HACETTEPE UNIVERSITY
FACULTY OF SCIENCE
TURKEY

HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS

A Bimonthly Publication
Volume 44 Issue 5
2015

ISSN 1303 5010

HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS

Volume 44 Issue 5
October 2015

A Peer Reviewed Journal
Published Bimonthly by the
Faculty of Science of Hacettepe University

Abstracted/Indexed in

SCI-EXP, Journal Citation Reports, Mathematical Reviews,
Zentralblatt MATH, Current Index to Statistics,
Statistical Theory & Method Abstracts,
SCOPUS, Tübitak-Ulakbim.

ISSN 1303 5010

This Journal is typeset using LaTeX.

Hacettepe Journal of Mathematics and Statistics

Cilt 44 Sayı 5 (2015)

ISSN 1303 – 5010

KÜNYE
YAYININ ADI: HACETTEPE JOURNAL OF MATHEMATICS AND STATISTICS
YIL : 2015   SAYI : 44 - 5   AY : Ekim
YAYIN SAHİBİNİN ADI : H. Ü. Fen Fakültesi Dekanlığı adına Prof. Dr. Bekir Salih
SORUMLU YAZI İŞL. MD. ADI : Prof. Dr. Yücel Tıraş
YAYIN İDARE MERKEZİ ADRESİ : H. Ü. Fen Fakültesi Dekanlığı
YAYIN İDARE MERKEZİ TEL. : 0 312 297 68 50
YAYININ TÜRÜ : Yaygın
BASIMCININ ADI : Hacettepe Üniversitesi Hastaneleri Basımevi.
BASIMCININ ADRESİ : 06100 Sıhhıye, ANKARA.
BASIMCININ TEL. : 0 312 305 1020
BASIM TARİHİ - YERİ : - ANKARA

Hacettepe Journal of Mathematics and Statistics

A Bimonthly Publication – Volume 44 Issue 5 (2015)
ISSN 1303 – 5010

Published by Hacettepe University
Faculty of Science

CONTENTS

Mathematics

Statistics

MATHEMATICS

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 991 – 1009

Orlicz-Lorentz spaces and their multiplication operators

René Erlin Castillo * , Héctor Camilo Chaparro† and Julio César Ramos Fernández‡

Abstract

The boundedness, closed range, invertibility, compactness and closedness of multiplication operators on Orlicz-Lorentz spaces are characterized in this paper.

Keywords: Compact operator, Multiplication operator, Orlicz-Lorentz spaces.

2000 AMS Classification: primary 47B33, 47B38, secondary 46E30.

Received 05/11/2013 : Accepted 27/06/2014 Doi : 10.15672/HJMS.2015449663

1. Introduction

Let f a complex-valued measurable function defined on a σ-finite measure space $(X, \mathcal{A}, \mu)$. For $\lambda \geq 0$, define $D_f(\lambda)$ the distribution function of f as

(1.1) $$D_f(\lambda) = \mu\left(\{x \in X : |f(x)| > \lambda\}\right).$$

Observe that D_f depends only on the absolute value $|f|$ of the function f and D_f may assume the value $+\infty$.

The distribution function D_f provides information about the size of f but not about the behavior of f itself near any given point. For instance, a function on $\mathbb{R}^n$ and each of its translates have the same distribution function. It follows from (1.1) that D_f is a decreasing function of λ (not necessarily strictly) and continuous from the right.

Let (X, μ) be a measurable space and f and g be a measurable functions on (X, μ) then D_f enjoy the following properties for all $\lambda_1, \lambda_2 \geq 0$:

(1) $|g| \leq |f|$ μ-a.e. implies that $D_g \leq D_f$;
(2) $D_{cf}(\lambda) = D_f\left(\frac{\lambda}{|c|}\right)$ for all $c \in \mathbb{C} \setminus \{0\}$;

---

*Universidad Nacional de Colombia, Departamento de Matemáticas, Bogotá, Colombia
Email: recastillo@unal.edu.co
†Universidad Nacional de Colombia, Departamento de Matemáticas, Bogotá, Colombia
Email:hcchaparrog@unal.edu.co
‡Universidad de Oriente, Departamento de Matemáticas, 6101 Cumaná, Estado Sucre, Venezuela
Email:jcramos@udo.edu.ve

(3) $D_{f+g}(\lambda_1+\lambda_2) \le D_f(\lambda_1) + D_g(\lambda_2)$;
(4) $D_{fg}(\lambda_1\lambda_2) \le D_f(\lambda_1) + D_g(\lambda_2)$.

For more details on distribution function see [7].

By f^* we mean the non-increasing rearrangement of f given as

$$f^*(t) = \inf\{\lambda > 0 : D_f(\lambda) \le t\}, \quad t \ge 0$$

where we use the convention that $\inf \emptyset = \infty$. f^* is decreasing and right-continuous. Notice

$$f^*(0) = \inf\{\lambda > 0 : D_f(\lambda) \le 0\} = \|f\|_\infty,$$

since

$$\|f\|_\infty = \inf\{\alpha \ge 0 : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\}.$$

Also observe that if D_f is strictly decreasing, then

$$f^*(D_f(t)) = \inf\{\lambda > 0 : D_f(\lambda) \le D_(f)t\} = t.$$

This fact demonstrates that f^* is the inverse function of the distribution function D_f. Let $\mathcal{F}(X,\mathcal{A})$ denote the set of all $\mathcal{A}$-measurable functions on X. Let $(X,\mathcal{A}_0,\mu)$ and $(Y,\mathcal{A}_1,\nu)$ be two measure spaces.

Two functions $f \in F(X,\mathcal{A}_0)$ and $g \in F(X,\mathcal{A}_1)$ are said to be equimeasurable if they have the same distribution function, that is, if

$$\mu\left(\{x \in X : |f(x)| > \lambda\}\right) = \nu\left(\{y \in Y : |g(y)| > \lambda\}\right), \quad \text{for all } \lambda \ge 0. \tag{1.2}$$

So then there exists only one right-continuous decreasing function f^* equimeasurable with f. Hence the decreasing rearrangement is unique.

In what follows, we gather some useful properties of the decreasing rearrangement function:

a) f^* is decreasing.
b) $f^*(t) > \lambda$ if and only if $D_f(\lambda) > t$.
c) f and f^* are equimeasurables, that is

$$D_f(\lambda) = D_{f^*}(\lambda) \quad \text{for all } \lambda \ge 0.$$

d) If $|f| \le \liminf_{n\to\infty} |f_n|$ then $f^* \le \liminf_{n\to\infty} f_n^*$.
e) If $E \in \mathcal{A}$, then $(\chi_E)^*(t) = \chi_{[0,\mu(E))}(t)$.
f) If $E \in \mathcal{A}$, then $(f\chi_E)^*(t) \le f^*(t)\chi_{[0,\mu(E))}(t)$.

A weight is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0,\infty)$ almost everywhere. Therefore, weights are allowed to be zero or infinite only on a set of Lebesgue measure zero.

Let $\varphi : [0,\infty) \to [0,\infty)$ be a convex function such that

(1) $\varphi(x) = 0$ if and only if $x = 0$;
(2) $\lim_{x\to\infty} \varphi(x) = \infty$.

Such as function is known as a Young function. A Young function is strictly increasing, in fact, let $0 < x < y$ then $0 < \frac{x}{y} < 1$ and hence, we might write

$$x = \left(1 - \frac{x}{y}\right)0 + \frac{x}{y}y.$$

Since φ is convex, we have

$$\begin{aligned}\varphi(x) &= \varphi\left(\left(1-\frac{x}{y}\right)0+\frac{x}{y}y\right)\\ &\leq \left(1-\frac{x}{y}\right)\varphi(0)+\frac{x}{y}\varphi(y)\\ &< \varphi(y).\end{aligned}$$

A Young function is said to satisfy the Δ_2-condition if there exists a nonnegative constant x_0 and k such that

$$\varphi(2x) \leq k\varphi(x) \quad \text{for } x \geq x_0. \tag{1.3}$$

If $x_0 = 0$, we say that φ satisfy globally the Δ_2-condition. The smaller constant k which satisfy (1.3) is denoted by k_Δ.

1.1. Claim. If φ is a Young function such that satisfy the Δ_2-condition, then for each $r \geq 0$ there exists a constant $k_\Delta(r)$ such that

$$\varphi(rx) \leq k_\Delta(r)\varphi(x) \tag{1.4}$$

for $x > 0$ large enough.

Proof of the claim. If $r > 0$, we can choose $n \in \mathbb{N}$ such that $r \leq 2^n$. Then we can applied (1.3) n-times and use the fact that φ is increasing to obtain

$$\varphi(rx) \leq \varphi(2^n x) \leq k^n\varphi(x),$$

and hence we have (1.4). □

1.2. Example. The function $\varphi_1(x) = \frac{x^p}{p}$ with $p > 1$ is a Young function which satisfy globally the Δ_2-condition with $k_\Delta = \frac{2^p}{p}$.

1.3. Example. The function $\varphi_2(t) = t^p \log(1+t)$ with $p \geq 1$ and $t \geq 0$ is a Young function which satisfy the Δ_2-condition, indeed, since

$$\lim_{t\to\infty} \frac{\varphi_2(2t)}{\varphi_2(t)} = \lim_{t\to\infty} \frac{2^p t^p \log(1+2t)}{t^p \log(1+t)} = 2^{p-1}.$$

Also, φ_2 satisfy globally the Δ_2-condition.

In fact, since for each $t \geq 0$ we have $(1+t)^2 \geq 1+2t$, then

$$\begin{aligned}\varphi_2(2t) &= 2^p t^p \log(1+2t)\\ &\leq 2^{p+1} t^p \log(1+2t)\\ &\leq 2^{p+1}\varphi_2(2t).\end{aligned}$$

1.4. Lemma. A Young function φ satisfy the Δ_2-condition if and only if there exist constants $\lambda > 1$ and $t_0 > 0$ such that

$$\frac{tp(t)}{\varphi(t)} < \lambda$$

for all $t \geq t_0$, where p is the right derivate of φ.

Proof. Suppose that φ satisfy the Δ_2-condition, then there exists a constant $k > 0$ such that

$$k\varphi(t) \geq \varphi(2t) = \int_0^{2t} p(s)\, ds > \int_t^{2t} p(s)\, ds$$

for t large enough, since p is increasing, then we have

$$\int_t^{2t} p(s)\,ds > tp(t);$$

hence, for t large enough, we obtain

$$\frac{tp(t)}{\varphi(t)} \leq k.$$

Conversely, if

$$\frac{tp(t)}{\varphi(t)} < \lambda$$

for all $t \geq t_0$, then

$$\int_t^{2t} \frac{p(s)}{\varphi(s)}\,ds < \lambda \int_t^{2t} \frac{ds}{s} = \lambda \log 2.$$

Since $p(s) = \varphi'(s)$, we have

$$\log\left(\frac{\varphi(2t)}{\varphi(t)}\right) < \lambda \log 2,$$

which implies that

$$\varphi(2t) < 2^\lambda \varphi(t). \qquad \square$$

The following result show us that the Young functions which satisfy the Δ_2-condition have a cross rate less than the function t^p for some $p > 1$.

1.5. Theorem. If φ is a Young function which satisfy the Δ_2-condition, then there exist constants $\lambda > 1$ and $C > 0$ such that

$$\varphi(t) \leq Ct^\lambda$$

for t large enough.

Proof. By (1.4) we can write

$$\int_{t_0}^{t} \frac{p(s)}{\varphi(s)}\,ds < \lambda \int_{t_0}^{t} \frac{ds}{s}$$

where $t \geq t_0$. Then

$$\log\left(\frac{\varphi(t)}{\varphi(t_0)}\right) < \lambda \log\left(\frac{t}{t_0}\right),$$

therefore

$$\varphi(t) < \frac{\varphi(t_0)}{t_0^\lambda} t^\lambda.$$

And the proof is complete. $\square$

1.6. Example. The following are Young functions:

(1) $\varphi(x) = \frac{|x|^p}{p}$ with $p > 1$.
(2) $\varphi(x) = e^{|x|} - |x| - 1$.
(3) $\varphi(x) = e^{|x|^\delta} - 1$ with $\delta > 1$.
(4) $\varphi(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1 \\ +\infty, & \text{otherwise.} \end{cases}$

Related with the Young function φ, we define, for $t \geq 0$ the complementary function of Young function as

$$\psi(t) = \sup\{ts - \varphi(s) : s \geq 0\}.$$

1.7. Example. If $\varphi(t) = \frac{1}{p}t^p$ with $p > 1$ and $t \geq 0$, then its complementary function is $\psi(t) = \frac{1}{q}t^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Indeed, by definition we have

$$\psi(t) = \sup\left\{ts - \frac{1}{p}s^p : s \geq 0\right\},$$

next, for $t > 0$ fixed, we can consider the function

$$g(s) = ts - \frac{1}{p}s^p, \quad \text{with } s \geq 0.$$

It is not hard to check that g achieved its maximum at $s = t^{\frac{1}{p-1}}$ which is given by

$$g\left(t^{\frac{1}{p-1}}\right) = \frac{1}{q}t^q.$$

Hence

$$\psi(t) = \sup\left\{ts - \frac{1}{p}s^p : s \geq 0\right\} = \frac{1}{q}t^q.$$

1.8. Proposition. If φ is a Young function, then its complementary function ψ is also a Young function.

Proof. It is clear that $\psi(0) = 0$ if and only if $x = 0$. Now, we just need to show that ψ is a convex function. To this end, let us choose $t_1, t_2 \in [0, +\infty)$ and $\lambda \in [0, 1]$. Then, by definition of ψ we have

$$\psi(\lambda t_1 + (1-\lambda)t_2) = \sup\{s(\lambda t_1 + (1-\lambda)t_2) - \varphi(s) : s \geq 0\}.$$

On the other hand

$$\lambda\psi(t_1) = \lambda \sup\{st_1 - \varphi(s) : s \geq 0\} \geq \lambda(st_1 - \varphi(s)) \ \forall\ s \geq 0$$

and

$$(1-\lambda)\psi(t_2) = (1-\lambda)\sup\{st_2 - \varphi(s) : s \geq 0\} \geq (1-\lambda)(st_2 - \varphi(s)) \ \forall\ s \geq 0.$$

From the last two inequalities, we have

$$\begin{aligned} s(\lambda t_1 + (1-\lambda)t_2) - \varphi(s) &= \lambda(st_1 - \varphi(s)) + (1-\lambda)(st_2 - \varphi(s)) \\ &\leq \lambda\psi(t_1) + (1-\lambda)\psi(t_2) \end{aligned}$$

for all $s \geq 0$. Which means that $\lambda\psi(t_1) + (1-\lambda)\psi(t_2)$ is an upper bound of the set

$$\{s(\lambda t_1 + (1-\lambda)t_2) - \varphi(s) : s \geq 0\},$$

then

$$\psi(\lambda t_1 + (1-\lambda)t_2)) \leq \psi(t_1) + (1-\lambda)\psi(t_2),$$

and so ψ is convex. □

1.9. Theorem (Young's Inequality)**.** Let ψ be the complementary function of φ. Then

$$ts \leq \varphi(s) + \psi(t)$$

where $t, s \in [0, +\infty)$.

Proof. Let $t, s \in [0, +\infty)$. Then

$$\begin{aligned}\psi(t) &= \sup\{st - \varphi(s) : s \geq 0\} \\ &\geq st - \varphi(s) \quad \forall\ s \geq 0,\end{aligned}$$

then

$$\psi(t) + \varphi(s) \geq st,$$

and the proof is complete. □

For more details on Young functions see [10].

2. Weighted Lorentz-Orlicz Spaces

The aim of this section is to present basic results about Lorentz-Orlicz spaces. We have tried to make the proofs as self-contained and synthetic as possible.

2.1. Definition (Luxemburg norm)**.** Let φ be a Young function. For any measurable function f on X,

$$\|f\|_{\varphi,w} = \inf\left\{\varepsilon > 0 : \int_0^\infty \varphi\left(\frac{f^*(t)}{\varepsilon}\right) w(t)\,dt \leq 1\right\} \in [0, \infty).$$

Where it is understood that $\inf(\emptyset) = +\infty$.

2.2. *Remark.* In this article, we will not always require that the Luxemburg norm actually be a norm. $\|\cdot\|_{\varphi,w}$ is indeed a quasinorm. A quasinorm is a functional that is like a norm except that it does only satisfy the triangle inequality with a constant $C \geq 1$, that is, $\|f + g\| \leq C(\|f\| + \|g\|)$ where $C \geq 1$.

2.3. Lemma. For any measurable function f on X, $\|f\|_{\varphi,w} = 0$ if and only if $f = 0$ μ-almost everywhere.

Proof. Clearly $\|f\|_{\varphi,w} = 0$ if and only if $\int_0^\infty \varphi\left(\frac{f^*(t)}{\varepsilon}\right) w(t)\,dt \leq 1\ \forall\ \varepsilon > 0$. It follows that

$$\begin{aligned}\|f\|_{\varphi,w} = 0 \text{ if and only if } &\int_0^\infty \varphi\left(\alpha f^*(t)\right) w(t)\,dt = 0\ \forall\ \alpha > 0 \\ \text{if and only if } &\varphi\left(\alpha f^*(t)\right) w(t) = 0\ \mu - \text{a.e.}\ \forall\ \alpha > 0 \\ \text{if and only if } &f^*(t) = 0\ \mu - \text{a.e.} \\ \text{if and only if } &D_f(\lambda) = 0\ \mu - \text{a.e.} \\ \text{if and only if } &f = 0\ \mu - \text{a.e.}\end{aligned}$$

□

Identification of almost everywhere equal functions. As with L_p spaces, one identifies the function which are μ-almost everywhere equal. This means that one works with the equivalence classes of the equivalence relation defined by the μ-almost everywhere equality. From now on, this will be done without further mention. Consequently, one write:

$$\|f\|_{\varphi,w} = 0 \text{ if and only if } f = 0. \tag{2.1}$$

2.4. Lemma. If $0 < \|f\|_{\varphi,w} < \infty$ then $\int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \leq 1$. In particular, $\|f\|_{\varphi,w} \leq 1$ is equivalent to $\int_0^\infty \varphi\left(f^*(t)\right) w(t)\,dt \leq 1$.

Proof. For all $b > \|f\|_{\varphi,w}$, we have

$$\int_0^\infty \varphi\left(\frac{f^*(t)}{b}\right) w(t)\,dt \leq 1.$$

Letting b decrease to $\|f\|_{\varphi,w}$, one obtains the first result by monotone convergence. The second statement follows from this and lemma 2.8. □

2.5. Proposition. The gauge $\|\cdot\|_{\varphi,w}$ is a quasinorm on the vector space of all the measurable functions f such that $\|f\|_{\varphi,w} < \infty$.

Proof. It is already seen that (2.1) holds under identification of a.e. equal functions.

It is clear that for all real λ, $\|\lambda f\|_{\varphi,w} = |\lambda|\|f\|_{\varphi,w}$.

It remains to prove the triangle inequality. Let f and g be two measurable functions such that $0 < \|f\|_{\varphi,w} + \|g\|_{\varphi,w} < \infty$. Then

$$\begin{aligned}
&\int_0^\infty \varphi\left(\frac{(f+g)^*(t)}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\right) w(t)\,dt \\
\leq &\int_0^\infty \varphi\left(\frac{f^*(t/2) + g^*(t/2)}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\right) w(t)\,dt \\
= &\int_0^\infty \varphi\left(\frac{\|f\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\frac{f^*(t/2)}{\|f\|_{\varphi,w}} + \frac{\|g\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\frac{g^*(t/2)}{\|g\|_{\varphi,w}}\right) w(t)\,dt \\
\leq &\frac{\|f\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\int_0^\infty \varphi\left(\frac{f^*(t/2)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
&+ \frac{\|g\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\int_0^\infty \varphi\left(\frac{g^*(t/2)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
= &\frac{\|f\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}2\int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(2t)\,dt \\
&+ \frac{\|g\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}2\int_0^\infty \varphi\left(\frac{g^*(t)}{\|f\|_{\varphi,w}}\right) w(2t)\,dt \\
\leq &\frac{\|f\|_{\varphi,w}}{\|f\|_{\varphi,w} + \|g\|_{\varphi,w}}\int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
&+ \frac{\|g\|_{\varphi,w}}{\|f\|_{\varphi,w} + \|g\|_{\varphi,w}}\int_0^\infty \varphi\left(\frac{g^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
\leq &1.
\end{aligned}$$

Where the last but one inequality follows from the convexity of φ and the fact that w is nonincreasing and the last inequality from lemma 2.4. Therefore

$$\|f+g\|_{\varphi,w} \leq 2\left(\|f\|_{\varphi,w} + \|g\|_{\varphi,w}\right).$$

As a consequence, the set of all measurable functions f such that $\|f\|_{\varphi,w} < \infty$ is a vector space. □

2.6. Definition. Let φ be a Young function. We define the weighted Lorenz-Orlicz spaces

$$L_{\varphi,w} = \left\{f : X \to \mathbb{C} \text{ measurable} : \int_0^\infty \varphi(\alpha f^*(t))w(t)\,dt < \infty, \text{ for some } \alpha > 0\right\}.$$

It follows from proposition 1.8 that if $L_{\varphi,w}$ is a weighted Lorentz-Orlicz space, then $L_{\psi,w}$ is also a weighted Lorenz-Orlicz space.

2.7. Proposition (Hölder's type inequality)**.** For $f \in L_{\varphi,1}$ and $g \in L_{\psi,1}$

$$\int_X |fg|\,d\mu \leq 2\|f\|_{\varphi,1}\|g\|_{\psi,1}.$$

In particular, $fg \in L_1$.

Proof. If $\|f\|_{\varphi,1} = 0$ or $\|g\|_{\psi,1} = 0$, one concludes with lemma 2.8.

Assume now that $0 < \|f\|_{\varphi,1}, \|g\|_{\psi,1}$. Because of Young's inequality: $st \leq \varphi(s) + \varphi(t)$ we have

$$\begin{aligned}\int_X \frac{|fg|}{\|f\|_{\varphi,1}\|g\|_{\psi,1}}\,d\mu &\leq \int_0^\infty \frac{f^*(t)g^*(t)}{\|f\|_{\varphi,1}\|g\|_{\psi,1}}\,dt\\ &\leq \int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,1}}\right)dt + \int_0^\infty \psi\left(\frac{g^*(t)}{\|g\|_{\psi,1}}\right)dt\\ &\leq 2.\end{aligned}$$

Therefore

$$\int_X |fg|\,d\mu \leq 2\|f\|_{\varphi,1}\|g\|_{\psi,1}.$$

□

2.8. Lemma. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $L_{\varphi,w}$. Then, the following assertions are equivalent:

(a) $\lim_{n\to\infty} \|f_n\|_{\varphi,w} = 0$;
(b) For all $\alpha > 0$, $\limsup_{n\to\infty} \int_0^\infty \varphi(\alpha f_n^*(t))w(t)\,dt \leq 1$;
(c) For all $\alpha > 0$, $\lim_{n\to\infty} \int_0^\infty \varphi(\alpha f_n^*(t))w(t)\,dt = 0$.

Proof. The equivalence $(a) \Leftrightarrow (b)$ is a direct consequence of the definition of $\|\cdot\|_{\varphi,w}$. Off course $(c) \Rightarrow (b)$ is obvious. As φ is convex and $\varphi(0) = 0$ for all $t \geq 0$ and $0 < \varepsilon \leq 1$, we have

$$\varphi(t) = \varphi\left((1-\varepsilon)0 + \varepsilon\frac{t}{\varepsilon}\right) \leq (1-\varepsilon)\varphi(0) + \varepsilon\varphi\left(\frac{t}{\varepsilon}\right),$$

that is

$$\varphi(t) \leq \varepsilon\varphi\left(\frac{t}{\varepsilon}\right) \quad t \geq 0, 0 < \varepsilon \leq 1.$$

From which $(b) \Rightarrow (c)$ follows easily. □

2.9. Theorem. The space $L_{\varphi,w}$ is a quasi-Banach space.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $L_{\varphi,w}$. Let us choose $\tilde{\varepsilon} > 0$ such that $\tilde{\varepsilon}\varphi^{-1}\left(\frac{\varepsilon}{k_0}\right) < \frac{1}{n+m}$ for $n, m \in \mathbb{N}$ and $\varepsilon > 0, k_0 > 0$. For such $\tilde{\varepsilon}$ there exists $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\|_{\varphi,w} < \tilde{\varepsilon}.$$

If $n, m \geq n_0$. By the definition of the Luxemburg quasi-norm we can use $k_0 > 0$ in such a way that $k_0 < \tilde{\varepsilon}$ and

$$\int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\,dt \leq 1.$$

Let $E = \{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}$, then

$$\varepsilon\chi_E(x) \leq |f_n(x) - f_m(x)|.$$

And hence

$$\varepsilon\chi_E^*(t) \le (f_n - f_m)^*(t),$$
$$\varepsilon\chi_{(0,\mu(E))}(t) \le (f_n - f_m)^*(t).$$

Therefore

$$\int_0^\infty \varphi\left(\frac{\varepsilon}{k_0}\chi_{(0,\mu(E))}(t)\right) w(t)\,dt \le \int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\,dt.$$

Then

$$\int_0^{\mu(E)} \varphi\left(\frac{\varepsilon}{k_0}\right) w(t)\,dt \le \int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\,dt$$
$$\Rightarrow \tilde{\varepsilon}\int_0^{D_{f_n-f_m}(\varepsilon)} w(t)\,dt \le \tilde{\varepsilon}\varphi^{-1}\left(\frac{\varepsilon}{k_0}\right)\int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\,dt$$
$$\Rightarrow \tilde{\varepsilon}\int_0^{D_{f_n-f_m}(\varepsilon)} w(t)\,dt \le \frac{1}{n+m}$$
$$\Rightarrow \tilde{\varepsilon}\lim_{n,m\to\infty}\int_0^{D_{f_n-f_m}(\varepsilon)} w(t) = 0.$$

Since $w > 0$, we must have $\lim_{n,m\to\infty} D_{f_n-f_m}(\varepsilon) = 0$ which means that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, then some subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges almost everywhere to a measurable function f, that is, $f_{n_k} \to f$ μ-a.e.

Let $\alpha > 0$. By lemma 2.8 there exists a large enough integer $n(\alpha)$ such that

$$\int_0^\infty \varphi\left(\alpha(f_n - f_m)^*(t)\right) w(t)\,dt \le 1, \quad \forall\ m, n \ge n(\alpha).$$

With Fatou's lemma this gives

$$\int_0^\infty \varphi\left(\alpha(f_n - f)^*(t)\right) w(t)\,dt \le \liminf \int_0^\infty \varphi\left(\alpha(f_n - f_m)^*(t)\right) w(t)\,dt \le 1$$

$\forall\ m \ge n(\alpha)$. Therefore $f_n - f$ belongs to $L_{\varphi,w}$, but $f_n \in L_{\varphi,w}$, so that $f \in L_{\varphi,w}$.

Moreover, as $\limsup_{m\to\infty}\int_0^\infty \varphi\left(\alpha(f_m - f)^*(t)\right) w(t)\,dt \le 1$ for all $\alpha > 0$, we have $\lim_{m\to\infty}\|f_m - f\|_{\varphi,w} = 0$. This proves that $L_{\varphi,w}$ is complete. □

2.10. Theorem. Simple functions are dense in $L_{\varphi,w}$.

Proof. Suppose $f \in L_{\varphi,w}$. We may assume that $f \ge 0$. Note that if $D_f(\lambda) = \infty$, then $\lim_{t\to\infty} f^*(t) = 0$. It follows that $D_f(\lambda) < \infty$.

Hence, given $\varepsilon, \delta > 0$, we can find a simple function $s_n \ge 0$ such that $s_n(x) = 0$ when $f(x) \le \varepsilon$ and $f(x) - \varepsilon \le s_n(x) \le f(x)$ when $f(x) > \varepsilon$ except on a set of measure less than δ. It follows that

$$\mu\left(\{x \in X : |f(x) - s_n(x)| > \varepsilon\}\right) < \delta.$$

Next, choose $n \in \mathbb{N}$ such that $n \ge \frac{1}{\varepsilon}$, then

$$(f - s_n)^*(t) = \inf\{\varepsilon > 0 : D_{f-s_n}(\varepsilon) < \delta \le t\}.$$

Thus

$$(f - s_n)^*(t) \le \frac{1}{n} \quad \text{for } t \ge \delta,$$

since $s_n \le f$, then $s_n^*(t) \le f^*(t)$, for each $t > 0$. Since $n > \frac{1}{\varepsilon}$, we have

$$(f - s_n)^*(t) \le \frac{1}{n} < \varepsilon,$$

next,

$$\int_0^\infty \varphi\left(\frac{(f-s_n)^*(t)}{k}\right) w(t)\,dt \le \int_0^\infty \varphi\left(\frac{1}{nk}\right) w(t)\,dt.$$

Let $a = \int_0^\infty w(t)\,dt$, then

$$\begin{aligned}\|f-s_n\|_{\varphi,w} &= \inf\left\{k>0 : \int_0^\infty \varphi\left(\frac{(f-s_n)^*(t)}{k}\right) w(t)\,dt \le 1\right\} \\ &= \frac{1}{n\varphi^{-1}\left(\frac{1}{a}\right)} \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

□

3. Multiplication Operator

Let $F(X)$ be a function space on non-empty set X. Let $u : X \to \mathbb{C}$ be a function such that $u \cdot f \in F(X)$ whenever $f \in F(X)$.

Then, the transformation $f \mapsto u \cdot f$ on F is denoted by M_u. In case $F(X)$ is a topological space and M_u is continuos, we call it a multiplication operator induced by u.

Multiplication operators generalize the notion of operator given by a diagonal matrix. More precisely, one of the results of operator theory is a spectral theorem, which states that every self-adjoint operator on a Hilbert space is unitarily equivalent to a multiplication operator on an L_2 space.

These operators received considerable attention over the past several decades specially on L_p spaces and Bergman spaces and they played an important role in the study of operators on Hilbert spaces.

For more details on these operators we refer to Abrahamese [1], Axler [4], Douglas [6], Halmos [8] and Takagi [12].

3.1. Example. Consider the Hilbert space $X = L_2[-1,3]$ of complex-valued square integrable functions on the interval $[-1,3]$. Define the operator

$$M_u(x) = u(x)x^2,$$

for any function $u \in X$. This will be a self-adjoint bounded linear operator with norm 9. Its spectrum will be the interval $[0,9]$ (the range of the function $x \to x^2$ defined on $[-1,3]$). Indeed, for any complex number λ, the operator $M_u - \lambda$ is given by

$$(M_u - \lambda)(x) = u(x)(x^2 - \lambda).$$

It is invertible if and only if λ is not in $[0,9]$, and then its inverse is

$$(M_u - \lambda)^{-1}(x) = \frac{u(x)}{x^2 - \lambda}.$$

which is another multiplication operator.

For a systematic study of the multiplication operators on different spaces we refer to [1, 3, 4, 5, 9, 11].

3.2. *Remark.* In general, the multiplication operators on measurable spaces is not 1–1. Indeed, let $(X, \mathcal{A}, \mu)$ be a measure space and

$$A = X \setminus \operatorname{supp}(u) = \{x \in X : u(x) = 0\}.$$

If $\mu(A) \neq 0$ and $f = \chi_A$ then for any $x \in X$ we have $f(x)u(x) = 0$ which implies that $M_u(f) = 0$, therefore $\ker(M_u) \neq \{0\}$ and hence M_u is not 1–1.

If, on the contrary, M_u is 1–1, then $\mu(X \setminus \operatorname{supp}(u)) = 0$. On the other hand, if $\mu(X \setminus \operatorname{supp}(u)) = 0$ and μ is a complete measure, then $M_u(f) = 0$ implies $f(x)u(x) = 0\ \forall\, x \in X$, then $\{x \in X : f(x) \neq 0\} \subseteq X \setminus \operatorname{supp}(u)$ and so $f = 0$ μ-a.e. on X.

Hence, if $\mu(X \setminus \operatorname{supp}(u)) = 0$ and μ is a complete measure, then M_u is 1–1.

3.3. Proposition. M_u is 1–1 on $Y = L_{\varphi,w}(\operatorname{supp} u)$.

Proof. Let $Y = L_{\varphi,w}(\operatorname{supp} u) = \{f\chi_{\operatorname{supp} u} : f \in L_{\varphi,w}\}$. Indeed, if $M_u(\tilde{f}) = 0$ with $\tilde{f} = f\chi_{\operatorname{supp} u} \in Y$, then $f(x)\chi_{\operatorname{supp} u}(x)u(x) = 0$ for all $x \in X$, and so

$$\begin{aligned} & f(x)u(x) = 0 \quad \forall\, x \in \operatorname{supp}(u), \\ \Rightarrow\ & f(x) = 0 \quad \forall\, x \in \operatorname{supp}(u), \\ \Rightarrow\ & f(x)\chi_{\operatorname{supp} u(x)}(x) = 0 \quad \forall\, x \in X. \end{aligned}$$

Then $\tilde{f} = 0$ and the proof is complete. □

In what follows, boundedness and invertibility of the multiplication M_u are characterized in terms of the boundedness and invertibility of the complex valued measurable function u respectively.

