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Some properties of ordered 0-minimal $(0, 2)$-bi-Γ-ideals in po-Γ-semigroups

M. Y. Abbasi* and Abul Basar †

Abstract

In this paper, we introduce ordered (generalized) (m, n)-Γ-ideals in po-Γ-semigroups. Then we characterize the po-Γ-semigroup through ordered (generalized) $(0, 2)$-Γ-ideals, ordered (generalized) $(1, 2)$-Γ-ideals and ordered (generalized) 0-minimal $(0, 2)$-Γ-ideals. Also, we investigate the notion of ordered (generalized) $(0, 2)$-bi-Γ-ideals, ordered 0-$(0, 2)$ bisimple po-Γ-semigroups and ordered 0-minimal (generalized) $(0, 2)$-bi-Γ-ideals in po-Γ-semigroups. It is proved that a po-Γ-semigroup S with a zero 0 is 0-$(0, 2)$-bisimple if and only if it is left 0-simple.

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1. Introduction and Preliminaries

The notion of the Γ-semigroup was introduced by M. K. Sen in [9] as a generalization of semigroups and ternary semigroups and the concept of the po-Γ-semigroup was given by Y. I. Kwon and S. K. Lee in [5]. Thereafter different aspects of ideal-theoretic results have been extensively studied in semigroups and po-Γ-semigroups in [1-4, 6].

The concept of the (m, n)-ideal in semigroups was given by S. Lajos in [8] as a generalization of one-sided ideals of semigroups. Thereafter, the notion of the generalized bi-ideal [(or generalized (1,1)-ideal] was introduced in semigroups by S. Lajos in [7] as a generalization of bi-ideals of semigroups. In this paper, we define and use the notion of ordered (generalized) (m, n)-Γ-ideals in po-Γ-semigroups to examine some important classical results and properties in po-Γ-semigroups. As an application of the results of

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this paper, the corresponding results of Γ-semigroups (without order) and semigroups (without order) can also be obtained.

Let S and Γ be two nonempty sets. Then a system $(S, \Gamma, \cdot)$ is called a Γ-semigroup, where $\cdot$ is a ternary operation $S \times \Gamma \times S \to S$ such that $(x \cdot \alpha \cdot y) \cdot \beta \cdot z = x \cdot \alpha \cdot (y \cdot \beta \cdot z)$, for all $x, y, z \in S$ and all $\alpha, \beta \in \Gamma$. Let A be a nonempty subset of $(S, \Gamma, \cdot)$. Then A is called a sub-Γ-semigroup of $(S, \Gamma, \cdot)$ if $a \cdot \gamma \cdot b \in A$, for all $a, b \in A$ and $\gamma \in \Gamma$. Furthermore, a Γ-semigroup S is said to be commutative if $a \cdot \gamma \cdot b = b \cdot \gamma \cdot a$, for all $a, b \in S$ and $\gamma \in \Gamma$.

A po-Γ-semigroup is an ordered set $(S, \leq)$ at the same time a Γ-semigroup $(S, \Gamma, \cdot)$ such that $a \leq b \Rightarrow a \cdot \alpha \cdot x \leq b \cdot \alpha \cdot x$ and $x \cdot \beta \cdot a \leq x \cdot \beta \cdot b$, for all $a, b, x \in S$ and $\alpha, \beta \in \Gamma$.

Notation 1: For subsets A, B of a po-Γ-semigroup S, the product set $A \cdot B$ of the pair (A, B) relative to S is defined as $A \cdot \Gamma \cdot B = \{a \cdot \gamma \cdot b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$ and for $A \subseteq S$, the product set $A \cdot A$ relative to S is defined as $A^2 = A \cdot A = A \cdot \Gamma \cdot A$.

Notation 2: For $M \subseteq S$, $(M] = \{s \in S \mid s \leq m, \text{ for some } m \in M\}$. Also, we write $(s]$ instead of $(\{s\}]$ for $s \in S$.

Notation 3: Let $B \subseteq S$. Then for a non-negative integer m, the power of $B^m = B\Gamma B\Gamma B\Gamma B \cdots$, where B occurs m times. Note that the power is suppressed when $m = 0$. So $B^0\Gamma S = S = S\Gamma B^0$.

In what follows we denote the po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ by S unless otherwise specified. Throughout the paper, for the sake of brevity, we denote $a \cdot \gamma \cdot b$ by $a\gamma b$.

1.1. Example. Let S be the set of all $m \times n$ matrices with entries from a field, where m, n are positive integers. Let $P(S)$ be the power set of S. Then it is easy to see that $P(S)$ is not a semigroup under multiplication of matrices because for A, $B \in P(S)$, the product AB is not defined. Let Γ be the set of $n \times m$ matrices with entries from the same field. Then for A, B, $C \in P(S)$ and P, $Q \in \Gamma$, we have $APB \in P(S)$, $AQB \in P(S)$ and since the matrix multiplication is associative, we get that S is a Γ-semigroup. Furthermore, define $A \leq B$ if and only if $A \subseteq B$ for all $A, B \in P(S)$, then $P(S)$ is a po-Γ-semigroup.

Suppose A and B are two nonempty subsets of S. Then we have the following (see [3]).

(1) $(A]\Gamma(B] \subseteq (A\Gamma B]$;

(2)$A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $((A]] = (A]$.

Suppose S is a po-Γ-semigroup and I is a nonempty subset of S. Then I is called an ordered right (resp. left) Γ-ideal of S if

(i) $I\Gamma S \subseteq I (rep.\ S\Gamma I \subseteq I)$,

(ii) $a \in I, b \leq a$ for $b \in S \Rightarrow b \in I$.

Equivalent definition:

(i) $I\Gamma S \subseteq I$ (resp. $S\Gamma I \subseteq I$).

(ii) $(I] = I$.

An ordered Γ-ideal I of S is both a right and a left ordered Γ-ideal of a po-Γ-semigroup S. A right, left or (two-sided) ordered Γ-ideal I of S is called proper if $I \neq S$.

Let S be a semigroup and A be a nonempty subset of S then A is called a generalized (m, n)-ideal of S if $A^m S A^n \subseteq A$, where m, n are arbitrary non-negative integers. Note that if A is a subsemigroup of S, then A is called an (m, n)-ideal of S. We now introduce the following definition.

1.2. Definition. Suppose B is a sub-Γ-semigroup (resp. nonempty subset) of a po-Γ-semigroup S. Then B is called an (resp. generalized) (m, n)-Γ-ideal of S if (i) $B^m \Gamma S \Gamma B^n \subseteq B$ and (ii) for $b \in B$, $s \in S$, $s \leq b \Rightarrow s \in B$.

Note that in the above Definition 1.2, if we set $m = n = 1$, then B is called a (generalized) bi-Γ-ideal of S. Moreover, if $m = 0$ and $n = 2$, then we obtain an ordered (generalized) $(0, 2)$-Γ-ideal of S. In a similar fashion, we can obtain an ordered (generalized) $(1, 2)$-Γ-ideal and an ordered (generalized) $(2, 1)$-Γ-ideal of S.

If B is a nonempty subset of S, then to see that $(B^2 \cup B\Gamma S\Gamma B^2]$ is an ordered (generalized) bi-Γ-ideal of S, we present the verification of it as follows:

$$
\begin{aligned}
((B^2 \cup B\Gamma S\Gamma B^2]] &= (B^2 \cup B\Gamma S\Gamma B^2] \text{ and}(\mathrm{B}^2 \cup \mathrm{B\Gamma S\Gamma B}^2]\Gamma\mathrm{S}\Gamma(\mathrm{B}^2 \cup \mathrm{B\Gamma S\Gamma B}^2] \\
&= (B^2 \cup B\Gamma S\Gamma B^2]\Gamma(S]\Gamma(B^2 \cup B\Gamma S\Gamma B^2] \\
&\subseteq (B^2\Gamma S\Gamma B^2 \cup B^2\Gamma S\Gamma B\Gamma S\Gamma B^2 \cup B\Gamma S\Gamma B^2\Gamma S\Gamma B^2 \cup B\Gamma S\Gamma B^2\Gamma S\Gamma B\Gamma S\Gamma B^2] \\
&\subseteq (B\Gamma S\Gamma B^2] \\
&\subseteq (B^2 \cup B\Gamma S\Gamma B^2].
\end{aligned}
$$

2. Main Results

We now develop ideal theory for po-Γ-semigroups. We begin our study with proving the following Lemma.

2.1. Lemma. The following assertions are equivalent for a subset B of a po-Γ-semigroup S.

(i) B is an ordered (generalized) $(0, 2)$-Γ-ideal of S;

(ii) B is an ordered left Γ-ideal of some ordered left Γ-ideal of S.

Proof. $(i) \Rightarrow (ii)$. Suppose B is an ordered (generalized) $(0, 2)$-Γ-ideal of a po-Γ-semigroup S. Then we obtain $(B \cup S\Gamma B]\Gamma B = (B^2 \cup S\Gamma B^2] \subseteq (B] = B$ and $((B]] = (B]$ and so B is an ordered left Γ-ideal of the ordered left Γ-ideal $(B \cup S\Gamma B]$ of S.
$(ii) \Rightarrow (i)$. Suppose L is an ordered left Γ-ideal of S and B is an ordered left Γ-ideal of L. Then, $S\Gamma B^2 \subseteq S\Gamma L\Gamma B \subseteq L\Gamma B \subseteq B$. Suppose $b \in B$ and $s \in S$ are such that $s \leq b$. As $b \in L$, we get $s \in L$ and so $s \in B$. Consequently, B is an ordered (generalized) $(0, 2)$-Γ-ideal of S.

2.2. Theorem. Suppose B is a subset of a po-Γ-semigroup S. Then the following results are equivalent:

(i) B is an ordered (generalized) $(1,2)$-Γ-ideal of S;

(ii) B is an ordered left Γ-ideal of some ordered (generalized) bi-Γ-ideal of S;

(iii) B is an ordered (generalized) bi-Γ-ideal of some left ordered Γ-ideal of S;

(iv) B is an ordered (generalized) $(0,2)$-Γ-ideal of some ordered right Γ-ideal of S;

(v) B is an ordered right-Γ-ideal of some ordered (generalized) $(0,2)$-Γ-ideal of S.

Proof. $(i) \Rightarrow (ii)$. Suppose B is an ordered (generalized) $(1,2)$-Γ-ideal of S. This means B is a sub-Γ-semigroup (nonempty subset) of S and $B\Gamma S\Gamma B^2 \subseteq B$. So $(B^2 \cup B\Gamma S\Gamma B^2]\Gamma B = (B^2 \cup B\Gamma S\Gamma B^2]\Gamma(B] \subseteq (B^3 \cup B\Gamma S\Gamma B^3] \subseteq (B^2 \cup B\Gamma S\Gamma B^2] \subseteq (B] = B$. Obviously, if $b \in B$, $s \in (S^2 \cup B\Gamma S\Gamma B^2]$ so that $s \leq b$ then $s \in B$. Hence, B is an ordered left Γ-ideal of the ordered (generalized) bi- Γ-ideal $(B^2 \cup B\Gamma S\Gamma B^2]$ of S.
$(ii) \Rightarrow (iii)$. Suppose B is an ordered left Γ-ideal of some ordered (generalized) bi-Γ-ideal A of S. Recall that $(B \cup S\Gamma B]$ is an ordered left Γ-ideal of S. According to our hypothesis, $B\Gamma(B \cup S\Gamma B]\Gamma B \subseteq (B]\Gamma(B \cup S\Gamma B]\Gamma(B] \subseteq (B^3 \cup B\Gamma S\Gamma B^2] \subseteq (B \cup A\Gamma S\Gamma A\Gamma B] \subseteq (B \cup A\Gamma B] \subseteq (B] = B$. Suppose $b \in B$, $s \in (B \cup S\Gamma B]$ such that $s \leq b$. As $b \in B$, $b \in A$. So $s \in A$ and therefore, $s \in B$. Hence, B is an ordered (generalized) bi-Γ-ideal of the left ordered Γ-ideal $(B \cup S\Gamma B]$ of S.
$(iii) \Rightarrow (iv)$. Suppose B is an ordered (generalized) bi-Γ-ideal of some left ordered Γ-ideal L of S. This implies that $B \subseteq L$, $B\Gamma L^1\Gamma B \subseteq B$ and $S\Gamma L \subseteq L$. Therefore $(B \cup B\Gamma S]\Gamma B^2 \subseteq (B \cup B\Gamma S]\Gamma(B^2] \subseteq (B^3 \cup B\Gamma S\Gamma B^2] \subseteq (B \cup B\Gamma S\Gamma L\Gamma B] \subseteq (B \cup B\Gamma L\Gamma B] \subseteq (B] = B$. Furthermore, suppose that $b \in B$, $s \in (B \cup B\Gamma S]$ such that $s \leq b$, so $b \in L$. Then $s \in L$, therefore $s \in B$. Hence, B is an ordered (generalized) $(0,2)$-Γ-ideal of the ordered right Γ-ideal $(B \cup B\Gamma S]$ of S.
$(iv) \Rightarrow (v)$. Suppose B is an ordered (generalized) $(0,2)$-Γ-ideal of some ordered right Γ-ideal R of S. This implies that $B \subseteq R$, $R\Gamma B^2 \subseteq B$ and $R\Gamma S \subseteq R$. Then $B\Gamma(B \cup S\Gamma B^2] \subseteq (B]\Gamma(B \cup S\Gamma B^2] \subseteq (B^2 \cup B\Gamma S\Gamma B^2] \subseteq (B \cup R\Gamma S\Gamma B^2] \subseteq (B \cup R\Gamma B^2] \subseteq (B] = B$. Let $b \in B$, $s \in (B \cup S\Gamma B^2]$ such that $s \leq b$. Then $b \in R$, so $s \in R$, thus $s \in B$. Hence, B is an ordered right Γ-ideal of the (generalized) $(0,2)$-Γ-ideal $(B \cup S\Gamma B^2]$ of S.
$(v) \Rightarrow (i)$. Suppose B is an ordered right Γ-ideal of an ordered (generalized) $(0,2)$-Γ-ideal R of S. This further implies that $B \subseteq R$, $B\Gamma R \subseteq B$ and $S\Gamma R^2 \subseteq R$. Then $B\Gamma S\Gamma B^2 \subseteq B\Gamma S\Gamma R^2 \subseteq B\Gamma R \subseteq B$. Suppose $b \in B$, $s \in S$ such that $s \leq b$. As $b \in R$, so $s \in B$. Hence B is an ordered (generalized) $(1,2)$-Γ-ideal of S. Hence, B is an ordered (generalized) bi-Γ-ideal of S.

2.3. Lemma. A sub-Γ-semigroup (nonempty subset) A of a po-Γ-semigroup S such that $A = (A]$ is an ordered (generalized) $(1,2)$-Γ-ideal of S if and only if there exists an ordered (generalized) $(0,2)$-Γ-ideal L of S and an ordered right Γ-ideal R of S so that $R\Gamma L^2 \subseteq A \subseteq R \cap L$.

Proof. Suppose A is an ordered (generalized)$(1,2)$-Γ-ideal of S. We know that $(A \cup S\Gamma A^2]$ and $(A \cup A\Gamma S]$ are an ordered (generalized) $(0,2)$-Γ-ideal and an ordered right Γ-ideal of S, respectively. Furthermore, assume $L = (A \cup S\Gamma A^2]$ and $R = (A \cup A\Gamma S]$. Then $R\Gamma L^2 \subseteq (A^3 \cup A^2\Gamma S\Gamma A^2 \cup A\Gamma S\Gamma A^2 \cup A\Gamma S\Gamma A\Gamma S\Gamma A^2] \subseteq (A^3 \cup A\Gamma S\Gamma A^2] \subseteq (A] = A$. Hence, $R \subseteq R \cap L$.

Conversely, suppose R is an ordered right Γ-ideal of S and L is an ordered (generalized) $(0,2)$-Γ-ideal of S so that $R\Gamma L^2 \subseteq A \subseteq R \cap L$. Then $A\Gamma S\Gamma A^2 \subseteq (R\cap L)\Gamma S\Gamma(R\cap L)\Gamma(R\cap L) \subseteq R\Gamma S\Gamma L^2 \subseteq R\Gamma L^2 \subseteq A$. Hence, A is an ordered (generalized) $(1,2)$-Γ-ideal of S.

2.4. Definition. An ordered (generalized) $(0,2)$-bi-Γ-ideal B of S is called 0-minimal if (i) $B \neq \{0\}$ and (ii) $\{0\}$ is the only ordered (generalized) $(0,2)$-bi-Γ-ideal of S properly contained in B.

2.5. Lemma. Suppose L is an ordered 0-minimal left Γ-ideal of a po-Γ-semigroup S with 0 and I is a sub-Γ-semigroup (nonempty subset) of L such that $I = (I]$. Then I is an ordered (generalized) $(0,2)$-Γ-ideal of S contained in L if and only if $(I\Gamma I] = \{0\}$ or $I = L$.

Proof. Suppose I is an ordered (generalized) $(0,2)$-Γ-ideal of S contained in L. As $(S\Gamma I^2]$ is an ordered left Γ-ideal of S and $(S\Gamma I^2] \subseteq I \subseteq L$, we obtain $(S\Gamma I^2] = \{0\}$ or $(S\Gamma I^2] = \{L\}$. If $(S\Gamma I^2] = L$, then $L = (S\Gamma I^2] \subseteq (I]$. So $I = L$. Suppose $(S\Gamma I^2] = \{0\}$. As $S\Gamma(I^2] \subseteq (S\Gamma I^2] = \{0\} \subseteq (I^2]$, then $(I^2]$ is an ordered left Γ-ideal of S contained in L. By the minimality of L, we obtain $(I^2] = \{0\}$ or $(I^2] = L$. If $(I^2] = L$, then $I = L$. Therefore, $I^2 = \{0\}$ or $I = L$.
The converse part is straightforward.

2.6. Lemma. Suppose M is an ordered 0-minimal (generalized) $(0,2)$-Γ-ideal of a po-Γ-semigroup S with a zero 0. Then $(M^2] = \{0\}$ or M is an ordered 0-minimal left Γ-ideal of S.

Proof. As $M^2 \subseteq M$ and $S\Gamma(M^2]^2 = S\Gamma(M^2]\Gamma(M^2] \subseteq (S\Gamma M^2]\Gamma(M^2] \subseteq (M]\Gamma(M^2] \subseteq (M^2]$. Then we obtain $(M^2]$ is an ordered (generalized) $(0,2)$-Γ-ideal of S contained in M. Therefore $(M^2] = \{0\}$ or $(M^2] = M$. Suppose $(M^2] = M$. As $S\Gamma M = S\Gamma(M^2] \subseteq (S\Gamma M^2] \subseteq (M] = M$, it follows that M is an ordered left Γ-ideal of S. Suppose B is an ordered left Γ-ideal of S contained in M. Therefore, $S\Gamma B^2 \subseteq B^2 \subseteq B \subseteq M$. Hence, B is an ordered (generalized) $(0,2)$-Γ-ideal of S contained in M and so $B = \{0\}$ or $B = M$.

2.7. Corollary. Suppose S is a po-Γ-semigroup without a zero 0. Then M is an ordered minimal (generalized) $(0,2)$-Γ-ideal of S if and only if M is an ordered minimal left Γ-ideal of S.

Proof. It follows by Lemma 2.5 and Lemma 2.6.

2.8. Lemma. Suppose S is a po-Γ-semigroup without a zero 0. Further suppose that M is a nonempty subset of S. Then the following results are equivalent:

(i) M is an ordered (generalized) minimal $(2,1)$-Γ-ideal of S;

(ii) M is an ordered (generalized) minimal bi-Γ-ideal of S.

Proof. Suppose S is a po-Γ-semigroup without zero and M is an ordered minimal (generalized) $(2,1)$-Γ-ideal of S. Then $(M^2\Gamma S\Gamma M] \subseteq M$ and so $(M^2\Gamma S\Gamma M]$ is an ordered (generalized) $(2,1)$-Γ-ideal of S. Therefore, we obtain $(M^2\Gamma S\Gamma M] = M$.
As $M\Gamma S\Gamma M = (M^2\Gamma S\Gamma M]\Gamma S\Gamma M \subseteq (M^2\Gamma S\Gamma M\Gamma S\Gamma M] \subseteq (M^2\Gamma S\Gamma M] = M$, we have that M is an ordered (generalized) bi-Γ-ideal of S. Let there exist an ordered (generalized) bi-Γ-ideal B of S contained in M. Then $B^2\Gamma S\Gamma B \subseteq B^2 \subseteq B \subseteq M$, therefore, B is an ordered (generalized) $(2,1)$-Γ-ideal of S contained in M. Applying the minimality of

M, we obtain $B = M$.
Conversely, suppose M is an ordered minimal (generalized) bi-Γ-ideal of S. Then M is an ordered (generalized) $(2,1)$-Γ-ideal of S. Suppose T is an ordered (generalized) $(2,1)$-Γ-ideal of S contained in M. As $(T^2\Gamma S\Gamma T]\Gamma S\Gamma(T^2\Gamma S\Gamma T] \subseteq (T^2\Gamma(S\Gamma T\Gamma S\Gamma T^2\Gamma S)\Gamma T] \subseteq (T^2\Gamma S\Gamma T]$, we obtain $(T^2\Gamma S\Gamma T]$ is an ordered (generalized) bi-Γ-ideal of S. This shows that $(T^2\Gamma S\Gamma T] = M$. As $M = (T^2\Gamma S\Gamma T] \subseteq (T] = T$, $M = T$. Hence, M is an ordered minimal (generalized) $(2,1)$-Γ-ideal of S.

2.9. Definition. A sub-Γ-semigroup (nonempty subset) B of a po-Γ-semigroup S is called an ordered (generalized) $(0,2)$-bi-Γ-ideal of S if B is an ordered (generalized) bi-Γ-ideal of S and also an ordered (generalized) $(0,2)$-Γ-ideal of S.

2.10. Lemma. Suppose B is a subset of a po-Γ-semigroup S. Then the following conditions are equivalent :

(i) B is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S;

(ii) B is an ordered Γ-ideal of some ordered left Γ-ideal of S.

Proof. $(i) \Rightarrow (ii)$. Suppose B is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S. This implies that $B\Gamma S\Gamma B \subseteq B$ and $S\Gamma B^2 \subseteq B$. Then $S\Gamma(B^2 \cup S\Gamma B^2] \subseteq (S\Gamma B^2 \cup S^2\Gamma B^2] \subseteq (S\Gamma B^2] \subseteq (B^2 \cup S\Gamma B^2]$ Therefore, $(B^2 \cup S\Gamma B^2]$ is an ordered left Γ-ideal of S. As $B\Gamma(B^2 \cup S\Gamma B^2] \subseteq (B^3 \cup B\Gamma S\Gamma B^2] \subseteq (B] = B$, $(B^2 \cup S\Gamma B^2]\Gamma B \subseteq (B^3 \cup S\Gamma B^3] \subseteq (B] = B$. Hence B is an ordered Γ-ideal of the left Γ-ideal $(B^2 \cup S\Gamma B^2]$ of S.
$(ii) \Rightarrow (i)$. Suppose B is an ordered Γ-ideal of some ordered left Γ-ideal L of S. By Lemma 2.1, B is an ordered (generalized) $(0,2)$-Γ-ideal of S and hence B is an ordered (generalized) bi-Γ-ideal of S.

2.11. Theorem. Suppose B is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal of a po-Γ-semigroup S with a zero 0. Then exactly one of the followings cases arises:

(i) $B = \{0, b\}$, $(b\Gamma S^1\Gamma b] = \{0\}$;

(ii) $B = (\{0, b\}]$, $b^2 = 0$, $(b\Gamma S\Gamma b] = B$;

(iii) $(S\Gamma b^2] = B$ for all $b \in B \setminus \{0\}$.

Proof. Suppose B is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal of a po-Γ-semigroup S. Furthermore, suppose $b \in B \setminus \{0\}$. Then $(S\Gamma b^2] \subseteq B$ and $(S\Gamma b\Gamma b]$ is an ordered left Γ-ideal of S, therefore $(S\Gamma b^2]$ is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S. Hence $(S\Gamma b^2] = \{0\}$ or $(S\Gamma b^2] = B$.
Let $(S\Gamma b^2] = \{0\}$. As $b^2 \in B$, we obtain either $b^2 = b$ or $b^2 = 0$ or $b^2 \in B \setminus \{0, b\}$. If $b^2 = b$, then $b = 0$. This is a contradiction. Let $b^2 \in B \setminus \{0, b\}$. Then $S^1\Gamma(\{0, b^2\}]^2 \subseteq (\{0, S\Gamma b^2\}] = (\{0\}] \cup (S\Gamma b^2] = \{0\} \subseteq (\{0\} \cup b^2]$, $(\{0\} \cup b^2]\Gamma S\Gamma(\{0\} \cup b^2] \subseteq (b^2\Gamma S\Gamma b^2] \subseteq (S\Gamma b^2] = \{0\} \subseteq \{0, b^2\}$. So $(\{0\} \cup b^2]$ is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S contained in B and we notice that $(\{0\} \cup b^2] \neq \{0\}$, $(\{0\} \cup b^2] \neq B$. This is too not possible since B is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal of S. So $b^2 = \{0\}$ and hence by Lemma 2.10, $B = (\{0, b\}]$. Now since we have $(b\Gamma S\Gamma b]$ is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S contained in B, we get $(b\Gamma S\Gamma b] = \{0\}$ or $(b\Gamma S\Gamma b] = B$. So, $(S\Gamma b^2] = \{0\}$ and it follows that either $B = \{0, b\}$ and $(b\Gamma S^1\Gamma b] = \{0\}$ or $B = \{0, b\}$,

$b^2 = \{0\}$ and $(b\Gamma S\Gamma b] = B$. If $(S\Gamma b^2] \neq \{0\}$, then $(S\Gamma b^2] = B$.

2.12. Corollary. Suppose B is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal of a po-Γ-semigroup S with a zero 0 so that $(B^2] \neq \{0\}$. Then $B = (S\Gamma b^2]$, for every $b \in B \setminus \{0\}$.

2.13. Definition. A po-Γ-semigroup S with a zero 0 is called 0-$(0,2)$- bisimple if (i) $(S^2] \neq \{0\}$ and $\{0\}$ is the only ordered proper (generalized) $(0,2)$-bi-Γ-ideal of S.

2.14. Corollary. A po-Γ-semigroup S with a zero 0 is 0-$(0,2)$-bisimple if and only if $(S\Gamma s^2] = S$, for every $s \in S \setminus \{0\}$.

Proof. If S is 0-$(0,2)$-bisimple, then $(S\Gamma S] \neq \{0\}$ and S is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal. By Corollary 2.12, we have $S = (S\Gamma s^2]$, for every $s \in S \setminus \{0\}$.
Conversely, suppose $S = (S\Gamma s^2]$, for every element $s \in S \setminus \{0\}$ and further suppose that B is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S such that $B \neq \{0\}$. Suppose $b \in B \setminus \{0\}$. Then $S = (S\Gamma b^2] \subseteq (S\Gamma B^2] \subseteq (B] = B$, therefore, $S = B$. As $S = (S\Gamma b^2] \subseteq (S\Gamma S] = (S^2]$, we obtain $\{0\} \neq S = (S\Gamma S] = (S^2]$. Hence S is 0-$(0,2)$-bi-simple.

2.15. Theorem. A po-Γ-semigroup S with a zero 0 is 0-$(0,2)$-bisimple if and only if S is left 0-simple.

Proof. We recall that every ordered left Γ-ideal B of a po-Γ-semigroup S is an ordered 0-$(0,2)$-bi-Γ-ideal of S. So $B = \{0\}$ or $B = S$. Therefore, if S is 0-$(0,2)$-bisimple then S is left 0-simple.
Conversely, if S is left 0-simple then $(S\Gamma s] = S$, for every $s \in S \setminus \{0\}$ from which it follows that $S = (S\Gamma s] = ((S\Gamma s]\Gamma s] \subseteq ((S\Gamma s^2]] = (S\Gamma s^2]$. Therefore, using Corollary 2.14, S is 0-$(0,2)$-bisimple.

2.16. Theorem. Suppose B is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal of a po-Γ-semigroup S. Then either $(B\Gamma B] = \{0\}$ or B is left 0-simple.

Proof. Suppose $(B\Gamma B] \neq \{0\}$. Then, by Corollary 2.12, we obtain $(S\Gamma b^2] = B$, for every $b \in B \setminus \{0\}$. As $b^2 \in B \setminus \{0\}$, for every $b \in B \setminus \{0\}$, we obtain $b^4 = (b^2)^2 \in B \setminus \{0\}$. Suppose $b \in B \setminus \{0\}$. As $(B\Gamma b^2]\Gamma S^1\Gamma(B\Gamma b^2] \subseteq (B\Gamma B\Gamma b^2] \subseteq (B\Gamma b^2]$ and $S\Gamma(B\Gamma b^2]^2 \subseteq (S\Gamma B\Gamma b^2\Gamma B\Gamma b^2] \subseteq (S\Gamma B^2\Gamma b^2] \subseteq (B\Gamma b^2]$, we get that $(B\Gamma b^2]$ is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S contained in B. Therefore, $(B\Gamma b^2] = \{0\}$ or $(B\Gamma b^2] = B$. As $b^4 \in B\Gamma b^2 \subseteq (B\Gamma b^2]$ and $b^4 \in B \setminus \{0\}$, we obtain $(B\Gamma b^2] = B$. By Corollary 2.14 and Theorem 2.15, it follows that B is left 0-simple.

References

[1] Abbasi, M. Y. and Basar Abul, On Ordered Quasi-Gamma-Ideals of Regular Ordered Gamma-Semigroups, *Algebra*, **2013**, Article ID 565848,(2013). doi:10.1155/2013/565848.

[2] Iampan, A. and Siripitukdet, M., On minimal and maximal ordered left ideals in po-Γ-semigroups, *Thai Journal of Mathematics*,2(2004), 275-282.

[3] Kehayopulu, N., On prime, weakly prime ideals in po-Γ- semigroups, *Lobachevskii Journal of Mathematics*, **30(4)**(2009), 257-262.

[4] Krgovic, D. N., On 0-Minimal $(0,2)$-bi-ideals of semigroups, *Publications De L' Institut Mathematique, Nouvelle serie, tome.*, **31(45)** (1982), 103-107.

[5] Kwon, Y. I. and Lee, S. K., Some special elements in ordered Γ-semigroups, *Kyungpook Math. J.*, **35**(1996), 679-685.

[6] Kwon, Y. I. and Lee, S. K., On weakly prime ideals of ordered-Γ-semigroups, *Comm. Korean Math. Soc.,* **13(2)**(1996), 251-256.

[7] Lajos, S., Notes on generalized bi-ideals in semigroups, *Soochow J. Math.,* **10** (1984), 55-59.

[8] Lajos, S., Generalized ideals in semigroups, *Acta Sci. Math.,* **2** (1961), 217-222.

[9] Sen, M. K., On Γ-semigroups, *Algebra and its applications, Int. Symp., New Delhi, (1981), Lecture Notes in Pure and Applied Mathematics 91, Decker, New York,(1984), 301-308.*

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Meromorphic subordination results for p-valent functions associated with convolution

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Abstract

In this paper, by making use of the convolution and subordination principals, we obtain some subordination results for certain family of meromorphic p-valent functions defined by using a new linear operator.

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Keywords: Meromorphic functions, subordination, convolution, linear operator.

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1. Introduction

Let $\sum_p$ be the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk $U^* = U \backslash \{0\}$, where $U = \{z : z \in \mathbb{C}\ ,\ |z| < 1\}$. If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence ([5] and [10]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f,\ g\ \in \Sigma_p$, Aouf et al. [3] defined the linear operator $D^n_{\lambda,p}(f * g)(z) : \Sigma_p \longrightarrow \Sigma_p$ $(\lambda \geq 0,\ p \in \mathbb{N},\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ by

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$$D^0_{\lambda,p}(f*g)(z) = (f*g)(z),$$

$$\begin{aligned} D^1_{\lambda,p}(f*g)(z) &= D_{\lambda,p}(f*g)(z) = (1-\lambda)(f*g)(z) + \lambda z^{-p}\ (z^{p+1}(f*g)(z))' \\ &= z^{-p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]a_k b_k z^k\ (\lambda \geq 0;\ p \in \mathbb{N}), \\ D^2_{\lambda,p}(f*g)(z) &= D_{\lambda,p}(D_{\lambda,p}(f*g))(z) \\ &= z^{-p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]^2 a_k b_k z^k\ (\lambda \geq 0;\ p \in \mathbb{N}) \end{aligned}$$

and (in general)

$$\begin{aligned} D^n_{\lambda,p}(f*g)(z) &= D_{\lambda,p}(D^{n-1}_{\lambda,p}(f*g)(z)) \\ &= z^{-p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]^n a_k b_k z^k\ (\lambda \geq 0;\ p \in \mathbb{N};\ n \in \mathbb{N}_0). \end{aligned}$$

(1.2)

From (1.2) it is easy to verify that [3]:

(1.3) $$z(D^n_{\lambda,p}(f*g)(z))' = \frac{1}{\lambda}D^{n+1}_{\lambda,p}(f*g)(z) - (p+\frac{1}{\lambda})D^n_{\lambda,p}(f*g)(z)\ (\lambda > 0).$$

Specializing the parameters n, l, p, λ and g in (1.2), we have:

(i) For $n = 0$ and $g(z)$ is in the form:

(1.4) $$g(z) = z^{-p} + \sum_{k=0}^{\infty}\frac{(\alpha_1)_{k+p}...(\alpha_q)_{k+p}}{(\beta_1)_{k+p}...(\beta_s)_{k+p}(1)_{k+p}}z^k,$$

$\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2, ..., \beta_s$ are complex or real ($\beta_j \notin Z_0^- = \{0, -1, -2, ...\}$, $j = 1, 2, ..., s$), we have, $D^n_{\lambda,p}(f*g)(z) = H_{p,q,s}(\alpha_1)f(z)$,where the linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [9] and Aouf [2] and contains in turn the operator $L_p(a,c)$ (see [8]) for $q = 2,\ s = 1,\ \alpha_1 = a > 0,\ \beta_1 = c\ (c \neq 0, -1, ...)$ and $\alpha_2 = 1$ and also contains the operator $D^{\nu+p-1}$ (see [13]) for $q = 2,\ s = 1$, $\alpha_1 = \nu + p\ (\nu > -p,\ p \in \mathbb{N})$ and $\alpha_2 = \beta_1 = p$;

(ii) For $n = 0$ and $g(z)$ is in the form:

(1.5) $$g(z) = z^{-p} + \sum_{k=1-p}^{\infty}[\frac{l+\lambda(k+p)}{l}]^m a_k b_k z^k\quad (\lambda, l \geq 0; m \in \mathbb{N}_0),$$

we have $D^0_{\lambda,p}(f*g)(z) = I^m_p(l,\lambda)f(z)$, where the operator $I^m_p(l,\lambda)$ was introduced and studied by El-Ashwah [6] and El-Ashwah and Aouf [7];

(iii) For $n = 0$ and $g(z)$ is in the form:

(1.6) $$g(z) = z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}z^k\ (\alpha \geq 0; \beta > -1),$$

we have $D^n_{\lambda,p}(f*g)(z) = Q^\alpha_{\beta,p}f(z)$ where the operator $Q^\alpha_{\beta,p}$ was introduced and studied by Aqlan et al.[4].

To prove our main results we need the next lemmas.

Lemma 1 [11]. *Let $q(z)$ be univalent in U and let $\varphi(z)$ be analytic in a domain containing $q(U)$. If $zq'(z)\varphi(q(z))$ is starlike and*

$$z\psi'(z)\varphi(\psi(z)) \prec zq'(z)\varphi(q(z)),$$

then $\psi(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Lemma 2 [12]. *Let* β, ν *be any complex numbers,* $\nu \neq 0$ and $q(z) = 1 + q_1 z + q_2 z^2 + ...$ *be univalent in* $U, q(z) \neq 0$. *Suppose that* $Q(z) = \gamma z q'(z)/q(z)$ *be starlike, and*

$$\Re\left\{\frac{\beta}{\nu} q(z) + \frac{zQ'(z)}{Q(z)}\right\} > 0.$$

If $\psi(z) = 1 + c_1 z + c_2 z^2 + ...$ *is analytic in* U *and satisfies*

$$\beta\psi(z) + \nu\frac{z\psi'(z)}{\psi(z)} \prec \beta q(z) + \nu\frac{zq'(z)}{q(z)},$$

then $\psi(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that, $\gamma \in \mathbb{C}, \lambda > 0, p \in \mathbb{N}, n \in \mathbb{N}_0, f,\ g \in \sum_p$ and the powers are the principal ones.

2.1. Theorem. *Let* $q(z) \neq 0$ *be univalent in* U *and* $zq'(z)/q(z)$, *be starlike. If* f *satisfies:*

$$\frac{1}{\lambda}\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} - \frac{\gamma}{\lambda}\frac{D^{n+2}_{\lambda,p}(f*g)(z)}{D^{n+1}_{\lambda,p}(f*g)(z)} \prec \frac{1-\gamma}{\lambda} + \frac{zq'(z)}{q(z)}, \tag{2.1}$$

then

$$\frac{z^{p(1-\gamma)} D^{n+1}_{\lambda,p}(f*g)(z)}{\left[D^{n}_{\lambda,p}(f*g)(z)\right]^{\gamma}} \prec q(z)$$

and $q(z)$ *is the best dominant.*

Proof. Let the function $p(z)$ defined by

$$p(z) = \frac{z^{p(1-\gamma)} D^{n+1}_{\lambda,p}(f*g)(z)}{\left[D^{n}_{\lambda,p}(f*g)(z)\right]^{\gamma}}\ (z \in U). \tag{2.2}$$

Differentiating (2.2) logarithmically with respect to z and using the identity (1.3), we have

$$\frac{zp'(z)}{p(z)} = \frac{1}{\lambda}\frac{D^{n+2}_{\lambda,p}(f*g)(z)}{D^{n+1}_{\lambda,p}(f*g)(z)} - \frac{\gamma}{\lambda}\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} - \frac{1}{\lambda}(1-\gamma),$$

that is, that

$$\frac{zp'(z)}{p(z)} + \frac{1}{\lambda}(1-\gamma) = \frac{1}{\lambda}\frac{D^{n+2}_{\lambda,p}(f*g)(z)}{D^{n+1}_{\lambda,p}(f*g)(z)} - \frac{\gamma}{\lambda}\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)}. \tag{2.3}$$

Therefore, in view of (2.3), the subordination (2.1) becomes

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}.$$

By an application of Lemma 1, with $\varphi(w) = \frac{1}{w}, w \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, we have $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Taking $n = 0$ and $g(z)$ of the form (1.4) and using the identity (see [9]):

$$z\left(H_{p,q,s}(\alpha_1)f(z)\right)' = \alpha_1 H_{p,q,s}(\alpha_1+1)f(z) - (\alpha_1+p)H_{p,q,s}(\alpha_1)f(z), \tag{2.4}$$

we have the following corollary. □

2.2. Corollary. *Let $q(z) \neq 0$ be univalent in U and $zq'(z)/q(z)$, be starlike . If f satisfies*

$$(\alpha_1+1)\frac{H_{p,q,s}(\alpha_1+2)f(z)}{H_{p,q,s}(\alpha_1+1)f(z)}-\gamma\alpha_1\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)}\prec\alpha_1(1-\gamma)+1+\frac{zq'(z)}{q(z)},$$

then

$$\frac{z^{p(1-\gamma)}H_{p,q,s}(\alpha_1+1)f(z)}{[H_{p,q,s}(\alpha_1)f(z)]^{\gamma}}\prec q(z)$$

and $q(z)$ is the best dominant.

Taking $q=2, s=1, \alpha_1=a>0, \beta_1=c>0$ and $\alpha_2=1$, in Corollary 1, we have the following result which correctes the result obtained by Ali and Ravichandran [1, Theorems 2.3].

2.3. Corollary. *Let $q(z) \neq 0$ be univalent in U and $zq'(z)/q(z)$, be starlike. If f satisfies*

$$(a+1)\frac{L_p(a+2;c)f(z)}{L_p(a+1;c)f(z)}-\gamma a\frac{L_p(a+1;c)f(z)}{L_p(a;c)f(z)}\prec a(1-\gamma)+1+\frac{zq'(z)}{q(z)},$$

then

$$\frac{z^{p(1-\gamma)}L_p(a+1;c)f(z)}{[L_p(a;c)f(z)]^{\gamma}}\prec q(z)$$

and $q(z)$ is the best dominant.

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 1, we have

2.4. Corollary. *Let $-1\le B<A\le 1$. If $f\in\sum_p$ satisfies*

$$\frac{1}{\lambda}\frac{D^{n+2}_{\lambda,p}(f*g)(z)}{D^{n+1}_{\lambda,p}(f*g)(z)}-\frac{\gamma}{\lambda}\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)}$$

$$\prec\frac{1}{\lambda}(1-\gamma)+\frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\frac{z^{p(1-\gamma)}D^{n+1}_{\lambda,p}(f*g)(z)}{\left[D^{n}_{\lambda,p}(f*g)(z)\right]^{\gamma}}\prec\frac{1+Az}{1+Bz}.$$

Taking $n=0$ and $g(z)$ in the form (1.4) with $q=2, s=1, \alpha_1=a, \beta_1=c, a,c>0$ and $\alpha_2=1$, in Corollary 3, we have the following result which corrects the result obtained by Ali and Ravichandran [1, Corollary 2.4].

2.5. Corollary. *Let $-1\le B<A\le 1$. If $f\in\sum_p$ satisfies*

$$(a+1)\frac{L_p(a+2;c)f(z)}{L_p(a+1;c)f(z)}-\gamma a\frac{L_p(a+1;c)f(z)}{L_p(a;c)f(z)}$$

$$\prec a(1-\gamma)+1+\frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\frac{z^{p(1-\gamma)}L_p(a+1;c)f(z)}{[L_p(a;c)f(z)]^{\gamma}}\prec\frac{1+Az}{1+Bz}.$$

By appealing to Lemma 2, we prove the following theorem.

2.6. Theorem. *Let $\gamma \neq 0$ and $q(z)$ be univalent in U, $q(z) \neq 0$, $Q(z) = \gamma z q'(z)/q(z)$ be starlike and*

$$\Re\left\{\frac{1}{\lambda\gamma}q(z) + \frac{zQ'(z)}{Q(z)}\right\} > 0 \ \ (z \in U). \tag{2.5}$$

If $f(z) \in \sum_p$ satisfies

$$(1-\gamma)\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} + \gamma\frac{D^{n+2}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} \prec q(z) + \lambda\gamma\frac{zq'(z)}{q(z)},$$

then

$$\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Let the function $p(z)$ defined by

$$p(z) = \frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} \ \ (z \in U). \tag{2.6}$$

Differentiating (2.6) logarithmically with respect to z and using the identity (1.3), we have

$$\frac{D^{n+2}_{\lambda,p}(f*g)(z)}{D^{n+1}_{\lambda,p}(f*g)(z)} = p(z) + \lambda\frac{zp'(z)}{p(z)},$$

therefore, we have

$$(1-\gamma)\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} + \gamma\frac{D^{n+2}_{\lambda,p}(f*g)(z)}{D^{n+1}_{\lambda,p}(f*g)(z)} = p(z) + \lambda\gamma\frac{zp'(z)}{p(z)}.$$

From (2.5), we have

$$p(z) + \lambda\gamma\frac{zp'(z)}{p(z)} \prec q(z) + \lambda\gamma\frac{zq'(z)}{q(z)}.$$

By an application of Lemma 2, it follows that $p(z) \prec q(z)$ and $q(z)$ is the best dominant. □

Taking $n = 0$ and $g(z)$ of the form (1.4) and using the identity (2.4) we have the following corollary.

2.7. Corollary. *Let $\gamma \neq 0, \alpha_1 \neq -1$ and $q(z)$ be univalent in U, $q(z) \neq 0$, $Q(z) = \gamma z q'(z)/q(z)$ be starlike and*

$$\Re\left\{\frac{\alpha_1 + 1 - \gamma}{\gamma}q(z) + \frac{zQ'(z)}{Q(z)}\right\} > 0 \ \ (z \in U).$$

If $f(z) \in \sum_p$ satisfies

$$(1-\gamma)\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} + \gamma\frac{H_{p,q,s}(\alpha_1+2)f(z)}{H_{p,q,s}(\alpha_1+1)f(z)}$$

$$\prec \frac{1}{\alpha_1+1}\left[\gamma + (1+\alpha_1-\gamma)q(z) + \gamma\frac{zq'(z)}{q(z)}\right], \tag{2.7}$$

then

$$\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Remarks. (i) *Taking* $n = 0$ *and* $g(z)$ *in the form* (1.4) *with* $q = 2, s = 1, \alpha_1 = a > 0, \beta_1 = c > 0$ *and* $\alpha_2 = 1$, *in Corollary 5, we have the result obtained by Ali and Ravichandran [1, Theorem 2.5];*

(ii) *Taking* $n = 0$ *and* $g(z)$ *of the form* (1.5) and using the identity [6]:

$$\lambda z \left(I_p^m(\lambda,l)f(z)\right)' = lI_p^{m+1}(\lambda,l)f(z) - (\lambda p + l)I_p^m(\lambda,l)f(z), \lambda > 0,$$

in our results, we have the results corresponding to the operator $I_p^m(\lambda,l)$*;*

(iii) *Taking* $n = 0$ *and* $g(z)$ *of the form* (1.6) *and using the identity [4]:*

$$z(Q^{\alpha}_{\beta,p}f(z))' = (\alpha+\beta-1)Q^{\alpha-1}_{\beta,p}f(z) - (\alpha+\beta+p-1)Q^{\alpha}_{\beta,p}f(z), \alpha \geq 0; \beta > -1,$$

in our results, we have the results corresponding to the operator $Q^{\alpha}_{\beta,p}$.

References

[1] R. Ali and V. Ravichandran, Differential subordination for meromorphic functions defined by a linear operator, J. Analysis Appl., 2 (2004), no. 3, 149-158.

[2] M. K. Aouf, Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function, Comput. Math. Appl., 55 (2008), no. 3, 494-509.

[3] M. K. Aouf, A. Shamandy, A. O. Mostafa and S. M. Madian, Properties of some families of meromorphic p-valent functions involving certain differential operator, Acta Univ. Apulensis, (2009), no. 20, 7-15.

[4] E. Aqlan, J. M. Jahangiri and S. R. Kulkarni, Certain integral operators applied to meromorphic p–valent functions, J. Nat. Geom., 24 (2003), 111–120.

[5] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[6] R.M. El-Ashwah, A note on certain meromorphic p-valent functions, Appl. Math. Letters, 22 (2009), 1756-1759.

[7] R.M. El-Ashwah and M.K. Aouf, Some properties of certain classes of meromorphically p-valent functions involving extended multiplier transformations, Comput. Math. Appl., 59 (2010), 2111-2120.

[8] J.-L. Liu and H. M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, J. Math. Anal. Appl., 259 (2000), 566-581.

[9] J.-L. Liu and H. M. Srivastava, Classes of meromorphically multivalent functions associated with generalized hypergeometric function, Math. Comput. Modelling, 39 (2004), 21-34.

[10] S. S. Miller and P. T. Mocanu, Differential Subordination : Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[11] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J., 28 (1981), no. 2, 157-171.

[12] V. Ravichandran and M. Jayamala, On sufficient conditions for Caratheodory functions, Far East J. Math. Sci., 12 (2004), 191-201.

[13] B. A. Uralegaddi and C. Somanatha, Certain classes of meromorphic multivalent functions, Tamkang J. Math., 23 (1992), 223-231.

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Near compactness of ditopological texture spaces

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Abstract

The authors consider the notions of near compactness, near cocompactness, near stability, near costability and near dicompactness in the setting of ditopological texture spaces. In particular preservation of these properties under surjective R-dimaps, co-R-dimaps and bi-R-dimaps is investigated and non-trivial characterizations of near dicompactness are given which generalize those for dicompactness. The notions of semiregularization, semicoregularization and semibiregularization are defined and used to give generalizations of Mrówka's Theorem for near compactness and near cocompactness, and of Tychonoff's Theorem for near compactness, near cocompactness and near dicompactness. Also, results related to pseudo-open and pseudo-closed sets are presented. Finally, examples are given of co-T_1 nearly dicompact ditopologies on textures which are not nearly plain and an open question is posed.

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Keywords: Texture, Ditopology, Near compactness, Near cocompactness, Near stability, Near costabiliy, Near dicompactness, R-dimaps, co-R-dimaps, Semiregularization, Semicoregularization, Semibiregularization, Tychonoff theorems.

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1. Introduction

The investigation of various types of generalized open set and generalized continuous function is a rich area of research in general topology, and there are interesting applications in various areas, particularly computer science. An early notion in this area is the now well known concept of regular-open set, and its counterpart regular-closed set. Closely related with this notion is the property of near compactness introduced by Singal and Mathur [16, 17], which has been studied by several other authors (see, for example, [11]). Important in this context is the notion of R-map, see [7]. In this paper these and related notions will be studied in the wider context of ditopological texture spaces.

Textures and ditopological texture spaces were first introduced by the first author as a point-based setting for the study of fuzzy sets, and work continues in this direction, see for example [1, 2, 3, 4, 5], and the recent work of Tiryaki [18], Özçağ and Brown [15], Yıldız and Brown [22]. On the other hand, textures provide a very convenient setting for the investigation of complement-free concepts in general, so much of the recent work does not involve fuzzy sets explicitly. In particular, the notions of diuniformity and dimetric have been introduced in [14], while a textural analogue of the notion of proximity, called a diextremity, is given in [23].

Compactness in ditopological texture spaces was introduced in [1], its study continued in [6, 20], and extended to real compactness in [20] and to strong compactness in [10]. In this paper we place near compactness in a ditopological setting. All the arguments for studying properties related to regular-openness and regular-closedness in the topological setting apply equally well to this case, and since bitopologies and $\mathbb{L}$-topologies, for $\mathbb{L}$ a Hutton algebra, are special cases of ditopologies, the new concepts of near cocompactness, near stability, near costability, and near dicompactness introduced here, may easily be carried over to these settings also.

To complete this introduction we recall various concepts from [3, 4] that will be needed later on in this paper.

Ditopological Texture Spaces: If S is a set, a *texturing* $\mathcal{S}$ of S is a subset of $\mathcal{P}(S)$ which is a point-separating, complete, completely distributive lattice with respect to containment containing S and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(S, \mathcal{S})$ is then called a *texture*.

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the *p-sets* and *q-sets*

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}, \; Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}.$$

The following are some basic examples of textures that we will need.

1.1. Examples. (1) If X is a set and $\mathcal{P}(X)$ the powerset of X, then $(X, \mathcal{P}(X))$ is the *discrete texture* on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$.

(2) Setting $\mathbb{I} = [0,1]$, $\mathcal{I} = \{[0,r), [0,r] \mid r \in \mathbb{I}\}$ gives the *unit interval texture* $(\mathbb{I}, \mathcal{I})$. For $r \in \mathbb{I}$, $P_r = [0,r]$ and $Q_r = [0,r)$.

(3) The texture $(L, \mathcal{L})$ is defined by $L = (0,1]$, $\mathcal{L} = \{(0,r] \mid r \in \mathbb{I}\}$. For $r \in L$, $P_r = (0,r] = Q_r$.

(4) The *real texture* $(\mathbb{R}, \mathcal{R})$ has $\mathcal{R} = \{(-\infty, r), \; (-\infty, r] \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. For $r \in \mathbb{R}$, $P_r = (-\infty, r]$ and $Q_r = (-\infty, r)$.

Since a texturing $\mathcal{S}$ need not be closed under the operation of taking set complement, the notion of topology is replaced by that of *dichotomous topology* or *ditopology*, namely a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

(1) $S, \emptyset \in \kappa$,
(2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
(3) $K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa$.

We assume no *a priori* relation between the open and closed sets. Note that in a ditopology, equal emphasis is placed on the open and closed sets.

For $A \in \mathcal{S}$ we define the *closure* $[A]$ and the *interior* $]A[$ of A under (τ, κ) by the equalities

$$[A] = \bigcap\{K \in \kappa \mid A \subseteq K\} \text{ and }]A[= \bigvee\{G \in \tau \mid G \subseteq A\}.$$

On the other hand, suppose that $(S, \mathcal{S})$ has a complementation σ, that is an inclusion reversing involution $\sigma : \mathcal{S} \to \mathcal{S}$. Then if τ and κ are related by $\kappa = \sigma(\tau)$ we say that (τ, κ) is a *complemented ditopology* on $(S, \mathcal{S}, \sigma)$. In this case we have $\sigma([A]) =]\sigma(A)[$ and $\sigma(]A[) = [\sigma(A)]$.

We recall the product of textures and of ditopological texture spaces. Let $(S_j, \mathcal{S}_j)$, $j \in J$, be textures and $S = \prod_{j \in J} S_j$. If $A_k \in \mathcal{S}_k$ for some $k \in J$ we write

$$E(k, A_k) = \prod_{j \in J} Y_j \text{ where } Y_j = \begin{cases} A_j, & \text{if } j = k \\ S_j, & \text{otherwise.} \end{cases}$$

Then the *product texturing* $\mathcal{S} = \bigotimes_{j \in J} \mathcal{S}_j$ of S consists of arbitrary intersections of elements of the set

$$\mathcal{E} = \left\{ \bigcup_{j \in J} E(j, A_j) \mid A_j \in \mathcal{S}_j \text{ for } j \in J \right\}.$$

Let $(S_j, \mathcal{S}_j)$, $j \in J$ be textures and $(S, \mathcal{S})$ their product. Then for $s = (s_j) \in S$,

$$P_s = \bigcap_{j \in J} E(j, P_{s_j}) = \prod_{j \in J} P_{s_j}, \text{ and } Q_s = \bigcup_{j \in J} E(j, Q_{s_j}).$$

It is easy to verify that for $A_j \in \mathcal{S}_j$, $j \in J$ we have $\prod_{j \in J} A_j \in \mathcal{S}$.

In case (τ_j, κ_j) is a ditopology on $(S_j, \mathcal{S}_j)$, $j \in J$, the *product ditopology* on the product texture $(S, \mathcal{S})$ has subbase $\{E(j, G) \mid G \in \tau_j, j \in J\}$, cosubbase $\gamma = \{E(j, K) \mid K \in \kappa_j, j \in J\}$.

Let $(S, \mathcal{S}), (T, \mathcal{T})$ be textures. In the following definition we consider the product texture $\mathcal{P}(S) \otimes \mathcal{T}$, and denote by $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$, respectively the p-sets and q-sets for the product texture $(S \times T, \mathcal{P}(S) \otimes \mathcal{T})$.

Direlations: Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures. Then

(1) $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
  R1 $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$.
  R2 $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.
(2) $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
  CR1 $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$.
  CR2 $\overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.
(3) A pair (r, R), where r is a relation and R a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$, is called a *direlation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$.

One of the most useful notions of (ditopological) texture spaces is that of difunction. A difunction is a special type of direlation.

Difunctions: Let (f, F) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called a *difunction from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies the following two conditions.

DF1 For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists\, t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.

DF2 For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

Image and Inverse Image: Let $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$ be a difunction.

(1) For $A \in \mathcal{S}$, the *image* $f^{\to}A$ and the *co-image* $F^{\to}A$ are defined by
$f^{\to}A = \bigcap\{Q_t \mid \forall\, s,\ f \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s\}$,
$F^{\to}A = \bigvee\{P_t \mid \forall\, s,\ \overline{P}_{(s,t)} \not\subseteq F \implies P_s \subseteq A\}$.

(2) For $B \in \mathcal{T}$, the *inverse image* $f^{\leftarrow}B$ and the *inverse co-image* $F^{\leftarrow}B$ are defined by
$f^{\leftarrow}B = \bigvee\{P_s \mid \forall\, t,\ f \not\subseteq \overline{Q}_{(s,t)} \implies P_t \subseteq B\}$,
$F^{\leftarrow}B = \bigcap\{Q_s \mid \forall\, t,\ \overline{P}_{(s,t)} \not\subseteq F \implies B \subseteq Q_t\}$.

For a difunction, the inverse image and the inverse co-image are equal, but the image and co-image are generally different. A difunction is called *surjective* if it satisfies

SUR. For $t, t' \in T$, $P_t \not\subseteq Q_{t'} \implies \exists\, s \in S$ with $f \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq F$.

Bicontinuity: The difunction $(f, F) : (S, \mathcal{S}, \tau_S, \kappa_S)) \to (T, \mathcal{T}, \tau_T, \kappa_T)$ is called *continuous* if $B \in \tau_T \implies F^{\leftarrow}B \in \tau_S$, *cocontinuous* if $B \in \kappa_T \implies f^{\leftarrow}B \in \kappa_S$, and *bicontinuous* if it is both continuous and cocontinuous.

On the other hand (f, F) is *open* (*co-open*) if $A \in \tau_S \implies f^{\to}A \in \tau_T$ ($F^{\to}A \in \tau_T$). Also, (f, F) is *closed* (*coclosed*) if $A \in \kappa_S \implies f^{\to}A \in \kappa_T$ ($F^{\to}A \in \kappa_T$).

If $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$, $j \in J$, are ditopological texture spaces and $(S, \mathcal{S}, \tau, \kappa)$ their product, the *projection difunctions* $(\pi_j, \Pi_j) : (S, \mathcal{S}, \tau, \kappa) \to (S_j, \mathcal{S}_j, \tau_j, \kappa_j)$ are important examples of surjective bicontinouous difunctions that satisfy $\pi_j^{\leftarrow}B = E(j, B) = \Pi_j^{\leftarrow}B$ for all $B \in \mathcal{S}_j$ (see [4] and [6]).

Separation Axioms A comprehensive treatment is given in [5], and the definitions will not be repeated here.

The layout of the paper is as follows. In Section 2 the concepts of near compactness and near cocompactness are given in the setting of ditopological texture spaces, while Section 3 deals with near stability and near costability, which are analogues for near compactness of the notion of stability in bitopological spaces introduced by Ralph Kopperman [12]. Section 4 deals with the preservation of these properties under surjective continuous difunctions via the notions of R-dimap and co-R-dimap, while Section 5 combines them in the notion of near dicompactness and provides important characterizations of this important concept. Finally Section 6 introduces the semibiregularization of a ditopological texture space, gives generalizations of the Mrówka characterization of compactness for near compactness and near cocompactness, discuses characterizations of near compact and near cocompact sets in near compact and near cocompact spaces using the notions of pseudo open and pseudo closed set, and gives versions of the Tychonoff product theorem for near compactness and near cocompactness. Here also we look at near dicompactness in the presence of point separation axioms.

This work includes results, many reformulated, from an unpublished section of the PhD thesis of the second author [9] together with interesting new material.

For terms from lattice theory not defined here the reader is referred to [8].

2. Near compactness and near cocompactness

The notion of *nearly compact* topological space was introduced by Singal and Mathur [16, 17], and has been studied by several other authors (see, for example, [11]). The definition of near compactness requires the concepts of regular-open and regular-closed sets in a topological space. We recall the definitions below:

2.1. Definition. Let $(X, \mathcal{T})$ be a topological space.

(1) If $A \subseteq X$ satisfies $A = \operatorname{int}(\operatorname{cl}(A))$ then A is called a *regular-open set*.
(2) If $B \subseteq X$ satisfies $B = \operatorname{cl}(\operatorname{int}(B)))$ then B is called a *regular-closed set*.

2.2. Examples. (1) Every regular-open set is open and every regular-closed set is closed. Moreover the complement of a regular-closed set is regular-open and the complement of a regular-open set is regular-closed.

(2) Let $X = \{a, b\}$ and let the topology on X be $\mathcal{T} = \{\emptyset, X, \{a\}\}$. Here $\{b\}$ is a closed set but $\operatorname{cl}(\operatorname{int}(\{b\})) = \emptyset$, so $\{b\}$ is not regular-closed.

2.3. Lemma. *Let $(X, \mathcal{T})$ be a topological space, $A, B \subseteq X$. Then* $\operatorname{int}(\operatorname{cl}(A))$ *is a regular-open set and* $\operatorname{cl}(\operatorname{int}(B))$ *a regular-closed set* □

In case of ditopological texture spaces we may give corresponding definitions, as follows.

2.4. Definition. Let (τ, κ) be a ditopology on the texture $(S, \mathcal{S})$.

(1) An element $A \in \mathcal{S}$ will be called *regular-open* if $]\,[A]\,[= A$.
(2) An element $B \in \mathcal{S}$ will be called *regular-closed* if $[\,]A[\,] = A$

It is clear that sets of the form $]\,[A]\,[$, $A \in \mathcal{S}$, are regular-open and those of the form $[\,]A[\,]$, $A \in \mathcal{S}$ are regular-closed. In general there is no relation between regular-open and regular-closed sets in a ditopological texture space, but for complemented ditopological texture spaces we do have the following result.

2.5. Proposition. *Let $(S, \mathcal{S}, \sigma, \tau, \kappa)$ be a complemented ditopological texture space and $A \in \mathcal{S}$. Then*

(1) *$A \in \mathcal{S}$ is regular-open if and only if $\sigma(A)$ is regular-closed.*
(2) *$A \in \mathcal{S}$ is regular-closed if and only if $\sigma(A)$ is regular-open .*

Proof. Straightforward. □

Now let us recall the definition of nearly compact topological space.

2.6. Definition. A topological space $(X, \mathcal{T})$ is said to be *nearly compact* if every open cover $\{U_i \mid i \in I\}$ of X admits a finite subfamily such that $X = \bigcup_{i=1}^{n} \operatorname{int}(\operatorname{cl}(U_i))$. That is, X is nearly compact if and only if every regular-open cover of X has a finite subcover.

The set $\mathbb{R}$ with its usual topology and $\mathbb{R}$ with the discrete topology are not nearly compact. However the indiscrete topology and the left ray topology $\{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ on $\mathbb{R}$ are examples of nearly compact topological spaces.

In the case of ditopological texture spaces we may give a corresponding definition of near compactness, and a dual notion of near cocompactness, as follows.

2.7. Definition. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathcal{S}$.

(1) A will be called *nearly compact in S* if whenever $A \subseteq \bigvee_{i \in I} G_i$, $G_i \in \tau$, there exists $I' \subseteq I$ finite, so that $A \subseteq \bigcup_{i \in I'}]\,[G_i]\,[$. The ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ will be called *nearly compact* if S is nearly compact in S.

(2) A will be called *nearly cocompact in* S if whenever $\bigcap_{i\in I} F_i \subseteq A$, $F_i \in \kappa$, there exists $I' \subseteq I$ finite, so that $\bigcap_{i\in I'}[\,]F_i[\,] \subseteq A$. The ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ will be called *nearly cocompact* if $\emptyset$ is nearly cocompact in S.

As in the topological case we have the following characterizations.

2.8. Proposition. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathcal{S}$.*

(1) *A is nearly compact in S if and only if every cover of A by regular-open sets in $\mathcal{S}$ has a finite subcover.*

(2) *A is nearly cocompact in S if every cocover of A by regular-open sets in $\mathcal{S}$ has a finite sub-cocover.*

Proof. (1). Let A be nearly compact in S and suppose G_i, $i \in I$ is cover of A by regular-open sets. Since in particular $G_i \in \tau$, we have I' finite with $A \subseteq \bigcup_{i\in I'}][G_i][= \bigcup_{i\in I'} G_i$ since the sets G_i are regular-open. Hence $\{G_i \mid i \in I\}$ has a finite subcover.

Conversely, suppose every regular-open cover of A has a finite subcover, and let $\{G_i \mid i \in I\}$ be an open cover of A. For each $i \in I$, $G_i \subseteq][G_i][$, so $\{][G_i][\mid i \in I\}$ is a cover of A by regular-open sets. By hypothesis there exists $I' \subseteq I$ finite with $A \subseteq \bigcup_{i\in I'}][G_i][$, which shows that A is nearly compact.

(2). The proof is dual to the above, and is therefore omitted. □

2.9. Corollary. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathcal{S}$.*

(1) *If A is compact in S it is nearly compact in S.*

(2) *If A is cocompact in S it is nearly cocompact in S.*

Proof. Clear from Proposition 2.8 since every regular-open set is open and every regular-closed set is closed. □

The following examples show that the converse of Corollary 2.9 is false. Also, in general, the properties of near compactness and near cocompactness are independent of one another.

2.10. Examples. Consider the texture $(L, \mathcal{L})$ as in Examples 1.1.

(1) Let $\tau = \mathcal{L}$ and $\kappa = \{L, \emptyset\}$. If we take r with $0 < r < 1$ and set $A = (0, r]$, then $[A] = L$ and so $][A][=]L[\neq A$, from which we see that A is not regular-open. It follows that the only regular-open sets are $\emptyset$ and L, so (τ, κ) is nearly compact because any regular-open cover of L is finite. On the other hand (τ, κ) is not compact by [6, Example 2.2 (2)].

(2) In just the same way, $\tau = \{\emptyset, L\}$ and $\kappa = \mathcal{L}$ define a ditopology which is nearly cocompact but not cocompact.

(3) Take $\tau = \{(0, r] \mid 0 \le r \le 1/2\} \cup \{L\}$ and $\kappa = \mathcal{L}$. This ditopology (τ, κ) is not nearly co-compact, since for example $\mathcal{C} = \{(0, 1/n] \mid n = 1, 2 \ldots\}$ is a regular-closed co-cover of $\emptyset$ which has no finite sub co-cover. However (τ, κ) is nearly compact because it is compact.

(4) In just the same way $\tau = \mathcal{L}$ and $\kappa = \{(0, r] \mid 1/2 \le r \le 1\} \cup \{\emptyset\}$ defines a ditopology which is nearly co-compact but not nearly compact.

For complemented ditopologies, however, near compactness and near cocompactness are equivalent.

2.11. Proposition. *Let $(S, \mathcal{S}, \sigma, \tau, \kappa)$ be a complemented ditopological texture space. Then (τ, κ) is nearly compact if and only if (τ, κ) is nearly cocompact.*

Proof. Straightforward. □

3. Near stability and near costability

We now wish to generalize the notions of stability and costability. The following definition would appear to be appropriate.

3.1. Definition. Let (τ, κ) be a ditopology on the texture space $(S, \mathcal{S})$.

(1) (τ, κ) will be called *nearly stable* if every set $F \in \mathcal{S} \setminus \{S\}$ of the form $F = \bigcap_{\alpha \in A} K_\alpha$, K_α regular-closed for each $\alpha \in A$, is nearly compact in S.
(2) (τ, κ) will be called *nearly costable* if every set $G \in \mathcal{S} \setminus \emptyset$ of the form $G = \bigvee_{\alpha \in A} G_\alpha$, G_α regular-open for each $\alpha \in A$, is nearly cocompact in S.

3.2. Example. Consider the texture $(L, \mathcal{L})$, as in previous examples,

(1) Take $\tau = \mathcal{L}$ and $\kappa = \{\emptyset, (0, 1/2], L\}$. As we have noted earlier, (τ, κ) is not stable because $(0, 1/2]$ is a closed set which is not compact. On the other hand the only regular-open sets in this space are $\emptyset$, $(0, 1/2]$ and L, so it is nearly stable.
(2) Dually, (τ, κ) defined by $\tau = \{\emptyset, (0, 1/2], L\}$ and $\kappa = \mathcal{L}$ is nearly costable but not costable.
(3) Let $\tau = \kappa = \{(0, 1/2 - 1/n] \mid n \geq 2\} \cup \{(0, 1/2], L\}$. The regular-closed set $(0, 1/2]$ is not nearly compact since $\{(0, 1/2 - 1/n] \mid n \geq 2\}$ is a cover by regular-open sets which has no finite subcover, so (τ, κ) is not nearly stable. On the other hand it is clearly nearly costable.
(4) Dually, (τ, κ) defined by $\tau = \kappa = \{(0, 1/2 + 1/n] \mid n \geq 2\} \cup \{\emptyset, (0, 1/2]\}$ is nearly stable but not nearly costable.

The last two examples show that in general near stability and near costability are independent of one another. However for complemented ditopological texture spaces these concepts are equivalent, as we now show.

3.3. Proposition. *Let $(S, \mathcal{S}, \sigma)$ be a texture with complementation σ and (τ, κ) a complemented ditopology on $(S, \mathcal{S}, \sigma)$. Then (τ, κ) is nearly stable if and only if (τ, κ) is nearly costable.*

Proof. Suppose that (τ, κ) is nearly stable. Let $\emptyset \neq G = \bigvee_{\alpha \in A} G_\alpha$ with G_α regular-open for each α. Also, let $\mathcal{D}$ be a regular-closed cocover of G, i.e.,

$$\bigcap \mathcal{D} \subseteq G = \bigvee G_\alpha.$$

If we let $K_\alpha = \sigma(G_\alpha)$ then K_α is regular-closed and $K = \sigma(G) = \sigma(\bigvee G_\alpha) = \bigcap \sigma(G_\alpha) = \bigcap_{\alpha \in A} K_\alpha$ satisfies $K \neq S$. Also,

$$\mathcal{C} = \{\sigma(F) \mid F \in \mathcal{D}\}$$

is a cover of K by regular-open sets. By hypothesis K is nearly compact so there exists $\alpha_1, \ldots \alpha_n \in A$ such that

$$K \subseteq \sigma(F_{\alpha_1}) \cup \sigma(F_{\alpha_2}) \cup \ldots \cup \sigma(F_{\alpha_n}) = \sigma(F_{\alpha_1} \cap F_{\alpha_2} \cap \ldots \cap F_{\alpha_n}),$$

so $F_{\alpha_1} \cap F_{\alpha_2} \cap \ldots \cap F_{\alpha_n} \subseteq \sigma(K) = G$, which proves that G is nearly cocompact. Hence, (τ, κ) is nearly costable.

Conversely, if (τ, κ) is nearly costable it is nearly stable. The proof is dual to the above, and is omitted. □

4. R-dimaps and co-R-dimaps

Now let us consider the preservation of the above properties under surjective difunctions. It is known [7] that near compactness of topological spaces is preserved under R-maps, so we begin by recalling the definition of R-map.

4.1. Definition. Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ be topological spaces and $f : X_1 \to X_2$ a function. If for every regular-open set $B \subseteq X_2$ the set $f^{-1}(B)$ is regular-open in X_1 the function f is called an *R-map*.

We make corresponding definitions for ditopological texture spaces.

4.2. Definition. Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$, $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ a difunction.

(1) (f, F) will be said to be an *R-dimap* if for every regular-open set $B \in \mathcal{S}_2$ the set $F^{\leftarrow}(B) \in \mathcal{S}_1$ is regular-open .
(2) (f, F) will be said to be a *co-R-dimap* if for every regular-closed set $B \in \mathcal{S}_2$ the set $f^{\leftarrow}(B) \in \mathcal{S}_1$ is regular-closed.
(3) (f, F) will be said to be a *bi-R-dimap* if it is an R-dimap and a co-R-dimap.

Now we may state and prove the following theorems, which generalize the topological case.

4.3. Theorem. *Let $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be a surjective R-dimap. If $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is nearly compact then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is nearly compact.*

Proof. Assume $S_2 = \bigvee_{\alpha \in A} G_\alpha$, where the sets $G_\alpha \in \mathcal{S}_2$ are regular-open . Since (f, F) is a difunction we have $F^{\leftarrow}(S_2) = S_1$ by [3, Proposition 2.28 (1 c)], so

$$S_1 = F^{\leftarrow}(S_2) = F^{\leftarrow}\Big(\bigvee_{\alpha \in A} G_\alpha\Big) = \bigvee_{\alpha \in A} F^{\leftarrow}(G_\alpha),$$

by [3, Corollary 2.12 (2)]. Also, $F^{\leftarrow}(G_\alpha)$ is regular-open for each $\alpha \in A$ since (f, F) is a R-bimap. Hence, by the near compactness of $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ there exists $A' \subseteq A$ finite such that $S_1 = \bigcup_{\alpha \in A'} F^{\leftarrow}(G_\alpha)$. Hence

$$f^{\leftarrow}(S_2) = S_1 = \bigcup_{\alpha \in A'} F^{\leftarrow}(G_\alpha) = F^{\leftarrow}\Big(\bigcup_{\alpha \in A'} G_\alpha\Big).$$

Since (f, F) is surjective $f^{\leftarrow}(S_2) \subseteq F^{\leftarrow}(\bigcup_{\alpha \in A'} G_\alpha)$ implies that $S_2 \subseteq \bigcup_{\alpha \in A'} G_\alpha$ by [3, Corollary 2.33 (1 ii)]. Therefore, $S_2 = \bigcup_{\alpha \in A'} G_\alpha$, and so $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is nearly compact. □

As expected, we have a dual theorem for near cocompactness.

4.4. Theorem. *Let $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be a surjective co R-dimap. If $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is nearly cocompact then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is nearly cocompact.*

Proof. This is dual to the proof of Theorem 4.3, and is omitted. □

Next let us investigate the preservation of near stability and near costability under surjective difunctions.

4.5. Theorem. *Let $(S_1, \mathcal{S}_1 \tau_1, \kappa_1)$, $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces with (τ_1, κ_1) nearly stable, and $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ a surjective bi-R-dimap. Then (τ_2, κ_2) is nearly stable.*

Proof. Take $K \in S_2 \setminus \{S_2\}$ of the form $K = \bigcap_{\alpha \in A} K_\alpha$ with K_α regular-closed for each $\alpha \in A$. We must prove that K is nearly compact for the ditopology (τ_2, κ_2) so take regular-open sets G_i, $i \in I$, satisfying

$$K \subseteq \bigvee \{G_i \mid i \in I\}. \tag{4.1}$$

Since (f, F) is a co-R-dimap, $f^{\leftarrow}(K) = f^{\leftarrow}(\bigcap_{\alpha \in A} K_\alpha) = \bigcap_{\alpha \in A} f^{\leftarrow}(K_\alpha)$ is an intersection of regular-closed sets in $(S_1, \mathcal{S}_1)$. As in the proof of [6, Theorem 3.17], $f^{\leftarrow}(K) \neq S_1$. Hence $f^{\leftarrow}(K)$ is nearly compact in $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$, since this space is nearly stable.

From (4.1) we have

$$f^{\leftarrow}(K) \subseteq f^{\leftarrow}\Big(\bigvee\{G_i \mid i \in I\}\Big) = F^{\leftarrow}\Big(\bigvee\{G_i \mid i \in I\}\Big) = \bigvee\{F^{\leftarrow}(G_i) \mid i \in I\}$$

by [3, Corollary 2.12 (2)], and $F^{\leftarrow}(G_i)$ is regular-open since (f, F) is an R-dimap. By near compactness we have $i_1, i_2, \ldots, i_n \in I$ with

$$\begin{aligned} f^{\leftarrow}(K) &\subseteq \bigcup_{k=1}^{n} F^{\leftarrow}(G_{i_k}) \\ &= F^{\leftarrow}\Big(\bigcup_{k=1}^{n} G_{i_k}\Big) = f^{\leftarrow}\Big(\bigcup_{k=1}^{n} G_{i_k}\Big). \end{aligned}$$

Since (f, F) is surjective we have

$$K = F(f^{\leftarrow}(K)) \subseteq F\Big(f^{\leftarrow}\Big(\bigcup_{k=1}^{n} G_{i_k}\Big)\Big) = \bigcup_{k=1}^{n} G_{i_k}.$$

This shows that K is nearly compact and completes the proof that (τ_2, κ_2) is nearly stable. □

4.6. Theorem. *Let $(S_1, \mathcal{S}_1 \tau_1, \kappa_1)$, $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces with (τ_1, κ_1) nearly costable, and $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ a surjective bi-R-dimap. Then (τ_2, κ_2) is nearly costable.*

Proof. The proof is dual to that of Theorem 4.5, and is omitted. □

5. Near dicompactness

The notion of dicompactness may be generalized for near compactness as follows.

5.1. Definition. A ditopological texture space will be called *nearly dicompact* if it is nearly compact, nearly cocompact, nearly stable and nearly costable.

As a consequence of Theorems 4.3, 4.4, 4.5 and 4.6 we may state the following:

5.2. Theorem. *Near dicompactness is preserved under a surjective bi-R-dimap.*

To give non-trivial characterizations of near dicompactness analogous to those for dicompact ditopological texture spaces we require the following definitions that are adapted from [1, 6].

5.3. Definition. Let (τ, κ) be a ditopology on $(S, \mathcal{S})$.

(1) A set $\mathcal{D} \subseteq \mathcal{S} \times \mathcal{S}$ is called a *difamily* on $(S, \mathcal{S})$. A difamily $\mathcal{D}$ is called *regular-closed, co-regular-open* if $A\, \mathcal{D}\, B$ implies A is regular-closed and B is regular-open, while it is called *regular-open, co-regular-closed* if A is regular-open and B is regular-closed.

(2) A difamily $\mathcal{D}$ has the *finite exclusion property* (fep) if whenever $(F_i, G_i) \in \mathcal{D}$, $i = 1, 2, \ldots, n$ we have $\bigcap_{i=1}^{n} F_i \not\subseteq \bigcup_{i=1}^{n} G_i$.

(3) A regular-closed, co-regular-open difamily $\mathcal{D}$ with $\bigcap\{F \mid F \in \operatorname{dom} \mathcal{D}\} \not\subseteq \bigvee\{G \mid G \in \operatorname{ran} \mathcal{D}\}$ is said to be *bound* in $(S, \mathcal{S}, \tau, \kappa)$.

(4) A difamily $\mathcal{D} = \{(G_i, F_i) \mid i \in I\}$ is called a *dicover* of $(S, \mathcal{S})$ if for all partitions I_1, I_2 of I (including the trivial partitions) we have

$$\bigcap_{i \in I_1} F_i \subseteq \bigvee_{i \in I_2} G_i.$$

(5) A difamily $\mathcal{D}$ is called *finite* (*co-finite*) if $\operatorname{dom} \mathcal{D}$ (resp. $\operatorname{ran} \mathcal{D}$) is finite.

5.4. Theorem. *The following are equivalent for a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$.*

(1) *$(S, \mathcal{S}, \tau, \kappa)$ is nearly dicompact.*
(2) *Every regular-closed, co-regular-open difamily with the finite exclusion property is bound.*
(3) *Every regular-open, co-regular-closed dicover has a sub-dicover which is finite and co-finite.*

Proof. (1) $\Longrightarrow$ (2). Suppose that (1) holds, but that we have a regular-closed, co-regular-open difamily $\mathcal{B} = \{(F_i, G_i) \mid i \in I\}$ with the fep, which is not bound in $(S, \mathcal{S}, \tau, \kappa)$. Let $F = \bigcap_{i \in I} F_i$. Then F is an intersection of regular-closed sets, and $F \subseteq \bigvee_{i \in I} G_i$ since $\mathcal{B}$ is not bound. According as $F \neq S$ or $F = S$ we may use near stability or near compactness, respectively, to show the existence of a finite subset J_1 of I with $F \subseteq \bigcup_{j \in J_1} G_j$. Now let $G = \bigvee_{j \in J_1} G_j$. Then G is a join of regular-open sets and $\bigcap_{i \in I} F_i \subseteq G$. Hence, according as $G \neq \emptyset$ or $G = \emptyset$, we may use near costability or near cocompactness, respectively, to show that $\bigcap_{j \in J_2} F_j \subseteq G$ for some finite subset J_2 of I. Since now $\bigcap_{j \in J_1 \cup J_2} F_j \subseteq \bigcup_{j \in J_1 \cup J_2} G_j$ we have a contradiction to the fact that $\mathcal{B}$ has the fep.

(2) $\Longrightarrow$ (3). Suppose that $\mathcal{C} = \{(G_i, F_i) \mid i \in I\}$ is a regular-open, co-regular-closed dicover with no finite, co-finite sub-dicover. As in the proof of [1, Theorem 3.5] we consider the set $\mathcal{F}$ of functions f satisfying

(a) $\operatorname{dom} f$ is a set of finite subsets of I.
(b) $\forall J \in \operatorname{dom} f,\ f(J) = (f_1(J), f_2(J)) \in \mathcal{P}_J^{\star\star}$.
(c) $J_1, \ldots, J_n \in \operatorname{dom} f \implies J_1 \cup \ldots \cup J_n \in \operatorname{dom} f$.
(d) $J,\, K \in \operatorname{dom} f,\ J \subseteq K \implies f_l(J) = J \cap f_l(K),\ l = 1, 2$.

Here

$$\mathcal{P}_J^{\star\star} = \{(J_1, J_2) \in \mathcal{P}_J^{\star} \mid \forall K \text{ finite}, J \subseteq K \subseteq I,\ \exists\, (K_1, K_2) \in \mathcal{P}_K^{\star} \text{ with } J \cap K_l = J_l,\ l = 1, 2\}.$$

where

$$\mathcal{P}_J = \{(J_1, J_2) \mid J = J_1 \sqcup J_2\}, \text{ and}$$
$$\mathcal{P}_J^{\star} = \{(J_1, J_2) \in \mathcal{P}_J \mid \bigcap_{j \in J_1} F_j \not\subseteq \bigvee_{j \in J_2} G_j\},$$

and $J = J_1 \sqcup J_2$ denotes that $\{J_1, J_2\}$ is a partition of J. Just as in the proof of [1, Theorem 3.5] we may establish that $\mathcal{F}$ contains an element g satisfying $\bigcup \operatorname{dom} g = I$.

Now consider the family $\mathcal{B} = \{(F_j, G_k) \mid j \in g_1(J), k \in g_2(J), J \in \operatorname{dom} g\}$. It is easy to show that $\mathcal{B}$ has the fep. Also F_j is regular-closed and G_k is regular-open, so by (2) we have

$$\bigcap_{J \in \operatorname{dom} g} \Big(\bigcap_{j \in g_1(J)} F_j \Big) \not\subseteq \bigvee_{J \in \operatorname{dom} g} \Big(\bigcup_{j \in g_2(J)} G_j \Big).$$

Let $I_1 = \bigcup \{g_1(J) \mid J \in \operatorname{dom} g\}$, $I_2 = I \setminus I_1$. Then (I_1, I_2) is a partition of I, and $I_2 \subseteq \bigcup \{g_2(J) \mid J \in \operatorname{dom} g\}$. This gives us

$$\bigcap_{J \in \operatorname{dom} g} \Big(\bigcap_{j \in g_1(J)} F_j \Big) = \bigcap_{i \in I_1} F_i \subseteq \bigvee_{i \in I_2} G_i \subseteq \bigvee_{J \in \operatorname{dom} g} \Big(\bigcup_{j \in g_2(J)} G_j \Big),$$

which is a contradiction.

(3) $\Longrightarrow$ (1). First take regular-open sets G_i, $i \in I$, with $S = \bigvee_{i \in I} G_i$. For $i \in I$ let $F_i = \emptyset$. Then $\mathcal{C} = \{(G_i, F_i) \mid i \in I\}$ is a regular-open , co-regular-closed dicover, so has

a finite, co-finite sub-dicover $\{(G_j, F_j) \mid j \in J\}$. For the partition $J_1 = \emptyset$, $J_2 = J$ of J,

$$S = \bigcap_{j \in J_1} F_j \subseteq \bigcup_{j \in J_2} G_j,$$

whence $S = \bigcup_{j \in J} G_j$, and $(\mathcal{S}, \tau, \kappa)$ is nearly compact. Near cocompactness is proved in an analogous way.

To establish near stability let $F \neq S$ have the form $F = \bigcap_{\alpha \in A} F_\alpha$ where each F_α is regular-closed and let G_i, $i \in I$, be regular-open sets with $F \subseteq \bigvee_{i \in I} G_i$. Define

$$\mathcal{C} = \{(S, F_\alpha) \mid \alpha \in A\} \cup \{(G_i, \emptyset) \mid i \in I\}.$$

It is clear that $\mathcal{C}$ is a regular-open , co-regular-closed dicover, and hence has a finite, co-finite sub-dicover $\mathcal{C}_1$. Without loss of generality we may assume

$$\mathcal{C}_1 = \{(S, F_\beta) \mid \beta \in B\} \cup \{(G_j, \emptyset) \mid j \in J\},$$

where $B \subseteq A$ and $J \subseteq I$ are finite. Now we obtain

$$F \subseteq \bigcap_{\beta \in B} F_\beta \subseteq \bigcup_{j \in J} G_j,$$

which shows that F is nearly compact in S. Hence (τ, κ) is nearly stable, and near costability can be proved in a similar way.

Hence (τ, κ) is nearly dicompact. □

6. Semibiregulization and the Tychonoff Theorems

As we have noted before, for any topological space $(X, \mathcal{T})$ the intersection of a finite number of regular-open sets is regular-open . It follows that the regular-open sets in any topological space $(X, \mathcal{T})$ form a base for a topology on X. This gives rise to the following definition.

6.1. Definition. The topology generated by the regular-open sets in the topological space $(X, \mathcal{T})$ is called the *semiregularization topology* of $(X, \mathcal{T})$ and is denoted by $(X, \mathcal{T}^\star)$.

It is clear that $\mathcal{T}^\star \subseteq \mathcal{T}$, but the converse is not true in general. Topologies for which $\mathcal{T} = \mathcal{T}^\star$ are given a special name.

6.2. Definition. The topological space $(X, \mathcal{T})$ is called *semiregular* if the regular-open sets form a base for the topology. Equivalently, if $\mathcal{T}^\star = \mathcal{T}$.

6.3. Example. For any set X, all the open sets in the discrete topology on X and in the indiscrete topology on X are regular-open . Hence these spaces are semiregular. For the usual topology on $\mathbb{R}$ the basic open sets (a, b) are also regular-open , so this topology is semiregular. However the only regular-open sets for the finite complement topology on an infinite set X are $\emptyset$ and X, which generate the indiscrete topology on X. Hence this space is not semiregular.

In the case of ditopological texture spaces we may define analogous concepts.

6.4. Definition. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space.

(1) Denote by $\tau^\star$ the topology generated by the regular-open sets of (τ, κ). Then $(\tau^\star, \kappa)$ is called the *semiregularization* of (τ, κ). The ditopology (τ, κ) is called *semiregular* if $\tau^\star = \tau$.

(2) Denote by $\kappa^\star$ the cotopology generated by the regular-closed sets of (τ, κ). Then $(\tau, \kappa^\star)$ is called the *semicoregularization* of (τ, κ). The ditopology (τ, κ) is called *semicoregular* if $\kappa^\star = \kappa$.

(3) The ditopology $(\tau^\star, \kappa^\star)$ is called the *semibiregularization* of (τ, κ) and (τ, κ) is called *semibiregular* if $(\tau^\star, \kappa^\star) = (\tau, \kappa)$.

An important result for topological spaces is that a topological space is nearly compact if and only if the semiregularization topology is compact. For ditopological texture spaces we have the following results.

6.5. Theorem. *Let (τ, κ) be a ditopology on $(S\mathcal{S})$.*

(1) *The following are equivalent:*
   (a) *(τ, κ) is nearly compact.*
   (b) *$(\tau^\star, \kappa)$ is compact.*
   (c) *$(\tau^\star, \kappa^\star)$ is compact.*

(2) *The following are equivalent:*
   (a) *(τ, κ) is nearly cocompact.*
   (b) *$(\tau, \kappa^\star)$ is cocompact.*
   (c) *$(\tau^\star, \kappa^\star)$ is cocompact.*

Proof. (1) (a) $\Longrightarrow$ (b). Let $\mathcal{C}$ be a cover of S by members of $\tau^\star$. Then $\mathcal{C}$ is also a cover of S by members of τ so there exists a finite subfamily of $\mathcal{C}$, say C_i, $i = 1 \ldots n$, such that

$$S = \bigcup_{i=1}^{n}][C_i][.$$

Since $][C_i][= C_i$ we obtain $S = \bigcup_{i=1}^{n} C_i$, so $(\tau^\star, \kappa)$ is compact.

(b) $\Longrightarrow$ (c). Clear since compactness depends only on the topology.

(c) $\Longrightarrow$ (a). Let $\mathcal{C}$ be a cover of S by member of τ. Then $C \subseteq][C][\forall C \in \mathcal{C}$, where the interior and closure are taken in (τ, κ), and therefore $\{][C][\mid C \in \mathcal{C}\}$ is a covering of S by members of $\tau^\star$. By (c) there exists a finite subfamily of $\mathcal{C}$, say C_i, $i = 1 \ldots n$, such that

$$S = \bigcup_{i=1}^{n}][C_i][.$$

Hence (τ, κ) is nearly compact.

(2). The proof is dual to (1), and is omitted. □

We may present the following generalization of Mrówka's characterization of compact topological spaces [13].

6.6. Theorem. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ be a ditopological texture space. The following are equivalent:*

(1) *$(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is nearly compact.*
(2) *For all ditopological spaces $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ the projection difunction*

$$(\pi_2, \Pi_2) : (S_1, \mathcal{S}_1, \tau_1^*, \kappa_1) \times (S_2, \mathcal{S}_2, \tau_2, \kappa_2) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$$

is co-open.
(3) *Condition* (2) *holds for all normal ditopological spaces $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$.*

Proof. In view of Theorem 6.5 (1 b) it is sufficient to apply [6, Corollary 2.18] with τ_1^* in place of τ_1. □

Dually,

6.7. Theorem. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ be a ditopological texture space. The following are equivalent:*

(1) *$(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is nearly cocompact.*

(2) *For all ditopological spaces* $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ *the projection difunction*

$$(\pi_2, \Pi_2) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1*) \times (S_2, \mathcal{S}_2, \tau_2, \kappa_2) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$$

is closed.

(3) *Condition* (2) *holds for all normal ditopological spaces* $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$. □

6.8. Theorem. *Let* (τ, κ) *be a ditopology on* $(S, \mathcal{S})$. *Then:*

(1) (τ, κ) *is nearly stable iff* $(\tau^\star, \kappa^\star)$ *is stable.*
(2) (τ, κ) *is nearly costable iff* $(\tau^\star, \kappa^\star)$ *is costable.*

Proof. (1). Suppose (τ, κ) is nearly stable. Take $F \in \kappa^\star$ with $F \neq S$. Then $F = \bigcap K_\alpha$, where the K_α are regular-closed sets. By near stability F is nearly compact in (τ, κ). However, as in the proof of Theorem 6.5 this means that F is compact in $(\tau^\star, \kappa^\star)$. Hence $(\tau^\star, \kappa^\star)$ is stable.

The converse is proved is exactly the same way.

(2). This is dual to (1), and the proof is omitted. □

6.9. Corollary. *The ditopological texture space* $(S, \mathcal{S}, \tau, \kappa)$ *is nearly dicompact if and only if* $(S, \mathcal{S}, \tau^\star, \kappa^\star)$ *is dicompact.* □

We recall the following notions from [6, Definition 3.10]:

6.10. Definition. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathcal{S}$.

(1) (a) $Q(A) = \bigcap\{\,]Q_s[\; \mid P_s \not\subseteq A\}$.
(b) A is called *pseudo open* if $Q(A) = \,]A[$.
(2) (a) $P(A) = \bigvee\{[P_s] \mid A \not\subseteq Q_s\}$.
(b) A is called *pseudo closed* if $P(A) = [A]$.

It is shown in [6, Lemma 3.11] that $]A[\, \subseteq Q(A) \subseteq A$ and $A \subseteq P(A) \subseteq [A]$ for all $A \in \mathcal{S}$. Now we have:

6.11. Theorem. *Let* $(S, \mathcal{S}, \tau, \kappa)$ *be a nearly compact nearly stable ditopological space for which* (τ^*, κ^*) *is* R_0. *Then every pseudo closed set in* (τ^*, κ^*) *is nearly compact in* (τ, κ).

Proof. By Theorem 6.5 (1c) we have that $(S, \mathcal{S}, \tau^*, \kappa^*)$ is compact, while by Theorem 6.8 it is stable. Hence by [6, Theorem 3.14 (1)] a pseudo closed set F in $(S, \mathcal{S}, \tau^*, \kappa^*)$ is compact. It follows as in the proof of Theorem 6.5 that F is nearly compact in $(S, \mathcal{S}, \tau, \kappa)$. □

Dually,

6.12. Theorem. *Let* $(S, \mathcal{S}, \tau, \kappa)$ *be a nearly cocompact nearly costable ditopological space for which* (τ^*, κ^*) *is co-*R_0. *Then every pseudo open set in* (τ^*, κ^*) *is nearly cocompact in* (τ, κ). □

In the opposite direction the following is an immediate consequence of [6, Theorem 3.15].

6.13. Theorem. *Let* $(S, \mathcal{S}, \tau, \kappa)$ *be a ditopological texture space.*

(1) *Suppose* (τ^*, κ^*) *is co-*R_1. *Then if* A *is nearly compact in* (τ, κ) *it is pseudo closed in* (τ^*, κ^*).
(2) *Suppose* (τ^*, κ^*) *is* R_1. *Then if* A *is cocompact in* (τ, κ) *it is pseudo open in* (τ^*, κ^*).

Now we may give Tychonoff Theorems for near compactness, near cocompactness and near dicompactness.

6.14. Theorem. *Let $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$, $i \in I$, be non-empty ditopological texture spaces and $(S, \mathcal{S}, \tau, \kappa)$ their product.*

(i) *$(S, \mathcal{S}, \tau, \kappa)$ is nearly compact if and only if $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$ is nearly compact for all $i \in I$.*
(ii) *$(S, \mathcal{S}, \tau, \kappa)$ is nearly cocompact if and only if $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$ is nearly cocompact for all $i \in I$.*
(iii) *$(S, \mathcal{S}, \tau, \kappa)$ is nearly dicompact if and only if $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$ is nearly dicompact for all $i \in I$.*

Proof. We will need the following lemma.

6.15. Lemma. *Let $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$, $i \in I$, be ditopological texture spaces and $(S, \mathcal{S}, \tau, \kappa)$ their product. Let $(S, \mathcal{S}, \tau', \kappa')$ denote the product of the spaces $(S_i, \mathcal{S}_i,\ \tau_i^\star, \kappa_i^\star)$. Then $(\tau^\star, \kappa^\star) = (\tau', \kappa')$.*

Proof. The sets $E(i, G)$, $i \in I$, $G \in \tau_i^\star$, form a subbase for the topology τ'. Since the regular-open sets in τ_i are a base for the topology $\tau_i^\star$ it follows that the family

$$\mathcal{G} = \{E(i, G) \mid i \in I, G \text{ regular-open in } \tau_i\}$$

is also a subbase for τ'. However $E(i, G)$ is regular-open in τ if and only if G is regular-open in τ_i, so it is not difficult to verify that $\mathcal{G}$ is also a subbase for the topology $\tau^\star$. Hence $\tau' = \tau^\star$, and the proof of $\kappa' = \kappa^\star$ is dual to this. □

We now turn to the proof of the theorem.

(i). Suppose that (τ, κ) is nearly compact. Then, by Theorem 6.5 (1 c), the ditopology $(\tau^\star, \kappa^\star)$ is compact. However, by Lemma 6.15, $(S, \mathcal{S}, \tau^\star, \kappa^\star)$ is the product of the spaces $(S_i, \mathcal{S}_i, \tau_i^\star, \kappa_i^\star)$ so by [6, Theorem 2.15 (i)] $(S_i, \mathcal{S}_i, \tau_i^\star, \kappa_i^\star)$ is compact for all $i \in I$. Hence, by Theorem 6.5 (1 c), $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$ is nearly compact for all $i \in I$.

Conversely, suppose that the spaces $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$ are nearly compact for all $i \in I$. Then by Theorem 6.5 (1 c) the spaces $(S_i, \mathcal{S}_i, \tau_i^\star, \kappa_i^\star)$ are compact, so by [6, Theorem 2.15 (i)], $(\tau', \kappa') = (\tau^\star, \kappa^\star)$ is a compact ditopology on $(S, \mathcal{S})$. The product ditopological texture $(S, \mathcal{S}, \tau, \kappa)$ is therefore nearly compact, again by Theorem 6.5 (1 c).

(ii). The proof is dual to (i), and is omitted.

(iii). Straightforward from (i) and (ii). □

We end by recalling an important result from [20]. First we will require the following definition.

6.16. Definition. Let $(S, \mathcal{S})$ be a texture.

(1) $s \in S$ will be called a *plain point* if $P_s \not\subseteq Q_s$. The set of plain points of S will be denoted by S_p.
(2) The texture $(S, \mathcal{S})$ will be called *nearly-plain* if given $s \in S$ there exists a point in S_p with the same q-set as s.

It is shown in [20, Proposition 4.1] that every dicompact bi-T_2 ditopological texture space has a nearly plain texture. This restriction rules out the possibility of a dicompact bi-T_2 ditopology on, for example, the texture $(L, \mathcal{L})$ of Examples 1.1 (1) for this texture has no plain points. Indeed, this texture does not even have the weaker almost plain property discussed in [21]. It is natural to ask if similar restrictions hold for near compactness. This is currently an open question, but the following examples show that there can exist co-T_1 nearly dicompact ditopologies on such textures.

6.17. Example. Consider $(L, \mathcal{L})$ with the ditopology (τ, κ) defined by $\tau = \mathcal{L}$, $\kappa = \{L, \emptyset\}$. As in Examples 2.10 (1) we see that the only regular-open sets are $L, \emptyset$, whence $\tau^* = \kappa^* = \{L, \emptyset\}$ and so $(L, \mathcal{L}, \tau, \kappa)$ is nearly dicompact. With regard to separation we see that $Q_s \not\subseteq Q_t \implies t < s$ so we may choose $t < r < s$ and then $G = (0, r] \in \tau$, $P_s \not\subseteq G \not\subseteq Q_t$, so $(L, \mathcal{L}, \tau, \kappa)$ is co-T_1 by [5, Theorem 4.11 (2 ii)]. On the other hand by [5, Theorem 4.11 (1 i)] it is clear that this space is not T_1 and hence, in particular not bi-T_2.

6.18. Example. Consider the Hutton texture of the real texture $(\mathbb{R}, \mathcal{R})$ given in Examples 1.1 (4). As shown in [22] this may be represented by $(M_{\mathbb{R}}, \mathcal{M}_{\mathbb{R}})$ where $M_{\mathbb{R}} = (\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \cup \{\infty\}$ and $\mathcal{M}_{\mathbb{R}} = \{U_r \mid r \in \mathbb{R}\} \cup \{V_r \mid r \in \mathbb{R}\} \cup \{M_{\mathbb{R}}\}$, where

$$U_r = (-\infty, r] \times \{0, 1\},\ V_r = (-\infty, r) \times \{0\} \cup (-\infty, r] \times \{1\}.$$

Moreover, $P_{(r,0)} = U_r$, $Q_{(r,0)} = V_r$; $P_{(r,1)} = Q_{(r,1)} = V_r$; $P_\infty = Q_\infty = M_{\mathbb{R}}$. As noted in [22] this texture is not nearly plain since ∞ is a non-plain point whose q-set is not equal to the q-set of any of the plain points. If we take $\tau = \{V_r \mid r \in \mathbb{R}\} \cup \{M_{\mathbb{R}}, \emptyset\}$, $\kappa = \{M_{\mathbb{R}}, \emptyset\}$ then much as in the above example we see that $\tau^* = \kappa^* = \{M_{\mathbb{R}}, \emptyset\}$, whence $(M_{\mathbb{R}}, \mathcal{M}_{\mathbb{R}}, \tau, \kappa)$ is nearly dicompact. Again, this space is co-T_1. To see this using [5, Theorem 4.11 (2 ii)] note that we have $Q_\infty \not\subseteq Q_{(s,k)}$ for any $s \in \mathbb{R}$, $k = 0, 1$, and $Q_{(s,m)} = V_s \not\subseteq V_t = Q_{(t,n)}$ for $t < s$ and $m, n = 0, 1$. In the first case we may take $r > s$, $G = V_r \in \tau$ to give $P_\infty \not\subseteq G \not\subseteq Q_{(s,k)}$ and in the second case $t < r < s$, $G = V_r$ to give $P_{(s,m)} \not\subseteq G \not\subseteq Q_{(t,n)}$.

As in the above example this space also is not T_1 and hence in particular not bi-T_2.

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References

[1] Brown, L. M. and Diker, M. *Ditopological texture spaces and intuitionistic sets*, Fuzzy Sets and Systems **98**, 217–224, 1998.

[2] Brown, L. M. and Ertürk, R. *Fuzzy Sets as Texture Spaces, I. Representation Theorems*, Fuzzy Sets and Systems **110** (2), 227–236, 2000.

[3] Brown, L. M., Ertürk, R. and Dost, Ş. *Ditopological texture spaces and fuzzy topology, I. Basic Concepts*, Fuzzy Sets and Systems **147** (2), 171–199, 2004.

[4] Brown, L. M., Ertürk, R. and Dost, Ş. *Ditopological texture spaces and fuzzy topology, II. Topological Considerations*, Fuzzy Sets and Systems **147** (2), 201–231, 204.

[5] Brown, L. M., Ertürk, R. and Dost, Ş. *Ditopological texture spaces and fuzzy topology, III. Separation Axioms*, Fuzzy Sets and Systems **157** (14), 1886–1912, 2006.

[6] Brown, L. M. and Gohar, M. M. *Compactness in ditopological texture spaces*, Hacet. J. Math. Stat. **38** (1), 21–43, 2009.

[7] Carnahan, D. *Some properties related to compactness in topological spaces* (PhD Dissertation, University of Arkansas, 1973).

[8] Gierz, G., Hofmann, K. H., Keimel, D., Lawson, J., Mislove, M. and Scott, D. *A compendium of continuous lattices* (Springer-Verlag, Berlin, Heidelberg, New York, 1980).

[9] Gohar, M. M. *Compactness in ditopological texture spaces* (PhD Thesis, Hacettepe University, 2002).

[10] Gohar, M. M. and Brown, L. M. *Strong compactness of ditopological texture spaces*, AIP Conference Proceedings **1309**, 153-168, 2010.

[11] Herrington, L. L. *Properties of nearly-compact spaces*, Proc. Amer. Math. Soc. **45**, 431–436, 1974.

[12] Kopperman, R. D. *Asymmetry and duality in topology*, Topology and its Applications **66**, 1–39, 1995.

[13] Mrówka, S. *Compactness and product spaces*, Coll.Math **7**, 19–22, 1959.

[14] Özçağ, S. and Brown, L. M. *Di-uniform texture spaces*, Appl. Gen. Top. **4** (1), 157–192, 2003.

[15] Özçağ, S., Brown, L. M. and Krsteska, B. *Di-uniformities and Hutton uniformities*, Fuzzy Sets and Systems **195**, 58–74, 2012.

[16] Singal, M. K. and Mathur, A. *On nearly compact spaces*, Boll Un. Mat. Ital. **6** (4), 702–710, 1969.

[17] Singal, M. K. and Mathur, A. *On nearly compact spaces (II)*, Boll. Un. Mat. Ital. **9** (4), 1974.

[18] Tiryaki, İ. U. *Fuzzy sets over the poset* $\mathbb{I}$, Hacet. J. Math. Stat. **37** (2), 143–166, 2008.

[19] Yıldız, F. and Brown, L. M. *Categories of dicompact bi-*T_2 *texture spaces and a Banach-Stone theorem*, Quaestions Mathematicae **30**, 167–192, 2007.

[20] Yıldız, F. and Brown, L. M. *Real dicompact textures*, Topology and its Applications **156** (11), 1970–1984, 2009.

[21] F. Yıldız and Brown, L. M. *Real dicompactifications of ditopological texture spaces*, Topology and its Applications **156** (18), 3041–3051, 2009.

[22] Yıldız, F. and Brown, L. M. *Extended real dicompactness and an application to Hutton spaces*, Fuzzy Sets and Systems **227**, 74–95, 2013.

[23] Yıldız, G. and Ertürk, R. *Di-extremities on textures*, Hacet. J. Math. Stat. **38** (3), 243–257, 2009.

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On universal central extensions of Hom-Lie algebras

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Abstract

We develop a theory of universal central extensions of Hom-Lie algebras. Classical results of universal central extensions of Lie algebras cannot be completely extended to Hom-Lie algebras setting, because of the composition of two central extensions is not central. This fact leads to introduce the notion of universal α-central extension. Classical results as the existence of a universal central extension of a perfect Hom-Lie algebra remains true, but others as the central extensions of the middle term of a universal central extension is split only holds for α-central extensions. A homological characterization of universal (α)-central extensions is given.

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1. Introduction

The Hom-Lie algebra structure was initially introduced in [4] motivated by examples of deformed Lie algebras coming from twisted discretizations of vector fields. Hom-Lie algebras are $\mathbb{K}$-vector spaces endowed with a bilinear skew-symmetric bracket satisfying a Jacobi identity twisted by a map. When this map is the identity map, then the definition of Lie algebra is recovered.

The study of this algebraic structure was the subject of several papers [4, 7, 8, 9, 11]. In particular, a (co)homology theory for Hom-Lie algebras, which generalizes the Chevalley-Eilenberg (co)homology for a Lie algebra, was the subject of [1, 2, 3, 10, 12].

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In the classical setting, homology theory is closely related with universal central extensions. Namely, the second homology with trivial coefficients group is the kernel of the universal central extension and universal central extensions are characterized by means of the first and second homologies with trivial coefficients.

Our goal in the present paper is to investigate if the homology for Hom-Lie algebras introduced in [10, 12] allows the characterization of universal central extensions of Hom-Lie algebras in terms of Hom-Lie homologies. A similar study for Hom-Leibniz algebras homology can be seen in [3]. But when we try to generalize the classical results of universal central extensions theory of Lie algebras to Hom-Lie algebras an important problem occurs, namely the composition of central extensions is not central in general, as Example 4.9 shows. This fact doesn't allow a complete generalization of classical results, however requires the introduction of a new concept of centrality for Hom-Lie algebra extensions.

To show our results, we organize the paper as follows: in Section 2 we recall some basic needed material on Hom-Lie algebras, the notions of center, commutator and module. In order to have examples, we include the classification of two-dimensional complex Hom-Lie algebras. In section 3 we recall the chain complex given in [12] and we prove its well-definition by means of the Generalized Cartan's formulas; the interpretation of low-dimensional homologies is given. In section 4 we present our main results on universal central extensions, namely we extend classical results and present a counter-example showing that the composition of two central extension is not a central extension (see Example 4.9). This fact lead us to define α-central extensions as extensions for which the image by the twisting endomorphism α of the kernel is included in the center of the middle Hom-Lie algebra. We can extend classical results as: a Hom-Lie algebra is perfect if and only if admits a universal central extension and the kernel of the universal central extension is the second homology with trivial coefficients of the Hom-Lie algebra. Nevertheless, other result as: if a central extension $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is universal, then (K, α_K) is perfect and every central extension of (K, α_K) is split only holds for universal α-central extensions, which means that only lifts on α-central extensions. Other relevant result, which cannot be extended in the usual way, is: if $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal α-central extension, then $H_1^\alpha(K) = H_2^\alpha(K) = 0$. Of course, when the twisting endomorphism is the identity morphism, then all the new notions and all the new results coincide with the classical ones.

2. Hom-Lie algebras

2.1. Definition. [4] A Hom-Lie algebra is a triple $(L, [-,-], \alpha_L)$ consisting of a $\mathbb{K}$-vector space L, a bilinear map $[-,-] : L \times L \to L$ and a $\mathbb{K}$-linear map $\alpha_L : L \to L$ satisfying:

a) $[x, y] = -[y, x]$ (skew-symmetry)
b) $[\alpha_L(x), [y, z]] + [\alpha_L(z), [x, y]] + [\alpha_L(y), [z, x]] = 0$ (Hom-Jacobi identity)

for all $x, y, z \in L$.

In terms of the adjoint representation $ad_x : L \to L, ad_x(y) = [x, y]$, the Hom-Jacobi identity can be written as follows [7]:

$$ad_{\alpha_L(z)} \circ ad_y = ad_{\alpha_L(y)} \circ ad_z + ad_{[z,y]} \circ \alpha_L$$

2.2. Definition. [10] A Hom-Lie algebra $(L, [-,-], \alpha_L)$ is said to be multiplicative if the linear map α_L preserves the bracket.

2.3. Example.

a) Taking $\alpha_L = Id$ in Definition 2.1 we obtain the definition of a Lie algebra. Hence Hom-Lie algebras include Lie algebras as a subcategory, thereby motivating the name "Hom-Lie algebras" as a deformation of Lie algebras twisted by an endomorphism. Moreover it is a multiplicative Hom-Lie algebra.
b) Let (A, μ_A, α_A) be a multiplicative Hom-associative algebra [7]. Then $HLie(A) = (A, [-,-], \alpha_A)$ is a multiplicative Hom-Lie algebra in which $[x, y] = \mu_A(x, y) - \mu_A(y, x)$, for all $x, y \in A$ [7, 10].
c) Let $(L, [-,-])$ be a Lie algebra and $\alpha : L \to L$ be a Lie algebra endomorphism. Define $[-,-]_\alpha : L \otimes L \to L$ by $[x, y]_\alpha = \alpha[x, y]$, for all $x, y \in L$. Then $(L, [-,-]_\alpha, \alpha)$ is a multiplicative Hom-Lie algebra [10, Th. 5.3].
d) Abelian or commutative Hom-Lie algebras are $\mathbb{K}$-vector spaces V with trivial bracket and any linear map $\alpha : V \to V$ [4].
e) The Jackson Hom-Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ is a Hom-Lie deformation of the classical Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ defined by $[h, f] = -2f, [h, e] = 2e, [e, f] = h$. The Jackson $\mathfrak{sl}_2(\mathbb{K})$ is related to derivations. As a $\mathbb{K}$-vector space is generated by e, f, h with multiplication given by $[h, j]_t = -2f - 2tf, [h, e]_t = 2e, [e, f]_t = h + \frac{t}{2}h$ and the linear map α_t is defined by $\alpha_t(e) = \frac{2+t}{2(1+t)}e = e + \sum_{k=0}^{\infty} \frac{(-1)^k}{2} t^k e, \alpha_t(h) = h, \alpha_t(f) = f + \frac{t}{2}f$ [8].
f) For examples coming from deformations we refer to [10].

