactness and sequential compactness coincide (see also [15]).

2. The Main Results

2.1. Theorem. *Let (X, d) be a (sequentially) compact cone metric space over a normal cone P with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d_s(Tx, Ty) < d_s(x, y), \text{ for all } x, y \in X, \ x \neq y. \tag{2.1}$$

Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0$, $x_2 = Tx_2, ..., x_{n+1} = Tx_n = T^{n+1}x_0, ...$ If for some n, $x_{n+1} = x_n$, then $\{x_n\}$ is a fixed point of T, the proof is complete. So we assume that for all n, $x_{n+1} \neq x_n$. Set $d_n = d_s(x_n, x_{n+1})$, then the contractive condition (2.1) implies

$$d_{n+1} = d_s(x_{n+1}, x_{n+2}) = d_s(Tx_n, Tx_{n+1}) < d_s(x_n, x_{n+1}).$$

Therefore, the sequence $\{d_n\}$ is a decreasing sequence of positive real numbers and hence convergent. Assume $d_n \to d^*$. From the compactness of X, we may assume that $x_n \to x^*$ in (X, d). Then, by the contractive condition (2.1) we have $Tx_n \to Tx^*$ and $T^2x_n \to T^2x^*$. By using Lemma 1.19 and uniform continuity of the norm $\|.\|$, we have $d_s(Tx_n, x_n) \to d_s(Tx^*, x^*)$ and $d_s(T^2x_n, Tx_n) \to d_s(T^2x^*, Tx^*)$. Hence, we have $d_s(Tx_n, x_n) \to d^* = d_s(Tx^*, x^*)$. Finally, we show that $Tx^* = x^*$. If not, then $d^* \neq 0$ and hence

$$d^* = d_s(Tx^*, x^*) > d_s(T^2x^*, Tx^*) = \lim_{n\to\infty} d_s(T^2x_n, Tx_n) = \lim_{n\to\infty} d_{n+1} = d^*.$$

Which is a contradiction, so $Tx^* = x^*$. The uniqueness of the fixed point is obvious from the contractive condition (2.1). □

It is desired to mention that Theorem 2.1 above can be applied for cone metric spaces over the cone defined in Example 1.9. However, Theorem 2 in [11] can not be applied for such kind of cone metric spaces.

2.2. Remark. If the cone P is strictly normal with constant $K = 1$ then the contractive condition (2.1) can be replaced by

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, \ x \neq y.$$

In this case the proof of Theorem 2.1 can be easily also achieved by applying Edelstein Theorem to the metric space (X, d_s).

2.3. Theorem. *Let (X, d) be a (sequentially) compact cone metric space over a normal cone P with normal constant $K \geq 1$ and let T be a mapping on X. Assume that*

$$\frac{1}{2}d_s(x, Tx) < Kd_s(x, y) \text{ implies } d_s(Tx, Ty) < d_s(x, y) \quad \text{for } x, y \in X. \tag{2.2}$$

Then, T has a unique fixed point.

Proof. Set $\alpha = \inf\{d_s(x, Tx) : x \in X\}$ and choose a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d_s(x_n, Tx_n) = \alpha$. Since X is compact, without loss of generality, we may assume that $\{x_n\}$ and $\{Tx_n\}$ converge to some elements v and w, respectively. We shall show that $\alpha = 0$. If not then $\alpha > 0$. By Lemma 1.19 we have

$$\lim_{n\to\infty} d_s(x_n, w) = d_s(v, w) = \lim_{n\to\infty} d_s(x_n, Tx_n) = \alpha$$

We can choose $n_0 \in \mathbb{N}$ such that

$$\frac{2}{3}\alpha < d_s(x_n, w) \text{ and } d_s(x_n, Tx_n) < \frac{4}{3}\alpha$$

for all $n \geq n_0$. Thus $\frac{1}{2}d_s(x_n, Tx_n) < d_s(x_n, w)$ for $n \geq n_0$. By (3.4), $d_s(Tx_n, Tw) < d_s(x_n, w)$ holds for $n \geq n_0$. This implies, again by means of Lemma 1.19, that

$$d_s(w, Tw) = \lim_{n\to\infty} d_s(Tx_n, Tw) \leq \lim_{n\to\infty} d_s(x_n, w) = \alpha.$$

From the definition of α, we obtain $d_s(w, Tw) = \alpha$. Since $\frac{1}{2}d_s(w, Tw) < d_s(w, Tw)$, then by (3.4), we have

$$d_s(Tw, T^2w) < d_s(w, Tw) = \alpha.$$

Which contradicts the definition of α. Therefore, we must have $\alpha = 0$. We next show that T has a fixed point. Arguing by contradiction, we assume that T does not have a fixed point. Hence, $0 < \frac{1}{2}d_s(x_n, Tx_n) < d_s(x_n, Tx_n)$ for all $n \in \mathbb{N}$ and so by (3.4) we have

$$d_s(Tx_n, T^2x_n) < d_s(x_n, Tx_n) \tag{2.3}$$

for all $n \in \mathbb{N}$. Also by Lemma 1.19 we have

$$\lim_{n\to\infty} d_s(v, Tx_n) = d_s(v, w) = \lim_{n\to\infty} d_s(x_n, Tx_n) = \alpha = 0,$$

which implies that $Tx_n \to v$. We also have

$$\lim_{n\to\infty} d_s(v, T^2x_n) \leq$$

$$K \lim_{n\to\infty} (d_s(v, Tx_n) + d_s(Tx_n, T^2x_n)) \leq K \lim_{n\to\infty} (d_s(v, Tx_n) + d_s(x_n, Tx_n)) = 0. \tag{2.4}$$

Thus, $T^2x_n \to v$. If we accept

$$\frac{1}{2} d_s(x_n, Tx_n) \geq K d_s(x_n, v) \text{ and } \frac{1}{2} d_s(Tx_n, T^2x_n) \geq K d_s(Tx_n, v),$$

then by the triangle inequality and (2.3) we have

$$d_s(x_n, Tx_n) \leq K d_s(x_n, v) + K d_s(Tx_n, v) \leq$$

$$\frac{1}{2} d_s(x_n, Tx_n) + \frac{1}{2} d_s(Tx_n, T^2x_n) < \frac{1}{2} d_s(x_n, Tx_n) + \frac{1}{2} d_s(x_n, Tx_n) = d_s(x_n, Tx_n), \tag{2.5}$$

which is a contradiction. Hence for every $n \in \mathbb{N}$, either

$$\frac{1}{2} d_s(x_n, Tx_n) < K d_s(x_n, v) \text{ or } \frac{1}{2} d_s(Tx_n, T^2x_n) < K d_s(Tx_n, v)$$

holds. Then by assumption (3.4), either

$$d_s(Tx_n, Tv) < d_s(x_n, v) \text{ or } d_s(T^2x_n, Tv) < d_s(Tx_n, v)$$

holds. Hence one of the following holds:

(i) There exists an infinite subset I of $\mathbb{N}$ such that

$$d_s(Tx_n, Tv) < d_s(x_n, v) \text{ for all } n \in I.$$

(ii) There exists an infinite subset J of $\mathbb{N}$ such that

$$d_s(T^2x_n, Tv) < d_s(Tx_n, v) \text{ for all } n \in J.$$

In the first case, we obtain

$$d_s(v, Tv) = \lim_{n\in I, n\to\infty} d_s(Tx_n, Tv) \leq \lim n \in I, n \to \infty d_s(x_n, v) = 0,$$

which implies that $Tv = v$. Also, in the second case, we obtain

$$d_s(v, Tv) = \lim_{n\in J, n\to\infty} d_s(T^2x_n, Tv) \leq \lim_{n\in J, n\to\infty} d_s(Tx_n, v) = 0,$$

which implies that $Tv = v$. That is v is a fixed point in both cases. This is a contradiction. Therefore there exists $z \in X$ such that $Tz = z$. To prove uniqueness, fix $y \in X$ with $y \neq z$. Then since $\frac{1}{2} d_s(z, Tz) = 0 < K d_s(z, y)$, we have

$$d_s(z, Ty) = d_s(Tz, Ty) < d_s(z, y)$$

and hence y is not a fixed point. □

Note that also, if in Theorem 2.3 the cone P is strictly normal with $K = 1$ then the assumption (3.4) can be replaced by the condition

$$\frac{1}{2} d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y) \text{ for } x, y \in X.$$

As a matter of great interest in Theorem 2.3, is that the author in [10] proved that the constant $\frac{1}{2}$ is the best constant. Namely, he proved that for every $\eta \in (\frac{1}{2}, \infty)$, there exists a compact metric space (X, d) and a mapping T on X satisfying the following:

(i) T has no fixed point.

(ii) $\eta d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$ for all $x,\ y \in X$.

3. Other Edelstein -type fixed point theorem in TVS-cone metric spaces

3.1. Lemma. *If (X, p) is a (sequentially) compact TVS-metric space then (X, d_p) is a (sequentially) compact metric space, where $d_p = \xi_e \circ p$.*

The proof followed by Lemma 1.13 , Lemma 1.16 (ii).

3.2. Theorem. *Let (X, p) be a (sequentially) compact TVS-cone metric space and let T be a mapping on X. Assume that*

$$\frac{1}{2}d_p(x, Tx) < d_p(x, y) \text{ implies } d(Tx, Ty) \ll d(x, y) \quad \text{for } x, y \in X. \tag{3.1}$$

Then, T has a unique fixed point.

The proof followed by Lemma 1.11 (viii), Lemma 3.1 and the classical Edelstein fixed point theorem.

3.3. Corollary. *Let (X, p) be a (sequentially) compact TVS-cone metric space and let T be a mapping on X. Assume that*

$$d(Tx, Ty) \ll d(x, y) \quad \text{for } x, y \in X. \tag{3.2}$$

Then, T has a unique fixed point.

Note that no regularity assumption is not required in Lemma 3.1 and hence in Corollary 3.3. Therefore, Corollary 3.3 improves Theorem 2 in [11] by replacing " $<$ " by " $\ll$ " and removing regularity.

3.4. Theorem. *Let (X, p) be a (sequentially) compact TVS-cone metric space and let T be a mapping on X. Assume that*

$$\frac{1}{2}d_p(x, Tx) < d_p(x, y) \text{ implies } d_p(Tx, Ty) < d_p(x, y) \quad \text{for } x, y \in X. \tag{3.3}$$

Then, T has a unique fixed point.

The proof followed by Lemma 3.1 and the classical Edelstein fixed point theorem.

3.5. Corollary. *Let (X, p) be a (sequentially) compact TVS-cone metric space and let T be a mapping on X. Assume that*

$$d_p(Tx, Ty) < d_p(x, y) \quad \text{for } x, y \in X. \tag{3.4}$$

Then, T has a unique fixed point.

Note that no normality assumptions are needed in Theorem 3.4 and Corollary 3.5.

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On multiplication lattice modules

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Abstract

In this paper we study multiplication lattice modules. Next we characterize hollow lattices modules. We also establish maximal elements in multiplication lattices modules.In [16] , we introduced the concept of a multiplication lattice L-module and we characterized it by principal elements.In this paper, we continue study on multiplication lattice L-module.

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Multiplicative lattices, Lattices modules,prime elements.

1.

A multiplicative lattice L is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins and has compact greatest element 1_L (least element 0_L) as a multiplicative identity (zero) . Multiplicative lattices have been studied extensively by E.W.Johnson, C.Jayaram, the current authors, and others, see, for example, $[1-16]$.

An element $a \in L$ is said to be proper if $a < 1$. An element $p < 1_L$ in L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. An element $m < 1$ in L is said to be maximal if $m < x \leq 1_L$ implies $x = 1_L$. It is easily seen that maximal elements are prime.

If a, b belong to L, $(a :_L b)$ is the join of all $c \in L$ such that $cb \leq a$. An element e of L is called meet principal if $a \bigwedge be = ((a :_L e) \bigwedge b)\, e$ for all $a, b \in L$. An element e of L is called join principal if $((ae \bigvee b) :_L e) = a \bigvee (b :_L e)$ for all $a, b \in L$. $e \in L$ is said to be principal if e is both meet principal and join principal. $e \in L$ is said to be weak meet (join) principal if $a \bigwedge e = e\,(a :_L e)$ $(a \bigvee (0_L :_L e) = (ea :_L e))$ for all $a \in L$. An element a of a multiplicative lattice L is called compact if $a \leq \bigvee b_\alpha$ implies $a \leq b_{\alpha_1} \bigvee b_{\alpha_2} \bigvee ... \bigvee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. If each element of L is a join of principal (compact) elements of L, then L is called a $PG-$lattice $(CG-$ lattice) .

Let M be a complete lattice. Recall that M is a lattice module over the multiplicative lattice L, or simply an L-module in the case there is a multiplication between elements of L and M, denoted by lB for $l \in L$ and $B \in M$, which satisfies the following properties :

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(i): $(lb)\,B = l\,(bB)$;

(ii): $\left(\bigvee_{\alpha} l_\alpha\right)\left(\bigvee_{\beta} B_\beta\right) = \bigvee_{\alpha,\beta} l_\alpha B_\beta$;

(iii): $1_L B = B$;

(iv): $0_L B = 0_M$ for all l, l_α, b in L and for all B, B_β in M.

Let M be an L-module. If N, K belong to M, $(N :_L K)$ is the join of all $a \in L$ such that $aK \leq N$. If $a \in L, (0_L :_M a)$ is the join of all $H \in M$ such that $aH = 0_M$. An element N of M is called meet principal if $(b \bigwedge (B :_L N))\, N = bN \bigwedge B$ for all $b \in L$ and for all $B \in M$. An element N of M is called join principal if $b \bigvee (B :_L N) = ((bN \bigvee B) :_L N)$ for all $b \in L$ and for all $N \in M$. N is said to be principal if it is both meet principal and join principal. In special case an element N of M is called weak meet principal (weak join principal) if $(B :_L N)\, N = B \bigwedge N$ $((bN :_L N) = b \bigvee (0_M :_L N))$ for all $B \in M$ and for all $b \in L$. N is said to be weak principal if N is both weak meet principal and weak join principal.

Let M be an L-module. An element N in M is called compact if $N \leq \bigvee_{\alpha} B_\alpha$ implies $N \leq B_{\alpha_1} \bigvee B_{\alpha_2} \bigvee ... \bigvee B_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. The greatest element of M will be denoted by 1_M. If each element of M is a join of principal (compact) elements of M, then M is called a $PG-$lattice (CG-lattice).

Let L be a multiplicative lattice and let M be an L-module. If M is CG-lattice, then any weak principal element N of M is compact [14, Corollary 2.2]. Especially, if L is a CG-lattice, then any weak principal element in L is compact [14, Corollary 2.3].

Let M be an L-module. An element $N \in M$ is said to be proper if $N < 1_M$. If $ann\,(1_M) = (0_M :_L 1_M) = 0_L$, M is called a faithful L-module. If $cm = 0_M$ implies $m = 0_M$ or $c = 0_L$ for any $c \in L$ and $m \in M$, then M is called a torsion-free L-module.

For various characterizations of lattice modules, the reader is referred to $[9 - 16]$.

H.M.Nakkar and I.A.Al-Khouja [13, 14] studied multiplicative lattice modules over multiplicative lattices. In [16], we introduced the concept of a multiplication lattice L-module and we characterized it by principal elements. In this study, we continue study on multiplication lattice L-module and we prove that many important theorems like Nakayama Lemma. We also prove that if L is a multiplicative $PG-$lattice and M is a multiplication PG- lattice module, then K is maximal element of M if and only if there exist a maximal element $p \in L$ such that $K = p1_M < 1_M$.

1.1. Definition Let L be a multiplicative lattice and $c \in L$. c is said to be a multiplication element if for every element a of L such that $a \leq c$ there exists an element $d \in L$ such that $a = cd$.

1.2. Definition Let L be a multiplicative lattice and M a lattice L-module. $N \in M$ is said to be a multiplication element if for every element K of M such that $K \leq N$ there exists an element $a \in L$ such that $K = aN$.

Note that, $a \in L$ is a multiplication element if and only if a is a weak meet principal element in L and $N \in M$ is a multiplication element if and only if N is a weak meet principal element in M. We say that M is a multiplication lattice L-module if 1_M is a multiplication element in M.

1.3. Theorem *Let L be a $PG-$lattice and M be a PG-lattice $L-$module. Then M is a multiplication lattice $L-$module if and only if for every maximal element $q \in L$,*

(i): *For every principal element $Y \in M$, there exists a principal element $q_Y \in L$ with $q_Y \not\leq q$ such that $q_Y Y = 0_M$ or*

(ii): *There exists a principal element $X \in M$ and a principal element $b \in L$ with $b \not\leq q$ such that $b1_M \leq X$.*

Proof. [see 16, Theorem 4] . □

1.4. *Let L be a PG-lattice, and M be a faithful multiplication PG-lattice L-module. Then the following conditions are equivalent.*

(i): 1_M is a compact element of M.
(ii): If $a, c \in L$ such that $a1_M \leq c1_M$, then $a \leq c$.
(iii): For each element N of M there exists a unique element a of L such that $N = a1_M$.
(iv): $1_M \neq a1_M$ for any proper element a of L.
(v): $1_M \neq p1_M$ for any maximal element p of L.

Proof. [see 16, Theorem 5] . □

1.5. *Let L be a PG−lattice and M be a faithful multiplication PG−lattice L-module such that 1_M compact. If $a \in L$ is a multiplication element, then $a1_M \in M$ is a multiplication element.*

Proof. Let $K \leq a1_M$. Since M is a multiplication module, $K = b1_M$ for some $b \in L$. Then $K = b1_M \leq a1_M$. Since 1_M is compact, $b \leq a$ by Theorem 2 (ii) . Since $a \in L$ is a multiplication element, we have $b = ac$ for some $c \in L$ and so $K = b1_M = (ac)\,1_M = c\,(a1_M)$. Consequently, $a1_M$ is a multiplication element. □

1.6. *Let L be a PG−lattice and M a faithful multiplication PG−lattice L-module such that 1_M is compact.*

(i): N is a multiplication element in M if and only if $(K :_L N)\,(N :_L 1_M) = (K :_L 1_M)$ for all $K \leq N$.
(ii): $a = (a1_M :_L 1_M)$ for all $a \in L$.
(iii): N is a multiplication element in M if and only if $(N :_L 1_M)$ is a multiplication element in L.
(iv): $a1_M$ is a multiplication element in M if and only if a is a multiplication element in L.

Proof. $(i) \Rightarrow$: Let N be a multiplication element in M and $K \leq N$. Then $K = bN$ for some $b \in L$. Since M is a multiplication lattice $L-$module,

$$K = bN = (bN :_L N)\,N = (bN :_L N)\,(N :_L 1_M)\,1_M = (K :_L 1_M)\,1_M.$$

Therefore $(K :_L N)\,(N :_L 1_M) = (K :_L 1_M)$ by Theorem 2 (ii).

$\Leftarrow$: Since

$$K = (K :_L 1_M)\,1_M = (K :_L N)\,(N :_L 1_M)\,1_M = (K :_L N)\,N$$

for all $K \leq N$, N is a multiplication element.

(ii) Since M is a multiplication lattice module,we have $a1_M = (a1_M :_L 1_M)\,1_M$ and so $a = (a1_M :_L 1_M)$ for all $a \in L$ by Theorem 2 (ii).

$(iii) \Rightarrow$: Let N be a multiplication element. If $a \leq (N :_L 1_M)$, then $a = (a1_M :_L 1_M)$ by (ii) and $a = (a1_M :_L 1_M) = (a1_M :_L N)\,(N :_L 1_M)$ by (i) . Therefore, $a = c\,(N :_L 1_M)$ where $c = (a1_M :_L N)$. Then $(N :_L 1_M)$ is a multiplication element in L.

$\Leftarrow$: Let $(N :_L 1_M)$ be a multiplication element in L. Then $(N :_L 1_M)\,1_M = N$ multiplication elemen in M by Proposition 1.

$(iv) \Rightarrow$: Let $N = a1_M$ be a multiplication element in M. Then $(N :_L 1_M) = (a1_M :_L 1_M) = a$ is a multiplication element in L by (iii) .

$\Leftarrow$: Let $a \in L$ be a multiplication element in L. Then $N = a1_M$ is a multiplication element in M by Proposition 1. □

1.7. Proposition *Let L be a multiplicative lattice and M a multiplication lattice L-module. If L is a Noetherian (Artinian) latice, then M is a Noetherian (Artinian) $L-$module.*

Proof. Suppose that $N_1 \leq N_2 \leq ....$ and L is Noetherian . Then $(N_1 :_L 1_M) \leq (N_2 :_L 1_M) \leq ...$. Since L is Noetherian, there is a positive integer $k > 0$ such that $(N_k :_L 1_M) = (N_{k+1} :_L 1_M) = ...$ and so $(N_k :_L 1_M) 1_M = (N_{k+1} :_L 1_M) 1_M = ...$. Therefore, $N_k = N_{k+1} = ...$. Similarly, if L is Artinian, then M is Artinian. □

1.8. Definition Let L be a multiplicative lattice and M be a lattice L-module. Let K be a proper element in M. K is said to be a small element if for every element N of M such that $K \vee N = 1_M$ implies $N = 1_M$.

1.9. Definition Let L be a multiplicative lattice and M be a lattice L-module. If every proper element of M is small, then M is called a hollow $L-$module.

1.10. Theorem *Let L be a PG-lattice and M be a faithful multiplication $PG-$lattice L-module with 1_M compact. Then M is a hollow $L-$module if and only if L is a hollow L-module.*

Proof. $\Rightarrow$: Suppose that M is hollow. Let $a < 1_L$ such that $a \vee b = 1_L$ for some $b \in L$. Then $(a \vee b) 1_M = a1_M \vee b1_M = 1_M$. Since $a < 1_L$, $a1_M < 1_M$ by Theorem 2. By hypothesis, $b1_M = 1_M$ and hence $b = 1_L$ by Theorem 2 (ii). Therefore a is a small element in L.

$\Leftarrow$: Suppose that L is a hollow L-module. Let $N < 1_M$ and K be a any element in M such that $N \vee K = 1_M$. Since M is a multiplication L-module, we have $N = (N :_L 1_M) 1_M$ and $K = (K :_L 1_M) 1_M$. Then,

$$1_M = N \vee K = (N :_L 1_M) 1_M \vee (K :_L 1_M) 1_M = [(N :_L 1_M) \vee (K :_L 1_M)] 1_M.$$

Therefore, $(N :_L 1_M) \vee (K :_L 1_M) = 1_L$ by Theorem 2 (ii). Since $N = (N :_L 1_M) 1_M < 1_M$, we have $(N :_L 1_M) < 1_L$ and so $(K :_L 1_M) = 1_L$ by hypothesis. This shows that $K = 1_M$. Consequently, M is hollow. □

1.11. Theorem *Let L be a PG-lattice and M be a faithful multiplication $PG-$lattice L-module with 1_M compact. Then, N is small if and only if there exists a small element $a \in L$ such that $N = a1_M$.*

Proof. $\Rightarrow$: Suppose that $N \in M$ is small and $N = a1_M$ for some proper element a in L. Suppose that $a \vee b = 1_L$ for some $b \in L$. Then

$$N \vee b1_M = a1_M \vee b1_M = (a \vee b) 1_M = 1_M$$

and so $b1_M = 1_M$ by hypothesis. Hence $b = 1_L$ by Theorem 2. This shows that a is small in L.

$\Leftarrow$: Suppose that $a \in L$ is small such that $N = a1_M$. Let $N \bigvee K = a1_M \vee K = 1_M$ for some $K \in M$. Since M is a multiplication L-module, there is an element $b \in L$ such that $K = b1_M$ and hence $(a \vee b) 1_M = a1_M \vee K = 1_M$. Then $a \vee b = 1_L$ by Theorem 2 (ii) and hence $b = 1_L$ by hypothesis. Therefore, $K = 1_M$. This shows that $a1_M$ is small. □

1.12. Definition Let M be a L-module. An element $N < 1_M$ in M is said to be prime if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$ i.e. $a \leq (N :_L 1_M)$ for every $a \in L, X \in M$.

1.13. Definition Let M be an L-module. M is said to be prime L-module if 0_M is prime element of M.

It is clear that 0_M is prime element in M if and only if $(0_M :_L 1_M) = (0_M :_L N)$ for all $0_M \neq N \in M$.

1.14. Definition Let M be an L-module. M is said to be coprime L-module if $(0_M :_L 1_M) = (N :_L 1_M)$ for all $N \in M$ such that $N < 1_M$.

Recall that a lattice L-module M is called simple if $M = \{0_M, 1_M\}$.

1.15. Proposition *If M is a multiplication and coprime L-module, then M is simple.*

Proof. Let $N \in M$ such that $N < 1_M$. Since M is a coprime L-module, we have $(0_M :_L 1_M) = (N :_L 1_M)$. Since M is a multiplication L-module, it follows that $N = (N :_L 1_M) 1_M = (0_M :_L 1_M) 1_M = 0_M$. Then M is simple. □

1.16. Definition Let L be a $PG-$lattice and M a $PG-$lattice L-module. Let p be a maximal element of L. M is called $p-$torsion provided for each principal element $X \in M$ there exists a principal element $q_X \in L$, $q_X \nleq p$ such that $q_X X = 0_M$.

1.17. Definition Let L be a $PG-$lattice and M be a $PG-$lattice L-module. M is called p-cyclic provided there exists a principal element $Z \in M$ and a principal element $q \in L$, $q \nleq p$ such that $q1_M \leq Z$.

Let M be an L-module. Let N and K be elements of M such that $N \leq K$. Define $[N, K] = \{A \in M : N \leq A \leq K\}$.Then $[N, K]$ is an L-module. It is clear that N is a multiplication element if and only if $[0_M, N]$ is a multiplication lattice L-module. Recall that $ann(X) = (0_M :_L X)$ for any $X \in M$.

1.18. Theorem *Let L be a PG-lattice and M be a PG-lattice L-module. Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a collection of elements of M such that $N = \bigvee\limits_{\lambda\in\Lambda} N_\lambda$ and $a = \bigvee\limits_{\lambda\in\Lambda} (N_\lambda :_L N)$.*

i) N is a multiplication element in M.
ii) $H = aH$ for all elements $H \leq N$.
iii) $1_L = a \bigvee ann(X)$ for all principal elements $X \leq N$.
iv) $(N :_L K) \bigvee ann(X) = \bigvee\limits_{\lambda\in\Lambda} (N_\lambda :_L K) \bigvee ann(X)$ for all principal elements $X \leq N$ and for all elements K in M.

Then, $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$. If all N_λ are multiplication elements, then the conditions are equivalent.

Proof. $(i) \Rightarrow (ii)$. Let $a = \bigvee\limits_{\lambda\in\Lambda} (N_\lambda :_L N)$. Then $aN = \bigvee\limits_{\lambda\in\Lambda} (N_\lambda : N) N = N$. Since N is a multiplication element, there exist an element $h \in L$ such that $H = hN$ for $H \leq N$. Therefore, $aH = ahN = h(aN) = hN = H$.

$(ii) \Rightarrow (iii)$.Suppose that $1_L \neq a \bigvee ann(X)$ for all principal elements $X \leq N$. There exists a maximal element $p \in L$ such that $a \bigvee ann(X) \leq p$ for each principal element $X \leq N$. Since $X = aX \leq pX \leq X$, we have $X = pX$. Then, $1_L = (pX :_L X) = p \bigvee ann(X)$. Since $ann(X) \leq a \bigvee ann(X) \leq p$, we get a contradiction.

$(iii) \Rightarrow (ii)$. Since $X = (a \bigvee ann(X)) X = aX$ for all principal elements $X \leq N$, it follows that $H = aH$ for every $H \leq N$.

$(iii) \Rightarrow (iv)$. Since $\bigvee\limits_{\lambda\in\Lambda} (N_\lambda :_L K) K \leq \bigvee\limits_{\lambda\in\Lambda} N_\lambda = N$, we have $\bigvee\limits_{\lambda\in\Lambda} (N_\lambda :_L K) \leq \left(\bigvee\limits_{\lambda\in\Lambda} N_\lambda :_L K\right) = (N :_L K)$. Therefore, $\bigvee\limits_{\lambda\in\Lambda} (N_\lambda :_L K) \bigvee ann(X) \leq (N :_L K) \bigvee ann(X)$ for all principal elements $X \leq N$. Conversely, $w_1$and w_2 be principal elements such that $w_1 \bigvee w_2 \leq (N : K) \bigvee ann(X)$ where $w_1 \leq (N : K)$ and $w_2 \leq ann(X)$. Then $1_L = a \bigvee ann(X) = \bigvee\limits_{\lambda\in\Lambda} (N_\lambda :_L N) \bigvee ann(X)$. Hence $w_1 = \bigvee\limits_{\lambda\in\Lambda} (N_\lambda :_L N) w_1 \bigvee ann(X) w_1$. Since $w_1 K \leq N$, we have $(N_\lambda :_L N) w_1 K \leq (N_\lambda :_L N) N \leq N_\lambda$ and so $(N_\lambda :_L N) w_1 \leq (N_\lambda :_L K)$. Therefore,

$$w_1 \bigvee w_2 = \bigvee_{\lambda\in\Lambda} (N_\lambda :_L N)\, w_1 \bigvee ann\,(X)\, w_1 \bigvee w_2$$
$$\leq \bigvee_{\lambda\in\Lambda} (N_\lambda :_L K) \bigvee ann\,(X)\,.$$

Hence, $(N :_L K) \bigvee ann\,(X) \leq \bigvee_{\lambda\in\Lambda} (N_\lambda :_L K) \bigvee ann\,(X)\,.$

$(iv) \Rightarrow (iii)$. If we take $K = N$, then $1_L = (N :_L N) \bigvee ann\,(X) = a \bigvee ann\,(X)$

$(iii) \Rightarrow (i)$. Suppose that $1_L = a \bigvee ann\,(X)$ for all principal elements $X \leq N$. Suppose that N is not p-torsion. There exists a principal element $X \leq N$ such that $ann\,(X) = (0_M :_L X) \leq p$. Since $1_L = a \bigvee ann\,(X)$, we have $a = \bigvee_{\lambda\in\Lambda} (N_\lambda :_L N) \nleq p$. There exists $\lambda \in \Lambda$ such that $(N_\lambda :_L N) \nleq p$. There exists a principal element $b \nleq p$ such that $b \leq (N_\lambda :_L N)$ .Since N_λ is a multiplication element and not $p-$torsion, it follows that N_λ is p-cyclic by Theorem 1. Indeed, if N_λ is a p-torsion, then $cN_\lambda = 0_M$ for some principal elemebt $c \nleq p$ and so $bN \leq N_\lambda$ implies $bcN \leq cN_\lambda = 0_M$. Then $bc \leq (0_M :_L N)$. Therefore $bcY = 0_M$ for all principal elements $Y \leq N$ and principal element $bc \nleq p$. Since N is not $p-$torsion, this is a contradiction. Hence N_λ is p-cyclic. Therefore, $dN_\lambda \leq Y_\lambda$ for some principal element $Y_\lambda \leq N_\lambda$ and principal element $d \nleq p$.Therefore, $bN \leq N_\lambda$ and so $bdN \leq dN_\lambda \leq Y_\lambda$ and $bd \nleq p$. Consequently, N is p-cyclic. □

1.19. Theorem *(Nakayama Lemma) Let M be a non-zero multiplication $PG-$lattice $L-$module. Let $a \in L$ such that for all maximal element $q \in L$, $a \leq q$. Then $a1_M < 1_M$.*

Proof. Let $a \in L$ such that $a \leq q$ for all maximal element $q \in L$ and suppose that $a1_M = 1_M$. Let consider a principal element $0_M \neq X \in M$.Since M is a multiplication $L-$module, we have $X = b1_M$ for some $b \in L$. Hence $X = b1_M = ab1_M = aX$. Thus $1_L = (aX : X) = a \vee (0_M :_L X)$ for all principal elements $X \in M$. Since $(0_M :_L X) < 1_L$, there exists a maximal element $p \in L$ such that $(0_M :_L X) \leq p$. By hypothesis $a \leq p$, hence $1_L = (aX : X) = a \vee (0_M :_L X) \leq p$ and we obtain a contradiction. □

1.20. Theorem *Let L be a multiplicative $PG-$lattice. Let M be a multiplication PG-lattice L-module and $(0_M : 1_M) \leq p$ for some prime element $p \in L$. If $a1_M \leq p1_M$ for some $a \in L$, then $a \leq p$ or $p1_M = 1_M$.*

Proof. If $a1_M \leq p1_M$ for some $a \in L$, then $aX \leq p1_M$ for all principal element $X \in M$. Hence $a \leq p$ or $X \leq p1_M$ [see 16, Theorem 6] . If $a \nleq p$, then $X \leq p1_M$ for all principal element $X \in M$, hence it is clear that $p1_M = 1_M$. □

1.21. Theorem *Let L be a multiplicative $PG-$lattice. Let M be a multiplication PG-lattice L-module. Then K is maximal element of M if and only if there exist a maximal element $p \in L$ such that $K = p1_M < 1_M$.*

Proof. $\Leftarrow$: If there exist a maximal element $p \in L$ such that $K = p1_M < 1_M$, then K is maximal [see16, Proposition 4] .

$\Rightarrow$: Let K be a maximal element of M and $(K :_L 1_M) = q$. Since M is a multiplication lattice module, we have $K = q1_M$. We show that $(K :_L 1_M) = q$ is a maximal element of L. If q is not a maximal element, there exists a maximal element p such that $q < p$. Then $p1_M \nleq K$. Indeed, if $p1_M \leq K$, then $p \leq (K :_L 1_M) = q$. This is a contradiction. Therefore, $1_M = K \vee p1_M = (q \vee p)1_M = p1_M$. Hence $X = pX$ for all principal elements X and so $1_L = (pX :_L X) = p \vee (0_M :_L X)$. This implies that $(0_M :_L X) \nleq p$. Therefore, there exists a principal element $p_X \in L$ such that $p_X \leq (0_M :_L X)$ and $p_X \nleq p$. If we take a principal element X such that $X \nleq K$, then $X \vee K = 1_M$. Hence $p_X X \vee p_X K = p_X 1_M$ and so $p_X K = p_X 1_M \leq K$. Therefore, $p_X \leq (K :_L 1_M) = q < p$. This is a contradiction. □

1.22. Theorem *Let L be a CG−lattice and M be a PG−lattice L−module. Then M is a multiplication lattice L−module if and only if for every maximal element $q \in L$,*

(i) For every principal element $Y \in M$,there exists a compact element $q_Y \in L$ with $q_Y \nleqslant q$ such that $q_Y Y = 0_M$ or

(ii) There exists a principal element $X \in M$ and a compact element $b \in L$ with $b \nleqslant q$ such that $b1_M \leqslant X$.

Proof. $\Longrightarrow$: Let M be a multiplication lattice L−module. We have two cases.

Case 1. Let $q1_M = 1_M$ where q is a maximal element of L. For every principal element $Y \in M$, there exists an element $a \in L$ such that $Y = a1_M$. Then $Y = a1_M = aq1_M = qY$. Therefore, $1_L = (qY :_L Y) = q \vee (0_M :_L Y)$.Hence $(0_M :_L Y) \nleqslant q$. There exists a compact element q_Y such that $q_Y \leqslant (0_M :_L Y)$ and $q_Y \nleqslant q$.

Case 2. Let $q1_M < 1_M$.There exists a principal element $X \in M$ such that $X = j1_M \nleqslant q1_M$, with $j \in L, j \nleqslant q$.There exists a compact element $b \in L$ with $b \leq j$ and $b \nleqslant q$.We obtain $b1_M \leq j1_M = X$.

$\Longleftarrow$:Let $N \in M$. Put $a = (N :_L 1_M)$. Clearly $a1_M = (N :_L 1_M)1_M \leq N$. Take any principal element $Y \leq N$. We will show that $(a1_M :_L Y) = 1_L$. Suppose there exists a maximal element $q \in L$ such that $(a1_M :_L Y) \leq q$.We have two case.

Case 1. Suppose that (i) is satisfied. There exists a compact element $q_Y \in L$ with $q_Y \nleqslant q$ such that $q_Y Y = 0_M$ for every principal element $Y \in M$. Then $q_Y \leqslant (0_M : Y) \leqslant (a1_M : Y) \leq q$.This is a contradiction.

Case 2. Suppose that (ii) is satisfied.There exists a principal element $X \in M$ and a compact element $b \in L$ with $b \nleqslant q$ such that $b1_M \leqslant X$. Then $bN \leq b1_M \leqslant X$ for any $N \in M$. Since X is a principal element of $M, bN = (bN :_L X)X$. Then $b(bN :_L X)1_M \leq (bN :_L X)X = bN \leq N$ and so $b(bN :_L X) \leq a = (N :_L 1_M)$. Therefore $b^2Y \leq b^2N = b(bN :_L X)X \leq aX \leq a1_M$ imply $b^2 \leq (a1_M :_L Y) \leq q$. Since q is maximal (and so prime) element of L, we have $b \leq q$.This is a contradiction. □

1.23. Definition Let M be an L-module. An element $N < 1_M$ in M is said to be primary, if $aX \leq N$ and $X \nleq N$ implies $a^k 1_M \leq N$, for some $k \geq 0$ *i.e* $a^k \leq (N : 1_M)$ for every $a \in L, X \in M$.

If a is an element of a multiplicative lattice L, we define $\sqrt{a} = \bigvee \{b \in L : b^n \leq a \text{ for some natural number } n\}$.

1.24. Theorem *Let L be a CG-lattice and M be a multiplication PG-lattice L-module. Suppose that p is a primary element in L with $(0_M :_L 1_M) \leq p$. If $aX \leq p1_M$, where $a \in L, X \in M$, then $X \leq p1_M$ or $a \leq \sqrt{p}$.*

Proof. We may suppose that X is principal in M. Suppose that $aX \leq p1_M$ with $a \nleq \sqrt{p}$. We will show that $(p1_M :_L X) = 1_L$. Suppose that there exists a maximal element $q \in L$ such that $(p1_M :_L X) \leq q$. By theorem 7, we have two cases.

Case 1. If there exists a compact element $q_x \in L$ with $q_x \nleq q$ such that $0_M = q_x X$, then $q_x \leq (0_M :_L X) \leq (p1_M :_L X) \leq q$. This is a contradiction.

Case 2. If there exists a principal element $Y \in M$ and a compact element $b \in L$ with $b \nleq q$ such that $b1_M \leq Y$, then $bX \leq b1_M \leq Y$. Since Y is principal, we have $bX = (bX :_L Y)Y$. Put $(bX :_L Y) = s$. Then $abX = asY$. Since Y is join principal, it follows that $(asY :_L Y) = as \vee (0_M :_L Y)$. Since Y is meet principal, we have $abX = (abX :_L Y)Y$. Put $c = (abX :_L Y)$. Since $cY = abX \leq bp1_M \leq pY$, it follows that $c \vee (0_M :_L Y) = (cY :_L Y) \leq (pY :_L Y) = p \vee (0_M :_L Y)$. Since $b(0_M :_L Y)1_M = (0_M :_L Y)b1_M \leq (0_M :_L Y)Y = 0_M$, we have $b(0_M :_L Y) \leq (0_M :_L 1_M) \leq p$. Hence $bc \vee b(0_M :_L Y) \leq bp \vee b(0_M :_L Y) \leq p$. Therefore, $bc \leq p$. On the other hand, $c = (abX :_L Y) = (asY :_L Y) = as \vee (0_M :_L Y)$ and so $abs \leq abs \vee b(0_M :_L Y) = bc \leq p$. If $b \leq \sqrt{p}$, since b is compact ,we obtain $b \leq a_1 \vee a_2 \vee ... \vee a_m$ such that $a_i^{n_i} \leq p$

for each $i = 1, ..m$. For $k = n_1 + n_2 + ... + n_m$, we have $b^k \leq a_1^k \vee a_2^k \vee ... \vee a_m^k \leq p$. Then $b^k \leq p \leq (p1_M :_L X) \leq q$. Since q is maximal, we obtain $b \leq q$. This is a contradiction. Therefore, $b \nleq \sqrt{p}$. Since $a \nleq \sqrt{p}, b \nleq \sqrt{p}$ and p is primary, we have $s \leq p$. So, $bX = sY \leq pY \leq p1_M$ and therefore $b \leq (p1_M :_L X) \leq q$. This is a contradiction.