3.4. Theorem. The linear transformation $M_u : f \to u \cdot f$ on the Orlicz-Lorentz space $L_{\varphi,w}$ is bounded if and only if u is essentially bounded. Moreover

$$\|M_u\| = \|u\|_\infty.$$

Proof. Let $u \in L_\infty(\mu)$, note $|(uf)(x)| \le \|u\|_\infty |f(x)|$, thus

$$\begin{aligned} \{x : |(uf)(x)| > \lambda\} &\subseteq \{x : \|u\|_\infty |f(x)| > \lambda\} \\ &= \left\{x : |f(x)| > \frac{\lambda}{\|u\|_\infty}\right\} \end{aligned}$$

then

$$D_{uf}(\lambda) \le D_f\left(\frac{\lambda}{\|u\|_\infty}\right)$$

and so

$$\left\{\lambda > 0 : D_f\left(\frac{\lambda}{\|u\|_\infty}\right) \le t\right\} \subseteq \{\lambda > 0 : D_{uf}(\lambda) \le t\}.$$

From this we have

$$\begin{aligned} \inf\{\lambda > 0 : D_{uf}(\lambda) \le t\} &\le \inf\left\{\lambda > 0 : D_f\left(\frac{\lambda}{\|u\|_\infty}\right) \le t\right\} \\ &\le \inf\{\alpha\|u\|_\infty > 0 : D_f(\alpha) \le t\} \\ &= \|u\|_\infty \inf\{\alpha > 0 : D_f(\alpha) \le t\}. \end{aligned}$$

Hence

$$(uf)^*(t) \le \|u\|_\infty f^*(t).$$

Then

$$\begin{aligned} \int_0^\infty \varphi\left(\frac{(uf)^*(t)}{\|u\|_\infty \|f\|_{\varphi,w}}\right) w(t)\, dt &\le \int_0^\infty \varphi\left(\frac{\|u\|_\infty f^*(t)}{\|u\|_\infty \|f\|_{\varphi,w}}\right) w(t)\, dt \\ &= \int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\, dt \le 1. \end{aligned}$$

Hence $f \in L_{\varphi,w}$ and

$$\|M_u f\|_{\varphi,w} \le \|u\|_\infty \|f\|_{\varphi,w}. \tag{3.1}$$

Conversely, suppose M_u is a bounded operator. If u is not essentially bounded function, then for every $n \in \mathbb{N}$, the set $E_n = \{x \in X : |u(x)| > n\}$ has a positive measure. Now, we know that

$$\chi^*_{E_n}(t) = \chi_{0,\mu(E_n)}(t),$$

and note

$$\{x : n\chi_{E_n}(x) > \lambda\} \subseteq \{x : |u\chi_{E_n}(x)| > \lambda\},$$

then

$$D_{n\chi_{E_n}}(\lambda) \leq D_{u\chi_{E_n}}(\lambda),$$

from this we have

$$\{\lambda > 0 : D_{u\chi_{E_n}}(\lambda) \leq t\} \subseteq \{\lambda > 0 : D_{n\chi_{E_n}}(\lambda) \leq t\}.$$

Hence

$$\inf\{\lambda > 0 : D_{n\chi_{E_n}}(\lambda) \leq t\} \leq \inf\{\lambda > 0 : D_{u\chi_{E_n}}(\lambda) \leq t\}.$$

That is,

$$(u\chi_{E_n})^*(t) \geq n(\chi_{E_n})^*(t).$$

This gives us

$$\begin{aligned} 1 &\geq \int_0^\infty \varphi\left(\frac{(u\chi_{E_n})^*(t)}{k}\right) w(t)\, dt \\ &\geq \int_0^\infty \varphi\left(\frac{(n\chi_{E_n})^*(t)}{k}\right) w(t)\, dt, \end{aligned}$$

and so

$$\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(u\chi_{E_n})^*(t)}{k}\right) w(t)\, dt \leq 1\right\} \subseteq \left\{k > 0 : \int_0^\infty \varphi\left(\frac{(n\chi_{E_n})^*(t)}{k}\right) w(t)\, dt \leq 1\right\},$$

thus

$$\inf\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(n\chi_{E_n})^*(t)}{k}\right) w(t)\, dt \leq 1\right\} \leq \inf\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(u\chi_{E_n})^*(t)}{k}\right) w(t)\, dt \leq 1\right\},$$

which means that

$$\|M_u\chi_{E_n}\|_{\varphi,w} \geq n\|\chi_{E_n}\|_{\varphi,w},$$

this contradicts the boundedness of M_u. Hence u must be essentially bounded.

Next, clearly by (3.1) we obtain

$$\|M_u\| \leq \|u\|_\infty. \tag{3.2}$$

For $\varepsilon > 0$, let $E = \{x \in X : |u(x)| \geq \|u\|_\infty - \varepsilon\}$ (observe that $\mu(E) > 0$), then

$$\{x \in X : (\|u\|_\infty - \varepsilon)\chi_E(x) > \lambda\} \subseteq \{x \in X : |u\chi_E(x)| > \lambda\},$$

then

$$D_{(\|u\|_\infty - \varepsilon)\chi_E}(\lambda) \leq D_{u\chi_E}(\lambda)$$

and so

$$\{\lambda > 0 : D_{u\chi_E}(\lambda) \leq t\} \subseteq \{\lambda > 0 : D_{(\|u\|_\infty - \varepsilon)\chi_E} \leq t\}$$

from this we have

$$\inf\{\lambda > 0 : D_{(\|u\|_\infty - \varepsilon)\chi_E} \leq t\} \leq \inf\{\lambda > 0 : D_{u\chi_E}(\lambda) \leq t\}.$$

Therefore

$$(u\chi_E)^*(t) \geq (\|u\|_\infty - \varepsilon)(\chi_E)^*(t),$$

then

$$\int_0^\infty \varphi\left(\frac{(\|u\|_\infty - \varepsilon)(\chi_E)^*(t)}{\|M_u\chi_E\|_{\varphi,w}}\right) w(t)\,dt \le \int_0^\infty \varphi\left(\frac{(u\chi_E)^*(t)}{\|M_u\chi_E\|_{\varphi,w}}\right) w(t)\,dt \le 1,$$

which implies that

$$\|(\|u\|_\infty - \varepsilon)\chi_E\|_{\varphi,w} \le \|M_u\chi_E\|_{\varphi,w},$$

and

$$(\|u\|_\infty - \varepsilon)\|\chi_E\|_{\varphi,w} \le \|M_u\chi_E\|_{\varphi,w},$$

hence

$$\|u\|_\infty - \varepsilon \le \frac{\|M_u\chi_E\|_{\varphi,w}}{\|\chi_E\|_{\varphi,w}},$$

which provide that

$$\|M_u\| \ge \|u\|_\infty - \varepsilon \quad \forall\, \varepsilon > 0$$

and so

$$\|M_u\| \ge \|u\|_\infty.$$

Therefore

$$\|M_u\| = \|u\|_\infty. \qquad \square$$

We will need the following well known result.

3.5. Theorem. Let $T \in B(X,Y)$ where X and Y are Banach spaces. Then T is bounded below if and only if T is 1–1 and has closed range.

For the proof of theorem 3.5 see [2].

3.6. Corollary. $M_u : L_{\varphi,w}(\operatorname{supp} u) \to L_{\varphi,w}(\operatorname{supp} u)$ has closed range if and only if M_u is bounded below on $L_{\varphi,w}(\operatorname{supp} u)$.

This result is clear since M_u is 1–1 on $L_{\varphi,w}(\operatorname{supp} u)$. Moreover, if $u \ne 0$ μ-a.e. on X with μ a complete measure, then we have the following result.

3.7. Corollary. If $\mu \ne 0$ μ-a.e. on X and μ is a complete measure, then

$$M_u : L_{\varphi,w}(X,\mathcal{A},u) \to L_{\varphi,w}(X,\mathcal{A},u)$$

has a closed range if and only if M_u is bounded below on $L_{\varphi,w}(X,\mathcal{A},u)$.

3.8. Theorem. $M_u : L_{\varphi,w}(\operatorname{supp} u) \to L_{\varphi,w}(\operatorname{supp} u)$ has a closed range if and only if there exists $\delta > 0$ such that $|u(x)| > \delta$ μ-a.e. on $\operatorname{supp}\mu$.

Proof. If there exists a $\delta > 0$ such that $|u(x)| \ge \delta$ μ-a.e. on supp(u), then for $f \in L_{\varphi,w}$ and $t > 0$ we have

$$\{x : |\delta f\chi_{\mathrm{supp}(u)}(x)| > \lambda\} \subseteq \{x : |uf\chi_{\mathrm{supp}(u)}(x)| > \lambda\},$$

and so

$$D_{\delta f\chi_{\mathrm{supp}(u)}}(\lambda) \le D_{uf\chi_{\mathrm{supp}(u)}}(\lambda),$$

then

$$\{\lambda > 0 : D_{uf\chi_{\mathrm{supp}(u)}}(\lambda) \le t\} \subseteq \{\lambda > 0 : D_{\delta f\chi_{\mathrm{supp}(u)}}(\lambda) \le t\},$$

from this we have

$$\inf\{\lambda > 0 : D_{\delta f\chi_{\mathrm{supp}(u)}}(\lambda) \le t\} \le \inf\{\lambda > 0 : D_{uf\chi_{\mathrm{supp}(u)}}(\lambda) \le t\},$$

thus

$$(uf\chi_{\mathrm{supp}(u)})^*(t) \geq \delta f\chi^*_{\mathrm{supp}(u)}(t),$$

then we shall note that

$$\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(uf\chi_{\mathrm{supp}(u)})^*(t)}{k}\right) w(t)\,dt \leq 1\right\} \subseteq \left\{k > 0 : \int_0^\infty \varphi\left(\frac{(\delta f\chi_{\mathrm{supp}(u)})^*(t)}{k}\right) w(t)\,dt \leq 1\right\}.$$

Hence

$$\inf\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(\delta f\chi_{\mathrm{supp}(u)})^*(t)}{k}\right) w(t)\,dt \leq 1\right\} \leq \inf\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(uf\chi_{\mathrm{supp}(u)})^*(t)}{k}\right) w(t)\,dt \leq 1\right\},$$

which means that

$$\|\delta f\chi_{\mathrm{supp}(u)}\|_{\varphi,w} \leq \|M_u f\chi_{\mathrm{supp}(u)}\|_{\varphi,w},$$

thus

$$\|M_u f\chi_{\mathrm{supp}(u)}\|_{\varphi,w} \geq \delta\|f\chi_{\mathrm{supp}(u)}\|_{\varphi,w}.$$

Therefore M_u has closed range.

Conversely, assume that M_u has closed range on $L_{\varphi,w}(\mathrm{supp}(u))$. Since $M_u : L_{\varphi,w}(\mathrm{supp}(u)) \to L_{\varphi,w}(\mathrm{supp}(u))$ is 1–1, then M_u is bounded below, then there exists an $\varepsilon > 0$ such that

$$\|M_u f\|_{\varphi,w} \geq \varepsilon\|f\|_{\varphi,w}$$

for all $f \in L_{\varphi,w}(\mathrm{supp}(u))$. Let $E = \{x \in \mathrm{supp}(u) : |u(x)| < \varepsilon/2\}$.

If $\mu(E) > 0$, then we can find a measurable set $F \subseteq E$ such that $\chi_F \in L_{\varphi,w}(\mathrm{supp}(u))$. Then

$$\{x : |u\chi_F| > \lambda\} \subseteq \left\{x : \left|\frac{\varepsilon}{2}\chi_F\right| > \lambda\right\}$$

and so

$$D_{u\chi_F}(\lambda) \leq D_{\frac{\varepsilon}{2}\chi_F}(\lambda),$$

from this we have

$$\{\lambda > 0 : D_{\frac{\varepsilon}{2}\chi_F}(\lambda) \leq t\} \subseteq \{\lambda > 0 : D_{u\chi_F}(\lambda) \leq t\},$$

then

$$\inf\{\lambda > 0 : D_{u\chi_F}(\lambda) \leq t\} \leq \inf\{\lambda > 0 : D_{\frac{\varepsilon}{2}\chi_F}(\lambda) \leq t\}$$

that is,

$$(u\chi_F)^*(t) \leq \left(\frac{\varepsilon}{2}\chi_F\right)^*(t),$$

and so

$$(u\chi_F)^*(t) \leq \frac{\varepsilon}{2}(\chi_F)^*(t).$$

Therefore

$$\begin{aligned}\|M_u\chi_F\|_{\varphi,w} &= \inf\left\{\varepsilon>0: \int_0^\infty \varphi\left(\frac{(u\chi_F)^*(t)}{\varepsilon}\right) w(t)\,dt \le 1\right\}\\ &\le \inf\left\{\varepsilon>0: \int_0^\infty \varphi\left(\frac{\frac{\varepsilon}{2}(\chi_F)^*(t)}{\varepsilon}\right) w(t)\,dt \le 1\right\}\\ &= \frac{\varepsilon}{2}\|\chi_F\|_{\varphi,w},\end{aligned}$$

which is a contradiction. Therefore $\mu(E)=0$. This completes the proof. □

3.9. Corollary. If $\mu \ne 0$ μ-a.e. on X, and μ is a complete measure, then M_u has a closed range on $L_{\varphi,w}(X,\mathcal{A},\mu)$ if and only if there exists $\delta>0$ such that $|u(x)|\ge\delta$ μ-a.e. on X.

Proof. The result follows as a consequence of

$$L_{\varphi,w}(X,\mathcal{A},\mu) = L_{\varphi,w}(\operatorname{supp} u)$$ □

3.10. Theorem. The set of all multiplication operators on $L_{\varphi,w}$ is a maximal abelian subalgebra of the set $B(L_{\varphi,w})$, the algebra of all bounded linear operators on $L_{\varphi,w}$.

Proof. Let

$$\mathcal{H} = \{M_u : u \in L_\infty\}$$

and consider the operator product

$$M_u \cdot M_v = M_{uv},$$

where $M_u, M_v \in \mathcal{H}$. Let us check that it is a Banach algebra. Let $u, v \in L_\infty$, then $|u| \le \|u\|_\infty$ and $|v| \le \|v\|_\infty$, therefore

$$\|uv\|_\infty \le \|u\|_\infty \|v\|_\infty,$$

this implies that the product is an inner operation, moreover the usual function product is associative, commutative and distributive respect to the sum and the scalar product, thus we conclude that $\mathcal{H}$ is a subalgebra of $B(L_{\varphi,w})$. Now, we like to check that it is a maximal subalgebra, that is, given $N \in B(L_{\varphi,w})$, if N commute with $\mathcal{H}$, we have to prove that $N \in \mathcal{H}$. Consider the unit function $e : X \to \mathbb{C}$ defined by $e(x) = 1$ for all $x \in X$. Let $N \in B(L_{\varphi,w})$ be an operator which commute with $\mathcal{H}$ and let χ_E the characteristic function of a measurable set E. Then

$$\begin{aligned}N(\chi_E) &= N[M_{\chi_E}(e)]\\ &= M_{\chi_E}[N(e)]\\ &= \chi_E \cdot N(e)\\ &= N(e)\cdot\chi_E\\ &= M_w \cdot \chi_E,\end{aligned}$$

where $w = N(e)$. Similarly

$$N(s) = M_w(s) \tag{3.3}$$

for any simple function.

Now, let us check that $w \in L_\infty$. By way of contradiction, assume that $w \notin L_\infty$, then the set

$$E_n = \{x \in X : |w(x)| > n\}$$

has a positive measure for each $n \in \mathbb{N}$. Note that

$$M_w(\chi_{E_n})(x) = w\chi_{E_n}(x) \ge n\chi_{E_n}(x)$$

for all $x \in X$. By the monotonicity property of the distribution function we have

$$D_{w\chi_{E_n}}(\lambda) \geq D_{\chi_{E_n}}\left(\frac{\lambda}{n}\right).$$

From this

$$\{\lambda > 0 : D_{w\chi_{E_n}}(\lambda) \leq t\} \subseteq \left\{\lambda > 0 : D_{\chi_{E_n}}\left(\frac{\lambda}{n}\right) \leq t\right\}.$$

Then

$$\inf\left\{\lambda > 0 : D_{\chi_{E_n}}\left(\frac{\lambda}{n}\right) \leq t\right\} \leq \inf\{\lambda > 0 : D_{w\chi_{E_n}}(\lambda) \leq t\}.$$

Putting $\alpha = \frac{\lambda}{n}$, we have

$$\|w\chi_{E_n}\|_{\varphi,w} \geq n\|\chi_{E_n}\|_{\varphi,w},$$

since χ_E is a simple function, then by (3.3) we have

$$M_w(\chi_{E_n}) = N(\chi_{E_n}).$$

Hence

$$\|N(\chi_{E_n})\|_{\varphi,w} \geq n\|\chi_{E_n}\|_{\varphi,w}.$$

Therefore N is an unbounded operator. This is a contradiction to the fact that N is bounded.

So then $w \in L_\infty$ and by theorem 3.4 M_w is bounded.

Next, given $f \in L_{\varphi,w}$, there exists a nondecreasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of measurable simple functions such that $\lim_{n\to\infty} s_n = f$, then by (3.3) we have

$$\begin{aligned} N(f) &= N(\lim s_n) \\ &= \lim N(s_n) \\ &= \lim M_w(s_n) \\ &= M_w(\lim s_n) \\ &= M_w(f). \end{aligned}$$

Therefore $N(f) = M_w(f)$ for all $f \in L_{\varphi,w}$ and thus we conclude that $N \in \mathcal{H}$. □

3.11. Corollary. The multiplication operator is invertible on $B(L_{\varphi,w})$ if and only if is invertible on L_∞.

Proof. Let M_u be invertible. Then there exists $N \in B(L_{\varphi,w})$ such that

$$M_u \cdot N = N \cdot M_u = I \tag{3.4}$$

where I represent the identity operator. Let us check that N commute with $\mathcal{H}$.

Let $M_w \in \mathcal{H}$, then

$$M_w \cdot M_u = M_u \cdot M_w. \tag{3.5}$$

Applying N to (3.5) and by (3.4) we obtain

$$\begin{aligned} N \cdot M_w \cdot M_u \cdot N &= N \cdot M_u \cdot M_w \cdot N, \\ N \cdot M_w \cdot I &= I \cdot M_w \cdot N, \\ N \cdot M_w &= M_w \cdot N, \end{aligned}$$

and thus we conclude that N commute with $\mathcal{H}$. By theorem 3.10 $N \in \mathcal{H}$, then there exists $g \in L_\infty$ such that $N = M_g$, hence

$$M_u \cdot M_g = M_g \cdot M_u = I,$$

this implies that $ug = gu = 1$ μ-a.e., which means that u is invertible on L_∞.

On the other hand, assume u is invertible on L_∞, that is, $\frac{1}{u} \in L_\infty$, then

$$\begin{aligned} M_u \cdot M_{\frac{1}{u}} &= M_{\frac{1}{u}} \cdot M_u \\ &= M_{(\frac{1}{u})u} \\ &= M_1 = I, \end{aligned}$$

which means that M_u is invertible on $B(L_{\varphi,w})$. □

For the sake of completeness and the convenience of the reader, we gives here one definition and one lemma which will play an important role on the coming results.

3.12. Definition. Let T be an operator. A subspace V of X is said to be invariant under T (or simply T-invariant) whenever

$$T(V) \subseteq V.$$

3.13. Lemma. Let $T : X \to X$ be an operator. If T is compact and M is a closed T-invariant subspace of X, then $T|_M$ is compact.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a subsequence in $M \subseteq X$. Then $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, thus there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $T(x_{n_k})$ converges in X, but $T(x_{n_k}) \subseteq T(M)$ since $\{x_{n_k}\}_{k\in\mathbb{N}} \subseteq M$. Then $T(x_{n_k})$ converges on $\overline{T(M)} \subseteq \overline{M} = M$. Therefore $T(x_{n_k})$ converges on M, hence $T|_M$ is compact. □

3.14. Theorem. Let M_u be a compact operator. For $\varepsilon > 0$ define

$$A_\varepsilon(u) = \{x \in X : |u(x)| \geq \varepsilon\},$$

and

$$L_{\varphi,w}(A_\varepsilon(u)) = \{f\chi_{A_\varepsilon(u)} : f \in L_{\varphi,w}\}.$$

Then $L_{\varphi,w}(A_\varepsilon(u))$ is a closed invariant subspace of $L_{\varphi,w}$ under M_u. Moreover

$$M_u\big|_{L_{\varphi,w}(A_\varepsilon(u))}$$

is a compact operator.

Proof. Let $h, s \in L_{\varphi,w}(A_\varepsilon(u))$ and $\alpha, \beta \in \mathbb{R}$. Then $h = f\chi_{A_\varepsilon(u)}$ and $s = g\chi_{A_\varepsilon(u)}$ where $f, g \in L_{\varphi,w}$ thus

$$\begin{aligned} \alpha h + \beta s &= \alpha(f\chi_{A_\varepsilon(u)}) + \beta(g\chi_{A_\varepsilon(u)}) \\ &= (\alpha f + \beta g)\chi_{A_\varepsilon(u)} \in L_{\varphi,w}(A_\varepsilon(u)), \end{aligned}$$

which means that $L_{\varphi,w}(A_\varepsilon(u))$ is a subspace of $L_{\varphi,w}$.

Next, for all $h \in L_{\varphi,w}(A_\varepsilon(u))$ we have

$$\begin{aligned} M_u h &= uh \\ &= u(f\chi_{A_\varepsilon(u)}) \\ &= (uf)\chi_{A_\varepsilon(u)}, \end{aligned}$$

where $uf \in L_{\varphi,w}$. Therefore $M_u \in L_{\varphi,w}(A_\varepsilon(u))$, which means that $L_{\varphi,w}(A_\varepsilon(u))$ is an invariant subspace of $L_{\varphi,w}$ under M_u.

Now, let us show that $L_{\varphi,w}(A_\varepsilon(u))$ is a closed set. Indeed, let g a function belonging to the closure of $L_{\varphi,w}(A_\varepsilon(u))$ then there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $L_{\varphi,w}(A_\varepsilon(u))$ such that

$$g_n \to g \text{ in } L_{\varphi,w}$$

Just remain to exhibit that g belongs to $L_{\varphi,w}(A_\varepsilon(u))$. Note that

$$g = g\chi_{A_\varepsilon(u)} + g\chi_{A^c_\varepsilon(u)}.$$

Next, we want to show that $g\chi_{A^c_\varepsilon(u)} = 0$. In fact, given $\varepsilon_1 > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\|g\chi_{A^c_\varepsilon(u)}\|_{\varphi,w} &= \|(g - g_{n_0} + g_{n_0})\chi_{A^c_\varepsilon(u)}\|_{\varphi,w}\\ &= \|(g - g_{n_0})\chi_{A^c_\varepsilon(u)}\|_{\varphi,w}\\ &\le \|g - g_{n_0}\|_{\varphi,w} < \varepsilon_1.\end{aligned}$$

Thus, $g\chi_{A^c_\varepsilon(u)} = 0$ which means that $g = g\chi_{A_\varepsilon(u)}$, that is, $g \in L_{\varphi,w}(A_\varepsilon(u))$. Finally by lemma 3.13 we have

$$M_u\,\big|_{L_{\varphi,w}(A_\varepsilon(u))}$$

is a compact operator. And the proof is now complete. □

3.15. Theorem. Let $M_u \in B(L_{\varphi,w})$. Then M_u is compact if and only if $L_{\varphi,w}(A_\varepsilon(u))$ is finite dimensional for each $\varepsilon > 0$.

Proof. If $|u(x)| \ge \varepsilon$, we should note that

$$|uf\chi_{A_\varepsilon}(x)| \ge \varepsilon f\chi_{A_\varepsilon(u)}(x)$$

and so

$$\{x : \varepsilon f\chi_{A_\varepsilon(u)}(x) > \lambda\} \subseteq \{x : \big|uf\chi_{A_\varepsilon(u)}(x)\big| > \lambda\},$$

thus

$$D_{\varepsilon f\chi_{A_\varepsilon(u)}}(\lambda) \le D_{uf\chi_{A_\varepsilon(u)}}(\lambda),$$

then

$$\{\lambda > 0 : D_{uf\chi_{A_\varepsilon(u)}}(\lambda) \le t\} \subseteq \{\lambda > 0 : D_{\varepsilon f\chi_{A_\varepsilon(u)}}(\lambda) \le t\}$$

from this we have

$$\inf\{\lambda > 0 : D_{\varepsilon f\chi_{A_\varepsilon(u)}}(\lambda) \le t\} \le \inf\{\lambda > 0 : D_{uf\chi_{A_\varepsilon(u)}}(\lambda) \le t\}$$

that is

$$(\varepsilon f\chi_{A_\varepsilon(u)})^*(t) \le (uf\chi_{A_\varepsilon(u)})^*(t).$$

Hence

$$\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(uf\chi_{A_\varepsilon(u)})^*(t)}{k}\right) w(t)\,dt \le 1\right\} \subseteq \left\{k > 0 : \int_0^\infty \varphi\left(\frac{\varepsilon(f\chi_{A_\varepsilon(u)})^*(t)}{k}\right) w(t)\,dt \le 1\right\}.$$

Therefore

$$\inf\left\{k > 0 : \int_0^\infty \varphi\left(\frac{\varepsilon(f\chi_{A_\varepsilon(u)})^*(t)}{k}\right) w(t)\,dt \le 1\right\} \le \inf\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(uf\chi_{A_\varepsilon(u)})^*(t)}{k}\right) w(t)\,dt \le 1\right\}.$$

And hence

$$\|M_u f\chi_{A_\varepsilon(u)}\|_{\varphi,w} \ge \varepsilon\|f\chi_{A_\varepsilon(u)}\|_{\varphi,w}. \tag{3.6}$$

Now, if M_u is a compact operator, then $L_{\varphi,w}(A_\varepsilon(u))$ is a closed invariant subspace of $L_{\varphi,w}$ under M_u and by lemma 3.13

$$M_u\,\big|_{L_{\varphi,w}(A_\varepsilon(u))}$$

is a compact operator. Then by (3.6) $M_u\big|_{L_{\varphi,w}(A_\varepsilon(u))}$ has a closed range in $L_{\varphi,w}(A_\varepsilon(u))$ and it is invertible, being compact $L_{\varphi,w}(A_\varepsilon(u))$ is finite dimensional.

Conversely, suppose that $L_{\varphi,w}(A_\varepsilon(u))$ is finite dimensional for each $\varepsilon > 0$. In particular, for each n, $L_{\varphi,w}(A_{\frac{1}{u}}(u))$ is finite dimensional, then for each n, define $u_n : X \to \mathbb{C}$ as

$$u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| \geq \frac{1}{n} \\ 0 & \text{if } |u(x)| < \frac{1}{n}. \end{cases}$$

Then we find that

$$((u_n - u) \cdot f)^*(t) \leq \|u_n - u\|_\infty f^*(t) \ \forall\ t > 0.$$

Consequently

$$\begin{aligned} \|M_{u_n} f - M_u f\|_{\varphi,w} &\leq \|u_n - u\|_\infty \|f\|_{\varphi,w} \\ &\leq \frac{1}{n}\|f\|_{\varphi,w}, \end{aligned}$$

which implies that M_{u_n} converges to M_u uniformly. As $L_{\varphi,w}(A_\varepsilon(u))$ is finite dimensional so M_{u_n} is a finite rank operator. Therefore, M_{u_n} is a compact operator and hence M_u is a compact operator. □

References

[1] M.B. Abrahamese, Multiplication operators, Lecture notes in Math., Vol. **693** (1978), 17-36, Springer Verlag, New York, 1978.

[2] Y.A. Abramovich, C.D. Aliprantis, *An Invitation to Operator Theory*, American Mathematical Society, (2002).

[3] S.C. Arora, Gopal Datt and Satish Verma, Multiplication operators on lorentz spaces, Indian Journal of Mathematics, Vol **48** (3) (2006), 317-329.

[4] A. Axler, Multiplication operators on Bergman space, J. Peine Angew Math., Vol. **33** (6) (1982), 26-44.

[5] R.E. Castillo, R. León; E. Trousselot. Multiplication operator on $L(p,q)$ spaces, Panamer. Math Journal Vol **19** (2009) No. 1, 37-44.

[6] R.G. Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, New York, 1972.

[7] Grafakos, Loukas. *Classical Fourier Analysis Second Edition*, Springer, (2008).

[8] P.R. Halmos, A Hilbert Space Problem Book, Van Nostrand, Princeton, N. J., 1961.

[9] B.S. Komal and Shally Gupta, Multiplication operators between Orlicz spaces, Integral Equations and Operator Theory. Vol. **41** (2001), 324-330.

[10] M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces*, Marcel Dekker Inc., New York, 1993.

[11] R.K. Singh and A. Kumar, Multiplication and composition operators with closed ranges, Bull. Aust. Math. Soc. Vol. **16** (1977), 247-252.

[12] H. Takagi, Fredholm Weighted Composition Operators, Integral equations and Operator Theory **16** (1993), 267-276.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1011 – 1022

Simplicial homology groups of certain digital surfaces

Emel ÜNVER DEMİR * and İsmet KARACA†

Abstract

In this paper we compute the simplicial homology groups of some digital surfaces.

Keywords: Digital topology, digital surface, simplicial homology groups.

2000 AMS Classification: 55N35, 68R10, 68U05, 68U10.

Received 26/03/2012 : Accepted 13/06/2014 Doi : 10.15672/HJMS.2015449657

1. INTRODUCTION

Digital topology [19, 17] has been used in different image processing and computer graphics algorithms for several decades. It addresses the fundamental properties of binary object connectivity in two dimensional (2D) and three dimensional (3D) digital images. Concepts and results of Digital Topology are used to specify and justify some important low-level image processing algorithms including algorithms for thinning, boundary extraction, object counting, and contour filling. The properties of digital images with tools from Topology (including Algebraic Topology) are used by many researchers [$1-12, 16, 17, 19$].

Homology is a powerful topological invariant which characterizes an object by its p-dimensional holes. Intuitively the 0-dimensional holes can be seen as "tiny holes", 1-dimensional holes can be seen as tunnels, and 2-dimensional holes can be seen as cavities. The usage of homology groups is a new topic and is not widely spread. Simplicial homology groups of digital images have been studied by several researchers [1, 10, 16]. Boxer et al. [10] extend results of [1] about computing simplicial homology groups of digital images. In this work, we compute simplicial homology groups of certain minimal simple closed surfaces.

This paper is organized as follows. Section 2 provides some basic notions used in this paper. In section 3, we compute the simplicial homology groups of certain digital surfaces.

*Department of Mathematics, Celal Bayar University, Muradiye Campus, Manisa, Turkey Email: `emel.unver@cbu.edu.tr`

†Department of Mathematics, Ege University, Bornova, Izmir 35100, TURKEY. Email:`ismet.karaca@ege.edu.tr`

2. PRELIMINARIES

Let $\mathbb{Z}^n$ be the set of lattice points in the n-dimensional Euclidean space where $\mathbb{Z}$ is the set of integers. For a positive integer l with $1 \leq l \leq n$ and two distinct points $p = (p_1, p_2, ..., p_n)$, $q = (q_1, q_2, ..., q_n) \in \mathbb{Z}^n$, p and q are c_l-*adjacent* [8] if

(1) there are at most l indices i such that $|p_i - q_i| = 1$; and

(2) for all other indices i such that $|p_i - q_i| \neq 1$, $p_i = q_i$.

Another commonly used notation for c_l-adjacency reflects the number of neighbors $q \in \mathbb{Z}^n$ that a given point $p \in \mathbb{Z}^n$ may have under the adjacency. For example, if $n = 1$ we have $c_1 = 2$-adjacency; if $n = 2$ we have $c_1 = 4$-adjacency and $c_2 = 8$-adjacency; if $n = 3$ we have $c_1 = 6$-adjacency, $c_2 = 18$-adjacency, and $c_3 = 26$-adjacency [8]. Given a natural number l in conditions (1) and (2) with $1 \leq l \leq n$, l determines each of the κ-adjacency relations of $\mathbb{Z}^n$ in terms of (1) and (2) [14] as follows.

$$\kappa \in \Big\{2n\ (n \geq 1),\ 3^n - 1\ (n \geq 2),\ 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1\ (2 \leq r \leq n-1, n \geq 3)\Big\} \tag{2.1}$$

The pair (X, κ) is considered in a digital picture $(\mathbb{Z}^n, \kappa, \overline{\kappa}, X)$ for $n \geq 1$ in [3, 4, 6, 13], which is called a *digital image* where $(\kappa, \overline{\kappa}) \in \{(\kappa, 2n), (2n, 3^n - 1)\}$. Each of κ and $\overline{\kappa}$ is one of the general κ-adjacency relations. We usually do not permit that κ and $\overline{\kappa}$ both equal $2n$ when $n > 1$, because of the digital connectivity paradox [18]. For instance, $(\kappa, \overline{\kappa}) \in \{(4, 8), (8, 4)\}$ and $\{(6, 18), (6, 26), (26, 6), (18, 6)\}$ are usually considered in $\mathbb{Z}^2$ and $\mathbb{Z}^3$, respectively [6, 13, 19, 20].

A *digital interval* is a set of the form $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ where $a, b \in \mathbb{Z}$ with $a < b$.

Let κ be an adjacency relation on $\mathbb{Z}^n$. A κ-neighbor of a lattice point p is κ-adjacent to p. A digital image $X \subset \mathbb{Z}^n$ is κ-*connected* [15] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, ..., x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0, 1, ..., r-1$. A κ-*component* of a digital image X is a maximal κ-connected subset of X.

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1-adjacency respectively. Then the function $f : X \to Y$ is called (κ_0, κ_1)-*continuous* [6, 20] if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y. We say that such a function is digitally continuous. Similar notions are defined on discrete manifolds in [11]: Let D_1 and D_2 be two discrete manifolds and $f : D_1 \to D_2$ be a mapping. f is said to be an *immersion* from D_1 to D_2 or a *gradually varied operator* if x and y are adjacent in D_1 implies either $f(x) = f(y)$ or $f(x)$, $f(y)$ are adjacent in D_2.