2.4. Definition. A homomorphism of Hom-Lie algebras $f : (L, [-,-], \alpha_L) \to (L', [-,-]', \alpha_{L'})$ is a $\mathbb{K}$-linear map $f : L \to L'$ such that

a) $f([x, y]) = [f(x), f(y)]'$
b) $f \circ \alpha_L(x) = \alpha_{L'} \circ f(x)$

for all $x, y \in L$.

The Hom-Lie algebras $(L, [-,-], \alpha_L)$ and $(L', [-,-]', \alpha_{L'})$ are isomorphic if there is a Hom-Lie algebras homomorphism $f : (L, [-,-], \alpha_L) \to (L', [-,-]', \alpha_{L'})$ such that $f : L \to L'$ is bijective.

A homomorphism of multiplicative Hom-Lie algebras is a homomorphism of the underlying Hom-Lie algebras.

So we have defined the category $\mathsf{Hom-Lie}$ (respectively, $\mathsf{Hom-Lie}_{\mathrm{mult}}$) whose objects are Hom-Lie (respectively, multiplicative Hom-Lie) algebras and whose morphisms are the homomorphisms of Hom-Lie (respectively, multiplicative Hom-Lie) algebras. There is an obvious inclusion functor $inc : \mathsf{Hom-Lie}_{\mathrm{mult}} \to \mathsf{Hom-Lie}$. This functor has as left adjoint the multiplicative functor $(-)_{\mathrm{mult}} : \mathsf{Hom-Lie} \to \mathsf{Hom-Lie}_{\mathrm{mult}}$ which assigns to a Hom-Lie algebra $(L, [-,-], \alpha_L)$ the Hom-Lie multiplicative algebra $(L/I, [-,-], \overline{\alpha})$, where I is the ideal of L spanned by the elements $\alpha_L[x, y] - [\alpha_L(x), \alpha_L(y)]$, for all $x, y \in L$ and $\overline{\alpha}$ is induced by α.

In the sequel we refer Hom-Lie algebra to a multiplicative Hom-Lie algebra and we will use the short notation (L, α_L) when there is not confusion with the bracket.

Let $(L, [-,-], \alpha_L)$ be an n-dimensional Hom-Lie algebra with basis $\{a_1, a_2, \ldots, a_n\}$ and endomorphism α_L represented by the matrix $A = (\alpha_{ij})$ with respect to the given basis. To determine its algebraic structure is enough to know its structure constants, i.e. the scalars c_{ij}^k such that $[a_i, a_j] = \sum_{k=1}^{n} c_{ij}^k a_k$, and the entries α_{ij} corresponding to the matrix A. These terms are related according to the following

2.5. Proposition. (see also [7]) Let $(L, [-,-], \alpha_L)$ be a Hom-Lie algebra with basis $\{a_1, a_2, \dots, a_n\}$. Let $c_{ij}^k, 1 \le i,j,k \le n$ be the structure constants relative to this basis and $\alpha_{ij}, 1 \le i,j \le n$ the entries of the matrix A associated to the endomorphism α_L with respect to the given basis. Then $(L, [-,-], \alpha_L)$ is a Hom-Lie algebra if and only if the structure constants and the entries α_{ij} satisfy the following properties:

a) $c_{ij}^k + c_{ji}^k = 0, 1 \le i,j,k \le n; \quad c_{ii}^k = 0, 1 \le i,k \le n$, $\mathrm{char}(\mathbb{K}) \ne 2$.

b) $\displaystyle\sum_{p=1}^n \alpha_{pi} \left(\sum_{q=1}^n c_{jk}^q c_{pq}^l \right) + \sum_{p=1}^n \alpha_{pk} \left(\sum_{q=1}^n c_{ij}^q c_{pq}^l \right) + \sum_{p=1}^n \alpha_{pj} \left(\sum_{q=1}^n c_{ki}^q c_{pq}^l \right) = 0,$
$1 \le i,j,k,l, \le n.$

Proof.

a) There is not difference with Lie-algebras case [6].

b) Applying Hom-Jacobi identity 2.1 b):

$$[\alpha(a_i),[a_j,a_k]] + [\alpha(a_k),[a_i,a_j]] + [\alpha(a_j),[a_k,a_i]] = 0$$

$$[\sum_{l=1}^n \alpha_{li} a_l, \sum_{m=1}^n c_{jk}^m a_m] + [\sum_{p=1}^n \alpha_{pk} a_p, \sum_{q=1}^n c_{ij}^q a_q] + [\sum_{r=1}^n \alpha_{rj} a_r, \sum_{s=1}^n c_{ki}^s a_s] = 0$$

$$\sum_{p=1}^n \alpha_{pi} \left(\sum_{q=1}^n c_{jk}^q [a_p, a_q] \right) + \sum_{p=1}^n \alpha_{pk} \left(\sum_{q=1}^n c_{ij}^q [a_p, a_q] \right) + \sum_{p=1}^n \alpha_{pj} \left(\sum_{q=1}^n c_{ki}^q [a_p, a_q] \right) = 0$$

$$\sum_{l=1}^n \left\{ \sum_{p=1}^n \alpha_{pi} \left(\sum_{q=1}^n c_{jk}^q c_{pq}^l \right) + \sum_{p=1}^n \alpha_{pk} \left(\sum_{q=1}^n c_{ij}^q c_{pq}^l \right) + \sum_{p=1}^n \alpha_{pj} \left(\sum_{q=1}^n c_{ki}^q c_{pq}^l \right) \right\} a_l = 0$$

□

2.6. Lemma. The Hom-Lie algebras $(L, [-,-], \alpha_L)$ and $(L, [-,-]', \alpha_{L'})$ with same underlying $\mathbb{K}$-vector space are isomorphic if and only if there exists a regular matrix P such that $A' = P^{-1}.A.P$ and $P.[a_i, a_j] = [P.a_i, P.a_j]'$, where A, A' and P denote the corresponding matrices representing $\alpha_L, \alpha_{L'}$ and f with respect to the basis $\{a_1, \dots, a_n\}$, respectively.

Proof. The fact comes directly from Definition 2.4. □

2.7. Proposition. The 2-dimensional complex multiplicative Hom-Lie algebras with basis $\{a_1, a_2\}$ are isomorphic to one in the following isomorphism classes:

a) Abelian.

b) $[a_1, a_2] = -[a_2, a_1] = a_1$ and α_l is represented by the matrix $\begin{pmatrix} 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix}$.

c) $[a_1, a_2] = -[a_2, a_1] = a_1$ and α_L is represented by the matrix $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & 1 \end{pmatrix}$, with $\alpha_{11} \ne 0$.

Proof. From the skew-symmetry condition we have that $[a_1, a_1] = [a_2, a_2] = 0$ and $[a_1, a_2] = -[a_2, a_1] = x.a_1 + y.a_2$. The Hom-Jacobi identity 2.1 b) is satisfied independently of the homomorphism α_L. So we only have restrictions coming from the fact that the $\mathbb{C}$-linear map $\alpha_L : L \to L$ represented by the matrix $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ must preserve the bracket.

First at all, we apply the change of basis given by the equations $\left\{ \begin{array}{l} a_1' = x.a_1 + y.a_2 \\ a_2' = \frac{1}{x}.a_2 \end{array} \right.$, if $x \ne 0$, and $\left\{ \begin{array}{l} a_1' = a_2 \\ a_2' = -\frac{1}{y}.a_1 \end{array} \right.$, if $x = 0$ and $y \ne 0$, to normalize the bracket, obtaining the bracket $[a_1', a_1'] = [a_2', a_2'] = 0, [a_1', a_2'] = -[a_2', a_1'] = p.a_1'$, for $p = 0, 1$.

From the fact that $\alpha_L : L \to L$ preserves the bracket, we derive the following equations:

$$\left.\begin{array}{rcl}(\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}).p &=& p.\alpha_{11} \\ p.\alpha_{21} &=& 0\end{array}\right\}$$

which reduces to the following system:

$$\left.\begin{array}{rcl}p.\alpha_{11}.(\alpha_{22} - 1) &=& 0 \\ p.\alpha_{21} &=& 0\end{array}\right\}$$

Hence, for $p = 0$ the system is trivially satisfied. All the matrices representing α_L are valid and the bracket is trivial, so $(L, [-,-], \alpha_L)$ is an abelian Hom-Lie algebra. In case $p = 1$, we derive the matrices corresponding to the cases b) and c).

The different classes obtained are not pairwise isomorphic thanks to Lemma 2.6. □

2.8. Remark.

a) Two algebras of the class b) in Proposition 2.7, with endomorphisms given by the matrices $\begin{pmatrix} 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix}$ and $\begin{pmatrix} 0 & \beta_{12} \\ 0 & \beta_{22} \end{pmatrix}$, are isomorphic if and only if $\alpha_{22} = \beta_{22}$ and $\beta_{12} = p.\alpha_{12} + q.\alpha_{22}, p, q \in \mathbb{C}, p \neq 0$.

b) Two algebras of the class c) in Proposition 2.7, with endomorphisms given by the matrices $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & 1 \end{pmatrix}$, are isomorphic if and only if $\alpha_{11} = \beta_{11}$ and $\beta_{12} = p.\alpha_{12} - q.\alpha_{11} + q, p, q \in \mathbb{C}, p \neq 0$.

c) Obviously if $\Phi : (L, [-,-], \alpha_L) \to (L, [-,-]', \alpha_{L'})$ is an isomorphism of Hom-Lie algebras, then $det(\alpha_L) = det(\alpha'_L)$. Consequently, if $det(\alpha_L) \neq det(\alpha'_L)$, then the Hom-Lie algebras are not isomorphic.

d) The following table shows by means of its algebraic properties that the classes given in Proposition 2.7 are not pairwise isomorphic.

	Abelian	$det(\alpha)$
2.7 a)	Yes	
2.7 b)	Non	0
2.7 c)	Non	$\neq 0$

Complex two-dimensional Hom-Lie algebras

2.9. Definition. Let $(L, [-,-], \alpha_L)$ be a Hom-Lie algebra. A Hom-Lie subalgebra (H, α_H) of $(L, [-,-], \alpha_L)$ is a linear subspace H of L, which is closed for the bracket and invariant by α_L, that is,

a) $[x, y] \in H$, for all $x, y \in H$.

b) $\alpha_L(x) \in H$, for all $x \in H$ $(\alpha_H = \alpha_{L|})$.

A Hom-Lie subalgebra (H, α_H) of $(L, [-,-], \alpha_L)$ is said to be a Hom-ideal if $[x, y] \in H$ for all $x \in H, y \in L$.

If (H, α_H) is a Hom-ideal of $(L, [-,-], \alpha_L)$, then $(L/H, [-,-], \overline{\alpha_L})$ naturally inherits a structure of Hom-Lie algebra, which is said to be the quotient Hom-Lie algebra.

2.10. Definition. Let (H, α_H) and (K, α_K) be Hom-ideals of a Hom-Lie algebra $(L, [-,-], \alpha_L)$. The commutator Hom-Lie subalgebra of (H, α_H) and (K, α_K), denoted by $([H, K], \alpha_{[H,K]})$, is the Hom-subalgebra of $(L, [-,-], \alpha_L)$ spanned by the brackets $[h, k], h \in H, k \in K$.

2.11. Lemma. Let (H, α_H) and (K, α_K) be Hom-ideals of a Hom-Lie algebra $(L, [-,-], \alpha_L)$. The following statements hold:

a) $(H \cap K, \alpha_{H \cap K})$ and $(H + K, \alpha_{H+K})$ are Hom-ideals of (L, α_L).

b) $[H, K] \subseteq H \cap K$.

c) $([H, K], \alpha_{[H,K]})$ is a Hom-ideal of (L, α_L) when α_L is surjective.

d) $([H, K], \alpha_{[H,K]})$ is a Hom-ideal of (H, α_H) and (K, α_K), respectively.

f) If $H = K = L$, then $([L, L], \alpha_{[L,L]})$ is a Hom-ideal of (L, α_L).

2.12. Lemma. Let (H, α_H) and (K, α_K) be Hom-ideals of a Hom-Lie algebra $(L, [-,-], \alpha_L)$ such that $H, K \subseteq \alpha_L(L)$, then $([H, K], \alpha_{[H,K]})$ is a Hom-ideal of $(\alpha_L(L), [-,-], \alpha_{L_|})$.

2.13. Definition. The center of a Hom-Lie algebra $(L, [-,-], \alpha_L)$ is the $\mathbb{K}$-vector subspace

$$Z(L) = \{x \in L \mid [x, y] = 0, \text{for all } y \in L\}$$

2.14. Remark. When $\alpha_L : L \to L$ is a surjective endomorphism, then $(Z(L), \alpha_{L|})$ is a Hom-ideal of $(L, [-,-], \alpha_L)$.

2.15. Definition. Let $(L, [-,-], \alpha_L)$ and $(M, [-,-], \alpha_M)$ be Hom-Lie algebras. A Hom-L-action from $(L, [-,-], \alpha_L)$ over $(M, [-,-], \alpha_M)$ consists in a bilinear map $\rho : L \otimes M \to M$, given by $\rho(x \otimes m) = x \centerdot m$, satisfying the following properties:

a) $[x, y] \centerdot \alpha_M(m) = \alpha_L(x) \centerdot (y \centerdot m) - \alpha_L(y) \centerdot (x \centerdot m)$
b) $\alpha_L(x) \centerdot [m, m'] = [x \centerdot m, \alpha_M(m')] + [\alpha_M(m), x \centerdot m']$
c) $\alpha_M(x \centerdot m) = \alpha_L(x) \centerdot \alpha_M(m)$

for all $x, y \in L$ and $m, m' \in M$.

Under these circumstances, we say that (L, α_L) Hom-acts over (M, α_M).

2.16. Remark. When (M, α_M) is an abelian Hom-Lie algebra, Definition 2.15 goes back to the definition of Hom-L-module in [10].

2.17. Example.

a) $(L, [-,-], \alpha_L)$ acts on itself by the action given by the bracket.
b) Let $\mathfrak{g}$ and $\mathfrak{m}$ be Lie algebras with a Lie action from $\mathfrak{g}$ over $\mathfrak{m}$. Then $(\mathfrak{g}, Id_{\mathfrak{g}})$ Hom-acts over $(\mathfrak{m}, Id_{\mathfrak{m}})$.
c) Let $\mathfrak{g}$ be a Lie algebra, $\alpha : \mathfrak{g} \to \mathfrak{g}$ an endomorphism and M a $\mathfrak{g}$-module in the usual sense, such that the action from $\mathfrak{g}$ over M satisfies the condition $\alpha(x) \centerdot m = x \centerdot m$, for all $x \in \mathfrak{g}$ and $m \in M$. Then (M, Id) is a Hom-$\mathfrak{g}$-module.

   An example of this situation is given by the 2-dimensional Lie algebra L generated by $\{e, f\}$ with bracket $[e, f] = -[f, e] = e$ and endomorphism α represented by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where M is the ideal spanned by $\{e\}$.
d) An abelian sequence of Hom-Lie algebras is an exact sequence of Hom-Lie algebras $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$, where (M, α_M) is an abelian Hom-Lie algebra, $\alpha_K \circ i = i \circ \alpha_M$ and $\pi \circ \alpha_K = \alpha_L \circ \pi$.

   The abelian sequence induces a Hom-L-module structure on (M, α_M) by means of the action given by $\rho : L \otimes M \to M, \rho(l, m) = [k, m], \pi(k) = l$.
e) For other examples we refer to Example 6.2 in [10].

3. Homology

Following [10, 12], for a Hom-Lie algebra (L, α_L) and a (left) Hom-L-module (M, α_M), one denotes by

$$C_n^{\alpha}(L, M) := M \otimes \Lambda^n L, \ n \geqslant 0$$

the n-chain module of (L, α_L) with coefficients in (M, α_M).

For $n \geqslant 1$, one defines the $\mathbb{K}$-linear map,

$$d_n : C_n^{\alpha}(L, M) \longrightarrow C_{n-1}^{\alpha}(L, M)$$

by

$$d_n(m \otimes x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^{n} (-1)^{i+1} x_i \centerdot m \otimes \alpha_L(x_1) \wedge \cdots \wedge \widehat{\alpha_L(x_i)} \wedge \cdots \wedge \alpha_L(x_n) +$$

$$\sum_{1\leqslant i<j\leqslant n}(-1)^{i+j}\,\alpha_M(m)\otimes[x_i,x_j]\wedge\alpha_L(x_1)\wedge\cdots\wedge\widehat{\alpha_L(x_i)}\wedge\cdots\wedge\widehat{\alpha_L(x_j)}\wedge\cdots\wedge\alpha_L(x_n)$$

Although in [10, 12] is proved that $(CL_n^\alpha(L,M),d_n)$ is a well-defined chain complex, we present an alternative proof by means of a generalization of Cartan's formulas. Firstly, we define for all $y\in L$ and $n\in\mathbb{N}$, two linear maps,

$$\theta_n(y):C_n^\alpha(L,M)\longrightarrow C_n^\alpha(L,M)$$

by

$$\theta_n(y)(m\otimes x_1\wedge\cdots\wedge x_n)=y\centerdot m\otimes\alpha_L(x_1)\wedge\cdots\wedge\alpha_L(x_n)+$$
$$\sum_{i=1}^{n}(-1)^i\,\alpha_M(m)\otimes[x_i,y]\wedge\alpha_L(x_1)\wedge\cdots\wedge\widehat{\alpha_L(x_i)}\wedge\cdots\wedge\alpha_L(x_n)$$

and

$$i_n(\alpha_L(y)):C_n^\alpha(L,M)\longrightarrow C_{n+1}^\alpha(L,M)$$

by

$$i_n(\alpha_L(y))(m\otimes x_1\wedge\cdots\wedge x_n)=(-1)^n\,m\otimes x_1\wedge\cdots\wedge x_n\wedge y$$

3.1. Proposition. (Generalized Cartan's formulas)
The following identities hold:

a) $d_{n+1}\circ i_n(\alpha_L(y))+i_{n-1}\left(\alpha_L^2(y)\right)\circ d_n=-\theta_n(y)$, for all $n\geqslant 1$.
b) $\theta_n(\alpha_L(x))\circ\theta_n(y)-\theta_n(\alpha_L(y))\circ\theta_n(x)=\theta_n([x,y])\circ(\alpha_M\otimes\alpha_L^{\wedge n})$, for all $n\geq 0$.
c) $\theta_n(x)\circ i_{n-1}(\alpha_L(y))-i_{n-1}\left(\alpha_L^2(y)\right)\circ\theta_{n-1}(x)=i_{n-1}(\alpha_L[x,y])\circ\left(\alpha_M\otimes\alpha_L^{\wedge(n-1)}\right)$, for all $n\geq 1$.
d) $\theta_{n-1}(\alpha_L(y))\circ d_n=d_n\circ\theta_n(y)$, for all $n\geq 1$.
e) $d_n\circ d_{n+1}=0$, for all $n\geq 1$.

Proof. The proof follows with a routine induction, so we omit it. □

In case $\alpha_L=Id_L,\alpha_M=Id_M$, the above formulas become Cartan's formulas for the Chevalley-Eilenberg homology [5].

Thanks to Proposition 3.1, $(C_\star^\alpha(L,M),d_\star)$ is a well-defined chain complex (an alternative proof can be seen in [12]). Its homology is said to be the homology of the Hom-Lie algebra (L,α_L) with coefficients in the Hom-L-module (M,α_M) and it is denoted by:

$$H_\star^\alpha(L,M):=H_\star(C_\star^\alpha(L,M),d_\star)$$

An easy computation in low-dimensional cycles and boundaries provides the following results:

$$H_0^\alpha(L,M)=\frac{Ker(d_0)}{\mathrm{Im}(d_1)}=\frac{M}{{}^LM}$$

where ${}^LM=\{l\centerdot m:m\in M,l\in L\}$.

Now let us consider M as a trivial Hom-L-module, i.e. $l\centerdot m=0$, then

$$H_1^\alpha(L,M)=\frac{Ker(d_1)}{\mathrm{Im}(d_2)}=\frac{M\otimes L}{\alpha_M(M)\otimes[L,L]}$$

In particular, if $M=\mathbb{K}$, then $H_1^\alpha(L,\mathbb{K})=\frac{L}{[L,L]}$.

4. Universal central extensions

Through this section we will deal with universal central extensions of Hom-Lie algebras. We will generalize classical results of universal central extensions theory of Lie algebras, but here an important problem appears, namely the composition of central extensions is not central in general, as Example 4.9 shows. This fact doesn't allow a complete generalization of classical results, however requires the introduction of a new concept of centrality for Hom-Lie algebra extensions.

4.1. Definition. A short exact sequence of Hom-Lie algebras $(K) : 0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is said to be central if $[M, K] = 0$. Equivalently, $M \subseteq Z(K)$.

The sequence (K) is said to be α-central if $[\alpha_M(M), K] = 0$. Equivalently, $\alpha_M(M) \subseteq Z(K)$.

4.2. Remark. Let us observe that both notions coincide when $\alpha_M = Id_M$. Obviously, every central extension is an α-central extension, but the converse doesn't hold as the following counter-example shows:

Consider the two-dimensional Hom-Lie algebra L with basis $\{a_1, a_2\}$, bracket given by

$$[a_1, a_2] = -[a_2, a_1] = a_1,$$

and endomorphism $\alpha_L = 0$.

Let K be the three-dimensional Hom-Lie algebra with basis $\{b_1, b_2, b_3\}$, bracket given by

$$[b_1, b_2] = -[b_2, b_1] = b_1, [b_1, b_3] = -[b_3, b_1] = b_1, [b_2, b_3] = -[b_3, b_2] = b_2,$$

and endomorphism $\alpha_K = 0$.

The surjective homomorphism $\pi : (K, 0) \to (L, 0)$ given by

$$\pi(b_1) = 0, \pi(b_2) = a_1, \pi(b_3) = a_2,$$

is an α-central extension, since Ker $(\pi) = \langle\{b_1\}\rangle$ and $[\alpha_K(Ker(\pi)), K] = 0$, but is not a central extension, since $[Ker(\pi), K] = \langle\{b_1\}\rangle$.

4.3. Definition. A central extension $(K) : 0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is said to be universal if for every central extension $(K') : 0 \to (M', \alpha_{M'}) \stackrel{i'}{\to} (K', \alpha_{K'}) \stackrel{\pi'}{\to} (L, \alpha_L) \to 0$ there exists a unique homomorphism of Hom-Lie algebras $h : (K, \alpha_K) \to (K', \alpha_{K'})$ such that $\pi' \circ h = \pi$.

A central extension $(K) : 0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is said to be universal α-central if for every α-central extension $(K') : 0 \to (M', \alpha_{M'}) \stackrel{i'}{\to} (K', \alpha_{K'}) \stackrel{\pi'}{\to} (L, \alpha_L) \to 0$ there exists a unique homomorphism of Hom-Lie algebras $h : (K, \alpha_K) \to (K', \alpha_{K'})$ such that $\pi' \circ h = \pi$.

4.4. Remark. Obviously, every universal α-central extension is a universal central extension. Let us observe that both notions coincide when $\alpha_M = Id_M$.

4.5. Definition. A Hom-Lie algebra (L, α_L) is said to be perfect if $L = [L, L]$.

4.6. Lemma. Let $\pi : (K, \alpha_K) \to (L, \alpha_L)$ be a surjective homomorphism of Hom-Lie algebras. If (K, α_K) is a perfect Hom-Lie algebra, then (L, α_L) is a perfect Hom-Lie algebra as well.

4.7. Lemma. Let $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ be a central extension and (K, α_K) a perfect Hom-Lie algebra. If there exists a homomorphism of Hom-Lie algebras $f : (K, \alpha_K) \to (A, \alpha_A)$ such that $\tau \circ f = \pi$, where $0 \to (N, \alpha_N) \stackrel{j}{\to} (A, \alpha_A) \stackrel{\tau}{\to} (L, \alpha_L) \to 0$ is a central extension, then f is unique.

The proofs of these two last Lemmas use classical arguments, so we omit them.

4.8. Lemma. If $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal central extension, then (K, α_K) and (L, α_L) are perfect Hom-Lie algebras.

Proof. Let us assume that (K, α_K) is not a perfect Hom-Lie algebra, then $[K, K] \subsetneq K$. Hence $(K/[K, K], \widetilde{\alpha})$, where $\widetilde{\alpha}$ is the induced homomorphism, is an abelian Hom-Lie algebra, consequently, it is a trivial Hom-L-module. Let us consider the central extension $0 \to (K/[K, K], \widetilde{\alpha}) \to (K/[K, K] \times L, \widetilde{\alpha} \times \alpha_L) \stackrel{pr}{\to} (L, \alpha_L) \to 0$. Then the homomorphisms of Hom-Lie algebras $\varphi, \psi : (K, \alpha_K) \to (K/[K, K] \times L, \widetilde{\alpha} \times \alpha_L)$ given by $\varphi(k) = (\overline{k}, \pi(k))$ and $\psi(k) = (0, \pi(k)), k \in K$, verify that $pr \circ \phi = \pi = pr \circ \psi$, so $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ cannot be a universal central extension.

Lemma 4.6 ends the proof. □

Classical categories as groups, Lie algebras, Leibniz algebras and other similar ones share the following property: the composition of two central extensions is a central extension, which is absolutely necessary in order to obtain characterizations of the universal central extensions. Unfortunately this property doesn't remain for the category of Hom-Lie algebras as the following counter-example 4.9 shows. This problem lead us to introduce the notion of α-central extensions in Definition 4.1, whose properties relative to the composition are given in Lemma 4.10.

4.9. Example. Consider the four-dimensional Hom-Lie algebra (L, α_L) with basis $\{a_1, a_2, a_3, a_4\}$, bracket operation given by

$$\left\{ \begin{array}{ll} [a_1, a_3] = -[a_3, a_1] = a_4, & [a_1, a_4] = -[a_4, a_1] = a_3, \\ [a_2, a_3] = -[a_3, a_2] = a_1, & [a_2, a_4] = -[a_4, a_2] = a_2, \end{array} \right.$$

(the non-written brackets are equal to zero) and endomorphism $\alpha_L = 0$.

Let (K, α_K) be the five-dimensional Hom-Lie algebra with basis $\{b_1, b_2, b_3, b_4, b_5\}$, bracket operation given by

$$\left\{ \begin{array}{ll} [b_2, b_3] = -[b_3, b_2] = b_1, & [b_2, b_4] = -[b_4, b_2] = b_5, \\ [b_2, b_5] = -[b_5, b_2] = b_4, & [b_3, b_4] = -[b_4, b_3] = b_2, \\ [b_3, b_5] = -[b_5, b_3] = b_3, & \end{array} \right.$$

(the non-written brackets are equal to zero) and endomorphism $\alpha_K = 0$.

Obviously (K, α_K) is a perfect Hom-Lie algebra since $K = [K, K]$. On the other hand, $Z(K, \alpha_K) =< \{b_1\} >$.

The linear map $\pi : (K, \alpha_K) \to (L, \alpha_L)$ given by

$$\pi(b_1) = 0, \pi(b_2) = a_1, \pi(b_3) = a_2, \pi(b_4) = a_3, \pi(b_5) = a_4,$$

is a central extension since π is a surjective homomorphism of Hom-Lie algebras and $\text{Ker}(\pi) =< \{b_1\} > \subseteq Z(K, \alpha_K)$.

Now let us consider the six-dimensional Hom-Lie algebra (F, α_F) with basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$, bracket operation given by

$$\left\{ \begin{array}{ll} [e_2, e_3] = -[e_3, e_2] = e_1, & [e_2, e_4] = -[e_4, e_2] = e_1, \\ [e_2, e_5] = -[e_5, e_2] = e_1, & [e_3, e_4] = -[e_4, e_3] = e_2, \\ [e_3, e_5] = -[e_5, e_3] = e_6, & [e_3, e_6] = -[e_6, e_3] = e_5, \\ [e_4, e_5] = -[e_5, e_4] = e_3, & [e_4, e_6] = -[e_6, e_4] = e_4, \\ [e_5, e_6] = -[e_6, e_5] = e_1, & \end{array} \right.$$

(the non-written brackets are equal to zero) and endomorphism $\alpha_F = 0$.

The linear map $\rho : (F, \alpha_F) \to (K, \alpha_K)$ given by

$$\rho(e_1) = 0, \rho(e_2) = b_1, \rho(e_3) = b_2, \rho(e_4) = b_3, \rho(e_5) = b_4, \rho(e_6) = b_5,$$

is a central extension since ρ is a surjective homomorphism of Hom-Lie algebras and $\mathrm{Ker}(\rho) =< \{e_1\} >= Z(F, \alpha_F)$.

The composition $\pi \circ \rho : (F, \alpha_F) \to (L, \alpha_L)$ is given by

$$\pi \circ \rho(e_1) = \pi(0) = 0, \quad \pi \circ \rho(e_2) = \pi(b_1) = 0, \quad \pi \circ \rho(e_3) = \pi(b_2) = a_1,$$
$$\pi \circ \rho(e_4) = \pi(b_3) = a_2, \quad \pi \circ \rho(e_5) = \pi(b_4) = a_3, \quad \pi \circ \rho(e_6) = \pi(b_5) = a_4,$$

Consequently, $\pi \circ \rho : (F, \alpha_F) \to (L, \alpha_L)$ is a surjective homomorphism, but is not a central extension, since $Z(F, \alpha_F) =< \{e_1\} >$ and $\mathrm{Ker}(\pi \circ \rho) = < \{e_1, e_2\} >$, i. e. $\mathrm{Ker}(\pi \circ \rho) \nsubseteq Z(F, \alpha_F)$.

4.10. Lemma. Let $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ and $0 \to (N, \alpha_N) \stackrel{j}{\to} (F, \alpha_F) \stackrel{\rho}{\to} (K, \alpha_K) \to 0$ be central extensions with (K, α_K) a perfect Hom-Lie algebra. Then the composition extension $0 \to (P, \alpha_P) = \mathrm{Ker}\ (\pi \circ \rho) \to (F, \alpha_F) \stackrel{\pi \circ \rho}{\to} (L, \alpha_L) \to 0$ is an α-central extension.

Moreover, if $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal α-central extension, then $0 \to (N, \alpha_N) \stackrel{j}{\to} (F, \alpha_F) \stackrel{\rho}{\to} (K, \alpha_K) \to 0$ is split, that is, there exists a Hom-Lie algebra homomorphism $\sigma : (K, \alpha_K) \to (F, \alpha_F)$ such that $\rho \circ \sigma = Id_K$.

Proof. We must prove that $[\alpha_P(P), F] = 0$.

Since (K, α_K) is a perfect Hom-Lie algebra, then every element $f \in F$ can be written as $f = \sum\limits_i \lambda_i [f_{i_1}, f_{i_2}] + n, n \in N, \lambda_i \in \mathbb{K}, f_{i_j} \in F, j = 1, 2$ since $\rho(f) \in K$, then $\rho(f) = \sum\limits_i \lambda_i [k_{i_1}, k_{i_2}] = \sum\limits_i \lambda_i [\rho(f_{i1}), \rho(f_{i_2})] = \rho \left(\sum\limits_i \lambda_i [f_{i_1}, f_{i_2}] \right)$, hence $f - \sum\limits_i \lambda_i [f_{i_1}, f_{i_2}] \in$ Ker (ρ).

So, for all $p \in P, f \in F$ we have that

$$[\alpha_P(p), f] = \sum_i \lambda_i \left([[p, f_{i_1}], \alpha_F(f_{i_2})] + [[f_{i_2}, p], \alpha_F(f_{i_1})] \right) + [\alpha_P(p), n] = 0$$

since $[p, f_{i_j}] \in \mathrm{Ker}\ (\rho) \subseteq Z(F)$.

For the second statement, if $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal α-central extension, then by the first statement, $0 \to (P, \alpha_P) = \mathrm{Ker}\ (\pi \circ \rho) \to (F, \alpha_F) \stackrel{\pi \circ \rho}{\to} (L, \alpha_L) \to 0$ is an α-central extension, then there exists a unique homomorphism of Hom-Lie algebras $\sigma : (K, \alpha_K) \to (F, \alpha_F)$ such that $\pi \circ \rho \circ \sigma = \pi$. On the other hand, $\pi \circ \rho \circ \sigma = \pi = \pi \circ Id$ and (K, α_K) is perfect, then Lemma 4.7 implies that $\rho \circ \sigma = Id$. □

4.11. Theorem.

a) If a central extension $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal α-central extension, then (K, α_K) is a perfect Hom-Lie algebra and every central extension of (K, α_K) is split.

b) Let $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ be a central extension.
If (K, α_K) is a perfect Hom-Lie algebra and every central extension of (K, α_K) is split, then $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal central extension.

c) A Hom-Lie algebra (L, α_L) admits a universal central extension if and only if (L, α_L) is perfect.

d) The kernel of the universal central extension is canonically isomorphic to $H_2^{\alpha}(L)$.

Proof.

a) If $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal α-central extension, then it is a universal central extension by Remark 4.4, so (K, α_K) is a perfect Hom-Lie algebra by Lemma 4.8 and every central extension of (K, α_K) is split by Lemma 4.10.

b) Consider a central extension $0 \to (N, \alpha_N) \xrightarrow{j} (A, \alpha_A) \xrightarrow{\tau} (L, \alpha_L) \to 0$. Construct the pull-back extension $0 \to (N, \alpha_N) \xrightarrow{\chi} (P, \alpha_P) \xrightarrow{\overline{\tau}} (K, \alpha_K) \to 0$, where $P = \{(a,k) \in A \times K \mid \tau(a) = \pi(k)\}$ and $\alpha_P(a,k) = (\alpha_A(a), \alpha_K(k))$, which is central, consequently is split, i.e. there exists a homomorphism $\sigma : (K, \alpha_K) \to (P, \alpha_P)$ such that $\overline{\tau} \circ \sigma = Id$.

Then $\overline{\pi} \circ \sigma$, where $\overline{\pi} : (P, \alpha_P) \to (A, \alpha_A)$ is induced by the pull-back construction, satisfies $\tau \circ \overline{\pi} \circ \sigma = \pi$. Lemma 4.8 ends the proof.

c) and d) For a Hom-Lie algebra (L, α_L) consider the homology chain complex $C_\star^\alpha(L)$, which is $C_\star^\alpha(L, \mathbb{K})$ where $\mathbb{K}$ is endowed with the trivial Hom-L-module structure.

As $\mathbb{K}$-vector spaces, let I_L be the subspace of $L \wedge L$ spanned by the elements of the form $-[x_1, x_2] \wedge \alpha_L(x_3) + [x_1, x_3] \wedge \alpha_L(x_2) - [x_2, x_3] \wedge \alpha_L(x_1), x_1, x_2, x_3 \in L$. That is, $I_L = \text{Im } (d_3 : C_3^\alpha(L) \to C_2^\alpha(L))$.

Now we denote the quotient $\mathbb{K}$-vector space $\frac{L \wedge L}{I_L}$ by $\mathfrak{uce}(L)$. Every class $x_1 \wedge x_2 + I_L$ is denoted by $\{x_1, x_2\}$, for all $x_1, x_2 \in L$.

By construction, the following identity holds:

$$\{[x_1, x_2], \alpha_L(x_3)\} + \{[x_2, x_3], \alpha_L(x_1)\} + \{[x_3, x_1], \alpha_L(x_2)\} = 0 \tag{4.1}$$

for all $x_1, x_2, x_3 \in L$.

Now $d_2(I_L) = 0$, so it induces a $\mathbb{K}$-linear map $u_L : \mathfrak{uce}(L) \to L$, given by $u_L(\{x_1, x_2\}) = [x_1, x_2]$. Moreover $(\mathfrak{uce}(L), \widetilde{\alpha})$, where $\widetilde{\alpha} : \mathfrak{uce}(L) \to \mathfrak{uce}(L)$ is defined by $\widetilde{\alpha}(\{x_1, x_2\}) = \{\alpha_L(x_1), \alpha_L(x_2)\}$, is a Hom-Lie algebra with respect to the bracket $[\{x_1, x_2\}, \{y_1, y_2\}] = \{[x_1, x_2], [y_1, y_2]\}$ and $u_L : (\mathfrak{uce}(L), \widetilde{\alpha}) \to (L, \alpha_L)$ is a homomorphism of Hom-Lie algebras. Actually, Im $u_L = [L, L]$, but (L, α_L) is a perfect Hom-Lie algebra, so u_L is a surjective homomorphism.

From the construction, it follows that $\text{Ker}(u_L) = H_2^\alpha(L)$, so we have the extension

$$0 \to (H_2^\alpha(L), \widetilde{\alpha}_|) \to (\mathfrak{uce}(L), \widetilde{\alpha}) \xrightarrow{u_L} (L, \alpha_L) \to 0$$

which is central, since $[\text{Ker}(u_L), \mathfrak{uce}(L)] = 0$, and universal, since for any central extension $0 \to (M, \alpha_M) \to (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ there exists the homomorphism of Hom-Lie algebras $\varphi : (\mathfrak{uce}(L), \widetilde{\alpha}) \to (K, \alpha_K)$ given by $\varphi(\{x_1, x_2\}) = [k_1, k_2], \pi(k_i) = x_i, i = 1, 2$, such that $\pi \circ \varphi = u_L$. Moreover, $(\mathfrak{uce}(L), \widetilde{\alpha})$ is a perfect Hom-Lie algebra, so by Lemma 4.7, φ is unique. □

4.12. Corollary.

a) Let $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ be a universal α-central extension, then $H_1^\alpha(K) = H_2^\alpha(K) = 0$.
b) Let $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ be a central extension such that $H_1^\alpha(K) = H_2^\alpha(K) = 0$, then $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is a universal central extension.

Proof.

a) If $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is a universal α-central extension, then (K, α_K) is perfect by Remark 4.4 and Lemma 4.8, so $H_1^\alpha(K) = 0$. By Lemma 4.10 and Theorem 4.11 c) and d) the universal central extension corresponding to (K, α_K) is split, so $H_2^\alpha(K) = 0$.

b) $H_1^\alpha(K) = 0$ implies that (K, α_K) is a perfect Hom-Lie algebra.

$H_2^\alpha(K) = 0$ implies that $(\mathfrak{uce}(K), \widetilde{\alpha}) \xrightarrow{\sim} (K, \alpha_K)$. Theorem 4.11 b) ends the proof. □

4.13. Definition. An α-central extension $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is said to be universal if for every central extension $0 \to (R, \alpha_R) \xrightarrow{j} (A, \alpha_A) \xrightarrow{\tau} (L, \alpha_L) \to 0$ there exists a unique homomorphism $\varphi : (K, \alpha_K) \to (A, \alpha_A)$ such that $\tau \circ \varphi = \pi$.

4.14. Proposition. Let $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ and $0 \to (N, \alpha_N) \xrightarrow{j} (F, \alpha_F) \xrightarrow{\rho} (K, \alpha_K) \to 0$ be central extensions. If $0 \to (N, \alpha_N) \xrightarrow{j} (F, \alpha_F) \xrightarrow{\rho} (K, \alpha_K) \to 0$ is a universal α-central extension, then $0 \to (P, \alpha_P) = \mathrm{Ker}(\pi \circ \rho) \xrightarrow{\chi} (F, \alpha_F) \xrightarrow{\pi\circ\rho} (L, \alpha_L) \to 0$ is an α-central extension which is universal in the sense of Definition 4.13.

Proof. $0 \to (P, \alpha_P) = \mathrm{Ker}(\pi \circ \rho) \xrightarrow{\chi} (F, \alpha_F) \xrightarrow{\pi\circ\rho} (L, \alpha_L) \to 0$ is an α-central extension by Lemma 4.10.

In order to obtain the universality, for any central extension $0 \to (S, \alpha_S) \to (A, \alpha_A) \xrightarrow{\sigma} (L, \alpha_L) \to 0$ construct the pull-back extension corresponding to σ and $\pi\circ\rho$, then Theorem 4.11 and Lemma 4.7 end the proof. □

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References

[1] Ammar, F., Ejbehi, Z. and Makhlouf, A. *Cohomology and deformations of hom-algebras*, J. Lie Theory **21** (4), 813–836, 2011.

[2] Ammar, F., Mabrouk, S. and Makhlouf, A. *Representations and cohomology of n-ary multiplicative Hom-Nambu-Lie algebras*, J. Geom. Phys. **61** (10), 1898–1913, 2011.

[3] Cheng, Y.-S. and Su, Y.-C. *(Co)homology and universal central extensions of Hom-Leibniz algebras*, Acta Math. Sin. **27** (5), 813–830, 2011.

[4] Hartwing, J. T., Larson, D. and Silvestrov, S. D. *Deformations of Lie algebras using σ-derivations*, J. Algebra **295**, 314–361, 2006.

[5] Hilton, P. J. and Stammbach, U. *A course in homological algebra*, Graduate Texts in Math. **4** (Springer 1971).

[6] Humphreys, J. E. *Introduction to Lie algebras and representation theory*, Graduate Texts in Math. **9** (Springer 1972).

[7] Makhlouf, A. and Silvestrov, S. *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2** (2), 51–64, 2008.

[8] Makhlouf, A. and Silvestrov, S. *Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras*, Forum Math. **22** (4), 715–739, 2010.

[9] Sheng, Y. *Representations of Hom-Lie algebras*, Algebr. Represent. Theory **16** (6), 1081–1098, 2012.

[10] Yau, D. *Hom-algebras as deformations and homology*, Arxiv: 0712.3515v1, 2007.

[11] Yau, D. *Enveloping algebras of Hom-Lie algebras*, J. Gen. Lie Theory Appl. **2** (2), 95–108, 2008.

[12] Yau, D. *Hom-algebras and homology*, J. Lie Theory **19** (2), 409–421, 2009.

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Hyperconnectedness and extremal disconnectedness in (a)topological spaces

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Abstract

The aim of this paper is to study hyperconnectedness and extremal disconnectedness in a space equipped with a countable number of topologies.

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1. Introduction

A bitopological space [7] is a nonempty set with two topologies. Kovár [8, 9] also studied the properties of a nonempty set equipped with three topologies. Datta and Roy Choudhuri [5], and Raut and Datta [13] introduced nontrivial infinitesimally small elements. With this, they defined a number system as an extension of real number system. Their study offers a natural framework for dealing with an infinite sequence of distinct topologies on a set. Also emergence of chaos in a deterministic system, in the theory of dynamical system, relates to an interplay of finite or infinite number of different topologies in the underlining set. All these matters motivated the authors to consider a countable number of topologies in (ω)topological spaces [2, 3, 4] and $(\aleph_0)$topological spaces [1]. In this paper, we introduce the notion of (a)topological spaces which is a set equipped with countable number of topologies. The notion of (a)topological spaces is more general than both the notions of (ω)topological spaces and $(\aleph_0)$topological spaces.

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Sarma [14] introduced the notion of pairwise extremal disconnectedness in bitopological spaces [7]. Among other results, she proved a result on pairwise extremal disconnectedness which is parallel to Urysohn's lemma on pairwise normal spaces. In [12], Mathew studied the hyperconnected topological spaces (Steen and Seebach [15]). In this paper, we study hyperconnectedness and extremal disconnectedness in the context of (a)topological spaces.

2. (a)topological spaces

2.1. Definition. If $\{\tau_n\}$ is a sequence of topologies on a set X, then the pair $(X, \{\tau_n\})$ is called an (a)topological space.

If there is no scope of confusion, we denote the (a)topological space $(X, \{\tau_n\})$, simply by X. Throughout the paper, N denotes the set of natural numbers. The elements of N are denoted by i, j, k, l, m, n etc. If a set G is open with respect to the topology τ_n i.e., if $G \in \tau_n$, then we say G is (τ_n)open. (τ_n)closed set, (τ_n)closure have the obvious meaning. The (τ_n)closure (resp. (τ_n)interior) of a set E is denoted by $(\tau_n)\mathrm{cl}E$ (resp. $(\tau_n)\mathrm{int}E$). Following Levine [11], we now introduce the following definitions.

2.2. Definition. Let X be an (a)topological space and let $m \neq n$. A set $A \subset X$ is said to be (m, n)semiopen if there exists a $U \in \tau_m$ such that $U \subset A \subset (\tau_n)\mathrm{cl}U$.

Thus any (τ_m)open set is an (m, n)semiopen set for any $n \neq m$. The class of all (m, n)semiopen sets with $m \neq n$, is denoted by $SO_{(m,n)}(X)$. We write $SO(X) = \bigcup_{(m,n)} SO_{(m,n)}(X)$. A set belonging to $SO(X)$ is called an (a)semiopen set.

2.3. Definition. An (a)topology $\{\sigma_n\}$ on X is said to be stronger (resp. weaker) than an (a)topology $\{\tau_n\}$ if $\tau_n \subset \sigma_n$ (resp. $\sigma_n \subset \tau_n$) for each n. If, in addition, $\tau_n \neq \sigma_n$ for at least one n, then $\{\sigma_n\}$ is said to be strictly stronger (resp. weaker) than $\{\tau_n\}$.

2.4. Definition. An (a)topological space $(X, \{\tau_n\})$ with property P is said to be maximal (resp. minimal) with respect to P if for any other (a)topology $\{\sigma_n\}$ strictly stronger (resp. weaker) than $\{\tau_n\}$, the space $(X, \{\sigma_n\})$ can not have this property.

3. Hyperconnected spaces

Recall that a topological space $(X, \mathscr{T})$ is hyperconnected if the intersection of any two nonempty open sets is nonempty.

3.1. Definition. An (a)topological space $(X, \{\tau_n\})$ is said to be hyperconnected if for any two nonempty sets U and V with $U \in \tau_m$, $V \in \tau_n$ and $m \neq n$, we have $U \cap V \neq \emptyset$.

It follows that if X is hyperconnected, then for any nonempty set $U \in \tau_m$, $(\tau_n)\mathrm{cl}U = X$ for any $n \neq m$.

3.2. Theorem. *Suppose for any (τ_m)open set G and (τ_n)open set H, $G \cap H \in \tau_l$ for some l. Then the (a)topological space $(X, \{\tau_n\})$ and the topological spaces (X, τ_n), $n \in N$ are hyperconnected iff the set D of all nonempty (a)semiopen sets is a filter.*

Proof. Let the (a)topological space $(X, \{\tau_n\})$ and the topological spaces (X, τ_n), $n \in N$ be hyperconnected and A, $B \in D$. Then there exist two pairs (m, n) and (k, l) with $m \neq n$ and $k \neq l$ such that for some $U \in \tau_m$ and $V \in \tau_k$, $U \subset A \subset (\tau_n)\mathrm{cl}U$, $V \subset B \subset (\tau_l)\mathrm{cl}V$. Since $(X, \{\tau_n\})$ and (X, τ_n), $n \in N$ are hyperconnected, we have $U \cap V \neq \emptyset$. Also $U \cap V \in \tau_i$ for some i. Therefore $(\tau_j)\mathrm{cl}(U \cap V) = X$ for any $j \neq i$. Hence $U \cap V \subset A \cap B \subset (\tau_j)\mathrm{cl}(U \cap V)$. Thus $A \cap B \in D$. Now let A be a nonempty (a)semiopen set and $B \supset A$. Then for some (m, n) with $m \neq n$, there exists $U \in \tau_m$ such that

$U \subset A \subset (\tau_n)\text{cl}U$. But $(\tau_n)\text{cl}U = X$. Therefore $U \subset B \subset (\tau_n)\text{cl}U$. Hence B is a nonempty (a)semiopen set. Thus D is a filter.

Since for any n, a (τ_n)open set is an (a)semiopen set, the converse follows. □

For a pair (m,n), the union of an arbitrary number of (m,n)semiopen sets is an (m,n)semiopen set. If X is hyperconnected, then the intersection of a finite number of (m,n)semiopen sets is (m,n)semiopen. Hence in this case, the class $SO_{(m,n)}(X)$ forms a topology on X. Since the class $\{SO_{(m,n)}(X) \mid m,n \in N\}$ is countable, and any countable class can be represented as a sequence, we rewrite the class $\{SO_{(m,n)}(X) \mid m,n \in N\}$ as a sequence $\{S_k\}$. So $(X,\{S_k\})$ is an (a)topological space. From Theorem 3.2, it follows that if $(X,\{\tau_n\})$ is hyperconnected and if for $G \in \tau_m$ and $H \in \tau_n$, $G \cap H \in \tau_l$ for some l, then $(X,\{S_k\})$ is hyperconnected.

3.3. Corollary. *If $(X,\{\tau_n\})$ is maximal hyperconnected, then $\{\tau_n\} = \{S_k\}$.*

Levine [10] introduced the concept of a simple extension of a topological space. Let $(X,\mathscr{P})$ be a topological space. A family $\mathscr{Q}$ of subsets of X is a simple extension of $\mathscr{P}$ if $\mathscr{Q}$ contains $\mathscr{P}$ and there exists a $P \notin \mathscr{P}$ such that $\mathscr{Q} = \{G \cup (H \cap P) \mid G,H \in \mathscr{P}\}$.

3.4. Theorem. *If for any (τ_m)open set G and (τ_n)open set H, $G \cap H \in \tau_l$ for some l and if the space $(X,\{\tau_n\})$ is maximal hyperconnected and the space (X,τ_n), $n \in N$ are hyperconnected, then the set D of all nonempty (a)semiopen sets is an ultrafilter.*

Proof. Since by Theorem 3.2, D is a filter, it is sufficient to show that for a nonempty set E, $X - E \in D$ if $E \notin D$. Let us suppose that $E \notin D$. Then $E \notin \bigcup_n \tau_n$. Let $\tau_n(E)$ denote a simple extension of τ_n. Then the space $(X,\{\tau_n(E)\})$ which is stronger than $(X,\{\tau_n\})$ is not hyperconnected. So for some nonempty set $U \in \tau_m(E)$ and nonempty set $V \in \tau_n(E)$ with $m \neq n$ we have $U \cap V = \emptyset$. By the definition of simple extension, $U = U_1 \cup (U_2 \cap E)$ and $V = V_1 \cup (V_2 \cap E)$ for some $U_i \in \tau_m$ and $V_i \in \tau_n$, $i = 1,2$. Since $U \cap V = \emptyset$, $U_1 \cap V_1 = \emptyset$. But $(X,\{\tau_n\})$ is hyperconnected and so either $U_1 = \emptyset$ or $V_1 = \emptyset$. Suppose without loss of generality, $U_1 = \emptyset$. Now we consider the cases (i) $V_1 = \emptyset$, and (ii) $V_1 \neq \emptyset$.

Case (i): $V_1 = \emptyset$. Since $U,V \neq \emptyset$, we have $U_2 \neq \emptyset$, $V_2 \neq \emptyset$. Therefore $U_2 \cap V_2 \neq \emptyset$, since $(X,\{\tau_n\})$ is hyperconnected. Now

$$\begin{aligned} & U \cap V = \emptyset \\ \Rightarrow\quad & U_2 \cap V_2 \cap E = \emptyset \\ \Rightarrow\quad & U_2 \cap V_2 \subset X - E \\ \Rightarrow\quad & X - E \in D, \text{ since } D \text{ is a filter.} \end{aligned}$$

Case (ii): $V_1 \neq \emptyset$. Since $U_2 \neq \emptyset$, we have $U_2 \cap V_1 \neq \emptyset$. Therefore $U_2 \cap V_1 \in D$. Again since $U \cap V = \emptyset$, we have $(U_2 \cap E) \cap V_1 = \emptyset \Rightarrow U_2 \cap V_1 \subset X - E$. Therefore $X - E \in D$. Thus D is an ultrafilter. □

Recall that a topological space $(X,\mathscr{T})$ is a door space if for each $A \subset X$, either $A \in \mathscr{T}$ or $X - A \in \mathscr{T}$.

3.5. Definition. The space $(X,\{\tau_n\})$ is said to be a door space if for every subset E of X, there exists an n_0 such that either $E \in \tau_{n_0}$ or $X - E \in \bigcup_{n \neq n_0} \tau_n$.

3.6. Theorem. *If the (a)topological space $(X,\{\tau_n\})$ is door and hyperconnected, then $\bigcap \tau_n - \{\emptyset\}$ is a filter.*

Proof. Let $A,B \in \bigcap_n \tau_n - \{\emptyset\}$. Then A,B are nonempty (τ_n)open sets for all n and so $A \cap B \in \bigcap_n \tau_n$. Since $(X,\{\tau_n\})$ is hyperconnected, $A \cap B \neq \emptyset$. Thus $A \cap B \in \bigcap_n \tau_n - \{\emptyset\}$. Now suppose $B \supset A \in \bigcap_n \tau_n - \{\emptyset\}$. Suppose that $B \notin \bigcap_n \tau_n - \{\emptyset\}$. Then $B \notin \tau_{n_0} - \{\emptyset\}$

for some n_0 and so $X - B \in \bigcup_{n\neq n_0}\tau_n$, since $(X, \{\tau_n\})$ is door. So we have $A \in \tau_{n_0}$ and $A \cap (X - B) = \emptyset$, which contradicts the hyperconnectivity of $(X, \{\tau_n\})$. Hence $B \in \bigcap_n \tau_n - \{\emptyset\}$. Therefore, $\bigcap_n \tau_n - \{\emptyset\}$ is a filter. □

3.7. Theorem. *If the* (a)*topological space* $(X, \{\tau_n\})$ *is door and hyperconnected and the topological spaces* (X, τ_n), $n \in N$ *are door* [6], *then* $\bigcap_n \tau_n - \{\emptyset\}$ *is an ultrafilter.*

Proof. By Theorem 3.6, $\bigcap_n \tau_n - \{\emptyset\}$ is a filter. If E is a nonempty set with $E \notin \bigcap_n \tau_n - \{\emptyset\}$, then $E \notin \tau_{n_0} - \{\emptyset\}$ for some n_0. Then $X - E \in \tau_{n_0}$, since (X, τ_{n_0}) is door. Therefore, $E \notin \bigcup_{n\neq n_0}\tau_n$, since $(X\{\tau_n\})$ is hyperconnected. And so $E \notin \tau_n$ for all $n \neq n_0 \Rightarrow X - E \in \tau_n$ for all $n \neq n_0$, since (X, τ_n), $n \in N$ are door. Therefore $X - E \in \bigcap_n \tau_n - \{\emptyset\}$. Hence $\bigcap_n \tau_n - \{\emptyset\}$ is an ultrafilter. □

3.8. Theorem. *If* $(X, \{\tau_n\})$ *is door and hyperconnected, then* $(X, \{\tau_n\})$ *is minimal door and maximal hyperconnected.*

Proof. Let $(X, \{\sigma_n\})$ be a door space which is weaker than $(X, \{\tau_n\})$. Suppose that $G \in \tau_m - \sigma_m$. Then $X - G \in \bigcup_{n\neq m}\sigma_n \subset \bigcup_{n\neq m}\tau_n$. But this is not possible, since $(X, \{\tau_n\})$ is hyperconnected. Hence $\sigma_n = \tau_n$ for all n.

Now let $(X, \{\rho_n\})$ be a hyperconnected space stronger than $(X, \{\tau_n\})$. Suppose that $G \in \rho_m - \tau_m$. Then $X - G \in \bigcup_{n\neq m}\tau_n \subset \bigcup_{n\neq m}\rho_n$ which is not possible, since $(X, \{\rho_n\})$ is hyperconnected. Thus $\rho_n = \tau_n$ for all n. □

The following example shows that for an (a)topological door space $(X, \{\tau_n\})$, the topological space (X, τ_n) may not be door even for a single n.

3.9. Example. Suppose R is the set of real numbers. Let τ_n be the topology on R, generated by the subbase $\{\emptyset\} \cup \{E \subset (-\infty, n) \mid 0 \in E\} \cup \{E \subset R \mid 0 \in E$ and E is not bounded above$\}$. Then the (a)topological space $(X, \{\tau_n\})$ is door but for no n, the topological space (X, τ_n) is door.

3.10. Definition. A set E in an (a)topological space $(X, \{\tau_n\})$ is said to be $(n \neq n_0)$dense if $\bigcap_{n\neq n_0}(\tau_n)\text{cl}E = X$.

3.11. Definition. The space $(X, \{\tau_n\})$ is said to be submaximal if for any n_0, every $(n \neq n_0)$dense subset is (τ_{n_0})open.

3.12. Theorem. *If the space* $(X, \{\tau_n\})$ *is hyperconnected and submaximal, then it is maximal hyperconnected.*

Proof. Let $(X, \{\sigma_n\})$ be a hyperconnected space stronger than $(X, \{\tau_n\})$. Let G be a nonempty set belonging to σ_{n_0} for some n_0. Then $(\sigma_n)\text{cl}G = X$ for all $n \neq n_0$. Hence $\bigcap_{n\neq n_0}(\sigma_n)\text{cl}G = X \Rightarrow \bigcap_{n\neq n_0}(\tau_n)\text{cl}G = X$. Therefore G is $(n \neq n_0)$dense in $(X, \{\tau_n\})$ and so $G \in \tau_{n_0}$. Hence $\tau_{n_0} = \sigma_{n_0}$. Thus $\tau_n = \sigma_n$ for all n. □

4. Extremally disconnected spaces

4.1. Definition. An (a)topological space $(X, \{\tau_n\})$ is said to be extremally disconnected if for any $G \in \tau_m$ and $H \in \tau_n$ with $G \cap H = \emptyset$ and $m \neq n$, there exist $k, l \in N$ with $m \neq k$, $n \neq l$ and $k \neq l$ such that for some (τ_k)closed set F and (τ_l)closed set K, we have $G \subset F$, $H \subset K$ and $F \cap K = \emptyset$.

If the sequence $\{\tau_n\}$ consists of two topologies $\mathscr{P}$ and $\mathscr{Q}$ only, and if the bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is pairwise extremally disconnected [14], then the space $(X, \{\tau_n\})$ is extremally disconnected.

4.2. Theorem. *The* (a)*topological space* $(X, \{\tau_n\})$ *is extremally disconnected iff for any* (τ_m)*open set* G *and* (τ_n)*closed set* C *with* $G \subset C$ *and* $m \neq n$*, there exist* $k, l \in N$ *with* $m \neq k$*,* $n \neq l$ *and* $k \neq l$ *such that for some* (τ_k)*closed set* F *and* (τ_l)*open set* U*, we have* $G \subset F \subset U \subset C$.

Proof. Suppose the space X is extremally disconnected. Let us consider a (τ_m)open set G and a (τ_n)closed set C with $G \subset C$ and $m \neq n$. Then $X - C$ is a (τ_n)open set and $G \cap (X - C) = \emptyset$. Therefore there exist k, l with $m \neq k$, $n \neq l$ and $k \neq l$ such that for some (τ_k)closed set F and (τ_l)closed set K, we have $G \subset F$, $X - C \subset K$, $F \cap K = \emptyset \Rightarrow G \subset F \subset U \subset C$ where $U = X - K \in \tau_l$.

To prove the converse, suppose $G \in \tau_m$ and $H \in \tau_n$ with $G \cap H = \emptyset$ and $m \neq n$. Then $G \subset X - H$ and $X - H$ is (τ_n)closed. Therefore there exist k, l with $m \neq k$, $n \neq l$ and $k \neq l$ such that for some (τ_k)closed set F and (τ_l)open set U, we have $G \subset F \subset U \subset X - H$. If $K = X - U$, then K is (τ_l)closed, $H \subset K$ and $F \cap K = \emptyset$. Thus the space is extremally disconnected. □

4.3. Theorem. *The space* $(X, \{\tau_n\})$ *is extremally disconnected if for every* m *and for every* $G \in \tau_m$*, we have* $(\tau_k)clG \in \tau_m$ *for all* $k \neq m$.

Proof. Let $G \in \tau_m$ and $H \in \tau_n$ with $G \cap H = \emptyset$ and $m \neq n$. Then $((\tau_n)\text{cl}G) \cap H = \emptyset$. Since $m \neq n$, by the given condition we have $(\tau_n)\text{cl}G \in \tau_m$. Therefore $((\tau_n)\text{cl}G) \cap ((\tau_m)\text{cl}H) = \emptyset$. □

Let $\mathscr{T}$ denote the smallest topology on X containing τ_n for all n.

4.4. Theorem. *Let* $(X, \{\tau_n\})$ *be extremally disconnected. Then for any* $G \in \tau_m$ *and* $H \in \tau_n$ *with* $m \neq n$ *and* $G \cap H = \emptyset$*, there exists a function* $f : X \to [0, 1]$ *such that*

(*i*) *f(G)={0}, f(H) ={1}*
(*ii*) f *is* $(\mathscr{T})$*continuous.*

Proof. Since $G \subset X - H$ and $C = X - H$ is (τ_n)closed, by Theorem 4.2, there exist $k(\frac{1}{2})$, $l(\frac{1}{2}) \in N$ with $k(\frac{1}{2}) \neq m$ and $l(\frac{1}{2}) \neq n$ such that for some $(\tau_{k(\frac{1}{2})})$closed set $F(\frac{1}{2})$ and $(\tau_{l(\frac{1}{2})})$open set $U(\frac{1}{2})$ we have $G \subset F(\frac{1}{2}) \subset U(\frac{1}{2}) \subset C$. Again applying Theorem 4.2 to the pair $(G, F(\frac{1}{2}))$ and $(U(\frac{1}{2}), C)$ there exist $k(\frac{1}{4})$, $l(\frac{1}{4})$, $k(\frac{3}{4})$, $l(\frac{3}{4}) \in N$ such that for some $(\tau_{k(\frac{1}{4})})$closed set $F(\frac{1}{4})$, $(\tau_{k(\frac{3}{4})})$closed set $F(\frac{3}{4})$, $(\tau_{l(\frac{1}{4})})$open set $U(\frac{1}{4})$ and $(\tau_{l(\frac{3}{4})})$open set $U(\frac{3}{4})$ we have $G \subset F(\frac{1}{4}) \subset U(\frac{1}{4}) \subset F(\frac{1}{2}) \subset U(\frac{1}{2}) \subset F(\frac{3}{4}) \subset U(\frac{3}{4}) \subset C$ and $k(\frac{1}{4}) \neq m$, $l(\frac{1}{4}) \neq k(\frac{1}{2})$, $k(\frac{3}{4}) \neq l(\frac{1}{2})$, $l(\frac{3}{4}) \neq n$. By repeating the process, we obtain $t \in D = \{\frac{i}{2^j} \mid 0 < i < 2^j, i, j \in N\}$, a $(\tau_{k(t)})$closed set $F(t)$ and a $(\tau_{l(t)})$open set $U(t)$ with $k(t), l(t) \in N$ such that if $s, t \in D$ with $s < t$, then $F(s) \subset U(s) \subset F(t) \subset U(t)$ and $k(t) \neq l(s)$. If we take $F(0) = \emptyset$ and $U(1) = X$, then the above relation is true when s, t coincide with 0 or 1. For $t \neq 0, 1$, we have, $G \subset F(t) \subset U(t) \subset C$. Now we define the function $f : X \to [0, 1]$ by $f(x) = sup\{t \mid x \notin U(t)\}$. Then $f(x) = 0$ for $x \in G$ and $f(x) = 1$ for $x \in H$. It is easy to verify that for $a \in (0, 1), \{x \in X \mid f(x) < a\} = \bigcup_{t<a} U(t)$ and $\{x \in X \mid f(x) > a\} = \bigcup_{t>a}(X - F(t))$. Since $U(t) \in \tau_{l(t)} \subset \mathscr{T}$ and $X - F(t) \in \tau_{k(t)} \subset \mathscr{T}$, it follows that f is $(\mathscr{T})$continuous. □

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References

[1] Bose, M. K. and Mukharjee, A. *On countable families of topologies on a set*, Novi Sad J. Math. **40** (2), 7–16, 2010.

[2] Bose, M. K. and Tiwari, R. (ω)*topological connectedness and hyperconnectedness*, Note Mat. **31**, 93–101, 2011.

[3] Bose, M. K. and Tiwari, R. *On* (ω)*topological spaces*, Riv. Mat. Univ. Parma. (7) **9**, 125–132, 2008.

[4] Bose, M. K. and Tiwari, R. *On increasing sequences of topologies on a set*, Riv. Mat. Univ. Parma. (7) **7**, 173–183, 2007.

[5] Datta, D. P. and Roy Choudhuri, A. *Scale-free analysis and the prime number theorem*, Fractals **18**, 171–184, 2010.

[6] Kelley, J. L. *General topology*, Van Nostrand, Princeton, 1955.

[7] Kelly, J. C. *Bitopological spaces*, Proc. Lond. Math. Soc. **13**, 71-89, 1963.

[8] Kovár, M. M. *A note on the comparison of topologies*, Int. J. Math. Math. Sci. **26**, 233–237, 2001.

[9] Kovár, M. M. *On* 3*-topological version of* θ*-regularity*, Int. J. Math. Math. Sci. **23**, 393–398, 2000.

[10] Levine, N. *Simple extensions of topologies*, Amer. Math. Monthly **71**, 22–25, 1964.

[11] Levine, N. *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70**, 36-41, 1963.

[12] Mathew P. M. *On hyperconnected spaces*, Indian J. Pure Appl. Math. **19**, 1180-1184, 1988.

[13] Raut, S. and Datta, D. P. *Non-archimedean scale invariance and cantor sets*, Fractals **18**, 111–118, 2010.

[14] Sarma, R. D. *A note on extremal disconnectedness in bitopological spaces*, Acta Math. Hungar. **120**, 141-146, 2008.

[15] Steen, L. A. and Seebach (Jr), J. A. *Counterexamples in Topology*, Holt, Rinehart, and Winston, New York, 1970.

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Atanassov's intuitionistic fuzzy grade of complete hypergroups of order less than or equal to 6

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Abstract

The length of the sequence of join spaces and Atanassov's intuitionistic fuzzy sets associated with a hypergroupoid H is called the intuitionistic fuzzy grade of H. In this paper, we consider the class of the complete hypergroups of order less than or equal to 6, determining their intuitionistic fuzzy grade.

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1. Introduction

The study of the connections between hyperstructures and fuzzy sets [31] (or Atanassov's intuitionistic fuzzy sets [1, 2]) opens a new field of research in fuzzy algebraic structures theory, theory initiated by Rosenfeld [28]: he showed that many results concerning groups may be extended in a natural way to fuzzy groups. The notion of fuzzy group has been generalized by Davvaz [19], introducing the concept of fuzzy subhypergroup of a hypergroup. Later on, this subject has been studied in depth also in connection with other structures, like rings [22], modules [20], n-ary hypergroups [21], complete hypergroups, etc. For example, Cristea and Darafsheh [16, 17], investigating a particular fuzzy subhypergroup of a complete hypergroup, have found a new decomposition of the group $\mathbb{Z}_n$, when $n \in \{p, p^2, pq\}$, for p and q prime numbers. The books [3, 10, 23, 30] are surveys of the theory of algebraic hyperstructures and their applications.

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Two fundamental relations between hyperstructures and fuzzy sets were considered by P.Corsini; he associated a join space with a fuzzy set [4], and then a fuzzy set with a hypergroupoid H [5]. These connections lead to a sequence of fuzzy sets and join spaces, which ends if two consecutive join spaces are isomorphic. The length of this sequence is called the fuzzy grade of the hypergroupoid H. Till now, one determined the fuzzy grade of the i.p.s. hypergroups of order less than or equal to 7 [6, 7], of the complete hypergroups or 1-hypergroups which are not complete [8, 14]. Moreover, several properties of the above sequence has been determined in the general case [29], and also for the direct product of two hypergroupoids [15]. Corsini et al. studied the same sequence associated with a hypergraph [11, 12], and with multivalued functions [13]. Cristea and Davvaz [18] extended the notion of fuzzy grade of a hypergroupoid to that of intuitionistic fuzzy grade.

The study of the fuzzy grade and intuitionistic fuzzy grade of remarkable classes of finite hypergroups helps us to identify some important properties which could be generalized for any finite hypergroup. For example, calculating intuitionistic fuzzy grade of the i.p.s. hypergroups of order 7, we noticed that, some times, the sequence of join spaces associated with an i.p.s. hypergroup is cyclic (see [24, 25]).

The study conducted in this note shows that the intuitionistic fuzzy grade of a complete hypergroup H of cardinality n depends, not only on the decomposition of n, as in the case of the fuzzy grade of a complete hypergroup, but also on the group used in the construction of H. We believe that the aspects treated in this particular case serve as a foundation, starting point for further research on the intuitionistic fuzzy grade of an arbitrary finite complete hypergroup.

Inspired and motivated by the above achievements, in this paper, we will construct the sequences of join spaces and Atanassov's intuitionistic fuzzy sets associated with the complete hypergroups of order less than or equal to 6. Our aim is to determine their intuitionistic fuzzy grades in order to make a comparison with their fuzzy grades determined by Cristea [14].

To do so, the paper is organized in the following way. In Section 2 we present some basic notions concerning hypergroups and a short description of the complete hypergroups. In Section 3 we present a brief introduction about the sequence of join spaces and Atanassov's intuitionistic fuzzy sets associated with a hypergroupoid. Section 4 includes the sequences of the intuitionistic fuzzy grades of the complete hypergroups of order less than or equal to 6. Finally, Section 5 concludes the paper, giving also some future lines of our research.

2. Preliminaries

In this paper, we adopt the terminology and notation used in [4, 5, 14, 18, 24, 25]. We consider $\langle H, \circ \rangle$ to be a hypergroupoid, where H denotes a non-empty set, $\mathcal{P}^*(H)$ stands for the set of all non-empty subsets of H and $\circ : H^2 \to \mathcal{P}^*(H)$ is a hyperoperation. The image of the pair $(x, y) \in H \times H$ is denoted by $x \circ y$. If A and B are nonempty subsets of H, then $A \circ B = \bigcup\limits_{\substack{a \in A \\ b \in B}} a \circ b$.

For the sake of convenience and completeness of our presentation, we recall some basic definitions and properties concerning hypergroups. More details on this argument can be found in the books [3, 10].

2.1. Definition. A *hypergroup* is a hypergroupoid $\langle H, \circ \rangle$ which satisfies the following conditions:

(i) For any $(a, b, c) \in H^3, (a \circ b) \circ c = a \circ (b \circ c)$ (the *associativity*),

(*ii*) For any $a \in H, H \circ a = a \circ H = H$ (the *reproducibility*).
If, for any $(x, y) \in H^2, x \circ y = H$, then the hypergroup H is called *total hypergroup*.

For each pair $(a, b) \in H^2$, we denote: $a/b = \{x \in H \mid a \in x \circ b\}$ and $b\backslash a = \{y \in H \mid a \in b \circ y\}$.

2.2. Definition. A commutative hypergroup $\langle H, \circ \rangle$ is called a *join space* if, for any four elements $a, b, c, d \in H$, such that $a/b \cap c/d \neq \emptyset$, it follows that $a \circ d \cap b \circ c \neq \emptyset$.

The notion of join space, introduced by Prenowitz, was used by Prenowitz and Jantosciak [27] for the reconstruction, from an algebraic point of view, of several branches of geometry: the projective, the descriptive and the spherical geometry.

2.3. Definition. Let $\langle H, \circ \rangle$ and $\langle H', \circ' \rangle$ be two hypergroups and $f : H \to H'$ an application from H in H'. We say that

(*i*) f is a *homomorphism* if, for all $(x, y) \in H^2$, $f(x \circ y) \subseteq f(x) \circ' f(y)$.
(*ii*) f is a *good homomorphism* if, for all $(x, y) \in H^2$, $f(x \circ y) = f(x) \circ' f(y)$.

We say that the two hypergroups are *isomorphic*, and we write $H \simeq H'$, if there is a good homomorphism between them which is also a bijection.

The relation β on a hypergroupoid $\langle H, \circ \rangle$ is defined as follows:

$$a\beta b \Longleftrightarrow \exists n \in \mathbb{N}^*, \exists (x_1, x_2, \ldots, x_n) \in H^n : a \in \prod_{i=1}^{n} x_i \ni b.$$

Notice that β is a reflexive and a symmetric relation on H, but generally, not a transitive one. Let us denote by β^* the transitive closure of β. It is well known that, if H is a hypergroup, then $\beta^* = \beta$ and H/β is a group[3].

One of the most important notions in hypergroup theory is that of the heart of a hypergroup H. Studying its properties one determines completely the structure of the hypergroup H.

2.4. Definition. The *heart* of a hypergroup H is $\omega_H = \{x \in H \mid \varphi_H(x) = 1\}$, where $\varphi_H : H \longrightarrow H/\beta$ is the canonical projection and 1 is the identity of the group H/β.

2.5. Definition. A hypergroup H is called 1-*hypergroup* if the cardinality of its heart equals 1.

2.6. Definition. Let $\langle H, \circ \rangle$ be a hypergroup and A be a non-empty subset of H. We say that A is a *complete part* of H if the following implication holds:

$$\forall n \in \mathbb{N}^*, \forall (x_1, x_2, \ldots, x_n) \in H^n, \prod_{i=1}^{n} x_i \cap A \neq \emptyset \Longrightarrow \prod_{i=1}^{n} x_i \subset A.$$

The *complete closure* of A in H is the intersection of all the complete parts of H, containing A; it is denoted by $\mathcal{C}(A)$.

2.7. Definition. A hypergroup $\langle H, \circ \rangle$ is called *complete* if, for any $(x, y) \in H^2$, $\mathcal{C}(x \circ y) = x \circ y$.

The following result concerning the complete hypergroups will be used in the sequel.

2.8. Theorem. *Any complete hypergroup may be constructed as the union* $H = \bigcup_{g \in G} A_g$, *where:*

(*i*) *G is a group.*
(*ii*) *The family $\{A_g \mid g \in G\}$ is a partition of G.*
(*iii*) *If $(a, b) \in A_{g_1} \times A_{g_2}$, then $a \circ b = A_{g_1 g_2}$.*

For a complete hypergroup H, it is known that $\omega_H = A_e$, where e is the identity of the group G, and it coincides with the set of identities of H. Therefore, by the above representation, we say that any complete hypergroup of order n is characterized by an m-tuple denoted $[k_1, k_2, \ldots, k_m]$, where $m = |G|$, $2 \leq m \leq n-1$, $G = \{g_1, g_2, \ldots, g_m\}$ and, for any $i \in \{1, 2, \ldots, m\}, k_i = |A_{g_i}|$. With other words, for determining all the non-isomorphic complete hypergroups of order n, it is enough to know the structure of the non-isomorphic groups of order m, $2 \leq m \leq n-1$, and all the m-decompositions of n, i.e. all the ordered systems of natural numbers $[k_1, k_2, \ldots, k_m]$ such that $k_i \geq 1$, $k_1 + k_2 + \ldots + k_m = n$ and $k_2 \leq k_3 \leq \ldots \leq k_m$, for $1 \leq i \leq m$.(see [14])

3. Intuitionistic fuzzy grade of hypergroups

In this section, first we recall the construction of the sequence of join spaces and Atanassov's intuitionistic fuzzy sets associated with a hypergroupoid H, and then the formulas for the membership functions associated with a complete hypergroup. In this paper H denotes a finite hypergroupoid.

For simplicity, we denote an Atanassov's intuitionistic fuzzy set (by short intuitionistic fuzzy set) $A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}$, where, for any $x \in X$, the *degree of membership* of x (namely $\mu_A(x)$) and the *degree of non-membership* of x (namely $\lambda_A(x)$) verify the relation $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$, by $A = (\mu, \lambda)$.

For any hypergroupoid $\langle H, \circ \rangle$, Cristea and Davvaz [18] defined an intuitionistic fuzzy set $A = (\bar{\mu}, \bar{\lambda})$ in the following way: for any $u \in H$, one considers:

$$(3.1) \qquad \bar{\mu}(u) = \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{n^2}, \quad \bar{\lambda}(u) = \frac{\displaystyle\sum_{(x,y)\in \bar{Q}(u)} \frac{1}{|x \circ y|}}{n^2},$$

where $Q(u) = \{(a, b) \in H^2 \mid u \in a \circ b\}, \bar{Q}(u) = \{(a, b) \in H^2 \mid u \notin a \circ b\}$. If $Q(u) = \emptyset$, we set $\bar{\mu}(u) = 0$ and similarly, if $\bar{Q}(u) = \emptyset$ we set $\bar{\lambda}(u) = 0$. It is clear that, for any $u \in H$, $0 \leq \bar{\mu}(u) + \bar{\lambda}(u) \leq 1$.

Now, let $A = (\bar{\mu}, \bar{\lambda})$ be an intuitionistic fuzzy set on H. One may associate with H two join spaces $\langle {}_0H, \circ_{\bar{\mu}\wedge\bar{\lambda}} \rangle$ and $\langle {}^0H, \circ_{\bar{\mu}\vee\bar{\lambda}} \rangle$, where, for any fuzzy set α on H, the hyperproduct "$\circ_\alpha$", introduced by Corsini [4], is defined as

$$(3.2) \qquad x \circ_\alpha y = \{u \in H \mid \alpha(x) \wedge \alpha(y) \leq \alpha(u) \leq \alpha(x) \vee \alpha(y)\}.$$

Using repeatedly the formulas (3.1) and (3.2), one obtains two sequences of join spaces and intuitionistic fuzzy sets associated with H, denoted by $({}_iH = \langle {}_iH, \circ_{\bar{\mu}_i\wedge\bar{\lambda}_i} \rangle; \bar{A}_i = (\bar{\mu}_i, \bar{\lambda}_i))_{i\geq 0}$ and $({}^iH = \langle {}^iH, \circ_{\bar{\mu}_i\vee\bar{\lambda}_i} \rangle; \bar{A}_i = (\bar{\mu}_i, \bar{\lambda}_i))_{i\geq 0}$.

The lengths of these sequences are called the lower, and respectively, the upper intuitionistic fuzzy grade of H, more exactly:

3.1. Definition. (see [18]) A set H endowed with an intuitionistic fuzzy set $A = (\mu, \lambda)$ has the *lower (upper) intuitionistic fuzzy grade* m, $m \in \mathbb{N}^*$, and we write $l.i.f.g.(H) = m$ (resp. $u.i.f.g.(H) = m$) if, for any i, $0 \leq i < m-1$, the join spaces $\langle {}_iH, \circ_{\bar{\mu}_i\wedge\bar{\lambda}_i} \rangle$ and $\langle {}_{i+1}H, \circ_{\bar{\mu}_{i+1}\wedge\bar{\lambda}_{i+1}} \rangle$ (resp. $\langle {}^iH, \circ_{\bar{\mu}_i\vee\bar{\lambda}_i} \rangle$ and $\langle {}^{i+1}H, \circ_{\bar{\mu}_{i+1}\vee\bar{\lambda}_{i+1}} \rangle$) associated with H are not isomorphic (where ${}_0H = \langle {}_0H, \circ_{\bar{\mu}\wedge\bar{\lambda}} \rangle$ and ${}^0H = \langle {}^0H, \circ_{\bar{\mu}\vee\bar{\lambda}} \rangle$) and for any s, $s \geq m$, ${}_sH$ is isomorphic with ${}_{m-1}H$ (resp. ${}^sH$ is isomorphic with ${}^{m-1}H$).

It is important to know that, if we start the construction of the above sequences with a hypergroupoid $\langle H, \circ \rangle$, and not with a set H endowed with an intuitionistic fuzzy set, then we obtain only one sequence of join spaces because, in this case, the join spaces $\langle {}_0H, \circ_{\bar{\mu}\wedge\bar{\lambda}} \rangle$ and $\langle {}^0H, \circ_{\bar{\mu}\vee\bar{\lambda}} \rangle$ are isomorphic (see [18]). In order to explain this situation, one introduces a new concept.

3.2. Definition. (see [18]) We say that a hypergroupoid H has the *intuitionistic fuzzy grade* $m, m \in \mathbb{N}^*$, and we write $i.f.g.(H) = m$, if $l.i.f.g.(H) = m$.

A natural question appears: When are these join spaces non-isomorphic? It is clear that it has to be answered for two consecutive join spaces in the built sequence, since in the case of isomorphism, the sequence ends. In order to solve this problem one introduces some notations. Let $({}_iH = \langle {}_iH, \circ_{\bar{\mu}_i \wedge \bar{\lambda}_i}\rangle; \bar{A}_i = (\bar{\mu}_i, \bar{\lambda}_i))_{i\geq 0}$ be the sequence of join spaces and intuitionistic fuzzy sets associated with a hypergroupoid H. Then, for any i, there are r, namely $r = r_i$, and a partition $\Pi = \{{}^iC_j\}_{j=1}^r$ of ${}_iH$ such that, for any $j \geq 1, x, y \in {}^iC_j \Longleftrightarrow \bar{\mu}_i(x) \wedge \bar{\lambda}_i(x) = \bar{\mu}_i(y) \wedge \bar{\lambda}_i(y)$. For $x \in H$, we denote $\lambda(x) = i_j$, when $x \in {}^iC_j$. On the set of the classes $\{{}^iC_j\}_{j=1}^r$ we define the following ordering relation:

$i_j < i_k$ if, for elements $x \in {}^iC_j$ and $y \in {}^iC_k$,

$$\bar{\mu}_i(x) \wedge \bar{\lambda}_i(x) < \bar{\mu}_i(y) \wedge \bar{\lambda}_i(y) \text{ (therefore } \lambda(x) < \lambda(y)).$$

With any ordered chain $({}^iC_{j_1}, {}^iC_{j_2}, \ldots, {}^iC_{j_r})$ one associates an ordered r-tuple of the type $(k_{j_1}, k_{j_2}, \ldots, k_{j_r})$, where $k_{j_l} = |{}^iC_{j_l}|$, for all $l, 1 \leq l \leq r$.

3.3. Theorem. *(see [9]) Let ${}_iH$ and ${}_{i+1}H$ be the join spaces associated with H determined by the membership functions $\bar{\mu}_i \wedge \bar{\lambda}_i$ and $\bar{\mu}_{i+1} \wedge \bar{\lambda}_{i+1}$, where ${}^iH = \bigcup_{l=1}^{r_1} C_l$, ${}^{i+1}H = \bigcup_{l=1}^{r_2} C'_l$ and $(k_1, k_2, \ldots, k_{r_1})$ is the r_1-tuple associated with ${}_iH$, $(k'_1, k'_2, \ldots, k'_{r_2})$ is the r_2-tuple associated with ${}_{i+1}H$. The join spaces ${}_iH$ and ${}_{i+1}H$ are isomorphic if and only if $r_1 = r_2$ and $(k_1, k_2, \ldots, k_{r_1}) = (k'_1, k'_2, \ldots, k'_{r_1})$ or $(k_1, k_2, \ldots, k_{r_1}) = (k'_{r_1}, k'_{r_1-1}, \ldots, k'_1)$.*

Now we recall the formulas for the membership functions $\bar{\mu}$ and $\bar{\lambda}$ associated with a complete hypergroup.

Let $H = \bigcup_{g\in G} A_g$ be a complete hypergroup of cardinality n. By Theorem 2.8, it is obvious that, for any $u \in H$, there exists a unique $g_u \in G$ such that $u \in A_{g_u}$. Moreover, we define on H the following equivalence $u \sim v \Longleftrightarrow \exists g \in G : u, v \in A_g$. Thereby one obtains that

$$(3.3) \qquad \bar{\mu}(u) = \frac{|Q(u)|}{|A_{g_u}|} \cdot \frac{1}{n^2}, \quad \bar{\lambda}(u) = \left(\sum_{v \notin \hat{u}} \frac{|Q(v)|}{|A_{g_v}|}\right) \cdot \frac{1}{n^2}.$$

We end this section with a useful result concerning the complete hypergroups generated by a group G isomorphic with the additive group $\mathbb{Z}_2$.

3.4. Proposition. *(see [18]) $H = \bigcup_{g\in G} A_g$ be a complete hypergroup of cardinality n. If the group G is isomorphic with the additive group $\mathbb{Z}_2$, then $i.f.g.(H) = 1$.*

4. Intuitionistic fuzzy grade of the complete hypergroups of order less than or equal to 6

Cristea [14] listed all the forty complete hypergroups of order less than or equal to 6, calculating their fuzzy grade. In this section we determine the intuitionistic fuzzy grade of them. When the group which generates the complete hypergroup is isomorphic with the additive group $\mathbb{Z}_2$, by Proposition 3.4 it follows that $i.f.g.(H) = 1$ and in this case we do not list the table of the complete hypergroups (the reader may see it in [14]).

4.1. Theorem. *Let H be a complete hypergroup of order $n \leq 6$.*

(*i*) *There are two non-isomorphic complete hypergroups of order 3 having* $i.f.g.(H) = 1$.
(*ii*) *There are five non-isomorphic complete hypergroups of order* 4*: for three of them, one finds that* $i.f.g.(H) = 1$*, and for other two that* $i.f.g.(H) = 2$.
(*iii*) *There are twelve non-isomorphic complete hypergroups of order* 5*: nine of them have* $i.f.g.(H) = 1$*, and three of them have* $i.f.g.(H) = 3$.
(*iv*) *There are twenty one non-isomorphic complete hypergroups of order* 6*: sixteen of them with* $i.f.g.(H) = 1$*, three of them with* $i.f.g.(H) = 2$ *and for two of them one finds that* $i.f.g.(H) = 3$.

Proof. We will denote, in the following tables, for any $s \in \{1, 2, \dots, 5\}, B_s = H \setminus \{a_s\}$ and $B_0 = H \setminus \{e\}$. Let H be a complete hypergroup of order $n \leq 6$, denoted by $H = \{e, a_1, \dots, a_n\}$, with $3 \leq n \leq 5$, that is $H = \bigcup_{g \in G} A_g$.

(*i*) If the hypergroup H is of order 3, then it is obvious that $G \simeq (\mathbb{Z}_2, +)$, so there are only two complete hypergroups with the associated 2-tuple of the form $[1, 2]$ or $[2, 1]$. Thus, by Proposition 3.4, it follows that $i.f.g.(H_i) = 1$, for $i \in \{1, 2\}$.

(*ii*) Let us suppose H of order 4.

(*a*) Setting $G \simeq (\mathbb{Z}_2, +)$, we obtain three complete hypergroups H_3, H_4, H_5, and by Proposition 3.4, it follows that $i.f.g.(H_i) = 1$, for $i \in \{3, 4, 5\}$.

(*b*) Setting $G \simeq (\mathbb{Z}_3, +)$, we distinguish two hypergroups, denoted by H_6, H_7.

(b_1) For H_6 represented here bellow

$\circ$	e	a_1	a_2	a_3
e	e	a_1	A_2	A_2
a_1		A_2	e	e
a_2			a_1	a_1
a_3				a_1

where $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2, a_3\}$, we calculate that

$$\begin{aligned} &\bar{\mu}(e) = 10/32, \quad \bar{\mu}(a_1) = 12/32, \quad \bar{\mu}(a_2) = \bar{\mu}(a_3) = 5/32, \\ &\bar{\lambda}(e) = 17/32, \quad \bar{\lambda}(a_1) = 15/32, \quad \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = 22/32. \end{aligned}$$

Therefore the associated join space ${}_0(H_6)$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3
e	e	$\{e, a_1\}$	B_1	B_1
a_1		a_1	H	H
a_2			A_2	A_2
a_3				A_2

and thus we obtain that

$$\begin{aligned} &\bar{\mu}_1(e) = \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = 13/48, \quad \bar{\mu}_1(a_1) = 9/48, \\ &\bar{\lambda}_1(e) = \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = 9/48, \quad \bar{\lambda}_1(a_1) = 13/48. \end{aligned}$$

Therefore the associated join space ${}_1(H_6)$ is the total hypergroup. Then, for any $r \geq 2$, ${}_r(H_6) \simeq {}_1(H_6)$ and thereby $i.f.g.(H_6) = 2$.

(b_2) Taking the complete hypergroup H_7

$\circ$	e	a_1	a_2	a_3
e	A_0	A_0	a_2	a_3
a_1		A_0	a_2	a_3
a_2			a_3	A_0
a_3				a_2

with $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3\}$, one gets that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = 3/16, \quad \bar{\mu}(a_2) = \bar{\mu}(a_3) = 5/16,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = 10/16, \quad \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = 8/16.$$

Therefore the associated join space ${}_0(H_7)$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3
e	A_0	A_0	H	H
a_1		A_0	H	H
a_2			$\{a_2, a_3\}$	$\{a_2, a_3\}$
a_3				$\{a_2, a_3\}$

and

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = 4/16,$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = 2/16.$$

Therefore the associated join space ${}_1(H_7)$ is the total hypergroup and, for any $r \geq 2$, ${}_r(H_7) \simeq {}_1(H_7)$, so $i.f.g.(H_7) = 2$.

(iii) We consider now the complete hypergroups of order 5.

(a) There are five complete 1-hypergroups of order 5, denoted here by $H_8, \ldots, H_{12}$.

(a_1) For the first one H_8 generated by the group $G \simeq (\mathbb{Z}_2, +)$, by Proposition 3.4, it follows that $i.f.g.(H_8) = 1$.

(a_2) For H_9 represented by the table

$\circ$	e	a_1	a_2	a_3	a_4
e	e	a_1	A_2	A_2	A_2
a_1		A_2	e	e	e
a_2			a_1	a_1	a_1
a_3				a_1	a_1
a_4					a_1

with $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2, a_3, a_4\}$, we find that

$$\bar{\mu}(e) = 21/75, \quad \bar{\mu}(a_1) = 33/75, \quad \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = 7/75,$$
$$\bar{\lambda}(e) = 40/75, \quad \bar{\lambda}(a_1) = 28/75, \quad \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 54/75.$$

Therefore the associated join space ${}_0(H_9)$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4
e	e	$\{e, a_1\}$	B_1	B_1	B_1
a_1		a_1	H	H	H
a_2			A_2	A_2	A_2
a_3				A_2	A_2
a_4					A_2

then

$$\bar{\mu}_1(e) = 47/250, \quad \bar{\mu}_1(a_1) = 32/250, \quad \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = 57/250,$$
$$\bar{\lambda}_1(e) = 40/250, \quad \bar{\lambda}_1(a_1) = 55/250, \quad \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = 30/250.$$

Then, for any $r \geq 1$, ${}_r(H_9) \simeq {}_0(H_9)$ and therefore $i.f.g.(H_9) = 1$.

(a_3) Set the complete hypergroup H_{10} as

$\circ$	e	a_1	a_2	a_3	a_4
e	e	A_1	A_1	A_2	A_2
a_1		A_2	A_2	e	e
a_2			A_2	e	e
a_3				A_1	A_1
a_4					A_1

where $A_0=\{e\}, A_1=\{a_1,a_2\}, A_2=\{a_3,a_4\}$. Then, for any $i\in\{1,2,3,4\}$, we calculate that $\bar{\mu}(e)=9/25,\quad \bar{\mu}(a_i)=4/25,\quad \bar{\lambda}(e)=8/25,\quad \bar{\lambda}(a_i)=13/25$.

It result the following join space ${}_0(H_{10})$

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4
e	e	H	H	H	H
a_1		B_0	B_0	B_0	B_0
a_2			B_0	B_0	B_0
a_3				B_0	B_0
a_4					B_0

and, for any $i\in\{1,2,3,4\}$, $\bar{\mu}_1(e)=13/125,\ \ \bar{\mu}_1(a_i)=28/125,\ \ \bar{\lambda}_1(e)=20/125,\ \ \bar{\lambda}_1(a_i)=5/125$. Therefore, we have, for any $r\geq 1$, ${}_r(H_{10})\simeq{}_0(H_{10})$ and $i.f.g.(H_{10})=1$.

(a_4) Let us consider H_{11} as

$\circ$	e	a_1	a_2	a_3	a_4
e	e	a_1	a_2	A_3	A_3
a_1		a_2	A_3	e	e
a_2			e	a_1	a_1
a_3				a_2	a_2
a_4					a_2

where $A_0=\{e\}, A_1=\{a_1\}, A_2=\{a_2\}, A_3=\{a_3,a_4\}$ (i.e. the 4-tuple associated with H is $[1,1,1,2]$) and $G\simeq(\mathbb{Z}_4,+)$, for which we calculate

$$\begin{array}{lll} \bar{\mu}(e)=\bar{\mu}(a_1)=6/25, & \bar{\mu}(a_2)=7/25, & \bar{\mu}(a_3)=\bar{\mu}(a_4)=3/25,\\ \bar{\lambda}(e)=\bar{\lambda}(a_1)=16/25, & \bar{\lambda}(a_2)=15/25, & \bar{\lambda}(a_3)=\bar{\lambda}(a_4)=19/25. \end{array}$$

Therefore the associated join space ${}_0(H_{11})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4
e	$\{e,a_1\}$	$\{e,a_1\}$	$\{e,a_1,a_2\}$	B_2	B_2
a_1		$\{e,a_1\}$	$\{e,a_1,a_2\}$	B_2	B_2
a_2			a_2	H	H
a_3				A_3	A_3
a_4					A_3

then

$$\begin{array}{lll} \bar{\mu}_1(e)=\bar{\mu}_1(a_1)=92/375, & \bar{\mu}_1(a_2)=47/375, & \bar{\mu}_1(a_3)=\bar{\mu}_1(a_4)=72/375,\\ \bar{\lambda}_1(e)=\bar{\lambda}_1(a_1)=45/375, & \bar{\lambda}_1(a_2)=90/375, & \bar{\lambda}_1(a_3)=\bar{\lambda}_1(a_4)=65/375. \end{array}$$

Therefore the associated join space ${}_1(H_{11})$ is as follows:

$\circ_{\bar{\mu}_1\wedge\bar{\lambda}_1}$	e	a_1	a_2	a_3	a_4
e	$\{e,a_1\}$	$\{e,a_1\}$	$\{e,a_1,a_2\}$	H	H
a_1		$\{e,a_1\}$	$\{e,a_1,a_2\}$	H	H
a_2			a_2	$\{a_2,a_3,a_4\}$	$\{a_2,a_3,a_4\}$
a_3				A_3	A_3
a_4					A_3

for which we find that

$$\begin{array}{ll} \bar{\mu}_2(e)=\bar{\mu}_2(a_1)=\bar{\mu}_2(a_3)=\bar{\mu}_2(a_4)=74/375, & \bar{\mu}_2(a_2)=79/375,\\ \bar{\lambda}_2(e)=\bar{\lambda}_2(a_1)=\bar{\lambda}_2(a_3)=\bar{\lambda}_2(a_4)=65/375, & \bar{\lambda}_2(a_2)=60/375. \end{array}$$

Therefore the associated join space ${}_2(H_{11})$ is as follows:

$\circ_{\bar{\mu}_2\wedge\bar{\lambda}_2}$	e	a_1	a_2	a_3	a_4
e	B_2	B_2	H	B_2	B_2
a_1		B_2	H	B_2	B_2
a_2			a_2	H	H
a_3				B_2	B_2
a_4					B_2

then

$$\bar{\mu}_3(e) = \bar{\mu}_3(a_1) = \bar{\mu}_3(a_3) = \bar{\mu}_3(a_4) \neq \bar{\mu}_3(a_2),$$
$$\bar{\lambda}_3(e) = \bar{\lambda}_3(a_1) = \bar{\lambda}_3(a_3) = \bar{\lambda}_3(a_4) \neq \bar{\lambda}_3(a_2).$$

Then, for any $r \geq 3$, ${}_r(H_{11}) \simeq {}_2(H_{11})$ and therefore $i.f.g.(H_{11}) = 3$.

(a_5) For the same 4-tuple $[1,1,1,2]$ associated with H, i.e. $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3, a_4\}$, but with $G \simeq (K, \cdot)$ the Klein four-group, it results the following complete hypergroup H_{12}

$\circ$	e	a_1	a_2	a_3	a_4
e	e	a_1	a_2	A_3	A_3
a_1		e	A_3	a_2	a_2
a_2			e	a_1	a_1
a_3				e	e
a_4					e

with

$$\bar{\mu}(e) = 7/25, \quad \bar{\mu}(a_1) = \bar{\mu}(a_2) = 6/25, \quad \bar{\mu}(a_3) = \bar{\mu}(a_4) = 3/25,$$
$$\bar{\lambda}(e) = 15/25, \quad \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 16/25, \quad \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 19/25.$$

Therefore the associated join space ${}_0(H_{12})$ is isomorphic with ${}_0(H_{11})$ and thereby we have that $i.f.g.(H_{12}) = 3$.

(b)The following complete hypergroups, denoted by $H_{13}, \ldots, H_{19}$, are not 1-hypergroups.

(b_1) There exist three complete hypergroups of order 5 (which are not 1-hypergroups) such that $G \simeq (\mathbb{Z}_2, +)$, (corresponding to the 2-tuples $[2,3]$, $[3,2]$, and $[4,1]$); for each of them we obtain, by Proposition 3.4, that $i.f.g.(H_i) = 1$, with $i \in \{13, 14, 15\}$.

(b_2) Let us consider H_{16} as the following complete hypergroup

$\circ$	e	a_1	a_2	a_3	a_4
e	A_0	A_0	A_0	a_3	a_4
a_1		A_0	A_0	a_3	a_4
a_2			A_0	a_3	a_4
a_3				a_4	A_0
a_4					a_3

where $A_0 = \{e, a_1, a_2\}, A_1 = \{a_3\}, A_2 = \{a_4\}$, for which we find that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = 11/75, \quad \bar{\mu}(a_3) = \bar{\mu}(a_4) = 21/75,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 42/75, \quad \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 32/75.$$

Therefore the associated join space ${}_0(H_{16})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4
e	A_0	A_0	A_0	H	H
a_1		A_0	A_0	H	H
a_2			A_0	H	H
a_3				$\{a_3, a_4\}$	$\{a_3, a_4\}$
a_4					$\{a_3, a_4\}$

and

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) \neq \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4),$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) \neq \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4).$$

Then, for any $r \geq 1$, ${}_r(H_{16}) \simeq {}_0(H_{16})$ and therefore $i.f.g.(H_{16}) = 1$.

(b_3) There exist two complete hypergroups H_{17} and H_{18} of order 5 generated by a group of order 4 and characterized by the 4-tuple $[2, 1, 1, 1]$. Setting $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3\}, A_3 = \{a_4\}$, if $G \simeq (\mathbb{Z}_4, +)$, then the hypergroup H_{17} is the following one

$\circ$	e	a_1	a_2	a_3	a_4
e	A_0	A_0	a_2	a_3	a_4
a_1		A_0	a_2	a_3	a_4
a_2			a_3	a_4	A_0
a_3				A_0	a_2
a_4					a_3

and if $G \simeq (K, \cdot)$ the Klein four-group, then the hypergroup H_{18} is represented by the table

$\circ$	e	a_1	a_2	a_3	a_4
e	A_0	A_0	a_2	a_3	a_4
a_1		A_0	a_2	a_3	a_4
a_2			A_0	a_4	a_3
a_3				A_0	a_2
a_4					A_0

In both cases one finds that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = 7/50, \quad \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = 12/50,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = 36/50, \quad \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 31/50.$$

Therefore the associated join space ${}_0(H_{17}) ={}_0 (H_{18})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4
e	A_0	A_0	H	H	H
a_1		A_0	H	H	H
a_2			$\{a_2, a_3, a_4\}$	$\{a_2, a_3, a_4\}$	$\{a_2, a_3, a_4\}$
a_3				$\{a_2, a_3, a_4\}$	$\{a_2, a_3, a_4\}$
a_4					$\{a_2, a_3, a_4\}$

and then

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) \neq \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4),$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) \neq \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4).$$

It follows that, for any $r \geq 1$, ${}_r(H_i) \simeq {}_0(H_i)$, with $i \in \{17, 18\}$, and therefore $i.f.g.(H_{17}) = i.f.g.(H_{18}) = 1$.

(b_4) For H_{19}

$\circ$	e	a_1	a_2	a_3	a_4
e	A_0	A_0	a_2	A_2	A_2
a_1		A_0	a_2	A_2	A_2
a_2			A_2	A_0	A_0
a_3				a_2	a_2
a_4					a_2

where $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3, a_4\}$, one finds the following membership functions

$$\bar{\mu}(e) = \bar{\mu}(a_1) = 8/50, \quad \bar{\mu}(a_2) = 16/50, \quad \bar{\mu}(a_3) = \bar{\mu}(a_4) = 9/50,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = 25/50, \quad \bar{\lambda}(a_2) = 17/50, \quad \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 24/50.$$

Therefore the associated join space ${}_0(H_{19})$ is as follows:

$\circ_{\bar\mu\wedge\bar\lambda}$	e	a_1	a_2	a_3	a_4
e	A_0	A_0	H	B_2	B_2
a_1		A_0	H	B_2	B_2
a_2			a_2	$\{a_2, a_3, a_4\}$	$\{a_2, a_3, a_4\}$
a_3				A_2	A_2
a_4					A_2

and

$$\begin{array}{lll} \bar\mu_1(e) = \bar\mu_1(a_1) = 72/375, & \bar\mu_1(a_2) = 47/375, & \bar\mu_1(a_3) = \bar\mu_1(a_4) = 92/375, \\ \bar\lambda_1(e) = \bar\lambda_1(a_1) = 65/375, & \bar\lambda_1(a_2) = 90/375, & \bar\lambda_1(a_3) = \bar\lambda_1(a_4) = 45/375. \end{array}$$

It is clear that the associated join space ${}_1(H_{19})$ is isomorphic with ${}_1(H_{11})$ and thus we obtain that $i.f.g.(H_{19}) = 3$.

(iv) Now we study the complete hypergroups of order 6. We denote the twenty one non-isomorphic complete hypergroups of order 6 by $H_{20}, H_{21}, \ldots, H_{40}$.

There are sixteen complete hypergroups of order 6 with the intuitionistic fuzzy grade equal to 1, listed in the sequel.

(a_1) Let us consider the complete hypergroup H_{20}

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	A_3	A_3	A_3
a_1		a_2	A_3	e	e	e
a_2			e	a_1	a_1	a_1
a_3				a_2	a_2	a_2
a_4					a_2	a_2
a_5						a_2

where $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3, a_4, a_5\}$ and $G \simeq (\mathbb{Z}_4, +)$. In particular, H_{20} is an 1-hypergroup. Then

$$\begin{array}{lll} \bar\mu(e) = \bar\mu(a_1) = 24/108, & \bar\mu(a_2) = 36/108, & \bar\mu(a_3) = \bar\mu(a_4) = \bar\mu(a_5) = 8/108, \\ \bar\lambda(e) = \bar\lambda(a_1) = 68/108, & \bar\lambda(a_2) = 56/108, & \bar\lambda(a_3) = \bar\lambda(a_4) = \bar\lambda(a_5) = 84/108. \end{array}$$

Therefore the associated join space ${}_0(H_{20})$ is as follows:

$\circ_{\bar\mu\wedge\bar\lambda}$	e	a_1	a_2	a_3	a_4	a_5
e	$\{e, a_1\}$	$\{e, a_1\}$	$\{e, a_1, a_2\}$	B_2	B_2	B_2
a_1		$\{e, a_1\}$	$\{e, a_1, a_2\}$	B_2	B_2	B_2
a_2			a_2	H	H	H
a_3				A_3	A_3	A_3
a_4					A_3	A_3
a_5						A_3

We obtain that

$$\begin{array}{lll} \bar\mu_1(e) = \bar\mu_1(a_1) = 101/540, & \bar\mu_1(a_2) = 50/540, & \bar\mu_1(a_3) = \bar\mu_1(a_4) = \bar\mu_1(a_5) = 96/540, \\ \bar\lambda_1(e) = \bar\lambda_1(a_1) = 60/540, & \bar\lambda_1(a_2) = 111/540, & \bar\lambda_1(a_3) = \bar\lambda_1(a_4) = \bar\lambda_1(a_5) = 65/540. \end{array}$$

Then, for any $r \geq 1$, ${}_r(H_{20}) \simeq {}_0(H_{20})$ and therefore $i.f.g.(H_{20}) = 1$.

(a_2) Let us see the complete hypergroup H_{21}

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	A_3	A_3	A_3
a_1		e	A_3	a_2	a_2	a_2
a_2			e	a_1	a_1	a_1
a_3				e	e	e
a_4					e	e
a_5						e

with $G \simeq (K, \cdot)$ the Klein four-group, $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3, a_4, a_5\}$. H_{21} is an 1-hypergroup, too. Then

$$\bar{\mu}(e) = 36/108, \quad \bar{\mu}(a_1) = \bar{\mu}(a_2) = 24/108, \quad \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 8/108,$$
$$\bar{\lambda}(e) = 56/108, \quad \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 68/108, \quad \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 84/108.$$

It follows that the associated join space ${}_0(H_{21})$ is isomorphic to ${}_0(H_{20})$ and thereby we have that $i.f.g.(H_{21}) = 1$.

(a_3) Setting now $G \simeq (\mathbb{Z}_2, +)$, it results five non-isomorphic complete hypergroups $H_{22}, \ldots, H_{26}$ corresponding to the 2-tuples $[1,5]$, $[2,4]$, $[3,3]$, $[4,2]$, $[5,1]$. By Proposition 3.4, it follows immediately that $i.f.g.(H_i) = 1$, for $i \in \{22, \ldots, 26\}$.

(a_4) For the complete hypergroup H_{27}, which is also an 1-hypergroup,

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	e	a_5	A_1	A_1	A_1	A_1
a_1		A_1	e	e	e	e
a_2			a_5	a_5	a_5	a_5
a_3				a_5	a_5	a_5
a_4					a_5	a_5
a_5						a_5

where $A_0 = \{e\}, A_1 = \{a_1, a_2, a_3, a_4\}, A_2 = \{a_5\}$, we calculate that

$$\bar{\mu}(e) = 36/144, \quad \bar{\mu}(a_1) = \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = 9/144, \quad \bar{\mu}(a_5) = 72/144,$$
$$\bar{\lambda}(e) = 81/144, \quad \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = 108/144, \quad \bar{\lambda}(a_5) = 45/144.$$

Therefore the associated join space ${}_0(H_{27})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4	a_5
e	e	B_5	B_5	B_5	B_5	$\{e, a_5\}$
a_1		A_1	A_1	A_1	A_1	H
a_2			A_1	A_1	A_1	H
a_3				A_1	A_1	H
a_4					A_1	H
a_5						a_5

We find

$$\bar{\mu}_1(e) = 74/540, \quad \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = 104/540, \quad \bar{\mu}_1(a_5) = 50/540,$$
$$\bar{\lambda}_1(e) = 75/540, \quad \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = 45/540, \quad \bar{\lambda}_1(a_5) = 99/540.$$

Then, for any $r \geq 1$, ${}_r(H_{27}) \simeq {}_0(H_{27})$ and therefore $i.f.g.(H_{27}) = 1$.

(a_5) For the complete hypergroup H_{28} represented here bellow

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	e	A_1	A_1	A_2	A_2	A_2
a_1		A_2	A_2	e	e	e
a_2			A_2	e	e	e
a_3				A_1	A_1	A_1
a_4					A_1	A_1
a_5						A_1

with $A_0 = \{e\}, A_1 = \{a_1, a_2\}, A_2 = \{a_3, a_4, a_5\}$ (in particular, H_{28} is an 1-hypergroup), we obtain that

$$\bar{\mu}(e) = 78/216, \quad \bar{\mu}(a_1) = \bar{\mu}(a_2) = 39/216, \quad \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 20/216,$$
$$\bar{\lambda}(e) = 59/216, \quad \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 98/216, \quad \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 117/216.$$

Therefore the associated join space ${}_0(H_{28})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4	a_5
e	e	$\{e, a_1, a_2\}$	$\{e, a_1, a_2\}$	H	H	H
a_1		A_1	A_1	B_0	B_0	B_0
a_2			A_1	B_0	B_0	B_0
a_3				A_2	A_2	A_2
a_4					A_2	A_2
a_5						A_2

and

$$\bar{\mu}_1(e) = 50/540, \quad \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = 101/540, \quad \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) = 96/540,$$
$$\bar{\lambda}_1(e) = 111/540, \quad \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = 60/540, \quad \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) = 65/540.$$

Then, for any $r \geq 1$, ${}_r(H_{28}) \simeq {}_0(H_{28})$ and therefore $i.f.g.(H_{28}) = 1$.

(a_6) Taking the complete hypergroup H_{29}

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	a_2	A_2	A_2	A_2
a_1		A_0	a_2	A_2	A_2	A_2
a_2			A_2	A_0	A_0	A_0
a_3				a_2	a_2	a_2
a_4					a_2	a_2
a_5						a_2

with $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3, a_4, a_5\}$, we calculate that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = 15/108, \quad \bar{\mu}(a_2) = 39/108, \quad \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 13/108,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = 52/108, \quad \bar{\lambda}(a_2) = 28/108, \quad \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 54/108.$$

Therefore the associated join space ${}_0(H_{29})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	$\{e, a_1, a_2\}$	B_2	B_2	B_2
a_1		A_0	$\{e, a_1, a_2\}$	B_2	B_2	B_2
a_2			a_2	H	H	H
a_3				A_2	A_2	A_2
a_4					A_2	A_2
a_5						A_2

We obtain that

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) = 101/540, \quad \bar{\mu}_1(a_2) = 50/540, \quad \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) = 96/540,$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = 60/540, \quad \bar{\lambda}_1(a_2) = 111/540, \quad \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) = 65/540.$$

Then, for any $r \geq 1$, ${}_r(H_{29}) \simeq {}_0(H_{29})$ and therefore $i.f.g.(H_{29}) = 1$.

(a_7) If we take the complete hypergroup H_{30} as

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_1	A_1	A_2	A_2
a_1		A_0	A_1	A_1	A_2	A_2
a_2			A_2	A_2	A_0	A_0
a_3				A_2	A_0	A_0
a_4					A_1	A_1
a_5						A_1

with $A_0 = \{e, a_1\}, A_1 = \{a_2, a_3\}, A_2 = \{a_4, a_5\}$, then it results that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 6/36,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 12/36.$$

Therefore ${}_0(H_{30})$ is a total hypergroup. Then, for any $r \geq 1$, ${}_r(H_{30}) \simeq {}_0(H_{30})$ and therefore $i.f.g.(H_{30}) = 1$.