1.25. Theorem *Let L be a CG-lattice and M be a PG-lattice L-module. Let M be a multiplication lattice L-module and $N < 1_M$. Then the following condition are equivalent.*

i) N is a primary element in M.

ii) $(N :_L 1_M)$ is a primary element in L.

iii) There exists a primary element p in L with $(0_M :_L 1_M) \leq p$ such that $N = p1_M$.

□

Proof. *i)* $\Longrightarrow$ *ii)* $\Rightarrow$ *iii)* : Clear.

iii) $\Longrightarrow$ *i)* : Let $aX \leq N$ and $X \nleq N$ for $a \in L, X \in M$. Since there exists a primary element p in L with $(0_M :_L 1_M) \leq p$ such that $N = p1_M$, we have $aX \leq p1_M$ and $X \nleq p1_M$. By theorem 8, $a \leq \sqrt{p}$ and so $a^k \leq p$ for some $k > 0$. Hence $a^k \leq (p1_M :_L 1_M) = (N :_L 1_M)$. □

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Properties Q and R for nonlinear contractions in g-metric spaces

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Abstract

We prove a general common xed point theorem for three operators in G-metric spaces satisfying a nonlinear contraction. Also we prove the uniqueness of the xed point, as well as studying the G-continuity of the xed point. Moreover, we show that these maps satisfy properties Q and R. Our results extend some recent works. An illustrative example is also discussed.

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1. Introduction and Preliminaries

The notion of G-metric spaces was introduced by Z. Mustafa and B. Sims [5, 8]. They generalized the concept of a metric space. Then, based on the notion of generalized metric spaces, several authors have obtained some xed point results for a self-mapping under various contractive conditions, (see [1, 2, 3, 6, 7, 9]). In this paper we prove a general common xed point theorem for three operators in a complete generalized metric space X involving a nonlinear contraction related to a function 2 where is given by the following:

1.1. De nition. Denote by the set of non-decreasing continuous functions : R^+ ! R^+

1.2. Definition ([8]). Let X be a nonempty set and let $G : X \times X \times X \to R^+$ a function satisfying the following axioms:

(G1) $G(x,y,z) = 0$ if $x = y = z$;
(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$;
(G3) $G(x,x,y) \geq G(x,y,z)$; for all $x,y,z \in X$; with $z \neq y$;
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ (symmetry in all three variables) ;
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$; for all $x,y,z,a \in X$;
(rectangle inequality) :

Then the function G is called a generalized metric, or, simply, a G-metric on X, and the pair (X,G) is called a G-metric space.

1.3. Definition ([8]). Let (X,G) be a G-metric space, let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to x if $\lim_{n,m \to \infty} G(x,x_n,x_m) = 0$; that is, for any $\varepsilon > 0$, there exists a $k \in N$ such that $G(x,x_n,x_m) < \varepsilon$ for all $n,m \geq k$ (throughout this paper we mean by N the set of all natural numbers). We call x the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.

1.4. Definition ([8]). Let (X,G) be a G-metric space. Then the following are equivalent:

(1) $\{x_n\}$ is G-convergent to x;
(2) $G(x_n,x_n,x) \to 0$; as $n \to \infty$;
(3) $G(x_n,x,x) \to 0$; as $n \to \infty$;
(4) $G(x_m,x_n,x) \to 0$; as $n,m \to \infty$:

1.5. Definition ([8]). Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called G-cauchy if for each $\varepsilon > 0$, there exists a $k \in N$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq k$, that is, if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

1.6. Proposition ([8]). Let (X,G) be a G-metric space.Then the following are equivalent:

(1) The sequence $\{x_n\}$ is G-cauchy:
(2) For every $\varepsilon > 0$; there exists a $k \in N$ such that $G(x_n,x_m,x_m) < \varepsilon$;
for all $n,m \geq k$:

1.7. Proposition ([8]). Let (X,G) be a G-metric space.Then $f : X \to X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convegent to x, $f(x_n)$ is G-convergent to $f(x)$.

1.8. Proposition ([8]). Let (X,G) be a G-metric space.Then the function $G(x,y,z)$ is jointly continuous in all the three variables.

1.9. Definition ([8]). A G-metric space (X,G) is called G-complete if every G-cauchy sequence is G-convergent in (X,G).

1.10. Proposition ([8]). Let (X, G) be a G-metric space. Then for any $x, y, z, a \in X$ it follows that:

(1) $G(x, y, z) = 0$, then $x = y = z$;

(2) $G(x, y, z) \geq G(x, x, y) + G(x, x, z)$;

(3) $G(x, y, y) \geq 2G(y, x, x)$;

(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;

(5) $G(x, y, z) \leq \frac{2}{3}[G(x, y, a) + G(x, a, z) + G(a, y, z)]$;

(6) $G(x, y, z) \leq [G(x, a, a) + G(y, a, a) + G(z, a, a)]$.

1.11. Definition. Let (X, G) be a G-metric space and $T, S : X \to X$ be two mappings with $F(T) \cap F(S) \neq \emptyset$. Then T and S have property Q if $F(T^n) \cap F(S^n) = F(T) \cap F(S)$, for each $n \in N$.

1.12. Definition. Let $\{T_i\}$ be a sequence of selfmaps of a G-metric space X. We shall say that this family has property R if $\cap_i F(T_i^n) = \cap_i F(T_i)$, for each $n \in N$.

In this paper we prove a general common fixed point theorem for three operators in G-metric spaces satisfying a nonlinear contraction. Also we prove the uniqueness of the fixed point, as well as studying the G-continuity of the fixed point. Moreover, we show that these maps satisfy properties Q and R. An interesting fact about maps satisfying properties Q and R is that they have no nontrivial periodic points. Some papers dealing with properties Q and R are ([4, 10]). Our results extend some recent works. An illustrative example is also discussed.

2. Fixed Point Theorem

2.1. Theorem. Let (X, G) be a complete G-metric space and let T_1, T_2, T_3 be selfmaps of X satisfying for all $x, y, z \in X$,

$$G(T_1x, T_2y, T_3z) \leq \alpha\,(M(x, y, z)), \tag{2.1}$$

where

$$M(x, y, z) = \max\{G(x, y, z), G(x, x, T_1x), G(y, y, T_2y), G(z, z, T_3z), G(x, x, T_2y), G(z, z, T_1x), G(y, y, T_1x), G(y, y, T_3z), G(z, z, T_2y)\},$$

and $\alpha < 1$. Then T_1, T_2 and T_3 have a unique common fixed point (say u) and each T_i is G-continuous at u.

Proof. We shall first show that, if u is a fixed point of one of the maps, then it is a common fixed point. Suppose that $u \in F(T_1)$. Then, from (2.1),

$$\begin{aligned} G(T_1u, T_2u, T_3u) &\leq \alpha(\max\{G(u, u, u), G(u, u, T_1u), G(u, u, T_2u), \\ &\quad G(u, u, T_3u), G(u, u, T_2u), G(u, u, T_1u), G(u, u, T_1u), \\ &\quad G(u, u, T_3u), G(u, u, T_2u)\}), \end{aligned} \tag{2.2}$$

which implies that

$$G(T_1u, T_2u, T_3u) \leq \alpha(\max\{G(u, u, T_3u), G(u, u, T_2u)\}).$$

If $T_2u \neq T_3u$, then, from properties (G3) and (G4), $G(u, u, T_3u) \leq G(T_2u, u, T_3u)$, and $G(u, u, T_2u) \leq G(T_2u, u, T_3u)$. Therefore, $G(T_1u, T_2u, T_3u) \leq \alpha(G(T_1u, T_2u, T_3u))$, which leads to a contradiction if $G(T_1u, T_2u, T_3u) > 0$. Therefore, $G(T_1u, T_2u, T_3u) = 0$ and from proposition (1.10), $u = T_1u = T_2u = T_3u$.

Suppose that $T_2u = T_3u$. Then (2.2) becomes

$$G(u; T_2u; T_2u) \quad (\max\{0; G(u; T_2u; T_2u)\});$$

which implies that $u = T_2u$, and therefore u is a common fixed point of $T_1; T_2$, and T_3.

A similar proof shows that, if u is a fixed point of T_2 or T_3, then it is a common fixed point.

Let x_0 be an arbitrary point in X. Take $x_1 = T_1x_0; x_2 = T_2x_1$ and $x_3 = T_3x_2$. We can then construct a sequence $\{x_n\}$ in X such that for any $n \in \mathbb{N}$

$$x_{3n+1} = T_1x_{3n}; \quad x_{3n+2} = T_2x_{3n+1}; \quad x_{3n+3} = T_3x_{3n+2}:$$

If there exists an integer p such that $x_{3p} = x_{3p+1}$, then $x_{3p} \in F(T_1)$, hence $x_{3p} \in F(T_2) \cap F(T_3)$.

Similarly, if there exists an integer p such that $x_{3p+1} = x_{3p+2}$ or $x_{3p+2} = x_{3p+3}$, then we have a common fixed point. Therefore, we shall assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Using (2.1) with $x = x_{3n}; y = x_{3n+1}$ and $z = x_{3n+2}$, we get

$$\begin{aligned}
G(x_{3n+1}; x_{3n+2}; x_{3n+3}) = {} & G(T_1x_{3n}; T_2x_{3n+1}; T_3x_{3n+2}) \\
& (\max\{G(x_{3n}; x_{3n+1}; x_{3n+2}); G(x_{3n}; x_{3n}; T_1x_{3n}); \\
& G(x_{3n+1}; x_{3n+1}; T_2x_{3n+1}); G(x_{3n+2}; x_{3n+2}; T_3x_{3n+2}); \\
& G(x_{3n}; x_{3n}; T_2x_{3n+1}); G(x_{3n+2}; x_{3n+2}; T_1x_{3n}); \\
& G(x_{3n+1}; x_{3n+1}; T_1x_{3n}); G(x_{3n+1}; x_{3n+1}; T_3x_{3n+2}); \\
& G(x_{3n+2}; x_{3n+2}; T_2x_{3n+1})\} \\
= {} & (\max\{G(x_{3n}; x_{3n+1}; x_{3n+2}); G(x_{3n}; x_{3n}; x_{3n+1}); \\
& G(x_{3n+1}; x_{3n+1}; x_{3n+2}); G(x_{3n+2}; x_{3n+2}; x_{3n+3}); \\
& G(x_{3n}; x_{3n}; x_{3n+2}); G(x_{3n+2}; x_{3n+2}; x_{3n+1}); \\
& G(x_{3n+1}; x_{3n+1}; x_{3n+1}); G(x_{3n+1}; x_{3n+1}; x_{3n+3}); \\
& G(x_{3n+2}; x_{3n+2}; x_{3n+2})\}:
\end{aligned}$$

Then, using (G1), (G3) and (G4), we get

$$G(x_{3n+1}; x_{3n+2}; x_{3n+3}) \quad (\max\{G(x_{3n}; x_{3n+1}; x_{3n+2}); G(x_{3n+1}; x_{3n+2}; x_{3n+3})\}:$$

Since the x_i are distinct, the above inequality implies that

$$G(x_{3n+1}; x_{3n+2}; x_{3n+3}) \quad (G(x_{3n}; x_{3n+1}; x_{3n+2})):$$

In a similar way, we obtain

$$G(x_{3n+2}; x_{3n+3}; x_{3n+4}) \quad (G(x_{3n+1}; x_{3n+2}; x_{3n+3})):$$

and

$$G(x_{3n+3}; x_{3n+4}; x_{3n+5}) \quad (G(x_{3n+2}; x_{3n+3}; x_{3n+4})):$$

From the above three inequalities, one can assert that

$$G(x_n; x_{n+1}; x_{n+2}) \quad (G(x_{n-1}; x_n; x_{n+1})):$$

If we define $t_n = G(x_n; x_{n+1}; x_{n+2})$, then $0 \quad t_n \quad t_{n-1}$, so that the sequence $\{t_n\}$ is non-increasing, hence convergent to some $r \quad 0$. Letting $n \to 1$ in (2.4), we have $r \quad (r)$, using the continuity of $$. If $r \neq 0$, we obtain the contradiction $r < (r) < r$. Hence $r = 0$. We rewrite this as

$$\lim_{n \to 1} G(x_n; x_{n+1}; x_{n+2}) = 0:$$

We next prove that $\{x_n\}$ is a G-Cauchy sequence. From (2.4),

$$G(x_n, x_{n+1}, x_{n+2}) \leq (G(x_{n-1}, x_n, x_{n+1})) \leq \cdots \leq \phi^n(G(x_0, x_1, x_2)).$$

Therefore, using conditions (G3), (G4), (G5) and (2.4), we have for any $k \in \mathbb{N}$

$$\begin{aligned}
G(x_n, x_{n+k}, x_{n+k}) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots \\
&\quad + G(x_{n+k-2}, x_{n+k-1}, x_{n+k-1}) + G(x_{n+k-1}, x_{n+k}, x_{n+k}) \\
&\leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots \\
&\quad + G(x_{n+k-2}, x_{n+k-1}, x_{n+k}) + G(x_{n+k-1}, x_{n+k}, x_{n+k+1}) \\
&\leq \phi^n(G(x_0, x_1, x_2)) + \phi^{n+1}(G(x_0, x_1, x_2)) + \cdots \\
&\quad + \phi^{n+k}(G(x_0, x_1, x_2)) \\
&= \sum_{i=n}^{n+k} \phi^i(G(x_0, x_1, x_2)) \\
&\leq \sum_{i=n}^{\infty} \phi^i(G(x_0, x_1, x_2)).
\end{aligned}$$

Using properties of ϕ, $\sum_{i=n}^{\infty} \phi^i(G(x_0, x_1, x_2))$ tends to 0 as $n \to \infty$. Then for any $k \in \mathbb{N}$

$$\lim_{n \to \infty} G(x_n, x_{n+k}, x_{n+k}) = 0,$$

and $\{x_n\}$ is a G-Cauchy sequence. Since X is a G-complete space, $\{x_n\}$ is a G-convergent to some $u \in X$; that is

$$\lim_{n \to \infty} G(x_n, x_n, u) = \lim_{n \to \infty} G(x_n, u, u) = 0.$$

To show that u is a common fixed point of the maps $T_i; i = 1, 2, 3$, it will be sufficient to show that $T_1 u = u$. From (2.1), we have

$$\begin{aligned}
G(T_1 u, x_{3n+2}, x_{3n+3}) &= G(T_1 u, T_2 x_{3n+1}, T_3 x_{3n+2}) \\
&\leq (\max\{G(u, x_{3n+1}, x_{3n+2}), G(u, u, T_1 u), \\
&\quad G(x_{3n+1}, x_{3n+1}, T_2 x_{3n+1}), G(x_{3n+2}, x_{3n+2}, T_3 x_{3n+2}), \\
&\quad G(u, u, T_2 x_{3n+1}), G(x_{3n+2}, x_{3n+2}, T_1 u), \\
&\quad G(x_{3n+1}, x_{3n+1}, T_1 u), G(x_{3n+1}, x_{3n+1}, T_3 x_{3n+2}), \\
&\quad G(x_{3n+2}, x_{3n+2}, T_2 x_{3n+1})\} \\
&= (\max\{G(u, x_{3n+1}, x_{3n+2}), G(u, u, T_1 u), \\
&\quad G(x_{3n+1}, x_{3n+1}, x_{3n+2}), G(x_{3n+2}, x_{3n+2}, x_{3n+3}), \\
&\quad G(u, u, x_{3n+2}), G(x_{3n+2}, x_{3n+2}, T_1 u), \\
&\quad G(x_{3n+1}, x_{3n+1}, T_1 u), G(x_{3n+1}, x_{3n+1}, x_{3n+3}), \\
&\quad G(x_{3n+2}, x_{3n+2}, x_{3n+2})\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
G(T_1 u, x_{3n+2}, x_{3n+3}) &\leq (\max\{G(u, x_{3n+1}, x_{3n+2}), G(T_1 u, u, u), \\
&\quad G(x_{3n+1}, x_{3n+1}, x_{3n+2}), G(x_{3n+2}, x_{3n+2}, x_{3n+3}), \\
&\quad G(u, u, x_{3n+2}), G(T_1 u, x_{3n+2}, x_{3n+2}), \\
&\quad G(T_1 u, x_{3n+1}, x_{3n+1}), G(x_{3n+1}, x_{3n+1}, x_{3n+3})\}.
\end{aligned}$$

Taking the limit as $n \to \infty$, we get $G(T_1 u, u, u) \leq (G(T_1 u, u, u))$, which implies that $T_1 u = u; T_2 u = u = T_3 u$, and u is a common fixed point of the three maps T_1, T_2 and T_3.

Let v be another common fixed point of each $T_i; i = 1, 2, 3$. Using (2.1)

$$\begin{aligned} G(u,u,v) &= G(T_1u, T_2u, T_3v) \\ &\quad (\max\{G(u,u,v), G(u,u,T_1u), G(u,u,T_2u), \\ &\quad G(v,v,T_3v), G(u,u,T_2u), G(v,v,T_1u), \\ &\quad G(u,u,T_1u), G(u,u,T_3v), G(v,v,T_2u)\} \\ &= (\max\{G(u,u,v), 0, 0, 0, 0, G(v,v,u), 0, \\ &\quad G(u,u,v), G(v,v,u)\}. \end{aligned}$$

This implies that

$$G(u,u,v) \quad (\max\{G(u,u,v), G(v,v,u)\}.$$

Then either

(1) $G(u,u,v) > G(v,v,u)$;

(2) $G(u,u,v) \quad G(v,v,u)$.

If (1) is true, then

$$G(u,u,v) \quad (G(u,u,v)) < G(u,u,v);$$

which implies that $u = v$,and u is the unique common fixed point of T_1, T_2 and T_3. If (2) is true, then we have

$$G(u,u,v) \quad G(v,v,u).$$

Now, again using (2.1) with $x = v, y = z = u$, we get

$$\begin{aligned} G(v,v,u) &= G(T_1v, T_2v, T_3u) \\ &\quad (\max\{G(v,v,u), G(v,v,T_1v), G(v,v,T_2v), \\ &\quad G(u,u,T_3u), G(v,v,T_2v), G(u,u,T_1v), \\ &\quad G(v,v,T_1v), G(v,v,T_3u), G(u,u,T_2v)\} \\ &= (\max\{G(v,v,u)G(u,u,v)\}. \end{aligned}$$

Then either

(3) $G(v,v,u) > G(u,u,v)$;

(4) $G(v,v,u) \quad G(u,u,v)$.

If (3) is true, then

$$G(v,v,u) \quad (G(v,v,u)) < G(v,v,u);$$

which implies that $u = v$,and u is the unique common fixed point of T_1, T_2 and T_3. If (4) is true, then we have

$$G(v,v,u) \quad G(u,u,v).$$

Combining (2.6) and (2.7), we get $G(v,v,u) = G(u,u,v)$. Using it in (2.5), we get that u is the unique common fixed point of T_1, T_2 and T_3.

We shall now show that each $T_i; i = 1, 2, 3$, is G-continuous at u. First we prove that T_1 is G-continuous at u. For this, let $\{u_n\}$ X be a sequence which isG-convergent to

u. Using (2.1), we have

$$\begin{aligned} G(T_1u_n;u;u) = {} & G(T_1u_n;T_2u;T_3u) \\ & (\max\{G(u_n;u;u);G(u_n;u_n;T_1u_n);G(u;u;T_2u); \\ & G(u;u;T_3u);G(u_n;u_n;T_2u);G(u;u;T_1u_n); \\ & G(u;u;T_1u_n);G(u;u;T_3u);G(u;u;T_2u)\} \\ = {} & (\max\{G(u_n;u;u);G(u_n;u_n;T_1u_n);0;0; \\ & G(u_n;u_n;u);G(u;u;T_1u_n);0;0\}: \end{aligned}$$

Therefore

$$\begin{aligned} G(T_1u_n;u;u) \quad & (\max\{G(u_n;u;u);G(T_1u_n;u_n;u_n); \\ & G(u_n;u_n;u);G(T_1u_n;u;u)\}: \end{aligned}$$

Taking the limit as $n \to 1$, we get $G(T_1u_n;u;u) = 0$, which implies that $\lim_{n\to 1} T_1u_n = u$ and T_1 is G-continuous at u. Similarly, one can show that each $T_i; i = 2;3$, is G-continuous at u. □

We now give some corollaries of Theorem 2.1. The first corresponds to $(t) = kt$ where $0 \quad k < 1$.

2.2. Corollary. Let $(X;G)$ be a complete G-metric space and let $T_1;T_2;T_3$ be selfmaps of X satisfying for all $x;y;z \in X$,

$$G(T_1x;T_2y;T_3z) \quad k(M(x;y;z));$$

where

$$\begin{aligned} M(x;y;z) = \max\{ & G(x;y;z);G(x;x;T_1x);G(y;y;T_2y);G(z;z;T_3z); \\ & G(x;x;T_2y);G(z;z;T_1x);G(y;y;T_1x);G(y;y;T_3z);G(z;z;T_2y)\}; \end{aligned}$$

where k is a constant satisfying $0 \quad k < 1$. Then $T_1;T_2$ and T_3 have a unique common fixed point (say u) and each T_i is G-continuous at u.

2.3. Corollary. Let $(X;G)$ be a complete G-metric space and let $T_1;T_2;T_3$ be selfmaps of X satisfying for all $x;y;z \in X$,

$$G(T_1^m x;T_2^m y;T_3^m z) \quad (M(x;y;z));$$

where

$$\begin{aligned} M(x;y;z) = \max\{ & G(x;y;z);G(x;x;T_1^m x);G(y;y;T_2^m y);G(z;z;T_3^m z); \\ & G(x;x;T_2^m y);G(z;z;T_1^m x);G(y;y;T_1^m x);G(y;y;T_3^m z);G(z;z;T_2^m y)\}; \end{aligned}$$

and 2. Then $T_1;T_2$ and T_3 have a unique common fixed point (say u) and each T_i is G-continuous at u.

Proof. From Theorem 2.1, we conclude that the maps $T_1^m;T_2^m$ and T_3^m have a unique common fixed point say u. For any $i = 1;2;3; T_iu = T_i(T_i^m u) = T_i^{m+1}u = T_i^m(T_iu)$, meaning that T_iu is also a fixed point of T_i^m. By the uniqueness of u, we get $T_iu = u$. □

By taking $T_1 = S$ and $T_2 = T_3 = T$ in Theorem 2.1, we obtain the following result:

2.4. Corollary. Let $(X; G)$ be a complete G-metric space and let $S; T$ be selfmaps of X satisfying for all $x; y; z \in X$,

$$G(Sx; Ty; Tz) \quad (M(x; y; z));$$

where

$$M(x; y; z) = \max\{G(x; y; z); G(x; x; Sx); G(y; y; Ty); G(z; z; Tz);$$
$$G(x; x; Ty); G(z; z; Sx); G(y; y; Sx); G(y; y; Tz); G(z; z; Ty)\};$$

and $\quad 2$. Then $S; T$ have a unique common xed point (say u). Also S and T are G-continuous at u.

By taking $T = T_1 = T_2 = T_3$ in Theorem 2.1 and in the above Corollaries, we obtain the following results:

2.5. Corollary. Let $(X; G)$ be a complete G-metric space and let T be selfmap of X satisfying for all $x; y; z \in X$,

$$G(Tx; Ty; Tz) \quad (M(x; y; z));$$

where

$$M(x; y; z) = \max\{G(x; y; z); G(x; x; Tx); G(y; y; Ty); G(z; z; Tz);$$
$$G(x; x; Ty); G(z; z; Tx); G(y; y; Tx); G(y; y; Tz); G(z; z; Ty)\};$$

and $\quad 2$. Then T has a unique xed point (say u). Also T is G-continuous at u.

2.6. Corollary. Let $(X; G)$ be a complete G-metric space and let T be selfmap of X satisfying for all $x; y; z \in X$,

$$G(Tx; Ty; Tz) \quad k(M(x; y; z));$$

where

$$M(x; y; z) = \max\{G(x; y; z); G(x; x; Tx); G(y; y; Ty); G(z; z; Tz);$$
$$G(x; x; Ty); G(z; z; Tx); G(y; y; Tx); G(y; y; Tz); G(z; z; Ty)\};$$

where k is a constant satisfying $0 \quad k < 1$. Then T has a unique xed point (say u). Also T is G-continuous at u.

2.7. Remark. Special cases of Corollary 2.6 are Theorems 2.1, 2.2 of [2], Theorems 2.1, 2.3, 2.5, 2.8, 2.9 of [5],Theorem 2.3 of [7] and Theorems 2.1, 2.4, 2.6, 2.8 of [9].

2.8. Example. Let $X = [0; 1]$ with G-metric de ned by $G(x; y; z) = |x - y| + |y - z| + |z - x|$. De ne $: R \to R$ as $(t) = \frac{2t}{3}$ and $T_1; T_2; T_3 : X \to X$ by

$$T_1x = \begin{cases} \frac{x}{15} & \text{if } x \in [0; \frac{1}{3}) \\ \frac{x}{8} & \text{if } x \in [\frac{1}{3}; 1] \end{cases};$$

$$T_2x = \begin{cases} \frac{x}{10} & \text{if } x \in [0; \frac{1}{3}) \\ \frac{x}{4} & \text{if } x \in [\frac{1}{3}; 1] \end{cases};$$

$$T_3x = \begin{cases} \frac{x}{5} & \text{if } x \in [0; \frac{1}{3}) \\ \frac{x}{2} & \text{if } x \in [\frac{1}{3}; 1] \end{cases}.$$

Now, we prove that the mappings $T_1; T_2$ and T_3 satisfy condition (2.1). We discuss the following eight cases:

Case-I: If $x; y; z \in [0; \frac{1}{3})$, then

$$M(x; y; z) = \max\left\{\max\left\{|x - y|, |y - z|, |z - x|\right\}, \frac{14x}{15}, \frac{9y}{10}, \frac{4z}{5}, \right.$$
$$\left. x + \frac{y}{10}, z + \frac{x}{15}, y + \frac{x}{15}, y + \frac{z}{5}, z + \frac{y}{10}\right\}.$$

So,

$$\begin{aligned} G(T_1x; T_2y; T_3z) &= \max\left\{\frac{x}{15}, \frac{y}{10}, \frac{y}{10}, \frac{z}{5}, \frac{z}{5}, \frac{x}{15}\right\} \\ &= \frac{1}{5}\max\left\{\frac{x}{3}, \frac{y}{2}, \frac{y}{2}, z, z, \frac{x}{3}\right\} \\ &\quad \frac{2}{3}\max\left\{\max\left\{|x - y|, |y - z|, |z - x|\right\}, \frac{14x}{15}, \frac{9y}{10}, \frac{4z}{5}, \right. \\ &\quad \left. x + \frac{y}{10}, z + \frac{x}{15}, y + \frac{x}{15}, y + \frac{z}{5}, z + \frac{y}{10}\right\} \\ &= \frac{2}{3}M(x; y; z) \\ &= (M(x; y; z)). \end{aligned}$$

Case-II: If $x; y; z \in [\frac{1}{3}; 1]$, then

$$M(x; y; z) = \max\left\{\max\left\{|x - y|, |y - z|, |z - x|\right\}, \frac{7x}{8}, \frac{3y}{4}, \frac{z}{2}, \right.$$
$$\left. x + \frac{y}{4}, z + \frac{x}{8}, y + \frac{x}{8}, y + \frac{z}{2}, z + \frac{y}{4}\right\}.$$

So,

$$\begin{aligned} G(T_1x; T_2y; T_3z) &= \max\left\{\frac{x}{8}, \frac{y}{4}, \frac{y}{4}, \frac{z}{2}, \frac{z}{2}, \frac{x}{8}\right\} \\ &= \frac{1}{2}\max\left\{\frac{x}{4}, \frac{y}{2}, \frac{y}{2}, z, z, \frac{x}{4}\right\} \\ &\quad \frac{2}{3}\max\left\{\max\left\{|x - y|, |y - z|, |z - x|\right\}, \frac{7x}{8}, \frac{3y}{4}, \frac{z}{2}, \right. \\ &\quad \left. x + \frac{y}{4}, z + \frac{x}{8}, y + \frac{x}{8}, y + \frac{z}{2}, z + \frac{y}{4}\right\} \\ &= \frac{2}{3}M(x; y; z) \\ &= (M(x; y; z)). \end{aligned}$$

Case-III: If $x \in [0; \frac{1}{3})$ and $y; z \in [\frac{1}{3}; 1]$, then

$$M(x; y; z) = \max\left\{\max\left\{|x - y|, |y - z|, |z - x|\right\}, \frac{14x}{15}, \frac{3y}{4}, \frac{z}{2}, \right.$$
$$\left. x + \frac{y}{4}, z + \frac{x}{15}, y + \frac{x}{15}, y + \frac{z}{2}, z + \frac{y}{4}\right\}.$$

So,

$$\begin{aligned}
G(T_1x; T_2y; T_3z) &= \max \left\{ \frac{x}{15}, \frac{y}{4}, \frac{y}{4}, \frac{z}{2}, \frac{z}{2}, \frac{x}{15} \right\} \\
&= \frac{1}{2} \max \left\{ \frac{2x}{15}, \frac{y}{2}, \frac{y}{2}, z, z, \frac{2x}{15} \right\} \\
&\quad \frac{2}{3} \max \left\{ \max\{x, y\}, \max\{y, z\}, \max\{x, \frac{14x}{15}, \frac{3y}{4}, \frac{z}{2}\}, \right. \\
&\qquad \left. x, \frac{y}{4}, z, \frac{x}{15}, y, \frac{x}{15}, y, \frac{z}{2}, z, \frac{y}{4} \right\} \\
&= \frac{2}{3} M(x; y; z) \\
&= (M(x; y; z)).
\end{aligned}$$

Case-IV: If $y \in [0; \frac{1}{3})$ and $x; z \in [\frac{1}{3}; 1]$, then

$$\begin{aligned}
M(x; y; z) = \max \{ & \max\{x, y\}, \max\{y, z\}, \max\{x, \frac{7x}{8}, \frac{9y}{10}, \frac{z}{2}\}, \\
& x, \frac{y}{10}, z, \frac{x}{8}, y, \frac{x}{8}, y, \frac{z}{2}, z, \frac{y}{10} \}.
\end{aligned}$$

So,

$$\begin{aligned}
G(T_1x; T_2y; T_3z) &= \max \left\{ \frac{x}{8}, \frac{y}{10}, \frac{y}{10}, \frac{z}{2}, \frac{z}{2}, \frac{x}{8} \right\} \\
&= \frac{1}{2} \max \left\{ \frac{x}{4}, \frac{y}{5}, \frac{y}{5}, z, z, \frac{x}{4} \right\} \\
&\quad \frac{2}{3} \max \left\{ \max\{x, y\}, \max\{y, z\}, \max\{x, \frac{7x}{8}, \frac{9y}{10}, \frac{z}{2}\}, \right. \\
&\qquad \left. x, \frac{y}{10}, z, \frac{x}{8}, y, \frac{x}{8}, y, \frac{z}{2}, z, \frac{y}{10} \right\} \\
&= \frac{2}{3} M(x; y; z) \\
&= (M(x; y; z)).
\end{aligned}$$

Case-V: If $z \in [0; \frac{1}{3})$ and $x; y \in [\frac{1}{3}; 1]$, then

$$\begin{aligned}
M(x; y; z) = \max \{ & \max\{x, y\}, \max\{y, z\}, \max\{x, \frac{7x}{8}, \frac{3y}{4}, \frac{4z}{5}\}, \\
& x, \frac{y}{4}, z, \frac{x}{8}, y, \frac{x}{8}, y, \frac{z}{5}, z, \frac{y}{4} \}.
\end{aligned}$$

So,

$$\begin{aligned}
G(T_1x; T_2y; T_3z) &= \max \left\{ \frac{x}{8}, \frac{y}{4}, \frac{y}{4}, \frac{z}{5}, \frac{z}{5}, \frac{x}{8} \right\} \\
&= \frac{1}{4} \max \left\{ \frac{x}{2}, y, y, \frac{4z}{5}, \frac{4z}{5}, \frac{x}{2} \right\} \\
&\quad \frac{2}{3} \max \left\{ \max\{x, y\}, \max\{y, z\}, \max\{x, z\}, \frac{7x}{8}, \frac{3y}{4}, \frac{4z}{5}, \right. \\
&\qquad \left. x, \frac{y}{4}, z, \frac{x}{8}, y, \frac{x}{8}, y, \frac{z}{5}, z, \frac{y}{4} \right\} \\
&= \frac{2}{3} M(x; y; z) \\
&= (M(x; y; z)).
\end{aligned}$$

Case-VI: If $x, y \in [0, \frac{1}{3})$ and $z \in [\frac{1}{3}, 1]$, then

$$\begin{aligned}
M(x; y; z) = \max &\left\{ \max\{x, y\}, \max\{y, z\}, \max\{x, z\}, \frac{14x}{15}, \frac{9y}{10}, \frac{z}{2}, \right. \\
&\left. x, \frac{y}{10}, z, \frac{x}{15}, y, \frac{x}{15}, y, \frac{z}{2}, z, \frac{y}{10} \right\}.
\end{aligned}$$

So,

$$\begin{aligned}
G(T_1x; T_2y; T_3z) &= \max \left\{ \frac{x}{15}, \frac{y}{10}, \frac{y}{10}, \frac{z}{2}, \frac{z}{2}, \frac{x}{15} \right\} \\
&= \frac{1}{2} \max \left\{ \frac{2x}{15}, \frac{y}{5}, \frac{y}{5}, z, z, \frac{2x}{15} \right\} \\
&\quad \frac{2}{3} \max \left\{ \max\{x, y\}, \max\{y, z\}, \max\{x, z\}, \frac{14x}{15}, \frac{9y}{10}, \frac{z}{2}, \right. \\
&\qquad \left. x, \frac{y}{10}, z, \frac{x}{15}, y, \frac{x}{15}, y, \frac{z}{2}, z, \frac{y}{10} \right\} \\
&= \frac{2}{3} M(x; y; z) \\
&= (M(x; y; z)).
\end{aligned}$$

Case-VII: If $x, z \in [0, \frac{1}{3})$ and $y \in [\frac{1}{3}, 1]$, then

$$\begin{aligned}
M(x; y; z) = \max &\left\{ \max\{x, y\}, \max\{y, z\}, \max\{x, z\}, \frac{14x}{15}, \frac{3y}{4}, \frac{4z}{5}, \right. \\
&\left. x, \frac{y}{4}, z, \frac{x}{15}, y, \frac{x}{15}, y, \frac{z}{5}, z, \frac{y}{4} \right\}.
\end{aligned}$$

So,

$$\begin{aligned} G(T_1x; T_2y; T_3z) &= \max \left\{ \frac{x}{15}, \frac{y}{4}, \frac{y}{4}, \frac{z}{5}, \frac{z}{5}, \frac{x}{15} \right\} \\ &= \frac{1}{4}\max \left\{ \frac{4x}{15}, y, y, \frac{4z}{5}, \frac{4z}{5}, \frac{4x}{15} \right\} \\ &\quad \frac{2}{3}\max \left\{ \max\{x, y\}, \{y, z\}, \{z, x\}, \frac{14x}{15}, \frac{3y}{4}, \frac{4z}{5} \right\}; \\ &\qquad \left\{ x, \frac{y}{4}, z, \frac{x}{15}, y, \frac{x}{15}, y, \frac{z}{5}, z, \frac{y}{4} \right\} \\ &= \frac{2}{3}M(x; y; z) \\ &= (M(x; y; z)). \end{aligned}$$

Case-VIII: If $y, z \in [0, \frac{1}{3})$ and $x \in [\frac{1}{3}, 1]$, then

$$\begin{aligned} M(x; y; z) = \max \Big\{ &\max\{x, y\}, \{y, z\}, \{z, x\}, \frac{7x}{8}, \frac{9y}{10}, \frac{4z}{5}; \\ &x, \frac{y}{10}, z, \frac{x}{8}, y, \frac{x}{8}, y, \frac{z}{5}, z, \frac{y}{10} \Big\}. \end{aligned}$$

So,

$$\begin{aligned} G(T_1x; T_2y; T_3z) &= \max \left\{ \frac{x}{8}, \frac{y}{10}, \frac{y}{10}, \frac{z}{5}, \frac{z}{5}, \frac{x}{8} \right\} \\ &= \frac{1}{5}\max \left\{ \frac{5x}{8}, \frac{y}{2}, \frac{y}{2}, z, z, \frac{5x}{8} \right\} \\ &\quad \frac{2}{3}\max \left\{ \max\{x, y\}, \{y, z\}, \{z, x\}, \frac{7x}{8}, \frac{9y}{10}, \frac{4z}{5}; \right. \\ &\qquad \left. x, \frac{y}{10}, z, \frac{x}{8}, y, \frac{x}{8}, y, \frac{z}{5}, z, \frac{y}{10} \right\} \\ &= \frac{2}{3}M(x; y; z) \\ &= (M(x; y; z)). \end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied for all $x, y, z \in X$ and 0 is the unique common fixed point of T_1, T_2 and T_3.

3. Properties Q and R

In this section, we shall show that maps satisfying the conditions of theorem (2.1) and Corollary (2.3) possess properties R and Q respectively.