Let X be a digital image with κ-adjacency. If $f : [0, m]_{\mathbb{Z}} \to X$ is a $(2, \kappa)$-continuous function such that $f(0) = x$ and $f(m) = y$, then f is called a *digital path* from x to y in X. If $f(0) = f(m)$ then the κ-path is said to be *closed*, and the function is called a κ-*loop*. Let $f : [0, m-1]_{\mathbb{Z}} \to X$ be a $(2, \kappa)$-continuous function such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = i \pm 1\ mod\ m$. Then the set $f([0, m-1]_{\mathbb{Z}})$ is called a *simple closed* κ-*curve*. A point $x \in X$ is called a κ-*corner*, if x is κ-adjacent to two and only two points $y, z \in X$ such that y and z are κ-adjacent to each other [4]. Moreover, the κ-corner x is called *simple* if y, z are not κ-corners and if x is the only point κ-adjacent to both y, z [3]. X is called a *generalized simple closed* κ-*curve* if what is obtained by removing all simple κ-corners of X is a simple closed κ-curve [4]. If (X, κ) is a κ-connected digital image in $\mathbb{Z}^3$, $|X|^x = N_3^*(x) \cap X$, where $N_3^*(x) = \{x' \in \mathbb{Z}^3 : x \text{ and } x' \text{ are 26-adjacent}\}$ [3, 4]. Generally, if (X, κ) is a κ-connected digital image in $\mathbb{Z}^n$, $|X|^x = N_n^*(x) \cap X$, where $N_n^*(x) = \{x' \in \mathbb{Z}^n : x \text{ and } x' \text{ are } c_n\text{-adjacent}\}$ [13].

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1-adjacency respectively. A function $f : X \to Y$ is a (κ_0, κ_1)-*isomorphism* [9] (called (κ_0, κ_1)-homeomorphism in

[5]) if f is (κ_0, κ_1)-continuous, bijective and $f^{-1} : Y \to X$ is (κ_1, κ_0)-continuous, in which case we write $X \approx_{(\kappa_0,\kappa_1)} Y$.

2.1. Definition. [13] Let $c^* := \{x_0, x_1, ..., x_n\}$ be a closed κ-curve in $\mathbb{Z}^2$ where $\{\kappa, \overline{\kappa}\} = \{4, 8\}$. A point x of the complement $\overline{c^*}$ of a closed κ-curve c^* in $\mathbb{Z}^2$ is said to be in the *interior* of c^* if it belongs to the bounded $\overline{\kappa}$-connected component of $\overline{c^*}$. The set of all interior points of c^* is denoted by $Int(c^*)$.

2.2. Definition. [13] Let (X, κ) be a digital image in $\mathbb{Z}^n$, $n \geq 3$ and $\overline{X} = \mathbb{Z}^n - X$. Then X is called a *closed κ-surface* if it satisfies the following.

(1) In case that $(\kappa, \overline{\kappa}) \in \{(\kappa, 2n), (2n, 3^n - 1)\}$, where the κ-adjacency is taken from (2.1) with $\kappa \neq 3^n - 2^n - 1$ and $\overline{\kappa}$ is the adjacency on $\overline{X}$, then

(a) for each point $x \in X$, $|X|^x$ has exactly one κ-component κ-adjacent to x;

(b) $|\overline{X}|^x$ has exactly two $\overline{\kappa}$-components $\overline{\kappa}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components; and

(c) for any point $y \in N_\kappa(x) \cap X$, $N_{\overline{\kappa}}(y) \cap C^{xx} \neq \emptyset$ and $N_{\overline{\kappa}}(y) \cap D^{xx} \neq \emptyset$, where $N_\kappa(x)$ means the κ-neighbors of x.

Further, if a closed κ-surface X does not have a simple κ-point, then X is called simple.

(2) In case that $(\kappa, \overline{\kappa}) = (3^n - 2^n - 1, 2n)$, then

(a) X is κ-connected,

(b) for each point $x \in X$, $|X|^x$ is a generalized simple closed κ-curve.

Further, if the image $|X|^x$ is a simple closed κ-curve, then the closed κ-surface X is called simple.

For a closed κ-surface S_κ, we denote by $\overline{S_\kappa}$ the complement of S_κ in $\mathbb{Z}^n$. Then a point x of $\overline{S_\kappa}$ is said to be *interior* of S_κ if it belongs to the bounded $\overline{\kappa}$-connected component of S_κ. The set of all interior points of S_κ is denoted by $int(S_\kappa)$.

The 3-dimensional digital images MSS^*_{18} and MSS^*_6 which are obtained from the minimal simple closed curves MSC_8 and MSC_4 in $\mathbb{Z}^2$, respectively, are essentially used in establishing the notion of a connected sum [13].

Figure 1. Minimal simple closed curves MSC_4 and MSC_8.

- $MSS^*_6 := MSS_6 \cup Int(MSS_6)$ where

$$MSS_6 \approx_{(6,6)} (MSC_4 \times [0, 2]_{\mathbb{Z}}) \cup (Int(MSC_4) \times \{0, 2\})$$

and MSC_4 is 4-isomorphic to the set

$$\{(1, 0), (1, 1), (0, 1), (-1, 1), (-1, 0), (-1, -1), (0, -1), (1, -1)\}.$$

- $MSS^*_{18} := MSS_{18} \cup Int(MSS_{18})$ where

$$MSS_{18} \approx_{(18,18)} (MSC_8 \times \{1\}) \cup (Int(MSC_8) \times \{0, 2\})$$

and MSC_8 is 8-isomorphic to the set

$$\{(0, 0), (-1, 1), (-2, 0), (-2, -1), (-1, -2), (0, -1)\}.$$

2.3. Definition. [13] Let S_{κ_0} be a closed κ_0-surface in $\mathbb{Z}^{n_0}$ and S_{κ_1} be a closed κ_1-surface in $\mathbb{Z}^{n_1}$ for $n_0, n_1 \geq 3$. Consider $A'_{\kappa_0} \subset A_{\kappa_0} \subset S_{\kappa_0}$ such that

$$A'_{\kappa_0} \approx_{(\kappa_0,8)} Int(MSC^*_8),\ A'_{\kappa_0} \approx_{(\kappa_0,4)} Int(MSC^*_4) \text{ or } A'_{\kappa_0} \approx_{(\kappa_0,8)} Int(MSC'^*_8).$$

Let $f : A_{\kappa_0} \to f(A_{\kappa_0}) \subset S_{\kappa_1}$ be a (κ_0, κ_1)-isomorphism. Let $S'_{\kappa_i} = S_{\kappa_i} \setminus A'_{\kappa_i}$, $i \in \{0, 1\}$. Then the connected sum, denoted by $S_{\kappa_0} \sharp S_{\kappa_1}$, is the quotient space $S'_{\kappa_0} \cup S'_{\kappa_1} / \sim$, where $i : A_{\kappa_0} \setminus A'_{\kappa_0} \to S'_{\kappa_0}$ is the inclusion map and $i(x) \sim f(x)$ for $x \in A_{\kappa_0} \setminus A'_{\kappa_0}$.

2.4. Definition. [21] Let S be a set of nonempty subsets of a digital image (X, κ). The members of S are called simplexes of (X, κ) if the following holds:

(i) If p and q are distinct points of $s \in S$, then p and q are κ-adjacent.

(ii) If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$ (note this implies every point p that belongs to a simplex determines a simplex $\{p\}$).

An m-simplex is a simplex S such that $|S| = m + 1$.

Let P be a digital m-simplex. If P' is a nonempty proper subset of P, then P' is called a face of P.

Since computing homology groups is easier than computing higher degree homotopy groups in algebraic topology, for the same reason computing homology groups of digital images is preferred to computing homotopy groups of digital images. The simplicial homology groups of n-dimensional digital images from algebraic topology have been introduced in [1].

2.5. Definition. [1] Let (X, κ) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some nonnegative integer d. If the following statements hold, then (X, κ) is called a finite digital simplicial complex:

(1) If P belongs to X, then every face of P also belongs to X.

(2) If $P, Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q.

The dimension of a digital simplicial complex X is the biggest integer m such that X has an m-simplex.

$C_q^{\kappa}(X)$ is a free abelian group with basis all digital (κ, q)-simplices in X [1].

2.6. Corollary. [10] Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then for all $q > m$, $C_q^{\kappa}(X)$ is a trivial group.

Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. The homomorphism $\partial_q : C_q^{\kappa}(X) \to C_{q-1}^{\kappa}(X)$ defined by

$$\partial_q(< p_0, p_1, ..., p_q >) = \begin{cases} \sum_{i=0}^{q} (-1)^i < p_0, p_1, ..., \widehat{p_i}, ..., p_q >, & q \leq m; \\ 0, & \text{q>m} \end{cases}$$

is called a boundary homomorphism where $\widehat{p_i}$ means deleting the point p_i. Then for all $1 \leq q \leq m$, we have $\partial_{q-1} \circ \partial_q = 0$ [1].

2.7. Theorem. [1] Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then

$$C_*^{\kappa}(X) : 0 \xrightarrow{\partial_{m+1}} C_m^{\kappa}(X) \xrightarrow{\partial_m} C_{m-1}^{\kappa}(X) \xrightarrow{\partial_{m-1}} ... \xrightarrow{\partial_1} C_0^{\kappa}(X) \xrightarrow{\partial_0} 0$$

is a chain complex.

Let (X, κ) be a digital simplicial complex. The group of digital simplicial q-cycles is $Z_q^{\kappa}(X) = Ker\ \partial_q = \{\sigma \in C_q^{\kappa}(X) | \partial_q(\sigma) = 0\}$ and the group of digital simplicial q-boundaries is $B_q^{\kappa}(X) = Im\ \partial_{q+1} = \{\tau \in C_q^{\kappa}(X) | \partial_{q+1}(\sigma) = \tau \text{ for } \sigma \in C_{q+1}^{\kappa}(X)\}$. The qth digital simplicial homology group is $H_q^{\kappa}(X) = Z_q^{\kappa}(X)/B_q^{\kappa}(X)$ [1].

2.8. Theorem. [1] If $f : X \to Y$ is a digital (κ_0, κ_1)-isomorphism, then for all q

$$H_q^{\kappa_0}(X) \cong H_q^{\kappa_1}(Y).$$

2.9. Theorem. [10] Let (X, κ) be a directed digital simplicial complex of dimension m.

(1) $H_q^{\kappa}(X)$ is a finitely generated abelian group for every $q \geq 0$.

(2) $H_q^{\kappa}(X)$ is a trivial group for all $q > m$.

(3) $H_q^{\kappa}(X)$ is a free abelian group, possibly zero.

2.10. Definition. [10] Let (X, κ) be a digital image of dimension m, and for each $q \geq 0$, let α_q be the number of digital (κ, q)-simplexes in X. The Euler characteristic of X, denoted by $\chi(X, \kappa)$, is defined by

$$\chi(X, \kappa) = \sum_{q=0}^{m} (-1)^q \alpha_q.$$

2.11. Theorem. [10] If (X, κ) is a digital image of dimension m, then

$$\chi(X, \kappa) = \sum_{q=0}^{m} (-1)^q rank\ H_q^{\kappa}(X).$$

2.12. Example. [10] By the definition of Euler characteristic, we have

$$\chi(MSS_6, 6) = \alpha_0 - \alpha_1 = 26 - 48 = -22$$
$$\chi(MSS_6 \sharp MSS_6, 6) = \alpha_0 - \alpha_1 = 42 - 80 = -38$$
$$\chi(MSS_{18}, 18) = \alpha_0 - \alpha_1 + \alpha_2 = 10 - 20 + 8 = -2$$
$$\chi(MSS_{18} \sharp MSS_{18}, 18) = \alpha_0 - \alpha_1 + \alpha_2 = 14 - 28 + 8 = -6$$

3. MAIN RESULTS

Simplicial homology groups of several digital surfaces have been computed in [10]. By using an argument similar to that of [10], we have the following theorems.

3.1. Theorem. The digital simplicial homology groups of $MSS_{18} \sharp MSS_{18}$ are

$$H_q^{18}(MSS_{18} \sharp MSS_{18}) = \begin{cases} \mathbb{Z}, & q = 0; \\ \mathbb{Z}^7, & q = 1; \\ 0, & q \geq 2. \end{cases}$$

Figure 2. $MSS_{18} \sharp MSS_{18}$

Proof. Let

$$\begin{aligned} MSS_{18} \sharp MSS_{18} = \{ & c_0 = (1,0,1), c_1 = (1,1,1), c_2 = (1,2,1), \\ & c_3 = (0,3,1), c_4 = (-1,2,1), c_5 = (-1,1,1), \\ & c_6 = (-1,0,1), c_7 = (0,-1,1), c_8 = (0,2,2), \\ & c_9 = (0,1,2), c_{10} = (0,0,2), c_{11} = (0,2,0), \\ & c_{12} = (0,1,0), c_{13} = (0,0,0) \}. \end{aligned}$$

Then we can direct $MSS_{18} \sharp MSS_{18}$ by the ordering $c_6 < c_5 < c_4 < c_7 < c_{13} < c_{10} < c_{12} < c_9 < c_{11} < c_8 < c_3 < c_0 < c_1 < c_2$. We have the following simplicial chain complexes:
$C_0^{18}(MSS_{18} \sharp MSS_{18})$ has for a basis $\{\langle c_0 \rangle, \langle c_1 \rangle, ..., \langle c_{13} \rangle\}$,
$C_1^{18}(MSS_{18} \sharp MSS_{18})$ has for a basis

$$\begin{aligned} \{ & \langle c_7 c_0 \rangle, \langle c_{10} c_0 \rangle, \langle c_{13} c_0 \rangle, \langle c_0 c_1 \rangle, \langle c_9 c_1 \rangle, \langle c_{12} c_1 \rangle, \langle c_1 c_2 \rangle, \langle c_8 c_2 \rangle, \langle c_{11} c_2 \rangle, \langle c_3 c_2 \rangle, \langle c_4 c_3 \rangle, \\ & \langle c_8 c_3 \rangle, \langle c_{11} c_3 \rangle, \langle c_5 c_4 \rangle, \langle c_4 c_8 \rangle, \langle c_4 c_{11} \rangle, \langle c_6 c_5 \rangle, \langle c_5 c_9 \rangle, \langle c_5 c_{12} \rangle, \langle c_6 c_7 \rangle, \langle c_6 c_{10} \rangle, \langle c_6 c_{13} \rangle, \\ & \langle c_7 c_{10} \rangle, \langle c_7 c_{13} \rangle, \langle c_9 c_8 \rangle, \langle c_{10} c_9 \rangle, \langle c_{12} c_{11} \rangle, \langle c_{13} c_{12} \rangle \}, \end{aligned}$$

and $C_2^{18}(MSS_{18} \sharp MSS_{18})$ has for a basis

$$\{\langle c_7 c_{13} c_0 \rangle, \langle c_7 c_{10} c_0 \rangle, \langle c_8 c_3 c_2 \rangle, \langle c_{11} c_3 c_2 \rangle, \langle c_4 c_8 c_3 \rangle, \langle c_4 c_{11} c_3 \rangle, \langle c_6 c_7 c_{10} \rangle, \langle c_6 c_7 c_{13} \rangle\}.$$

Thus, we obtain the following short sequence:

$$0 \xrightarrow{\partial_3} C_2^{18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\partial_2} C_1^{18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\partial_1} C_0^{18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\partial_0} 0.$$

By Theorem 2.9, $H_q^{18}(MSS_{18}\sharp MSS_{18})$ is a trivial group for all $q > 2$.

We determine the kernel of ∂_2. If

$$\begin{aligned}
&\partial_2(a_1\langle c_7c_{13}c_0\rangle + a_2\langle c_7c_{10}c_0\rangle + a_3\langle c_8c_3c_2\rangle + a_4\langle c_{11}c_3c_2\rangle + a_5\langle c_4c_8c_3\rangle + a_6\langle c_4c_{11}c_3\rangle\\
&\quad + a_7\langle c_6c_7c_{10}\rangle + a_8\langle c_6c_7c_{13}\rangle) = a_1\langle c_{13}c_0\rangle + (-a_1 - a_2)\langle c_7c_0\rangle + (a_1 + a_8)\langle c_7c_{13}\rangle\\
&\quad + a_2\langle c_{10}c_0\rangle + (a_2 + a_7)\langle c_7c_{10}\rangle + (a_3 + a_4)\langle c_3c_2\rangle - a_3\langle c_8c_2\rangle + (a_3 + a_5)\langle c_8c_3\rangle\\
&\quad - a_4\langle c_{11}c_2\rangle + (a_4 + a_6)\langle c_{11}c_3\rangle + (-a_5 - a_6)\langle c_4c_3\rangle + a_5\langle c_4c_8\rangle\\
&\quad + a_6\langle c_4c_{11}\rangle - a_7\langle c_6c_{10}\rangle + (a_7 + a_8)\langle c_6c_7\rangle - a_8\langle c_6c_{13}\rangle = 0,
\end{aligned}$$

then one easily sees that $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = 0$. Therefore, $Z_2^{18}(MSS_{18}\sharp MSS_{18}) = \{0\}$ and hence $H_2^{18}(MSS_{18}\sharp MSS_{18}) = \{0\}$.

Since $Ker\ \partial_2 = Z_2^{18}(MSS_{18}\sharp MSS_{18}) = \{0\}$, $Im\ \partial_2 \cong C_2^{8}(MSS_{18}\sharp MSS_{18})$, and so $B_1^{18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}^8$.

We can use standard methods to determine that $Z_1^{18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}^{15}$, from which it follows easily that $B_0^{18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}^{13}$. However, the direct calculation of $Z_1^{18}(MSS_{18}\sharp MSS_{18})$ is very long. Since our goal is to calculate $H_1^{18}(MSS_{18}\sharp MSS_{18})$, we will do so below without showing a direct calculation of $Z_1^{18}(MSS_{18}\sharp MSS_{18})$.

By using the short sequence again, we have

$$Z_0^{18}(MSS_{18}\sharp MSS_{18}) = \Big\{ \sum_{i=0}^{13} a_i\langle c_i\rangle \mid a_i \in \mathbb{Z}, i = 0, 1, ..., 13\Big\} \cong \mathbb{Z}^{14}$$

Any 0-cycle $w_0 = \sum_{i=0}^{13} a_i\langle c_i\rangle$ can be written as

$$\begin{aligned}
w_0 = \partial_1((&-a_7)\langle c_7c_0\rangle + (a_1 + a_2 + a_3)\langle c_0c_1\rangle + (a_2 + a_3)\langle c_1c_2\rangle\\
&+ (-a_3)\langle c_3c_2\rangle + a_{11}\langle c_4c_{11}\rangle + (a_4 + a_{11})\langle c_5c_4\rangle + a_{12}\langle c_5c_{12}\rangle\\
&+ (a_4 + a_5 + a_{11} + a_{12})\langle c_6c_5\rangle + a_{13}\langle c_6c_{13}\rangle\\
&+ (-a_4 - a_5 - a_6 - a_{11} - a_{12} - a_{13})\langle c_6c_{10}\rangle + a_8\langle c_9c_8\rangle\\
+ (a_8 + a_9)&\langle c_{10}c_9\rangle + (a_0 + a_1 + a_2 + a_3 + a_7)\langle c_{10}c_0\rangle) + \sum_{i=0}^{13} a_i\langle c_{10}\rangle.
\end{aligned}$$

So w_0 is homologous to 0-chain $\sum_{i=0}^{13} a_i\langle c_{10}\rangle$. Hence the 0-chain is homologous to an integral multiple of $\langle c_{10}\rangle$. Thus we deduce $H_0^{18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}$.

To compute the $H_1^{18}(MSS_{18}\sharp MSS_{18})$, we can use the results in [10]. By Example 2.12, we know that $\chi(MSS_{18}\sharp MSS_{18}, 18) = -6$. From Theorem 2.11,

$$\begin{aligned}
\chi(MSS_{18}\sharp MSS_{18}, 18) &= \sum_{q=0}^{2}(-1)^q rank\ H_q^{18}(MSS_{18}\sharp MSS_{18})\\
-6 &= 1 - rank\ H_1^{18}(MSS_{18}\sharp MSS_{18}) + 0
\end{aligned}$$

Thus we get $rank\ H_1^{18}(MSS_{18}\sharp MSS_{18}) = 7$ which in turn gives us

$$H_1^{18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}^7.$$

□

3.2. Theorem. The digital simplicial homology groups of MSS_6 are

$$H_q^6(MSS_6) = \begin{cases} \mathbb{Z}, & q = 0; \\ \mathbb{Z}^{23}, & q = 1; \\ 0, & q \neq 0, 1. \end{cases}$$

Figure 3. MSS_6

Proof. If we take

$$\begin{aligned} MSS_6 = \{ & c_0 = (-1,-1,0), c_1 = (0,-1,0), c_2 = (1,-1,0), c_3 = (1,0,0), \\ & c_4 = (0,0,0), c_5 = (-1,0,0), c_6 = (-1,1,0), c_7 = (0,1,0), \\ & c_8 = (1,1,0), c_9 = (1,1,1), c_{10} = (0,1,1), c_{11} = (-1,1,1), \\ & c_{12} = (-1,0,1), c_{13} = (1,0,1), c_{14} = (1,-1,1), c_{15} = (0,-1,1), \\ & c_{16} = (-1,-1,1), c_{17} = (-1,-1,2), c_{18} = (0,-1,2), c_{19} = (1,-1,2), \\ & c_{20} = (1,0,2), c_{21} = (0,0,2), c_{22} = (-1,0,2), c_{23} = (-1,1,2), \\ & c_{24} = (0,1,2), c_{25} = (1,1,2)\}, \end{aligned}$$

then we can direct MSS_6 by the ordering $c_0 < c_{16} < c_{17} < c_5 < c_{12} < c_{22} < c_6 < c_{11} < c_{23} < c_1 < c_{15} < c_{18} < c_4 < c_{21} < c_7 < c_{10} < c_{24} < c_2 < c_{14} < c_{19} < c_3 < c_{13} < c_{20} < c_8 < c_9 < c_{25}$.

We have the following simplicial chain complexes:
$C_0^6(MSS_6)$ has for a basis $\{\langle c_0\rangle, \langle c_1\rangle, ..., \langle c_{25}\rangle\}$, and $C_1^6(MSS_6)$ has for a basis

$$\begin{aligned} \{ & \langle c_0c_1\rangle, \langle c_0c_5\rangle, \langle c_0c_{16}\rangle, \langle c_1c_2\rangle, \langle c_1c_4\rangle, \langle c_1c_{15}\rangle, \langle c_2c_{14}\rangle, \langle c_2c_3\rangle, \langle c_4c_3\rangle, \langle c_3c_8\rangle, \langle c_3c_{13}\rangle, \\ & \langle c_5c_4\rangle, \langle c_4c_7\rangle, \langle c_5c_6\rangle, \langle c_5c_{12}\rangle, \langle c_6c_7\rangle, \langle c_6c_{11}\rangle, \langle c_7c_8\rangle, \langle c_7c_{10}\rangle, \langle c_8c_9\rangle, \langle c_{10}c_9\rangle, \langle c_{13}c_9\rangle, \\ & \langle c_9c_{25}\rangle, \langle c_{11}c_{10}\rangle, \langle c_{10}c_{24}\rangle, \langle c_{12}c_{11}\rangle, \langle c_{11}c_{23}\rangle, \langle c_{16}c_{12}\rangle, \langle c_{12}c_{22}\rangle, \langle c_{14}c_{13}\rangle, \langle c_{13}c_{20}\rangle, \\ & \langle c_{15}c_{14}\rangle, \langle c_{14}c_{19}\rangle, \langle c_{16}c_{15}\rangle, \langle c_{15}c_{18}\rangle, \langle c_{16}c_{17}\rangle, \langle c_{17}c_{18}\rangle, \langle c_{17}c_{22}\rangle, \langle c_{18}c_{19}\rangle, \langle c_{18}c_{21}\rangle, \\ & \langle c_{19}c_{20}\rangle, \langle c_{21}c_{20}\rangle, \langle c_{20}c_{25}\rangle, \langle c_{22}c_{21}\rangle, \langle c_{21}c_{24}\rangle, \langle c_{22}c_{23}\rangle, \langle c_{23}c_{24}\rangle, \langle c_{24}c_{25}\rangle\}. \end{aligned}$$

Thus we get the following short sequence:

$$0 \xrightarrow{\partial_2} C_1^6(MSS_6) \xrightarrow{\partial_1} C_0^6(MSS_6) \xrightarrow{\partial_0} 0.$$

By Theorem 2.9, we have $H_q^6(MSS_6) = \{0\}$ for every $q > 1$.

Direct calculation yields that $Z_1^6(MSS_6) \cong \mathbb{Z}^{23}$, from which it follows easily that $B_0^6(MSS_6) \cong \mathbb{Z}^{25}$. However, direct calculation of $Z_1^6(MSS_6)$ is very long. Since our goal is to calculate $H_1^6(MSS_6)$, we do so below without showing a direct calculation of $Z_1^6(MSS_6)$.

By using the short sequence, we have

$$Z_0^6(MSS_6) = \Big\{ \sum_{i=0}^{25} a_i\langle c_i\rangle \mid a_i \in \mathbb{Z},\ i = 0, 1, ..., 25 \Big\} \cong \mathbb{Z}^{26}.$$

Any 0-cycle $w_0 = \displaystyle\sum_{i=0}^{25} a_i\langle c_i\rangle$ can be written as

$$\begin{aligned}
w_0 = \partial_1((&-a_6)\langle c_6c_{11}\rangle + (-a_6 - a_{11})\langle c_{11}c_{23}\rangle + (a_6 + a_{11} + a_{23})\langle c_{22}c_{23}\rangle \\
&+ (a_6 + a_{11} + a_{22} + a_{23})\langle c_{12}c_{22}\rangle + (a_6 + a_{11} + a_{12} + a_{22} + a_{23})\langle c_5c_{12}\rangle \\
&+ (a_5 + a_6 + a_{11} + a_{12} + a_{22} + a_{23})\langle c_0c_5\rangle \\
&+ (-a_0 - a_5 - a_6 - a_{11} - a_{12} - a_{22} - a_{23})\langle c_0c_{16}\rangle \\
&+ (-a_0 - a_5 - a_6 - a_{11} - a_{12} - a_{16} - a_{22} - a_{23})\langle c_{16}c_{17}\rangle \\
&+ (-a_0 - a_5 - a_6 - a_{11} - a_{12} - a_{16} - a_{17} - a_{22} - a_{23})\langle c_{17}c_{18}\rangle \\
&+ a_{15}\langle c_1c_{15}\rangle + (-a_1 - a_{15})\langle c_1c_4\rangle \\
&+ (-a_1 - a_4 - a_{15})\langle c_4c_7\rangle + (-a_1 - a_4 - a_7 - a_{15})\langle c_7c_{10}\rangle \\
&+ (-a_1 - a_4 - a_7 - a_{10} - a_{15})\langle c_{10}c_{24}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{15} + a_{24})\langle c_{21}c_{24}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{15} + a_{21} + a_{24})\langle c_{18}c_{21}\rangle + (-a_8)\langle c_8c_9\rangle \\
&+ (-a_8 - a_9)\langle c_9c_{25}\rangle + (a_8 + a_9 + a_{25})\langle c_{20}c_{25}\rangle \\
&+ (a_8 + a_9 + a_{20} + a_{25})\langle c_{13}c_{20}\rangle + (a_8 + a_9 + a_{13} + a_{20} + a_{25})\langle c_3c_{13}\rangle \\
&+ (a_3 + a_8 + a_9 + a_{13} + a_{20} + a_{25})\langle c_2c_3\rangle \\
&+ (-a_2 - a_3 - a_8 - a_9 - a_{13} - a_{20} - a_{25})\langle c_2c_{14}\rangle \\
&+ (-a_2 - a_3 - a_8 - a_9 - a_{13} - a_{14} - a_{20} - a_{25})\langle c_{14}c_{19}\rangle \\
&+ (a_2 + a_3 + a_8 + a_9 + a_{13} + a_{14} + a_{19} + a_{20} + a_{25})\langle c_{18}c_{19}\rangle) + \sum_{i=0}^{25} a_i\langle c_{18}\rangle.
\end{aligned}$$

So w_0 is homologous to 0-chain $\sum_{i=0}^{25} a_i\langle c_{18}\rangle$. Hence the 0-chain is homologous to an integral multiple of $\langle c_{18}\rangle$. Thus we get

$$H_0^6(MSS_6) \cong \mathbb{Z}.$$

We use the results in [10] to compute the $H_1^6(MSS_6)$. From Example 2.12, we have $\chi(MSS_6, 6) = -22$. From Theorem 2.11,

$$\begin{aligned}
\chi(MSS_6, 6) &= \sum_{q=0}^{1}(-1)^q rank\ H_q^6(MSS_6) \\
-22 &= 1 - rank\ H_1^6(MSS_6)
\end{aligned}$$

Thus we get $rank\ H_1^6(MSS_6) = 23$ which gives us

$$H_1^6(MSS_6) \cong \mathbb{Z}^{23}.$$

□

3.3. Theorem. The digital simplicial homology groups of $MSS_6\sharp MSS_6$ are

$$H_q^6(MSS_6\sharp MSS_6) = \begin{cases} \mathbb{Z}, & q = 0; \\ \mathbb{Z}^{39}, & q = 1; \\ 0, & q \neq 0, 1. \end{cases}$$

Figure 4. $MSS_6 \sharp MSS_6$

Proof. Let

$$\begin{aligned} MSS_6\sharp MSS_6 = \{ & c_0=(0,0,0), c_1=(1,0,0), c_2=(2,0,0), c_3=(2,1,0),\\ & c_4=(1,1,0), c_5=(0,1,0), c_6=(0,2,0), c_7=(1,2,0),\\ & c_8=(2,2,0), c_9=(2,3,0), c_{10}=(1,3,0), c_{11}=(0,3,0),\\ & c_{12}=(0,4,0), c_{13}=(1,4,0), c_{14}=(2,4,0), c_{15}=(2,4,1),\\ & c_{16}=(1,4,1), c_{17}=(0,4,1), c_{18}=(0,3,1), c_{19}=(2,3,1),\\ & c_{20}=(2,2,1), c_{21}=(0,2,1), c_{22}=(0,1,1), c_{23}=(2,1,1),\\ & c_{24}=(2,0,1), c_{25}=(1,0,1), c_{26}=(0,0,1), c_{27}=(0,0,2),\\ & c_{28}=(1,0,2), c_{29}=(2,0,2), c_{30}=(2,1,2), c_{31}=(1,1,2),\\ & c_{32}=(0,1,2), c_{33}=(0,2,2), c_{34}=(1,2,2), c_{35}=(2,2,2),\\ & c_{36}=(2,3,2), c_{37}=(1,3,2), c_{38}=(0,3,2), c_{39}=(0,4,2),\\ & c_{40}=(1,4,2), c_{41}=(2,4,2)\}. \end{aligned}$$

We can direct $MSS_6\sharp MSS_6$ by the ordering $c_0 < c_{26} < c_{27} < c_5 < c_{22} < c_{32} < c_6 < c_{21} < c_{33} < c_{11} < c_{18} < c_{38} < c_{12} < c_{17} < c_{39} < c_1 < c_{25} < c_{28} < c_4 < c_{31} < c_7 < c_{34} < c_{10} < c_{37} < c_{13} < c_{16} < c_{40} < c_2 < c_{24} < c_{29} < c_3 < c_{23} < c_{30} < c_8 < c_{20} < c_{35} < c_9 < c_{19} < c_{36} < c_{14} < c_{15} < c_{41}$.