(a_8) Let us consider H_{31} given by the following table

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	a_3	A_2	A_2
a_1		A_0	A_0	a_3	A_2	A_2
a_2			A_0	a_3	A_2	A_2
a_3				A_2	A_0	A_0
a_4					a_3	a_3
a_5						a_3

with $A_0 = \{e, a_1, a_2\}, A_1 = \{a_3\}, A_2 = \{a_4, a_5\}$. We calculate that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = 26/216, \quad \bar{\mu}(a_3) = 60/216, \quad \bar{\mu}(a_4) = \bar{\mu}(a_5) = 39/216,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 99/216, \quad \bar{\lambda}(a_3) = 65/216, \quad \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 86/216.$$

Therefore the associated join space ${}_0(H_{31})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	H	B_3	B_3
a_1		A_0	A_0	H	B_3	B_3
a_2			A_0	H	B_3	B_3
a_3				a_3	$\{a_3, a_4, a_5\}$	$\{a_3, a_4, a_5\}$
a_4					A_2	A_2
a_5						A_2

Then

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = 96/540, \quad \bar{\mu}_1(a_3) = 50/540, \quad \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) = 101/540,$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = 65/540, \quad \bar{\lambda}_1(a_3) = 111/540, \quad \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) = 60/540.$$

Then, for any $r \geq 1$, ${}_r(H_{31}) \simeq {}_0(H_{31})$ and therefore $i.f.g.(H_{31}) = 1$.

(a_9) Let us consider the following complete hypergroup H_{32}

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	A_0	a_4	a_5
a_1		A_0	A_0	A_0	a_4	a_5
a_2			A_0	A_0	a_4	a_5
a_3				A_0	a_4	a_5
a_4					a_5	A_0
a_5						a_4

with $A_0 = \{e, a_1, a_2, a_3\}, A_1 = \{a_4\}, A_2 = \{a_5\}$. One gets that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = \bar{\mu}(a_3) = 9/72, \quad \bar{\mu}(a_4) = \bar{\mu}(a_5) = 18/72,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = 36/72, \quad \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 27/72.$$

Therefore the associated join space ${}_0(H_{32})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	A_0	H	H
a_1		A_0	A_0	A_0	H	H
a_2			A_0	A_0	H	H
a_3				A_0	H	H
a_4					$\{a_4, a_5\}$	$\{a_4, a_5\}$
a_5						$\{a_4, a_5\}$

with

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) \neq \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5),$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) \neq \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5).$$

Then, for any $r \geq 1$, ${}_r(H_{32}) \simeq {}_0(H_{32})$ and therefore $i.f.g.(H_{32}) = 1$.

(a_{10}) Let us consider H_{33} given by the following table

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	A_2	A_2	A_3	A_3
a_1		e	A_3	A_3	A_2	A_2
a_2			e	e	a_1	a_1
a_3				e	a_1	a_1
a_4					e	e
a_5						e

with $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2, a_3\}, A_3 = \{a_4, a_5\}$ and $G \simeq (K, \cdot)$ the Klein four-group. It is obvious that H_{33} is an 1-hypergroup. It results that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = 10/36, \quad \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 4/36,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = 18/36, \quad \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 24/36.$$

Therefore the associated join space ${}_0(H_{33})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4	a_5
e	$\{e, a_1\}$	$\{e, a_1\}$	H	H	H	H
a_1		$\{e, a_1\}$	H	H	H	H
a_2			$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$
a_3				$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$
a_4					$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$
a_5						$\{a_2, a_3, a_4, a_5\}$

and

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) \neq \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5),$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) \neq \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5).$$

Then, for any $r \geq 1$, ${}_r(H_{33}) \simeq {}_0(H_{33})$ and therefore $i.f.g.(H_{33}) = 1$.

(a_{11}) Let us consider the following complete hypergroup H_{34}

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	a_2	a_3	A_3	A_3
a_1		A_0	a_2	a_3	A_3	A_3
a_2			A_0	A_3	a_3	a_3
a_3				A_0	a_2	a_2
a_4					A_0	A_0
a_5						A_0

with $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3\}, A_3 = \{a_4, a_5\}$ and $G \simeq (K, \cdot)$ the Klein four-group. We calculate that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 5/36, \quad \bar{\mu}(a_2) = \bar{\mu}(a_3) = 8/36,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 21/36, \quad \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = 18/36.$$

Therefore the associated join space ${}_0(H_{34})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4	a_5
e	$\{e, a_1, a_4, a_5\}$	$\{e, a_1, a_4, a_5\}$	H	H	$\{e, a_1, a_4, a_5\}$	$\{e, a_1, a_4, a_5\}$
a_1		$\{e, a_1, a_4, a_5\}$	H	H	$\{e, a_1, a_4, a_5\}$	$\{e, a_1, a_4, a_5\}$
a_2			$\{a_2, a_3\}$	$\{a_2, a_3\}$	H	H
a_3				$\{a_2, a_3\}$	H	H
a_4					$\{e, a_1, a_4, a_5\}$	$\{e, a_1, a_4, a_5\}$
a_5						$\{e, a_1, a_4, a_5\}$

then

$$\bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) \neq \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3),$$
$$\bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) \neq \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3).$$

Then, for any $r \geq 1$, ${}_r(H_{34}) \simeq {}_0(H_{34})$ and therefore $i.f.g.(H_{34}) = 1$.

(a_{12}) For the complete hypergroup H_{35}

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	a_2	a_3	a_4	a_5
a_1		A_0	a_2	a_3	a_4	a_5
a_2			a_3	a_4	a_5	A_0
a_3				a_5	A_0	a_2
a_4					a_2	a_3
a_5						a_4

with $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3\}, A_3 = \{a_4\}, A_4 = \{a_5\}$, we calculate that

$$\bar{\mu}(e) = \bar{\mu}(a_1) = 4/36, \quad \bar{\mu}(a_2) = \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 7/36,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = 28/36, \quad \bar{\lambda}(a_2) = \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 25/36.$$

We notice that ${}_0(H_{35})$ is isomorphic to ${}_0(H_{33})$ and therefore, for any $r \geq 1$, ${}_r(H_{35}) \simeq {}_0(H_{35})$ and so $i.f.g.(H_{35}) = 1$.

Now we present the complete hypergroups of order 6 which have the intuitionistic fuzzy grade equal to 2.

(b_1) The complete hypergroup H_{36} is the following one

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	a_3	a_4	a_5
a_1		A_0	A_0	a_3	a_4	a_5
a_2			A_0	a_3	a_4	a_5
a_3				a_4	a_5	A_0
a_4					A_0	a_3
a_5						a_4

where $A_0 = \{e, a_1, a_2\}, A_1 = \{a_3\}, A_2 = \{a_4\}, A_3 = \{a_5\}$, and $G \simeq (\mathbb{Z}_4, +)$. Then

$$\begin{array}{ll} \bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = 4/36, & \bar{\mu}(a_3) = \bar{\mu}(a_4) = \bar{\mu}(a_5) = 8/36, \\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 24/36, & \bar{\lambda}(a_3) = \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 20/36. \end{array}$$

Therefore the associated join space $_0(H_{36})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	H	H	H
a_1		A_0	A_0	H	H	H
a_2			A_0	H	H	H
a_3				$\{a_3, a_4, a_5\}$	$\{a_3, a_4, a_5\}$	$\{a_3, a_4, a_5\}$
a_4					$\{a_3, a_4, a_5\}$	$\{a_3, a_4, a_5\}$
a_5						$\{a_3, a_4, a_5\}$

We find that

$$\begin{array}{l} \bar{\mu}_1(e) = \bar{\mu}_1(a_1) = \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = \bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) = 6/36, \\ \bar{\lambda}_1(e) = \bar{\lambda}_1(a_1) = \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) = 3/36. \end{array}$$

It follows that $_1(H_{36})$ is a total hypergroup. Then, for any $r \geq 2$, $_r(H_{36}) = {}_1(H_{36})$ and therefore $i.f.g.(H_{36}) = 2$.

(b_2) The complete hypergroup H_{37} has the following table

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	a_3	a_4	a_5
a_1		A_0	A_0	a_3	a_4	a_5
a_2			A_0	a_3	a_4	a_5
a_3				A_0	a_5	a_4
a_4					A_0	a_3
a_5						A_0

with $G \simeq (K, .)$ the Klein four-group, $A_0 = \{e, a_1, a_2\}, A_1 = \{a_3\}, A_2 = \{a_4\}, A_3 = \{a_5\}$. We obtain the same membership functions as in the previous case. So, $i.f.g.(H_{37}) = 2$.

(b_3) Taking the complete hypergroup H_{38} as the following 1-hypergroup

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	a_3	A_4	A_4
a_1		a_2	a_3	A_4	e	e
a_2			A_4	e	a_1	a_1
a_3				a_1	a_2	a_2
a_4					a_3	a_3
a_5						a_3

where $A_0 = \{e\}, A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3\}, A_4 = \{a_4, a_5\}$, then

$$\begin{array}{lll} \bar{\mu}(e) = \bar{\mu}(a_1) = \bar{\mu}(a_2) = 14/72, & \bar{\mu}(a_3) = 16/72, & \bar{\mu}(a_4) = \bar{\mu}(a_5) = 7/72, \\ \bar{\lambda}(e) = \bar{\lambda}(a_1) = \bar{\lambda}(a_2) = 51/72, & \bar{\lambda}(a_3) = 49/72, & \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 58/72. \end{array}$$

Therefore the associated join space ${}_0(H_{38})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4	a_5
e	$\{e,a_1,a_2\}$	$\{e,a_1,a_2\}$	$\{e,a_1,a_2\}$	$\{e,a_1,a_2,a_3\}$	B_3	B_3
a_1		$\{e,a_1,a_2\}$	$\{e,a_1,a_2\}$	$\{e,a_1,a_2,a_3\}$	B_3	B_3
a_2			$\{e,a_1,a_2\}$	$\{e,a_1,a_2,a_3\}$	B_3	B_3
a_3				a_3	H	H
a_4					A_4	A_4
a_5						A_4

and we obtain that

$$\begin{array}{lll}\bar{\mu}_1(e)=\bar{\mu}_1(a_1)=\bar{\mu}_1(a_2)=227/1080, & \bar{\mu}_1(a_3)=95/1080, & \bar{\mu}_1(a_4)=\bar{\mu}_1(a_5)=152/1080,\\ \bar{\lambda}_1(e)=\bar{\lambda}_1(a_1)=\bar{\lambda}_1(a_2)=90/1080, & \bar{\lambda}_1(a_3)=222/1080, & \bar{\lambda}_1(a_4)=\bar{\lambda}_1(a_5)=165/1080.\end{array}$$

Therefore the associated join space ${}_1(H_{38})$ is as follows:

$\circ_{\bar{\mu}_1\wedge\bar{\lambda}_1}$	e	a_1	a_2	a_3	a_4	a_5
e	$\{e,a_1,a_2\}$	$\{e,a_1,a_2\}$	$\{e,a_1,a_2\}$	$\{e,a_1,a_2,a_3\}$	H	H
a_1		$\{e,a_1,a_2\}$	$\{e,a_1,a_2\}$	$\{e,a_1,a_2,a_3\}$	H	H
a_2			$\{e,a_1,a_2\}$	$\{e,a_1,a_2,a_3\}$	H	H
a_3				a_3	$\{a_3,a_4,a_5\}$	$\{a_3,a_4,a_5\}$
a_4					A_4	A_4
a_5						A_4

for which we find

$$\begin{array}{lll}\bar{\mu}_2(e)=\bar{\mu}_2(a_1)=\bar{\mu}_2(a_2)=39/216, & \bar{\mu}_2(a_3)=35/216, & \bar{\mu}_2(a_4)=\bar{\mu}_2(a_5)=32/216,\\ \bar{\lambda}_2(e)=\bar{\lambda}_2(a_1)=\bar{\lambda}_2(a_2)=26/216, & \bar{\lambda}_2(a_3)=30/216, & \bar{\lambda}_2(a_4)=\bar{\lambda}_2(a_5)=33/216.\end{array}$$

Then, for any $r\geq 2$, ${}_r(H_{38})\simeq {}_1(H_{38})$ and therefore $i.f.g.(H_{38})=2$.

The last two complete hypergroups of order 6 have the intuitionistic fuzzy grade equal to 3.

(c_1) For the 1-hypergroup H_{39}

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	A_2	A_2	A_3	A_3
a_1		A_2	A_3	A_3	e	e
a_2			e	e	a_1	a_1
a_3				e	a_1	a_1
a_4					A_2	A_2
a_5						A_2

where $A_0=\{e\}, A_1=\{a_1\}, A_2=\{a_2,a_3\}, A_3=\{a_4,a_5\}$, and $G\simeq(\mathbb{Z}_4,+)$, we find that

$$\begin{array}{llll}\bar{\mu}(e)=18/72, & \bar{\mu}(a_1)=20/72, & \bar{\mu}(a_2)=\bar{\mu}(a_3)=9/72, & \bar{\mu}(a_4)=\bar{\mu}(a_5)=8/72,\\ \bar{\lambda}(e)=37/72, & \bar{\lambda}(a_1)=35/72, & \bar{\lambda}(a_2)=\bar{\lambda}(a_3)=46/72, & \bar{\lambda}(a_4)=\bar{\lambda}(a_5)=47/72.\end{array}$$

Therefore the associated join space ${}_0(H_{39})$ is as follows:

$\circ_{\bar{\mu}\wedge\bar{\lambda}}$	e	a_1	a_2	a_3	a_4	a_5
e	e	$\{e,a_1\}$	$\{e,a_2,a_3\}$	$\{e,a_2,a_3\}$	B_1	B_1
a_1		a_1	$\{e,a_1,a_2,a_3\}$	$\{e,a_1,a_2,a_3\}$	H	H
a_2			A_2	A_2	$\{a_2,a_3,a_4,a_5\}$	$\{a_2,a_3,a_4,a_5\}$
a_3				A_2	$\{a_2,a_3,a_4,a_5\}$	$\{a_2,a_3,a_4,a_5\}$
a_4					A_3	A_3
a_5						A_3

and

$$\bar{\mu}_1(e) = 87/540, \quad \bar{\mu}_1(a_1) = 55/540, \quad \bar{\mu}_1(a_2) = \bar{\mu}_1(a_3) = 117/540,$$
$$\bar{\lambda}_1(e) = 105/540, \quad \bar{\lambda}_1(a_1) = 137/540, \quad \bar{\lambda}_1(a_2) = \bar{\lambda}_1(a_3) = 75/540,$$
$$\bar{\mu}_1(a_4) = \bar{\mu}_1(a_5) = 82/540, \quad \bar{\lambda}_1(a_4) = \bar{\lambda}_1(a_5) = 110/540.$$

Therefore the associated join space ${}_1(H_{39})$ is as follows:

$\circ_{\bar{\mu}_1 \wedge \bar{\lambda}_1}$	e	a_1	a_2	a_3	a_4	a_5
e	e	H	B_1	B_1	$\{e, a_4, a_5\}$	$\{e, a_4, a_5\}$
a_1		a_1	$\{a_1, a_2, a_3\}$	$\{a_1, a_2, a_3\}$	B_0	B_0
a_2			A_2	A_2	$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$
a_3				A_2	$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$
a_4					A_3	A_3
a_5						A_3

and

$$\bar{\mu}_2(e) = \bar{\mu}_2(a_1) = 52/540, \quad \bar{\mu}_2(a_2) = \bar{\mu}_2(a_3) = \bar{\mu}_2(a_4) = \bar{\mu}_2(a_5) = 109/540,$$
$$\bar{\lambda}_2(e) = \bar{\lambda}_2(a_1) = 137/540, \quad \bar{\lambda}_2(a_2) = \bar{\lambda}_2(a_3) = \bar{\lambda}_2(a_4) = \bar{\lambda}_2(a_5) = 80/540.$$

Therefore the associated join space ${}_2(H_{39})$ is as follows:

$\circ_{\bar{\mu}_2 \wedge \bar{\lambda}_2}$	e	a_1	a_2	a_3	a_4	a_5
e	$\{e, a_1\}$	$\{e, a_1\}$	H	H	H	H
a_1		$\{e, a_1\}$	H	H	H	H
a_2			$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$
a_3				$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$
a_4					$\{a_2, a_3, a_4, a_5\}$	$\{a_2, a_3, a_4, a_5\}$
a_5						$\{a_2, a_3, a_4, a_5\}$

for which we calculate

$$\bar{\mu}_3(e) = \bar{\mu}_3(a_1) \neq \bar{\mu}_3(a_3) = \bar{\mu}_3(a_4) = \bar{\mu}_3(a_4) = \bar{\mu}_3(a_5),$$
$$\bar{\lambda}_3(e) = \bar{\lambda}_3(a_1) \neq \bar{\lambda}_3(a_2) = \bar{\lambda}_3(a_3) = \bar{\lambda}_3(a_4) = \bar{\lambda}_3(a_5).$$

Then, for any $r \geq 3$, ${}_r(H_{39}) \simeq {}_2(H_{39})$ and therefore $i.f.g.(H_{39}) = 3$.

(c_2) Let us consider H_{40} as the following complete hypergroup

$\circ$	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	a_2	a_3	A_3	A_3
a_1		A_0	a_2	a_3	A_3	A_3
a_2			a_3	A_3	A_0	A_0
a_3				A_0	a_2	a_2
a_4					a_3	a_3
a_5						a_3

with $A_0 = \{e, a_1\}, A_1 = \{a_2\}, A_2 = \{a_3\}, A_3 = \{a_4, a_5\}$, and $G \simeq (\mathbb{Z}_4, +)$ (the 4-tuple associated with H_{40} is $[2, 1, 1, 2]$). Then, we obtain the following membership functions

$$\bar{\mu}(e) = \bar{\mu}(a_1) = 9/72, \quad \bar{\mu}(a_2) = 16/72, \quad \bar{\mu}(a_3) = 18/72,$$
$$\bar{\lambda}(e) = \bar{\lambda}(a_1) = 44/72, \quad \bar{\lambda}(a_2) = 37/72, \quad \bar{\lambda}(a_3) = 35/72,$$
$$\bar{\mu}(a_4) = \bar{\mu}(a_5) = 10/72, \quad \bar{\lambda}(a_4) = \bar{\lambda}(a_5) = 43/72.$$

It is clear that the associated join space ${}_0(H_{40})$ is isomorphic to the join space ${}_0(H_{39})$ and therefore $i.f.g.(H_{40}) = 3$.

□

Making a short comparison with the fuzzy grade of the same hypergroups, we notice that there are no complete hypergroups of order less than or equal to 6 with the fuzzy grade equal to 3, instead there are 5 such hypergroups with the intuitionistic fuzzy grade equal to 3. Moreover, for the complete hypergroups of order 3 or 4, the fuzzy grade coincides with the intuitionistic fuzzy grade.

5. Conclusions and future work

In this paper, we have presented the join spaces and the membership functions of the intuitionistic fuzzy sets associated with all forty non-isomorphic complete hypergroups of order less than or equal to 6, determining their intuitionistic fuzzy grades. A similar work has been done by Cristea [14], regarding the fuzzy grades of the same hypergroups.

The fuzzy grade of a complete hypergroup H constructed from a group G does not depend on the group G, but only on the m-decomposition of $n = |H|$. More exactly, if G_1 and G_2 are non-isomorphic groups of the same order m, and H_1 and H_2 are the correspondent complete hypergroups of order n, then $f.g.(H_1) = f.g.(H_2)$. This is an immediate consequence of Theorem 2.3 [14]. In this paper, we noticed that the intuitionistic fuzzy grade of a complete hypergroup does not have the same property. For example, let H be a complete hypergroup of order 6 such that $[1, 1, 2, 2]$ is the 4-tuple associated with it. Therefore, there exist two non-isomorphic hypergroups of such type: the hypergroup denoted in this article with H_{39} (obtained with the group $G \simeq (\mathbb{Z}_4, +)$) and the hypergroup H_{33} (obtained with the group $G \simeq (K, \cdot)$ the Klein four group). We have obtained that $i.f.g.(H_{39}) = 3$ and $i.f.g.(H_{33}) = 1$. Thereby the intuitionistic fuzzy grade of a complete hypergroup depends also on the group G. It seems interesting to find conditions connected with the group G (with $|G| = m$) such that $i.f.g.(H)$ depends only on the m-decomposition of $n = |H|$. This theme will be discussed in a future work.

References

[1] Atanassov, K.T. *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87-96.

[2] Atanassov, K.T. *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **61**(1994), 137-142.

[3] Corsini, P. *Prolegomena of hypergroup theory*, Aviani Editore, 1993.

[4] Corsini, P. *Join spaces, power sets, fuzzy sets*, in: Proc. Fifth Int. Cong. A.H.A. 1993, Iasi, Romania, Hadronic Press (1994), 45-52.

[5] Corsini, P. *A new connection between hypergroups and fuzzy sets*, Southeast Asian Bull. Math., **27**(2003), 221-229.

[6] Corsini, P., Cristea, I. *Fuzzy grade of i.p.s. hypergroups of order less than or equal to* 6, PU. M. A., **14**(2003), no.4, 275-288.

[7] Corsini, P., Cristea, I. *Fuzzy grade of i.p.s. hypergroups of order* 7, Iran. J. Fuzzy Syst., **1**(2004), 15-32.

[8] Corsini, P., Cristea, I. *Fuzzy sets and non complete 1-hypergroups*, An. St. Univ. Ovidius Constanta Ser. Mat., **13**(2005), no.1, 27-53.

[9] Corsini, P., Leoreanu, V. *Join spaces associated with fuzzy sets*, J. Combin. Inform. Syst. Sci., **20**(1995), no. 1-4, 293-303.

[10] Corsini, P., Leoreanu, V. *Applications of hyperstructure theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordercht, 2003.

[11] Corsini, P., Leoreanu-Fotea, V. *On the grade of a sequence of fuzzy sets and join spaces determined by a hypergraph*, Southeast Asian Bull. Math., **34**(2010), no.2, 231-242.

[12] Corsini, P., Leoreanu, V., Iranmanesh, A. *On the sequence of hypergroups and membership functions determined by a hypergraph*, J. Mult.-Valued Logic Soft Comput., **14**(2008), no.6, 565-577.

[13] Corsini, P., Mahjoob, R. *Multivalued functions, fuzzy subsets and join spaces*, Ratio Math., **20**(2010), 1-41.

[14] Cristea, I. *Complete hypergroups,* 1*-hypergroups and fuzzy sets*, An. St. Univ. Ovidius Constanta Ser. Mat., **10**(2002), no. 2, 25-37.
[15] Cristea, I. *About the fuzzy grade of the direct product of two hypergroupoids*, Iran. J. Fuzzy Syst., **7**(2010), no.2, 95-108.
[16] Cristea, I., Darafsheh, M.R. *A decomposition of the cyclic groups of order p and p^2, where p is a prime number*, J. Intell. Fuzzy Syst., **21**(2010), no.4, 255-259.
[17] Cristea, I., Darafsheh, M.R. *About a decomposition of the group $\mathbb{Z}_{pq}$, where p and q are distinct odd primes*, European J. Combin., **31**(2010), no. 7, 1880-1886.
[18] Cristea, I., Davvaz, B. *Atanassov's intuitionistic fuzzy grade of hypergroups*, Inform. Sci., **180**(2010), 1506-1517.
[19] Davvaz, B. *Fuzzy H_v-groups*, Fuzzy Sets and Systems, **101** (1999), 191-195.
[20] Davvaz, B. *Fuzzy H_v-sumodules*, Fuzzy Sets and Systems, **117**(2001), 477-484.
[21] Davvaz, B., Corsini, P., Leoreanu-Fotea, V. *Atanassov's intuitionistic (S,T)-fuzzy n-ary sub-hypergroups and their properties*, Inform. Sci., **179**(2009), 654-666.
[22] Davvaz, B., Leoreanu-Fotea, V. *Intuitionistic fuzzy n-ary hypergroups*, J. Mult.-Valued Logic Soft Comput., **16**(1-2) (2010), 87-104.
[23] Davvaz, B., Leoreanu-Fotea, V. *Hyperring theory and Applications*, International Academic Press, USA, 2007.
[24] Davvaz, B., Hassani Sadrabadi, E., Cristea, I. *Atanassov's intuitionistic fuzzy grade of i.p.s. hypergroups of order* 7, J. Mult.-Valued Logic Soft Comput., **20**(2013), no.5-6, 467-506.
[25] Davvaz, B., Hassani Sadrabadi, E., Cristea, I. *Atanassov's intuitionistic fuzzy grade of i.p.s. hypergroups of order less than or equal to* 6, Iran. J. Fuzzy Syst., **9**(2012), no.4, 71-97.
[26] Migliorato, R. *On the complete hypergroups*, Riv. Mat. Pura Appl., **14** (1994), 21-31.
[27] Prenowitz, W., Jantosciak, J. *Geometries and join spaces*, J. Reind und Angew Math., **257**(1972), 100-128.
[28] Rosenfeld, A. *Fuzzy groups*, J. Math. Anal. Appl., **35**(1971), 512-517.
[29] Ştefănescu, M., Cristea, I. *On the fuzzy grade of hypergroups*, Fuzzy Sets and Systems, **159**(2008), 1097-1106.
[30] Vougiouklis, T. *Hyperstructures and their representations*, Hadronic Press, Inc, 115 Palm Harbor, USA. 1994.
[31] Zadeh, L.A. *Fuzzy sets*, Inform. Control, **8**(1965), 338-353.

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Representation and characterization of rapidly varying functions

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Abstract

In this paper representations and characterizations of the class of rapidly varying functions in the sense of de Haan, for index $+\infty$, will be proved. The statements of this theorems will be given in a form that is used by Karamata. Also, some characterization of normalized rapidly varying functions are proved.

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1. Introduction and Results

Karamata's theory of regular variation (see e.g. [6]) was appeared during the thirties of last century as a result of the first serious study of Tauberian type theorems for integral transformations (see e.g. [7] and [8]). The main object in this theory is the class of slowly varying functions in the sense of Karamata which is denoted by SV.

A measurable function $f : [a,\infty) \mapsto (0,\infty)$ $(a > 0)$ is called slowly varying in the sense of Karamata if it satisfies the following condition

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = 1, \tag{1.1}$$

for every $\lambda > 0$,

L. de Haan in [5] introduced the class of rapidly varying functions (denoted by R_∞), with the index of variability $+\infty$. In fact, this notion has already appeared in some Karamata's papers (see e.g. [11]), but in a less distinctly form. In recent years, the Theory of rapid variability and its generalizations have experienced great development in asymptotic analysis and in mathematics in general (see e.g. [1], [2], [3], [4] and [10]),

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simultaneously with Karamata's theory of regular variability (see [1]). Important properties of the class R_∞ can be seen in [4].

A measurable function $f : [a,\infty) \mapsto (0,\infty)$ $(a > 0)$ is called rapidly varying in the sense of de Haan with the index of variability ∞, if it satisfies the following condition

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \infty, \tag{1.2}$$

for every $\lambda > 1$.

1.1. Remark. In this paper we will consider a function $f \in R_\infty$ defined on the interval $(0,\infty)$. Analogous results can be obtained if the domain of a function f is interval $[a,\infty)$, $a > 0$.

According to results from [1], rapidly varying function f satisfies condition

$$\lim_{x\to\infty} \inf_{\mu\geq\lambda} \frac{f(\mu x)}{f(x)} = \infty \tag{1.3}$$

for every $\lambda > 1$ and it follows that for some $x_0 > 0$ function f is bounded on every interval (x_0, x). Also, $f(x) \to \infty$ for $x \to \infty$ holds.

Now, we give the representation of a functions from functional class R_∞ in Karamata's form.

1.2. Remark. In the following theorem, operator $\underline{D}$ is lower Dini derivative (see [9])

$$\underline{D}g(x) = \varliminf_{y\to x} \frac{g(y)-g(x)}{y-x}, \quad \text{for } g : \mathbb{R} \to \mathbb{R},\ x \in \mathbb{R},$$

and denotation $\sim$ represents strong asymptotic equivalence relation.

1.3. Theorem. *For a function $f : (0,\infty) \mapsto (0,\infty)$ the next assertions are mutually equivalent:*

(a) *function f belongs to the class R_∞;*

(b) *there is a non-decreasing, absolutely continuous function $g : \mathbb{R} \mapsto \mathbb{R}$ such that $\lim\limits_{x\to\infty} \underline{D}g(x) = \infty$ and there is a measurable function $j : (0,\infty) \mapsto (0,\infty)$ such that $j(x) \sim x$ for $x \to \infty$, so that*

$$f(x) = \exp\big(g(\log(j(x)))\big),$$

for all $x > 0$;

(c) *there are a measurable functions $j : (0,\infty) \mapsto (0,\infty)$ and $h : (0,\infty) \mapsto [0,\infty)$, such that $\lim\limits_{x\to\infty} h(x) = \infty$ and $j(x) \sim x$ for $x \to \infty$, for which holds*

$$f(x) = exp\left\{c + \int_0^{j(x)} h(u)\frac{du}{u}\right\},$$

for all $x > 0$ and for some $c \in \mathbb{R}$.

Now, we give the characterization of a elements from the class R_∞ in Karamata's form.

1.4. Theorem. *Let $f : (0,\infty) \mapsto (0,\infty)$ be a measurable function. Then $f \in R_\infty$ if and only if for all $\alpha > 0$ there is a measurable function $j_\alpha : (0,\infty) \mapsto (0,\infty)$ such that $j(x) \sim x$, for $x \to \infty$, and there is a non-decreasing function $k_\alpha : (0,\infty) \mapsto (0,\infty)$, so that*

$$f(x) = x^\alpha \cdot k_\alpha\big(j_\alpha(x)\big), \qquad \text{for } x > 0.$$

The following theorem gives a few characterization of elements of one proper subclass of class R_∞, which could be called class of normalized rapidly varying functions (see, e.g., [1]).

1.5. Theorem. *For a measurable function $f:(0,\infty)\mapsto(0,\infty)$ the following assertions are mutually equivalent:*

(a) $\lim\limits_{\substack{x\to\infty\\ \lambda\to 1_+}} \log_\lambda \dfrac{f(\lambda x)}{f(x)}=\infty$*;*

(b) $\dfrac{f(x)}{x}=o\,(\underline{D}f(x))$*, for $x\to\infty$ (o is Landau symbol [1]);*

(c) *there exists a function $g:\mathbb{R}\mapsto\mathbb{R}$ such that $\lim\limits_{x\to\infty}\underline{D}g(x)=\infty$ so that holds:*

$$f(x)=\exp(g(\log(x)))$$

for all $x>0$;

(d) *for all $\alpha\in\mathbb{R}$ function $\frac{f(x)}{x^\alpha}$ is increasing on some interval $[x_\alpha,\infty)$.*

1.6. Remark. 1) Theorem 1.5 holds even without assumption that the function f is measurable, but this assumption should be included because in Theorem 1.5 one important subclass of class R_∞ is characterized.

2) The fact that for a measurable function $f:(0,\infty)\mapsto(0,\infty)$ exists a measurable function $h:(0,\infty)\mapsto\mathbb{R}$ such that $\lim\limits_{x\to\infty}h(x)=\infty$, and for which is $f(x)=\exp\{c+\int_0^x h(u)\frac{du}{u}\}$ for all $x>0$ and some $c\in\mathbb{R}$, implicates (c) from Theorem 1.5 (and, also implicates (a), (b) and (d) from Theorem 1.5). The proof is analog to the proof (c) $\Rightarrow$ (a) $\Rightarrow$ (b) from Theorem 1.3. The opposite direction need not to be true without additional conditions.

3) If f is absolutely continuous function, opposite direction in 2) is true. That can be proved analogously to the proof (b) $\Rightarrow$ (c) from Theorem 1.3.

2. Proofs

Proof of Theorem 1.3. (a) $\Rightarrow$ (b) Let $f\in R_\infty$. Let construct sequence (x_n) of positive real numbers with the following properties:

1° (x_n) is strictly increasing sequence and $\lim\limits_{n\to\infty}x_n=\infty$,

2° $\lim\limits_{n\to\infty}\dfrac{x_{n+1}}{x_n}=1$, and

3° $\dfrac{f(x)}{f(y)}>2$ for all $x>0$ and all $y>0$, for which $x\geq x_{n+1}>x_n\geq y\geq x_1$, where $n\in\mathbb{N}$.

Let $x_1>0$ so that f is locally bounded on the interval $[x_1,\infty)$ and let $x_{n+1}=(\lambda_n+\frac{1}{n})x_n$ for $n\in\mathbb{N}$, where $\lambda_n=\sup\{\lambda\geq 1\mid f(\lambda x_n)\leq 2\sup\limits_{x_1\leq t\leq x_n}f(t)\}$. Clearly, for all $n\in\mathbb{N}$ there exists λ_n in $\mathbb{R}$ and $x_n\leq\lambda_n x_n<x_{n+1}$. If $x\geq x_{n+1}>x_n\geq y>0$, then $x>\lambda_n x_n$ for $n\in\mathbb{N}$, and according to the definition of the sequence (λ_n) it is $f(x)>2f(y)$ for $x_1\leq y\leq x_n$. Especially, $f(x_{n+1})>2f(x_n)$ for $n\in\mathbb{N}$, which yields $\lim\limits_{n\to\infty}f(x_n)=\infty$. As f is locally bounded function on the interval $[x_1,\infty)$, it follows $\lim\limits_{n\to\infty}x_n=\infty$.

According to the definition of sequence (λ_n) it can be concluded that sequences (μ_n) and (y_n) are such that, for every $n\in\mathbb{N}$, it follows that $\mu_n\in(\lambda_n-\frac{1}{n},\lambda_n)$ and $y_n\in[x_1,x_n]$, for which it is $\dfrac{f(\mu_n x_n)}{f(y_n)}\leq 2$. Then, according to the theorem of uniform convergence for rapidly varying functions (see (1.3) or [1]), it follows $\limsup_{n\to\infty} \frac{\mu_n x_n}{y_n} \le 1$, i.e., $\limsup_{n\to\infty} \mu_n \le 1$, so it follows that $\lim_{n\to\infty} \lambda_n = 1$. Thus, $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$.

Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a linear function on $[t_n, t_{n+1}]$ such that $g(t_n) = \ln f(x_n)$, where $t_n = \ln x_n$, for every $n \in \mathbb{N}$. Also, $g(t) = e^t - x_1 + g(t_1)$, for $t < t_1$. Now, we have that g is a continuous, piecewise smooth and strictly increasing (hence, absolutely continuous and non-decreasing) function, and (from 1° and 3°) it satisfies

$$g'(t) = \frac{g(t_{n+1}) - g(t_n)}{t_{n+1} - t_n} = \frac{\ln f(x_{n+1}) - \ln f(x_n)}{\ln x_{n+1} - \ln x_n} > 0,$$

for any $t \in (t_n, t_{n+1})$, $n \in \mathbb{N}$. Furthermore, (from 2° and 3°) it satisfies

$$\lim_{t\to\infty} g'(t) = \lim_{n\to\infty} \frac{\ln \frac{f(x_{n+1})}{f(x_n)}}{\ln \frac{x_{n+1}}{x_n}} \ge \lim_{n\to\infty} \frac{\ln 2}{\ln \frac{x_{n+1}}{x_n}} = \infty,$$

for $\mathbb{R} \ni t \ne t_n$, for every $n \in \mathbb{N}$. Thus, $\lim_{x\to\infty} \underline{D}g(x) = \infty$.

Now, let $j(x) = e^{g^{-1}(\ln f(x))}$, for $x > 0$. A function $j(x)$ is measurable, because $f(x)$ is a measurable function, and $\exp\big(g^{-1}(\log(t))\big)$ is a piecewise smooth function (and hence it is absolutely continuous function), for $t > 0$.

Now, we will show that $j(x) \sim x$, for $x \to \infty$. From condition 3° it follows that $\frac{f(x)}{f(x_{n-1})} > 2$ and $\frac{f(x_{n+2})}{f(x)} > 2$, for some $x \in [x_n, x_{n+1})$ and $n \in \mathbb{N}$, $n \ge 2$. For those x and n we obtain

$$f(x_{n-1}) < f(x) < f(x_{n+2}),$$

so, we have

$$g^{-1}\big(\ln f(x_{n-1})\big) < g^{-1}\big(\ln f(x)\big) < g^{-1}\big(\ln f(x_{n+2})\big).$$

Furthermore, for those x and n, we have

$$t_{n-1} = \ln x_{n-1} < \ln j(x) < \ln x_{n+2} = t_{n+2},$$

and finally we obtain that

$$\frac{x_{n-1}}{x_{n+1}} < \frac{x_{n-1}}{x} < \frac{j(x)}{x} < \frac{x_{n+2}}{x} < \frac{x_{n+2}}{x_n}.$$

Hence, from 2° it is satisfied that $j(x) \sim x$, for $x \to \infty$, and $f(x) = \exp(g(\log(j(x))))$ for $x > 0$.

(b) $\Rightarrow$ (c) Let functions g and j have properties given in (b). Let

$$g_0 = \begin{cases} g(x), & \text{for } x > 0, \\ g(0), & \text{for } x \le 0. \end{cases}$$

Let $h(x) = \underline{D}g_0(\ln x)$ for $x > 0$. Then h is measurable, locally integrable, and $\lim_{x\to\infty} h(x) = \infty$ holds. Also, it is

$$\int_0^{j(x)} h(u)\frac{du}{u} = \int_0^{j(x)} \underline{D}g_0(\ln u)\frac{du}{u} = \int_{-\infty}^{\ln j(x)} \underline{D}g_0(t)dt =$$

$$= \int_0^{\ln j(x)} \underline{D}g(t)dt = g(\ln j(x)) - g(0) = \ln f(x) - c,$$

for a constant $c = g(0) \in \mathbb{R}$, and for all $x > 0$.

Thus, $f(x) = \exp\left\{c + \int_0^{j(x)} h(u)\frac{du}{u}\right\}$, for all $x > 0$ and for mentioned $c \in \mathbb{R}$.

(c) $\Rightarrow$ (a) Let $\lambda > 1$ and $M \in \mathbb{R}$. Then, there is $x_0 > 0$ such that $h(x) > \dfrac{2M}{\ln \lambda}$ and $\lambda^{-\frac{1}{4}} < \dfrac{j(x)}{x} < \lambda^{\frac{1}{4}}$, for $x > \dfrac{x_0}{\lambda}$. Hence, it follows

$$\ln \frac{f(\lambda x)}{f(x)} = \int_0^{j(\lambda x)} h(u)\frac{du}{u} - \int_0^{j(x)} h(u)\frac{du}{u} = \int_{j(x)}^{j(\lambda x)} h(u)\frac{du}{u} >$$
$$> \frac{2M}{\ln \lambda} \cdot \big(\ln j(\lambda x) - \ln j(x)\big) = \frac{2M}{\ln \lambda} \cdot \ln\left(\lambda \cdot \frac{j(\lambda x)}{\lambda x} \cdot \frac{x}{j(x)}\right) >$$
$$> \frac{2M}{\ln \lambda} \cdot \ln\left(\lambda \cdot \lambda^{-\frac{1}{4}} \cdot \lambda^{-\frac{1}{4}}\right) = M,$$

for $x > x_0$. Therefore, it holds that $\lim\limits_{x\to\infty} \dfrac{f(\lambda x)}{f(x)} = \infty$, for every $\lambda > 1$. Also, f is a measurable function, as a composition of three function: a measurable function j, an absolutely continuous function $\int\limits_0^x h(u)\dfrac{du}{u}$ and an exponential function. Hence, $f \in R_\infty$. □

Proof of Theorem 1.4. ($\Rightarrow$) If $f \in R_\infty$, then $\dfrac{f(x)}{x^\alpha} \in R_\infty$, for $x > 0$, and every fixed $\alpha > 0$. From Theorem 1.3, it follows that $\frac{f(x)}{x^\alpha} = \exp\Big\{ g_\alpha\big(\log(j_\alpha(x))\big)\Big\}$, for that α and every $x > 0$, where $g_\alpha : \mathbb{R} \mapsto \mathbb{R}$ is a non-decreasing function and $j_\alpha : (0,\infty) \mapsto (0,\infty)$ is a measurable function such that $j_\alpha(x) \sim x$, for $x \to \infty$. If we take that $k_\alpha(t) = \exp\big\{g_\alpha(\log(t))\big\}$, for $t > 0$, we obtain that Theorem holds for this direction.
($\Leftarrow$) For arbitrary $\alpha > 0$, if there is a measurable function $j_\alpha : (0,\infty) \mapsto (0,\infty)$ such that $\lim\limits_{x\to\infty} \dfrac{j_\alpha(x)}{x} = 1$ and a non-decreasing function $k_\alpha : (0,\infty) \mapsto (0,\infty)$, that is satisfied $f(x) = x^\alpha \cdot k_\alpha(j_\alpha(x))$, for $x > 0$ we obtain that

$$\frac{f(\lambda x)}{f(x)} = \lambda^\alpha \cdot \frac{k_\alpha\big(j_\alpha(\lambda x)\big)}{k_\alpha\big(j_\alpha(x)\big)} \geq \lambda^\alpha,$$

for $\lambda > 1$ and sufficiently large x. The previous inequality holds because $j_\alpha(\lambda x) \geq \sqrt{\lambda}x \geq j_\alpha(x)$ for mentioned α, λ and sufficiently large x. Therefore, it follows that $\lim\limits_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \infty$, for $\lambda > 1$. Also, f is a measurable function. Finally, $f \in R_\infty$. □

Proof of Theorem 1.5. (a) $\Leftrightarrow$ (c) Let introduce function f in the following way: $f(x) = \exp(g(\log(x)))$, for $x > 0$. Then, equivalence (a) $\Leftrightarrow$ (c) follows from:

$$\varliminf_{\substack{x\to\infty\\ \lambda\to 1_+}} \log_\lambda \frac{f(\lambda x)}{f(x)} = \varliminf_{\substack{t\to\infty\\ \delta\to 0_+}} \frac{g(t+\delta) - g(t)}{\delta} = \varliminf_{t\to\infty} \underline{D}g(t).$$

(b) $\Leftrightarrow$ (c) Again, let introduce function f as $f(x) = \exp(g(\log(x)))$, for $x > 0$. Then, from the fact that $\underline{D}g(x) = \underline{D}(\ln f(e^x)) = \dfrac{\underline{D}f(e^x)e^x}{f(e^x)}$, for all $x \in \mathbb{R}$, it follows $\varliminf\limits_{x\to\infty} \underline{D}g(x) = \infty$ is equal to the fact that $\frac{f(t)}{t} = o(\underline{D}g(t))$, for $t \to \infty$. This proves the equivalence (b) $\Leftrightarrow$ (c).

(c) $\Leftrightarrow$ (d) Once more, let introduce function f by $f(x) = \exp(g(\log(x)))$, for $x > 0$. Then the function $\frac{f(x)}{x^\alpha}$, $x > 0$ and $\alpha \in \mathbb{R}$, is increasing on an interval $[x_\alpha, \infty)$ if and only if the function $\ln \dfrac{f(e^t)}{e^{\alpha t}} = g(t) - \alpha t$, for $t \in \mathbb{R}$ and the same $\alpha \in \mathbb{R}$, is increasing on an interval $[t_\alpha, \infty)$, and this last condition is equivalent to the fact that $\varliminf_{t\to\infty} Dg(t) \geq \alpha$ for all $\alpha \in \mathbb{R}$. The last fact is equivalent to the fact that $\varliminf_{t\to\infty} Dg(t) = \infty$. $\square$

References

[1] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular variation*, Cambridge Univ. Press, Cambridge, 1987.

[2] D. Djurčić, D. R. Kočinac, M. R. Žižović, Some properties of rapidly varying sequences, *J. Math. Anal. Appl.* **327(2)** (2007), 1297–1306.

[3] D. Djurčić, A. Torgašev, Some asymptotic relations for generalized inverse, *J. Math. Anal. Appl.*, **335(2)** (2007), 1397–1402.

[4] N. Elez, D. Djurčić, Some properties of rapidly varying functions, *J. Math. Anal. Appl.*, **401(2)** (2013), 888–895.

[5] L. de Haan, *On Regular variation and its applications to the weak convergence of sample extremes*, Math. Centre tracts, Vol. **32**, CWI, Amsterdam, 1970.

[6] J. Karamata, Sur un mode de croissance régulière des functions, *Mathematica (Cluj)* **4** (1930), 38–53.

[7] J. Karamata, Neuer Beweis und Verallgemeinerung der Taubersche Sätz, welche die Laplacessche und Stieltjessche Transformationen betreffen, *J. Reine Angew. Math.* **164** (1931), 27–39.

[8] J. Karamata, Sur un mode de croissance régulière. Théorèmes fondamenteaux, *Bull Soc. Math. France* **61** (1933), 55–62.

[9] I.P. Natanson, *Theory of functions of a real variable*, (Translated by Leo F. Boron with the collaboration of Edwin Hewitt), Frederick Ungar Publishing Co., New York, 1955.

[10] S. Matucci, P. Řehák, Rapidly varying decreasing solutions of half-linear difference equations, Math. and Com. Modelling, **49(7-8)**, (2009), 1692–1699.

[11] E. Seneta, Functions of regular variation, *LNM* Vol. **506**, Springer, New York, 1976.

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Notes on near-ring ideals with (σ,τ)-derivation

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Abstract

In the present paper, we extend some well known results concerning derivations of prime near-rings in [4], [5] and [13] to $(\sigma,\tau)-$derivations and semigroup ideals of prime near-rings.

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1. Introduction

An additively written group $(N,+)$ equipped with a binary operation $. : N \times N \rightarrow N, (x,y) \rightarrow xy$, such that $x(yz) = (xy)z$ and $x(y+z) = xy + xz$ for all $x,y,z \in N$ is called a left near-ring. A near-ring N is called zero symmetric if $0x = 0$ for all $x \in N$ (recall that left distributive yields $x0 = 0$). An element x of N is said to be distributive if $(y+z)\,x = yx + zx$ for all $x,y,z \in N$. In what follows all near-rings are zero symmetric left near-rings. A near-ring N is said to be $3-$prime if $xNy = \{0\}$ implies $x = 0$ or $y = 0$. For any $x,y \in N$, as usual $[x,y] = xy - yx$ and $xoy = xy + yx$ will denote the well-known Lie and Jordan products respectively, while the symbol (x,y) will denote the additive commutator $x + y - x - y$. Given an element a of N, we put $C(a) = \{x \in N \mid ax = xa\}$. The set $Z = \{x \in N \mid yx = xy \text{ for all } y \in N\}$ is called multiplicative center of N. A nonempty subset U of N will be said a semigroup right ideal (resp. a semigroup left ideal) if $UN \subseteq U$ (resp. $NU \subseteq U$) and U is both a semigroup right ideal and a semigroup left ideal, it will be called a semigroup ideal. An additive mapping $d : N \rightarrow N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x,y \in N$ or equivalently, as noted in [13, Proposition 1], if $d(xy) = d(x)y + xd(y)$ for all $x,y \in N$. An element $x \in N$ for which $d(x) = 0$ is called constant. Following [8], an additive mapping d of N is called $(\sigma,\tau)-$derivation if there exist automorphisms $\sigma,\tau : N \rightarrow N$ such that

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$d(xy) = \tau(x)\, d(y) + d(x)\, \sigma(y)$ for all $x, y \in N$ or equivalently, as noted in [8, Lemma 1], if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in N$. Of course a $(1,1)-$derivation where 1 is the identity map on N is a derivation.

As well known that derivations or $(\sigma, \tau)-$derivations are important both in algebra and ring theory. These topics have many implications such as generalizations of Lie algebra, differantial and homological algebra. Some researchers have studied on these topics.(see [6], [7], [10] and [12]). Since E. C. Posner published his paper [11] in 1957, many authors have investigated properties of derivations of prime and semiprime rings. In view of these results it is natural to look for comparable results on near-rings. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in 1987 [3], but thus far only few papers on this subject in near-rings have been published (see refererences for a partial bibliography).

In the present paper, we shall attempt to generalize some known results for derivations to $(\sigma, \tau)-$derivations and semigroup ideals of a left prime near-ring N. In Theorem 3.3, we extend [13, Theorem 1]. Theorem 3.7 is a generalization of [4, Lemma 3.2] to $(\sigma, \tau)-$derivation and semigroup ideals of N. Finally, it is shown that under appropriatiate additional hypothesis near-ring N must be a commutative ring.

2. Preliminaries

We begin with the following known results.

2.1. Lemma. [3, Lemma 3]*Let N be a prime near-ring.*

(i) *If $z \in Z\setminus\{0\}$, then z is not a zero divisor.*

(ii) *If Z contains a nonzero element z for which $z + z \in Z$, then $(N, +)$ is abelian.*

(iii) *Let d be a nonzero derivation on N. Then $xd(N) = (0)$ implies $x = 0$, and $d(N)x = (0)$ implies $x = 0$.*

(iv) *If N is $2-$torsion free and d is a derivation on N such that $d^2 = 0$, then $d = 0$.*

2.2. Lemma. [4, Lemma 1.3]*Let N be a $3-$prime near-ring and d be a nonzero derivation on N.*

(i) *If U is a nonzero semigroup right ideal (resp. semigroup left ideal) and $x \in N$ such that $Ux = (0)$ (resp. $xU = (0)$), then $x = 0$.*

(ii) *If U is a nonzero semigroup right ideal or semigroup left ideal, then $d(U) \neq (0)$.*

(iii) *If U is a nonzero semigroup right ideal and $x \in N$ which centralizes U, then $x \in Z$.*

2.3. Lemma. [4, Lemma 1.4]*Let N be a $3-$prime near-ring and U be a nonzero semigroup ideal of N. Let d be a nonzero derivation on N.*

(i) *If $x, y \in N$ and $xUy = (0)$, then $x = 0$ or $y = 0$.*

(ii) *If $x \in N$ and $d(U)x = (0)$, then $x = 0$.*

(iii) *If $x \in N$ and $xd(U) = (0)$, then $x = 0$.*

2.4. Lemma. [4, Theorem 2.1]*Let N be a $3-$prime near-ring and U be a nonzero semigroup right ideal or a nonzero semigroup left ideal of N. If N admits a nonzero derivation d for which $d(U) \subset Z$, then N is a commutative ring.*

2.5. Lemma. [4, Lemma 3.2]*Let N be a $3-$prime near-ring and U a nonzero semigroup ideal of N. Let d be a nonzero derivation on N such that $d^2(U) \neq (0)$. If $a \in N$ and $[a, d(U)] = (0)$, then $a \in Z$.*

2.6. Lemma. [5, Lemma 1.8]*Let N be a $3-$prime near-ring with $2N \neq (0)$, and U a nonzero semigroup ideal. If d is a derivation on N such that $d^2(U) = (0)$, then $d = 0$.*

2.7. Lemma. [5, Lemma 2.4]*Let N be an arbitrary near-ring. Let S and T be nonempty subsets of N such that $st = -ts$ for all $s \in S$ and $t \in T$. If $a, b \in S$ and c is an element of T for which $-c \in T$, then $(ab)\,c = c\,(ab)$.*

2.8. Lemma. [8, Lemma 1]*Let N be a $3-$prime near-ring and d be a (σ, τ)-derivation on N. Then $d\,(xy) = d\,(x)\,\sigma\,(y) + \tau\,(x)\,d\,(y)$, for all $x, y \in N$.*

2.9. Lemma. [9, Lemma 4]*Let N be a $3-$prime near-ring, d a (σ, τ)-derivation of N and U a nonzero semigroup right ideal (resp. semigroup left ideal). If $d\,(U) = (0)$, then $d = 0$.*

2.10. Lemma. [9, Theorem 1]*Let N be a 3-prime near-ring, d a nonzero (σ, τ)-derivation of N and U a nonzero semigroup right ideal of N. If $d\,(U) \subset Z$, then N is a commutative ring.*

2.11. Lemma. [9, Theorem 3]*Let N be a 3-prime near-ring, d a nonzero (σ, τ)-derivation of N such that $\sigma d = d\sigma, \tau d = d\tau$ and U a nonzero semigroup ideal of N. If $d^2\,(U) = (0)$, then $d = 0$.*

2.12. Lemma. [1, Lemma 2.2]*Let d be a (σ, τ)-derivation on the near-ring N. Then N satisfies the following partial distributive laws:*
(i) $(\tau\,(x)\,d\,(y) + d\,(x)\,\sigma\,(y))\,z = \tau\,(x)\,d\,(y)\,z + d\,(x)\,\sigma\,(y)\,z$, for all $x, y, z \in N$.
(ii) $(d\,(x)\,\sigma\,(y) + \tau\,(x)\,d\,(y))\,z = d\,(x)\,\sigma\,(y)\,z + \tau\,(x)\,d\,(y)\,z$ for all $x, y, z \in N$.

3. The Main Results

3.1. Theorem. *Let N be a $3-$prime near-ring and U a nonzero semigroup ideal of N. If d_1 is a nonzero (σ, τ)-derivation and d_2 a nonzero derivation of N such that $d_1\,(x)\,\sigma\,(d_2\,(y)) = -\tau\,(d_2\,(x))\,d_1\,(y)$ for all $x, y \in U$, then $(N, +)$ is abelian.*

Proof. Writing yr, $y \in U$, $r \in N$ instead of y, we have

$$d_1\,(x)\,\sigma\,(d_2\,(y)\,r + yd_2\,(r)) = -\tau\,(d_2\,(x))\,(\tau\,(y)\,d_1\,(r) + d_1\,(y)\,\sigma\,(r))$$

and so

$$d_1\,(x)\,\sigma\,(d_2\,(y))\sigma(r) + d_1\,(x)\,\sigma(y)\sigma(d_2\,(r)) = -\tau\,(d_2\,(x))\,d_1\,(y)\,\sigma\,(r) - \tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(r)$$

Using the hypothesis, we get

(3.1) $$d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(r)) = -\tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(r)\,, \text{ for all } x, y \in U, r \in N.$$

Replacing r by $r + t$, $t \in N$ in (3.1), we get

$$d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(r)) + d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(t))$$
$$= -\tau\,(d_2\,(x))\,\tau\,(y)\,(d_1\,(r) + d_1\,(t)).$$

Using $-(a + b) = (-b) + (-a)$, for all $a, b \in N$, we have

$$d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(r)) + d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(t))$$
$$= -\tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(t) - \tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(r)$$

and so

$$d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(r)) + d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(t)) + \tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(r) + \tau\,(d_2\,(x))\,\tau\,(y)\,d_1\,(t) = 0.$$

Using the (3.1) and $(r, t) = r + t - r - t$ in the last equation, we arrive at

$$d_1\,(x)\,\sigma\,(y)\,\sigma\,(d_2\,(r, t)) = 0, \text{ for all } x, y \in U, r, t \in N.$$

That is

(3.2) $$\sigma^{-1}(d_1\,(x))Ud_2\,(r, t) = (0), \text{ for all } x \in U, r, t \in N.$$

By Lemma 2.3 (i), we get $d_1(U) = (0)$ or $d_2(r,t) = 0$, for all $r,t \in N$. If $d_1(U) = (0)$, then $d_1 = 0$ by Lemma 2.9. This is a contradiction. So that $d_2(r,t) = 0$ for all $r,t \in N$. For any $w \in N$, we have $d_2(wr, wt) = 0$. Hence we obtain that $d_2(w)(r,t) = 0$, for all $w,r,t \in N$. From Lemma 2.1 (iii) and $d_2 \neq 0$, we get $(r,t) = 0$, for all $r,t \in N$. Thus the proof is completed. □

3.2. Theorem. *Let N be a $2-$torsion free $3-$prime near-ring and U a nonzero semigroup ideal of N. If d_1 is a (σ,τ)-derivation and d_2 a derivation of N such that $d_1(x)\sigma(d_2(y)) = -\tau(d_2(x))d_1(y)$ for all $x,y \in U$, then $d_1 = 0$ or $d_2 = 0$.*

Proof. Assume that $d_2 \neq 0$. Using the same method as in the proof of Theorem 3.1, we have

$$d_1(x)\sigma(y)\sigma(d_2(r)) = -\tau(d_2(x))\tau(y)d_1(r), \text{ for all } x,y,r \in U. \tag{3.3}$$

Replacing y by $yd_2(z)$ in (3.3), we have

$$d_1(x)\sigma(y)\sigma(d_2(z))\sigma(d_2(r)) = -\tau(d_2(x))\tau(y)\tau(d_2(z))d_1(r), \text{ for all } x,y,r,z \in U. \tag{3.4}$$

Using (3.3) in (3.4), we arrive at

$$-\tau(d_2(x))\tau(y)d_1(z)\sigma(d_2(r)) = -\tau(d_2(x))\tau(y)\tau(d_2(z))d_1(r)$$

and so

$$\tau(d_2(x))\tau(y)(d_1(z)\sigma(d_2(r)) - \tau(d_2(z))d_1(r)) = 0, \text{ for all } x,y,r,z \in U.$$

Since τ is an automorphism of N, we get

$$d_2(x)U\tau^{-1}(d_1(z)\sigma(d_2(r)) - \tau(d_2(z))d_1(r)) = (0), \text{ for all } x,r,z \in U. \tag{3.5}$$

By Lemma 2.3 (i), we get $d_2(x) = 0$ or $d_1(z)\sigma(d_2(r)) = \tau(d_2(z))d_1(r)$, for all $x,r,z \in U$. The first case contradicts $U \neq (0)$ by Lemma 2.2 (ii). So we must have $d_1(z)\sigma(d_2(r)) = \tau(d_2(z))d_1(r)$, for all $r,z \in U$. Hence we obtain that $2d_1(z)\sigma(d_2(r)) = 0$ by the hypothesis. Since N is $2-$torsion free, we have $\sigma^{-1}(d_1(z))d_2(r) = 0$, for all $r,z \in U$. Hence $d_1(U) = (0)$ by Lemma 2.3 (iii) . This gives us $d_1 = 0$ by Lemma 2.9. This completes the proof. □

3.3. Theorem. *Let N be a $2-$torsion free $3-$prime near-ring and U a nonzero semigroup ideal of N. If d_1 is a (σ,τ)-derivation and d_2 a derivation of N such that d_1d_2 acts as a (σ,τ)-derivation on U, then $d_1 = 0$ or $d_2 = 0$.*

Proof. By calculating $d_1d_2(xy)$ in two different ways, we see that

$$d_1d_2(xy) = d_1d_2(x)\sigma(y) + \tau(x)d_1d_2(y)$$

and

$$d_1d_2(xy) = d_1d_2(x)\sigma(y) + \tau(d_2(x))d_1(y) + d_1(x)\sigma(d_2(y)) + \tau(x)d_1d_2(y).$$

Equating these two expressions for $d_1d_2(xy)$, we obtain that

$$d_1(x)\sigma(d_2(y)) = -\tau(d_2(x))d_1(y) \text{ for all } x,y \in U.$$

Then $d_1 = 0$ or $d_2 = 0$ by Theorem 3.2. □

3.4. Theorem. *Let N be a $3-$prime near-ring and U a nonzero semigroup ideal of N. If d is a (σ,τ)-derivation of N such that $d(x)\sigma(y) = \tau(x)d(y)$ for all $x,y \in U$, then $d = 0$.*

Proof. Assume that

(3.6) $d(x)\sigma(y) = \tau(x)d(y), \text{ for all } x, y \in U.$

Replacing y by $yz, z \in U$ in (3.6), we have

$$d(x)\sigma(y)\sigma(z) = \tau(x)(d(y)\sigma(z) + \tau(y)d(z))$$

and so

$$d(x)\sigma(y)\sigma(z) = \tau(x)d(y)\sigma(z) + \tau(x)\tau(y)d(z), \text{ for all } x, y, z \in U.$$

Applying (3.6) in this equation, we get $\tau(x)\tau(y)d(z) = 0$, for all $x, y, z \in U$. Hence

$$xU\tau^{-1}(d(z)) = (0), \text{ for all } x, z \in U.$$

By Lemma 2.3 (i) and $U \neq (0)$, we get $d(z) = 0$, for all $z \in U$, and so $d = 0$ by Lemma 2.9. □

In [4], Bell and Argaç studied commutativity in 3−prime near-rings with a nonzero derivation d for which $d(xy) = d(yx)$ for all x, y in some nonzero one sided ideal. Ashraf and Ali showed this result for (σ, σ)-derivation on N in [2]. Now, we continue this study for a (σ, τ)-derivation d and a semigroup ideal U of near-rings without any restriction on U.

3.5. Theorem. *Let N be a 3−prime near-ring and U a nonzero semigroup ideal of N. If d is a (σ, τ)-derivation of N such that $d([x, y]) = 0$ for all $x, y \in U$, then N is a commutative ring.*

Proof. In the view of our hypothesis, we have

(3.7) $d([x, y]) = 0, \text{ for all } x, y \in U.$

Replacing y by xy in (3.7), we get

$$\begin{aligned} 0 &= d([x, xy]) = d(x[x, y]) \\ &= d(x)\sigma([x, y]) + \tau(x)d([x, y]), \end{aligned}$$

Using (3.7) in this equation, we have

$$d(x)\sigma([x, y]) = 0, \text{ for all } x, y \in U.$$

That is

(3.8) $d(x)\sigma(x)\sigma(y) = d(x)\sigma(y)\sigma(x), \text{ for all } x, y \in U.$

Writing yr, $r \in N$ instead of y in (3.8), we get

$$d(x)\sigma(x)\sigma(y)\sigma(r) = d(x)\sigma(y)\sigma(r)\sigma(x), \text{ for all } x, y \in U.$$

Using (3.8) in this equation, we arrive at

$$d(x)\sigma(y)\sigma(x)\sigma(r) = d(x)\sigma(y)\sigma(r)\sigma(x)$$

and so

$$d(x)\sigma(y)\sigma([x, r]) = 0.$$

That is

$$\sigma^{-1}(d(x))U[x, r] = (0), \text{ for all } x \in U, r \in N.$$

This yields that for each fixed $x \in U$ either $d(x) = 0$ or $x \in Z$ by Lemma 2.3 (i). But $x \in Z$ also implies that $d(x) \in Z$. Therefore, for any cases we find that $d(x) \in Z$, for any $x \in U$. By Lemma 2.10, we obtain that N is a commutative ring. This completes proof of our theorem. □

3.6. Theorem. *Let N be a $3-$prime near-ring and U a nonzero semigroup ideal of N. If d is a (σ,τ)-derivation of N such that $d(xoy)=0$ for all $x,y\in U$, then N is a commutative ring.*

Proof. Replacing x by yx in the hypothesis, we get

$$\begin{aligned}0&=d(yxoy)=d(y(xoy))\\&=d(y)\sigma(xoy)+\tau(y)d(xoy).\end{aligned}$$

Using the hypothesis, we find that

$$d(y)\sigma(xoy)=0.$$

That is

$$d(y)\sigma(x)\sigma(y)=-d(y)\sigma(y)\sigma(x),\ \text{for all } x,y\in U. \tag{3.9}$$

Taking $xr, r\in N$ instead of x in (3.9) and using (3.9), we obtain

$$\sigma^{-1}(d(y))U[r,y]=(0),\ \text{for all } y\in U, r\in N.$$

Now using the same arguments in the last paragraph of the proof of Theorem 3.5, we get the required result. □

3.7. Theorem. *Let N be a $3-$prime near-ring and U be a nonzero semigroup ideal of N. If $a\in N, d$ is a nonzero (σ,τ)-derivation on N such that $\sigma d=d\sigma, \tau d=d\tau$ and $[a,d\left(U\right)]=\left(0\right)$, then $a\in Z$.*

Proof. Note that $d\left(U\right)\subseteq C\left(a\right)$ by the hypothesis. Assume that $\tau\left(y\right)\in C\left(a\right)$. Then it is obvious that $\tau\left(y\right)d\left(x\right), d\left(x\right)\in C\left(a\right)$. Also, we get $ad(yx)=d(yx)a$ by the hypothesis. That is

$$a\left(\tau\left(y\right)d\left(x\right)+d\left(y\right)\sigma\left(x\right)\right)=\left(\tau\left(y\right)d\left(x\right)+d\left(y\right)\sigma\left(x\right)\right)a,\ \text{for all } x\in U.$$

We can apply Lemma 2.12 (i) to get

$$a\tau\left(y\right)d\left(x\right)+ad\left(y\right)\sigma\left(x\right)=\tau\left(y\right)d\left(x\right)a+d\left(y\right)\sigma\left(x\right)a,\ \text{for all } x\in U.$$

Since $\tau\left(y\right)d\left(x\right)\in C\left(a\right)$, we get

$$ad\left(y\right)\sigma\left(x\right)=d\left(y\right)\sigma\left(x\right)a,\ \text{for all } x\in U. \tag{3.10}$$

Thus we obtain $d\left(y\right)\sigma\left(x\right)\in C(a)$, for all $x\in U$. That is

$$d\left(y\right)\sigma\left(U\right)\subseteq C\left(a\right),\ \text{for all } \tau\left(y\right)\in C\left(a\right). \tag{3.11}$$

On the other hand, if $d^2(U)=(0)$, then $d=0$ by Lemma 2.11. Since $d\neq 0$, we can choose any $z\in U$ such that $d^2\left(z\right)\neq 0$. Let $\tau\left(y\right)=d\left(z\right)$. Since $d\left(y\right)\sigma\left(x\right)\in C\left(a\right)$, for all $x\in U, \tau\left(y\right)\in C\left(a\right)$, we have $d(y)\sigma(xr)=d\left(y\right)\sigma\left(x\right)\sigma(r)\in C\left(a\right)$ for all $x\in U, r\in N$. Thus

$$ad\left(y\right)\sigma\left(x\right)\sigma\left(r\right)=d\left(y\right)\sigma\left(x\right)\sigma\left(r\right)a,\ \text{for all } x\in U, r\in N.$$

Using the equation (3.10), we have

$$d\left(y\right)\sigma\left(x\right)a\sigma\left(r\right)=d\left(y\right)\sigma\left(x\right)\sigma\left(r\right)a,$$

and so

$$d\left(y\right)\sigma\left(x\right)[a,\sigma(r)]=0,\ \text{for all } x\in U, r\in N.$$

That is

$$\sigma^{-1}(d(y))U[\sigma^{-1}(a),r])=(0),\ \text{for all } r\in N.$$

This yields $d(y)=0$ or $a\in Z$ by Lemma 2.3 (i). If $d(y)=0$, then $d(\tau^{-1}(d(z))=0$. Using $\tau d=d\tau$, we have $d^2(z)=0$. But this contradicts $d^2(z)\neq 0$. Thus we must have $a\in Z$. This completes the proof. □

As immediate corollaries of Theorem 3.7 and Lemma 2.10 we give the following theorem.

3.8. Theorem. *Let N be a $3-$prime near-ring and U be a nonzero semigroup ideal of N. If $a\in N, d$ is a nonzero (σ,τ)-derivation on N such that $\sigma d=d\sigma, \tau d=d\tau$ and $[d(U),d(U)]=(0)$, then N is a commutative ring.*

3.9. Theorem. *Let N be a $2-$torsion free $3-$prime near-ring and U be a nonzero semigroup ideal of N. If d_1 is a derivation and d_2 is a (σ,τ)-derivation on N such that $d_2\tau=\tau d_2, d_2\sigma=\sigma d_2$ and $d_1(x)d_2(y)=-d_2(y)d_1(x)$, for all $x,y\in U$, then $d_1=0$ or $d_2=0$.*

Proof. Assume that $d_1\neq 0$ and $d_2\neq 0$. By the hypothesis, we have

(3.12) $\quad d_1(x)d_2(y)=-d_2(y)d_1(x)$, for all $x,y\in U$.

We may assume $d_1^2(U)\neq(0)\neq d_2^2(U^2)$ by Lemma 2.6 and Lemma 2.11. Let $w\in d_2(U^2)$. It is easy to see $w,-w\in d_2(U)$. If we take $T=d_2(U), S=d_1(U)$, then $[uv, d_2(U^2)]=(0)$, for all $u,v\in d_1(U)$ by Lemma 2.7. Thus $uv\in Z$ for all $u,v\in d_1(U)$ by Theorem 3.7. Also we have $d_1(x)d_1(y)\in Z$, for all $x,y\in U$. It follows that

$$d_1(x)d_1(x)d_1(y)=d_1(x)d_1(y)d_1(x).$$

Multipliying this equation by $d_1(y)$ from right hand, we have

$$d_1(x)d_1(x)d_1(y)d_1(y)=d_1(x)d_1(y)d_1(x)d_1(y).$$

Using $d_1(x)d_1(y), d_1(y)d_1(x)\in Z$ respectively in the last equation, we find that

$$d_1(x)d_1(y)d_1(x)d_1(y)=d_1(y)d_1(x)d_1(x)d_1(y).$$

Again using $d_1(y)d_1(x), d_1(x)d_1(y)\in Z$ respectively, we arrive at

$$d_1(y)d_1(x)d_1(x)d_1(y)=d_1(y)d_1(x)d_1(y)d_1(x)$$

and so

$$d_1(y)d_1(x)(d_1(x)d_1(y)-d_1(y)d_1(x))=0.$$

Since $d_1(y)d_1(x)$ is central, Lemma 2.1 (i) shows that for any $x,y\in U$, either $d_1(x)d_1(y)-d_1(y)d_1(x)=0$ or $d_1(y)d_1(x)=0$. Hence we get $d_1(x)d_1(y)=d_1(y)d_1(x)=0$, for all $x,y\in U$. That is $[d_1(U),d_1(U)]=(0)$. By Lemma 2.5, we get $d_1^2(U)=0$ or $d_1(U)\subset Z$. In the first case, we find that $d_1=0$ by Lemma 2.6. In the second case, we have N is a commutative ring by Lemma 2.4. But this fact that together with (3.12) shows that $2d_2(y)d_1(x)=0$ for all $x,y\in U$, i.e. $d_2(U)d_1(U)=(0)$. Therefore we get $d_1=0$ or $d_2=0$ from Lemma 2.1 (iii) and Lemma 2.9. Thus we must have $d_1=0$ or $d_2=0$. □

3.10. Theorem. *Let N be a $2-$torsion free $3-$prime near-ring and U be a nonzero semigroup ideal of N. If d_1 is a nonzero derivation and d_2 is a nonzero (σ,τ)-derivation on N such that $d_2\tau=\tau d_2, d_2\sigma=\sigma d_2$.*

(i) If $d_1(x)d_2(y)+d_2(y)d_1(x)\in Z$, for all $x,y\in U$ and at least one of $d_1(U)\cap Z$ and $d_2(U)\cap Z$ is nonzero, then N is a commutative ring.

(ii) If $xd_2(y)+d_2(y)x\in Z$, for all $x,y\in U$, and $U\cap Z\neq(0)$, then N is a commutative ring.

Proof. (i) Let $d_1(U) \cap Z \neq (0)$. Let $x \in U$ such that $d_1(x) \in Z \setminus \{0\}$ and $y \in U$. Then $d_1(x) d_2(y) + d_2(y) d_1(x) = 2d_1(x) d_2(y) \in Z$. Since N is a $2-$torsion free $3-$prime near-ring and $d_1(x) \in Z \setminus \{0\}$, we have $d_2(U) \subseteq Z$. Hence N is a commutative ring by Lemma 2.10.

(ii) Let $x \in U \cap Z$ and $y \in U$. Then, $xd_2(y) + d_2(y)x = 2xd_2(y) \in Z$. Thus, $d_2(U) \subseteq Z$. As in the proof of (i), we get N is a commutative ring. □

3.11. Theorem. *Let N be a $2-$torsion free $3-$prime near-ring, U be a nonzero semigroup ideal of N which is closed under addition. Suppose that N has nonzero derivation d_1 and nonzero (σ,τ)-derivation d_2 such that $d_1(x) d_2(y) + d_2(y) d_1(x) \in Z$ for all $x, y \in U$ and $d_1(U) \cap Z \neq (0)$ or $d_2(U) \cap Z \neq (0)$. Then N is a commutative ring.*

Proof. By Theorem 3.9, we cannot have $d_1(x) d_2(y) + d_2(y) d_1(x) = 0$ for all $x, y \in U^2$. Since $d_1(U^2) \subseteq U$, there exist $x_0, y_0 \in U^2$ such that $u_0 = d_1(x_0) d_2(y_0) + d_2(y_0) d_1(x_0)$ is a nonzero central element in U. If either of $d_1(u_0)$ and $d_2(u_0)$ is nonzero, our conclusion follows from Theorem 3.10. On the other hand, if $d_1(u_0) = d_2(u_0) = 0$, then $d_1(u_0x) d_2(u_0y) + d_2(u_0y) d_1(u_0x) = u_0d_1(x) \tau(u_0) d_2(y) + \tau(u_0) d_2(y) u_0d_1(x)$. Using $u_0 \in Z, \tau(u_0) \in Z$ in the last equation, we get

$$u_0\tau(u_0)(d_1(x) d_2(y) + d_2(y) d_1(x)) \in Z.$$

Since $0 \neq u_0\tau(u_0) \in Z$, we obtain that $d_1(x) d_2(y) + d_2(y) d_1(x) \in Z$. Hence N is a commutative ring by Theorem 3.10. □

References

[1] Ashraf, M., Asma A. and Ali, S. (σ,τ)-*Derivations on prime near rings*, Archivum Mathematicum (Brno), Tomus **40**, 281-286, (2004).

[2] Ashraf, M., Ali, S. *On* (σ,τ)-*Derivations of prime near-rings-II*, Sarajevo Journal of Math., **4** (16), 23-30, (2008).

[3] Bell, H. and Mason. *G. On derivations in near rings, Near rings and Near fields*, North-Holland Mathematical Studies **137**, 31-35, (1987).

[4] Bell, H. E. *On derivations in near-rings II*, Kluwer Academic Pub. Math. Appl., Dordr. **426**, 191-197, (1997).

[5] Bell, H. E. and Argaç, N. *Derivations, products of derivations and commutativity in near-rings*, Algebra Colloquium **8** (4), 399-477, (2001).

[6] Murray, F. J., Neumann, J. *On ring of operators*, Annals of Math. **37**, 116-229, (1936).

[7] Neumann, J. *On ring of operators III*, Annals of Math. **41**, 94-161, (1940).

[8] Gölbaşı, Ö., Aydın, N. *Results on prime near-ring with* (σ,τ)-*derivation*, Mathematical J. of Okayama Univ., **46**, 1-7, (2004).

[9] Gölbaşı, Ö., Aydın, N. *On near-ring ideals with* (σ,τ)-*derivation*, Archivum Math. (Brno), Tomus **43**, 87-92, 2007.

[10] Hartwig, J. T., Larsson, D. and Silvestrov, S. D. *Deformations of Lie algebras using* σ-*derivations*, Journal of Algebra, **295** (2), 314-361, (2006).

[11] Posner, E. C. *Derivations in Prime Rings*, Proc. Amer. Math. Soc. **8**, 1093-1100, (1957).

[12] Richard, L. Silvestrov, S. D. *Quasi-Lie structure of* σ-*derivations of* $C[t]$, Journal of Algebra, **319** (2), 1285-1304, (2008).

[13] Wang, X. K. *Derivations in Prime Near-rings*, Proc. Amer. Math. Soc. **121** (2), 361-366, (1994).

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Oscillation of fourth-order nonlinear neutral delay dynamic equations

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Abstract

In this work we establish some new sufficient conditions for oscillation of fourth-order nonlinear neutral delay dynamic equations of the form

$$(a(t)([x(t)-p(t)x(h(t))]^{\Delta\Delta\Delta})^{\alpha})^{\Delta}+q(t)x^{\beta}(g(t))=0, \quad t\in[t_0,\infty)_{\mathbb{T}},$$

where α and β are quotients of positive odd integers with $\beta \le \alpha$.

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1. Introduction

In this work we investigate the oscillatory behaviour of solutions of fourth order half-linear and sub-half-linear neutral delay dynamic equations of the form

$$(a(t)((x(t)-p(t)x(h(t)))^{\Delta\Delta\Delta})^{\alpha})^{\Delta}+q(t)x^{\beta}(g(t))=0, \quad t\in[t_0,\infty)_{\mathbb{T}} \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$ with the property that $0 \le t_0 \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$.

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$, introduced by Hilger [12] in order to unify the continuous and discrete analysis. By a time scale interval $[t_*,\infty)_{\mathbb{T}}$ it is meant $[t_*,\infty)\cap\mathbb{T}$.

We assume that the following conditions are satisfied:

(i) α, β are quotients of positive odd integers with $\beta \le \alpha$,

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(ii) $a : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}_+ := (0, \infty)$ satisfies $a^{\Delta}(t) \geq 0$ and

$$\int_{t_0}^{\infty} a^{-1/\alpha}(t)\Delta t = \infty, \tag{1.2}$$

(iii) $p, q : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}_+$ are rd-continous,

(iv) $g, h : \mathbb{T} \to \mathbb{T}$ satisfy $g(t) \leq t$, $h(t) \leq t$, $g^{\Delta}(t) > 0$, $h^{\Delta}(t) > 0$, and $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} h(t) = \infty$,

(v) $l(t) := h^{-1} \circ g(t)$ satisfies $l^{\Delta}(t) \geq 0$ and $\lim_{t\to\infty} l(t) = \infty$.

By a solution of (1.1), we mean a nontrivial at infinity function $x \in C_{rd}[t_{-1}, \infty)_{\mathbb{T}}$ that satisfies (1.1), where $t_{-1} = \inf\{g(t) : t \geq t_0\} \cap \inf\{h(t) : t \geq t_0\}$. We tacitly assume that (1.1) possesses such solutions. Recall that such a solution of Eq. (1.1) is called nonoscillatory if there exists a $t_0^* \geq t_0$ such that $x(t)x^{\sigma}(t) > 0$ for all $t \in [t_0^*, \infty)_{\mathbb{T}}$; otherwise, it is said to be oscillatory. Eq. (1.1) is oscillatory if all its solutions are oscillatory.

We note that there is an extensive study concerning the oscillation of second-order dynamic equations on time scales [1, 3, 7, 11, 14]. For some works on oscillation and asymptotic behaviour of third-order dynamic equations, see [5, 10]. Fourth-order dynamic equations are rarely considered in the literature due to difficulties peculiar to such equations [8, 13]. Our aim in this paper is to make a contribution to the oscillation of fourth-order equations of the form (1.1).

2. Preliminaries

To start, we first provide some notations of the time scale calculus to be used in this work. For more details we refer the reader to [3].

2.1. Definition. Let $\mathbb{T}$ be a time scale and $t \in \mathbb{T}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$.

2.2. Definition. A point $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is called *right-scattered, right-dense, left-scattered* and *left-dense* if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$ and $\rho(t) = t$ holds, respectively. Points that are left-dense and right-dense at the same time are called *dense*. The set $\mathbb{T}^{\kappa}$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

2.3. Definition. Let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of f at $t \in \mathbb{T}^{\kappa}$, denoted by $f^{\Delta}(t)$, to be the number (provided it exists) with the property such that for every $\epsilon > 0$, there exists a neighbourhood $\mathbb{U}$ of t with

$$\left| f^{\sigma}(t) - f(s) - f^{\Delta}(t)\left[\sigma(t) - s\right] \right| \leq \epsilon \left|\sigma(t) - s\right| \text{ for all } s \in \mathbb{U},$$

where and in the sequel $f^{\sigma}(t) := f(\sigma(t))$ is used.

2.4. Definition. A function $f : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous (rd-continuous) if f is continuous at right-dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at left-dense points in $\mathbb{T}$.

Every rd-continuous function has an antiderivative. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of a function $f : \mathbb{T}^{\kappa} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$. In this case the integral of f is defined by

$$\int_a^b f(s)\Delta s = F(b) - F(a).$$

Riemann and Lebesgue integrals on an arbitrary time scale as well as improper integrals are introduced in [2, 9].

2.5. Definition. Taylor monomials $h_n(t,s) : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ are defined [3] by

$$h_{n+1}(t,s) = \int_s^t h_n(\tau,s)\Delta\tau, \quad n = 1, 2, \dots$$
$$h_0(t,s) = 1.$$

We need the following lemmas.

2.6. Lemma. *Let (1.2) hold and*

(2.1) $\quad 0 < p(t) \le 1$ *eventually.*

If x is an eventually positive solution of (1.1), then for $z(t) = x(t) - p(t)x(h(t))$ there are only the following three possibilities:

(a) $z(t) > 0,\ z^{\Delta}(t) > 0,\ z^{\Delta\Delta}(t) > 0,\ z^{\Delta\Delta\Delta}(t) > 0,\ z^{\Delta^4}(t) < 0$;
(b) $z(t) > 0,\ z^{\Delta}(t) > 0,\ z^{\Delta\Delta}(t) < 0,\ z^{\Delta\Delta\Delta}(t) > 0,\ z^{\Delta^4}(t) < 0$;
(c) $z(t) < 0,\ z^{\Delta}(t) > 0,\ z^{\Delta\Delta}(t) < 0,\ z^{\Delta\Delta\Delta}(t) > 0,\ z^{\Delta^4}(t) < 0$

for t sufficiently large.

Proof. We see from Eq. (1.1) that $(a(t)(z^{\Delta\Delta\Delta})^{\alpha})^{\Delta} < 0$ eventually. So, $a\,(z^{\Delta\Delta\Delta})^{\alpha}$ is eventually monotone. Suppose that $z^{\Delta\Delta\Delta}(t) < 0$ for t sufficiently large. By using (1.2) we see that $z^{\Delta\Delta}(t) \to -\infty$ as $t \to \infty$ and hence $z(t) \to -\infty$ as $t \to \infty$. In particular, we have

$$x(t) \le p(t)x(h(t)) \le x(h(t))$$

and so x is bounded. But this implies that z is also bounded, a contradiction.

Thus we must have $z^{\Delta\Delta\Delta}(t) > 0$. Since $a^{\Delta}(t) \ge 0$, it follows from

$$(a(t)(z^{\Delta\Delta\Delta})^{\alpha})^{\Delta} = a^{\sigma}(t)((z^{\Delta\Delta\Delta})^{\alpha})^{\Delta} + a^{\Delta}(t)(z^{\Delta\Delta\Delta})^{\alpha}$$

that $((z^{\Delta\Delta\Delta}(t))^{\alpha})^{\Delta} < 0$ eventually. Now using the time scales chain rule [3] with $y = z^{\Delta\Delta\Delta}$, we obtain

$$(y^{\alpha})^{\Delta}(t) = \alpha y^{\Delta}(t) \int_0^1 (hy^{\sigma}(t) + (1-h)y(t))^{\alpha-1} dh$$

Since the integral is nonnegative, we have $y^{\Delta}(t) < 0$, i.e.,

(2.2) $\quad z^{\Delta^4}(t) < 0.$

If $z(t) > 0$, then in view of (2.2) it follows from Kiguradze's lemma, see [4, 6], that either (a) or (b) holds. In case $z(t) < 0$, we see as above that $z(t)$ is bounded, and hence (c) holds.

□

2.7. Lemma. *Let (1.2) hold and*

(2.3) $\quad -1 < p(t) \le 0$ *eventually.*

If x is an eventually positive solution of (1.1), then for $z(t) = x(t) - p(t)x(h(t))$ there are only two possibilities (a) and (b) of Lemma 2.6.

Proof. It suffices to note that $z(t)$ is eventually positive due to $z(t) \ge x(t)$. □

2.8. Lemma. [5, Lemma 4]. *If a function* y *satisfies*

$$y(t) > 0,\ y^{\Delta}(t) > 0,\ y^{\Delta\Delta}(t) > 0,\ y^{\Delta\Delta\Delta} \leq 0$$

eventually for $t \in \mathbb{T}$, *then*

$$\liminf_{t\to\infty} \frac{ty(t)}{h_2(t,t_0)y^{\Delta}(t)} \geq 1. \tag{2.4}$$

2.9. Lemma. [5]. *Suppose that* $|y|^{\Delta}$ *is of one sign on* $[t_0,\infty)_{\mathbb{T}}$ *and* $0 < \lambda < 1$. *Then*

$$\frac{(|y|^{1-\lambda})^{\Delta}}{1-\lambda} \leq \frac{|y|^{\Delta}}{|y|^{\lambda}} \quad \text{on } [t_0,\infty)_{\mathbb{T}}. \tag{2.5}$$

3. Main Results

Our first result is as follows.

3.1. Theorem. *Let (i)–(v) be satisfied, and* (2.1) *hold. Suppose that*

$$\limsup_{t\to\infty} \frac{1}{a(g(t))} \int_{g(t)}^{t} q(s)h_3^{\beta}(g(s),t_0)\Delta s > 1, \tag{3.1}$$

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \left[\frac{1}{a(u)} \int_{u}^{t} (g(t)-g(s))^{\beta} g^{\beta}(s) q(s)\Delta s\right]^{1/\alpha} \Delta u > 1, \tag{3.2}$$

and

$$\limsup_{t\to\infty} \frac{1}{a(l(t))} \int_{l(t)}^{t} q(s)h_3^{\beta}(l(t),l(s))\Delta s > 1 \tag{3.3}$$

when $\beta = \alpha$, *while*

$$\int_{t_0}^{\infty} a^{-\beta/\alpha}(s)q(s)h_3^{\beta}(g(s),t_0)\Delta s = \infty, \tag{3.4}$$

$$\limsup_{t\to\infty} \int_{t_0}^{\infty} \left[\frac{1}{a(u)} \int_{u}^{t} (t-g(s))^{\beta} g^{\beta}(s) q(s)\Delta s\right]^{1/\alpha} \Delta u = \infty, \tag{3.5}$$

and

$$\int_{t_0}^{\infty} a^{-\beta/\alpha}(s)q(s)h_3^{\beta}(s,l(s))\Delta s = \infty \tag{3.6}$$

when $\beta < \alpha$.

Then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). We may assume that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) > 0$ eventually. Let $z(t) = x(t) - p(t)x(h(t))$. By Lemma 2.6, we need to consider three possible cases.

Suppose (a) holds. For any given $c_1 \in (0,1)$ there exists $t_1 > t_0$ such that

$$z^{\Delta\Delta}(t) \geq c_1 t z^{\Delta\Delta\Delta}(t), \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.7}$$

From (2.4) with $y = z^{\Delta}$, for any given c_2, $0 < c_2 < 1$, there exists $t_1' > t_1$ such that

$$y(t) \geq c_2 \frac{h_2(t,t_0)}{t} y^{\Delta}(t), \quad t \in [t_1',\infty)_{\mathbb{T}}. \tag{3.8}$$

Combining (3.7) and (3.8) we get

$$z^{\Delta}(t) \geq c\, h_2(t,t_0) z^{\Delta\Delta\Delta}(t), \quad t \in [t_1',\infty)_{\mathbb{T}}, \tag{3.9}$$

where $c := c_1c_2 \in (0,1)$ is an arbitrary constant. In view of the monotonicity of $z^{\Delta\Delta\Delta}$, it follows from (3.9) that there is a $t_2 \geq t_1'$ such that

$$z(g(t)) \geq c\, h_3(g(t),t_0) z^{\Delta\Delta\Delta}(g(t)), \quad t \in [t_2,\infty)_{\mathbb{T}}. \tag{3.10}$$

Using (3.10) and the fact that $x(t) \geq z(t)$ in Eq. (1.1), we obtain

$$(3.11) \quad (a(t)w^{\alpha}(t))^{\Delta} + c^{\beta}q(t)h_3^{\beta}(g(t), t_0)w^{\beta}(g(t)) \leq 0, \quad t \in [t_2, \infty)_{\mathbb{T}},$$

where $w = z^{\Delta\Delta\Delta}$. Integrating (3.11) from $g(t)$ to t leads to

$$\begin{aligned} a(g(t))w^{\alpha}(g(t)) &\geq c^{\beta}\int_{g(t)}^{t} q(s)h_3^{\beta}(g(s), t_0)w^{\beta}(g(s))\Delta s \\ &\geq c^{\beta}w^{\beta}(g(t))\int_{g(t)}^{t} q(s)h_3^{\beta}(g(s), t_0)\Delta s \end{aligned}$$

and hence

$$w^{\alpha-\beta}(g(t)) \geq \frac{c^{\beta}}{a(g(t))}\int_{g(t)}^{t} q(s)h_3^{\beta}(g(s), t_0)\Delta s,$$

which contradicts (3.1) when $\beta = \alpha$ by taking the limsup of both sides as $t \to \infty$ and then letting $c \to 1^-$. Let $\beta < \alpha$. Setting $v(t) = a(t)w^{\alpha}(t)$ in (3.11) and increasing the size of t_2 if necessary we have

$$v^{\Delta} + c^{\beta}a^{-\beta/\alpha}(t)q(t)h_3^{\beta}(g(t), t_0)v^{\beta/\alpha} \leq 0, \quad t \in [t_2, \infty)_{\mathbb{T}}$$

or

$$-\frac{v^{\Delta}}{v^{\beta/\alpha}} \geq c^{\beta}a^{-\beta/\alpha}(t)q(t)h_3^{\beta}(g(t), t_0), \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

Integrating this inequality from t_2 to t and applying Lemma 2.9 we get

$$-\frac{v^{1-\beta/\alpha}(t)}{1-\beta/\alpha} + \frac{v^{1-\beta/\alpha}(t_2)}{1-\beta/\alpha} \geq -\int_{t_2}^{t} c^{\beta}a^{-\beta/\alpha}(s)q(s)h_3^{\beta}(g(s), t_0)\Delta s.$$

Letting $t \to \infty$ we obtain a contradiction to (3.4).

Suppose (b) holds. For any given $k \in (0,1)$ there exists $t_1 \geq t_0$ such that

$$(3.12) \quad z(g(t)) \geq k\, g(t)z^{\Delta}(g(t)), \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Using $x(t) \geq z(t)$ and (3.12) in Eq. (1.1) leads to

$$(3.13) \quad (a(t)(y^{\Delta\Delta}(t))^{\alpha})^{\Delta} + k^{\beta}g^{\beta}(t)q(t)y^{\beta}(g(t)) \leq 0, \quad t \in [t_1, \infty)_{\mathbb{T}},$$

where $y = z^{\Delta}$. In view of $y(t) > 0$, $y^{\Delta}(t) < 0$, and $y^{\Delta\Delta}(t) > 0$, we have from (3.13)

$$(3.14) \quad y^{\Delta\Delta}(u) \geq \left[\frac{k^{\beta}}{a(u)}\int_{u}^{t} g^{\beta}(s)q(s)y^{\beta}(g(s))\Delta s\right]^{1/\alpha}, \quad t \geq u \geq t_1.$$

Clearly,

$$(3.15) \quad -y(g(s)) \leq \int_{g(s)}^{g(t)} y^{\Delta}(u)\Delta u \leq (g(t) - g(s))y^{\Delta}(g(t)), \quad t \geq s \geq t_1.$$

From (3.14) and (3.15) we get

$$y^{\Delta\Delta}(u) \geq \left[\frac{k^{\beta}}{a(u)}\int_{u}^{t} g^{\beta}(s)(g(t) - g(s))^{\beta}q(s)(-y^{\Delta}(g(t)))^{\beta}\Delta s\right]^{1/\alpha},$$

and hence

$$-y^{\Delta}(g(t)) \geq k^{\beta/\alpha}\int_{g(t)}^{t}\left[\frac{1}{a(u)}\int_{u}^{t} g^{\beta}(s)(g(t) - g(s))^{\beta}q(s)\Delta s\right]^{1/\alpha}\Delta u\,(-y^{\Delta}(g(t)))^{\beta/\alpha}$$

or

$$(-y^{\Delta}(g(t)))^{1-\beta/\alpha} \geq k^{\beta/\alpha}\int_{g(t)}^{t}\left[\frac{1}{a(u)}\int_{u}^{t} g^{\beta}(s)(g(t) - g(s))^{\beta}q(s)\Delta s\right]^{1/\alpha}\Delta u,$$

which as in case (a) results in a contradiction with (3.2) when $\beta = \alpha$. Let $\beta < \alpha$. From (3.15) we have

$$-y(g(s)) \le (t - g(s))y^{\Delta}(t), \quad t \ge s \ge t_1.$$

Using this inequality in (3.14) and setting $v = -y^{\Delta}$ we have

$$-\frac{v^{\Delta}(u)}{v^{\beta/\alpha}(u)} \ge \left[\frac{k^{\beta}}{a(u)} \int_u^t g^{\beta}(s)(t - q(s))^{\beta} \Delta s\right]^{1/\alpha}, \quad t \ge u \ge t_1.$$

The rest is similar to that of case (a) above and hence is omitted.

Suppose (c) holds. It follows that

$$x(g(t)) \ge y(h^{-1} \circ g(t)), \quad t \in [t_1, \infty)_{\mathbb{T}},$$

where $y = -z$. By Eq. (1.1), we may write that

$$(a(t)(y^{\Delta\Delta\Delta})^{\alpha})^{\Delta} \ge q(t)y^{\beta}(l(t)), \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.16}$$

In view of

$$y^{\Delta}(t) < 0, \quad y^{\Delta\Delta}(t) > 0, \quad y^{\Delta\Delta\Delta}(t) < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}$$

and

$$y^{\Delta\Delta}(s) \ge h_1(t, s)(-y^{\Delta\Delta\Delta}(t)),$$

we see that

$$y(s) \ge h_3(t, s)(-y^{\Delta\Delta\Delta}(l(t))), \quad t \ge s \ge t_1.$$

Replacing s by $l(s)$ and t by $l(t)$ in the above inequality leads to

$$y(l(s)) \ge h_3(l(t), l(s))(-y^{\Delta\Delta\Delta}(t)), \quad l(t) \ge l(s) \ge t_1. \tag{3.17}$$

Integrating (3.16)) and using (3.17) we obtain

$$a(l(t))w^{\alpha}(l(t)) \ge \int_{l(t)}^t q(s)h_3^{\beta}(l(t), l(s))w^{\beta}(l(s))\Delta s, \quad w := -y^{\Delta\Delta\Delta},$$

and hence

$$w^{\alpha-\beta}(l(t)) \ge \frac{1}{a(l(t))} \int_{l(t)}^t q(s)h_3^{\beta}(l(t), l(s))\Delta s.$$

This inequality contradicts (3.3) when $\beta = \alpha$ by taking the limsup of both sides as $t \to \infty$. It remains to consider $\beta < \alpha$. Set $v = a(t)w^{\alpha}$ and $w = -y^{\Delta\Delta\Delta}$ in (3.16). Then we have

$$-v^{\Delta}(t) \ge q(t)y^{\beta}(l(t)), \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

As in (3.17) we can obtain

$$y(l(t)) \ge h_3(t, l(t))w(t), \quad t \ge s \ge t_1.$$

Thus

$$-v^{\Delta}(t) \ge q(t)h_3^{\beta}(t, l(t))w^{\beta}(t), \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

The remainder is similar to that of cases (a) and (b) and hence is omitted.

□

3.2. Theorem. *Let (i)–(v) be satisfied, and* (2.1) *hold. In addition to* (3.3) *and* (3.6), *suppose that for* $k \in \{1, 3\}$,

$$\limsup_{t\to\infty} h_k^{\beta}(g(t), t_0)h_{3-k}^{\beta}(t, g(t))\frac{1}{a(t)} \int_t^{\infty} q(s)\Delta s > 1 \quad \textit{when } \beta = \alpha \tag{3.18}$$

and

$$\int_{t_0}^{\infty} h_k^{\beta}(g(t), t_0)h_{3-k}^{\beta}(t, g(t))a^{-\beta/\alpha}(t)q(t)\Delta t = \infty \quad \textit{when } \beta < \alpha. \tag{3.19}$$

Then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$, and let $z(t) = x(t) - p(t)x(h(t))$. As in the proof of Theorem 3.1 we need to distinguish three possible cases.

We may claim that

$$z(g(t)) \geq h_k(g(t), t_1)h_{3-k}(t, g(t))z^{\Delta\Delta\Delta}(t) \quad \text{for } k \in \{1, 3\} \text{ and } g(t) \geq t_1 \tag{3.20}$$

when (a) and (b) are satisfied.

Indeed, if (a) holds, then integrating the inequality

$$z^{\Delta\Delta}(v) \geq z^{\Delta\Delta}(v) - z^{\Delta\Delta}(u_1) = \int_{u_1}^{v} z^{\Delta\Delta\Delta}(s)\Delta s \geq h_1(v, u_1)z^{\Delta\Delta\Delta}(v)$$

twice leads to

$$z(v) \geq h_3(v, u_1)z^{\Delta\Delta\Delta}(v), \quad v \geq u_1 \geq t_1. \tag{3.21}$$

Also

$$z^{\Delta\Delta}(v) \geq z^{\Delta\Delta}(v) - z^{\Delta\Delta}(u) = \int_{u}^{v} z^{\Delta\Delta\Delta}(s)\Delta s \geq h_1(v, u)z^{\Delta\Delta\Delta}(v)$$

gives

$$z^{\Delta}(v) \geq h_2(v, u)z^{\Delta\Delta\Delta}(v). \tag{3.22}$$

Putting $v = g(t) \geq t_1$ and $v = t$ into (3.22), we have

$$z^{\Delta}(g(t)) \geq h_2(t, g(t))z^{\Delta\Delta\Delta}(t),$$

i.e.,

$$z^{\Delta_k}(g(t)) \geq h_{3-k}(t, g(t))z^{\Delta\Delta\Delta}(t) \quad \text{for } k \in \{1, 3\} \text{ and } g(t) \geq t_1. \tag{3.23}$$

If (b) holds, then

$$z(v) \geq z(v) - z(u_1) = \int_{u_1}^{v} z^{\Delta}(s)\Delta s \geq h_1(v, u_1)z^{\Delta}(v). \tag{3.24}$$

Clearly, (3.21) and (3.24) can be written at once

$$z(v) \geq h_k(v, u_1)z^{\Delta^k}(v) \quad \text{for } k \in \{1, 3\} \text{ and } v \geq u_1 \geq t_1,$$

and hence

$$z(g(t)) \geq h_k(g(t), t_1)z^{\Delta^k}(g(t)) \quad \text{for } k \in \{1, 3\} \text{ and } g(t) \geq t_1 \tag{3.25}$$

when $g(t) \geq t_1$. From (3.23) and (3.25) the claim follows.

Now by using $x(t) \geq z(t)$ in (1.1) and integrating the resulting inequality from t to u and then letting $u \to \infty$, we have

$$w^{\alpha}(t) \geq \left[\frac{1}{a(t)} \int_{t}^{\infty} q(s)\Delta s\right] z^{\beta}(g(t)), \quad t \geq t_1, \tag{3.26}$$

where $w = z^{\Delta\Delta\Delta}$. Using (3.20) in (3.26) leads to

$$w^{\alpha-\beta}(t) \geq h_k(g(t), t_1)h_{3-k}(t, g(t)) \left[\frac{1}{a(t)} \int_{t}^{\infty} q(s)\Delta s\right], \quad t \geq t_1,$$

which contradicts (3.18) when $\beta = \alpha$. If $\beta < \alpha$, we first write from (1.1) that

$$(a(t)w^{\alpha}(t))^{\Delta} + q(t)z^{\beta}(g(t)) \leq 0, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.27}$$

Using (3.20) and $v = a(t)w^{\alpha}$ in (3.27), we have

$$-v^{\Delta} \geq h_k^{\beta}(g(t), t_1)h_{3-k}^{\beta}(t, g(t))a^{-\beta/\alpha}(t)q(t)v^{\beta/\alpha}, \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

The rest of the proof is similar to that above and hence is omitted.

Finally, we need to consider the case (c). Since the proof for this case is similar to that of Theorem 3.1-case (c), we omit the details.

□

In case (2.3) holds, we have the following similar theorems.

3.3. Theorem. *Let (i)–(v) be satisfied, and* (2.3) *hold. Suppose that*

$$\limsup_{t\to\infty} \frac{1}{a(g(t))} \int_{g(t)}^{t} q(s)[1+p(s)]^{\beta} h_3^{\beta}(g(s)t_0)\Delta s > 1, \tag{3.28}$$

and

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \left[\frac{1}{a(u)} \int_{u}^{t} (g(t)-g(s))^{\beta}[1+p(s)]^{\beta} q(s)\Delta s\right]^{1/\alpha} \Delta u > 1 \tag{3.29}$$

when $\beta = \alpha$*, while*

$$\int_{t_0}^{\infty} a^{-\beta/\alpha}(s)[1+p(s)]^{\beta} q(s) h_3^{\beta}(g(s),t_0)\Delta s = \infty \tag{3.30}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{\infty} \left[\frac{1}{a(u)} \int_{u}^{t} (t-g(s))^{\beta} g^{\beta}(s)[1+p(s)]^{\beta} q(s)\Delta s\right]^{1/\alpha} \Delta u = \infty \tag{3.31}$$

when $\beta < \alpha$.

Then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(g(t)) > 0$, and $x(\tau(t)) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. It is clear that

$$z(t) = x(t) - p(t)x(h(t)) \geq x(t).$$

By Lemma 2.7, $z^{\Delta}(t)$ is eventually positive, and so

$$x(t) = z(t) + p(t)x(h(t)) \geq z(t) + p(t)z(h(t)) \geq [1+p(t)]z(t)$$

for $t \in [t_1,\infty)_{\mathbb{T}}$ for some $t_1 \geq t_0$. In view of this inequality and Eq. (1.1) we get

$$(a(t)(z^{\Delta\Delta\Delta}(t))^{\alpha})^{\Delta} + q(t)[1+p(g(t))]^{\beta} z^{\beta}(g(t)) \leq 0, \quad t \in [t_0,\infty)_{\mathbb{T}}. \tag{3.32}$$

The remainder of the proof proceeds from (3.32) similarly as in the cases (a) and (b) of that of Theorem 3.1. □

3.4. Theorem. *Let (i)–(v) be satisfied, and* (2.3) *hold. Suppose that for* $k \in \{1,3\}$,

$$\limsup_{t\to\infty} h_k^{\beta}(g(t),t_0) h_{3-k}^{\beta}(t,g(t)) \frac{1}{a(t)} \int_{t}^{\infty} [1+p(g(s))]^{\beta} q(s)\Delta s > 1 \quad \textit{when } \beta = \alpha \tag{3.33}$$

and

$$\int_{t_0}^{\infty} h_k^{\beta}(g(t),t_0) h_{3-k}^{\beta}(t,g(t)) a^{-\beta/\alpha}(t)[1+p(g(t))]^{\beta} q(t)\Delta t = \infty \quad \textit{when } \beta < \alpha. \tag{3.34}$$

Then Eq. (1.1) *is oscillatory.*

The results of this paper are presented in a form which is essentially new and of high degree of generality. We note that the obtained results when $\beta = \alpha$ (half-linear case) are not applicable to equations of type (1.1) with $g(t) = t$. This means that the delays generate oscillation.

It is possible to formulate the corresponding theorems and give illustrative examples for special time scales such as $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}}$ with $q > 1$, $\mathbb{T} = \mathbb{N}^2$. The details are left to the reader.

It would be of interest to study the oscillatory behaviour of all solutions of (1.1) when $\beta > \alpha$ (super half-linear case) or $p(t) \leq -1$ or $p(t) > 1$.

References

[1] D. R. Anderson and A. Zafer, Interval criteria for second-order super-half-linear functional dynamic equations with delay and advance arguments. J. Difference Equ. Appl. **16** (2010), 917–930.

[2] M. Bohner, G. Sh. Guseinov, Improper integrals on time scales. Special issue: dynamic equations on time scales. Dynam. Systems Appl. **12** (2003), 45–65.

[3] M. Bohner and A. Peterson, Dynamic Equations on Time-Scales : An Introduction with Applications, Birkhauser, Boston 2001.

[4] M. Bohner and R. P. Agarwal, Basic calculus on time scales and some of its applications. Result Math. **35** (1999), 3–22

[5] L. Erbe, A. Peterson, and S. Saker, Hille and Nehari type criteria for third order dynamic equations, J. Math. Anal. Appl. **333** (2007), 505–522.

[6] L. Erbe, R. Mert, A. Peterson, and A, Zafer, Oscillation of even order nonlinear delay dynamic equations on time scales. In: Czech Math J

[7] S.R. Grace, M. Bohner, and R.P. Agarwal, On the oscillation of second order half linear dynamic equations, J. Diff. Eqns. Appl. **15** (2009), 451–460.

[8] S. R. Grace, M. Bohner, and S. Sun, Oscillation of fourth-order dynamic equations, Hacet. J. Math. Stat. **39** (4) (2010), 545–553

[9] G. Sh. Guseionov, Integration on time scales. J. Math. Anal. Appl. **285** (2003), 107?127.

[10] Z. Han, T. Li, S. Sun, and C. Zhang, Oscillation behaviour of third order neutral Emden-Fowler delay dynamic equations on time-scales, Adv. Diff. Eqns. **2010** (2010), Article ID.586312, 23 pages.

[11] Z. Han, S. Sun and B. Shi, Oscillation criteria for a class of second order Emden-Fowler delay dynamic equations on time-scales, J. Math. Anal. Appl. **334** (2007), 847–858.

[12] S. Hilger, *Ein Maβkettenkalkül mit A nwendung auf Zentrumsmannigfaltigkeiten*, PhD thesis, Universität Würzburg, 1988.

[13] T. Li, E. Thandapani, and S. Tang, Oscillation theorems for fourth-order delay dynamic equations on time scales, Bull. Math. Anal. Appl. **3** (2011), 190–199.

[14] A. Zafer, On oscillation and nonoscillation of second-order dynamic equations. Appl. Math. Lett. 22 (2009), no. 1, 136–141,

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Chebyshev-type matrix polynomials and integral transforms

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Abstract

In this study we introduce a new type generalization of Chebyshev matrix polynomials of second kind by using the integral representation. We obtain their matrix recurrence relations, matrix differential equation and generating matrix functions. We investigate operational rules associated with operators corresponding to Chebyshev-type matrix polynomials of second kind. Furthermore, in order to give qualitative properties of this integral transform, we introduce the Chebyshev-type matrix polynomials of first kind.

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1. Introduction

Matrix generalization of special functions has become important in the last two decades. Extension of the matrix framework of the classical families of Hermite, Laguerre, Jacobi, Bessel, Gegenbauer and Pincherle matrix polynomials are introduced in [11, 14, 15, 19, 25, 21] and some generalized forms are studied in [1, 20, 22, 26, 28]. Moreover, some properties of the these matrix polynomials are given in [3, 4, 6, 7, 16, 24]. Chebyshev matrix polynomials of first kind are introduced by Defez and Jódar starting from the hypergeometric matrix function. Some properties such as Rodrigues formula, three-term recurrences relation and orthogonality property are studied in [10]. Second kind Chebyshev matrix polynomials are defined in [5] by using integral representation method. Furthermore generating matrix function and some families of bilinear and bilateral generating matrix functions for Chebyshev matrix polynomials of the second kind are derived in [2].

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Throughout this paper, the zero matrix and identity matrix will be denoted by $\mathbf{0}$ and I, respectively. If A is a matrix in $\mathbb{C}^{r\times r}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. Its 2-norm is denoted by $\|A\|$ and defined by

$$\|A\| = \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2},$$

where for a vector y in $\mathbb{C}^r$, $\|y\|_2 = \left(y^T, y\right)^{\frac{1}{2}}$ is the Euclidean norm of y. If $f\left(z\right)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{r\times r}$ such that $\sigma(A)\subset\Omega$, then from the properties of matrix functional calculus in [13, p. 558], it follows that $f(A)g(A) = g(A)f(A)$. Hence, if $B\in\mathbb{C}^{r\times r}$ is a matrix for which $\sigma(B)\subset\Omega$ and $AB = BA$, then $f(A)g(B) = g(B)f(A)$. Let A be a matrix in $\mathbb{C}^{r\times r}$ such that $\mathrm{Re}\left(z\right) > 0$ for every eigenvalues $z\in\sigma(A)$. Then we say that A is a positive stable matrix.

Let A be a positive stable matrix. Then two-variable Hermite matrix polynomials are defined in [5] by

$$H_n\left(x,y,A\right) = \sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\left(-1\right)^k n!\left(x\sqrt{2A}\right)^{n-2k}y^k}{k!\left(n-2k\right)!}, \tag{1.1}$$

which satisfy the recurrences

$$\frac{\partial}{\partial x}H_n\left(x,y,A\right) = \sqrt{2A}nH_{n-1}\left(x,y,A\right), \tag{1.2}$$

$$\frac{\partial}{\partial y}H_n\left(x,y,A\right) = -n\left(n-1\right)H_{n-2}\left(x,y,A\right), \tag{1.3}$$

$$H_{n+1}\left(x,y,A\right) = \left(x\sqrt{2A} - 2\left(\sqrt{2A}\right)^{-1}y\frac{\partial}{\partial x}\right)H_n\left(x,y,A\right). \tag{1.4}$$

Also, second order matrix differential equation

$$\left[y\frac{\partial^2}{\partial x^2}I - xA\frac{\partial}{\partial x} + nA\right]H_n\left(x,y,A\right) = \mathbf{0} \tag{1.5}$$

and the expression

$$\sum_{n=0}^{\infty}\frac{H_n\left(x,y,A\right)}{n!}t^n = \exp\left(xt\sqrt{2A} - yt^2 I\right) \tag{1.6}$$

are given in [5].

For a positive stable matrix A, the second kind Chebyshev matrix polynomials with two variables are defined in [5] by

$$U_n\left(x,y,A\right) = \sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\left(-1\right)^k\left(n-k\right)!\left(x\sqrt{2A}\right)^{n-2k}y^k}{k!\left(n-2k\right)!}. \tag{1.7}$$

These matrix polynomials satisfy integral representation

$$U_n\left(x,y,A\right) = \frac{1}{n!}\int_0^{\infty}e^{-t}t^n H_n\left(x,\frac{y}{t},A\right)dt. \tag{1.8}$$

It has already been shown that most of the properties of $U_n\left(x,y,A\right)$, linked to the ordinary case by

$$U_n\left(x,y,A\right) = y^{\frac{n}{2}}U_n\left(\frac{x}{\sqrt{y}},A\right),$$

can be directly inferred from those of the $H_n(x,y,A)$ and from the integral representation given in (1.8).

The aim of this paper is to introduce a generalization for Chebyshev matrix polynomials by modifiying the integral transform. The organization of this paper is as follows. In section 2, we define Chebyshev-type matrix polynomials of second kind and give an explicit expression, recurrence relations, matrix differential equation and generating matrix functions. Besides, we focus on two index two variable second kind Chebyshev-type matrix polynomials. Section 3 deals with operational identities which yield different view for Chebyshev-type matrix polynomials of second kind. Finally in section 4, we give the definition of the Chebyshev-type matrix polynomials of the first kind.