3.1. Theorem. Under the conditions of Theorem 2.1, T_1, T_2 and T_3 have Property R.

Proof. From Theorem 2.1, T_1, T_2 and T_3 have a fixed point. Therefore, $(F(T_1^n) \cap F(T_2^n) \cap F(T_3^n))$ is non empty for each $n \in N$. Let $n > 1$ and suppose that $p \in F(T_1^n) \cap F(T_2^n) \cap F(T_3^n)$. We claim that $p \in F(T_1) \cap F(T_2) \cap F(T_3)$. To prove this, it is sufficient to show

that p is a fixed point of T_1. Suppose that $T_1 p \neq p$. Then using (2.1)

$$\begin{aligned} G(T_1p;p;p) = {} & G(T_1^{n+1}p;T_2^n p;T_3^n p) \\ & (\max\{G(T_1^n p;T_2^{n-1}p;T_3^{n-1}p);G(T_1^n p;T_1^n p;T_1^{n+1}p); \\ & G(T_2^{n-1}p;T_2^{n-1}p;T_2^n p);G(T_3^{n-1}p;T_3^{n-1}p;T_3^n p); \\ & G(T_1^n p;T_1^n p;T_2^n p);G(T_3^{n-1}p;T_3^{n-1}p;T_1^{n+1}p); \\ & G(T_2^{n-1}p;T_2^{n-1}p;T_1^{n+1}p);G(T_2^{n-1}p;T_2^{n-1}p;T_3^n p); \\ & G(T_3^{n-1}p;T_3^{n-1}p;T_2^n p)\} \\ = {} & (\max\{G(p;T_2^{n-1}p;T_3^{n-1}p);G(p;p;T_1p);G(T_2^{n-1}p;T_2^{n-1}p;p); \\ & G(T_3^{n-1}p;T_3^{n-1}p;p);G(p;p;p);G(T_3^{n-1}p;T_3^{n-1}p;T_1p); \\ & G(T_2^{n-1}p;T_2^{n-1}p;T_1p);G(T_2^{n-1}p;T_2^{n-1}p;p);G(T_3^{n-1}p;T_3^{n-1}p;p)\}). \end{aligned}$$

If $p = T_2^{n-1}p$, then $G(T_2^{n-1}p;T_2^{n-1}p;p) = 0$. If $p \neq T_2^{n-1}p$, then from (G3) and (G4),

$$G(T_2^{n-1}p;T_2^{n-1}p;p) \quad G(T_2^{n-1}p;p;T_3^{n-1}p).$$

Similarly, either

$$G(T_3^{n-1}p;T_3^{n-1}p;p) = 0 \text{ or}$$

$$G(T_3^{n-1}p;T_3^{n-1}p;p) \quad G(p;T_2^{n-1}p;T_3^{n-1}p);$$

$$G(T_2^{n-1}p;T_2^{n-1}p;T_1p) = 0 \text{ or}$$

$$G(T_2^{n-1}p;T_2^{n-1}p;T_1p) \quad G(T_1p;T_2^{n-1}p;T_3^{n-1}p);$$

and

$$G(T_3^{n-1}p;T_3^{n-1}p;T_1p) = 0 \text{ or}$$

$$G(T_3^{n-1}p;T_3^{n-1}p;T_1p) \quad G(T_1p;T_2^{n-1}p;T_3^{n-1}p).$$

Therefore

$$G(T_1p;p;p) \quad (\max\{G(T_1p;p;p);G(p;T_2^{n-1}p;T_3^{n-1}p);G(T_1p;T_2^{n-1}p;T_3^{n-1}p)\}).$$

Then either

$$G(T_1p;p;p) \quad (G(T_1p;p;p));$$

$$\text{or } G(T_1p;p;p) \quad (G(p;T_2^{n-1}p;T_3^{n-1}p));$$

$$\text{or } G(T_1p;p;p) \quad (G(T_1p;T_2^{n-1}p;T_3^{n-1}p)).$$

The condition

$$G(T_1p;p;p) \quad (G(p;T_2^{n-1}p;T_3^{n-1}p));$$

implies that

$$\begin{aligned} G(T_1p;p;p) = {} & G(T_1^{n+1}p;T_2^n p;T_3^n p) \\ & (G(T_1^n p;T_2^{n-1}p;T_3^{n-1}p)) \quad :::: \quad (G(T_1p;p;p)); \end{aligned}$$

and if

$$G(T_1p;p;p) \quad (G(T_1p;T_2^{n-1}p;T_3^{n-1}p));$$

implies that

$$G(T_1p; p; p) = G(T_1p; T_2^n p; T_3^n p)$$
$$(G(T_1p; T_2^{n-1}p; T_3^{n-1}p)) \quad :::: \quad (G(T_1p; p; p)):$$

Thus in either case, we obtain

$$G(T_1p; p; p) \quad (G(T_1p; p; p));$$

which is a contradiction. Hence $T_1p = p$. Then, by Theorem 2.1, $T_2p = T_3p = p$. Therefore, $p \in F(T_1) \setminus F(T_2) \setminus F(T_3)$, and $T_1; T_2$ and T_3 satisfy property R.

□

3.2. Corollary. Under the conditions of Corollary 2.3, S and T have Property Q.

3.3. Remark. For $T_1 = T_2 = T_3 = T$ in Theorem 3.1, we obtain Theorems 3.1, 3.2 of [2] as special cases of our result.

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On the sum of best simultaneously proximinal subspaces

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Abstract

Let X be a Banach space and G a subspace of X. A point $g_0 \in G$ is said to be a best simultaneous approximation for a bounded set $A \subseteq X$ if $d(A,G) = \inf_{g\in G} \sup_{a\in A} \|a-g\| = \sup_{a\in A} \|a-g_0\|$. In this paper we prove that if F and G are two subspaces of a Banach space X such that G is reflexive and F is simultaneously proximinal, then $F+G$ is simultaneously proximinal provided that $F \cap G$ is finite dimensional and $F+G$ is closed. Some other related results are also presented.

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1. Introduction

Let X be a normed space and G a closed subspace of X. For $x \in X$, a point $g_0 \in G$ satisfying $\|x-g_0\| = d(x,G) = \inf_{g\in G} \|x-g\|$ is called a point of best approximation to x from G. The set $P_G(x) = \{g \in G : \|x-g\| = d(x,G)\}$ is called the set of all best approximations to x from G. The set G is called proximinal in X if for each $x \in X$, $P_G(x) \neq \phi$, see [12] . Note that for $x \in \overline{G}$, $d\left(x,\overline{G}\right) = 0$. Consequently if G is proximinal it follows that G is closed. If a bounded set A is given in X one might like to approximate all elements of A simultaneously by a single element of G. This type of problem arises when a function being approximated is not known precisely but it is known to belong to

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a set. For the definition regarding simultaneous approximation one may refer to Narang [12] .

1.1. Definition. Let G be a closed subspace of a normed space X and A be a bounded subset of X. A point $g_0 \in G$ is called a point of simultaneous approximation to A from G if

$$d(A,G) = \inf_{g\in G}\sup_{a\in A}\|a-g\| = \sup_{a\in A}\|a-g_0\|.$$

A subspace $G \subset X$ is called simultaneously proximinal if every bounded set $A \subset X$ admits a point of best simultaneous approximation.

Taking the set A to be a singleton it follows that simultaneously proximinal subspaces are proximinal and hence closed.

The problem of best simultaneous approximation has been studied by many authors [1, 11, 12, 13, 14, 15]. Most of these works deal with characterization of best simultaneous approximation in spaces of continuous functions with values in a Banach space X. In [11], some existence and uniqueness of best simultaneous approximation of bounded functions with values in a uniformly convex Banach space by corresponding continuous function were obtained. In [13], some results were obtained in the spaces of $L^p(I,X)$, $1 \le p < \infty$. In [5], Cheney and Wulbert raised the question:

Problem: If F and G are proximinal subspaces of a Banach space X and $F+G$ is closed, does it follow that $F+G$ is proximinal in X ?

In [7] , Feder gave a negative answer to this problem. Further Feder proved that if G is reflexive and F is proximinal such that $E \cap F$ is finite dimensional then $F+G$ is proximinal. In [10], P. K. Lin proved that a Banach space G is reflexive if and only if for every Banach space X with $G \subseteq X$, F is proximinal in X and $G+F$ is closed implies $G+F$ is proximinal in X. In [2] , it has been proved that if F and G are reflexive proximinal subspaces of a Banach spaces X such that F and G are p-orthogonal, then $L^p(I,F)+L^p(I,G)$ is proximinal in $L^p(I,X)$.

In this paper we generalize Feder's result to simultaneous approximation of bounded sets. Some other related results are presented.

Throughout this paper we write X/Y for the quotient space and the coset in X/Y will be denoted by $\overline{x}$ for $x \in X$, where X is a normed space and X^* the space of all bounded linear functionals on X.

2. Best Simultaneous Approximation

In this section we generalize Feder's result to simultaneous approximation of bounded sets. Some other results concerning simultaneous approximations of the sum of two simultaneously proximinal subspaces will be proved.

2.1. Theorem. *Let F be a simultaneously proximinal subspace of a normed space X and E be a subspace of X. Assume $E+F$ is closed. If $E+F$ is simultaneously proximinal in X, then $(E+F)/F$ is simultaneously proximinal in X/F.*

Proof. Let B be a nonempty bounded subset of X/F. Then $B = H/F$ for some $H \subset X$. We will show that there exists a bounded subset A of X such that $B = A/F$, similar assertion as in Lemma 3.2,[8] . Let $C = \bigcup_{b\in B} b = \bigcup_{h\in H} h+F$. Claim $B = \{\overline{x} = x+F : x \in C\}$. Indeed if $b \in B$, then $b = h_b+F$ for some $h_b \in H$. But F is a subspace. Thus $h_b = h_b+0 \in h_b+F \subset C$. Hence $b = h_b+F \in \{\overline{x} = x+F : x \in C\}$ and $B \subseteq \{\overline{x} = x+F : x \in C\}$. Similarly if $x \in C$, then $x \in h_x+F$ for some $h_x \in H$. This implies that $x = h_x+f_x$ for some $f_x \in F$. Hence $x+F = h_x+f_x+F = h_x+F \in B$. Therefore $\{\overline{x} = x+F : x \in C\} \subseteq B$.

for every $e \in E$ and $f \in F$. Therefore,

$$\begin{aligned}\sup_{a\in A}\|a-(e_1+f_1)\|^p &= \sup_{a\in A}\|a_1+a_2-(e_1+f_1)\|^p \\ &= \sup_{a\in A}\left(\|a_1-e_1\|^p+\|a_2-f_1\|^p\right)\end{aligned}$$

Now using Proposition 6 and inequalities (1) and (2) we get:

$$\begin{aligned}\sup_{a\in A}\left(\|a_1-e_1\|^p+\|a_2-f_1\|^p\right) &= \sup_{a\in A}\|a_1-e_1\|^p+\sup_{a\in A}\|a_2-f_1\|^p \\ &\leq \sup_{a_1\in A_1}\|a_1-e\|^p+\sup_{a_2\in A_2}\|a_2-f\|^p \\ &= \sup_{a\in A}\left(\|a_1-e\|^p+\|a_2-f\|^p\right) \\ &= \sup_{a\in A}\|a_1+a_2-(e+f)\|^p,\end{aligned}$$

for all $e \in E$ and $f \in F$. Hence e_1+f_1 is a best simultaneous approximation of $A \subseteq X$ in $E \oplus F$.

Now we prove the main result of this paper.

2.7. Theorem. *Let F and G be two subspaces of a Banach space X. Assume that F is simultaneously proximinal and G is reflexive such that $F \cap G$ is finite dimensional, and $F+G$ is closed. Then $F+G$ is simultaneously proximinal.*

Proof. First we prove the theorem in the case $F \cap G = \{0\}$. Let A be a bounded subset of X. Then there exists a sequence $\{h_n\}$in F and a sequence $\{g_n\}$ in G such that

$$\sup_{a\in A}\|a-(h_n+g_n)\| \longrightarrow d(A, F+G).$$

Note that $S_n = \sup_{a\in A}\|a-(h_n+g_n)\|$ is a bounded sequence in R, being convergent. Hence there exists $M_1 > 0$ such that $|S_n| \leq M_1$. The inequality

$$\|h_n+g_n\| \leq \|h_n+g_n-a\|+\|a\|,$$

implies that

$$\begin{aligned}\|h_n+g_n\| &\leq \sup_{a\in A}\|h_n+g_n-a\|+\sup_{a\in A}\|a\| \\ &\leq M_1+M_2 = M\end{aligned}$$

and so (h_n+g_n) is a bounded sequence in $F+G$.

Since the projection $P : F+G \longrightarrow G$, $P(h+g) = g$ is bounded and $P(F) = 0$, by the closed graph theorem it follows that:

$$\begin{aligned}\|g_n\| &= \|P(h_n+g_n)\| \\ &\leq \|h_n+g_n\| \leq M,\end{aligned}$$

which implies that g_n is bounded and so is h_n. Since G is reflexive $\{g_n\}$ has a weakly convergent subsequence $g_{n_k} \xrightarrow{w} g_0$ for some $g_0 \in G$. Using Corollary 14 [6, p 422] there is a sequence of convex combinations $\hat{g}_n = \sum_{i\in I_n}\lambda_i g_i$, where $I_n = \{i : p_n < i \leq p_{n+1}\}$, p_n is an increasing sequence of integers $\lambda_i \geq 0$, $\sum_{i\in I_n}\lambda_i = 1$ such that $\|\hat{g}_n - g_0\| \longrightarrow 0$. Now

let $\hat{h}_n = \sum_{i\in I_n} \lambda_i h_i$. Then

$$\begin{aligned}\sup_{a\in A}\left\|a-\left(\hat{h}_n+\hat{g}_n\right)\right\| &= \sup_{a\in A}\left\|\sum_{i\in I_n}\lambda_i\left(a-h_i-g_i\right)\right\| \\ &\leq \sup_{a\in A}\sum_{i\in I_n}\lambda_i\left\|a-h_i-g_i\right\| \\ &\leq \sum_{i\in I_n}\lambda_i\sup_{a\in A}\left\|a-h_i-g_i\right\|.\end{aligned}$$

Hence

$$\sup_{a\in A}\left\|a-\hat{h}_n-\hat{g}_n\right\| \longrightarrow d\left(A,F+G\right).$$

Also

$$\sup_{a\in A}\left\|a-\left(\hat{h}_n+g_0\right)\right\| \leq \sup_{a\in A}\left(\left\|a-\hat{h}_n-\hat{g}_n\right\|+\left\|\hat{g}_n-g_0\right\|\right),$$

implies that

$$\sup_{a\in A}\left\|a-\hat{h}_n-g_0\right\| \longrightarrow d\left(A,F+G\right).$$

Let h_0 be a point of simultaneous approximation to the set $A-g_0$ in F. Then

$$\sup_{a\in A}\left\|a-\left(h_0+g_0\right)\right\| = \sup_{a\in A}\left\|a-h_0-g_0\right\| \leq \sup_{a\in A}\left\|a-\hat{h}_n-g_0\right\| \longrightarrow d\left(A,F+G\right).$$

This implies that

$$\sup_{a\in A}\left\|a-\left(h_0+g_0\right)\right\| = d\left(A,F+G\right).$$

Hence $F+G$ is simultaneously proximinal in X.

Finally if $F\cap G$ is finite dimensional using Corollary 34-10 [4, p 137] , we can find a closed subspace H of G such that $F+G=F+H$ with $F\cap G=\{0\}$.

We note that for two closed subspaces G and H of a Banach space X, $G+H$ need not be closed but if G is of finite dimension then $G+H$ is closed, Indeed let us consider the quotient space X/H, equipped with the quotient norm

$\|.\|_{X/H}$ and the quotient map $\pi : X \to X/H$. Since the linear subspace $V=\pi(G)\subset X/H$ is finite dimensional, it follows that V is closed in X/H. By continuity of the quotient map π, its preimage $\pi^{-1}(V)=G+H$ is closed, see $[3,p\ 160,167]$, $[9,p\ 82]$. As a corollary of Theorem 8, we have:

2.8. Corollary. *Let F and G be two subspaces of a Banach space X. Assume that F is simultaneously proximinal and G is of finite dimension. Then $F+G$ simultaneously proximinal.*

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On property $(gw1)$

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Abstract

In this note we introduce and study the property $(gw1)$, which extend property (gw) introduced by Amouch and Berkani in [7]. We investigate the property $(gw1)$ in connection with Weyl type theorems, and establish for abounded linear operator defined on a Banach space the sufficient and necessary conditions for which property $(gw1)$ holds. We also study the property $(gw1)$ for operators satisfying the single valued extension property (SVEP). Classes of operators are considered as illustrating examples.

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1. Introduction

Throughout this paper, $\mathbf{B}(\mathcal{X})$ denote the algebra of all bounded linear operators acting on a Banach space $\mathcal{X}$. For $T \in \mathbf{B}(\mathcal{X})$, let T^*, $\ker(T)$, $\mathcal{R}(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote respectively the adjoint, the null space, the range, the spectrum, the point spectrum and the approximate point spectrum of T. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by

$$\alpha(T) := dim\ker(T) \qquad \text{and} \qquad \beta(T) := codim\mathcal{R}(T).$$

If the range $\mathcal{R}(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator. In the sequel $SF_+(\mathcal{X})$ (resp. $SF_-(\mathcal{X})$) will denote the set of all upper (resp. lower) semi-Fredholm operators. If $T \in \mathbf{B}(\mathcal{X})$ is either upper or lower semi- Fredholm, then T is called a semi-Fredholm operator, and the index of T is defined by

$$ind(T) = \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is a Fredholm operator. An operator T is called Weyl if it is Fredholm of index zero.
Let $a := asc(T)$ be the ascent of an operator T ; i.e., the smallest nonnegative integer p such that $\ker(T^a) = \ker(T^{a+1})$. If such integer does not exist we put $asc(T) = \infty$. Analogously, let $d := dsc(T)$ be descent of an operator T ; i.e., the smallest nonnegative integer d such that $\mathcal{R}(T^d) = \mathcal{R}(T^{d+1})$, and if such integer does not exist we put $d(T) = \infty$. It is well known that if $asc(T)$ and $dsc(T)$ are both finite then $asc(T) = dsc(T)$ [23, Proposition 38.3]. Moreover, $0 < asc(T - \lambda I) = dsc(T - \lambda I) < \infty$ precisely when λ is a pole of the resolvent of T , see Heuser [23, Proposition 50.2].

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An operator $T \in \mathbf{B}(\mathcal{X})$ is called Browder if it is Fredholm "of finite ascent and descent". The Weyl spectrum of T is defined by $\sigma_W(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not Weyl}\}$. For $T \in \mathbf{B}(\mathcal{X})$, let $SF_+^-(\mathcal{X}) := \{T \in SF_+(\mathcal{X}) : ind(T) \leq 0\}$. Then the upper Weyl spectrum of T is defined by $\sigma_{SF_+^-}(T) := \left\{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(\mathcal{X})\right\}$. Let $\Delta(T) = \sigma(T) \setminus \sigma_W(T)$ and $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Following Coburn [17], we say that Weyl's theorem holds for $T \in \mathbf{B}(\mathcal{X})$ (in symbols, $T \in \mathcal{W}$) if $\Delta(T) = E^0(T)$, where $E^0(T) = \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$ and that Browder's theorem holds for T (in symbols, $T \in \mathcal{B}$) if $\sigma_b(T) = \sigma_W(T)$, where

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not Browder}\}.$$

Here and elsewhere in this paper, for $K \subset \mathbb{C}$, $isoK$ is the set of isolated points of K.

According to Rakočević [28], an operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy a-Weyl's theorem (in symbols, $T \in a\mathcal{W}$) if $\Delta_a(T) = E_a^0(T)$, where

$$E_a^0(T) = \{\lambda \in iso\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}.$$

It is known [28] that an operator satisfying a- Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in \mathbf{B}(\mathcal{X})$ and a nonnegative integer n define T_n to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_0 = T$). If for some integer n the range space $\mathcal{R}(T^n)$ is closed and T_n is an upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp. a lower) semi- B-Fredholm operator. In this case the index of T is defined as the index of the semi-B-Fredholm operator T_n, see [8]. Moreover, if T_n is a Fredholm operator, then T is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator $T \in \mathbf{B}(\mathcal{X})$ is said to be a B-Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not a B-Weyl operator}\}.$$

Given $T \in \mathbf{B}(\mathcal{X})$, we say that the generalized Weyl's theorem holds for T (and we write $T \in g\mathcal{W}$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T),$$

where $E(T)$ is the set of all isolated eigenvalues of T, and that the generalized Browder's theorem holds for T (in symbols, $T \in g\mathcal{B}$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T),$$

where $\pi(T)$ is the set of all poles of T, see [11, Definition 2.13]. It is known [11, 21] that

$$g\mathcal{W} \subseteq g\mathcal{B} \cap \mathcal{W} \quad \text{and that} \quad g\mathcal{B} \cup \mathcal{W} \subseteq \mathcal{B}.$$

Moreover, given $T \in g\mathcal{B}$, it is clear that $T \in g\mathcal{W}$ if and only if $E(T) = \pi(T)$. Generalized Weyl's theorem has been studied in [6, 12, 9, 10, 11, 19] and the references therein.

Let $SBF_+(\mathcal{X})$ be the class of all upper semi-B-Fredholm operators,

$$SBF_+^-(\mathcal{X}) = \{T \in SBF_+(\mathcal{X}) : ind(T) \leq 0\}.$$

The upper B-Weyl spectrum of T

$$\sigma_{SBF_+^-}(T) := \left\{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_+^-(\mathcal{X})\right\}.$$

We say that T obeys generalized a-Weyl's theorem (in symbols, $T \in ga\mathcal{W}$), if

$$\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E_a(T);$$

where $E_a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ ([11, Definition 2.13]). Generalized a-Weyl's theorem has been studied in [11, 13, 14].

2. Results

We will say that $T \in \mathbf{B}(\mathcal{X})$ has the single-valued extension property at λ_0, (SVEP for short) if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation: $(T - \lambda I)f(\lambda) = 0$, for all $\lambda \in U$ is the function $f \equiv 0$. $T \in \mathbf{B}(\mathcal{X})$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$(see [26]).

2.1. Remark. For $T \in \mathbf{B}(\mathcal{X})$, let $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$ and $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. If T^* has the SVEP, then it is known [24, page 35] that $\sigma(T) = \sigma_a(T)$ and from [5, Theorem 2.9] we have $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. Thus $E^a(T) = E(T)$ and $\Delta_a^g(T) = \Delta^g(T)$.

2.2. Definition. ([28]) A bounded linear operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy property (w) if

$$\Delta^g(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T).$$

2.3. Definition. ([7]) A bounded linear operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy property (gw) if

$$\Delta_a^g(T) = E(T).$$

2.4. Definition. ([16]) A bounded linear operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy property $(w1)$ if

$$\Delta^g(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subseteq E^0(T).$$

2.5. Definition. A bounded linear operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy property $(gw1)$ if

$$\Delta_a^g(T) \subseteq E(T).$$

2.6. Theorem. Let $T \in \mathbf{B}(\mathcal{X})$. If property $(gw1)$ holds for T, then property $(w1)$ holds for T.

Proof. Assume that T satisfies property $(gw1)$ and let $\lambda \in \Delta_a(T)$. Since $\sigma_{SBF_+^-}(T) \subseteq \sigma_{SF_+^-}(T)$, then $\lambda \in \Delta_a^g(T) \subseteq E(T)$. As $\alpha(T - \lambda I) < \infty$, then $\lambda \in E^0(T)$ and $\Delta^g(T) \subseteq E^0(T)$. □

2.7. Theorem. Let $T \in \mathbf{B}(\mathcal{X})$. Then

(1) property (gw) holds for T if and only if T satisfies property $(gw1)$ and $E(T) = \pi(T)$.
(2) property (gw) holds for T if and only if T satisfies property $(gw1)$ and $\sigma_{SBF_+^-}(T) \cap E(T) = \emptyset$.

Proof. (1). Suppose that T has property (gw), then property $(gw1)$ holds for T. Let $\lambda \in E(T)$, then $\lambda \in \Delta_a^g(T)$, thus $T - \lambda I \in SBF_+^-(\mathcal{X})$. Since $\lambda \in iso\sigma(T)$, we know that $T - \lambda I \in g\mathcal{B}$ and hence $\lambda \in \pi(T)$. Conversely, suppose T satisfies property $(gw1)$ and $E(T) = \pi(T)$. Let $\lambda \in E(T)$, which means that $\lambda \in \Delta_a^g(T)$, thus property (gw) holds for T.
(2). Suppose that T has property (gw) and this implies that property $(gw1)$ holds for T, and $\sigma_{SBF_+^-}(T) \cap E(T) = \emptyset$. For the converse, if $\lambda \in E(T)$, $\lambda \notin \sigma_{SBF_+^-}(T)$ since $\sigma_{SBF_+^-}(T) \cap E(T) = \emptyset$. Then $\lambda \in \Delta_a^g(T)$, hence $\Delta_a^g(T) = E(T)$. □

The following example shows that property $(gw1)$ does not implies property (gw) in general.

2.8. Example. Let $S \in \mathbf{B}(\mathcal{X})$ be any quasi-nilpotent operator acting on an infinite dimensional Banach space $\mathcal{X}$ such that $\mathcal{R}(S^n)$ is non-closed for all$n$. Let $T = 0 \oplus S$ defined on the Banach space $\mathcal{X} \oplus \mathcal{X}$. Since $\mathcal{R}(T^n) = \mathcal{R}(S^n)$ is non-closed for all n, then T is not a semi-B-Fredholm operator, so $\sigma_{SBF_+^-}(T) = \{0\}$. Since $\sigma_a(T) = \{0\}$ and $E(T) = \{0\}$, then T does not satisfies property (gw). But T satisfies property $(gw1)$, since $\Delta_a^g(T) = \emptyset \subseteq E(T)$.

The following example shows that property $(gw1)$ does not implies that $\sigma_{SBF_+^-}(T) \cap E(T) = \emptyset$.

2.9. Example. Let $\mathcal{X} = \ell^p$, let $T_1, T_2 \in \mathbf{B}(\mathcal{X})$ be given by

$$T(x_1, x_2, \cdots) := \left(0, \frac{1}{2}x_1, \frac{1}{3}x_2, \frac{1}{4}x_3, \cdots\right) \quad \text{and} \quad T_2 := 0,$$

and let

$$T := \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \in \underline{(}\mathcal{X} \oplus \mathcal{X}).$$

Then

$$\sigma(T) = \sigma_a(T) = \sigma_{BW}(T) = \sigma_{SBF_+^-}(T) = E(T) = \{0\}$$

and

$$\Delta_a^g(T) = \emptyset.$$

Therefore, property $(gw1)$ holds for T but $\sigma_{SBF_+^-}(T) \cap E(T) = \{0\}$.

2.10. Theorem. Let $T \in \mathbf{B}(\mathcal{X})$. Then the following statements are equivalent:

(a) Property $(gw1)$ holds for T;
(b) $\sigma_{SBF_+^-}(T) = \sigma_a(T) \cap \sigma_{BW}(T)$;
(c) $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup E(T)$;
(d) $\Delta_a^g(T) \subseteq \pi(T)$.

Proof. $(a) \Leftrightarrow (b)$. Suppose T has property $(gw1)$. Clearly, $\sigma_{SBF_+^-}(T) \subseteq \sigma_{BW}(T) \cap \sigma_a(T)$. We only need to prove that $\sigma_{SBF_+^-}(T) \supseteq \sigma_{BW}(T) \cap \sigma_a(T)$. Let $\lambda \notin \sigma_{SBF_+^-}(T)$, then $T - \lambda I \in SBF_+^-(\mathcal{X})$, thus $T - \lambda I$ is semi-B-Fredholm and $ind(T - \lambda I) \leq 0$ or $\lambda \in \Delta_a^g(T)$. Since T has property $(gw1)$, we know that if $\lambda \in \Delta_a^g(T)$, $T - \lambda I \in g\mathcal{B}$, which means that $\lambda \notin \sigma_a(T) \cap \sigma_{BW}(T)$. Conversely, let $\lambda \in \Delta_a^g(T)$, since $\sigma_{SBF_+^-}(T) = \sigma_a(T) \cap \sigma_{BW}(T)$, it follows that $T - \lambda I \in g\mathcal{B}$, hence $\lambda \in E(T)$, which means that property $(gw1)$ holds for T.
$(a) \Leftrightarrow (c)$. Suppose T satisfies property $(gw1)$. $\sigma_a(T) \supseteq \sigma_{SBF_+^-}(T) \cup E(T)$ is clear. Let $\lambda \notin \sigma_{SBF_+^-}(T) \cup E(T)$, then $T - \lambda I \in SBF_+^-(\mathcal{X})$ and $ind(T - \lambda I) \leq 0$. If $\alpha(T - \lambda I) = 0$, then $\lambda \notin \sigma_a(T)$; if $\alpha(T - \lambda I) > 0$, then $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, since T satisfies property $(gw1)$, it follows that $\lambda \in E(T)$. It is in contradiction to the fact that $\lambda \notin E(T) \cup \sigma_{SBF_+^-}(T)$. Thus $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup E(T)$. For the converse, if $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup E(T)$, then $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E(T)$, which means that property $(gw1)$ holds for T.
$(a) \Leftrightarrow (d)$. Suppose T has property $gw1$. Let $\lambda \in \Delta_a^g(T)$, then $\lambda \in E(T)$, since $T - \lambda I$ is upper semi-B-Fredholm and $\lambda \in iso\sigma(T)$, we know that $T - \lambda I \in g\mathcal{B}$, hence $\lambda \in \pi(T)$. Conversely, using the fact that $\pi(T) \subseteq E(T)$, if $\Delta_a^g(T) \subseteq \pi(T)$, then T has property $(gw1)$. □

In the following, let $H(T)$ be the class of all complex-valued functions which are analytic on a neighborhood of $\sigma(T)$ and are not constant on any component of $\sigma(T)$.

2.11. Theorem. Let $T \in \mathbf{B}(\mathcal{X})$. Suppose that property $(gw1)$ holds for T. Then the following statements are equivalent:

(i) For any $f \in H(T)$, property $(gw1)$ holds for $f(T)$;
(ii) For any $f \in H(T)$, $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$, and if $\sigma_a(T) \neq \sigma_{SBF_+^-}(T)$, then $\sigma(T) = \sigma_a(T)$;
(ii) For each pair $\lambda, \mu \in \mathbb{C} \setminus \sigma_{SBF_+^-}(T)$, $ind(T-\lambda I)ind(T-\mu I) \geq 0$, and if $\sigma_a(T) \neq \sigma_{SBF_+^-}(T)$, then $\sigma(T) = \sigma_a(T)$.

Proof. $(i) \Rightarrow (ii)$. $\sigma_{SBF_+^-}(f(T)) \subseteq f(\sigma_{SBF_+^-}(T))$ is clear. We need to prove $\sigma_{SBF_+^-}(f(T)) \supseteq f(\sigma_{SBF_+^-}(T))$. Let $\mu_0 \notin \sigma_{SBF_+^-}(f(T))$, then $f(T) - \mu_0 I \in SBF_+^-(\mathcal{X})$. Let

$$f(T) - \mu_0 I = (T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2} \cdots (T-\lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Thus $T - \lambda_j I$ is upper semi-B-Fredholm operator and $\mu_0 \notin \sigma_a(f(T))$ or $\mu_0 \in \Delta_a^g(f(T))$. If $\mu_0 \notin \sigma_a(f(T))$, then $f(T) - \mu_0 I$ is bounded from below, which means that each $T - \lambda_j I$ is bounded from below. Then $\mu_0 \notin f(\sigma_{SBF_+^-}(T))$. If $\mu_0 \in \Delta_a^g(f(T))$, since property $(gw1)$ holds for $f(T)$, we know that $f(T) - \mu_0 I \in g\mathcal{B}$. Hence $T - \lambda_j I \in g\mathcal{B}$ and $\lambda_j \notin \sigma_{SBF_+^-}(T)$. Then $\mu_0 \notin f(\sigma_{SBF_+^-}(T))$. Next we will prove if $\sigma_a(T) \neq \sigma_{SBF_+^-}(T)$, then $\sigma(T) = \sigma_a(T)$. Let $\lambda_0 \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Then $T - \lambda_0 I \in g\mathcal{B}$ because property $(gw1)$ holds for T. For any $\mu_0 \notin \sigma_a(T)$, $\alpha(T - \mu_0 I) = 0$. Let $f(T) = (T-\mu_0 I)(T-\lambda_0 I)$, then $0 \in \sigma_a(f(T)) \setminus \sigma_{SBF_+^-}(f(T))$. Since $f(T)$ has property $(gw1)$, we know that $f(T) \in g\mathcal{B}$. This implies that $f(T) - \mu_0 I \in g\mathcal{B}$. The fact $\alpha(T - \mu_0 I) = 0$ tell us that $T - \mu_0 I$ is invertible, which means that $\mu_0 \notin \sigma(T)$. Hence $\sigma(T) = \sigma_a(T)$.
$(ii) \Rightarrow (i)$. Let $\mu_0 \in \Delta_a^g(f(T))$, then $f(T) - \mu_0 I \in SBF_+^-(\mathcal{X})$ and $\alpha(f(T) - \mu_0 I) > 0$. Let

$$f(T) - \mu_0 I = (T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2} \cdots (T-\lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Since $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$ and $\mu_0 \notin \sigma_{SBF_+^-}(f(T))$, it follows that $\lambda_j \notin \sigma_{SBF_+^-}(T)$. Then $T - \lambda_j I \in SBF_+^-(\mathcal{X})$. Let $\alpha(T - \lambda_j I = 0)$ if $1 \leq j \leq i$ and $\alpha(T - \lambda_j I > 0)$ if $i < j \leq k$. Then $T - \lambda_j I$ is bounded from below if $1 \leq j \leq i$. Using the fact $\sigma(T) = \sigma_a(T)$ we know that $T - \lambda_j I$ is invertible. If $i < j \leq k$, then $\lambda_j \notin \Delta_a^g(T)$, since T has property $(gw1)$, $T - \lambda_j I \in g\mathcal{B}$. Thus $f(T) - \mu_0 I \in g\mathcal{B}$ and $\mu_0 \in E(f(T))$. Hence property $(gw1)$ holds for $f(T)$.
$(i) \Rightarrow (iii)$. Suppose that there exist $\lambda, \mu \in \sigma_{SBF_+^-}(T)$ such that $ind(T-\lambda I)ind(T-\mu I) < 0$. Let $ind(T-\lambda I) = k > 0$, then $T - \lambda I$ is B-Fredholm. If $ind(T-\mu I) = -t < 0$, $t \neq \infty$, then let $f(T) = (T-\lambda I)^t (T-\mu I)^k$; if $ind(T-I) = \infty$, then let $f(T) = (T-\lambda I)(T-\mu I)$. Thus $0 \in \Delta_a^g(f(T))$, since $f(T)$ has property $(gw1)$, we know that $f(T) \in g\mathcal{B}$. Thus $T - \lambda I \in g\mathcal{B}$ and $T - \mu I \in g\mathcal{B}$. It is in contradiction to the fact that $ind(T - \lambda I) > 0$. Hence for each pair $\lambda, \mu \in \mathbb{C} \setminus \sigma_{SBF_+^-}(T)$, $ind(T - \lambda I)ind(T - \mu I) \geq 0$.
$(iii) \Rightarrow (i)$. Let $\mu_0 \notin \Delta_a^g(f(T))$, then $f(T) - \mu_0 \in SBF_+^-(\mathcal{X})$. Let

$$f(T) - \mu_0 I = (T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2} \cdots (T-\lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Then for any λ_j, $T - \lambda_j I$ is upper semi-B-Fredholm and

$$\sum_{j=1}^{k} ind(T-\lambda_j)^{n_j} \leq 0.$$

By condition $ind(T-\lambda_j I)\leq 0$, we get that $T-\lambda_j I\in SBF_+^-(\mathcal{X})$. Since the rest of the proof is similar to the proof of $(ii)\Rightarrow(i)$ we omit it, and hence property $(gw1)$ holds for $f(T)$. □

2.12. Theorem. Let $T\in\mathbf{B}(\mathcal{X})$. Then the following statements are equivalent:

(I) T satisfies property $(gw1)$ and $E(T)=\pi(T)$

(II) T satisfies generalized Weyl's theorem and $ind(T-\lambda I)=0$ for all $\lambda\in\Delta_a^g(T)$.

Proof. $(I)\Rightarrow(II)$. Suppose that T satisfies property (gw) and $E(T)=\pi(T)$. Let $\lambda\in\Delta^g(T)$. Since $\sigma_{SBF_+^-}(T)\subseteq\sigma_{BW}(T)$, then $\lambda\notin\sigma_{SBF_+^-}(T)$. If $\alpha(T-\lambda I)=0$, as $\lambda\notin\sigma_{BW}(T)$, then $T-\lambda I$ will be invertible. But this is impossible since $\lambda\in\sigma(T)$. Hence $\alpha(T-\lambda I)>0$ and $\lambda\in\sigma_a(T)$. As T satisfies property $(gw1)$, then $\lambda\in E(T)$. This implies that $\Delta^g(T)\subseteq E(T)$. To show the opposite inclusion, let $\lambda\in E(T)$ be arbitrary. Since T satisfies property $(gw1)$, and $E(T)=\pi(T)$, then $\lambda\notin\sigma_{SBF_+^-}(T)$ and hence $ind(T-\lambda I)\leq 0$. On the other hand, as $\lambda\in E(T)$, then $\lambda\in iso\sigma(T)$, and hence T^* has the SVEP at λ. By [3], we have $ind(T-\lambda I\geq 0$. So $ind(T-\lambda I)=0$, and $\lambda\notin\sigma_{BW}(T)$. Hence $\sigma(T)\setminus\sigma_{BW}(T)=E(T)$ and $ind(T-\lambda I)=0$ for all $\lambda\in\Delta_a^g(T)$.
$(II)\Rightarrow(I)$. Conversely, assume that T satisfies generalized Weyl's theorem and $ind(T-\lambda I)=0$ for all $\lambda\in\Delta_a^g(T)$. If $\lambda\in\Delta_a^g(T)$, then $T-\lambda I$ is a semi-B-Fredholm operator such that $ind(T-\lambda I)=0$. Hence $T-\lambda I$ is a B-Weyl operator. Since T satisfies generalized Weyl's theorem, then $\lambda\in E(T)$ and hence $\Delta_a^g(T)\subseteq E(T)$. To show $E(T)=\pi(T)$, let $\lambda\in E(T)$, then $T-\lambda I$ is a B-Weyl operator and since $\lambda\in\sigma(T)$, then $\alpha(T-\lambda I)>0$. Thus $\lambda\in\pi(T)$. Consequently T satisfies property $(gw1)$ and $E(T)=\pi(T)$. □

The following example shows that generalized a-Weyl's theorem and generalized Weyl's theorem does not imply property $(gw1)$.

2.13. Example. Let $\mathcal{X}=\ell^2(\mathbb{N})$ and $S\in\mathbf{B}(\mathcal{X})$ be the unilateral right shift and let V defined by

$$V(x_1,x_2,\cdots)=(0,x_2,x_3,\cdots),\quad (x_n)\in\ell^2(\mathbb{N}).$$

If $T=S\oplus V$, then $\sigma(T)=\mathbf{D}(0,1)$ the closed unit disc in $\mathbb{C}$, $iso\sigma(T)=\emptyset$ and $\sigma_a(T)=\mathbf{C}(0,1)\cup\{0\}$, where $\mathbf{C}(0,1)$ is unit circle of $\mathbb{C}$. This implies that

$$\sigma_{SBF_+^-}(T)=\mathbf{C}(0,1)\quad\text{and}\quad\Delta_a^g(T)=\{0\}.$$

Moreover we have $E(T)=\emptyset$ and $E_a(T)=\{0\}$. Hence T satisfies generalized a- Weyl's theorem and so T satisfies generalized Weyl's theorem. But T does not satisfy property $(gw1)$.

2.14. Theorem. Let $T\in\mathbf{B}(\mathcal{X})$. If T^* has the SVEP, then the following statements are equivalent:

(1) Property (gw) holds for T ;
(2) generalized Weyl's theorem holds for T ;
(3) generalized a-Weyl's theorem holds for T;
(4) Property $(gw1)$ holds for T and $E(T)=\pi(T)$

Proof. Suppose that T^* has the SVEP, then as it had been already mentioned by Remark 2.1, we have

$$\sigma_a(T)=\sigma(T),\sigma_{SBF_+^-}(T)=\sigma_{BW}(T),E_a(T)=E(T)$$

and $\Delta_a^g(T)=\Delta^g(T)$. The equivalence between (1), (2), and (3) follows from [7, Theorem 2.8]. Since (1) ⇔ (4) has already been proved, the proof is complete. □

2.15. Theorem. Let $T\in\mathbf{B}(\mathcal{X})$. If T has SVEP then $\sigma_{SBF_+^-}(T^*)=\sigma_{BW}(T)$.

Proof. The inclusion $\sigma_{SBF_+^-}(T^*) \subseteq \sigma_{BW}(T)$ holds for every $T \in \mathbf{B}(\mathcal{X})$. Suppose that $\lambda \notin \sigma_{SBF_+^-}(T^*)$. Then $T^* - \lambda I^* \in SBF_+(\mathcal{X})$ with $ind(T^* - \lambda I^*) \leq 0$. By duality $T - \lambda I$ is lower semi-B-Fredholm and the SVEP of T entails that $asc(T - \lambda I) < \infty$. By [1, Theorem 3.4] we have $ind(T - \lambda I) \leq 0$, thus $ind(T^* - \lambda I^*) = -ind(T - \lambda I) \geq 0$. Therefore, $ind(T^* - \lambda I^*) = ind(T - \lambda I) = 0$, and, again by [1, Theorem 3.4], we have $dsc(T - \lambda I) < \infty$, thus $\lambda \notin \sigma_{BW}(T)$. □

A bounded operator $T \in \mathbf{B}(\mathcal{X})$ is said to be polaroid if $iso\sigma(T) = \emptyset$ or every isolated point of $\sigma(T)$ is a pole of the resolvent of T (see [4]).