We have the following simplicial chain complexes:
$C_0^6(MSS_6\sharp MSS_6)$ has for a basis $\{\langle c_0\rangle, \langle c_1\rangle, ..., \langle c_{41}\rangle\}$, and
$C_1^6(MSS_6\sharp MSS_6)$ has for a basis

$$\begin{aligned} \{ & \langle c_0c_1\rangle, \langle c_0c_5\rangle, \langle c_0c_{26}\rangle, \langle c_1c_4\rangle, \langle c_1c_2\rangle, \langle c_1c_{25}\rangle, \langle c_2c_3\rangle, \langle c_2c_{24}\rangle, \langle c_4c_3\rangle, \langle c_3c_8\rangle, \langle c_3c_{23}\rangle,\\ & \langle c_4c_7\rangle, \langle c_5c_4\rangle, \langle c_5c_6\rangle, \langle c_5c_{22}\rangle, \langle c_6c_{11}\rangle, \langle c_6c_{21}\rangle, \langle c_6c_7\rangle, \langle c_7c_{10}\rangle, \langle c_7c_8\rangle, \langle c_8c_9\rangle, \langle c_8c_{20}\rangle,\\ & \langle c_9c_{14}\rangle, \langle c_{10}c_9\rangle, \langle c_9c_{19}\rangle, \langle c_{10}c_{13}\rangle, \langle c_{11}c_{10}\rangle, \langle c_{11}c_{12}\rangle, \langle c_{11}c_{18}\rangle, \langle c_{12}c_{13}\rangle, \langle c_{12}c_{17}\rangle,\\ & \langle c_{13}c_{16}\rangle, \langle c_{13}c_{14}\rangle, \langle c_{14}c_{15}\rangle, \langle c_{16}c_{15}\rangle, \langle c_{19}c_{15}\rangle, \langle c_{15}c_{41}\rangle, \langle c_{17}c_{16}\rangle, \langle c_{16}c_{40}\rangle, \langle c_{18}c_{17}\rangle,\\ & \langle c_{17}c_{39}\rangle, \langle c_{21}c_{18}\rangle, \langle c_{18}c_{38}\rangle, \langle c_{20}c_{19}\rangle, \langle c_{19}c_{36}\rangle, \langle c_{23}c_{20}\rangle, \langle c_{20}c_{35}\rangle, \langle c_{22}c_{21}\rangle, \langle c_{21}c_{33}\rangle,\\ & \langle c_{26}c_{22}\rangle, \langle c_{22}c_{32}\rangle, \langle c_{24}c_{23}\rangle, \langle c_{23}c_{30}\rangle, \langle c_{25}c_{24}\rangle, \langle c_{24}c_{29}\rangle, \langle c_{26}c_{25}\rangle, \langle c_{25}c_{28}\rangle, \langle c_{26}c_{27}\rangle,\\ & \langle c_{27}c_{28}\rangle, \langle c_{27}c_{32}\rangle, \langle c_{28}c_{29}\rangle, \langle c_{28}c_{31}\rangle, \langle c_{29}c_{30}\rangle, \langle c_{30}c_{35}\rangle, \langle c_{31}c_{30}\rangle, \langle c_{32}c_{31}\rangle, \langle c_{31}c_{34}\rangle,\\ & \langle c_{32}c_{33}\rangle, \langle c_{33}c_{34}\rangle, \langle c_{33}c_{38}\rangle, \langle c_{34}c_{35}\rangle, \langle c_{34}c_{37}\rangle, \langle c_{35}c_{36}\rangle, \langle c_{31}c_{34}\rangle, \langle c_{32}c_{33}\rangle, \langle c_{33}c_{34}\rangle,\\ & \langle c_{33}c_{38}\rangle, \langle c_{34}c_{35}\rangle, \langle c_{34}c_{37}\rangle, \langle c_{35}c_{36}\rangle, \langle c_{31}c_{34}\rangle, \langle c_{32}c_{33}\rangle, \langle c_{33}c_{34}\rangle, \langle c_{33}c_{38}\rangle, \langle c_{34}c_{35}\rangle,\\ & \langle c_{34}c_{37}\rangle, \langle c_{35}c_{36}\rangle, \langle c_{27}c_{32}\rangle, \langle c_{28}c_{29}\rangle, \langle c_{28}c_{31}\rangle, \langle c_{29}c_{30}\rangle, \langle c_{30}c_{35}\rangle, \langle c_{31}c_{30}\rangle, \langle c_{32}c_{31}\rangle,\\ & \langle c_{31}c_{34}\rangle, \langle c_{32}c_{33}\rangle, \langle c_{33}c_{34}\rangle, \langle c_{33}c_{38}\rangle, \langle c_{34}c_{35}\rangle, \langle c_{34}c_{37}\rangle, \langle c_{35}c_{36}\rangle, \langle c_{37}c_{36}\rangle, \langle c_{36}c_{41}\rangle,\\ & \langle c_{38}c_{37}\rangle, \langle c_{37}c_{40}\rangle, \langle c_{38}c_{39}\rangle, \langle c_{39}c_{40}\rangle, \langle c_{40}c_{41}\rangle\}. \end{aligned}$$

Thus we obtain the following short sequence:

$$0 \xrightarrow{\partial_2} C_1^6(MSS_6\sharp MSS_6) \xrightarrow{\partial_1} C_0^6(MSS_6\sharp MSS_6) \xrightarrow{\partial_0} 0.$$

By Theorem 2.9, $H_q^6(MSS_6\sharp MSS_6)$ is a trivial group for $q > 1$.

Direct calculation yields that $Z_1^6(MSS_6\sharp MSS_6) \cong \mathbb{Z}^{39}$, from which it follows easily that $B_0^6(MSS_6\sharp MSS_6) \cong \mathbb{Z}^{41}$. However, direct calculation of the group $Z_1^6(MSS_6\sharp MSS_6)$

of digital simplicial 1-cylces is very long. Since our goal is to calculate $H_1^6(MSS_6\sharp MSS_6)$, we do so below without showing a direct calculation of $Z_1^6(MSS_6\sharp MSS_6)$.

By using the short sequence again, we have

$$Z_0^6(MSS_6\sharp MSS_6) = \Big\{ \sum_{i=0}^{41} a_i\langle c_i\rangle \mid a_i \in \mathbb{Z},\ i = 0, 1, ..., 41\Big\} \cong \mathbb{Z}^{42}.$$

Any 0-cycle $w_0 = \sum_{i=0}^{41} a_i\langle c_i\rangle$ can be written as

$$\begin{aligned} w_0 = \partial_1(&- a_{12}\langle c_{12}c_{17}\rangle + (-a_{12} - a_{17})\langle c_{17}c_{39}\rangle + (a_{12} + a_{17} + a_{39})\langle c_{38}c_{39}\rangle \\ &+ (a_{12} + a_{17} + a_{38} + a_{39})\langle c_{18}c_{38}\rangle \\ &+ (a_{12} + a_{17} + a_{18} + a_{38} + a_{39})\langle c_{11}c_{18}\rangle \\ &+ (a_{11} + a_{12} + a_{17} + a_{18} + a_{38} + a_{39})\langle c_6c_{11}\rangle \\ &+ (-a_6 - a_{11} - a_{12} - a_{17} - a_{18} - a_{38} - a_{39})\langle c_6c_{21}\rangle \\ &+ (-a_6 - a_{11} - a_{12} - a_{17} - a_{18} - a_{21} - a_{38} - a_{39})\langle c_{21}c_{33}\rangle \\ &+ (a_6 + a_{11} + a_{12} + a_{17} + a_{18} + a_{21} + a_{33} + a_{38} + a_{39})\langle c_{32}c_{33}\rangle \\ &+ (a_6 + a_{11} + a_{12} + a_{17} + a_{18} + a_{21} + a_{32} + a_{33} + a_{38} + a_{39})\langle c_{22}c_{32}\rangle \end{aligned}$$

$$
\begin{aligned}
&+ (a_6 + a_{11} + a_{12} + a_{17} + a_{18} + a_{21} + a_{22} + a_{32} + a_{33} + a_{38} + a_{39})\langle c_5c_{22}\rangle \\
&+ (a_5 + a_6 + a_{11} + a_{12} + a_{17} + a_{18} + a_{21} + a_{22} + a_{32} + a_{33} \\
&+ a_{38} + a_{39})\langle c_0c_5\rangle + (-a_0 - a_5 - a_6 - a_{11} - a_{12} - a_{17} - a_{18} - a_{21} \\
&- a_{22} - a_{32} - a_{33} - a_{38} - a_{39})\langle c_0c_{26}\rangle + (-a_0 - a_5 - a_6 - a_{11}- \\
&- a_{12} - a_{17} - a_{18} - a_{21} - a_{22} - a_{26} - a_{32} - a_{33} - a_{38} - a_{39})\langle c_{26}c_{27}\rangle \\
&+ (-a_0 - a_5 - a_6 - a_{11} - a_{12} - a_{17} - a_{18} - a_{21} - a_{22} - a_{26} - a_{27} - a_{32} \\
&- a_{33} - a_{38} - a_{39})\langle c_{27}c_{28}\rangle + a_{25}\langle c_1c_{25}\rangle + (-a_1 - a_{25})\langle c_1c_4\rangle \\
&+ (-a_1 - a_4 - a_{25})\langle c_4c_7\rangle + (-a_1 - a_4 - a_7 - a_{25})\langle c_7c_{10}\rangle \\
&+ (-a_1 - a_4 - a_7 - a_{10} - a_{25})\langle c_{10}c_{13}\rangle \\
&+ (-a_1 - a_4 - a_7 - a_{10} - a_{13} - a_{25})\langle c_{13}c_{16}\rangle \\
&+ (-a_1 - a_4 - a_7 - a_{10} - a_{13} - a_{16} - a_{25})\langle c_{16}c_{40}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} + a_{25} + a_{40})\langle c_{37}c_{40}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} + a_{25} + a_{37} + a_{40})\langle c_{34}c_{37}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} + a_{25} + a_{34} + a_{37} + a_{40})\langle c_{31}c_{34}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} + a_{25} + a_{31} + a_{34} + a_{37} + a_{40})\langle c_{28}c_{31}\rangle \\
&+ (-a_{14})\langle c_{14}c_{15}\rangle + (-a_{14} - a_{15})\langle c_{15}c_{41}\rangle + (a_{14} + a_{15} + a_{41})\langle c_{36}c_{41}\rangle \\
&+ (a_{14} + a_{15} + a_{36} + a_{41})\langle c_{19}c_{36}\rangle + (a_{14} + a_{15} + a_{19} + a_{36} + a_{41})\langle c_9c_{19}\rangle \\
&+ (a_9 + a_{14} + a_{15} + a_{19} + a_{36} + a_{41})\langle c_8c_9\rangle \\
&+ (-a_8 - a_9 - a_{14} - a_{15} - a_{19} - a_{36} - a_{41})\langle c_8c_{20}\rangle \\
&+ (-a_8 - a_9 - a_{14} - a_{15} - a_{19} - a_{20} - a_{36} - a_{41})\langle c_{20}c_{35}\rangle \\
&+ (a_8 + a_9 + a_{14} + a_{15} + a_{19} + a_{20} + a_{35} + a_{36} + a_{41})\langle c_{30}c_{35}\rangle \\
&+ (a_8 + a_9 + a_{14} + a_{15} + a_{19} + a_{20} + a_{30} + a_{35} + a_{36} + a_{41})\langle c_{23}c_{30}\rangle \\
&+ (a_8 + a_9 + a_{14} + a_{15} + a_{19} + a_{20} + a_{23} + a_{30} + a_{35} + a_{36} + a_{41})\langle c_3c_{23}\rangle \\
&+ (a_3 + a_8 + a_9 + a_{14} + a_{15} + a_{19} + a_{20} + a_{23} + a_{30} + a_{35} + a_{36} + a_{41})\langle c_2c_3\rangle \\
&+ (-a_2 - a_3 - a_8 - a_9 - a_{14} - a_{15} - a_{19} - a_{20} - a_{23} - a_{30} - a_{35} - a_{36} \\
&- a_{41})\langle c_2c_{24}\rangle + (-a_2 - a_3 - a_8 - a_9 - a_{14} - a_{15} - a_{19} - a_{20} - a_{23} - a_{24} - a_{30} \\
&- a_{35} - a_{36} - a_{41})\langle c_{24}c_{29}\rangle + (a_2 + a_3 + a_8 + a_9 + a_{14} + a_{15} + a_{19} \\
&\quad + a_{20} + a_{23} + a_{24} + a_{29} + a_{30} + a_{35} + a_{36} + a_{41})\langle c_{28}c_{29}\rangle) + \sum_{i=0}^{41} a_i\langle c_{28}\rangle.
\end{aligned}
$$

So w_0 is homologous to 0-chain $\sum_{i=0}^{41} a_i\langle c_{28}\rangle$. Hence the 0-cycle is homologous to an integral multiple of $\langle c_{28}\rangle$. Thus we get $H_0^6(MSS_6\sharp MSS_6) \cong \mathbb{Z}$.

From Example 2.12, Theorem 2.11, and the above, we have

$$
\begin{aligned}
-38 = \chi(MSS_6\sharp MSS_6) =& rank\ H_0^6(MSS_6\sharp MSS_6) - rank\ H_1^6(MSS_6\sharp MSS_6) \\
=& 1 - rank\ H_1^6(MSS_6\sharp MSS_6).
\end{aligned}
$$

Therefore, $rank\ H_1^6(MSS_6\sharp MSS_6) = 39$. It follows from Theorem 2.9 that $H_1^6(MSS_6\sharp MSS_6) \cong \mathbb{Z}^{39}$. □

Acknowledgement

The authors would like to express their sincere thanks to the referee for the valuable suggestions and comments which help to clarify the paper.

References

[1] H. Arslan, I. Karaca, and A. Öztel, Homology groups of n-dimensional digital images, XXI. Turkish National Mathematics Symposium 2008, B1-13.

[2] R. Ayala, E. Domingues, A.R. Frances, A. Quintero, Digital homotopy with obstacles, Discrete Applied Mathematics 139(2004), 5-30.

[3] G. Bertrand, Simple points, topological numbers and geodesic neighborhoods in cubic grids, Pattern Recognition Letters 15(1994), 1003-1011.

[4] G. Bertrand and R. Malgouyres, Some topological properties of discrete surfaces, Journal of Mathematical Imaging and Vision 11(1999), 207-211.

[5] L. Boxer, Digitally continuous functions, Pattern Recognition Letters 15 (1994), 833-839.

[6] L. Boxer, A classical construction for the digital fundamental group, Journal of Mathematical Imaging and Vision 10(1999), 51-62.

[7] L. Boxer, Properties of digital homotopy, Journal of Mathematical Imaging and Vision 22 (2005), 19-26.

[8] L. Boxer, Homotopy properties of sphere-like digital images, Journal of Mathematical Imaging and Vision 24(2006), 167-175.

[9] L. Boxer, Digital products, wedges, and covering spaces, Journal of Mathematical Imaging and Vision 25 (2006), 169-171.

[10] L. Boxer, I. Karaca and A. Oztel, Topological invariants in Digital Images, Journal of Mathematical Sciences: Advances and Application, 11(2011) no 2, 109-140.

[11] L. Chen, Discrete surfaces and manifolds, Scientific Practical Computing, Rockville, MD, 2004.

[12] L. Chen and Y. Rong, Linear time recognition algorithms for topological invariants in 3D, Proceedings of International Conference on Pattern Recognition,2008.

[13] S.E. Han, Connected sum of digital closed surfaces, Information Sciences 176(2006), 332-348.

[14] S.E. Han, An extended digital (k_0, k_1)-continuity, Journal of Applied Mathematics and Computing 16(1-2)(2004), 445-452.

[15] G.T.Herman, Oriented surfaces in digital spaces, CVGIP: Graphical Models and Image Processing 55(1993), 381-396.

[16] T. Kacynski, K. Mischaikow, and M. Mrozek, *Computational Homlogy*, Springer-Verlag, New York, 2010.

[17] T.Y. Kong, A digital fundamental group, Computers and Graphics 13(1989), 159-166.

[18] T.Y. Kong and A. Rosenfeld, Digital topology - A brief introduction and bibliography, Topological Algorithms for the Digital Image Processing, Elsevier Science, Amsterdam, 1996.

[19] A. Rosenfeld, Digital topology, American Mathematical Monthly 86(1979), 76-87.

[20] A. Rosenfeld, 'Continuous' functions on digital pictures, Pattern Recognition Letters 4(1986), 177-184.

[21] Edwin H. Spanier, *Algebraic Topology*, Springer-Verlag, New York, 1966.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1023 – 1031

One step iterative scheme for a pair of nonexpansive mappings in a convex metric space

Hafiz Fukhar-ud-din *

Abstract

We propose and analyse a one step explicit iteration scheme for a pair of nonexpansive mappings in a uniformly convex metric space. Our results refine and generalize several recent and comparable results in uniformly convex Banach spaces and $CAT(0)$ spaces, simultaneously.

Keywords: Convex metric space, one step iterative scheme, nonexpansive mapping, common fixed point, strong convergence and $\triangle-$convergence.

2000 AMS Classification: Primary 47H09, 47H10; Secondary 49M05.

Received 18/02/2014 : Accepted 19/06/2014 Doi : 10.15672/HJMS.2015449658

1. Introduction and preliminaries

The fixed point theory of nonexpansive mappings proposed in the setting of Banach spaces extremly depends on the linear structure of the underlying space. A nonlinear framework for theory of iterative construction of fixed points of nonexpansive mappings is a metric space embedded with a "convex structure". In the literature, different notions of convexity in metric spaces are provided (see, for example, Kirk [10, 11], Penot [15] and Takahashi [20]).

Takahashi [20] introduced the notion of a convex structure in a metric space X as a mapping $W : X^2 \times I \to X$ satisfying

$$d\left(u, W\left(x, y, \alpha\right)\right) \leq \alpha d(u, x) + (1 - \alpha) d(u, y) \tag{1.1}$$

for all $x, y, u \in X$ and $\alpha \in I = [0, 1]$. A metric space X together with a convex structure W is known as a convex metric space. For the sake of simplicity, we also denote a convex metric space by X. A nonempty subset C of X is convex if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in I$.There are many examples of convex metric spaces which cannot be imbedded in any Banach space (see [20]). Some other examples of convex metric spaces are Hadamard manifolds [3] and $CAT\left(0\right)$ spaces [2, 9].

*Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan
Email: hfdin@kfupm.edu.sa

A convex metric space X is uniformly convex [5, 19] if for any $\varepsilon > 0$, there exists $\alpha > 0$ such that $d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq r\left(1-\alpha\right) < r$ for all $r > 0$ and $x, y, z \in X$ with $d\left(z, x\right) \leq r, d\left(z, y\right) \leq r$ and $d\left(x, y\right) \geq r\varepsilon$.

A closed subset X of the unit ball $S_1\left(0\right) = \{x \in H : \|x\| \leq 1\}$ in a Hilbert space H with diameter $\delta\left(X\right) \leq \sqrt{2}$, turns out to be a uniformly convex metric space with $d\left(x, y\right) = \cos^{-1}\left\langle x, y\right\rangle$ for all $x, y \in X$ and $W\left(x, y, \alpha\right) = \frac{\alpha x+(1-\alpha)y}{\|\alpha x+(1-\alpha)y\|}$ for all $x, y \in X$ and $\alpha \in I$.

A mapping T on a subset C of X is nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T if $Tx = x$. Denote by $F(T)$, the set of all fixed points of T.

Ishikawa iterative scheme [6] is a two step iterative scheme and has been extensively used to approximate common fixed points of nonexpansive mappings by a number of researchers (see, for example, [7, 13, 21, 22]).

In order to reduce the computational cost of a two step iterative scheme, we propose a one step iterative scheme for a pair of nonexpansive mappings $S, T : C \to C$ in a convex metric space as follows:

$$(1.2) \qquad x_{n+1} = W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right)$$

where $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$ and satisfy $\alpha_n + \beta_n < 1$(see also [1]).

In Banach space setting, (1.2) becomes one step iterative scheme [23]:

$$(1.3) \qquad x_{n+1} = \alpha_n Tx_n + \beta_n Sx_n + \left(1-\alpha_n-\beta_n\right)x_n.$$

When $S = I$ in (1.2), it reduces to Mann iterative scheme [14]:

$$(1.4) \qquad x_{n+1} = W\left(Tx_n, x_n, \alpha_n\right).$$

One of the interesting and important aspect of approximation theory of fixed points is to consider an iterative scheme with bounded error term and therefore such an iterative scheme has been widely studied by a number of researchers in various frames of work; see, for instance, [7] and references therein. It is remarked that the scheme (1.2) can be reshaped as Mann iteration scheme with errors by replacing $\{Sx_n\}$ or $\{Tx_n\}$ with $\{u_n\}$(*i.e.*, the error term).

Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, define $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. Then (i) $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$ is called the asymptotic radius of $\{x_n\}$ with respect to $C \subseteq X$, (ii) For any $y \in C$, the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\} \leq r(y, \{x_n\})\}$ is called the asymptotic center of $\{x_n\}$ with respect to $C \subseteq X$.

A subset C of a metric space X is Chebyshev if for every $x \in X$, there exists $z \in C$ such that $d\left(z, x\right) < d\left(c, x\right)$ for all $c \in C$ and $c \neq z$. If C is a Chebyshev subset of a metric space X, then we define the nearest point projection $P : X \to C$ by sending x to z. This is consistent with the notion of orthogonal projection onto a subspace of a Euclidean space. It has been shown in [4] that every closed convex subset of a uniformly convex metric space is Chebyshev.

A sequence $\{x_n\}$ in X is said to $\triangle-$converge to $x \in X$ [12] if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle-\lim_n x_n = x$.

It has been shown in the literature that the notion of $\triangle-$convergence and weak convergence in Banach spaces share many useful properties.

In this manuscript, we approximate the common fixed points of two nonexpansive mappings by one step iterative scheme (1.2) in a convex metric space.

For the development of our main results, some key results are listed in the form of lemmas:

1.1. Lemma. *([4]). Let C be a nonempty closed convex subset of a uniformly convex metric space and $\{x_n\}$ a bounded sequence in C such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in C such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

1.2. Lemma. *([18]). Let X be a uniformly convex metric space with continuous convex structure W. Then for any $\varepsilon > 0$ and $r > 0$, there exists $\delta > 0$ such that*

$$d\left(z, W\left(x, y, \alpha\right)\right) \leq r\left(1 - 2\min\left\{\alpha, 1-\alpha\right\}\delta\right)$$

for all $x, y, z \in X, d(z, x) \leq r, d(z, y) \leq r, d(x, y) \geq r\varepsilon$ and $\alpha \in I$.

From now onwards, for a pair of nonexpansive mappings $S, T : C \to C$, we set $F = F(T) \cap F(S)$.

2. Main Results

We start with the following lemma.

2.1. Lemma. *Let C be a closed and convex subset of a convex metric space X and let S, T be nonexpansive mappings on C such that $F \neq \phi$. Then for the sequence $\{x_n\}$ defined in (1.2), $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.*

Proof. Let $p \in F$. Applying (1.1) to (1.2), we have

$$\begin{aligned}
d\left(x_{n+1}, p\right) &= d\left(W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\
&\leq \alpha_n d\left(Tx_n, p\right) + \left(1-\alpha_n\right) d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\
&\leq \alpha_n d\left(x_n, p\right) + \left(1-\alpha_n\right)\left[\frac{\beta_n}{1-\alpha_n} d\left(Sx_n, p\right) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d\left(Sx_n, p\right)\right] \\
&\leq \alpha_n d\left(x_n, p\right) + \left(1-\alpha_n\right)\left[\frac{\beta_n}{1-\alpha_n} d\left(x_n, p\right) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d\left(x_n, p\right)\right] \\
&= \alpha_n d\left(x_n, p\right) + \beta_n d\left(x_n, p\right) + \left(1-\alpha_n-\beta_n\right) d\left(x_n, p\right) \\
&= d\left(x_n, p\right).
\end{aligned}$$

That is,

$$d\left(x_{n+1}, p\right) \leq d\left(x_n, p\right) \text{ for all } p \in F. \tag{2.1}$$

This gives that $\{x_n\}$ is a decreasing and bounded below sequence of nonnegative real numbers, therefore $\lim_{n\to\infty} d(x_n, p)$ exists. ∎

The following lemma provides an analogue of Schu Lemma [16] in the setting of convex metric spaces and is needed in the next lemma.

2.2. Lemma. *Let X be a uniformly convex metric space with continuous convex structure W. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{u_n\}$ and $\{v_n\}$ are sequences in X such that $\limsup_{n\longrightarrow\infty} d(u_n, x) \leq r$, $\limsup_{n\longrightarrow\infty} d(v_n, x) \leq r$ and $\lim_{n\longrightarrow\infty} d(W(u_n, v_n, a_n), x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

Proof. The case $r = 0$ is trivial. Suppose $r > 0$ and assume $\lim_{n\to\infty} d(u_n, v_n) \neq 0$. If $n_0 \geq 1$, then $d(u_{n_i}, v_{n_i}) \geq \frac{\alpha}{2} > 0$ for some $\alpha \in (0, r]$ and for $n_i \geq n_0$. Since $\limsup_{i\longrightarrow\infty} d(u_{n_i}, x) \leq r$ and $\limsup_{n\longrightarrow\infty} d(v_{n_i}, x) \leq r$, so $\max\{d(u_{n_i}, x), d(v_{n_i}, x)\} \leq$

$r+\frac{1}{n_i}$ for $n_i \geq n_0$ and $d(u_{n_i}, v_{n_i}) \geq \frac{\alpha}{2} = \left(r+\frac{1}{n_i}\right)\frac{\alpha_{n_i}}{2(n_i r+1)} \geq \left(r+\frac{1}{n_i}\right)\frac{\alpha}{2(r+1)}$. Therefore Lemma 1.2 gives that

$$\begin{aligned} d(W\left(u_{n_i}, v_{n_i}, a_{n_i}\right), x) &\leq \left(r+\frac{1}{n_i}\right)\left(1-2\min\left\{a_{n_i}, 1-a_{n_i}\right\}\delta\right) \\ &\leq \left(r+\frac{1}{n_i}\right)\left(1-2a_{n_i}\left(1-a_{n_i}\right)\delta\right) \\ &\leq \left(r+\frac{1}{n_i}\right)\left(1-2b\left(1-c\right)\delta\right). \end{aligned}$$

Thus, by letting $i \to \infty$, we obtain

$$\lim_{i\to\infty} d(W\left(u_{n_i}, v_{n_i}, a_{n_i}\right), x) \leq \left(1-2b\left(1-c\right)\delta\right) r < r,$$

a contradiction. ∎

2.3. Lemma. *Let C be a nonempty, closed and convex subset of a uniformly convex metric space X with continuous convex structure W and let S, T be nonexpansive mappings on C such that $F \neq \phi$. Then for the sequence $\{x_n\}$ in (1.2), we have*

$$\lim_{n\to\infty} d\left(x_n, Sx_n\right) = 0 = \lim_{n\to\infty} d\left(x_n, Tx_n\right).$$

Proof. It follows from Lemma 2.1 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Assume that $\lim_{n\to\infty} d(x_n, p) = c$. If $c = 0$, the result is trivial. For $c > 0, \lim_{n\to\infty} d(x_{n+1}, p) = c$ gives that

$$\lim_{n\to\infty} d\left(W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) = c. \tag{2.2}$$

Nonexpansiveness of T gives that

$$\limsup_{n\to\infty} d(Tx_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = c. \tag{2.3}$$

Since

$$\begin{aligned} d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) &\leq \frac{\beta_n}{1-\alpha_n} d(Sx_n, p) + \left(1-\frac{\beta_n}{1-\alpha_n}\right) d(x_n, p) \\ &\leq d(x_n, p), \end{aligned}$$

therefore

$$\limsup_{n\to\infty} d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \leq c. \tag{2.4}$$

Using Lemma 2.2 $\left(\text{with } x = p, r = c, a_n = \alpha_n, u_n = Tx_n, v_n = W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)\right)$ together with (2.2-2.4), we get

$$\lim_{n\to\infty} d\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)\right) = 0. \tag{2.5}$$

Now the estimate

$$\begin{aligned} d(x_{n+1}, Tx_n) &\leq d\left(W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), Tx_n\right) \\ &\leq (1-\alpha_n) d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), Tx_n\right) \\ &\leq (1-b) d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), Tx_n\right), \end{aligned}$$

together with (2.5) implies that

$$\lim_{n\to\infty} d(x_{n+1}, Tx_n) = 0. \tag{2.6}$$

Since S is nonexpansive, $\limsup_{n\to\infty} d(Sx_n, p) \leq c$.
By triangle inequality, we have

$$\begin{aligned} d(x_{n+1}, p) &\leq d(x_{n+1}, Tx_n) + d\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)\right) \\ &\quad + d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right). \end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides in the above inequality, we have

$$c \leq \liminf_{n\to\infty} d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right).$$

Therefore

$$\lim_{n\to\infty} d\left(W\left(Sx_n, x_n, \frac{\alpha_n}{1-\beta_n}\right), p\right) = c. \tag{2.7}$$

Again by Lemma 2.2 (with $x = p, r = c, a_n = \frac{\alpha_n}{1-\beta_n}, u_n = S_n x_n, v_n = x_n$), we get

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{2.8}$$

Further note that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(x_{n+1}, Tx_n) + d\left(Tx_n, W\left(Sx_n, x_n, \frac{\alpha_n}{1-\beta_n}\right)\right) \\ &\quad + d\left(W\left(Sx_n, x_n, \frac{\alpha_n}{1-\beta_n}\right), x_n\right) \\ &\leq d(x_{n+1}, Tx_n) + d\left(Tx_n, W\left(Sx_n, x_n, \frac{\alpha_n}{1-\beta_n}\right)\right) \\ &\quad + (1 - \frac{\alpha_n}{1-\beta_n}) d(x_n, Sx_n) \\ &\leq d(x_{n+1}, Tx_n) + d\left(Tx_n, W\left(Sx_n, x_n, \frac{\alpha_n}{1-\beta_n}\right)\right) \\ &\quad + \left(\frac{1-2a}{1-b}\right) d(x_n, Sx_n). \end{aligned}$$

Letting $n \to \infty$ in the above estimate, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.9}$$

As a direct consequence of (2.6) and (2.9), the inequality

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n)$$

provides that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Hence

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n).$$

∎

The conclusion of Lemma 2.3 is interesting because the sequence generated by (1.2) gives an approximate fixed point sequence for both S and T without assuming that these mappings commute.

Now we state a result concerning $\triangle-$convergence of the iterative scheme (1.2). The method of proof is closely related to Theorem 3.1 in [8].

2.4. Theorem. *Let C be a nonempty, closed and convex subset of a uniformly convex complete metric space X with continuous convex structure W and $S, T : C \to C$ be nonexpansive mappings with $F \neq \phi$. Then the sequence $\{x_n\}$ in (1.2), $\triangle-$converges to an element of F.*

Proof. In the proof of Lemma 2.1, it has been shown that $\{x_n\}$ is bounded. Therefore $\{x_n\}$ has a unique asymptotic centre, that is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. First, we show that $u \in C$. Suppose $u \notin C$. As C is a Chebyshev set, we can define a nearest point projection $P : X \to C$.Therefore $d(Pu, u_n) < d(u, u_n) \Longrightarrow r(Pu, \{u_n\}) < r(u, \{u_n\}) \Longrightarrow u$ is not the asymptotic center of $\{u_n\}$, a contradiction. Hence $u \in C$. Also by Lemma 2.2, we have $\lim_{n\to\infty} d(u_n, Tu_n) = 0 = \lim_{n\to\infty} d(u_n, Su_n)$. Define $\{z_m\}$ in C by $z_m = T^m u$.
Observe that

$$\begin{aligned} d(z_m, u_n) &\leq d(T^m u, T^m u_n) + \sum_{j=1}^{m} d(T^j u_n, T^{j-1} u_n) \\ &\leq d(u, u_n) + m d(T u_n, u_n). \end{aligned}$$

Therefore, we have

$$r(z_m, \{u_n\}) = \limsup_{n\to\infty} d(z_m, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 1.1 that $\lim_{m\to\infty} T^m u = u$. Since C is closed, so $\lim_{m\to\infty} T^m u = u \in C$ and $\lim_{m\to\infty} T^{m+1} u = Tu$.That is, $Tu = u$. Similarly we have $Su = u$. Therefore $\lim_{n\to\infty} d(x_n, u)$ exists by Lemma 2.1. If $x \neq u$, then by the uniqueness of asymptotic centres, we have

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

a contradiction. Hence $x = u$.
Therefore, $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\triangle-$converges to x. ∎

Using the concept of near point projection, we establish the following theorem.

2.5. Theorem. *Let C be a nonempty, closed and convex subset of a complete uniformly convex metric space X and $S, T : C \to C$ be nonexpansive mappings. Let P be the nearest point projection of C onto F. For an initial value x_1, define $\{x_n\}$ as given in (1.2) where $\alpha_n, \beta_n \in [a, b]$ for some $a, b \in \mathbb{R}$ with $0 < a \leq b < 1$. Then $\{Px_n\}$ converges strongly to a point of F.*

Proof. It follows from (2.1) that, for any $n \geq 1, m \geq 1$, we have

$$d(Px_n, x_{n+m}) \leq d(Px_n, x_{n+m-1}) \leq d(Px_n, x_{n+m-2}) \leq ... \leq d(Px_n, x_{n+1}).$$

That is,

(2.10) $\quad d(Px_n, x_{n+m}) \le d(Px_n, x_n)$ for $n \ge 1, m \ge 1$.

In order to prove the result, we show that $\{Px_n\}$ is a Cauchy sequence. By definition of nearest point projection and (2.10), we have

$$d(Px_{n+1}, x_{n+1}) \le d(Px_n, x_{n+1}) \le d(Px_n, x_n).$$

Hence $d(Px_n, x_n) \to c$(say). If $c = 0$, then for an arbitrary $\varepsilon > 0$, there exists an integer $n_0 \ge 1$ such that

(2.11) $\quad d(Px_n, x_n) < \varepsilon$ for all $n \ge n_0$.

By (2.11), for $m > n \ge n_0$,we have

$$\begin{aligned} d(Px_n, Px_m) &\le d(Px_n, Px_{n_0}) + d(Px_{n_0}, Px_m) \\ &\le d(Px_n, x_n) + d(x_n, Px_{n_0}) + d(Px_{n_0}, x_m) + d(x_m, Px_m) \\ &< 4\varepsilon. \end{aligned}$$

This proves that $\{Px_n\}$ is a Cauchy sequence. Assume that $c > 0$ and $\{Px_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and two subsequences $\{Px_{n_i}\}$ and $\{Px_{m_i}\}$ of $\{Px_n\}$ such that $d(Px_{n_i}, Px_{m_i}) \ge \varepsilon$ for all $i \ge 1$. Since $\{d(Px_n, x_n)\}$ is a decreasing sequence and $d(Px_n, x_n) \to c$, therefore we have

$$c \le d(Px_n, x_n) \le c + \frac{1}{n} \text{ for } n \ge n_0.$$

Let $n_0 \le n_i, m_i \le l$. By (2.10), we have

$$d(Px_{n_i}, x_l) \le d(Px_{n_i}, x_{n_i}) < c + \frac{1}{n} \text{ and } d(Px_{m_i}, x_l) \le d(Px_{m_i}, x_{m_i}) < c + \frac{1}{n}.$$

Moreover,

$$d(Px_{n_i}, Px_{m_i}) \ge \left(\frac{\varepsilon}{c + \frac{1}{n}}\right)\left(c + \frac{1}{n}\right) \ge \left(\frac{\varepsilon}{c+1}\right)\left(c + \frac{1}{n}\right).$$

By uniform convexity of X, there exists $\delta\left(\frac{\varepsilon}{c+1}\right) > 0$ such that

$$d\left(x_l, \frac{1}{2}Px_{n_i} \oplus \frac{1}{2}Px_{m_i}\right) \le \left(c + \frac{1}{n}\right)\left(1 - \delta\left(\frac{\varepsilon}{c+1}\right)\right).$$

Let $n \to \infty$ in the above inequality, we have

$$c \le d(Px_l, x_l) \le d\left(x_l, \frac{1}{2}Px_{n_i} \oplus \frac{1}{2}Px_{m_i}\right) \le c\left(1 - \delta\left(\frac{\varepsilon}{c+1}\right)\right) < c,$$

a contradiction.