2. Second Kind Chebyshev-type Matrix Polynomials with Two-Variable

As already remarked, integral transform relating Chebyshev and Hermite matrix polynomials are not new. Therefore we can introduce a new generalization for the second kind Chebyshev matrix polynomials with two variables by modifiying the integral transfrom as:

$$U_n(x,y,A,B) = \frac{1}{n!}\int_0^\infty e^{-Bt}t^n H_n\left(x,\frac{y}{t},A\right)dt, \tag{2.1}$$

where A and B are positive stable matrices in $\mathbb{C}^{r\times r}$ and $AB = BA$.

We note that for the case $A = [2]_{1\times 1}$ and $B = [1]_{1\times 1}$, the expression (2.1) coincides with the formula which was proved by Dattoli ([8]) for the scalar second kind Chebyshev polynomials with two variables.

The use of the identity (1.1) allows us the explicit expression for $U_n(x,y,A,B)$ in the form

$$U_n(x,y,A,B) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n-k)! B^{k-n-1}\left(x\sqrt{2A}\right)^{n-2k} y^k}{k!\,(n-2k)!}. \tag{2.2}$$

It is clear from (2.2) that

$$U_{-1}(x,y,A,B) = \mathbf{0}, \quad U_0(x,y,A,B) = B^{-1}, \quad U_1(x,y,A,B) = x\sqrt{2A}B^{-2}.$$

In addition, we can write

$$U_n(x,y,A,I) = U_n(x,y,A), \qquad U_n(x,1,A,I) = U_n(x,A),$$
$$U_n(x,0,A,B) = B^{-(n+1)}\left(x\sqrt{2A}\right)^n, \quad U_{2n}(0,y,A,B) = (-1)^n B^{-(n+1)} y^n.$$

In order to investigate some important properties, we give the generating matrix function of second kind Chebyshev-type matrix polynomials with two variables in the following proposition.

2.1. Proposition. *Let A and B be positive stable matrices in $\mathbb{C}^{r\times r}$ and $AB = BA$. Then the second kind Chebyshev-type matrix polynomials with two variables have the generating matrix function*

$$\sum_{n=0}^\infty U_n(x,y,A,B)\,z^n = \left(B - xz\sqrt{2A} + yz^2 I\right)^{-1}, \tag{2.3}$$

where $\left\|xz\sqrt{2A} - yz^2 I\right\| < \|B\|$.

Proof. Multiplying both sides of (2.1) by z^n, summing up over n, using (1.6) and then integrating over t, we have (2.3). □

2.2. Theorem. *Let A and B be positive stable matrices in $\mathbb{C}^{r\times r}$ and $AB = BA$. Then the second kind Chebyshev-type matrix polynomials with two variables have the generating matrix function*

$$\sum_{n=0}^{\infty}\frac{(n+m)!}{n!m!}U_{n+m}\left(x,y,A,B\right)z^n$$
$$=\left(B-x\sqrt{2A}z+yz^2I\right)^{-(m+1)}U_m\left(xI-\left(\sqrt{\frac{A}{2}}\right)^{-1}yz,\left(B-x\sqrt{2A}z+yz^2I\right)y,A\right),$$

where $\left\|xz\sqrt{2A}-yz^2I\right\|<\left\|B\right\|.$

Proof. From (2.1) we have,

$$\sum_{n=0}^{\infty}\frac{(n+m)!}{n!m!}U_{n+m}\left(x,y,A,B\right)z^n=\frac{1}{m!}\int_0^{\infty}e^{-Bt}t^m\sum_{n=0}^{\infty}H_{n+m}\left(x,\frac{y}{t},A\right)\frac{(zt)^n}{n!}dt.$$

By using generalized form of the identity [18]:

$$\sum_{n=0}^{\infty}H_{n+m}\left(x,y,A\right)\frac{t^n}{n!}=\exp\left(x\sqrt{2A}t-yt^2I\right)H_m\left(xI-\left(\sqrt{\frac{A}{2}}\right)^{-1}yt,y,A\right),$$

we have

$$\sum_{n=0}^{\infty}\frac{(n+m)!}{n!m!}U_{n+m}\left(x,y,A,B\right)z^n$$
$$=\frac{1}{m!}\int_0^{\infty}e^{-t\left(B-x\sqrt{2A}z+yz^2I\right)}t^mH_m\left(xI-\left(\sqrt{\frac{A}{2}}\right)^{-1}yz,\frac{y}{t},A\right)dt.$$

This completes the proof. □

2.3. Corollary. *Let A be a positive stable matrix in $\mathbb{C}^{r\times r}$. Then the second kind Chebyshev matrix polynomials with two variables have the generating matrix function*

$$\sum_{n=0}^{\infty}\frac{(n+m)!}{n!m!}U_{n+m}\left(x,y,A\right)z^n$$
$$=\left(I-xz\sqrt{2A}+yz^2I\right)^{-(m+1)}U_m\left(xI-\left(\sqrt{\frac{A}{2}}\right)^{-1}yz,\left(I-xz\sqrt{2A}+yz^2I\right)y,A\right),$$

where $\left\|xz\sqrt{2A}-yz^2I\right\|<1.$

Now, let us get matrix recurrence relations for Chebyshev-type matrix polynomials with two-variable by using the integral representation.

2.4. Proposition. *Let A and B be positive stable matrices in $\mathbb{C}^{r\times r}$ and $AB = BA$. Then the second kind Chebyshev-type matrix polynomials with two variables satisfy*

$$y\frac{\partial}{\partial x}U_{n-1}\left(x,y,A,B\right)=\sqrt{\frac{A}{2}}\left(x\frac{\partial}{\partial x}-n\right)U_n\left(x,y,A,B\right)\tag{2.4}$$

and

$$x\frac{\partial}{\partial x}U_n\left(x,y,A,B\right)=\left(n-2y\frac{\partial}{\partial y}\right)U_n\left(x,y,A,B\right).\tag{2.5}$$

Proof. From (2.1) and (1.2), we have

$$y\frac{\partial}{\partial x}U_{n-1}(x,y,A)=\frac{1}{(n-1)!}\int_0^\infty e^{-Bt}t^{n-1}y\frac{\partial}{\partial x}H_{n-1}\left(x,\frac{y}{t},A\right)dt$$
$$=\frac{\left(\sqrt{2A}\right)^{-1}}{n!}\int_0^\infty e^{-Bt}t^{n}\frac{y}{t}\frac{\partial^2}{\partial x^2}H_{n}\left(x,\frac{y}{t},A\right)dt.$$

Using (1.5), we get (2.4). (2.5) can be proved similarly. □

2.5. Proposition. *Let A and B be positive stable matrices in $\mathbb{C}^{r\times r}$ and $AB=BA$. Then the second kind Chebyshev-type matrix polynomials with two variables satisfy the three-term recurrence relation*

(2.6) $$BU_{n+1}(x,y,A,B)=x\sqrt{2A}U_n(x,y,A,B)-yU_{n-1}(x,y,A,B).$$

Proof. Equation (2.6) follows from differentiating both side of (2.3) with respect to z, making the necessary arrangements and identification of the coefficients of z^n. □

Now, let us get the matrix differential equation of second kind Chebyshev-type matrix polynomials with two variables. The recurrences given by (2.4) and (2.6) can be expressed as the definition of rising and lowering operators for $U_n(x,y,A,B)$. We can write

(2.7) $$U_{n-1}(x,y,A,B)=\sqrt{\frac{A}{2}}\frac{1}{y}\widehat{D}_x^{-1}\left[x\frac{\partial}{\partial x}-n\right]U_n(x,y,A,B)$$

and

(2.8) $$U_{n+1}(x,y,A,B)=\left[xB^{-1}\sqrt{2A}-B^{-1}\sqrt{\frac{A}{2}}\widehat{D}_x^{-1}\left[x\frac{\partial}{\partial x}-n\right]\right]U_n(x,y,A,B),$$

where $\widehat{D}_x^{-1}$ denotes the inverse derivative operator and is defined by

$$\widehat{D}_x^{-n}[f(x)]=\frac{1}{(n-1)!}\int_0^x(x-\xi)^{n-1}f(\xi)\,d\xi.$$

(see [12] for details). So that for $f(x)=1$, we have

$$\widehat{D}_x^{-n}[1]=\frac{x^n}{n!}.$$

Equations (2.7) and (2.8) allow us to introduce of the rising and lowering operators

(2.9) $$\widehat{M}=\left[xB^{-1}\sqrt{2A}-B^{-1}\sqrt{\frac{A}{2}}\widehat{D}_x^{-1}\left[x\frac{\partial}{\partial x}-\widehat{n}\right]\right],$$
$$\widehat{P}=\left[\sqrt{\frac{A}{2}}\frac{1}{y}\widehat{D}_x^{-1}\left[x\frac{\partial}{\partial x}-\widehat{n}\right]\right],$$

where $\widehat{n}$ is a number operator in the sense $\widehat{n}u_s(x,y,A,B)=su_s(x,y,A,B)$. Using (2.9) $U_n(x,y,A,B)$ can be rewriten as

$$\widehat{M}\widehat{P}U_n(x,y,A,B)=U_n(x,y,A,B),$$

namely,

$$U_n(x,y,A,B) = \sqrt{\frac{A}{2}}\frac{1}{y}\widehat{D}_x^{-1}\left[x\frac{\partial}{\partial x}-(n+1)\right] \times\left\{xB^{-1}\sqrt{2A}-B^{-1}\sqrt{\frac{A}{2}}\widehat{D}_x^{-1}\left[x\frac{\partial}{\partial x}-n\right]\right\}U_n(x,y,A,B).$$

After some arrangements and use of the obvious identity

$$\partial x\widehat{D}_x^{-1}=\widehat{1},$$

we arrive at the following theorem.

2.6. Theorem. *Let A and B be positive stable matrices in $\mathbb{C}^{r\times r}$ and $AB = BA$. Then the second kind Chebyshev-type matrix polynomials with two variables are a solution of the second order matrix differential equation of the form:*

$$\left[\left(2yB-x^2A\right)\frac{\partial^2}{\partial x^2}-3Ax\frac{\partial}{\partial x}+An(n+2)\right]U_n(x,y,A,B)=\mathbf{0}.$$

2.7. Corollary. *Let A be a positive stable matrix in $\mathbb{C}^{r\times r}$. Then the second kind Chebyshev matrix polynomials are a solution of the second order matrix differential equation of the form:*

$$\left[\left(2I-x^2A\right)\frac{d^2}{dx^2}-3Ax\frac{d}{dx}+An(n+2)\right]U_n(x,A)=\mathbf{0}. \tag{2.10}$$

It is now interesting to extend the above results to generalized forms of Chebyshev-type matrix polynomials with two-variable. We define generalized Chebyshev-type matrix polynomials with two-variable by

$$U_{n,m}(x,y,A,B)=\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-1)^k(n-mk+k)!B^{(m-1)k-n-1}\left(x\sqrt{mA}\right)^{n-mk}y^k}{k!(n-mk)!},$$

which can be written in terms of $H_{n,m}(x,y,A)$ as

$$U_{n,m}(x,y,A,B)=\frac{1}{n!}\int_0^\infty e^{-Bt}t^nH_{n,m}\left(x,\frac{y}{t^{m-1}},A\right)dt, \tag{2.11}$$

where $H_{n,m}(x,y,A)$ is two-index two-variable Hermite matrix polynomials, defined by [23]:

$$H_{n,m}(x,y,A)=\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-1)^k n!\left(x\sqrt{mA}\right)^{n-mk}y^k}{k!(n-mk)!}. \tag{2.12}$$

Since $H_{n,m}(x,y,A)$ has the generating matrix function as

$$\sum_{n=0}^{\infty}\frac{H_{n,m}(x,y,A)}{n!}t^n=\exp\left(xt\sqrt{mA}-yt^mI\right), \tag{2.13}$$

we find from (2.11) that the generating matrix function of $U_{n,m}(x,y,A,B)$ is

$$\sum_{n=0}^{\infty}U_{n,m}(x,y,A,B)z^n=\left(B-xz\sqrt{mA}+yz^mI\right)^{-1}, \tag{2.14}$$

where A, B are positive stable matrices in $\mathbb{C}^{r\times r}$, $AB=BA$ and $\left\|xz\sqrt{mA}-yz^mI\right\|<\|B\|$.

Taking $A=[m]_{1\times 1}$ and $B=[b]_{1\times 1}$in (2.14), the polynomials $U_{n,m}(x,y,m,b)$ reduce to the special case of the generalized Humbert polynomials (see [27]). The properties of this special matrix polynomials can be studied in further research.

3. Different Considerations for Chebyshev-type Matrix Polynomials of Second Kind

We will try to understand more deeply the role played by the integral transform connecting Hermite and the second kind Chebyshev matrix polynomials. It is obvious that both $H_n(x,y,A)$ and $U_n(x,y,A,B)$ reduce to ordinary form for $y=1$.

It is easy to find that

$$U_n(x,A,B)=\frac{1}{n!}\int_0^\infty e^{-Bt}t^{\frac{n}{2}}H_n\left(x\sqrt{t},A\right)dt. \tag{3.1}$$

After suitable change of variable, (3.1) yields

$$U_n(x,A,B)=\frac{2}{n!x^{n+2}}\int_0^\infty s^{n+1}\exp\left(-\frac{Bs^2}{x^2}\right)H_n(s,A)\,ds.$$

So, $U_n(x,A,B)$ can be viewed as a kind of Mellin transform of the function

$$f(\xi,A,B)=\exp\left(-\frac{B\xi^2}{x^2}\right)H_n(\xi,A).$$

Let us now consider the problem from an operational point of view. Let $f(x)$ be an appropriate function. Then one can easily get

$$\exp\left(\lambda x\frac{d}{dx}\right)f(x)=f(x\exp\lambda).$$

So, we obtain from (3.1) that

$$U_n(x,A,B)=\frac{1}{n!}\int_0^\infty e^{-Bt}t^{\frac{n}{2}}t^{\frac{1}{2}x\frac{d}{dx}}dtH_n(x,A). \tag{3.2}$$

Using the well-known definition of the Γ- function

$$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt,$$

we can rewrite (3.2) in the form

$$n!U_n(x,A,B)=B^{-\widehat{Q}}\Gamma\left(\widehat{Q}\right)H_n(x,A),$$

where $\widehat{Q}=\left[1+\frac{1}{2}\left(n+x\frac{d}{dx}\right)\right]$.

We conclude this section giving another representation for the second kind Chebyshev-type matrix polynomials. The use of the identities (1.2) and (1.3) in (2.1) for $B=\alpha I$ allow to conclude that

$$\frac{\partial}{\partial y}U_n(x,y,A,\alpha I)=\frac{\partial}{\partial\alpha}U_{n-2}(x,y,A,\alpha I),$$

$$\frac{\partial}{\partial x}U_n(x,y,A,\alpha I)=-\sqrt{2A}\frac{\partial}{\partial\alpha}U_{n-1}(x,y,A,\alpha I)$$

which can be combined to give

$$2A\frac{\partial^2}{\partial\alpha\partial y}U_n(x,y,A,\alpha I)=\frac{\partial^2}{\partial x^2}U_n(x,y,A,\alpha I).$$

Last identity and the fact that

$$U_n(x,0,A,\alpha I)=\frac{\left(x\sqrt{2A}\right)^n}{\alpha^{n+1}},$$

allow to define $U_n(x,y,A,\alpha I)$ as

$$U_n(x,y,A,\alpha I) = \exp\left[y(2A)^{-1}\,\widehat{D}_\alpha^{-1}\frac{\partial^2}{\partial x^2}\right]\frac{\left(x\sqrt{2A}\right)^n}{\alpha^{n+1}},$$

where A is a positive stable matrix in $\mathbb{C}^{r\times r}$ and α is a complex number such that $\mathrm{Re}(\alpha) > 0$.

4. First Kind Chebyshev-type Matrix Polynomials with Two-Variable

The two-variable Hermite matrix polynomials will be used here to define Chebyshev-type matrix polynomials of first kind. The Chebyshev polynomials of the first kind are defined by [9]:

$$T_n(x) = \frac{n}{2}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(n-k-1)!\,(2x)^{n-2k}}{k!\,(n-2k)!}. \tag{4.1}$$

Let A and B be positive stable matrices in $\mathbb{C}^{r\times r}$ and $AB = BA$. Then the first kind Chebyshev-type matrix polynomials can be defined by

$$T_n(x,A,B) = n\left(\sqrt{2A}\right)^{-1}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k B^{k-n}(n-k-1)!\left(x\sqrt{2A}\right)^{n-2k}}{k!\,(n-2k)!}, \tag{4.2}$$

or by using (1.1)

$$T_n(x,A,B) = \frac{\left(\sqrt{2A}\right)^{-1}}{(n-1)!}\int_0^\infty e^{-Bt}t^{n-1}H_n\left(x,\frac{1}{t},A\right)dt. \tag{4.3}$$

For the case $A = [2]_{1\times1}$ and $B = [1]_{1\times1}$, (4.2) coincides with (4.1).

In a similar way, we define the Chebyshev-type matrix polynomials of the first kind with two variables as

$$T_n(x,y,A,B) = n\left(\sqrt{2A}\right)^{-1}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k B^{k-n}(n-k-1)!\left(x\sqrt{2A}\right)^{n-2k}y^k}{k!\,(n-2k)!},$$

or

$$T_n(x,y,A,B) = \frac{\left(\sqrt{2A}\right)^{-1}}{(n-1)!}\int_0^\infty e^{-Bt}t^{n-1}H_n\left(x,\frac{y}{t},A\right)dt.$$

In this article, new special polynomials are introduced using integral representation. The possibility of combining these two approaches in order to study new families of special matrix polynomials is a problem for further research.

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References

[1] Aktaş, R., Çekim, B. and Çevik, A. *Extended Jacobi matrix polynomials*, Utilitas Mathematica **92**, 47-64, 2013.

[2] Altın, A. and Çekim, B. *Generating matrix functions for Chebyshev matrix polynomials of the second kind*, Hacettepe Journal of Mathematics and Statistics **41** (1), 25–32, 2012.

[3] Altın, A. and Çekim, B. Some properties associated with Hermite matrix polynomials, Utilitas Mathematica **88**, 171-181, 2012.

[4] Altın, A. and Çekim, B. *Some miscellaneous properties for Gegenbauer matrix polynomials*, Utilitas Mathematica **92**, 377-387, 2013.

[5] Batahan, R. S. *A new extension of Hermite matrix polynomials and its applications*, Linear Algebra and its Applications **419,** 82–92, 2006.

[6] Çekim, B., Altın, A. and Aktaş, R. Some new results for Jacobi matrix polynomials, Filomat **27** (4), 713-719, 2013.

[7] Çekim, B., Altın, A. and Aktaş, R. Some relations satisfied by orthogonal matrix polynomials, Hacettepe Journal of Mathematics and Statistics **40** (2), 241-253, 2011.

[8] Dattoli, G. *Integral transforms and Chebyshev-like polynomials*, Applied Mathematics and Computation **148**, 225-234, 2004.

[9] Davis, P.J. *Interpolation and Approximation* (Dover, New York, 1975).

[10] Defez, E. and Jódar, L. *Chebyshev matrix polynomials and second order matrix differential equations*, Utilitas Mathematica **61**, 107-123, 2002.

[11] Defez, E. and Jódar, L. *Jacobi matrix differential equation, polynomial solutions, and their properties*, Computers and Mathematics with Applications **48**, 789-803, 2004.

[12] Ditkin, V. A. and Prudnikov, A. *Integral Transforms and Operational Calculus* (Pergamon-Press, Oxford, 1965).

[13] Dunford, N. and Schwartz, J. T. *Linear Operators*, *part I, General Theory,* (Interscience, New York, 1963).

[14] Jódar, L., Company, R. and Navarro, E. *Laguerre matrix polynomials and systems of second order differential equations*, Applied Numerical Mathematics **15**, 53-63, 1994.

[15] Jódar, L. and Company, R. *Hermite matrix polynomials and second order matrix differential equations*, *Journal of Approximation Theory and its Applications* **12** (2), 20-30, 1996.

[16] Jódar, L. and Sastre, J. *On the Laguerre matrix polynomials*, Utilitas Mathematica **53**, 37-48,1998.

[17] Lancaster, P. *Theory of Matrices* (Academic Press, New York, 1969).

[18] Kargin, L. and Kurt, V. *Some relations on Hermite matrix polynomials*, Mathematical and Computational Applications **18** (3), 323-329, 2013.

[19] Khammash, G.S. and Shehata, A. *On Humbert matrix polynomials*, Asian Journal of Current Engineering and Maths **5**, 232-240, 2012.

[20] Khammash, G.S. and Shehata, A. *On Humbert matrix polynomials of two variables*, Advances in Pure Mathematics **2**, 423-427, 2012.

[21] Kishka, Z.M.G., Shehata, A. and Abul-Dahab, M. *The generalized Bessel matrix polynomials*, Journal of Mathematical and Computational Science **2** (2), 305-316, 2012.

[22] Metwally, M.S., Mohamed, M.T. and Shehata, A. *On Hermite-Hermite matrix polynomials*, Mathematica Bohemica **133** (4), 421-434, 2008.

[23] Metwally, M.S., Mohamed, M.T. and Shehata, A. *Generalizations of two-index two-variable Hermite matrix polynomials*, *Demonstratio Mathematica* **42** (4), 687-701, 2009.

[24] Metwally, M.S. *Operational rules and arbitrary order two-index two-variable Hermite matrix generating functions*, Acta Mathematica Academiae Paedagogicae Nyíregyháziensis **27** (1), 41-49, 2011.

[25] Sayyed, K.A.M., Metwally, M.S. and Batahan, R.S. *Gegenbauer matrix polynomials and second order matrix differential equations*, *Divulgaciones Matematicas* **12** (2), 101-115, 2004.

[26] Shehata, A. *A new extension of Gegenbauer matrix polynomials and their properties*, Bulletin of International Mathematical Virtual Institute **2**, 29-42, 2012.

[27] Srivastava, H.M. and Manocha, H.L. *A Treatise on Generating Functions* (Ellis Harwood, New York, 1985).

[28] Taşdelen, F., Aktaş, R. and Çekim, B. *On a multivariable extension of Jacobi matrix polynomials*, Computers and Mathematics with Applications **61** (9), 2412-2423, 2011.

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Periodic and subharmonic solutions for a $2n$th-order nonlinear difference equation

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Abstract

By using the critical point method, some new criteria are obtained for the existence and multiplicity of periodic and subharmonic solutions to a $2n$th-order nonlinear difference equation. The proof is based on the Linking Theorem in combination with variational technique. Our results generalize and improve some known ones.

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1. Introduction

Existence of periodic solutions of higher-order differential equations has been the subject of many investigations [8,19-21,34,38,39]. By using various methods and techniques, such as fixed point theory, the Kaplan-Yorke method, critical point theory, coincidence degree theory, bifurcation theory and dynamical system theory etc., a series of existence results for periodic solutions have been obtained in the literature. Difference equations, the discrete analogs of differential equations, occur widely in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology and other fields. For the general background of difference equations, one can refer to monographs [1,3,4,31]. Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity [22,31,33,48] and results on oscillation and other topics [1-4,7,11-15,17,18,28-30,32,44-47]. Only a few papers discuss the periodic solutions of higher-order difference equations. Therefore, it is worthwhile to explore this topic.

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a \leq b$. $*$ denotes the transpose of a vector.

In this paper, we consider the following forward and backward difference equation

$$\Delta^n\left(r_{k-n}\Delta^n u_{k-n}\right) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}),\ n \in \mathbf{Z}(3),\ k \in \mathbf{Z}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta u_k = u_{k+1} - u_k$, $\Delta^n u_k = \Delta(\Delta^{n-1}u_k)$, r_k is real valued for each $k \in \mathbf{Z}$, $f \in C(\mathbf{Z} \times \mathbf{R}^3, \mathbf{R})$, r_k and $f(k, v_1, v_2, v_3)$ are T-periodic in k for a given positive integer T.

We may think of (1.1) as a discrete analogue of the following $2n$th-order functional differential equation

$$\frac{d^n}{dt^n}\left[r(t)\frac{d^n u(t)}{dt^n}\right] = (-1)^n f(t, u(t+1), u(t), u(t-1)),\ t \in \mathbf{R}. \tag{1.2}$$

Equations similar in structure to (1.2) arise in the study of the existence of solitary waves of lattice differential equations, see Smets and Willem [42].

The widely used tools for the existence of periodic solutions of difference equations are the various fixed point theorems in cones [1,3,4,27]. It is well known that critical point theory is a powerful tool that deals with the problems of differential equations [8,10,16,25,26,43]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [28-30] and Shi *et al.*[41] established sufficient conditions on the existence of periodic solutions of second-order nonlinear difference equations. Compared to first-order or second-order difference equations, the study of higher-order equations has received considerably less attention (see, for example, [1,5,6,11-15,17,18,23,31,35,37] and the references contained therein). Ahlbrandt and Peterson [5] in 1994 studied the $2n$th-order difference equation of the form,

$$\sum_{i=0}^{n}\Delta^i\left(r_i(k-i)\Delta^i u(k-i)\right) = 0 \tag{1.3}$$

in the context of the discrete calculus of variations, and Peil and Peterson [37] studied the asymptotic behavior of solutions of (1.3) with $r_i(k) \equiv 0$ for $1 \leq i \leq n-1$. In 1998, Anderson [6] considered (1.3) for $k \in \mathbf{Z}(a)$, and obtained a formulation of generalized zeros and (n,n)-disconjugacy for (1.3). Migda [35] in 2004 studied an mth-order linear difference equation. In 2007, Cai and Yu [9] have obtained some criteria for the existence of periodic solutions of a $2n$th-order difference equation

$$\Delta^n \left(r_{k-n}\Delta^n u_{k-n}\right) + f(k, u_k) = 0,\ n \in \mathbf{Z}(3),\ k \in \mathbf{Z}, \tag{1.4}$$

for the case where f grows superlinearly at both 0 and ∞. However, to the best of our knowledge, the results on periodic solutions of higher-order nonlinear difference equations are very scarce in the literature. Furthermore, since (1.1) contains both advance and retardation, there are very few manuscripts dealing with this subject. The main purpose of this paper is to give some sufficient conditions for the existence and multiplicity of periodic and subharmonic solutions to a $2n$th-order nonlinear difference equation. The main approach used in our paper is a variational technique and the Linking Theorem. Particularly, our results not only generalize the results in the literature [9], but also improve them. In fact, one can see the following Remarks 1.2 and 1.4 for details. The motivation for the present work stems from the recent papers in [13,24].

Let

$$\underline{r} = \min_{k\in\mathbf{Z}(1,T)}\{r_k\},\ \bar{r} = \max_{k\in\mathbf{Z}(1,T)}\{r_k\}.$$

Our main results are as follows.

Theorem 1.1. *Assume that the following hypotheses are satisfied:*
(r) $r_k > 0, \forall k \in \boldsymbol{Z}$;

(F_1) *there exists a functional* $F(k, v_1, v_2) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *with* $F(k, v_1, v_2) \geq 0$ *and it satisfies*

$$F(k+T, v_1, v_2) = F(k, v_1, v_2),$$
$$\frac{\partial F(k-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(k, v_1, v_2)}{\partial v_2} = f(k, v_1, v_2, v_3);$$

(F_2) *there exist constants* $\delta_1 > 0$, $\alpha \in \left(0, \frac{1}{4}\underline{r}\lambda_{\min}^n\right)$ *such that*

$$F(k, v_1, v_2) \leq \alpha\left(v_1^2 + v_2^2\right),\ \textit{for } k \in \boldsymbol{Z} \textit{ and } v_1^2 + v_2^2 \leq \delta_1^2;$$

(F_3) *there exist constants* $\rho_1 > 0$, $\zeta > 0$, $\beta \in \left(\frac{1}{4}\bar{r}\lambda_{\max}^n, +\infty\right)$ *such that*

$$F(k, v_1, v_2) \geq \beta\left(v_1^2 + v_2^2\right) - \zeta,\ \textit{for } k \in \boldsymbol{Z} \textit{ and } v_1^2 + v_2^2 \geq \rho_1^2,$$

where $\lambda_{\min}$, $\lambda_{\max}$ *are constants which can be referred to (2.7).*

Then for any given positive integer $m > 0$, *(1.1) has at least three* mT*-periodic solutions.*

Remark 1.1. By (F_3) it is easy to see that there exists a constant $\zeta' > 0$ such that

(F_3') $F(k, v_1, v_2) \geq \beta\left(v_1^2 + v_2^2\right) - \zeta',\ \forall (k, v_1, v_2) \in \mathbf{Z}\times\mathbf{R}^2.$

As a matter of fact, let $\zeta_1 = \max\left\{\left|F(k, v_1, v_2) - \beta\left(v_1^2 + v_2^2\right) + \zeta\right| : k \in \mathbf{Z}, v_1^2 + v_2^2 \leq \rho_1^2\right\}$, $\zeta' = \zeta + \zeta_1$, we can easily get the desired result.

Corollary 1.1. *Assume that* (r) *and* $(F_1) - (F_3)$ *are satisfied. Then for any given positive integer* $m > 0$, *(1.1) has at least two nontrivial* mT*-periodic solutions.*

Remark 1.2. Corollary 1.1 reduces to Theorem 1.1 in [9].

Theorem 1.2. *Assume that* (r), (F_1) *and the following conditions are satisfied:*

(F_4) $\lim_{\rho\to 0} \frac{F(k, v_1, v_2)}{\rho^2} = 0,\ \rho = \sqrt{v_1^2 + v_2^2},\ \forall (k, v_1, v_2) \in \boldsymbol{Z} \times \boldsymbol{R}^2$;

(F_5) *there exist constants* $R_1 > 0$ *and* $\theta > 2$ *such that for* $k \in \boldsymbol{Z}$ *and* $v_1^2 + v_2^2 \geq R_1^2$,

$$0 < \theta F(k, v_1, v_2) \leq \frac{\partial F(k, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(k, v_1, v_2)}{\partial v_2} v_2.$$

Then for any given positive integer $m > 0$, (1.1) has at least three mT-periodic solutions.

Remark 1.3. Assumption (F_5) implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that

(F_5') $F(k, v_1, v_2) \geq a_1 \left(\sqrt{v_1^2 + v_2^2}\right)^{\theta} - a_2, \ \forall (k, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2.$

Corollary 1.2. *Assume that (r) and $(F_1), (F_4), (F_5)$ are satisfied. Then for any given positive integer $m > 0$, (1.1) has at least two nontrivial mT-periodic solutions.*

If $f(k, u_{k+1}, u_k, u_{k-1}) = q_k g(u_k)$, (1.1) reduces to the following $2n$th-order nonlinear equation,

$$\Delta^n (r_{k-n} \Delta^n u_{k-n}) = (-1)^n q_k g(u_k), \ k \in \mathbf{Z}, \tag{1.5}$$

where $g \in C(\mathbf{R}, \mathbf{R}), q_{k+T} = q_k > 0$, for all $k \in \mathbf{Z}$. Then, we have the following results.

Theorem 1.3. *Assume that (r) and the following hypotheses are satisfied:*
(G_1) *there exists a functional $G(v) \in C^1(\boldsymbol{R}, \boldsymbol{R})$ with $G(v) \geq 0$ and it satisfies*

$$G'(v) = g(v),$$

(G_2) *there exist constants $\delta_2 > 0$, $\alpha \in \left(0, \frac{1}{2}\underline{r}\lambda_{\min}^n\right)$ such that*

$$G(v) \leq \alpha |v|^2, \text{ for } |v| \leq \delta_2;$$

(G_3) *there exist constants $\rho_2 > 0$, $\zeta > 0$, $\beta \in \left(\frac{1}{2}\bar{r}\lambda_{\max}^n, +\infty\right)$ such that*

$$G(v) \geq \beta |v|^2 - \zeta, \text{ for } |v| \geq \rho_2,$$

where $\lambda_{\min}$, $\lambda_{\max}$ are constants which can be referred to (2.7).

Then for any given positive integer $m > 0$, (1.5) has at least three mT-periodic solutions.

Corollary 1.3. *Assume that (r) and $(G_1) - (G_3)$ are satisfied. Then for any given positive integer $m > 0$, (1.5) has at least two nontrivial mT-periodic solutions.*

Remark 1.4. Corollary 1.3 reduces to Corollary 1.1 in [9].

The rest of the paper is organized as follows. First, in Section 2, we shall establish the variational framework associated with (1.1) and transfer the problem of the existence of periodic solutions of (1.1) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give an example to illustrate the main result.

For the basic knowledge of variational methods, the reader is referred to [27,34,36,40].

2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for (1.1) and give some lemmas which will be of fundamental importance in proving our main results. First, we state some basic notations.

Let S be the set of sequences $u = (\cdots, u_{-k}, \cdots, u_{-1}, u_0, u_1, \cdots, u_k, \cdots) = \{u_k\}_{k=-\infty}^{+\infty}$, that is

$$S = \{\{u_k\} | u_k \in \mathbf{R}, \ k \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_k + bv_k\}_{k=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T, E_{mT} is defined as a subspace of S by

$$E_{mT} = \{u \in S | u_{k+mT} = u_k,\ \forall k \in \mathbf{Z}\}.$$

Clearly, E_{mT} is isomorphic to $\mathbf{R}^{mT}$. E_{mT} can be equipped with the inner product

$$(2.1)\qquad \langle u, v\rangle = \sum_{j=1}^{mT} u_j v_j,\ \forall u, v \in E_{mT},$$

by which the norm $\|\cdot\|$ can be induced by

$$(2.2)\qquad \|u\| = \left(\sum_{j=1}^{mT} u_j^2\right)^{\frac{1}{2}},\ \forall u \in E_{mT}.$$

It is obvious that E_{mT} with the inner product (2.1) is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbf{R}^{mT}$.

On the other hand, we define the norm $\|\cdot\|_s$ on E_{mT} as follows:

$$(2.3)\qquad \|u\|_s = \left(\sum_{j=1}^{mT} |u_j|^s\right)^{\frac{1}{s}},$$

for all $u \in E_{mT}$ and $s > 1$.

Since $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants $c_1,\ c_2$ such that $c_2 \geq c_1 > 0$, and

$$(2.4)\qquad c_1\|u\|_2 \leq \|u\|_s \leq c_2\|u\|_2,\ \forall u \in E_{mT}.$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_{mT}$, define the functional J on E_{mT} as follows:

$$(2.5)\qquad J(u) = \frac{1}{2}\sum_{k=1}^{mT} r_{k-1}\left(\Delta^n u_{k-1}\right)^2 - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k),$$

where

$$\frac{\partial F(k-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(k, v_1, v_2)}{\partial v_2} = f(k, v_1, v_2, v_3).$$

Clearly, $J \in C^1(E_{mT}, \mathbf{R})$ and for any $u = \{u_k\}_{k\in\mathbf{Z}} \in E_{mT}$, by using $u_0 = u_{mT},\ u_1 = u_{mT+1}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_k} = (-1)^n \Delta^n \left(r_{k-n}\Delta^n u_{k-n}\right) - f(k, u_{k+1}, u_k, u_{k-1}).$$

Thus, u is a critical point of J on E_{mT} if and only if

$$\Delta^n \left(r_{k-n}\Delta^n u_{k-n}\right) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}),\ \forall k \in \mathbf{Z}(1, mT).$$

Due to the periodicity of $u = \{u_k\}_{k\in\mathbf{Z}} \in E_{mT}$ and $f(k, v_1, v_2, v_3)$ in the first variable k, we reduce the existence of periodic solutions of (1.1) to the existence of critical points of J on E_{mT}. That is, the functional J is just the variational framework of (1.1).

Let P be the $mT \times mT$ matrix defined by

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

By matrix theory, we see that the eigenvalues of P are

(2.6) $$\lambda_j = 2\left(1-\cos\frac{2j}{mT}\pi\right), j = 0,1,2,\cdots, mT-1.$$

Thus, $\lambda_0 = 0, \lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_{mT-1} > 0$. Therefore,

(2.7) $$\left.\begin{aligned} &\lambda_{\min} = \min\{\lambda_1, \lambda_2, \cdots, \lambda_{mT-1}\} = 2\left(1-\cos\tfrac{2}{mT}\pi\right), \\ &\lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_{mT-1}\} = \begin{cases} 4, & \text{when mT is even,} \\ 2\left(1+\cos\frac{1}{mT}\pi\right), & \text{when mT is odd.} \end{cases} \end{aligned}\right\}$$

Let

$$W = \ker P = \{u \in E_{mT} | Pu = 0 \in \mathbf{R}^{mT}\}.$$

Then

$$W = \{u \in E_{mT} | u = \{c\},\ c \in \mathbf{R}\}.$$

Let V be the direct orthogonal complement of E_{mT} to W, i.e., $E_{mT} = V \oplus W$. For convenience, we identify $u \in E_{mT}$ with $u = (u_1, u_2, \cdots, u_{mT})^*$.

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\left\{u^{(i)}\right\} \subset E$ for which $\left\{J\left(u^{(i)}\right)\right\}$ is bounded and $J'\left(u^{(i)}\right) \to 0 (i \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Linking Theorem [40]). *Let E be a real Banach space, $E = E_1 \oplus E_2$, where E_1 is finite dimensional. Suppose that $J \in C^1(E, \boldsymbol{R})$ satisfies the P.S. condition and*
(J_1) there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap E_2} \geq a$;
(J_2) there exists an $e \in \partial B_1 \cap E_2$ and a constant $R_0 \geq \rho$ such that $J|_{\partial Q} \leq 0$, where $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{se | 0 < s < R_0\}$.
Then J possesses a critical value $c \geq a$, where

$$c = \inf_{h\in\Gamma} \sup_{u\in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, E) \mid h|_{\partial Q} = id\}$, where id denotes the identity operator.

Lemma 2.2. *Assume that (r), (F_1) and (F_3) are satisfied. Then the functional J is bounded from above in E_{mT}.*

Proof. By (F_3') and (2.4), for any $u \in E_{mT}$,

$$\begin{aligned} J(u) &= \frac{1}{2}\sum_{k=1}^{mT} r_{k-1}\left(\Delta^n u_{k-1}, \Delta^n u_{k-1}\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\ &= \frac{1}{2}\sum_{k=1}^{mT} r_k\left(\Delta^n u_k, \Delta^n u_k\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\ &\leq \frac{\bar{r}}{2}x^* P x - \sum_{k=1}^{mT}\left[\beta(u_{k+1}^2 + u_k^2) - \zeta'\right] \\ &\leq \frac{\bar{r}}{2}\lambda_{\max}\|x\|_2^2 - 2\beta\|u\|_2^2 + mT\zeta', \end{aligned}$$

where $x = (\Delta^{n-1}u_1, \Delta^{n-1}u_2, \cdots, \Delta^{n-1}u_{mT})^*$. Since

$$\|x\|_2^2 = \sum_{k=1}^{mT}\left(\Delta^{n-2}u_{k+1} - \Delta^{n-2}u_k\right)^2 \leq \lambda_{\max}\sum_{k=1}^{mT}\left(\Delta^{n-2}u_k\right)^2 \leq \lambda_{\max}^{n-1}\|u\|_2^2,$$

we have

$$J(u) \leq \left(\frac{\bar{r}}{2}\lambda_{\max}^{n} - 2\beta\right)\|u\|_2^2 + mT\zeta' \leq mT\zeta'.$$

The proof of Lemma 2.2 is complete. □

Remark 2.1. The case $mT = 1$ is trivial. For the case $mT = 2$, P has a different form, namely,

$$P = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

However, in this special case, the argument need not to be changed and we omit it.

Lemma 2.3. *Assume that* (r), (F_1) *and* (F_3) *are satisfied. Then the functional* J *satisfies the P.S. condition.*

Proof. Let $\left\{J\left(u^{(i)}\right)\right\}$ be a bounded sequence from the lower bound, i.e., there exists a positive constant M_1 such that

$$-M_1 \leq J\left(u^{(i)}\right),\ \forall i \in \mathbf{N}.$$

By the proof of Lemma 2.2, it is easy to see that

$$-M_1 \leq J\left(u^{(i)}\right) \leq \left(\frac{\bar{r}}{2}\lambda_{\max}^{n} - 2\beta\right)\left\|u^{(i)}\right\|_2^2 + mT\zeta',\ \forall i \in \mathbf{N}.$$

Therefore,

$$\left(2\beta - \frac{\bar{r}}{2}\lambda_{\max}^{n}\right)\left\|u^{(i)}\right\|_2^2 \leq M_1 + mT\zeta'.$$

Since $\beta > \frac{1}{4}\bar{r}\lambda_{\max}^{n}$, it is not difficult to know that $\left\{u^{(i)}\right\}$ is a bounded sequence in E_{mT}. As a consequence, $\left\{u^{(i)}\right\}$ possesses a convergence subsequence in E_{mT}. Thus the P.S. condition is verified. □

3. Proof of the main results

In this Section, we shall prove our main results by using the critical point theory.

3.1. Proof of Theorem 1.1

Assumptions (F_1) and (F_2) imply that $F(k, 0) = 0$ and $f(k, 0) = 0$ for $k \in \mathbf{Z}$. Then $u = 0$ is a trivial mT-periodic solution of (1.1).

By Lemma 2.2, J is bounded from the upper on E_{mT}. We define $c_0 = \sup\limits_{u \in E_{mT}} J(u)$. The proof of Lemma 2.2 implies $\lim\limits_{\|u\|_2 \to +\infty} J(u) = -\infty$. This means that $-J(u)$ is coercive. By the continuity of $J(u)$, there exists $\bar{u} \in E_{mT}$ such that $J(\bar{u}) = c_0$. Clearly, $\bar{u}$ is a critical point of J.

We claim that $c_0 > 0$. Indeed, by (F_2), for any $u \in V,\ \|u\|_2 \leq \delta_1$, we have

$$\begin{aligned} J(u) &= \frac{1}{2}\sum_{k=1}^{mT} r_{k-1}\left(\Delta^n u_{k-1}, \Delta^n u_{k-1}\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\ &= \frac{1}{2}\sum_{k=1}^{mT} r_k\left(\Delta^n u_k, \Delta^n u_k\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\ &\geq \frac{1}{2}\underline{r}x^* P x - \alpha\sum_{k=1}^{mT}(u_{k+1}^2 + u_k^2) \end{aligned}$$

$$\geq \frac{1}{2}\underline{r}\lambda_{\min}\|x\|_2^2 - 2\alpha\|u\|_2^2,$$

where $x = (\Delta^{n-1}u_1, \Delta^{n-1}u_2, \cdots, \Delta^{n-1}u_{mT})^*$. Since

$$\|x\|_2^2 = \sum_{k=1}^{mT}\left(\Delta^{n-2}u_{k+1} - \Delta^{n-2}u_k\right)^2 \geq \lambda_{\min}\sum_{k=1}^{mT}\left(\Delta^{n-2}u_k\right)^2 \geq \lambda_{\min}^{n-1}\|u\|_2^2,$$

we have

$$J(u) \geq \left(\frac{1}{2}\underline{r}\lambda_{\min}^n - 2\alpha\right)\|u\|_2^2.$$

Take $\sigma = \left(\frac{1}{2}\underline{r}\lambda_{\min}^n - 2\alpha\right)\delta_1^2$. Then

$$J(u) \geq \sigma, \ \forall u \in V \cap \partial B_{\delta_1}.$$

Therefore, $c_0 = \sup_{u \in E_{mT}} J(u) \geq \sigma > 0$. At the same time, we have also proved that there exist constants $\sigma > 0$ and $\delta_1 > 0$ such that $J|_{\partial B_{\delta_1} \cap V} \geq \sigma$. That is to say, J satisfies the condition (J_1) of the Linking Theorem.

Noting that $\sum_{k=1}^{mT} r_{k-1}\left(\Delta^n u_{k-1}\right)^2 = 0$, for all $u \in W$, we have

$$J(u) = \frac{1}{2}\sum_{k=1}^{mT} r_{k-1}\left(\Delta^n u_{k-1}\right)^2 - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) = -\sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \leq 0.$$

Thus, the critical point $\bar{u}$ of J corresponding to the critical value c_0 is a nontrivial mT-periodic solution of (1.1).

In order to obtain another nontrivial mT-periodic solution of (1.1) different from $\bar{u}$, we need to use the conclusion of Lemma 2.1. We have known that J satisfies the P.S. condition on E_{mT}. In the following, we shall verify the condition (J_2).

Take $e \in \partial B_1 \cap V$, for any $z \in W$ and $s \in \mathbf{R}$, let $u = se + z$. Then

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{k=1}^{mT} r_k\left(\Delta^n u_k, \Delta^n u_k\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
&\leq \frac{\bar{r}}{2}s^2\sum_{k=1}^{mT}\left(\Delta^n e_k, \Delta^n e_k\right) - \sum_{k=1}^{mT} F(k, se_{k+1} + z_{k+1}, se_k + z_k) \\
&\leq \frac{\bar{r}}{2}s^2 y^* P y - \sum_{k=1}^{mT}\left\{\beta\left[(se_{k+1} + z_{k+1})^2 + (se_k + z_k)^2\right] - \zeta'\right\} \\
&\leq \frac{\bar{r}}{2}s^2\lambda_{\max}\|y\|_2^2 - 2\beta\sum_{k=1}^{mT}(se_k + z_k)^2 + mT\zeta' \\
&= \frac{\bar{r}}{2}s^2\lambda_{\max}\|y\|_2^2 - 2\beta s^2 - 2\beta\|z\|_2^2 + mT\zeta',
\end{aligned}$$

where $y = (\Delta^{n-1}e_1, \Delta^{n-1}e_2, \cdots, \Delta^{n-1}e_{mT})^*$. Since

$$\|y\|_2^2 = \sum_{k=1}^{mT}\left(\Delta^{n-2}e_{k+1} - \Delta^{n-2}e_k\right)^2 \leq \lambda_{\max}\sum_{k=1}^{mT}\left(\Delta^{n-2}e_k\right)^2 \leq \lambda_{\max}^{n-1},$$

we have

$$J(u) \leq \left(\frac{\bar{r}}{2}\lambda_{\max}^n - 2\beta\right)s^2 - 2\beta\|z\|_2^2 + mT\zeta' \leq -2\beta\|z\|_2^2 + mT\zeta'.$$

Thus, there exists a positive constant $R_2 > \delta_1$ such that for any $u \in \partial Q$, $J(u) \leq 0$, where $Q = (\bar{B}_{R_2} \cap W) \oplus \{se|0 < s < R_2\}$. By the Linking Theorem, J possesses a critical value $c \geq \sigma > 0$, where

$$c = \inf_{h\in\Gamma} \sup_{u\in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, E_{mT}) \mid h|_{\partial Q} = id\}$.

Let $\tilde{u} \in E_{mT}$ be a critical point associated to the critical value c of J, i.e., $J(\tilde{u}) = c$. If $\tilde{u} \neq \bar{u}$, then the conclusion of Theorem 1.1 holds. Otherwise, $\tilde{u} = \bar{u}$. Then $c_0 = J(\bar{u}) = J(\tilde{u}) = c$, that is $\sup\limits_{u\in E_{mT}} J(u) = \inf\limits_{h\in\Gamma} \sup\limits_{u\in Q} J(h(u))$. Choosing $h = id$, we have $\sup\limits_{u\in Q} J(u) = c_0$. Since the choice of $e \in \partial B_1 \cap V$ is arbitrary, we can take $-e \in \partial B_1 \cap V$. Similarly, there exists a positive number $R_3 > \delta_1$, for any $u \in \partial Q_1$, $J(u) \leq 0$, where $Q_1 = (\bar{B}_{R_3} \cap W) \oplus \{-se|0 < s < R_3\}$.

Again, by the Linking Theorem, J possesses a critical value $c' \geq \sigma > 0$, where

$$c' = \inf_{h\in\Gamma_1} \sup_{u\in Q_1} J(h(u)),$$

and $\Gamma_1 = \{h \in C(\bar{Q}_1, E_{mT}) \mid h|_{\partial Q_1} = id\}$.

If $c' \neq c_0$, then the proof is finished. If $c' = c_0$, then $\sup\limits_{u\in Q_1} J(u) = c_0$. Due to the fact $J|_{\partial Q} \leq 0$ and $J|_{\partial Q_1} \leq 0$, J attains its maximum at some points in the interior of sets Q and Q_1. However, $Q \cap Q_1 \subset W$ and $J(u) \leq 0$ for any $u \in W$. Therefore, there must be a point $u' \in E_{mT},\ u' \neq \tilde{u}$ and $J(u') = c' = c_0$. The proof of Theorem 1.1 is complete. □

Remark 3.1. Similarly to above argument, we can also prove Theorems 1.2 and 1.3. For simplicity, we omit their proofs.

Remark 3.2. Due to Theorems 1.1, 1.2 and 1.3, the conclusion of Corollaries 1.1, 1.2 and 1.3 is obviously true.

4. Example

As an application of Theorem 1.1, we give an example to illustrate our main result.

Example 4.1. For all $n \in \mathbf{Z}(3),\ k \in \mathbf{Z}$, assume that

(4.1)
$$\Delta^n (r_{k-n}\Delta^n u_{k-n}) = (-1)^n \mu u_k \left[\left(8 + \sin^2\left(\frac{\pi k}{T}\right)\right)(u_{k+1}^2 + u_k^2)^{\frac{\mu}{2}-1} + \left(8 + \sin^2\left(\frac{\pi(k-1)}{T}\right)\right)(u_k^2 + u_{k-1}^2)^{\frac{\mu}{2}-1}\right],$$

where r_k is real valued for each $k \in \mathbf{Z}$ and $r_{k+T} = r_k > 0$, $\mu > 2$, T is a given positive integer.

We have

$$f(k, v_1, v_2, v_3) = \mu v_2 \left[\left(8 + \sin^2\left(\frac{\pi k}{T}\right)\right)(v_1^2 + v_2^2)^{\frac{\mu}{2}-1} + \left(8 + \sin^2\left(\frac{\pi(k-1)}{T}\right)\right)(v_2^2 + v_3^2)^{\frac{\mu}{2}-1}\right]$$

and

$$F(k, v_1, v_2) = \left[8 + \sin^2\left(\frac{\pi k}{T}\right)\right](v_1^2 + v_2^2)^{\frac{\mu}{2}}.$$

Then

$$\frac{\partial F(k-1,v_2,v_3)}{\partial v_2}+\frac{\partial F(k,v_1,v_2)}{\partial v_2}$$

$$=\mu v_2\left[\left(8+\sin^2\left(\frac{\pi k}{T}\right)\right)\left(v_1^2+v_2^2\right)^{\frac{\mu}{2}-1}+\left(8+\sin^2\left(\frac{\pi(k-1)}{T}\right)\right)\left(v_2^2+v_3^2\right)^{\frac{\mu}{2}-1}\right].$$

It is easy to verify all the assumptions of Theorem 1.1 are satisfied. Consequently, for any given positive integer $m > 0$, (4.1) has at least three mT-periodic solutions.

References

[1] R.P. Agarwal. *Difference Equations and Inequalities: Theory, Methods and Applications.* Marcel Dekker, New York (1992).

[2] R.P. Agarwal, K. Perera and D. O'regan. *Multiple positive solutions of singular and nonsingular discrete problems via variational methods.* Nonlinear Anal., **58**(1-2) (2004) 69-73.

[3] R.P. Agarwal and P.J.Y. Wong. *Advanced Topics in Difference Equations.* Kluwer Academic Publishers, Dordrecht (1997).

[4] C.D. Ahlbrandt and A.C. Peterson. *Discrete Hamiltonian Systems: Difference Equations, Continued Fraction and Riccati Equations.* Kluwer Academic Publishers, Dordrecht (1996).

[5] C.D. Ahlbrandt and A.C. Peterson. *The (n,n)-disconjugacy of a 2nth-order linear difference equation.* Comput. Math. Appl., **28**(1-3) (1994) 1-9.

[6] D. Anderson. *A 2nth-order linear difference equation.* Comput. Math. Appl., **2**(4) (1998) 521-529.

[7] R.I. Avery and A.C. Pererson. *Three positive fixed points of nonlinear operators on ordered Banach space.* Comput. Math. Appl., **42**(3-5) (2001) 313-322.

[8] V. Benci and P.H. Rabinowitz. *Critical point theorems for indefinite functionals.* Invent Math., **52**(3) (1979) 241-273.

[9] X.C. Cai and J.S.Yu. *Existence of periodic solutions for a 2nth-order nonlinear difference equation.* J. Math. Anal. Appl., **329**(2) (2007) 870-878.

[10] K.C. Chang. *Infinite Dimensional Morse Theory and Multiple Solution Problems.* Birkhäuser, Boston (1993).

[11] P. Chen. *Existence of homoclinic orbits in discrete Hamilton systems without Palais-Smale condition.* J. Difference Equ. Appl., **19**(11) (2013) 1781-1794.

[12] P. Chen and H. Fang. *Existence of periodic and subharmonic solutions for second-order p-Laplacian difference equations.* Adv. Difference Equ., **2007** (2007) 1-9.

[13] P. Chen and X.H. Tang. *Existence and multiplicity of homoclinic orbits for 2nth-order nonlinear difference equations containing both many advances and retardations.* J. Math. Anal. Appl., **381**(2) (2011) 485-505.

[14] P. Chen and X.H. Tang. *Existence of infinitely many homoclinic orbits for fourth-order difference systems containing both advance and retardation.* Appl. Math. Comput., **217**(9) (2011) 4408-4415.

[15] P. Chen and X.H.Tang. *Existence of homoclinic solutions for the second-order discrete p-Laplacian systems.* Taiwanese J. Math., **15**(5) (2011) 2123-2143.

[16] P. Chen and X.H. Tang. *New existence and multiplicity of solutions for some Dirichlet problems with impulsive effects.* Math. Comput. Modelling, **55**(3-4) (2012) 723-739.

[17] P. Chen and X.H.Tang. *Infinitely many homoclinic solutions for the second-order discrete p-Laplacian systems.* Bull. Belg. Math. Soc., **20**(2) (2013) 193-212.

[18] P. Chen and X.H. Tang. *Existence of homoclinic solutions for some second-order discrete Hamiltonian systems.* J. Difference Equ. Appl., **19**(4) (2013) 633-648.

[19] F.H. Clarke. *Periodic solutions to Hamiltonian inclusions.* J. Differential Equations, **40**(1) (1981) 1-6.

[20] F.H. Clarke. *Periodic solutions of Hamilton's equations and local minima of the dual action.* Trans. Amer. Math. Soc., **287**(1) (1985) 239-251.

[21] G. Cordaro. *Existence and location of periodic solution to convex and non coercive Hamiltonian systems.* Discrete Contin. Dyn. Syst., **12**(5) (2005) 983-996.

[22] L.H. Erbe, H. Xia and J.S. Yu. *Global stability of a linear nonautonomous delay difference equations.* J. Difference Equ. Appl., **1**(2) (1995) 151-161.

[23] H. Fang and D.P. Zhao. *Existence of nontrivial homoclinic orbits for fourth-order difference equations.* Appl. Math. Comput., **214**(1) (2009) 163-170.

[24] C.J. Guo, D. O'Regan and R.P. Agarwal. *Existence of multiple periodic solutions for a class of first-order neutral differential equations.* Appl. Anal. Discrete Math., **5**(1) (2011) 147-158.

[25] C.J. Guo, D. O'Regan, Y.T. Xu and R.P. Agarwal. *Existence and multiplicity of homoclinic orbits of a second-order differential difference equation via variational methods.* Appl. Math. Inform. Mech., **4**(1) (2012) 1-15.

[26] C.J. Guo and Y.T. Xu. *Existence of periodic solutions for a class of second order differential equation with deviating argument.* J. Appl. Math. Comput., **28**(1-2) (2008) 425-433.

[27] D.J. Guo. *Nonlinear Functional Analysis.* Shandong Scientific Press, Jinan (1985).

[28] Z.M. Guo and J.S. Yu. *Applications of critical point theory to difference equations.* Fields Inst. Commun., **42** (2004) 187-200.

[29] Z.M. Guo and J.S. Yu. *Existence of periodic and subharmonic solutions for second-order superlinear difference equations.* Sci. China Math, **46**(4) (2003) 506-515.

[30] Z.M. Guo and J.S. Yu. *The existence of periodic and subharmonic solutions of subquadratic second order difference equations.* J. London Math. Soc., **68**(2) (2003) 419-430.

[31] V.L. Kocic and G. Ladas. *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications.* Kluwer Academic Publishers, Dordrecht (1993).

[32] Y.J. Liu and W.G. Ge. *Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator.* J. Math. Anal. Appl., **278**(2) (2003) 551-561.

[33] H. Matsunaga, T. Hara and S. Sakata. *Global attractivity for a nonlinear difference equation with variable delay.* Computers Math. Appl., **41**(5-6) (2001) 543-551.

[34] J. Mawhin and M. Willem. *Critical Point Theory and Hamiltonian Systems.* Springer, New York (1989).

[35] M. Migda. *Existence of nonoscillatory solutions of some higher order difference equations.* Appl. Math. E-notes, **4**(2) (2004) 33-39.

[36] A. Pankov and N. Zakhrchenko. *On some discrete variational problems.* Acta Appl. Math., **65**(1-3) (2001) 295-303.

[37] T. Peil and A. Peterson. *Asymptotic behavior of solutions of a two-term difference equation.* Rocky Mountain J. Math., **24**(1) (1994) 233-251.

[38] P.H. Rabinowitz. *Periodic solutions of Hamiltonian systems.* Comm. Pure Appl. Math., **31**(2) (1978) 157-184.

[39] P.H. Rabinowitz. *On subharmonic solutions of Hamiltonian systems.* Comm. Pure Appl. Math., **33**(5) (1980) 609-633.

[40] P.H. Rabinowitz. *Minimax Methods in Critical Point Theory with Applications to Differential Equations.* Amer. Math. Soc., Providence, RI, New York (1986).

[41] H.P. Shi, W.P. Ling, Y.H. Long and H.Q. Zhang. *Periodic and subharmonic solutions for second order nonlinear functional difference equations.* Commun. Math. Anal., **5**(2) (2008) 50-59.

[42] D. Smets and M. Willem. *Solitary waves with prescribed speed on infinite lattices.* J. Funct. Anal., **149**(1) (1997) 266-275.

[43] Y.T. Xu and Z.M. Guo. *Applications of a Z_p index theory to periodic solutions for a class of functional differential equations.* J. Math. Anal. Appl., **257**(1) (2001) 189-205.

[44] J.S. Yu and Z.M. Guo. *On boundary value problems for a discrete generalized Emden-Fowler equation.* J. Differential Equations, **231**(1) (2006) 18-31.

[45] J.S. Yu, Y.H. Long and Z.M. Guo. *Subharmonic solutions with prescribed minimal period of a discrete forced pendulum equation.* J. Dynam. Differential Equations, **16**(2) (2004) 575-586.

[46] Z. Zhou, J.S. Yu and Y. M. Chen. *Homoclinic solutions in periodic difference equations with saturable nonlinearity.* Sci. China Math, **54**(1) (2011) 83-93.

[47] Z. Zhou, J.S. Yu and Z.M. Guo. *Periodic solutions of higher-dimensional discrete systems.* Proc. Roy. Soc. Edinburgh (Section A), **134**(5) (2004) 1013-1022.

[48] Z. Zhou and Q. Zhang. *Uniform stability of nonlinear difference systems.* J. Math. Anal. Appl., **225**(2) (1998) 486-500.

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Incomplete generalized Fibonacci and Lucas polynomials

José L. Ramírez*

Abstract

In this paper, we define the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials, we study the recurrence relations, some properties of these polynomials and the generating function of the incomplete Fibonacci and Lucas polynomials.

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1. Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications in many fields of science and art (see, e.g., [7]). The Fibonacci numbers F_n are defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geqslant 1.$$

The incomplete Fibonacci and Lucas numbers were introduced by Filipponi [6]. The incomplete Fibonacci numbers $F_n(k)$ and the incomplete Lucas numbers $L_n(k)$ are defined by

$$F_n(k) = \sum_{j=0}^{k} \binom{n-1-j}{j} \quad \left(n = 1, 2, 3, \dots ; 0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \right),$$

and

$$L_n(k) = \sum_{j=0}^{k} \frac{n}{n-j} \binom{n-j}{j} \quad \left(n = 1, 2, 3, \dots ; 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \right).$$

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Is is easily seen that [7]

$$F_n\left(\left\lfloor\frac{n-1}{2}\right\rfloor\right)=F_n \quad\text{and}\quad L_n\left(\left\lfloor\frac{n}{2}\right\rfloor\right)=L_n.$$

Pintér and Srivastava [9] determined the generating functions of the incomplete Fibonacci and Lucas numbers. Djordjević [1] introduced the incomplete generalized Fibonacci and Lucas numbers. Djordjević and Srivastava [2] defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers. Tasci and Cetin Firengiz [14] defined the incomplete Fibonacci and Lucas p-numbers. Tasci et al. [15] defined the incomplete bivariate Fibonacci and Lucas p-polynomials. Ramírez [11] introduced the incomplete k-Fibonacci and k-Lucas numbers, the bi-periodic incomplete Fibonacci sequences [10]. Ramírez and Sirvent introduced the incomplete tribonacci numbers and polynomials [12].

A large classes of polynomials can also be defined by Fibonacci-like recurrence relations such yield Fibonacci numbers. Such polynomials are called Fibonacci polynomials [7]. They were studied in 1883 by Catalan and Jacobsthal. The polynomials $F_n(x)$ studied by Catalan are defined by the recurrence relation

$$F_0(x)=0,\quad F_1(x)=1,\quad F_{n+1}(x)=xF_n(x)+F_{n-1}(x),\ n\geqslant 1.$$

The Fibonacci polynomials studied by Jacobsthal are defined by

$$J_0(x)=1,\quad J_1(x)=1,\quad J_{n+1}(x)=J_n(x)+xJ_{n-1}(x),\ n\geqslant 1.$$

The Lucas polynomials $L_n(x)$, originally studied in 1970 by Bicknell, are defined by

$$L_0(x)=2,\quad L_1(x)=x,\quad L_{n+1}(x)=xL_n(x)+L_{n-1}(x),\ n\geqslant 1.$$

Nalli and Haukkanen [8] introduced the $h(x)$-Fibonacci polynomials that generalize Catalan's Fibonacci polynomials $F_n(x)$ and the k-Fibonacci numbers $F_{k,n}$ [5]. Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Fibonacci polynomials $\{F_{h,n}(x)\}_{n\in\mathbb{N}}$ are defined by the recurrence relation

$$F_{h,0}(x)=0,\ F_{h,1}(x)=1,\ F_{h,n+1}(x)=h(x)F_{h,n}(x)+F_{h,n-1}(x),\ n\geqslant 1. \tag{1.1}$$

For $h(x)=x$ we obtain Catalan's Fibonacci polynomials, and for $h(x)=k$ we obtain k-Fibonacci numbers. For $k=1$ and $k=2$ we obtain the usual Fibonacci numbers and the Pell numbers.

Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Lucas polynomials $\{L_{h,n}(x)\}_{n\in\mathbb{N}}$ are defined by the recurrence relation

$$L_{h,0}(x)=2,\ L_{h,1}(x)=h(x),\ L_{h,n+1}(x)=h(x)L_{h,n}(x)+L_{h,n-1}(x), n\geqslant 1.$$

For $h(x)=x$ we obtain the Lucas polynomials, and for $h(x)=k$ we have the k-Lucas numbers [3]. For $k=1$ we obtain the usual Lucas numbers. Nalli and Haukkanen [8] obtained some relations for these polynomials sequences. In particular, they found an explicit formula to $h(x)$-Fibonacci polynomials and $h(x)$-Lucas polynomials respectively

$$F_{h,n}(x)=\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-i-1}{i}h^{n-2i-1}(x), \tag{1.2}$$

$$L_{h,n}(x)=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n}{n-i}\binom{n-i}{i}h^{n-2i}(x). \tag{1.3}$$

From Equations (1.2) and (1.3), we introduce the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials and we obtain new recurrence relations, new identities and the generating function of the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials.

2. Some Properties of $h(x)$-Fibonacci and $h(x)$-Lucas Polynomials

The characteristic equation associated with the recurrence relation (1.1) is $v^2 = h(x)v + 1$. The roots of this equation are

$$\alpha(x) = \frac{h(x) + \sqrt{h(x)^2 + 4}}{2}, \qquad \beta(x) = \frac{h(x) - \sqrt{h(x)^2 + 4}}{2}.$$

Then we have the following basic identities:

$$\alpha(x) + \beta(x) = h(x), \qquad \alpha(x) - \beta(x) = \sqrt{h(x)^2 + 4}, \qquad \alpha(x)\beta(x) = -1.$$

The $h(x)$-Fibonacci polynomials and the $h(x)$-Lucas numbers verify the following properties (see [8] for the proofs).

- Binet formula: $F_{h,n}(x) = (\alpha(x)^n - \beta(x)^n)/(\alpha(x) - \beta(x))$, $L_{h,n}(x) = \alpha(x)^n + \beta(x)^n$.
- Generating function: $g_f(t) = t/(1 - h(x)t - t^2)$.
- Relation with $h(x)$-Fibonacci polynomials:

$$L_{h,n}(x) = F_{h,n-1}(x) + F_{h,n+1}(x), \ n \geqslant 1.$$

3. The incomplete $h(x)$-Fibonacci Polynomials

3.1. Definition. The incomplete $h(x)$-Fibonacci polynomials are defined by

$$F^l_{h,n}(x) = \sum_{i=0}^{l} \binom{n-1-i}{i} h^{n-2i-1}(x), \quad 0 \le l \le \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{3.1}$$

In Table 1, some polynomials of incomplete $h(x)$-Fibonacci polynomials are provided.

$n \setminus l$	0	1	2	3
1	1			
2	h			
3	h^2	h^2+1		
4	h^3	h^3+2h		
5	h^4	h^4+3h^2	h^4+3h^2+1	
6	h^5	h^5+4h^3	h^5+4h^3+3h	
7	h^6	h^6+5h^4	$h^6+5h^4+6h^2$	$h^6+5h^4+6h^2+1$

Table 1. The polynomials $F^l_{h,n}(x)$, for $1 \leqslant n \leqslant 7$.

Note that

$$F_{1,n}^{\lfloor \frac{n-1}{2} \rfloor}(x) = F_n.$$

For $h(x) = 1$, we get incomplete Fibonacci numbers [6]. If $h(x) = k$ we obtained incomplete k-Fibonacci numbers [11].

Some special cases of (3.1) are

$$\begin{aligned}
&F^0_{h,n}(x) = h^{n-1}(x),\ (n \geq 1);\\
&F^1_{h,n}(x) = h^{n-1}(x) + (n-2)h^{n-3}(x),\ (n \geq 3);\\
&F^2_{h,n}(x) = h^{n-1}(x) + (n-2)h^{n-3}(x) + \frac{(n-4)(n-3)}{2}h^{n-5}(x),\ (n \geq 5);\\
&F^{\lfloor \frac{n-1}{2} \rfloor}_{h,n}(x) = F_{h,n}(x),\ (n \geq 1);\\
&F^{\lfloor \frac{n-3}{2} \rfloor}_{h,n}(x) = \begin{cases} F_{h,n}(x) - \frac{nh(x)}{2}, & \text{if } n \geq 3 \text{ and even;}\\ F_{h,n}(x) - 1, & \text{if } n \geq 3 \text{ and odd.}\end{cases}
\end{aligned}$$

3.2. Proposition. *The recurrence relation of the incomplete $h(x)$-Fibonacci polynomials $F^l_{h,n}(x)$ is*

$$F^{l+1}_{h,n+2}(x) = h(x)F^{l+1}_{h,n+1}(x) + F^l_{h,n}(x), \quad 0 \leq l \leq \left\lfloor \frac{n-2}{2} \right\rfloor. \tag{3.2}$$

The relation (3.2) can be transformed into the non-homogeneous recurrence relation

$$F^l_{h,n+2}(x) = h(x)F^l_{h,n+1}(x) + F^l_{h,n}(x) - \binom{n-1-l}{l}h^{n-1-2l}(x). \tag{3.3}$$

Proof. From Definition 3.1 we get

$$\begin{aligned}
&h(x)F^{l+1}_{h,n+1}(x) + F^l_{h,n}(x)\\
&\qquad = h(x)\sum_{i=0}^{l+1}\binom{n-i}{i}h^{n-2i}(x) + \sum_{i=0}^{l}\binom{n-i-1}{i}h^{n-2i-1}(x)\\
&\qquad = \sum_{i=0}^{l+1}\binom{n-i}{i}h^{n-2i+1}(x) + \sum_{i=1}^{l+1}\binom{n-i}{i-1}h^{n-2i+1}(x)\\
&\qquad = h^{n-2i+1}(x)\left(\sum_{i=0}^{l+1}\left[\binom{n-i}{i} + \binom{n-i}{i-1}\right]\right) - h^{n+1}(x)\binom{n}{-1}\\
&\qquad = \sum_{i=0}^{l+1}\binom{n-i+1}{i}h^{n-2i+1}(x) - 0\\
&\qquad = F^l_{h,n+2}(x).
\end{aligned}$$

□

3.3. Proposition. *The following equality holds:*

$$\sum_{i=0}^{s}\binom{s}{i}F^{l+i}_{h,n+i}(x)h^i(x) = F^{l+s}_{h,n+2s}(x), \quad 0 \leq l \leq \frac{n-s-1}{2}. \tag{3.4}$$

Proof. We proceed by induction on s. The sum (3.4) clearly holds for $s = 0$ and $s = 1$; see (3.2). Now suppose that the result is true for all $j < s+1$. We prove it for $s+1$:

$$\sum_{i=0}^{s+1} \binom{s+1}{i} F_{h,n+i}^{l+i}(x)h^i(x) = \sum_{i=0}^{s+1} \left[\binom{s}{i} + \binom{s}{i-1}\right] F_{h,n+i}^{l+i}(x)h^i(x)$$
$$= \sum_{i=0}^{s+1} \binom{s}{i} F_{h,n+i}^{l+i}(x)h^i(x) + \sum_{i=0}^{s+1} \binom{s}{i-1} F_{h,n+i}^{l+i}(x)h^i(x)$$
$$= F_{h,n+2s}^{l+s}(x) + \binom{s}{s+1} F_{h,n+s+1}^{l+s+1}(x)h^{s+1}(x) + \sum_{i=-1}^{s} \binom{s}{i} F_{h,n+i+1}^{l+i+1}(x)h^{i+1}(x)$$
$$= F_{h,n+2s}^{l+s}(x) + 0 + \sum_{i=0}^{s} \binom{s}{i} F_{h,n+i+1}^{l+i+1}(x)h^{i+1}(x) + \binom{s}{-1} F_{h,n}^{l}(x)$$
$$= F_{h,n+2s}^{l+s}(x) + h(x)\sum_{i=0}^{s} \binom{s}{i} F_{h,n+i+1}^{l+i+1}(x)h^{i}(x) + 0$$
$$= F_{h,n+2s}^{l+s}(x) + h(x)F_{h,n+2s+1}^{l+s+1}(x) = F_{h,n+2s+2}^{l+s+1}(x).$$

□

3.4. Proposition. *For $n \geq 2l+2$,*

$$\sum_{i=0}^{s-1} F_{h,n+i}^{l}(x)h^{s-1-i}(x) = F_{h,n+s+1}^{l+1}(x) - h^s(x)F_{h,n+1}^{l+1}(x). \tag{3.5}$$

Proof. We proceed by induction on s. The sum (3.5) clearly holds for $s = 1$; see (3.2). Now suppose that the result is true for all $j < s$. We prove it for s:

$$\sum_{i=0}^{s} F_{h,n+i}^{l}(x)h^{s-i}(x) = h(x)\sum_{i=0}^{s-1} F_{h,n+i}^{l}(x)h^{s-i-1}(x) + F_{h,n+s}^{l}(x)$$
$$= h(x)\left(F_{h,n+s+1}^{l+1}(x) - h^s(x)F_{h,n+1}^{l+1}(x)\right) + F_{h,n+s}^{l}(x)$$
$$= \left(h(x)F_{h,n+s+1}^{l+1}(x) + F_{h,n+s}^{l}(x)\right) - h^{s+1}(x)F_{h,n+1}^{l+1}(x)$$
$$= F_{h,n+s+2}^{l+1}(x) - h^{s+1}(x)F_{h,n+1}^{l+1}(x).$$

□

3.5. Lemma. *The following equality holds:*

$$F'_{h,n}(x) = h'(x)\left(\frac{nL_{h,n}(x) - h(x)F_{h,n}(x)}{h^2(x)+4}\right). \tag{3.6}$$

Proof. By deriving into the Binet's formula it is obtained:

$$F'_{h,n}(x) = \frac{n\left[\alpha^{n-1}(x) - (-\alpha(x))^{-n-1}\right]\alpha'(x)}{\alpha(x)+\alpha(x)^{-1}}$$
$$- \frac{\left[\alpha^{n}(x) - (-\alpha(x))^{-n}\right](1-\alpha^{-2}(x))\alpha'(x)}{\left[\alpha(x)+\alpha^{-1}(x)\right]^2},$$

where $\alpha(x) = (h(x) + \sqrt{h^2(x)+4})/2$. Then $\alpha'(x) = (h'(x)\alpha(x))/(\alpha(x)+\alpha^{-1}(x))$, $1 - \alpha^{-2}(x) = h(x)/\alpha(x)$. Therefore

$$F'_{h,n}(x) = \frac{n\left[\alpha^n(x) + (-\alpha(x))^{-n}\right] h'(x)}{\left[\alpha(x)+\alpha^{-1}(x)\right]^2} - \frac{\left[\alpha^n(x) - (-\alpha(x))^{-n}\right]}{\alpha(x)+\alpha^{-1}(x)} \cdot \frac{h(x)h'(x)}{\left[\alpha(x)+\alpha^{-1}(x)\right]^2}.$$

On the other hand, $F_{h,n+1}(x) + F_{h,n-1}(x) = \alpha^n(x) + \beta^n(x) = \alpha^n(x) + (-\alpha(x))^{-n} = L_{h,n}(x)$.
From where, after some algebra Equation (3.6) is obtained. □

Lemma 3.5 generalizes Proposition 13 of [4].

3.6. Lemma. *The following equality holds:*

$$\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} i\binom{n-1-i}{i} h^{n-1-2i}(x) = \frac{((h(x)^2+4)n-4)F_{h,n}(x) - nh(x)L_{h,n}(x)}{2(h^2(x)+4)}. \tag{3.7}$$

Proof. From Equation (1.2) we have

$$h(x)F_{h,n}(x) = \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1-i}{i} h^{n-2i}(x).$$

By deriving into the above equation:

$$\begin{aligned} h'(x)F_{h,n}(x) + h(x)F'_{h,n}(x) &= \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} (n-2i)\binom{n-1-i}{i} h^{n-2i-1}(x)h'(x) \\ &= nF_{h,n}(x)h'(x) - 2\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} i\binom{n-1-i}{i} h^{n-2i-1}(x)h'(x). \end{aligned}$$

From Lemma 3.5

$$\begin{aligned} h'(x)F_{h,n}(x) + h(x)h'(x)\left(\frac{nL_{h,n}(x) - h(x)F_{h,n}(x)}{h^2(x)+4}\right) \\ = nF_{h,n}(x)h'(x) - 2\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} i\binom{n-1-i}{i} h^{n-2i-1}(x)h'(x). \end{aligned}$$

From where, after some algebra Equation (3.7) is obtained.

□

3.7. Proposition. *The following equality holds:*

$$\sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor} F^l_{h,n}(x) = \begin{cases} \dfrac{4F_{h,n}(x) + nh(x)L_{h,n}(x)}{2(h^2(x)+4)}, & \text{if } n \text{ is even;} \\ \dfrac{(h^2(x)+8)F_{h,n}(x) + nh(x)L_{h,n}(x)}{2(h^2(x)+4)}, & \text{if } n \text{ is odd.} \end{cases} \tag{3.8}$$

Proof. We have

$$\begin{aligned}
&\sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} F_{h,n}^{l}(x)\\
&= \binom{n-1-0}{0} h^{n-1}(x) + \left[\binom{n-1-0}{0} h^{n-1}(x) + \binom{n-1-1}{1} h^{n-3}(x)\right]\\
&+ \cdots + \left[\binom{n-1-0}{0} h^{n-1}(x) + \binom{n-1-1}{1} h^{n-3}(x)\right.\\
&\left. + \cdots + \binom{n-1-\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{n-1}{2} \rfloor} h^{n-1-2\lfloor \frac{n-1}{2} \rfloor}(x)\right]\\
&= \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right) \binom{n-1-0}{0} h^{n-1}(x) + \left\lfloor \frac{n-1}{2} \right\rfloor \binom{n-1-1}{1} h^{n-3}(x)\\
&+ \cdots + \binom{n-1-\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{n-1}{2} \rfloor} h^{n-1-2\lfloor \frac{n-1}{2} \rfloor}(x)\\
&= \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 - i\right) \binom{n-1-i}{i} h^{n-1-2i}(x)\\
&= \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right) \binom{n-1-i}{i} h^{n-1-2i}(x)\\
&- \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} i \binom{n-1-i}{i} h^{n-1-2i}(x)\\
&\qquad = \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right) F_{h,n}(x) - \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} i \binom{n-1-i}{i} h^{n-1-2i}(x).
\end{aligned}$$

From Lemma 3.6 the Equation (3.8) is obtained. □

4. The incomplete $h(x)$-Lucas Polynomials

4.1. Definition. The incomplete $h(x)$-Lucas polynomials are defined by

$$L_{h,n}^{l}(x) = \sum_{i=0}^{l} \frac{n}{n-i} \binom{n-i}{i} h^{n-2i}(x), \quad 0 \le l \le \left\lfloor \frac{n}{2} \right\rfloor. \tag{4.1}$$

In Table 2, some polynomials of incomplete $h(x)$-Lucas polynomials are provided. Note that

$$L_{1,n}^{\lfloor \frac{n}{2} \rfloor}(x) = L_n.$$

$n \setminus l$	0	1	2	3
1	h			
2	h^2	h^2+2		
3	h^3	h^3+3h		
4	h^4	h^4+4h^2	h^4+4h^2+2	
5	h^5	h^5+5h^3	h^5+5h^3+5h	
6	h^6	h^6+6h^4	$h^6+6h^4+9h^2$	$h^6+6h^4+9h^2+2$
7	h^7	h^7+7h^5	$h^7+7h^5+14h^3$	$h^7+7h^5+14h^3+7h$

Table 2. The polynomials $L^l_{h,n}(x)$, for $1 \leqslant n \leqslant 7$.

Some special cases of (4.1) are

$$\begin{aligned}
&L^0_{h,n}(x) = h^n(x),\ \ (n \geq 1);\\
&L^1_{h,n}(x) = h^n(x) + nh^{n-2}(x),\ \ (n \geq 2);\\
&L^2_{h,n}(x) = h^n(x) + nh^{n-2}(x) + \frac{n(n-3)}{2}h^{n-4}(x),\ \ (n \geq 4);\\
&L^{\lfloor \frac{n}{2} \rfloor}_{h,n}(x) = L_{h,n}(x),\ \ (n \geq 1);\\
&L^{\lfloor \frac{n-2}{2} \rfloor}_{h,n}(x) = \begin{cases} L_{h,n}(x) - 2, & \text{if } n \geq 2 \text{ and even;}\\ L_{h,n}(x) - nh(x), & \text{if } n \geq 2 \text{ and odd.}\end{cases}
\end{aligned}$$

4.2. Proposition. *The following equality holds:*

$$L^l_{h,n}(x) = F^{l-1}_{h,n-1}(x) + F^l_{h,n+1}(x); \quad 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor. \tag{4.2}$$

Proof. Applying Definition 3.1 to the right-hand side (RHS) of (4.2) results

$$\begin{aligned}
(RHS) &= \sum_{i=0}^{l-1} \binom{n-2-i}{i} h^{n-2-2i}(x) + \sum_{i=0}^{l} \binom{n-i}{i} h^{n-2i}(x)\\
&= \sum_{i=1}^{l} \binom{n-1-i}{i-1} h^{n-2i}(x) + \sum_{i=0}^{l} \binom{n-i}{i} h^{n-2i}(x)\\
&= \sum_{i=0}^{l} \left[\binom{n-1-i}{i-1} + \binom{n-i}{i} \right] h^{n-2i}(x) - \binom{n-1}{-1}\\
&= \sum_{i=0}^{l} \frac{n}{n-i} \binom{n-i}{i} h^{n-2i}(x) + 0 = L^l_{h,n}(x).
\end{aligned}$$

□

4.3. Proposition. *The recurrence relation of the incomplete $h(x)$-Lucas polynomials $L^l_{h,n}(x)$ is*

$$L^{l+1}_{h,n+2}(x) = h(x)L^{l+1}_{h,n+1}(x) + L^l_{h,n}(x), \quad 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor. \tag{4.3}$$

The relation (4.3) can be transformed into the non-homogeneous recurrence relation

$$L^l_{h,n+2}(x) = h(x)L^l_{h,n+1}(x) + L^l_{h,n}(x) - \frac{n}{n-l}\binom{n-l}{l} h^{n-2l}(x). \tag{4.4}$$

Proof. It is clear from (4.2) and (3.2). □

4.4. Proposition. *The following equality holds:*

$$h(x)L^{l}_{h,n}(x) = F^{l}_{h,n+2}(x) - F^{l-2}_{h,n-2}(x), \quad 0 \le l \le \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof. By (4.2),

$$F^{l}_{h,n+2}(x) = L^{l}_{h,n+1}(x) - F^{l-1}_{h,n}(x) \quad \text{and} \quad F^{l-2}_{h,n-2}(x) = L^{l-1}_{h,n-1}(x) - F^{l-1}_{h,n}(x),$$

whence, from (4.3)

$$F^{l}_{h,n+2}(x) - F^{l-2}_{h,n-2}(x) = L^{l}_{h,n+1}(x) - L^{l-1}_{h,n-1}(x) = h(x)L^{l}_{h,n}(x).$$

□

4.5. Proposition. *The following equality holds:*

$$\sum_{i=0}^{s} \binom{s}{i} L^{l+i}_{h,n+i}(x)h^{i}(x) = L^{l+s}_{h,n+2s}(x), \quad 0 \le l \le \frac{n-s}{2}.$$

Proof. Using (4.2) and (3.4), we get

$$\sum_{i=0}^{s} \binom{s}{i} L^{l+i}_{h,n+i}(x)h^{i}(x) = \sum_{i=0}^{s} \binom{s}{i} \left[F^{l+i-1}_{h,n+i-1}(x) + F^{l+i}_{h,n+i+1}(x)\right] h^{i}(x)$$
$$= \sum_{i=0}^{s} \binom{s}{i} F^{l+i-1}_{h,n+i-1}(x)h^{i}(x) + \sum_{i=0}^{s} \binom{s}{i} F^{l+i}_{h,n+i+1}(x)h^{i}(x)$$
$$= F^{l-1+s}_{h,n-1+2s}(x) + F^{l+s}_{h,n+1+2s}(x) = L^{l+s}_{h,n+2s}(x).$$

□

4.6. Proposition. *For $n \ge 2l+1$,*

$$\sum_{i=0}^{s-1} L^{l}_{h,n+i}(x)h^{s-1-i}(x) = L^{l+1}_{h,n+s+1}(x) - h^{s}(x)L^{l+1}_{h,n+1}(x).$$

The proof can be done by using (4.3) and induction on s.

4.7. Lemma. *The following equality holds:*

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} i\frac{n}{n-i}\binom{n-i}{i} h^{n-2i}(x) = \frac{n}{2}\left[L_{h,n}(x) - h(x)F_{h,n}(x)\right].$$

The proof is similar to Lemma 3.6.

4.8. Proposition. *The following equality holds:*

$$\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} L^{l}_{h,n}(x) = \begin{cases} L_{h,n}(x) + \frac{nh(x)}{2}F_{h,n}(x), & \text{if } n \text{ is even;} \\ \frac{1}{2}\left(L_{h,n}(x) + nh(x)F_{h,n}(x)\right), & \text{if } n \text{ is odd.} \end{cases} \tag{4.5}$$

Proof. An argument analogous to that of the proof of Proposition 3.7 yields

$$\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} L^{l}_{h,n}(x) = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) L_{h,n}(x) - \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} i\frac{n}{n-i}\binom{n-i}{i} h^{n-2i}(x).$$

From Lemma 4.7 the Equation (4.5) is obtained. □

5. Generating functions of the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials

In this section, we give the generating functions of incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials.

5.1. Lemma. *(See [9], p. 592). Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the followin non-homogeneous recurrence relation:*

$$s_n = as_{n-1} + bs_{n-2} + r_n, \quad n > 1,$$

where a and b are complex numbers and $\{r_n\}$ is a given complex sequence. Then the generating function $U(t)$ of the sequence $\{s_n\}$ is

$$U(t) = \frac{G(t) + s_0 - r_0 + (s_1 - s_0 a - r_1)t}{1 - at - bt^2},$$

where $G(t)$ denotes the generating function of $\{r_n\}$.

5.2. Theorem. *The generating function of the incomplete $h(x)$-Fibonacci polynomials $F^l_{h,n}(x)$ is given by*

$$\begin{aligned} R_{h,l}(x) &= \sum_{i=0}^{\infty} F^l_{h,i}(x)t^i \\ &= t^{2l+1}\left[F_{h,2l+1}(x) + (F_{h,2l+2}(x) - h(x)F_{h,2l+1}(x))\, t \right. \\ &\qquad \left. - \frac{t^2}{(1-h(x)t)^{l+1}}\right]\left[1 - h(x)t - t^2\right]^{-1}. \end{aligned}$$

Proof. Let l be a fixed positive integer. From (3.1) and (3.3), $F^l_{h,n}(x) = 0$ for $0 \le n < 2l+1$, $F^l_{h,2l+1}(x) = F_{h,2l+1}(x)$, and $F^l_{h,2l+2}(x) = F_{h,2l+2}(x)$, and that

$$F^l_{h,n}(x) = h(x)F^l_{h,n-1}(x) + F^l_{h,n-2}(x) - \binom{n-3-l}{l} h^{n-3-2l}(x).$$

Now let

$$s_0 = F^l_{h,2l+1}(x), s_1 = F^l_{h,2l+2}(x), \quad \text{and} \quad s_n = F^l_{h,n+2l+1}(x).$$

Also let $r_0 = r_1 = 0$, and

$$r_n = \binom{n+l-1}{n-2} h^{n-2}(x).$$

The generating function of the sequence $\{r_n\}$ is $G(t) = t^2/(1-h(x)t)^{l+1}$; see [13, p. 355]. Thus, from Lemma 5.1, we get the generating function $R_{h,l}(x)$ of sequence $\{s_n\}$. □

5.3. Theorem. *The generating function of the incomplete $h(x)$-Lucas polynomials $L^l_{h,n}(x)$ is given by*

$$\begin{aligned} S_{h,l}(x) &= \sum_{i=0}^{\infty} L^l_{h,i}(x)t^i \\ &= t^{2l}\left[L_{h,2l}(x) + (L_{h,2l+1}(x) - h(x)L_{h,2l}(x))\, t \right. \\ &\qquad \left. - \frac{t^2(2-t)}{(1-h(x)t)^{l+1}}\right]\left[1 - h(x)t - t^2\right]^{-1}. \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 5.2. Let l be a fixed positive integer. From (4.1) and (4.4), $L^l_{h,n}(x) = 0$ for $0 \leq n < 2l$, $L^l_{h,2l}(x) = L_{h,2l}(x)$, and $L^l_{h,2l+1}(x) = L_{h,2l+1}(x)$, and that

$$L^l_{h,n}(x) = h(x)L^l_{h,n-1}(x) + L^l_{h,n-2}(x) - \frac{n-2}{n-2-l}\binom{n-2-l}{n-2-2l}h^{n-2-2l}(x).$$

Now let

$$s_0 = L^l_{h,2l}(x), \quad s_1 = L^l_{h,2l+1}(x), \quad \text{and} \quad s_n = L^l_{h,n+2l}(x).$$

Also let $r_0 = r_1 = 0$, and

$$r_n = \binom{n+2l-2}{n+l-2}h^{n+2l-2}(x).$$

The generating function of the sequence $\{r_n\}$ is $G(t) = t^2(2-t)/(1-h(x)t)^{l+1}$; see [13, p. 355]. Thus, from Lemma 5.1, we get the generating function $S_{h,l}(x)$ of sequence $\{s_n\}$. □

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References

[1] Djordjević, G. B. *Generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers*, Fibonacci Quart. **42**(2), 106–113, 2004.

[2] Djordjević, G. B. and Srivastava, H. M. *Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers*, Math. Comput. Modelling **42**(9-10), 1049–1056, 2005.

[3] Falcón, S. *On the k-Lucas Numbers*, Int. J. Contemp. Math. Sci. **21**(6), 1039–1050, 2011.

[4] Falcón, S. and Plaza, A. *On k-Fibonacci sequences and polynomials and their derivatives*, Chaos Solitons Fractals **39**(3), 1005–1019, 2009.

[5] Falcón, S. and Plaza, A. *On the Fibonacci k-numbers*, Chaos Solitons Fractals **32**(5), 1615–1624, 2007.

[6] Filipponi, P. *Incomplete Fibonacci and Lucas numbers*, Rend. Circ. Mat. Palermo **45**(2), 37–56, 1996.

[7] Koshy, T. *Fibonacci and Lucas Numbers with Applications*, A Wiley-Interscience, New York, 2001.

[8] Nalli, A. and Haukkanen, P. *On generalized Fibonacci and Lucas polynomials*, Chaos Solitons Fractals **42**(5), 3179–3186, 2009.

[9] Pintér, Á. and Srivastava, H. M. *Generating functions of the incomplete Fibonacci and Lucas numbers*, Rend. Circ. Mat. Palermo **48**(2), 591–596, 1999.

[10] Ramírez, J. *Bi-periodic incomplete Fibonacci sequence* Ann. Math. Inform. **42**, 83–92, 2013.

[11] Ramírez, J. *Incomplete k-Fibonacci and k-Lucas numbers*, Chinese J. of Math. Article ID 107145, 2013.

[12] Ramírez, J. and Sirvent, V. *Incomplete tribonacci numbers and polynomials*, J. Integer Seq. **17**, 2014.

[13] Srivastava, H. M. and Manocha, H. L. *A Treatise on Generating Functions*, Halsted Pres (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1984.

[14] Tasci, D. and Cetin Firengiz, M. *Incomplete Fibonacci and Lucas p-numbers*, Math. Comput. Modelling **52**(9-10), 1763–1770, 2010.

[15] Tasci, D., Cetin Firengiz, M. and Tuglu, N. *Incomplete bivariate Fibonacci and Lucas p-polynomials*, Discrete Dyn. Nat. Soc., 2012.

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Feckly reduced rings

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Abstract

Let R be a ring with identity and $J(R)$ denote the Jacobson radical of R. In this paper, we introduce a new class of rings called feckly reduced rings. The ring R is called *feckly reduced* if $R/J(R)$ is a reduced ring. We investigate relations between feckly reduced rings and other classes of rings. We obtain some characterizations of being a feckly reduced ring. It is proved that a ring R is feckly reduced if and only if every cyclic projective R-module has a feckly reduced endomorphism ring. Among others we show that every left Artinian ring is feckly reduced if and only if it is 2-primal, R is feckly reduced if and only if $T(R, R)$ is feckly reduced if and only if $R[x]/ < x^2 >$ is feckly reduced.

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1. Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. A ring is *reduced* if it has no nonzero nilpotent elements. It is well known that the structure of rings with Jacobson radical zero is easy to handle with, namely Artinian rings with Jacobson radical zero are direct sums of matrix rings. For any ring R, the ring $R/J(R)$ has zero Jacobson radical. Therefore it will be useful to study the rings with Jacobson radical zero. Some properties of rings are common with a ring R and $R/J(R)$, such as being Dedekind finite, stably finite, right (left) quasi-duo, and having stable range one. Invertible elements in $R/J(R)$ have invertible preimages in R and vice versa. Also, R and $R/J(R)$ have the same simple modules. By this motivation we introduce

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a class of rings, namely, feckly reduced rings. We supply some examples to show that there is no implication between the classes of reduced rings and feckly reduced rings. We show that a ring R is feckly reduced if and only if every cyclic projective R-module has a feckly reduced endomorphism ring. Apart from this, we obtain a characterization of feckly reduced rings in terms of its Jacobson radical. On the other hand, we prove that being a feckly reduced ring is not Morita invariant. In addition to these, we study trivial extensions and Dorroh extensions of feckly reduced rings.

Throughout this paper, $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the ring of rational numbers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. We write $R[x]$, $R[[x]]$, $N(R)$ and $J(R)$ for the polynomial ring, the power series ring over a ring R, the set of all nilpotent elements and the Jacobson radical of R, respectively.

2. Feckly Reduced Rings

In this section, we introduce the concept of a feckly reduced ring. We show that there is no implication between the classes of reduced rings and feckly reduced rings.

2.1. Definition. A ring R is called *feckly reduced* if $R/J(R)$ is a reduced ring.

Note that feckly reduced rings need not be reduced and reduced rings may not be feckly reduced as the following examples show.

2.2. Example. Let F be a field. Consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. Then $J(R) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ and $R/J(R) \cong \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$. Since $R/J(R)$ is a reduced ring, R is feckly reduced but it is not reduced.

2.3. Example. Let R denote the localization of $\mathbb{Z}$ at $3\mathbb{Z}$, that is, $R = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, 3 \nmid n\}$. Let Q denote the set of quaternions over the ring R, that is, a free R-module with basis $1, i, j, k$. Then Q is a noncommutative domain, and so it is reduced. On the other hand, $J(Q) = 3Q$, and $Q/J(Q)$ is isomorphic to 2×2 full matrix ring over $\mathbb{Z}_3$ via an isomorphism f defined by $f\big((a_0/b_0)1 + (a_1/b_1)i + (a_2/b_2)j + (a_3/b_3)k + 3Q\big) = \begin{bmatrix} a_0b_0^{-1} + a_1b_1^{-1} - a_2b_2^{-1} & a_1b_1^{-1} + a_2b_2^{-1} - a_3b_3^{-1} \\ a_1b_1^{-1} + a_2b_2^{-1} + a_3b_3^{-1} & a_0b_0^{-1} - a_1b_1^{-1} + a_2b_2^{-1} \end{bmatrix}$ for any $(a_0/b_0)1 + (a_1/b_1)i + (a_2/b_2)j + (a_3/b_3)k + 3Q \in Q/3Q$ where the entries of the matrix are read modulo the ideal (3) of $\mathbb{Z}$. Hence $Q/J(Q)$ has a nonzero nilpotent element. Therefore Q is not feckly reduced.

Note that obviously, being a reduced ring and a feckly reduced ring coincide when the ring is semisimple.

2.4. Remark. Let R be a ring with $R/J(R)$ semisimple. By Weddernburn-Artin Theorem, $R/J(R)$ is isomorphic to $A_1 \times \cdots \times A_n$ where A_i is isomorphic to the ring of all $(m_i \times m_i)$-matrices over division rings D_i $(i = 1, \cdots, n)$. If the aforementioned matrix rings' types are $m_i \times m_i$ with $m_i \geq 2$, then $R/J(R)$ is not reduced. Therefore R is not feckly reduced. If $m_i = 1$ for all i, then this is not true. For example, let R denote the localization of $\mathbb{Z}$ at $3\mathbb{Z}$, i.e., $R = \{x/y \in \mathbb{Q} : 3 \nmid y\}$. Then $J(R) = \{x/y \in R : 3 \mid x\}$, and so $R/J(R)$ is a semisimple reduced ring, also $R/J(R)$ is isomorphic to $\mathbb{Z}_3$. Therefore R is feckly reduced.

Let $J^{\#}(R)$ denote the subset $\{x \in R \mid \exists\, n \in \mathbb{N}$ such that $x^n \in J(R)\}$ of R. It is obvious that $J(R) \subseteq J^{\#}(R)$, but the following example shows that the reverse inclusion does not hold in general.

2.5. Example. Let R denote the ring $M_2(\mathbb{Z}_2)$. Then

$$J^{\#}(R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\},$$

while $J(R) = 0$.

We now give a characterization of feckly reduced rings in terms of its Jacobson radical.

2.6. Proposition. *A ring R is feckly reduced if and only if $J(R) = J^{\#}(R)$.*

Proof. Let R be a feckly reduced ring. We always have $J(R) \subseteq J^{\#}(R)$. For the converse inclusion, if $x \in J^{\#}(R)$, then $x^n \in J(R)$ for some $n \geq 1$ and so $x \in J(R)$. Thus $J(R) = J^{\#}(R)$. For the sufficiency, let $x \in R$ such that $x^n \in J(R)$ for some positive integer n. Then $x \in J^{\#}(R)$. Since $J(R) = J^{\#}(R)$, $x \in J(R)$ and so R is feckly reduced. □

By 2.6. Proposition, we can say that commutative rings and local rings are feckly reduced. The following result is an easy consequence of 2.6. Proposition.

2.7. Corollary. *Let R be a feckly reduced ring. Then all nilpotent elements of R belong to $J(R)$.*

In a ring R, $N(R) \subset J(R)$ is not an adequate condition in order that R being feckly reduced as is seen from 2.3. Example.

2.8. Lemma. *Let R be a ring with $N(R) = J(R)$. Then it is feckly reduced.*

Proof. Since $R/N(R)$ does not have any nonzero nilpotent elements, $R/J(R)$ is reduced. □

3. Examples

The purpose of this section is to supply several examples of feckly reduced rings. We see that feckly reduced rings are abundant.

3.1. Example. Let $N_2(R)$ be the set of all nilpotent elements of index two of a ring R. Assume that $J(R)$ contains $N_2(R)$. By [2, Corollary 4], we have the following.

(1) If R is a semiperfect ring, then it is feckly reduced.
(2) If R is a right or left self-injective ring, then it is feckly reduced.
(3) If R is an I-ring, i.e., every non-nil right ideal of R contains a nonzero idempotent, then it is feckly reduced.

3.2. Proposition. Every semi-abelian π-regular ring is feckly reduced.

Proof. Let R be a semi-abelian π-regular ring. According to [1, Corollary 3.13], $J(R) = N(R)$, and so R is feckly reduced by 2.8. Lemma. □

Recall that a left ideal L of a ring R is called *GW-ideal* if for any $a \in L$, there exists a positive integer n such that $a^n R \subseteq L$ and the ring R is called *left WQD* if every maximal left ideal of R is a GW-ideal.

3.3. Example. Every left WQD ring is feckly reduced by [12, Theorem 2.7].

3.4. Proposition. *Every locally finite abelian ring is feckly reduced.*

Proof. Let R be a locally finite abelian ring. Due to [4, Proposition 2.5], we have $N(R) = J(R)$. Then 2.8. Lemma completes the proof. □

Recall that a ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. Let R be a left morphic ring, that is, for any $a \in R$ there exists $b \in R$ such that $Ra = l(b)$ and $l(a) = Rb$. Then $J(R) = Z(R_R)$ ([8]).

3.5. Theorem. *Every semicommutative left and right morphic ring is feckly reduced.*

Proof. Let R be a semicommutative left and right morphic ring. By [8, Theorem 24], R being right morphic implies that it is left p-injective. We first note that R is right duo. In fact, for any $a \in R$, in view of left p-injectivity $aR = rl(a)$. By semicommutativity, $l(a)$ is a two sided ideal and so is $rl(a) = aR$. Because of this fact, every right ideal of R is also a left ideal. On the other hand, again by [8, Theorem 24], R being left morphic implies that $Z(R_R) = J(R)$. To complete the proof it is enough to show that $a^2 \in J(R)$ implies $a \in J(R)$. Otherwise, since $Z(R_R) = J(R)$, $r(a^2)$ is essential in R but $r(a)$ is neither essential in R nor in $r(a^2)$. There exists a right ideal $K \leq r(a^2)$ such that $r(a) \oplus K$ is essential in $r(a^2)$. Since K is also a left ideal, $aK \leq K$. Hence $a(aK) = 0$ since $K \leq r(a^2)$, and then $aK \leq r(a) \cap K$. It follows that $K \leq r(a) \cap K = 0$. Thus $K = 0$ and $r(a)$ is essential in R. This is the required contradiction. □

3.6. Theorem. *Every semicommutative left morphic ring with ACC on right annihilators is feckly reduced.*

Proof. Let R be a semicommutative left morphic ring with ACC on right annihilators. Then R is right p-injective and so it is left duo. Also we have $Z({}_RR) = J(R)$ by [8, Theorem 31]. The rest is similar to the proof of 3.5. Theorem. □

A ring R with involution $*$ is called a *$*$-ring*. An element p in a $*$-ring R is called a *projection* if $p^2 = p = p^*$. A $*$-ring R is said to be *$*$-clean* if each of its elements is the sum of a unit and a projection, and R is called *strongly $*$-clean* if each of its elements is the sum of a unit and a projection that commute with each other. If the preceding projection is unique, we call R *uniquely strongly $*$-clean*.

We call a $*$-ring R *strongly nil-$*$-clean* if every element of R is the sum of a nilpotent element and a projection that commute with each other.

3.7. Theorem. *Let R be a strongly nil-$*$-clean ring. Then*

(1) *Every idempotent in R is a projection.*
(2) *$N(R)$ forms an ideal.*
(3) *$R/N(R)$ is Boolean.*
(4) *$N(R) = J(R)$.*
(5) *R is feckly reduced.*

Proof. Let $e^2 = e \in R$. There exist a projection p and a nilpotent v in R such that $e = p+v$ and $pv = vp$. Then it is easily proved that e is also projection, that is $e = e^* = e^2$ and e is central. For any $x \in R$, there exist an idempotent $g \in R$ and a nilpotent $v \in N(R)$ such that $x = g + v$. Thus $x^2 = g + (2g + v)v$, and so $x - x^2 = (-2g + 1 - v)v \in R$ is nilpotent. Write $(x - x^2)^m = 0$, and so $x^m \in x^{m+1}R$ and $x^m = x^{m+1}y = yx^{m+1}$. Clearly, $xy = yx$ and $x^n y^n$ is an idempotent. This shows that R is strongly π-regular. It is well known that $N(R)$ forms an ideal of R. Hence $N(R) \subseteq J(R)$ since $J(R)$ contains all nil left or nil right ideals. Further, $x - x^2 \in N(R)$, and so $R/N(R)$ is Boolean. Let $x \in J(R)$. There exists an idempotent $e \in R$ such that $x - e \in N(R) \subseteq J(R)$. Hence $e \in J(R)$. Thus $e = 0$ and so $x \in N(R)$. It follows that $J(R) = N(R)$. Therefore R is feckly reduced. □

Recall that R is called a *gsr-ring* [10] if for any $x \in R$, there exists some integer $n(x) \geq 2$ such that $xRx = x^{n(x)}Rx^{n(x)}$.

3.8. Proposition. *Every gsr-ring is feckly reduced.*

Proof. Let R be a gsr-ring and $x \in R$ with $x^2 \in J(R)$. Then $xRx = x^{n(x)}Rx^{n(x)}$ for some integer $n(x)$ with $n(x) \geq 2$. This implies that $xRx = x^2Rx^2$. Hence $xRx \subseteq J(R)$, and so $(RxR)^2 \subseteq J(R)$. Also $J(R)$ is a semiprime ideal of R by [6, Ex. 10.20]. It follows that $RxR \subseteq J(R)$, thus $x \in J(R)$. This completes the proof. □

4. Further Results

A ring R is said to be *right continuous* [11] if (1) every right ideal of R isomorphic to a direct summand of R is a direct summand of R and (2) every complement right ideal of R is a direct summand of R. Thus if R is right continuous, then $J(R) = Z(R_R)$ and $R/Z(R_R)$ is von Neumann regular.

4.1. Theorem. *Let R be a ring with $J(R) = Z(R_R)$. If R is reduced, then it is feckly reduced.*

Proof. To complete the proof it is enough to show that $x^2 \in J(R)$ implies $x \in J(R)$. Let $x \in R$ with $x^2 \in J(R) = Z(R_R)$ and so $r(x^2)$ is an essential right ideal of R. Let $t \in r(x^2)$. So $x^2t = 0$. Since R is reduced, we have $xt = 0$. Hence $t \in r(x)$. It follows that $r(x) = r(x^2)$ and $r(x)$ is an essential right ideal of R and so $x \in Z(R_R) = J(R)$. This completes the proof. □

An ideal of a feckly reduced ring need not be feckly reduced, as the following example shows.

4.2. Example. Let F be a field and R the ring $\left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & e \end{bmatrix} : a, b, c, d, e \in F \right\}$ and I an ideal $\left\{ \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{bmatrix} : b, c, d, e \in F \right\}$ of R. Then it can be shown $J(R) = \left\{ \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} : b, c, d \in F \right\}$ and $J(I) = \left\{ \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} : c, d \in F \right\}$. Since F is a field, $R/J(R)$ is reduced, and so R is feckly reduced. On the other hand, consider the element $x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in I$. Since x is nilpotent, we have $x \in J^{\#}(I)$, but $x \notin J(I)$. Hence $J^{\#}(I) \neq J(I)$. By 2.6. Proposition, I is not feckly reduced.

4.3. Theorem. *Let I be an ideal of a ring R with $I \subseteq J(R)$. Then R is feckly reduced if and only if R/I is feckly reduced.*

Proof. Let $\overline{R} = R/I$. Since $I \subseteq J(R)$, $J(\overline{R}) = J(R)/I$. Suppose that R is feckly reduced. Since $\overline{R}/J(\overline{R}) \cong R/J(R)$, $\overline{R}$ is feckly reduced. Conversely, assume that $\overline{R}$ is feckly reduced and $a \in R$ with $a^2 \in J(R)$. Then $\overline{a}^2 \in J(R)/I = J(\overline{R})$ and so $\overline{a} \in J(\overline{R})$. Hence $a \in J(R)$, as desired. □

4.4. Theorem. *If R is feckly reduced, then eRe is also feckly reduced for any $e^2 = e \in R$.*

Proof. Assume that $a \in eRe$ with $\overline{a}^2 = \overline{0}$. Then $a^2 \in J(eRe) = eJ(R)e \subseteq J(R)$ and so $a \in J(R)$. Thus $eae = a \in eJ(R)e$, so $\overline{a} = \overline{0}$ in $eRe/J(eRe)$. □

4.5. Corollary. *Let M be a module with its endomorphism ring feckly reduced. Then every direct summand of M has a feckly reduced endomorphism ring.*

We now give a characterization of feckly reduced rings.

4.6. Theorem. *A ring R is feckly reduced if and only if every cyclic projective R-module has a feckly reduced endomorphism ring.*

Proof. Let R be a feckly reduced ring and mR a projective R-module. Then mR is isomorphic to a direct summand I of R as an R-module. 4.5. Corollary implies that the endomorphism ring of mR is feckly reduced. The sufficiency is clear due to $R \cong \mathrm{End}_R(R)$. □

4.7. Proposition. *Let M_1 and M_2 be R-modules for a ring R. If M_1 and M_2 have feckly reduced endomorphism rings and $Hom(M_1, M_2) = 0$, then $M = M_1 \oplus M_2$ has a feckly reduced endomorphism ring.*

Proof. Let $S_i = \mathrm{End}_R(M_i)$ for $i = 1, 2$ and $S = \mathrm{End}_R(M)$. We may write S as $\begin{bmatrix} S_1 & Hom(M_2, M_1) \\ 0 & S_2 \end{bmatrix}$. Then $J(S) = \begin{bmatrix} J(S_1) & Hom(M_2, M_1) \\ 0 & J(S_2) \end{bmatrix}$. Thus $S/J(S) \cong S_1/J(S_1) \times S_2/J(S_2)$. By assumption, S_1 and S_2 are feckly reduced. This implies that S is also feckly reduced. □

Note that every field is feckly reduced and every matrix ring over any field contains nilpotent elements. Therefore feckly reduced property is not Morita invariant. Also, the full matrix ring $M_n(R)$ over a ring R is never feckly reduced for all $n \geq 2$ because of $M_n(R)/J(M_n(R)) = M_n(R)/M_n(J(R)) \cong M_n(R/J(R))$.

If R is feckly reduced, then it need not be abelian, semicommutative, symmetric, reversible, and reduced (see 2.2. Example). In this direction we have the following.

4.8. Proposition. *Every feckly reduced ring is directly finite.*

Proof. Let R be a feckly reduced ring and $x, y \in R$ with $xy = 1$. Then yx is an idempotent. Since all nilpotents belong to $J(R)$, $yxy - yxyyx = y - y^2x \in J(R)$. Multiplying the latter from the left by x, $xy - xy^2x = 1 - yx \in J(R)$. Hence $yx = 1$. □

Recall that a ring R is called *2-primal* if $P(R) = N(R)$ where $P(R)$ is the prime radical of R.

4.9. Proposition. *Let R be a left Artinian ring. Then R is feckly reduced if and only if it is 2-primal.*

Proof. By [5, p.449], we have $P(R) = J(R)$. If R is feckly reduced, then $J(R) = N(R)$, and so it is 2-primal. If R is 2-primal, then $N(R) = P(R)$, and so it is feckly reduced due to 2.8. Lemma. This completes the proof. □

Note that direct products of reduced ring is again reduced.