2.16. Theorem. Suppose that $T \in \mathbf{B}(\mathcal{X})$. Then the following statements hold:

(i) If T is polaroid and T^* has SVEP then property (gw) holds for T.
(ii) If T is polaroid and T has SVEP then property (gw) holds for T^*.

Proof. (i). Note that by Remark 2.1 we have $\sigma(T) = \sigma_a(T)$. Suppose first that $iso\sigma(T) = \emptyset$. Then $E(T) = \emptyset$. We show that also $\Delta_a^g(T)$ is empty. By Remark 2.1 we have $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T)$ and the last set is empty, since $\sigma(T)$ has no isolated points. Therefore, T satisfies property (gw). Consider the other case, $iso\sigma(T) \neq \emptyset$. Suppose that $\lambda \in E(T)$. Then λ is isolated in $\sigma(T)$ and hence, by the polaroid condition, λ is a pole of the resolvent of T, i.e. $asc(T - \lambda I) = dsc(T - \lambda I) < \infty$. By assumption $\alpha(T - \lambda I) < \infty$, so by [1, Theorem 3.1] $\beta(T - \lambda I) < \infty$,and hence $T - \lambda I$ is a Fredholm operator. Therefore, by 2.1, $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T)$.. Conversely, if $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T)$ then λ is an isolated point of $\sigma(T)$. Clearly, $0 < \alpha(T - \lambda I)$, so $\lambda \in E(T)$ and hence T satisfies property (gw).
(ii). First note that since T has SVEP then $\sigma(T^*) = \sigma(T) = \sigma(T^*)$, see Corollary 2.45 of [1]. Suppose first that $iso\sigma(T) = \sigma(T^*) = \emptyset$. Then $E(T^*) = \emptyset$. By Theorem 2.15 we have $\sigma_a(T^*) \setminus \sigma_{SBF_+^-}(T^*) = \sigma(T) \setminus \sigma_{BW}(T) = \emptyset$, so T^* satisfies property (gw). Suppose that $iso\sigma(T) \neq \emptyset$ and let $\lambda \in E(T^*)$. Then λ is an isolated point of $\sigma(T^*) = \sigma(T)$, hence a pole of the resolvent of T^*, since T^* is polaroid by Theorem 2.5 of [4]. By assumption $\alpha(T^* - \lambda I)^p$ and since the ascent and the descent of $T^* - \lambda I^*$ are both finite it then follows by Theorem 3.1 of [1] that $\beta(T - \lambda I) = \alpha(T - \lambda I) < \infty$, so $T^* - \lambda I^* \in g\mathcal{B}$ and hence also $T - \lambda I \in g\mathcal{B}$. Therefore, $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$ and by Theorem 2.15and Remark 2.1 it then follows that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Conversely, if $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T)$, then λ is an isolated point of the spectrum of $\sigma(T) = \sigma(T^*)$, $T - \lambda I \in g\mathcal{B}$, or equivalently $T^* - \lambda I^* \in g\mathcal{B}$. Since $\alpha(T^* - \lambda I^*) = \beta(T^* - \lambda I^*)$ we then have $\alpha(T^* - \lambda I^*) > 0$ (otherwise $\lambda \notin \sigma(T^*)$). Clearly, $\alpha(T^* - \lambda I^*) > 0$, since by assumption $T^* - \lambda I^* \in SF_+^-(\mathcal{X}^*)$, so that $\lambda \in E(T^*)$. Thus T^* satisfies property (gw). □

2.17. Corollary. Suppose that $T \in \mathbf{B}(\mathcal{X})$. Then the following statements hold:

(i) If T is polaroid and T^* has SVEP then property $(gw1)$ holds for T.
(ii) If T is polaroid and T has SVEP then property $(gw1)$ holds for T^*.

The following example shows that in the statements (i) of Theorem 2.16 and Corollary 2.17 the assumption that T^* has SVEP cannot be replaced by the assumption that T has SVEP.

2.18. Example. Denote by S the unilateral right shift on $\ell^2(\mathbb{N})$ and define

$$V(x_1, x_2, \cdots) = (0, x_2, x_3, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Clearly, V is a quasi-nilpotent operator. Let $T = S \oplus V$. We have $\sigma(T) = \mathbf{D}, D$ the closed unit disc of $\mathbb{C}$, so $iso\sigma(T) = E(T) = \emptyset$ and hence T is polaroid. Moreover,

$\sigma_a(T) = \partial\mathbf{D} \cup \{0\}$. Since $\sigma_a(T)$ does not cluster at λ T has SVEP at 0, as well as at the points $\lambda \notin \sigma_a(T)$. Since T has SVEP at all points $\partial\sigma(T)$ it then follows that T has SVEP. Finally, $\sigma_{SBF_+^-}(T) = \partial\sigma(T)$ so $\Delta_a^g(T) = \{0\} \neq E(T) = \emptyset$, thus T does not satisfy property $(gw1)$ and hence does not satisfy property (gw).

Analogously, in the statements (ii) of Theorem 2.16 and Corollary 2.17 the assumption that T has SVEP cannot be replaced by the assumption that T^* has SVEP.

2.19. Example. Let us consider the left shift $L \in (\ell^2(\mathbb{N}))$, and let V^* be the adjoint of the quasi-nilpotent operator V defined in Example 2.18. We have $L^* = R$, R the unilateral right shift. If we define $W := L \oplus V^*$ then, as observed in Example 2.18 $W^* = R \oplus V$ has SVEP. From Example 2.18 we also know that $\sigma(W) = \overline{\sigma(W^*)} = \mathbf{D}$, so $iso\sigma(W^*) = E(W^*) = \emptyset$ and hence W is polaroid. Moreover, $\sigma_a(W^*) = \partial\mathbf{D} \cup \{0\}$. Finally, $\sigma_{SBF_+^-}(W^*) = \partial\sigma(W^*)$ so $\Delta_a^g(W^*) = \{0\} \neq E(W^*) = \emptyset$, thus W^* does not satisfy property $(gw1)$ and hence does not satisfy property (gw).

2.20. Example. Let $\mathcal{H}$ be a Hilbert space, an operator T acting on $\mathcal{H}$ is said to be paranormal if $\|Tx\|^2 \leq \left\|T^2x\right\| \|x$

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1.5. Definition. ([8]) Let (X, d) be a $b-$metric space. If Y is a nonempty subset of X, then the closure $\overline{Y}$ of Y is the set of limits of all convergent sequences of points in Y, i.e.,

$$\overline{Y} = \left\{ x \in X : \text{there exists a sequence } \{x_n\} \text{ in } Y \text{ such that } \lim_{n\to\infty} x_n = x \right\}.$$

1.6. Definition. ([8]) Let (X, d) be a $b-$metric space. Then a subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, we have $x \in Y$ (i.e., $\overline{Y} = Y$).

1.7. Definition. ([8]) The $b-$metric space (X, d) is complete if every Cauchy sequence in X converges.

In general a $b-$metric function d for $k > 1$ is not jointly continuous in all of its two variables. Following is an example of a $b-$metric which is not continuous.

1.8. Example. ([16]) Let $X = \mathbb{N} \cup \{\infty\}$ and $D : X \times X \to \mathbb{R}$ defined by

$$D(m,n) = \begin{cases} 0, & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right|, & \text{if } m, n \text{ are even or } mn = \infty, \\ 5, & \text{if } m \text{ and } n \text{ are odd and } m \neq n, \\ 2, & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$, we have

$$D(m,p) \leq 3(D(m,n) + D(n,p)).$$

Thus, (X, D) is a $b-$metric space with $k = 3$. If $x_n = 2n$, for each $n \in \mathbb{N}$, then

$$D(2n, \infty) = \frac{1}{2n} \to 0, \text{ as } n \to \infty,$$

that is, $x_n \to \infty$, but $D(x_{2n}, 1) = 2 \nrightarrow D(\infty, 1)$, as $n \to \infty$.

As $b-$metric is not continuous in general, so we need the following simple lemma about the b-convergent sequences.

1.9. Lemma. ([2]) Let (X, d) be a $b-$metric space with $k \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y, respectively. Then, we have

$$\frac{1}{k^2} d(x,y) \leq \liminf_{n\longrightarrow\infty} d(x_n, y_n) \leq \limsup_{n\longrightarrow\infty} d(x_n, y_n) \leq k^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0$. Moreover for each $z \in X$ we have

$$\frac{1}{k} d(x,z) \leq \liminf_{n\longrightarrow\infty} d(x_n, z) \leq \limsup_{n\longrightarrow\infty} d(x_n, z) \leq k d(x,z).$$

Also, we present the following simple lemma needed in the proof of our main result.

1.10. Lemma. Let (X, d) be a $b-$metric space. If there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = t$ for some $t \in X$ then $\lim_{n\to\infty} y_n = t$.

Proof. By a triangle inequality in a $b-$metric space, we have

$$d(y_n, t) \leq k(d(y_n, x_n) + d(x_n, t)).$$

Now, by taking the upper limit when $n \to \infty$ in the above inequality we get

$$\limsup_{n\longrightarrow\infty} d(y_n, t) \leq k(\limsup_{n\longrightarrow\infty} d(x_n, y_n) + \limsup_{n\longrightarrow\infty} d(x_n, t)) = 0.$$

Hence $\lim_{n\to\infty} y_n = t$.

1.11. Definition. ([19]) Let (X, d) be a $b-$metric space. A pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

2. Common fi ed point results

2.1. Theorem. Suppose that f, g, S and T are self mappings on a complete $b-$metric space (X,d) such that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$. If

$$d(fx,gy) \leq \frac{q}{k^4}\max\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{1}{2}(d(Sx,gy)+d(fx,Ty))\}, \tag{2.1}$$

holds for each $x, y \in X$ with $0 < q < 1$, then f, g, S and T have a unique common fixed point in X provided that S and T are continuous and and pairs $\{f,S\}$ and $\{g,T\}$ are compatible.

Proof. Let $x_0 \in X$. As $f(X) \subseteq T(X)$, there exists $x_1 \in X$ such that $fx_0 = Tx_1$. Since $gx_1 \in S(X)$, we can choose $x_2 \in X$ such that $gx_1 = Sx_2$. In general, x_{2n+1} and x_{2n+2} are chosen in X such that $fx_{2n} = Tx_{2n+1}$ and $gx_{2n+1} = Sx_{2n+2}$. Define a sequence $\{y_n\}$ in X such that $y_{2n} = fx_{2n} = Tx_{2n+1}$, and $y_{2n+1} = gx_{2n+1} = Sx_{2n+2}$, for all $n \geq 0$. Now, we show that $\{y_n\}$ is a Cauchy sequence. Consider

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \\
&\leq \frac{q}{k^4}\max\{d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n+1}, Tx_{2n+1}), \\
&\quad \frac{1}{2}(d(Sx_{2n}, gx_{2n+1}) + d(fx_{2n}, Tx_{2n+1}))\} \\
&= \frac{q}{k^4}\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \frac{1}{2}(d(y_{2n-1}, y_{2n+1}) + \\
&\quad + d(y_{2n}, y_{2n}))\} \\
&= \frac{q}{k^4}\max\{(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{d(y_{2n-1}, y_{2n+1})}{2}\} \\
&\leq \frac{q}{k^4}\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{k}{2}(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}))\}.
\end{aligned}$$

If $d(y_{2n}, y_{2n+1}) > d(y_{2n-1}, y_{2n})$ for some n, then from the above inequality we have

$$d(y_{2n}, y_{2n+1}) < \frac{q}{k^3} d(y_{2n}, y_{2n+1}),$$

a contradiction. Hence $d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$ for all $n \in \mathbb{N}$. Also, by the above inequality we obtain

$$d(y_{2n}, y_{2n+1}) \leq \frac{q}{k^3} d(y_{2n-1}, y_{2n}). \tag{2.2}$$

Similarly,

$$d(y_{2n-1}, y_{2n}) \leq \frac{q}{k^3} d(y_{2n-2}, y_{2n-1}). \tag{2.3}$$

From 2.2 and 2.3 we have

$$d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2}),$$

where $\lambda = \frac{q}{k^3} < 1$ and $n \geq 2$. Hence, for all $n \geq 2$, we obtain

$$d(y_n, y_{n-1}) \leq \cdots \leq \lambda^{n-1} d(y_1, y_0). \tag{2.4}$$

So for all $n > m$, we have

$$d(y_n, y_m) \leq k d(y_m, y_{m+1}) + k^2 d(y_{m+1}, y_{m+2}) + \cdots + k^{n-m-1} d(y_{n-1}, y_n).$$

Now from (4), we have

$$\begin{aligned} d(y_n, y_m) &\leq (k\lambda^m + k^2\lambda^{m+1} + \cdots + k^{n-m-1}\lambda^{n-1})d(y_1, y_0) \\ &\leq k\lambda^m[1 + k\lambda + (k\lambda)^2 + \cdots]d(y_1, y_0) \\ &\leq \frac{k\lambda^m}{1-k\lambda}d(y_1, y_0) \end{aligned}$$

On taking limit as $m, n \to \infty$, we have $d(y_n, y_m) \to 0$ as $k\lambda < 1$. Therefore $\{y_n\}$ is a Cauchy sequence. Since X is a complete $b-$metric space, there is some y in X such that

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y.$$

We show that y is a common fixed point of f, g, S and T.

Since S is continuous, therefore $\lim_{n\to\infty} S^2x_{2n+2} = Sy$ and $\lim_{n\to\infty} Sfx_{2n} = Sy$.

Since a pair $\{f, S\}$ is compatible, $\lim_{n\to\infty} d(fSx_{2n}, Sfx_{2n}) = 0$. So by lemma 1.10, we have $\lim_{n\to\infty} fSx_{2n} = Sy$.

Putting $x = Sx_{2n}$ and $y = x_{2n+1}$ in 2.1, we obtain

$$\begin{aligned} d(fSx_{2n}, gx_{2n+1}) \quad \leq \quad & \frac{q}{k^4}\max\{d(S^2x_{2n}, Tx_{2n+1}), d(fSx_{2n}, S^2x_{2n}), (gx_{2n+1}, Tx_{2n+1}), \\ & \frac{1}{2}(d(S^2x_{2n}, gx_{2n+1}) + d(fSx_{2n}, Tx_{2n+1}))\}. \end{aligned} \tag{2.5}$$

Taking the upper limit as $n \to \infty$ in 2.5 and using lemma 1.9, we get

$$\begin{aligned} \frac{d(Sy, y)}{k^2} \quad \leq \quad & \limsup_{n\longrightarrow\infty} d(fSx_{2n}, gx_{2n+1}) \\ \leq \quad & \frac{q}{k^4}\max\{\limsup_{n\longrightarrow\infty} d(S^2x_{2n}, Tx_{2n+1}), \limsup_{n\longrightarrow\infty} d(fSx_{2n}, S^2x_{2n}), \\ & \limsup_{n\longrightarrow\infty} d(gx_{2n+1}, Tx_{2n+1}), \\ & \frac{1}{2}(\limsup_{n\longrightarrow\infty} d(S^2x_{2n}, gx_{2n+1}) + \limsup_{n\longrightarrow\infty} d(fSx_{2n}, Tx_{2n+1}))\}. \\ \leq \quad & \frac{q}{k^4}\max\{k^2d(Sy, y), 0, 0, \frac{k^2}{2}(d(Sy, y) + d(Sy, y))\} \\ = \quad & \frac{q}{k^4}k^2d(Sy, y) = \frac{q}{k^2}d(Sy, y). \end{aligned}$$

Consequently, $d(Sy, y) \leq qd(Sy, y)$. As $0 < q < 1$, so $Sy = y$.

Using continuity of T, we obtain $\lim_{n\to\infty} T^2x_{2n+1} = Ty$ and $\lim_{n\to\infty} Tgx_{2n+1} = Ty$.

Since g and T are compatible, $\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$. So, by lemma 1.10, we have $\lim_{n\to\infty} gTx_{2n} = Ty$.

Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in 2.1, we obtain

$$\begin{aligned} d(fx_{2n}, gTx_{2n+1}) \quad \leq \quad & \frac{q}{k^4}\max\{d(Sx_{2n}, T^2x_{2n+1}), d(fx_{2n}, Sx_{2n}), d(gTx_{2n+1}, T^2x_{2n+1}), \\ & \frac{1}{2}(d(Sx_{2n}, gTx_{2n+1}) + d(fx_{2n}, T^2x_{2n+1}))\}. \end{aligned} \tag{2.6}$$

Taking upper limit as $n \to \infty$ in 2.6 and using lemma 1.9, we obtain

$$\begin{aligned} \frac{d(y, Ty)}{k^2} \quad \leq \quad & \limsup_{n\longrightarrow\infty} d(fx_{2n}, gTx_{2n+1}) \leq \frac{q}{k^4}\max\{k^2d(y, Ty), 0, 0, \frac{k^2}{2}(d(y, Ty) + d(y, Ty))\} \\ = \quad & \frac{qd(y, Ty)}{k^2}, \end{aligned}$$

which implies that $Ty = y$. Also, we can apply condition 2.1 to obtain

$$\begin{aligned} d(fy, gx_{2n+1}) &\leq \frac{q}{k^4} \max\{d(Sy, Tx_{2n+1}), d(fy, Sy), d(gx_{2n+1}, Tx_{2n+1}), \\ &\quad \frac{1}{2}(d(Sy, gx_{2n+1}) + d(fy, Tx_{2n+1}))\}. \end{aligned} \tag{2.7}$$

Taking upper limit $n \to \infty$ in 2.7, and using $Sy = Ty = y$, we have

$$\begin{aligned} \frac{d(fy, y)}{k^2} &\leq \frac{q}{k^4} \max\{k^2 d(Sy, y), k^2 d(fy, Sy), k^2 d(y, y), \frac{k^2}{2}(d(Sy, y) + d(fy, y))\} \\ &= \frac{q}{k^2} d(fy, y). \end{aligned}$$

which implies that $d(fy, y) = 0$ and $fy = y$ as $0 < q < 1$.

Finally, from condition 2.1, and the fact $Sy = Ty = fy = y$, we have

$$\begin{aligned} d(y, gy) &= d(fy, gy) \\ &\leq \frac{q}{k^4} \max\{d(Sy, Ty), d(fy, Sy), d(gy, Ty), \frac{1}{2}(d(Sy, gy) + d(fy, Ty))\} \\ &= \frac{q}{k^4} d(y, gy) \leq q d(y, gy), \end{aligned}$$

which implies that $d(y, gy) = 0$ and $gy = y$. Hence $Sy = Ty = fy = gy = y$.

If there exists another common fixed point x in X for f, g, S and T, then

$$\begin{aligned} d(x, y) =& d(fx, gy) \\ &\leq \frac{q}{k^4} \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}(d(Sx, gy) + d(fx, Ty))\} \\ &= \frac{q}{k^4} \max\{d(x, y), d(x, x), d(y, y), \frac{1}{2}(d(x, y) + d(x, y))\} \\ &= \frac{q}{k^4} d(x, y) \leq q d(x, y), \end{aligned}$$

which further implies that $d(x, y) = 0$ and hence, $x = y$. Thus, y is a unique common fixed point of f, g, S and T. □

Now, we give two examples to support our result.

2.2. Example. Let $X = [0, \infty)$ be endowed with $b-$metric $d(x, y) = |x - y|^2 = (x - y)^2$, where $k = 2$. Define f, g, S and T on X by

$$f(x) = \ln(1 + \tfrac{x}{4}), \quad g(x) = \ln(1 + \tfrac{x}{5}),$$

$$S(x) = e^{5x} - 1, \quad T(x) = e^{4x} - 1.$$

Obviously, $f(X) = T(X) = g(X) = S(X) = [0, \infty)$. We show that the pair $\{f, S\}$ is compatible: Let $\{x_n\}$ be a sequence in X such that for some $t \in X$, $\lim_{n\to\infty} d(fx_n, t) = 0$ and $\lim_{n\to\infty} d(Sx_n, t) = 0$. That is, $\lim_{n\to\infty} |fx_n - t| = \lim_{n\to\infty} |Sx_n - t| = 0$. Since f and S are continuous, we have

$$\begin{aligned} \lim_{n\to\infty} d(fSx_n, Sfx_n) &= (\lim_{n\to\infty} |fSx_n - Sfx_n|)^2 = (|ft - St|)^2 \\ &= (\left|\ln(1 + \frac{t}{4}) - e^{5t} + 1\right|)^2. \end{aligned}$$

But $(\left|\ln(1 + \frac{t}{4}) - e^{5t} + 1\right|)^2 = 0 \iff t = 0$, so the pair $\{f, S\}$ is compatible. Similarly $\{g, T\}$ is compatible.

For each $x, y \in X$, the mean value theorem gives

$$\begin{aligned} d(fx, gy) &= (fx - gy)^2 = [\ln(1 + \frac{x}{4}) - \ln(1 + \frac{y}{5})]^2 \\ &\leq (\frac{x}{4} - \frac{y}{5})^2 \leq \frac{1}{20^2}(5x - 4y)^2 \\ &\leq \frac{1}{20^2}(e^{5x} - e^{4y})^2 = \frac{1}{20^2}d(Sx, Ty) \\ &\leq \frac{\frac{1}{25}}{2^4}\max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}(d(Sx, gy) + d(fx, Ty))\}, \end{aligned}$$

where $\frac{1}{25} \leq q < 1$ and $k = 2$. Thus, f, g, S and T satisfy all conditions of Theorem 2.1. Moreover 0 is the unique common fixed point of f, g, S and T.

2.3. Example. Let $X = [0, 1]$ be endowed with $b-$metric $d(x, y) = |x - y|^2 = (x - y)^2$, where $k = 2$. Define f, g, S and T on X by

$$f(x) = (\tfrac{x}{2})^8, \quad g(x) = (\tfrac{x}{2})^4,$$

$$S(x) = (\tfrac{x}{2})^4, \quad T(x) = (\tfrac{x}{2})^2.$$

Obviously, $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. Furthermore, the pairs $\{f, S\}$ and $\{g, T\}$ are compatible.

For each $x, y \in X$, we have

$$\begin{aligned} d(fx, gy) &= (fx - gy)^2 = ((\frac{x}{2})^8 - (\frac{y}{2})^4)^2 = ((\frac{x}{2})^4 + (\frac{y}{2})^2)^2.((\frac{x}{2})^4 - (\frac{y}{2})^2)^2 \\ &\leq (\frac{1}{16} + \frac{1}{4})^2 d(Sx, Ty) = \frac{\frac{5}{16}}{2^4}d(Sx, Ty) \\ &\leq \frac{q}{k^4}\max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}(d(Sx, gy) + d(fx, Ty))\}, \end{aligned}$$

where $\frac{5}{16} \leq q < 1$ and $k = 2$. Thus, f, g, S and T satisfy all conditions of Theorem 2.1. Moreover 0 is the unique common fixed point of f, g, S and T.

Now, we get the special cases of Theorem 2.1 as follows:

2.4. Corollary. Let (X, d) be a complete $b-$metric space and $f, g : X \to X$ two mappings such that

$$d(fx, gy) \leq \frac{q}{k^4}\max\{d(x, y), d(fx, x), d(gy, y), \frac{1}{2}(d(x, gy) + d(fx, y))\},$$

holds for all $x, y \in X$ with $0 < q < 1$. Then, there exists a unique point $y \in X$ such that $fy = gy = y$.

Proof. If we take $S = T = I_X$ (identity mapping on X), then Theorem 2.1 gives that f and g have a unique common fixed point.

2.5. Corollary. Let (X, d) be a complete $b-$metric space and $S, T : X \to X$ continuous mappings such that

$$d(x, y) \leq q\max\{d(Sx, Ty), d(x, Sx), d(y, Ty), \frac{d(Sx, y) + d(x, Ty)}{2}\},$$

holds for all $x, y \in X$ with $0 < q < 1$. Then, S and T have a unique common fixed point.

Proof. If we take f and g as identity maps on X, then Theorem 2.1 gives that S and T have a unique common fixed point.

2.6. Corollary. Let (X, d) be a complete $b-$metric space and $f : X \to X$ a mapping such that

$$d(fx, fy) \leq \frac{q}{k^4}\max\{d(x, y), d(fx, x), d(fy, y), \frac{1}{2}(d(x, fy) + d(fx, y))\},$$

holds for all $x, y \in X$, with $0 < q < 1$. Then f has a unique fixed point in X.

Proof. Take S and T as identity maps on X and $f = g$ and then apply Theorem 2.1.

2.7. Theorem. Let $\{f,\ S\}$ and$\{g, T\}$ be compatible self mappings on a complete $b-$metric space (X, d) satisfying

(2.8)
$$d(fx, gy) \leq \frac{1}{k^4}(a_1 d(Sx, Ty)+a_2 d(fx, Ty)+a_3 d(Sx, gy)+a_4 d(fx, Sx)+a_5 d(gy, Ty)),$$

for all $x, y \in X$, where $a_i \geq 0$ $(i = 1, 2, 3, 4, 5)$ are real constants with $a_1 + \alpha a_2 + \beta a_3 + a_4 + a_5 < 1$, where $\alpha + \beta = 2$, for $\alpha, \beta \in \mathbb{N} \cup \{0\}$. If $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ and S and T are continuous, then f, g, S and T have a unique common fixed point.

Proof. Let x_0 in X. Since $f(X) \subseteq T(X)$, Choose $x_1, x_2 \in X$ such that $Tx_1 = fx_0$, and $Sx_2 = gx_1$. This can be done as $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. In general, $x_{2n+1} \in X$ is chosen such that $Tx_{2n+1} = fx_{2n}$ and $x_{2n+2} \in X$ such that $Sx_{2n+2} = gx_{2n+1}$; $n = 0, 1, 2, \cdots$. Denote $y_{2n} = Tx_{2n+1} = fx_{2n}$, and $y_{2n+1} = Sx_{2n+2} = gx_{2n+1}$, for all $n \geq 0$. Now, we show that $\{y_n\}$ is a Cauchy sequence. For this, consider

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \\
&\leq \frac{1}{k^4}(a_1 d(Sx_{2n}, Tx_{2n+1}) + a_2 d(fx_{2n}, Tx_{2n+1}) + a_3 d(Sx_{2n}, gx_{2n+1}) \\
&+ a_4 d(fx_{2n}, Sx_{2n}) + a_5 d(gx_{2n+1}, Tx_{2n+1})) \\
&= \frac{1}{k^4}(a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n}, y_{2n}) + a_3 d(y_{2n-1}, y_{2n+1}) \\
&+ a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n+1}, y_{2n})) \\
&\leq \frac{1}{k^4}(a_1 d(y_{2n-1}, y_{2n}) + ka_3 d(y_{2n-1}, y_{2n}) + ka_3 d(y_{2n}, y_{2n+1}) \\
&+ a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n+1}, y_{2n})).
\end{aligned}$$

Hence,

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &\leq \frac{1}{k^4}(a_1 d(y_{2n-1}, y_{2n}) + ka_3 d(y_{2n-1}, y_{2n}) + ka_3 d(y_{2n}, y_{2n+1}) + \\
&\quad +a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n}, y_{2n+1})) \\
&\leq \frac{1}{k^3}(a_1 d(y_{2n-1}, y_{2n}) + a_3 d(y_{2n-1}, y_{2n}) \qquad (2.9) \\
&+ a_3 d(y_{2n}, y_{2n+1}) + a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n}, y_{2n+1})).
\end{aligned}$$

Now, we prove that $d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$, for each $n \in \mathbb{N}$. If $d(y_{2n-1}, y_{2n}) < d(y_{2n}, y_{2n+1})$, for some $n \in \mathbb{N}$ then from the above inequality we have

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &< \frac{1}{k^3}(a_1 d(y_{2n}, y_{2n+1}) + 2a_3 d(y_{2n}, y_{2n+1}) + a_4 d(y_{2n}, y_{2n+1}) + a_5 d(y_{2n}, y_{2n+1})) \\
&= \frac{1}{k^3}(a_1 + 2a_3 + a_4 + a_5) d(y_{2n}, y_{2n+1}) \\
&< d(y_{2n}, y_{2n+1}),
\end{aligned}$$

a contradiction. So, we have $d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$, for each $n \in \mathbb{N}$. Thus

$$d(y_{2n}, y_{2n+1}) \leq \frac{1}{k^3}(a_1 + 2a_3 + a_4 + a_5) d(y_{2n-1}, y_{2n}). \qquad (2.10)$$

Also, we have

$$\begin{aligned} d(y_{2n}, y_{2n-1}) &= d(fx_{2n}, gx_{2n-1}) \\ &\leq \frac{1}{k^4}(a_1 d(Sx_{2n}, Tx_{2n-1}) + a_2 d(fx_{2n}, Tx_{2n-1}) + a_3 d(Sx_{2n}, gx_{2n-1}) \\ &+ a_4 d(fx_{2n}, Sx_{2n}) + a_5 d(gx_{2n-1}, Tx_{2n-1})) \\ &= \frac{1}{k^4}(a_1 d(y_{2n-1}, y_{2n-2}) + a_2 d(y_{2n}, y_{2n-2}) + a_3 d(y_{2n-1}, y_{2n-1}) \\ &+ a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n-1}, y_{2n-2})) \\ &\leq \frac{1}{k^4}(a_1 d(y_{2n-1}, y_{2n-2}) + k a_2 d(y_{2n}, y_{2n-1}) + k a_2 d(y_{2n-1}, y_{2n-2}) \\ &+ a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n-1}, y_{2n-2})). \end{aligned}$$

Hence,

$$\begin{aligned} d(y_{2n}, y_{2n-1}) &\leq \frac{1}{k^3}(a_1 d(y_{2n-1}, y_{2n-2}) + a_2 d(y_{2n}, y_{2n-1}) \\ &+ a_2 d(y_{2n-1}, y_{2n-2}) + a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n-1}, y_{2n-2})). \end{aligned} \tag{2.11}$$

Similarly, if $d(y_{2n-1}, y_{2n-2}) \leq d(y_{2n}, y_{2n-1})$, for some $n \in \mathbb{N}$ then from 2.11 we obtain

$$\begin{aligned} d(y_{2n}, y_{2n-1}) &\leq \frac{1}{k^3}(a_1 d(y_{2n}, y_{2n-1}) + 2a_2 d(y_{2n}, y_{2n-1}) + a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n}, y_{2n-1})) \\ &\leq \frac{1}{k^3}(a_1 + 2a_2 + a_4 + a_5) d(y_{2n}, y_{2n-1}) \\ &< d(y_{2n}, y_{2n-1}), \end{aligned}$$

a contradiction. So, we have $d(y_{2n}, y_{2n-1}) \leq d(y_{2n-1}, y_{2n-2})$ for each $n \in \mathbb{N}$. Now from 2.11 we get

$$d(y_{2n}, y_{2n-1}) \leq \frac{1}{k^3}(a_1 + 2a_2 + a_4 + a_5) d(y_{2n-1}, y_{2n-2}). \tag{2.12}$$

Now, from 2.10 and 2.12 we have

$$d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2}), \qquad n \geq 2,$$

where $\lambda = \max\{\frac{a_1+2a_2+a_4+a_5}{k^3}, \frac{a_1+2a_3+a_4+a_5}{k^3}\}$. As $k \geq 1$, so $\lambda \in (0, 1)$. Now for $n \geq 2$, we have

$$d(y_n, y_{n-1}) \leq \ldots \leq \lambda^{n-1} d(y_1, y_0). \tag{2.13}$$

For $n > m$, we have

$$d(y_n, y_m) \leq k d(y_m, y_{m+1}) + k^2 d(y_{m+1}, y_{m+2}) + \ldots + k^{n-m-1} d(y_{n-1}, y_n).$$

Hence from 2.13 and $k\lambda < 1$, we have

$$\begin{aligned} d(y_n, y_m) &\leq (k\lambda^m + k^2\lambda^{m+1} + \ldots + k^{n-m-1}\lambda^{n-1}) d(y_1, y_0) \\ &\leq k\lambda^m [1 + k\lambda + (k\lambda)^2 + \cdots] d(y_1, y_0) \\ &\leq \frac{k\lambda^m}{1 - k\lambda} d(y_1, y_0) \\ &= \frac{k\lambda^m}{1 - k\lambda} d(y_1, y_0) \to 0, \quad \text{as} \quad m \to \infty. \end{aligned}$$

So $\{y_n\}$ is a Cauchy sequence. Let $y \in X$ be such that $\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y$. Since S is continuous, so $\lim_{n\to\infty} S^2 x_{2n+2} = Sy$ and

$\lim_{n\to\infty} Sfx_{2n} = Sy$. Using compatibility of a pair $\{f, S\}$, we have $\lim_{n\to\infty} d(fSx_n, Sfx_n) = 0$. So, by Lemma 1.10, we obtain $\lim_{n\to\infty} fSx_{2n} = Sy$. From 2.8, we have

$$d(fSx_{2n}, gx_{2n+1}) \leq \frac{1}{k^4}(a_1 d(S^2x_{2n}, Tx_{2n+1}) + a_2 d(fSx_{2n}, Tx_{2n+1}) + a_3 d(S^2x_{2n}, gx_{2n+1}) + a_4 d(fSx_{2n}, S^2x_{2n}) + a_5 d(gx_{2n+1}, Tx_{2n+1})).$$

Taking the upper limit as $n \to \infty$, and using Lemma 1.9 we get

$$\begin{aligned}\frac{d(Sy,y)}{k^2} &\leq \frac{1}{k^4}(k^2a_1 d(Sy,y) + k^2a_2 d(Sy,y) + k^2a_3 d(Sy,y) + k^2a_4 d(Sy,Sy) + k^2a_5 d(y,y)) \\ &\leq \frac{1}{k^4}(k^2a_1 d(Sy,y) + k^2a_2 d(Sy,y) + k^2a_3 d(Sy,y)) \\ &= \frac{1}{k^2}(a_1+a_2+a_3)d(Sy,y) \\ &\leq \frac{(a_1+a_2+a_3+a_4+a_5)}{k^2} d(Sy,y).\end{aligned}$$

Therefore, $d(Sy, y) \leq (a_1 + a_2 + a_3 + a_4 + a_5)d(Sy, y)$. As $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, so $Sy = y$.

Similarly, using continuouty of T , we obtain that $\lim_{n\to\infty} T^2x_{2n+1} = Ty$ and $\lim_{n\to\infty} Tgx_{2n+1} = Ty$. Since g and T are compatible, $\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$, so by Lemma 1.10, we have $\lim_{n\to\infty} gTx_{2n} = Ty$.

From 2.8, it follows that

$$d(fx_{2n}, gT_{2n+1}) \leq \frac{1}{k^4}(a_1 d(Sx_{2n}, T^2x_{2n+1}) + a_2 d(fx_{2n}, T^2x_{2n+1}) + a_3 d(Sx_{2n}, gTx_{2n+1}) + a_4 d(fx_{2n}, Sx_{2n}) + a_5 d(gTx_{2n+1}, T^2x_{2n+1})).$$

Taking upper limit as $n \to \infty$ and using Lemma 1.9 we get

$$\begin{aligned}\frac{d(y,Ty)}{k^2} &\leq \frac{1}{k^4}(k^2a_1 d(y,Ty) + k^2a_2 d(y,Ty) + k^2a_3 d(y,Ty) + k^2a_4 d(y,y) + k^2a_5 d(Ty,Ty) \\ &= \frac{1}{k^2}(a_1+a_2+a_3)d(y,Ty) \\ &\leq \frac{(a_1+a_2+a_3+a_4+a_5)}{k^2} d(y,Ty),\end{aligned}$$

that is, $d(y, Ty) \leq (a_1 + a_2 + a_3 + a_4 + a_5)d(y, Ty)$. Therefore, $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ implies that $Ty = y$. Again from 2.8, it follows that

$$d(fy, gx_{2n+1}) \leq \frac{1}{k^4}(a_1 d(Sy, Tx_{2n+1}) + a_2 d(fy, Tx_{2n+1}) + a_3 d(Sy, gx_{2n+1}) + a_4 d(fy, Sy) + a_5 d(gx_{2n+1}, Tx_{2n+1})).$$

Taking upper limit as $n \to \infty$ and using $Sy = y$ and $Ty = y$, we get

$$\begin{aligned}\frac{d(fy,y)}{k^2} &\leq \frac{1}{k^4}(k^2a_1 d(Sy,y) + k^2a_2 d(fy,y) + k^2a_3 d(Sy,y) + k^2a_4 d(fy,y) + k^2a_5 d(y,y) \\ &\leq \frac{a_1+a_2+a_3+a_4+a_5}{k^2} d(fy,y).\end{aligned}$$

Therefore, $d(fy, y) \leq (a_1+a_2+a_3+a_4+a_5)d(fy, y)$. As $a_1+a_2+a_3+a_4+a_5 < 1$, so $fy = y$. Again, from 2.8 we have $d(fy, gy) = 0$, hence $fy = gy$. Thus, $fy = gy = Sy = Ty = y$.

If there exists another common fixed point x in X for f, g, S and T, then

$$\begin{aligned} d(x,y) &= d(fx,gy) \\ &\leq \frac{1}{k^4}(a_1 d(Sx,Sy) + a_2 d(fx,Ty) + a_3 d(Sx,gy) + a_4 d(fx,Sx) + a_5 d(gy,Ty)) \\ &= \frac{1}{k^4}(a_1 + a_2 + a_3) d(x,y) \\ &\leq (a_1 + a_2 + a_3 + a_4 + a_5) d(x,y), \end{aligned}$$

Therefore, $d(x,y) \leq (a_1 + a_2 + a_3 + a_4 + a_5) d(x,y)$. As $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, so $d(x,y) = 0$, i.e., $x = y$. Therefore y is a unique common fixed point of f, g, S and T. □

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Fuzzy Γ-ideals in Γ-AG-groupoids

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Abstract

In this paper we study fuzzy Γ-ideals and prime, semiprime fuzzy Γ-ideals of a Γ-AG-groupoid S. We prove that, if S is a Γ-AG-groupoid with left identity, then every fuzzy Γ-ideal of S is idempotent if and only if every fuzzy Γ-ideal of S is semiprime. We also show that, if S is a Γ-AG-groupoid with left identity e, then every fuzzy Γ-ideal of S is prime if and only if every fuzzy Γ-ideal of S is idempotent and the set of fuzzy Γ-ideals of S is totally ordered by inclusion.

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1. Introduction

In [17], Zadeh introduced the notion of a fuzzy subset f of a set S as a function from S into unit interval $[0, 1]$. The notion of a fuzzy ideal in Γ-rings was first introduced by Jun and Lee [5]. They studied some preliminary properties of fuzzy ideals of Γ-rings. Dutta and Chanda [4], studied the structures of fuzzy ideals of a Γ-ring and characterized Γ-field and Noetherian Γ-ring. Jun [6] defined fuzzy prime ideal of a Γ-ring and obtained a number of characterizations for a fuzzy ideal to be a fuzzy prime ideal.

Kazim and naseerudin [8], have introduced the concept of LA-semigroups (also known as AG-groupoids). An LA-semigroup S is a groupoid which satisfies the left invertive law $(ab)c = (cb)a$ for all $a, b, c \in S$. It is a midway structure between a commutative semigroup and a groupoid. It is a non-associative structure which has wide applications in the theory of flocks and in automata theory.