This proves that $\{Px_n\}$ is a Cauchy sequence in F. As F is closed, therefore it converges to a point of F. ∎

Recall that a mapping $T : C \to C$ is semi-compact if every bounded sequence $\{x_n\}$ has a convergent subsequence whenever $d(x_n, Tx_n) \to 0$.

Let $f : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$.The mappings $S, T : C \to C$ with $F \neq \phi$, satisfy Condition (I) [7] (see also [17]) if

$$\frac{1}{2}\left[d(x, Tx) + d(x, Sx)\right] \ge f(d(x, F)) \text{ for } x \in C,$$

where $d(x, F) = \inf_{p \in F} d(x, p)$.

Using Lemma 2.3, we obtain the following strong convergence theorem.

2.6. Theorem. *Let C be a nonempty, closed and convex subset of a uniformly convex complete metric space with continuous convex structure W and let $S, T : C \to C$ be nonexpansive mappings with $F \neq \phi$. If S and T satisfy Condition (I), then the sequence $\{x_n\}$ defined in (1.2), converges strongly to an element of F.*

Proof. By Lemma 2.3, we have

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n).$$

Using Condition (I), we get that that $\lim_{n\to\infty} d(x_n, F) = 0$. For a given $\varepsilon > 0$, there exists $N_\varepsilon \geq 1$ and $y_\varepsilon \in F$ such that $d(x_n, y_\varepsilon) < \varepsilon$ for all $n \geq N_\varepsilon$.Thus, if $\varepsilon_k = 2^{-k}$ for $k \geq 1$, then corresponding to each ε_k, there exist $N_k \geq 1$ and $y_k \in F$ such that $d(x_n, y_k) \leq \frac{\varepsilon_k}{4}$ for all $n \geq N_k$. On choosing $N_{k+1} \geq N_k$ for any $k \geq 1$,we have that

$$\begin{aligned} d(y_k, y_{k+1}) &\leq d\left(y_k, x_{N_{k+1}}\right) + d\left(x_{N_{k+1}}, y_{k+1}\right) \\ &< \frac{\varepsilon_k}{4} + \frac{\varepsilon_{k+1}}{4} = \frac{3}{4}\varepsilon_{k+1}. \end{aligned}$$

If $x \in S[y_{k+1}, \varepsilon_{k+1}]$, then

$$\begin{aligned} d(x, y_k) &\leq d(x, y_{k+1}) + d(y_{k+1}, y_k) \\ &< \varepsilon_{k+1} + \frac{3}{4}\varepsilon_{k+1} = \frac{7}{4}\varepsilon_{k+1} < 2\varepsilon_{k+1} = \varepsilon_k. \end{aligned}$$

That is, $x \in S[y_k, \varepsilon_k]$.Hence $\{S[y_k, \varepsilon_k] : k \geq 1\}$ is a decreasing sequence of nonempty, bounded, closed and convex subsets in a uniformly convex complete metric space and so $\cap_{k=1}^{\infty} S[y_k, \varepsilon_k] \neq \emptyset$ by Theorem 1([19], p. 200). Now there exists a $p \in X$ such that

$$d(y_k, p) \leq \frac{1}{2^k} \to 0 \text{ as } k \to \infty.$$

That is, $y_k \to p$. Since F is closed, therefore $p \in F$.

In view of the inequality

$$d(x_n, y_k) \leq \frac{\varepsilon_k}{4} \text{ for all } n \geq N_k,$$

we get that $x_n \to p$. ∎

We can also prove the following strong convergence theorem.

2.7. Theorem. *Let C be a closed and convex subset of a uniformly convex complete metric space X and let $S, T : C \to C$ be nonexpansive mappings with $F \neq \phi$. If, either S or T is semi-compact, then the sequence $\{x_n\}$ defined in (1.2), converges strongly to an element of F.*

2.8. Remark. (1) Our results can be extended for two finite families of nonexpansive mappings (2) Our results are valid in uniformly convex Banach spaces and $CAT(0)$ spaces, simultaneously.

Acknowledgement. *The author is grateful to King Fahd University of Petroleum & Minerals for supporting the research project IN121023.*

References

[1] M. Abbas and S. H. Khan, *Some $\triangle-$convergence theorems in $CAT(0)$ spaces*, Hacet. J. Math. Stat., 40(4), 563–569, 2011.

[2] Bridson, M and A. Haefliger, A. *Metric spaces of non-positive curvature* (Springer-Verlag, Berlin, Heidelberg, New York, 1999).

[3] Busemann, H. *Spaces with non-positive curvature*, Acta Math. 80 (1), 259–310, 1948.

[4] Fukhar-ud-din, H. *Existence and approximation of fixed points in convex metric spaces,* Carpathian J. Math. 30 (2014), 175-185.

[5] Fukhar-ud-din, H, Khan, A. R. and Akhtar, Z. *Fixed point results for a generalized nonexpansive map in uniformly convex metric spaces,* Nonlinear Anal., 75 (13), 4747–4760, 2012.

[6] Ishikawa, S. *Fixed points by a new iteration method,* Proc. Amer. Math. Soc., 44 (1), 147-150, 1974.

[7] Khan, S. H and Fukhar-ud-din, H. *Weak and strong convergence of a scheme with errors for two nonexpansive mappings,* Nonlinear Anal., 61 (8), 1295 – 1301, 2005.

[8] Khan, A.R, Fukhar-ud-din, H and Khan, M.A.A. *An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces*, Fixed Point Theory Appl., 2012:54, 2012.

[9] Khan, A. R., Khamsi, M. A. and Fukhar-ud-din, H. *Strong convergence of a general iteration scheme in CAT(0)-spaces,* Nonlinear Anal. 74 (3), 783-791, 2011.

[10] Kirk, W. A. *An abstract fixed point theorem for nonexpansive mappings,* Proc. Amer. Math. Soc., 82 (4), 640-642, 1981.

[11] Kirk, W.A. *Fixed point theory for nonexpansive mappings II,* Contemporary Math.,18(1), 121-140, 1983.

[12] Kirk, W and Panyanak, B. *A concept of convergence in geodesic spaces,* Nonlinear Anal., 68 (12), 3689-3696, 2008.

[13] Maiti, M and Ghosh, M. K. *Approximating fixed points by Ishikawa iterates*, Bull. Austral. Math. Soc. 40 (1), 113-I17, 1989.

[14] Mann, W. R. *Mean value methods in iterations*, Proc. Amer. Math. Soc. 4 (3), 506-510, 1953.

[15] Penot, J. P. *Fixed point theorems without convexity,* Bull. Soc. Math. France, 60(1), 129-152, 1979.

[16] Schu, J. *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. 43(1), 153–159, 1991.

[17] Senter, H. F. and Dotson, W. G. *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., 44(2) , 375–380, 1974.

[18] Shimizu, T. *A convergence theorem to common fixed points of families of nonexpansive mappings in convex metric spaces,* in Proceedings of the International Conference on Nonlinear and Convex Analysis, 2005, 575–585.

[19] Shimizu,T. and Takahashi, W. *Fixed points of multivalued mappings in certain convex metric spaces*, Topol. Methods Nonlinear Anal. 8 (1), 197–203, 1996.

[20] Takahashi, W. *A convexity in metric spaces and nonexpansive mappings*, Kodai Math. Sem. Rep., 22 (2), 142-149, 1970.

[21] Takahashi,W. and Tamura, T. *Convergence theorems for a pair of nonexpansive mappings,* J. Convex Anal., 5 (1), 45–56,1998.

[22] Tan, K. K. and Xu, H. K. *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process,* J. Math. Anal. Appl. 178(2), 301-308,1993.

[23] Yao, Y. and Chen, R. *Weak and strong convergence of a modified Mann iteration for asymptotically nonexpansive mappings,* Nonlinear Funct. Anal. Appl. 12 (2), 307-315, 2007.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1033 – 1044

Suborbital graphs for the group Γ^2

Bahadır Özgür Güler*†, Murat Beşenk‡, Yavuz Kesicioğlu§, Ali Hikmet Değer¶

Abstract

In this paper, we investigate suborbital graphs formed by the action of Γ^2 which is the group generated by the second powers of the elements of the modular group Γ on $\hat{\mathbb{Q}}$. Firstly, conditions for being an edge, self-paired and paired graphs are provided, then we give necessary and sufficient conditions for the suborbital graphs to contain a circuit and to be a forest. Finally, we examine the connectivity of the subgraph $F_{u,N}$ and show that it is connected if and only if $N \leq 2$.

Keywords: Modular group, Group action, Suborbital graphs

2000 AMS Classification: 20H05, 05C25

Received 13/11/2013 : Accepted 27/05/2014 Doi : 10.15672/HJMS.2015449656

1. Introduction

Let PSL(2,$\mathbb{R}$) denote the group of all linear fractional transformations

$$T : z \to \frac{az+b}{cz+d}\ \text{, where } a, b, c \text{ and } d \text{ are real and } ad - bc = 1.$$

In terms of matrix representation, the elements of PSL(2,$\mathbb{R}$) correspond to the matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1.$$

This is the automorphism group of the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im(z)} > 0\}$.

The modular group Γ=PSL$(2, \mathbb{Z})$, is the subgroup of PSL(2,$\mathbb{R}$) such that a, b, c and d are integers. It is generated by the matrices

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad ; \quad V = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

*Dept. of Math., Karadeniz Tech. Uni., Turkey, Email: boguler@ktu.edu.tr
†Corresponding Author.
‡Dept. of Math., Karadeniz Tech. Uni., Turkey, Email: mbesenk@ktu.edu.tr
§Dept. of Math., Recep Tayyip Erdogan Uni., Turkey, Email: yavuzkesicioglu@yahoo.com
¶Dept. of Math., Karadeniz Tech. Uni., Turkey, Email: ahikmetd@ktu.edu.tr

with defining relationships $U^2 = V^3 = I$, where I is the identity matrix. Γ is a Fuchsian group of signature $(0; 2, 3, \infty)$, so it is isomorphic to a free product $C_2 * C_3$.

Define Γ^m as the subgroup of Γ generated by the m^{th} powers of all elements of Γ. Especially, Γ^2 and Γ^3 have been studied extensively by Newman [13,14,15]. It turns out that,

$$\Gamma^2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : ab + bc + cd \equiv 0 \ (mod\ 2) \right\},$$

by Rankin [Eq. 1.7.1, 16]. From the equation $ab + bc + cd \equiv 0\ (mod\ 2)$, we see that at least one of the integers a, b, c, d must be even. Suppose first that $a = 2a_0$. Then using the determinant, we have that b and c are odd. So, d must be odd as well. Hence, we get the elements of Γ^2 as the matrices $\begin{pmatrix} 2a & b \\ c & d \end{pmatrix}$. Similarly, supposing $d = 2d_0$, we can get the elements of the form $\begin{pmatrix} a & b \\ c & 2d \end{pmatrix}$. Lastly, if a or d is not even, then both b and c will be even. To sum up, Γ^2 has three types of elements

$$\begin{pmatrix} 2a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & 2b \\ 2c & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & 2d \end{pmatrix}$$

where b, c and d of the first, a and d of the second and a, b, c of the third matrix are odd.

1.1. Theorem. [13] *The group Γ^2 is the free product of two cyclic groups of order 3, and*

$$|\Gamma : \Gamma^2| = 2\ ,\ \Gamma = \Gamma^2 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Gamma^2.$$

The elements of Γ^2 may be characterized by the requirement that the sum of the exponents of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be divisible by 2.

The idea of a suborbital graph has been used mainly by finite group theorists. In [7], Jones, Singerman and Wicks showed that this idea is also useful in the study of the modular group, where they proved that the well-known Farey Graph is an example of a suborbital graph. Furthermore, they proved the following result:

Theorem A. The suborbital graph $G_{u,n}$ of Γ contains directed triangles if and only if $u^2 \pm u + 1 \equiv 0\ (mod\ n)$.

Morever they posed the conjecture: $G_{u,n}$ is a forest if and only if it contains no triangles, that is, if and only if $u^2 \pm u + 1 \not\equiv 0\ (mod\ n)$. Akbas proved in [2] that this conjecture is true. By similar arguments, we concern with suborbital graphs of Picard group $\mathbf{P}$, which is the subgroup of PSL$(2, \mathbb{C})$ with entries coming from $\mathbb{Z}[i]$ in [3]. Since $\mathbb{Z}[i]$ is a unique factorization domain with finitely many units, our expectation was to find similar formulas. Consequently, theorem A was improved as

Theorem B. The suborbital graph $G_{u,N}$ of $\mathbf{P}$ contains directed triangles if and only if $\varepsilon^2 u^2 \mp \varepsilon u \pm 1 \equiv 0\ (mod\ N)$.

In this study, we will continue to investigate the combinatorial properties of these graphs for the group Γ^2. It is an important subgroup of Γ since all the groups Γ^m can be expressed in the terms of $\Gamma, \Gamma^2, \Gamma^3$. The purpose of this paper is to characterize all circuits in the suborbital graph and connectedness for Γ^2. As it can be seen from Section 3, we show that the main difference is in connectedness of related graphs.

2. The action of Γ^2 on $\hat{\mathbb{Q}}$

Every element of $\hat{\mathbb{Q}}$ can be represented as a reduced fraction $\frac{x}{y}$ with $x, y \in \mathbb{Z}$ and $(x, y) = 1$. This representation is not unique, because $\frac{x}{y} = \frac{-x}{-y}$. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$. The action of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\frac{x}{y}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \longrightarrow \frac{ax+by}{cx+dy}.$$

Hence, the actions of a matrix on $\frac{x}{y}$ and on $\frac{-x}{-y}$ are identical. If the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 1 and $(x, y) = 1$, then $(ax + by, cx + dy) = 1$.

2.1. *Transitive Action.*

2.1. Lemma. (i) *The action of Γ^2 on $\hat{\mathbb{Q}}$ is transitive.*
(ii) *The stabilizer of a point is an infinite cyclic group.*

Proof. (i) Here we only prove the case that any element of the form $\frac{a}{2b}$ of $\hat{\mathbb{Q}}$ is sent ∞ by an element of Γ^2. The rest are similar. Let $\frac{a}{2b} \in \hat{\mathbb{Q}}$, $(a, 2b) = 1$. There exist integers x_0 and y_0 such that $ay_0 - 2bx_0 = 1$ (known as Bezout's identity [8]). Hence, we have that $T := \begin{pmatrix} a & x_0 \\ 2b & y_0 \end{pmatrix} \in \Gamma$. All solutions of the equation $ay - 2bx = 1$ are $x = x_0 + an$, $y = y_0 + 2bn$ for $n \in \mathbb{Z}$. If x_0 is odd, x would be even by taking n-odd. So, x_0 can be chosen as an even number. Hence, $T \in \Gamma^2$ and $T(\infty) = \frac{a}{2b}$ means that $\frac{a}{2b}$ is in the orbit of ∞.

(ii) By (i), since the stabilizers of any two points in $\hat{\mathbb{Q}}$ are conjugate in Γ^2, it is sufficient to consider the stabilizer Γ^2_∞ of ∞. It is clear that $\Gamma^2_\infty = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle$. ∎

We remark that Lemma 2.1 (i) can be proven by using the signature of Γ^2 as well. There is a homomorphism $\theta : \Gamma \longrightarrow C_2 = \{e, \alpha\}$ defined by $\theta(U) = \alpha$, and $\theta(V) = e$. The kernel is Γ^2. By the permutation theorem [19], Γ^2 has signature $(0; 3, 3, \infty)$. It means that there is only one orbit, so the action is transitive.

2.2. ***Imprimitive Action.*** We now discuss the imprimitivity of the action of Γ^2 on $\hat{\mathbb{Q}}$. For this, let (G, Ω) be a transitive permutation group, consisting of a group G acting on a set Ω transitively. An equivalence relation $\approx$ on Ω is called *G-invariant* if, whenever $\alpha, \beta \in \Omega$ satisfy $\alpha \approx \beta$, then $g(\alpha) \approx g(\beta)$ for all $g \in G$. The equivalence classes are called blocks.

We call (G, Ω) *imprimitive* if Ω admits some G-invariant equivalence relation different from

- **(i)** the identity relation, $\alpha \approx \beta$ if and only if $\alpha = \beta$;
- **(ii)** the universal relation, $\alpha \approx \beta$ for all $\alpha, \beta \in \Omega$.

Otherwise, (G, Ω) is called *primitive*. These two relations are supposed to be trivial relations.

2.2. Lemma. [4] *Let (G, Ω) be a transitive permutation group. (G, Ω) is primitive if and only if G_α, the stabilizer of $\alpha \in \Omega$, is a maximal subgroup of G for each $\alpha \in \Omega$.*

From the above lemma we see that whenever, for some α, $G_\alpha \lneq H \lneq G$, then Ω admits some G-invariant equivalence relation other than the trivial one and the universal one.

Because of the transitivity, every element of Ω has the form $g(\alpha)$ for some $g \in G$. Thus one of the non-trivial G-invariant equivalence relations on Ω by H is given as follows:

$$g(\alpha) \approx g'(\alpha) \text{ if and only if } g' \in gH.$$

The number of blocks (equivalence classes) is the index $|G : H|$ and the block containing α is just the orbit $H(\alpha)$.

Let $N \in \mathbb{N}$ and let $\Gamma_0^2(N)$ be defined by

$$\Gamma_0^2(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^2 : c \equiv 0 \ (mod\ N) \right\}.$$

Then $\Gamma_0^2(N)$ is a subgroup of Γ^2. It is clear that $\Gamma_\infty^2 \leq \Gamma_0^2(N) \leq \Gamma^2$ for $N \in \mathbb{N}$ and $\Gamma_\infty^2 \lneq \Gamma_0^2(N) \lneq \Gamma^2$ for $N > 1$.

2.3. Lemma. $|\Gamma_0(N) : \Gamma_0^2(N)| = 2$. *In fact,*

$$\Gamma_0(N) = \begin{cases} \Gamma_0^2(N) \cup \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \Gamma_0^2(N), & N \text{ is odd} \\ \Gamma_0^2(N) \cup \begin{pmatrix} N+1 & -1 \\ N & 1 \end{pmatrix} \Gamma_0^2(N), & N \text{ is even} \end{cases} \blacksquare$$

Proof. First, we suppose that N is even. Let's show that $\Gamma_0^2(N) \cup \begin{pmatrix} N+1 & -1 \\ N & 1 \end{pmatrix} \Gamma_0^2(N) = \Gamma_0(N)$. We have that $T := \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$ with $ad - bcN = 1$. Here, a and d are odd. If b is even, T would be an element of $\Gamma_0^2(N)$. We suppose that b is odd. Hence, it can be written as $T = \begin{pmatrix} N+1 & -1 \\ N & 1 \end{pmatrix} \begin{pmatrix} x & y \\ cN & z \end{pmatrix}$. Then, we have that $\begin{pmatrix} 1 & 1 \\ -N & N+1 \end{pmatrix} \begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} x & y \\ cN & z \end{pmatrix}$. Let's say that

$$\underbrace{\begin{pmatrix} a + cN & b + d \\ -aN + cN(N+1) & -bN + dN(N+1) \end{pmatrix}}_{A} = \begin{pmatrix} x & y \\ cN & z \end{pmatrix}.$$

As $b + d$ is even, $A \in \Gamma_0^2(N)$.

Now, let N be odd. In this case, assume that b and c are even in T. Then a and d are odd. Hence, T is an element of $\Gamma_0^2(N)$. Moreover, it can be written as $T = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \begin{pmatrix} x & y \\ cN & z \end{pmatrix}$. As above, let's say that $\underbrace{\begin{pmatrix} a & b \\ (c-a)N & d - bN \end{pmatrix}}_{B} = \begin{pmatrix} x & y \\ cN & z \end{pmatrix}$. Since $d - bN$ is even, $B \in \Gamma_0^2(N)$. In the case: b-even and c-odd, it is clear that $B \in \Gamma_0^2(N)$. If a and d are even in the equation $ad - bcN = 1$, $B \in \Gamma_0^2(N)$ again. Finally if a is odd and d is even (or vice versa), the result is the same. Consequently, we obtain that $|\Gamma_0(N) : \Gamma_0^2(N)| = 2$. $\blacksquare$

Therefore, from the above constructed equivalence relation "$\approx$", we get Γ^2-invariant equivalence relation on $\hat{\mathbb{Q}}$ by $\Gamma_0^2(N)$. It is clear that, by Lemma 2.3, Γ^2 acts imprimitively on $\hat{\mathbb{Q}}$.

Let $v = \frac{r}{s}$ and $w = \frac{x}{y}$ be elements of $\hat{\mathbb{Q}}$. Because of the transitive action, we have that $v = g_1(\infty)$ and $w = g_2(\infty)$ for some elements $g_1, g_2 \in \Gamma^2$ of the form

$$g_1 := \begin{pmatrix} r & * \\ s & * \end{pmatrix}, \quad g_2 := \begin{pmatrix} x & * \\ y & * \end{pmatrix}.$$

From the relation

$$v \approx w \text{ if and only if } g_1^{-1} g_2 \in \Gamma_0^2(N),$$

we get

$$v \approx w \text{ if and only if } ry - sx \equiv 0 \ (mod\ N).$$

By our general discussion of imprimitivity, the number of blocks under $\approx$ is given by $\Psi(N) = \left|\Gamma^2 : \Gamma_0^2(N)\right|$. So the block of ∞ is obtained as

$$[\infty] := \left\{ \frac{x}{y} \in \hat{\mathbb{Q}} \mid y \equiv 0 \ (mod\ N) \right\}.$$

2.4. Lemma. $\Psi(N) = N \prod_{p|N} \left(1 + \dfrac{1}{p}\right)$ *where the product is over the distinct primes p dividing N.*

Proof. To calculate $\Psi(N)$ we use two decomposition of the index $\left|\Gamma : \Gamma_0^2(N)\right|$ as the following

$$|\Gamma : \Gamma^2||\Gamma^2 : \Gamma_0^2(N)| = |\Gamma : \Gamma_0(N)||\Gamma_0(N) : \Gamma_0^2(N)|.$$

Here, $|\Gamma : \Gamma^2| = 2$ and $|\Gamma : \Gamma_0(N)| = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$ are well-known by [13,16] and [16,17] respectively. We prove that the index of $|\Gamma_0(N) : \Gamma_0^2(N)|$ is equal to 2 in Lemma 2.3. Writing these values in above equation, the result is obvious.

3. Suborbital Graphs for Γ^2 on $\hat{\mathbb{Q}}$

In[18], Sims introduced the idea of the suborbital graphs of a permutation group G acting on a set Δ , these are graphs with vertex-set Δ, on which G induces automorphisms. We summarise Sims'theory as follows:

Let (G, Δ) be transitive permutation group. Then G acts on $\Delta \times \Delta$ by $g(\alpha, \beta) = (g(\alpha), g(\beta))(g \in G, \alpha, \beta \in \Delta)$. The orbits of this action are called *suborbitals* of G. The orbit containing (α, β) is denoted by $O(\alpha, \beta)$. From $O(\alpha, \beta)$ we can form a *suborbital graph* $G(\alpha, \beta)$: its vertices are the elements of Δ, and there is a directed edge from γ to δ if $(\gamma, \delta) \in O(\alpha, \beta)$. A directed edge from γ to δ is denoted by $\gamma \to \delta$. If $(\gamma, \delta) \in O(\alpha, \beta)$, then we will say that there exists an edge $\gamma \to \delta$ in $G(\alpha, \beta)$. In this paper our calculation concerns Γ^2, so we can draw this edge as a hyperbolic geodesic in the upper half-plane $\mathbb{H}$, that is, as euclidean semi-circles or half-lines perpendicular to the real line.

The orbit $O(\beta, \alpha)$ is also a suborbital graph and it is either equal to or disjoint from $O(\alpha, \beta)$. In the latter case $G(\beta, \alpha)$ is just $G(\alpha, \beta)$ with the arrows reversed and we call, in this case, $G(\alpha, \beta)$ and $G(\beta, \alpha)$ *paired suborbital graphs*. In the former case $G(\alpha, \beta) = G(\beta, \alpha)$ and the graph consists of pairs of oppositely directed edges; it is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call *self paired*.

3.1. Definition. By a directed circuit in a graph we mean a sequence $v_1, v_2, \ldots, v_m$ of different vertices such that $v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_m \longrightarrow v_1$, where $m \geq 3$.

If $m = 3$, then the circuit, directed or not, is called a triangle.
If $m = 2$, then we will say the configuration $v_1 \longrightarrow v_2 \longrightarrow v_1$ is self paired.

A graph which contains no circuit is called a forest.

The above ideas are also described in a paper by Neumann [12] and in books by Tsuzuku [20] and by Biggs and White [4], the emphasis being on applications to finite groups. The reader is refereed to [1, 2, 3, 6, 7, 9, 10, 11] for some relevant previous work on suborbital graphs.

If $\alpha = \beta$, then $O(\alpha,\alpha) = \{(\gamma,\gamma) \mid \gamma \in \Delta\}$ is the diagonal of $\Delta \times \Delta$. The corresponding suborbital graph $G(\alpha,\alpha)$, called the trivial suborbital graph, is self-paired: it consists of a loop based at each vertex $\gamma \in \Delta$. We shall be mainly interested in the remaining non-trivial suborbital graphs.

Since Γ^2 acts transitively on $\hat{\mathbb{Q}}$, each suborbital contains a pair (∞, v) for some $v \in \hat{\mathbb{Q}}$; writing $v = \frac{u}{N}$, $(u,N) = 1$ and $N \geq 0$. We denote this suborbital by $O_{u,N}$ and the corresponding suborbital graph by $G_{u,N}$.

3.1. *Graph* $G_{u,N}$. If $v = \infty$, we would have the simplest suborbital graph, namely $G_{1,0} = G_{-1,0}$. Therefore, we can take $v \in \mathbb{Q}$. Let $v' = \frac{u'}{N'} \in \mathbb{Q}$. The necessary and sufficient condition for $O(\infty, v) = O(\infty, v')$ is that v and v' are in the same orbit of Γ^2_∞. Since Γ^2_∞ is generated by $z : v \to v+2$, then $z\left(\frac{u}{N}\right) = \frac{u+2N}{N} = \frac{u'}{N'}$. Therefore, we have that $N = N'$ and $u \equiv u'$ $(mod\ 2N)$. Hence, $G_{u,N} = G_{u',N'}$ if and only if $N = N'$ and $u \equiv u'$ $(mod\ 2N)$. Consequently, for a fixed N there are $2\varphi(N)$ distinct suborbital graphs $G_{u,N}$ where $\varphi(N)$ is Euler's phi function.

3.2. Theorem. $\frac{r}{s} \to \frac{x}{y} \in G_{u,N}$ *if and only if*

(i) *If r is even, then $x \equiv \pm ur$ $(mod\ N)$, $y \equiv \pm us$ $(mod\ N)$, $y \not\equiv \pm us$ $(mod\ 2N)$ and $ry - sx = \mp N$.*

(ii) *If s is even, then $x \equiv \pm ur$ $(mod\ 2N)$, $y \equiv \pm us$ $(mod\ N)$ and $ry - sx = \mp N$.*

(iii) *If r and s are odd, then $x \equiv \pm ur$ $(mod\ N)$, $y \equiv \pm us$ $(mod\ 2N)$ and $ry - sx = \mp N$.*

Proof. (i) Let r be even. By the transitivity of Γ^2, without loss of generality, we assume that $\frac{r}{s} < \frac{x}{y}$ where all letters are positive integers. Thus, we have that $ry - sx < 0$. Since $\frac{r}{s} \to \frac{x}{y} \in G_{u,N}$, there exist some $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^2$ such that $T\left(\frac{1}{0}, \frac{u}{N}\right) = \left(\frac{r}{s}, \frac{x}{y}\right)$. As $ry - sx < 0$, the multiplication of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix}$ is equal to $\begin{pmatrix} -r & x \\ -s & y \end{pmatrix}$ or $\begin{pmatrix} r & -x \\ s & -y \end{pmatrix}$. If the first case is valid, we have that $a = -r$, $c = -s$, $au + bN = x$ and $cu + dN = y$. That is, $x \equiv -ur$ $(mod\ N)$ and $y \equiv -us$ $(mod\ N)$. Since r is even, then a is also even. To have $T \in \Gamma^2$, d must be odd. From $-us + dN = y$, we have that $y \not\equiv \pm us$ $(mod\ 2N)$.

(ii) Suppose s is even. In a similar way, we see that b and c must be even because $T\left(\frac{1}{0}\right) = \frac{-r}{-s} = \frac{a}{c}$. As in (i), we may assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} = \begin{pmatrix} -r & x \\ -s & y \end{pmatrix}$. Hence, we have that $a = -r$, $c = -s$, $au+bN = x$, $cu+dN = y$ and $ry-sx = -N$. That is, $-ur + bN = x$ and $-us + dN = y$. Since b is even, we have that $x \equiv -ur$ $(mod\ 2N)$ and $y \equiv -us$ $(mod\ N)$.

(iii) Let r and s be odd. With similar argument, it can be seen that d must be even. From the same matrix equation in (ii), we obtain that $x \equiv -ur$ $(mod\ N)$ and $y \equiv -us$ $(mod\ 2N)$.

In the opposite direction, we shall prove (i) for minus sign. Suppose that r is even, $x \equiv -ur\ (mod\ N)$, $y \equiv -us\ (mod\ N)$, $y \not\equiv -us\ (mod\ 2N)$ and $ry - sx = -N$. In this case, there exist integers b, d such that $x = -ur - bN$, $y = -us - dN$. So, it is clear that $\begin{pmatrix} -r & -b \\ -s & -d \end{pmatrix} \in \Gamma^2$ which means $\frac{r}{s} \to \frac{x}{y} \in G_{u,N}$. Because $-N = ry - sx = r(-us - dN) - s(-ur - bN)$. This implies $-rd + sb = 1$. As r is even, d must be even. Otherwise, it contradicts our hypothesis. With similar argument, we obtain the elements of Γ^2 which are $\begin{pmatrix} -r & 2b \\ -s & d \end{pmatrix}$ and $\begin{pmatrix} -r & -b \\ -s & -2d \end{pmatrix}$ for (ii) and (iii) respectively. ∎

3.3. Theorem. *All suborbital graphs for Γ^2 on $\hat{\mathbb{Q}}$ are paired.*

Proof. Because of the transitivity of Γ^2, it is sufficient to show that $G\left(\infty, \frac{u}{N}\right) \neq G\left(\frac{u}{N}, \infty\right)$. It means that there is no $T \in \Gamma^2$ which sends the pair $\left(\infty, \frac{u}{N}\right)$ to the pair $\left(\frac{u}{N}, \infty\right)$. On the contrary, assume that $T(\infty) = \frac{u}{N}$ and $T\left(\frac{u}{N}\right) = \infty$. By comparing the determinants, we have that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} = \begin{pmatrix} -u & 1 \\ -N & 0 \end{pmatrix} \text{ or } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} = \begin{pmatrix} u & -1 \\ N & 0 \end{pmatrix}.$$

In the first case, we obtain $a = -u$, $c = -N$, $au + bN = 1$ and $cu + dN = 0$. That is, $d = u$ and $u^2 = -1 + bN$. Taking $T = \begin{pmatrix} -u & b \\ -N & u \end{pmatrix}$ we see that the only case for T to be an element of Γ^2 is that N and b must be even. Since $u^2 = -1 + bN$, then $u^2 \equiv -1\ (mod\ bN)$. As N and b are even, $u^2 \equiv -1\ (mod\ 4)$ which has no solution. For the second case, taking $T = \begin{pmatrix} u & b \\ N & -u \end{pmatrix}$, similar contradiction is obtained.

3.4. Corollary. *There are no self-paired suborbital graphs for Γ^2 on $\hat{\mathbb{Q}}$.*

In section 2 we introduced, for each integer N, a Γ^2-invariant equivalence relation $\underset{N}{\approx}$ on $\hat{\mathbb{Q}}$, with $\frac{r}{s} \underset{N}{\approx} \frac{x}{y}$ if and only if $ry - sx \equiv 0\ (mod\ N)$. If $\frac{r}{s} \to \frac{x}{y}$ in $G_{u,N}$, then Theorem 3.2 implies that $ry - sx = \pm N$, so $\frac{r}{s} \underset{N}{\approx} \frac{x}{y}$. Thus, each connected component of $G_{u,N}$ lies in a single block for $\underset{N}{\approx}$, of which there are $\Psi(N)$, so we have:

3.5. Corollary. *The graph $G_{u,N}$ is a disjoint union of $\Psi(N)$ subgraphs.*

3.2. *Subgraph* $F_{u,N}$. We represent the subgraph of $G_{u,N}$ whose vertices form the block $[\infty] = \{x/y \in \hat{\mathbb{Q}} \mid y \equiv 0\ (mod\ N)\}$ by $F_{u,N}$.