4.10. Proposition. *Let $\{R_i\}_{i \in I}$ be a class of rings for an index set I. Then $\prod_{i \in I} R_i$ is feckly reduced if and only if for each $i \in I$, R_i is feckly reduced.*

Proof. If R_i is feckly reduced for each $i \in I$, then $\prod_{i \in I} R_i$ is a feckly reduced ring since $\prod_{i \in I} R_i / J(\prod_{i \in I} R_i) \cong \prod_{i \in I} \big(R_i/J(R_i)\big)$. Suppose that $\prod_{i \in I} R_i$ is feckly reduced and let $a_i \in R_i$ with $a_i^2 \in J(R_i)$ for $i \in I$. Then $(0, \dots, a_i^2, \dots, 0) = (0, \dots, a_i, \dots, 0)^2 \in J(\prod_{i \in I} R_i) = \prod_{i \in I} J(R_i)$, and so $(0, \dots, a_i, \dots, 0) \in J(\prod_{i \in I} R_i)$. Hence $a_i \in J(R_i)$, as asserted. □

Let S and T be any rings, M an S-T-bimodule and R the formal triangular matrix ring $\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. It is well-known that $J(R) = \begin{bmatrix} J(S) & M \\ 0 & J(T) \end{bmatrix}$ and $R/J(R) \cong S/J(S) \times T/J(T)$.

4.11. Proposition. *Let* $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. *Then R is feckly reduced if and only if S and T are feckly reduced.*

Proof. The necessity is obvious from 4.10. Proposition. Assume that S and T are feckly reduced. Then $S/J(S)$ and $T/J(T)$ are feckly reduced, by the remark above, $R/J(R) \cong S/J(S) \times T/J(T)$. Since a direct product of reduced rings is again reduced, $R/J(R)$ is reduced and so R is feckly reduced. □

For a ring R, let $R \propto R$ denote the ring $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in R \right\}$. Then $J(R \propto R) = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a \in J(R),\ b \in R \right\}$.

4.12. Theorem. *Let R be a ring. Then $R \propto R$ is feckly reduced if and only if R is feckly reduced.*

Proof. Let R be a feckly reduced ring and $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^2 = \begin{bmatrix} a^2 & ab+ba \\ 0 & a^2 \end{bmatrix} \in J(R \propto R)$. By the remark above, $a^2 \in J(R)$ and so $a \in J(R)$. Hence $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in J(R \propto R)$. Assume that $R \propto R$ is feckly reduced and let $a \in R$ with $a^2 \in J(R)$. Then $\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}^2 \in J(R \propto R)$. Therefore, $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in J(R \propto R)$ and so $a \in J(R)$, as asserted. □

For a ring R, let $T(R,R) = \{(a,b) \mid a, b \in R\}$ with the addition componentwise and multiplication defined by $(a_1,b_1)(a_2,b_2) = (a_1a_2, a_1b_2 + b_1a_2)$. Then $T(R,R)$ is a ring which is called the *trivial extension of R by R*. Clearly, $T(R,R)$ is isomorphic to the ring $R \propto R$ and $T(R,R)$ is also isomorphic to the ring $R[x]/ < x^2 >$. Hence by 4.12. Theorem, we have the following.

4.13. Corollary. *The following conditions are equivalent for a ring R.*

(1) *R is feckly reduced.*
(2) *$T(R,R)$ is feckly reduced.*
(3) *$R[x]/ < x^2 >$ is feckly reduced.*

4.14. Theorem. *Let R be a ring. Then the following are equivalent.*

(1) *R is feckly reduced.*
(2) *$T_n(R)$ is feckly reduced for all $n \in \mathbb{N}$.*

Proof. Let $I = \{[a_{ij}] \in T_n(R) : a_{ii} = 0, i = 1, 2, \ldots, n\}$. Then $I \subseteq J(T_n(R))$ and $T_n(R)/I \cong \bigoplus_{i=1}^{n} R_i$ where each $R_i = R$. So by 4.3. Theorem and 4.10. Proposition, we have (1) $\Leftrightarrow$ (2). Therefore the proof is completed. □

Let R be a ring and V an R-R-bimodule which is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then *ideal-extension* (it is also called *Dorroh extension*) $I(R;V)$ of R by

V is defined to be the additive abelian group $I(R;V) = R \oplus V$ with multiplication $(r,v)(s,w) = (rs, rw + vs + vw)$.

4.15. Proposition. *Suppose that for any $v \in V$ there exists $w \in V$ such that $v+w+vw = 0$. Then the following are equivalent for a ring R.*

(1) *R is feckly reduced.*
(2) *An ideal-extension $S = I(R;V)$ is feckly reduced.*

Proof. (1) $\Rightarrow$ (2) Let $s = (r,v) \in S$ with $s^2 = (r^2, rv+vr+v^2) \in J(S)$. It is easy to verify that $r^2 \in J(R)$ and so $r \in J(R)$ by (1). Note that $(0,V) \subseteq J(S)$ by hypothesis. Since $s = (r,v) = (r,0)+(0,v)$, it suffices to show that $(r,0) \in J(S)$. For any $(x,y) \in S$, $(1,0)-(r,0)(x,y) = (1-rx, -ry) \in U(S)$ because $(1-rx,-ry) = (1-rx,0)(1,(1-rx)^{-1}(-ry))$ and $(1,(1-rx)^{-1}(-ry)) = (1,0) + (0,(1-rx)^{-1}(-ry)) \in U(S)$ by $(0,V) \subseteq J(S)$. Thus $s = (r,v) \in J(S)$.

(2) $\Rightarrow$ (1) Suppose that S is feckly reduced and let $a \in R$ with $a^2 \in J(R)$. Then $(a,0)^2 = (a^2,0) \in S$. By the preceding discussion, $(a^2,0) \in J(S)$ and so $(a,0) \in J(S)$ by (2). Therefore $a \in J(R)$, as desired. □

4.16. Example. Let R be a feckly reduced ring, n a positive integer and $S = \{[a_{ij}] \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. If $V = \{[a_{ij}] \in T_n(R) \mid a_{11} = \cdots = a_{nn} = 0\}$, then $S \cong I(R;V)$. Since $V \subseteq J(S)$, S is feckly reduced by 4.15. Proposition and noncommutative if $n \geq 3$.

If R is a ring and $\sigma : R \to R$ is a ring homomorphism, let $R[[x,\sigma]]$ denote the ring of skew formal power series over R; that is all formal power series in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over R. Note that $J(R[[x,\sigma]]) = J(R) + < x >$. Since $R[[x,\sigma]] \cong I(R; < x >)$ where $< x >$ is the ideal generated by x, 4.15. Proposition gives the next result.

4.17. Corollary. *Let R be a ring and $\sigma : R \to R$ a ring homomorphism. Then the following are equivalent.*

(1) *R is feckly reduced.*
(2) *$R[[x,\sigma]]$ is feckly reduced.*

4.18. Remark. Let R be a ring. Then the ring $R[[x]]$ of formal power series is feckly reduced if and only if R is feckly reduced.

We now investigate some relations between clean rings, exchange rings and feckly reduced rings.

4.19. Proposition. *Every clean ring is exchange. The converse holds for feckly reduced rings.*

Proof. By [7], it is known that every clean ring is exchange. Let R be a feckly reduced exchange ring. Then $R/J(R)$ is exchange and abelian. Hence it is clean. On the other hand, since R is exchange, by [7], idempotents lift modulo $J(R)$. Therefore R is clean. □

Recall that a ring R is called *J-clean (nil clean)* if for every $a \in R$, there exist $e^2 = e \in R$ and $b \in J(R)(b \in N(R))$ such that $a = e + b$.

4.20. Theorem. *Consider the following conditions for a ring R.*

(1) *R is an abelian exchange ring.*
(2) *R is a J-clean ring.*
(3) *R is a feckly reduced ring.*

Then (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3). The converse statements hold if R is nil-clean.

Proof. (1) $\Rightarrow$ (3) Let R be an abelian exchange ring. Since R is exchange, $R/J(R)$ is also exchange and idempotents lift modulo $J(R)$. Then $R/J(R)$ is abelian. The rest follows from [14, Corollary 3.12].
(2) $\Rightarrow$ (3) Clear from [9].
The converse statements hold by noting that every nil-clean ring is clean and abelian, and every clean ring is exchange. □

The converse statements (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) do not hold in general.

4.21. Examples. (1) Let F be a field. Then $M_2(F)$ is an exchange ring which is not feckly reduced.
(2) Consider the ring $\mathbb{Z}$ of integers. For $3 \in \mathbb{Z}$, there is no any idempotent e of $3\mathbb{Z}$ such that $1 - e \in 2\mathbb{Z}$. Therefore $\mathbb{Z}$ is not exchange. But clearly, it is feckly reduced.
(3) Let R be the ring $\{m/n \in \mathbb{Q} : gcd(m,n) = 1,\ 2 \nmid n,\ 3 \nmid n\}$. Then $R/J(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$. This implies that R is feckly reduced. On the other hand, $4 \in R$ can not be written as the sum of an idempotent and a unit in R. Hence R is not clean.
(4) The ring $\mathbb{Z}_5$ is feckly reduced but not J-clean.

4.22. Proposition. *Every semiregular feckly reduced ring is clean.*

Proof. Let R be a semiregular feckly reduced ring. Then $R/J(R)$ is strongly regular. Hence it is clean and idempotents of R lift modulo $J(R)$. This implies that R is clean. □

4.23. Proposition. *Every right (left) quasi-duo ring is feckly reduced.*

Proof. Let R be a right (left) quasi-duo ring and $a \in R$ with $a^2 \in J(R)$. Every factor ring of a right (left) quasi-duo ring is again right (left) quasi-duo and by [13, Lemma 2.3] every nilpotent element of a right (left) quasi-duo ring is in Jacobson radical. Accordingly, we have $a \in J(R)$. Therefore R is feckly reduced. □

On the contrary of 4.23. Proposition, there is a feckly reduced ring which is not right quasi-duo, for example, consider the Hamilton quaternion over the field of real numbers and let R denote this ring. Since R is a division ring, we have $J(R) = 0$, and so $J(R[x]) = 0$ due to $J(R[x]) \subseteq J(R)[x]$. Also $R[x]$ is a domain and so it is reduced. This implies that $R[x]$ is a feckly reduced ring. On the other hand, consider the maximal right ideal $I = (1+ix)R[x]$ of $R[x]$. If I were a left ideal, then $\big((1+ix)k + k(1+ix)\big)(2k)^{-1} = 1 \in I$, this is a contradiction. Therefore $R[x]$ is not right quasi-duo. Nevertheless, for exchange rings these notions are equivalent as the following theorem shows.

4.24. Theorem. *Let R be an exchange ring. Then the following are equivalent.*

(1) *R is feckly reduced.*
(2) $N(R) \subseteq J(R)$.
(3) $N_2(R) \subseteq J(R)$.
(4) *R is right quasi-duo.*

Proof. (1) $\Rightarrow$ (2) From 2.7. Corollary. (2) $\Rightarrow$ (3) Clear. (3) $\Leftrightarrow$ (4) From [3, Proposition 2.3]. (3) $\Rightarrow$ (1) Since R is exchange, $R/J(R)$ is also exchange. Then $R/J(R)$ is reduced by [2, Theorem 2]. □

We say that B is a *subring* of a ring A if $\emptyset \neq B \subseteq A$ and for any $x, y \in B$, $x - y, xy \in B$ and $1_A \in B$. Let A be a ring and B a subring of A and $R[A, B]$ denote the set $\{(a_1, a_2, \cdots, a_n, b, b, \cdots) : a_i \in A, b \in B, n \geq 1, 1 \leq i \leq n\}$. Then $R[A, B]$ is a ring under the componentwise addition and multiplication. Also $J(R[A, B]) = R[J(A), J(A) \cap J(B)]$.

4.25. Proposition. *Consider the following conditions for a ring A and a subring B of A.*

(1) *A and B are feckly reduced.*
(2) *$R[A,B]$ is feckly reduced.*
(3) *A is feckly reduced and $N(B) \subseteq J(B)$.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Let $(a_1, \cdots, a_n, b, b, \cdots) \in R[A,B]$ with $(a_1, \cdots, a_n, b, b, \cdots)^2 \in J(R[A,B])$ for some $n \geq 1$. Then $(a_1^2, \cdots, a_n^2, b^2, b^2, \cdots) \in J(R[A,B])$. This implies that $a_i^2, b^2 \in J(A)$ for $i = 1, \ldots, n$ and $b^2 \in J(B)$. By assumption, $a_i, b \in J(A)$ for $i = 1, \ldots, n$ and $b \in J(B)$. Therefore $(a_1, \cdots, a_n, b, b, \cdots) \in J(R[A,B])$.

(2) $\Rightarrow$ (3) Let $a \in A$ with $a^2 \in J(A)$. Then $(a, 0, 0, \cdots)^2 = (a^2, 0, 0, \cdots) \in J(R[A,B])$. By (1), we have $(a, 0, 0, \cdots) \in J(R[A,B])$, and so $a \in J(A)$. Therefore A is feckly reduced. In order to show $N(B) \subseteq J(B)$, let $b \in B$ with $b^n = 0$ for some positive integer n. Then $(0, b, b, \cdots)^n = (0, 0, 0, \cdots) \in J(R[A,B])$. Since $R[A,B]$ is feckly reduced, $(0, b, b, \cdots) \in J(R[A,B])$. Hence $b \in J(B)$, as desired. □

The following result is an immediate consequence of 4.24. Theorem and 4.25. Proposition.

4.26. Corollary. *Let B be a subring of a ring A. If B is an exchange ring, then the following are equivalent.*

(1) *$R[A,B]$ is feckly reduced.*
(2) *A and B are feckly reduced.*

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References

[1] W. Chen, *On Semi-abelian π-Regular Rings*, Int. J. Math. Math. Sci. Volume 2007, Article ID 63171, 10 pages, doi:10.1155/2007/63171.
[2] Y. Hirano, D. V. Huynh and J. K. Park, *On rings whose prime radical contains all nilpotent elements of index two*, Arch. Math. (Basel) 66(5)(1996), 360-365.
[3] C. Y. Hong, N. K. Kim and T. K. Kwak, *On the maximality of prime ideals in exchange rings*, Commun. Korean Math. Soc. 17(3)(2002), 409-422.
[4] C. Huh, N. K. Kim and Y. Lee, *Examples of strongly π-regular rings*, J. Pure Appl. Algebra 189(2004), 195-210.
[5] T. W. Hungerford, Algebra, Springer-Verlag, New York, 1974.
[6] T. Y. Lam, A first course in noncommutative rings, Springer-Verlag, New York, 2001.
[7] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. 229(1977), 269-278.
[8] W. K. Nicholson and E. Sanchez Campos, *Rings with the dual of the isomorphism theorem*, J. Algebra 271(2004), 391-406.
[9] W. K. Nicholson and Y. Zhou, *Rings in which elements are uniquely the sum of an idempotent and a unit*, Glasg. Math. J. 46(2004), 227-236.
[10] F. A. Szasz, *On generalized subcommutative regular rings*, Monatsh. Math. 77(1973), 67-71.
[11] Y. Utumi, *On continuous and self-injective rings,* Trans. Amer. Math. Soc. 118(1965), 158-173.
[12] J. Wei and L. Li, *MC2 Rings and WQD rings*, Glasg. Math. J. 51(2009), 691-702.
[13] H. P. Yu, *On quasi-duo rings*, Glasg. Math. J. 37(1995), 21-31.
[14] H. P. Yu, *On The Structure of Exchange Rings*, Comm. Algebra 25(2)(1997), 661-670.

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On soft continuous mappings and soft connectedness of soft topological spaces

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Abstract

The notion of soft topological space which is defined over an initial universe with a fixed set of parameters was introduced by Shabir and Naz. In this paper, the concept of soft continuous mapping between two soft topological spaces is first proposed. Then the main properties of soft continuous mappings are studied. Finally, the notion of soft connectedness of soft topological spaces is proposed, and some related properties are discussed.

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1. Introduction

Uncertainty is an attribute of information. To solve the complicated problems in economics, engineering and environment, we cannot successfully use classical methods because of various uncertainties. A wide range of theories such as probability theory, fuzzy set theory, intuitionistic fuzzy set theory, rough set theory, vague set theory and the interval mathematics are well known and often useful mathematical approaches for modeling uncertainties. Each of these theories has its inherent difficulties as pointed out by Molodtsov [29]. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theories. Molodtsov [29] initiated the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties

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that have troubled the existing theoretical approaches. This theory has proven useful in many fields such as decision making [8,14,26,32,38], data analysis [50], forecasting [44] and simulation [28].

The concept and basic properties of soft set theory were presented in [29,33]. In the classical soft set theory, a situation may be complex in the real world because of the fuzzy nature of the parameters. With this point of view, the classical soft sets have been extended to fuzzy soft sets [30,34], intuitionistic fuzzy soft sets [31], vague soft sets [46], interval-valued fuzzy soft sets [47] and interval-valued intuitionistic fuzzy soft sets [23].

Algebraic nature of soft sets has been studied by some authors. Maji et al. [33] presented some definitions on soft sets such as a soft subset, the complement of a soft set. Based on the analysis of several operations on soft sets introduced in [33], Ali et al. [3] presented some new algebraic operations for soft sets and proved that certain De Morgan's laws hold in soft set theory with respect to these new definitions. Qin and Hong introduced the concept of soft equality and some related properties were derived in [37]. Kharal and Ahmad [27] introduced the notion of a mapping on the classes of soft sets which is a pivotal notion for the advanced development of any new area of mathematical sciences. Babitha and Sunil [7] studied soft set relations and many related concepts were discussed. As a continuation of [7], kernels and closures of soft set relations, and soft set relation mappings were studied in [49]. In [39], Sezgin and Atag*ü*n presented a detailed theoretical study of operations on soft sets. Ali et al. [4] discussed algebraic structures of soft sets associated with new operations.

Up to the present, soft set theory has also been applied to several algebra structures: groups [1,2,40], semirings [12], rings [5,6,10], BCK/BCI-algebras [17-19], BCH-algebras [25], d-algebras [20], Hilbert algebras [21], ordered semigroups [22], BL-algebras [51] and fuzzy semigroup [48]. Xiao et al. [45] proposed the notions of exclusive disjunctive soft sets and studied some of its operations. Gong et al. [15] studied the bijective soft set with its operations. Ontology-based (or DL-based) soft set theory was presented in [24]. An idea of soft mappings is given and some of their properties are studied in [35]. Ge et al. [16] characterized some properties of topological spaces by using soft set theory. Recently, in [41], Shabir and Naz proposed the notion of soft topological space (defined over an initial universe with a fixed set of parameters) and investigated the basic properties. Tanay and Kandemir [42] studied topological structure of fuzzy soft sets. Çağman et al. defined the soft topology on a soft set, and presented its related properties in [9]. Shabir and Naz [41] pointed out that it will be necessary to carry out more theoretical research to establish a general framework for the practical application of soft topological spaces. In the present paper, we attempt to make some efforts in this aspect.

This paper will attempt to construct the basic theories about soft continuous mappings and soft connectedness of soft topological spaces. The rest of this paper is organized as follows. The next section briefly recalls the notions of soft set, topology and soft topology. In Section 3, based on soft set mapping, we define soft continuous mapping from one soft topological space to another soft topological space and give some equivalence characterizations of soft continuous mapping. Section 4 gives the concept of soft connectedness, and in Section 4, some related properties are discussed. The last section summarizes the conclusions and presents some topics for future research.

2. Preliminaries

In this section we will briefly recall the notions of soft set, topology and soft topology. See especially [11,29,33,41] for further details and background.

2.1. Definition. [29] Let U be a common universe and E be a set of parameters. Let $P(U)$ denote the power set of U and $A \subset E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \longrightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For each $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A).

2.2. Definition. [13] For two soft sets (F, A) and (G, B) over a common universe U we say that (F, A) is a soft subset of (G, B) if
(1) $A \subset B$,
(2) $\forall \varepsilon \in A$, $F(\varepsilon) \subset G(\varepsilon)$.
We write $(F, A)\widetilde{\subset}(G, B)$.

(F, A) is said to be a soft superset of (G, B) if (G, B) is a soft subset of (F, A). We denote it by $(F, A)\widetilde{\supset}(G, B)$.

2.3. Remark. In [41], Shabir and Naz cited another notion of a soft subset as follows:
(F, A) is a soft subset of (G, B) iff (1) $A \subset B$ and (2) $\forall \varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations [33].
However, by the analysis of Definition 23, Theorem 1, Proposition 7 and Example 4 et al. in [41], all of these were obtained based on Definition 2.2 instead of the notion of a soft subset in [33]. Therefore, in the present paper, we will use the notion given in Definition 2.2.

2.4. Definition. [3,41] The relative complement of a soft set (F, A) is denoted by $(F, A)'$ and is defined by $(F, A)' = (F', A)$ where $F' : A \longrightarrow P(U)$ is a mapping given by $F'(\alpha) = U - F(\alpha)$ for all $\alpha \in A$.

Obviously, $(F')' = F$ and $((F, A)')' = (F, A)$.

2.5. Definition. [33] A soft set (F, A) over U is said to be a NULL soft set and is denoted by Φ, if $F(\varepsilon) = \emptyset$ for all $\varepsilon \in A$.

2.6. Definition. [33] The union of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$ and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(F, A) \cup (G, B) = (H, C)$.

2.7. Definition. [36] The intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cap B$ and $\forall e \in C$, $H(e) = F(e) \cap G(e)$. We write $(F, A) \cap (G, B) = (H, C)$.

2.8. Proposition. *Let (F, A), (G, B) and (H, C) be three soft sets over a common universe U. Then*
(1) $(F, A) \cup ((G, B) \cap (H, C)) = ((F, A) \cup (G, B)) \cap ((F, A) \cup (H, C))$.
(2) $((F, A) \cap (G, B)) \cup (H, C) = ((F, A) \cup (H, C)) \cap ((G, B) \cup (H, C))$.
(3) $(F, A) \cap ((G, B) \cup (H, C)) = ((F, A) \cap (G, B)) \cup ((F, A) \cap (H, C))$.
(4) $((F, A) \cup (G, B)) \cap (H, C) = ((F, A) \cap (H, C)) \cup ((G, B) \cap (H, C))$.
(5) $(F, A) \cap ((G, B) \cap (H, C)) = ((F, A) \cap (G, B)) \cap (H, C)$.
(6) $(F, A) \cup ((G, B) \cup (H, C)) = ((F, A) \cup (G, B)) \cup (H, C)$.

Here the first five statements are from [36] and (6) is from [33]. It is easy to see that $(F, A) \cap (\bigcap_{i \in J}(G_i, B_i)) = \bigcap_{i \in J}((F, A) \cap (G_i, B_i))$ and $(F, A) \cup (\bigcup_{i \in J}(G_i, B_i)) = \bigcup_{i \in J}((F, A) \cup (G_i, B_i))$, where J is an index set, (F, A) and (G_i, B_i), $i \in J$ are soft sets over a common universe U.

2.9. Definition. [11,43] Let X be an initial universe set and τ be the collection of subsets of X then τ is said to be a topology on X if
(1) $\emptyset$, X belong to τ,
(2) the union of any number of soft sets in τ belongs to τ,
(3) the intersection of any two soft sets in τ belongs to τ.

The pair (X, τ) is called a topological space. The members of τ are said to be open sets in X.

2.10. Definition. [11] (1) Let (X, τ) be a topological space. A subset A of X is said to be a closed set in X, if its complement A' belongs to τ, where $A' = X - A$.
(2) Let (X, τ_1), (Y, τ_2) be two topological spaces and f be a mapping from X to Y. If $f^{-1}(B) \in \tau_1$ for all $B \in \tau_2$ then f is called a continuous mapping from (X, τ_1) to (Y, τ_2), where $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$.

2.11. Proposition. *[43] Let (X, τ) be a topological space. Then (X, τ) is connected if and only if there exists no $A, B \in \tau - \{\emptyset\}$ such that $A \cap B = \emptyset$ and $A \cup B = X$.*

In the following, let X be an initial universe set and E be a non-empty set of parameters.

2.12. Definition. [41] Let Y be a non-empty subset of X. Then $\widetilde{Y}$ denotes the soft set (Y, E) over X for which $Y(\alpha) = Y$ for all $\alpha \in E$.

In particular, (X, E) will be denoted by $\widetilde{X}$.

2.13. Definition. [41] Let τ be the collection of soft sets over X. Then τ is said to be a soft topology on X if
(1) Φ, $\widetilde{X}$ belong to τ,
(2) the union of any number of soft sets in τ belongs to τ,
(3) the intersection of any two soft sets in τ belongs to τ.

The triple (X, τ, E) is called a soft topological space over X. The members of τ are said to be soft open sets in X.

2.14. Definition. [41] Let (X, τ, E) be a soft topological space over X. A soft set (F, E) over X is said to be a soft closed set in X, if its relative complement $(F, E)'$ belongs to τ.

2.15. Proposition. *[41] Let (X, τ, E) be a soft topological space over X. Then the collection $\tau_\alpha = \{F(\alpha) \mid (F, E) \in \tau\}$ for each $\alpha \in E$, is a topology on X.*

2.16. Definition. [41] Let (X, τ, E) be a soft topological space over X and (F, E) be a soft set over X. Then the soft closure of (F, E), denoted by $\overline{(F, E)}$ is the intersection of all soft closed super sets of (F, E).

$\overline{(F, E)}$ is the smallest soft closed set over X which contains (F, E) (see [41]).

2.17. Proposition. *[41] Let (X, τ, E) be a soft topological space over X and $(F, E), (G, E)$ be two soft sets over X. Then*
(1) $\overline{\Phi} = \Phi$, $\overline{\widetilde{X}} = \widetilde{X}$.
(2) $(F, E) \widetilde{\subset} \overline{(F, E)}$.
(3) (F, E) *is a soft closed set if and only if* $(F, E) = \overline{(F, E)}$.
(4) $\overline{\overline{(F, E)}} = \overline{(F, E)}$.
(5) $(F, E) \widetilde{\subset} (G, E)$ *implies* $\overline{(F, E)} \widetilde{\subset} \overline{(G, E)}$.

3. Soft continuous mappings between soft topological spaces

In this section, we will introduce the notion of soft continuous mapping between soft topological spaces and discuss some related properties. Let X, Y be two initial universe sets and E be a non-empty set of parameters. In what follows, the set of all soft sets over X (resp., Y) will be denoted by $S(X)$ (resp., $S(Y)$).

In order to give the notion of soft continuous mapping, we first need to introduce the following notion of soft set mapping and inverse soft set mapping which can be regarded as special cases of Definitions 8 and 9 in the paper of Kharal and Ahmad [27], called "Mappings on soft classes".

3.1. Definition. Let f be a mapping from X to Y,
(1) The soft set mapping induced by f, denoted by the notation $f^{\rightarrow}$, is a mapping from $S(X)$ to $S(Y)$ that maps (F, E) to $f^{\rightarrow}((F, E)) = (f^{\rightarrow}(F), E)$, where $f^{\rightarrow}(F)$ is defined by $f^{\rightarrow}(F)(e) = \{f(x) \mid x \in F(e)\}$, $\forall e \in E$.
(2) The inverse soft set mapping induced by f, denoted by the notation $f^{\leftarrow}$, is a mapping from $S(Y)$ to $S(X)$ that maps (G, E) to $f^{\leftarrow}((G, E))$, where $f^{\leftarrow}((G, E)) = (f^{\leftarrow}(G), E)$ is defined by $f^{\leftarrow}(G)(e) = \{x \mid f(x) \in G(e)\}$, $\forall e \in E$.

3.2. Example. Let $X = \{h_1, h_2, h_3\}$, $Y = \{p_1, p_2\}$, and $E = \{e_1, e_2\}$. The mapping f is given by $f(h_1) = p_1$, $f(h_2) = p_1$, $f(h_3) = p_2$.
(1) If $(F, E) \in S(X)$ is defined by $\{F(e_1) = \{h_1, h_2\}, F(e_2) = \{h_1, h_3\}\}$, then
$f^{\rightarrow}((F, E)) = (f^{\rightarrow}(F), E) = \{f^{\rightarrow}(F)(e_1) = \{p_1\}, f^{\rightarrow}(F)(e_2) = Y\} \in S(Y)$.
(2) If $(G, B) \in S(Y)$ is defined by $\{G(e_1) = \{p_2\}, G(e_2) = \{p_1\}\}$, then
$f^{\leftarrow}((G, E)) = (f^{\leftarrow}(G), E) = \{f^{\leftarrow}(G)(e_1) = \{h_3\}, f^{\leftarrow}(G)(e_2) = \{h_1, h_2\}\} \in S(X)$.

The following Propositions 3.3 and 3.4 give some basic properties of soft set mappings and inverse soft set mappings which can be regarded as special cases of Theorems 14 and 16 in [27].

3.3. Proposition. *[27] Let f be a mapping from X to Y, $(F_1, E), (F_2, E) \in S(X)$. Then*
(1) $f^{\rightarrow}(\Phi) = \Phi$.
(2) $(F_1, E)\widetilde{\subset}(F_2, E) \Longrightarrow f^{\rightarrow}((F_1, E))\widetilde{\subset} f^{\rightarrow}((F_2, E))$.
(3) $f^{\rightarrow}((F_1, E) \cup (F_2, E)) = f^{\rightarrow}((F_1, E)) \cup f^{\rightarrow}((F_2, E))$.
(4) $f^{\rightarrow}((F_1, E) \cap (F_2, E))\widetilde{\subset} f^{\rightarrow}((F_1, E)) \cap f^{\rightarrow}((F_2, E))$.

3.4. Proposition. *[27] Let f be a mapping from X to Y and $(G_1, E), (G_2, E) \in S(Y)$. Then*
(1) $f^{\leftarrow}(\Phi) = \Phi$, $f^{\leftarrow}(\widetilde{Y}) = \widetilde{X}$.
(2) $(G_1, E)\widetilde{\subset}(G_2, E) \Longrightarrow f^{\leftarrow}((G_1, E))\widetilde{\subset} f^{\leftarrow}((G_2, E))$.
(3) $f^{\leftarrow}((G_1, E) \cup (G_2, E)) = f^{\leftarrow}((G_1, E)) \cup f^{\leftarrow}((G_2, E))$.
(4) $f^{\leftarrow}((G_1, E) \cap (G_2, E)) = f^{\leftarrow}((G_1, E)) \cap f^{\leftarrow}((G_2, E))$.
(5) $f^{\leftarrow}((G_1, E)') = (f^{\leftarrow}((G_1, E)))'$.

3.5. Proposition. *Let f be a mapping from X to Y, $(F, E) \in S(X)$, $(G, E) \in S(Y)$. Then*
(1) $f^{\leftarrow}(f^{\rightarrow}((F, E)))\widetilde{\supset}(F, E)$. *If f is one-one, then* $f^{\leftarrow}(f^{\rightarrow}((F, E))) = (F, E)$.
(2) $f^{\rightarrow}(f^{\leftarrow}((G, E)))\widetilde{\subset}(G, E)$. *If f is surjective, then* $f^{\rightarrow}(f^{\leftarrow}((G, E))) = (G, E)$.

Proof. (1) Let $f^{\rightarrow}((F, E)) = (G, E)$. Then $\forall e \in E$, $f^{\leftarrow}(G)(e) = \{x \mid f(x) \in G(e)\} = \{x \mid f(x) \in \{f(t) \mid t \in F(e)\}\} \supset F(e)$, which implies that $f^{\leftarrow}(f^{\rightarrow}((F, E))) \widetilde{\supset}(F, E)$.

If f is one-one, notice that $\{x \mid f(x) \in \{f(t) \mid t \in F(e)\}\} = F(e)$, thus $f^{\leftarrow}(f^{\rightarrow}((F, E))) = (F, E)$.

(2) Let $f^{\leftarrow}((G, E)) = (F, E)$. Then $\forall e \in E$, $f^{\rightarrow}(F)(e) = \{f(x) \mid x \in F(e)\} = \{f(x) \mid x \in \{t \mid f(t) \in G(e)\}\} \subset G(e)$, which implies that $f^{\rightarrow}(f^{\leftarrow}((G, E)))\widetilde{\subset}(G, E)$.

If f is surjective, notice that $\{f(x) \mid x \in \{t \mid f(t) \in G(e)\}\} = G(e)$, thus $f^{\rightarrow}(f^{\leftarrow}((G, E))) = (G, E)$. □

3.6. Definition. Let (X, τ_1, E) and (Y, τ_2, E) be two soft topological spaces over X and Y, respectively, and f be a mapping from X to Y. If $\forall (G, E) \in \tau_2$, we have $f^{\leftarrow}((G, E)) \in \tau_1$ then f is called a soft continuous mapping from (X, τ_1, E) to (Y, τ_2, E).

Next, we will give an example about soft continuous mapping.

3.7. Example. Let $X = \{h_1, h_2, h_3\}$, $Y = \{p_1, p_2, p_3\}$ and $E = \{e_1, e_2\}$.

$\tau_1 = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E)\}$, where (F_1, E) and (F_2, E) are two soft sets over X, defined as follows:

$F_1(e_1) = \{h_2\}$, $F_1(e_2) = \{h_1\}$,

$F_2(e_1) = \{h_2, h_3\}$, $F_2(e_2) = \{h_1, h_2\}$.

Then τ_1 is a soft topology on X and hence (X, τ_1, E) is a soft topological space over X.

$\tau_2 = \{\Phi, \widetilde{Y}, (G_1, E), (G_2, E)\}$, where (G_1, E) and (G_2, E) are two soft sets over Y, defined as follows:

$G_1(e_1) = \{p_1\}$, $G_1(e_2) = \{p_2\}$,

$G_2(e_1) = \{p_1, p_3\}$, $G_2(e_2) = \{p_1, p_2\}$.

Then τ_2 is a soft topology on Y and hence (Y, τ_2, E) is a soft topological space over Y.

If f is a mapping from X to Y, defined as follows:

$f(h_1) = p_2$, $f(h_2) = p_1$, $f(h_3) = p_3$,

then it is easy to verify that $f^{\leftarrow}((G, E)) \in \tau_1$ for all $(G, E) \in \tau_2$. Thus f is a soft continuous mapping from (X, τ_1, E) to (Y, τ_2, E).

3.8. Proposition. *Let (X, τ_1, E) and (Y, τ_2, E) be two soft topological spaces over X and Y, respectively. If f is a soft continuous mapping from (X, τ_1, E) to (Y, τ_2, E), then f is a continuous mapping from $(X, (\tau_1)_\alpha)$ to $(Y, (\tau_2)_\alpha)$ for all $\alpha \in E$.*

Proof. By Proposition 2.15, $(X, (\tau_2)_\alpha)$ and $(Y, (\tau_2)_\alpha)$ are two topological spaces for all $\alpha \in E$. If $B \in (\tau_2)_\alpha$, then there exists a soft set $(G, E) \in \tau_2$ such that $B = G(\alpha)$. Since f is a soft continuous mapping from (X, τ_1, E) to (Y, τ_2, E), then $f^{\leftarrow}((G, E)) \in \tau_1$. Thus

$$f^{-1}(B) = f^{-1}(G(\alpha)) = \{x \mid f(x) \in G(\alpha)\} = f^{\leftarrow}(G)(\alpha) \in (\tau_1)_\alpha,$$

and by Definition 2.10, f is a continuous mapping from $(X, (\tau_1)_\alpha)$ to $(Y, (\tau_2)_\alpha)$. □

Proposition 3.8 shows that a soft continuous mapping gives a parameterized family of continuous mappings.

3.9. Example. Let (X, τ_1, E) and (Y, τ_2, E) be two soft topological spaces and f be the soft continuous mapping from (X, τ_1, E) to (Y, τ_2, E) given in Example 3.7. By Proposition 2.15,

$(\tau_1)_{e_1} = \{\emptyset, X, \{h_2\}, \{h_2, h_3\}\}$ and $(\tau_1)_{e_2} = \{\emptyset, X, \{h_1\}, \{h_1, h_2\}\}$ are two topologies on X,

$(\tau_2)_{e_1} = \{\emptyset, Y, \{p_1\}, \{p_1, p_3\}\}$ and $(\tau_2)_{e_2} = \{\emptyset, Y, \{p_2\}, \{p_1, p_2\}\}$ are two topologies on Y.

It is easy to verify that f is a continuous mapping from $(X, (\tau_1)_{e_1})$ to $(Y, (\tau_2)_{e_1})$ and f is also a continuous mapping from $(X, (\tau_1)_{e_2})$ to $(Y, (\tau_2)_{e_2})$.

Now we give an example to show that the inverse of Proposition 3.8 does not hold in general.

3.10. Example. Let (X, τ_1, E) be the soft topological space given in Example 3.7 and $Y = \{p_1, p_2, p_3\}$, $\tau_2 = \{\Phi, \widetilde{Y}, (G_3, E)\}$, where the soft set (G_3, E) over Y is defined by $\{G_3(e_1) = \{p_1\}, G_3(e_2) = \{p_1, p_2\}\}$. If f is the mapping from X to Y, given in Example 3.7 then by Proposition 2.15,

$(\tau_1)_{e_1} = \{\emptyset, X, \{h_2\}, \{h_2, h_3\}\}$ and and $(\tau_1)_{e_2} = \{\emptyset, X, \{h_1\}, \{h_1, h_2\}\}$,
$(\tau_2)_{e_1} = \{\emptyset, Y, \{p_1\}\}$ and $(\tau_2)_{e_2} = \{\emptyset, Y, \{p_1, p_2\}\}$ are two topologies on Y.
It can be easily seen that f is a continuous mapping from $(X, (\tau_1)_{e_1})$ to $(Y, (\tau_2)_{e_1})$ continuous mapping from $(X, (\tau_1)_{e_2})$ to $(Y, (\tau_2)_{e_2})$. However,
$f^{\leftarrow}((G_3, E)) = \{f^{\leftarrow}(G_3)(e_1) = \{h_2\}, f^{\leftarrow}(G_3)(e_2) = \{h_1, h_2\}\} \notin \tau_1$, which implies that f is not a soft continuous mapping from (X, τ_1, E) to (Y, τ_2, E).

Next, we will give some equivalence characterizations of soft continuous mappings.

3.11. Proposition. *Let (X, τ_1, E) (resp., (Y, τ_2, E)) be a soft topological space over X (resp., Y) and f be a mapping from X to Y. The following conditions are equivalent:*
(1) f is a soft continuous mapping from (X, τ_1, E) to (Y, τ_2, E).
(2) For each soft closed set (G, E) in Y, $f^{\leftarrow}((G, E))$ is a soft closed set in X.
(3) For each soft set (F, E) over X, $f^{\rightarrow}(\overline{(F, E)}) \widetilde{\subset} \overline{f^{\rightarrow}((F, E))}$.
(4) For each soft set (G, E) over Y, $f^{\leftarrow}(\overline{(G, E)}) \widetilde{\supset} \overline{f^{\leftarrow}((G, E))}$.

Proof. (1) $\Longrightarrow$ (2) Let (G, E) be a soft closed set in Y. Then $(G, E)'$ be a soft open set in Y. By (1) and Proposition 3.4, $f^{\leftarrow}((G_1, E)') = (f^{\leftarrow}((G_1, E)))'$ is a soft open set in X. Hence $f^{\leftarrow}((G, E))$ is a soft closed set in X.

(2) $\Longrightarrow$ (3) Let (F, E) be a soft set over X. By Proposition 2.17 (2), $f^{\rightarrow}((F, E)) \widetilde{\subset} \overline{f^{\rightarrow}((F, E))}$. Then by Propositions 3.4 (2) and 3.5(1), $(F, E) \widetilde{\subset} f^{\leftarrow}(f^{\rightarrow}((F, E))) \widetilde{\subset} f^{\leftarrow}(\overline{f^{\rightarrow}((F, E))})$. Since $\overline{f^{\rightarrow}((F, E))}$ is a soft closed set in Y, then by (2), $f^{\leftarrow}(\overline{f^{\rightarrow}((F, E))})$ is a soft closed set in X. Thus $\overline{(F, E)} \widetilde{\subset} f^{\leftarrow}(\overline{f^{\rightarrow}((F, E))})$. Also by Propositions 3.3 (2) and 3.5 (2),
$f^{\rightarrow}(\overline{(F, E)}) \widetilde{\subset} f^{\rightarrow}(f^{\leftarrow}(\overline{f^{\rightarrow}((F, E))})) \widetilde{\subset} \overline{f^{\rightarrow}((F, E))}$. So $f^{\rightarrow}(\overline{(F, E)}) \widetilde{\subset} \overline{f^{\rightarrow}((F, E))}$.

(3) $\Longrightarrow$ (4) Let (G, E) be a soft set over Y. By (3), Propositions 3.5 (2) and 2.17 (5), $f^{\rightarrow}(\overline{f^{\leftarrow}((G, E))}) \widetilde{\subset} \overline{f^{\rightarrow}(f^{\leftarrow}((G, E)))} \widetilde{\subset} \overline{(G, E)}$. Then by Propositions 3.4 (2) and 3.5 (1), $f^{\leftarrow}(\overline{(G, E)}) \widetilde{\supset} f^{\leftarrow}(f^{\rightarrow}(\overline{f^{\leftarrow}((G, E))})) \widetilde{\supset} \overline{f^{\leftarrow}((G, E))}$.

(4) $\Longrightarrow$ (1) If (G, E) is a soft open set in Y, then $(G, E)'$ is a soft closed set in Y. By (4) and Proposition 2.17 (3), $\overline{f^{\leftarrow}((G, E)')} \widetilde{\subset} f^{\leftarrow}((G, E)')$. Obviously, $\overline{f^{\leftarrow}((G, E)')} \widetilde{\supset} f^{\leftarrow}((G, E)')$. Thus $\overline{f^{\leftarrow}((G, E)')} = f^{\leftarrow}((G, E)')$, which implies that $f^{\leftarrow}((G, E)') = (f^{\leftarrow}((G, E)))'$ (by Proposition 3.4 (5)) is a soft closed set in X. Therefore, $f^{\leftarrow}((G, E))$ is a soft open set in X. So f is a soft continuous mapping from (X, τ_1, E) to (Y, τ_2, E). □

4. Soft connectedness of soft topological spaces

In this section, we will introduce the notion of soft connectedness of soft topological spaces and discuss some related properties.

4.1. Definition. Let (X, τ, E) be a soft topological space over X. If there exists no $(F, E), (G, E) \in \tau - \{\Phi\}$ such that $(F, E) \cap (G, E) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{X}$, then (X, τ, E) is called soft connected.

4.2. Definition. [41] Let X be an initial universe set, E be a set of parameters and $\tau = \{\Phi, \widetilde{X}\}$. Then τ is called the soft indiscrete topology on X and (X, τ, E) is said to be the soft indiscrete space over X.

4.3. Example. Any soft indiscrete space is soft connected.

4.4. Example. Let $X = \{h_1, h_2, h_3\}$ and $E = \{e_1, e_2\}$. $\tau = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$, where (F_1, E), (F_2, E), and (F_3, E) are three soft sets over X, defined as follows:
$F_1(e_1) = \{h_2\}$, $F_1(e_2) = \emptyset$,
$F_2(e_1) = \{h_1, h_3\}$, $F_2(e_2) = \emptyset$,

$F_3(e_1) = X,\ F_3(e_2) = \emptyset$.
Then (X, τ, E) is a soft topological space over X. It is easy to verify that there exists no $(F, E), (G, E) \in \tau - \{\Phi\}$ such that $(F, E) \cap (G, E) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{X}$, so (X, τ, E) is soft connected. By the definition of τ_α, we have $\tau_{e_1} = \{\emptyset, X, \{h_2\}, \{h_1, h_3\}\}$, and $\tau_{e_2} = \{\emptyset, X\}$. Obviously, (X, τ_{e_1}) is not connected, but (X, τ_{e_2}) is connected.

It is worth pointing out that this example shows that (X, τ, E) is soft connected but (X, τ_α) $(\alpha \in E)$ may not be connected.

4.5. Remark. (X, τ, E) may not be soft connected even if (X, τ_α) is connected for every parameter $\alpha \in E$.

4.6. Example. Let X be a non-empty initial universe set and $E = \{e_1, e_2\}$ be the set of parameters. $\tau = \{\Phi, \widetilde{X}, (F, E), (G, E)\}$, where (F, E) and (G, E) are two soft sets over X, defined as follows:
$F(e_1) = \emptyset,\ F(e_2) = X$,
$G(e_1) = X,\ G(e_2) = \emptyset$.
Then (X, τ, E) is a soft topological space over X. Obviously, $(F, E), (G, E) \in \tau - \{\Phi\}$, $(F, E) \cap (G, E) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{X}$, so (X, τ, E) is not soft connected. However, $\tau_{e_1} = \tau_{e_2} = \{\emptyset, X\}$ (indiscrete topology), which implies that (X, τ_{e_1}) and (X, τ_{e_2}) are connected.

The following proposition gives some equivalence characterizations of soft connectedness.

4.7. Proposition. *Let (X, τ, E) be a soft topological space over X. Then the following conditions are equivalent:*
(1) (X, τ, E) is soft connected.
(2) There exists no $(F, E), (G, E) \in \tau' - \{\Phi\}$ such that $(F, E) \cap (G, E) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{X}$, where $\tau' = \{(F, E)' \mid (F, E) \in \tau\}$.
(3) There exists no $(F, E), (G, E) \in S(X) - \{\Phi\}$ such that $((F, E) \cap \overline{(G, E)}) \cup (\overline{(F, E)} \cap (G, E)) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{X}$.
(4) There exists no $(F, E) \in S(X) - \{\Phi, \widetilde{X}\}$ such that $(F, E) \in \tau \cap \tau'$.

Proof. (1) $\Longrightarrow$ (2) Assume there exist $(F, E), (G, E) \in \tau' - \{\Phi\}$ such that $(F, E) \cap (G, E) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{X}$. Then $\forall e \in E$, $F(e) \cap G(e) = \emptyset$ and $F(e) \cup G(e) = X$. Thus $\forall e \in E$, $F'(e) = X - F(e) = G(e)$ and $G'(e) = X - G(e) = F(e)$, which implies that $(F, E)' = (G, E) \in \tau' - \{\Phi\}$ and $(G, E)' = (F, E) \in \tau' - \{\Phi\}$. Then there exist $(F, E), (G, E) \in \tau - \{\Phi\}$ such that $(F, E) \cap (G, E) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{X}$. However, (X, τ, E) is soft connected. This is a contradiction.

(2) $\Longrightarrow$ (3) Assume there exist $(F, E), (G, E) \in S(X) - \{\Phi\}$ such that $((F, E) \cap \overline{(G, E)}) \cup (\overline{(F, E)} \cap (G, E)) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{X}$. Obviously, $(F, E) \cap (G, E) = \Phi$. By Propositions 2.8 and 2.17, $\overline{(G, E)} = \overline{(G, E)} \cap \widetilde{X} = \overline{(G, E)} \cap ((F, E) \cup (G, E)) = (\overline{(G, E)} \cap (F, E)) \cup (\overline{(G, E)} \cap (G, E)) = (G, E)$, which implies that (G, E) is a soft closed set. By using the same methods, we can show that (F, E) is also a soft closed set. Hence there exist $(F, E), (G, E) \in \tau' - \{\Phi\}$ such that $(F, E) \cap (G, E) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{X}$. This is a contradiction. So (3) holds.

(3) $\Longrightarrow$ (4) Assume there exists $(F, E) \in \tau \cap \tau' - \{\Phi, \widetilde{X}\}$. If take $(G, E) = (F, E)'$ then $(F, E), (G, E) \in \tau \cap \tau' - \{\Phi\}$ $(\subset S(X) - \{\Phi\})$. Besides, we have $((F, E) \cap \overline{(G, E)}) \cup (\overline{(F, E)} \cap (G, E)) = (F, E) \cap (G, E) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{X}$. This is a contradiction. So (4) holds.

(4) $\Longrightarrow$ (1) Assume (X, τ, E) is not soft connected. Then there exist $(F, E), (G, E) \in \tau - \{\Phi\}$ such that $(F, E) \cap (G, E) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{X}$. It is easy to see that

$(F,E)' = (G,E)$ and $(G,E)' = (F,E)$. Thus $(F,E),(G,E) \in \tau \cap \tau' - \{\Phi, \widetilde{X}\}$. This is a contradiction. □

4.8. Definition. (1) The difference (H,E) of two soft sets (F,E) and (G,E) over X, denoted by $(F,E) - (G,E)$ (or $(F,E) \setminus (G,E)$) is defined as $H(e) = F(e) - G(e)$ for all $e \in E$.
(2) Let (F,E) be a soft set over X and Y be a non-empty subset of X. Then the soft subset of (F,E) over Y denoted by $({}^Y F, E)$ is defined as ${}^Y F(e) = Y \cap F(e)$, for all $e \in E$. In other words $({}^Y F, E) = \widetilde{Y} \cap (F,E)$.
(3) Let (X, τ, E) be a soft topological space over X and Y be a non-empty subset of X. Then $\tau_Y = \{({}^Y F, E) \mid (F,E) \in \tau\}$ is said to be the soft relative topology on Y and (Y, τ_Y, E) is called a soft subspace of (X, τ, E).
In fact, τ_Y is a soft topology on Y.

4.9. Proposition. *Let (Y, τ_Y, E) be a soft subspace of a soft topological space (X, τ, E). If $(Z, (\tau_Y)_Z, E)$ is a soft subspace of (Y, τ_Y, E) then $(Z, (\tau_Y)_Z, E)$ is also a soft subspace of (X, τ, E).*

Proof.
$$\begin{aligned}(\tau_Y)_Z &= \{\widetilde{Y} \cap (F,E) \mid (F,E) \in \tau\}_Z \\ &= \{\widetilde{Z} \cap \widetilde{Y} \cap (F,E) \mid (F,E) \in \tau\} \\ &= \{\widetilde{Z} \cap (F,E) \mid (F,E) \in \tau\} \\ &= \tau_Z.\end{aligned}$$
So $(Z, (\tau_Y)_Z, E)$ is a soft subspace of (X, τ, E). □

4.10. Definition. Let (X, τ, E) be a soft topological space over X and Y be a non-empty subset of X. If (Y, τ_Y, E) is soft connected then Y is called a soft connected subset of X.

Let (Y, τ_Y, E) be a soft subspace of a soft topological space (X, τ, E). For a soft set $(F,E) \in S(Y)$, $\overline{(F,E)}$ and $\overline{(F,E)}_Y$ will denote the soft closures of (F,E) in (X, τ, E) and (Y, τ_Y, E), respectively.

Now we discuss some basic properties about soft connectedness.

4.11. Proposition. *Let (X, τ, E) be a soft topological space over X. If Y is a soft connected subset of X, then there exists no $(F,E),(G,E) \in \tau - \{\Phi\}$ such that $(F,E) \cap (G,E) = \Phi$ and $(F,E) \cup (G,E) = \widetilde{Y}$.*

Proof. If there exist $(F,E),(G,E) \in \tau - \{\Phi\}$ such that $(F,E) \cap (G,E) = \Phi$ and $(F,E) \cup (G,E) = \widetilde{Y}$, then $(F,E) = \widetilde{Y} \cap (F,E) \in \tau_Y - \{\Phi\}$ and $(G,E) = \widetilde{Y} \cap (G,E) \in \tau_Y - \{\Phi\}$. However, Y is a soft connected subset of X. This is a contradiction. □

The following example shows that the inverse of Proposition 4.11 may not hold in general:

4.12. Example. Let $X = \{a,b,c\}$, $Y = \{b,c\}$, and $E = \{e_1, e_2\}$. $\tau = \{\Phi, \widetilde{X}, (F_1,E), (F_2,E), (F_3,E)\}$, where (F_1,E), (F_2,E), and (F_3,E) are three soft sets over X, defined as follows:
$F_1(e_1) = \{a,b\}$, $F_1(e_2) = \{a,c\}$,
$F_2(e_1) = \{a,c\}$, $F_2(e_2) = \{a,b\}$,
$F_3(e_1) = \{a\}$, $F_3(e_2) = \{a\}$.
Then (X, τ, E) is a soft topological space over X.
By Definition 4.8, we have
${}^Y(F_1,E) = \{\ {}^Y F_1(e_1) = \{b\},\ {}^Y F_1(e_2) = \{c\}\ \}$,
${}^Y(F_2,E) = \{\ {}^Y F_2(e_1) = \{c\},\ {}^Y F_2(e_2) = \{b\}\ \}$,
${}^Y(F_3,E) = \{\ {}^Y F_3(e_1) = \emptyset,\ {}^Y F_3(e_2) = \emptyset\ \} = \Phi$.

Then $\tau_Y = \{\Phi, \widetilde{Y}, {}^Y(F_1, E), {}^Y(F_2, E)\}$.

It is easy to verify that there exists no $(F,E),(G,E) \in \tau - \{\Phi\}$ such that $(F,E) \cap (G,E) = \Phi$ and $(F,E) \cup (G,E) = \widetilde{Y}$. However, ${}^Y(F_1,E), {}^Y(F_2,E) \in \tau_Y - \{\Phi\}$, ${}^Y(F_1,E) \cap {}^Y(F_2,E) = \Phi$ and ${}^Y(F_1,E) \cup {}^Y(F_2,E) = \widetilde{Y}$, i.e. (Y, τ_Y, E) is not soft connected.

By Proposition 4.11, we can easily show the following corollary:

4.13. Corollary. *Let (X,τ,E) be a soft topological space over X and Y be a soft connected subset of X. If there exist $(F,E),(G,E) \in \tau$ such that $(F,E) \cap (G,E) = \Phi$ and $\widetilde{Y} \widetilde{\subset} (F,E) \cup (G,E)$ then $\widetilde{Y} \widetilde{\subset} (F,E)$ or $\widetilde{Y} \widetilde{\subset} (G,E)$.*

4.14. Lemma. *[41] Let (Y,τ_Y,E) be a soft subspace of a soft topological space (X,τ,E) and (F,E) be a soft set over X, then (F,E) is soft closed in Y if and only if $(F,E) = \widetilde{Y} \cap (G,E)$ for some soft closed set (G,E) in X.*

4.15. Proposition. *Let (X,τ,E) be a soft topological space over X and Y be a nonempty subset of X. Then Y is a soft connected subset of X if and only if there exists no $(F,E),(G,E) \in S(Y) - \{\Phi\}$ such that $((F,E) \cap \overline{(G,E)}) \cup (\overline{(F,E)} \cap (G,E)) = \Phi$ and $(F,E) \cup (G,E) = \widetilde{Y}$.*

Proof. For all $(F,E),(G,E) \in S(Y) - \{\Phi\}$, by Definition 2.16, Proposition 2.8 and Lemma 4.14, we have

$$\begin{aligned}
(F,E) \cap \overline{(G,E)}_Y &= (F,E) \cap (\bigcap\{(H,E) \mid (H,E) \in (\tau_Y)', (H,E) \widetilde{\supset} (G,E)\})^{\S} \\
&= (F,E) \cap (\bigcap\{\widetilde{Y} \cap (Q,E) \mid (Q,E) \in \tau', (Q,E) \widetilde{\supset} (G,E)\})^{\P} \\
&= ((F,E) \cap \widetilde{Y}) \cap (\bigcap\{(Q,E) \mid (Q,E) \in \tau', (Q,E) \widetilde{\supset} (G,E)\}) \\
&= (F,E) \cap (\bigcap\{(Q,E) \mid (Q,E) \in \tau', (Q,E) \widetilde{\supset} (G,E)\}) \\
&= (F,E) \cap \overline{(G,E)}.
\end{aligned}$$

Similarly, we can show that $\overline{(F,E)}_Y \cap (G,E) = \overline{(F,E)} \cap (G,E)$.

By Proposition 4.7, (Y,τ_Y,E) is soft connected if and only if there exists no $(F,E),(G,E) \in S(Y) - \{\Phi\}$ such that $((F,E) \cap \overline{(G,E)}_Y) \cup (\overline{(F,E)}_Y \cap (G,E)) = \Phi$ and $(F,E) \cup (G,E) = \widetilde{Y}$. Then (Y,τ_Y,E) is soft connected if and only if there exists no $(F,E),(G,E) \in S(Y) - \{\Phi\}$ such that $((F,E) \cap \overline{(G,E)}) \cup (\overline{(F,E)} \cap (G,E)) = \Phi$ and $(F,E) \cup (G,E) = \widetilde{Y}$, i.e. Y is a soft connected subset of X if and only if there exists no $(F,E),(G,E) \in S(Y) - \{\Phi\}$ such that $((F,E) \cap \overline{(G,E)}) \cup (\overline{(F,E)} \cap (G,E)) = \Phi$ and $(F,E) \cup (G,E) = \widetilde{Y}$. □

4.16. Proposition. *Let (X,τ,E) be a soft topological space over X and Y be a soft connected subset of X. If there exist $(F,E),(G,E) \in S(Y)$ such that $((F,E) \cap \overline{(G,E)}) \cup (\overline{(F,E)} \cap (G,E)) = \Phi$ and $\widetilde{Y} \widetilde{\subset} (F,E) \cup (G,E)$ then $\widetilde{Y} \widetilde{\subset} (F,E)$ or $\widetilde{Y} \widetilde{\subset} (G,E)$.*

Proof. If there exist $(F,E),(G,E) \in S(Y)$ such that $((F,E) \cap \overline{(G,E)}) \cup (\overline{(F,E)} \cap (G,E)) = \Phi$ and $\widetilde{Y} \widetilde{\subset} (F,E) \cup (G,E)$ then by Propositions 2.8 and 2.17, we have

$$\begin{aligned}
&((\widetilde{Y} \cap (F,E)) \cap \overline{(\widetilde{Y} \cap (G,E))}) \cup (\overline{(\widetilde{Y} \cap (F,E))} \cap (\widetilde{Y} \cap (G,E))) \\
&\widetilde{\subset} ((\widetilde{Y} \cap (F,E)) \cap \overline{(G,E)}) \cup (\overline{(F,E)} \cap (\widetilde{Y} \cap (G,E))) \\
&= \widetilde{Y} \cap ((F,E) \cap \overline{(G,E)}) \cup (\overline{(F,E)} \cap (G,E)) \\
&= \widetilde{Y} \cap \Phi \\
&= \Phi.
\end{aligned}$$

Besides, we have $\widetilde{Y} \cap (F,E), \widetilde{Y} \cap (G,E) \in S(Y)$, and $(\widetilde{Y} \cap (F,E)) \cup (\widetilde{Y} \cap (G,E)) = \widetilde{Y} \cap ((F,E) \cup (G,E)) = \widetilde{Y}$. Since Y is a soft connected subset of X, $\widetilde{Y} \cap (F,E) = \Phi$ or $\widetilde{Y} \cap (G,E) = \Phi$ by Proposition 4.15. If $\widetilde{Y} \cap (F,E) = \Phi$ then by $(\widetilde{Y} \cap (F,E)) \cup (\widetilde{Y} \cap (G,E)) = \widetilde{Y}$, we have $\widetilde{Y} \widetilde{\subset} (G,E)$. Similarly, if $\widetilde{Y} \cap (G,E) = \Phi$ then $\widetilde{Y} \widetilde{\subset} (F,E)$. □

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§ $(\tau_Y)' = \{\widetilde{Y} - (W,E) \mid (W,E) \in \tau_Y\}$.

¶ $\tau' = \{\widetilde{X} - (V,E) \mid (V,E) \in \tau\}$.

4.17. Proposition. *Let (X, τ, E) be a soft topological space over X, Y be a soft connected subset of X and Z be a non-empty subset of X. If $\widetilde{Y} \widetilde{\subset} \widetilde{Z} \widetilde{\subset} \overline{\widetilde{Y}}$ then Z is also a soft connected subset of X.*

Proof. Assume that Z is not a soft connected subset of X. By Proposition 4.15, there exist $(F, E), (G, E) \in S(Z) - \{\Phi\}$ $(\subset S(Y))$ such that $((F, E) \cap \overline{(G, E)}) \cup (\overline{(F, E)} \cap (G, E)) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{Z}$. Then $\widetilde{Y} \widetilde{\subset} (F, E) \cup (G, E)$. Since Y is a soft connected subset of X, we have $\widetilde{Y} \widetilde{\subset} (F, E)$ or $\widetilde{Y} \widetilde{\subset} (G, E)$ by Proposition 4.16. If $\widetilde{Y} \widetilde{\subset} (F, E)$, then $\widetilde{Z} \cap (G, E) \widetilde{\subset} \overline{(F, E)} \cap (G, E) = \Phi$ for $\widetilde{Z} \widetilde{\subset} \overline{\widetilde{Y}} \widetilde{\subset} \overline{(F, E)}$. Thus $(G, E) = \widetilde{Z} \cap (G, E) = \Phi$. This is a contradiction. Similarly, if $\widetilde{Y} \widetilde{\subset} (G, E)$ then $(F, E) = \Phi$. This is also a contradiction. □

The following proposition shows that the soft connectedness is an invariant property under a soft continuous mapping.

4.18. Proposition. *Let f be a soft continuous mapping from soft topological space (X, τ_1, E) to soft topological space (Y, τ_2, E). If (X, τ_1, E) is soft connected and $f(X) \neq \emptyset$ then $f(X)$ is a soft connected subset of Y.*

Proof. Assume $f(X)$ is not a soft connected subset of Y. By Proposition 4.15, there exist $(F, E), (G, E) \in S(f(X)) - \{\Phi\}$ $(\subset S(Y) - \{\Phi\})$ such that $((F, E) \cap \overline{(G, E)}) \cup (\overline{(F, E)} \cap (G, E)) = \Phi$ and $(F, E) \cup (G, E) = \widetilde{f(X)}$. Then $f^{\leftarrow}((F, E))$, $f^{\leftarrow}((G, E)) \in S(X) - \{\Phi\}$ and by Propositions 3.11 and 3.4,

$$\begin{aligned}
&(f^{\leftarrow}((F, E)) \cap \overline{f^{\leftarrow}((G, E))}) \cup (\overline{f^{\leftarrow}((F, E))} \cap f^{\leftarrow}((G, E))) \\
\widetilde{\subset}\, &(f^{\leftarrow}((F, E)) \cap f^{\leftarrow}(\overline{(G, E)})) \cup (f^{\leftarrow}(\overline{(F, E)}) \cap f^{\leftarrow}((G, E))) \\
= \, &f^{\leftarrow}(((F, E) \cap \overline{(G, E)}) \cup (\overline{(F, E)} \cap (G, E))) \\
= \, &f^{\leftarrow}(\Phi) \\
= \, &\Phi.
\end{aligned}$$

Besides, by Propositions 3.4, and 3.5, we have

$$f^{\leftarrow}((F, E)) \cup f^{\leftarrow}((G, E)) = f^{\leftarrow}((F, E) \cup (G, E)) = f^{\leftarrow}(\widetilde{f(X)}) = f^{\leftarrow}(f^{\rightarrow}(\widetilde{X})) = \widetilde{X}.$$

It follows that (X, τ_1, E) is not soft connected. This is a contradiction. So $f(X)$ is a soft connected subset of Y. □

To illustrate the idea of Proposition 4.18, we give the following example:

4.19. Example. Let $X = \{h_1, h_2, h_3\}$, $Y = \{p_1, p_2, p_3\}$ and $E = \{e_1, e_2\}$. $\tau_1 = \{\Phi, \widetilde{X}, (F_1, E), (F_2, E)\}$, where (F_1, E) and (F_2, E) are two soft sets over X, defined as $F_1(e_1) = \{h_2\}$, $F_1(e_2) = \emptyset$, $F_2(e_1) = \{h_1, h_3\}$, $F_2(e_2) = \emptyset$. Then (X, τ_1, E) is a soft topological space over X and it is easy to verify that (X, τ_1, E) is soft connected.

Let $\tau_2 = \{\Phi, \widetilde{Y}, (G, E)\}$, where $(G, E) = \{G(e_1) = \{p_1\}, G(e_2) = \{p_2, p_3\}\}$. Then (Y, τ_2, E) is a soft topological space over Y.

If $f : X \longrightarrow Y$ is a mapping defined as $f(h_1) = f(h_2) = f(h_3) = p_1$, obviously, f is a soft continuous mapping from (X, τ_1, E) to (Y, τ_2, E). In this case, $f(X) = \{p_1\} \neq \emptyset$ and it is easy to verify that $f(X)$ is a soft connected subset of Y.

5. Conclusion

In this paper, we construct the basic theories about soft continuous mappings and soft connectedness of soft topological spaces. We also give some examples to explain these new notions. Moreover the concepts and some results proposed in this paper can be extended into fuzzy cases. In our future study on soft topological spaces, may be the following topics should be considered:

(1) To discuss soft basis, soft sub-basis of soft topological spaces,

(2) To study soft compactness of soft topological spaces,
(3) To describe the real world application of soft topological spaces.

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References

[1] Aktaş, H. and Çağman, N. *Soft sets and soft groups*, Information Science **177** (13), 2726-2735, 2007.

[2] Aygunoglu, A. and Aygun, H. *Introduction to fuzzy soft groups* , Computers and Mathematics with Applications **58** (6), 1279-1286, 2009.

[3] Ali, M.I., Feng, F., Liu, X., Min, W.K. and Shabir, M. *On some new operations in soft set theory*, Computers and Mathematics with Applications **57** (9), 1547-1553, 2009.

[4] Ali, M.I., Shabir, M. and Naz, M. *Algebraic structures of soft sets associated with new operations*, Computers and Mathematics with Applications **61** (9), 2647-2654, 2011.

[5] Acar, U., Koyuncu, F. and Tanay, B. *Soft sets and soft rings*, Computers and Mathematics with Applications **59** (11), 3458-3463, 2010.

[6] Atagün, A.O. and Sezgin, A. *Soft substructures of rings, fields and modules*, Computers and Mathematics with Applications **61** (3), 592-601, 2011.

[7] Babitha, K.V. and Sunil, J.J. *Soft set relations and functions*, Computers and Mathematics with Applications **60** (7), 1840-1849, 2010.

[8] Çağman, N. and Enginoğlu, S. *Soft matrix theory and its decision making*, Computers and Mathematics with Applications **59** (10), 3308-3314, 2010.

[9] Çağman, N., Karataş, S. and Enginoglu, S. *Soft topology*, Computers and Mathematics with Applications **62** (1), 351-358, 2011.

[10] Çelik, Y., Ekiz C. and Yamak, S. *A new view on soft rings*, Hacettepe Journal of Mathematics and Statistics **40** (2), 273-286, 2011.

[11] Engelking, R. *General Topology*, Panstwowe Wydawnictwo Naukowe, 1975.

[12] Feng, F., Jun, Y.B. and Zhao, X.Z. *Soft semirings*, Computers and Mathematics with Applications **56**, 2621-2628, 2008.

[13] Feng, F., Li, C., Davvaz, B. and Ali, M.I. *Soft sets combined with fuzzy sets and rough sets: a tentative approach*, Soft Computing **14**, 8999-9911, 2010.

[14] Feng, F., Jun, Y.B., Liu, X. and Li, L. *An adjustable approach to fuzzy soft set based decision making*, J. Comput. Appl. Math. **234** (1), 10-20, 2010.

[15] Gong, K., Xiao, Z. and Zhang, X. *The bijective soft set with its operations*, Computers and Mathematics with Applications **60** (8), 2270-2278, 2010.

[16] Ge, X., Li, Z. and Ge, Y. *Topological spaces and soft sets*, Journal of Computational Analysis and Applications **13** (5), 881-885, 2011.

[17] Jun, Y.B. *Soft BCK/BCI-algebras*, Computers and Mathematics with Applications **56** (5), 1408-1413, 2008.

[18] Jun, Y.B. and Park, C.H. *Applications of soft sets in ideal theory of BCK/BCI-algebras*, Information Science **178** (11), 2466-2475, 2008.

[19] Jun, Y.B., Lee, K.J. and Zhan, J. *Soft p-ideals of soft BCI-algebras*, Computers and Mathematics with Applications **58** (10), 2060-2068, 2009.

[20] Jun, Y.B., Lee, K.J. and Park, C.H. *Soft set theory applied to ideals in d-algebras*, Computers and Mathematics with Applications **57** (3), 367-378, 2009.

[21] Jun, Y.B. and Park, C.H. *Applications of soft sets in Hilbert algebras*, Iranian Journal of Fuzzy Systems **6** (2), 75-86, 2009.

[22] Jun, Y.B., Lee, K.J. and Khan, A. *Soft ordered semigroups*, Math. Logic Quart. **56** (1), 42-50, 2010.

[23] Jiang, Y., Tang, Y., Chen, Q., Liu, H. and Tang, J. *Interval-valued intuitionistic fuzzy soft sets and their properties*, Computers and Mathematics with Applications **60** (3), 906-918, 2010.
[24] Jiang, Y., Tang, Y., Chen, Q., Wang, J. and Tang, S. *Extending soft sets with description logics,* Computers and Mathematics with Applications **59** (6), 2087-2096, 2010.
[25] Kazancı O., Yılaz, Ş and Yamak, S. *Soft sets and soft BCH-algebras*, Hacettepe Journal of Mathematics and Statistics **39** (2), 205-217, 2010.
[26] Kong, Z., Gao, L. and Wang, L. *Comment on "A fuzzy soft set theoretic approach to decision making problems"*, J. Comput. Appl. Math. **223** (2), 540-542, 2009.
[27] Kharal, A. and Ahmad, B. *Mappings on fuzzy soft classes*, Advances in Fuzzy Systems, **2009**, 1-6, 2009.
[28] Kalayathankal, S.J. and Singh, G.S. *A fuzzy soft flood alarm model*, Math. Comput. Simul. **80** (5), 887-893, 2010.
[29] Molodtsov, D. *Soft set theory—first results*, Computers and Mathematics with Applications **37**, 19-31, 1999.
[30] Maji, P.K., Biswas, R. and Roy, A.R. *Fuzzy soft sets*, J. Fuzzy Math. **9** (3), 589-602, 2001.
[31] Maji, P.K., Biswas, R. and Roy, A.R. *Intuitionistic fuzzy soft sets*, J. Fuzzy Math. **9** (3), 677-692, 2001.
[32] Maji, P.K., Roy, A.R. and Biswas, R. *An application of soft sets in a decision making problem*, Computers and Mathematics with Applications **44** (8-9), 1077-1083, 2002.
[33] Maji, P.K., Biswas, R. and Roy, A.R. *Soft set theory*, Computers and Mathematics with Applications **45**, 555-562, 2003.
[34] Majumdar, P. and Samanta, S.K. *Generalised fuzzy soft sets*, Computers and Mathematics with Applications **59** (4), 1425-1432, 2010.
[35] Majumdar, P. and Samanta, S.K. *On soft mappings*, Computers and Mathematics with Applications **60** (9), 2666-2672, 2010.
[36] Pei, D. and Miao, D. *From soft sets to information systems*, in: Granular Computing, IEEE International Conference, 2, 617-621, 2005.
[37] Qin, K. and Hong, Z. *On soft equality*, J. Comput. Appl. Math. **234** (5), 1347-1355, 2010.
[38] Roy, A.R. and Maji, P.K. *A fuzzy soft set theoretic approach to decision making problems*, J. Comput. Appl. Math. **203** (2), 412-418, 2007.
[39] Sezgin, A. and Atagün, A.O. *On operations of soft sets*, Computers and Mathematics with Applications **61**, 1457-1467, 2011.
[40] Sezgin, A. and Atagün, A.O. *Soft groups and normalistic soft groups*, Computers and Mathematics with Applications **62** (2), 685-698, 2011.
[41] Shabir, M. and Naz, M. *On soft topological spaces*, Computers and Mathematics with Applications **61**, 1786-1799, 2011.
[42] Tanay, B. and Kandemir, M.B. *Topological structure of fuzzy soft sets*, Computers and Mathematics with Applications **61** (10), 2952-2957, 2011.
[43] Xiong, J.C. *Point set topology teaching materials*, Higher Education Press, 2001 (in Chinese).
[44] Xiao, Z., Gong, K. and Zou, Y. *A combined forecasting approach based on fuzzy soft sets*, J. Comput. Appl. Math. **228** (1), 326-333, 2009.
[45] Xiao, Z., Gong, K., Xia, S. and Zou, Y. *Exclusive disjunctive soft sets*, Computers and Mathematics with Applications **59** (6), 2128-2137, 2010.
[46] Xu, W., Ma, J., Wang, S. and Hao, G. *Vague soft sets and their properties*, Computers and Mathematics with Applications **59** (2), 787-794, 2010.
[47] Yang, X.B., Lin, T.Y., Yang, J.Y., Li, Y. and Yu, D. *Combination of interval-valued fuzzy set and soft set*, Computers and Mathematics with Applications **58** (3), 521-527, 2009.
[48] Yang, C.-F. *Fuzzy soft semigroups and fuzzy soft ideals*, Computers and Mathematics with Applications **61** (2), 255-261, 2011.
[49] Yang, H.-L. and Guo, Z.-L. *Kernels and closures of soft set relations, and soft set relation mappings*, Computers and Mathematics with Applications **61** (3), 651-662, 2011.
[50] Zou, Y. and Xiao, Z. *Data analysis approaches of soft sets under incomplete information*, Knowledge-Based Systems **21** (8), 941-945, 2008.

[51] Zhan, J. and Jun, Y.B. *Soft BL-algebras based on fuzzy sets*, Computers and Mathematics with Applications **59** (6), 2037-2046, 2010.

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Interval estimation for the two-parameter bathtub-shaped lifetime distribution based on records

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Abstract

In this paper, we study the estimation problems for the two-parameter bathtub-shaped lifetime distribution based on upper record values. Exact confidence intervals and exact joint confidence regions for the parameters are constructed. Approximate confidence intervals and regions are also discussed based on the asymptotic normality of the maximum likelihood estimators. A simulation study is done for the performance of all proposed confidence intervals and regions. Two numerical examples with real data set and simulated data, are presented to illustrate the methods proposed here.

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1. Introduction

The failure rate function is an important characteristic of a lifetime distribution and the shapes of the failure rate functions are qualitatively different. In practice, units in a population are followed from actual birth to death, a bathtub-shaped failure rate function is often seen. In recent years, some lifetime distributions with bathtub-shaped failure rate function have been investigated by several authors. For example, Bebbington et al. [5], Gurvich et al. [10], Haynatzki et al. [11], Hjorth [12], Mudholkar and Srivastava [15], Pham and Lai [17], Smith and Bain [18], Wang [19] and Xie et al. [22]. A recent account on bathtub-shaped failure rate functions can be found in the review article by Nadarajah [16].

In this paper, we discuss the two-parameter lifetime distribution with bathtub-shaped or increasing failure rate function proposed by Chen [7]. The cumulative distribution function (cdf) of this distribution is given by

$$F(x) = 1 - e^{\lambda(1-e^{x^\beta})}, \qquad x > 0, \quad \lambda, \beta > 0, \tag{1.1}$$

and hence the probability density function (pdf) is given by

$$f(x) = \lambda\beta x^{\beta-1} e^{[x^\beta + \lambda(1-e^{x^\beta})]} \qquad x > 0, \quad \lambda, \beta > 0.$$

The reliability function $R(x)$ and hazard (failure rate) function $H(x)$ of this distribution are given, respectively, by

$$R(x) = e^{\lambda(1-e^{x^\beta})}, \qquad x > 0, \quad \lambda, \beta > 0,$$

and

$$H(x) = \lambda\beta x^{\beta-1} e^{x^\beta} \qquad x > 0, \quad \lambda, \beta > 0.$$

The parameter β is the shape parameter which also affects the shape of the failure rate function. When $\beta < 1$, the failure rate function of this distribution has a bathtub shape. When $\beta \geq 1$, this distribution has an increasing failure rate (see, Chen [7] and Wu [21]).

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) random variables with cdf $F(x)$ and pdf $f(x)$. An observation X_j is called an upper record value if its value exceeds that of all previous observations. That is, X_j is an upper record values if $X_j > X_i$ for every $i < j$. If $\{U(n), n \geq 1\}$ is defined by

$$U(1) = 1 \quad \text{and} \quad U(n) = \min\{j : j > U(n-1), X_j > X_{U(n-1)}\},$$

for $n \geq 2$, then the sequence $\{X_{U(n)}, n \geq 1\}$ provides a sequence of upper record statistics. The sequence $\{U(n), n \geq 1\}$ represents the record times.

The definition of record values was formulated by Chandler [6]. A record value or record statistic is the largest or smallest value obtained from a sequence of random variables. The theory of record values relies largely on the theory of order statistics. As mentioned by Ahsanullah and Nevzorov [3] records are very popular because they arise naturally in many fields of studies such as climatology, sports, medicine, traffic, industry and so on. Such records are memorials of their time. The annals of records reflect the progress in science and technology and enable us to study the evaluation of mankind on the basic of record achievements in various areas of its activity. For example, in industry and reliability studies, many products fail under stress. A wooden beam breaks when sufficient perpendicular force is applied to it, an electronic component ceases to function in an environment of too high temperature, and a battery dies under the stress of time. However, the precise breaking stress or failure point varies even among identical items. Hence, in such experiments, measurements may be made sequentially and only the record values are observed. Lee et al. [14] indicated that there are some situations in lifetime testing experiments in which a failure time of a product is recorded if it exceeds all preceding failure times. These recorded failure times are the upper record value sequence. As mentioned by Ahmadi and Balakrishnan [1], there is a connection between record values and minimal repair process, which is as follows. Let X be a lifetime of a component with cdf $F(x)$ and $X(m)$ denote the lifetime if m minimal repairs are allowed. Then, $X(m)$ has the same distribution as the m-th upper record derived from iid observations from $F(x)$. For more details and applications of record values, see, for example, Ahsanullah [2] and Arnold et al. [4].

The purpose of this paper is to construct the interval estimation for the parameters of the bathtub-shaped distribution based on record values. The rest of this paper is organized as follows. Section 2 provides the maximum likelihood estimators (MLEs) of the parameters β and λ, and also establishes the approximate confidence intervals and region for the parameters. Furthermore, the exact confidence intervals for the parameter β and exact joint confidence regions for the parameters β and λ are obtained by using some pivotal quantities. Section 3 conducts some simulations to study the performance of the proposed confidence intervals and regions. Section 4 discusses two numerical examples for illustration. Section 5 makes some conclusions.

2. Main Results

In this section, we will derive the approximate confidence intervals and region for the parameters based on the asymptotic normality of the MLEs. The exact confidence intervals for β and exact joint confidence regions for β and λ will also be discussed.

2.1. Maximum Likelihood Estimation. Let $X_{U(1)} < X_{U(2)} < \cdots < X_{U(m)}$ be the first m observed upper record values from two parameter bathtub-shaped lifetime distribution in (1.1). For notation simplicity, we will write X_i for $X_{U(i)}$. The likelihood function is given by (see Arnold et al. [4])

$$\begin{aligned} L(\beta,\lambda) &= f(x_m)\prod_{i=1}^{m-1}\frac{f(x_i)}{1-F(x_i)} \\ &= (\lambda\beta)^m e^{\lambda(1-e^{x_m^\beta})}\prod_{i=1}^{m} x_i^{\beta-1}e^{x_i^\beta}. \end{aligned}$$

The log-likelihood function is then

$$\begin{aligned} l(\beta,\lambda) &= \ln L(\beta,\lambda) \\ &= m\ln\lambda + m\ln\beta + \lambda(1-e^{x_m^\beta}) + (\beta-1)\sum_{i=1}^{m}\ln x_i + \sum_{i=1}^{m} x_i^\beta. \end{aligned}$$

The MLEs of (β,λ) can be obtained by solving the likelihood equations

$$\frac{\partial l(\beta,\lambda)}{\partial\beta} = \frac{m}{\beta} - \lambda x_m^\beta e^{x_m^\beta}\ln x_m + \sum_{i=1}^{m}\ln x_i + \sum_{i=1}^{m} x_i^\beta \ln x_i = 0,$$

and

$$\frac{\partial l(\beta,\lambda)}{\partial\lambda} = \frac{m}{\lambda} + (1-e^{x_m^\beta}) = 0.$$

The approximate confidence intervals and region for the unknown parameters have been discussed by some authors. See for example, Doostparast et al. [8] and Gupta and Kundu [9]. Here we will use the asymptotic normality of the MLEs to construct the confidence intervals and region for the parameters. To obtain the Fisher information matrix, we need

$$\frac{\partial^2 l(\beta,\lambda)}{\partial \beta^2} = -\frac{m}{\beta^2} - \lambda x_m^\beta (\ln x_m)^2 e^{x_m^\beta}[1+x_m^\beta] + \sum_{i=1}^{m} x_i^\beta (\ln x_i)^2,$$

$$\frac{\partial^2 l(\beta,\lambda)}{\partial \beta \partial \lambda} = \frac{\partial^2 l(\beta,\lambda)}{\partial \lambda \partial \beta} = -x_m^\beta (\ln x_m) e^{x_m^\beta},$$

and

$$\frac{\partial^2 l(\beta,\lambda)}{\partial \lambda^2} = -\frac{m}{\lambda^2}.$$

Under suitable regularity conditions, we know that $\sqrt{m}(\widehat{\beta}-\beta,\widehat{\lambda}-\lambda)'$ is approximately bivariate normal with mean $(0,0)$ and covariance matrix $I^{-1}(\beta,\lambda)$ evaluated at the MLEs $(\widehat{\beta},\widehat{\lambda})$, where

$$I(\beta,\lambda) = -\frac{1}{m}\begin{bmatrix} \frac{\partial^2 l(\beta,\lambda)}{\partial \beta^2} & \frac{\partial^2 l(\beta,\lambda)}{\partial \beta \partial \lambda} \\ \frac{\partial^2 l(\beta,\lambda)}{\partial \lambda \partial \beta} & \frac{\partial^2 l(\beta,\lambda)}{\partial \lambda^2} \end{bmatrix}.$$

Thus, the approximate confidence intervals for β and λ can be obtained in the usual way. Furthermore, note that $m[\widehat{\beta}-\beta,\widehat{\lambda}-\lambda]I(\widehat{\beta},\widehat{\lambda})[\widehat{\beta}-\beta,\widehat{\lambda}-\lambda]'$ is asymptotically chi-square distributed with 2 degrees of freedom. Now, using this result, the $100(1-\alpha)\%$ approximate joint confidence region for (β,λ) is given by

$$\left\{(\beta,\lambda):\ m[\widehat{\beta}-\beta,\widehat{\lambda}-\lambda]I(\widehat{\beta},\widehat{\lambda})[\widehat{\beta}-\beta,\widehat{\lambda}-\lambda]' \le \chi^2_\alpha(2)\right\},$$

where $\chi^2_\alpha(2)$ is the percentile of chi-square distribution with right-tail probability α and 2 degrees of freedom.

2.2. Exact Interval Estimations. Let $X_1 < X_2 < \cdots < X_m$ be the first m upper record values from the two-parameter bathtub-shaped lifetime distribution in (1.1). Set

$$Y_i = -\ln[1-F(X_i)] = \lambda(e^{X_i^\beta}-1), \qquad i=1,2,\ldots,m.$$

Then, $Y_1 < Y_2 < \cdots < Y_m$ are the first m upper record values from a standard exponential distribution. Moreover, $Z_1 = Y_1$ and $Z_i = Y_i - Y_{i-1}$, for $i=2,\ldots,m$, are iid standard exponential random variables (see Arnold et al. [4]). Hence,

$$V_j = 2\sum_{i=1}^{j} Z_i = 2\ Y_j$$

has a chi-square distribution with $2j$ degrees of freedom and

$$U_j = 2\sum_{i=j+1}^{m} Z_i = 2\ (Y_m - Y_j)$$

has a chi-square distribution with $2(m-j)$ degrees of freedom, where $j=1,\ldots,m-1$. We can also find that U_j and V_j are independent random variables for each j. Let

$$T_j = \frac{U_j/2(m-j)}{V_j/2j} = \frac{j\ U_j}{(m-j)V_j} = \frac{j}{m-j}\left(\frac{Y_m - Y_j}{Y_j}\right). \tag{2.1}$$

It is easy to show that T_j has an F distribution with $2(m-j)$ and $2j$ degrees of freedom for $j=1,\ldots,m-1$. Therefore, using the pivotal quantities T_j, $j=1,\ldots,m-1$, we can provide $m-1$ confidence intervals for β. To obtain the confidence interval for β, we further need the following lemmas.

2.1. Lemma. *For any $0 < c_1 < c_2$, the function*

$$g(\beta) = \frac{e^{c_2^\beta} - 1}{e^{c_1^\beta} - 1}$$

is a strictly increasing function of β for any $\beta > 0$.

Proof. The proof of Lemma 2.1 can be found in Chen [7]. □

2.2. Lemma. *Suppose that $0 < c_1 < c_2 < \cdots < c_m$. Let*

$$T_j(\beta) = \frac{j}{m-j}\left[\frac{e^{c_m^\beta} - 1}{e^{c_j^\beta} - 1} - 1\right], \quad j = 1, \ldots, m-1.$$

Then for all $j = 1, \ldots, m-1$,

(a) *$T_j(\beta)$ is strictly increasing in β for any $\beta > 0$.*
(b) *For $t > 0$, the equation, $T_j(\beta) = t$ has a unique solution in $\beta > 0$.*

Proof. (a) By Lemma 2.1, it is easy to show that $T_j(\beta)$ is a strictly increasing function of β.
(b) Since the function $T_j(\beta)$ is strictly increasing in $\beta > 0$ with $\lim_{\beta\to 0} T_j(\beta) = 0$ and $\lim_{\beta\to\infty} T_j(\beta) = \infty$, then the lemma follows.

□

Let $F_{(\alpha),(\upsilon_1,\upsilon_2)}$ denote the upper α percentile of F distribution with υ_1 and υ_2 degrees of freedom. Lemma 2.2 can be used to construct $m-1$ exact confidence intervals for the shape parameter β based on the pivotal quantities $T_j(\beta)$, $j = 1, 2, \ldots, m-1$. These exact confidence intervals are given in the following theorem.

2.3. Theorem. *Suppose that $X_1 < X_2 < \cdots < X_m$ be the first m observed upper record values from the two-parameter bathtub-shaped distribution. Then, for any $0 < \alpha < 1$ and for each $j = 1, 2, \ldots, m-1$,*

$$\left(\varphi(X_1, \ldots, X_m, F_{1-\frac{\alpha}{2}(2(m-j),2j)}), \varphi(X_1, \ldots, X_m, F_{\frac{\alpha}{2}(2(m-j),2j)})\right),$$

is a $100(1-\alpha)\%$ confidence interval for β, where $\varphi(X_1, \ldots, X_m, t)$ is the solution of β for the equation

$$\frac{j}{m-j}\left[\frac{e^{X_m^\beta} - 1}{e^{X_j^\beta} - 1} - 1\right] = t.$$

Proof. From (2.1), we know that the pivot

$$T_j(\beta) = \frac{j}{m-j}\left[\frac{e^{X_m^\beta} - 1}{e^{X_j^\beta} - 1} - 1\right],$$

has an F distribution with $2(m-j)$ and $2j$ degrees of freedom. Hence, the event

$$F_{1-\frac{\alpha}{2}(2(m-j),2j)} < \frac{j}{m-j}\left[\frac{e^{X_m^\beta} - 1}{e^{X_j^\beta} - 1} - 1\right] < F_{\frac{\alpha}{2}(2(m-j),2j)},$$

is equivalent to the event

$$\varphi(X_1, \ldots, X_m, F_{1-\frac{\alpha}{2}(2(m-j),2j)}) < \beta < \varphi(X_1, \ldots, X_m, F_{\frac{\alpha}{2}(2(m-j),2j)}).$$

This completes the proof. □

Now, let us consider another pivotal quantity to construct the confidence interval for parameter β as

$$W(\beta, m) = \frac{\frac{1}{m}\sum_{i=1}^{m} Y_i}{\left[\prod_{i=1}^{m} Y_i\right]^{1/m}} = \frac{\frac{1}{m}\sum_{i=1}^{m}(e^{x_i^\beta} - 1)}{\left[\prod_{i=1}^{m}(e^{x_i^\beta} - 1)\right]^{1/m}}.$$

It is easy to show that the distribution of $W(\beta, m)$ does not depend on (β, λ) and hence it provides a pivotal quantity for β. To derive the confidence interval for β based on this pivotal quantity, one need the following lemma.

2.4. Lemma. *Suppose that* $0 < c_1 < c_2 < \cdots < c_m$. *Let*

$$W(\beta, m) = \frac{\frac{1}{m}\sum_{i=1}^{m}(e^{c_i^\beta} - 1)}{\left[\prod_{i=1}^{m}(e^{c_i^\beta} - 1)\right]^{1/m}}.$$

Then,

(a) $W(\beta, m)$ *is strictly increasing in* β *for any* $\beta > 0$.
(b) *For* $t > 1$, *the equation,* $W(\beta, m) = t$ *has a unique solution in* $\beta > 0$.

Proof. (a) The proof can be found in Wu et al. [20].
(b) Since the function $W(\beta, m)$ is strictly increasing in $\beta > 0$ with $\lim_{\beta\to 0} W(\beta, m) = 1$ and $\lim_{\beta\to\infty} W(\beta, m) = \infty$, then the lemma follows. □

Let $W_{\alpha(m)}$ be the upper α percentile of the distribution of the pivotal quantity $W(\beta, m)$. We have the following theorem.

2.5. Theorem. *Suppose that* $X_1 < X_2 < \cdots < X_m$ *be the first* m *observed upper record values from the two-parameter bathtub-shaped distribution. Then, for any* $0 < \alpha < 1$,

$$\psi(X_1, \ldots, X_m, W_{1-\frac{\alpha}{2}(m)}) < \beta < \psi(X_1, \ldots, X_m, W_{\frac{\alpha}{2}(m)})$$

is a $100(1-\alpha)\%$ *confidence interval for* β, *where* $\psi(X_1, \ldots, X_m, t)$ *is the solution of* β *for the equation*

$$\frac{\frac{1}{m}\sum_{i=1}^{m}(e^{X_i^\beta} - 1)}{\left[\prod_{i=1}^{m}(e^{X_i^\beta} - 1)\right]^{1/m}} = t.$$

Proof. Note that

$$P\left(W_{1-\frac{\alpha}{2}(m)} < W(\beta, m) < W_{\frac{\alpha}{2}(m)}\right) = 1 - \alpha,$$

for any $0 < \alpha < 1$. Then, by Lemma 2.4, one can construct an exact confidence interval for β. □

It should be mentioned here that since the exact distribution of the pivotal quantity $W(\beta, m)$ is too hard to derive algebraically, we need to compute the percentiles of $W(\beta, m)$ by using Monte Carlo simulation. In Table 1, we present the upper percentiles $W_{\alpha(m)}$ of $W(\beta, m)$ for $m = 2, 3, \ldots, 20$ and various values of α, over 50000 replications.

Now, in order to derive the exact joint confidence region for (β, λ), let

$$S = U_j + V_j = 2Y_m. \tag{2.2}$$

It is easy to show that S has a chi-square distribution with $2m$ degrees of freedom. Furthermore, by Johnson et al. [13], T_j defined in (2.1) and S are independent for each j. Using the joint pivots $(S, T_1), \ldots, (S, T_{m-1})$, we can construct $m - 1$ exact joint confidence regions for (β, λ).

Table 1. Upper percentile $W_{\alpha(m)}$ of $W(\beta, m)$

	α									
m	0.995	0.005	0.99	0.01	0.975	0.025	0.95	0.05	0.90	0.10
2	1.0000	7.1897	1.0000	4.9480	1.0001	3.2949	1.0003	2.4000	1.0013	1.7691
3	1.0004	4.9643	1.0010	3.9014	1.0028	2.8395	1.0061	2.2901	1.0131	1.8074
4	1.0028	4.2341	1.0043	3.5076	1.0092	2.7647	1.0161	2.2669	1.0296	1.8387
5	1.0071	3.4388	1.0100	3.0886	1.0182	2.5200	1.0277	2.1453	1.0461	1.8156
6	1.0096	3.2875	1.0149	2.8818	1.0253	2.4163	1.0382	2.0899	1.0586	1.7836
7	1.0169	2.9389	1.0227	2.6522	1.0344	2.2674	1.0489	1.9794	1.0725	1.7498
8	1.0207	2.8689	1.0276	2.5550	1.0433	2.2014	1.0608	1.9594	1.0862	1.7358
9	1.0278	2.7457	1.0352	2.4686	1.0513	2.1332	1.0684	1.9258	1.0955	1.7185
10	1.0325	2.6099	1.0425	2.3344	1.0607	2.0542	1.0788	1.8808	1.1066	1.6924
11	1.0362	2.4933	1.0452	2.2812	1.0641	2.0258	1.0836	1.8503	1.1107	1.6784
12	1.0417	2.4335	1.0528	2.2743	1.0702	2.0070	1.0922	1.8420	1.1209	1.6655
13	1.0462	2.3429	1.0574	2.1808	1.0762	1.9588	1.0991	1.8082	1.1292	1.6606
14	1.0530	2.2814	1.0658	2.1009	1.0842	1.9174	1.1059	1.7818	1.1349	1.6489
15	1.0546	2.2565	1.0650	2.0953	1.0886	1.9038	1.1091	1.7598	1.1404	1.6315
16	1.0623	2.1736	1.0733	2.0553	1.0940	1.8935	1.1155	1.7673	1.1456	1.6357
17	1.0641	2.1703	1.0768	2.0406	1.0979	1.8751	1.1194	1.7487	1.1482	1.6220
18	1.0679	2.1394	1.0792	2.0114	1.1001	1.8594	1.1235	1.7355	1.1544	1.6173
19	1.0779	2.1080	1.0873	1.9745	1.1071	1.8248	1.1300	1.7167	1.1607	1.6043
20	1.0753	2.0807	1.0883	1.9579	1.1096	1.8176	1.1322	1.7005	1.1635	1.5874

Let $\chi^2_{\alpha(v)}$ be the upper α percentile of the χ^2 distribution with v degrees of freedom. The following theorem provide $m-1$ exact joint confidence regions for (β, λ).

2.6. Theorem. *Suppose that $X_1 < X_2 < \cdots < X_m$ be the first m observed upper record values from the two-parameter bathtub-shaped distribution. Then, for any $j = 1, 2, \ldots, m-1$, the following inequalities determine a $100(1-\alpha)\%$ joint confidence region for (β, λ):*

$$\varphi\big(X_1, \ldots, X_m, F_{\frac{1+\sqrt{1-\alpha}}{2}(2(m-j),2j)}\big) < \beta < \varphi\big(X_1, \ldots, X_m, F_{\frac{1-\sqrt{1-\alpha}}{2}(2(m-j),2j)}\big),$$

and

$$\frac{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}(2m)}}{2(e^{X_m^\beta}-1)} < \lambda < \frac{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}(2m)}}{2(e^{X_m^\beta}-1)}.$$

where $0 < \alpha < 1$, and $\varphi(X_1, \ldots, X_m, t)$ is the solution of β for the equation

$$\frac{j}{m-j}\left[\frac{e^{X_m^\beta}-1}{e^{X_j^\beta}-1} - 1\right] = t.$$

Proof. From (2.2), we know that

$$S = 2\lambda(e^{X_m^\beta}-1),$$

has a chi-square distribution with $2m$ degrees of freedom, and it is independent of T_j for each j. Next, for $0 < \alpha < 1$, we have

$$P\left(F_{\frac{1+\sqrt{1-\alpha}}{2}(2(m-j),2j)} < T_j < F_{\frac{1-\sqrt{1-\alpha}}{2}(2(m-j),2j)}\right) = \sqrt{1-\alpha},$$

and

$$P\left(\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}(2m)} < S < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}(2m)}\right) = \sqrt{1-\alpha}.$$

From these relationships, we conclude that

$$P\Big(F_{\frac{1+\sqrt{1-\alpha}}{2}(2(m-j),2j)} < T_j < F_{\frac{1-\sqrt{1-\alpha}}{2}(2(m-j),2j)}\,, \quad \chi^2_{\frac{1+\sqrt{1-\alpha}}{2}(2m)} < S < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}(2m)}\Big) = 1-\alpha,$$

or equivalently

$$P\bigg(\varphi(X_1,\ldots,X_m,F_{\frac{1+\sqrt{1-\alpha}}{2}(2(m-j),2j)}) < \beta < \varphi(X_1,\ldots,X_m,F_{\frac{1-\sqrt{1-\alpha}}{2}(2(m-j),2j)})\,, \frac{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}(2m)}}{2(e^{X_m^\beta}-1)} < \lambda < \frac{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}(2m)}}{2(e^{X_m^\beta}-1)}\bigg) = 1-\alpha.$$

This completes the proof. □

3. Simulation Results

In this section, we carry out a Monte Carlo simulation to study the performance of our proposed confidence intervals and regions. In this simulation, we randomly generate upper record sample $X_1, X_2, \ldots, X_m$ from a two-parameter bathtub-shaped lifetime distribution with the values of parameters $(\beta,\lambda) = (0.5, 0.02)$, $(1, 0.1)$, and $(1.2, 0.05)$ and sample sizes $m = 5, 7, 10, 15$. We then compute the 95% confidence intervals and regions using Theorems 2.3, 2.5, and 2.6. We also provide the approximate joint confidence region obtained by the asymptotic normality of the MLEs. We replicate the process 5000 times. We present, in Tables 2 and 3, the average confidence lengths and confidence areas. The simulation results show that:

(1) The coverage probabilities of the exact confidence intervals for β and joint confidence regions for (β,λ) are close to the desired level of 0.95 for different parameters and sample sizes. But, the coverage probabilities of the approximate joint confidence region for (β,λ) are very low.

(2) The pivot $W(\beta,m)$ works better than the pivots $T_j(\beta)$, $j = 1,\ldots,m-1$ to establish confidence interval for the parameter β. This is because the average confidence lengths based on $W(\beta,m)$ are smaller than those based on $T_j(\beta)$, $j = 1,\ldots,m-1$.

(3) If we consider $m-1$ pivotal quantities $T_1(\beta),\ldots,T_{m-1}(\beta)$ to establish the confidence intervals for the parameter β, We find that the pivotal quantity $T_j(\beta)$ provides the shortest confidence length when j is around $[\frac{m}{2}]$, where $[y]$ denotes the largest integer which is less than or equal to y.

(4) From Table 3, we observe that in the most of cases considered, the first joint pivot (S, T_1) provides the smallest confidence area for (β,λ). Thus, the first joint confidence region is the best exact joint confidence region.

(5) In most of the cases considered, the approximate method does not work well to establish the joint confidence region for (β,λ). It provides the low coverage probabilities. Also, the average confidence area based on the approximate method is bigger than those obtained based on the exact methods.

Table 2. The average confidence length (CL) and coverage probability (CP) of the 95% confidence interval for β

		(β,λ)=(0.5, 0.02)		(β,λ)=(1, 0.1)		(β,λ)=(1.2, 0.05)	
m	*Method*	CL	CP	CL	CP	CL	CP
5	$W(\beta,m)$	1.6223	0.947	3.7091	0.949	3.9695	0.952
	$T_1(\beta)$	6.6763	0.954	16.2102	0.948	18.1848	0.954
	$T_2(\beta)$	1.9591	0.956	5.1808	0.949	5.4815	0.946
	$T_3(\beta)$	1.9323	0.957	4.9852	0.952	5.6397	0.948
	$T_4(\beta)$	8.8491	0.951	15.1237	0.953	17.5400	0.951
7	$W(\beta,m)$	0.8227	0.953	2.0708	0.948	2.1691	0.953
	$T_1(\beta)$	5.0157	0.950	12.5267	0.952	13.2268	0.950
	$T_2(\beta)$	1.3491	0.951	3.2347	0.952	3.6846	0.943
	$T_3(\beta)$	1.0302	0.951	2.2803	0.951	2.6532	0.945
	$T_4(\beta)$	1.0676	0.950	2.6369	0.942	2.6851	0.945
	$T_5(\beta)$	1.4285	0.948	3.4875	0.953	4.0611	0.941
	$T_6(\beta)$	14.2993	0.950	21.7503	0.937	39.4215	0.948
10	$W(\beta,m)$	0.4918	0.951	1.1508	0.947	1.2793	0.957
	$T_1(\beta)$	3.9378	0.954	9.9073	0.950	10.1735	0.958
	$T_2(\beta)$	1.0393	0.943	2.4302	0.953	2.7124	0.949
	$T_3(\beta)$	0.6972	0.948	1.6400	0.950	1.8349	0.946
	$T_4(\beta)$	0.6120	0.954	1.3761	0.955	1.5010	0.947
	$T_5(\beta)$	0.6077	0.955	1.4188	0.948	1.5628	0.952
	$T_6(\beta)$	0.6669	0.946	1.5013	0.946	1.7014	0.948
	$T_7(\beta)$	0.7960	0.958	1.8017	0.950	2.0555	0.952
	$T_8(\beta)$	1.1632	0.954	2.8045	0.952	3.1162	0.955
	$T_9(\beta)$	6.7247	0.948	13.1327	0.947	15.3015	0.951
15	$W(\beta,m)$	0.3065	0.952	0.7895	0.950	0.8206	0.953
	$T_1(\beta)$	3.3448	0.950	7.8014	0.952	8.4577	0.947
	$T_2(\beta)$	0.7912	0.950	1.9128	0.941	2.0974	0.946
	$T_3(\beta)$	0.5138	0.950	1.1995	0.954	1.3766	0.948
	$T_4(\beta)$	0.4226	0.949	0.9857	0.947	1.1265	0.944
	$T_5(\beta)$	0.3960	0.946	0.9055	0.951	0.9934	0.955
	$T_6(\beta)$	0.3879	0.944	0.8782	0.947	0.9948	0.951
	$T_7(\beta)$	0.3795	0.946	0.8580	0.954	0.9915	0.951
	$T_8(\beta)$	0.3916	0.953	0.9160	0.950	1.0350	0.945
	$T_9(\beta)$	0.4071	0.952	0.9896	0.949	1.0614	0.946
	$T_{10}(\beta)$	0.4661	0.954	1.0737	0.937	1.1867	0.949
	$T_{11}(\beta)$	0.5410	0.947	1.1941	0.951	1.3402	0.948
	$T_{12}(\beta)$	0.6802	0.948	1.5637	0.944	1.7454	0.946
	$T_{13}(\beta)$	1.0401	0.949	2.5110	0.952	2.7780	0.949
	$T_{14}(\beta)$	5.5032	0.948	16.648	0.950	13.9555	0.957

Table 3. The average confidence area (CA) and coverage probability (CP) of the 95% confidence region for (β, λ) obtained by the exact and approximate methods.

		(β,λ)=(0.5, 0.02)		(β,λ)=(1, 0.1)		(β,λ)=(1.2, 0.05)	
m	$Method$	CA	CP	CA	CP	CA	CP
5	(S, T_1)	0.0103	0.951	0.1252	0.952	0.0699	0.950
	(S, T_2)	0.0156	0.958	0.1609	0.956	0.0904	0.953
	(S, T_3)	0.0226	0.960	0.2172	0.937	0.1256	0.954
	(S, T_4)	0.0360	0.957	0.2989	0.954	0.1861	0.948
	approx	0.0216	0.837	0.2041	0.896	0.1013	0.397
7	(S, T_1)	0.0067	0.955	0.0826	0.952	0.0454	0.952
	(S, T_2)	0.0090	0.948	0.1001	0.952	0.0549	0.944
	(S, T_3)	0.0119	0.951	0.1196	0.955	0.0692	0.950
	(S, T_4)	0.0158	0.951	0.1435	0.946	0.0880	0.952
	(S, T_5)	0.0223	0.956	0.1848	0.955	0.1167	0.947
	(S, T_6)	0.0336	0.949	0.2507	0.938	0.1658	0.951
	approx	0.0137	0.599	0.1376	0.596	0.0812	0.309
10	(S, T_1)	0.0046	0.954	0.0549	0.944	0.0297	0.956
	(S, T_2)	0.0056	0.950	0.0640	0.956	0.0350	0.954
	(S, T_3)	0.0068	0.955	0.0712	0.953	0.0412	0.952
	(S, T_4)	0.0082	0.954	0.0808	0.956	0.0472	0.943
	(S, T_5)	0.0099	0.954	0.0940	0.946	0.0555	0.952
	(S, T_6)	0.0123	0.950	0.1100	0.947	0.0671	0.954
	(S, T_7)	0.0158	0.956	0.1325	0.953	0.0836	0.941
	(S, T_8)	0.0218	0.949	0.1652	0.956	0.1083	0.956
	(S, T_9)	0.0312	0.951	0.2193	0.946	0.1509	0.954
	approx	0.0094	0.652	0.0952	0.666	0.0534	0.345
15	(S, T_1)	0.0030	0.953	0.0367	0.947	0.0198	0.948
	(S, T_2)	0.0035	0.951	0.0399	0.945	0.0222	0.950
	(S, T_3)	0.0040	0.952	0.0441	0.954	0.0247	0.954
	(S, T_4)	0.0045	0.952	0.0473	0.950	0.0271	0.948
	(S, T_5)	0.0051	0.950	0.0508	0.955	0.0303	0.949
	(S, T_6)	0.0056	0.946	0.0555	0.944	0.0330	0.950
	(S, T_7)	0.0065	0.944	0.0613	0.947	0.0362	0.951
	(S, T_8)	0.0075	0.956	0.0664	0.946	0.0406	0.952
	(S, T_9)	0.0087	0.951	0.0745	0.946	0.0472	0.950
	(S, T_{10})	0.0101	0.954	0.0845	0.946	0.0545	0.952
	(S, T_{11})	0.0122	0.948	0.0991	0.954	0.0649	0.946
	(S, T_{12})	0.0157	0.951	0.1180	0.944	0.0781	0.950
	(S, T_{13})	0.0209	0.946	0.1501	0.949	0.1009	0.941
	(S, T_{14})	0.0294	0.944	0.1941	0.942	0.1377	0.956
	approx	0.0057	0.586	0.0591	0.591	0.0338	0.315

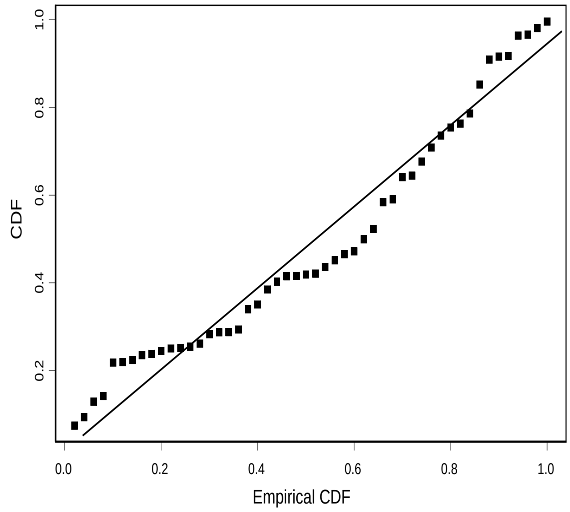

Figure 1. PP-plot of the real data set in Example 4.1.

4. Illustrative Examples

To illustrate the use of our proposed estimation methods, the following two numerical examples are discussed.

4.1. Example. (**Real data set**) Here we consider the real data of the amount of annual rainfall (in inches) recorded at the Los Angeles Civic Center for the 50 years, from 1959 to 2009. (see the website of Los Angeles Almanac: www.laalmanac.com/weather/we08aa.htm) The data are as follows:

8.180	4.850	18.790	8.380	7.930	13.680	20.440	22.000	16.580
27.470	7.740	12.320	7.170	21.260	14.920	14.350	7.210	12.300
33.440	19.670	26.980	8.960	10.710	31.280	10.430	12.820	17.860
7.660	2.480	8.081	7.350	11.990	21.000	7.360	8.110	24.350
12.440	12.400	31.010	9.090	11.570	17.940	4.420	16.420	9.250
37.960	13.190	3.210	13.530	9.080				

We check the validity of the two-parameter bathtub-shaped distribution based on the parameters $\widehat{\beta} = 0.4721$ and $\widehat{\lambda} = 0.0212$ using the Kolmogorov-Smirnov (K-S) test. It is observed that the K-S distance is 0.1385 with a corresponding p-value 0.2715. This indicates that the two-parameter bathtub-shaped distribution provides a good fit to the data. Figure 1 also shows the probability plot (PP) of the data. This figure supports our conclusion. During this period, we observe the following seven upper record values:

$$8.18 \quad 18.79 \quad 20.44 \quad 22.00 \quad 27.47 \quad 33.44 \quad 37.96$$

The MLEs of β and λ are $\widehat{\beta} = 0.432798$ and $\widehat{\lambda} = 0.0566$, respectively. Let us now obtain the approximate joint confidence region. Based on the result in Section 2.1, a

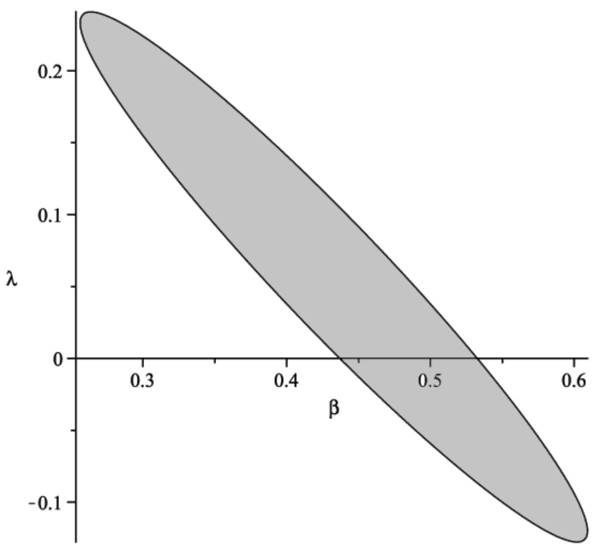

Figure 2. The 95% approximate joint confidence region in Example 4.1

95% approximate joint confidence region is the ellipse

$$A = \left\{(\beta, \lambda) : m \begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\lambda} - \lambda \end{pmatrix}' \begin{bmatrix} 340.3978 & 312.4234 \\ 312.4234 & 311.9288 \end{bmatrix} \begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\lambda} - \lambda \end{pmatrix} - 5.9991 \leq 0 \right\},$$

where $m = 7$. This ellipse is provided in Figure 2. The area of this approximate joint confidence region is 0.0291. Now, we use the methods proposed in Section 2.2 to construct the exact confidence intervals for β and exact joint confidence region for (β, λ). To obtain the 95% confidence intervals for β, we consider the pivots $W(\beta, m)$, $T_1(\beta), \ldots, T_6(\beta)$. We need the percentiles

$$\begin{aligned}
&W_{0.025(7)} = 2.2674, \quad W_{0.975(7)} = 1.0344, \quad F_{0.025(12,2)} = 39.41462,\\
&F_{0.975(12,2)} = 0.1962375, \quad F_{0.025(10,4)} = 8.843881, \quad F_{0.975(10,4)} = 0.2237967,\\
&F_{0.025(8,6)} = 5.599623, \quad F_{0.975(8,6)} = 0.2149754, \quad F_{0.025(6,8)} = 4.651696,\\
&F_{0.975(6,8)} = 0.1785835, \quad F_{0.025(4,10)} = 4.468342, \quad F_{0.975(4,10)} = 0.1130725,\\
&F_{0.025(2,12)} = 5.095867, \quad \text{and} \quad F_{0.975(2,10)} = 0.0253713.
\end{aligned}$$

Here, the percentiles of $W_{0.025(7)}$ and $W_{0.975(7)}$ are obtained from Table 1. By Theorems 2.3 and 2.5 and using the S-PLUS package, the 95% confidence intervals and corresponding confidence lengths for β are given in Table 4.