In 1981, the notion of Γ-semigroups was introduced by Sen [13, 14]. Let M and Γ be any nonempty sets. M is said to be a Γ-semigroup if there exists a mapping $M \times \Gamma \times M \longrightarrow M$, such that M satisfies the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$

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Define $f : S \longrightarrow [0,1]$ by $f(1) = 0.9$, $f(2) = 0.8$, $f(3) = 0.5$, $f(4) = 0.5$, $f(5) = 0.5$.

$$\mathsf{U}(f;t) := \begin{cases} S & \text{if } t \in (0, 0.5] \\ \{1,2\} & \text{if } t \in (0.5, 0.8] \\ \{1\} & \text{if } t \in (0.8, 0.9] \\ \emptyset & \text{if } t \in (0.9, 1] \end{cases}$$

Then by Proposition 3.8, f is a fuzzy ideal of S.

3.10. Lemma. If S is a Γ-AG-groupoid with left identity e. Then every fuzzy right Γ-ideal of S is a fuzzy left Γ-ideal of S.

Proof. : Let S be a Γ-AG-groupoid with left identity e and f a fuzzy right Γ-ideal of S. Let $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned} f(a\alpha b) &= f((e\gamma a)\alpha b) \text{ because } e \text{ is left identity} \\ &= f((b\gamma a)\alpha e) \text{ by left invertive law} \\ &\geq f(b\gamma a) \text{ because } f \text{ is a fuzzy right } \Gamma\text{-ideal} \\ &\geq f(b) \text{ because } f \text{ is a fuzzy right } \Gamma\text{-ideal.} \end{aligned}$$

Thus $f(a\alpha b) \geq f(b)$ for all $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Hence f is a fuzzy left Γ-ideal.

Note that the intersection and union of any family of fuzzy Γ-ideals of a Γ-AG-groupoid S is a fuzzy Γ-ideal of S. □

3.11. Proposition. If S is a Γ-AG-groupoid and f, g, h are fuzzy subsets of S. Then $(f\Gamma g)\Gamma h = (h\Gamma g)\Gamma f$.

Proof. Let $x \in S$ and $\alpha, \beta \in \Gamma$, then

$$\begin{aligned} ((f\Gamma g)\Gamma h)(x) &= \bigvee_{x=y\alpha z} \{(f\Gamma g)(y) \wedge h(z)\} \\ &= \bigvee_{x=y\alpha z} \left\{ \begin{array}{ll} \left(\bigvee\limits_{y=a\beta b} (f(a) \wedge g(b)) \right) \wedge h(z) & \\ 0 & \text{if } y \neq a\beta b \end{array} \right\} \\ &= \bigvee_{x=(a\beta b)\alpha z} \{(f(a) \wedge g(b)) \wedge h(z)\} \end{aligned}$$

Since $x = (a\beta b)\alpha z = (z\beta b)\alpha a$, so $((f\Gamma g)\Gamma h)(x) = \bigvee\limits_{x=(z\beta b)\alpha a} [(h(z) \wedge g(b)) \wedge f(a)]$.

Also $h(z) \wedge g(b) \leq \bigvee\limits_{z\alpha b = c\alpha d} \{h(c) \wedge g(d)\})$, so

$$\begin{aligned} ((f\Gamma g)\Gamma h)(x) &\leq \bigvee_{x=(z\beta b)\alpha a} \left[\left\{ \bigvee_{z\alpha b = c\alpha d} \{h(c) \wedge g(d)\} \right\} \wedge f(a) \right] \\ &= \bigvee_{x=(z\beta b)\alpha a} \{(h\Gamma g)(z\beta b) \wedge f(a)\} \\ &\leq \bigvee_{x=s\gamma t} \{(h\Gamma g)(s) \wedge f(t)\} \\ &= ((h\Gamma g)\Gamma f)(x). \end{aligned}$$

Hence $(f\Gamma g)\Gamma h \leq (h\Gamma g)\Gamma f$. Similarly, $(h\Gamma g)\Gamma f \leq (f\Gamma g)\Gamma h$. Thus $(f\Gamma g)\Gamma h = (h\Gamma g)\Gamma f$.

3.12. Remark. The above Proposition shows that if S is a Γ-AG-groupoid and $F(S)$ is the collection of all fuzzy subsets of S, then $(F(S), \Gamma')$ is a Γ-AG-groupoid.

3.13. Lemma. Let S be a Γ-AG-groupoid with left identity e and f, g, h be fuzzy subsets of S, then

$$f\Gamma(g\Gamma h) = g\Gamma(f\Gamma h).$$

Proof. The proof follows from Proposition 3.11 and from a generalized form of Lemma 4 in [12].

3.14. Lemma. Let S be a Γ-AG-groupoid with left identity e and f a fuzzy right Γ-ideal of S, then $f\Gamma f$ is a fuzzy Γ-ideal of S.

Proof: Since f is a fuzzy right Γ-ideal of S, by Lemma 3.10, f is a fuzzy left Γ-ideal of S.

Let $a, b \in S$ and $\alpha \in \Gamma$. If $(f\Gamma f)(a) = 0$, then $(f\Gamma f)(a\alpha b) \geq (f\Gamma f)(a)$. Otherwise

$$\begin{aligned}
(f\Gamma f)(a\alpha b) &= \bigvee_{a=y\gamma z} \{f(y) \wedge f(z)\} \\
(\text{If } a &= y\gamma z, \text{ then } a\alpha b = (y\gamma z)\alpha b = (b\gamma z)\alpha y \text{ by left invertive law}) \\
\text{So, } (f\Gamma f)(a\alpha b) &= \bigvee_{a=y\gamma z} \{f(y) \wedge f(z)\} \\
&\leq \bigvee_{a=y\gamma z} \{f(b\gamma z) \wedge f(y)\}, \text{ since } f \text{ is a fuzzy left } \Gamma\text{-ideal} \\
&\leq \bigvee_{a\alpha b=c\gamma d} \{f(c) \wedge f(d)\} = (f\Gamma f)(a\alpha b).
\end{aligned}$$

Thus $(f\Gamma f)(a\alpha b) \geq (f\Gamma f)(a)$. Hence $f\Gamma f$ is a fuzzy right Γ-ideal of S and by Lemma 3.10, a fuzzy Γ-ideal of S.

3.15. Lemma. If S is a Γ-AG-groupoid and f, g are fuzzy Γ-ideals of S, then $f\Gamma g \subseteq f \cap g$.

proof: Let f and g be fuzzy Γ-ideals of S and $x \in S$. If $(f\Gamma g)(x) = 0$, then $(f\Gamma g)(x) \leq (f \cap g)(x)$, otherwise

$$\begin{aligned}
(f\Gamma g)(x) &= \bigvee_{x=y\alpha z} (f(y) \wedge g(z)) \\
&\leq \bigvee_{x=y\alpha z} (f(y) \wedge g(z))(f(y\alpha z) \wedge g(y\alpha z)), \text{ since } f \text{ and } g \text{ are fuzzy } \Gamma\text{-ideals of } S \\
&= \bigvee_{x=y\alpha z} (f(x) \wedge g(x)) \\
&= (f \cap g)(x).
\end{aligned}$$

Thus $f\Gamma g \subseteq f \cap g$.

3.16. Remark. If S is a Γ-AG-groupoid with left identity e and f and g are fuzzy right Γ-ideals of S, then $f\Gamma g \subseteq f \cap g$.

3.17. Remark. If S is a Γ-AG-groupoid and f a fuzzy Γ-ideal of S, then $f\Gamma f \subseteq f$.

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\tilde{M}$. They are related as

$$g(h(X,Y),N) = g(A_N X, Y), \tag{2:7}$$

where g denotes the Riemannian metric on $\tilde{M}$ as well as induced on M. The mean curvature vector H on M is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) \tag{2:8}$$

where n is the dimension of M and $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame of vector fields on M.

A submanifold M of a Riemannian manifold $\tilde{M}$ is said to be *totally umbilical* if

$$h(X,Y) = g(X,Y)H. \tag{2:9}$$

If $h(X,Y) = 0$ for any $X, Y \in \Gamma(TM)$ then M is said to be *totally geodesic submanifold*. If $H = 0$, then it is called *minimal submanifold*.

If M is totally umbilical, then from (2.9), the equations (2.5) and (2.6) reduce to the following equations, respectively;

$$\tilde{\nabla}_X Y = \nabla_X Y + g(X,Y)H, \tag{2:10}$$

$$\tilde{\nabla}_X N = -g(H,N)X + \nabla^{\perp}_X N. \tag{2:11}$$

Now, for any $X \in \Gamma(TM)$, we write

$$JX = PX + FX, \tag{2:12}$$

where PX is the tangential component and FX is the normal component of JX.

Similarly for any $N \in \Gamma(T^{\perp}M)$, we write

$$JN = BN + CN, \tag{2:13}$$

where BN is the tangential component and CN is the normal component of JN. The covariant derivatives of the tensor fields J, P and F are respectively defined as

$$(\tilde{\nabla}_X J)Y = \tilde{\nabla}_X JY - J\tilde{\nabla}_X Y; \quad \forall X, Y \in \Gamma(T\tilde{M}); \tag{2:14}$$

$$(\tilde{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y; \quad \forall X, Y \in \Gamma(TM); \tag{2:15}$$

$$(\tilde{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y; \quad \forall X, Y \in \Gamma(TM). \tag{2:16}$$

3. Contact CR-submanifolds

In this section we consider the submanifold M tangent to the structure vector field ξ and defined as follows: A submanifold M tangent to ξ is called a *contact CR-submanifold* if it admits a pair of differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that $\mathcal{D}$ is invariant and its orthogonal complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant i.e., $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle \xi \rangle$ with $J(\mathcal{D}_x) = \mathcal{D}_x$ and $J(\mathcal{D}^{\perp}_x) = T^{\perp}_x M$, for every $x \in M$. Thus, a contact CR-submanifold M tangent to ξ is *invariant* if $\mathcal{D}^{\perp}$ is identically zero and an *anti-invariant* if $\mathcal{D}$ is identically zero, respectively. If neither $\mathcal{D} = \{0\}$ nor $\mathcal{D}^{\perp} = \{0\}$, then M is *proper contact CR-submanifold*.

Let M be a proper contact CR-submanifold of an almost contact metric manifold $\tilde{M}$, then for any $X \in \Gamma(TM)$, we have

$$X = P_1 X + P_2 X + \eta(X)\xi, \tag{3:1}$$

where P_1 and P_2 are the orthogonal projections from TM to $\mathcal{D}$ and $\mathcal{D}^{\perp}$, respectively. For a contact CR-submanifold, from (2.12) and (3.1), we obtain

$$PX = P_1X \quad \text{and} \quad FX = P_2X:$$

Let M be a contact CR-submanifold of an almost contact metric manifold $\tilde{M}$. Then the normal bundle $T^{\perp}M$ is decomposed as

$$T^{\perp}M = \mathcal{D}^{\perp} \quad ; \tag{3:2}$$

where is the orthogonal complemetary distribution of $\mathcal{D}^{\perp}$ in $T^{\perp}M$ and is a invariant subbundle of $T^{\perp}M$.

Let M be a contact CR-submanifold of a cosymplectic manifold $\tilde{M}$, then for any $Z; W \in \Gamma(\mathcal{D}^{\perp})$ and $U \in \Gamma(TM)$, we have

$$g(A_{\phi W}Z; U) = g(h(Z; U); W):$$

Using (2.5), we obtain

$$g(A_{\phi W}Z; U) = g(\tilde{r}_U Z; W) = g(\tilde{r}_U Z; W):$$

By the structure equation (2.3), we get

$$g(A_{\phi W}Z; U) = g(\tilde{r}_U Z; W):$$

Thus, from (2.6), we derive

$$g(A_{\phi W}Z; U) = g(A_{\phi Z}U; W) = g(h(W; U); Z):$$

Again Using (2.5), we obtain

$$g(A_{\phi W}Z; U) = g(\tilde{r}_W U; Z) = g(U; \tilde{r}_W Z):$$

Then from (2.6), we get

$$g(A_{\phi W}Z; U) = g(A_{\phi Z}W; U):$$

Hence, for a contact CR-submanifold of a cosymplectic manifold we conclude that

$$A_{\phi W}Z = A_{\phi Z}W \quad \forall Z; W \in \Gamma(\mathcal{D}^{\perp}): \tag{3:3}$$

Now, for any $X \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ and $Z; W \in \Gamma(\mathcal{D}^{\perp})$, we have

$$g([Z; W]; X) = g(\tilde{r}_Z W - \tilde{r}_W Z; X)$$

$$= g(\tilde{r}_W Z - \tilde{r}_Z W; X):$$

Thus, from (2.14) and (2.3), we obtain

$$g([Z; W]; X) = g(\tilde{r}_W Z - \tilde{r}_Z W; X)$$

$$= g(A_{\phi W}Z - A_{\phi Z}W; X).$$

Thus, from (3.3), we obtain $g([Z; W]; X) = 0$. This means that $[Z; W] \in \Gamma(\mathcal{D}^{\perp})$, for any $Z; W \in \Gamma(\mathcal{D}^{\perp})$, that is, $\mathcal{D}^{\perp}$ is integrable. Now for any $X; Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$, we have

$$h(X; PY) + r_X PY = \tilde{r}_X PY = \tilde{r}_X Y:$$

As $\tilde{M}$ is cosymplectic, then by (2.14) and the structure equation (2.3), we obtain

$$h(X; PY) + r_X PY = \tilde{r}_X Y:$$

Using (2.5), (2.12) and (2.13), we derive

$$h(X; PY) + r_X PY = Pr_X Y + Fr_X Y + Bh(X; Y) + Ch(X; Y):$$

Equating the normal components, we get

$$Fr_X Y = h(X; PY) - Ch(X; Y): \tag{3:4}$$

Similarly,

$$\mathsf{F}\mathsf{r}_{Y}\mathsf{X} = \mathsf{h}(\mathsf{Y};\mathsf{P}\mathsf{X}) \quad \mathsf{C}\mathsf{h}(\mathsf{X};\mathsf{Y}): \tag{3:5}$$

Thus from (3.4) and (3.5), we obtain

$$\mathsf{F}[\mathsf{X};\mathsf{Y}] = \mathsf{h}(\mathsf{X};\mathsf{P}\mathsf{Y}) \quad \mathsf{h}(\mathsf{Y};\mathsf{P}\mathsf{X}): \tag{3:6}$$

Hence, we conclude that $\mathsf{F}[\mathsf{X};\mathsf{Y}] = 0$ if and only if $\mathsf{h}(\mathsf{X};\mathsf{P}\mathsf{Y}) = \mathsf{h}(\mathsf{Y};\mathsf{P}\mathsf{X})$, that is the distribution $\mathcal{D}$ h i is integrable if and only if $\mathsf{h}(\mathsf{X};\mathsf{P}\mathsf{Y}) = \mathsf{h}(\mathsf{Y};\mathsf{P}\mathsf{X})$, for all $\mathsf{X};\mathsf{Y}$ 2 $\Gamma(\mathcal{D}$ h $\mathsf{i})$.

We give the following main result of this section.

3.1. Theorem *Let* M *be a totally umbilical contact CR-submanifold of a cosymplectic manifold* $\tilde{\mathsf{M}}$*. Then at least one of the following statements is true*

(i) M *is totally geodesic,*
(ii) *the anti-invariant distribution* $\mathcal{D}^{\perp}$ *is one-dimensional, i.e.,* $\dim\ \mathcal{D}^{\perp} = 1$*,*
(iii) *the mean curvature vector* H 2 $\Gamma(\ )$*.*

Proof. For a cosympectic manifold, we have

$$\tilde{\mathsf{r}}_{Z}\mathsf{W} = \tilde{\mathsf{r}}_{Z}\mathsf{W};$$

for any $\mathsf{Z};\mathsf{W}$ 2 $\Gamma(\mathcal{D}^{\perp})$. Using (2.10) and (2.11), we derive

$$\mathsf{g}(\mathsf{H};\ \mathsf{W}\ )\mathsf{Z} + \mathsf{r}^{\perp}_{Z}\mathsf{W} = \mathsf{r}_{Z}\mathsf{W} + \mathsf{g}(\mathsf{Z};\mathsf{W})\,\mathsf{H}: \tag{3:7}$$

Taking the product with Z 2 $\Gamma(\mathcal{D}^{\perp})$ in (3.7), we get

$$\mathsf{g}(\mathsf{H};\ \mathsf{W}\ )\mathsf{k}\mathsf{Z}\mathsf{k}^2 = \mathsf{g}(\mathsf{Z};\mathsf{W})\mathsf{g}(\mathsf{H};\ \mathsf{Z}\ ): \tag{3:8}$$

Interchanging Z and W in (3.8), we obtain

$$\mathsf{g}(\mathsf{H};\ \mathsf{Z}\ )\mathsf{k}\mathsf{W}\mathsf{k}^2 = \mathsf{g}(\mathsf{Z};\mathsf{W})\mathsf{g}(\mathsf{H};\ \mathsf{W}\ ): \tag{3:9}$$

Thus, from (3.8) and (3.9), we deduce that

$$\mathsf{g}(\mathsf{H};\ \mathsf{Z}\ ) = \frac{\mathsf{g}(\mathsf{Z};\mathsf{W})^2}{\mathsf{k}\mathsf{Z}\mathsf{k}^2\mathsf{k}\mathsf{W}\mathsf{k}^2}\mathsf{g}(\mathsf{H};\ \mathsf{Z}\ ):$$

That is

$$\mathsf{g}(\mathsf{H};\ \mathsf{Z}\ )\mathsf{f}\,1 \quad \frac{\mathsf{g}(\mathsf{Z};\mathsf{W})^2}{\mathsf{k}\mathsf{Z}\mathsf{k}^2\mathsf{k}\mathsf{W}\mathsf{k}^2}\mathsf{g} = 0: \tag{3:10}$$

Hence, the equation (3.10) has a solution if at least one of the followings holds

$$(\mathsf{i})\ \mathsf{H} = 0 \quad \text{or} \quad (\mathsf{ii})\ \mathsf{Z}\ \mathsf{k}\ \mathsf{W} \quad \text{or} \quad (\mathsf{iii})\ \mathsf{H}\ ?\quad \mathcal{D}^{\perp}:$$

That is either M is totally geodesic or as Z and W are parallel to each other for any $\mathsf{Z};\mathsf{W}$ 2 $\Gamma(\mathcal{D}^{\perp})$ that is these two vectors are linearly dependent and hence $\dim\ \mathcal{D}^{\perp} = 1$ or H 2 $\Gamma(\ )$, this proves the theorem completely. □

3.2. Example Consider a flat manifold of real dimension 6 which have a complex Kaehler structure of dimension 3, that is $(\mathbb{C}^3;\mathsf{J};\mathsf{h})$ be a Kaehler manifold with complex structure J and Euclidean Hermitian metric h. Then $\tilde{\mathsf{M}} = \mathbb{C}^3$ $\mathbb{R}$ is a cosymplectic manifold with the structure vector field $= \frac{\partial}{\partial t}$, dual 1-form $= \mathsf{dt}$ and the metric $\mathsf{g} = \mathsf{h} + \mathsf{dt}^2$. Now, consider $\mathsf{M} = \mathbb{R}^3$ S^1, where S^1 is a unit circle being taken as totally real submanifold of $\mathbb{C}^3$. Then M is a contact CR-submanifold of $\tilde{\mathsf{M}}$ with the invariant distribution $\mathcal{D} = \mathbb{R}^2$, anti-invariant distribution $\mathcal{D}^{\perp} = \Gamma(\mathsf{S}^1)$ and the 1-dimensional distribution $\mathsf{h}\ \mathsf{i} = \mathbb{R}$.

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charges interest against this loan. Now the retailer is in dilemma for optimal procurement and also for the amount for immediate part payment. Here an amount, borrowed from the money lending source as a loan with interest, is paid to the wholesaler at the beginning on receipt of goods. In return, the wholesaler/supplier offers a relaxed credit period as permissible delay in payment of rest amount and a reduced unit purchasing price depending on the amount of immediate part payment. Inflation also plays an important role for the optimal order policy and influences the demand of certain products. The effect of inflation and time value of the money cannot be ignored for determining the optimal inventory policy. In the present paper, an inventory control system in which immediate part payment and the delay-payment for the restare allowed by the wholesaler for an item over a finite planning horizon or random planning horizon with selling price and inflation induced demand is considered. In addition, against an immediate part payment (variable) to the wholesaler, there is a provision for (i) borrowing money from a money lending source and (ii) earning some relaxation on credit period from the wholesaler. The models are developed with respect to the retailer for maximum profit. Also in this present paper we have developed a continuous production control inventory model for deteriorating items with shortages in which two different rates of production are available and it is possible that production started at one rate and after some time it may be switched over to another rate .Such a situation is desirable in the sense that by starting at a low rate of production , a large quantum stock of manufactured item at the initial stage is avoided ,leading to reduction in the holding cost . The randomness in the planning horizon has been removed using the chance-constraint technique. Single objective problems incorporating immediate part payment, delay in payment for the rest, and selling price, inflation dependent demand with two production rate are formulated to maximize the profit function with shortages and solved using Generalized Reduced Gradient (GRG)method. The decision variables for these inventory models are the immediate part payment and number of cycles. These models are illustrated with numerical examples. Finally, the sensitivity analyses for the profit function and immediate part payment with respect to some parameters are carried out and the results are presented graphically.

2. Literature Review

First, Goyal [12] introduce the concept of permissible delay in payment in an EOQ model. Since then, lots of literature is available in this area of study. The important papers related to such studies are Chu et al. [6], Chung [7], Jamal et al.[17], Sarker et al. [24] and others. Effect of inflation and time value of money is also well established in inventory problems. The many authors such as Moon and Lee [21], Chen [10], Dey et al. [11], Padmanavan and Vrat[22] and others worked in these area. Jaggi et al. [16] developed an inventory model with shortages, in which units are deteriorating at constant rate and demand rate is increasing exponentially due to inflation over a finite planning horizon using discount cash flow approach. Tripathi et al. [26] developed a cash flow oriented EOQ model under permissible delay in payments for non-deteriorating items and time-dependent demand rate under inflation and time discounting. Huang[15] introduce an EPQ model under two level of trade credit policy. Liao[19] introduce an EPQ model of deteriorating item under permissible delay in payments. Chung ,Cßrdenas- Barr$\frac{3}{4}$n [5] present a simple easy solution procedures to locate the optimal solutions of an inventory model that considers deteriorating items under stock-dependent demand and two-level trade credit. Cßrdenas[1-3] introduce an EOQ model with cash discount offer. Cßrdenas-Barr$\frac{3}{4}$n, Smith and Goyal[4] introduce an EOQ model with planned backorders when the supplier offers a temporary fixed- percentage discount and has specified a minimum quantity of additional units purchase. Widyadana, Cßrdenas- Barr$\frac{3}{4}$n and Wee[28]

proposed an EOQ model of deteriorating with a simplified approach. Cßrdenas- Barr$\frac{3}{4}$n [2] introduce an EOQ model with imperfect quality and quantity discounts. Taleizadeh, Mohammadi, Cßrdenas- Barr$\frac{3}{4}$n and Samimi [27] proposed an EOQ model for perishable product with special sale and shortage. Chen, Cßrdenas- Barr$\frac{3}{4}$n,Teng [9] extend EOQ model under conditionally permissible delay in payment. Jana, Das and Roy [18] introduce a partial backlogging inventory model for deteriorating item under Fuzzy inflation and discounting over random planning horizon. Singh et al. [25] proposed a two warehouse model under inflation with large quantity of purchase orders. Chung and Huang [8] studied ordering policy with permissible delay in payments to show the convexity of total annual variable cost function. Guria et al. [14] proposed a pricing model for petrol/diesel and determined the optimal ordering policy for an existing petrol/diesel retailing station under permissible delay in payment. Several authors like Panda and Maiti[23] investigated the inventory models of this type of item. Maiti et al. [20] introduced the concept of advanced payment for determining the optimal ordering policy under stochastic lead-time and price dependent demand condition. Recently Maiti and Guria [13] introduce an EOQ model with inflation reduced purchasing price, selling price and demand with immediate part payment.
In this paper we introduce an EPQ model for a deteriorating item with inflation reduced selling price and demand with immediate part payment.

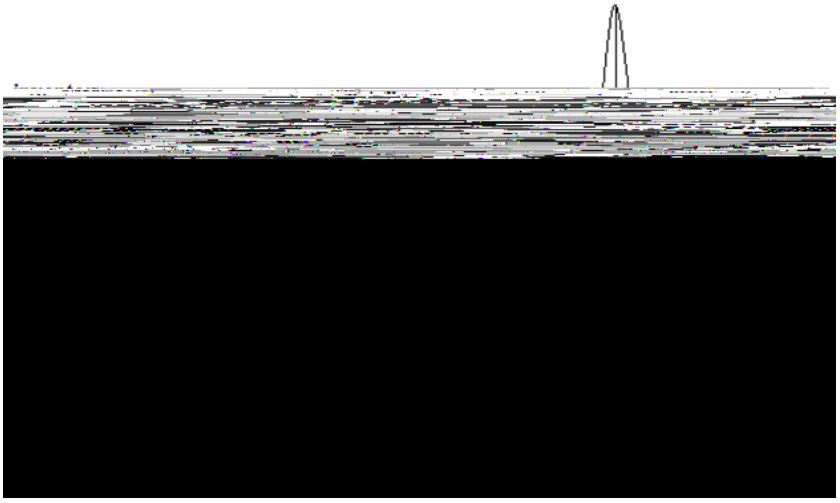

Figure 1. Graphical representation of finite time horizon inventory system

3. Mathematical Model Formulation:

To formulate the mathematical model for the proposed inventory system, the following notations and assumptions are made.

3.1. Notations

(i) $q_i(t)$=inventory level at time t for the ith cycle.
(ii) Q_{1i} = inventory level at time $t = t_{s_{1i}}$ for the ith cycle (t_{i-1}, t_i).

(iii) Q_{2i} = inventory level at time $t = t_{s_{2i}}$ for the ith cycle (t_{i-1}, t_i).
(iv) Q_{3i} = shortage quantity for the ith cycle (t_{i-1}, t_i).
(v) $Q_i = (Q_{1i} + Q_{2i})$ = total inventory for the ith cycle (t_{i-1}, t_i).
(vi) c_3 = set up cost for each cycle.
(vii) A = immediate part payment for each cycle .
(viii) n = number of cycles.
(ix) H= finite planning horizon for the crisp model.
(x) $\hat{H}$ =random planning horizon that follows normal distribution with mean $m_{\hat{H}}$ and standard deviation $\sigma_{\hat{H}}$ for the stochastic model.
(xi) T = length of each cycle.
(xii) M = permissible delay period for each cycle.
(xiii) $M^{'}$= rest period of each cycle after the credit period, i.e $(\frac{H}{n} - M)$.
(xiv) r= the unit raw material cost.
(xv) s_i = selling price per unit quantity for the ith cycle.
(xvi) D_0= original demand at t = 0
(xvii) $D_i(t)$= rate of demand (variable) for the ith cycle.
(xviii) R_i= The unit production cost in the ith cycle.
(xix) $P_1(> D_i(t))$and $P_2(> P_1)$ constant production rates started at time t_{i-1} and at time $t_{s_{1i}}$ respectively for the ith cycle.
(xx) I_e= rate of interest per unit to be earned by the retailer.
(xxi) I_b= rate of interest per unit to be paid by the retailer to money lender against immediate part payment A.
(xxii) c_1 = inventory holding cost per unit quantity per unit time for each cycle.
(xxiii) c_2 = inventory shortage cost per unit quantity per unit time for each cycle.
(xxiv) α = rate of inflation.
(xxv) t_s= time of beginning of shortages which is taken as $t_s = r^{'}\frac{H}{n}$;$0 < r^{'} < 1.0$
(xxvi) β and p_r are two given numbers where $\beta > 0$,$0 \leq \beta \leq 1.0$ and $\varepsilon_{\hat{H}}$ is a real number whose standard normal value is p_r.
(xxvii) Z (n,A) is the profit function for the whole period.
(xxviii) t_p= time of beginning the second production P_2 which is taken as $t_p = \frac{r^{'}}{4}\frac{H}{n}$.
(xxix) t_q= time of beginning the demand after the second production $t_q = r^{'}\frac{H}{n}$.
(xxx) t_r= time of beginning the second production P_2 after shortage which is taken as $t_r = r^{''}\frac{H}{n}$;$0 < r^{'} < r^{''} < 1.0$.
(xxxi) $t_{s_{1i}} = t_{i-1} + t_p$, $t_{s_{2i}} = t_{i-1} + t_q$, $t_{s_{3i}} = t_{i-1} + t_s$,$t_{s_{4i}} = t_{i-1} + t_r$.

3.2. Assumption

(i) Lead time is zero.
(ii) Shortages are allowed and fully backlogged.
(iii) Immediate part payment A is made and same for all the cycles and it is paid at the beginning of each cycle. For the 1stcycle, it is borrowed from a money lending source with the condition that it will be paid at the end of business period H with interest at the rate of I_b and for the remaining cycles, the immediate part payment A will be given from the revenue earned up to that time. So, for ith cycle $(i \geq 2)$ the immediate part payment A must be less than the total revenue at t_{i-1}.
(iv) The rest payment for the wholesaler will be done at the end of the credit period M where $t_{i-1} + M \leq t_i$ for each cycle. There is a fixed credit period M_0 and it is assumed that it will be enhanced depending upon the amount of the immediate advanced payment A and is given in the form $M = M_0 + t^{'}A$ where $t^{'} << 1$ and $A < rQ_1$.

Lemma 3: Prove that $\sum_{i=1}^{n} e^{\alpha_1 t_{i-1}+\alpha(1-\gamma)t_{i-1}} = \frac{(e^{nw}-1)}{(e^{w}-1)}$

Where $\frac{H}{n} = t_{i-1} - t_i$,$w = [\alpha_1 + \alpha(1-\gamma)]\frac{H}{n}$

Therefore **TS**=$\frac{(rm)^{1-\gamma}}{\alpha_1} D_0 \frac{e^{w}(e^{nw}-1)}{(e^{w}-1)}(e^{-x}+e^{-w})$ (Using lemma 2 & 3) (16)

The holding cost for ith cycle (t_{i-1}, t_i) and during the whole business period (0,H) are respectively obtained as

$HC_i = c_1 \int_{t_{i-1}}^{t_i} q_i(t)dt = c_1\{\frac{P_1}{\theta}(\theta t_p + e^{-\theta t_p}+1) + \frac{P_2}{\theta}(\theta(t_p - t_q) + e^{\theta(t_p-t_q)}-1) + \frac{D_0}{(mr)^{\gamma}(\alpha_1+\theta)}\frac{1}{\alpha_1\theta}e^{(\alpha_1 t_{i-1}-\alpha\gamma t_{i-1})}$

$[(\alpha_1+\theta) - \alpha_1 e^{-\theta t_p} - \theta e^{\alpha_1 t_s} + \alpha_1 e^{\alpha_1 t_p}(1-e^{\theta(t_p-t_q)}) + \alpha_1 e^{\alpha_1 t_q}(1-e^{\theta(t_q-t_s)})] - \frac{Q_{1i}}{\theta}(1 - e^{\theta(t_p-t_q)}) + \frac{Q_{2i}}{\theta}(1-e^{\theta(t_q-t_s)})\}$

$\mathbf{THC} = \sum_{i=1}^{n} HC_i = c_1\{n\frac{P_1}{\theta}(\theta t_p + e^{-\theta t_p}+1) + n\frac{P_2}{\theta}(\theta(t_p-t_q) + e^{\theta(t_p-t_q)}-1) + \frac{D_0}{(mr)^{\gamma}(\alpha_1+\theta)}\frac{1}{\alpha_1\theta}[(\alpha_1 +$

$\theta) - \alpha_1 e^{-\theta t_p} - \theta e^{\alpha_1 t_s} + \alpha_1 e^{\alpha_1 t_p}(1-e^{\theta(t_p-t_q)}) + \alpha_1 e^{\alpha_1 t_q}(1-e^{\theta(t_q-t_s)})]\sum_{i=1}^{n} e^{(\alpha_1 t_{i-1}-\alpha\gamma t_{i-1})} -$

$\frac{1}{\theta}(1-e^{\theta(t_p-t_q)})\sum_{i=1}^{n} Q_{1i} + \frac{1}{\theta}(1-e^{\theta(t_q-t_s)}\sum_{i=1}^{n} Q_{2i}\}$

$= c_1\{n\frac{P_1}{\theta}(\theta t_p + e^{-\theta t_p} + 1) + n\frac{P_2}{\theta}(\theta(t_p - t_q) + e^{\theta(t_p-t_q)} - 1) + \frac{D_0}{(mr)^{\gamma}(\alpha_1+\theta)}\frac{1}{\alpha_1\theta}[(\alpha_1 +$

$\theta) - \alpha_1 e^{-\theta t_p} - \theta e^{\alpha_1 t_s} + \alpha_1 e^{\alpha_1 t_p}(1-e^{\theta(t_p-t_q)}) + \alpha_1 e^{\alpha_1 t_q}(1-e^{\theta(t_q-t_s)})]\frac{(e^{nv}-1)}{(e^{v}-1)} + \frac{nP_1}{\theta^2}(1 -$

$e^{\theta(t_p-t_q)})(1-e^{-\theta t_p}) - \frac{D_0}{(rm)^{\gamma}(\alpha_1+\theta)}\frac{1}{\theta}(1-e^{\theta(t_p-t_q)}(e^{-\theta t_p} - e^{\alpha_1 t_q})\frac{(e^{nv}-1)}{(e^{v}-1)} + \frac{P_2 n}{\theta^2}(1-e^{\theta(t_q-t_s)})(1-$

$e^{\theta(t_p-t_q)}) + \frac{P_1 n}{\theta^2}(1-e^{\theta(t_q-t_s)})(e^{\theta(t_p-t_q)} + e^{-\theta t_q}) + \frac{D_0}{(rm)^{\gamma}(\alpha_1+\theta)}\frac{1}{\theta}(1-e^{\theta(t_q-t_s)})(e^{\theta(t_p-t_q)} -$

$1)e^{\alpha_1 t_p}\frac{(e^{nv}-1)}{(e^{v}-1)} + \frac{D_0}{(rm)^{\gamma}(\alpha_1+\theta)}\frac{1}{\theta}(1-e^{\theta(t_q-t_s)})(e^{-\theta t_p} - e^{\alpha_1 t_p})\frac{(e^{nv}-1)}{(e^{v}-1)}\}$ (17)

The shortage cost for the i-th cycle (t_{i-1}, t_i) and during the whole business period (0,H) are respectively obtained as

$SHC_i = c_2[\int_{t_{s_{3i}}}^{t_i} -q_i(t)dt + \int_{t_{s_{4i}}} -q_i(t)dt]$

$= -c_2[\frac{D_0}{s_i^{\gamma}\alpha_1^2}e^{\alpha_1 t_{i-1}}(\alpha_1 t_r e^{\alpha_1 t_s} - \alpha_1 t_s e^{\alpha_1 t_s} + e^{\alpha_1 t_s}) + \frac{D_0}{s_i^{\gamma}\alpha_1^2}e^{\alpha_1 t_i}(\alpha_1\frac{H}{n} - \alpha t_r - 1) - \frac{P_2}{2}(\frac{H}{n} - t_r)^2]$

$\mathbf{TSHC} = -c_2[\frac{D_0}{(mr)^{\gamma}\alpha_1^2}(\alpha_1 t_r e^{\alpha_1 t_s} - \alpha_1 t_s e^{\alpha_1 t_s} + e^{\alpha_1 t_s})\frac{(e^{nv}-1)}{(e^{v}-1)} + \frac{D_0}{(mr)^{\gamma}\alpha_1^2}(\alpha_1\frac{H}{n} - \alpha t_r -$

$1)e^{v}\frac{(e^{nv}-1)}{(e^{v}-1)} - \frac{P_2}{2}(\frac{H}{n} - t_r)^2 n]$ (18)

The raw material cost of all units purchased for the i-th cycle (t_{i-1}, t_i) and during the whole business period (0,H) are respectively obtained as

$Rc_i = rQ_i$

$\mathbf{TRC} = r\sum_{i=1}^{n} Q_i = r[\frac{P_2 n}{\theta}(1-e^{(t_p-t_q)}) + \frac{P_1 n}{\theta}(e^{(t_p-t_q)} - e^{-\theta t_q}) + \frac{D_0(e^{\theta(t_p-t_q)}-1)}{(mr)^{\gamma}(\alpha_1+\theta)}e^{\alpha_1 t_p}\frac{(e^{nv}-1)}{(e^{v}-1)} +$

$\frac{D_0(e^{-\theta t_p} - e^{\alpha_1 t_p})}{(mr)^{\gamma}(\alpha_1+\theta)}$

$\frac{(e^{nv}-1)}{(e^{v}-1)} + \frac{D_0}{(mr)^{\gamma}\alpha_1}(e^{\alpha_1 t_r} - e^{\alpha_1 t_s})\frac{(e^{nv}-1)}{(e^{v}-1)}]$

(19)

In the finite time horizon, total interest can be earned in two ways:

(i) interest is earned from the revenue due to the continuous sale during the whole time horizon (i.e., TIECS) and

(ii)interest is earned from the part of the revenue remaining at the end of each cycle after paying due amount of this cycle and the advance (immediate part payment) for the next cycle from the total sale revenue for the present cycle. This is followed for all cycles except the first one (i.e., IET_n).