3.6. Corollary. *The graph $G_{u,N}$ consists of $\Psi(N)$ disjoint copies of $F_{u,N}$.*

Proof. The vertices of each subgraph form a single block with respect to the Γ^2-invariant equivalence relation $\underset{N}{\approx}$ defined by $ry - sx \equiv 0\ (mod\ N)$. Therefore, if $x_1 \to x_2$ is an edge in the subgraph $F_{u,N}$, $T(x_1) \to T(x_2)$ is also an edge in any other subgraph with $T \in \Gamma^2$ because of the transitivity of Γ^2 on $\hat{\mathbb{Q}}$.

Now, Theorem 3.2 immediately gives:

3.7. Theorem. $\frac{r}{s} \to \frac{x}{y} \in F_{u,N}$ *if and only if*

(i) *If r is even, then $x \equiv \pm ur\ (mod\ N)$, $y \equiv \pm us\ (mod\ N)$, $y \not\equiv \pm us\ (mod\ 2N)$ and $ry - sx = \mp N$.*
(ii) *If s is even, then $x \equiv \pm ur\ (mod\ 2N)$, $y \equiv \pm us\ (mod\ N)$ and $ry - sx = \mp N$.*
(iii) *If r and s are odd, then $x \equiv \pm ur\ (mod\ N)$, $y \equiv \pm us\ (mod\ 2N)$ and $ry - sx = \mp N$.*

3.8. Theorem. *$\Gamma_0^2(N)$ permutes the vertices and the edges of $F_{u,N}$ transitively.*

Proof. Let v, w be any vertices of $F_{u,N}$. Since Γ^2 acts on $\hat{\mathbb{Q}}$ transitively, there exist $T \in \Gamma^2$ such that $T(v) = w$. Taking $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $v = \frac{k_1}{l_1 N}$ and $w = \frac{k_2}{l_2 N}$ we see that $N|c$. It means that $\Gamma_0^2(N)$ permutes the vertices of $F_{u,N}$.

Let $\frac{x_1}{y_1 N} \xrightarrow{e_1} b_1$ and $\frac{x_2}{y_2 N} \xrightarrow{e_2} b_2$ be any edges of $F_{u,N}$. We can give following diagram:

$$\begin{array}{lll} \left(\frac{1}{0}, \frac{u}{N}\right) & \xrightarrow{T_2} & \left(\frac{x_2}{y_2 N}, b_2\right) \\ \downarrow T_1 & \nearrow T_2 \circ T_1^{-1} & \\ \left(\frac{x_1}{y_1 N}, b_1\right) & & \end{array}$$

By this representation, we have $T_1 = \begin{pmatrix} x_1 & * \\ y_1 N & * \end{pmatrix}$ and $T_2 = \begin{pmatrix} x_2 & * \\ y_2 N & * \end{pmatrix}$. Since $T_2 \circ T_1^{-1}$ has the form $\begin{pmatrix} * & * \\ kN & * \end{pmatrix}$ for some integer k, then $T := T_2 \circ T_1^{-1} \in \Gamma_0^2(N)$. It is clear that $T\left(\frac{x_1}{y_1 N}\right) = \frac{x_2}{y_2 N}$ and $T(b_1) = b_2$. Since T is an element of a group of hyperbolic isometries of $\mathbb{H}$, geodesics are sent to geodesics under its action. So, T transform the edges e_1 to e_2. Consequently, $\Gamma_0^2(N)$ permutes the edges of $F_{u,N}$.

3.9. Lemma. *There is an isomorphism $F_{u,N} \longrightarrow F_{-u,N}$ given by $v \longrightarrow -v$.*

Proof. It is clear that $v \longrightarrow -v$ is one-to-one and onto. Let's show that the structure is preserved. Here, it means that if $a \to b \in F_{u,N}$, then $-a \to -b \in F_{-u,N}$. Suppose that $\frac{r}{s} \to \frac{x}{y} \in F_{u,N}$ and r is even. By Theorem 3.7(i), taking $\frac{r}{s} < \frac{x}{y}$, we have that $x \equiv -ur \ (mod\ N)$, $y \equiv -us \ (mod\ N)$, $y \not\equiv -us \ (mod\ 2N)$ and $ry - sx = -N$. Since $\frac{r}{s} < \frac{x}{y}$, then $\frac{-r}{s} > \frac{-x}{y}$. Taking $-x \equiv (-u)(-r) \ (mod\ N)$, $y \equiv (-u)s \ (mod\ N)$, $y \not\equiv (-u)s \ (mod\ 2N)$ and $-ry + sx = N$, we have that $\frac{-r}{s} \to \frac{-x}{y} \in F_{u,N}$. For other conditions, the rest are similar.

3.10. Lemma. *If $M|N$, then there is a homomorphism $F_{u,N} \longrightarrow F_{-u,M}$ given by $v \longrightarrow -Nv/M$.*

Proof. We suppose that $\frac{r}{sN}$, $\frac{x}{yN}$ are adjacent vertices in $F_{u,N}$ and $\frac{r}{sN} < \frac{x}{yN}$ and that is written as $\frac{r}{sN} \xrightarrow{<} \frac{x}{yN} \in F_{u,N}$. If r is even, then $x \equiv -ur \ (mod\ N)$, $yN \equiv -usN \ (mod\ N)$, $yN \not\equiv -us \ (mod\ 2N)$ and $ry - sx = -1$. Since $M|N$, $x \equiv -ur \ (mod\ M)$, $yM \equiv -usM \ (mod\ M)$, $yM \not\equiv -us \ (mod\ 2M)$. $ry - sx = -1$ is also true for M. For other conditions, the rest are similar.

3.11. Theorem. *$F_{u,N}$ contains directed triangles if and only if $u^2 \mp u + 1 \equiv 0 \ (mod\ N)$.*

Proof. Suppose that $F_{u,N}$ contains a directed triangle. Because of the transitive action, the form of directed triangle can be taken as $\infty \to \frac{u}{N} \xrightarrow{<} \frac{r}{N} \to \infty$ for some integer r. First, let u be even. From the second edge, we have $u - r = -1$ and $r \equiv -u^2 \ (mod\ N)$ by Theorem 3.2. So, we obtain $u^2 + u + 1 \equiv 0 \ (mod\ N)$. Similarly, if $\frac{u}{N} \xrightarrow{>} \frac{r}{N}$, then we see that $u^2 - u + 1 \equiv 0 \ (mod\ N)$. Now, N is even. By applying Theorem 3.2 to the second edge, we have $u - r = -1$ and $r \equiv -u^2 \ (mod\ 2N)$, giving $u^2 + u + 1 \equiv 0 \ (mod\ 2N)$. It is impossible, because there is no solution for this equivalence. Finally, suppose that u, N are odd. Again, from the second edge, we have $u - r = -1$ and $r \equiv -u^2 \ (mod\ N)$, giving $u^2 + u + 1 \equiv 0 \ (mod\ N)$. If $\frac{u}{N} \xrightarrow{>} \frac{r}{N}$, it would be $u^2 - u + 1 \equiv 0 \ (mod\ N)$. Combining all of the equivalences, we obtain $u^2 \mp u + 1 \equiv 0 \ (mod\ N)$.

Conversely, if $u^2 \mp u + 1 \equiv 0 \ (mod\ N)$, we see that either $u + 1 \equiv -u^2 \ (mod\ N)$ or $-u + 1 \equiv -u^2 \ (mod\ N)$. Theorem 3.2. implies that there is an edge $\frac{u}{N} \to \frac{u+1}{N}$ with

$\frac{u}{N} < \frac{u+1}{N}$ in $F_{u,N}$ or $\frac{u}{N} \to \frac{u-1}{N}$ with $\frac{u}{N} > \frac{u+1}{N}$ in $F_{u,N}$. Consequently, there is a directed triangle $\infty \to \frac{u}{N} \to \frac{u\pm 1}{N} \to \infty$ in $F_{u,N}$. ∎

Let us give some examples. For u, N-odd, $\frac{1}{0} \to \frac{3}{13} \to \frac{4}{13} \to \frac{1}{0}$ or $\frac{1}{13} \to \frac{10}{9\cdot 13} \to \frac{9}{8\cdot 13} \to \frac{1}{13}$ is a directed triangle in $F_{3,13}$. For u-even and N-odd, $\frac{1}{0} \to \frac{2}{7} \to \frac{3}{7} \to \frac{1}{0}$ or $\frac{1}{7} \to \frac{5}{4\cdot 7} \to \frac{4}{3\cdot 7} \to \frac{1}{7}$ is a directed triangle in $F_{2,7}$. For N-even, we know that there is no triangle.

Observation. We know that there is no triangle in $F_{u,2N_0}$ for N-even by Theorem 3.11. Because of the relationships between elliptic elements with circuits, our expectation is that there is no elliptic element of order 3 of the form $\begin{pmatrix} u & 2b \\ 2N_0 & d \end{pmatrix} \in \Gamma^2$. Indeed, being an elliptic element of order 3, it is well-known that $u + d = \pm 1$. Taking determinant of $\begin{pmatrix} 1-d & 2b \\ 2N_0 & d \end{pmatrix}$, we have $d - d^2 - 4bN_0 = 1$. It is clear that there is no solution for $d - d^2 \equiv 1 \ (mod\ 4)$. ∎

On the other hand, we know that the suborbital graph for modular group is a forest if and only if it contains no triangles [2]. Using this fact, we can give the following result;

3.12. Corollary. *The graph $G_{u,N}$ is a forest if and only if $u^2 \pm u + 1 \not\equiv 0 \ (mod\ N)$.*

3.3. *Connectedness*. In this last section, we examine the connectedness of $F_{u,N}$.

3.13. Definition. A subgraph K of $G_{u,N}$ is called connected if any pair of its vertices can be joined by a path in K.

3.14. Theorem. *The subgraphs $F_{0,1}$ and $F_{1,1}$ are connected.*

Proof. Here, to see the situation better, we write the edge conditions for $F_{0,1}$ and $F_{1,1}$ by Theorem 3.2 explicitly.

Case $F_{0,1}$: $\frac{r}{s} \to \frac{x}{y} \in F_{0,1}$ if and only if

(i) If r-even, then y-odd and $ry - sx = \mp 1$.
(ii) If s-even, then x-even and $ry - sx = \mp 1$.
(iii) If r, s-odd, then y-even and $ry - sx = \mp 1$.

We will show that each vertex $\frac{a}{b}$ of $F_{0,1}$ can be joined to ∞ by a path in $F_{0,1}$. It is clear for $b = 1$. Since $(a, b) = 1$, we can write the equation $ad - bc = -1$ by Bezout's identity. For this pair (c, d) satisfying the equation we claim that $\frac{a}{b}$ can be joined with $\frac{c}{d}$ by above edge condition.

Subcase1. Suppose a-even. By the equation we have that b, c must be odd and there are two possibilities for d. If d-odd, then $\frac{a}{b} \xrightarrow{i} \frac{c}{d}$ (means that we have $\frac{c}{d} \to \frac{a}{b}$ by (i)). If d-even, then $\frac{c}{d} \xrightarrow{ii} \frac{a}{b}$.

Subcase2. Let b-even. By the equation we have that a, d must be odd and there are two possibilities for c. If c-odd, then $\frac{c}{d} \xrightarrow{iii} \frac{a}{b}$. If d-even, then $\frac{a}{b} \xrightarrow{ii} \frac{c}{d}$.

Subcase3. Assume that a-odd and b-odd. By the equation it is impossible that c, d are odd or even at once, so there are two possibilities. If c-odd and d-even, then $\frac{a}{b} \xrightarrow{iii} \frac{c}{d}$. If c-even and d-odd, then $\frac{c}{d} \xrightarrow{i} \frac{a}{b}$.

Consequently $F_{0,1}$ is connected.

Case $F_{1,1}$: $\frac{r}{s} \to \frac{x}{y} \in F_{1,1}$ if and only if

(i) If r-even, then y-even and $ry - sx = \mp 1$.

(ii) If s-even, then x-odd and $ry - sx = \mp 1$.
(iii) If r, s-odd, then y-odd and $ry - sx = \mp 1$.

Taking a vertex $\frac{a}{b}$ in $F_{1,1}$, there exists the equation $ad - bc = -1$ by Bezout's identity. We shall show that $\frac{a}{b}$ is adjacent to vertex $\frac{c}{d}$ in $F_{1,1}$.

Subcase1. Suppose a-even. By the equation we have that b, c must be odd and there are two possibilities for d. If d-odd, then $\frac{c}{d} \xrightarrow{iii} \frac{a}{b}$. If d-even, then $\frac{a}{b} \xrightarrow{i} \frac{c}{d}$.

Subcase2. Let b-even. By the equation we have that a, d must be odd and there are two possibilities for c. If c-odd, then $\frac{a}{b} \xrightarrow{ii} \frac{c}{d}$. If c-even, then $\frac{c}{d} \xrightarrow{i} \frac{a}{b}$.

Subcase3. Assume that a-odd and b-odd. By the equation it is impossible that c, d are odd or even at once, so there are two possibilities. If c-odd and d-even, then $\frac{c}{d} \xrightarrow{ii} \frac{a}{b}$. If c-even and d-odd, then $\frac{a}{b} \xrightarrow{iii} \frac{c}{d}$.

Consequently, $F_{1,1}$ is connected.

3.15. Theorem. *The subgraphs $F_{1,2}$ and $F_{3,2}$ are connected.*

Proof. We shall show that each vertex $v = \frac{a}{2b}$ $(b \geq 1)$ of $F_{1,2}$ is joined to ∞ by a path in $F_{1,2}$. Since the pattern is periodic with period 2, we can show by induction on b. If $b = 1$, then $v = \frac{a}{2}$ can be joined with ∞. If $a = 1$, it is clear that $\frac{1}{0} \to \frac{1}{2}$. If $a = 1$, then $\frac{3}{2} \to \frac{1}{0}$ because $1 \equiv -3$ $(mod\ 4)$ and $3 \cdot 0 - 2 \cdot 1 = -2$. If $a = 5$, then $\frac{1}{0} \to \frac{5}{2}$. The same holds for the rest periodically. So we can assume that $b \geq 2$.

To complete the proof, we show that v is adjacent to a vertex $w = \frac{a}{2b_1}$ with $b_1 < b$. It means that, w is connected by a path to ∞, and hence so is v. As $(a, b) = 1$, there exist integers c, d such that $ad - bc = 1$. For some $k \in \mathbb{Z}$, replacing c and d by $c + ka$ and $d + kb$, we can suppose $0 < d < b$.

(i) If c is odd, then $w = \frac{c}{2d}$ can be joined with $\frac{a}{2b}$. Indeed, $\frac{a}{2b} \xrightarrow{>} \frac{c}{2d}$ gives that $a \cdot 2d - c \cdot 2b = 2$ and $c \equiv a$ $(mod\ 4)$. If $c \not\equiv a$ $(mod\ 4)$, taking $c \equiv -a$ $(mod\ 4)$ we obtain $\frac{a}{2b} \xleftarrow{<} \frac{c}{2d}$ by $2bc - 2ad = -2$. Hence, if c is odd, $\frac{a}{2b}$ is adjacent to $\frac{c}{2d}$ in $F_{1,2}$.

(ii) If c is even, then $a - c$ is odd. As $0 < b - d < b$, we can take $w = \frac{a-c}{2(b-d)}$, adjacent to $\frac{a}{2b}$ because $2(bc - cd) = -2$. Here, if $2a - c \not\equiv 0$ $(mod\ 4)$, then we have that $a - c \equiv a$ $(mod\ 4)$ and $2(ad - bc) = 2$.

Consequently, $F_{1,2}$ is connected. By the isomorphism $\underset{v}{F_{1,2}} \underset{\to}{\longrightarrow} \underset{-v}{F_{-1,2}} = F_{3,2}$, $F_{3,2}$ is also connected.

3.16. Corollary. *All graphs $F_{u,2}$ are connected.*

3.17. Corollary. *The graph $G_{u,2}$ has $2 \cdot \psi(2) = 6$ connected components. Its blocks are $[\infty], [1], [0]$. The connected components of $[\infty]$ are $F_{1,2}$ and $F_{3,2}$.*

3.18. Theorem. *The subgraphs $F_{1,3}$, $F_{2,3}$, $F_{4,3}$ and $F_{5,3}$ are not connected.*

Proof. It is sufficient to study with $F_{1,3}$ and $F_{2,3}$. Because there is an isomorphism from $F_{1,3}(F_{2,3})$ to $F_{5,3}(F_{4,3})$ respectively.

Case $F_{1,3}$: If $F_{1,3}$ is connected, then each vertex $v = \frac{a}{3b}$ would be joined to ∞. We shall show that no vertices of $F_{1,3}$ where $1 < v < 2$ are adjacent to ∞. Further, we assert that there is no such a vertex v adjacent to vertices outside this interval. Of course, there is at least some vertex of $F_{1,3}$ in this strip. Suppose $\frac{2}{3} \leq \frac{c}{3d} < 1 < \frac{a}{3b} < 2$. Then we have $\frac{c}{d} < 3 < \frac{a}{b}$. This is impossible because $cd - ad = -1$. Similarly, if $1 < \frac{k}{3l} < \frac{f}{3e} \leq \frac{7}{3}$, then $\frac{k}{l} < 4 < \frac{f}{e}$ contradicts $ke - lf = -1$. It means that no vertices of $F_{1,3}$ between 1 and 2 are adjacent to ∞ and that $F_{1,3}$ is not connected.

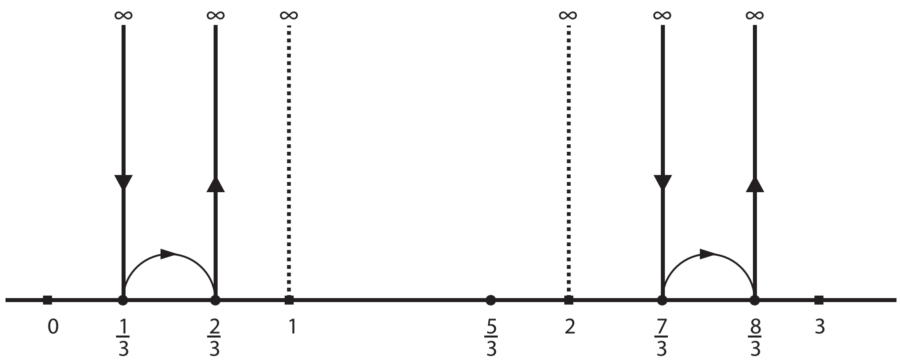


Figure 1. The subgraph $F_{1,3}$

Case $F_{2,3}$: As above, let's show that no vertices of $F_{2,3}$ between $\frac{3}{2}$ and 2 are adjacent to vertices outside this interval. Suppose that $1 \le \frac{x}{3y} < \frac{3}{2} < \frac{a}{3b} < 2$ and $\frac{x}{3y} \overset{<}{\longrightarrow} \frac{a}{3b} \in F_{2,3}$. Then we have that $\frac{x}{y} < \frac{9}{2} < \frac{a}{b}$ and $xb - ay = -1$. By [7], we obtain that $x = 4$, $y = 1$, $a = 5$ and $b = 1$. But $\frac{4}{3} \to \frac{5}{3}$ is not in $F_{2,3}$. If $\frac{2}{3} < \frac{x}{3y} < 2 < \frac{a}{3b} < \frac{8}{3}$ and $\frac{x}{3y} \overset{<}{\longrightarrow} \frac{a}{3b} \in F_{2,3}$, then we would have $\frac{x}{y} < 6 < \frac{a}{b}$ and $xb - ay = -1$. It is impossible because of well-known Farey sequence. Consequently, $F_{2,3}$ is not connected.

3.19. Corollary. *All graphs $F_{u,3}$ are not connected.*

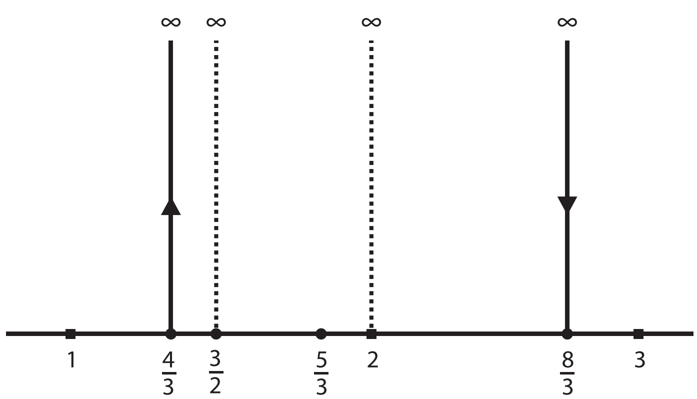


Figure 2. The subgraph $F_{2,3}$

3.20. Theorem. *The subgraphs $F_{1,4}$, $F_{3,4}$, $F_{5,4}$ and $F_{7,4}$ are not connected.*

Proof. As remarked in the proof of Theorem 3.18, it is sufficient to study with $F_{1,4}$ and $F_{3,4}$.

Case $F_{1,4}$: We will show that no vertices in $F_{1,3}$ between $\frac{1}{2}$ and 1 are adjacent to vertices outside this interval. Suppose $\frac{1}{4} \le \frac{a}{4b} < \frac{1}{2} < \frac{x}{4y} < 1$. Then we have $\frac{a}{b} < 2 < \frac{x}{y}$. This is impossible because $ay - bx = -1$. Similarly, if $\frac{a}{4b} < 1 < \frac{x}{4y} \leq \frac{7}{4}$, then $\frac{a}{b} < 4 < \frac{x}{y} < 7$ is a contradiction. So $F_{1,4}$ is not connected.

Case $F_{3,4}$: As above, it is seen that no vertices of $F_{3,4}$ between 1 and 2 are adjacent to vertices outside this interval. Consequently, $F_{3,4}$ is not connected.

3.21. Theorem. *The subgraph $F_{u,N}$ is connected if and only if $N \leq 2$.*

Proof. If $F_{u,N}$ is connected, we know that $N \leq 4$ by [7]. For $N = 3, 4$, we proved that $F_{u,N}$ is not connected by Theorem 3.18 and 3.20. Conversely, if $N \leq 2$, the result immediately follows from Theorem 3.14 and 3.15. ∎

References

[1] Akbaş, M. and Başkan, T. *Suborbital graphs for the normalizer of* $\Gamma_0(N)$, Tr. J. of Mathematics, **20**, 379-387, 1996.

[2] Akbaş, M. *On suborbital graphs for the modular group*, Bull. London Math. Soc., **33**, 647-652, 2001.

[3] Beşenk, M. et al. *Circuit lengths of graphs for the Picard group*, J. Inequal. Appl., 2013:**106**, 8 pp., 2013.

[4] Bigg, N.L. and White, A.T. *Permutation groups and combinatorial structures*, London Mathematical Society Lecture Note Series, **33**, CUP, Cambridge, 140 pp.,1979.

[5] Dixon, J. D. and Mortimer, B. *Permutation Groups*, Graduate Texts in Mathematics 163, Springer-Verlag, 1996.

[6] Güler, B.Ö. et al. *Elliptic elements and circuits in suborbital graphs*, Hacet. J. Math. Stat., **40** (2), 203-210, 2011.

[7] Jones, G.A., Singerman, D. and Wicks, K. *The modular group and generalized Farey graphs*, London Mathematical Society Lecture Note Series, **160**, CUP, Cambridge, 316-338, 1996.

[8] Jones, G.A. and Jones, J.M. *Elementary Number Theory*, Springer Undergraduate Mathematics Series, Springer-Verlag, 1998.

[9] Kader, S., Güler, B. Ö. and Değer, A. H. *Suborbital graphs for a special subgroup of the normalizer*, IJST. Trans A., **34** (A4), 305-312, 2010.

[10] Kader, S. and Güler, B. Ö. *On suborbital graphs for the extended Modular group* $\hat{\Gamma}$, Graphs and Combinatorics, **29**, no. 6, 1813-1825, 2013.

[11] Keskin, R. *Suborbital graphs for the normalizer of* $\Gamma_0(m)$, European J. Combin., **27**, no. 2, 193-206, 2006.

[12] Neumann, P.M. *Finite Permutation Groups, Edge-Coloured Graphs and Matrices*, Topics in Group Theory and Computation, Ed. M. P. J. Curran, Academic Press, 1977.

[13] Newman, M. *The Structure of some subgroups of the modular group*, Illinois J. Math., **6**, 480-487, 1962.

[14] Newman, M. *Free subgroups and normal subgroups of the modular group*, Illinois J. Math., **8**, 262-265, 1964.

[15] Newman, M. *Classification of normal subgroups of the modular group*, Transactions of the American Math. Society, Vol.126, **2**, 267-277, 1967.

[16] Rankin, R. A. *Modular Forms and Functions*, Cambridge University Press, 2008.

[17] Schoeneberg, B. *Elliptic modular functions*, Springer Verlag, 1974.

[18] Sims, C.C. *Graphs and finite permutation groups*, Math. Z., **95**, 76-86, 1967.

[19] Singerman, D. *Subgroups of Fuchsian groups and finite permutation groups*, Bull. London Math. Soc., **2**, 319-323, 1970.

[20] Tsuzuku, T. *Finite Groups and Finite Geometries*, Cambridge University Press, Cambridge, 1982.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1045 – 1055

On some new converses of the Jensen and the Lah-Ribarič operator inequality

Rozarija Jakšić* , Mario Krnić† and Josip Pečarić‡

Abstract

In this paper we study some new converses of the Jensen and the Lah-Ribarič operator inequality regarding convex functions. First we give two series of converses in a general setting. The general results are then applied to quasi-arithmetic operator means with a particular emphasis to power operator means. The obtained results are also compared with some related results, known from the literature.

Keywords: Jensen operator inequality, Lah-Ribarič operator inequality, convexity, operator convexity, converse

2000 AMS Classification: Primary 47A63, 47A64.

Received 24/09/2013 : Accepted 21/01/2014 Doi : 10.15672/HJMS.2015449089

1. Introduction

The Jensen inequality is one of the most important inequalities in modern mathematics since it implies the whole series of other classical inequalities (e.g. those by Hölder, Minkowski, Beckenbach-Dresher, Young, the arithmetic-geometric mean inequality etc.). Applications of this inequality in various branches of mathematics, especially in mathematical analysis and statistics, have certainly contributed to its importance. During decades, the Jensen inequality was extensively studied by some famous authors and was generalized in numerous directions. For a comprehensive inspection of the Jensen inequality including history, proofs and diverse applications, the reader is referred to [10].

In this paper we refer to a quite general operator form of the Jensen inequality. In order to present such result, we first introduce the appropriate setting.

*University of Zagreb, Faculty of Textile Technology, Prilaz baruna Filipovića 28a, 10000 Zagreb, CROATIA
Email: jaksic.rozarija@gmail.com

†University of Zagreb, Faculty of Electrical Engineering and Computing, Unska 3, 10000 Zagreb, CROATIA
Email:mario.krnic@fer.hr

‡University of Zagreb, Faculty of Textile Technology, Prilaz baruna Filipovića 28a, 10000 Zagreb, CROATIA
Email:pecaric@element.hr

Let T be a locally compact Hausdorff space and let $\mathcal{A}$ be a C^*-algebra. We say that a field $(x_t)_{t\in T}$ of elements in $\mathcal{A}$ is continuous if the function $t \to x_t$ is norm continuous on T. Additionally, if T is equipped with a Radon measure μ and the function $t \to ||x_t||$ is integrable, then, the so-called Bochner integral $\int_T x_t d\mu(t)$ can be formed. More precisely, the Bochner integral is the unique element in $\mathcal{A}$ such that the relation

$$\varphi\left(\int_T x_t d\mu(t)\right) = \int_T \varphi\left(x_t\right) d\mu(t)$$

holds for every linear functional φ in the norm dual $\mathcal{A}^*$ (see [5]).

Assume furthermore that there is a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$. Such field is said to be continuous if the function $t \to \phi_t(x)$ is continuous for every $x \in \mathcal{A}$. If the C^*-algebras are unital and the field $t \to \phi_t(\mathbf{1})$ is integrable with integral $\mathbf{1}$, we say that $(\phi_t)_{t\in T}$ is unital. We assume that such field is continuous.

If $f : I \to \mathbb{R}$ is operator convex function, where I is a real interval of any type, and $(\phi_t)_{t\in T}$ is a unital field, then the Jensen operator inequality (see Hansen *et.al.*, [6]) asserts that

$$f\left(\int_T \phi_t(x_t) d\mu(t)\right) \leq \int_T \phi_t(f(x_t)) d\mu(t) \tag{1.1}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in I. If $f : I \to \mathbb{R}$ is operator concave function, then the sign of inequality in (1.1) is reversed.

Observe that the above inequality refers to an operator convex function. Recall that a continuous function $f : I \to \mathbb{R}$ is operator convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for each $\lambda \in [0, 1]$ and every pair of self-adjoint operators x and y (acting) on an infinite dimensional Hilbert space $\mathcal{H}$ with spectra in I (the ordering is defined by setting $x \leq y$ if $y - x$ is positive semi-definite).

In the same paper, Hansen *et.al.* obtained the following inequality which holds for an usual convex function $f : [m, M] \to \mathbb{R}$ (see [6], proof of Theorem 2):

$$\int_T \phi_t(f(x_t)) d\mu(t) \leq \alpha_f \int_T \phi_t(x_t) d\mu(t) + \beta_f \mathbf{1}. \tag{1.2}$$

In this matter, the usual notation is used:

$$\alpha_f = \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad \beta_f = \frac{Mf(m) - mf(M)}{M - m}.$$

Inequality (1.2) will be referred to as the Lah-Ribarič operator inequality. Observe that the operator inequality (1.2) is established by applying the functional calculus to the well-known inequality

$$f(t) \leq \alpha_f t + \beta_t, \tag{1.3}$$

which holds for every convex function on the interval $[m, M]$. Recall that $l(t) = \alpha_f t + \beta_t$ is the linear function limiting convex function $f(t)$ on interval $[m, M]$ from the above.

The main objective of this paper is to derive converses of the above inequalities (1.1) and (1.2). Although inequality (1.1) holds for an operator convex function, both series of converses will be established for convex functions in the classical real sense.

The paper is organized in the following way: after this Introduction, in Section 2 we derive our main results, that is, we obtain two series of converses that correspond to the Jensen and the Lah-Ribarič operator inequality. Further, in Sections 3 and 4 general results are then applied to quasi-arithmetic operator means, with a particular emphasis to power operator means. In such a way, we obtain converse inequalities for quasi-arithmetic and power operator means.

The techniques that will be used in the proofs are mainly based on the classical real and functional calculus, especially on the well-known monotonicity principle for self-adjoint elements of a C^*-algebra $\mathcal{A}$: If $x \in \mathcal{A}$ with a spectra $\mathrm{Sp}(x)$, then

$$f(t) \geq g(t),\ t \in \mathrm{Sp}(x) \quad \Longrightarrow \quad f(x) \geq g(x), \tag{1.4}$$

where f and g are real valued continuous functions.

2. Basic results

In this section we give our main results, that is, converses of the Jensen and the Lah-Ribarič operator inequality in a general setting presented in the Introduction. As we have already discussed, the results that follow refer to an usual convex function. Although regarding different inequalities, it appears that these two series of converses are closely connected.

First we give a series of converses for the Jensen operator inequality. It should be noticed here that the following theorem in the classical real case was proved by Dragomir in the recent paper [2]. In fact, such series of scalar inequalities will be exploited in establishing the corresponding operator form.

2.1. Theorem. *Let $f : I \to \mathbb{R}$ be a continuous convex function, and let $m, M \in \mathbb{R}$, $m < M$, be such that interval $[m, M]$ belongs to the interior of interval I. Further, suppose $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the series of inequalities*

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right)\\
&\leq \frac{f'_-(M) - f'_+(m)}{M - m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right)\\
&\leq \frac{1}{4}(M - m)(f'_-(M) - f'_+(m))\mathbf{1}
\end{aligned} \tag{2.1}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in $[m, M]$. If f is concave on I, then the signs of inequalities in (2.1) *are reversed.*

Proof. Taking into account the operator version of the Lah-Ribarič inequality (1.2), it follows that

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right)\\
&\leq \alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f\mathbf{1} - f\left(\int_T \phi_t(x_t)d\mu(t)\right).
\end{aligned} \tag{2.2}$$

On the other hand, regarding convexity of f, we have the so-called gradient inequality,

$$f(t) - f(M) \geq f'_-(M)(t - M),$$

which holds for every $t \in [m, M]$, that is,

$$(t - m)f(t) - (t - m)f(M) \geq f'_-(M)(t - M)(t - m), \quad t \in [m, M],$$

after multiplying with $t - m$. In the same way, it follows that

$$(M - t)f(t) - (M - t)f(m) \geq f'_+(m)(M - t)(t - m), \quad t \in [m, M].$$

Now, adding the above two inequalities, and then, dividing by $m-M$, we have

$$\alpha_f t+\beta_f-f(t)\le\frac{f'_-(M)-f'_+(m)}{M-m}(M-t)(t-m). \tag{2.3}$$

Moreover, taking into account the arithmetic-geometric mean inequality, the following series of inequalities holds for all $t\in[m,M]$ (see also [2]):

$$\begin{aligned}\alpha_f t+\beta_f-f(t)&\le\frac{f'_-(M)-f'_+(m)}{M-m}(M-t)(t-m)\\&\le\frac{1}{4}(M-m)(f'_-(M)-f'_+(m)).\end{aligned} \tag{2.4}$$

Now, since $m\mathbf{1}\le x_t\le M\mathbf{1}$ for every $t\in T$, it follows that $m\phi_t(\mathbf{1})\le\phi_t(x_t)\le M\phi_t(\mathbf{1})$, that is, $m\mathbf{1}\le\int_T\phi_t(x_t)d\mu(t)\le M\mathbf{1}$. Hence, applying the functional calculus to the above series of inequalities, that is, setting $\int_T\phi_t(x_t)d\mu(t)$ instead of t, we have

$$\begin{aligned}&\alpha_f\int_T\phi_t(x_t)d\mu(t)+\beta_f\mathbf{1}-f\left(\int_T\phi_t(x_t)d\mu(t)\right)\\&\le\frac{f'_-(M)-f'_+(m)}{M-m}\left(M\mathbf{1}-\int_T\phi_t(x_t)d\mu(t)\right)\left(\int_T\phi_t(x_t)d\mu(t)-m\mathbf{1}\right)\\&\le\frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1}.\end{aligned} \tag{2.5}$$

Finally, comparing (2.2) and (2.5), we obtain (2.1), as claimed. □

2.2. Remark. Observe that in the statement of Theorem 2.1 the interval $[m,M]$ belongs to the interior of the interval I. This condition assures finiteness of the one-sided derivatives in (2.1). Without this assumption these derivatives might be infinite.