From the simulation result in Section 3, we know that, on the average, the pivot $W(\beta, m)$ works better than the pivots $T_j(\beta)$, $j = 1, \ldots, 6$. It is not the best one in this example because the result here is based on only one sample. Among the pivots $T_j(\beta)$, $j = 1, \ldots, 6$, we observe that, in this example, the pivot $T_4(\beta)$ provides the shortest confidence interval length and hence, $(0.3723, 0.5095)$ is an optimal 95% confidence interval for β.

Table 4. The 95% confidence interval (CI) for β and corresponding confidence length (CL)

Pivot	CI	CL
$W(\beta, m)$	(0.3859,0.6861)	0.3001
$T_1(\beta)$	(0.4100,1.2874)	0.8774
$T_2(\beta)$	(0.3909,0.9163)	0.5254
$T_3(\beta)$	(0.3811,0.6094)	0.2283
$T_4(\beta)$	(0.3723,0.5095)	0.1372
$T_5(\beta)$	(0.3511,0.5321)	0.1810
$T_6(\beta)$	(0.3214, 0.6568)	0.3354

To obtain the 95% joint confidence regions for (β, λ), we need the percentiles

$$F_{0.0127(12,2)} = 78.15579, \quad F_{0.9873(12,2)} = 0.15572, \quad F_{0.0127(10,4)} = 12.79912,$$
$$F_{0.9873(10,4)} = 0.179534, \quad F_{0.0127(8,6)} = 7.37466, \quad F_{0.9873(8,6)} = 0.1699301,$$
$$F_{0.0127(6,8)} = 5.884774, \quad F_{0.9873,(6,8)} = 0.135599, \quad F_{0.0127(4,10)} = 5.569966,$$
$$F_{0.9873,(4,10)} = 0.07813, \quad F_{0.0127(2,12)} = 6.421784, \quad F_{0.9873,(2,10)} = 0.01279496,$$
$$\chi^2_{0.0127(14)} = 28.37037, \quad \text{and} \quad \chi^2_{0.9873(14)} = 4.888863.$$

By Theorem 2.6 and using the S-PLUS package for solving non-linear equation, we obtain the following 95% joint confidence regions for (β, λ) based on the joint pivots (S, T_j), $j = 1, \ldots, 6$:

$$A_1 = \left\{(\beta, \lambda) : 0.4091 < \beta < 2.1540, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\},$$
$$A_2 = \left\{(\beta, \lambda) : 0.3882 < \beta < 1.1574, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\},$$
$$A_3 = \left\{(\beta, \lambda) : 0.3792 < \beta < 0.6847, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\},$$
$$A_4 = \left\{(\beta, \lambda) : 0.3709 < \beta < 0.5474, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\},$$
$$A_5 = \left\{(\beta, \lambda) : 0.3496 < \beta < 0.5779, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\},$$
$$A_6 = \left\{(\beta, \lambda) : 0.3206 < \beta < 0.7445, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\}.$$

Figure 3 shows the above joint confidence regions for parameters β and λ. The areas of above joint confidence regions are 0.0073, 0.0109, 0.0128, 0.0145, 0.0210, and 0.0331, respectively. Thus, in this example, A_1 is the optimal joint confidence region (β, λ) since the joint pivot (S, T_1) provides the smallest confidence area. Note that the confidence areas based on all the joint pivots (S, T_j), $j = 1, \ldots, 5$ are all smaller than the area based on the approximate method. However, the area based on joint pivot (S, T_6) is larger than that based on the approximate method.

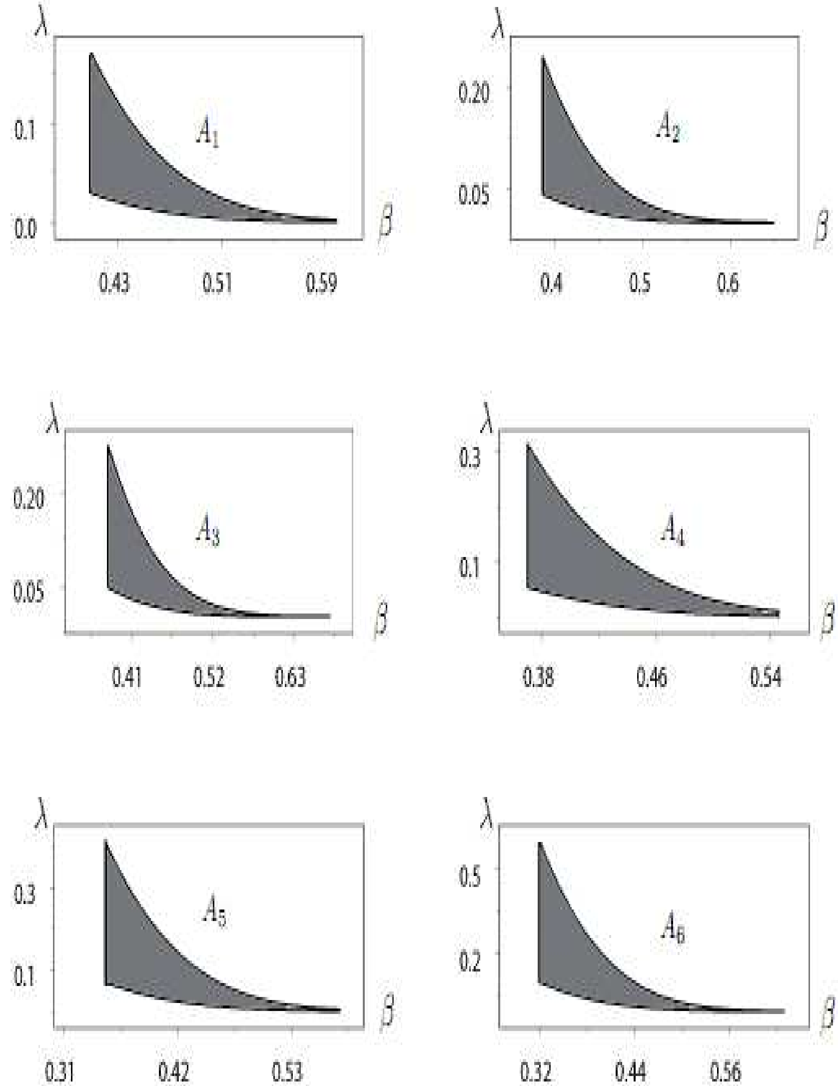

Figure 3. The 95% joint confidence region for (β, λ) in Example 4.1

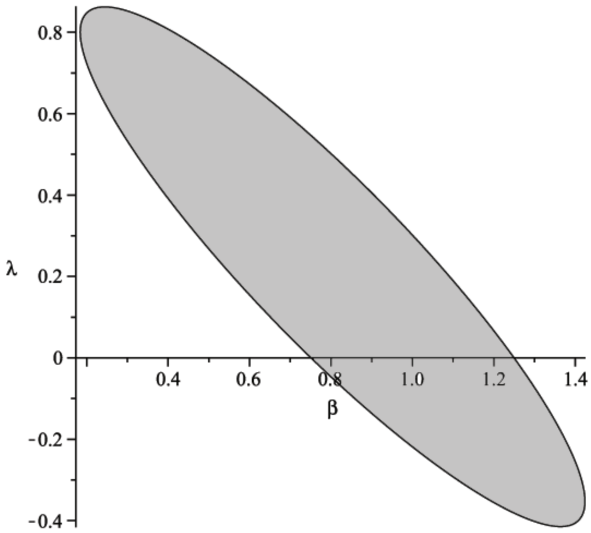

Figure 4. The 95% approximate joint confidence region in Example 4.2

4.2. Example. (**Simulated data set**) Let us consider the first four upper record values simulated from the two-parameter bathtub-shaped distribution with $\beta = 1.2$ and $\lambda = 0.05$. The simulated data are as follows:

$$1.351052 \quad 1.989847 \quad 3.030312 \quad 3.821197$$

The MLEs of β and λ are $\widehat{\beta} = 0.8039041$ and $\widehat{\lambda} = 0.2237688$, respectively. For 95% approximate joint confidence region, we have the ellipse

$$B = \left\{(\beta, \lambda) : m \begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\lambda} - \lambda \end{pmatrix}' \begin{bmatrix} 21.19736 & 18.58491 \\ 18.58491 & 19.97105 \end{bmatrix} \begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\lambda} - \lambda \end{pmatrix} - 5.9991 \leq 0 \right\},$$

where $m = 4$. The ellipse is provided in Figure 4. The area of above joint confidence region is 0.5331. To obtain the 95% confidence intervals for β, we need the percentiles

$$W_{0.025(4)} = 2.7647, \quad W_{0.975(4)} = 1.0092, \quad F_{0.025(6,2)} = 39.33146,$$
$$F_{0.975(6,2)} = 0.137744, \quad F_{0.025(4,4)} = 9.60453, \quad F_{0.975(4,4)} = 0.1041175,$$
$$F_{0.025(2,6)} = 7.259856 \text{ and } F_{0.975(2,6)} = 0.0254249.$$

By Theorems 2.3 and 2.5 and using the S-PLUS package, the 95% confidence intervals for β are given in Table 5.

To obtain the 95% joint confidence regions for (β, λ), we need the percentiles

$$F_{0.0127(6,2)} = 78.07254, \quad F_{0.9873(6,2)} = 0.101436, \quad F_{0.0127(4,4)} = 14.02461,$$
$$F_{0.9873(4,4)} = 0.0713032, \quad F_{0.0127(2,6)} = 9.858393, \quad F_{0.9873(2,6)} = 0.012809,$$
$$\chi^2_{0.0127(8)} = 19.4347, \text{ and } \chi^2_{0.9873(8)} = 1.768713.$$

One can obtain the 95% joint confidence regions for (β, λ) as follows:

$$B_1 = \left\{(\beta, \lambda) : 1.0427 < \beta < 1.8677, \quad \frac{1.768713}{2(e^{(3.821197)^\beta} - 1)} < \lambda < \frac{19.4347}{2(e^{(3.821197)^\beta} - 1)}\right\},$$

Table 5. The 95% confidence interval (CI) for β and corresponding confidence length (CL)

Pivot	CI	CL
$W(\beta, m)$	(0.9705,1.2487)	0.2782
$T_1(\beta)$	(1.0431,1.4578)	0.4147
$T_2(\beta)$	(1.0130,1.1272)	0.1142
$T_3(\beta)$	(0.9124,1.1999)	0.2875

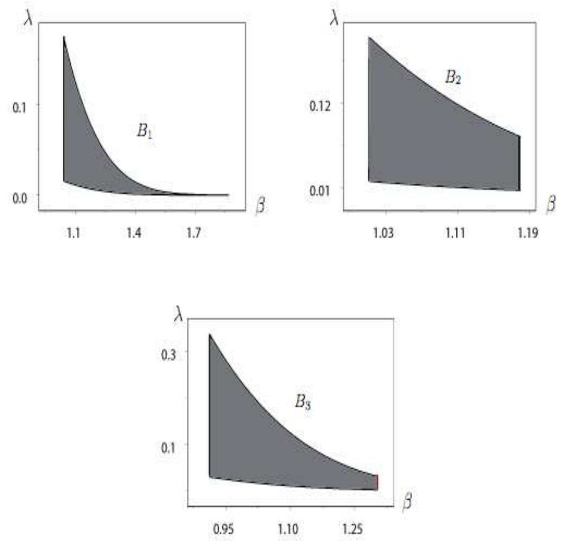

Figure 5. The 95% joint confidence region for (β, λ) in Example 4.2

$$B_2 = \left\{ (\beta, \lambda) : 1.0126 < \beta < 1.1802, \quad \frac{1.768713}{2(e^{(3.821197)^\beta} - 1)} < \lambda < \frac{19.4347}{2(e^{(3.821197)^\beta} - 1)} \right\},$$
$$B_3 = \left\{ (\beta, \lambda) : 0.9119 < \beta < 1.3032, \quad \frac{1.768713}{2(e^{(3.821197)^\beta} - 1)} < \lambda < \frac{19.4347}{2(e^{(3.821197)^\beta} - 1)} \right\}.$$

Figure 5 shows the above joint confidence regions. The areas of the above joint confidence regions are 0.05005, 0.0200, and 0.0238, respectively. Thus, in this example, B_2 is the optimal joint confidence regions for parameters β and λ.

5. Conclusions

The subject of record values has received attention in the past few decades. The two-parameter bathtub-shaped lifetime distribution can be widely used in reliability applications because of its failure rate function. We study the interval estimation of parameters of the two-parameter bathtub-shaped distribution based on record values. We provide three theorems based on the method of pivotal quantity to establish the exact confidence intervals and regions for the parameters. Two numerical examples are used to illustrate the proposed methods, and we also assess the confidence intervals and regions by performing a Monte Carlo simulation. The simulation results provide us some idea to choose the optimal pivots for constructing confidence intervals and regions.

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References

[1] Ahmadi, J. and Balakrishnan, N. *Prediction of order statistics and record values from two independent sequences*, Statistics **44**, 417-430, 2010.
[2] Ahsanullah, M. *Introduction to Record Statistics*, NOVA Science Publishers Inc., Huntington, New York, 1995.
[3] Ahsanullah, M. and Nevzorov, V. B. *Record statistics. In:International Encyclopedia of Statistical Science*, Springer, 1195-1202, 2011.
[4] Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. *Records*, Wiley, New York, 1998.
[5] Bebbington, M., Lai, C.-D. and Zitikis, R. *Useful periods for lifetime distributions with bathtub shaped hazard rate functions*, IEEE Transactions on Reliability **55**, 245-251, 2006.
[6] Chandler, K. N. *The distribution and frequency of record values.* Journal of the Royal Statistical Society: Series B **14**, 220-228, 1952.
[7] Chen, Z. A. *A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function*, Statistics and Probability Letters **49**, 155-161, 2000.
[8] Doostparast, M., Deepak, S. and Zangoie A. *Estimation with the lognormal distribution on the basis of records*, Journal of Statisticl Computation and Simulation **83**, 2339-2351, 2013.
[9] Gupta, R. D. and Kundu, D. *Generalized exponential distributions: Statistical inferences* , Technical Report, The University of New Brunswick, Saint John, 1999.
[10] Gurvich, M. R., Dibenedetto, A. T. and Rande, S. V.A *new statistical distribution for characterizing the random strength of brittle materials*, Journal of Materials Science **32**, 2559-2564, 1997.
[11] Haynatzki, G. R., Weron, K. and Haynatzka, V. R. *A new statistical model of tumor latency time*, Mathematical and Computer Modelling **32**, 251-256, 2000.
[12] Hjorth, U. *A reliability distribution with increasing, decreasing, and bathtub-shaped failure rate*, Technometrics **22**, 99-107, 1980.
[13] Johnson, N. L., Kotz, S. and Balakrishnan, N. *Continuous Univariate Distribution*, vol. 1. Wiley, New York, 1994.
[14] Lee, H. M., Lee, W. C., Lei, C. L. and Wu, J. W. *Computational procedure of assessing lifetime performance index of Weibull lifetime products with the upper record values*, Mathematics and Computers in Simulation **81**, 1177-1189, 2011.
[15] Mudholkar, G. S. and Srivastava, D. K. *Exponentiated Weibull family for analyzing bathtub failure-rate data*, IEEE Transactions on Reliability **42**, 299-302, 1993.
[16] Nadarajah, S. *Bathtub-shaped failure rate functions*, Quality and Quantity **43**, 855-863, 2009.
[17] Pham, H. and Lai, C.-D. *On recent generalizations of the Weibull distribution*, IEEE Transactions on Reliability **56**, 454-458, 2007.
[18] Smith, R. M. and Bain, L. J. *An exponential power life-testing distribution*, Communications in Statistics **4**, 469-481, 1975.
[19] Wang, F. K. *A new model with bathtub-shaped failure rate using an additive Burr XII distribution*, Reliability Engineering and System Safety **70**, 305-312, 2000.
[20] Wu, J. W., Lu, H. L., Chen, C. H. and Wu, C. H. *Statistical inference about the shape parameter of the new two-parameter bathtub-shaped lifetime distribution*, Quality and Reliability Engineering International **20**, 607-616, 2004.

[21] Wu, S.-J. *Estimation of the two-parameter bathtub-shaped lifetime distribution with progressive censoring*, Journal of Applied Statistics **35**, 1139-1150, 2008.

[22] Xie, M., Tang, Y. and Goh, T. N. *A modified Weibull extension with bathtub-shaped failure rate function*, Reliability Engineering and System Safety **76**, 279-285, 2002.

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Modified systematic sampling in the presence of linear trend

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Abstract

A new systematic sampling design called "Modified Systematic Sampling (MSS)", proposed by [2] is more general than Linear Systematic Sampling (LSS) and Circular Systematic Sampling (CSS). In the present paper, this scheme is further extended for populations having a linear trend. Expressions for mean and variance of sample mean are obtained for the population having perfect linear trend among population values. Expression for the average variance is also obtained for super population model. Further, efficiency of MSS with respect to CSS is obtained for different sample size.

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1. Introduction

In survey sampling, Linear Systematic Sampling (LSS) is a commonly used design. Generally, it is useful when population size N is a multiple of sample size n, i.e. $N = nk$ (where k is the sampling interval). Thus, we have k samples each of size n. However, LSS is not beneficial when population size N is not a multiple of the sample size n, i.e. $N \neq nk$. Because in this case, LSS cannot provide a constant sample size n, thus, estimate of population mean (total) is biased. Therefore, Circular Systematic Sampling (CSS) was introduced by Lahiri in 1952 (cited in [1, p.139]). Contrary to LSS, CSS is not advantageous when population size N is a multiple of the sample size n, i.e. $N = nk$ as in this case, CSS produces n replicates of k samples. Further, in CSS, the number of samples also rapidly increase to N as compared to k samples of LSS.

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To improve the efficiency of systematic sampling, researchers proposed several modifications in the selection procedure. The considerable work is done by [4], [6] and [7]. In the recent years, [8] proposed Diagonal Systematic Sampling (DSS) under the condition that $n \leq k$ as a competitor of LSS. Later, the condition $n \leq k$ for DSS have relaxed by [9]. The generalization of DSS is suggested by [10]. Some modification in LSS are proposed by [11], in which odd and even sample sizes are dealt separately. Further modification on LSS is also proposed by [12]. Diagonal Circular Systematic Sampling (DCSS) proposed by [5] is an extension of DSS to the circular version of systematic sampling. A note on DCSS has been proposed by [3]. However, some of these schemes are applicable when $N = nk$ while other can be used only when $N \neq nk$.

A new systematic sampling design called "Modified Systematic Sampling (MSS)" proposed by [2], which is applicable in both situations, whether $N = nk$ or $N \neq nk$. According to this design, first compute least common multiple of N and n, i.e. L, then find k_1, m, s and k, where $k_1 = \frac{L}{n}$, $m = \frac{L}{N}$, $s = \frac{N}{k_1}$ and $k = [k_1/m]$ or $k = [N/n]$ is rounded off to integer. Consequently, $ms = n$, which means that there are m sets and each set contains s units in a sample. Thus, in MSS the j^{th} unit of the i^{th} set of a sample of n units can be written as:

$$(1.1) \qquad y_{ij}^{(r)} = r + (i-1)k + (j-1)k_1 - hN \text{ if } hN < r + (i-1)k + (j-1)k_1 \leq (h+1)N$$

for $h = 0, 1, 2;\ i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., s$.

This sampling scheme reduces to LSS if $L = N$ or $N = nk$ and CSS if $L = N \times n$, the detail is given below.

If $N = nk$ then $L = N$, $k_1 = k$, $m = 1$ and $s = n$. Thus, Equation (1.1) reduces to

$$(1.2) \qquad y_j^{(r)} = r + (j-1)k, \ \ j = 1, 2, 3, ..., n$$

which is LSS.

Similarly, if $L = N \times n$, then $k_1 = N$, $m = n$ and $s = 1$. So, Equation (1.1) can be written as

$$(1.3) \qquad y_i^{(r)} = r + (i-1)k - hN \ \text{ if } \ hN < r + (i-1)k \leq (h+1)N$$

for $h = 0, 1, 2;\ i = 1, 2, 3, ..., n$.

Which is CSS.

To study the characteristics of MSS, we use an alternative method by partitioning the total number of samples into different sets of similar samples. To develop an alternative method, let us assume that k_1 can be written as $k_1 = qk + r_m$, where q and r_m are quotient and remainder respectively. Further, we assume that $w = 1$ if $(m-q) \leq 1$ and, $w = (m-q)$ if $(m-q) > 1$. In both cases, there are two types of partitioning, i.e. between samples and within samples(see detail in Subsections 1.1 and 1.2).

1.1. When $w = 1$. In this case partitioning between samples and within samples are given in the Subsections 1.1.1 and 1.1.2.

1.1.1. *Partitioning between samples.* In this case, k_1 possible samples are mainly partitioned into two groups. The first group consists of initial $\{k_1 - (m-1)k\}$ samples and second group contains last $(m-1)k$ samples. However, in the second group, there are $(m-1)$ subgroups, each attains k samples. If a random number r is selected from the first k_1 units of a population, there is a possibility that it is selected from the first group, i.e. $\{k_1 - (m-1)k\}$ or it is selected from the $(m-1)$ subgroups of the second group, i.e. $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$ such that $u = 1, 2, ..., (m-1)$, where integer u is selected corresponding to a random number r.

1.1.2. *Partitioning within samples.* Furthermore, in the first group, all s units of all m sets in each sample are labeled as $r+(i-1)k+(j-1)k_1$, such that $i=1,2,...,m$ and $j=1,2,...,s$; while in the second group, all s units of the first $(m-u)$ sets are labeled as $r+(i-1)k+(j-1)k_1$ such that $i=1,2,...,(m-u)$ and $j=1,2,...,s$ and in each of the last u sets, first $(s-1)$ units are labeled as $r+(i-1)k+(j-1)k_1$ such that $i=(m-u+1),...,m$; $j=1,2,...,(s-1)$ and last unit is labeled as $r+(i-1)k+(j-1)k_1-N$ such that $i=(m-u+1),...,m$ and $j=s$.

1.2. When $w=(n-q)>1$. In this case partitioning between samples and within samples are given in the Subsections 1.2.1 and 1.2.2.

1.2.1. *Partitioning between samples.* In this case k_1 samples are mainly partitioned into two groups, the first group consists of the number of samples in which $r \leq \{k_1-(w-1)k+r_m\}$. The econd group contains the number of samples in which $r > \{k_1-(w-1)k+r_m\}$. The first group is further partitioned into $\{(m-w)-(w-1)+2\}$ subgroups in which, there are r_m number of samples in each of the first and the last subgroups, and k samples in each of the middle $\{(m-w)-(w-1)\}$ subgroups. In each subgroup of the first group, corresponding to a random number r, an integer u is picked in such a way that $u=(w-1)$ if $1 \leq r \leq \{k_1-(m-u-1)k\}$, $u=w,w+1,w+2,...,(m-w)$ if $\{k_1-(m-u)k\} < r \leq \{k_1-(m-u-1)k\}$ and $u=(m-w+1)$ if $\{k_1-(m-u)k\} < r \leq \{k_1-(m-u)k+r_m\}$.
The second group consists of last $\{(w-1)k-r_m\}=\{(k-r_m)+(w-2)k\}$ samples, which is the combination of the first $(k-r_m)$ samples and the last $(w-2)$ sets of k samples. These $(w-2)$ sets of k samples further partitioned in such a way that the first r_m of every k samples forms the first subgroup and the last $(k-r_m)$ samples of every k samples together with the first $(k-r_m)$ samples of this group forms the second subgroup. However, when $w=2$, then we have only $(k-r_m)$ samples in the second group.

1.2.2. *Partitioning within samples.* In each sample of the first group, all s units of the first $(m-u)$ sets are labeled as $r+(i-1)k+(j-1)k_1$ such that $i=1,2,...,(m-u)$ and $j=1,2,...,s$, and in each of the last u sets, the first $(s-1)$ units are labeled as $r+(i-1)k+(j-1)k_1$ such that $i=(m-u+1),...,m$, $j=1,2,...,(s-1)$ and the last unit is labeled as $r+(i-1)k+(j-1)k_1-N$ such that $i=(m-u+1),...,m$ and $j=s$.

In each sample of the first subgroup of the second group, all s units of the first $(w-x)$ sets are labeled as $r+(i-1)k+(j-1)k_1$ such that $i=1,2,...,(w-x)$ and $j=1,2,...,s$; the units of middle $(m-w+1)$ sets are labeled in such a way that, the first $(s-1)$ units of each set are labeled as $r+(i-1)k+(j-1)k_1$ such that $i \in (w-x+1),...,(m-x+1)$, $j=1,2,...,(s-1)$ and the last unit of each set is labeled as $r+(i-1)k+(j-1)k_1-N$ such that $i \in (w-x+1),...,(m-x+1)$ and $j=s$; the units of the last $(x-1)$ sets are labeled in such a way that, the first $(s-2)$ units are labeled as $r+(i-1)k+(j-1)k_1$ such that $i \in (m-x+2),...,m$, $j=1,2,...,(s-2)$ and the last two units in each set is labeled as $r+(i-1)k+(j-1)k_1-N$ such that $i \in (m-x+2),...,m$ and $j=(s-1),s$. However, when $s=1$, the units in these $(x-1)$ sets are labeled as $r+(i-1)k+(j-1)k_1-2N$. The possible values of x are $2,3,...,(w-1)$.
Note: If $w=2$, then this set of samples does not exist.
In the second subgroup of the second group, all s units of the the first $(w-x)$ sets are labeled as $r+(i-1)k+(j-1)k_1$ such that $i=1,2,...,(w-x)$ and $j=1,2,...,s$; The units of middle $(m-w)$ sets are labeled in such a way that the first $(s-1)$ units of each set are labeled as $r+(i-1)k+(j-1)k_1$ such that $i \in (w-x+1),...,(m-x)$ and $j=1,2,...,(s-1)$, the last unit of each set is labeled as $r+(i-1)k+(j-1)k_1-N$ such that $i \in (w-x+1),...,(m-x)$ and $j=s$, the units of the last $(x-1)$ sets are labeled

in such a way that, the first $(s-2)$ units are labeled as $r+(i-1)k+(j-1)k_1$ such that $i \in (m-x+2), ..., m$, $j = 1, 2, ..., (s-2)$ and the last two units in each set is labeled as $r+(i-1)k+(j-1)k_1 - N$ such that $i \in (m-x+2), ..., m$ and $j = (s-1), s$. However, when $s = 1$, the units in these $(x-1)$ sets are labeled as $r+(i-1)k+(j-1)k_1 - 2N$, the possible values of x are $1, 2..., (w-1)$.

2. Mean and variance of MSS for population having linear trend

The following linear model of hypothetical population is to be considered as

$$(2.1) \qquad Y_t = \alpha + \beta\, t, \qquad t = 1, 2, 3, ..., N$$

where α and β are the intercept and slope of the model respectively.

2.1. Mean of MSS. The sample mean for both cases, i.e. $w = 1$ and $w > 1$ are given below (see detial in Appendix A.1).

Case (i) when $w = 1$

$$(2.2) \qquad \bar{y}_{MSS} = \alpha + \beta \begin{cases} \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\}\right], \\ \quad \text{if} \quad r \le \{k_1 - (m-1)k\} \\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{k_1}{m}\right], \\ \quad \text{where} \left| \begin{array}{l} u = 1, 2, ..., (m-1) \text{ if} \\ \{k_1 - (m-u)k\} < r \le \{k_1 - (m-u-1)k\}. \end{array} \right. \end{cases}$$

Case (ii) when $w > 2$

$$(2.3) \qquad \bar{y}_{MSS} = \alpha + \beta \begin{cases} \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{N}{n}\right], \\ \quad \text{where} \left| \begin{array}{l} u = (w-1) \; if \quad r \le \{k_1 - (m-u-1)k \\ u = w, w+1, ..., (m-w) \; if \\ \{N - (m-u)k\} < r \le \{k_1 - (m-u-1)k\}, \\ u = (m-w+1) \; if \\ \{k_1 - (m-m)k\} < r \le \{k_1 - (m-u)k + r_m\} \end{array} \right. \\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w-1+2x)\frac{N}{n}\right], \\ \quad \text{where} \left| \begin{array}{l} x = 2, ..., (w-1) \; if \\ \{k_1 - (w-x)k\} < r \le \{k_1 - (w-x)k + r_m\} \end{array} \right. \\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w+2x)\frac{N}{n}\right], \\ \quad \text{where} \left| \begin{array}{l} x = 1, 2, 3, ..., (w-1) \;\; if \\ \{k_1 - (w-x)k\} + r_m < r \le \{k_1 - (w-x)k + k\} \end{array} \right. \end{cases}$$

If $w = 2$, the Equation (2.3) will reduce to

$$(2.4) \qquad \bar{y}_{MSS} = \alpha + \beta \begin{cases} \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{N}{n}\right] \\ \quad \text{where} \left| \begin{array}{l} u = (w-1) \; if \quad r \le \{k_1 - (m-u-1)k, \\ u = w, w+1, ..., (m-w) \; if \\ \{N - (m-u)k\} < r \le \{k_1 - (m-u-1)k\}, \\ u = (m-w+1) \; if \\ \{k_1 - (m-m)k\} < r \le \{k_1 - (m-u)k + r_m\} \end{array} \right. \\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w+2x)\frac{N}{n}\right], \\ \quad \text{where} \left| \begin{array}{l} x = 1, 2, 3, ..., (w-1) \;\; if \\ \{k_1 - (w-x)k\} + r_m < r \le \{k_1 - (w-x)k + k\} \end{array} \right. \end{cases}$$

2.1.1. *Unbiasedness of $\bar{y}_{MSS}$.* (see detial in Appendix A.1.1).
The sample mean ($\bar{y}_{MSS}$) is an unbiased estimator of population mean ($\bar{Y}$)

$$E(\bar{y}_{MSS}) = \alpha + \beta \left\{ \frac{N+1}{2} \right\} = \bar{Y}. \tag{2.5}$$

2.1.2. *Variance of $\bar{y}_{MSS}$.* (see detial in Appendix A.2)

(i) when $w = 1$

$$V(\bar{y}_{MSS}) = \frac{1}{12m^2} b^2 \left[m^2(k_1^2 - 1) + m(m^2-1)k(mk - 2k_1) \right]. \tag{2.6}$$

Note: In this case, if $N = nk$ then $L = N$, so $m = 1$, thus,

$$V(\bar{y}_{MSS}) = \frac{1}{12} b^2 (k^2 - 1).$$

This is a variance of linear systematic sampling.

(ii) when $w > 1$

$$\begin{aligned}(\bar{y}_{MSS}) = {} & \tfrac{1}{12m^2} b^2 \left[m^2(k_1^2 - 1) + m(m^2-1)k(mk - 2k_1) \right. \\ & \left. + 4w(w-1)k_1 \left\{ 3k_1 - (3m - 2w + 1)k \right\} \right].\end{aligned} \tag{2.7}$$

3. Yates corrected estimator

Yates corrected estimator of population mean for MSS is derived below.

3.1. Yates corrected estimator for MSS. The corrected estimator $\bar{y}^c_{MSS}$ of population mean using MSS is given by

$$\bar{y}^c_{MSS} = \tfrac{1}{n} \left[\lambda_{1l} Y_{r1} + \textstyle\sum_{l=2}^{n-1} Y_{rl} + \lambda_{2l} Y_{rn} \right], \tag{3.1}$$

where λ_{1l} and λ_{2l} are selected so that sample mean coincides with the population mean in the presence of linear trend for all choices of $r \in 1, 2, ..., k_1$.
Alternatively Equation (3.1) can be written as

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + a_l(r) \left(Y_{r1} - Y_{rn} \right), \tag{3.2}$$

where $a_l(r) = \frac{\lambda_{1l} - 1}{n} = \frac{1 - \lambda_{2l}}{n}$.
Under the model given in (2.1), the population mean is

$$\bar{Y} = \alpha + \beta \frac{N+1}{2}. \tag{3.3}$$

As mentioned earlier, that there are two cases, i.e. (i) $w = 1$ and (ii) $w > 1$.
First, we consider the Case (i).

3.1.1. *Case (i): when $w = 1$.* In this situation, a random start r is selected from k_1 units such that $r \leq \{k_1 - (m-1)k\}$ or $r > \{k_1 - (m-1)k\}$.
If $r \leq \{k_1 - (m-1)k\}$, then $l = 1$ and the last value of each sample is labeled $\{r + (m-1)k + (s-1)k_1\}$. Thus, (3.2) becomes

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + a_1(r) \left(Y_r - Y_{r+(m-1)k+(s-1)k_1} \right). \tag{3.4}$$

Under the linear model given in (2.1), we have $\bar{y}_{MSS} = \alpha + \beta \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} \right]$, $Y_r = \alpha + \beta\, r$ and $Y_{r+(m-1)k+(s-1)k_1} = \alpha + \beta \{ r + (m-1)k + (s-1)k_1 \}$.

Putting these values in (3.4), we have

$$\bar{y}^c_{MSS} = \alpha + \beta \left[r + \frac{1}{2}\{(m-1)k + (s-1)k_1\} - a_1(r)\{(m-1)k + (s-1)k_1\} \right]. \tag{3.5}$$

Comparing the coefficients of α and β in (3.3) and (3.5) and solving for $a_1(r)$, we have

$$a_1(r) = \left\{ \frac{2r - 1 + (m-1)k - k_1}{2\{(m-1)k + (s-1)k_1\}} \right\}.$$

Putting $a_1(r)$ in (3.4), we have

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + \frac{2r - 1 + (m-1)k - k_1}{2\{(m-1)k + (s-1)k_1\}} \left(Y_r - Y_{r+(m-1)k+(s-1)k_1} \right). \tag{3.6}$$

If $r > \{k_1 - (m-1)k\}$, then $l = 2$ and the last value of each sample is labeled $\{r + (m-1)k + (s-1)k_1 - N\}$. Thus, (3.2) becomes

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + a_2(r) \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N} \right). \tag{3.7}$$

Under the linear model (2.1), we have $\bar{y}_{MSS} = \alpha + \beta \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{k_1}{m} \right]$, where $u = 1, 2, ..., (m-1)$ if $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$, $Y_r = \alpha + \beta r$ and $Y_{r+(s-1)k_1+(m-1)k-N} = \alpha + \beta\{r + (s-1)k_1 + (m-1)k - N\}$. Putting these values in (3.7), we have

$$\bar{y}^c_{MSS} = \alpha + \beta \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{k_1}{m} - a_2(r)\{(m-1)k - k_1\} \right]. \tag{3.8}$$

Comparing the coefficients of α and β in (3.3) and (3.8) and solving for $a_2(r)$, we have

$$a_2(r) = \left\{ \frac{2r - (k_1+1) + (m-1)k - 2uk_1/m}{2\{(m-1)k - k_1\}} \right\}, \tag{3.9}$$

where $u = 1, 2, ..., (m-1)$, which are picked corresponding to a random number r such that $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$.
Putting $a_2(r)$ in (3.7), we have

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + \left\{ \frac{2r - (k_1+1) + (m-1)k - 2uk_1/m}{2\{(m-1)k - k_1\}} \right\} \times \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N} \right). \tag{3.10}$$

3.1.2. *Case (ii): when* $w > 1$. As mentioned earlier in Section 1 , when $s = 1$, MSS becomes CSS (see [2]). Therefore, we focus the MSS for $s > 1$. It is also mentioned in Subsection 1.2, all k_1 samples are partitioned into two groups. The first group contains the samples where $r \leq k_1 - (w-1)k + r_m$ and the second group consist of the samples in which $r > k_1 - (w-1)k + r_m$.
The corrected sample mean for each sample in the first group is similar to the corrected sample mean found in Subsection 3.1.1, where $r > k_1 - (m-1)k$, because the pattern of samples in both situations is similar. Further, the weights assigned to the first and the last units of each sample in this group will be similar to the weights given in (3.9), i.e.

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + a_2(r) \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N} \right),$$

$$a_2(r) = \left\{ \frac{2r - (k_1+1) + (m-1)k - 2uk_1/n}{2\{(m-1)k - k_1\}} \right\},$$

where $u = (w-1)$ corresponding to a random number r such that $1 \leq r \leq \{k_1 - (m-u-1)k\}$, $u = w, w+1, ..., (m-w)$ if $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$ and

$u = (m - w + 1)$ if $\{k_1 - (m - u)k\} < r \leq \{k_1 - (m - u)k + r_m\}$. Thus,

$$y^c_{MSS} = \bar{y}_{MSS} + \left\{\frac{2r - (k_1+1) + (m-1)k - 2uk_1/n}{2\{(m-1)k - k_1\}}\right\} \times \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N}\right). \tag{3.11}$$

The second group having samples in which $r > k_1 - (w-1)k + r_m$, and the first subgroup consists of the number of samples in which $\{k_1 - (w - x)k\} < r \leq \{k_1 - (w - x)k + r_m\}$ such that $x = 2, ..., (w - 1)$. The Yates corrected estimator with $l = 3$ in (3.2), for the samples of the first subgroup can be written as

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + a_3(r)\left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N}\right). \tag{3.12}$$

Under a linear model (2.1) $\bar{y}_{MSS} = \alpha + \beta[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w-1+2x)\frac{k_1}{m}]$, where $x = 2, ..., (w-1)$ if $\{k_1 - (w - x)k\} < r \leq \{k_1 - (w - x)k + r_m\}$, $Y_r = \alpha + \beta\, r$ and $Y_{r+(s-1)k_1+(m-1)k-N} = \alpha + \beta\{r + (s - 1)k_1 + (m - 1)k - N\}$. Putting these values in (3.12), we have

$$\bar{y}^c_{MSS} = \alpha + \beta\Big[r + \tfrac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m - w - 1 + 2x)\tfrac{k_1}{m} + a_3(r)\{(m-1)k - k_1\}\Big]. \tag{3.13}$$

Comparing the coefficients of α and β given in (3.3) and (3.13) and solving for $a_3(r)$, we have

$$a_3(r) = \left\{\frac{2r - (k_1 + 1) + (m - 1)k - 2k_1(m - w + 2x - 1)/m}{2\{(m-1)k - k_1\}}\right\}. \tag{3.14}$$

Putting $a_3(r)$ in the corrected estimator given in (3.12), we have

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + \left\{\frac{2r - (k_1+1) + (m-1)k - 2k_1(m-w+2x-1)/m}{2\{(m-1)k - k_1\}}\right\} \times \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N}\right), \tag{3.15}$$

where $x = 2, ..., (w - 1)$, which are picked corresponding to a random number r such that $\{k_1 - (w - x)k\} < r \leq \{k_1 - (w - x)k + r_m\}$. Similarly, the second subgroup consists of the number of samples in which $\{k_1 - (w - x)k + r_m\} < r \leq \{k_1 - (w - x)k + k\}$ such that $x = 1, 2, ..., (w - 1)$. The Yates corrected estimator with $l = 4$ in (3.2), for samples of this subgroup, can be written as

$$\bar{y}^c_{MSS} = \bar{y}_{MSS} + a_4(r)\left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N}\right). \tag{3.16}$$

Under the linear model (2.1), $\bar{y}_{MSS} = \alpha + \beta[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m - w + 2x)\frac{k_1}{m}]$, where $x = 1, 2, ..., (w - 1)$ if $\{k_1 - (w - x)k\} < r \leq \{k_1 - (w - x)k + r_m\}$, $Y_r = \alpha + \beta r$ and $Y_{r+(s-1)k_1+(m-1)k-N} = \alpha + \beta\{r + (s - 1)k_1 + (m - 1)k - N\}$. Putting these values in (3.16), we have

$$\bar{y}^c_{MSS} = \alpha + \beta\Big[r + \tfrac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m - w + 2x)\tfrac{k_1}{m} + a_4(r)\{(m-1)k - k_1\}\Big]. \tag{3.17}$$

Comparing the coefficients of α and β given in (3.3) and (3.17) and solving for $a_4(r)$, we have

$$a_4(r) = \left\{\frac{2r - (k_1 + 1) + (m - 1)k - 2k_1(m - w + 2x)/m}{2\{(m-1)k - k_1\}}\right\}. \tag{3.18}$$

Putting $a_4(r)$ in the corrected estimator given in (3.16), we have

$$y^c_{MSS} = \bar{y}_{MSS} + \left\{ \frac{2r-(k_1+1)+(m-1)k-2k_1(m-w+2x)/m}{2\{(m-1)k-k_1\}} \right\} \times \left(Y_r - Y_{r+(m-1)k+(s-1)k_1-N}\right), \tag{3.19}$$

where $x = 1, 2, ..., (w-1)$, which are picked corresponding to a random number r such that $\{k_1 - (w-x)k\} < r \leq \{k_1 - (w-x)k + r_m\}$.

4. Average variance

In real life application, we hardly found such population exhibiting perfect linear trend. Therefore, it is necessary to study the average variance of the corrected estimator under MSS using following super population model.

$$Y_t = \alpha + \beta\, t + e_t, \tag{4.1}$$

where $E(e_t) = 0$, $V(e_t) = E(e_t^2) = \sigma^2 t^g$, $Cov(e_t, e_v) = 0,\ t \neq v = 1, 2, 3, ..., N$ and g is the predetermined constant.
The average variance of $\bar{y}^{(r)}_{MSS}$ under modified systematic sampling for population modeled by $Y_t = \alpha + \beta\, t + e_t$ is given by

Case (i) when $w = 1$ (see detial in Appendix B)

$$\begin{aligned} E\left\{V(\bar{y}^{(r)}_{MSS})\right\} = \frac{\sigma^2}{k_1}\Bigg\{ & \sum_{r=1}^{k_1-(m-1)k} \chi_1(u,r) \\ & + \sum_{u=1}^{m-1} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} \chi_2(u,r) \\ & + k_1 \sum_{t=1}^{N} t^g \Big/ N^2 \Bigg\}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \chi_1(u,r) = {} & \delta_1^+(r) r^g + \theta \sum_{i=1}^{m} \sum_{j=1}^{s} \{r + (i-1)k + (j-1)k_1\}^g \\ & + \delta_1^-(r)\{r + (m-1)k + (s-1)k_1\}^g, \end{aligned}$$

$$\begin{aligned} \chi_2(u,r) = {} & \delta_2^+(r) r^g + \theta \frac{1}{n^2} \Bigg\{ \sum_{i=1}^{m-u} \sum_{j=1}^{s} (r + (i-1)k + (j-1)k_1)^g \\ & + \sum_{i=m-u+1}^{m} \Bigg(\sum_{j=1}^{s-1} (r + (i-1)k + (j-1)k_1)^g + \Big(r + (i-1)k \\ & + (s-1)k_1 - N\Big)^g \Bigg) \Bigg\} + \delta_2^-(r)\left(r + (m-1)k + (s-1)k_1 - N\right)^g, \end{aligned}$$

$\delta_l^+(r) = a_l(r)\{a_l(r) + 2\left(\frac{1}{n} - \frac{1}{N}\right)\}$, $\delta_l^-(r) = a_l(r)\{a_l(r) - 2\left(\frac{1}{n} - \frac{1}{N}\right)\}$ and $\theta = \frac{1}{n}\left(\frac{1}{n} - \frac{2}{N}\right)$, such that $l = 1, 2$.

Case (ii) When $w > 1$ (see detial in Appendix B)

$$\begin{aligned} E\left\{V\left(\bar{y}^{(r)}_{MSS}\right)\right\} = \frac{\sigma^2}{k_1}\Bigg[& \sum_{u=w-1}^{w-1} \sum_{r=1}^{k_1-(m-u-1)k} \chi_2(u,r) \\ & + \sum_{u=w}^{m-w} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} \chi_2(u,r) \\ & + \sum_{u=m-w+1}^{m-w+1} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} \chi_2(u,r) \\ & + \sum_{x=2}^{w-1} \sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_m} \chi_3(x,r) \\ & + \sum_{x=1}^{w-1} \sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k} \chi_4(x,r) \\ & + k_1 \sum_{t=1}^{N} t^g \Big/ N^2 \Bigg], \end{aligned} \tag{4.3}$$

where

$$\chi_2(u,r) = \delta_2^+(r)r^g + \theta\frac{1}{n^2}\Big\{\sum_{i=1}^{m-u}\sum_{j=1}^{s}(r+(i-1)k+(j-1)k_1)^g$$
$$+\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s-1}(r+(i-1)k+(j-1)k_1)^g$$
$$+(r+(i-1)k+(s-1)k_1-N)^g\Big)\Big\} + \delta_2^-(r)\,(r+(m-1)k+(s-1)k_1-N)^g,$$

$$\chi_3(x,r) = \delta_3^+(r)r^g + \theta\Big\{\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g$$
$$+\sum_{i=w-x+1}^{m-x+1}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g$$
$$+\{r+(i-1)k+(s-1)k_1-N\}^g\Big)$$
$$+\sum_{i=m-x+2}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g$$
$$+\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\}$$
$$+\delta_3^-(r)\{r+(m-1)k+(s-1)k_1-N\}^g,$$

$$\chi_4(x,r) = \delta_4^+(r)r^g + \theta\Big\{\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g$$
$$+\sum_{i=w-x+1}^{m-x}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g$$
$$+\{r+(i-1)k+(s-1)k_1-N\}^g\Big)$$
$$+\sum_{i=m-x+1}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g$$
$$+\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\}$$
$$+\delta_4^-(r)\{r+(m-1)k+(s-1)k_1-N\}^g,$$

$\delta_l^+(r) = a_l(r)\{a_l(r) + 2\left(\frac{1}{n}-\frac{1}{N}\right)\}$, $\delta_l^-(r) = a_l(r)\{a_l(r) - 2\left(\frac{1}{n}-\frac{1}{N}\right)\}$ and $\theta = \frac{1}{n}\left(\frac{1}{n}-\frac{2}{N}\right)$, for $l = 2,\ 3,\ 4$

5. Empirical study

Due to the complex nature of the derived expressions, the average variances of MSS and CSS cannot be theoretically compared. Therefore, in this paper, a computer based efficiency comparison of MSS and CSS is made numerically under super population model (4.1). The numerical comparison has been made for $N = 21$, $N = 30$, $N = 50$ and $N = 78$. As mentioned earliar, if $L = N$ then MSS reduces to LSS and if $L = (N \times n)$ then MSS becomes CSS. Therefore, the choice of a sample size considered in this paper is based on the fact that $N < L < (N \times n)$.

The relative efficiency of MSS over CSS is presented in Table 1 under $g = 0, 1, 2, 3$. This table includes 40 different combinations of N and n each at $g = 0, 1,\ 2$ and 3 which are to be considered for efficiency comparison, and it is observed that CSS is not applicable for 4 combinations. Thus, we have $36 \times 4 = 144$ results of efficiency comparison and found that MSS is more efficient than CSS in 135 cases. Further, it is to be noted, whenever $\frac{N}{n} = \left(\frac{n}{2}+\frac{1}{2}\right)$, the efficiency of MSS over CSS is dramatically increased.

5.1. Natural Population. We use the following natural population for efficiency comparison. The results are given in Table 2. Population 1: [Source: [1, page.228]. Table 2 reflacts that MSS is more efficient than CSS.

Table 1. Rercent Relative Efficiency (PRE) of MSS over CSS under linear trend

N	n	$g=0$	$g=1$	$g=2$	$g=3$	N	n	$g=0$	$g=1$	$g=2$	$g=3$
21	6	1235.016	1933.767	2709.724	3246.609	78	4	117.427	136.092	146.405	147.684
	9	117.385	151.130	174.582	186.989		8	129.769	182.162	229.578	259.786
	12	240.000	591.045	1375.565	2521.240		9	192.866	316.722	411.684	454.488
30	4	135.923	162.510	180.204	185.741		10	139.707	208.739	281.749	337.633
	8	246.131	376.018	544.687	700.611		12	6766.915	13989.818	19161.993	21879.417
	9	103.060	138.491	160.844	170.657		14	-	-	-	-
	12	-	-	-	-		15	90.595	152.753	201.481	223.241
	14	103.355	144.513	188.675	222.580		16	191.878	323.211	512.333	718.994
50	4	122.565	143.552	155.795	158.042		18	105.243	205.796	258.754	278.058
	6	101.443	123.039	137.125	142.782		20	257.844	455.482	780.262	1198.619
	8	97.891	125.605	146.267	156.789		21	236.637	633.532	1610.276	3245.550
	12	93.959	130.154	163.181	184.581		22	7.521	14.757	28.913	56.125
	14	43.631	84.386	161.900	305.136		24	114.967	260.473	359.195	399.452
	15	113.396	192.752	237.112	255.203		27	-	-	-	-
	16	93.665	136.682	182.496	217.408		28	129.784	203.929	293.560	381.668
	18	176.253	339.329	667.273	1275.240		30	-	-	-	-
	20	83.942	360.224	621.068	698.071		32	127.980	210.134	322.967	448.077
	22	128.892	193.530	275.040	353.302		33	153.748	309.110	503.525	655.560
	24	101.712	159.836	238.027	313.959		34	129.642	215.246	339.183	482.954
							36	151.165	418.855	673.244	790.552
							38	100.656	170.712	280.950	410.473

The symbol (-) indicates that CSS is not possible

Table 2. Percent Relative Efficiency (PRE) of MSS over CSS for Population 1

$N=80$	Variance		$Eff = \frac{V(CSS)}{V(MSS)} \times 100$
n	MSS	CSS	
6	148053.500	148326.200	100.184
12	37312.340	46858.630	125.585
14	33277.620	37824.400	113.663
15	28206.520	39716.150	140.805
18	29362.610	50787.490	172.967
24	9108.401	37832.020	415.353
25	7983.309	19399.580	243.002
26	7471.210	8915.224	119.328
28	6836.277	12549.070	183.566

Here, $V(MSS)$ = Variance of modified systematic sampling and $V(CSS)$ = Variance of circular systematic sampling.

6. Conclusion

Modified Systematic Sampling (MSS) is a more general scheme than LSS and CSS. Because, when least common multiple of N and n is equal to lower extreme, i.e. $L = N$, MSS coincides with LSS. If it is equal to upper extreme, i.e. $L = (N \times n)$, then MSS coincides with CSS. However, when L lies between these two extreme values, i.e. $N < L < (N \times n)$, MSS is advantageous over CSS. In this case, the number of samples is considerably reduced in MSS as compared to CSS, i.e. minimum reduction is half of the samples.

Contrary to the CSS, the explicit expressions for mean and variance of mean are derived for population having perfect linear trend among the population values. Further, numerical comparison is carried out in this paper clearly favors the use of MSS over CSS for population modeled by a super population model with linear trend as well as for natural population.

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References

[1] Murthy, M.N. *Sampling Theory and Method*, Statistical Publishing Society, Calcutta, 1967.

[2] Khan, Z. Shabbir, J. and Gupta, S. *A new sampling design for systematic sampling*, Communications in Statistics - Theory and Methods,**42**, 2659-2670, 2013.

[3] Khan, Z. Gupta, S and Shabbir, J. *A Note on Diagonal Circular Systematic Sampling*, Journal of Statistical Theory and Practice, **8**, 439-443, 2014.

[4] Madow, W.G. *On the theory of systematic sampling III*, Annals of Mathematics and Statistics, **24**, 101-106, 1953.

[5] Sampath, S and Varalakshmi, V. *Diagonal circular systematic sampling*, Model Assisted Statistics and Applications, **3** (4), 345-352, 2008.

[6] Sethi, V. K. *On optimum pairing of units*, Sankhya **B, 27**, 315-320, 1965.

[7] Singh, D. Jindal, K.K. and Grag, J.N.*On modified systematic sampling*, Biometrika, **55**, 41-546, 1968.

[8] Subramani, J. *Diagonal systematic sampling scheme for finite populations*, Journal of the Indian Society of Agriculture Statistics, **53** (2), 187-195, 2000.

[9] Subramani, J. *Further results on diagonal systematic sampling scheme for finite populations*, Journal of the Indian Society of Agricultural Statistics, **63** (3), 277-282, 2009.

[10] Subramani, J.*Generalization of diagonal systematic sampling scheme for finite populations*, Model Assisted Statistics and Applications, **5**, 117-128, 2010.

[11] Subramani, J. *A Modification on linear systematic sampling*, Model Assisted Statistics and Applications, **8** (3), 215-227, 2013a.

[12] Subramani, J. *A further modification on linear systematic sampling for finite populations*, Journal of Statistical Theory and Practice, **7**, 471-479, 2013b.

Appendix A. Mean and variance of MSS for population having linear trend

The following linear model of hypothetical population is to be considered

$$Y_t = \alpha + \beta\, t, \qquad t = 1, 2, 3, ..., N, \tag{A.1}$$

where α and β are the intercept and slope of the model respectively.

A.1. Mean of MSS. The sample mean for both cases, i.e. $w = 1$ and $w > 1$ are separately discussed below:

Case (i) when $w = 1$
If $r \leq (k_1 - (m-1)k$, the mean, $\bar{y}_{MSS}$ can be written as

$$\bar{y}_{MSS} = \alpha + \beta \frac{1}{ms} \sum_{i=1}^{m-w+1} \sum_{j=1}^{s} \{r + (i-1)k + (j-1)k_1\}$$

After simplification, we have

$$\bar{y}_{MSS} = \alpha + \beta \left[r + \frac{1}{2} \{(s-1)k_1 + (m-1)k\} \right].$$

If $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$ for $u = 1, 2, ..., m-1$, then

$$\begin{aligned} \bar{y}_{MSS} = \alpha + \beta \tfrac{1}{ms} \Big[& \textstyle\sum_{i=1}^{m-u} \sum_{j=1}^{s} \{r + (i-1)k + (j-1)k_1\} \\ & + \textstyle\sum_{i=m-u+1}^{m} \Big\{ \sum_{j=1}^{s-1} \{r + (i-1)k + (j-1)k_1\} \\ & + \{r + (i-1)k + (s-1)k_1 - N\} \Big\} \Big]. \end{aligned}$$

After simplification, we have

$$\bar{y}_{MSS} = \alpha + \beta \left[r + \frac{1}{2} \{(s-1)k_1 + (m-1)k\} - u \frac{k_1}{m} \right].$$

Thus $\bar{y}_{MSS}$ is a piecewise function of r, i.e.

$$\bar{y}_{MSS} = \alpha + \beta \begin{cases} \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} \right] \\ \qquad \text{if} \quad r \leq \{k_1 - (m-1)k\} \\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{k_1}{m} \right] \\ \qquad \text{where} \; \left| \begin{array}{l} u = 1, 2, ..., (m-1) \text{ if} \\ \{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}. \end{array} \right. \end{cases} \tag{A.2}$$

Case (ii) when $w > 1$
If $r \leq \{k_1 - (w-1)k + r_m\}$, then r must belongs to any one of the three subgroups which have been discussed in Section 1. Therefore, corresponding to a random number r, an integer u is picked in such a way that $u = (w-1)$ if $1 \leq r \leq \{k_1 - (m-u-1)k\}$; $u = w, w+1, w+2, ..., (m-w)$ if $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u-1)k\}$ and $u = (m-w+1)$ if $\{k_1 - (m-u)k\} < r \leq \{k_1 - (m-u)k + r_m\}$.

For each subgroup, $\bar{y}_{MSS}$ can be written as

$$\begin{aligned}\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[&\textstyle\sum_{i=1}^{m-u}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}\\&+\textstyle\sum_{i=m-u+1}^{m}\Big\{\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}\\&+\{r+(i-1)k+(s-1)k_1-N\}\Big].\end{aligned}$$

After few steps, we have

$$\bar{y}_{MSS} = \alpha + \beta\left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{k_1}{m}\right].$$

If $w > 2$, then it is also possible that $\{k_1-(w-x)k\} < r \le \{k_1-(w-x)k+r_1\}$, such that $x = 2, 3, ..., w-1$. So,

$$\begin{aligned}\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[&\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}\\&+\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}\\&+\{r+(i-1)k+(s-1)k_1-N\}\\&+\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2}\{r+(i-1)k+(j-1)k_1\}\\&+\textstyle\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}.\end{aligned} \tag{A.3}$$

When $s = 2$, then Equation (A.3) can be expressed as

$$\begin{aligned}\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[&\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}\\&+\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}\\&+\{r+(i-1)k+(s-1)k_1-N\}\Big\}\\&+\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}\Big\}\Big].\end{aligned}$$

Also, when $s = 1$, then Equation (A.3) can be expressed as

$$\begin{aligned}\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[&\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}\\&+\textstyle\sum_{i=w-x+1}^{m-x+1}\{r+(i-1)k+(s-1)k_1-N\}\\&+\textstyle\sum_{i=m-x+2}^{m}\{r+(i-1)k+(s-1)k_1-2N\}\Big].\end{aligned}$$

After simplifying of Equation (A.3) for each case, i.e. $s = 1$, $s = 2$ and $s > 2$, we have

$$\bar{y}_{MSS} = \alpha + \beta\left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w-1+2x)\frac{k_1}{m}\right].$$

If $\{k_1-(w-x)k+r_1\} < r \le \{k_1-(w-x)k+k\}$, then

$$\begin{aligned}\bar{y}_{MSS} = \alpha + \beta\tfrac{1}{ms}\Big[&\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}\\&+\textstyle\sum_{i=w-x+1}^{m-x}\Big\{\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}\\&+\{r+(i-1)k+(s-1)k_1-N\}\Big\}\\&+\textstyle\sum_{i=m-x+1}^{m}\Big\{\sum_{j=1}^{s-2}\{r+(i-1)k+(j-1)k_1\}\\&+\textstyle\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}\Big\}\Big].\end{aligned} \tag{A.4}$$

When $s = 2$, then Equation (A.4) can be expressed as

$$\begin{aligned}\bar{y}_{MSS} = \alpha + \beta \tfrac{1}{ms}\Big[&\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r + (i-1)k + (j-1)k_1\}\\ &+ \textstyle\sum_{i=w-x+1}^{m-x}\Big\{\sum_{j=1}^{s-1}\{r + (i-1)k + (j-1)k_1\}\\ &+ \{r + (i-1)k + (s-1)k_1 - N\}\Big\}\\ &+ \textstyle\sum_{i=m-x+1}^{m}\sum_{j=s-1}^{s}\{r + (i-1)k + (j-1)k_1 - N\}\Big].\end{aligned}$$

When $s = 1$, then Equation (A.4) can be expressed as,

$$\begin{aligned}\bar{y}_{MSS} = \alpha + \beta \tfrac{1}{ms}\Big[&\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r + (i-1)k + (j-1)k_1\}\\ &+ \textstyle\sum_{i=w-x+1}^{m-x}\{r + (i-1)k + (s-1)k_1 - N\}\\ &+ \textstyle\sum_{i=m-x+1}^{m}\{r + (i-1)k + (s-1)k_1 - 2N\}\Big].\end{aligned}$$

After simplification of Equation (A.4) for each case, i.e. $s = 1$, $s = 2$ and $s > 2$, we have

$$\bar{y}_{MSS} = \alpha + \beta\Big[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m - w + 2x)\frac{k_1}{m}\Big].$$

Thus, mean of MSS for above model of hypothetical population with random start r is given by:

$$\text{(A.5)}\quad \bar{y}_{MSS} = \alpha + \beta\begin{cases}\left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{N}{n}\right]\\ \quad\text{where}\ \left|\begin{array}{l}u = (w-1)\ if\quad r \le \{k_1 - (m-u-1)k,\\ u = w, w+1, ..., (m-w)\ if\\ \{N - (m-u)k\} < r \le \{k_1 - (m-u-1)k\},\\ u = (m-w+1)\ if\\ \{k_1 - (m-u)k\} < r \le \{k_1 - (m-u)k + r_m\}\end{array}\right.\\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m - w - 1 + 2x)\frac{N}{n}\right],\\ \quad\text{where}\ \left|\begin{array}{l}x = 2, ..., (w-1)\ if\\ \{k_1 - (w-x)k\} < r \le \{k_1 - (w-x)k + r_m\}\end{array}\right.\\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m - w + 2x)\frac{N}{n}\right],\\ \quad\text{where}\ \left|\begin{array}{l}x = 1, 2, 3, ..., (w-1)\ \ if\\ \{k_1 - (w-x)k\} + r_m < r \le \{k_1 - (w-x)k + k\}\end{array}\right.\end{cases}$$

If $w = 2$, then Equation (A.5) reduces to

$$\bar{y}_{MSS} = \alpha + \beta\begin{cases}\left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - u\frac{N}{n}\right]\\ \quad\text{where}\ \left|\begin{array}{l}u = (w-1)\ if\quad r \le \{k_1 - (m-u-1)k,\\ u = w, w+1, ..., (m-w)\ if\\ \{N - (m-u)k\} < r \le \{k_1 - (m-u-1)k\},\\ u = (m-w+1)\ if\\ \{k_1 - (m-u)k\} < r \le \{k_1 - (m-u)k + r_m\}\end{array}\right.\\ \left[r + \frac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m - w + 2x)\frac{N}{n}\right],\\ \quad\text{where}\ \left|\begin{array}{l}x = 1, 2, 3, ..., (w-1)\ \ if\\ \{k_1 - (w-x)k\} + r_m < r \le \{k_1 - (w-x)k + k\}\end{array}\right.\end{cases}$$

A.1.1. *Unbiasedness of sample mean $\bar{y}_{MSS}$.* We have two cases:

Case (i) when $w=1$: Taking the expected value of (A.2), we have

$$E(\bar{y}_{MSS}) = \frac{1}{k_1}\Big[\sum_{r=1}^{(k_1-(m-1)k)}\left[r+\tfrac{1}{2}\{(s-1)k_1+(m-1)k\}\right] + \sum_{u=1}^{m-1}\sum_{r=(k_1-(m-u)k)+1}^{k_1-(m-u-1)k)}\left[r+\tfrac{1}{2}\{(s-1)k_1+(m-1)k\} - u\tfrac{k_1}{m}\right]\Big].$$

As $sk_1 = N$, then

$$E(\bar{y}_{MSS}) = \frac{1}{k_1}\Big[\sum_{r=1}^{(k_1-(m-1)k)}\left[\alpha+\beta\left\{r+\tfrac{1}{2}\left(N-k_1+(m-1)k\right)\right\}\right] + \sum_{u=1}^{m-1}\sum_{r=(k_1-(m-u)k)+1}^{k_1-(m-u-1)k)}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left(N-k_1+(m-1)k\right) - u\tfrac{k_1}{m}\Big\}\Big]\Big].$$

After a little algebra, we have

$$E(\bar{y}_{MSS}) = \alpha+\beta\left\{\frac{N+1}{2}\right\} = \bar{Y},$$

which shows that $\bar{y}_{MSS}$ is an unbiased estimator of $\bar{Y}$.

Case (ii) when $w>1$:

If $w>2$, we take the expected value of (A.5), we have

$$\begin{aligned}E(\bar{y}_{MSS}) = \frac{1}{k_1}\Big[&\sum_{u=w-1}^{w-1}\sum_{r=1}^{k_1-(m-u-1)k}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((s-1)k_1\right.\\ &\left.+(m-1)k-u\tfrac{k_1}{m}\Big\}\right]\\ &+\sum_{u=w}^{m-w}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((s-1)k_1\right.\\ &\left.+(m-1)k-u\tfrac{k_1}{m}\Big\}\right]\\ &+\sum_{u=m-w+1}^{m-w+1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((s-1)k_1\right.\\ &\left.+(m-1)k-u\tfrac{k_1}{m}\Big\}\right]\\ &+\sum_{x=2}^{w-1}\sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_m}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((s-1)k_1\right.\\ &\left.+(m-1)k-\tfrac{(m-w-1+2x)k_1}{m}\Big\}\right]\\ &+\sum_{x=1}^{w-1}\sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((s-1)k_1\right.\\ &\left.+(m-1)k-\tfrac{(m-w+2x)k_1}{m}\Big\}\right]\Big].\end{aligned}$$

But if $w=2$, we take the expected value of (A.5), given by

$$\begin{aligned}E(\bar{y}_{MSS}) = \frac{1}{k_1}\Big[&\sum_{u=w-1}^{w-1}\sum_{r=1}^{k_1-(m-u-1)k}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((s-1)k_1\right.\\ &\left.+(m-1)k-u\tfrac{k_1}{m}\Big\}\right]\\ &+\sum_{u=w}^{m-w}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((s-1)k_1\right.\\ &\left.+(m-1)k-u\tfrac{k_1}{m}\Big\}\right]\\ &+\sum_{u=m-w+1}^{m-w+1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((s-1)k_1\right.\\ &\left.+(m-1)k-u\tfrac{k_1}{m}\Big\}\right]\\ &+\sum_{x=1}^{w-1}\sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((s-1)k_1\right.\\ &\left.+(m-1)k-\tfrac{(m-w+2x)k_1}{m}\Big\}\right]\Big].\end{aligned}$$

After few steps, we have

$$E(\bar{y}_{MSS}) = \tfrac{1}{2m}\Big[2\alpha m + \beta\Big\{2k_1 - 2r_1 - 2km + 2wk + 2wr_1 + 2kwm - 2k_1 w - 2kw^2 + Nm + m\Big\}\Big]$$

$$E(\bar{y}_{MSS}) = \Big[\alpha + \beta\tfrac{1}{2m}\Big\{m\,(N+1) - 2k_1(w-1) + 2r_1(w-1) - 2wk(w-1) + 2km(w-1)\Big\}\Big]$$

$$\text{(A.6)}\quad E(\bar{y}_{MSS}) = \left[\alpha + \beta\left\{\frac{N+1}{2} + \frac{(w-1)}{m}\,(r_1 - k_1 - wk + km)\right\}\right].$$

When $w = (m-q)$ in (A.6), we have

$$E(\bar{y}_{MSS}) = \left[\alpha + \beta\left\{\frac{N+1}{2} + \frac{(m-q-1)}{m}\,(r_1 - k_1 - (m-q)k + km)\right\}\right],$$

$$E(\bar{y}_{MSS}) = \left[\alpha + \beta\left\{\frac{N+1}{2} + \frac{(m-q-1)}{m}\,(qk + r_1 - k_1 - mk + km)\right\}\right],$$

$$E(\bar{y}_{MSS}) = \alpha + \beta\left\{\frac{N+1}{2} + \frac{(m-q-1)}{m}\,(qk + r_1 - k_1)\right\},$$

$$E(\bar{y}_{MSS}) = \alpha + \beta\left\{\frac{N+1}{2}\right\} = \bar{Y}$$

.

the above equation shows that $\bar{y}_{MSS}$ is unbiased estimator of $\bar{Y}$ as $k_1 = qk + r_1$.
Note: Putting $w = 1$ in (A.6), we also have

$$E(\bar{y}_{MSS}) = \alpha + \beta\left\{\frac{N+1}{2}\right\} = \bar{Y}.$$

A.2. The variance of $\bar{y}_{MSS}$.

$$V(\bar{y}_{MSS}) = E(\bar{y}_{MSS} - \bar{Y})^2 = \tfrac{1}{k_1}\textstyle\sum_{r=1}^{k_1}(\bar{y}_{r(MSS)} - \bar{Y})^2.$$

(i) when $w = 1$

$$V(\bar{y}_{MSS}) = \tfrac{1}{k_1}\Big[\textstyle\sum_{r=1}^{k_1-(m-1)k}\left\{\bar{y}_{r(MSS)} - \bar{Y}\right\}^2 + \sum_{u=1}^{m-1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\left\{\bar{y}_{r(MSS)} - \bar{Y}\right\}^2\Big].$$

$$V(\bar{y}_{MSS}) = \tfrac{1}{k_1}\Big[\textstyle\sum_{r=1}^{k_1-(m-1)k}\Big[\alpha + \beta\left\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k\right)\right\} - \left\{\alpha + \beta\tfrac{N+1}{2}\right\}\Big]^2 + \sum_{u=1}^{m-1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\alpha + \beta\{r + \tfrac{1}{2}\left((s-1)k_1 + (m-1)k\right) - u\tfrac{k_1}{m}\Big\} - \left\{\alpha + \beta\tfrac{N+1}{2}\right\}\Big]^2\Big].$$

$$V(\bar{y}_{MSS}) = \tfrac{1}{k_1}\Big[\textstyle\sum_{r=1}^{k_1-(m-1)k}\Big[\beta\left\{r + \tfrac{1}{2}\left((m-1)k - k_1\right)\right\} - \tfrac{\beta}{2}\Big]^2 + \sum_{u=1}^{m-1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\beta\Big\{r + \tfrac{1}{2}\left((m-1)k - k_1\right) - u\tfrac{k_1}{m}\Big\} - \tfrac{\beta}{2}\Big]^2\Big].$$

After simplification, we have

$$V(\bar{y}_{MSS}) = \frac{1}{12m^2} b^2 \left[m^2(k_1^2 - 1) + m(m^2 - 1)k(mk - 2k_1) \right], \tag{A.7}$$

Note: If $N = nk$, then $L = N$, so $m = 1$, thus

$$V(\bar{y}_{MSS}) = \frac{1}{12} b^2 (k^2 - 1).$$

This is a variance of linear systematic sampling.

(ii) when $w > 1$
If $w > 2$, then $V(\bar{y}_{MSS})$ will be expressed as:

$$\begin{aligned} V(\bar{y}_{MSS}) = \tfrac{1}{k_1} \Big[& \textstyle\sum_{u=w-1}^{w-1} \sum_{r=1}^{k_1-(m-u-1)k} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2 \\ & + \textstyle\sum_{u=w}^{m-w} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2 \\ & + \textstyle\sum_{u=m-w+1}^{m-w+1} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2 \\ & + \textstyle\sum_{x=2}^{w-1} \sum_{r=(k_1-(w-x)k)+1}^{k_1-(w-x)k+r_m)} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2 \\ & + \textstyle\sum_{x=1}^{w-1} \sum_{r=(k_1-(w-x)k)+r_m+1}^{k_1-(w-x)k+k)} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2 \Big]. \end{aligned}$$

If $w = 2$, then the term $\sum_{x=2}^{w-1} \sum_{r=(k_1-(w-x)k)+1}^{k_1-(w-x)k+r_m)} \left\{ \bar{y}_{r(MSS)} - \bar{Y} \right\}^2$ will be omitted from $V(\bar{y}_{MSS})$.

$$\begin{aligned} V(\bar{y}_{MSS}) = \tfrac{1}{k_1} \Big[& \textstyle\sum_{u=w-1}^{w-1} \sum_{r=1}^{k_1-(m-u-1)k} \Big[\alpha + \beta \Big\{ r + \tfrac{1}{2} \Big((s-1)k_1 \\ & +(m-1)k \Big) - u\tfrac{k_1}{m} \Big\} - \left\{ \alpha + \beta \tfrac{N+1}{2} \right\} \Big]^2 \\ & + \textstyle\sum_{u=w}^{m-w} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} \Big[\alpha + \beta \Big\{ r + \tfrac{1}{2} \Big((s-1)k_1 \\ & +(m-1)k \Big) - u\tfrac{k_1}{m} \Big\} - \left\{ \alpha + \beta \tfrac{N+1}{2} \right\} \Big]^2 \\ & + \textstyle\sum_{u=m-w+1}^{m-w+1} \sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m} \Big[\alpha + \beta \Big\{ r + \tfrac{1}{2} \Big((s-1)k_1 \\ & +(m-1)k \Big) - u\tfrac{k_1}{m} \Big\} - \left\{ \alpha + \beta \tfrac{N+1}{2} \right\} \Big]^2 \\ & + \textstyle\sum_{x=2}^{w-1} \sum_{r=(k_1-(w-x)k)+1}^{k_1-(w-x)k+r_1)} \Big[\alpha + \beta \Big\{ r + \tfrac{1}{2} \Big\{ (s-1)k_1 \\ & +(m-1)k \Big\} - (m-w-1+2x)\tfrac{k_1}{m} \Big\} - \left\{ \alpha + \beta \tfrac{N+1}{2} \right\} \Big]^2 \\ & + \textstyle\sum_{x=1}^{w-1} \sum_{r=(k_1-(w-x)k)+r_1+1}^{k_1-(w-x)k+k)} \Big[\alpha + \beta \Big\{ r + \tfrac{1}{2} \Big\{ (s-1)k_1 \\ & +(m-1)k \Big\} - (m-w+2x)\tfrac{k_1}{m} \Big\} - \left\{ \alpha + \beta \tfrac{N+1}{2} \right\} \Big]^2 \Big], \end{aligned}$$

$$
\begin{aligned}
V(\bar{y}_{MSS}) = \tfrac{1}{k_1}\Big[&\textstyle\sum_{u=w-1}^{w-1}\sum_{r=1}^{k_1-(m-u-1)k}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right)\\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2\\
&+\textstyle\sum_{u=w}^{m-w}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right)\\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}^2\\
&+\textstyle\sum_{u=m-w+1}^{m-w+1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right)\\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2\\
&+\textstyle\sum_{x=2}^{w-1}\sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_m}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right)\\
&-(m-w-1+2x)\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2\\
&+\textstyle\sum_{x=1}^{w-1}\sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right)\\
&-(m-w+2x)\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2.
\end{aligned}
\tag{A.8}
$$

If $w=2$, then Equation (A.8) reduces to

$$
\begin{aligned}
V(\bar{y}_{MSS}) = \tfrac{1}{k_1}\Big[&\textstyle\sum_{u=w-1}^{w-1}\sum_{r=1}^{k_1-(m-u-1)k}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right)\\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2\\
&+\textstyle\sum_{u=w}^{m-w}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right)\\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2\\
&+\textstyle\sum_{u=m-w+1}^{m-w+1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m}\Big[\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right)\\
&-u\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2\\
&+\textstyle\sum_{x=1}^{w-1}\sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k}\Big[\alpha+\beta\Big\{r+\tfrac{1}{2}\left((m-1)k-k_1\right)\\
&-(m-w+2x)\tfrac{k_1}{m}\Big\}-\tfrac{\beta}{2}\Big]^2\Big].
\end{aligned}
$$

After simplification, we have

$$
\begin{aligned}
V(\bar{y}_{MSS}) = \tfrac{1}{12m^2}b^2\Big[&m^2(k_1^2-1)+m(m^2-1)k(mk-2k_1)\\
&+4(w-1)\Big\{3k(m-q-w)\left\{m(k_1-qk)+(k_1-mk)\right\}\\
&+k_1w\ \left\{3k_1-(3m-2w+1)k\right\}\Big\}\Big].
\end{aligned}
$$

The term $(w-1)\left[3k(m-q-w)\left\{m(k_1-qk)+(k_1-mk)\right\}\right]$ will be vanished in both situations, when $w=1$ or $w=(m-q)$. So, we are left with

$$
\begin{aligned}
V(\bar{y}_{MSS}) = \tfrac{1}{12m^2}b^2&\left[m^2(k_1^2-1)+m(m^2-1)k(mk-2k_1)\right.\\
&\left.+4w(w-1)k_1\left\{3k_1-(3m-2w+1)k\right\}\right].
\end{aligned}
\tag{A.9}
$$

Appendix B. Average variance

In real life application, we hardly found such population exhibiting perfect linear trend, therefore, it is necessary to study the average variance of the corrected estimator under MSS using following super population model.

$$Y_t = \alpha + \beta t + e_t, \tag{B.1}$$

where $E(e_t) = 0$, $V(e_t) = E(e_t^2) = \sigma^2 t^g$, $Cov(e_t, e_v) = 0$, $t \neq v = 1, 2, 3, ..., N$and g is a predetermine constant.
Under the above super population model (B.1), the average variance expression of MSS is given below:

Case (i) when $w = 1$
Consider that l^{th} sum of squares (SSl) are given by

$$SSl = \left[\bar{y}_{MSS}^{(r)} - \bar{Y}\right]^2 = \left[\{\bar{y}_{MSS} + a_l(r)\left(Y_{r1} - Y_{rn}\right)\} - \bar{Y}\right]^2, \tag{B.2}$$

where $l = 1$ if $r \leq k_1 - (m-1)k$ and $l = 2$ if $r > k_1 - (m-1)k$.
When $r \leq k_1 - (m-1)k$, the expressions of $\bar{y}_{MSS}$, $\bar{Y}$, $Y_{r1}$and Y_{rn} under the model $Y_t = \alpha + \beta\, t + e_t$, can be expressed as:
$\bar{y}_{MSS} = \alpha + \beta\left[r + \frac{1}{2}\left\{(s-1)k_1 + (m-1)k\right\}\right] + \frac{1}{ms}\sum_{i=1}^{m}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1}$,
$\bar{Y} = \alpha + \beta\frac{N+1}{2} + \frac{1}{N}\sum_{t=1}^{N} e_t$, $Y_{r1} = \alpha + \beta r + e_r$ and $Y_{rn} = \alpha + \beta\left\{r + (m-1)k + (s-1)k_1\right\} + e_{r+(m-1)k+(s-1)k_1}$.
Substituting these expressions in (B.2), we have

$$SS1 = \left[\bar{y}_{MSS}^{(r)} - \bar{Y}\right]^2 = \left[\frac{1}{n}\left\{\sum_{i=1}^{m}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1} + na_1(r)\left(e_r - e_{r+(m-1)k+(s-1)k_1}\right)\right\} - \frac{1}{N}\sum_{t=1}^{N} e_t\right]^2.$$

Similarly, if $r > k_1 - (m-1)k$ the expressions of $\bar{y}_{MSS}$, $\bar{Y}$, Y_{r1} and Y_{rn} under the super population model $Y_t = \alpha + \beta\, t + e_t$, can be written as

$$\bar{y}_{MSS} = \alpha + \beta\left[r + \frac{1}{2}\left\{(s-1)k_1 + (m-1)k\right\} - u\frac{k_1}{m}\right] + \frac{1}{ms}\left[\sum_{i=1}^{m-u}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1} + \sum_{i=m-u+1}^{m}\left\{\sum_{j=1}^{s-1} e_{r+(i-1)k+(j-1)k_1} + e_{r+(i-1)k+(s-1)k_1-N}\right\}\right],$$

$\bar{Y} = \alpha + \beta\frac{N+1}{2} + \frac{1}{N}\sum_{t=1}^{N} e_t$, $Y_{r1} = \alpha + \beta r + e_r$ and $Y_{rn} = \alpha + \beta\left\{r + (m-1)k + (s-1)k_1 - N\right\} + e_{r+(m-1)k+(s-1)k_1-N}$.
Thus,

$$SS2 = \left[\bar{y}_{MSS}^{(r)} - \bar{Y}\right]^2 = \left[\frac{1}{n}\left\{\sum_{i=1}^{m-u}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1} + \sum_{i=m-u+1}^{m}\left(\sum_{j=1}^{s-1} e_{r+(i-1)k+(j-1)k1} + e_{r+(i-1)k+(s-1)k_1-N}\right) + na_2(r)(e_r - e_{r+(m-1)k+(s-1)k_1-N})\right\} - \frac{1}{N}\sum_{t=1}^{N} e_t\right]^2.$$

The average variance of the corrected sample mean can be written as:

$$E\left\{V(\bar{y}_{MSS}^{(r)})\right\} = \frac{1}{k_1}\left\{\sum_{r=1}^{k_1-(m-1)k} E\left(SS1\right) + \sum_{u=1}^{m-1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k} E\left(SS2\right)\right\} \tag{B.3}$$

under the assumptions of super population model.

$$\begin{aligned}E\left(SS1\right) = &\Big[\tfrac{1}{n^2}\Big\{\textstyle\sum_{i=1}^{m}\sum_{j=1}^{s}E(e^2_{r+(i-1)k+(j-1)k_1}) + n^2a_1^2(r)\Big\{E(e_r^2)\\ &+E(e^2_{r+(m-1)k+(s-1)k_1})\Big\} + 2na_1(r)\Big\{E(e_r^2)\\ &-E(e^2_{r+(m-1)k+(s-1)k_1})\Big\}\Big\}\\ &+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}E(e_t^2) - \tfrac{2}{nN}\Big\{\textstyle\sum_{i=1}^{m}\sum_{j=1}^{s}E(e^2_{r+(i-1)k+(j-1)k_1})\\ &+na_1(r)\Big\{E(e_r^2) - E(e^2_{r+(m-1)k+(s-1)k_1})\Big\}\Big\}\Big],\end{aligned}$$

$$\begin{aligned}E\left(SS1\right) = \sigma^2&\Big[\tfrac{1}{n^2}\Big\{\textstyle\sum_{i=1}^{m}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g\\ &+n^2a_1^2(r)\left\{r^g + (r+(m-1)k+(s-1)k_1)^g\right\}\\ &+2na_1(r)\left\{r^g - (r+(m-1)k+(s-1)k_1)^g\right\}\Big\}\\ &-\tfrac{2}{nN}\Big\{\textstyle\sum_{i=1}^{m}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g\\ &+na_1(r)\left\{r^g - (r+(m-1)k+(s-1)k_1)^g\right\}\Big\}\\ &+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g\Big].\end{aligned}$$

(B.4)
$$\begin{aligned}E\left(SS1\right) = \sigma^2&\Big\{a_1(r)\left(a_1(r) + 2\left(\tfrac{1}{n} - \tfrac{1}{N}\right)\right)r^g\\ &+\tfrac{1}{n}\left(\tfrac{1}{n} - \tfrac{2}{N}\right)\textstyle\sum_{i=1}^{m}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g\\ &+a_1(r)\left(a_1(r) - 2\left(\tfrac{1}{n} - \tfrac{1}{N}\right)\right)\{r+(m-1)k+(s-1)k_1\}^g\\ &+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g\Big\}.\end{aligned}$$

Similarly,

$$\begin{aligned}E\left(SS2\right) = &\Big[\tfrac{1}{n^2}\Big\{\textstyle\sum_{i=1}^{m-u}\sum_{j=1}^{s}E(e^2_{r+(i-1)k+(j-1)k_1})\\ &+\textstyle\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s}E(e^2_{r+(i-1)k+(j-1)k_1})\\ &+E(e^2_{r+(i-1)k+(s-1)k_1-N})\Big) + n^2a_2^2(r)\Big\{E(e_r^2)\\ &+E(e^2_{r+(m-1)k+(s-1)k_1-N})\Big\} + 2na_2(r)\Big\{(E(e_r^2)\\ &-E(e^2_{r+(m-1)k+(s-1)k_1-N})\Big\}\Big\}\\ &-\tfrac{2}{nN}\Big\{\textstyle\sum_{i=1}^{m-u}\sum_{j=1}^{s}E(e^2_{r+(i-1)k+(j-1)k_1})\\ &+\textstyle\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s}E(e^2_{r+(i-1)k+(j-1)k_1})\\ &+E(e^2_{r+(i-1)k+(s-1)k_1-N})\Big)\\ &+na_2(r)\Big\{(E(e_r^2) - E(e^2_{r+(m-1)k+(s-1)k_1-N})\Big\}\Big\}\\ &+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}E(e_t^2)\Big],\end{aligned}$$

$$
\begin{aligned}
E\left(SS2\right) = &\Big[\tfrac{1}{n^2}\Big\{\textstyle\sum_{i=1}^{m-u}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\textstyle\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\left(r+(i-1)k+(s-1)k_1-N\right)^g\Big)+n^2a_2^2(r)\Big\{r^g \\
&+\left(r+(m-1)k+(s-1)k_1-N\right)^g\Big\} \\
&+2na_2(r)\left\{r^g-\left(r+(m-1)k+(s-1)k_1-N\right)^g\right\}\Big\} \\
&-\tfrac{2}{nN}\Big\{\textstyle\sum_{i=1}^{m-u}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g. \\
&+\textstyle\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\left(r+(i-1)k+(s-1)k_1-N\right)^g\Big) \\
&+na_2(r)\left\{r^g-\left(r+(m-1)k+(s-1)k_1-N\right)^g\right\}\Big\} \\
&+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g\Big],
\end{aligned}
$$

$$
\begin{aligned}
E\left(SS2\right) = &\Big[a_2(r)\left(a_2(r)+2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\right)r^g \\
&+\tfrac{1}{n}\left(\tfrac{1}{n}-\tfrac{2}{N}\right)\tfrac{1}{n^2}\Big\{\textstyle\sum_{i=1}^{m-u}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\textstyle\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\left(r+(i-1)k+(s-1)k_1-N\right)^g\Big)\Big\} \\
&+a_2(r)\left(a_2(r)-2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\right)\Big(r+(m-1)k+(s-1)k_1 \\
&-N\Big)^g+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g\Big].
\end{aligned} \tag{B.5}
$$

Equations (B.4) and (B.5) can be written as:

$$
\begin{aligned}
E\left(SS1\right) = \sigma^2&\Big\{\delta_1^+(r)r^g+\theta\textstyle\sum_{i=1}^{m}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g \\
&+\delta_1^-(r)\{r+(m-1)k+(s-1)k_1\}^g+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(SS2\right) = \sigma^2&\Big[\delta_2^+(r)r^g+\theta\tfrac{1}{n^2}\Big\{\textstyle\sum_{i=1}^{m-u}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\textstyle\sum_{i=m-u+1}^{m}\Big(\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g. \\
&+\left(r+(i-1)k+(s-1)k_1-N\right)^g\Big\} \\
&+\delta_2^-(r)\left(r+(m-1)k+(s-1)k_1-N\right)^g+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g\Big],
\end{aligned}
$$

where $\delta_l^+(r)=a_l(r)\{a_l(r)+2\left(\frac{1}{n}-\frac{1}{N}\right)\}$, $\delta_l^-(r)=a_l(r)\{a_l(r)-2\left(\frac{1}{n}-\frac{1}{N}\right)\}$ and $\theta=\frac{1}{n}\left(\frac{1}{n}-\frac{2}{N}\right)$, such that $l=1,2$. Also

$$
E\left(SS1\right)=\sigma^2\left\{\chi_1\left(u,r\right)+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g\right\} \tag{B.6}
$$

and

$$
E\left(SS2\right)=\sigma^2\left\{\chi_2\left(u,r\right)+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g\right\}, \tag{B.7}
$$

where

$$\chi_1(u,r) = \delta_1^+(r)r^g + \theta\sum_{i=1}^{m}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g + \delta_1^-(r)\{r+(m-1)k+(s-1)k_1\}^g$$

,
and

$$\chi_2(u,r) = \delta_2^+(r)r^g + \theta\frac{1}{n^2}\Bigg\{\sum_{i=1}^{m-u}\sum_{j=1}^{s}(r+(i-1)k+(j-1)k_1)^g + \sum_{i=m-u+1}^{m}\Bigg(\sum_{j=1}^{s-1}(r+(i-1)k+(j-1)k_1)^g + (r+(i-1)k+(s-1)k_1-N)^g\Bigg)\Bigg\} + \delta_2^-(r)\left(r+(m-1)k+(s-1)k_1-N\right)^g.$$

Substituting the values of $E(SS1)$ and $E(SS2)$ in (B.3), we have

(B.8)
$$E\left\{V(\bar{y}_{MSS}^{(r)})\right\} = \frac{\sigma^2}{k_1}\Bigg\{\sum_{r=1}^{k_1-(m-1)k}\chi_1(u,r) + \sum_{u=1}^{m-1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\chi_2(u,r) + k_1\sum_{t=1}^{N}t^g\Big/N^2\Bigg\}.$$

Case (ii) when $w > 1$
We can write

(B.9)
$$SSl = \left[\bar{y}_{MSS}^{(r)} - \bar{Y}\right]^2 = \left[\{\bar{y}_{MSS} + a_l(r)(Y_{r1} - Y_{rn})\} - \bar{Y}\right]^2,$$

where $l = 2$ if $r \le k_1 - (w-1)k + r_m$, $l = 3$ if $k_1 - (w-x)k < r \le k_1 - (w-x)k + r_m$ such that $x = 2, ..., (m-1)$
and $l = 4$ if $k_1-(w-x)k+r_m < r \le k_1-(w-x)k+k$ such that $x = 1, 2, ..., m-1$. Furthermore, when $r \le k_1 - (w-1)k + r_m$, we realize whether $1 \le r \le k_1 - (m-u-1)k$ such that $u = w-1$, $k_1 - (m-u)k < r \le k_1 - (m-u-1)k$ such that $u = w, w+1, ..., (m-w)$ or $k_1 - (m-u)k < r \le k_1 - (m-u)k + r_m$ such that $u = (m-w+1)$. However, for each of these subgroups $E(SS2)$ will be used. Thus, the average variance of the corrected sample mean can be expressed as

(B.10)
$$E\left[V\left(\bar{y}_{MSS}^{(r)}\right)\right] = \frac{1}{N}\Bigg[\sum_{u=w-1}^{w-1}\sum_{r=1}^{k_1-(m-u-1)k}E[SS2] + \sum_{u=w}^{n-w}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}E[SS2] + \sum_{u=m-w+1}^{m-w+1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u)k+r_m}E[SS2] + \sum_{x=2}^{w-1}\sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_m}E[SS3] + \sum_{x=1}^{w-1}\sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k}E[SS4]\Bigg].$$

The $E(SS2)$ is already obtained in case of $w = 1$, i.e.

(B.11)
$$E(SS2) = \chi_2(u,r) + \frac{1}{N^2}\sum_{t=1}^{N}t^g.$$

Now consider

(B.12)
$$E(SS3) = E\left[\{\bar{y}_{MSS} + a_3(r)(Y_{r1} - Y_{rn})\} - \overline{Y}\right]^2.$$

Under the super population model, we have

$$\begin{aligned}\bar{y}_{MSS} = \alpha + \beta\Big\{r + \tfrac{1}{2}\{(s-1)k_1 + (m-1)k\} - (m-w-1+2x)\tfrac{k_1}{m}\Big\}\\
+\tfrac{1}{n}\Big[\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1}\\
+\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1} e_{r+(i-1)k+(j-1)k_1} + e_{r+(i-1)k+(s-1)k_1-N}\Big\}\\
+\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2} e_{r+(i-1)k+(j-1)k_1}\\
+\textstyle\sum_{j=s-1}^{s} e_{r+(i-1)k+(s-1)k_1-N}\Big\}\Big],\end{aligned}$$

$\bar{Y} = \alpha + \beta\frac{N+1}{2} + \frac{1}{N}\sum_{t=1}^{N} e_t$, $Y_{r1} = \alpha + \beta r + e_r$ and $Y_{rn} = \alpha + \beta\{r + (m-1)k + (s-1)k_1\} + e_{r+(m-1)k+(s-1)k_1}$.
Substituting these expressions in (B.11), we have

$$\begin{aligned}E(SS3) = E\Big[\tfrac{1}{n}\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s} e_{r+(i-1)k+(j-1)k_1}\\
+\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1} e_{r+(i-1)k+(j-1)k_1} + e_{r+(i-1)k+(s-1)k_1-N}\Big\}\\
+\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2} e_{r+(i-1)k+(j-1)k_1} + \sum_{j=s-1}^{s} e_{r+(i-1)k+(j-1)k_1-N}\Big\}\\
+ na_2(r)(e_r - e_{r+(m-1)k+(s-1)k_1-N})\Big\} - \tfrac{1}{N}\textstyle\sum_{t=1}^{N} e_t\Big]^2.\end{aligned}$$

Applying the assumption of super population model, we have

$$\begin{aligned}E(SS3) = \Big[\tfrac{1}{n^2}\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1})\\
+\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1} E(e^2_{r+(i-1)k+(j-1)k_1})\\
+E(e^2_{r+(i-1)k+(s-1)k_1-N})\Big\}\\
+\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2} E(e^2_{r+(i-1)k+(j-1)k_1})\\
+\textstyle\sum_{j=s-1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1-N})\Big\}\\
+n^2a_2^2(r)\Big\{E(e_r^2) + E(e^2_{r+(m-1)k+(s-1)k_1-N})\Big\}\\
+2na_2(r)\Big\{E(e_r^2) - E(e^2_{r+(m-1)k+(s-1)k_1-N})\Big\}\Big\}\\
-2\tfrac{1}{nN}\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1})\\
+\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1} E(e^2_{r+(i-1)k+(j-1)k_1})\\
+E(e^2_{r+(i-1)k+(s-1)k_1-N})\\
+\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2} E(e^2_{r+(i-1)k+(j-1)k_1})\\
+\textstyle\sum_{j=s-1}^{s} E(e^2_{r+(i-1)k+(j-1)k_1-N})\Big\}\\
+na_2(r)\Big\{E(e_r^2) - E(e^2_{r+(m-1)k+(s-1)k_1-N})\Big\}\Big\}\\
+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N} E(e_t^2)\Big].\end{aligned}$$

$$
\begin{aligned}
E\left(SS3\right) = \Big[&\tfrac{1}{n^2}\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\left(r+(i-1)k+(j-1)k_1-N\right)^g\Big\} \\
&+\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\textstyle\sum_{j=s-1}^{s}\left(r+(i-1)k+(j-1)k_1-N\right)^g\Big\} \\
&+n^2a_2^2(r)\left\{r^g+\left(r+(i-1)k+(j-1)k_1-N\right)^g\right\} \\
&+2na_2(r)\left\{r^g-\left(r+(i-1)k+(j-1)k_1-N\right)^g\right\}\Big\} \\
&-2\tfrac{1}{nN}\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\textstyle\sum_{i=w-x+1}^{m-x+1}\Big\{\sum_{j=1}^{s-1}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\left(r+(i-1)k+(j-1)k_1-N\right)^g\Big\} \\
&+\textstyle\sum_{i=m-x+2}^{m}\Big\{\sum_{j=1}^{s-2}\left(r+(i-1)k+(j-1)k_1\right)^g \\
&+\textstyle\sum_{j=s-1}^{s}\left(r+(i-1)k+(j-1)k_1-N\right)^g\Big\} \\
&+na_2(r)\left\{r^g-\left(r+(i-1)k+(j-1)k_1-N\right)^g\right\}\Big\} \\
&+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g\Big],
\end{aligned}
$$

$$
\begin{aligned}
E\left(SS3\right) = {}& a_3(r)\left(a_3(r)+2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\right)r^g \\
&+\textstyle\sum_{x=2}^{w-1}\sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_1}\Big\{\tfrac{1}{n}\left(\tfrac{1}{n}-\tfrac{2}{N}\right)\Big\{\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r \\
&+(i-1)k+(j-1)k_1\}^g \\
&+\textstyle\sum_{i=w-x+1}^{m-x+1}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g \\
&+\{r+(i-1)k+(s-1)k_1-N\}^g\Big) \\
&+\textstyle\sum_{i=m-x+2}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g \\
&+\textstyle\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\} \\
&+a_3(r)\left(a_3(r)-2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\right)\{r+(m-1)k+(s-1)k_1-N\}^g \\
&+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g,
\end{aligned}
$$

$$
\begin{aligned}
E\left(SS3\right) = {}& \delta_3^+(r)r^g+\theta\Big\{\textstyle\sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g \\
&+\textstyle\sum_{i=w-x+1}^{m-x+1}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g \\
&+\{r+(i-1)k+(s-1)k_1-N\}^g\Big) \\
&+\textstyle\sum_{i=m-x+2}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g \\
&+\textstyle\sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\} \\
&+\delta_3^-(r)\{r+(m-1)k+(s-1)k_1-N\}^g+\tfrac{1}{N^2}\textstyle\sum_{t=1}^{N}t^g,
\end{aligned}
$$

where

$$\delta_3^+(r)=a_3(r)\{a_3(r)+2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\} \text{ and } \delta_3^-(r)=a_3(r)\{a_3(r)-2\left(\tfrac{1}{n}-\tfrac{1}{N}\right)\}.$$

Also

(B.13) $E\,(SS3) = \chi_3\,(x,r) + \frac{1}{N^2}\sum_{t=1}^{N} t^g,$

where

$$\chi_3\,(x,r) = \delta_3^{+}(r) r^g + \theta \Big\{ \sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g \\ + \sum_{i=w-x+1}^{m-x+1}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g \\ +\{r+(i-1)k+(s-1)k_1-N\}^g\Big) \\ + \sum_{i=m-x+2}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g \\ + \sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\} \\ +\delta_3^{-}(r)\{r+(m-1)k+(s-1)k_1-N\}^g.$$

Similarly,

(B.14) $E\,(SS4) = \chi_4\,(x,r) + \frac{1}{N^2}\sum_{t=1}^{N} t^g,$

where

$$\chi_4\,(x,r) = \delta_4^{+}(r) r^g + \theta \Big\{ \sum_{i=1}^{w-x}\sum_{j=1}^{s}\{r+(i-1)k+(j-1)k_1\}^g \\ + \sum_{i=w-x+1}^{m-x}\Big(\sum_{j=1}^{s-1}\{r+(i-1)k+(j-1)k_1\}^g \\ +\{r+(i-1)k+(s-1)k_1-N\}^g\Big) \\ + \sum_{i=m-x+1}^{m}\Big(\sum_{j=1}^{s-2}\{r+(m-1)k+(j-1)k_1\}^g \\ + \sum_{j=s-1}^{s}\{r+(i-1)k+(j-1)k_1-N\}^g\Big)\Big\} \\ +\delta_4^{-}(r)\{r+(m-1)k+(s-1)k_1-N\}^g.$$

Putting $E\,(SSl)$ for $l = 2,\ 3,\ 4$ in (B.10), we have

$$E\left\{V\left(\bar{y}_{MSS}^{(r)}\right)\right\} = \frac{\sigma^2}{k_1}\Big[\sum_{u=w-1}^{w-1}\sum_{r=1}^{k_1-(m-u-1)k}\chi_2\,(u,r) \\ + \sum_{u=w}^{m-w}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\chi_2\,(u,r) \\ + \sum_{u=m-w+1}^{m-w+1}\sum_{r=k_1-(m-u)k+1}^{k_1-(m-u-1)k}\chi_2\,(u,r) \\ + \sum_{x=2}^{w-1}\sum_{r=k_1-(w-x)k+1}^{k_1-(w-x)k+r_m}\chi_3\,(x,r) \\ + \sum_{x=1}^{w-1}\sum_{r=k_1-(w-x)k+r_m+1}^{k_1-(w-x)k+k}\chi_4\,(x,r) \\ + k_1\sum_{t=1}^{N} t^g \Big/ N^2\Big].$$

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Generalized class of estimators for population median using auxiliary information

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Abstract

This article suggests a generalized class of estimators for population median of the study variable in simple random sampling using information on an auxiliary variable. Asymptotic expressions of bias and mean square error of the proposed class of estimators have been obtained. Asymptotic optimum estimator has been investigated along with its approximate mean square error. It has been shown that proposed generalized class of estimators are more efficient than estimators considered by [26] , [5],[6], [22], [1] , [19] and other estimators . In addition theoretical findings are supported by an empirical study based on two populations to show the superiority of the constructed estimators over others.

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1. Introduction

In the sampling literature, Statisticians are often interested in dealing with variables that have highly skewed distributions such as consumptions and incomes. In such situations median is considered the more appropriate measure of location than mean. It has been well recognised that use of auxiliary information results in efficient estimators of population parameters. Initially, estimation of median without auxiliary variable analyzed, after that some authors including [6], [9], [24] and [7] used the auxiliary information in median estimation. [6], proposed the problem of estimating the population median M_y of study variable Y using the auxiliary variable X for the unites in the sample and its median M_x for the whole population. Some other important references in this context

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are [3], [11], [8], [15], [2], [4], [21, 20],[25] and [19].
Let Yi and Xi (i =1,2,....N) be the values of the population unites for the study variable Y and auxiliary variable X respectively .Further suppose that y_i and x_i (i=1,2.....n) be the values of the unites including in the sample say, s_n of size n drawn by simple random sampling without replacement scheme. [6] suggested a ratio estimator for population median M_y of the study variable Y, assuming population median of auxiliary variable X, M_x is known, given as

$$(1.1) \qquad \hat{M}_r = \hat{M}_y \frac{M_x}{\hat{M}_x}$$

where $\hat{M}_y$ (due to [5]) and are the sample estimators of M_y and M_x respectively. Suppose that $Y_{(1)}$,$Y_{(2)}$,.....$Y_{(n)}$ are the y values of sample unites in ascending order. Further, suppose t be an integer satisfying and p=t/n be the proportion of y values in the sample that are less than or equal to the median value M_y, an unknown population parameter. If $Q_y(t)$ denote the t-quantile of Y then $\hat{M}_y = Q_y(0.5)$. [6] defined a matrix of proportion (p_{ij}) is

	$Y \leq M_y$	$Y \leq M_y$	Total
$X \leq M_x$	p_{11}	p_{21}	$p_{.1}$
$X > M_x$	p_{12}	p_{21}	$p_{.2}$
Total	p_1	p_2	1

Following [14] and [10], the product estimator for population median M_y is defined as

$$(1.2) \qquad \hat{M}_p = \hat{M}_y \frac{\hat{M}_x}{M_x}$$

The usual difference estimator for population median M_y is given by

$$(1.3) \qquad \hat{M}_d = \hat{M}_y + d(M_x - \hat{M}_x)$$

Where d is a constant to be determined such that the mean square error of $\hat{M}_d$ is minimum.
[21] proposed the following modified product and ratio estimator for population median M_y , respectively, as

$$(1.4) \qquad \hat{M}_1 = \hat{M}_y \frac{a - \hat{M}_x}{a + M_x}$$

and

$$(1.5) \qquad \hat{M}_2 = \hat{M}_y \frac{a + M_x}{a - \hat{M}_x}$$

Where a is suitably chosen scalar.
[26] type estimator for median estimation is

$$(1.6) \qquad \hat{M}_3 = \hat{M}_y \frac{M_x}{\hat{M}_x}$$

[12, 13] and [30]-type estimator is given by

$$(1.7) \qquad \hat{M}_4 = \hat{M}_y \left[\frac{M_x}{M_x + \beta(\hat{M}_x - M_x)} \right]$$

[16]-type esatimator is given by

$$\hat{M}_5 = \hat{M}_y \left[2 - \left(\frac{M_x}{\hat{M}_x}\right)^{\nu}\right] \quad (1.8)$$

[29]- type esatimator is given by

$$\begin{cases} \hat{M}_6 = w\hat{M}_y + (1-w)\hat{M}_y \dfrac{\hat{M}_x}{M_x} \\ \hat{M}_7 = w\hat{M}_y + (1-w)\hat{M}_y \dfrac{M_x}{\hat{M}_x} \end{cases} \quad (1.9)$$

Where w is suitably chosen scalar .
All the estimator considered from (1.1) to (1.9) and conventional estimator $\hat{M}_y$ are members of the [27] and [28]-type class of estimators

$$G = \left\{ \hat{M}_y^{(G)} : \hat{M}_y^{(G)} = G\left(\hat{M}_y, \frac{\hat{M}_x}{M_x}\right) \right\} \quad (1.10)$$

Where the function G assumes a value in a bounded closed convex subset $Q \subset R_2$, which contains the point (M_y ,1) and is such that G$(M_y, 1) = 1$
Using first order Taylor's series expansion about the point$(\hat{M}_y, 1)$, we have

$$(\hat{M}_y^{(G)} = G(M_y, 1) + (\hat{M}_y - M_y)G_{10}(M_y, 1) + O(n^{-1})) \quad (1.11)$$

Where $U = \dfrac{\hat{M}_x}{M_x}$
and $G_{01}(M_y, 1) = \dfrac{\partial G(.)}{\partial U}\Big|_{(M_y,1)}$
Using condition we have

$$\hat{M}_y^{(G)} = M_y + (\hat{M}_y - M_y) + (U-1)G_{01}(M_y, 1) + O(n^{-1})$$

or

$$(\hat{M}_y^{(G)} - M_y) = (\hat{M}_y - M_y) + (U-1)G_{01}(M_y, 1) + O(n^{-1}) \quad (1.12)$$

Squaring and taking expectations both sides of (1.12),we get the MSE of $\hat{M}_y^{(G)}$ to the first order of approximation as

$$MSE(\hat{M}_y^{(G)}) = \left[V(\hat{M}_y) + \frac{V(\hat{M}_y)}{{M_x}^2} G_{01}^2(\hat{M}_y, 1) + 2\frac{Cov(\hat{M}_y, \hat{M}_x)}{M_x} G_{01}^2(\hat{M}_y, 1)\right] \quad (1.13)$$

Here as $N \to \infty$,$n \to \infty$ then n/N$\to$ f and we assumed that as $N \to \infty$ the distribution of (X, Y) approaches a continues distribution with marginal densities $f_x(x)$ and $f_y(y)$ of X and Y respectively. Super population model framework is necessary for treating the values of X and Y in a realization of N independent observation from a continuous distribution. It is also assumed that $f_x(M_x)$ and $f_y(M_y)$ are positive. Under these conditions, sample median $\hat{M}_y$ is consistent and asymptotically normal (due to [5]) with mean M_y and variance

$$V(\hat{M}_y) = \gamma M_y^2 C_y^2 \quad (1.14)$$

$$and \;\; V(\hat{M}_x) = \gamma M_x^2 C_x^2 \quad (1.15)$$

$$Cov(\hat{M}_y, \hat{M}_x) = \gamma \rho_c M_y M_x C_y C_x \quad (1.16)$$

Where $\gamma = (1-f)/4n, f = n/N, C_y = [M_y f_y(M_y)]^{-1}, C_x = [M_x f_x(M_x)]^{-1}$ and $\rho_c = (4p_{11} - 1))$ with $p_{11} = P(M_x, M_y)$goes from -1 to $+1$ as p_{11}increase from 0 to 0.5.

Substituting these values we get the MSE of $\hat{M}_y^{(G)}$ to the first degree of approximation as

(1.17) $MSE(\hat{M}_y^{(G)}) = \gamma\left[M_y^2C_y^2 + C_x^2\{G_{01}(M_y,1)\}^2 + 2\rho_c C_x C_y M_y G_{01}(M_y,1)\right]$

The MSE is minimum when

(1.18) $G_{01} = (M_y,1) = -k_c M_y$

Where $k_c = \rho_c\left(\dfrac{C_y}{C_x}\right)$

Thus the minimum MSE of $\hat{M}_y^{(G)}$ is given by

(1.19) $MSE_{min}(\hat{M}_y^{(G)}) = \gamma C_y^2 M_y^2(1-\rho_c^2) = MSE_{min}(\hat{M}_d)$

Which equal to the minimum MSE of the estimator $\hat{M}_d$ defined at (1.3).
It is to be mentioned that minimum MSEs of the estimators $\hat{M}_r$,$\hat{M}_p$ and $\hat{M}_i(i=1,2,...,7)$ are equal to MSE expression given in equation (1.19). It is obvious from (1.19) that the estimators of the form $\hat{M}_y^{(G)}$ are asymptotically no more efficient than the difference estimator at its optimum value or the regression type estimator given as

(1.20) $\hat{M}_{lr} = \hat{M}_y + \hat{d}(M_x - \hat{M}_x)$

where $\hat{d} = \dfrac{\hat{f}_x\hat{M}_x}{\hat{f}_y\hat{M}_y}(4\hat{p}_{11}-1)$

[19] Suggested following Classes of estimator

(1.21) $\hat{M}_d^{(1)} = d_1\hat{M}_y + (1-d_1)(M_x - \hat{M}_x)$

(1.22) $\hat{M}_d^{(2)} = d_1\hat{M}_y + d_2(M_x - \hat{M}_x)$

(1.23) $\hat{M}_d^{(3)} = d_1\hat{M}_y + d_2\hat{M}_x + (1-d_1-d_2)M_x$

(1.24) $\hat{M}_d^{(4)} = \left[d_1\hat{M}_y + d_2(M_x - \hat{M}_x)\right]\left(\dfrac{(\phi M_x+\delta)}{(\phi\hat{M}_x+\delta)}\right)^{\beta}$

where d_1 and d_2 are suitable constants to be determined such that MSEs of the estimators considered in (1.21) to (1.24) are minimum, ϕ and δ are either real numbers or the functions of the known parameters of auxiliary variable X.
Biases and minimum MSEs of the estimators considered in (1.21) to (1.24) are given as

(1.25) $B\left(\hat{M}_d^{(1)}\right) = (d_1-1)M_y$

(1.26) $B\left(\hat{M}_d^{(2)}\right) = (d_1-1)M_y$

(1.27) $B\left(\hat{M}_d^{(3)}\right) = (d_1-1)(1-R)M_y$

(1.28) $B\left(\hat{M}_d^{(4)}\right) = M_y\left[d_1^2\{1+\gamma\delta C_x^2(\delta-k_c)\} + d_2R\gamma\delta C_x^2 - 1\right]$

(1.29) $MSE_{min}\left(\hat{M}_d^{(1)}\right) = M_y^2\left[1 + R^2\gamma C_x^2 - \dfrac{\{1+R\gamma C_x^2(R+k_c)\}^2}{\{1+\gamma(C_y^2+RC_x^2(R+2k_c))\}}\right]$

(1.30) $MSE_{min}\left(\hat{M}_d^{(2)}\right) = \dfrac{M_y^2\gamma C_y^2\left(1-\rho_c^2\right)}{\left[1+\gamma C_y^2\left(1-\rho_c^2\right)\right]}$

(1.31) $MSE_{min}\left(\hat{M}_d^{(3)}\right) = \dfrac{M_y^2\gamma C_y^2\left(1-\rho_c^2\right)(1-R)^2}{\left[(1-R)^2+\gamma C_y^2\left(1-\rho_c^2\right)\right]}$

(1.32) $MSE_{min}\left(\hat{M}_d^{(4)}\right) = \dfrac{\left(1-\delta^2\gamma C_x^2\right)M_y^2\gamma C_y^2\left(1-\rho_c^2\right)}{\left[\left(1-\delta^2\gamma C_x^2\right)+\gamma C_y^2\left(1-\rho_c^2\right)\right]}$

2. The Suggested Generalised Class of Estimators

We propose a generalized family of estimators for population median of the study variable Y, as

$$(2.1) \qquad t_m = \left\{ w_1 \hat{M}_y \left(\frac{M_x}{\hat{M}_x}\right)^{\alpha} exp\left(\frac{\eta(M_x - \hat{M}_x)}{\eta(M_x + \hat{M}_x) + 2\lambda}\right)\right\} + w_2\hat{M}_x + (1 - w_1 - w_2)M_x$$

where w_1 and w_2 are suitable constants to be determined such that MSE of t_m is minimum, η and λ are either real numbers or the functions of the known parameters of auxiliary variables such as coefficient of variation C_x, skewness $\beta_{1(x)}$, kurtosis $\beta_{2(x)}$ and correlation coefficientρ_c (see [17]).
It is to be mentioned that
(i) For (w_1, w_2) =(1,0), the class of estimator t_m reduces to the class of estimator as

$$(2.2) \qquad t_{mp} = \left\{ \hat{M}_y \left(\frac{M_x}{\hat{M}_x}\right)^{\alpha} exp\left(\frac{\eta(M_x - \hat{M}_x)}{\eta(M_x + \hat{M}_x) + 2\lambda}\right)\right\}$$

(ii) For $(w_1, w_2) = (w_1, 0)$, the class of estimator t_m reduces to the class of estimator as

$$(2.3) \qquad t_{mq} = \left\{ w_1\hat{M}_y \left(\frac{M_x}{\hat{M}_x}\right)^{\alpha} exp\left(\frac{\eta(M_x - \hat{M}_x)}{\eta(M_x + \hat{M}_x) + 2\lambda}\right)\right\}$$

A set of new estimators generated from (2.1) using suitable values of w_1, w_2, α, η and λ are listed in Table 2.1.