Therefore, for the first case, the interest earned ($IECS_i$) by continuous sale during the interval $(t_{i-1}, t_i + t_s)$ is given by

$$IECS_i = I_e[\int_{t_{i-1}}^{t_i - \mathbb{R}^{+M}} s_i D_i(t_{i-1} + M - t)dt + \int_{t_{i-1}+M}^{t_i - \mathbb{R}^{+t_s}} s_i D_i(t_{i-1} + t_s - t)dt]$$

$$= I_e[(t_{i-1} + M)\int_{t_{i-1}}^{t_i - \mathbb{R}^{+M}} s_i D_i dt + (t_{i-1} + t_s)\int_{t_{i-1}+M}^{t_i - \mathbb{R}^{+t_s}} s_i D_i dt - \int_{t_{i-1}}^{t_i - \mathbb{R}^{+t_s}} s_i D_i t dt]$$

$$= I_e \frac{(mr)^{1-\gamma} D_0}{\alpha_1}[(M - r^{'}\frac{H}{n})e^{\alpha_1 M} - M + \frac{1}{\alpha}(e^{\alpha_1 r^{'}\frac{H}{n}} - 1)e^{\{\alpha_1 + \alpha(1-\gamma)\}t_{i-1}}$$

Therefore, the total interest (TIECS) earned by continuous sale during the whole business period (0,H) is given by

$$\textbf{TIECS} = I_e \frac{(mr)^{1-\gamma} D_0}{\alpha_1}[(M - r^{'}\frac{H}{n})e^{\alpha_1 M} - M + \frac{1}{\alpha}(e^{\alpha_1 r^{'}\frac{H}{n}} - 1)\frac{(e^{nw}-1)}{(e^w-1)}] \quad (20)$$

Similarly, for the second case, the interest earned (IET_n) is given by

$$IET_n = I_e[M^{'}\{\int_{t_0}^{t_0 \mathbb{R}^M} s_1 D_1 dt + \int_{t_1}^{t_1 \mathbb{R}^M} s_1 D_1 dt + \int_{t_2}^{t_2 \mathbb{R}^M} s_2 D_2 dt + ... + \int_{t_{n-1}}^{t_n - \mathbb{R}^{+M}} s_1 D_1 dt -$$

$$(Q_1 R_1 + Q_2 R_2 + ... + Q_n R_n) + nA\} + M\{(Ts_1 - Q_1 P_1) + (Ts_1 + Ts_2 - Q_1 P_1 - Q_2 P_2) +$$

$$..... + (Ts_1 + Ts_2 + Ts_3 + Ts_4 + Ts_5 + ... + Ts_{n-1} - Q_1 P_1 - Q_2 P_2 - Q_3 P_3 - Q_4 P_4 - .... -$$

$$Q_{n-1} P_{n-1})\} + M^{'}\{(Ts_1 - Q_1 R_1) + (Ts_1 + Ts_2 - Q_1 R_1 - Q_2 R_2) + ... + (Ts_1 + Ts_2 +$$

$$Ts_3 + Ts_4 + Ts_5 + ... + Ts_{n-1} - Q_1 P_1 - Q_2 P_2 - Q_3 P_3 - Q_4 P_4 - .... - Q_{n-1} P_{n-1})\}]$$

$$= I_e[M^{'}\{\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j - \mathbb{R}^{+M}} s_j D_j dt - \sum_{j=1}^{n} Q_j P_j + nA\} + (M + M^{'})\{\sum_{j=1}^{n-1}(n-j)(Ts_j - Q_j R_j)\}]$$

Now $\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j - \mathbb{R}^{+M}} s_j D_j dt = \frac{(mr)^{1-\gamma} D_0 (e^{\alpha_1 M} - 1)}{\alpha_1} \sum_{j=1}^{n} e^{\alpha_1 t_{i-1} + \alpha(1-\gamma)t_{i-1}}$

$$= \frac{(mr)^{1-\gamma} D_0 (e^{\alpha_1 M} - 1)}{\alpha_1} \frac{(e^{nw}-1)}{(e^w-1)} \quad \text{(using Lemma 3)}$$

$$\sum_{j=1}^{n-1}(n-j)(Ts_j - Q_j R_j) = \sum_{j=1}^{n-1}(n-j)(s_j Q_j - Q_j R_j) = \sum_{j=1}^{n-1}(n-j)(s_j - R_j)Q_j$$

$$= (mr - \frac{r}{A^{\gamma_1}})[\{\frac{P_2}{\theta}(1 - e^{\theta(t_p - t_q)}) + \frac{P_1}{\theta}(e^{\theta(t_p - t_q)} - e^{\theta t_q})\sum_{j=1}^{n-1}(n-j)e^{\alpha t_{j-1}} + \frac{D_0}{(mr)^{\gamma}(\alpha_1 + \theta)}(e^{\theta t_q} -$$

$$e^{\alpha_1 t_p})\sum_{j=1}^{n-1}(n-j)e^{\{\alpha_1 + (1-\gamma\alpha)\}t_{j-1}} + \frac{D_0}{(rm)^{\gamma}\alpha_1}\sum_{j=1}^{n-1}(n-j)e^{\{\alpha_1 + (1-\gamma\alpha)\}t_{j-1}}\}]$$

Lemma 4: Prove that $\sum_{j=1}^{n} je^{jx} = \frac{ne^{(n+2)x} - (n+1)e^{(n+1)x} + e^x}{(e^x - 1)^2}$

Now $\sum_{j=1}^{n-1}(n-j)e^{\alpha t_{j-1}} = \sum_{j=1}^{n-1}(n-j)e^{\alpha(j-1)\frac{H}{n}} = e^{-\alpha\frac{H}{n}}\sum_{j=1}^{n-1}(n-j)e^{\alpha j\frac{H}{n}}$

$$= e^{-x}\sum_{j=1}^{n-1}(n-j)e^{jx}, \text{ where } x = \alpha\frac{H}{n}$$

$$= e^{-x}[n\sum_{j=1}^{n-1}e^{jx} - \sum_{j=1}^{n-1}je^{jx}]$$

$$= e^{-x}[n\frac{e^x(e^{(n-1)x}-1)}{(e^x-1)} - \frac{((n-1)e^{(n+1)x} - ne^{nx} + e^x)}{(e^x-1)^2}]$$

$$= [n\frac{(e^{(n-1)x}-1)}{(e^x-1)} - \frac{((n-1)e^{nx} - ne^{(n-1)x} + 1)}{(e^x-1)^2}] \quad \text{(using Lemma 4)}$$

Lemma 5: Prove that $\sum_{j=1}^{n-1}(n-j)e^{\{\alpha_1 + (1-\gamma)\alpha\}t_{j-1}} = [n\frac{(e^{(n-1)w}-1)}{(e^w-1)} - \frac{((n-1)e^{nw} - ne^{(n-1)w} + 1)}{(e^w-1)^2}]$

Also $\sum_{j=1}^{n} Q_j R_j = \frac{r}{A^{\gamma_1}}[\{\frac{P_2}{\theta}(1 - e^{\theta(t_p - t_q)}) + \frac{P_1}{\theta}(e^{\theta(t_p - t_q)} - e^{-\theta t_q})\}\sum_{j=1}^{n} e^{\alpha t_{j-1}} + \frac{D_0}{(rm)^{\gamma}(\alpha_1 + \theta)}$

$$(e^{-\theta t_q} - e^{\alpha_1 t_p})\sum_{j=1}^{n} e^{\{\alpha_1 + (1-\gamma)\alpha\}t_{j-1}} + \frac{D_0}{(rm)^{\gamma}\alpha_1}(e^{\alpha_1 t_r} - e^{\alpha_1 t_s})\sum_{j=1}^{n} e^{\{\alpha_1 + (1-\gamma)\alpha\}t_{j-1}}]$$

Now, $\sum_{j=1}^{n} e^{\alpha t_{j-1}} = \sum_{j=1}^{n} e^{\alpha(j-1)\frac{H}{n}} = e^{-\alpha\frac{H}{n}} \sum_{j=1}^{n} e^{\alpha j\frac{H}{n}} = e^{-x} \sum_{j=1}^{n} e^{jx} = e^{-x}\frac{e^x(e^{nx}-1)}{(e^x-1)}$
$=\frac{(e^{nx}-1)}{(e^x-1)}$
And $\sum_{j=1}^{n} e^{\alpha_1 t_{j-1}+\alpha(1-\gamma)t_{i-1}} = \frac{(e^{nw}-1)}{(e^w-1)}$
Therefore $\sum_{j=1}^{n} Q_j R_j = \frac{r}{A^{\gamma_1}}[\{\frac{P_2}{\theta}(1-e^{\theta(t_p-t_q)})+\frac{P_1}{\theta}(e^{\theta(t_p-t_q)}-e^{-\theta t_q})\}\frac{(e^{nx}-1)}{(e^x-1)}+\frac{D_0}{(rm)^\gamma(\alpha_1+\theta)}$
$(e^{-\theta t_q}-e^{\alpha_1 t_p})\frac{(e^{nw}-1)}{(e^w-1)}+\frac{D_0}{(rm)^\gamma\alpha_1}(e^{\alpha_1 t_r}-e^{\alpha_1 t_s})\frac{(e^{nw}-1)}{(e^w-1)}]$
Using **Lemma 5** IET_n is given by
$IET_n= I_e\{M^{'}[\frac{(mr)^{1-\gamma}D_0}{\alpha_1}(e^{\alpha_1 M}-1)\frac{(e^{nw}-1)}{(e^w-1)}-\frac{r}{A^{\gamma_1}}(\{\frac{P_2}{\theta}(1-e^{\theta(t_p-t_q)})+\frac{P_1}{\theta}(e^{\theta(t_p-t_q)}-$
$e^{-\theta t_q})\}\frac{(e^{nx}-1)}{(e^x-1)}+\frac{D_0}{(mr)^\gamma(\alpha_1+\theta)}(e^{-\theta t_q}-e^{\alpha_1 t_p})\frac{(e^{nw}-1)}{(e^w-1)}+\frac{D_0}{(mr)^\gamma\alpha_1}(e^{\alpha_1 t_r}-e^{\alpha_1 t_s})\frac{(e^{nw}-1)}{(e^w-1)})+$
$nA]+(M+M^{'})[(mr-\frac{r}{A^{\gamma_1}})(\{\frac{P_2}{\theta}(1-e^{\theta(t_p-t_q)})+\frac{P_1}{\theta}(e^{\theta(t_p-t_q)}-e^{-\theta t_q})\}(n\frac{(e^{(n-1)x}-1)}{(e^x-1)}-$
$\frac{((n-1)e^{nx}-ne^{(n-1)x}+1)}{(e^x-1)^2})+\frac{D_0}{(mr)^\gamma(\alpha_1+\theta)}(e^{-\theta t_q}-e^{\alpha_1 t_p})$
$(n\frac{(e^{(n-1)w}-1)}{(e^w-1)}-\frac{((n-1)e^{nw}-ne^{(n-1)w}+)}{(e^w-1)^2}))]\}$ (21)
The interest paid at the end of business period (0,H) is given by
TIP $= I_b$ AH
(22)
Total set-up cost at the end of the business period (0,H) is given by
TC_3=nc_3 (23)
Hence the total profit function for the retailer is given over the whole business period is given by
$Z(n, A) = TS+TIECS+IET_n-TRC-THC-TSHC-TC_3-TIP$ (24)
Here the optimization problem for this model is
Max Z(n,A)
(25)
Subject to the constrains
Assumption (iii) & (iv)
(26)

5.2 Case -II: Random time horizon

Chance Constraint for random time horizon

Here, the time horizon $\hat{H}$ is considered as random but the total time horizon is partitioned into n cycles with length T for each cycle. Therefore, following Section 4, corresponding chance constraint for the whole business period is given by $Prob(|(nT - \hat{H})| \leq \beta) \geq p_r$ which is reduced to $m_{\hat{H}} - \beta + \varepsilon_{\hat{H}}\sigma_{\hat{H}} \leq nT \leq m_{\hat{H}} + \beta - \varepsilon_{\hat{H}}\sigma_{\hat{H}}$
(27)

where $p_r = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{R} e^{-\frac{t^2}{2}} dt$ is a cumulative probability $Prob[t \leq \epsilon]$ available in standard table for different values of $\varepsilon_{\hat{H}}$.

Model-IIA: model with shortages

Considering $H = m_{\hat{H}}$ in Model-IA, the model with fully backlogged shortages in random time horizon is formulated. Hence, total profit function for the retailer over the whole business period is given by the Eq.(24).
Here the optimization problem for this model is
Max Z(n,A)
(28)
Subject to the constrains (26) & (27) (29)

6. Numerical Experiments

For illustration, we use following data for **Model-IA:**
D_0=15 units/year,$r = 2.5, r^{'} = 0.8, r^{''} = 0.9, t^{'} = 0.001, P_1 = 530, P_2 = 550\ \gamma = 0.08, \gamma_1 = 0.05, \alpha = 0.1, k_2 = 0.7, \theta = 0.3, m = 1.3, H = 12, c_3 = \$6/order,\ m_0 = 0.2, I_e = \$1.7/dollar/year,\ I_b = \$0.8/dollar/year, c_2 = \$4/unit/year, c_1 = \$3/unit/year$
Considering $\beta = 0.5, p_r = 0.6554, \varepsilon_{\hat{H}} = 0.4, m_{\hat{H}} = 11.1, \sigma_{\hat{H}} = 0.6$, together with the above numerical parameters the results of **Model-IIA** are derived.

Table 1:-Profit of different shortage period for the crisp Inventory model(i.e effect of $r^{'}$ and $r^{''}$on profit in Model-IA)

$r^{'}$	$r^{''}$	Z	A	n	M	THC	$TSHC$	TRC
0.55	0.85	34493.43	2759.615	3	2.959615	27282.22	2916.143	3869.383
0.56	0.84	34392.44	2766.780	3	2.966780	27657.29	2659.746	3912.416
0.57	0.83	34304.64	2773.887	3	2.973887	28028.99	2393.959	3955.110
0.58	0.82	34229.99	2780.938	3	2.980938	28397.35	2118.790	3997.467
0.59	0.81	34168.42	2787.932	3	2.987932	28762.41	1834.244	4039.488
0.6	0.8	34119.88	2794.872	3	2.994872	29124.22	1540.325	4081.176
0.61	0.79	34084.33	2801.756	3	3.001756	29482.82	1237.040	4122.533
0.62	0.78	34061.71	2808.585	3	3.008585	29838.24	924.3951	4163.559

Table 2:-Variation of profit with different $\varepsilon_{\hat{H}}$&$\sigma_{\hat{H}}$ parameters in random time horizon for stochastic inventory model (i.e effect of $\varepsilon_{\hat{H}}$&$\sigma_{\hat{H}}$ on profit -Model IIA)

$\varepsilon_{\hat{H}}$	$\sigma_{\hat{H}}$	Z	A	n	nT	M	T
	0.6	24333.74	2800.193	3	11.42	3.000193	3.806667
	0.7	24333.74	2792.192	3	11.39	2.992192	3.796667
0.3	0.8	24333.74	2784.202	3	11.36	2.984202	3.786667
	0.9	24333.74	2776.223	3	11.33	2.976223	3.776667
	1.0	24333.74	22768.256	3	11.30	2.968256	3.766667
	0.6	24248.77	2784.202	3	11.36	2.984202	3.786667
	0.7	24193.03	2773.566	3	11.32	2.973566	3.773333
0.4	0.8	24138.02	2762.951	3	11.28	3.962951	3.760000
	0.9	24083.73	2752.355	3	11.24	2.952355	3.746667
	1.0	24030.17	2741.779	3	11.20	2.941779	3.733333
	0.6	24165.43	2768.256	3	11.30	2.968256	3.766667
	0.7	24097.23	2755.002	3	11.25	2.955002	3.750000
0.5	0.8	24030.17	2741.779	3	11.20	2.941779	3.733333
	0.9	23964.23	2728.587	3	11.15	2.928587	30716667
	1.0	23899.42	2715.425	3	11.10	2.915425	3.700000

Table 3:-Variation of profit with different values of Interest earned in random time horizon for stochastic inventory model (i.e effect of I_e and $I_b$on profit-Model-II A)

I_e	I_b	Z	A	n	M
	0.9	21143.74	2682.486	3	2.882486
	0.8	24248.77	2784.202	3	2.984202
1.7	0.7	27469.66	2886.476	3	3.086476
	0.6	30807.04	2989.263	3	3.189263
	0.9	25991.75	2733.271	3	2.933271
	0.8	29151.41	2829.591	3	3.029591
1.8	0.7	32420.76	2926.390	3	3.126390
	0.6	35800.34	3023.631	3	3.223631
	0.9	30867.12	2778.834	3	2.978834
	0.8	34075.79	2870.293	3	3.070293
1.9	0.7	37388.59	2962.166	3	3.162166
	0.6	40805.97	3054.425	3	3.254425

7. Results and Discussions:

Optimal solutions for Model-IA is given in Table 1 for above set of input parameters. In Table 1, with different shortage periods in finite time horizon crisp inventory model (i.e., effect of $r^{'}$ & $r^{''}$on profit function of Model-IA), values of maximum profit are presented. It is noted from Table 1 and fig 3, that as the period of shortages decreases (i.e., $r^{'}$ increases), the total profit decreases as expected. It is also noted that with the increase in $r^{'}$ optimal values of immediate part payment also increases in Model-IA. Further increase in $r^{'}$ also increases the optimal profit and delay period.

Table 2 presents the values of profit function with different $\varepsilon_{\hat{H}}$ & $\sigma_{\hat{H}}$ in random time horizon (i.e., effects of $\varepsilon_{\hat{H}}$ & $\sigma_{\hat{H}}$ on profit: Model-IIA). From Table 2 and fig 1, it is revealed that as the total time horizon is nearer to 12.0 (the crisp value of H), profit becomes maximum. Here, as the dispersion about the mean value of $\hat{H}(\sigma_{\hat{H}})$ is small, time horizon is nearer to 12.0 and hence the profit is maximum. Thus with the increase

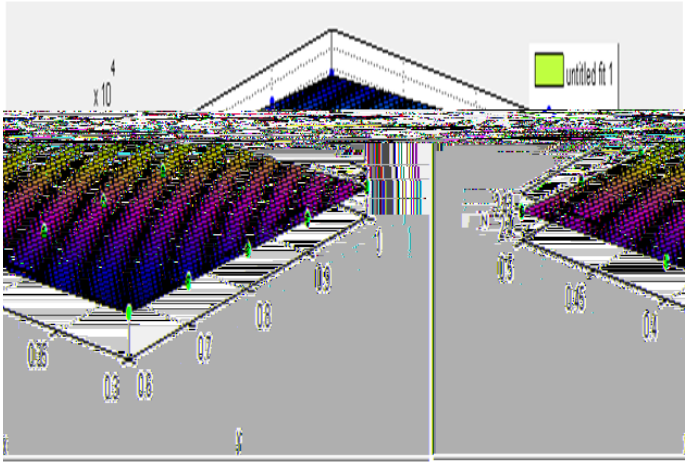

Figure 2. Effect of $\varepsilon_{\hat{H}}$ & $\sigma_{\hat{H}}$ on profit Z

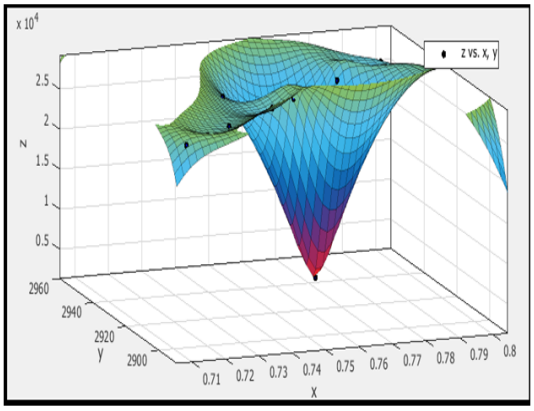

Figure 3. Effect of $r^{'}$ on profit Z and A(Model-IA)

in dispersion, profit decreases. Same trend is observed with the variation of $\varepsilon_{\hat{H}}$ .Smaller $\varepsilon_{\hat{H}}$ indicates that the chance constraint (27) is strictly satisfied and with the decrease of $\varepsilon_{\hat{H}}$ profit increases. Therefore, all these trends are as per the formulation of the model. Table 3 presents the values of profit function with different values of Interest earned in random time horizon (i.e., effect of I_e on profit function of Model-IIA). It is noted from Table 3 that as rate of interest earned increases the total profit increases as expected. It is also noted that with the increase in I_e optimal values of immediate part payment also increases in Model-IIA. Further decrease of I_b (interest payable) increases the optimal profit immediate part payment and delay period.

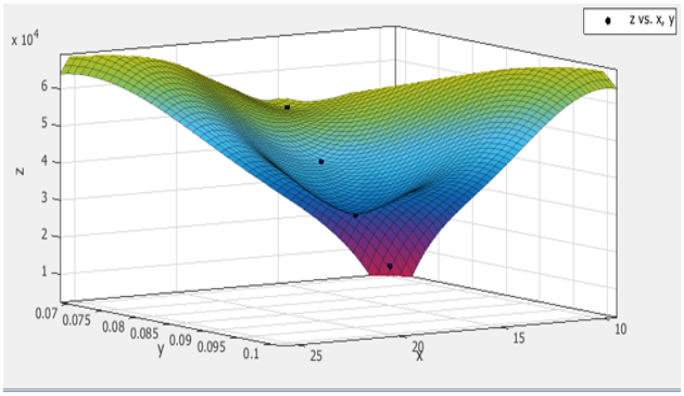

Figure 4. Effect of D_0, gama on profit Z (Model-IA)

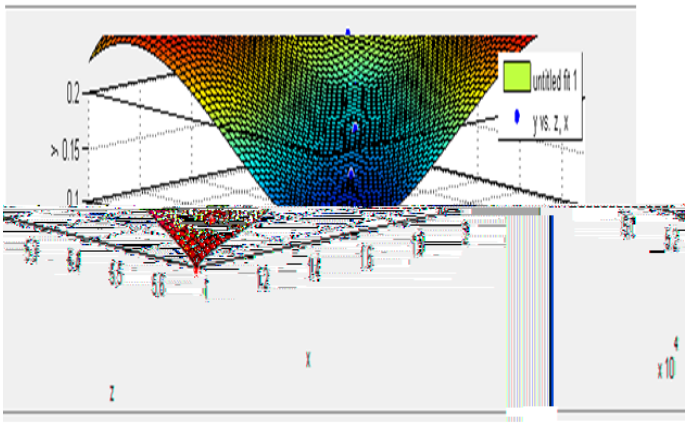

Figure 5. Effect of c_1,$r^{'}$ on profit Z (Model-IIA)

8. Sensitivity analyses:
The effects of change in the demand parameter (γ) on the immediate part payment (A) and the profit (Z) are studied and presented in Fig. 9. From this graph, it is observed that, when the value of γ increases, the profit and the immediate part payment decrease with same number of replenishment and almost same value of credit period, but the effect of changes in γ is significant, as with the increase in γ demand decreases and hence the optimum profit and advance payment decrease.

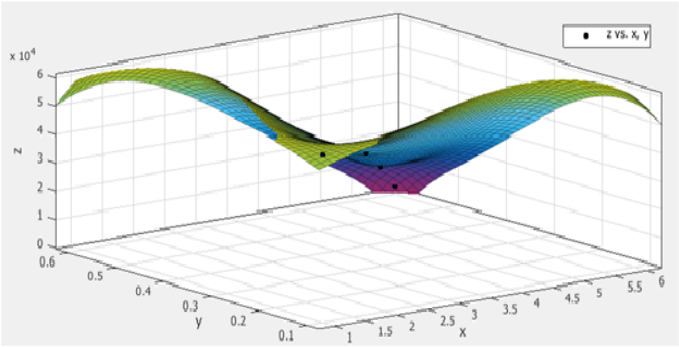

Figure 6. Effect of c_1,$r^{'}$ on profit Z (Model-IA)

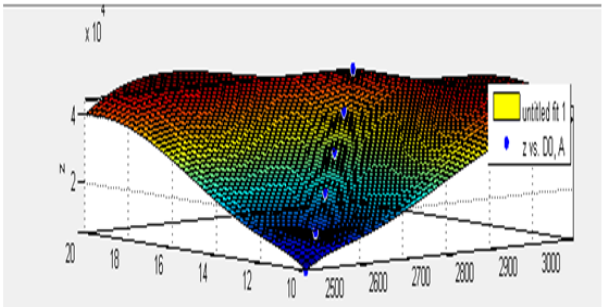

Figure 7. Effect of D_0 on profit Z and immediate part payment A(Model-IIA)

The effects of initial demand D_0 on the immediate part payment (A) and the profit (Z) are studied and presented in Fig. 7. From this graph, it is observed that, when the value of D_0 increases, the profit and the immediate part payment increase with same number of replenishment and almost same value of credit period, but the effect of changes in D_0 is significant, as with the increase of demand the profit also increase.
The effects of change of holding cost and setup cost on the profit (Z) are studied and presented in Fig. 8. From this graph, it is observed that, with the increase of holding

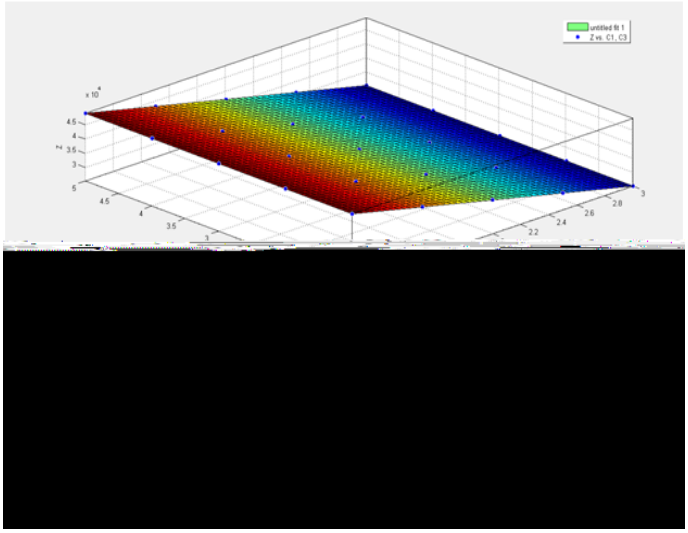

Figure 8. Effect of c_1,c_3 on profit Z (Model-IA)

Figure 9. Effect of γ on profit Z and immediate part payment A (Model-IA)

cost and setup cost the total profit increase.

9. Conclusions:

In this paper we introduce the concept of immediate part payment in EPQ model in both deterministic and random planning horizon under inflation with allowing shortages.

This study presents deterministic EPQ model for inflation rate and selling price dependent demand under a situation in which the supplier offers a trade credit period to his retailer to settle down the account for purchased quantities and reduced unit production cost against an immediate part payment paid on the receipt of the raw material by the retailer who, in this situation, makes loan from the money lending source. Numerical results for all models with immediate part payment show that immediate part payment plays an important role for the retailer to earn more profit. The intuitive reason is that, when the immediate part payment increases then credit period also increases. In this situation the retailer earn more by taking loan from money lending source. Hence this analysis answers to the retailer's dilemma how much to make for immediate part payment to enjoy the wholesaler's concessions for maximum profit in spite of the fact that more part payment means more loan and more interest paid.

The above model can be extended in various ways. We should extend this model for two trade credit; taking partial trade credit; taking demand, selling price as a fuzzy number etc.

A. Appendix:

Lemma 1: Prove that $\sum_{i=1}^{n} e^{(\alpha_1 t_{i-1}-\alpha\gamma t_{i-1})} = \frac{(e^{nv}-1)}{(e^{v}-1)}$

Where $\frac{H}{n} = t_{i-1} - t_i$,$v = (\alpha_1 - \alpha\gamma)\frac{H}{n}$

Proof: $\sum_{i=1}^{n} e^{(\alpha_1 t_{i-1}-\alpha\gamma t_{i-1})} = \sum_{i=1}^{n} e^{(\alpha_1-\alpha\gamma)t_{i-1}}$

$= \sum_{i=1}^{n} e^{(\alpha_1-\alpha\gamma)(i-1)\frac{H}{n}}$

$= \sum_{i=1}^{n} e^{(\alpha_1-\alpha\gamma)i\frac{H}{n}} e^{-(\alpha_1-\alpha\gamma)\frac{H}{n}}$

$= \sum_{i=1}^{n} e^{-v}e^{iv} = e^{-v}\sum_{i=1}^{n} e^{iv} = e^{-v}\frac{e^{v}(e^{nv}-1)}{(e^{v}-1)} = \frac{(e^{nv}-1)}{(e^{v}-1)}$ (A.1)

Lemma 2: Prove that $\sum_{i=1}^{n} e^{\alpha_1 t_i+\alpha(1-\gamma)t_{i-1}} = e^{-x}\frac{e^{w}(e^{nw}-1)}{(e^{w}-1)}$

Where $\frac{H}{n} = t_{i-1} - t_i$,$x = \alpha(1-\gamma)\frac{H}{n}$,$w = [\alpha_1 + \alpha(1-\gamma)]\frac{H}{n}$

Proof: $\sum_{i=1}^{n} e^{\alpha_1 t_i+\alpha(1-\gamma)t_{i-1}} = \sum_{i=1}^{n} e^{\alpha_1 i\frac{H}{n}+\alpha(1-\gamma)(i-1)\frac{H}{n}}$

$= \sum_{i=1}^{n} e^{-\alpha(1-\gamma)\frac{H}{n}} e^{i[\alpha_1+\alpha(1-\gamma)\frac{H}{n}]}$

$= e^{-\alpha(1-\gamma)\frac{H}{n}} \sum_{i=1}^{n} e^{i[\alpha_1+\alpha(1-\gamma)\frac{H}{n}]}$

$= e^{-x}\sum_{i=1}^{n} e^{iw} = e^{-x}\frac{e^{w}(e^{nw}-1)}{(e^{w}-1)}$ (A.2)

Lemma 3: Prove that $\sum_{i=1}^{n} e^{\alpha_1 t_{i-1}+\alpha(1-\gamma)t_{i-1}} = \frac{(e^{nw}-1)}{(e^{w}-1)}$

Where $\frac{H}{n} = t_{i-1} - t_i$,$w = [\alpha_1 + \alpha(1-\gamma)]\frac{H}{n}$

Proof: $\sum_{i=1}^{n} e^{\alpha_1 t_{i-1}+\alpha(1-\gamma)t_{i-1}} = \sum_{i=1}^{n} e^{(i-1)[\alpha_1+\alpha(1-\gamma)]\frac{H}{n}} = \sum_{i=1}^{n} e^{-[\alpha_1+\alpha(1-\gamma)]\frac{H}{n}} e^{i[\alpha_1+\alpha(1-\gamma)]\frac{H}{n}}$

$= e^{-[\alpha_1+\alpha(1-\gamma)]\frac{H}{n}} \sum_{i=1}^{n} e^{i[\alpha_1+\alpha(1-\gamma)]\frac{H}{n}} = e^{-w}\frac{e^{w}(e^{nw}-1)}{(e^{w}-1)} = \frac{(e^{nw}-1)}{(e^{w}-1)}$ (A.3)

Lemma 4: Prove that $\sum_{j=1}^{n} je^{jx} = \frac{ne^{(n+2)x}-(n+1)e^{(n+1)x}+e^{x}}{(e^{x}-1)^2}$

Proof: We have $\sum_{j=1}^{n} e^{jx} = \frac{e^{x}(e^{nx}-1)}{(e^{x}-1)}$ (A.4)

Differentiating both sides of (A.4) with respect to x

$$\sum_{j=1}^{n} je^{jx} = \frac{(e^x-1)\{e^x(e^{nx}-1)+e^x ne^{nx}\}-e^x(e^{nx}-1)e^x}{(e^x-1)^2} = \frac{ne^{(n+2)x}-(n+1)e^{(n+1)x}+e^x}{(e^x-1)^2} \tag{A.5}$$

Lemma 5: Prove that $\sum_{j=1}^{r-1}(n-j)e^{\{\alpha_1+(1-\gamma)\alpha\}t_{j-1}} = [n\frac{(e^{(n-1)w}-1)}{(e^w-1)} - \frac{((n-1)e^{nw}-ne^{(n-1)w}+1)}{(e^w-1)^2}]$

Proof: $\sum_{j=1}^{r-1}(n-j)e^{\{\alpha_1+(1-\gamma)\alpha\}t_{j-1}} = \sum_{j=1}^{r-1}(n-j)e^{\{\alpha_1+(1-\gamma)\alpha\}(j-1)\frac{H}{n}}$

$=e^{-\{\alpha_1+(1-\gamma)\alpha\}\frac{H}{n}}\sum_{j=1}^{r-1}(n-j)e^{\{\alpha_1+(1-\gamma)\alpha\}j\frac{H}{n}}$

$=e^{-w}\sum_{j=1}^{r-1}(n-j)e^{jw},$

$=e^{-w}[n\sum_{j=1}^{r-1}e^{jw} - \sum_{j=1}^{r-1}je^{jw}]$

$=e^{-w}[n\frac{e^w(e^{(n-1)w}-1)}{(e^w-1)} - \frac{((n-1)e^{(n+1)w}-ne^{nw}+e^w)}{(e^w-1)^2}]$

$=[n\frac{(e^{(n-1)w}-1)}{(e^x-1)} - \frac{((n-1)e^{nw}-ne^{(n-1)w}+1)}{(e^w-1)^2}]$ (Using Lemma 4) (A.6)

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Bayesian estimation and prediction for the generalized Lindley distribution under asymmetric loss function

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Abstract

The paper develops the Bayesian estimation procedure for the generalized Lindley distribution under squared error and general entropy loss functions in case of complete sample of observations. For obtaining the Bayes estimates, both non-informative and informative priors are used. Monte Carlo simulation is performed to compare the behaviour of the proposed estimators with the maximum likelihood estimators in terms of their estimated risks. Discussion is further extended to Bayesian prediction problem based on an informative sample where an attempt is made to derive the prediction intervals for future observations. Numerical illustrations are provided based on a real data example.

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1. Introduction

The one parameter exponential distribution is the most popular distribution for analysing the lifetime data due to its simplicity and has also been widely applied in many areas. But its applicability is restricted to constant hazard rate. It seems quite unrealistic to assume time independent hazard rate for any real-life system. [15] introduced a distribution in the context of Bayes theorem, as a counter example of fudicial statistics named as Lindley distribution which overcomes the drawback of exponential distribution as it does not permit the constant hazard rate. The various papers have been published on the parameter estimation for Lindley distribution under different set-up. [11] have provided a comprehensive treatment of the mathematical properties of Lindley distribution. [14] have discussed the procedures of reliability estimation for progressively type-II censored

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sample under both classical and Bayesian set-up when the data follows the Lindley distribution. They have also suggested that this distribution can be used as a life time model.

The generalizations of exponential distribution are Weibull and Gamma distributions which are most frequently used lifetime models in survival analysis. However, these distributions have their own limitations. The Weibull distributions are more popular than the gamma because the survival function of the latter cannot be expressed in a closed form and one needs numerical integration. But, the Weibull distribution also has its own disadvantages. [2] have pointed out that the maximum likelihood estimators of the Weibull parameters may not behave properly for the whole parameter space. The above three distributions (exp, Gamma, and Weibull) are inappropriate when the failure rate is indicated to be uni-modal or bath-tub shaped.

Recently, [16] have introduced a two parameter generalization of Lindley distribution as an extended model for modelling of bathtub data alternative to gamma, lognormal, Weibull, and exponentiated exponential distributions. The generalized Lindley distribution has the following probability density function (PDF)

$$f(x) = \frac{\alpha\lambda^2}{(1+\lambda)}(1+x)\left[1 - \frac{1+\lambda+\lambda x}{(1+\lambda)}\right]^{\alpha-1} e^{-\lambda x} \tag{1.1}$$

It is denoted by $GLD(\alpha, \lambda)$. The corresponding CDF is given by

$$F(x) = \left[1 - \frac{1+\lambda+\lambda x}{(1+\lambda)} e^{-\lambda x}\right]^{\alpha} \tag{1.2}$$

For $\alpha = 1$, it reduces to one parameter Lindley distribution (see Figure 1).

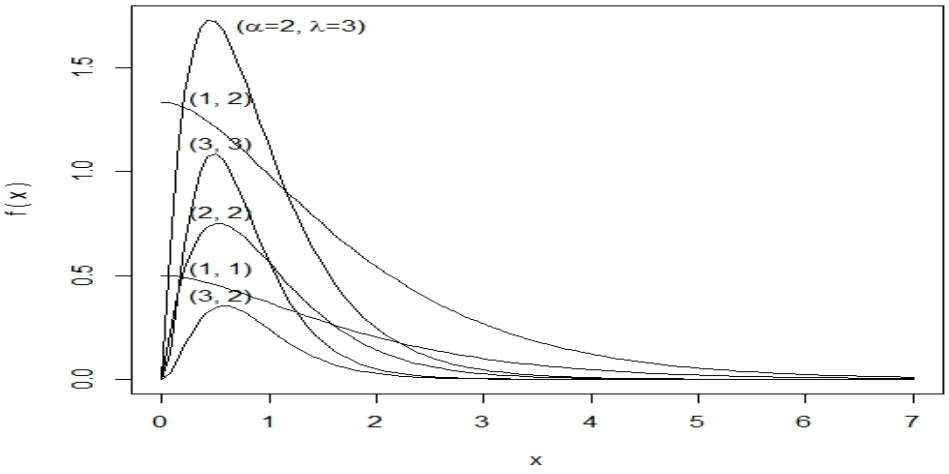

Figure 1: Density function for different values of α and λ

[16] have discussed different properties and the inferential procedure for this distribution under classical paradigm. They have also checked in many ways the goodness-of-fit of this distribution to real data set over gamma, Weibull, and log-normal distributions.

In Bayesian approach, a prior distribution for the parameter is considered and there is no clear cut rule by which one can say that one prior is better than any other. The prior information is purely subjective assessment of an expert before any data have been observed. Also [3] argues that when information is not in compact form the Bayesian

analysis uses non-informative prior consideration.

For some distributions, the posterior density is not in tractable form as it contains integrals. The Bayes estimation cannot be performed explicitly and becomes quite complicated. In such a situation, some approximate methods are inevitable such as numerical integration or analytic approximation techniques. Alternatively, many authors use Monte Carlo morkov chain (MCMC) methods namely Gibbs sampling [22] and Metropolis Hasting [13, 4] algorithms to simulate the deviates from posterior density so that sample based inference can be done.

It is often seen that usual squared error loss function (SELF) is considered for Bayesian procedure which is symmetrical and associates equal importance to the losses due to overestimation and underestimation of equal magnitude. But it may be illogical to assume symmetric loss function. In some situation overestimation is more serious than underestimation or vice-versa. For example, in the estimation of reliability and failure rate functions, an overestimation is usually much more serious than an underestimation. A more appropriate asymmetric loss function, general entropy loss function (GELF) is introduced by [5], defined as

$$L\left(\theta,\hat{\theta}\right)=\left(\frac{\hat{\theta}}{\theta}\right)^{C}-C\log\left(\frac{\hat{\theta}}{\theta}\right)-1 \tag{1.3}$$

Where, $\hat{\theta}$ is the estimate of θ and C is the loss parameter which reflects the departure from symmetry. The loss parameter C allows different shapes of this loss function. For $C>0$, a positive error $\left(\hat{\theta}>\theta\right)$causes more serious consequences than a negative error and vice versa. The Bayes estimate $\hat{\theta_G}$ of θ under GELF is given by

$$\hat{\theta}_G=\left[E_\pi\left(\theta^{-C}|\underset{\sim}{x}\right)\right]^{(-1/C)} \tag{1.4}$$

provided that the expectation $E_\pi\left(\theta^{-C}|\underset{\sim}{x}\right)$ exits and is finite. Here, E_π denote the expectations with respect to posterior pdf of θ. The Bayes estimate of θ under SELF is the posterior mean of θ. For $C=-1$, equation (4) provides the Bayes estimator under SELF for θ.

The performance of the Bayes estimators depends upon the prior distribution and the loss functions to be considered. Many authors have been studying the performance of the Bayes estimator under various assumptions regarding the prior belief and loss functions (symmetric and asymmetric). [21] and [7] have obtained the Bayes estimators under different loss functions for inverse Gaussian distribution and generalized exponential distribution respectively. [20] have proposed to obtain the Bayes estimator for the parameter of exponentiated gamma distribution under general entropy loss function as they have found that the Bayes estimator obtained under GELF have the smaller risk than considered existing estimators.

Prediction of future observations based on information from existing data is concerned topic in Statistics, arising in many contexts in a natural way. [9] have developed the Bayesian procedure to the prediction problems of future observables. Prediction problems can be generally classified into two types. 1.) One sample prediction 2.) Two sample prediction. In earlier case, the variable to be predicted comes from the same sequence of variables observed and is dependent on observed data, and arises in case of

censored data. In two sample prediction, the variable to be predicted comes from another independent future sample follow the same distribution function. Bayesian prediction for future observations based on certain distributions has been discussed by several authors [see 8, 23, 19, 1, 17, 10, 18, 12, and references cited therein].

This distribution is not considered earlier in literature through Bayesian setup. Therefore in this paper, our aim is to perform the Bayesian estimation for the unknown parameters of the generalized Lindley distribution under different forms of the gamma priors in case of symmetric and asymmetric loss functions. We have also discussed the two sample prediction problems for considered model.

The rest of the paper is organized as follows. In section 2, the maximum likelihood estimators under type II censored data, along with its asymptotic properties are discussed. In section 3, for obtaining the Bayes estimates for α and λ the MCMC techniques have been utilized. To implement the Gibbs algorithm full condition posteriors for α and λ are derived. Bayesian prediction is discussed in section 4. We have also constructed the predictive bounds for future observations. In section 5, a simulation study is conducted to check the behaviour of our proposed estimators. In section 6, a real data set is analyzed. Finally, conclusions have been given in section 7.