2.3. Remark. It should be noticed here that the first expression in the series of inequalities (2.1), that is, the element $\int_T\phi_t(f(x_t))d\mu(t)-f\left(\int_T\phi_t(x_t)d\mu(t)\right)$ is not positive in general. This element is positive if f is in addition operator convex function, due to the Jensen operator inequality (1.1).

The following result represents converses of the Lah-Ribarič operator inequality (1.2):

2.4. Theorem. *Suppose $f:I\to\mathbb{R}$ is a continuous convex function, and $m,M\in\mathbb{R}$, $m<M$, are such that interval $[m,M]$ belongs to the interior of interval I. Further, let $(\phi_t)_{t\in T}$ be a unital field of positive linear mappings $\phi_t:\mathcal{A}\to\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras, defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the series of inequalities*

$$\begin{aligned}0&\le\alpha_f\int_T\phi_t(x_t)d\mu(t)+\beta_f\mathbf{1}-\int_T\phi_t(f(x_t))d\mu(t)\\&\le\frac{f'_-(M)-f'_+(m)}{M-m}\int_T\phi_t\left([M\mathbf{1}-x_t][x_t-m\mathbf{1}]\right)d\mu(t)\\&\le\frac{f'_-(M)-f'_+(m)}{M-m}\left(M\mathbf{1}-\int_T\phi_t(x_t)d\mu(t)\right)\left(\int_T\phi_t(x_t)d\mu(t)-m\mathbf{1}\right)\\&\le\frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1}\end{aligned} \tag{2.6}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in $[m,M]$. If f is concave on I, then the signs of inequalities in (2.6) are reversed.

Proof. The first inequality in (2.6) holds by virtue of the Lah-Ribarič inequality (1.2). Further, starting from the scalar inequality (2.3), it follows that relation

$$\alpha_f x_t + \beta_f \mathbf{1} - f(x_t) \le \frac{f'_-(M) - f'_+(m)}{M-m}(M\mathbf{1} - x_t)(x_t - m\mathbf{1})$$

holds for every $t \in T$. Now, applying the positive linear mappings ϕ_t to the above relation, we obtain

$$\alpha_f \phi_t(x_t) + \beta_f \phi_t(\mathbf{1}) - \phi_t(f(x_t)) \le \frac{f'_-(M) - f'_+(m)}{M-m}\phi_t([M\mathbf{1} - x_t][x_t - m\mathbf{1}]),$$

while integrating yields

$$\begin{aligned} &\alpha_f \int_T \phi_t(x_t) d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t)) d\mu(t) \\ &\le \frac{f'_-(M) - f'_+(m)}{M-m} \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t), \end{aligned}$$

so that the second inequality in (2.6) holds.

Taking into account Theorem 2.1, it is enough to justify the third inequality sign in (2.6). To prove our assertion, we note that the function

$$h(t) = (M-t)(t-m) = -t^2 + (M+m)t - Mm, \quad t \in [m, M]$$

is operator concave (see e.g. [3]). Finally, applying the Jensen operator inequality (1.1) to the above function h, it follows that

$$\begin{aligned} &\int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t) \\ &\le \left(M\mathbf{1} - \int_T \phi_t(x_t) d\mu(t)\right)\left(\int_T \phi_t(x_t) d\mu(t) - m\mathbf{1}\right), \end{aligned}$$

and the proof is completed. □

Below, series of inequalities in (2.1) and (2.6) will be applied to quasi-arithmetic and power operator means.

3. Applications to quasi-arithmetic operator means

Roughly speaking, an arbitrary C^*-algebra is isomorphic to a C^*-algebra of bounded operators on a Hilbert space $\mathcal{H}$, denoted by $\mathfrak{B}(\mathcal{H})$. It is a consequence of the well-known Gelfand-Naimark theorem (see [4]). Hence, for the reader convenience, from now on, C^*-algebras will be regarded as algebras of bounded operators on a Hilbert space.

Now, for the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let $P\left[\mathfrak{B}(\mathcal{H}), \mathfrak{B}(\mathcal{K})\right]$ denotes the set of all fields $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{K})$, defined on a locally compact Hausdorff space T with a bounded Radon measure μ, which are unital.

A generalized quasi-arithmetic operator mean is defined by

$$M_\psi(x, \phi) = \psi^{-1}\left(\int_T \phi_t(\psi(x_t)) d\mu(t)\right), \tag{3.1}$$

where $(x_t)_{t\in T}$ is a continuous field of operators in $\mathfrak{B}(\mathcal{H})$ with spectra in $[m, M] \subseteq \mathbb{R}$, $(\phi_t)_{t\in T} \in P\left[\mathfrak{B}(\mathcal{H}), \mathfrak{B}(\mathcal{K})\right]$, and $\psi : [m, M] \to \mathbb{R}$ is a continuous strictly monotone function.

Throughout this section we also use the notation

$$\psi_m = \min\{\psi(m), \psi(M)\}, \quad \psi_M = \max\{\psi(m), \psi(M)\},$$

for a continuous strictly monotone function $\psi : [m, M] \to \mathbb{R}$.

In paper [9], Mićić *et.al.* investigated an order among the above quasi-arithmetic means. More precisely, they obtained that the inequality

$$M_{\psi}(x,\phi) \le M_{\chi}(x,\phi) \tag{3.2}$$

holds if one of the following two conditions is fulfilled:
(i) $\chi \circ \psi^{-1}$ is operator convex and χ^{-1} is operator monotone,
(ii) $\chi \circ \psi^{-1}$ is operator concave and $-\chi^{-1}$ is operator monotone.
On the other hand, if
(i') $\chi \circ \psi^{-1}$ is operator concave and χ^{-1} is operator monotone,
(ii') $\chi \circ \psi^{-1}$ is operator convex and $-\chi^{-1}$ is operator monotone,
then the sign of inequality in (3.2) is reversed.

Moreover, if ψ^{-1} is operator convex and χ^{-1} is operator concave, then

$$M_{\psi}(x,\phi) \le M_{1}(x,\phi) \le M_{\chi}(x,\phi)\,, \tag{3.3}$$

while for operator concave function ψ^{-1} and operator convex function χ^{-1} the signs of inequalities in series (3.3) are reversed.

As we see, the above relations (3.2) and (3.3), regarding order among quasi-arithmetic means, are derived via operator convexity and operator monotonicity. For more details about an order among operator means, the reader is referred to papers [7], [8] and [9].

As distinguished from the above relations (3.2) and (3.3), converses of quasi-arithmetic operator means are derived by virtue of the convexity and monotonicity in the classical real sense. The corresponding result can be carried out by virtue of our Theorem 2.1.

3.1. Theorem. *Let $\chi, \psi : I \to \mathbb{R}$ be continuous strictly monotone functions and let the interval $[m,M]$ belongs to the interior of interval I. Further, suppose that $\chi \circ \psi^{-1}$ is well-defined and convex on $\psi(I)$. If $(\phi_t)_{t\in T} \in P\left[\mathfrak{B}(\mathcal{H}), \mathfrak{B}(\mathcal{K})\right]$, where $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces and T is a locally compact Hausdorff space with a bounded Radon measure μ, then the series of inequalities*

$$\begin{aligned}
&\chi\left(M_{\chi}(x,\phi)\right) - \chi\left(M_{\psi}(x,\phi)\right)\\
&\le \frac{(\chi\circ\psi^{-1})'_{-}(\psi_M) - (\chi\circ\psi^{-1})'_{+}(\psi_m)}{\psi_M - \psi_m}\left[\psi_M \mathbf{1} - \psi\left(M_{\psi}(x,\phi)\right)\right]\\
&\quad \times \left[\psi\left(M_{\psi}(x,\phi)\right) - \psi_m \mathbf{1}\right]\\
&\le \frac{1}{4}\left(\psi_M - \psi_m\right)\left[(\chi\circ\psi^{-1})'_{-}(\psi_M) - (\chi\circ\psi^{-1})'_{+}(\psi_m)\right]\mathbf{1}
\end{aligned} \tag{3.4}$$

holds for every continuous field $(x_t)_{t\in T}$ of operators in $\mathfrak{B}(\mathcal{H})$ with spectra in $[m,M]$. Further, if $\chi\circ\psi^{-1}$ is concave on $\psi(I)$, then the signs of inequalities in (3.4) *are reversed.*

Proof. Since $\psi : I \to \mathbb{R}$ is a continuous strictly monotone function, it follows that $\psi_m \le \psi(t) \le \psi_M$, for all $t \in [m,M]$. Moreover, by virtue of the functional calculus, it follows that $\psi_m \mathbf{1} \le \psi(x_t) \le \psi_M \mathbf{1}$ for every $t \in T$. This means that the spectra of the field $(y_t)_{t\in T} = (\psi(x_t))_{t\in T}$ is contained in the interval $[\psi_m, \psi_M]$.

On the other hand, since the function $\chi\circ\psi^{-1}$ is obviously continuous on $\psi(I)$, the interval $[\psi_m, \psi_M]$ belongs to the interior of $\psi(I)$.

Finally, utilizing Theorem 2.1, that is, the series of inequalities in (2.1) with ψ_m, ψ_M, $\chi\circ\psi^{-1}$, $(y_t)_{t\in T}$ respectively instead of m, M, f, $(x_t)_{t\in T}$, and with definition (3.1) of quasi-arithmetic means, we obtain (3.4). □

3.2. Remark. Clearly, with assumptions as in Theorem 3.1, the operator $\chi\left(M_{\chi}(x,\phi)\right) - \chi\left(M_{\psi}(x,\phi)\right)$ is not positive in general. It is positive if the function $\chi\circ\psi^{-1}$ is operator convex on the corresponding interval. Moreover, applying operator convexity and monotonicity to suitable functions, one obtains relations (3.2) and (3.3). For more details the reader is referred to [9].

With the same setting as in the previous result, Theorem 2.4 can also be exploited in deriving converses of the Lah-Ribarič operator inequality involving quasi-arithmetic means.

3.3. Theorem. *Let $\chi, \psi : I \to \mathbb{R}$ be continuous strictly monotone functions and let the interval $[m, M]$ belongs to the interior of interval I. Further, suppose that $\chi \circ \psi^{-1}$ is well-defined and convex on $\psi(I)$. If $(\phi_t)_{t\in T} \in P\left[\mathfrak{B}(\mathcal{H}), \mathfrak{B}(\mathcal{K})\right]$, where $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces and T is a locally compact Hausdorff space with a bounded Radon measure μ, then the series of inequalities*

$$\begin{aligned}
0 &\le \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\psi\left(M_\psi(x,\phi)\right) + \frac{\psi(M)\chi(m)-\psi(m)\chi(M)}{\psi(M)-\psi(m)}\mathbf{1} - \chi\left(M_\chi(x,\phi)\right) \\
&\le \frac{(\chi\circ\psi^{-1})'_-(\psi_M) - (\chi\circ\psi^{-1})'_+(\psi_m)}{\psi_M-\psi_m} \\
&\quad \times \int_T \phi_t\left([\psi_M\mathbf{1} - \psi(x_t)][\psi(x_t)-\psi_m\mathbf{1}]\right) d\mu(t) \\
&\le \frac{(\chi\circ\psi^{-1})'_-(\psi_M) - (\chi\circ\psi^{-1})'_+(\psi_m)}{\psi_M-\psi_m}\left[\psi_M\mathbf{1} - \psi\left(M_\psi(x,\phi)\right)\right] \\
&\quad \times \left[\psi\left(M_\psi(x,\phi)\right) - \psi_m\mathbf{1}\right] \\
&\le \frac{1}{4}\left(\psi_M-\psi_m\right)\left[(\chi\circ\psi^{-1})'_-(\psi_M) - (\chi\circ\psi^{-1})'_+(\psi_m)\right]\mathbf{1}
\end{aligned} \tag{3.5}$$

holds for every continuous field $(x_t)_{t\in T}$ of operators in $\mathfrak{B}(\mathcal{H})$ with spectra in $[m, M]$. If $\chi \circ \psi^{-1}$ is concave on $\psi(I)$, then the signs of inequalities in (3.5) *are reversed.*

Proof. Considering the same setting as in the proof of Theorem 3.1 and with notation as in Theorem 2.4, we have

$$\alpha_{\chi\circ\psi^{-1}} = \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}, \quad \beta_{\chi\circ\psi^{-1}} = \frac{\psi(M)\chi(m)-\psi(m)\chi(M)}{\psi(M)-\psi(m)},$$

so the result is an immediate consequence of the series of inequalities in (2.6). □

3.4. Remark. The first inequality in (3.5) can be rewritten in the following form:

$$\begin{aligned}
&(\psi(M)-\psi(m))\chi\left(M_\chi(x,\phi)\right) - (\chi(M)-\chi(m))\psi\left(M_\psi(x,\phi)\right) \\
&\le (\psi(M)\chi(m)-\psi(m)\chi(M))\mathbf{1}.
\end{aligned} \tag{3.6}$$

The above inequality (3.6) can be regarded as an operator analogue of the corresponding relation for linear functionals (see [10], Theorem 4.3, p. 108).

3.5. Remark. With notations as in Theorems 3.1 and 3.3, suppose that the function $\chi \circ \psi^{-1}$ is differentiable in points ψ_m and ψ_M. In this case expressions ψ_m and ψ_M in (3.4) and (3.5) can respectively be replaced by $\psi(m)$ and $\psi(M)$, due to the symmetry. In addition, utilizing a chain rule, the expression

$$(\chi\circ\psi^{-1})'_-(\psi(M)) - (\chi\circ\psi^{-1})'_+(\psi(m))$$

can be rewritten in a more suitable form, that is,

$$(\chi\circ\psi^{-1})'_-(\psi(M)) - (\chi\circ\psi^{-1})'_+(\psi(m)) = \frac{\chi'(M)}{\psi'(M)} - \frac{\chi'(m)}{\psi'(m)}. \tag{3.7}$$

4. Applications to power operator means

As a particular case of a quasi-arithmetic mean defined by (3.1), we may consider a power operator mean (see e.g. [8]):

$$M_r(x,\phi) = \begin{cases} \left(\int_T \phi_t(x_t^r)d\mu(t)\right)^{\frac{1}{r}}, & r \neq 0 \\ \exp\left(\int_T \phi_t(\log x_t)d\mu(t)\right), & r = 0. \end{cases} \tag{4.1}$$

By virtue of relations (3.2) and (3.3), Mićić *et.al.* [9], established the following order among power operator means:

$$M_r(x,\phi) \le M_s(x,\phi), \tag{4.2}$$

for either $r \le s$, $r, s \in \mathbb{R} \setminus [-1,1]$ or $\frac{1}{2} \le r \le 1 \le s$ or $r \le -1 \le s \le -\frac{1}{2}$. However, a class of inequalities in (4.2) is a consequence of operator convexity and monotonicity of the corresponding power functions.

On the other hand, regarding the method developed in this paper, converses for power operator means are established via the classical convexity. The following result appears to be a consequence of Theorem 3.1 when considering the above power operator means.

4.1. Theorem. *Let $(\phi_t)_{t\in T} \in P[\mathfrak{B}(\mathcal{H}), \mathfrak{B}(\mathcal{K})]$, where $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces and T is a locally compact Hausdorff space with a bounded Radon measure μ, and let $(x_t)_{t\in T}$ be a continuous field of positive operators in $\mathfrak{B}(\mathcal{H})$ with spectra in $[m, M] \subseteq \mathbb{R}_+$.*

(i) *If either $s \le 0 < r$ or $r < 0 \le s$ or $0 < r < s$ or $s < r < 0$, then the following series of inequalities holds:*

$$\begin{aligned} &[M_s(x,\phi)]^s - [M_r(x,\phi)]^s \\ &\le \frac{s}{r} \cdot \frac{M^{s-r} - m^{s-r}}{M^r - m^r} \left[M^r\mathbf{1} - [M_r(x,\phi)]^r\right]\left[[M_r(x,\phi)]^r - m^r\mathbf{1}\right] \\ &\le \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right)\mathbf{1}. \end{aligned} \tag{4.3}$$

Further, if $0 \le s < r$ or $r < s \le 0$, then the signs of inequalities in (4.3) are reversed.

(ii) *If $r < 0$ then*

$$\begin{aligned} 0 &\le \log[M_0(x,\phi)] - \log[M_r(x,\phi)] \\ &\le -\frac{1}{rM^rm^r}\left[M^r\mathbf{1} - [M_r(x,\phi)]^r\right]\left[[M_r(x,\phi)]^r - m^r\mathbf{1}\right] \\ &\le -\frac{(M^r - m^r)^2}{4rM^rm^r}\mathbf{1}, \end{aligned} \tag{4.4}$$

while for $r > 0$ the signs of inequalities in (4.4) are reversed.

(iii) *If $s \in \mathbb{R}$, then the following series of inequalities holds:*

$$\begin{aligned} &[M_s(x,\phi)]^s - [M_0(x,\phi)]^s \\ &\le \frac{s(M^s - m^s)}{\log M - \log m}\left[\log M\mathbf{1} - \log[M_0(x,\phi)]\right]\left[\log[M_0(x,\phi)] - \log m\mathbf{1}\right] \\ &\le \frac{s}{4}\left(\log M - \log m\right)\left(M^s - m^s\right)\mathbf{1}. \end{aligned} \tag{4.5}$$

Proof. The proof is a simple consequence of Theorem 3.1, that is, the series of inequalities in (3.4) with particular choices of functions χ and ψ.

More precisely, let $\chi(t) = t^s$ and $\psi(t) = t^r$, where s and r are mutually different real parameters not equal to zero. Then the function $\left(\chi \circ \psi^{-1}\right)(t) = t^{\frac{s}{r}}$ is convex on $\mathbb{R}_+$ if $\frac{s}{r} \le 0$ or $\frac{s}{r} \ge 1$. It is possible in each of the following four cases: $s < 0 < r$ or $r < 0 < s$

or $0 < r < s$ or $s < r < 0$. Finally, since $\left(\chi \circ \psi^{-1}\right)'(t) = \frac{s}{r} t^{\frac{s-r}{r}}$, considering (3.4) with the above functions χ and ψ on the interval $[m, M]$, we obtain (4.3).

On the other hand, the function $\left(\chi \circ \psi^{-1}\right)(t) = t^{\frac{s}{r}}$ is concave on $\mathbb{R}_+$ if $0 \le \frac{s}{r} \le 1$, hence if $0 < s < r$ or $r < s < 0$ we obtain series (4.3) with reversed signs of inequalities. Clearly, the series of inequalities in (4.3), as well as the series with reversed signs of inequalities, holds also for $s = 0$.

It remains to consider the cases when one of the parameters r and s is equal to zero. If $s = 0$, then setting $\chi(t) = \log t$ and $\psi(t) = t^r$, it follows that $\left(\chi \circ \psi^{-1}\right)(t) = \frac{1}{r}\log t$. Clearly, this function is convex for $r < 0$, while it is concave for $r > 0$. Moreover, since $\left(\chi \circ \psi^{-1}\right)'(t) = \frac{1}{rt}$, after a straightforward computation we obtain (4.4) without the first inequality sign in the convex case, while in the concave case the reversed series of inequalities holds. The first inequality sign in (4.4), as well as in the reversed series of inequalities, holds due to the operator convexity of the function $\frac{1}{r}\log t$ when $r < 0$, that is, operator concavity when $r > 0$.

Finally, if $r = 0$, then setting $\chi(t) = t^s$ and $\psi(t) = \log t$, it follows that the function $\left(\chi \circ \psi^{-1}\right)(t) = \exp(st)$ is convex for every $s \ne 0$. In addition, $\left(\chi \circ \psi^{-1}\right)'(t) = s\exp(st)$, which yields (4.5) after a straightforward computation. Of course, the series of inequalities in (4.5) holds also for $s = 0$. □

4.2. Remark. Observe that the function $\left(\chi \circ \psi^{-1}\right)(t) = \frac{1}{r}\log t$ is simultaneously convex and operator convex, that is, concave and operator concave depending on whether $r < 0$ or $r > 0$. Hence, the first expression in (4.4) is the positive operator yielding the inequality

$$\log\left[M_r(x,\phi)\right] \le \log\left[M_0(x,\phi)\right]$$

for $r < 0$. On the other hand, if $r > 0$ then the following inequality holds:

$$\log\left[M_0(x,\phi)\right] \le \log\left[M_r(x,\phi)\right].$$

It is well-known that the function $f(t) = t^r$ is operator convex on $\mathbb{R}_+$ if either $1 \le r \le 2$ or $-1 \le r \le 0$, and is operator concave on $\mathbb{R}_+$ when $0 \le r \le 1$. Hence, discussing the operator convexity of the function $\left(\chi \circ \psi^{-1}\right)(t) = t^{\frac{s}{r}}$ (see the proof of Theorem 4.1), we obtain conditions on parameters r and s under which the operator $\left[M_s(x,\phi)\right]^s - \left[M_r(x,\phi)\right]^s$ is positive in the series of inequalities (4.3).

4.3. Corollary. *With the same assumptions as in the statement of Theorem* 4.1*, the series of inequalities*

$$\begin{aligned} 0 &\le \left[M_s(x,\phi)\right]^s - \left[M_r(x,\phi)\right]^s \\ &\le \frac{s}{r}\cdot\frac{M^{s-r}-m^{s-r}}{M^r-m^r}\left[M^r\mathbf{1} - \left[M_r(x,\phi)\right]^r\right]\left[\left[M_r(x,\phi)\right]^r - m^r\mathbf{1}\right] \\ &\le \frac{s}{4r}\left(M^r-m^r\right)\left(M^{s-r}-m^{s-r}\right)\mathbf{1} \end{aligned} \tag{4.6}$$

holds if either $0 < r \le s \le 2r$ *or* $2r \le s \le r < 0$ *or* $0 \le s+r \le r \ne 0$ *or* $0 \ne r \le r+s \le 0$. *Further, if* $0 \ne r \le s \le 0$ *or* $0 \le s \le r \ne 0$*, then the signs of inequalities in* (4.6) *are reversed.*

Proof. Regarding the proof of Theorem 4.1, it follows that the first inequality sign in (4.6) holds when $\left(\chi \circ \psi^{-1}\right)(t) = t^{\frac{s}{r}}$ is operator convex function. This function is operator convex if either $1 \le \frac{s}{r} \le 2$ or $-1 \le \frac{s}{r} \le 0$, that is, when either $0 < r \le s \le 2r$ or $2r \le s \le r < 0$ or $0 \le s + r \le r \ne 0$ or $0 \ne r \le r + s \le 0$. Moreover, since the operator convexity of the function $\left(\chi \circ \psi^{-1}\right)(t) = t^{\frac{s}{r}}$ implies its usual convexity, it follows that the remaining signs of inequalities in (4.6) are also valid under the above conditions.

On the other hand, function $\left(\chi \circ \psi^{-1}\right)(t) = t^{\frac{s}{r}}$ is operator concave if $0 \le \frac{r}{s} \le 1$, that is, when $0 \ne r \le s \le 0$ or $0 \le s \le r \ne 0$. Under these conditions $\left(\chi \circ \psi^{-1}\right)(t) =$

$t^{\frac{s}{r}}$ is concave in the classical sense, as well. This gives (4.6) with reversed signs of inequalities. □

4.4. Remark. With the conditions as in Corollary 4.3, we obtain the order among operators $[M_s(x,\phi)]^s$ and $[M_r(x,\phi)]^s$. Moreover, applying the operator monotonicity of suitable power functions, one obtains conditions as in (4.2). In fact, it is a more specific use of relations (3.2) and (3.3), for more details see [9].

4.5. Remark. It should be noticed here that the above discussion as in Corollary 4.3 and Remark 4.2 can not be applied to the series of inequalities in (4.5) since the exponential function $f(t) = \exp t$ is not operator convex (see e.g. [1]).

Guided by the proof of Theorem 4.1, we also obtain an interesting consequence of Theorem 3.3, that is, the converses of the Lah-Ribarič inequality that correspond to power operator means.

4.6. Theorem. *Let $(\phi_t)_{t\in T} \in P[\mathfrak{B}(\mathcal{H}), \mathfrak{B}(\mathcal{K})]$, where $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces and T is a locally compact Hausdorff space with a bounded Radon measure μ, and let $(x_t)_{t\in T}$ be a continuous field of positive operators in $\mathfrak{B}(\mathcal{H})$ with spectra in $[m, M] \subseteq \mathbb{R}_+$.*

(i) *If either $s \le 0 < r$ or $r < 0 \le s$ or $0 < r < s$ or $s < r < 0$, then the following series of inequalities holds:*

$$\begin{aligned}
0 &\le \frac{M^s - m^s}{M^r - m^r}\left[M_r(x,\phi)\right]^r + \frac{M^r m^s - m^r M^s}{M^r - m^r}\mathbf{1} - [M_s(x,\phi)]^s \\
&\le \frac{s}{r}\cdot\frac{M^{s-r} - m^{s-r}}{M^r - m^r}\int_T \phi_t\left([M^r\mathbf{1} - x_t^r][x_t^r - m^r\mathbf{1}]\right) d\mu(t) \\
&\le \frac{s}{r}\cdot\frac{M^{s-r} - m^{s-r}}{M^r - m^r}\left[M^r\mathbf{1} - [M_r(x,\phi)]^r\right]\left[[M_r(x,\phi)]^r - m^r\mathbf{1}\right] \\
&\le \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right)\mathbf{1}. \qquad (4.7)
\end{aligned}$$

Moreover, if $0 \le s < r$ or $r < s \le 0$, then the signs of inequalities in (4.7) are reversed.

(ii) *If $r < 0$ then*

$$\begin{aligned}
0 &\le \frac{\log M - \log m}{M^r - m^r}\left[M_r(x,\phi)\right]^r + \frac{M^r \log m - m^r \log M}{M^r - m^r}\mathbf{1} - \log\left[M_0(x,\phi)\right] \\
&\le -\frac{1}{rM^r m^r}\int_T \phi_t\left([M^r\mathbf{1} - x_t^r][x_t^r - m^r\mathbf{1}]\right) d\mu(t) \\
&\le -\frac{1}{rM^r m^r}\left[M^r\mathbf{1} - [M_r(x,\phi)]^r\right]\left[[M_r(x,\phi)]^r - m^r\mathbf{1}\right] \\
&\le -\frac{(M^r - m^r)^2}{4rM^r m^r}\mathbf{1}, \qquad (4.8)
\end{aligned}$$

while for $r > 0$ the signs of inequalities in (4.8) are reversed.

(iii) *The series of inequalities*

$$\begin{aligned}
0 &\le \frac{M^s - m^s}{\log M - \log m}\log\left[M_0(x,\phi)\right] + \frac{m^s \log M - M^s \log m}{\log M - \log m}\mathbf{1} - [M_s(x,\phi)]^s \\
&\le \frac{s\,(M^s - m^s)}{\log M - \log m}\int_T \phi_t\left([\log M\mathbf{1} - \log x_t][\log x_t - \log m\mathbf{1}]\right) d\mu(t) \\
&\le \frac{s\,(M^s - m^s)}{\log M - \log m}\left[\log M\mathbf{1} - \log[M_0(x,\phi)]\right]\left[\log[M_0(x,\phi)] - \log m\mathbf{1}\right] \\
&\le \frac{s}{4}\left(\log M - \log m\right)\left(M^s - m^s\right)\mathbf{1} \qquad (4.9)
\end{aligned}$$

holds for all $s \in \mathbb{R}$.

Proof. We use the same procedure as in the proof of Theorem 4.1, applied to the series of inequalities in (3.5). □

References

[1] R. Bhatia, *Matrix Analysis*, Springer-Verlag, 1997.

[2] S.S. Dragomir, *Reverses of the Jensen's inequality in terms of first derivative and applications*, RGMIA Res. Rep. Coll. (2011), preprint.

[3] T. Furuta, J. Mićić Hot, J. Pečarić, Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Element, Zagreb, 2005.

[4] I. Gelfand, M. Naimark, *On the imbedding of normed rings into the ring of operators on a Hilbert space*, Math. Sbornik **12** (1943), 197-217.

[5] F. Hansen, G. Pedersen, *Jensen's operator inequality*, Bull. London Math. Soc. **35** (2003), 553–564.

[6] F. Hansen, J. Pečarić, I. Perić, *Jensen's operator inequality and its converses*, Math. Scand. **100** (2007), 61–73.

[7] J. Mićić, J. Pečarić, *Order among power means of positive operators, II*, Sci. Math. Japon. **71** (2010), 93–109.

[8] J. Mićić, J. Pečarić, Y. Seo, *Converses of Jensen's operator inequality*, Oper. Matrices **4** (2010), 385-403.

[9] J. Mićić, J. Pečarić, Y. Seo, *Order among quasi-arithmetic means of positive operators*, Math. Reports, in press.

[10] J.E. Pečarić, F. Proschan, Y.L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press Inc., San Diego, 1992.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1057 – 1069

Toric ideals of simple surface singularities

Gülay Kaya* , Pınar Mete† and Mesut Şahin‡

Abstract

In this paper, we study a class of toric ideals obtained by using some geometric data of ADE trees which are the minimal resolution graphs of rational surface singularities. We compute explicit Gröbner bases for these toric ideals that are also minimal generating sets consisting of large number of binomials of degree ≤ 4. In particular, they give rise to squarefree initial ideals as well.

Keywords: toric ideals, simple surface singularities, semigroup of Lipman

2000 AMS Classification: Primary: 14M25, 13A50; Secondary: 32S45, 15C25

Received 13/12/2011 : Accepted 06/01/2014 Doi : 10.15672/HJMS.2015449094

1. Introduction

Algebraic varieties having squarefree initial ideals are of special interest. Many authors have presented squarefree initial ideals arising from different contexts, see for instance [5, 11, 13, 14, 16]. Normal toric ideals are known to have at least one squarefree term in each minimal binomial generator by [19, Proposition 4.1] and [17, Lemma 6.1]. They have Cohen-Macaulay initial ideals when their configurations are Δ-normal, see [18]. These suggest that they have (at least simplicial ones) squarefree initial ideals with respect to a term order. The challenge lies in the choice of a correct term order. Motivated by fundamental questions in combinatorial commutative algebra and its applications to statistics and optimization, recently, with the aid of Gale diagrams, Dueck et al. [8] have succeeded to show the existence of a term order with respect to which normal toric ideals of codimension 2 have squarefree initial ideals. They have also proven that the Gröbner bases giving rise to these initial ideals constitute minimal generating sets for the toric ideals.

*Department of Mathematics, Galatasaray University, İstanbul, Turkey
Email: gukaya@gsu.edu.tr
†Department of Mathematics, Balıkesir University, Balıkesir, 10145 Turkey
Email:pinarm@balikesir.edu.tr
‡Department of Mathematics, Çankırı Karatekin University, Çankırı, 18100 Turkey
Email:mesutsahin@gmail.com
Corresponding Author.

The aim of the present paper is to extend the discussion to certain examples of normal toric ideals of higher codimension. As a case study, we concentrate on certain toric ideals of higher codimension arising from singularity theory that are promising because of the speciality of the corresponding singularities. These are the simplicial normal toric ideals corresponding to the simple or ADE surface singularities. In section 3, we prove that toric ideals of DE type singularities have squarefree initial ideals. Our methods are computational and use the configurations given in [1]. The reduced Gröbner bases we obtain are also shown to be minimal generating sets containing a large number of binomials of degree at most 4, see section 4. In the last section, we speculate on initial ideals of A_n-type trees whose configurations seem impossible to give a closed form.

2. Preliminaries

2.1. Gröbner basis. Let $\mathcal{A} = \{\mathbf{a_1}, \dots, \mathbf{a_N}\}$ be a configuration in $\mathbb{Z}^n$ and $K[\mathcal{A}] := K[\{x_\mathbf{a} \,|\, \mathbf{a} \in \mathcal{A}\}]$ denote the polynomial ring in variables $x_\mathbf{a}$ with $\mathbf{a} \in \mathcal{A}$ over the field K. Consider the affine semigroup $\mathbb{N}\mathcal{A} = \{\lambda_1\mathbf{a_1} + \cdots + \lambda_N\mathbf{a_N} \;:\; \lambda_i \in \mathbb{N}\}$ and let $K[\mathbb{N}\mathcal{A}] := K[\{\mathbf{u}^\mathbf{a} \,|\, \mathbf{a} \in \mathcal{A}\}]$ be the associated semigroup ring. The *toric ideal* $I_\mathcal{A}$ of $\mathcal{A}$ is the kernel of the following K-algebra epimorphism:

$$\pi : K[\mathcal{A}] \to K[\mathbb{N}\mathcal{A}], \quad \pi(x_\mathbf{a}) := \mathbf{u}^\mathbf{a} = u_1^{a_1} \cdots u_n^{a_n}.$$

It is known that $I_\mathcal{A}$ is a prime ideal generated by binomials $x_\mathbf{a} - x_\mathbf{b}$ with $\pi(x_\mathbf{a}) = \pi(x_\mathbf{b})$ [20]. The zero set of $I_\mathcal{A}$ is called the toric variety $V_\mathcal{A}$ of $\mathcal{A}$.