Table 2.1:Set of estimators generated from the class of estimators t_m

Subset of proposed estimator	w_1	w_2	α	η	λ
$t_{m1} = \hat{M}_y$ [5]	1	0	0	0	1
$t_{m2} = \hat{M}_y\left(\frac{M_x}{\hat{M}_x}\right) = \hat{M}_r$ [6]	1	0	1	0	1
$t_{m3} = \hat{M}_y\left(\frac{M_x}{\hat{M}_x}\right)^{\alpha} = \hat{M}_3$ [26]	1	0	α	0	1
$t_{m4} = \hat{M}_y\left(\frac{\hat{M}_x}{M_x}\right) = M_p$ [10]	1	0	-1	0	1
$t_{m5} = w_1\hat{M}_y\left(\frac{M_x}{\hat{M}_x}\right)$ [1]	1	0	1	0	1
$t_{m6} = w_1\hat{M}_y\left(\frac{\hat{M}_x}{M_x}\right)$	w_1	0	-1	0	1
$t_{m7} = w_1\hat{M}_y$ [1]	w_1	0	0	0	1
*$t_{m8} = w_1\hat{M}_y + w_2\hat{M}_x + (1 - w_1 - w_2)M_x = \hat{M}_d^3$	w_1	w_2	0	0	1

*Estimator proposed by [19] given in equation (1.23). Another set of estimators generated from class of estimator t_{mq} given in (2.3) using suitable values of η and λ are summarized in table 2.2

Table 2.2:Set of estimators generated from the estimator t_{mq}

Subset of proposed estimator	α	η	λ
$t_{mq}^{(1)}=\left\{w_1\hat{M}_y\left(\frac{M_x}{\hat{M}_x}\right)exp\left(\frac{(M_x-\hat{M}_x)}{(M_x+\hat{M}_x)+2}\right)\right\}$	1	1	1
$t_{mq}^{(2)}=\left\{w_1\hat{M}_y\left(\frac{M_x}{\hat{M}_x}\right)exp\left(\frac{(M_x-\hat{M}_x)}{(M_x+\hat{M}_x)+2\rho_c}\right)\right\}$	1	1	ρ_c
$t_{mq}^{(3)}=\left\{w_1\hat{M}_y\left(\frac{M_x}{\hat{M}_x}\right)exp\left(\frac{(M_x-\hat{M}_x)}{(M_x+\hat{M}_x)+2M_x}\right)\right\}$	1	1	M_x
$t_{mq}^{(4)}=\left\{w_1\hat{M}_y\left(\frac{M_x}{\hat{M}_x}\right)exp\left(\frac{(M_x-\hat{M}_x)}{(M_x+\hat{M}_x)}\right)\right\}$	1	1	0
$t_{mq}^{(5)}=\left\{w_1\hat{M}_y\left(\frac{\hat{M}_x}{M_x}\right)exp\left(\frac{(M_x-\hat{M}_x)}{(M_x+\hat{M}_x)}\right)\right\}$	-1	1	0
$t_{mq}^{(6)}=\left\{w_1\hat{M}_y\left(\frac{M_x}{\hat{M}_x}\right)exp\left(\frac{M_x(M_x-\hat{M}_x)}{M_x(M_x+\hat{M}_x)+2\rho_c}\right)\right\}$	1	M_x	ρ_c
$t_{mq}^{(7)}=\left\{w_1\hat{M}_y exp\left(\frac{M_x(M_x-\hat{M}_x)}{M_x(M_x+\hat{M}_x)+2\rho_c}\right)\right\}$	0	M_x	ρ_c
$t_{mq}^{(8)}=\left\{w_1\hat{M}_y\left(\frac{M_x}{\hat{M}_x}\right)exp\left(\frac{\rho_c(M_x-\hat{M}_x)}{\rho_c(M_x+\hat{M}_x)+2M_x}\right)\right\}$	1	ρ_c	M_x
$t_{mq}^{(9)}=\left\{w_1\hat{M}_y\left(\frac{\hat{M}_x}{M_x}\right)exp\left(\frac{\rho_c(M_x-\hat{M}_x)}{\rho_c(M_x+\hat{M}_x)+2M_x}\right)\right\}$	-1	ρ_c	M_x

Expressing (2.1) in terms of e's, we have

(2.4) $t_m = w_1 M_y(1+e_0)(1+e_1)^{-\alpha} exp\{-ke_1(1+ke_1)^{-1}\}+w_2M_x(1+e_1)+(1-w_1-w_2)M_x$

where $k=\dfrac{\eta M_x}{2(\eta M_x+\lambda)}$

Up to the first order of approximation we have,

(2.5) $$(t_m - M_y) = \left[(w_1-1)b + w_2M_y\{e_0 - ae_1 + de_1^2 - ae_0e_1\} + w_2M_xe_1\right]$$

where $a=(\alpha+k)$, $b=(M_y-M_x)$ and $d=\left\{\dfrac{3}{2}k^2+\alpha k+\dfrac{\alpha(\alpha+1)}{2}\right\}$

Squaring both sides of equation (2.5) and neglecting terms of e's having power greater than two, we have

(2.6)
$$(t_m-\bar{Y})^2 = \left[(1-2w_1)b^2 + w_1^2\{b^2+m_y^2(e_0^2+a^2e_1^2-2ae_0e_1)\} + w_2^2M_x^2e_1^2 + 2w_1w_2M_yM_x(e_0e_1-ae_1^2)\right]$$

Taking expectations both sides, we get the MSE of the estimator t_m to the first order of approximation as

(2.7) $$MSE(t_m) = \left[(1-2w_1)b^2 + w_1^2A + w_2^2B + 2w_1w_2C\right]$$

where,

$A = b^2 + M_y^2\gamma(C_y^2 + a^2C_x^2 - 2a\rho_cC_yC_x)$

$B = M_x^2\gamma C_x^2$

$C = M_yM_x\gamma(\rho_cC_y - aC_x)C_x$

The optimum values of w_1 and w_2 are obtained by minimizing (2.7) and is given by

$$w_1^* = \frac{b^2B}{(AB-C^2)} \quad and \quad w_2^* = \frac{-b^2C}{(AB-C^2)} \tag{2.8}$$

Substituting the optimal values of w_1 and w_2 in equation (2.7) we obtain the minimum MSE of the estimator t_m as

$$MSE_{min}(t_m) = b^2\left[1 - \frac{b^2B}{(AB-C^2)}\right] \tag{2.9}$$

Putting the values of A, B, C and b and simplifying, we get the minimum MSE of estimator t_m as

$$MSE_{min}(t_m) = \left[\frac{M_y^2(1-R)^2\gamma C_y^2(1-\rho_c^2)}{(1-R)^2\gamma C_y^2(1-\rho_c^2)}\right] \tag{2.10}$$

where $R = \dfrac{M_x}{M_y}$

MSE expression given in (2.10) is same as the minimum MSE of Estimator $\hat{M}_d^3$ given in (1.31)

Similarly, the minimum MSE of the class of estimators t_{mq} is given by

$$MSE_{min}(t_{mq}) = M_y^2\left[\frac{(\gamma C_y^2 + a^2\gamma C_x^2 - 2a\gamma\rho_c C_y C_x)}{(1+\gamma C_y^2 + a^2\gamma C_x^2 - 2a\gamma\rho_c C_y C_x)}\right] \tag{2.11}$$

3. Efficiency Comparisons

From equations (1.19) and (2.10) we have

(3.1)

$$\left\{MSE_{min}\left(\hat{M}_y^{(G)}\right) = MSE_{min}\left(\hat{M}_d\right)\right\} - MSE_{min}\left(\hat{t}_m\right) = \frac{(1-R)^2 MSE_{min}\left(\hat{M}_d\right)}{(1-R)^2 + \dfrac{MSE_{min}\left(\hat{M}_d\right)}{M_y^2}} > 0$$

From equations (1.19) and (2.11) we have

$$\left\{MSE_{min}\left(\hat{M}_y^{(G)}\right) = MSE_{min}\left(\hat{M}_d\right)\right\} - MSE_{min}\left(t_m\right) > 0$$

$$\gamma C_y^2 M_y^2(1-\rho_c^2) - M_y^2\left[\frac{(\gamma C_y^2 + a^2\gamma C_x^2 - 2a\gamma\rho_c C_y C_x)}{(1+1+\gamma C_y^2 + a^2\gamma C_x^2 - 2a\gamma\rho_c C_y C_x)}\right] > 0$$

$$\gamma C_y^2 M_y^2(1-\rho_c^2)(1+1+\gamma C_y^2 + a^2\gamma C_x^2 - 2a\gamma\rho_c C_y C_x) > \gamma C_y^2 + a^2\gamma C_x^2 - 2a\gamma\rho_c C_y C_x) \tag{3.2}$$

From equations (1.30) and (2.10)

$$MSE_{min}\left(t_m\right) - MSE_{min}\left(\hat{M}_d\right)$$

(3.3)

$$= \frac{M_y^2 R(R-2) MSE_{min}\left(\hat{M}_d\right)}{M_y^2 + MSE_{min}\left(\hat{M}_d\right)\left\{M_y^2(1-R)^2 + MSE_{min}\left(\hat{M}_d\right)\right\}} < 0, \qquad When \;\; 0 < R < 2$$

Since, $MSE_{min}\left(\hat{M}_d^{(2)}\right) - MSE_{min}\left(\hat{M}_d^{(4)}\right) > 0$

$$\frac{\delta^2\gamma C_x^2 M_y^2\left\{MSE_{min}\left(\hat{M}_d\right)\right\}^2}{M_y^2 + MSE_{min}\left(\hat{M}_d\right)\left\{M_y^2(1-\delta^2\gamma C_x^2) + MSE_{min}\left(\hat{M}_d\right)\right\}} > 0 \tag{3.4}$$

and from (3.3) we have, $MSE_{min}\left(t_m\right) - MSE_{min}\left(\hat{M}_d^{(2)}\right) < 0$

$$\text{(3.5)} \qquad Therefore, MSE_{min}\left(t_m\right) - MSE_{min}\left(\hat{M}_d^{(4)}\right) < 0, When \;\; 0 < R < 2$$

It follows from (3.1), (3.2),(3.3), (3.4) and (3.5) that the proposed class of estimators t_m is better than the Conventional difference estimator $\hat{M}_d$, the class of estimators ${M_y}^{(G)}$ and estimator belonging to the class of estimators ${M_y}^{(G)}$ i.e. usual unbiased estimator $\hat{M}_y$,due to [5], usual ratio-type estimator $\hat{M}_y$ due to [6], product estimator $\hat{M}_p$ and $\hat{M}_i$ (i=3,4...7) at their optimum conditions. Further it is shown that the proposed class of estimators t_m is better than the estimators $M_d^{(2)}, M_d^{(4)}$ and $M_d^{(1)}$ considered by [19].

Remark 3.1: Estimator Based on optimum values

Putting the optimum values of w_1^* and w_2^* in the equation (2.1) we get the optimum estimator as:

$$\text{(3.6)} \qquad t'_m = \left\{ w_1^* \hat{M}_y \left(\frac{M_x}{\hat{M}_x}\right)^{\alpha} exp\left(\frac{\eta(M_x - \hat{M}_x)}{\eta(M_x + \hat{M}_x) + 2\lambda}\right)\right\} + w_2^* \hat{M}_x + (1 - w_1^* - w_2^*) M_x$$

If the experimenter is not able to specify the value precisely, then it may be desirable to estimate the optimum values from the samples, therefore the values of w_1^* and w_2^* are given as: $w_1^* = \dfrac{\hat{b}^2 \hat{B}}{(\hat{A}\hat{B} - \hat{C}^2)}$ $\;and\;$ $w_2^* = \dfrac{\hat{b}^2 \hat{C}}{(\hat{A}\hat{B} - \hat{C}^2)}$

where, $\quad A = \hat{b}^2 + \hat{M_y}^2 \gamma(\hat{C_y}^2 + \hat{a}^2 \hat{C_x}^2 - 2a\hat{\rho}_c \hat{C}_y \hat{C}_x)$

$B = \hat{M_x}^2 \gamma \hat{C_x}^2, \hat{\rho}_c = 4(4\hat{p_{11}} - 1)$

$C = \hat{M}_y \hat{M}_x \gamma(\hat{\rho}_c \hat{C}_y - a\hat{C}_x)\hat{C}_x, \hat{C}_x = \left\{\hat{M}_x \hat{f}_x\left(\hat{M}_x\right)\right\}^{-1}, \hat{C}_y = \left\{\hat{M}_y \hat{f}_y\left(\hat{M}_y\right)\right\}^{-1}$

$\hat{a} = (\alpha + \hat{k})$, $\hat{b} = (\hat{M}_y - \hat{M}_x)$ and $\hat{k} = \dfrac{\eta \hat{M}_x}{2(\eta \hat{M}_x + \lambda)}$

Here, we have assumed that the population median of auxiliary variable x is known, therefore $\hat{M}_x$ can also be remain as M_x .

Expressing (3.6) in terms of e's, we have

$t'_m = w_1^* M_y(1+e_0)(1+e_1)^{-\alpha} exp\{-\hat{k}e_1(1+\hat{k}e_1)^{-1}\} + w_2^* M_x(1+e_1) + (1 - w_1^* - w_2^*) M_x$

Proceeding as above, we get the minimum MSE of the estimator t'_m given as:

$$\text{(3.7)} \qquad MSE_{min}(t'_m) = \left[\frac{\hat{M_y}^2 (1-\hat{R})^2 \hat{\gamma} \hat{C_y}^2 (1 - \hat{\rho_c}^2)}{(1-\hat{R})^2 \hat{\gamma} \hat{C_y}^2 (1 - \hat{\rho_c}^2)}\right]$$

Remark 3.2. It may be noted here that the minimum MSEs of the estimators considered in (2.10) and (2.11) are usable only if we know the exact values of C_x, C_y, R, k_c and ρ_c . If these values are unknown then we can estimate them from samples as $\hat{C}_y = \left\{\hat{M}_y \hat{f}_y\left(\hat{M}_y\right)\right\}^{-1}$,$\hat{C}_x = \left\{\hat{M}_x \hat{f}_x\left(\hat{M}_x\right)\right\}^{-1}$, $\hat{R} = \hat{M}_x/\hat{M}_y$, $\hat{k_c} = \hat{\rho}_c\left(\hat{C}_y/\hat{C}_x\right)$ and $\hat{\rho}_c = 4(4\hat{p}_{11} - 1)$ with $\hat{p}_{11}$ being the sample values analogues of p_{11} ([18]; [24]).

4. Empirical study

Data Statistics: To illustrate the efficiency of proposed generalized class of estimators in the application, we consider the following two population data sets.

Population I. (Source [23])

y : The number of fish caught by marine recreational fisherman in 1995.

x : The number of fish caught by marine recreational fisherman in 1964.

The values of the required parameters are :

N=69, n=17, $M_y = 2068$, $M_x = 2011$, $f_y(M_y) = 0.00014$,$f_x(M_x) = 0.00014$, $\rho_c = 0.1505$, R=0.97244

Population II. (Source [23])

y : The number of fish caught by marine recreational fisherman in 1995.

x : The number of fish caught by marine recreational fisherman in 1993.

The values of the required parameters are:

N=69, n=17, $M_y = 2068$, $M_x = 2307$, $f_y(M_y) = 0.00014$,$f_x(M_x) = 0.00013$, $\rho_c = 0.3166$, R=1.11557

Table 3.1: Variances / MSEs/minimum MSEs of different Estimators

Estimators	Population I	Population II
$V\left(\hat{M}_y\right)$	565443.57	565443.57
$MSE\left(\hat{M}_r\right)$	988372.76	536149.50
$MSE_{min}\left(\hat{M}_d\right)$	552636.13	508766.02
$MSE_{min}\left(\hat{M}_d^{(G)}\right)$	552636.13	508766.02
$MSE_{min}\left(\hat{M}_i\right)$	552636.13	508766.02
$MSE_{min}\left(\hat{M}_d^1\right)$	485969.06	495484.97
$MSE_{min}\left(\hat{M}_d^2\right)$	489395.24	454675.78
$MSE_{min}\left(\hat{M}_d^3\right)$	3229.34	51355.17
$MSE_{min}\left(\hat{M}_d^3\right)$	480458.97	454616.15
$MSE_{min}\left(t_m\right)$	3229.34	51355.17
$MSE_{min}\left(t_{mq}^1\right)$	3267.42	58727.72
$MSE_{min}\left(t_{mq}^2\right)$	3267.43	58729.63
$MSE_{min}\left(t_{mq}^3\right)$	3254.89	55919.25
$MSE_{min}\left(t_{mq}^4\right)$	3267.43	58730.48
$MSE_{min}\left(t_{mq}^5\right)$	3238.55	55037.68
$MSE_{min}\left(t_{mq}^6\right)$	3267.43	58730.48
$MSE_{min}\left(t_{mq}^7\right)$	3232.56	51514.08
$MSE_{min}\left(t_{mq}^8\right)$	3247.25	54709.03
$MSE_{min}\left(t_{mq}^9\right)$	3253.88	59211.32

(for i=1,2,....,7)

Analysing table 3.1 we conclude that the estimators based on auxiliary information are more efficient than the one which does not use the auxiliary information as $\hat{M}_y$ .The members of the class of estimators t_{mq} ,obtained from generalized class of estimators t_m , are almost equally efficient but more than the usual unbiased estimator $\hat{M}_y$ (due to [5]), usual ratio estimator $\hat{M}_r$ (due to [6]), difference type estimator $\hat{M}_d$, the class of estimators $\hat{M}_y^{(G)}$, the estimators $\hat{M}_i$ (i=1,2,...7) and the estimators $\hat{M}_d^{(1)}$, $\hat{M}_d^{(2)}$ and $\hat{M}_d^{(3)}$ (due to [19]). Among the proposed estimators t_m and t_{mq}^j (j=1,2,...9) the performance of the estimator t_m ,which is equal efficient to the estimator $\hat{M}_d^{(3)}$ (due to [19]) , is best in the sense of having the least MSE followed by the estimator $\hat{M}_{mq}^{(7)}$ which utilize the information on population median M_x and ρ_c

5. Conclusion

In this article we have suggested a generalized class of estimators for the population median of study variable y when information is available on an auxiliary variable in simple random sampling without replacement (SRSWOR). In addition, some known estimators of population median such as usual unbiased estimator for population median $\hat{M}_y$ due to [5], estimators due to [6], [26], [10] , [1] and [19] are found to be members of the proposed generalized class of estimators. Some new members are also generated from the proposed generalized class of estimators. We have determined the biases and mean square errors of the proposed class of estimators up to the first order of approximation. The proposed generalized class of estimators are advantageous in the sense that the properties of the estimators, which are members of the proposed class of estimators, can be easily obtained from the properties of the proposed generalized class. Thus the study unifies properties of several estimators for population median. In theoretical and empirical efficiency comparisons, it has been shown that the proposed generalized class of estimators are more efficient than the estimators considered here and equally efficient to the estimator $\hat{M}_d^{(3)}$

References

[1] Al, S. and Cingi, H. *New estimators for the population median in simple random sampling*, Tenth Islamic Countries Conference on Statistical Sciences, held in New Cairo, Egypt. **2**009.

[2] Arcos, A., Rueda, M., Martinez-Miranda, M. D. *Using multiparametric auxiliary information at the estimation stage.*, Statist. Pap. **46** 339-358, 2009.

[3] Chambers, R. L. and Dunstan, R. *Estimating distribution functions from survey data.*, Biometrika **73** 597-604., 1986.

[4] Garcia, M. R. and Cebrian, A. A. *On estimating the median from survey data using multiple auxiliary information.* , Metrika **54** 59-76., 2001.

[5] Gross, T. S. *Median estimation in sample surveys.*, Proc. Surv. Res. Meth. Sect. Amer.Statist. Assoc. **1**81-184., 1980.

[6] Kuk, A. Y. C. and Mak, T. K. *Median estimation in the presence of auxiliary information.*, J. Roy. Statist. Soc. Ser. **B51** 261-269., 1989.

[7] Koyuncu, N. and Kadilar, C. *A new Calibration estimator in stratified double sampling.*, HJMS **43(2)** 337-346., 2014.

[8] Mak, T. K. and Kuk, A. Y. C. *A new method for estimating finite population quantiles using auxiliary information.*, Cana. J. Statist. **21** 29-38., 1993.

[9] Meeden, G. *Median estimation using auxiliary information.*,Surv. Methodol. **21** 71-77., 1995.

[10] Murthy, M. N. *Product method of estimation.* , Sankhya **26** 294-307., 1964.

[11] Rao, J. N. K., Kovar, J. G. and Mantel, H. J. *On estimating distribution functions and quantiles from survey data using auxiliary information.*, Biometrika **77** 365-375., 1990.

[12] Reddy, V. N.*On ratio and product method of estimation.*, Sankhya **B35** 307-317., 1973.

[13] Reddy, V. N. *On a transformed ratio method of estimation.*, Sankhya **C36** 59-70., 1974.

[14] Robson, D. S. *Application of multivariate polykays to the theory of unbiased ratiotype estimators.* , J. Amer. Statist. Assoc. **52** 511-522., 1957.

[15] Rueda, M., Arcos, A., Arte's, E. *Quantile interval estimation in finite population using a multivariate ratio estimator.* , Metrika **47** 203-213., 1998.

[16] Sahai, A., Ray, S. K. *An efficient estimator using auxiliary information.*, Metrika **27** 27-275., 1980.

[17] Sharma, P. And Singh, R. *Efficient estimator of population mean in stratified random sampling using auxiliary attribute.* , World Applied Sciences Journal **27(12)** 1786-1791., 2013.

[18] Silverman, B.W. *Density Estimation for Statistics and Data Analysis . London* ,Chapman and Hall., 1986.

[19] Singh, H.P. and Solanki, R. S. *Some Classes of estimators for the Population Median Using Auxiliary Information.*,Com. in Stat. **42** 4222-4238., 2013.

[20] Singh, H. P., Sidhu, S. S., and Singh, S. *Median estimation with known interquartile range of auxiliary variable.*,Int. J. Appl. Math. Statist. **4** 68-80., 2006.

[21] Singh, H. P., Singh, S. and Joarder, A. H. *Estimation of population median when mode of an auxiliary variable is known.*,J. Statist. Res. **37(1)** 57-63., 2003a.

[22] Singh, H. P., Singh, S., Puertas, S. M. *Ratio type estimators for the median of finite populations.* , Allgemeines Statistisches Archiv. **87** 369-382., 2003b.

[23] Singh, S. *Advanced sampling Theory with Applications: How Michael 'Selected' Amy.*,The Netherlands: Kluwer Academic Publishers., 2003.

[24] Singh, S., Joarder, A. H., and Tracy, D. S. *Median estimation using double sampling.*, Austral. NZ J. Statist. **43** 33-46., 2001.

[25] Singh, S., Singh, H. P. and Upadhyaya, L. N. *Chain ratio and regression type estimators for median estimation in survey sampling.*, Statist. Pap. **48(1)** 23-46., 2007.

[26] Srivastava, S. K. *An estimator using auxiliary information in sample surveys.*, Calcutta Statist. Assoc. Bull. **6** 121-132., 1967.

[27] Srivastava, S. K. *A generalized estimator for the mean of a finite population using multi-auxiliary information.*, J. Amer. Statist. Assoc. **66** 404-407., 1971.

[28] Srivastava, S. K. and Jhajj, H. S. *A class of estimators of the population mean in survey sampling using auxiliary information.*, Biometrika **68** 341-343., 1981.

[29] Vos, J. W. E. *Mixing of direct ratio and product method estimators.*, Statististica Neerlandica **34** 209-218., 1980.

[30] Walsh, J. E. *Generalization of ratio estimator for population total.*, Sankhya **A42** 99-106., 1970.

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The Weibull-Lomax distribution: properties and applications

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Abstract

We introduce a new model called the Weibull-Lomax distribution which extends the Lomax distribution and has increasing and decreasing shapes for the hazard rate function. Various structural properties of the new distribution are derived including explicit expressions for the moments and incomplete moments, Bonferroni and Lorenz curves, mean deviations, mean residual life, mean waiting time, probability weighted moments, generating and quantile function. The Rényi and q entropies are also obtained. We provide the density function of the order statistics and their moments. The model parameters are estimated by the method of maximum likelihood and the observed information matrix is determined. The potentiality of the new model is illustrated by means of two real life data sets. For these data, the new model outperforms the McDonald-Lomax, Kumaraswamy-Lomax, gamma-Lomax, beta-Lomax, exponentiated Lomax and Lomax models.

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1. Introduction

The Lomax or Pareto II (the shifted Pareto) distribution was pioneered to model business failure data by Lomax [45]. This distribution has found wide application in a variety of fields such as income and wealth inequality, size of cities, actuarial science, medical and biological sciences, engineering, lifetime and reliability modeling. It has been applied to model data obtained from income and wealth [37, 16], firm size [23], size distribution of computer files on servers [40], reliability and life testing [38], receiver operating characteristic (ROC) curve analysis [21] and Hirsch-related statistics [34].

The characterization of the Lomax distribution is described in a number of ways. It is known as a special form of Pearson type VI distribution and has also considered as a mixture of exponential and gamma distributions. In the lifetime context, the Lomax model belongs to the family of decreasing failure rate [24] and arises as a limiting distribution of residual lifetimes at great age [18]. This distribution has been suggested as heavy tailed alternative to the exponential, Weibull and gamma distributions [19]. Further, it is related to the Burr family of distributions [55] and as a special case can be obtained from compound gamma distributions [30]. Some details about the Lomax distribution and Pareto family are given in Arnold [12] and Johnson *et al.* [41].

The distributional properties, estimation and inference of the Lomax distribution are described in the literature as follows. In record value theory, some properties and moments for the Lomax distribution have been discussed in [7, 17, 43, 11]. The comparison of Bayesian and non-Bayesian estimation from the Lomax distribution based on record values have been made in [4, 49]. The moments and inference for the order statistics and generalized order statistics (gos) are given in [52, 25] and [47], respectively. The estimation of parameters in case of progressive and hybrid censoring have been investigated in [13, 28, 10, 39] and [14]. The problem of Bayesian prediction bounds for future observation based on uncensored and type-I censored sample from the Lomax model are dealt in [3] and [9]. Further, the Bayesain and non-Bayesian estimators of the sample size in case of type-I censored samples for the Lomax distribution are obtained in [1], and the estimation under step-stress accelerated life testing for the Lomax distribution is considered in [38]. The parameter estimation through generalized probability weighted moments (PWMs) is addressed in [2]. More recently, the second-order bias and bias-correction for the maximum likelihood estimators (MLEs) of the parameters of the Lomax distribution are determined in [33].

The main aim of this paper is to provide another extension of the Lomax distribution using the *Weibull-G* generator defined by Bourguignon *et al.* [20]. So, we propose the new *Weibull-Lomax* ("WL" for short) distribution by adding two extra shape parameters to the Lomax model. The objectives of the research are to study some structural properties of the proposed distribution.

A random variable Z has the Lomax distribution with two parameters α and β, if it has cumulative distribution function (cdf) (for $x > 0$) given by

$$H_{\alpha,\beta}(x) = 1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}, \tag{1.1}$$

where $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters, respectively. The probability density function (pdf) corresponding to (1.1) reduces to

$$h_{\alpha,\beta}(x) = \frac{\alpha}{\beta}\left[1 + \left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)}. \tag{1.2}$$

The survival function $S(t)$ and the hazard rate function (hrf) $h(t)$ at time t for the Lomax distribution are given by

$$S(t) = \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha} \quad \text{and} \quad h(t) = \frac{\alpha}{\beta}\left[1 + \left(\frac{x}{\beta}\right)\right]^{-1},$$

respectively.

The rth moment of Z (for $r < k$) comes from (1.2) as $\mu^{'}_{r<Z} = \alpha\,\beta^r\; B(r+1, \alpha - r)$, where $B(p,q) = \int_0^1 w^{p-1}\,(1-w)^{q-1}dw$ is the complete beta function. The mean of Z can be expressed as $E(Z) = \beta/(\alpha-1)$, for $\alpha > 1$, and the variance is $Var(Z) = \beta^2/[(\alpha-1)^2\,(\alpha-2)]$, for $\alpha > 2$. As α tends to infinity, the mean tends to β, the variance tends to β^2, the skewness tends to 36 and the excess kurtosis approaches 21.

The trend of parameter(s) induction to the baseline distribution has received increased attention in recent years to explore properties and for efficient estimation of the parameters. In the literature, some extensions of the Lomax distribution are available such as the exponentiated Lomax (EL) [6], Marshall-Olkin extended-Lomax (MOEL) [32, 35], beta-Lomax (BL), Kumaraswamy-Lomax (KwL), McDonald-Lomax (McL) [44] and gamma-Lomax (GL) [27].

The first parameter induction to the Lomax distribution was suggested by [6] using Lehmann alternative type I proposed by Gupta *et al.* [36]. The three-parameter EL cdf (for $x > 0$) is defined by

$$G_{a,\alpha,\beta}(x) = \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{a}, \tag{1.3}$$

where $a > 0$ is a shape power parameter. The pdf corresponding to (1.3) (for $x > 0$) is given by

$$g_{a,\alpha,\beta}(x) = \frac{a\alpha}{\beta}\left[1 + \left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)}\left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{a-1}, \tag{1.4}$$

with two shape parameters and one scale parameter.

Let Y be a random variable having the EL distribution (1.4) with parameters a, α and β. Using the transformation $t = 1 - [1 + (x/\beta)]^{-\alpha}$ and the binomial expansion, the rth moment of Y (for $r < \alpha$) is obtained from (1.4) as

$$\mu^{'}_{r,Y}(x) = a\,\beta^r \sum_{m=0}^{r} (-1)^m \binom{r}{m} B\big(a, \tfrac{m-r}{\alpha} + 1\big). \tag{1.5}$$

The rth incomplete moment of Y is given by

$$\mu^{'}_{(r,Y)}(z) = \int_0^z y^r\, g_{a,\alpha,\beta}(y)\,dy = a\,\beta^r \sum_{m=0}^{r} (-1)^m \binom{r}{m} B_y\big(a, \tfrac{m-r}{\alpha} + 1\big), \tag{1.6}$$

where $B_y(p,q) = \int_0^y w^{p-1}\,(1-w)^{q-1}dw$ is the incomplete beta function. Some other mathematical quantities of Y are obtained in [5, 6, 42].

The second parameter extension to the Lomax model, named the MOEL distribution, was proposed by [32] using a flexible generator pioneered by Marshall and Olkin [46]. The three-parameter MOEL cdf is given by

$$F_{\alpha,\beta,\delta}(x) = \delta\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^{\alpha} - \overline{\delta}\right\}^{-1}. \tag{1.7}$$

The pdf corresponding to (1.7) becomes

$$f_{\alpha,\beta,\delta}(x) = \alpha\beta\delta\left[1 + \left(\frac{x}{\beta}\right)\right]^{\alpha-1}\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^{\alpha} - \overline{\delta}\right\}^{-2}, \tag{1.8}$$

where $\overline{\delta} = 1 - \delta$ and $\delta > 0$ is a shape (or tilt) parameter.

The properties and the estimation of the reliability for the MOEL distribution are studied in [32] and [35]. The acceptance sampling plans (double and grouped) based on non-truncated and truncated samples for the MOEL distribution has been considered by [15, 53, 54, 50].

Lemonte and Cordeiro [44] discussed three parameter inductions to the Lomax distributions, namely the BL, KwL and McL by including two, two and three extra shape parameters using the beta-G, Kumaraswamy-G and McDonald-G generators defined by Eugene *et al.* [31], Cordeiro and de Castro [26] and Alexander *et al.* [8], respectively. The cdfs of the BL, KwL and McL distributions are given by

$$F_{BL}(x;a,b,\alpha,\beta) = I_{\left\{1-\left[1+(\frac{x}{\beta})\right]^{-\alpha}\right\}(x)}(a,b), \tag{1.9}$$

$$F_{KwL}(x;a,b,\alpha,\beta) = 1-\left[1-\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{a}\right]^{b} \tag{1.10}$$

and

$$F_{McL}(x;a,b,c,\alpha,\beta) = I_{\left\{1-\left[1+(\frac{x}{\beta})\right]^{-\alpha}\right\}^{c}(x)}(a,b), \tag{1.11}$$

respectively, where $I_w(p,q) = B_x(p,q)/B(p,q)$ is the incomplete beta function ratio, and $a>0$, $b>0$ and $c>0$ are extra shape parameters whose role is to govern the skewness and tail weights.

The density functions corresponding to (1.9), (1.10) and (1.11) are given by

$$f_{BL}(x;a,b,\alpha,\beta) = \frac{\alpha}{\beta\, B(a,b)}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha b+1)}\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{a-1}, \tag{1.12}$$

$$f_{KwL}(x;a,b,\alpha,\beta) = \frac{a\,b\,\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)}\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{a-1} \times \left[1-\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{a}\right]^{b-1} \tag{1.13}$$

and

$$f_{McL}(x;a,b,c,\alpha,\beta) = \frac{c\,\alpha}{\beta\, B(a\,c^{-1},b)}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \times \left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{a-1}\left[1-\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{c}\right]^{b-1}, \tag{1.14}$$

respectively.

Recently, Cordeiro *et al.* [27] introduced a three-parameter *gamma-Lomax* (GL) distribution based on a versatile and flexible gamma generator proposed by Zagrafos and Balakrishnan [56] using Stacy's generalized gamma distribution and record value theory. The GL cdf is given by

$$F(a,\alpha,\beta)(x) = \frac{\Gamma\Big(a,\alpha\,\log\left[1+(\frac{x}{\beta})\right]\Big)}{\Gamma(a)}, \quad x>0, \tag{1.15}$$

where $\alpha>0$ and $a>0$ are shape parameters and $\beta>0$ is a scale parameter. The pdf corresponding to (1.15) is given by

$$f(a,\alpha,\beta)(x) = \frac{\alpha^{a}}{\beta\,\Gamma(a)}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)}\left\{\log\left[1+\left(\frac{x}{\beta}\right)\right]\right\}^{a-1}, \quad x>0. \tag{1.16}$$

More recently, Bourguignon *et al.* [20] proposed the Weibull-G class influenced by the Zografos-Balakrishnan-G class. Let $G(x;\Theta)$ and $g(x;\Theta)$ denote the cumulative and density functions of the baseline model with parameter vector Θ and consider the Weibull cdf $F_W(x) = 1-e^{-a\,x^{b}}$ (for $x>0$ and $a,b>0$). Bourguignon *et al.* [20] replaced the argument x by $G(x;\Theta)/\overline{G}(x;\Theta)$, where $\overline{G}(x;\Theta)=1-G(x;\Theta)$, and defined their class of distributions, say Weibull-G(a,b,Θ), by the cdf

$$F(x;a,b,\Theta)=a\,b\int_0^{\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]} x^{b-1}\exp\left(-a\,x^b\right)dx=1-\exp\left\{-a\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^b\right\}. \tag{1.17}$$

The *Weibull-G* density function is given by

$$f(x;a,b,\Theta)=a\,b\;g(x;\Theta)\left[\frac{G(x;\Theta)^{b-1}}{\overline{G}(x;\Theta)^{b+1}}\right]\exp\left\{-a\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^b\right\},\quad x\in\Re. \tag{1.18}$$

In this context, we propose and study the WL distribution based on equations (1.17) and (1.18). The paper is outlined as follows. In Section 2, we define the WL distribution. We provide a mixture representation for its density function in Section 3. Structural properties such as the ordinary and incomplete moments, Bonferroni and Lorenz curves, mean deviations, mean residual life, mean waiting time, probability weighted moments, generating function and quantile function are derived in Section 4. In Section 5, we obtain the Rényi and q entropies. The density of the order statistics is determined in Section 6. The maximum likelihood estimation of the model parameters is discussed in Section 7. We explore its usefulness by means of two real data sets in Section 8. Finally, Section 9 offers some concluding remarks.

2. The WL distribution

Inserting (1.1) in equation (1.17) yields the four-parameter WL cdf

$$F(x;a,b,\alpha,\beta)=1-\exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\alpha}-1\right\}^b\right\}. \tag{2.1}$$

The pdf corresponding to (2.1) is given by

$$\begin{aligned} f(x;a,b,\alpha,\beta) &= \frac{ab\,\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{b\,\alpha-1}\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1} \\ &\times\ \exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\alpha}-1\right\}^b\right\}, \end{aligned} \tag{2.2}$$

where $a>0$ and $b>0$ are two additional shape parameters.

Plots of the WL pdf for some parameter values are displayed in Figure 1. Henceforth, we denote by $X\sim$WL(a,b,α,β) a random variable having the pdf (2.2). The survival function (sf) ($S(x)$), hrf ($h(x)$), reversed-hazard rate function (rhrf) ($r(x)$) and cumulative hazard rate function (chrf) ($H(x)$) of X are given by

$$S(x;a,b,\alpha,\beta)=\exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\alpha}-1\right\}^b\right\}, \tag{2.3}$$

$$h(x)=\frac{a\,b\,\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{b\,\alpha-1}\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1},$$

$$r(x)=\frac{\frac{ab\,\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{b\,\alpha-1}\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1}\exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\alpha}-1\right\}^b\right\}}{1-\exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\alpha}-1\right\}^b\right\}}$$

and

$$H(x)=-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\alpha}-1\right\}^b,$$

respectively. Plots of the WL hrf for some parameter values are displayed in Figure 2.

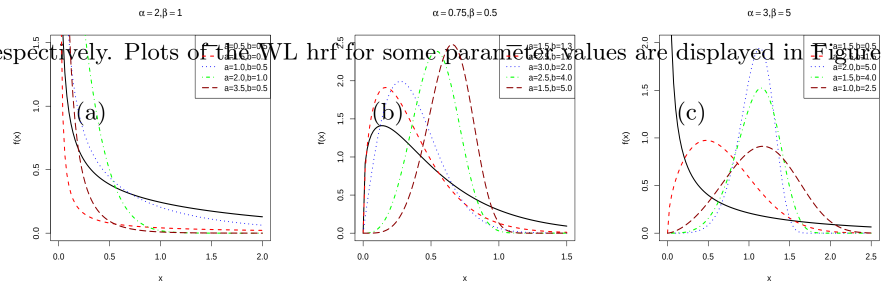

FIGURE 1. Plots of the WL pdf for some parameter values

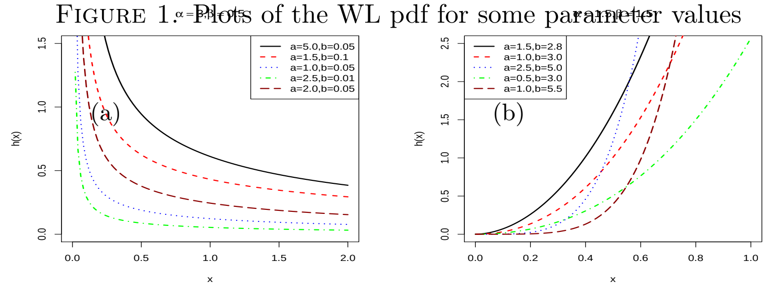

FIGURE 2. Plots of the WL hrf for some parameter values

3. Mixture representation

The WL density function can be expressed as

$$f(x;a,b,\alpha,\beta) = a\,b\,g(x)\,\frac{G(x)^{b-1}}{\overline{G}(x)^{b+1}}\,\exp\left\{-a\left[\frac{G(x)}{\overline{G}(x)}\right]^{b}\right\}. \tag{3.1}$$

Inserting (1.1) and (1.2) in equation (3.1), we obtain

$$\begin{aligned} f(x;a,b,\alpha,\beta) &= \frac{ab\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \frac{\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1}}{\left[1-\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}\right]^{b+1}} \\ &\times \quad \exp\left\{-a\left\{\frac{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}}{1-\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}}\right\}^{b}\right\}. \end{aligned} \tag{3.2}$$

In order to obtain a simple form for the WL pdf, we can expand (3.1) in power series. Let $A = \exp\left\{-a\left\{\frac{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}}{1-\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}}\right\}^{b}\right\}$.

By expanding the exponential function in A, we have

$$A = \sum_{k=0}^{\infty} \frac{(-1)^k\, a^k}{k!}\, \frac{\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{kb}}{\left[1-\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}\right]^{kb}}.$$

Inserting this expansion in (3.2) and, after some algebra, we obtain

$$\begin{aligned} f(x;a,b,\alpha\beta) &= \sum_{k=0}^{\infty} \frac{(-1)^k\, a^k}{k!}\, \frac{a\, b\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \\ &\times \left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b(k+1)-1} \\ &\times \underbrace{\left[1-\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}\right]^{-[b(k+1)-1]}}_{B_k}. \end{aligned}$$

After a power series expansion, the quantity B_k in the last equation becomes

$$B_k = \sum_{j=0}^{\infty} (-1)^j \binom{-[(k+1)b+1]}{j} \left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^j.$$

Combining the last two results, we can write

$$f(x;a,b,\alpha,\beta) = \sum_{k,j=0}^{\infty} \underbrace{\frac{(-1)^k\, a^{k+1}}{k!\, j!}\, \frac{b}{[(k+1)b+j]}\, \frac{\Gamma([k+1]b+j+1)}{\Gamma([k+1]b+1)}}_{v_{k,j}}$$

$$\times \underbrace{[(k+1)b+j]\, \frac{\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{[(k+1)b+j]-1}}_{g_{a,\alpha,(k+1)b+j}}.$$

The last equation can be rewritten as

$$f(x;a,b,\alpha,\beta) = \sum_{k,j=0}^{\infty} v_{k,j}\; g_{a,\alpha,(k+1)b+j}(x). \tag{3.3}$$

Equation (3.3) reveals that the WL density function has a double mixture representation of EL densities. So, several of its structural properties can be derived form those of the EL distribution. The coefficients $v_{k,j}$ depend only on the generator parameters. This equation is the main result of this section.

4. Some Structural Properties

Established algebraic expansions to determine some structural properties of the WL distribution can be more efficient than computing those directly by numerical integration of its density function, which can be prone to rounding off errors among others.

4.1. Quantile Function. Quantile functions are in widespread use in general statistics and often find representations in terms of lookup tables for key percentiles. The quantile function (qf) of X is obtained by inverting (2.1) as

$$Q(u) = \beta\left\{\left[\left\{-a^{-1}\log(1-u)\right\}^{1/b}+1\right]^{1/\alpha}-1\right\}. \tag{4.1}$$

Simulating the WL random variable is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the random variable $X = Q(U)$ follows (2.2), i.e. $X \sim$WL(a,b,α,β).

4.2. Moments. Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis). The rth moment of X can be obtained from (3.3) as

$$\mu'_r = E(X^r) = \sum_{k,j=0}^{\infty} v_{k,j} \int_0^{\infty} x^r\, g_{a,\alpha,(k+1)b+j}(x)\, dx.$$

Using (3.3), we obtain (for $r \leq \alpha$)

$$\mu'_r = \beta^r \sum_{m=0}^{r} \sum_{k,j=0}^{\infty} (-1)^m \left[(k+1)b+j\right] \binom{r}{m} v_{k,j}\, B\left([k+1]b+j, \tfrac{m-r}{\alpha}+1\right). \tag{4.2}$$

Setting $r = 1$ in (4.2), we have the mean of X. Further, the central moments (μ_n) and cumulants (κ_n) of X are obtained from (4.2) as

$$\mu_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mu_1'^k\, \mu'_{n-k} \qquad \text{and} \qquad \kappa_n = \mu'_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \kappa_k\, \mu'_{n-k},$$

respectively, where $\kappa_1 = \mu'_1$. Thus, $\kappa_2 = \mu'_2 - \mu_1'^2$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$, etc. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.

The nth descending factorial moment of X (for $n = 1, 2, \ldots$) is

$$\mu'_{(n)} = E(X^{(n)}) = E[X(X-1)\times\cdots\times(X-n+1)] = \sum_{j=0}^{n} s(n,j)\, \mu'_j,$$

where $s(n,j) = (j!)^{-1}\,[d^j j^{(n)}/dx^j]_{x=0}$ is the Stirling number of the first kind.

4.3. Incomplete moments. The answers to many important questions in economics require more than just knowing the mean of the distribution, but its shape as well. This is obvious not only in the study of econometrics but in other areas as well. The rth incomplete moment of X ($r \leq \alpha$) follows from (3.3) as

$$m_r(z) = \beta^r \sum_{m=0}^{r} \sum_{k,j=0}^{\infty} (-1)^m \left[(k+1)b+j\right] v_{k,j} \binom{r}{m} B_z\left([k+1]b+j, \tfrac{m-r}{\alpha}+1\right). \tag{4.3}$$

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. For a given probability π, they are defined by $B(\pi) = m_1(q)/(\pi\, \mu'_1)$ and $L(\pi) = m_1(q)/\mu'_1$, respectively, where $m_1(q)$ can be determined from (4.3) with $r = 1$ and $q = Q(\pi)$ is calculated from (4.1).

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean and median defined by $\delta_1 = \int_0^\infty |x-\mu| f(x)dx$ and $\delta_2(x) = \int_0^\infty |x-M| f(x)dx$, respectively, where $\mu'_1 = E(X)$ is the mean and $M = Q(0.5)$ is the median. These measures can be determined from $\delta_1 = 2\mu'_1 F(\mu'_1) - 2m_1(\mu'_1)$ and $\delta_2 = \mu'_1 - 2m_1(M)$, where $F(\mu'_1)$ comes from (2.1).

A further application of the first incomplete moment is related to the mean residual life and the mean waiting time given by $m(t; a, b, \alpha, \beta) = [1 - m_1(t)]/S(t) - t$ and $\mu(t; a, b, \alpha, \beta) = t - [m_1(t)/F(t; a, b, \alpha, \beta)]$, respectively, where $F(\cdot;\cdot)$ and $S(\cdot;\cdot) = 1 - F(\cdot;\cdot)$ are obtained from (2.1).

4.4. Probability weighted moments. The probability weighted moments (PWMs) are used to derive estimators of the parameters and quantiles of generalized distributions. These moments have low variance and no severe bias, and they compare favorably with estimators obtained by the maximum likelihood method. The (s,r)th PWM of X (for $r \geq 1, s \geq 0$) is formally defined by $\rho_{r,s} = E[X^r\, F(X)^s] = \int_0^\infty x^r\, F(x)^s\, f(x)dx$. We can write from (2.1)

$$F(x;a,b,\alpha,\beta)^s = \sum_{i=0}^{\infty} (-1)^i \binom{s}{i} \exp\left\{-i\,a\left\{\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right\}^{b}\right\}.$$

Then, we can express $\rho_{s,r}$ after some algebra from (2.1) and (2.2) as

$$\rho_{r,s} = \sum_{i=0}^{\infty} \frac{(-1)^i \binom{s}{i}}{i+1} \int_0^\infty x^r\, f(x;(i+1)a,b,\alpha,\beta)dx.$$

By using (4.2), we obtain (for $r < \alpha$)

$$\rho_{r,s} = \beta^r \sum_{i,j,k=0}^{\infty} \frac{(-1)^i \binom{s}{i}}{(i+1)}\, s_{i,k,j} \sum_{m=0}^{r} B\left([k+1]b+j, \tfrac{(m-r)}{\alpha}+1\right),$$

where

$$s_{i,k,j} = \frac{(-1)^k\, a^k\, (i+1)^k\, \Gamma((k+1)b+j+1)}{[(b+1)k+1]\, \Gamma((k+1)b+1)\; j!\, k!}.$$

4.5. Generating function. The moment generating function (mgf) $M_X(t)$ of a random variable X provides the basis of an alternative route to analytical results compared with working directly with the pdf and cdf of X. We obtain the mgf of the WL distribution from equation (3.3) as

$$\begin{aligned} M_X(t) &= \sum_{k,j=0}^{\infty} v_{k,j} \int_0^\infty [(k+1)b+j]\, \frac{\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \\ &\times \left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{[(k+1)b+j]-1} \mathrm{e}^{tx}\, dx. \end{aligned}$$

By expanding the binomial terms, we can write

$$\begin{aligned} M_X(t) &= \frac{\alpha}{\beta} \sum_{k,j=0}^{\infty} v_{k,j} \sum_{m=0}^{\infty} (-1)^m \binom{[(k+1)b+j]-1}{m} \int_0^\infty [(k+1)b+j] \\ &\times \left(1+\frac{x}{\beta}\right)^{-(m+1)\alpha-1} \mathrm{e}^{tx}\, dx. \end{aligned}$$

By expanding the binomial terms again, we obtain (for $t < 0$)

$$\begin{aligned} M_X(t) &= \alpha \sum_{k,j,m,n=0}^{\infty} \frac{(-1)^m\, [(k+1)b+j]\, v_{k,j}\, n!}{\beta^{n+1}} \binom{[(k+1)b+j]-1}{m} \\ &\times \binom{-(1+m)\alpha-1}{n} (-t)^{-(n+1)}, \end{aligned}$$

which is the main result of this section.

5. Rényi and q-Entropies

The entropy of a random variable X is a measure of the uncertain variation. The Rényi entropy is defined by

$$I_R(\delta) = \frac{1}{1-\delta} \log\left[I(\delta)\right],$$

where $I(\delta) = \int_{\Re} f^{\delta}(x)\, dx, \delta > 0$ and $\delta \neq 1$. We have

$$\begin{aligned} I(\theta) &= \left(\frac{a\, b\, \alpha}{\beta}\right)^{\delta} \int_0^{\infty} \left(1+\frac{x}{\beta}\right)^{\delta(b\alpha-1)} \left\{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\}^{\delta(b-1)} \\ &\times \exp\left\{-a\delta\left\{\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right\}^{b}\right\} dx. \end{aligned}$$

By expanding the exponential term of the above integrand, we can write

$$\begin{aligned} I(\theta) = \left(\frac{a\, b\, \alpha}{\beta}\right)^{\delta} \int_0^{\infty} \left(1+\frac{x}{\beta}\right)^{\delta(b\alpha-1)} \left\{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\}^{\delta(b-1)} \\ \times \sum_{k=0}^{\infty} \frac{(-1)^k\, (\delta\, a)^k}{k!} \left\{\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right\}^{bk} dx. \end{aligned}$$

Using the binomial expansion twice in the last equation and integrating, we obtain

$$I(\theta) = \left(\frac{a\, b\, \alpha}{\beta}\right)^{\delta} \sum_{m=0}^{\infty} t_m. \tag{5.1}$$

Hence, the Rényi entropy reduces to

$$I_R(\delta) = \frac{1}{1-\delta} \log\left[\left(\frac{a\, b\, \alpha}{\beta}\right)^{\delta} \left(\sum_{m=0}^{\infty} t_m\right)\right], \tag{5.2}$$

where

$$t_m = \sum_{k,j=0}^{\infty} \frac{(-1)^k\, \beta^{m+1}\, \Gamma(\delta(b+1)+bk+j)\, \Gamma(m-\delta(b-1)+bk+j)}{k!\, j!\, m!\, [m\alpha+\delta(\alpha+1)-1]\, \Gamma(\delta(b+1)+bk)\, \Gamma(bk-\delta(b+1)+j)}.$$

The q-entropy, say $H_q(f)$, is defined by

$$H_q(f) = \frac{1}{q-1} \log\left[1-I_q(f)\right],$$

where $I_q(f) = \int_{\Re} f^{q}(x)\, dx$, $q > 0$ and $q \neq 1$. From equation (5.2), we can easily obtain

$$H_q(f) = \frac{1}{q-1} \log\left[1-\left(\frac{a\, b\, \alpha}{\beta}\right)^{q} \left(\sum_{m=0}^{\infty} t_m\right)\right].$$

6. Order Statistics

Here, we provide the density of the ith order statistic $X_{i:n}$, $f_{i:n}(x)$ say, in a random sample of size n from the WL distribution. By suppressing the parameters, we have (for $i = 1, \ldots, n$)

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{i+j-1}. \tag{6.1}$$

Thus, we can write

$$F(x)^{i+j-1} = \sum_{k=0}^{\infty} (-1)^k \binom{i+j-1}{k} \exp\left\{-ak\left\{\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right\}^{b}\right\}$$

and then by inserting (2.2) in equation (6.1), we obtain

$$f_{i:n}(x) = \sum_{m=0}^{\infty} t_{m+1} \; f(x;(m+1)a,b,\alpha,\beta), \tag{6.2}$$

where

$$t_{m+1} = \frac{1}{(m+1)\,B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^{j+m} \binom{n-i}{j} \binom{i+j-1}{m}$$

and $f(x;(m+1)a,b,\alpha,\beta)$ denotes the WL density function with parameters $(m+1)a, b$, α and β. So, the density function of the WL order statistics is a mixture of WL densities. Based on equation (6.2), we can obtain some structural properties of $X_{i:n}$ from those WL properties.

The rth moment of $X_{i:n}$ (for $r < \alpha$) follows from (4.2) and (6.2) as

$$E(X_{i:n}^r) = \beta^r \sum_{m=0}^{r} (-1)^m \binom{r}{m} t_{m+1} \sum_{k,j=0}^{\infty} [(k+1)b+j]\, v_{k,j}\, B\Big([k+1]b+j, \tfrac{m-r}{\alpha}+1\Big), \tag{6.3}$$

where $v_{k,j}$ is given in Section 3.

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. Based upon the moments (6.3), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable WL distributions. They are linear functions of expected order statistics defined by (for $s \geq 1$) $\lambda_s = s^{-1} \sum_{p=0}^{s-1} (-1)^p \binom{s-1}{p} E(X_{s-p:p})$.

The first four L-moments are: $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$. We can easily obtain the λ's for X from (6.3) with $r = 1$.

7. Estimation

Here, we consider the estimation of the unknown parameters of the WL distribution by the maximum likelihood method. Let $x_1, \ldots, x_n$ be a sample of size n from the WL distribution given by (2.2). The log-likelihood function for the vector of parameters $\Theta = (a, b, \alpha, \beta)^\intercal$ can be expressed as

$$\ell = \ell(\Theta) = n \log(ab\,\alpha) - n \log \beta - (\alpha - 1) \sum_{i=1}^{n} \log\Big[1 + \frac{x_i}{\beta}\Big] + (b-1) \sum_{i=1}^{n} \log\Big\{\Big[1 + \frac{x_i}{\beta}\Big]^\alpha - 1\Big\} - a \sum_{i=1}^{n} \Big\{\Big[1 + \frac{x_i}{\beta}\Big]^\alpha - 1\Big\}^b. \tag{7.1}$$

Let $z_i = \big(1 + \frac{x_i}{\beta}\big)^\alpha - 1$. Then, we can write ℓ as

$$\ell = n \log(ab\alpha) - n \log \beta - \big(1 - \tfrac{1}{\alpha}\big) \sum_{i=1}^{n} \log\big(z_i + 1\big) + (b-1) \sum_{i=1}^{n} \log(z_i) - a \sum_{i=1}^{n} z_i^b. \tag{7.2}$$

The log-likelihood function can be maximized either directly by using the R-package (AdequecyModel), SAS (PROC NLMIXED) or the Ox program (sub-routine MaxBFGS) (see Doornik, [29]) or by solving the nonlinear likelihood equations obtained by differentiating (7.1) or (7.2). In *AdequecyModel* package, there exists many maximization algorithms like

NR (Newton-Raphson), BFGS (Broyden-Fletcher-Goldfarb-Shanno), BHHH (Berndt-Hall-Hall-Hausman), NM (Nelder-Mead), SANN (Simulated-Annealing) and Limited-Memory quasi-Newton code for Bound-constrained optimization (L-BFGS-B). However, the MLEs here are computed using L-BFGS-B method.

The components of the score vector $U(\Theta)$ are given by

$$U_a = \tfrac{n}{a} - \textstyle\sum_{i=1}^n z_i^b,$$

$$U_b = \tfrac{n}{b} - \textstyle\sum_{i=1}^n \log z_i - a \sum_{i=1}^n z_i^b \log z_i,$$

$$\begin{aligned} U_\alpha = {} & \tfrac{n}{\alpha} + \tfrac{1}{\alpha^2} \textstyle\sum_{i=1}^n \log\,(z_i+1) + \left(1-\tfrac{1}{\alpha}\right) \sum_{i=1}^n \log\,(z_i+1)^{\frac{1}{\alpha}} \\ & +(b-1)\textstyle\sum_{i=1}^n \left[\left(1+z_i^{-1}\right)\log\,(z_i+1)^{\frac{1}{\alpha}}\right] \\ & -ab \textstyle\sum_{i=1}^n \left(z_i^b + z_i^{b-1}\right) \log\,(z_i+1)^{\frac{1}{\alpha}}\,, \end{aligned}$$

$$\begin{aligned} U_\beta = {} & -\tfrac{n}{\beta} - \tfrac{\alpha}{\beta^2}\left(1-\tfrac{1}{\alpha}\right)\textstyle\sum_{i=1}^n (z_i+1)^{-\frac{1}{\alpha}} \\ & -\tfrac{\alpha(b-1)}{\beta^2}\textstyle\sum_{i=1}^n z_i^{-1}\,(z_i+1)^{1-\frac{1}{\alpha}} + \tfrac{ab\alpha}{\beta^2}\sum_{i=1}^n z_i^{b-1}\,(z_i+1)^{1-\frac{1}{\alpha}}\,. \end{aligned}$$

Setting these above equations to zero and solving them simultaneously also yield the MLEs of the four parameters.

For interval estimation of the model parameters, we require the 4×4 observed information matrix $J(\Theta)=\{J_{rs}\}$ (for $r,s=a,b,\alpha,\beta$) given in Appendix A. Under standard regularity conditions, the multivariate normal $N_4(0, J(\widehat{\Theta})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. Then, the $100(1-\gamma)\%$ confidence intervals for a, b, α and β are given by $\hat{a}\pm z_{\gamma/2}\times\sqrt{var(\hat{a})}$, $\hat{b}\pm z_{\gamma/2}\times\sqrt{var(\hat{b})}$, $\hat{\alpha}\pm z_{\gamma/2}\times\sqrt{var(\hat{\alpha})}$ and $\hat{\beta}\pm z_{\gamma/2}\times\sqrt{var(\hat{\beta})}$, respectively, where the $var(\cdot)$'s denote the diagonal elements of $J(\widehat{\Theta})^{-1}$ corresponding to the model parameters, and $z_{\gamma/2}$ is the quantile $(1-\gamma/2)$ of the standard normal distribution. Two problems that can be addressed in a future research are: (i) how large are the correlations between the parameter estimates? and (ii) how about the sample size required in order for the asymptotic standard errors to be reasonable approximations? The answer to problem (i) could be investigated through simulation studies. The answer to (ii) is related to the adequacy of the normal approximation to the MLE $\widehat{\Theta}$. Clearly, some asymptotic techniques could be adopted to improve the normal approximation for $\widehat{\Theta}$.

The likelihood ratio (LR) statistic can be used to check if the fitted new distribution is strictly "superior" to the fitted Lomax distribution for a given data set. Then, the test of $H_0: a=b=1$ versus $H_1: H_0$ *is not true* is equivalent to compare the WL and Lomax distributions and the LR statistic becomes $w=2\{\ell(\widehat{a},\widehat{b},\widehat{\alpha},\widehat{\beta})-\ell(1,1,\tilde{\alpha},\tilde{\beta})\}$, where $\widehat{a}$, $\widehat{b}$, $\widehat{\alpha}$ and $\widehat{\beta}$ are the MLEs under H_1 and $\tilde{\alpha}$ and $\tilde{\beta}$ are the estimates under H_0.

8. Applications

In this section, we illustrate the usefulness of the WL model. We fit the WL distribution to two data sets and compare the results with those of the fitted McL, KwL, GL, BL, EL and Lomax models.

8.1. Aircraft Windshield data sets. The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the nonstructural outer ply or failure of the heating system. These failures do not result in damage to the aircraft but do result in replacement of the windshield.

We consider the data on failure and service times for a particular model windshield given in Table 16.11 of Murthy *et al.* [48]. These data were recently studied by Ramos *et al.* [51]. The data consist of 153 observations, of which 88 are classified as failed windshields, and the remaining 65 are service times of windshields that had not failed at the time of observation. The unit for measurement is 1000 h.

First data set: Failure times of 84 Aircraft Windshield
0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

Second data set: Service times of 63 Aircraft Windshield
0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

We estimate the unknown parameters of each model by maximum likelihood using L-BFGS-B method and the goodness-of-fit statistics Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (A^*) and Cramér–von Mises (W^*) are used to compare the five models. The statistics $A^*$and W^* are described in details in [22]. In general, the smaller the values of these statistics, the better the fit to the data. The required computations are carried out using the R-script AdequacyModel developed by Pedro Rafael Diniz Marinho, Cícero Rafael Barros Dias and Marcelo Bourguignon. It is freely available from
http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf.

Tables 1 and 3 give the MLEs and their corresponding standard errors (in parentheses) of the model parameters. The model selection is carried out using the AIC , BIC, CAIC and HQIC statistics defined by:

$$AIC = -2\ell(\cdot) + 2p, \quad BIC = -2\ell(\cdot) + p\log(n),$$

$$CAIC = -2\ell(\cdot) + \frac{2pn}{n-p-1}, \quad \text{and} \quad HQIC = 2\,\log\big[\log(n)\big(k - 2\,\ell(\cdot)\big)\big],$$

where $\ell(\cdot)$ denotes the log-likelihood function evaluated at the MLEs, p is the number of parameters, and n is the sample size. The figures in Tables 1 and 3 indicate that the fitted Lomax models have huge parameter estimates, although they are accurate compared with their standard errors. Sometimes, the log-likelihood can become quite flat by fitting special models of the WL distribution leading to numerical maximization problems. For these cases, we can obtain different MLEs for the model parameters using alternative algorithms of maximization since they correspond to local maximums of the log-likelihood function. Thus, it is important to investigate the global maximum.

The values of the AIC, CAIC, BIC, HQIC, A^* and W^* are listed in Tables 2 and 4.

TABLE 1. MLEs and their standard errors (in parentheses) for failure times of 84 Aircraft Windshield data

Distribution	a	b	c	α	β
WL	0.0128	0.5969	-	6.7753	1.5324
	(0.0114)	(0.3590)	-	(3.9049)	(1.3863)
McL	2.1875	119.1751	12.4171	19.9243	75.6606
	(0.5211)	(140.2970)	(20.8446)	(38.9601)	(147.2422)
KwL	2.6150	100.2756	-	5.2771	78.6774
	(0.3822)	(120.4856)	-	(9.8116)	(186.0052)
GL	3.5876	-	-	52001.4994	37029.6583
	(0.5133)	-	-	(7955.0003)	(81.1644)
BL	3.6036	33.6387	-	4.8307	118.8374
	(0.6187)	(63.7145)	-	(9.2382)	(428.9269)
EL	3.6261	-	-	20074.5097	26257.6808
	(0.6236)	-	-	(2041.8263)	(99.7417)
Lomax	-	-	-	51425.3500	131789.7800
	-	-	-	(5933.4892)	(296.1198)

TABLE 2. The statistics $\ell(\cdot)$, AIC, BIC , CAIC , HQIC, A^* and W^* for failure times of 84 Aircraft Windshield data

Distribution	$\ell(\cdot)$	AIC	CAIC	BIC	HQIC	A^*	W^*
WL	**-127.8652**	**263.7303**	**264.2303**	**273.5009**	**267.6603**	**0.6185**	0.0932
McL	-129.8023	269.6045	270.3640	281.8178	274.5170	0.6672	0.0858
KwL	-132.4048	272.8096	273.3096	282.5802	276.7396	0.6645	0.0658
GL	-138.4042	282.8083	283.1046	290.1363	285.7559	1.3666	0.1618
BL	-138.7177	285.4354	285.9354	295.2060	289.3654	1.4084	0.1680
EL	-141.3997	288.7994	289.0957	296.1273	291.7469	1.7435	0.2194
Lomax	-164.9884	333.9767	334.1230	338.8620	335.9417	1.3976	0.1665

Tables 2 and 4 compare the WL model with the McL, KwL, GL, BL, EL and Lomax models. We note that the WL model gives the lowest values for the AIC, BIC and CAIC, HQIC and A^* statistics (except W^* for the first data set) among all fitted models. So, the WL model could be chosen as the best model. The histogram of the data and the estimated pdfs and cdfs for the fitted models are displayed in Figure 3. It is clear from Tables 2 and 4 and Figure 3 that the WL distribution provides a better fit to the histogram and therefore could be chosen as the best model for both data sets.

Table 3. MLEs and their standard errors (in parentheses) for service times of 63 Aircraft Windshield data

Distribution	a	b	c	α	β
WL	0.1276	0.9204	-	3.9136	3.0067
	(0.2964)	(0.4277)	-	(3.8489)	(8.2769)
McL	1.3230	53.7712	5.7144	7.4371	42.8972
	(0.2517)	(199.2803)	(5.3853)	(34.7310)	(150.8150)
KwL	1.6691	60.5673	-	2.5649	65.0640
	(0.2570)	(86.0131)	-	(4.7589)	(177.5919)
GL	1.9073	-	-	35842.4330	39197.5715
	(0.3213)	-	-	(6945.0743)	(151.6530)
BL	1.9218	31.2594	-	4.9684	169.5719
	(0.3184)	(316.8413)	-	(50.5279)	(339.2067)
EL	1.9145	-	-	22971.1536	32881.9966
	(0.3482)	-	-	(3209.5329)	(162.2299)
Lomax	-	-	-	99269.78 00	207019.3700
	-	-	-	(11863.5222)	(301.2366)

Table 4. The statistics $\ell(\cdot)$, AIC, BIC, CAIC, HQIC, A^* and W^* for service times of 63 Aircraft Windshield data

Distribution	$\ell(.)$	AIC	CAIC	BIC	HQIC	A^*	W^*
WL	**-98.11712**	**204.2342**	**204.9239**	**212.8068**	**207.6059**	**0.2417**	**0.0356**
McL	-98.5883	207.1766	208.2292	217.8923	211.3911	0.3560	0.0573
KwL	-100.8676	209.7353	210.4249	218.3078	213.1069	0.7391	0.1219
GL	-102.8332	211.6663	212.0731	218.0958	214.1951	1.112	0.1836
BL	-102.9611	213.9223	214.6119	222.4948	217.2939	1.1336	0.1872
EL	-103.5498	213.0995	213.5063	219.5289	215.6282	1.2331	0.2037
Lomax	-109.2988	222.5976	222.7976	226.8839	224.2834	1.1265	0.1861

9. Concluding remarks

In this paper, we propose a four-parameter Weibull-Lomax (WL) distribution. We study some structural properties of the WL distribution including an expansion for the density function and explicit expressions for the ordinary and incomplete moments, mean residual life, mean waiting time, probability weighted moments, generating function and quantile function. Further, the explicit expressions for the Rényi entropy, q entropy and order statistics are also derived. The maximum likelihood method is employed for estimating the model parameters. We also obtain the observed information matrix. We fit the WL model to two real life data sets to show the usefulness of the proposed distribution. The new model provides consistently a better fit than the other models, namely: the McDonald-Lomax, Kumaraswamy-Lomax, gamma-Lomax, beta-Lomax, exponentiated-Lomax and Lomax distributions. We hope that the proposed model will attract wider application in areas such as engineering, survival and lifetime data, hydrology, economics (income inequality) and others.

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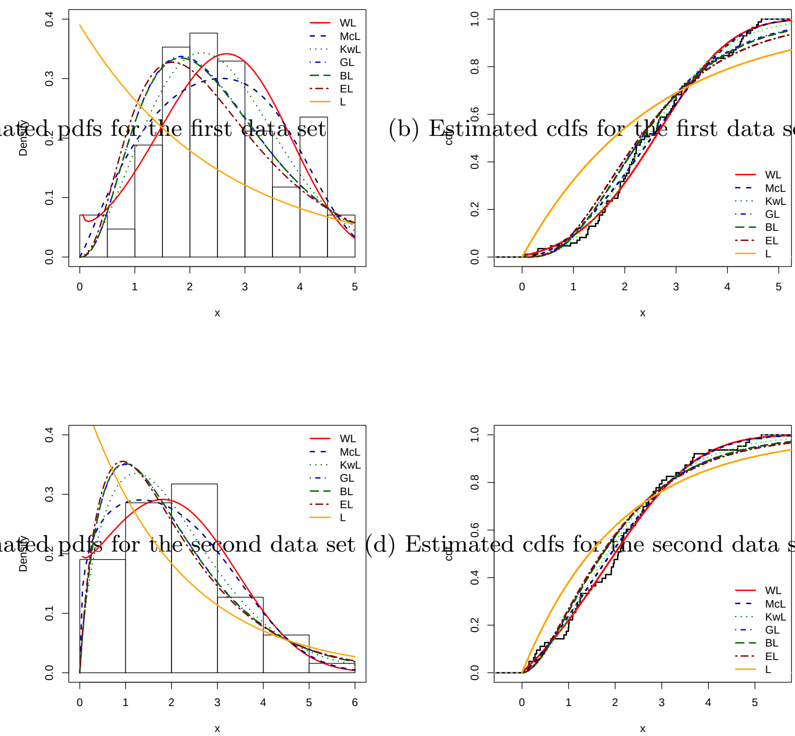

(a) Estimated pdfs for the first data set (b) Estimated cdfs for the first data set.

(c) Estimated pdfs for the second data set (d) Estimated cdfs for the second data set

FIGURE 3. Plots of the estimated pdfs and cdfs for the WL, McL, KwL, GL, BL, EL and Lomax models to the first and second data sets.

Appendix A

The elements of the 4×4 observed information matrix $J(\Theta) = \{J_{rs}\}$ (for $r, s = a, b, \alpha, \beta$) are given by

$$J_{aa} = -\tfrac{n}{a^2},$$

$$J_{ab} = -\textstyle\sum_{i=1}^n z_i^b \log z_i,$$

$$J_{a\alpha} = -b \textstyle\sum_{i=1}^n z_i^{b-1}\,(z_i+1)\log\,(z_i+1)^{\frac{1}{\alpha}},$$

$$J_{a\beta} = \tfrac{\alpha b}{\beta^2} \textstyle\sum_{i=1}^n z_i^{b-1}\,(z_i+1)^{1-\frac{1}{\alpha}},$$

$$J_{bb} = -\tfrac{n}{b^2} - a \textstyle\sum_{i=1}^n z_i^b\,[\log(z_i)]^2,$$

$$J_{b\alpha} = \textstyle\sum_{i=1}^n \left(1+z_i^{-1}\right)\log\,(z_i+1)^{\frac{1}{\alpha}} - a(b+1)\textstyle\sum_{i=1}^n \left[z_i^{b-1}\,(z_i+1)\log\,(z_i+1)^{\frac{1}{\alpha}}\right],$$

$$J_{b\beta} = -\tfrac{\alpha}{\beta^2}\textstyle\sum_{i=1}^n z_i^{-1}(z_i+1)^{1-\frac{1}{\alpha}} - \tfrac{a\alpha}{\beta^2}\textstyle\sum_{i=1}^n z_i^{b-1}\,(z_i+1)^{1-\frac{1}{\alpha}}\,[1+b\log z_i],$$

$$J_{\alpha\alpha} = -\tfrac{n}{\alpha^2} + \textstyle\sum_{i=1}^n (z_i+1)\left[\alpha + (b-1)\left\{\alpha\left(1+z_i^{-1}\right) - z_i^{-2}\right\} - ab\left\{\alpha\left(z_i^b + z_i^{b-1}\right) + b\,z_i^b + (b-1)\,z_i^{b-2}\right\}\right],$$

$$J_{\alpha\beta} = -\tfrac{1}{\beta^2}\textstyle\sum_{i=1}^n (z_i+1)^{-\frac{1}{\alpha}}$$

References

[1] Abd-Elfattah, A.M., Alaboud, F.M. and Alharby, A.H. *On sample size estimation for Lomax distribution*, Australian Journal of Basic and Applied Sciences **1**, 373–378, 2007.

[2] Abd-Elfattah, A.M. and Alharby, A.H. *Estimation of Lomax distribution based on generalized probability weighted moments*, Journal of King Abdulaziz University (Science) **22**, 171–184, 2010.

[3] Abd-Ellah, A.H. *Bayesian one sample prediction bounds for the Lomax distribution*, Indian Journal of Pure and Applied Mathematics **34**, 101–109, 2003.

[4] Abd-Ellah, A.H. *Comparision of estimates using record statistics from Lomax model: Bayesian and non-Bayesian approaches*, Journal of Statistical Research (Iran) **3**, 139–158, 2006.

[5] Abdul-Moniem, I.B. *Recurrence realtions for moments of lower generalized order statistics from exponentiated Lomax distribution and its characterization*, International Journal of Mathematical Archive **3**, 2144–2150, 2012.

[6] Abdul-Moniem, I.B. and Abdel-Hameed, H.F. *On exponentiated Lomax distribution*, International Journal of Mathematical Archive **3**, 2144–2150, 2012.

[7] Ahsanullah, M. *Recoed values of the Lomax distribution*, Statistica Neerlandica **45**, 21–29, 1991.

[8] Alexander, C., Cordeiro, G.M., Ortega, E.M.M. and Sarabia, J.M. *Generalized beta-generated distributions*, Computational Statistics and Data Analysis **56**, 1880–1897., 2012.

[9] Al-Hussaini, E.K., Jaheen, Z.F. and Nigm, A.M. *Bayesian prediction based on finite mixture of Lomax component model anf type I censoring*, Statistics **35**, 259–268, 2001.

[10] Al-Zahrani, B. and Al-Sobhi, M. *On parameters estimation of Lomax distribution under general progressive censoring*, Journal of Quality and Reliability Engineering Article ID 431541, 7 pages, 2013.

[11] Amin, E.A. *Kth upper record values and their moments*, International Mathematical Forum **6**, 3013–3021, 2011.

[12] Arnold, B.C. *Pareto Distributions* (International Cooperative Publishing House, Maryland, 1983).

[13] Asgharzadeh, A. and Valiollahi, R. *Estimation of the scale parameter of the Lomax distribution under progressive censoring*, International Journal for Business and Economics **6**, 37–48, 2011.

[14] Ashour, S.K., Abdelfattah, A.M. and Mohammad, B.S.K. *Parameter estimation of the hybrid censored Lomax distribution*, Pakistan Journal of Statistics and Operations Research **7**, 1–19, 2011.

[15] Aslam, M., Shoaib, M., Lio, Y.L. and Jun, C-H. *Improved grouped acceptance sampling plans for Marshall-Olkin extended Lomax distribution percentiles*, International Journal of Current Research and Review **3**, 10–25, 2011.

[16] Atkinson, A.B. and Harrison, A.J. *Distribution of Personal Wealth in Britain* (Cambridge University Press, Cambridge, 1978).

[17] Balakrishnan, N. and Ahsanullah, M. *Relations for single and product moments of record values from Lomax distribution*, Sankhyã Series B **17**, 140–146, 1994.

[18] Balkema, A.A. and de Hann, L. *Residual life at great age*, Annals of Probability **2**, 972–804, 1974.

[19] Bryson, M.C. *Heavy-tailed distribution: properties and tests*, Technometrics **16**, 161–68, 1974.

[20] Bourguignon, M., Silva, R.B. and Cordeiro, G.M. *The Weibull–G family of probability distributions*, Journal of Data Science **12**, 53–68, 2014.

[21] Campbell, G. and Ratnaparkhi, M.V. *An application of Lomax distributions in receiver operating characteristic (ROC) curve analysis*, Communications in Statistics–Theory Methods **22**, 1681–1697, 1993.

[22] Chen, G. and Balakrishnan, N. *A general purpose approximate goodness-of-fit test*, Journal of Quality Technology **27**, 154–161, 1995.

[23] Corbellini, A., Crosato, L., Ganugi, P and Mazzoli, M. *Fitting Pareto II distributions on firm size: Statistical methodology and economic puzzles*. Paper presented at the International Conference on Applied Stochastic Models and Data Analysis, Chania, Crete, 2007.

[24] Chahkandi, M. and Ganjali, M. *On some lifetime distributions with decrasing failure rate*, Computational Statistics and Data Analysis **53**, 4433–4440, 2009.

[25] Childs, A., Balakrishnan, N. and Moshref, M. *Order statistics from non-identical right-truncated Lomax random variables with applications*, Statistical Papers **42**, 187–206, 2001.

[26] Cordeiro, G.M. and de Castro, M. *A new family of generalized distributions*, Journal of Statistical Computation and Simulation **81**, 883–893, 2011.

[27] Cordeiro, G.M., Ortega, E.M.M. and Popović, B.V. *The gamma-Lomax distribution*, Journal of Statistical computation and Simulation iFirst, doi:10.1080/00949655.2013.822869, 2013.

[28] Cramer, E. and Schemiedt, A.B. *Progressively type-II censored competing risks data from Lomax distribution*, Computational Statistics and Data Analysis **55**, 1285–1303, 2011.

[29] Doornik, J.A. *Ox 5: An Object-Oriented Matrix Programming Language*, fifth edition (Timberlake Consultants, London, 2007).

[30] Durbey, S.D. *Compound gamma, beta and F distributions*, Metrika **16**, 27–31, 1970.

[31] Eugene, N., Lee, C. and Famoye, F. *Beta-normal distribution and its applications*, Communications in Statistics–Theory and Methods **31**, 497–512, 2002.

[32] Ghitany, M. E., AL-Awadhi, F. A and Alkhalfan, L. A. *Marshall-Olkin extended Lomax distribution and its applications to censored data*, Communications in Statistics–Theory and Methods **36**, 1855–1866, 2007.

[33] Giles, D.E., Feng, H. and Godwin, R.T. *On the bias of the maximum likelihood estimator for the two-parameter Lomax distribution*, Communications in Statistics–Theory and Methods **42**, 1934–1950, 2013.

[34] Glänzel, W. *On some new bibliometric applications of statistics related to the h-index*, Scientometrics **77**, 187–196, 2008.

[35] Gupta, R.C., Ghitany, M. E. and Al-Mutairi, D. K. *Estimation of reliability from Marshall-Olkin extended Lomax distributions*, Journal of Statistical Computation and Simulation **80**, 937–947, 2010.

[36] Gupta, R. C., Gupta, P. I. and Gupta, R. D. *Modeling failure time data by Lehmann alternatives*, Communications in statistics–Theory and Methods **27**, 887–904, 1998.

[37] Harris, C.M. *The Pareto distribution as a queue service descipline*, Operations Research **16**, 307–313, 1968.

[38] Hassan, A.S. and Al-Ghamdi, A.S. *Optimum step stress accelerated life testing for Lomax distibution*, Journal of Applied Sciences Research **5**, 2153–2164, 2009.

[39] Helu, A., Samawi, H. and Raqab, M.Z. *Estimation on Lomax progressive censoring using the EM algorithm*, Journal of Statistical Computation and Simulation iFirst, doi:10.1080/00949655.2013.861837, 2013.

[40] Holland, O., Golaup, A. and Aghvami, A.H. *Traffic characteristics of aggregated module downloads for mobile terminal reconfiguration*, IEE proceedings on Communications **135**, 683–690.

[41] Johnson, N.L., Kotz, S. and Balakrishnan, N. *Contineous Univariate Distributions: Vol. 1, 2nd edition* (Wiley, New York, 1994).

[42] Kumar, D. *On moments of lower generalized order statistics from exponentiated Lomax distribution*, American Journal of Mathematical and Management Sciences **32**, 238–256, 2013.

[43] Lee, M-Y. and Lim, E-K. *Characterization of the Lomax, exponential and Pareto distributions by conditional expectations of recoed values*, Journal of the Chungcheong Mathematical Society **22**, 149–153, 2009.

[44] Lemonte, A. J. and Cordeiro, G. M. *An extended Lomax distribution*, Statistics **47**, 800–816, 2013.

[45] Lomax, K.S. *Business failures: Another example of the analysis of failure data*, Journal of the American Statistical Association **49**, 847–852, 1954.

[46] Marshall, A. N. and Olkin, I. *A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families*, Biometrika **84**, 641–652, 1997.

[47] Moghadam, M.S., Yahmaei, F. and Babanezhad, M. *Inference for Lomax distribution under generalized order statistics*, Applied Mathematical Sciences **6**, 5241–5251, 2012.

[48] Murthy, D.N.P. Xie, M. and Jiang, R. *Weibull Models* (Wiley, 2004).

[49] Nasiri, P. and Hosseini, S. *Statistical inferences for Lomax distribution based on record values (Bayesian and classical)*, Journal of Modern Applied Statistical Methods **11**, 179–189, 2012.

[50] Ramaswamy, A.R.S. and Sutharani, R. *Designing double acceptance sampling plans based on truncated life tests in Marshall-Olkin extended Lomax distribution using minimum angle method*, The International Innovative Research Journal for Advanced Mathematics **1**, 47–54, 2013.

[51] Ramos, M.W.A., Marinho, P.R.D., da Silva R.V. and Cordeiro, G.M. *The exponentiated Lomax Poisson distribution with an application to lifetime data*, Advances and Applications in Statistics **34**, 107–135, 2013.

[52] Saran, J. and Pushkarna, N. *Moments of order statistics from doubly truncated Lomax distribution*, Journal of Statistical Research **33**, 57–66, 1999.

[53] Srinivasa Rao, G. *A group acceptance sampling plans based on truncated life tests for Marshall-Olkin extended Lomax distribution*, Electronic Journal of Applied Statistical Analysis **3**, 18–27, 2010.

[54] Srinivasa Rao, G. *Double acceptance sampling plans based on truncated life tests for Marshall-Olkin extended Lomax distribution*, Journal of Modern Applied Statistical Methods **10**, 116–122, 2011.

[55] Tadikamalla, P.R. *A look at the Burr and realted distributions*, International Statistical Review **48**, 337–344, 1980.

[56] Zografos K. and Balakrishnan, N. *On families of beta- and generalized gamma-generated distributions and associated inference*, Statistical Methodology **6**, 344–362, 2009.

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Some results on the extreme distributions of surplus process with nonhomogeneous claim occurrences

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Abstract

In this paper; survival (non-ruin) probability after a definite time period of an insurance company is studied in a discrete time model based on non-homogenous claim occurrences. Furthermore, distributions of the minimum and maximum levels of surplus in compound binomial risk model with non-homogeneous claim occurrences are obtained and some of its characteristics are given.

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1. Introduction

Surplus process (or risk process) is a model of accumulation of insurer's capital and the premium incomes during the periods. So, the surplus process is one of the most important stochastic process for an insurance company which can be defined as discrete or continuous time in actuarial risk theory. Ruin occurs when surplus is zero or negative value which means that the total claim amounts equal or exceed the surplus at a certain time for insurance companies. Furthermore, the estimation of the surplus at a certain time is essential for the insurance companies due to their future investment strategies and actions to be taken just before ruin. In this regard, it is vital importance for controlling the maximum and minimum level of the surplus and its related quantities.

The compound binomial model has been first proposed by Gerber (1988 a). Distributional properties of some actuarial quantities associated with compound binomial model

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have been studied in De Vylder and Goovaerts (1984,1988), Shiu (1989), Willmot (1993), Dickson (1994) and De Vylder and Marceau (1996). The compound binomial model, as a discrete time version of the classical compound Poisson model of risk theory has been widely studied in the recent literature (see, e.g. Yuen and Guo (2001), Cosette and Marceau (2000), Cossette et al. (2003, 2004 and 2006), Liu and Zhao (2007), Tuncel and Tank (2014)).

In classical risk model, the number of the claims is assumed to have a Poisson process $\{N_t : t \geq 0\}$ with parameter λ and the claim amounts $Y_1, Y_2, \ldots$ are non-negative, independent and identically distributed random variables with same distribution function. The total claim amounts process $\{S_t : t \geq 0\}$ is a compound Poisson process with parameter λ where $S_t = \sum_{i=1}^{N_t} Y_i$ designates the total claim amounts up to time t. In this regards, surplus of the insurers at time t can be defined as follows

$$(1.1) \qquad U_t = u + ct - S_t$$

where $U_0 = u$ is the amount of initial reserve, the premium income is c per each period, and Y_i is the eventual claim amount in period i. For simplicity, throughout the paper we assume that $c = 1$.

Let I_i be a binary random variable representing the claim occurrence. That is $I_i = 1$ if a claim occurs in period i and $I_i = 0$, otherwise. For $i \geq 1$, define $Y_i = I_i X_i$, where the random variable I_i and the individual claim amount random variable X_i are independent in each time period. The random variable X_i is strictly positive and $\{X_i, i \geq 1\}$ forms a sequence of iid random variables with probability mass function (p.m.f.) $f(x) = P\{X = x\}$. Under this assumptions, the process (1.1) can be rewritten as

$$(1.2) \qquad U_t = u + t - \sum_{i=1}^{t} I_i X_i,$$

where u is non negative integer, N_n is the number of claims up to time n and X_i is the amount of ith claim. It is assumed that X_i random variables are independent and identically distributed (i.i.d.) and independent of the claim number process. Ruin of insurer's occurs when $U_t \leq 0$ for some $t \geq 1$. The random time to ruin is defined as

$$(1.3) \qquad T = \inf\{t > 0 : U_t \leq 0\}.$$

by Gerber (1988). Thus, ultimate ruin probability and survival probability can be defined as

$$\psi(u) = P(T < \infty | U_o = u)$$
$$\phi(u) = 1 - \psi(u)$$

respectively. Similarly, ruin probability and survival probability in finite time can be defined as

$$\psi(u, n) = P(T \leq n | U_o = u)$$
$$\phi(u, n) = 1 - \psi(u, n)$$

respectively.

Let the random indicators $I_1, I_2, \ldots$ be independent with $p = P\{I_i = 1\}$, then the model (1.2) is called the compound binomial model and

$$P\{N_n = k\} = C_n^k p^k (1 - p)^{n-k}, k = 0, 1, \ldots, n.$$

In here, distribution of N_n is classical binomial distribution.Tuncel and Tank (2014) suggested a recursive formula when the claim occurrences probabilities are non-homogeneous such as $p_i = P\{I_i = 1\}$.

Let M_n and K_n denote respectively the maximum and minimum levels of the surplus process up to period n,

$$M_n = \max_{1\leq t\leq n} U_t \quad , \quad K_n = \min_{1\leq t\leq n} U_t.$$

These quantities may be useful tools on possible future investment or borrowing strategies for their consistent financial statement in an insurance company. Recursive equations are given for both marginal and joint distributions of the M_n and K_n values under the condition that insurance company survives at time n for homogenous case by Eryilmaz et.al. (2012).

The remainder of the present paper is organized as follows: Section 2 presents recursive equations to compute marginal and joint distributions of M_n and K_n under the condition $T > n$. Section 3 gives means and variances of M_n and K_n for zero truncated geometric claim size distribution. Finally, discussions are given in Section 4.

2. Distributions of Extremes Surplus Process

For $u = 1, 2, ...$ and $n \geq 0$, define

$$\begin{aligned}\phi^{(1,n)}(u) &= P_u(T > n)\\ \theta^{(1,n)}(u;k) &= P_u(M_n \leq k, T > n)\\ \gamma^{(1,n)}(u;k) &= P_u(K_n \geq k, T > n)\end{aligned}$$

where

$$\begin{aligned}P_u(T > n) &= \phi^{(1,n)}(u)\\ &= \begin{cases} 1 & ,n = 0\\ \sum_{t=1}^{n} p_t \prod_{i=1}^{t-1} q_i \sum_{x=1}^{u+t-1} f(x) P_{u+t-x}(T^{(t+1,n)} > n - t) + \left(1 - \sum_{t=1}^{n} p_t \prod_{i=1}^{t-1} q_i\right) & ,n > 0\end{cases}\end{aligned}$$

and k is a positive threshold which can be also considered as an upper barrier for surviving of the insurance company. In here, $T^{(t+1,n)}$ and $\phi^{(1,n)}(u)$ represents ruin time after the t-th period and non-ruin probability when the claim occurrences have nonhomogeneous probabilities respectively (Tuncel and Tank(2014)).

2.1. Theorem. *For* $u = 1, 2, ...$

$$P_u(M_n \leq k \mid T > n) = \frac{\theta^{(1,n)}(u;k)}{\phi^{(1,n)}(u)} \tag{2.1}$$

where

a. *If* $k \geq u + n$ *and* $n \geq 0$ *then*

$$\theta^{(1,n)}(u;k) = \phi^{(1,n)}(u)$$

b. *If* $u \leq k < u + n$ *and* $n \geq 0$ *then*

$$\theta^{(1,n)}(u;k) = \sum_{t=1}^{k-u+1} p_t \prod_{i=1}^{t-1} q_i \sum_{x=\max(1,u+t-k)}^{u+t-1} f(x) \theta^{(t+1,n-t)}(u + t - x; k) \tag{2.2}$$

Proof. It is clear that $P_u(M_n \leq k \mid T > n) = 1$ for $k \geq u+n$. So $\theta^{(1,n)}(u;k) = \phi^{(1,n)}(u)$ is trivial.

By conditioning on W_1, the time of the first claim, for $u \leq k < u+n$ and $n \geq 0$ then

$$P_u\left(M_n \leq k, T > n\right) = \sum_{t=1}^{\infty} P_u\left(U_1 \leq k, ..., U_n \leq k \mid W_1 = t\right) P\left(W_1 = t\right)$$

where $P(W_1 = t) = \prod_{i=1}^{t-1} q_i p_t$.

If $t \leq n$ then

$$P_u\left(U_1 = u+1, ..., U_{t-1} = u+t-1 | W_1 = t\right) = 1$$

and

$$P_u\left(U_1 \leq k, ..., U_n \leq k \mid W_1 = t\right) = P_u\left(U_t \leq k, ..., U_n \leq k, T^{(t+1,n)} > n-t\right) \tag{2.3}$$

for $t \leq k-u+1$. Noting that $U_t > 0$ for $t \leq n$ since the ruin occurs after period n and than by conditioning on the value of the first claim one obtains

$$\begin{aligned} & P_u\left(U_t \leq k, ..., U_n \leq k, T > n-t | W_1 = t\right) \\ & = \sum_{x=1}^{\infty} P_u\left(u+t-X > 0, X = x, M_{n-t}^{(t+1,n)} \leq k, T^{(t+1,n)} > n-t\right) \\ & = \sum_{x=\max(1,u+t-k)}^{u+t-1} f(x) P_{u+t-x}\left(M_{n-t}^{(t+1,n)} \leq k, T^{(t+1,n)} > n-t\right) \qquad (2.4) \end{aligned}$$

for $t \leq k-u+1$. For $t > n$, $P(M_n = u+n) = 1$. Thus $P(M_n \leq k, T > n) = 0$, if $k < u+n$ and $t > n$. Thus, for $u \leq k < u+n$,

$$\theta^{(1,n)}\left(u;k\right) = \sum_{t=1}^{k-u+1} p_t \prod_{i=1}^{t-1} q_i \sum_{x=\max(1,u+t-k)}^{u+t-1} f(x) \theta^{(t+1,n-t)}\left(u+t-x;k\right)$$

can be obtained by using (2.3) and (2.4). Hence the proof is completed. □

Expansion of (2.2), which is recursive formula given in Theorem 2.1, as in follows:

- For $n = 1$ and $u \leq k < u+1$

$$\theta^{(1,1)}\left(u;k\right) = \frac{p_1 \sum\limits_{x=u+1-k}^{u} f(x)}{\phi^{(1,1)}\left(u\right)}$$

- For $n = 2$ and $u \leq k < u+2$

$$\theta^{(1,2)}\left(u;k\right) = \begin{cases} \frac{1}{\phi^{(1,2)}(u)}\left[A_1\right] & , k = u \\ \frac{1}{\phi^{(1,2)}(u)}\left[A_2\right] & , k = u+1 \end{cases} \tag{2.5}$$

where

$$A_1 = p_1 p_2 \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+1-x} f(y) + p_1 q_2 \sum_{x=\max(1,u+2-k)}^{u} f(x).$$

$$\begin{aligned} A_2 = & \ p_1 p_2 \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+1-x} f(y) \\ & + p_1 q_2 \sum_{x=\max(1,u+2-k)}^{u} f(x) + q_1 p_2 \sum_{x=\max(1,u+2-k)}^{u+1} f(x). \end{aligned}$$

- For $n = 3$ and $u \le k < u + 3$

(2.6) $$\theta^{(1,3)}(u;k) = \begin{cases} \frac{1}{\phi^{(1,3)}(u)}[A_3] & ,k = u \\ \frac{1}{\phi^{(1,3)}(u)}[A_4] & ,k = u+1 \\ \frac{1}{\phi^{(1,3)}(u)}[A_5] & ,k = u+2 \end{cases}$$

where

$$\begin{aligned} A_3 = & p_1p_2p_3 \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+1-x} f(y) \sum_{z=\max(1,u+3-k-x-y)}^{u+2-x-y} f(z) \\ & + p_1p_2q_3 \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+3-k-x)}^{u+1-x} f(y) \\ & + p_1q_2p_3 \sum_{x=\max(1,u+2-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+2-x} f(y) \\ & + p_1q_2q_3 \sum_{x=\max(1,u+3-k)}^{u} f(x) \end{aligned}$$

$$\begin{aligned} A_4 = & p_1p_2p_3 \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+1-x} f(y) \sum_{z=\max(1,u+3-k-x-y)}^{u+2-x-y} f(z) \\ & + p_1p_2q_3 \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+3-k-x)}^{u+1-x} f(y) \\ & + p_1q_2q_3 \sum_{x=\max(1,u+3-k)}^{u} f(x) + q_1p_2q_3 \sum_{x=\max(1,u+3-k)}^{u+1} f(x) \\ & + p_1q_2p_3 \sum_{x=\max(1,u+2-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+2-x} f(y) \\ & + q_1p_2p_3 \sum_{x=\max(1,u+2-k)}^{u+1} f(x) \sum_{y=\max(1,u+3-k-x)}^{u+2-x} f(y) \end{aligned}$$

$$\begin{aligned} A_5 = & \ p_1p_2p_3 \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+1-x} f(y) \sum_{z=\max(1,u+3-k-x-y)}^{u+2-x-y} f(z) \\ & + p_1q_2q_3 \sum_{x=\max(1,u+3-k)}^{u} f(x) + q_1p_2q_3 \sum_{x=\max(1,u+2-k)}^{u+2} f(x) \\ & + p_1q_2p_3 \sum_{x=\max(1,u+2-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+2-x} f(y) \\ & + q_1p_2p_3 \sum_{x=\max(1,u+2-k)}^{u+1} f(x) \sum_{y=\max(1,u+3-k-x)}^{u+2-x} f(y) \\ & + p_1p_2q_3 \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+3-k-x)}^{u+1-x} f(y). \end{aligned}$$

2.2. Theorem. *For* $u = 1, 2, ...$

$$P_u\left(K_n \geq k \mid T > n\right) = \frac{\gamma^{(1,n)}\left(u;k\right)}{\phi^{(1,n)}\left(u\right)} \tag{2.7}$$

where

a. *If* $k \leq n$ *and* $n = 0$ *then*

$\gamma^{(1,n)}\left(u;k\right) = 1$

b. *If* $k \leq u + 1$ *and* $n \geq 0$ *then,*

$$\gamma^{(1,n)}\left(u;k\right) = \sum_{t=\max(1,k-u+1)}^{n} p_t \prod_{i=1}^{t-1} q_i \sum_{x=1}^{u+t-k} f(x)\gamma^{(t+1,n-t)}\left(u+t-x;k\right) + \sum_{t=n+1}^{\infty} p_t \prod_{i=1}^{t-1} q_i$$

Proof. The proof is clear for $k \leq n$ and $n = 0$.

By conditioning on the time of first claim, for $k \leq u + 1$

$$P_u\left(K_n \geq k \mid T > n\right) = \sum_{t=1}^{\infty} P_u\left(U_1 \geq k, ..., U_n \geq k | W_1 = t\right) P\left(W_1 = t\right) \tag{2.8}$$

where $P(W_1 = t) = \prod_{i=1}^{t-1} q_i p_t$. If $t \leq n$ and $k \leq u + 1$ then

$$P_u\left(U_1 \geq k, ..., U_{t-1} \geq k \mid W_1 = t\right) = 1 \tag{2.9}$$

and

$$P_u\left(U_1 \geq k, ..., U_{t-1} \geq k, T > n | W_1 = t\right) = P_u\left(U_t \geq k, ..., U_n \geq k, T^{(t+1,n)} > n - t | W_1 = t\right)$$

If $t > n$ and $k \leq u + 1$ then

$$P_u\left(U_1 \geq k, ..., U_n \geq k \mid W_1 = t\right) = 1. \tag{2.10}$$

By conditioning on the time of first claim, for $t \leq n$ and $k \leq u + 1$

(2.11)

$$P_u\left(U_t \geq k, ..., U_n \geq k, T > n - t \mid W_1 = t\right) = \sum_{x=1}^{u+t-k} P_{u+t-x}\left(K_{n-t}^{(t+1,n)} \geq k, T^{(t+1,n)} > n - t\right)$$

for $t \leq n$ and $k \leq u + 1$. Hence,

$$\gamma^{(1,n)}\left(u;k\right) = \sum_{t=\max(1,k-u+1)}^{n} p_t \prod_{i=1}^{t-1} q_i \sum_{x=1}^{u+t-k} f(x)\gamma^{(t+1,n-t)}\left(u+t-x;k\right) + \sum_{t=n+1}^{\infty} p_t \prod_{i=1}^{t-1} q_i$$

can be obtained by using (2.9),(2.10) and (2.11). Thus the proof is completed. □

Expansion of (2.7) for $n = 1, 2, 3$, which is also recursive formula given in Theorem 2.2, as in follows:

- For $n = 1$

$$\gamma^{(1,1)}\left(u;k\right) = \begin{cases} 1 & , k = u + 1 \\ \frac{1}{\phi^{(1,1)}(u)}\left[a_1\right] & , k < u + 1 \end{cases} \tag{2.12}$$

where

$$a_1 = p_1 \sum_{x=1}^{u+1-k} f(x) + q_1$$

- For $n = 2$

$$\gamma^{(1,2)}(u;k) = \begin{cases} \frac{1}{\phi^{(1,2)}(u)}[a_2] & ,k = u+1 \\ \frac{1}{\phi^{(1,2)}(u)}[a_3] & ,k < u+1 \end{cases} \tag{2.13}$$

where

$$a_2 = q_1 p_2 \sum_{x=1}^{u+2-k} f(x) + q_1 q_2$$

and

$$a_3 = p_1 p_2 \sum_{x=1}^{u+1-k} f(x) \sum_{y=1}^{u+2-x-k} f(y) + p_1 q_2 \sum_{x=1}^{u+1-k} f(x) + q_1 p_2 \sum_{x=1}^{u+2-k} f(x) + q_1 q_2$$

- For $n = 3$

$$\gamma^{(1,3)}(u;k) = \begin{cases} \frac{1}{\phi^{(1,3)}(u)}[a_4] & ,k = u+1 \\ \frac{1}{\phi^{(1,3)}(u)}[a_5] & ,k < u+1 \end{cases} \tag{2.14}$$

where

$$a_4 = q_1 q_2 p_3 \sum_{y=1}^{u+3-k-x} f(y) + q_1 p_2 p_3 \sum_{x=1}^{u+2-k} f(x) \sum_{y=1}^{u+3-k-x} f(y) + q_1 q_2 q_3 + q_1 p_2 q_3 \sum_{y=1}^{u+2-k} f(y)$$

and

$$\begin{aligned} a_5 = {} & p_1 p_2 p_3 \sum_{x=1}^{u+1-k} f(x) \sum_{y=1}^{u+2-k-x} f(y) \sum_{z=1}^{u+3-k-x-y} f(z) \\ & + p_1 q_2 q_3 \sum_{x=1}^{u+1-k} f(x) + p_1 q_2 p_3 \sum_{x=1}^{u+1-k} f(x) \sum_{y=1}^{u+3-k-x} f(y) \\ & + p_1 p_2 q_3 \sum_{x=1}^{u+1-k} f(x) \sum_{y=1}^{u+2-k-x} f(y) + q_1 q_2 p_3 \sum_{y=1}^{u+3-k} f(y) \\ & + q_1 p_2 q_3 \sum_{x=1}^{u+2-k} f(x) + q_1 p_2 p_3 \sum_{x=1}^{u+2-k} f(x) \sum_{y=1}^{u+3-k-x} f(y) \\ & + q_1 q_2 q_3 \end{aligned}$$

3. Case study

As mentioned before, insurance company may face nonhomogeneous claim occurrences probabilities in different periods (e.g. month). For this reason, in this section four different cases are considered for different values of α and u in finite time model and given in Table 1 where $P(I_i = 1) = p_i$ for $i = 1, ..., 12$.

Table 1. Claim occurrence probabilities

Case1 $p_i = 0.01 * i$, $i = 1, ..., 12$	Case2 $p_i = 0.02 * i$, $i = 1, ..., 12$
Case3 $p_i = 0.03 * i$, $i = 1, ..., 12$	Case4 $p_i = 0.04 * i$, $i = 1, ..., 12$

Let claim size distribution be geometric with the following cdf and pmf

$$F(x) = 1 - \alpha^x \ , \ x = 1, 2, ... \quad (3.1)$$

$$f(x) = (1 - \alpha)\alpha^{x-1} \ , \ x = 1, 2, ... \quad (3.2)$$

respectively. It is clear that

$$E(X) = \frac{1}{1 - \alpha} \ , 0 < \alpha < 1. \quad (3.3)$$

According to the cases which are given in Table 1, we obtained expected values and variances of M_n and K_n for the cases, which are given in Table 2 where the claim amount distribution as in (3.2) and $\mu_1 = E(M_n \mid T > n)$,$\sigma_1^2 = Var(M_n \mid T > n)$,$\mu_2 = E(K_n \mid T > n)$ and $\sigma_2^2 = Var(K_n \mid T > n)$.

Table 2. Expected values and variances of M_n and K_n for $\alpha = 9/10$ in cases

u	Cases	μ_1	σ_1^2	μ_2	σ_2^2
4	Case1	14.7363	3.6557	4.9425	0.1632
	Case2	13.7613	5.0934	4.8865	0.3170
	Case3	12.9846	5.4676	4.8333	0.4561
	Case4	12.3376	5.3540	4.7801	0.5920
8	Case1	18.3975	5.2155	8.8418	0.8073
	Case2	17.2249	6.9138	8.6875	1.5438
	Case3	16.3182	7.1493	8.5432	2.2048
	Case4	15.6258	6.7602	8.4036	2.8110

We sketch the graphics of cumulative distribution function of M_n and K_n for cases in Figure 1 and Figure 2 respectively. In the Figures 1 and 2 solid line represents for $u = 4$ and dashed line represents $u = 8$.

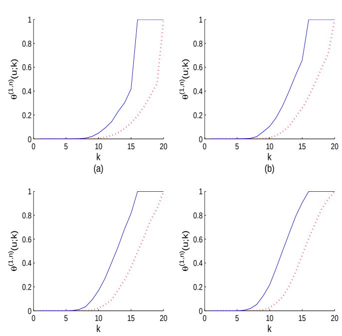

Figure 1. Cumulative Distribution function of M_n given $T > n$
a) Case 1 b) Case 2
c) Case 3 d) Case 4

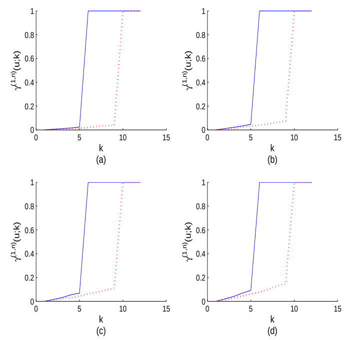

Figure 2. Cumulative Distribution function of K_n given $T > n$
a) Case 1 b) Case 2
c) Case 3 d) Case 4

4. Conclusions

This study presents some characteristical results and distributions of maximum and minimum levels of surplus in compound binomial risk model with nonhomogeneous claim occurrences by different cases which have critical importance for an insurance company. This study may also lead to future studies with stochastic premium income in continuous time model.

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References

[1] Asmussen, S. Ruin Probabilities. World Scientific, Singapore, 2000.

[2] Cheng, S., Gerber, H.U. and Shiu, E.S.W. *Discounted probabilities and ruin theory in the compound Binomial model*, Insurance: Mathematics and Economics, **26**, 239-250, 2000.

[3] Cossette, H., Landriault, D. and Marceau, E. *Ruin probabilities in the compound Markov Binomial model*, Scandinavian Actuarial Journal **4**, 301-323, 2003.

[4] Cossette, H., Landriault, D. and Marceau, E.. *The compound Binomial model defined in a markovian environment*, Insurance: Mathematics and Economics **35** (2), 425-443, 2004.

[5] Cossette, H., Landriault, D. and Marceau, E. *Ruin probabilities in the discrete-time renewal risk model*, Insurance: Mathematics and Economics **38** (2), 309-323, 2006.

[6] Cossette, H. and Marceau, E. *The discrete-time risk model with correlated classes of business*, Insurance: Mathematics and Economics **26** (2), 133-149, 2000.

[7] De Vylder, F. and Goovaerts, M.J. *Bounds for classical ruin probabilities*, Insurance: Mathematics and Economics, **3** (2), 121-131, 1984.

[8] De Vylder, F. and Goovaerts, M.J. *Recursive calculation of finite time ruin probabilities*, Insurance: Mathematics and Economics, **7** (1), 1-7, 1988.

[9] De Vylder, F. and Marceau, E. *Classical numerical ruin probabilities*, Scandinavian Actuarial Journal **2**, 109-123, 1996.

[10] Dickson, D.C.M. *Some comments on the compound Binomial model*, Astin Bulletin **24**, 33-35,1994

[11] Eryilmaz, S. *On distributions of runs in the compound Binomial risk model*, Methodology and Computing in Applied Probability, **16**, 149-159, 2014.

[12] Eryilmaz, S., Tuncel, A. and Tank, F. *On the extremes of surplus process in compound Binomial model*, Selcuk Journal of Applied Mathematics **13** (2), 69-78, 2012.
[13] Gerber, H.U. *Mathematical fun with the compound Binomial process.* Astin Bulletin, **18**, 161-168, 1988.
[14] Liu, G. and Zhao, J. *Joint distributions of some actuarial random vectors in the compound Binomial model*, Insurance: Mathematicsand Economics **40** (1), 95-103, 2007.
[15] Shiu, E.S.W. *The probability of eventual ruin in the compound Binomial model*, Astin Bulletin **19** (2), 179-190, 1989.
[16] Tuncel, A. and Tank, F. *Computational results on the compound binomial risk model with nonhomogeneous claim occurrences*, Journal of Computational and Applied Mathematics, **263**, 69-77, 2014.
[17] Willmot, G.E. *Ruin probabilities in the compound Binomial model*, Insurance: Mathematics and Economics **12** (2), 133-142, 1993.
[18] Yuen, K.C. and Guo, J. *Ruin probabilieties for time-correlated claims in compound Binomial model*, Insurance: Mathematics and Economics **29** (1), 47-57, 2001.

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On the expected discounted penalty function for a risk model with two classes of claims and random incomes

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Abstract

In this paper, we consider a risk model with two independent classes of insurance risks and random incomes. We assume that the two independent claim counting processes are, respectively, the Poisson and the Erlang(2) process. When the individual premium sizes are exponentially distributed, the explicit expressions for the Laplace transforms of the expected discounted penalty functions are derived. We prove that the expected discounted penalty functions satisfy some defective renewal equations. By employing an associated compound geometric distribution, the analytic expressions for the solutions of the defective renewal equations are obtained. Assuming that the distributions of premium sizes have rational Laplace transforms, we also give the explicit representations for the Laplace transforms of the expected discounted penalty functions.

2000 AMS Classification: 60H10, 62P05.

Keywords: Compound Poisson risk model, Erlang risk process, Expected discounted penalty function, Random income.

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1. Introduction

In the actuarial literature, the surplus process of an insurance company is often modeled by the following classic risk process

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0, \tag{1.1}$$

where $u = U(0) \geq 0$ is the initial surplus, $c > 0$ is the constant premium income rate, $N(t)$ counting the number of claims that occurred before time t is a Poisson process, and $\{Y_i\}_{i\geq 1}$ is a sequence of strictly positive random variables (r.v.) representing the claim amounts.