2. Maximum likelihood estimates

Suppose that $\{x_1, x_2, ..., x_n\}$ is the independent and identical (IID) random sample of size n obtained from (1). Then the likelihood equation is given by

$$L\left(\underset{\sim}{x}|\alpha,\lambda\right) = \left[\frac{\alpha\lambda^2}{(1+\lambda)}\right]^n e^{-\lambda\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}(1+x_i)\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}\right]^{\alpha-1} \tag{2.1}$$

Then, the log likelihood function can be written as

$$LogL = n\log\left[\frac{\alpha\lambda^2}{(1+\lambda)}\right] - \lambda\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log(1+x_i) + (\alpha-1)\sum_{i=1}^{n}\log\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}\right] \tag{2.2}$$

The MLEs of α and λ can be obtained as the simultaneous solution of the following equations

$$\frac{n}{\alpha} + \sum_{i=1}^{n}\log\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}\right] = 0 \tag{2.3}$$

$$\frac{2n}{\lambda} - \frac{n}{(1+\lambda)} + \frac{\lambda(\alpha-1)}{(1+\lambda)^2}\sum_{i=1}^{n}\frac{(2+x_i+\lambda(1+x_i))}{\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}\right]} x_i e^{\lambda x_i} \tag{2.4}$$

In order to solve the above equations, we can apply any suitable iterative procedure such as Newton RaphsonÔÇÖs method.
Under some regularity conditions, the MLEs $(\hat{\alpha},\hat{\lambda})$ is approximately bi-variate normal with mean $(\hat{\alpha},\hat{\lambda})$ and covariance matrix $I^{-1}\left(\hat{\alpha},\hat{\lambda}\right)$ i.e.$(\hat{\alpha},\hat{\lambda}) \sim N_2\left(\left[\hat{\alpha}\ \hat{\lambda}\right]', I^{-1}\left(\hat{\alpha},\hat{\lambda}\right)\right)$.
where $I\left(\hat{\alpha},\hat{\lambda}\right)$ is the observed FisherÔÇÖs information matrix, and defined as

$$I\left(\hat{\alpha},\hat{\lambda}\right) = \begin{bmatrix} -\frac{dlogL}{d\alpha^2} & -\frac{dlogL}{d\lambda d\alpha} \\ -\frac{dlogL}{d\alpha d\lambda} & -\frac{dlogL}{d\lambda^2} \end{bmatrix}_{\left(\hat{\alpha},\hat{\lambda}\right)}$$

The diagonal elements of $I^{-1}\left(\hat{\alpha},\hat{\lambda}\right)$ provides the asymptotic variances for the parameters α and λ respectively. Then two-sided $100(1-\gamma)\%$ normal approximation confidence interval of α and λ can be defined as $\left\{\hat{\alpha}\mp Z_{\gamma/2}\sqrt{var(\hat{\alpha})}\right\}$ and $\left\{\hat{\lambda}\mp Z_{\gamma/2}\sqrt{var(\hat{\lambda})}\right\}$ respectively.

3. Bayes estimation

In this section, we have developed the Bayesian estimation procedure for the parameters of $GLD(\alpha,\lambda)$ assuming independent informative priors for the unknown parameters. It is assumed that the parameters α and λ are independent. The variation in the parameters α and λ have been incorporated in the analysis through Gamma priors having the following density function

$$g(\lambda)=\frac{a_1^{b_1}}{\Gamma b_1}e^{-a_1\lambda}\lambda^{b_1-1};\lambda,a_1,b_1>0 \tag{3.1}$$

$$g(\alpha)=\frac{a_2^{b_2}}{\Gamma b_2}e^{-a_2\alpha}\alpha^{b_2-1};\alpha,a_2,b_2>0 \tag{3.2}$$

Where, a_1,a_2,b_1, and b_2 are the hyper-parameters. Then, the joint posterior PDF of α and λ is defined as

$$\pi\left(\alpha,\lambda\mid\underset{\sim}{x}\right)=\frac{L\left(\underset{\sim}{x}|\alpha,\lambda\right)g(\alpha)g(\lambda)}{\int_0^\infty\int_0^\infty L\left(\underset{\sim}{x}|\alpha,\lambda\right)g(\alpha)g(\lambda)d\alpha d\lambda} \tag{3.3}$$

The Gibbs sampling techniques is used for simulating the sample from the joint posterior density. The full conditional posterior density, for implementing the Gibbs algorithm, of each of unknowns is obtained by regarding all other parameters in (11) as known. Then, we have

$$\pi_1\left(\alpha\mid\lambda,\underset{\sim}{x}\right)\propto\alpha^{n+b_2-1}e^{-\alpha a_2}\prod_{i=1}^{n}\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}e^{-\lambda x_i}\right]^{\alpha-1} \tag{3.4}$$

$$\pi_2\left(\lambda\mid\alpha,\underset{\sim}{x}\right)\propto\frac{\lambda^{2n+b_1-1}}{(1+\lambda)^n}e^{-\lambda(a_1+\sum_{i=1}^{n}x_i)}\prod_{i=1}^{n}\left[1-\frac{1+\lambda+\lambda x_i}{(1+\lambda)}e^{-\lambda x_i}\right]^{\alpha-1} \tag{3.5}$$

The Gibbs algorithm consist the following steps

(1) Start with j=1, and initial values of $\{\alpha^{(0)},\lambda^{(0)}\}$.
(2) Generate new draws for α and λ as follows:

   (a) $\alpha^{(j)}\sim\pi_1\left(\alpha\mid\lambda^{(j-1)},\underset{\sim}{x}\right)$
   (b) $\lambda^{(j)}\sim\pi_2\left(\lambda\mid\alpha^{(j)},\underset{\sim}{x}\right)$

   In this step, the Metropolis algorithm is used to generate deviates from the full conditional posterior densities by choosing any arbitrary proposal distribution.
(3) Repeat step 2 for all j= 1, 2,..., M and obtained $(\alpha_1,\lambda_1),(\alpha_2,\lambda_2),...(\alpha_M,\lambda_M)$.

Once we have the posterior sample, we can use discrete formulas, applied to these samples to approximate the integrals of interest. By this sampling approach, the Bayes estimates of α and λ with respect to the general entropy loss function are as follows

$$\hat{\alpha}_G=\left[E_\pi\left(\alpha^{(-C)}\mid\underset{\sim}{x}\right)\right]^{(-1/C)}\approx\left(\frac{1}{M-M_0}\sum_{k=1}^{M-M_0}\alpha_k^{-C}\right)^{(-1/C)}$$

$$\hat{\lambda}_G = \left[E_\pi\left(\lambda^{(-C)} \mid \underset{\sim}{x}\right)\right]^{(-1/C)} \approx \left(\frac{1}{M-M_0}\sum_{k=1}^{M-M_0}\lambda_k^{-C}\right)^{(-1/C)}$$

Where, M_0 is the burn-in-period of Markov Chain. The HPD credible intervals for α and λ can be constructed by using algorithm given in [6]. Let $\{(\alpha_{(i)}, \lambda_{(i)}); i = 1, 2, ..., M\}$ be the corresponding ordered MCMC sample of $\{(\alpha_i, \lambda_i); i = 1, 2, ..., M\}$. Then construct all the $100(1-\psi)\%$ credible intervals for of α and λ as

$(\alpha_{[1]}, \alpha_{[M(1-\psi)]}), ..., (\alpha_{[M\psi]}, \alpha_{[M]})$ and $(\lambda_{[1]}, \lambda_{[M(1-\psi)]}), ..., (\lambda_{[M\psi]}, \lambda_{[M]})$.

Here, $[x]$ denotes the largest integer less than or equal to x. Then the HPD credible interval is that interval which has the shortest length.

4. Bayesian Prediction

In this section, we have discussed the Bayesian prediction of future ordered sample on the basis of observed data. Let $y_{(1)}, y_{(2)}, ..., y_{(m)}$ be another sample (called future sample) of size m, independent of the informative sample $x_{(1)}, x_{(2)}, ..., x_{(m)}$ from the same distribution. The density function of $y_{(s)}$ th ordered future sample is

(4.1)
$$p\left(y_{(s)} \mid \alpha, \lambda\right) = \frac{m!}{(s-1)!(m-s)!}\left[F\left(y_{(s)} \mid \alpha, \lambda\right)\right]^{s-1}\left[1-F\left(y_{(s)} \mid \alpha, \lambda\right)\right]^{m-s} f\left(y_{(s)} \mid \alpha, \lambda\right)$$

By substituting (1) and (2) in (14), we get

(4.2)
$$p\left(y_{(s)} \mid \alpha, \lambda\right) = \frac{m!\alpha\lambda^2}{(s-1)!(m-s)!}\frac{(1+y_{(s)})}{(1+\lambda)}e^{-\lambda y_{(s)}} \sum_{j=0}^{m-s}(-1)^j\binom{m-s}{j}\left(1-\frac{1+\lambda+\lambda y_{(s)}}{(1+\lambda)}e^{-\lambda y_{(s)}}\right)^{\alpha(s+j)-1}$$

The Bayesian predictive density of $y_{(s)}th$ ordered future sample can be obtained as

(4.3)
$$p\left(y_{(s)} \mid \underset{\sim}{x}\right) = \int_0^\infty\int_0^\infty p\left(y_{(s)} \mid \alpha, \lambda\right)\pi\left(\alpha, \lambda \mid \underset{\sim}{x}\right)d\alpha d\lambda$$

It is to be noted that $p\left(y_{(s)} \mid \underset{\sim}{x}\right)$ cannot be expressed in closed form and hence it cannot be evaluated analytically. The MCMC sample $\{(\alpha_i, \lambda_i); i = 1, 2, ..., M\}$ obtained from $\pi\left(\alpha, \lambda \mid \underset{\sim}{x}\right)$ using Gibbs sampling has been utilized to obtain the consistent estimate of $p\left(y_{(s)} \mid \underset{\sim}{x}\right)$. It can be obtained as

(4.4)
$$\hat{p}\left(y_{(s)} \mid \underset{\sim}{x}\right) = \frac{1}{M}\sum_{i=1}^{M}p\left(y_{(s)} \mid \alpha_i, \lambda_i\right)$$

Thus, we can obtain the two sided $100(1-\psi)\%$ prediction interval (L, U) for future sample by solving the following two equations:

$$P\left(Y_{(s)} > U \mid \underset{\sim}{x}\right) = \frac{\psi}{2}$$

and

$$P\left(Y_{(s)} > L \mid \underset{\sim}{x}\right) = 1 - \frac{\psi}{2}$$

It is not possible to obtain the solutions analytically. We need to apply suitable numerical techniques for solving these non-linear equations.

5. Simulation Study

This section contains the simulation study which is carried out to compare the behaviour of the different Bayes estimators and Maximum likelihood estimators of the shape and scale parameters. The comparisons are made on the basis of simulated risk (expectation of loss function over sample space) and width of the confidence/HPD intervals. The risk function of the estimators cannot be expressed in closed form. Therefore, the risks of the estimators are estimated on the basis of simulated sample. For this purpose, we generate 1000 samples of size n (small sample size n = 20, moderate sample size n = 30, and large sample size n = 50) from generalized Lindley distribution by using inversion method suggested by [16]. The estimators $\hat{\alpha}_M$ and $\hat{\lambda}_M$denote the MLEÔÇÖs of the parameters α and λ respectively.$(\hat{\alpha}_G, \hat{\lambda}_G)$ and $(\hat{\alpha}_s, \hat{\lambda}_s)$ are the corresponding Bayes estimators under GELF and SELF respectively. The risks of the estimators under GELF and SELF are denoted by $R_G(\cdot)$ and $R_s(\cdot)$ respectively. We have looked into many ways to view the clear picture of the performance of the considered competing estimators.

(1) For varying sample size when parameters are fixed (see Tables 1 and 2).
(2) For different values of hyper parameters along with increasing sample size when both shape and scale parameters are fixed (see Tables 1 and 2).
(3) For the choice of the loss parameter C of GELF when sample size and model parameters are fixed (see Figures 2, 3, 4 and 5).
(4) For variation in the model parameters for fixed sample size, loss parameter, and hyper parameters (see Tables 3, 4, 5 and 6).

The MCMC sample have been utilize for Bayesian analysis so that sample based inference can be done. We check the convergence of the MCMC sample and it is found that Markov chain converges rapidly with any arbitrary initial starting. All computational algorithms are calculated with help of R software.

In case of non-informative prior, we take $a_1 = a_2 = b_1 = b_2 = 0$. For informative prior, the hyper parameters can be chosen easily if we take prior mean (say, ν) equals to the true value of the parameter with varying prior variance (say, η). The prior variance indicates the confidence of our prior guess. A large prior variance shows less confidence in prior guess and resulting prior distribution is relatively flat. On other hand, small prior variance indicates greater confidence in prior guess. Another case is also considered where prior mean is taken away from the true value of the parameter with moderate variance.

From the Table 1, it is observed that the risks of all the estimators decrease as sample size increases in all the considered cases. The width of the confidence/HPD intervals also decreases as sample size increases. As confidence of prior guess decreases, the risks of the estimators and width of the HPD intervals both increases. We can also conclude that the maximum likelihood estimator is less efficient than the Bayes estimator since MLE shows the larger Risk/MSE in all the cases.

The risks of all the estimators under GELF increases as loss parameter C increase in all the considered cases, although the increase is more for maximum likelihood estimators. The behaviour of all considered estimator remain same for all combinations of

model parameters and hyper parameters. ThatÔÇÖs why; we are presenting selected figures. When over estimation is more serious than under estimation i.e. $C > 0$, the risks of $\hat{\alpha}_G$ and $\hat{\lambda}_G$ under GELF are found to be least among the considered competing estimators (see Figure 3). When under estimation is more serious than over estimation i.e. $C < 0$, the magnitudes of the risks of the estimators obtained under SELF and GELF for both parameters are more or less same but least than that of MLEs (see Figure 2).

Under SELF, the risks of $\hat{\alpha}_G$ and $\hat{\alpha}_G$ are going to be smaller than that of $\hat{\alpha}_s$ and $\hat{\alpha}_s$ when $C > 0$. When under estimation is more serious than over estimation, the Bayes estimators obtained under GELF have smaller risk than that of obtained under SELF for $C < -1$, and for $C > 1$, the Bayes estimators obtained under GELF become less efficient than Bayes estimators obtained under SELF (see Figures 4 and 5).

For comparing the performance of the estimators for different combination of model parameters, we fix $C = 1.5$ when over estimation is more serious than under estimation and $C = -0.5$ in case of under estimation is more serious than over estimation.

Table 3, 4, and 5 shows the performance of the proposed estimators for the variations of the model parameters α and λ under for fixed values of $n = 20$ and C. it is to be noticed that as the value of α increases the magnitude of the risk of the estimators of α increases but reveres trend have found in the magnitude of the risk of the estimators of λ for fixed values of λ under both loss functions. Further, it is also noticed that as λ increases for fixed α the magnitude of the risk of α decreases but increment in the magnitude of the risk of λ is noticed under both functions for any given values of n and C. The similar trends are found in case of the width of the confidence/HPD intervals (see Table 6).

6. Real data analysis

In this section, a real data set is analyzed to verify our proposed estimation procedure. The data set of relief times of twenty patients has taken from [16]. It has been observed by [16] that the generalized Lindley distribution can be effectively used to analyse this data set. The relief times of the twenty patients are as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1,1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The maximum likelihood estimates, Bayes estimates under SELF along with their confidence/HPD intervals and Bayes estimates under GELF are presented in Table 7. The MLEs of shape and scale parameters are same as obtained by [16]. The width of the HPD interval is smaller than that of classical interval for both parameters. For this data set, Bayesian analysis is carried out in case of non-informative priors. The MCMC iterations of α and λ are plotted in Figure 6(a) and 6(b) respectively. Trace plots are indicating that the MCMC samples are well mixed and stationary achieved. The $3 - D$ histogram of the posterior samples of α and λ is plotted in Figure 7. This figure indicates that posterior distributions of α and λ are symmetric. [16] have checked the goodness-of-fit to the real data set using maximum likelihood estimates. For demonstrating the goodness-of-fit of the this real data set for the Bayes estimates, We have applied Kolmogorov-Smirnov (KS) one sample test and KS distances are presented in the Table 9. The empirical CDF and fitted CDF (for maximum likelihood estimates and Bayes estimates) have been plotted in figure 10. It is observed that the goodness-of-fit to the real data set is quite acceptable even with the Bayes estimates.

We also predict the first three and last observations of independent future sample of size $m(= 5, 10, 15, 20, and\ 30)$ corresponding to real data set. Bayesian point predictions and along with their confidence bounds are presented in Table 8. It is to be noticed that the width of the predictive bounds increases as s increases. The density function (16) for first ordered statistics of future sample is plotted for different values of m in Figure 8. Figures 9(a), 9(c), and 9(e) are the MCMC trace plots of $y_{(1)}, y_{(2)}, and\ y_{(3)}$ future sample for $m = 5$. The posterior densities of simulated $y_{(1)}, y_{(2)}, and\ y_{(3)}$ are plotted in Figures 9(b), 9(d), and 9(f) respectively.

7. Discussion

In this paper, we have considered the problem of estimation of the parameters of the generalized Lindley distribution under Bayesian paradigm. On the basis of compression of simulated risks of estimators, it is found that Bayes estimators perform better than maximum likelihood estimation and Bayes estimator under GELF is also more efficient than the Bayes estimator under SELF in most of the situation. From the above mentioned discussion, we may conclude that the Bayes procedure discussed in this paper can be recommended for their use.

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Table 1. Average estimates (risks under SELF) for fixed $\alpha = 2$ and $\lambda = 2$.

n	Parameters	MLE	Bayes				
			$a_1 = b_1 = 0$	$\eta = 2$	$\eta = 2$	$\eta = 2$	$\eta = 3$
			$a_2 = b_2 = 0$	$\nu = 8$	$\nu = 0.5$	$\nu = 100$	$\nu = 8$
20	α	2.41643	2.33918	2.31967	2.15601	2.33746	2.3568
		(1.0155)	(0.8966)	(0.8144)	(0.2945)	(0.8891)	(0.8444)
	λ	2.17122	2.13785	2.13387	2.0932	2.13749	2.1516
		(0.2511)	(0.2346)	(0.2256)	(0.1432)	(0.2339)	(0.2303)
30	α	2.28203	2.23343	2.22504	2.13827	2.23275	2.25006
		(0.6173)	(0.5657)	(0.5341)	(0.2692)	(0.563)	(0.5494)
	λ	2.12204	2.10035	2.09862	2.07749	2.1002	2.11056
		(0.1543)	(0.1469)	(0.1434)	(0.1057)	(0.1467)	(0.1458)
50	α	2.16842	2.14088	2.13817	2.10441	2.14079	2.15339
		(0.3091)	(0.2924)	(0.2842)	(0.1961)	(0.2920)	(0.2907)
	λ	2.07563	2.06292	2.06238	2.05392	2.06292	2.0696
		(0.0862)	(0.0835)	(0.0824)	(0.0688)	(0.0835)	(0.0834)

Table 2. The average 95% confidence intervals and HPD credible intervals when $\alpha = 2$ and $\lambda = 2$.

n	Parameters	MLE	Bayes				
			$a_1 = b_1 = 0$ $a_2 = b_2 = 0$	$\eta = 2$ $\nu = 8$	$\eta = 2$ $\nu = 0.5$	$\eta = 2$ $\nu = 100$	$\eta = 3$ $\nu = 8$
20	α	0.66531	1.43837	1.43402	1.43266	1.43881	1.47635
		4.16755	3.28692	3.25047	2.92185	3.28363	3.28279
	λ	1.29705	1.65981	1.6584	1.65404	1.65913	1.67981
		3.04538	2.61849	2.61175	2.53719	2.61718	2.62601
30	α	0.95444	1.5371	1.53588	1.53209	1.53742	1.5638
		3.60962	2.95623	2.94134	2.77382	2.95495	2.96393
	λ	1.42057	1.7162	1.71586	1.71469	1.71665	1.73011
		2.82352	2.48552	2.48254	2.44366	2.48559	2.49306
50	α	1.20733	1.62981	1.62974	1.63008	1.63024	1.64592
		3.12951	2.6680	2.66209	2.59496	2.66732	2.67594
	λ	1.54166	1.77032	1.77072	1.77143	1.77076	1.77881
		2.6096	2.35619	2.35533	2.33853	2.3564	2.36166

Table 3. Average estimates and (risks of the estimators under SELF) for fixed $n = 20, a_1 = \alpha/8, a_2 = \lambda/8, b_1 = \alpha^2/8$ and $b_2 = \lambda^2/8$

α	λ	$R_s(\hat{\alpha_M})$	$R_s(\hat{\alpha_s})$	$R_s(\hat{\alpha_G})$		$R_s(\hat{\lambda_M})$	$R_s(\hat{\lambda_s})$	$R_s(\hat{\lambda_G})$	
				$C = 1.5$	$C = -0.5$			$C = 1.5$	$C = -0.5$
1	1	1.18926	1.1496	1.09209	1.13836	1.10603	1.08415	1.05623	1.0787
		(0.2413)	(0.2068)	(0.1676)	(0.1985)	(0.0881)	(0.0808)	(0.0746)	(0.0794)
	2	1.18016	1.14091	1.08628	1.13022	2.22522	2.1779	2.11736	2.16609
		(0.2222)	(0.1895)	(0.1551)	(0.1821)	(0.3907)	(0.3494)	(0.3193)	(0.3429)
	3	1.17671	1.13645	1.08309	1.12601	3.35374	3.27324	3.17764	3.25459
		(0.2146)	(0.1804)	(0.1480)	(0.1735)	(0.9502)	(0.8122)	(0.7374)	(0.7961)
2	1	2.51594	2.39503	2.24553	2.36759	1.08876	1.06803	1.0493	1.0648
		(1.6743)	(1.2070)	(0.9205)	(1.1514)	(0.0635)	(0.0564)	(0.0532)	(0.0558)
	2	2.48721	2.37191	2.23164	2.34598	2.18462	2.1422	2.10237	2.13519
		(1.5061)	(1.1019)	(0.8525)	(1.0532)	(0.2723)	(0.2394)	(0.2247)	(0.2368)
	3	2.47464	2.3591	2.2233	2.3854	3.28637	3.21833	3.15587	3.23048
		(1.4331)	(1.0434)	(0.8121)	(1.0944)	(0.6494)	(0.5572)	(0.5209)	(0.5652)
3	1	3.93621	3.6472	3.40706	3.59098	1.08289	1.06056	1.04617	1.05653
		(5.4476)	(2.9496)	(2.2087)	(2.7431)	(0.0557)	(0.0463)	(0.0442)	(0.0454)
	2	3.88236	3.61435	3.38874	3.55775	2.17115	2.1265	2.09612	2.11717
		(4.8483)	(2.7246)	(2.0681)	(2.5149)	(0.2354)	(0.1962)	(0.1861)	(0.1912)
	3	3.85709	3.60264	3.39322	3.5421	3.26405	3.19776	3.15249	3.18051
		(4.5762)	(2.6409)	(2.0425)	(2.4092)	(0.5567)	(0.4619)	(0.4382)	(0.4458)

Table 4. Table 4: Risks of the estimators of α under GELF for fixed $n = 20, a_1 = \alpha/8, a_2 = \lambda/8, b_1 = \alpha^2/8$ and $b_2 = \lambda^2/8$

				$C = -0.5$			$C = 1.5$
α	λ	$R_G(\hat{\alpha_M})$	$R_G(\hat{\alpha_s})$	$R_G(\hat{\alpha_G})$	$R_G(\hat{\alpha_M})$	$R_G(\hat{\alpha_s})$	$R_G(\hat{\alpha_G})$
1	1	0.01478	0.01393	0.01374	0.19217	0.17097	0.14699
	2	0.01404	0.01317	0.01299	0.17968	0.15908	0.13774
	3	0.01374	0.01275	0.01259	0.17467	0.15278	0.13257
2	1	0.02052	0.01825	0.01793	0.29859	0.23729	0.19487
	2	0.01931	0.01717	0.01687	0.27468	0.21975	0.18232
	3	0.01876	0.01654	0.01684	0.26415	0.20996	0.17491
3	1	0.02514	0.01999	0.01958	0.39748	0.2604	0.21068
	2	0.02362	0.01883	0.01846	0.36345	0.24328	0.19875
	3	0.02289	0.01823	0.01787	0.39748	0.2604	0.21068

Table 5. Risks of the estimators of λ under GELF for fixed $n = 20, a_1 = \alpha/8, a_2 = \lambda/8, b_1 = \alpha^2/8$ and $b_2 = \lambda^2/8$

				$C = -0.5$			$C = 1.5$
α	λ	$R_G(\hat{\lambda_M})$	$R_G(\hat{\lambda_s})$	$R_G(\hat{\lambda_G})$	$R_G(\hat{\lambda_M})$	$R_G(\hat{\lambda_s})$	$R_G(\hat{\lambda_G})$
1	1	0.00773	0.0075	0.00748	0.08351	0.07823	0.07452
	2	0.00834	0.00793	0.00791	0.09137	0.08371	0.07912
	3	0.00884	0.00813	0.0081	0.09784	0.08617	0.08102
2	1	0.00593	0.0056	0.00559	0.06212	0.0565	0.05448
	2	0.00626	0.00586	0.00585	0.06608	0.05957	0.05719
	3	0.00656	0.00601	0.00603	0.06966	0.06141	0.05879
3	1	0.00531	0.00473	0.00472	0.05497	0.04725	0.0458
	2	0.00554	0.00493	0.00492	0.05783	0.04967	0.04801
	3	0.00578	0.00507	0.00506	0.05497	0.04725	0.04580

Table 6. The average 95% confidence intervals and HPD credible intervals for fixed $n = 20, a_1 = \alpha/8, a_2 = \lambda/8, b_1 = \alpha^2/8$ and $b_2 = \lambda^2/8$

α	λ	$\hat{\alpha}_M$	$\hat{\lambda}_M$	$\hat{\alpha}_s$	$\hat{\lambda}_s$
1	1	0.43419, 1.94432	0.61290, 1.59915	0.72157, 1.59942	0.78885, 1.38292
	2	0.45081, 1.90950	1.19498, 3.25545	0.72461, 1.57792	1.56223, 2.80175
	3	0.45795, 1.89547	1.75541, 4.95207	0.72605, 1.56817	2.32549, 4.23455
2	1	0.60628, 4.42559	0.66304, 1.51448	1.40878, 3.44031	0.82630, 1.31148
	2	0.66147, 4.31296	1.30680, 3.06245	1.41862, 3.37987	1.64299, 2.64491
	3	0.68665, 4.26262	1.93453, 4.63822	1.42277, 3.34938	2.45219, 3.99012
3	1	0.58566, 7.28676	0.68251, 1.48326	2.11234, 5.28990	0.84815, 1.27310
	2	0.69299, 7.07173	1.35003, 2.99225	2.12769, 5.19606	1.68985, 2.56415
	3	0.58566, 7.28676	0.68251, 1.48326	2.11234, 5.28990	0.84815, 1.27310

Table 7. MLE and Bayes estimates along with their confidence /HPD intervals for real data set.

	MLE	Bayes		
		SELF	GELF	
			$C = -0.5$	$C = 1.5$
α	27.876 -9.714 = 0, 65.4664	26.5648 15.4246, 38.9235	26.2139	24.7238
λ	2.5395 1.6581, 3.4209	2.51002 2.1715, 2.8020	2.5074	2.49682

Table 8. Bayesian point prediction of future observations and corresponding predictive bounds for varying m and s for real data set

Size (m)	S	SELF	GELF		Predictive bounds
			$C = -0.5$	$C = 1.5$	
5	1	1.28808	1.28085	1.25130	0.908644, 1.652350
	2	1.54413	1.53781	1.51235	1.179738, 1.951884
	3	1.75999	1.75403	1.73012	1.380534, 2.174262
	5	2.53550	2.51722	2.44541	1.715948, 3.367726
10	1	1.15339	1.14781	1.12496	0.836256, 1.415345
	2	1.33655	1.33221	1.31459	1.047185, 1.634530
	3	1.48345	1.47947	1.46325	1.189963, 1.781339
	10	2.82503	2.80789	2.74140	2.019081, 3.720739
15	1	1.10226	1.09658	1.07293	0.794486, 1.359548
	2	1.21866	1.21436	1.19681	0.936517, 1.506165
	3	1.35761	1.35421	1.34043	1.077485, 1.653755
	15	2.99531	2.97918	2.91633	2.165752, 3.874245
20	1	1.05936	1.03511	1.05464	0.794045, 1.315908
	2	1.19595	1.19213	1.17651	0.928652, 1.454710
	3	1.30105	1.29783	1.28471	1.041361, 1.567923
	20	3.14707	3.13062	3.06668	2.326170, 4.067767

Table 9. Kolmogorov-Smirnov test statistics summary for real data set

MLE	SELF	GELF		$D_{(20,0.05)}$
		C=-0.5	C=1.5	
0.1377	0.1389	0.1359	0.1224	0.294

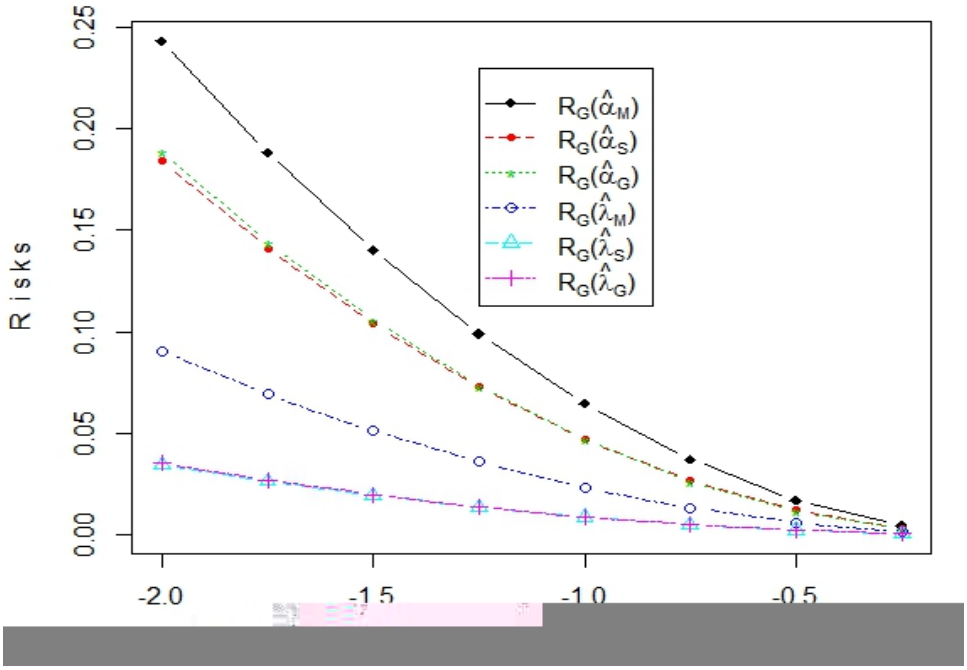

Figure 2. Risks plot of the estimators of α and λ under GELF ($C < 0$) for fixed $n = 20,\ \alpha = 2,\ \lambda = 2,\ a_1 = a_2 = 0.25$ and $b_1 = b_2 = 0.5$.

Figure 3. Risks plot of the estimators of α and λ under GELF ($C > 0$) for fixed $n = 20,\ \alpha = 2,\ \lambda = 2,\ a_1 = a_2 = 0.25$ and $b_1 = b_2 = 0.5$.

Figure 4. Risks plot of the estimators of α and λ under SELF for fixed $C < 0,\ n = 20,\ \alpha = 2,\ \lambda = 2,\ a_1 = a_2 = 0.25$ and $b_1 = b_2 = 0.5$.

Figure 5. Risks plot of the estimators of α and λ under SELF for fixed $C > 0,\ n = 20,\ \alpha = 2,\ \lambda = 2,\ a_1 = a_2 = 0.25$ and $b_1 = b_2 = 0.5$.

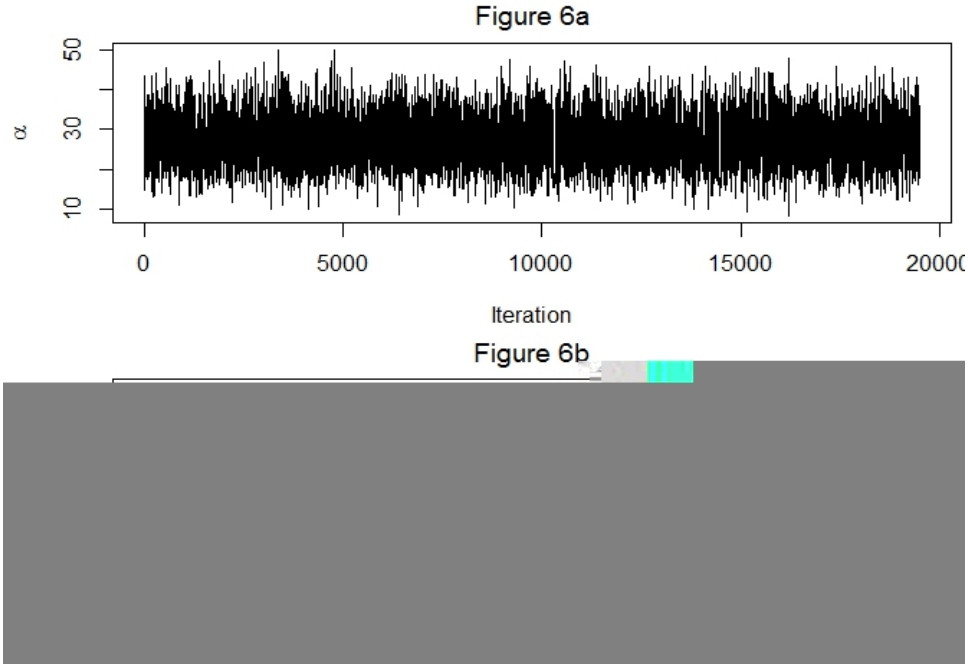

Figure 6. MCMC trace plots of simulated samples (a) for α and (b) for λ for real data set.

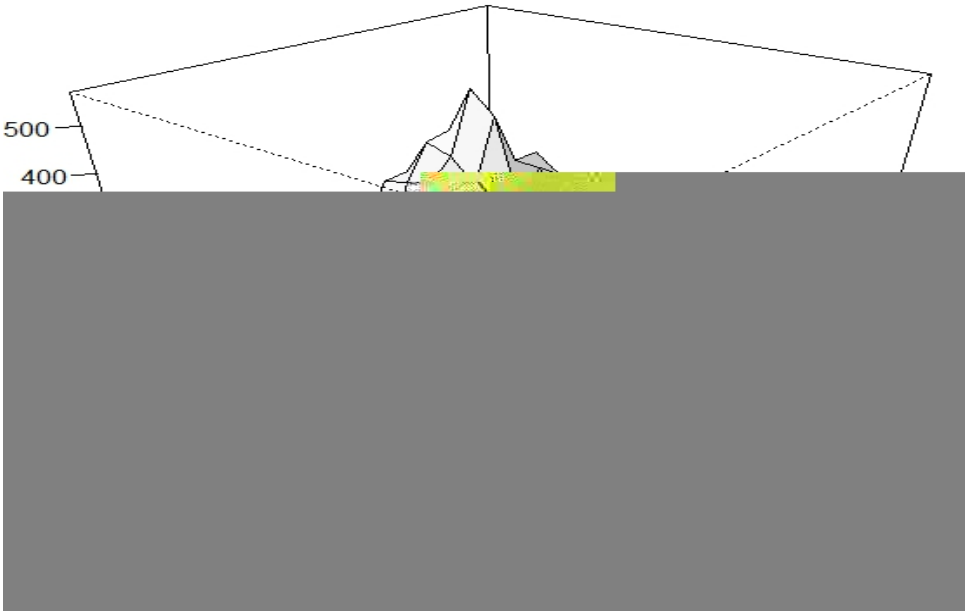

Figure 7. 3-D histogram of simulated α and λ for real data set.

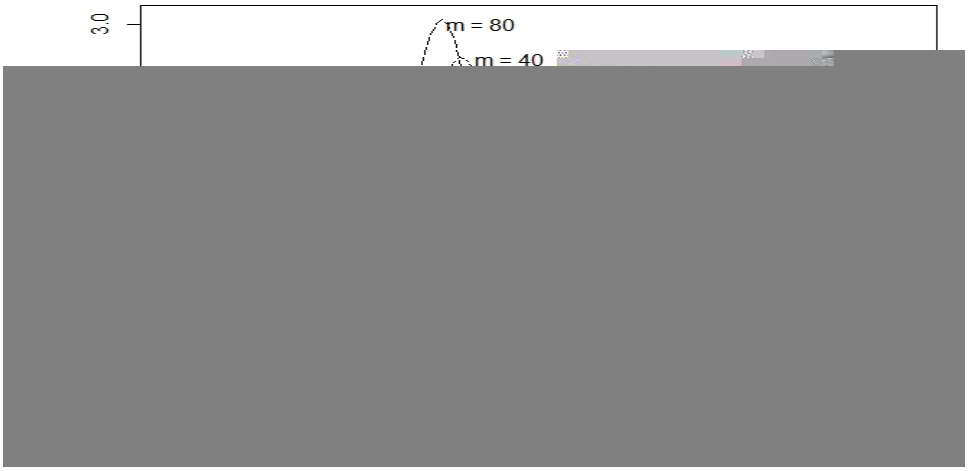

Figure 8. Density function plot of the first future observation with varying m for real data set.

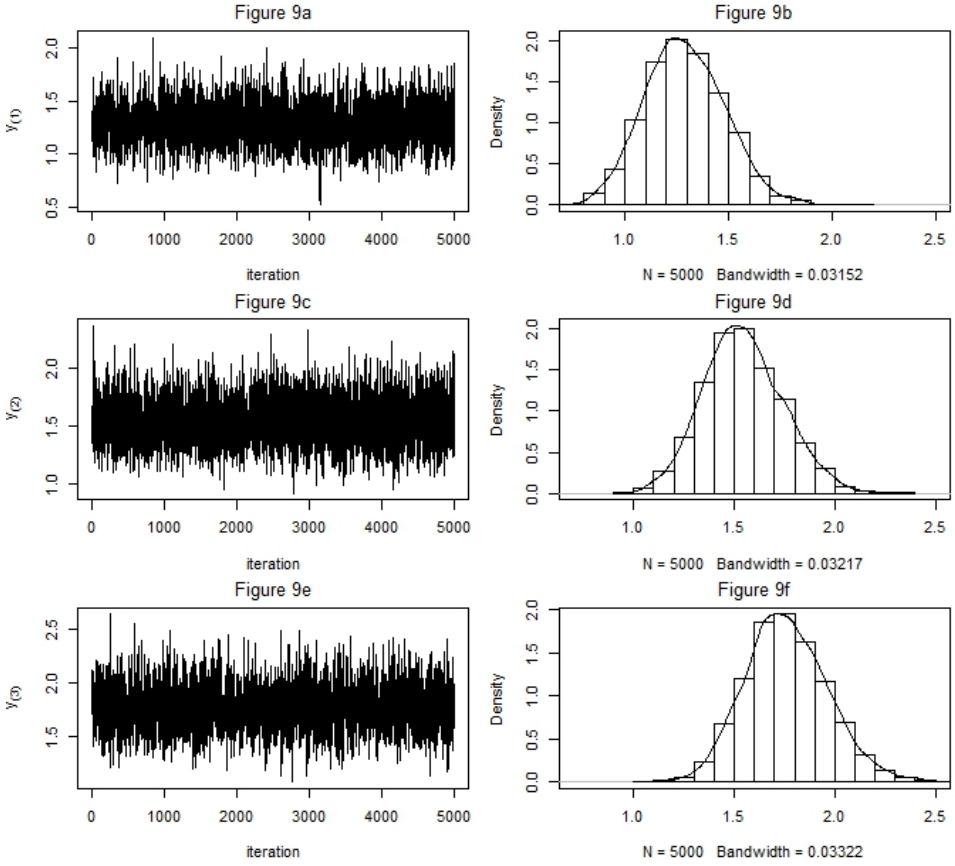

Figure 9. Trace and density plots of simulated $y_{(1)}$, $y_{(2)}$, and $y_{(3)}$ the first three future observations when $m = 5$ for real data set.

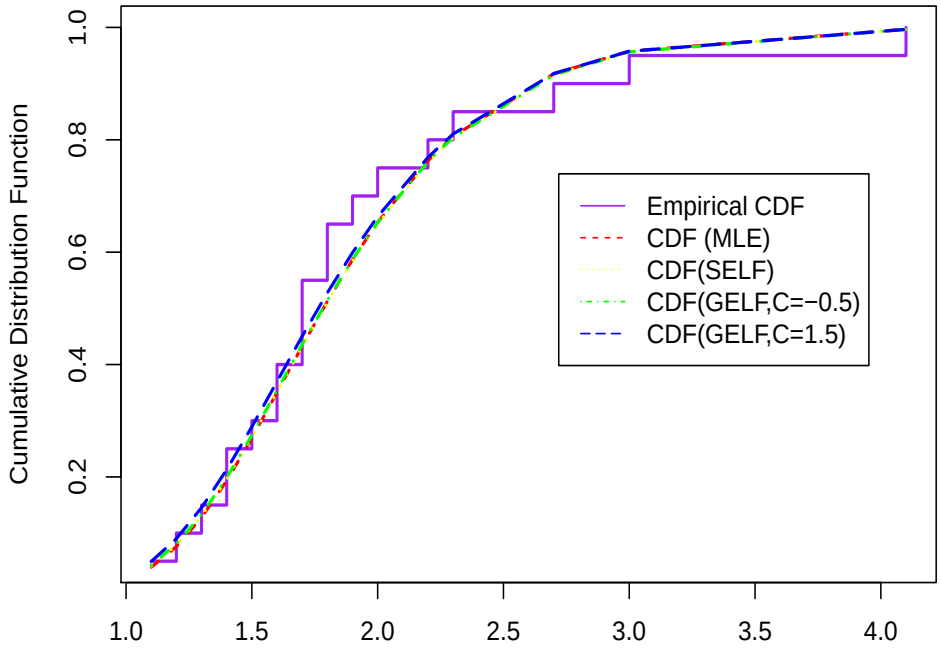

Figure 10. Empirical and fitted distribution function for real data set.

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Some conditional and unconditional expectation identities for the multivariate normal with non-zero mean

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Abstract

We give formulas for the conditional and unconditional expectations of products of multivariate Hermite and modified Hermite polynomials, each with a multivariate normal argument. A unified approach is given that covers both of these polynomials, each associated with a covariance matrix. This *extended* Hermite polynomial is associated with a matrix which is the difference between two covariance matrices, in other words, with any symmetric matrix.

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1. Introduction

Conditional expectation identities have a fundamental role in contingency table analysis and its applications, see, for example, Lancaster (1957). A well known identity is that $\mathbb{E}[H_n(X_2) \mid X_1] = \rho^n H_n(X_1)$, where $H_n(\cdot)$ is a Hermite polynomial and (X_1, X_2) has the standard bivariate normal distribution with correlation coefficient ρ. Conditional expectation identities are also useful for characterizing distributions, see Gupta and Ahsanullah (2004). Applications of conditional expectation identities are numerous. Some recent applications include: multivariate input processes (Biller and Ghosh, 2006); mechanisms that modulate the transfer of spiking correlations (Rosenbaum and Josić, 2011); linear-feedback sum-capacity for Gaussian multiple access channels (Ardestanizadeh *et al.*, 2012).

In this short note, we derive conditional expectation identities involving a product of multivariate Hermite and/or modified Hermite polynomials with multivariate normal arguments. This is done by extending the family of these polynomials to a polynomial associated with a matrix whose eigenvalues may be both positive and negative. Our identities generalize that due to Lancaster (1957).

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We illustrate the main results of this short note by several examples, see Example 1.1 and Examples 2.1 to 2.5. These examples give new expressions for expectations of Hermite polynomials with random arguments and expectations of products of Hermite polynomials with random arguments. Such expectations crop up in many theoretical as well as applied areas. We mention: non-central limit theorems for non-linear functionals of Gaussian fields (Dobrushin and Major, 1979, Section 4); combinatorial problems (Azor *et al.*, 1982); Wiener analysis of binary hysteresis systems (Nakayama and Omori, 1982); level crossings for regularized Gaussian processes (Berzin *et al.*, 1998); central limit theorems for functionals of level overshoot by a Gaussian field with dependence (Jeon (1998), see, for example, equation (4)); nonlinear time series analysis (Terdik (1999), see, for example, equation (2.20)); locating human faces in a cluttered scene (Rajagopalan *et al.* (2000), see, for example, equation (3)); influence of the order of input expansions in spectral stochastic finite element methods (Gaignaire *et al.* (2006), see, for example, equations (4)-(5)); stochastic finite element based on stochastic linearization for stochastic nonlinear ordinary differential equations with random coefficients (Saleh *et al.*, 2006, Section 3). Hence, the expressions given in the examples can be very useful.

Let $\mathbb{N}_+$ and $\mathbb{R}$ denote the set of non-negative integers and the set of real numbers. Suppose that

$$X = N_V \sim \mathcal{N}_p(0, V), \tag{1.1}$$

a p-dimensional normal random variable with zero means and covariance V. If $V > 0$, that is, if V is positive-definite, then its density is

$$\phi_V(x) = (2\pi)^{-p/2} \det(V)^{-1/2} \exp\left(-x'V^{-1}x/2\right)$$

for $x \in \mathbb{R}^p$. For $n \in \mathbb{N}_+^p$, $t \in \mathbb{R}^p$, $x \in \mathbb{R}^p$ and $D_j = \partial/\partial x_j$, set $n! = \prod_{j=1}^p n_j!$, $t^n = \prod_{j=1}^p t_j^{n_j}$, $(-D)^n = \prod_{j=1}^p (-D_j)^{n_j}$ and define the nth *multivariate Hermite polynomial* as

$$H_n(x, V) = \phi_V(x)^{-1}(-D)^n \phi_V(x) = \exp(q/2)(-D)^n \exp(-q/2)$$

for $q = x'V^{-1}x$ and $n \in \mathbb{N}_+^p$. This is shown in Withers (2000) to be given simply by

$$H_n(x, V) = \mathbb{E}\ \left[V^{-1}(x + \mathrm{i}X)\right]^n,$$

where $\mathrm{i} = \sqrt{-1}$, see also Withers and McGavin (2003). Its exponential generating function (egf) is

$$\sum_{n \in \mathbb{N}_+^p} H_n(x, V) t^n/n! = \mathbb{E}\left\{\exp\left[t'V^{-1}(x + \mathrm{i}X)\right]\right\} = \exp\left(t'V^{-1}x - t'V^{-1}t/2\right) \tag{1.2}$$

for $x, t \in \mathbb{R}^p$. The nth *modified multivariate Hermite polynomial* is defined by

$$H_n^*(x, V) = \phi_V(x) D^n \phi_V(x)^{-1} = \mathbb{E}\left\{\left[V^{-1}(x + X)\right]^n\right\}$$

for $n \in \mathbb{N}_+^p$. This is just $H_n(x, V)$ with all its signs positive. Its egf is

$$\sum_{n \in \mathbb{N}_+^p} H_n^*(x, V) t^n/n! = \mathbb{E}\left[\exp\left(t'V^{-1}(x + X)\right)\right] = \exp\left(t'V^{-1}x + t'V^{-1}t/2\right) \tag{1.3}$$

for $x, t \in \mathbb{R}^p$. There is a problem with notation: $H_n(x, V^{-1}) = \mathbb{E}[(Vx + \mathrm{i}X)^n]$ and its modified form exist for $V \geq 0$ (positive semi-definite), not just for $V > 0$ (positive definite). Some authors get around this by using V rather than V^{-1} as the second argument: see page 273 of Willink (2005).

We prefer to work with what we shall call the *extended Hermite polynomial*

$$h_n(x, C) = \mathbb{E}\left[(x + Y + \mathrm{i}Z)^n\right]$$

for $n \in \mathbb{N}_+^p$, $x \in \mathbb{R}^p$ and $C = A - B \in \mathbb{R}^{p\times p}$, where $Y \sim \mathcal{N}_p(0, A)$ and $Z \sim \mathcal{N}_p(0, B)$ are independent. That this polynomial only depends on $(A,\ B)$ through C, follows from its egf,

$$\sum_{n\in\mathbb{N}_+^p} h_n(x, C)t^n/n! = \exp\left(t'x + t'Ct/2\right)$$

for $t \in \mathbb{R}^p$. So, $h_n(x, 0) = x^n$.

1.1. Theorem. *Using the egf, it follows that*

$$\begin{aligned} &h_n\left(x_1 + x_2, C_1 + C_2\right)/n! = h_n\left(x_1, C_1\right)/n!\ \otimes h_n\left(x_2, C_2\right)/n!, \\ &h_n\left(x, A - B\right)/n! = h_n\left(x_1, A\right)/n!\ \otimes h_n\left(x_3, -B\right)/n!,\ x = x_1 + x_3. \end{aligned} \tag{1.4}$$

For $A > 0$, $B > 0$ we shall see that this is essentially a convolution of a Hermite polynomial and a modified Hermite polynomial. Here, $a_n \otimes b_n = \sum_{0\le k\le n,\ k\in\mathbb{N}_+^r} a_k b_{n-k}$ is the convolution of a_n and b_n in $\mathbb{N}_+^r$. The obvious choice of x_1 is x. In this case, $h_n(x_3, -B) = h_n(0, -B) = \mathrm{i}^n\mathbb{E}[Z^n]$ is essentially just a moment of a multivariate normal.

The choice of A, B is not unique. We can write $C = H'\Lambda H$, where $H'H = I$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p) = \mathrm{diag}(\Lambda_1, 0, -\Lambda_3)$, where $\lambda_1 \ge \cdots \ge \lambda_p$ and $\Lambda_j > 0$ for $j = 1, 3$. That is, Λ_1 consists of the positive eigenvalues of C and $-\Lambda_3$ consists of the negative eigenvalues of C. Then the obvious choice of A, B is the *minimal choice*,

$$A = H'\mathrm{diag}\left(\Lambda_1, 0, 0\right)H,\ B = H'\mathrm{diag}\left(0, 0, \Lambda_3\right)H. \tag{1.5}$$

If $C = 0$, that is $A = B$, then $Y + \mathrm{i}Z \sim \mathcal{C}N_p(0, V)$, the complex normal distribution with *complex covariance* $\mathbb{E}[(YY' + ZZ')] = 2A$. Its real moments are all zero: $\mathbb{E}[(Y + \mathrm{i}Z)^n] = 0$ for $n \in \mathbb{N}_+^p$.

By (1.2) and (1.3) we have the other special cases

$$h_n(x, V) = \mathbb{E}\left[(x + X)^n\right] \text{ if } V \ge 0, \tag{1.6}$$

$$= H_n^*\left(V^{-1}x, V^{-1}\right) \text{ if } V > 0, \tag{1.7}$$

$$h_n(x, -V) = \mathbb{E}\left[(x + \mathrm{i}X)^n\right] \text{ if } V \ge 0, \tag{1.8}$$

$$= H_n\left(V^{-1}x, V^{-1}\right) \text{ if } V > 0. \tag{1.9}$$

Equivalently when $V > 0$, $H_n^*(x, V^{-1}) = h_n(Vx, V)$ and $H_n(x, V^{-1}) = h_n(Vx, -V)$. So, the class of functions $h_n(x, C)$ includes x^n, $H_n(x, V)$, $H_n^*(x, V)$ as well as the mixed case, where C has both positive and negative eigenvalues.

Note that for $p = 1$, $n \in \mathbb{N}_+$, $x \in \mathbb{R}$ and $N \sim \mathcal{N}(0, 1)$, $h_n(x, 1) = \mathbb{E}[(x + N)^n] = H_n^*(x)$ and $h_n(x, -1) = \mathbb{E}[(x + \mathrm{i}N)^n] = H_n(x)$, where $H_n(x) = H_n(x, 1)$ and $H_n^*(x) = H_n^*(x, 1)$ are the usual univariate Hermite and modified Hermite polynomials. Also

$$H_n\left(x, \sigma^{-2}\right) = \sigma^n H_n(\sigma x) = \sigma^n h_n(\sigma x, -1), \tag{1.10}$$

$$H_n^*\left(x, \sigma^{-2}\right) = \sigma^n H_n^*(\sigma x) = \sigma^n h_n(\sigma x, 1). \tag{1.11}$$

We now partition X of (1.1) as $\binom{X_1}{X_2}$ with $X_j \in \mathbb{R}^{p_j}$, so that $p_1 + p_2 = p$. We denote conditional expectation by $\mathbb{E}^{X_1}[f(X)] = \mathbb{E}[f(X)|X_1]$. Partition V into $(V_{jk} : j, k = 1, 2)$, where V_{jk} is a $p_j \times p_k$ block matrix, and set $V_{2\cdot1} = covar(X_2|X_1) = V_{22} - V_{21}V_{11}^{-1}V_{12}$ if $p_1 > 0$, and $V_{2\cdot1} = covar(X_2) = V_{22}$ if $p_1 = 0$. The latter corresponds to not conditioning. We assume that $V_{2\cdot1} > 0$, that is, X_2 is not just a multiple of X_1.

Now consider $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{r\times p}$ with $\alpha_j \in \mathbb{R}^{r\times p_j}$ so that $\alpha X = \sum_{j=1}^2 \alpha_j X_j$. We have the following.

1.2. Theorem. *Suppose that* $Y = \binom{Y_1}{Y_2} = X + \mu \sim \mathcal{N}_p(\mu, V)$ *with* $Y_j \in \mathbb{R}^{p_j}$, $j = 1, 2$. *Partition* μ *in the same way. Set*

$$\delta = \left(\alpha_1 + \alpha_2 V_{21}V_{11}^{-1}\right)\left(Y_1 - \mu_1\right),\ F_0 = \alpha_2 V_{2\cdot1}\alpha_2',\ F = F_0 + C. \tag{1.12}$$

Then

$$\mathbb{E}^{Y_1}\left[h_n\left(\alpha Y, C\right)\right] = h_n\left(\alpha\mu + \delta, F\right). \tag{1.13}$$

We now give some special cases with $\mu = 0$, that is $Y = X$ of (1.1), in terms of the Hermite polynomials H_n, H_n^*. Taking $C = 0$ gives

$$\mathbb{E}^{X_1}\left[(\alpha X)^n\right] = H_n^*\left(F_0^{-1}\delta, F_0^{-1}\right) \tag{1.14}$$

for $F_0 > 0$, that is, for $r \le p_2$ and α_2 of rank r.

By (1.6) for $C \ge 0$, $h_n(x, C) = \mathbb{E}[(x + N_C)^n]$ so that by (1.13), the left hand side of (1.13) is equal to $h_n(\delta, F) = \mathbb{E}[(\delta + N_F)^n]$. So, by (1.7),

$$\mathbb{E}^{X_1}\left[H_n^*\left(C^{-1}\alpha X, C^{-1}\right)\right] = H_n^*\left(F^{-1}\delta, F^{-1}\right) \tag{1.15}$$

for $C > 0$. By (1.8) for $D = -C \ge 0$, $h_n(x, C) = \mathbb{E}[(x + \mathrm{i}N_D)^n]$ so that by (1.13), the left hand side of (1.13) is equal to $h_n(\delta, F) = \mathbb{E}[(\delta + iN_G)^n]$ if $G = -F = D - F_0 \ge 0$. So, by (1.9),

$$\mathbb{E}^{X_1}\left[H_n\left(D^{-1}\alpha X, D^{-1}\right)\right] = H_n\left(G^{-1}\delta, G^{-1}\right) \tag{1.16}$$

for $G = D - F_0 > 0$.

Taking $p_1 = 0$ gives $\mathbb{E}[h_n(\alpha Y, C)] = h_n(\alpha\mu, F)$, where $F = C + \alpha V_{22}\alpha'$. For example, taking $C > 0$, then $\mathbb{E}[H_n^*(C^{-1}\alpha Y, C^{-1})] = H_n^*(F^{-1}\alpha\mu, F^{-1})$.

1.3. Example. Take $r = 1$ and set $\beta = \alpha' \in \mathbb{R}^p$ and $\beta_j = \alpha_j' \in \mathbb{R}^{p_j}$. Set $v = \beta_2' V_{2\cdot1}\beta_2$. Then by (1.14), (1.15) at $C = 1$, (1.11), (1.16) at $C = -1$, and (1.10),

$$\mathbb{E}^{X_1}\left[\left(\beta' X\right)^n\right] = \sigma^n H_n^*(\delta/\sigma) \text{ for } \sigma^2 = v > 0, \tag{1.17}$$

$$\mathbb{E}^{X_1}\left[H_n^*\left(\beta' X\right)\right] = \sigma^n H_n^*(\delta/\sigma) \text{ for } \sigma^2 = 1 + v, \tag{1.18}$$

$$\begin{aligned}\mathbb{E}^{X_1}\left[H_n\left(\beta' X\right)\right] &= \sigma^n H_n(\delta/\sigma) \text{ for } \sigma^2 = 1 - v > 0,\\ &= \sigma^n H_n^*(\delta/\sigma) \text{ for } \sigma^2 = v - 1 > 0,\\ &= \delta^n \text{ for } v = 1.\end{aligned} \tag{1.19}$$

For example, taking $p_1 = 0$ and setting $v = \beta' V \beta$ and $N \sim \mathcal{N}(0, 1)$ gives

$$\begin{aligned}\mathbb{E}\left[\left(\beta' X\right)^n\right] &= \sigma^n \mathbb{E}\left[N^n\right] \text{ for } \sigma^2 = v > 0,\\ \mathbb{E}\left[H_n^*\left(\beta' X\right)\right] &= \sigma^n \mathbb{E}\left[N^n\right] \text{ for } \sigma^2 = 1 + v,\\ \mathbb{E}\left[H_n\left(\beta' X\right)\right] &= \sigma^n \mathbb{E}\left[(\mathrm{i}N)^n\right] \text{ for } \sigma^2 = 1 - v > 0,\\ &= \sigma^n \mathbb{E}\left[N^n\right] \text{ for } \sigma^2 = v - 1 > 0,\\ &= \delta_{0n} \text{ for } v = 1,\end{aligned}$$

where $\delta_{jk} = 1$ or 0 for $j = k$ or $j \ne k$. Consider the standardized bivariate normal, $p_j \equiv 1$, $V = \binom{1\ \rho}{\rho\ 1}$, where $|\rho| < 1$. Then in (1.17)-(1.19), $\delta = (\beta_1 + \rho\beta_2)X_1$ and σ^2 is given by $(1 - \rho^2)\beta_2^2$, $1 + (1 - \rho^2)\beta_2^2$, $1 - (1 - \rho^2)\beta_2^2$, respectively. If also $\beta_2 = 1$ then one can take $\sigma = \rho$ in (1.19) since ρN has the same distribution as $|\rho|N$ for $N \sim \mathcal{N}(0, 1)$, giving $\mathbb{E}^{X_1}[H_n(\beta_1 X_1 + X_2)] = \rho^n H_n((\beta_1/\rho + 1)X_1)$.

In the notation of (1.1), we can write $N_{C_1} = H' N_{L_1}$, $N_{C_3} = H' N_{L_3}$, where $L_1 = \mathrm{diag}(\Lambda_1, 0, 0)$ and $L_3 = \mathrm{diag}(0, 0, \Lambda_3)$. In our result, (1.13), the form of $h_n(x, C)$ is determined by the eigenvalues of C, while the form of $h_n(\delta, F)$ is determined by the eigenvalues of F. We now show the following.

1.4. Theorem. *An eigenvalue of F of (1.12) is either an eigenvalue of C or it satisfies*

$$\det\left(V_{2\cdot1}^{-1} - J_\lambda\right) = 0, \tag{1.20}$$

where $J_\lambda = \alpha_2'(\lambda I_r - C)^{-1}\alpha_2$.

In Section 2, we extend our result, (1.13), to products of extended Hermite polynomials. Some conclusions and future work are noted in Section 3. The proofs of all main results are provided in the appendix.

2. An extension to products

We now give an extension of Theorem 1.2 to products. Suppose that for $1 \le k \le K$, $n_k \in \mathbb{N}_+^{r_k}$, $t_k \in \mathbb{R}^{r_k}$, C_k in $\mathbb{R}^{r_k \times r_k}$, $\alpha_k = (\alpha_{k1}, \alpha_{k2}) \in \mathbb{R}^{r_k \times p}$, $\alpha_{kj} \in \mathbb{R}^{r_k \times p_j}$.

2.1. Theorem. *Suppose that* $Y = \binom{Y_1}{Y_2} \sim \mathcal{N}_p(\mu, V)$ *with* $Y_j \in \mathbb{R}^{p_j}$, $j = 1, 2$. *Partition* μ *in the same way. Set* $\delta_k = (\alpha_{k1} + \alpha_{k2} V_{21} V_{11}^{-1})(Y_1 - \mu_1)$, $\widetilde{\delta}_k = \alpha_k \mu + \delta_k$, $\widetilde{\delta}' = (\widetilde{\delta}_1', \ldots, \widetilde{\delta}_K')$, $n' = (n_1, \ldots, n_K)$, $\gamma' = (\alpha_{12}', \ldots, \alpha_{K2}')$, $\Lambda = \operatorname{diag}(C_1, \ldots, C_K)$ *and* $D = \Lambda + \gamma V_{2\cdot 1} \gamma'$. *Then*

$$\mathbb{E}^{Y_1}\left[\prod_{k=1}^{K} h_{n_k}(\alpha_k Y, C_k)\right] = h_n\left(\widetilde{\delta}, D\right). \tag{2.1}$$

2.2. Example. Find $a_n = \mathbb{E}[\prod_{k=1}^{K} H_{n_k}(N)]$ for $N \sim \mathcal{N}(0,1)$. So, $\mu = p_1 = 0$, $p_2 = V = \alpha_k = 1$, $C_k = -1$, $\widetilde{\delta} = 0$, $\gamma = 1_K$, the K-vector of ones, $D = 1_K 1_K' - I_K$.
If $K = 1$ then $D = 0$ so $a_n = \delta_{n0}$.
If $K = 2$ then $D = \binom{01}{10}$ and $q = t_1 t_2$. The coefficient of t^n in $\exp(q)$ is $\delta_{n_1 n_2}/n_1!$ so that $a_n = h_n(0, D) = \delta_{n_1 n_2} n_1!$. So, $\{H_n(x)/n!^{1/2}\}$ are orthonormal with respect to $\phi(x)$, as is well-known.
If $K = 3$ then $q = t_1 t_2 + t_2 t_3 + t_3 t_1$, $\exp(q) = \sum_{m_1, m_2, m_3 = 0}^{\infty} t^n/m!$ at $n_1 = m_2 + m_3$, $n_2 = m_1 + m_3$, $n_3 = m_1 + m_2$. Set $|n| = \sum_{k=1}^{3} n_k$. So, $a_n = 0$ if $|n|$ is odd, while if $|n|$ is even, then $a_n = n!/m!$ at $2m_1 = n_2 + n_3 - n_1$, $2m_2 = n_1 + n_3 - n_2$, $2m_3 = n_1 + n_2 - n_3$.
If $K = 4$ then $q = \sum_{1 \le j < k \le 4} t_j t_k$ has six terms. Also

$$\exp(q) = \sum_{m_{12}, \ldots, m_{34} = 0}^{\infty} t^n / (m_{12}! \cdots m_{34}!)$$

at $n_j = \sum_{k \ne j} m_{jk}$, where $m_{kj} = m_{jk}$, $j = 1, \ldots, 4$. This gives four equations in six ms, so we can make m_{12} and m_{13} arbitrary, giving

$$\begin{aligned} &m_{14} = n_1 - m_{12} - m_{13},\ 2m_{23} = n_2' + n_3' - n_4', \\ &2m_{24} = n_2' + n_4' - n_3',\ 2m_{34} = n_3' + n_4' - n_2', \end{aligned} \tag{2.2}$$

where $n_j' = n_j - m_{1j}$. So,

$$a_n = n! \sum_{m_{12}, m_{13}} 1/\left[m_{12}! \cdots m_{34}!\right]_{(2.2)}.$$

For example, take $n_1 = 1$. Then $(m_{12}, m_{13}) = (0,0)$, $(0,1)$ or $(1,0)$. So, setting $n_{23} = n_2 + n_3 - n_4$, $n_{24} = n_2 + n_4 - n_3$, $n_{34} = n_3 + n_4 - n_2$, $M_{jk} = (n_{jk} - 1)/2$ and $N_{jk} = (n_{jk} + 1)/2$, we have $a_n/n! = 1/N_{23}!M_{24}!M_{34}! + 1/M_{23}!N_{24}!M_{34}! + 1/M_{23}!M_{24}!N_{34}!$.

2.3. Example. Find $a_n = \mathbb{E}[\prod_{k=1}^{K} H_{n_k}^*(N)]$ for $N \sim \mathcal{N}(0,1)$. The parameters are as for Example 2.1 except that $C_k = 1$, $D = 1_K 1_K' + I_K$ and $q = \sum_{k=1}^{K} t_k^2 + \sum_{1 \le j < k \le K} t_j t_k$.
If $K = 1$ then $q = t^2$ and $\exp(q) = \sum_{m=0}^{\infty} t^{2m}/m!$, giving $a_{2m+1} = 0$ and $a_{2m} = (2m)!/m!$.
If $K = 2$ then $q = t_1^2 + t_2^2 + t_1 t_2$ and $\exp(q) = \sum_{m_1, m_2, m_3 = 0}^{\infty} t^n/m!$ at $n_j = 2m_j + m_3$. So, $a_n = 0$ if $|n|$ is odd, while if $|n|$ is even and $n_1 \le n_2$, then

$$a_n = n! \sum_{k=0}^{\min(n_1, n_2)} 1/m_1!m_2!m_3! \Big|_{2m_1 = n_1 - k,\ 2m_2 = n_2 - k},$$

writing k for m_3. For example,

$$a_{0n_2} = \mathbb{E}\left[H^*_{n_2}(N)\right] = n_2!/\left(n_2/2\right)!,$$
$$a_{1n_2} = \mathbb{E}\left[NH^*_{n_2}(N)\right] = n_2!/\left(\left[n_2-1\right]/2\right)!,\ n_2 \geq 1,$$
$$a_{3n_2} = \mathbb{E}\left[\left(N^3+3N\right)H^*_{n_2}(N)\right] = 3!n_2!\left\{1/\left(\left[n_2-1\right]/2\right)! + 1/3!\left(\left[n_2-3\right]/2\right)!\right\},\ n_2 \geq 3.$$

2.4. Example. Find $a_n = \sigma^n\mathbb{E}[H^*_{n_1}(N/\sigma_1)\ H_{n_2}(N/\sigma_2)] = \mathbb{E}[h_{n_1}(N,\sigma_1^2)\ h_{n_2}(N,-\sigma_2^2)]$. So, $K=2$, $C_1=\sigma_1^2$, $C_2=-\sigma_2^2$, $D = 1_2 1_2' + \mathrm{diag}(\sigma_1^2,-\sigma_2^2)$ and $q = \sum_{j=1}^2 c_j t_j^2 + t_1 t_2$, where $c_1 = (1+\sigma_1^2)/2$ and $c_2 = (1-\sigma_2^2)/2$. So, in a derivation similar to that of Example 2.2 for $K=2$ one obtains $a_n = 0$ for $|n|$ odd, while if $|n|$ is even and $n_1 \leq n_2$, then

$$a_n = n!\sum_{k=0}^{\min(n_1,n_2)} c_1^{m_1}c_2^{m_2}/m_1!m_2!m_3!\Big|_{2m_1=n_1-k,\ 2m_2=n_2-k}. \tag{2.3}$$

For example,

$$a_{0n_2} = n_2!c_2^{n_2/2}/\left(n_2/2\right)!,$$
$$a_{1n_2} = n_2!c_2^{(n_2-1)/2}/\left(\left[n_2-1\right]/2\right)!,\ n_2 \geq 1,$$
$$a_{3n_2} = 3!n_2!\left\{c_1c_2^{(n_2-1)/2}/\left(\left[n_2-1\right]/2\right)! + c_2^{(n_2-3)/2}/3!\left(\left[n_2-3\right]/2\right)!\right\},\ n_2 \geq 3.$$

2.5. Example. Find $a_n = \sigma^n\mathbb{E}[H_{n_1}(N/\sigma_1)\cdots H_{n_K}(N/\sigma_K)] = \mathbb{E}[h_{n_1}(N,C_1)\cdots h_{n_K}(N,C_K)]$, where $C_k = -\sigma_k^2$, $\mu=0$, $V=1$, $\gamma=1_K$, $D = 1_K1_K' - \mathrm{diag}(\sigma_k^2)$, $q = \sum_{j=1}^K c_jt_j^2 + \sum_{1\leq j<k\leq K} t_jt_k$ and $c_k = (1-\sigma_k^2)/2$. So,

$$\exp(q) = \sum_m \prod_{k=1}^K \left(c_kt_k^2\right)^{m_k}/m_k!\ \sum_M \prod_{j<k}\left(t_jt_k\right)^{M_{jk}}/M_{jk}!,\ n_k = 2m_k + \sum_{j\neq k} M_{jk}.$$

If $K=2$ then $a_n = 0$ if $|n|$ is odd, while if $|n|$ is even and $n_1 \leq n_2$, then a_n is given by (2.3).

By Theorem 2.1, Examples 2.1-2.4 can be extended by (i) replacing N by $\mu+N$; and (ii) replacing expectation by conditional expectation.

According to (1.4), the form of h_n depends on the positive and negative parts of D, and these are determined by the positive and negative eigenvalues of D, which we now obtain. Suppose that x is an eigenvector of D with eigenvalue λ. Then $Dx = \lambda x$ implies $\gamma V_{2\cdot1}\gamma' x = (\lambda I - \Lambda)x$. So, either λ is an eigenvalue of Λ, that is an eigenvalue of C_k for some $k \in \{1,\ldots,K\}$, or $\det(\lambda I - \Lambda) \neq 0$ and $z = V_{2\cdot1}\gamma' x$ satisfies $z = V_{2\cdot1}\gamma'(\lambda I - \Lambda)^{-1}\gamma z$, which implies that

$$\det\left(V_{2\cdot1}^{-1} - J_\lambda\right) = 0, \tag{2.4}$$

where $J_\lambda = \sum_{k=1}^K \alpha_{k2}'(\lambda I_{r_k} - C_k)^{-1}\alpha_{k2}$. That is, the r eigenvalues of D are given by (2.4) and the eigenvalues of $\{C_k\}$. The number of roots of (2.4) depends on the number of distinct $\{C_k\}$. We illustrate this with an important example that takes up the rest of this section.

2.6. Example. Suppose that $r_k \equiv 1$, that is, the arguments of the h_{n_k} functions in (2.1) are all scalar. Consider the case when $C_k = -1$, 0 or 1. Then

$$J_\lambda = \sum_{j=-1}^1 (\lambda - j)^{-1} w_j, \tag{2.5}$$

where $w_j = \sum_{C_k=j} \alpha_{k2}'\alpha_{k2} \geq 0 \in \mathbb{R}^{p_2\times p_2}$ and $w = \sum_{k=1}^K w_j > 0$, assuming that $\alpha_{k2} \neq 0$ for $k = 1,\ldots,K$. (We can always assume this, since if $\alpha_{k2} = 0$, then the corresponding

term in the product on the left hand side of (2.1) can be factored out.) There are a number of cases to consider when evaluating the roots of (2.4).

Consider the situation, where $p_2 = 1$. Then $\{w_j\}$ are scalar and the equation (2.4) for the eigenvalues of D of (2.1) becomes the cubic $v = \sum_{j=-1}^{1}(\lambda - j)^{-1}w_j$, where $v = V_{2\cdot 1}^{-1}$.

The case $w_{-1} > 0 = w_0 = w_1$: The only root is $\lambda = w/v - 1$.

The case $w_0 > 0 = w_{-1} = w_1$: The only root is $\lambda = w/v$.

The case $w_1 > 0 = w_{-1} = w_0$: The only root is $\lambda = w/v + 1$.

The case $w_{-1} > 0$, $w_0 > 0$, $w_1 = 0$: There are two roots, $\lambda = (w - v \pm \epsilon_1^{1/2})/(2v)$, where $\epsilon_1 = (w - v)^2 + 4vw_0 > 0$. So, one root is positive and the other negative.

The case $w_{-1} > 0$, $w_1 > 0$, $w_0 = 0$: There are two roots, $\lambda = (w \pm \epsilon_0^{1/2})/(2v)$, where $\epsilon_0 = w^2 + 4v(v + w_1 - w_{-1}) = (w - 2v)^2 + 8vw_1 > 0$. If $v + w_1 < w_{-1}$ then both roots are positive. If $v + w_1 > w_{-1}$ then one root is positive and the other negative.

The case $w_0 > 0$, $w_1 > 0$, $w_{-1} = 0$: There are two positive roots, $\lambda = (v + w \pm \epsilon_{-1}^{1/2})/(2v)$, where $\epsilon_{-1} = (v + w)^2 - 4vw_0 = (w - v)^2 + 4vw_1 > 0$.

The general case $w_{-1} > 0$, $w_0 > 0$, $w_1 > 0$: There are three roots, those of $v\lambda^3 - w\lambda^2 + (w_{-1} - w_1 - v)\lambda + w_0 = 0$. So, one or two roots are positive and the other two or one are negative.

3. Conclusions

We have given new expressions for conditional and unconditional expectations of products of multivariate Hermite polynomials with multivariate normal arguments. This required development of an extended Hermite polynomial. Some possible applications of the expressions are noted.

Future work is to extend the results to matric variate Hermite polynomials and complex variate Hermite polynomials. The future work could also consider multivariate non-normal arguments, matric variate non-normal arguments and complex variate non-normal arguments.

Appendix: Proofs

Proof of Theorem 1.2: We prove (1.13). We use the well-known representation

$$X_2 = N_A + BX_1,\ A = V_{2\cdot 1},\ B = V_{21}V_{11}^{-1}, \tag{A.1}$$

where $N_A \sim \mathcal{N}(0, A)$ is independent of X_1. (This works since it implies that $X_2 \sim \mathcal{N}(0, V_{22})$, $\mathbb{E}[X_2X_1'] = V_{21}$, and X is normal.) So, for $\alpha = (\alpha_1, \alpha_2)$, $\alpha_j \in \mathbb{R}^{r\times p_j}$, $\alpha X = \sum_{j=1}^{2}\alpha_j X_j = \delta + \alpha_2 N_A$, where $\delta = (\alpha_1 + \alpha_2 B)X_1$. (1.13) now follows since the egf of $\mathbb{E}^{Y_1}[h_n(\alpha Y, C)]$ is $\mathbb{E}^{Y_1}\left[\exp\left(t'\alpha Y + t'Ct/2\right)\right] = \exp\left[t'(\alpha\mu + \delta) + t'Ft/2\right]$.

Proof of Theorem 1.1: Using the minimal choice of A, B of (1.5), for $t \in \mathbb{R}^p$ set $u = Ht$ so that $t = H'u$, $t'Ct = u'\Lambda u = u_1'\Lambda_1 u_1 - u_3'\Lambda_3 u_3$, where $u_j = H_j t$, partitioning H' as $(H_1'H_2'H_3')$. For any $x_1 \in \mathbb{R}^p$, set $x_3 = x - x_1$,

$$\begin{aligned} q_1 &= t'x_1 + u_1'\Lambda_1 u_1/2 = t'x_1 + u'V_1 u/2 = t'x_1 + t'C_1 t/2, \\ q_3 &= t'x_3 - u_3'\Lambda_3 u_3/2 = t'x_3 - u'V_3 u/2 = t'x_3 - t'C_3 t/2, \end{aligned}$$

say with $C_j = H_j'\Lambda_j H_j \geq 0$. So, $\exp(q_1)$ is the egf of $h_n(x_1, C_1) = \mathbb{E}[(x_1 + N_{C_1})^n]$ and $\exp(q_3)$ is the egf of $h_n(x_3, -C_3) = \mathbb{E}[(x_3 + \mathrm{i}N_{C_3})^n]$. But $\exp(q_1 + q_3)$ is the egf of $h_n(x, C)$. So, we obtain

$$h_n(x, C) = \sum_{n_1+n_3=n} \binom{n}{n_1} h_{n_1}\left(x_1, C_1\right) h_{n_3}\left(x_3, -C_3\right),$$

that is, $h_n(x, C)/n!$ can be written as (1.4), the convolution of $h_n(x_1, C_1)/n$ and $h_n(x_3, -C_3)/n!$. □

Proof of Theorem 1.3: For λ an eigenvalue of F with eigenvector x, $Fx = Cx + \alpha_2 z$, where $z = V_{2\cdot 1}\alpha_2' x \in \mathbb{R}^{p_2}$. In the special case that $\alpha_2' x = 0$, that is, $z = 0$ then λ is an eigenvalue of C with eigenvector x and $\alpha_2' x = 0$, implying p_2 constraints on x so that $p_2 < r$ and we can take $r - p_2$ orthogonal solutions for the eigenvalues x from the nullspace $\{x : \alpha_2' x = 0\}$. But if $z \neq 0$ then $\lambda x = Fx = Cx + \alpha_2 z$ implies that $z = V_{2\cdot 1}\alpha_2'(\lambda I_r - C)^{-1}\alpha_2 z$ with $z \neq 0$ so that (1.20) holds. □

Proof of Theorem 2.1: For A of (A.1), $\alpha_k Y = \widetilde{\delta}_k + \alpha_{k2} N_A$. So, for $t' = (t_1', \ldots, t_K') \in \mathbb{R}^r$, $r = \sum_{k=1}^K r_k$, the egf of $\mathbb{E}^{Y_1}[\prod_{k=1}^K h_{n_k}(\alpha_k Y, C_k)]$ as a function of $(n_1, \ldots, n_K) \in \mathbb{R}^r$ is

$$\mathbb{E}^{Y_1}\left[\exp\left\{\sum_{k=1}^K \left(t_k'\alpha_k Y + t_k' C_k t_k/2\right)\right\}\right] = \exp(q), \tag{A.2}$$

where $q = \sum_{k=1}^K q_k$, $q_k = t_k'\widetilde{\delta}_k + t_k' D_k t_k/2$ and $D_k = \alpha_{k2} V_{2\cdot 1}\alpha_{k2}' + C_k$. The exponent in the left hand side of (A.2) is

$$\sum_{k=1}^K \left[t_k'\left(\widetilde{\delta}_k + \alpha_{k2} N_A\right) + t_k' C_k t_k/2\right] = \sum_{k=1}^K \left[t_k'\widetilde{\delta}_k + t_k' C_k t_k/2\right] + u' N_A,$$

where $u = \sum_{k=1}^K \alpha_{k2}' t_k = \gamma' t$ and $t' = (t_1', \ldots, t_K')$. So,

$$q = \sum_{k=1}^K \left(t_k'\widetilde{\delta}_k + t_k' C_k t_k/2\right) + u' A u/2 = t'\widetilde{\delta} + t' D t/2.$$

This completes the proof. □

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