Classic risk models rely on assumption that there is only one class of claims. Although this hypothesis simplifies the study of many risk quantities, it has been proven to be too restrictive in different contexts. In recent years, many authors have studied various aspects of the so-called correlated aggregate claims risk model. Yuen et al. [18] considered the non-ruin probability for a correlated risk process involving two dependent classes of insurance risks, with exponential claims, which can be transformed into a surplus process with two independent classes of insurance risks, for which one claim number process is Poisson and the other is a renewal process with Erlang(2) claim inter-arrival times. Li and Garrido [9] considered a risk process with two classes of independent risks, namely, the compound Poisson process and the renewal process with generalized Erlang(2) inter-arrival times. A system of integro-differential equations for the non-ruin probabilities was derived and explicit results for claim amounts having distributions belonging to the rational family were obtained. A further extension was given by Li and Lu [10]. They derived a system of integro-differential equations for the Gerber-Shiu discounted penalty functions, when the ruin is caused by a claim belonging either to the first or to the second class and obtained explicit results when the claim sizes are exponentially distributed. Recently, Zhang et al. [19] extended the model of Li and Lu [10], by considering the claim number process of the second class to be a renewal process with generalized Erlang(n) inter-arrival times. The authors derived an integro-differential equation system for the Gerber-Shiu functions, and obtained their Laplace transforms when the corresponding Lundberg equation has distinct roots. Chadjiconstantinidis and Papaioannou [4] studied a risk model with two independent classes of insurance risks in the presence of a constant dividend barrier. A system of integro-differential equations with certain boundary conditions for the Gerber-Shiu function was derived and solved. Using systems of integro-differential equations for the moment-generating function as well as for the arbitrary moments of the discounted sum of the dividend payments until ruin, a matrix version of the dividends-penalty was derived. Under the risk models involving two classes of insurance risks described above, the premiums are assumed to be received at a constant rate over time.

Sometimes, the insurance company may have lump sums of income. In order to describe the stochastic income, Boucherie et al. [3] added a compound Poisson process with positive jumps to the Cramér-Lundberg model. The (non-)ruin probabilities for the risk models with stochastic premiums were studied in Boikov [2] and Temnov [12]. Assuming that the premium process is a Poisson process, Bao [1] studied the Gerber-Shiu function in the compound Poisson risk model. Yang and Zhang [17] extended the compound Poisson risk model in Bao [1] to a Sparre Andersen risk model with generalized Erlang(n) interclaim time distribution. Labbe and Sendova [7] considered a risk model with stochastic premiums income, where both the premium size distribution and the claim size distribution are non-lattice. Zhang and Yang [20] extended the model in Labbe and Sendova [7] by assuming that there exists a specific dependence structure among the claim sizes, interclaim times and premium sizes. Xie and Zou [16] construct a risk model with a dependence setting where there exists a specific structure among the time between two claim occurrences, premium sizes and claim sizes. When the claims are subexponentially distributed, the asymptotic formulae for ruin probabilities are obtained.

All risk models with random incomes described in the paragraph above focus on risk model with only one class of insurance risk. Motivated by these papers, we explore analogue problems, but in a risk model with random incomes involving two independent classes of insurance risks. Moreover, we assume that the two independent claim counting processes are, respectively, the Poisson and the Erlang(2) process.

Studying the risk model with two classed of claims and random incomes is of interest in ruin theory for two reasons. The first is to make predictions and give risk measures for smaller business whose premium income is more fluctuant than what is received in well establish and large insurance company. The second reason for study this risk model is to provide insight about how the randomness in premiums' process influences the risk process with two classes of claims. Replacing the constant premium income in risk model with two classes of claims by a stochastic income can also be interpreted as a stepping stone for risk models with two classes of claims that are closer to real phenomena.

The paper is structured as follows: the risk model with two independent classes of insurance risks and random incomes is introduced in Section 2. Assuming that the premiums are exponentially distributed, the explicit expressions for the Laplace transforms of the expected discounted penalty are derived in Section 3 and the defective renewal equations for the expected discounted penalty are obtained in Section 4. By employing an associated compound geometric distribution, the analytic expressions for the solutions of the defective renewal equations are also given in Section 4. In Section 5, given that distributions of the premium sizes have rational Laplace transforms, we derive the explicit representations for the Laplace transforms of the expected discounted penalty functions. Finally, in Section 6, two numerical examples are given.

2. The model

Let us consider the surplus process $U(t)$ of an insurance company,

$$U(t) = u + \sum_{j=1}^{M(t)} X_j - S(t), \quad t \geq 0, \tag{2.1}$$

where $u = U(0)$ is the initial capital and X_j is the jth premium income with distribution function G, probability density function (p.d.f.) f_G, mean μ_G and Laplace transform (LT) $\tilde{f}_G(s) = \int_0^\infty e^{-sx} f_G(x)\mathrm{d}x$. We assume that $M(t)$ is a Poisson process with intensity $\lambda > 0$, then the corresponding premium income inter-arrival times, denoted by $\{W_i\}_{i\geq 1}$, are independent and identically distributed (i.i.d.) exponentially distributed r.v. with parameter λ.

In this paper, we assume that $S(t)$ is generated by two classes of insurance risks, namely

$$S(t) = S_1(t) + S_2(t) = \sum_{i=1}^{N_1(t)} Y_i + \sum_{i=1}^{N_2(t)} Z_i, \quad t \geq 0, \tag{2.2}$$

where $S_i(t)$, $i = 1, 2$, represents the aggregate claims up to time t from the i-th class. Although such models are usually studied in the context of correlated aggregate claims, here we assume that $S_1(t)$ and $S_2(t)$ are stochastically independent.

The r.v. $\{Y_i\}_{i\geq 1}$ are the nonnegative claim severities from the first class, which are i.i.d. random variables with common distribution function F_1, p.d.f. f_1, mean μ_{F_1} and LT $\tilde{f}_1(s) = \int_0^\infty e^{-sx} f_1(x)\mathrm{d}x$. Similarly, $\{Z_i\}_{i\geq 1}$ are the positive claim severities from the second class, also assumed i.i.d. r.v., with common distribution function F_2, p.d.f. f_2, mean μ_{F_2} and LT $\tilde{f}_2(s) = \int_0^\infty e^{-sx} f_2(x)\mathrm{d}x$. The claim number process $N_1(t)$ is assumed to be Poisson with parameter $\lambda_1 > 0$. More specifically, the corresponding claim inter-arrival times, denoted by $\{V_i\}_{i\geq 1}$, are i.i.d. exponentially distributed r.v. with parameter λ_1. In addition, $N_2(t)$ is a renewal process with i.i.d. claim inter-arrival times $\{L_i\}_{i\geq 1}$, which are independent of $\{V_i\}_{i\geq 1}$ and Erlang(2) distributed r.v., i.e. $L_i = L_{i1} + L_{i2}$, where $\{L_{ij}\}_{i\geq 1, j\geq 1}$ are i.i.d. exponentially distributed r.v. with parameter λ_2.

We finally assume that $\{X_i\}_{i\geq 1}$, $\{Y_i\}_{i\geq 1}$ and $\{Z_i\}_{i\geq 1}$ are mutually independent, also independent of $M(t)$, $N_1(t)$ and $N_2(t)$, and $\lambda\mu_G > \lambda_1\mu_{F_1} + \lambda_2/2\mu_{F_2}$, providing a net profit condition.

Denote the ruin time by $T = \inf\{t \geq 0 : U(t) < 0\}$ and ∞ if $U(t) \geq 0$ for all $t \geq 0$. The ruin probability is defined as $\phi(u) = P(T < \infty | U(0) = u), \quad u \geq 0$. Further define J to be the cause-of-ruin r.v., i.e., $J = j$, if the ruin is caused by a claim of class j, $j = 1, 2$, then ruin probability $\phi(u)$ can be decomposed as $\phi(u) = \phi_1(u) + \phi_2(u)$, where $\phi_j(u) = P(T < \infty, J = j | U(0) = u), \quad u \geq 0, \quad j = 1, 2$, is the ruin probability due to a claim of class j. For $\delta \geq 0$, and $j = 1, 2$, the expected discounted penalty (Gerber-Shiu) function at ruin, if the ruin is caused by a claim of class j is defined as

$$\Phi_j(u) = E[e^{-\delta T} w_j(U(T-), |U(T)|) I(T < \infty, J = j) | U(0) = u], \quad u \geq 0, \tag{2.3}$$

where $\delta \geq 0$ is interpreted as the force of interest, $U(T-)$ is the surplus immediately before ruin, $|U(T)|$ is the deficit at ruin, $I(.)$ is the indicator function, and $w_j(x_1, x_2), 0 \leq x_1, x_2 < \infty$, $j = 1, 2$, be the non-negative measurable function defined on $[0, \infty) \times (0, \infty)$. The financial explanations on $w(x_1, x_2)$ can be found in Gerber and Shiu [6]. It is easy to see that choosing different forms of the function $w_j(x_1, x_2)$ in Eq.(2.3) yields different information relating to the deficit at ruin and the surplus immediately before ruin.

In the classical risk model, due to the strong Markov property of the surplus process, the expected discounted penalty function is time homogenous, i.e., it is independent of the time at which the surplus process is observed. However, for our risk model, the expected discounted penalty function functions are no longer time homogeneous, due to the assumption that the claim inter-arrival times from the second class are Erlang(2) distributed. Therefore, for the expected discounted penalty functions, defined in (2.3), we assume that a claim from the second class occurs exactly at time 0. More generally, we can define the expected discounted penalty functions, denoted by $\Phi_j(u, \tau)$, as bivariate functions of current reserve u and the length of time τ, elapsed since the time of the last claim from the second class (the surplus process renews itself at these points). The quantities we are interested in are $\Phi_j(u, 0) = \Phi_j(u)$, $j = 1, 2$, and $u \geq 0$,

$$\Psi_j(u) = E[e^{-\delta(T-t)} w_j(U(T-), |U(T)|) I(T < \infty, J = j) | L_{11} = t, U(t) = u], \tag{2.4}$$

the expected discounted penalty functions at the time of the realization of $\{L_{i1}\}_{i\geq 1}$. Then by the law of total probability, for $j = 1, 2$, we have

$$\Phi_j(u, \tau) = \Phi_j(u)\Pr(L_{11} > \tau) + \Psi_j(u)\Pr(L_{11} < \tau) = e^{-\lambda_2 \tau}\Phi_j(u) + (1 - e^{-\lambda_2 \tau})\Psi_j(u).$$

3. Laplace transforms

Throughout this paper, we will use a hat ˜ to designate the Laplace transform of a function. Given that the premium size is exponentially distributed, the explicit expressions for the Laplace transforms of the expected discounted penalty functions can be derived. For this purpose, we first consider the integral equation satisfied by the expected discounted penalty function.

Let $J=\min\{V_1, L_{11}, W_1\}$, then for $u\geq 0$,

$$\Phi_1(u)=\int_0^\infty\int_0^\infty \Pr(J=t, J=W_1)e^{-\delta t}\Phi_1(u+x)\mathrm{d}G(x)\mathrm{d}t$$

$$+\int_0^\infty \Pr(J=t, J=V_1)e^{-\delta t}\left[\int_0^u \Phi_1(u-y)\mathrm{d}F_1(y)+\int_u^\infty w_1(u,y-u)\mathrm{d}F_1(y)\right]\mathrm{d}t$$

$$+\int_0^\infty \Pr(J=t, J=L_{11})e^{-\delta t}\Psi_1(u)\mathrm{d}t. \tag{3.1}$$

Note that $\Pr(J=W_1)=\lambda/(\lambda+\lambda_1+\lambda_2)$, $\Pr(J=V_1)=\lambda_1/(\lambda+\lambda_1+\lambda_2)$, $\Pr(J=L_{11})=\lambda_2/(\lambda+\lambda_1+\lambda_2)$, $\Pr(J>t|J=W_1)=\Pr(J>t|J=V_1)=\Pr(J>t|J=L_{11})=\exp(-(\lambda+\lambda_1+\lambda_2)t)$.

Plugging the expressions above into (3.1) and making some simplifications, we can get

$$\Phi_1(u)=\frac{\lambda}{\lambda^*+\delta}\int_0^\infty \Phi_1(u+x)\mathrm{d}G(x)+\frac{\lambda_2}{\lambda^*+\delta}\Psi_1(u)$$
$$+\frac{\lambda_1}{\lambda^*+\delta}\left[\int_0^u \Phi_1(u-y)\mathrm{d}F_1(y)+w_1(u)\right], \tag{3.2}$$

where $\lambda^*=\lambda+\lambda_1+\lambda_2$, $w_1(u)=\int_u^\infty w_1(u,y-u)\mathrm{d}F_1(y)$. Similarly, we derive

$$\Psi_1(u)=\frac{\lambda}{\lambda^*+\delta}\int_0^\infty \Psi_1(u+x)\mathrm{d}G(x)+\frac{\lambda_2}{\lambda^*+\delta}\int_0^u \Phi_1(u-y)\mathrm{d}F_2(y)$$
$$+\frac{\lambda_1}{\lambda^*+\delta}\left[\int_0^u \Psi_1(u-y)\mathrm{d}F_1(y)+w_1(u)\right], \tag{3.3}$$

Assume $A_1(u)=\int_0^\infty \Phi_1(u+x)\mathrm{d}G(x)$ and $\bar{A}_1(u)=\int_0^\infty \Psi_1(u+x)\mathrm{d}G(x)$. Taking Laplace transforms in (3.2) and (3.3) and making some simplifications, we have

$$\tilde{\Phi}_1(s)=\frac{\lambda}{\lambda^*+\delta}\tilde{A}_1(s)+\frac{\lambda_2}{\lambda^*+\delta}\tilde{\Psi}_1(s)+\frac{\lambda_1}{\lambda^*+\delta}\left[\tilde{\Phi}_1(s)\tilde{f}_1(s)+\tilde{w}_1(s)\right], \tag{3.4}$$

$$\tilde{\Psi}_1(s)=\frac{\lambda}{\lambda^*+\delta}\tilde{\bar{A}}_1(s)+\frac{\lambda_2}{\lambda^*+\delta}\tilde{\Phi}_1(s)\tilde{f}_2(s)+\frac{\lambda_1}{\lambda^*+\delta}\left[\tilde{\Psi}_1(s)\tilde{f}_1(s)+\tilde{w}_1(s)\right], \tag{3.5}$$

Similar analysis gives

$$\Phi_2(u)=\frac{\lambda}{\lambda^*+\delta}\int_0^\infty \Phi_2(u+x)\mathrm{d}G(x)+\frac{\lambda_2}{\lambda^*+\delta}\Psi_2(u)+\frac{\lambda_1}{\lambda^*+\delta}\int_0^u \Phi_2(u-y)\mathrm{d}F_1(y), \tag{3.6}$$

$$\Psi_2(u)=\frac{\lambda}{\lambda^*+\delta}\int_0^\infty \Psi_2(u+x)\mathrm{d}G(x)+\frac{\lambda_2}{\lambda^*+\delta}\left[\int_0^u \Phi_2(u-y)\mathrm{d}F_2(y)+w_2(u)\right]$$
$$+\frac{\lambda_1}{\lambda^*+\delta}\int_0^u \Psi_2(u-y)\mathrm{d}F_1(y). \tag{3.7}$$

Assume $A_2(u)=\int_0^\infty \Phi_2(u+x)\mathrm{d}G(x)$ and $\bar{A}_2(u)=\int_0^\infty \Psi_2(u+x)\mathrm{d}G(x)$. Taking Laplace transforms in (3.6) and (3.7) and making some simplifications, we obtain

$$\tilde{\Phi}_2(s)=\frac{\lambda}{\lambda^*+\delta}\tilde{A}_2(s)+\frac{\lambda_2}{\lambda^*+\delta}\tilde{\Psi}_2(s)+\frac{\lambda_1}{\lambda^*+\delta}\tilde{\Phi}_2(s)\tilde{f}_1(s), \tag{3.8}$$

$$\tilde{\Psi}_2(s)=\frac{\lambda}{\lambda^*+\delta}\tilde{\bar{A}}_2(s)+\frac{\lambda_2}{\lambda^*+\delta}\left[\tilde{\Phi}_2(s)\tilde{f}_2(s)+\tilde{w}_2(s)\right]+\frac{\lambda_1}{\lambda^*+\delta}\tilde{\Psi}_2(s)\tilde{f}_1(s). \tag{3.9}$$

Now, we introduce the Dickson-Hipp operator T_r provided by Dickson and Hipp [5]. Define the Dickson-Hipp operator T_r as be

$$T_rf(x)=\int_x^\infty e^{-r(y-x)}f(y)\mathrm{d}y, \quad x\geq 0,$$

where $f(x)$ is a real-valued function, r is a complex number. It is easy to see that $T_s f(0) = \tilde{f}(s)$, and that for distinct r_1 and r_2,

$$T_{r_1}T_{r_2}f(x) = T_{r_2}T_{r_1}f(x) = \frac{T_{r_1}f(x) - T_{r_2}f(x)}{r_2 - r_1}, \quad x \geq 0.$$

If $r_1 = r_2 = r$,

$$T_{r_1}T_{r_2}f(x) = \int_x^\infty (y-x)e^{-r(y-x)}f(y)\mathrm{d}y, \quad x \geq 0.$$

The properties for the Dickson-Hipp operator can also be found in Dickson and Hipp [5], Li and Garrido [8], Xie and Zou [15].

Suppose the premium sizes are exponentially distributed, i.e., $G(x) = 1 - e^{-\frac{x}{\mu_G}}$, for $\mu_G > 0$. Taking Laplace transform of $A_i(u)$, $i = 1, 2$, and using the Dickson-Hipp operator, we can get

$$\begin{aligned}\tilde{A}_i(s) &= \int_0^\infty e^{-su}\int_0^\infty \Phi_i(u+x)\frac{e^{-\frac{x}{\mu_G}}}{\mu_G}\mathrm{d}x\mathrm{d}u = \int_0^\infty\int_0^\infty e^{-su}\Phi_i(u+x)\mathrm{d}u\frac{e^{-\frac{x}{\mu_G}}}{\mu_G}\mathrm{d}x\\ &= \int_0^\infty T_s\Phi_i(x)\frac{e^{-\frac{x}{\mu_G}}}{\mu_G}\mathrm{d}x = \frac{1}{\mu_G}T_{\frac{1}{\mu_G}}T_s\Phi_i(0) = \frac{\tilde{\Phi}_i(s) - \tilde{\Phi}_i(\frac{1}{\mu_G})}{1 - s\mu_G}, \quad i = 1, 2. \qquad (3.10)\end{aligned}$$

Similarly,

$$\tilde{\tilde{A}}_i(s) = \frac{\tilde{\Psi}_i(s) - \tilde{\Psi}_i(\frac{1}{\mu_G})}{1 - s\mu_G}, \quad i = 1, 2. \qquad (3.11)$$

Combining the above results with (3.4), (3.5), (3.8) and (3.9), respectively, we derive

$$\tilde{\Phi}_1(s) = \frac{-\left(1 - \frac{\lambda}{(\lambda^*+\delta)(1-s\mu_G)} - \frac{\lambda_1\tilde{f}_1(s)}{\lambda^*+\delta}\right)\left(\frac{\lambda\tilde{\Phi}_1(\frac{1}{\mu_G})}{(\lambda^*+\delta)(1-s\mu_G)} - \frac{\lambda_1\tilde{w}_1(s)}{\lambda^*+\delta}\right) - \frac{\lambda_2\varpi_1(s)}{\lambda^*+\delta}}{\left(1 - \frac{\lambda}{(\lambda^*+\delta)(1-s\mu_G)} - \frac{\lambda_1\tilde{f}_1(s)}{\lambda^*+\delta}\right)^2 - \frac{\lambda_2^2\tilde{f}_2(s)}{(\lambda^*+\delta)^2}}, \qquad (3.12)$$

$$\tilde{\Phi}_2(s) = \frac{-\frac{\lambda\tilde{\Phi}_2(\frac{1}{\mu_G})(1 - \frac{\lambda}{(\lambda^*+\delta)(1-s\mu_G)} - \frac{\lambda_1\tilde{f}_1(s)}{\lambda^*+\delta})}{(\lambda^*+\delta)(1-s\mu_G)} - \frac{\lambda_2\varpi_2(s)}{\lambda^*+\delta}}{\left(1 - \frac{\lambda}{(\lambda^*+\delta)(1-s\mu_G)} - \frac{\lambda_1\tilde{f}_1(s)}{\lambda^*+\delta}\right)^2 - \frac{\lambda_2^2\tilde{f}_2(s)}{(\lambda^*+\delta)^2}}. \qquad (3.13)$$

where $\varpi_i(s) = \frac{\lambda\tilde{\Psi}_i(\frac{1}{\mu_G})}{(\lambda^*+\delta)(1-s\mu_G)} - \frac{\lambda_i\tilde{w}_i(s)}{\lambda^*+\delta}$, $i = 1, 2$.

To obtain $\tilde{\Phi}_1(s)$ and $\tilde{\Phi}_2(s)$, we still have to determine $\tilde{\Phi}_1(\frac{1}{\mu_G})$, $\tilde{\Psi}_1(\frac{1}{\mu_G})$, $\tilde{\Phi}_2(\frac{1}{\mu_G})$, and $\tilde{\Psi}_2(\frac{1}{\mu_G})$. For this purpose, we discuss analytically the zeros of the common denominators of (3.12) and (3.14), i.e., the roots of following equation

$$\left(1 - \frac{\lambda}{(\lambda^*+\delta)(1-s\mu_G)} - \frac{\lambda_1\tilde{f}_1(s)}{\lambda^*+\delta}\right)^2 - \frac{\lambda_2^2\tilde{f}_2(s)}{(\lambda^*+\delta)^2} = 0. \qquad (3.14)$$

3.1. Lemma. 1. *For $\delta > 0$, Eq.(3.14) has exactly two roots, say, $\rho_1(\delta)$, and $\rho_2(\delta)$, in the right half complex plane, i.e.,* **Re** $\rho_i(\delta) > 0$ *for $i = 1, 2$.*

Proof. Eq.(3.14) can be simplified as

$$\left(1 - s\mu_G - \frac{\lambda}{\lambda^*+\delta} - \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}\right)^2 - \frac{(\lambda_2(1-s\mu_G))^2\tilde{f}_2(s)}{(\lambda^*+\delta)^2} = 0.$$

Let $r > 0$ be a sufficiently large number, and define $\mathbf{C}_r$ as the contour containing the imaginary axis running from $-\mathbf{i}r$ to $\mathbf{i}r$ and a half circle with radius r running clockwise from $\mathbf{i}r$ to $-\mathbf{i}r$. Firstly, we apply Rouché's theorem to prove that equation

$$1 - s\mu_G - \frac{\lambda}{\lambda^*+\delta} - \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta} = 0,$$

has exactly one root inside $\mathbf{C}_r$. When s on the imaginary axis, we have,

$$\frac{\left|\frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}\right|}{\left|1-s\mu_G-\frac{\lambda}{\lambda^*+\delta}\right|}=\frac{\left|\frac{\lambda_1(1-s\mu_G)}{\lambda^*+\delta}\right||\tilde{f}_1(s)|}{\left|\frac{\lambda_1+\lambda_2+\delta}{\lambda^*+\delta}-s\mu_G\right|}\leq\frac{\left|\frac{\lambda_1(1-s\mu_G)}{\lambda^*+\delta}\right|}{\left|\frac{\lambda_1+\lambda_2+\delta}{\lambda^*+\delta}-s\mu_G\right|}<1.$$

For s on the half circle, we have for $\forall\ \varepsilon>0$,

$$\frac{\left|\frac{1}{\mu_G}-s\right|}{\left|\frac{\lambda_1+\lambda_2+\delta}{(\lambda^*+\delta)\mu_G}-s\right|}<1+\varepsilon,$$

when r is sufficiently large. In particular, for $\varepsilon=\frac{\lambda+\lambda_2+\delta}{\lambda_1}$, there exists $r_0>0$ such that when $r>r_0$, we get

$$\frac{\left|\frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}\right|}{\left|1-s\mu_G-\frac{\lambda}{\lambda^*+\delta}\right|}\leq\frac{\lambda_1}{\lambda^*+\delta}\frac{\left|\frac{1}{\mu_G}-s\right|}{\left|\frac{\lambda_1+\lambda_2+\delta}{(\lambda^*+\delta)\mu_G}-s\right|}<\frac{\lambda_1}{\lambda^*+\delta}(1+\varepsilon)\leq 1.$$

That is to say, we show that for $s\in\mathbf{C}_r$, the module $|1-s\mu_G-\frac{\lambda}{\lambda^*+\delta}|>|\frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}|$. Using Rouché's theorem, we conclude that the number of roots of the equation $1-s\mu_G-\frac{\lambda}{\lambda^*+\delta}-\frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}=0$ equals the number of roots of the equation $1-s\mu_G-\frac{\lambda}{\lambda^*+\delta}=0$ inside $\mathbf{C}_r$. Moreover, the latter has exactly one root inside $\mathbf{C}_r$. It follows that $1-s\mu_G-\frac{\lambda}{\lambda^*+\delta}-\frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}=0$ has exactly one positive real root inside $\mathbf{C}_r$.

Secondly, we apply Rouché's theorem and the result above to prove this Lemma. When s on the imaginary axis

$$\begin{aligned}&\left|\left(1-s\mu_G-\frac{\lambda}{\lambda^*+\delta}-\frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}\right)^2\right|\\&\geq\left(\left|\frac{\lambda_1+\lambda_2+\delta}{\lambda^*+\delta}-s\mu_G\right|-\left|\frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}\right|\right)^2\\&>\left(\frac{\lambda_2\,|1-s\mu_G|}{\lambda^*+\delta}\right)^2\geq\left|\frac{(\lambda_2(1-s\mu_G))^2\tilde{f}_2(s)}{(\lambda^*+\delta)^2}\right|.\end{aligned}$$

For s on the half circle, we get for $\varepsilon=\frac{\lambda}{\lambda_1+\lambda_2+\delta}$, there exists $r_1>0$ such that when $r>r_1$,

$$\frac{\left|\frac{(\lambda_1+\lambda_2+\delta)(1-s\mu_G)}{\lambda^*+\delta}\right|}{\left|1-s\mu_G-\frac{\lambda}{\lambda^*+\delta}\right|}\leq\frac{\lambda_1+\lambda_2+\delta}{\lambda^*+\delta}\frac{\left|\frac{1}{\mu_G}-s\right|}{\left|\frac{\lambda_1+\lambda_2+\delta}{(\lambda^*+\delta)\mu_G}-s\right|}<\frac{\lambda_1+\lambda_2+\delta}{\lambda^*+\delta}(1+\varepsilon)\leq 1,$$

then

$$\begin{aligned}&\left|\left(1-s\mu_G-\frac{\lambda}{\lambda^*+\delta}-\frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}\right)^2\right|\\&>\left(\frac{(\lambda_1+\lambda_2+\delta)\,|1-s\mu_G|}{\lambda^*+\delta}-\frac{\lambda_1\,|1-s\mu_G|}{\lambda^*+\delta}\right)^2>\left|\frac{(\lambda_2(1-s\mu_G))^2\tilde{f}_2(s)}{(\lambda^*+\delta)^2}\right|.\end{aligned}$$

That is to say, for $s\in\mathbf{C}_r$,

$$\left|\left(1-s\mu_G-\frac{\lambda}{\lambda^*+\delta}-\frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}\right)^2\right|>\left|\frac{(\lambda_2(1-s\mu_G))^2\tilde{f}_2(s)}{(\lambda^*+\delta)^2}\right|.$$

Using Rouché's theorem, we conclude that the number of roots of the equation

$$\left(1-s\mu_G-\frac{\lambda}{\lambda^*+\delta}-\frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}\right)^2-\frac{(\lambda_2(1-s\mu_G))^2\tilde{f}_2(s)}{(\lambda^*+\delta)^2}=0$$

equals the number of roots of the equation $\left(1 - s\mu_G - \frac{\lambda}{\lambda^*+\delta} - \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta}\right)^2 = 0$ inside $\mathbf{C}_r$. Moreover, by the discussion in the paragraph above, the latter has exactly two roots with positive real parts inside $\mathbf{C}_r$. It follows from all above that Eq.(3.14) has exactly two distinct positive real roots, say, $\rho_1(\delta)$, and $\rho_2(\delta)$, inside $\mathbf{C}_r$. Finally, letting $r \to \infty$ completes the proof. □

Denote the root with the smaller module by $\rho_1(\delta)$. It is easily seen that $\rho_1(\delta) \to 0^+$ as $\delta \to 0^+$. In the rest of the paper, we denote these two roots $\rho_i(\delta)$ by ρ_i, $i = 1, 2$, for simplicity.

Since $\tilde{\Phi}(s)$ is finite for all s with $\mathbf{Re}\ s \geq 0$, we know ρ_1 and ρ_2 must be zeros of the numerators of (3.12) and (3.13). From Eq.(3.12), we can give the following equations for $\tilde{\Phi}_1(\frac{1}{\mu_G})$ and $\tilde{\Psi}_1(\frac{1}{\mu_G})$,

$$-(1 - \frac{\lambda}{(\lambda^*+\delta)(1-\rho_i\mu_G)} - \frac{\lambda_1\tilde{f}_1(\rho_i)}{\lambda^*+\delta})(\frac{\lambda\tilde{\Phi}_1(\frac{1}{\mu_G})}{(\lambda^*+\delta)(1-\rho_i\mu_G)} - \frac{\lambda_1\tilde{w}_1(\rho_i)}{\lambda^*+\delta})$$
$$= \frac{\lambda_2(\frac{\lambda\tilde{\Psi}_1(\frac{1}{\mu_G})}{(\lambda^*+\delta)(1-\rho_i\mu_G)} - \frac{\lambda_1\tilde{w}_1(\rho_i)}{\lambda^*+\delta})}{\lambda^*+\delta}, \tag{3.15}$$

where $i = 1, 2$. From Eq.(3.13), we can give the following equations for $\tilde{\Phi}_2(\frac{1}{\mu_G})$ and $\tilde{\Psi}_2(\frac{1}{\mu_G})$,

$$-\frac{\lambda\tilde{\Phi}_2(\frac{1}{\mu_G})\left(1 - \frac{\lambda}{(\lambda^*+\delta)(1-\rho_i\mu_G)} - \frac{\lambda_1\tilde{f}_1(\rho_i)}{\lambda^*+\delta}\right)}{(\lambda^*+\delta)(1-\rho_i\mu_G)} = \frac{\lambda_2\left(\frac{\lambda\tilde{\Psi}_2(\frac{1}{\mu_G})}{(\lambda^*+\delta)(1-\rho_i\mu_G)} - \frac{\lambda_2\tilde{w}_2(\rho_i)}{\lambda^*+\delta}\right)}{\lambda^*+\delta}. \tag{3.16}$$

By solving linear equations (3.15) and (3.16), we can get $\tilde{\Phi}_i(\frac{1}{\mu_G})$ and $\tilde{\Psi}_i(\frac{1}{\mu_G})$, $i = 1, 2$. Then $\tilde{\Phi}_1(s)$ and $\tilde{\Phi}_2(s)$ can also be obtained.

4. Defective renewal equations

In this section, we study the defective renewal equations satisfied by the two expected discounted penalty functions in the risk model with two classes of claims and random income.

Based on the results of (3.12) and (3.13), the Laplace transforms of $\Phi_1(u)$ and $\Phi_2(u)$ can be simplified as

$$\tilde{\Phi}_1(s) = \frac{\tilde{f}_{1,1}(s) + \tilde{f}_{1,2}(s)}{\tilde{h}_1(s) - \tilde{h}_2(s)}, \tag{4.1}$$

$$\tilde{\Phi}_2(s) = \frac{\tilde{f}_{2,1}(s) + \tilde{f}_{2,2}(s)}{\tilde{h}_1(s) - \tilde{h}_2(s)}, \tag{4.2}$$

where $\tilde{h}_1(s) = (1 - s\mu_G - \frac{\lambda}{\lambda^*+\delta})^2$, $\tilde{h}_2(s) = \frac{s^2\mu_G^2}{(\lambda^*+\delta)^2}(2\lambda_1(\lambda^*+\delta)\tilde{f}_1(s) + \lambda_2^2\tilde{f}_2(s) - \lambda_1^2\tilde{f}_1^2(s)) + \frac{2s\mu_G}{(\lambda^*+\delta)^2}((2\lambda_1(\lambda^*+\delta) - \lambda\lambda_1)\tilde{f}_1(s) + \lambda_2^2\tilde{f}_2(s) - \lambda_1^2\tilde{f}_1^2(s)) + \frac{1}{(\lambda^*+\delta)^2}((2\lambda_1(\lambda^*+\delta) - 2\lambda\lambda_1)\tilde{f}_1(s) + \lambda_2^2\tilde{f}_2(s) - \lambda_1^2\tilde{f}_1^2(s))$, $\tilde{f}_{1,1}(s) = -(1 - s\mu_G - \frac{\lambda}{\lambda^*+\delta})\frac{\lambda\tilde{\Phi}_1(\frac{1}{\mu_G})}{\lambda^*+\delta} - \frac{\lambda\lambda_2(1-s\mu_G)\tilde{\Psi}_1(\frac{1}{\mu_G})}{(\lambda^*+\delta)^2}$, $\tilde{f}_{1,2}(s) = \frac{\lambda_1}{(\lambda^*+\delta)^2}((\lambda^*+\lambda_2-\lambda_1+\delta)\tilde{w}_1(s) - \lambda_1\tilde{f}_1(s)\tilde{w}_1(s) + \lambda\tilde{\Phi}_1(\frac{1}{\mu_G})\tilde{f}_1(s)) - \frac{s\mu_G\lambda_1}{(\lambda^*+\delta)^2}((2(\lambda^*+\lambda_2+\delta) - \lambda)\tilde{w}_1(s) - 2\lambda_1\tilde{f}_1(s)\tilde{w}_1(s) + \lambda\tilde{\Phi}_1(\frac{1}{\mu_G})\tilde{f}_1(s)) + \frac{s^2\mu_G^2\lambda_1}{(\lambda^*+\delta)^2}((\lambda^*+\lambda_2+\delta)\tilde{w}_1(s) - \lambda_1\tilde{f}_1(s)\tilde{w}_1(s))$, $\tilde{f}_{2,1}(s) = -(1 - s\mu_G - \frac{\lambda}{\lambda^*+\delta})\frac{\lambda\tilde{\Phi}_2(\frac{1}{\mu_G})}{\lambda^*+\delta} - \frac{\lambda\lambda_2(1-s\mu_G)\tilde{\Psi}_2(\frac{1}{\mu_G})}{(\lambda^*+\delta)^2}$, $\tilde{f}_{2,2}(s) = \frac{1}{(\lambda^*+\delta)^2}(\lambda_2^2\tilde{w}_2(s) + \lambda\lambda_1\tilde{f}_1(s)\tilde{\Phi}_2(\frac{1}{\mu_G})) - \frac{s\mu_G}{(\lambda^*+\delta)^2}(2\lambda_2^2\tilde{w}_2(s) + \lambda\lambda_1\tilde{f}_1(s)\tilde{\Phi}_2(\frac{1}{\mu_G})) + \frac{s^2\mu_G^2\lambda_2^2\tilde{w}_2(s)}{(\lambda^*+\delta)^2}$. Define $f_{1,1}(u)$, $f_{1,2}(u)$, $f_{2,1}(u)$, $f_{2,2}(u)$, $h_1(u)$ and $h_2(u)$ as the inverse image functions of $\tilde{f}_{1,1}(s)$, $\tilde{f}_{1,2}(s)$,

$\tilde{f}_{2,1}(s)$, $\tilde{f}_{2,2}(s)$, $\tilde{h}_1(s)$, and $\tilde{h}_2(s)$, i.e., $T_s f_{i,j}(0) = \tilde{f}_{i,j}(s)$ and $T_s h_i(0) = \tilde{h}_i(s)$, $i = 1, 2$, $j = 1, 2$.

4.1. Proposition. 1. *The Laplace transform $\tilde{\Phi}_i(s)$ of the expected discounted penalty function satisfies*

$$\tilde{\Phi}_i(s) = \frac{T_s T_{\rho_2} T_{\rho_1} h_2(0)}{\mu_G^2} \tilde{\Phi}_i(s) + \frac{T_s T_{\rho_2} T_{\rho_1} f_{i,2}(0)}{\mu_G^2}, \quad i = 1, 2. \tag{4.3}$$

Proof. Since $\tilde{\Phi}_i(s)$ $(i = 1, 2)$ is analytic for all s with $\mathbf{Re}\ s \geq 0$, we know ρ_1 and ρ_2 are zeros of the numerators of which means that $\tilde{f}_{i,1}(\rho_j) = -\tilde{f}_{i,2}(\rho_j)$ for $i = 1, 2$, $j = 1, 2$. Because $\tilde{f}_{i,1}(s)$ is a polynomial of degree 1, applying the Lagrange interpolating theorem, we have

$$\tilde{f}_{i,1}(s) = -\frac{\tilde{f}_{i,2}(\rho_1)(s - \rho_2) - \tilde{f}_{i,2}(\rho_2)(s - \rho_1)}{\rho_1 - \rho_2},$$

which yields

$$\begin{aligned}
\tilde{f}_{i,1}(s) + \tilde{f}_{i,2}(s) &= \frac{(s - \rho_2)\left(\tilde{f}_{i,2}(s) - \tilde{f}_{i,2}(\rho_1)\right) - (s - \rho_1)\left(\tilde{f}_{i,2}(s) - \tilde{f}_{i,2}(\rho_2)\right)}{\rho_1 - \rho_2} \\
&= (s - \rho_1)(s - \rho_2) T_s T_{\rho_2} T_{\rho_1} f_{i,2}(0).
\end{aligned} \tag{4.4}$$

Obviously, an simple expression for the denominator of $\tilde{\Phi}_i(s)$, $i = 1, 2$, can be dealt with in a similar way. Due to Lemma 1, we get that $\tilde{h}_1(\rho_i) = \tilde{h}_2(\rho_i)$ for $i = 1, 2$. Similarly, because $\tilde{h}_1(s)$ is a polynomial of degree 2, using the Lagrange interpolating theorem, we have

$$\begin{aligned}
\tilde{h}_1(s) &= \tilde{h}_1(0) \frac{(s - \rho_1)(s - \rho_2)}{\rho_1 \rho_2} + s \left(\frac{\tilde{h}_1(\rho_1)}{\rho_1} \frac{s - \rho_2}{\rho_1 - \rho_2} + \frac{\tilde{h}_1(\rho_2)}{\rho_2} \frac{s - \rho_1}{\rho_2 - \rho_1} \right) \\
&= \tilde{h}_1(0) \frac{(s - \rho_1)(s - \rho_2)}{\rho_1 \rho_2} + (s - \rho_1)(s - \rho_2) \left(\frac{\tilde{h}_2(\rho_1)}{\rho_1} \frac{1}{\rho_1 - \rho_2} + \frac{\tilde{h}_2(\rho_2)}{\rho_2} \frac{1}{\rho_2 - \rho_1} \right) \\
&\quad + \tilde{h}_2(\rho_1) \frac{s - \rho_2}{\rho_1 - \rho_2} + \tilde{h}_2(\rho_2) \frac{s - \rho_1}{\rho_2 - \rho_1}.
\end{aligned}$$

Using the result above and recalling the Property 6 of the Dickson-Hipp operator derived in Li and Garrido [8], $\tilde{h}_1(s) - \tilde{h}_2(s)$ can be rewritten as

$$\begin{aligned}
\tilde{h}_1(s) - \tilde{h}_2(s) &= \tilde{h}_1(0) \frac{(s - \rho_1)(s - \rho_2)}{\rho_1 \rho_2} + (s - \rho_1)(s - \rho_2) \left(\frac{\tilde{h}_2(\rho_1)}{\rho_1(\rho_1 - \rho_2)} + \frac{\tilde{h}_2(\rho_2)}{\rho_2(\rho_2 - \rho_1)} \right) \\
&\quad - \left(\tilde{h}_2(s) - \tilde{h}_2(\rho_1) \frac{s - \rho_2}{\rho_1 - \rho_2} - \tilde{h}_2(\rho_2) \frac{s - \rho_1}{\rho_2 - \rho_1} \right) \\
&= (s - \rho_1)(s - \rho_2) \left(T_0 T_{\rho_2} T_{\rho_1} h_1(0) - T_s T_{\rho_2} T_{\rho_1} h_2(0) \right).
\end{aligned} \tag{4.5}$$

It is easy to check that $T_0 T_{\rho_2} T_{\rho_1} h_1(0) = \mu_G^2$ which makes (4.5) become

$$\tilde{h}_1(s) - \tilde{h}_2(s) = (s - \rho_1)(s - \rho_2) \left(\mu_G^2 - T_s T_{\rho_2} T_{\rho_1} h_2(0) \right). \tag{4.6}$$

Invoking (4.4) and (4.6) into (4.1) and (4.2), we can derive $\tilde{\Phi}_i(s) = \frac{T_s T_{\rho_2} T_{\rho_1} f_{i,2}(0)}{\mu_G^2 - T_s T_{\rho_2} T_{\rho_1} h_2(0)}$ which gives (4.3). The result of Proposition 1 is proved. □

Now, we are ready to obtain the defective renewal equations for $\Phi_i(u)$, $i = 1, 2$.

4.2. Proposition. 2. $\Phi_i(u)$ *satisfies the following defective renewal equation*

$$\Phi_i(u) = \kappa_\delta \int_0^u \Phi_i(u-y)\varsigma(y)\mathrm{d}y + \xi_i(u), \quad i=1,2, \tag{4.7}$$

where

$$\kappa_\delta = \frac{(2\lambda_1(\lambda^*+\delta)-2\lambda\lambda_1)T_0T_{\rho_2}T_{\rho_1}f_1(0)+\lambda_2^2T_0T_{\rho_2}T_{\rho_1}f_2(0)-\lambda_1^2T_0T_{\rho_2}T_{\rho_1}f_1*f_1(0)}{(\lambda^*+\delta)^2\mu_G^2}$$
$$+\frac{2}{(\lambda^*+\delta)^2\mu_G}((2\lambda_1(\lambda^*+\delta)-\lambda\lambda_1)(\rho_1T_0T_{\rho_2}T_{\rho_1}f_1(0)-T_0T_{\rho_2}f_1(0))$$
$$+\lambda_2^2(\rho_1T_0T_{\rho_2}T_{\rho_1}f_2(0)-T_0T_{\rho_2}f_2(0))-\lambda_1^2(\rho_1T_0T_{\rho_2}T_{\rho_1}f_1*f_1(0)-T_0T_{\rho_2}f_1*f_1(0)))$$
$$+\frac{1}{(\lambda^*+\delta)^2}(2\lambda_1(\lambda^*+\delta)(1-(\rho_2+\rho_1)T_0T_{\rho_2}f_1(0)$$
$$+\rho_1^2T_0T_{\rho_2}T_{\rho_1}f_1(0))+\lambda_2^2(1-(\rho_2+\rho_1)T_0T_{\rho_2}f_2(0)+\rho_1^2T_0T_{\rho_2}T_{\rho_1}f_2(0))$$
$$-\lambda_1^2(1-(\rho_2+\rho_1)T_0T_{\rho_2}f_1*f_1(0)+\rho_1^2T_0T_{\rho_2}T_{\rho_1}f_1*f_1(0))),$$

$$\varsigma(y) = \frac{1}{\kappa_\delta}\{\frac{(2\lambda_1(\lambda^*+\delta)-2\lambda\lambda_1)T_{\rho_2}T_{\rho_1}f_1(y)+\lambda_2^2T_{\rho_2}T_{\rho_1}f_2(y)-\lambda_1^2T_{\rho_2}T_{\rho_1}f_1*f_1(y)}{(\lambda^*+\delta)^2\mu_G^2}$$
$$+\frac{2}{(\lambda^*+\delta)^2\mu_G}((2\lambda_1(\lambda^*+\delta)-\lambda\lambda_1)(\rho_1T_{\rho_2}T_{\rho_1}f_1(y)-T_{\rho_2}f_1(y))$$
$$+\lambda_2^2(\rho_1T_{\rho_2}T_{\rho_1}f_2(y)-T_{\rho_2}f_2(y))-\lambda_1^2(\rho_1T_{\rho_2}T_{\rho_1}f_1*f_1(y)-T_{\rho_2}f_1*f_1(y)))$$
$$+\frac{1}{(\lambda^*+\delta)^2}(2\lambda_1(\lambda^*+\delta)(f_1(y)-(\rho_2+\rho_1)T_{\rho_2}f_1(y)$$
$$+\rho_1^2T_{\rho_2}T_{\rho_1}f_1(y))+\lambda_2^2(f_2(y)-(\rho_2+\rho_1)T_{\rho_2}f_2(y)+\rho_1^2T_{\rho_2}T_{\rho_1}f_2(y))$$
$$-\lambda_1^2(f_1*f_1(y)-(\rho_2+\rho_1)T_{\rho_2}f_1*f_1(y)+\rho_1^2T_{\rho_2}T_{\rho_1}f_1*f_1(y)))\ \}.$$

and

$$\xi_1(u) = \frac{\lambda_1((\lambda^*+\lambda_2-\lambda_1+\delta)T_{\rho_2}T_{\rho_1}w_1(u)-\lambda_1T_{\rho_2}T_{\rho_1}f_1*w_1(u)+\lambda\tilde{\Phi}_1(\frac{1}{\mu_G})T_{\rho_2}T_{\rho_1}f_1(u))}{(\lambda^*+\delta)^2\mu_G^2}$$
$$-\frac{\lambda_1}{(\lambda^*+\delta)^2\mu_G}(((2(\lambda^*+\lambda_2+\delta)-\lambda)(\rho_1T_{\rho_2}T_{\rho_1}w_1(u)-T_{\rho_2}w_1(u))$$
$$-2\lambda_1(\rho_1T_{\rho_2}T_{\rho_1}f_1*w_1(u)-T_{\rho_2}f_1*w_1(u))+\lambda\tilde{\Phi}_1(\frac{1}{\mu_G})(\rho_1T_{\rho_2}T_{\rho_1}f_1(u)-T_{\rho_2}f_1(u)))$$
$$+\frac{\lambda_1}{(\lambda^*+\delta)^2}((\lambda^*+\lambda_2+\delta)(w_1(u)-(\rho_2+\rho_1)T_{\rho_2}w_1(u)+\rho_1^2T_{\rho_2}T_{\rho_1}w_1(u))$$
$$-\lambda_1(f_1*w_1(u)-(\rho_2+\rho_1)T_{\rho_2}f_1*w_1(u)+\rho_1^2T_{\rho_2}T_{\rho_1}f_1*w_1(u))),$$

$$\xi_2(u) = \frac{1}{(\lambda^*+\delta)^2\mu_G^2}(\lambda_2^2T_{\rho_2}T_{\rho_1}w_2(u)+\lambda\lambda_1\tilde{\Phi}_2(\frac{1}{\mu_G})T_{\rho_2}T_{\rho_1}f_1(u))$$
$$-\frac{2\lambda_2^2(\rho_1T_{\rho_2}T_{\rho_1}w_2(u)-T_{\rho_2}w_2(u))+\lambda\lambda_1\tilde{\Phi}_2(\frac{1}{\mu_G})(\rho_1T_{\rho_2}T_{\rho_1}f_1(u)-T_{\rho_2}f_1(u))}{(\lambda^*+\delta)^2\mu_G}$$
$$+\frac{\lambda_2^2}{(\lambda^*+\delta)^2}(w_2(u)-(\rho_2+\rho_1)T_{\rho_2}w_2(u)+\rho_1^2T_{\rho_2}T_{\rho_1}w_2(u)).$$

Proof. By employing the property of the Dickson-Hipp operator, we deduce

$$\frac{\frac{\tilde f(s)-\tilde f(\rho_2)}{s-\rho_2}-\frac{\tilde f(s)-\tilde f(\rho_1)}{s-\rho_1}}{\rho_2-\rho_1}=\frac{T_sT_{\rho_1}f(0)-T_sT_{\rho_2}f(0)}{\rho_2-\rho_1}=T_sT_{\rho_2}T_{\rho_1}f(0), \tag{4.8}$$

$$\frac{\frac{s\tilde f(s)-\rho_2\tilde f(\rho_2)}{s-\rho_2}-\frac{s\tilde f(s)-\rho_1\tilde f(\rho_1)}{s-\rho_1}}{\rho_2-\rho_1}=\rho_1T_sT_{\rho_2}T_{\rho_1}f(0)-T_sT_{\rho_2}f(0), \tag{4.9}$$

$$\frac{\frac{s^2\tilde f(s)-\rho_2^2\tilde f(\rho_2)}{s-\rho_2}-\frac{s^2\tilde f(s)-\rho_1^2\tilde f(\rho_1)}{s-\rho_1}}{\rho_2-\rho_1}=\tilde f(s)-(\rho_2+\rho_1)T_sT_{\rho_2}f(0)+\rho_1^2T_sT_{\rho_2}T_{\rho_1}f(0). \tag{4.10}$$

Recalling the definition of the Dickson-Hipp operator T_r and together with (4.8)-(4.10), one finds

$$\begin{aligned}
&T_sT_{\rho_2}T_{\rho_1}h_2(0)\\
&=\frac{(2\lambda_1(\lambda^*+\delta)-2\lambda\lambda_1)T_sT_{\rho_2}T_{\rho_1}f_1(0)+\lambda_2^2T_sT_{\rho_2}T_{\rho_1}f_2(0)-\lambda_1^2T_sT_{\rho_2}T_{\rho_1}f_1*f_1(0)}{(\lambda^*+\delta)^2}\\
&+\frac{2\mu_G}{(\lambda^*+\delta)^2}((2\lambda_1(\lambda^*+\delta)-\lambda\lambda_1)(\rho_1T_sT_{\rho_2}T_{\rho_1}f_1(0)-T_sT_{\rho_2}f_1(0))\\
&+\lambda_2^2(\rho_1T_sT_{\rho_2}T_{\rho_1}f_2(0)-T_sT_{\rho_2}f_2(0))-\lambda_1^2(\rho_1T_sT_{\rho_2}T_{\rho_1}f_1*f_1(0)-T_sT_{\rho_2}f_1*f_1(0)))\\
&+\frac{\mu_G^2}{(\lambda^*+\delta)^2}(2\lambda_1(\lambda^*+\delta)(\tilde f_1(s)-(\rho_2+\rho_1)T_sT_{\rho_2}f_1(0)\\
&+\rho_1^2T_sT_{\rho_2}T_{\rho_1}f_1(0))+\lambda_2^2(\tilde f_2(s)-(\rho_2+\rho_1)T_sT_{\rho_2}f_2(0)+\rho_1^2T_sT_{\rho_2}T_{\rho_1}f_2(0))\\
&-\lambda_1^2(\tilde f_1^2(s)-(\rho_2+\rho_1)T_sT_{\rho_2}f_1*f_1(0)+\rho_1^2T_sT_{\rho_2}T_{\rho_1}f_1*f_1(0))).
\end{aligned} \tag{4.11}$$

Similarly, we find

$$\begin{aligned}
T_sT_{\rho_2}T_{\rho_1}f_{1,2}(0)&=\frac{\lambda_1}{(\lambda^*+\delta)^2}((\lambda^*+\lambda_2-\lambda_1+\delta)T_sT_{\rho_2}T_{\rho_1}w_1(0)\\
&-\lambda_1T_sT_{\rho_2}T_{\rho_1}f_1*w_1(0)+\lambda\tilde\Phi_1(\frac{1}{\mu_G})T_sT_{\rho_2}T_{\rho_1}f_1(0))\\
&-\frac{\mu_G\lambda_1}{(\lambda^*+\delta)^2}(((2(\lambda^*+\lambda_2+\delta)-\lambda)(\rho_1T_sT_{\rho_2}T_{\rho_1}w_1(0)-T_sT_{\rho_2}w_1(0))\\
&-2\lambda_1(\rho_1T_sT_{\rho_2}T_{\rho_1}f_1*w_1(0)-T_sT_{\rho_2}f_1*w_1(0))\\
&+\lambda\tilde\Phi_1(\frac{1}{\mu_G})(\rho_1T_sT_{\rho_2}T_{\rho_1}f_1(0)-T_sT_{\rho_2}f_1(0)))+\frac{\mu_G^2\lambda_1}{(\lambda^*+\delta)^2}((\lambda^*+\lambda_2+\delta)(\tilde w_1(s)\\
&-(\rho_2+\rho_1)T_sT_{\rho_2}w_1(0)+\rho_1^2T_sT_{\rho_2}T_{\rho_1}w_1(0))-\lambda_1(\tilde f_1(s)\tilde w_1(s)\\
&-(\rho_2+\rho_1)T_sT_{\rho_2}f_1*w_1(0)+\rho_1^2T_sT_{\rho_2}T_{\rho_1}f_1*w_1(0)))=\mu_G^2T_s\xi_1(0),
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
&T_sT_{\rho_2}T_{\rho_1}f_{2,2}(0)=\frac{1}{(\lambda^*+\delta)^2}(\lambda_2^2T_sT_{\rho_2}T_{\rho_1}w_2(0)+\lambda\lambda_1\tilde\Phi_2(\frac{1}{\mu_G})T_sT_{\rho_2}T_{\rho_1}f_1(0))\\
&-\frac{\mu_G}{(\lambda^*+\delta)^2}(2\lambda_2^2(\rho_1T_sT_{\rho_2}T_{\rho_1}w_2(0)-T_sT_{\rho_2}w_2(0))\\
&+\lambda\lambda_1\tilde\Phi_2(\frac{1}{\mu_G})(\rho_1T_sT_{\rho_2}T_{\rho_1}f_1(0)-T_sT_{\rho_2}f_1(0)))\\
&+\frac{\mu_G^2\lambda_2^2}{(\lambda^*+\delta)^2}(\tilde w_2(s)-(\rho_2+\rho_1)T_sT_{\rho_2}w_2(0)+\rho_1^2T_sT_{\rho_2}T_{\rho_1}w_2(0))=\mu_G^2T_s\xi_2(0),
\end{aligned} \tag{4.13}$$

where the operator $*$ is denoted as convolution.

Therefore, plugging (4.11), (4.12) and (4.13) into (4.3), we can get

$$\tilde\Phi_i(s)=\frac{T_sT_{\rho_2}T_{\rho_1}h_2(0)}{\mu_G^2}\tilde\Phi_i(s)+T_s\xi_i(0),\quad i=1,2. \tag{4.14}$$

Inverting the Laplace transform in (4.14) gives

$$\Phi_i(u) = \frac{T_0T_{\rho_2}T_{\rho_1}h_2(0)}{\mu_G^2}\int_0^u \Phi_i(u-y)\frac{T_{\rho_2}T_{\rho_1}h_2(y)}{T_0T_{\rho_2}T_{\rho_1}h_2(0)}\mathrm{d}y + \xi_i(u),$$

which corresponds to (4.7).

To show that (4.7) to be a defective renewal equation, we need to verified $\kappa_\delta < 1$. We first consider the case $\delta > 0$. Comparing (4.11) at $s = 0$ to the expression of κ_δ gives $\kappa_\delta = \frac{T_0T_{\rho_1}T_{\rho_2}h_2(0)}{\mu_G^2}$. Because of $\rho_1(\delta) > 0$ and $\rho_2(\delta) > 0$, putting (4.6) at $s = 0$ in (4.6), one deduces

$$\kappa_\delta = \frac{T_0T_{\rho_1}T_{\rho_2}h_2(0)}{\mu_G^2} = 1 - \frac{\tilde{h}_1(0) - \tilde{h}_2(0)}{\mu_G^2\rho_1\rho_2} = 1 - \frac{\delta^2 + 2\lambda_2\delta}{\mu_G^2(\lambda^* + \delta)^2\rho_1\rho_2} < 1.$$

For $\delta = 0$, putting $s = \rho_1(\delta)$ in (3.14) yields

$$\Big(\lambda^* + \delta - (\lambda^* + \delta)\mu_G\rho_1(\delta) - \lambda - \lambda_1(1 - \mu_G\rho_1(\delta))\tilde{f}_1(\rho_1(\delta))\Big)^2$$
$$= \lambda_2^2(1 - \mu_G\rho_1(\delta))^2\tilde{f}_2(\rho_1(\delta)).$$

Note the fact that $\rho_1(0) = 0$. Differentiating the equation above with respect to δ and then putting $\delta = 0$, one finds

$$\rho_1^{'}(0) = \frac{1}{\lambda\mu_G - \lambda_1\mu_{F_1} - \frac{\lambda_2\mu_{F_2}}{2}} > 0,$$

where the inequality above follows from the net profit condition. Then takeing the limit $\delta \to 0^+$ in κ_δ and using L'Hôpital's rule, we can get

$$\kappa_0 = \frac{T_0T_0T_{\rho_2(0)}h_2(0)}{\mu_G^2} = 1 - \frac{1}{\mu_G^2(\lambda^*)^2\rho_2} \times \lim_{\delta\to 0^+}\frac{\delta^2 + 2\lambda_2\delta}{\rho_1(\delta)}$$
$$= 1 - \frac{2\lambda_2}{\mu_G^2(\lambda^*)^2\rho_2\rho_1^{'}(0)} < 1.$$

Thus, Eq.(4.7) is defective renewal equation. This completes the proof. □

In order to derive the analytic expression for $\Phi_i(u)$, an associated compound geometric distribution function are defined as

$$\overline{H}(u) = \frac{\zeta}{1+\zeta}\sum_{n=1}^{\infty}\left(\frac{1}{1+\zeta}\right)^n \overline{K}^{*n}(u), \quad u \geq 0,$$

where $\zeta = (1-\kappa_\delta)/\kappa_\delta$, $\overline{K}^{*n}(u)$ is the tail of the n-fold convolution of $K(u) = 1-\overline{K}(u) = \int_0^u \varsigma(y)\mathrm{d}y$. By employing the Theorem 2.1 of Lin and Willmot [11], we can derive the following Proposition.

4.3. Proposition. 3. *The expected discounted penalty function $\Phi_i(u)$ satisfying the defective renewal equation (4.7) can be rewritten as*

$$\Phi_i(u) = \frac{1}{\zeta}\int_0^u [1 - \overline{H}(u-y)]\mathrm{d}B_i(y) + \frac{B_i(0)}{\zeta}[1 - \overline{H}(u)], \quad i = 1, 2, \tag{4.15}$$

or

$$\Phi_i(u) = \frac{1}{\zeta}\int_0^u B_i(u-y)\mathrm{d}H(y) + \frac{1}{1+\zeta}B_i(u), \quad i = 1, 2, \tag{4.16}$$

where $B_i(u) = \xi_i(u)/\kappa_\delta$.

Proof. Using the Eq.(4.7) and the result of Theorem 2.1 obtained in Lin and Willmot [11], the proof is straightforward. □

5. Premium sizes with rational Laplace transforms

In this section, we consider the situation in which the premium size have the following rational Laplace transforms, i.e.,

$$\tilde{f}_G(s) = \frac{\varrho(s)}{\prod_{i=1}^{N}(s+\varrho_i)^{n_i}}, \tag{5.1}$$

where $N, n_i \in \mathbf{N}^+$ with $n_1 + n_2 + \cdots + n_N = n$, $\varrho_i > 0$, $i = 1, 2, \cdots, N$, and $\varrho_i \neq \varrho_j$ for $i \neq j$. $\varrho(s)$ is a polynomial function of degree $n-1$ or less and satisfying $\varrho(0) = \prod_{i=1}^{N} \varrho_i^{n_i}$. Using partial fraction, Eq.(5.1) can be rewritten as

$$\tilde{f}_G(s) = \sum_{i=1}^{N}\sum_{j=1}^{n_i} \frac{\alpha_{ij}\varrho_i^j}{(s+\varrho_i)^j}, \tag{5.2}$$

where

$$\alpha_{ij} = \frac{1}{\varrho_i^j (n_i - j)!} \frac{\mathrm{d}^{n_i - j}}{\mathrm{d}s^{n_i - j}} \left\{ \prod_{k=1, k\neq i}^{N} \frac{\varrho(s)}{(s+\varrho_k)^{n_k}} \right\} |_{s=-\varrho_i}$$

Taking the inverse Laplace transform of Eq.(5.2), one deduces

$$f_G(x) = \sum_{i=1}^{N}\sum_{j=1}^{n_i} \alpha_{ij} \frac{x^{j-1}\varrho_i^j e^{-\varrho_i x}}{(j-1)!}, \tag{5.3}$$

which is a density function of a combination of Erlangs. Define

$$\beta_{ij}(x) = \frac{x^{j-1}\varrho_i^j e^{-\varrho_i x}}{(j-1)!}, \quad x > 0, \quad j \in \mathbf{N}^+,$$

as the density function of Erlang(j) with parameter ϱ_i, χ_{ij} is a random variable with density function $\beta_{ij}(x)$. Thus, χ_{ij} can be defined as $\chi_{ij} = \vartheta_{i1} + \vartheta_{i2} + \cdots + \vartheta_{ij}$, where $\vartheta_{i1}, \vartheta_{i2}, \cdots, \vartheta_{ij}$ are i.i.d. exponentials with mean $1/\varrho_i$. For $\mathbf{Re}\ s > \max(\varrho_i)$, we get, for $k = 1, 2$,

$$\tilde{A}_k(s) = \int_0^\infty e^{-su} \int_0^\infty \Phi_k(u+x) f_G(x) \mathrm{d}x \mathrm{d}u$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{n_i} \alpha_{ij} \int_0^\infty \beta_{ij}(x) T_s \Phi_k(x) \mathrm{d}x = \sum_{i=1}^{N}\sum_{j=1}^{n_i} \alpha_{ij} \mathrm{E}[T_s \Phi_k(\vartheta_{i1} + \vartheta_{i2} + \cdots + \vartheta_{ij})]$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{n_i} \alpha_{ij}\varrho_i^j \mathrm{E}\left[T_s T_{\varrho_i}^j \Phi_k(0)\right],$$

where $T_{\varrho_i}^j = \underbrace{T_{\varrho_i} \cdots T_{\varrho_i}}_{j}$. Furthermore, by the Property 5 of the Dickson-Hipp operator provided in Li and Garrido [8], we get, $k = 1, 2$,

$$\tilde{A}_k(s) = \sum_{i=1}^{N}\sum_{j=1}^{n_i} \alpha_{ij}\varrho_i^j \left(\frac{\tilde{\Phi}_k(s)}{(\varrho_i - s)^j} - \sum_{l=1}^{j} \frac{T_{\varrho_i}^j \Phi_k(0)}{(\varrho_i - s)^{j+1-l}} \right) = \tilde{f}_G(-s)\tilde{\Phi}_k(s) - Q_k(s), \tag{5.4}$$

where $Q_k(s) = \sum_{i=1}^{N}\sum_{j=1}^{n_i} \alpha_{ij}\varrho_i^j \sum_{l=1}^{j} \frac{T_{\varrho_i}^j \Phi_k(0)}{(\varrho_i - s)^{j+1-l}}$. Similarly, after some careful calculations, we can find $k = 1, 2$,

$$\tilde{\bar{A}}_k(s) = \sum_{i=1}^{N}\sum_{j=1}^{n_i} \alpha_{ij}\varrho_i^j \left(\frac{\tilde{\Psi}_k(s)}{(\varrho_i - s)^j} - \sum_{l=1}^{j} \frac{T_{\varrho_i}^j \Psi_k(0)}{(\varrho_i - s)^{j+1-l}} \right) = \tilde{f}_G(-s)\tilde{\Psi}_k(s) - \bar{Q}_k(s), \tag{5.5}$$

where $\bar{Q}_k(s) = \sum_{i=1}^{N}\sum_{j=1}^{n_i} \alpha_{ij}\varrho_i^j \sum_{l=1}^{j} \frac{T_{\varrho_i}^j \Psi_k(0)}{(\varrho_i - s)^{j+1-l}}$.

Substituting (5.4) and (5.5) into (3.4), (3.5), (3.8) and (3.9), one finds

$$\tilde{\Phi}_1(s) = \frac{\lambda_1 L(s)\tilde{w}_1(s) - \lambda L(s)Q_1(s) - \lambda\lambda_2\bar{Q}_1(s) + \lambda_1\lambda_2\tilde{w}_1(s)}{L^2(s) - \lambda_2^2\tilde{f}_2(s)}, \tag{5.6}$$

$$\tilde{\Phi}_2(s) = \frac{-\lambda L(s)Q_2(s) - \lambda\lambda_2\bar{Q}_2(s) + \lambda_2^2\tilde{w}_2(s)}{L^2(s) - \lambda_2^2\tilde{f}_2(s)}, \tag{5.7}$$

where $L(s) = \lambda^* + \delta - \lambda\tilde{f}_G(-s) - \lambda\tilde{f}_1(s)$.

Note that the common denominator of (5.6) and (5.7) is analytic for s in the right half complex plane expect the points ϱ_i's. To make it analytic for all s with **Re** $s \geq 0$, we assume $\Lambda(s) = \prod_{i=1}^{N}(s - \varrho_i)^{n_i}$ and multiply both the numerators and denominators of (5.6) and (5.7) by $\Lambda(s)$. Then, one finds

$$\tilde{\Phi}_1(s) = \frac{\lambda_1 L(s)\Lambda(s)\tilde{w}_1(s) - \lambda L(s)\Lambda(s)Q_1(s) - \lambda\lambda_2\Lambda(s)\bar{Q}_1(s) + \lambda_1\lambda_2\Lambda(s)\tilde{w}_1(s)}{L^2(s)\Lambda(s) - \lambda_2^2\tilde{f}_2(s)\Lambda(s)}, \tag{5.8}$$

$$\tilde{\Phi}_2(s) = \frac{-\lambda L(s)\Lambda(s)Q_2(s) - \lambda\lambda_2\Lambda(s)\bar{Q}_2(s) + \lambda_2^2\Lambda(s)\tilde{w}_2(s)}{L^2(s)\Lambda(s) - \lambda_2^2\tilde{f}_2(s)}. \tag{5.9}$$

From (5.8) and (5.9), in order to determine $\tilde{\Phi}_1(s)$ and $\tilde{\Phi}_2(s)$, we need to find $\Lambda(s)Q_k(s)$ and $\Lambda(s)\bar{Q}_k(s)$, $k = 1, 2$. Note that $\Lambda(s)Q_k(s)$ and $\Lambda(s)\bar{Q}_k(s)$, $k = 1, 2$ are polynomials of degree $n - 1$, i.e.,

$$\Lambda(s)Q_k(s) = \sum_{i=1}^{n} L_{k,i}s^{i-1}, \quad \Lambda(s)\bar{Q}_k(s) = \sum_{i=1}^{n} \bar{L}_{k,i}s^{i-1}.$$

Then, we need to find n unknown coefficients $L_{k,i}$'s and n unknown coefficients $\bar{L}_{k,i}$'s. For this purpose, we give without proof the following Lemma. The result of the following Lemma can be proved by the same technique provided in Lemma 1.

5.1. Lemma. 2 *For $\delta > 0$, the common denominator of (5.8) and (5.9) has exactly $2n$ zeros, say $\rho_1(\delta), \cdots, \rho_{2n}(\delta)$, in the right half complex plane.*

Assume that $\rho_1(\delta), \cdots, \rho_{2n}(\delta)$ are distinct. Since $\tilde{\Phi}_1(s)$ and $\tilde{\Phi}_2(s)$ are analytic for all s with **Re** $s \geq 0$, then the roots $\rho_1(\delta), \cdots, \rho_{2n}(\delta)$ are zeros of the numerators of (5.8) and (5.9). Thus we can get the following $2n$ linear equations satisfied by $L_{1,i}$ and $\bar{L}_{1,i}$, $i = 1, 2, \cdots, 2n$,

$$\lambda_1 L(\rho_i(\delta))\Lambda(\rho_i(\delta))\tilde{w}_1(\rho_i(\delta)) - \lambda L(\rho_i(\delta))\Lambda(\rho_i(\delta))Q_1(\rho_i(\delta)) - \lambda\lambda_2\Lambda(\rho_i(\delta))\bar{Q}_1(\rho_i(\delta)) + \lambda_1\lambda_2\Lambda(\rho_i(\delta))\tilde{w}_1(\rho_i(\delta)) = 0. \tag{5.10}$$

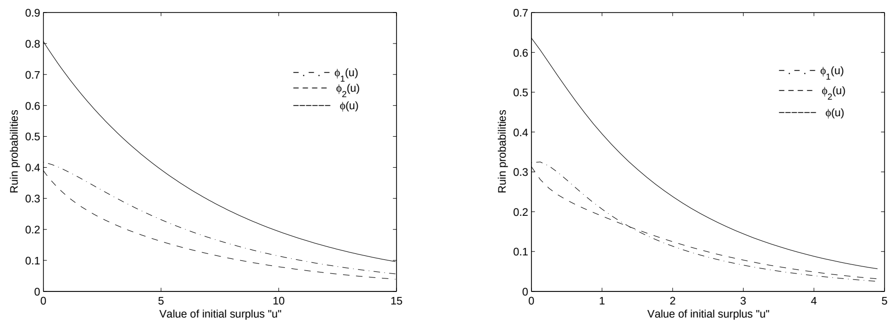
Similarly, we also get $2n$ linear equations satisfied by $L_{2,i}$ and $\bar{L}_{2,i}$, $i = 1, 2, \cdots, 2n$,

$$-\lambda L(\rho_i(\delta))\Lambda(\rho_i(\delta))Q_2(\rho_i(\delta)) - \lambda\lambda_2\Lambda(\rho_i(\delta))\bar{Q}_2(\rho_i(\delta)) + \lambda_2^2\Lambda(\rho_i(\delta))\tilde{w}_2(\rho_i(\delta)) = 0. \tag{5.11}$$

After solving linear equations (5.10) and (5.11), $L_{k,i}$ and $\bar{L}_{k,i}$, $k = 1, 2, i = 1, 2, \cdots, n$ can be determined. Then, we can derive the Laplace transforms (5.8) and (5.9).

6. Numerical examples

In this section, we give two numerical examples to show how to find the ruin probabilities $\phi_1(u)$, $\phi_2(u)$ and $\phi(u)$ and illustrate the behavior of these ruin probabilities.



(a) (b)

Figure 1. (a)Ruin probabilities in Example 1. (b)Ruin probabilities in Example 2.

6.1. Example 1. We illustrate the ruin probabilities when the claim sizes and the premium sizes are exponentially distributed. For illustration purpose, we set $\mu_G = 1.8$, $\mu_{F_1} = 1.5$, $\mu_{F_2} = 1$, $\lambda = 2.5$, $\lambda_1 = 1$ and $\lambda_2 = 3$. The net profit condition is obviously fulfilled. Let $\delta = 0$, $w_i(x_1, x_2) = 1 (i = 1, 2)$, then the expected penalty function $\Phi_i(u) (i = 1, 2)$ simplifies to the ruin probability $\phi_i(u) (i = 1, 2)$.

Eq. (3.14) can be simplified as

$$\left(1 - \frac{5}{13(1-1.8s)} - \frac{1}{6.5(1+1.5s)}\right)^2 = \left(\frac{3}{6.5}\right)^2 \frac{1}{(1+s)}.$$

After solving the equation above, we obtain five roots, 0, 0.390477, -0.879945, -0.600289, -0.141669. Then, we derive $\tilde{\Phi}_1(\frac{1}{\mu_G}) = 0.640091$, $\tilde{\Psi}_1(\frac{1}{\mu_G}) = 0.559909$, $\tilde{\Phi}_2(\frac{1}{\mu_G}) = 0.510574$, and $\tilde{\Psi}_1(\frac{1}{\mu_G}) = 0.689426$. Finally, the inversion of the Laplace transforms in (3.12) and (3.13) yields

$$\phi_1(u) = -0.071144e^{-0.879945u} + 0.018003e^{-0.600289u} + 0.469982e^{-0.141669u},$$

$$\phi_2(u) = 0.079266e^{-0.879945u} - 0.016749e^{-0.600289u} + 0.327589e^{-0.141669u}.$$

Figure 1(a) shows the behavior of ruin probabilities $\phi_1(u)$, $\phi_2(u)$ and $\phi(u)$ in Example 1, for different values of $u \in [0, 15]$.

6.2. Example 2. In this numerical example, we illustrate the ruin probabilities ruin probabilities when claim sizes from one class are distributed as Erlang(2) and claim sizes from the other class are distributed as a mixture of two exponentials, i.e., $f_1(x) = 6.76xe^{-2.6x}$ and $f_2(x) = 0.15e^{-x} + 1.5e^{-2x}$, for $x \geq 0$. For illustration purpose, we also assume that $\delta = 0$, $w_i(x_1, x_2) = 1 (i = 1, 2)$ and the premium sizes are exponentially distributed with $\mu_G = 1.8$. Let $\lambda = 2.5$, $\lambda_1 = 1$, $\lambda_2 = 3$. Solving Eq.(3.14) yields eight roots, 0, 0.390985, -1.194764, -0.489621, -3.578668-0.234714**i**, -3.578668+0.234714**i**, -1.946361-0.1240087**i**, -1.946361+0.1240087**i**. Then, we derive $\tilde{\Phi}_1(\frac{1}{\mu_G}) = 0.307941$, $\tilde{\Psi}_1(\frac{1}{\mu_G}) = 0.307444$, $\tilde{\Phi}_2(\frac{1}{\mu_G}) = 0.453832$, and $\tilde{\Psi}_1(\frac{1}{\mu_G}) = 0.296168$. Furthermore, the inversion of the Laplace transforms in (3.12) and (3.13) gives

$$\phi_1(u) = 0.081003e^{-1.946361u}\cos(0.124009u) + 0.088605e^{-1.946361u}\sin(0.124009u)$$
$$-0.125463e^{-3.578668u}\cos(0.234714u) + 0.117397e^{-3.578668u}\sin(0.234714u)$$

$$
\begin{aligned}
&+0.093966e^{-1.194764u} + 0.273590e^{-0.489621u}.\\
\phi_2(u) &= -0.072828e^{-1.946361u}\cos(0.124009u) - 0.084349e^{-1.946361u}\sin(0.124009u)\\
&+0.084415e^{-3.578668u}\cos(0.234714u) - 0.108023e^{-3.578668u}\sin(0.234714u)\\
&-0.046824e^{-1.194764u} + 0.348229e^{-0.489621u}.
\end{aligned}
$$

Figure 1(b) shows the behavior of the ruin probabilities $\phi_1(u)$, $\phi_2(u)$ and $\phi(u)$ in Example 2, for different values of $u \in [0, 5]$.

7. Concluding remarks

In this paper, we analyze the ruin problems in a risk model with two independent classes of insurance risks and random incomes, one is from the classical risk process, the other is from a Erlang(2) risk process. The expected discounted penalty functions are studied through some analytic methods. Assuming that the premium sizes are exponentially distributed, we show the defective renewal equations for the expected discounted penalty functions can be derived. While for the distributions of premium sizes have rational Laplace transforms, the Laplace transforms for the discounted penalty functions are also be derived.

The model considered in this paper can be extended in the more general framework. For example, the model can be a risk process with two independent classes, one being compound Poisson, the other being generalized Erlang(2), and such extension will only lead to a little computation involvement.

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References

[1] Bao,Z.H. *The expected discounted penalty at ruin in the risk process with random income*, Applied Mathematics and Computation **179**(2), 559-566, 2006.

[2] Boikov,A.V. *The Cramér-Lundberg model with stochastic premium process*, Theory of Probabilistic Applications **47**(3), 489-493, 2002.

[3] Boucherie,R.J., Boxma,O.J. and Sigman,K. *A note on negative customers, GI/G/I workload, and risk processes*, Probability in Engineering and Informational System **11**(3), 305-311, 1997.

[4] Chadjiconstantinidis, S. and Papaioannou, A.D. *Analysis of the Gerber-Shiu function and dividend barrier problems for a risk process with two classes of claims*, Insurance: Mathematics and Economics **45**(3), 470-484, 2009.

[5] Dickson, D.C.M. and Hipp, C. *On the time to ruin for Erlang(2) risk processes*, Insurance: Mathematics and Economics **29**(3), 333-344, 2001.

[6] Gerber, H.U. and Shiu, E.S.W. *On the time value of ruin*, North American Actuarial Journal **2**(1), 48-78, 1998.

[7] Labbé, C. and Sendova, K.P. *The expected discounted penalty function under a risk model with stochastic income*, Applied Mathematics and Computation **215**(5), 1852-1867, 2009.

[8] Li,S. and Garrido, J. *On ruin for Erlang(n) risk process*, Insurance: Mathematics and Economics **34**(3), 391-408, 2004.

[9] Li,S. and Garrido, J. *Ruin probabilities for two classes of risk processes*, ASTIN Bulletin **35**(1), 61-77, 2005.

[10] Li,S. and Lu, Y. *On the expected discounted penalty functions for two classes of risk processes*, Insurance: Mathematics and Economics **36**(2), 179-193, 2005.

[11] Lin, X.S. and Willmot, G.E. *The moments of the time of ruin, the surplus before ruin, and the deficit at ruin*, Insurance: Mathematics and Economics **27**(1), 19-44, 2000.

[12] Temnov,G. *Risk processes with random income*, Journal of Mathematical Sciences **123**(1), 3780-3794, 2004.

[13] Willmot, G.E. *A Laplace transform representation in a class of renewal queuing and risk process*, Journal of Applied Probability **36**(2), 570-584, 1999.

[14] Willmot, G.E. *On the discounted penalty function in the renewal risk model with general interclaim times*, Insurance: Mathematics and Economics **41**(1), 17-31, 2007.

[15] Xie,J.H. and Zou,W. *On the expected discounted penalty function for the compound Poisson risk model with delayed claims*, Journal of Computational and Applied Mathematics **235**(8), 2392-2404, 2011.

[16] Xie,J.H. and Zou,W. *On a risk model with random incomes and dependence between claim sizes and claim intervals*, Indagationes Mathematicae **24**(3), 557-580, 2013.

[17] Yang,H. and Zhang,Z.M. *On a class of renewal risk model with random income*, Applied Stochastic Models in Business and Industry **25**(6), 678-695, 2009.

[18] Yuen, K.C., Guo, J. and Wu, X. *On a correlated aggregate claims model with Poisson and Erlang risk processes*, Insurance: Mathematics and Economics **31**(2), 205-214, 2002.

[19] Zhang, Z., Li, S. and Yang, H. *The Gerber-Shiu discounted penalty function for a risk model with two classes of claims*, Journal of Computational and Applied Mathematics **230**(2), 643-655, 2009.

[20] Zhang,Z.M. and Yang, H. *On a risk model with stochastic premiums income and dependence between income and loss*, Journal of Computational and Applied Mathematics **234** (1), 44-57, 2010.

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Multi-sample test based on bootstrap methods for second order stochastic dominance

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Abstract

Statistical inferences under second order stochastic dominance for two sample case has a long and rich history. But the $k(\geq 2)$ sample case has not been well studied. In this article we consider $k(\geq 2)$ sample test for the equality of distribution functions against second order stochastic dominance alternative. A test statistic is constructed with isotonic regression estimates of stop-loss transform functions, and the asymptotic distribution of the proposed test is given. A bootstrap procedure is employed to obtain the p-value of the test, and some simulation results are presented to illustrate the proposed test method.

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Keywords: second order stochastic dominance, stop-loss transform, isotonic regression estimation, bootstrap critical values.

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1. Introduction

Ordering of distribution functions play an important role in many scientific areas including lifetime testing, reliability and economics, (see, for example, Alzaid et al. [1], Boland and Samaniego [6], Li and Lu [14], Shaked and Shanthikumar [18]). Many types of orderings of varying degrees of strength for comparing univariate distributions are discussed in the literature, including likelihood ratio ordering (Dykstra et al. [8]), uniform stochastic ordering or hazard rate ordering (Dykstra et al. [7]), and first- and second-order stochastic ordering (Feng and Wang [11], Klonner [13], Schnid and Trede [17]). Among them, first- and second-order stochastic ordering are the weakest, and used widely

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in practice (see, for example, Fong et al. [12], Klonner [13], Sriboonchita et al. [20], Wong [23]).

Second-order stochastic ordering is often called as second-order stochastic dominance or concave stochastic order, especially in economics. Since the beginning of the 1970's, stochastic dominance rules have been an essential tool in the comparison and analysis of poverty and income inequality. More recently, stochastic dominance has also been employed in the development of the theory of decision under risk and in actuarial sciences. The influential articles by Atkinson [2] and Shorrocks [19] are examples of theoretical works that provided a far-reaching insight into the importance of the stochastic dominance rules. And in economics and finance, second order stochastic dominance plays a major role in developing a general framework to establish a criterion for selecting one option over another. Therefore, it is of major interest to acquire a deep understanding of the meaning and implications of the second order stochastic dominance assumptions. This is why we focus on the statistical test of second order stochastic dominance in this article.

Testing against second order stochastic dominance of two distributions has a rich history and has been studied by many authors, for example, Liu and Wang [15], Bai et al.[3] and Berrendero and Carcamo [5], among others. In practice, we may be faced to compare multiple distributions in the mean of second order stochastic dominance. However, as far as we know, the multi-sample comparisons have not been well studied. In this article, we consider the test of stochastic equality of multiple distributions against the stochastic monotonicity under second order stochastic dominance.

The rest of the article is organized as follows. In section 2, as preparation we define some estimators for the unknown functionals of distribution functions, and discuss their consistency. In section 3 we provide test for the stochastic equality against second order stochastic dominance of k distributions and give the asymptotic distribution of the test statistic. In section 4 we establish a bootstrap procedure to implement the proposed test. In section 5 we present simulation to illustrate the performance of the proposed method. Some conclusion remarks are given in section 6.

2. Preliminaries

In this section, we first recall the definition of second order stochastic dominance for the convenience of statement, and then present estimators of the integrated distribution functions which satisfy the ordering restrictions.

2.1. Second order stochastic dominance.

2.1. Definition. Let X and Y be independent random variables with corresponding cumulative distribution functions F and G respectively. We say that Y dominates X in the sense of second order stochastic dominance, and denote by $X \leq_{SSD} Y$ or $F \leq_{SSD} G$, if for every nondecreasing and concave function $u(\cdot)$, we have

(2.1) $E(u(X)) \leq E(u(Y))$

or if

(2.2) $E(X-t)_{-} \leq E(Y-t)_{-}, \quad \forall t \in R.$

The equivalence of (2.1) and (2.2) refers to Stoyan [21]. In addition, a straightforward application of Fubini's theorem leads to yet another equivalent expression. In fact, define the transform W_F associated with a distribution function F by

(2.3) $W_F(t) = \int_{-\infty}^{t} F(y)dy, \quad \forall t \in R,$

then (2.1) is equivalent to

(2.4) $W_F(t) \geq W_G(t), \quad \forall t \in R.$

see Theorem 4.A.2 in Shaked and Shanthikumar [18].

2.2. Isotonic regression estimators of the integrated distribution functions. Assume that there are k independent samples $X_{i1}, X_{i2}, \cdots, X_{in_i}$, where $X_{ij}, j = 1, \cdots, n_i$ have common distribution function $F_i, i = 1, 2, \cdots, k$. We are interested in how to test with the samples that

(2.5) $F_1 \geq_{SSD} F_2 \geq_{SSD} \cdots \geq_{SSD} F_k$

or, equivalently

(2.6) $W_{F_1}(t) \leq W_{F_2}(t) \leq \cdots \leq W_{F_k}(t), \quad \forall t \in R.$

For this purpose, we first estimate the integrated distribution functions $W_{F_i}(t)$.

As is well known, a suitable estimator of F_i is the empirical distribution function

$$\hat{F}_i(x) = \frac{1}{n_i} \sum_{j=1}^{n_i} I_{[X_{ij}, \infty)}(x)$$

where $I_A(\cdot)$ denotes the indicator function associated with the set A. An immediate estimator of W_{F_i}, denoted by $W_{\hat{F}_i}$, can be obtained by substituting $F_i(x)$ with $\hat{F}_i(x)$, $W_{\hat{F}_i}(t) = \int_{-\infty}^{t} \hat{F}_i(y)dy, \quad \forall t \in R$. Let

$$W_{\hat{F}}(t) = (W_{\hat{F}_1}(t), W_{\hat{F}_2}(t), \cdots, W_{\hat{F}_k}(t)), \qquad t \in R.$$

It is obvious that the vector $W_{\hat{F}}(t)$ need not satisfy inequality (2.6), even if the inequality holds. To get such estimators, we employ isotonic regression. Let $N_{rs} = \sum\limits_{j=r}^{s} n_j$, and $Av_n[W_{\hat{F}}(t), r, s] = \sum\limits_{j=r}^{s} n_j W_{\hat{F}_j}(t)/N_{rs}$ for $r \leq s$. Define the estimator of $W_{F_i}(t)$ by

(2.7) $W^*_{\hat{F}_i}(t) = \max\limits_{r \leq i} \min\limits_{s \geq i} Av_n[W_{\hat{F}}(t), r, s], \quad i = 1, \cdots, k.$

$Av_n[W_{\hat{F}}(t), r, s]$ is the weighted average of $W_{\hat{F}_r}(t), \cdots, W_{\hat{F}_s}(t)$, and for each t, $W^*_{\hat{F}_i}(t)$ is the isotonic regression estimator of $W_{\hat{F}_i}(t)$ with weights $\{n_1, \cdots, n_k\}$ (see Robertson et al.[16]).

Let $|| \cdot ||$ denote the sup norm. The following lemma gives the consistency of the estimators, and thus the reasonability to construct a test statistic with them.

2.2. Lemma. $P[\| W_{\hat{F}_i} - W_{F_i} \| \to 0, \quad n_i \to \infty,\ i = 1, \cdots, k] = 1.$

Furthermore, if inequality (2.6) holds, then

$$P[||W^*_{\hat{F}_i} - W_{F_i}|| \to 0, \quad n_i \to \infty,\ i = 1, 2, \cdots, k] = 1.$$

The first conclusion of Lemma 2.2 is a straightforward consequence of Glivenko-Cantelli Theorem in van der Vaart and Wellner [22], and the second one can be proved easily by combining the first one and the properties of isotonic regression (Robertson et al. [16]). We omit the proof (see also, for example, El Barmi and Marchev [9]).

3. Hypothesis Tests

In this section, we discuss the tests of hypotheses under second order stochastic dominance. The hypotheses are defined as

$$H_0: \ F_1 = F_2 = \cdots = F_k,$$

and

$$H_1: \ F_1 \geq_{SSD} F_2 \geq_{SSD} \cdots \geq_{SSD} F_k,$$

We first set the notation in Subsection 3.1, then study the tests of H_0 versus $H_1 - H_0$ in Subsection 3.2.

3.1. Notation and lemmas. Let $n = \sum_{i=1}^{k} n_i$,

$$\begin{aligned} a_{in} &= \frac{n_i}{n}, \\ Z_{in_i}(t) &= \sqrt{n_i}[W_{\hat{F}_i}(t) - W_{F_i}(t)], \\ Z^*_{in_i}(t) &= \sqrt{n_i}[W^*_{\hat{F}_i}(t) - W_{F_i}(t)], \ i = 1, 2, \cdots, k, \end{aligned}$$

and

$$A_{rsn} \ = \ \sum_{j=r}^{s} a_{jn}, \ 1 \leq r \leq s \leq k.$$

When limits

$$(3.1) \quad \lim_{n\to\infty} a_{in} = a_i > 0, \quad i = 1, 2, \cdots, k$$

exist, denote

$$A_{rs} = \lim_{n\to\infty} A_{rsn} = \sum_{j=r}^{s} a_j.$$

For standard Brownian bridge $B = (B(t))_{0\leq t\leq 1}$ and distribution function H on R, denote $B_H(x) = \int\limits_{-\infty}^{x} B(H(s))ds, \ x \in R$. If $\int x^2 H(dx) < \infty$, then $B_H = (B_H(x))_{x\in R}$ is a centered Gaussian process with covariance function

$$\rho_H(x, y) = \int\limits_{-\infty}^{x} \int\limits_{-\infty}^{y} (H(u \wedge v) - H(u)H(v))dudv, \quad x, y \in R.$$

See Berrendero and Carcamo [5].

In this paper, we use " $\stackrel{w}{\to}$ " to denote weak convergence (or convergence in distribution).

The following result is helpful to derive the asymptotic distributions of test statistics. Its proof is similar to that of Lemma 1 in Baringhaus and Gr$\ddot{u}$bel [4].

3.1. Lemma. *Let $f_n, g_n (n \in \mathbb{N}), g, h$ be continuous real functions on $K = [-\infty, \infty]$ such that $f_n = g_n + c_n h$, where $(c_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers with $\lim\limits_{n\to\infty} c_n = \infty$. Assume further that $h \leq 0$, $A = \{h = 0\} \neq \emptyset$, and g_n converges uniformly to g. Then*

$$\lim_{n\to\infty} \sup_{t\in K} f_n(t) = \sup_{t\in A} g(t).$$

3.2. Test of H_0 versus $H_1 - H_0$. In this subsection, we consider the problem of testing H_0 versus $H_1 - H_0$. To this end, define test statistic T_n by

$$T_n = \sqrt{n}\sup_{t\in R}(W^*_{\hat{F}_k}(t) - W^*_{\hat{F}_1}(t)).$$

It is easy to see from Lemma 2.2 that when the alternative hypotheses holds, $W^*_{\hat{F}_k}$ and $W^*_{\hat{F}_1}$ would have different limits, thus T_n would take large values with large probability. To obtain the properties of T_n more explicitly, we next study its asymptotic distribution.

3.2. Theorem. *If for all F_is have finite second moments, then*

$$(Z_{1n_1}(t), Z_{2n_2}(t), \cdots, Z_{kn_k}(t))' \xrightarrow{w} (B_{F_1}(t), B_{F_2}(t), \cdots, B_{F_k}(t))', \quad \forall t \in R,$$

as $\min n_i \to \infty$.

The theorem is an easy result of empirical process theory (van der Vaart and Wellner [22], see also Theorem 1 in Baringhaus and Grübel [4]). From Theorem 3.2, it may be shown the following theorem.

Let $S_i = \{j : W_{F_j}(t) = W_{F_i}(t), \forall t \in R, j = 1, \cdots, k\}$, c_i and d_i be the left and right endpoints of the support of $F_i, i = 1, 2, \cdots, k$. The following condition will be employed to give the asymptotic distribution of $Z^*_{in_i}$s.

$$\inf_{c_i+\eta\le t\le d_i-\eta}[W_{F_j}(t) - W_{F_i}(t)] > 0, \tag{3.2}$$

for some $\eta > 0$ and all $j > S_i$, $i = 1, 2, \cdots, k$, where $\inf_\emptyset(.) = \infty$, and $j > S_i$ means $j > l$ for all $l \in S_i$.

3.3. Theorem. *Suppose all the k distributions have finite second moments, and (3.1) and (3.2) hold. Then under H_1 it holds that*

$$(Z^*_{1n_1}(t), Z^*_{2n_2}(t), \cdots, Z^*_{kn_k}(t))' \xrightarrow{w} (Z^*_1(t), Z^*_2(t), \cdots, Z^*_k(t))', \quad t \in R$$

as $n \to \infty$, where

$$Z^*_i(t) = \sqrt{a_i}\max_{r\le i, r\in S_i}\min_{i\le s, s\in S_i}\frac{\sum_{\{r\le j\le s\}}\sqrt{a_j}B_{F_j}(t)}{A_{rs}}.$$

We omit its proof, which is similar to that of Theorem 4 in El Barmi and Mukerjee [10]. Based on the conclusion, we may obtain the asymptotic distribution of the test statistic.

3.4. Theorem. *Suppose the conditions of Theorem 3.3 are satisfied. Then under H_0 it holds that*

$$T_n \xrightarrow{w} T = \sup_{t\in R}\{\max_{r\le k}\frac{\sum_{j=r}^{k}\sqrt{a_j}B_{F_j}(t)}{A_{rk}} - \min_{s\ge 1}\frac{\sum_{j=1}^{s}\sqrt{a_j}B_{F_j}(t)}{A_{1s}}\}.$$

Proof. Define stochastic processes $V_n(t) = \sqrt{n}(W^*_{\hat{F}_k}(t) - W^*_{\hat{F}_1}(t))$, then

$$\begin{aligned} V_n(t) &= \sqrt{n}(W^*_{\hat{F}_k}(t) - W_{F_k}(t)) - \sqrt{n}(W^*_{\hat{F}_1}(t) - W_{F_1}(t)) \\ &\quad + \sqrt{n}(W_{F_k}(t) - W_{F_1}(t)) \\ &= \sqrt{\tfrac{n}{n_k}}\sqrt{n_k}(W^*_{\hat{F}_k}(t) - W_{F_k}(t)) - \sqrt{\tfrac{n}{n_1}}\sqrt{n_1}(W^*_{\hat{F}_1}(t) - W_{F_1}(t)) \\ &\quad + \sqrt{n}(W_{F_k}(t) - W_{F_1}(t)) \\ &= \sqrt{\tfrac{n}{n_k}}Z^*_{kn_k}(t) - \sqrt{\tfrac{n}{n_1}}Z^*_{1n_1}(t) + \sqrt{n}(W_{F_k}(t) - W_{F_1}(t)). \end{aligned} \tag{3.3}$$

Under H_0, the third term on the right-hand side is just zero. By Theorem 3.3 and Slutsky theorem, we obtain

$$\begin{aligned}\sqrt{\tfrac{n}{n_k}}Z^*_{k,n_k}(t)-\sqrt{\tfrac{n}{n_1}}Z^*_{1,n_1}(t) &\xrightarrow{w} \sqrt{1/a_k}Z^*_k(t)-\sqrt{1/a_1}Z^*_1(t)\\ &=\max_{r\le k}\frac{\sum_{j=r}^{k}\sqrt{a_j}B_{F_j}(t)}{A_{rk}}-\min_{s\ge 1}\frac{\sum_{j=1}^{s}\sqrt{a_j}B_{F_j}(t)}{A_{1s}}.\end{aligned}$$

By Lemma 3.1 and continuous mapping theorem, we have

$$T_n\xrightarrow{w}T = \sup_{t\in R}\{\max_{r\le k}\frac{\sum_{j=r}^{k}\sqrt{a_j}B_{F_j}(t)}{A_{rk}}-\min_{s\ge 1}\frac{\sum_{j=1}^{s}\sqrt{a_j}B_{F_j}(t)}{A_{1s}}\}.\quad\square$$

3.5. Theorem. *Suppose that H_1-H_0 does hold. Then $P(T_n\to\infty)=1$.*

Proof. The first two terms on the right-hand side of (3.3) are stochastically bounded. If H_1-H_0 does hold, then there is at least one i which satisfies $W_{F_i}(t)<W_{F_{i+1}}(t)$ for all t in some non-empty interval $(a,b)\subset R$. As $\sqrt{n}\to\infty$, we obtain

$$\sup_{t\in R}V_n(t)\to\infty$$

with probability one. $\square$

Theorem 3.4 gives the null asymptotic distribution of T_n, thus the feasibility of the test theoretically. Theorem 3.5 reveals that the proposed test is consistent.

4. Bootstrap Procedure

To use the statistic T_n to make a decision in practice, we require the p-value of the test statistic. Although the asymptotic distribution of T_n under the null hypothesis is given, however, it is very complicated, and depends on the underlying unknown distributions F_i, thus is difficult to be used directly to compute the critical value. In this section, we give a bootstrap method to compute an approximated p-value for T_n.

4.1. Asymptotic behavior of Bootstrap statistic. Recall that $\hat{F}_i$ are the empirical distribution functions associated with the samples $X_{i1},\cdots,X_{in_i}$ from $F_i, i=1,\cdots,k$. These random variables are the initial segments of k infinite sequences $(X_{ij})_{j\in\mathbb{N}}$ of random variables defined on some background probability space $(\Omega,\mathcal{A},P)$; the almost sure statements below refer to P. Given the initial segments, let $\hat{\zeta}_{n,1}\cdots,\hat{\zeta}_{n,n}$ be a sample of size n from the (random) distribution function

(4.1) $H_n=\frac{n_1}{n}\hat{F}_1+\frac{n_2}{n}\hat{F}_2+\cdots+\frac{n_k}{n}\hat{F}_k$.

Let

$$\begin{aligned}\hat{F}_{n,n_i}(x) &= \frac{1}{n_i}\sum_{j=n_1+\cdots+n_{i-1}+1}^{n_1+\cdots+n_i} I_{[\hat{\zeta}_{n,j},\infty)}(x),\\ W^*_{\hat{F}_{n,n_i}}(x) &= \max_{r\le i}\min_{s\ge i}Av_n[W_{\hat{F}_{n,n_i}}(x),r,s],\\ \hat{Z}_{n,n_i} &= \sqrt{n_i}(W_{\hat{F}_{n,n_i}}-W_{H_n}),\\ \hat{Z}^*_{n,n_i} &= \sqrt{n_i}(W^*_{\hat{F}_{n,n_i}}-W_{H_n}),\quad i=1,\cdots,k,\end{aligned}$$

and define the bootstrap version of T_n by

(4.2) $\hat{T}_n=\sup_{t\in R}\sqrt{n}(W^*_{\hat{F}_{n,n_k}}(t)-W^*_{\hat{F}_{n,n_1}}(t))$

The following theorem shows that, with probability 1, the limit distribution of $\hat{T}_n$ is the same as that of T_n.

4.1. Theorem. *Suppose that the conditions of Theorem 3.4 hold, then with probability one,*

$$(4.3)\quad \hat{T}_n \xrightarrow{w} \sup_{t\in R}\{\max_{r\le k} \frac{\sum_{j=r}^{k}\sqrt{a_j}B_{F_j}(t)}{A_{rk}} - \min_{s\ge 1}\frac{\sum_{j=1}^{s}\sqrt{a_j}B_{F_j}(t)}{A_{1s}}\}$$

where $a_i, A_{rk}, A_{1s}, i, r, s = 1, \cdots, k$ *are the same as in Theorem 3.4.*

Proof. Let $\mathcal{F} = \{\phi_t(x) = (x-t)_- : t \in R\}$, and $\hat{U}^{F_i}_{n,n_i} = (\hat{U}^{F_i}_{n_i}(\phi))_{\phi\in\mathcal{F}}$,

$$\hat{U}^{F_i}_{n,n_i}(\phi) := \sqrt{n_i}(\int \phi d\hat{F}_{n,n_i} - \int \phi dH_n)$$

be the empirical processes associated with the k parts of the resamples. See van der Vaart and Wellner [22], with probability one, we have

$$(4.4)\quad \hat{U}^{F_i}_{n,n_i} \xrightarrow{w} B_{F_i}, \quad i = 1\cdots, k.$$

In analogy to (3.2), we now define the stochastic processes $\hat{V}_n(t)$ by

$$\begin{aligned}
\hat{V}_n(t) &= \sqrt{n}(W^*_{\hat{F}_{n,n_k}}(t) - W^*_{\hat{F}_{n,n_1}}(t)) \\
&= \sqrt{n}[(W^*_{\hat{F}_{n,n_k}}(t) - \int \phi_t dH_n) - (W^*_{\hat{F}_{n,n_1}}(t) - \int \phi_t dH_n)] \\
&= \sqrt{\frac{n}{n_k}}\sqrt{n_k}(W^*_{\hat{F}_{n,n_k}}(t) - W_{H_n}(t)) - \sqrt{\frac{n}{n_1}}\sqrt{n_1}(W^*_{\hat{F}_{n,n_1}}(t) - W_{H_n}(t)) \\
&= \sqrt{\frac{n}{n_k}}\hat{Z}^*_{n,n_k}(t) - \sqrt{\frac{n}{n_1}}\hat{Z}^*_{n,n_1}(t)
\end{aligned}$$

Note that $\hat{Z}^*_{n,n_k}(t)$ may be obtained from the isotonic regression of $\hat{U}^{F_i}_{n,n_i}(\phi_t), i = 1, \cdots, k$. By continuous mapping theorem, the conditional independence of the subsamples and (4.4), we obtain

$$\hat{V}_n(t) \xrightarrow{w} \max_{r\le k}\frac{\sum_{j=r}^{k}\sqrt{a_j}B_{F_j}(t)}{A_{rk}} - \min_{s\ge 1}\frac{\sum_{j=1}^{s}\sqrt{a_j}B_{F_j}(t)}{A_{1s}}$$

with probability one. This leads to that

$$\hat{T}_n = \sup_{t\in R}\hat{V}_n(t) \xrightarrow{w} \sup_{t\in R}\{\max_{r\le k}\frac{\sum_{j=r}^{k}\sqrt{a_j}B_{F_j}(t)}{A_{rk}} - \min_{s\ge 1}\frac{\sum_{j=1}^{s}\sqrt{a_j}B_{F_j}(t)}{A_{1s}}\}$$

with probability one, by continuous mapping theorem and Lemma 3.1. □

4.2. Determination of the p-value. To apply the test in practice, we propose a bootstrap approximation to the p-value of the test as follows.

Step 1. Compute test statistic T_n from the original samples $X_{i1}, \cdots, X_{in_i}, i = 1, \cdots, k$;

Step 2. Let $\hat{\zeta}_{n,1}\cdots, \hat{\zeta}_{n,n}$ be a bootstrap sample of size n from the pooled empirical distribution function $H_n = \frac{n_1}{n}\hat{F}_1 + \frac{n_2}{n}\hat{F}_2 + \cdots + \frac{n_k}{n}\hat{F}_k$, where $\hat{F}_i$ is the empirical distribution function associated with the sample $X_{i1}, \cdots, X_{in_i}, i = 1, \cdots, k$. Divide this bootstrap sample into k parts $\hat{\zeta}_{n,n_1+\cdots+n_{i-1}+1}, \cdots, \hat{\zeta}_{n,n_1+\cdots+n_i}, i = 1, \cdots, k$. Use these k parts to compute a bootstrap version of the test statistic $\hat{T}_n$ by (4.2);

Step 3. Repeat step 2 a large number B of times, yielding B bootstrap test statistics $\hat{T}^{(b)}_n, b = 1, \cdots, B$;

Step 4. The p-value of the proposed test is given by $p = \frac{Card\{b:\hat{T}_n^{(b)}>T_n, b=1,\cdots,B\}}{B}$.
We reject H_0 at a given level α when $p < \alpha$. Theorem 4.1, Theorem 3.4 and Theorem 3.5 ensure that the true level of the proposed test would be closed to the nominal significant level under H_0, and the power (the rejection probability) should be high under $H_1 - H_0$ when the sample sizes are enough large. The simulation in next section will confirm the intuition statements.

5. Simulation Study

To investigate the properties of the tests, we carried out a simulation study for $k = 3$. The empirical rejection rates of T_n in 1000 replications are recorded for various scenarios. For each of the scenarios, the number of resampling is taken as 1000; the sample sizes of the three distributions are taken as the same, and they are set at 100, 200 in different simulations for evaluating the effect of sample size.

Table 1 reports the simulation results for scenarios for which H_0 is true. Two different significance levels are considered. In Table 2, the empirical rejection rates of the test are given for the scenarios for which $H_1 - H_0$ is true. The significance level is taken as $\alpha = 0.05$.

TABLE 1: Empirical rejection rates of the test under H_0

Distributions	$n_1 = n_2 = n_3 = 100$		$n_1 = n_2 = n_3 = 200$	
$F_1 = F_2 = F_3$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$
U(0,1)	0.012	0.060	0.015	0.051
Exp(1)	0.013	0.057	0.010	0.056
N(0,1)	0.012	0.062	0.010	0.048
$\chi^2(2)$	0.016	0.037	0.012	0.047
Beta(2,2)	0.007	0.055	0.009	0.052

TABLE 2: Empirical rejection rates of the test under $H_1 - H_0$

Distributions			$n_1 = n_2 = n_3 = 100$	$n_1 = n_2 = n_3 = 200$
F_1	F_2	F_3	$\hat{p}$	$\hat{p}$
Uni(0,1.1)	Uni(0,1)	Uni(0,1)	0.732	0.946
Uni(0,1.1)	Uni(0,1.1)	Uni(0,1)	0.721	0.985
Exp(1)	Exp(1)	Exp(1.1)	0.213	0.236
Exp(1)	Exp(1.1)	Exp(1.1)	0.134	0.291
Exp(1)	Exp(1.1)	Exp(1.2)	0.324	0.569
N(0.1,1)	N(0,1)	N(0,1)	0.191	0.275
N(0.1,1)	N(0.1,1)	N(0,1)	0.174	0.253
N(0.5,1)	N(0.25,1)	N(0,1)	0.976	1.000
Uni(0,1)	Uni(0,1)	Beta(2,2)	0.478	0.853
Uni(0,1)	Beta(2,2)	Beta(2,2)	0.629	0.860

From Table 1, we see that the simulated size of the proposed test is reasonable and gets closer to α with the sample size n increasing. For fixed sample sizes, the performance of the test vary slightly with the population distributions.

Furthermore, from Table 2, we could have the following observations.

(1) With the increasing of sample sizes, the power (empirical rejection rate) of the proposed test increases fast.

(2) The power is related to how the probability distributions going against the null hypothesis. It is lower when the differences of the population distributions are slight, and goes higher when the differences become significant. In addition, the test looks more

sensitive to the differences of the distributions with bounded supports than that with unbounded supports.

6. Concluding Remarks

In this article, we present an extension of Baringhaus and Grübel [4] through isotonic regression, and give a test for the homogeneity of multiple populations against the second stochastic dominance ordering. The method can be used also to test the null hypothesis of second stochastic dominance ordering, even umbrella ordering in the sense of second stochastic dominance ordering, with an appropriate estimators of the distributions under umbrella ordering restriction.

Bootstrap method is employed to give the p-value of the proposed test. Generally, the approximated p-value is good for limited sample sizes. However, when the null hypothesis is not the homogeneity of the distributions, based on our simulations those are not presented here, the obtained p-value may not enough accurate (conservative in general), and how to improve the approximation should be studied further.

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References

[1] Alzaid, A., Kim, J.S. and Proschan, F. *Laplace ordering and its application*, J. Appli. Probability **28** (1), 116-130, 1991.

[2] Atkinson, A.B. *On the measurement of Inequality*, Journal of Economic Theory **2** (3), 244-263, 1970.

[3] Bai, Z.D., Li, H., Liu, H.X., Wong, W.K. *Test statistics for prospect and markowitz stochastic dominance with applications*, Econometrics Journal **14** (2), 278-303, 2011.

[4] Baringhaus, L. and Grübel, R. *Nonparametric two-sample tests for increasing convex order*, Bernoulli **15** (1), 99-123, 2009.

[5] Berrendero, J.R. and Carcamo, J. *Test for the second order stochastic dominance based on L-statistics*, Journal of Business and Economic Statistics **29** (2), 260-270, 2011.

[6] Boland, P.J. and Samaniego, F.J. *Stochastic ordering results for consecutive k-out-of-n: F systems*, IEEE Transactions on reliability **53** (1), 7-10, 2004.

[7] Dykstra, R., Kochar, S., Robertson, T. *Statistical inference for uniform stochastic ordering in several populations*, Annals of statistics **19** (2), 870-888, 1991.

[8] Dykstra, R., Kochar, S., Robertson, T. *Inference for likelihood ratio ordering in the two sample problem*, Journal of American Statistical Association **90** (431), 1034-1040, 1995.

[9] El Barmi, H. and Marchev, D. *New and improved estimators of distribution functions under second-order stochastic dominance*, Journal of Nonparametric Statistics **21** (2), 143-153, 2009.

[10] El Barmi, H. and Mukerjee, H. *Inferences under a stochastic ordering constraint: the k-sample case*, J. Amer. Statist. Assoc **469** (100), 252-261, 2005.

[11] Feng, Y.Q. and Wang, J.D. *Likelihood ratio test against simple stochastic ordering among several multinomial populations*, J. Statist. Plann. Inference **137** (4), 1362-1374, 2007.

[12] Fong, W.M., Lean, H.H., Wong, W.K. *Stochastic dominance and behavior towards risk: the market for internet stocks*, Journal of Economic Behavior and Organization **68** (1), 194-208, 2008.

[13] Klonner, S. *The first order stochastic dominance ordering of the singh-maddala distribution*, Economics Letters **69** (2), 123-128, 2000.

[14] Li, X.H. and Lu, J.Y. *Stochastic comparisons on residual life and inactivity time of series and parallel systems*, Probability in Engineering and Informational Science **17** (2), 267-275, 2003.

[15] Liu, X.S. and Wang, J.D. *Testing the equality of multinomial populations ordered by increasing convexity under the alternative*, The Canadian Journal of Statistics **32** (2), 159-168, 2004.

[16] Roberbson, T., Wright, T., Dykstra, R.L. *Order-restricted inferences*, Wiley, New York, 1988.

[17] Schnid, F. and Trede, M. *A kolmogorov-type test for second order stochastic dominance*, Statistics and Probability Letters **37** (2), 183-193, 1998.

[18] Shaked, M. and Shanthikumar, J.G. *Stochastic orders and their applications*, Springer-Verlag:New York, 2007.

[19] Shorrocks, A.F. *Ranking income distributions*, Economics **50** (197), 3-17, 1983.

[20] Sriboonchita, S., Wong, W.K., Dhompongsa, S., Nguyen, H.T. *Stochastic dominance and applications to finance, risk and economics, boca raton*, FL:Chapmand and Hall, 2009.

[21] Stoyan, D. *Comparison methods for queues and other stochastic models*, New York,Wiley, 1983.

[22] van der Vaart A.W. and Wellner J.A. *Weak convergence and empirical processes*, Springer: New York, 1996.

[23] Wong, W.K. *Stochastic dominance theory for location-scale family*, Journal of Applied Mathematics and Decision Sciences. 2006, 1-10, 2006.

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[2] Ehrig, H. and Herrlich, H. *The construct PRO of projection spaces: its internal structure*, in: Categorical methods in Computer Science, Lecture Notes in Computer Science **393** (Springer-Verlag, Berlin, 1989), 286–293.

[3] Hurvich, C. M. and Tsai, C. L. *Regression and time series model selection in small samples*, Biometrika **76** (2), 297–307, 1989.

[4] Papoulis, A. *Probability random variables and stochastic process* (McGraw-Hill, 1965).

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