The *initial monomial*, $in(f)$, of a polynomial $f \in I_\mathcal{A} \setminus \{0\}$ is the greatest monomial of f with respect to a term order on the monomials of $K[\mathcal{A}]$. The *initial ideal*, $in(I_\mathcal{A})$, of $I_\mathcal{A}$ is a *monomial* ideal generated by all initial monomials of polynomials in $I_\mathcal{A}$. A finite subset $\mathcal{G} \subset I_\mathcal{A}$ is called a Gröbner basis of $I_\mathcal{A}$ if $in(I_\mathcal{A}) = in(\mathcal{G})$, where $in(\mathcal{G})$ is the monomial ideal generated by initial monomials of polynomials in $\mathcal{G}$. The following is the key in proving our main results.

2.1. Lemma. [2, Lemma 1.1] *With the preceding notation, let M and M' be monomials in $K[\mathcal{A}]$. The finite set $\mathcal{G}$ is a Gröbner basis of $I_\mathcal{A}$ if and only if $\pi(M) \neq \pi(M')$ for all $M \notin in(\mathcal{G})$ and $M' \notin in(\mathcal{G})$ with $M \neq M'$.*

2.2. ADE-trees. Here, we briefly review basics of ADE-trees, see [4, 3, 23, 9, 10] for more details. Let Γ be a weighted graph without loops, with vertices $C_1, \dots, C_n$ and with weight $w_i \geq 2$ at each vertex C_i. The incidence matrix $\mathcal{M}(\Gamma) = [c_{ij}]$, associated with Γ is a symmetric matrix and defined in the following way: $c_{ii} = -w_i$ and c_{ij} is the number of edges linking the vertices C_i and C_j whenever $i \neq j$. On the free abelian group $\mathcal{L}$ generated by the vertices C_i of Γ, $\mathcal{M}(\Gamma)$ defines a symmetric bilinear form $(Y \cdot Z)$ for a pair (Y, Z) of elements in $\mathcal{L}$ via $(C_i \cdot C_j) := c_{ij}$. The elements $C = \sum_{i=1}^n m_i C_i$ of $\mathcal{L}$ will be called *cycles* of the graph Γ where $m_i \in \mathbb{Z}$. A *positive cycle* is a non-zero cycle with non-negative coefficients.

If $w_i = 2$ for all i and $C \cdot C \leq -2$ for any cycle then Γ is of type A_n, D_n, E_6, E_7 and E_8. It is well known that these are the Dynkin diagrams obtained as the minimal resolution graphs of the rational singularities of complex surfaces. The semigroup of Lipman is the set

$$\mathcal{E}^+(\Gamma) := \{C \in \mathcal{L} \mid (C \cdot C_i) \leq 0 \quad \text{for} \quad 1 \leq i \leq n\}.$$

which is not empty since $M(\Gamma)$ is negative definite in this case. By [15], each element of this set corresponds to a function in the maximal ideal of the local ring of the singularity on the surface having Γ as the minimal resolution graph.

In [22] and [1], the authors have studied the structure of this semigroup and provided an algorithm to find a generating set over $\mathbb{Z}$ by associating an affine toric variety $V_{\mathcal{A}}$, c.f. also [21]. This toric variety corresponds to the configuration $\mathcal{A}$ of the smallest n-tuples $(d_1, \dots, d_n) \in \mathbb{N}^n$ such that $(C \cdot C_i) = -d_i$ for $C \in \mathcal{E}^+(\Gamma)$. The interested reader can see [1] for the details.

3. Squarefree initial ideals

In this section, we obtain reduced Gröbner bases for toric ideals of affine toric varieties corresponding to DE-type singularities. Throughout the section, we assume that the first term of a binomial is its initial monomial for a fixed term order. In order to find the set $\mathcal{A}$ which determines the parametrization of the toric variety $V_{\mathcal{A}}$, we use Proposition 3.9 and 3.12 in [1].

3.1. D_n-type singularities. We have $n \geq 4$. Since toric ideals behave in a different manner when n is even and odd, we discuss two cases separately.

When $n = 2m$: Let $J = \{3, 5, \dots, n-1\}$ and $J^c = \{2, 4, \dots, n-2\}$. Consider the subset

$$D_{2m} := \{2\mathbf{e}_i, \mathbf{e}_j, 2\mathbf{e}_1, 2\mathbf{e}_n, \mathbf{e}_k + \mathbf{e}_\ell, \mathbf{e}_i + \mathbf{e}_1 + \mathbf{e}_n \,|\, i, k, \ell \in J,\ j \in J^c \text{ and } k < \ell\},$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{Z}^n$. Then we introduce one variable for each element in the set D_{2m} and define the polynomial ring $K[D_{2m}]$ to be the K-algebra generated by the set of these varibeles

$$\{x_1, \dots, x_n, x_{j,k}, y_i \mid \text{where } i, j, k \in J \text{ and } j < k\}.$$

Similarly we define the semigroup ring $K[\mathbb{N}D_{2m}]$ to be the K-algebra generated by

$$\{u_i^2, u_j, u_1^2, u_n^2, u_k u_\ell, u_i u_1 u_n \,|\, i, k, \ell \in J,\ j \in J^c \text{ and } k < \ell\}.$$

The toric ideal $I_{D_{2m}}$ is thus the kernel of $\pi : K[D_{2m}] \to K[\mathbb{N}D_{2m}]$ which is defined as:

$$\pi(x_i) = u_i^2,\ \pi(x_j) = u_j,\ \pi(x_1) = u_1^2,\ \pi(x_n) = u_n^2,\ \pi(x_{k,\ell}) = u_k u_\ell,$$

$$\pi(y_i) = u_i u_1 u_n$$

for all $i, k, \ell \in J$, $j \in J^c$ with $k < \ell$.

We next define the ordering $\succ^{even}$ to be the reverse lexicographic ordering imposed by:

$$x_1 \succ \cdots \succ x_{n-1} \succ x_n \succ x_{j_1,j_2} \succ x_{j_3,j_4} \succ y_{k_1} \succ y_{k_2}$$

where $j_1, j_2, j_3, j_4, k_1, k_2 \in J$ with $j_2 < j_4$ or $j_2 = j_4$, $j_1 < j_3$; and $k_1 < k_2$.

Then a squarefree initial ideal for $I_{D_{2m}}$ is given by the following theorem, since the first monomial of a binomial is its initial term.

3.1. Theorem. *The following set $\mathcal{G}_{D_{2m}}$*

$x_{i,k}x_{j,\ell} - x_{i,j}x_{k,\ell}$	$x_{i,\ell}x_{j,k} - x_{i,j}x_{k,\ell}$	$i < j < k < \ell$
$x_{i,j}x_{i,k} - x_i x_{j,k}$	$x_j x_{i,k} - x_{i,j}x_{j,k}$	$i < j < k$
$x_k x_{i,j} - x_{i,k}x_{j,k}$	$x_{j,k}y_i - x_{i,j}y_k$	$i < j < k$
$x_{i,k}y_j - x_{i,j}y_k$		$i < j < k$
$x_i x_j - x_{i,j}^2$	$x_j y_i - x_{i,j}y_j$	$i < j$
$x_{i,j}y_i - x_i y_j$	$x_{i,j}x_1 x_n - y_i y_j$	$i < j$
$x_i x_1 x_n - y_i^2$		$i \in J$

is a Gröbner basis of $I_{D_{2m}}$ with respect to the ordering $\succ^{even}$ defined above.

Proof. Let M and M' be two monomials in $K[D_{2m}]$ with $M \notin in(\mathcal{G}_{D_{2m}})$ and $M' \notin in(\mathcal{G}_{D_{2m}})$, where $in(\mathcal{G}_{D_{2m}})$ is the monomial ideal generated by initial terms of binomials in $\mathcal{G}_{D_{2m}}$. Since $x_i x_j \in in(\mathcal{G}_{D_{2m}})$, we may assume that

$$M = x_a^p x_1^{\alpha_1} x_n^{\alpha_n} x_{b_1,c_1} \cdots x_{b_q,c_q} y_{d_1} \cdots y_{d_r} \quad \text{and}$$
$$M' = x_{a'}^{p'} x_1^{\alpha'_1} x_n^{\alpha'_n} x_{b'_1,c'_1} \cdots x_{b'_{q'},c'_{q'}} y_{d'_1} \cdots y_{d'_{r'}}, \quad \text{where}$$
$$x_a \succ x_{b_1,c_1} \succ \cdots \succ x_{b_q,c_q} \succ y_{d_1} \succ \cdots \succ y_{d_r},$$
$$x_{a'} \succ x_{b'_1,c'_1} \succ \cdots \succ x_{b'_{q'},c'_{q'}} \succ y_{d'_1} \succ \cdots \succ y_{d'_{r'}}.$$

First, we observe that the ordering above implies that $c_1 \le \cdots \le c_q$, $c'_1 \le \cdots \le c'_{q'}$ and $d_1 \le \cdots \le d_r$, $d'_1 \le \cdots \le d'_{r'}$. Moreover, we have $b_1 < c_1 \le b_2 < c_2 \le \cdots \le b_q < c_q$ and $b'_1 < c'_1 \le b'_2 < c'_2 \le \cdots \le b'_{q'} < c'_{q'}$, since $x_{i,k}x_{j,\ell}, x_{i,\ell}x_{j,k}, x_{i,j}x_{i,k} \in in(\mathcal{G}_{D_{2m}})$.

The images of M and M' are found easily as

$$\pi(M) = u_a^{2p} u_1^{2\alpha_1 + r} u_n^{2\alpha_n + r} u_{b_1} u_{c_1} \cdots u_{b_q} u_{c_q} u_{d_1} \cdots u_{d_r}$$
$$\pi(M') = u_{a'}^{2p'} u_1^{2\alpha'_1 + r'} u_n^{2\alpha'_n + r'} u_{b'_1} u_{c'_1} \cdots u_{b'_{q'}} u_{c'_{q'}} u_{d'_1} \cdots u_{d'_{r'}}.$$

In what follows we will prove that $\pi(M) = \pi(M') \Rightarrow M = M'$, by the virtue of Lemma 2.1. It follows from $\pi(M) = \pi(M')$ that we have the following identities

(3.1) $\quad 2\alpha_1 + r = 2\alpha'_1 + r'$

(3.2) $\quad 2\alpha_n + r = 2\alpha'_n + r'$

(3.3) $\quad 2p + 2q + r = 2p' + 2q' + r'$

(3.4) $\quad \alpha_1 - \alpha_n = \alpha'_1 - \alpha'_n$ (follows directly from (3.1) and (3.2)).

To accomplish our goal $M = M'$, we will assume now that $M \neq M'$ to obtain a contradiction in all possible cases considered below. Since $M \neq M'$, we may suppose further that they have no variable in common without loss of generality. This is because $in(\mathcal{G}_{D_{2m}})$ is an ideal and $M, M' \notin in(\mathcal{G}_{D_{2m}})$ implies that the new monomials obtained by dividing M and M' by their greatest common divisor will also lie outside of $in(\mathcal{G}_{D_{2m}})$.

If $\alpha_1 > 0$ and $\alpha_n > 0$ then $\alpha'_1 = \alpha'_n = 0$, as M and M' have no common variable. Since $x_{i,j}x_1x_n$, $x_kx_1x_n \in in(\mathcal{G}_{D_{2m}})$, we have $p = q = 0$. This implies that $r = 2p' + 2q' + r'$ by (3.3) and thus $2p' + 2q' + 2\alpha_1 = 0$ by (3.1), a contradiction.

If $\alpha_1 > 0$ and $\alpha_n = 0$ then $\alpha'_1 = 0$ which implies together with (3.4) that $\alpha_1 = -\alpha'_n \le 0$, contradiction. The case $\alpha_1 = 0$ and $\alpha_n > 0$ is done similarly. So, we have only the case where $\alpha_1 = 0$ and $\alpha_n = 0$. A similar argument shows that $\alpha'_1 = \alpha'_n = 0$. In this case $r = r'$ by (3.1).

Case I: Assume $r = r' > 0$. Since $x_j y_i \in in(\mathcal{G}_{D_{2m}})$, for all $i < j$, it follows that $a \le d_r$. Again by $x_{i,j}y_i, x_{j,k}y_i, x_{i,k}y_j \in in(\mathcal{G}_{D_{2m}})$, for all $i < j < k$, we have $(b_q <) c_q \le d_r$ and $(b'_{q'} <) c'_{q'} \le d'_{r'}$. Hence, d_r (resp. d'_r) is the biggest index appearing in $\pi(M)$ (resp. $\pi(M')$). Since $\pi(M) = \pi(M')$, it follows that $d_r = d'_r$. But this implies that y_{d_r} is a variable appearing in both M and M', contradiction.

Case II: Assume $r = r' = 0$. If $q = 0$ then $\pi(M) = \pi(M')$ implies that $u_a^{2p} = u_{a'}^{2p'} u_{b'_1} u_{c'_1} \cdots u_{b'_{q'}} u_{c'_{q'}}$, which is possible only if $q' = 0$ as $b'_{q'} < c'_{q'}$. But in this case $a = a'$ and x_a is a common variable of M and M', a contradiction. Thus $q > 0$ and $q' > 0$.

Since $x_j x_{i,k}, x_k x_{i,j} \in in(\mathcal{G}_{D_{2m}})$, we have $a \le c_q$ and $a' \le c'_{q'}$. Since $b_q < c_q$ and $b'_{q'} < c'_{q'}$, we observe that c_q (resp. $c'_{q'}$) is the biggest index appearing in $\pi(M)$ (resp. $\pi(M')$) which yields together with $\pi(M) = \pi(M')$ that $c_q = c'_{q'}$. In this case u_{b_q} and $u_{b'_{q'}}$ appear in $\pi(M) = \pi(M')$. Clearly $b_q > b'_{q'}$ or $b_q < b'_{q'}$, as otherwise M and M' would have a common variable x_{b_q,c_q}. If $b_q > b'_{q'} (> \cdots > b'_1)$ then $b_q = a'$ as u_{b_q}

appears in $\pi(M')$. This forces that $b'_{q'} < b_q = a' < c_q = c'_{q'}$ which is impossible, since $x_j x_{i,k} \in in(\mathcal{G}_{D_{2m}})$. The other case $b_q < b'_{q'}$ is impossible by a similar argument. $\square$

3.2. Remark. Note that we have

$$|\mathcal{G}_{D_{2m}}| = 2\binom{m-1}{4} + 5\binom{m-1}{3} + 4\binom{m-1}{2} + \binom{m-1}{1}.$$

$\dim V_{D_{2m}} = 2m$, $\operatorname{codim} V_{D_{2m}} = m - 1 + \binom{m-1}{2}$.

When $n = 2m+1$ **:** Let $J = \{2, 4, \ldots, n-1\}$ and $J^c = \{3, 5, \ldots, n-2\}$. Consider the subset D_{2m+1} defined by

$$\{2\mathbf{e}_i, \mathbf{e}_j, 4\mathbf{e}_1, 4\mathbf{e}_n, \mathbf{e}_k + \mathbf{e}_\ell, \mathbf{e}_1 + \mathbf{e}_n, \mathbf{e}_i + 2\mathbf{e}_1, \mathbf{e}_i + 2\mathbf{e}_n, \mathbf{e}_i + 3\mathbf{e}_1 + \mathbf{e}_n, \mathbf{e}_i + \mathbf{e}_1 + 3\mathbf{e}_n \\ \mid i, k, \ell \in J,\, j \in J^c \text{ and } k < \ell\},$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{Z}^n$. As before we introduce one variable for each member of D_{2m+1} and define the polynomial ring $K[D_{2m+1}]$ to be the K-algebra generated by the set

$$\{x_1, \ldots, x_n, x_{j,k}, x_{1,n}, x_{i,1}, x_{i,n}, y_{i,1}, y_{i,n} \mid \text{where } i, j, k \in J \text{ and } j < k\}$$

and the semigroup ring $K[\mathbb{N}D_{2m+1}]$ to be the K-algebra generated by

$$\{u_i^2, u_j, u_1^4, u_n^4, u_k u_\ell, u_1 u_n, u_i u_1^2, u_i u_n^2, u_i u_1^3 u_n, u_i u_1 u_n^3 \mid i, k, \ell \in J,\, j \in J^c \text{ and } k < \ell\}.$$

The toric ideal $I_{D_{2m+1}}$ is thus the kernel of $\pi : K[D_{2m+1}] \to K[\mathbb{N}D_{2m+1}]$ which is defined as follows:

$$\pi(x_i) = u_i^2,\ \pi(x_j) = u_j,\ \pi(x_1) = u_1^4,\ \pi(x_n) = u_n^4,\ \pi(x_{k,\ell}) = u_k u_\ell,\ \pi(x_{1,n}) = u_1 u_n \\ \pi(x_{i,1}) = u_i u_1^2,\ \pi(x_{i,n}) = u_i u_n^2,\ \pi(y_{i,1}) = u_i u_1^3 u_n,\ \pi(y_{i,n}) = u_i u_1 u_n^3$$

for all $i, k, \ell \in J$, $j \in J^c$ with $k < \ell$.

Finally, we define the ordering $\succ^{odd}$ to be the reverse lexicographic ordering imposed by:

$$y_{i_1,1} \succ y_{i_2,1} \succ y_{i_1,n} \succ y_{i_2,n} \succ x_1 \succ \cdots \succ x_n \succ \\ \succ x_{j_1,j_2} \succ x_{j_3,j_4} \succ x_{k_1,1} \succ x_{k_2,1} \succ x_{\ell_1,n} \succ x_{\ell_2,n} \succ x_{1,n}$$

where $j_1, j_2, j_3, j_4, k_1, k_2, \ell_1, \ell_2 \in J$ with $j_2 < j_4$ or $j_2 = j_4,\ j_1 < j_3$ and $k_1 < k_2$ and $\ell_1 < \ell_2$.

Then a squarefree initial ideal for $I_{D_{2m+1}}$ is given by the following theorem as the first monomials are the initial terms with respect to the ordering $\succ^{odd}$.

3.3. Theorem. *The following set* $\mathcal{G}_{D_{2m+1}}$

$x_{i,k}x_{j,\ell} - x_{i,j}x_{k,\ell}$	$x_{i,\ell}x_{j,k} - x_{i,j}x_{k,\ell}$	$i < j < k < \ell \in J$
$x_{j,k}x_{i,n-1} - x_{i,j}x_{k,n-1}$	$x_{i,k}x_{j,n-1} - x_{i,j}x_{k,n-1}$	$i < j < k \in J$
$x_{j,k}x_{i,n} - x_{i,j}x_{k,n}$	$x_{i,k}x_{j,n} - x_{i,j}x_{k,n}$	$i < j < k \in J$
$x_j x_{i,k} - x_{i,j}x_{j,k}$	$x_{i,j}x_{i,k} - x_i x_{j,k}$	$i < j < k \in J$
$x_k x_{i,j} - x_{i,k}x_{j,k}$		$i < j < k \in J$
$x_i x_j - x_{i,j}^2$	$x_{i,j}x_1 - x_{i,n-1}x_{j,n-1}$	$i < j \in J$
$x_{i,j}x_n - x_{i,n}x_{j,n}$	$x_{i,j}x_{i,n-1} - x_i x_{j,n-1}$	$i < j \in J$
$x_{i,j}x_{i,n} - x_i x_{j,n}$	$x_j x_{i,n-1} - x_{i,j}x_{j,n-1}$	$i < j \in J$
$x_j x_{i,n} - x_{i,j}x_{j,n}$	$x_{j,n-1}x_{i,n} - x_{i,n-1}x_{j,n}$	$i < j \in J$
$x_{i,n-1}x_{j,n} - x_{1,n}^2 x_{i,j}$		$i < j \in J$
$x_i x_1 - x_{i,n-1}^2$	$x_i x_n - x_{i,n}^2$	$i \in J$
$x_{i,n-1}x_{i,n} - x_{1,n}^2 x_i$	$x_{i,n}x_1 - x_{1,n}^2 x_{i,n-1}$	$i \in J$
$y_{i,1} - x_{1,n}x_{i,1}$	$y_{i,n} - x_{1,n}x_{i,n}$	$i \in J$
$x_{i,n-1}x_n - x_{1,n}^2 x_{i,n}$	$x_1 x_n - x_{1,n}^4$	$i \in J$

is a Gröbner basis of $I_{\mathcal{D}_{2m+1}}$ with respect to the ordering $\succ^{odd}$.

Proof. Let M and M' be two monomials in $K[D_{2m+1}]$ with $M \notin in(\mathcal{G}_{D_{2m+1}})$ and $M' \notin in(\mathcal{G}_{D_{2m+1}})$, where $in(\mathcal{G}_{D_{2m+1}})$ is the monomial ideal generated by initial terms of binomials in $\mathcal{G}_{D_{2m+1}}$. Since $y_{i,1}, y_{i,n}, x_i x_j \in in(\mathcal{G}_{D_{2m+1}})$, we may assume that

$$\begin{aligned} M &= x_a^p x_1^{\alpha_1} x_n^{\alpha_n} x_{1,n}^{\beta} x_{b_1,c_1} \cdots x_{b_q,c_q} x_{d_1,n-1} \cdots x_{d_r,n-1} x_{e_1,n} \cdots x_{e_s,n} \quad \text{and} \\ M' &= x_{a'}^{p'} x_1^{\alpha_1'} x_n^{\alpha_n'} x_{1,n}^{\beta'} x_{b_1',c_1'} \cdots x_{b_{q'}',c_{q'}'} x_{d_1',n-1} \cdots x_{d_{r'}',n-1} x_{e_1',n} \cdots x_{e_{s'}',n}, \end{aligned}$$

where the variables are ordered with respect to

$$x_a \succ x_{b_1,c_1} \succ \cdots \succ x_{b_q,c_q} \succ x_{d_1,n-1} \succ \cdots \succ x_{d_r,n-1} \succ x_{e_1,n} \succ \cdots \succ x_{e_s,n},$$

$$x_{a'} \succ x_{b_1',c_1'} \succ \cdots \succ x_{b_{q'}',c_{q'}'} \succ x_{d_1',n-1} \succ \cdots \succ x_{d_{r'}',n-1} \succ x_{e_1',n} \succ \cdots \succ x_{e_{s'}',n}.$$

First, we observe that the ordering above implies that $c_1 \le \cdots \le c_q$, $c_1' \le \cdots \le c_{q'}'$, $d_1 \le \cdots \le d_r$, $d_1' \le \cdots \le d_{r'}'$ and $e_1 \le \cdots \le e_r$, $e_1' \le \cdots \le e_{r'}'$. Moreover, we have $b_1 < c_1 \le b_2 < c_2 \le \cdots \le b_q < c_q$ and $b_1' < c_1' \le b_2' < c_2' \le \cdots \le b_{q'}' < c_{q'}'$, since $x_{i,k}x_{j,\ell}, x_{i,\ell}x_{j,k}, x_{i,j}x_{i,k} \in in(\mathcal{G}_{D_{2m+1}})$.

The images of M and M' are found as follows

$$\begin{aligned} \pi(M) &= u_a^{2p} u_1^{4\alpha_1+\beta+2r} u_n^{4\alpha_n+\beta+2s} u_{b_1} u_{c_1} \cdots u_{b_q} u_{c_q} u_{d_1} \cdots u_{d_r} u_{e_1} \cdots u_{e_s} \\ \pi(M') &= u_{a'}^{2p'} u_1^{4\alpha_1'+\beta'+2r'} u_n^{4\alpha_n'+\beta'+2s'} u_{b_1'} u_{c_1'} \cdots u_{b_{q'}'} u_{c_{q'}'} u_{d_1'} \cdots u_{d_{r'}'} u_{e_1'} \cdots u_{e_{s'}'}. \end{aligned}$$

It follows from $\pi(M) = \pi(M')$ that we have the following identities

$$\begin{aligned} &(3.5) \quad & 2p + 2q + r + s &= 2p' + 2q' + r' + s' \\ &(3.6) \quad & 4\alpha_1 + \beta + 2r &= 4\alpha_1' + \beta' + 2r' \\ &(3.7) \quad & 4\alpha_n + \beta + 2s &= 4\alpha_n' + \beta' + 2s' \\ &(3.8) \quad & 2\alpha_1 - 2\alpha_n + r - s &= 2\alpha_1' - 2\alpha_n' + r' - s' \text{ (follows from (3.6) and (3.7)).} \end{aligned}$$

To accomplish our goal $M = M'$, we will assume contrarily that $M \neq M'$ and obtain a contradiction in all possible cases considered below. Since $M \neq M'$, we may suppose further that they have no variable in common without loss of generality.

Since $x_1 x_n \in in(\mathcal{G}_{D_{2m+1}})$, it follows that α_1 and α_n can not be positive simultaneously. If $\alpha_1 > 0$ then $\alpha_n = 0$ and $\alpha_1' = 0$ immediately. That $p = q = s = 0$ follows respectively from $x_i x_1, x_{i,j} x_1, x_{i,n} x_1 \in in(\mathcal{G}_{D_{2m+1}})$. Thus equations 3.5 and 3.8 become

$$\begin{aligned} r &= 2p' + 2q' + r' + s' \\ 2\alpha_1 + r &= -2\alpha_n' + r' - s' \end{aligned}$$

and we have $2\alpha_1 = -2(p' + q' + s' + \alpha_n') \le 0$, contradiction. If $\alpha_n > 0$ then clearly $\alpha_n' = 0$ and $\alpha_1 = 0$. That $p = q = r = 0$ follows respectively from $x_i x_n, x_{i,j} x_n, x_{i,n-1} x_n \in in(\mathcal{G}_{D_{2m+1}})$. Thus equations 3.5 and 3.8 become

$$\begin{aligned} s &= 2p' + 2q' + r' + s' \\ -2\alpha_n - s &= 2\alpha_1' + r' - s' \end{aligned}$$

and we have $2\alpha_n = -2(p' + q' + r' + \alpha_1') \le 0$, contradiction. So, both $\alpha_1 = \alpha_n = 0$. One can show that $\alpha_1' = 0$ and $\alpha_n' = 0$ by a similar argument.

Now, $x_{j,n-1}x_{i,n}, x_{i,n-1}x_{j,n}, x_{i,n-1}x_{i,n} \in in(\mathcal{G}_{D_{2m+1}})$ implies that r and s (resp. r' and s') can not be positive at the same time.

If $r > 0$, then $s = 0$ in which case equation 3.8 becomes $r = r' - s'$. If $r' > 0$, then $s' = 0$ and we have $r = r' > 0$, which is impossible as in this case, d_r would be equal to $d_{r'}'$ since these are the biggest indices of variables in M and M', x_{d_r} would be a common variable. If $s' > 0$, then $r' = 0$ and we have $r = -s'$, contradiction as $r > 0$ and $s' > 0$.

If $s > 0$, then $r = 0$ in which case equation 3.8 becomes $-s = r' - s'$. If $r' > 0$, then $s' = 0$ and we have $-s = r'$, which contradicts the assumption that $s > 0$ and $r' > 0$. If $s' > 0$, then $r' = 0$ and we have $s = s' > 0$, which is impossible as in this case e_s would be $e'_{s'}$ and since these are the biggest indices of variables in M and M', x_{e_s} would be a common variable.

Hence, $r = s = 0$ and this implies together with equation 3.8 that $r' = s'$. Since they can not be positive simultaneously, $r' = s' = 0$ as well. After all these observations, equation 3.6 reveals that $\beta = \beta'$. Since M and M' have no common variable, it follows that $\beta = \beta' = 0$.

If $q = 0$ then $\pi(M) = \pi(M')$ implies that $u_a^{2p} = u_{a'}^{2p'} u_{b'_1} u_{c'_1} \cdots u_{b'_{q'}} u_{c'_{q'}}$, which is possible only if $q' = 0$ as $b'_{q'} < c'_{q'}$. But in this case $a = a'$ and x_a is a common variable of M and M', a contradiction. Similarly, $q' = 0$ gives rise to a contradiction. Thus $q > 0$ and $q' > 0$.

Since $x_j x_{i,k}, x_k x_{i,j} \in in(\mathcal{G}_{D_{2m+1}})$, we have $a \leq c_q$ and $a' \leq c'_{q'}$. Since $b_q < c_q$ and $b'_{q'} < c'_{q'}$, we observe that c_q (resp. $c'_{q'}$) is the biggest index appearing in $\pi(M)$ (resp. $\pi(M')$) which yields together with $\pi(M) = \pi(M')$ that $c_q = c'_{q'}$. In this case u_{b_q} and $u_{b'_{q'}}$ appear in $\pi(M) = \pi(M')$. Clearly $b_q > b'_{q'}$ or $b_q < b'_{q'}$, as otherwise M and M' would have a common variable x_{b_q,c_q}. If $b_q > b'_{q'} (> \cdots > b'_1)$ then $b_q = a'$ as u_{b_q} appears in $\pi(M')$. This forces that $b'_{q'} < b_q = a' < c_q = c'_{q'}$ which is impossible, since $x_j x_{i,k} \in in(\mathcal{G}_{D_{2m+1}})$. The other case $b_q < b'_{q'}$ is impossible by a similar argument. □

3.4. Remark. Note that if $n = 2m + 1$ we have,

$$|\mathcal{G}_{D_{2m+1}}| = 2\binom{m}{4} + 7\binom{m}{3} + 9\binom{m}{2} + 7\binom{m}{1} + \binom{m}{0}.$$

$\dim V_{D_{2m+1}} = 2m + 1$, $\operatorname{codim} V_{D_{2m+1}} = 2m + 1 + \binom{m}{2}$.

3.2. E_n-type Singularities. We will give Gröbner bases of toric ideals $I_{\mathcal{E}_n}$, where $n = 6, 7, 8$, without proofs, as they can easily be checked by a computation in Cocoa [7]. To begin with, let us define the set $\mathcal{E}_6 \subset \mathbb{Z}^6$:

$$\{3\mathbf{e}_1, 3\mathbf{e}_2, \mathbf{e}_3, 3\mathbf{e}_4, 3\mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_1{+}\mathbf{e}_2, \mathbf{e}_1{+}\mathbf{e}_5, \mathbf{e}_2{+}\mathbf{e}_4, \mathbf{e}_4{+}\mathbf{e}_5, 2\mathbf{e}_2{+}\mathbf{e}_5, \mathbf{e}_2{+}2\mathbf{e}_5, 2\mathbf{e}_1{+}\mathbf{e}_4, \mathbf{e}_1{+}2\mathbf{e}_4\}.$$

Let $K[\mathcal{E}_6]$ be the polynomial ring $K[x_1, \ldots, x_{14}]$ with 14 variables and $K[\mathbb{N}\mathcal{E}_6]$ be the semigroup ring generated over K by monomials $u^{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{E}_6$. Then, as before, the toric ideal $I_{\mathcal{E}_6}$ is the kernel of the epimorphism defined by sending the i-th variable x_i to $u^{\mathbf{a}_i}$, where $\mathbf{a}_i$ denotes the i-th element in $\mathcal{E}_6$, for all $i = 1, \ldots, 14$. Similarly, we define the set $\mathcal{E}_7 \subset \mathbb{Z}^7$:

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, 2\mathbf{e}_4, \mathbf{e}_5, 2\mathbf{e}_6, 2\mathbf{e}_7, \mathbf{e}_4 + \mathbf{e}_6, \mathbf{e}_4 + \mathbf{e}_7, \mathbf{e}_6 + \mathbf{e}_7\}.$$

Again, $K[\mathcal{E}_7]$ denotes the polynomial ring $K[x_1, \ldots, x_{10}]$ with 10 variables and $K[\mathbb{N}\mathcal{E}_7]$ be the semigroup ring generated over K by monomials $u^{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{E}_7$. Thus, the toric ideal $I_{\mathcal{E}_7}$ is the kernel of the epimorphism defined by sending the i-th variable x_i to $u^{\mathbf{a}_i}$, where $\mathbf{a}_i$ denotes the i-th element in $\mathcal{E}_7$, for all $i = 1, \ldots, 10$. Finally, the set $\mathcal{E}_8 \subset \mathbb{Z}^8$ is defined as $\{\mathbf{e}_1, \ldots, \mathbf{e}_8\}$.

3.5. Theorem. *With the notations above we have the following:*

(1) A Gröbner basis for $I_{\mathcal{E}_6}$ with respect to lexicographic ordering with $x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > x_{11} > x_{12} > x_{13} > x_{14} > x_7 > x_8 > x_9 > x_{10}$ is given by

$$\begin{array}{llll}
x_7x_{10} - x_8x_9, & x_{13}x_{10} - x_{14}x_8, & x_{13}x_9 - x_{14}x_7, & x_{12}x_{14} - x_8x_9x_{10}, \\
x_{12}x_{13} - x_8^2x_9, & x_{11}x_{10} - x_{12}x_9, & x_{11}x_8 - x_{12}x_7, & x_{11}x_{14} - x_8x_9^2, \\
x_{11}x_{13} - x_7x_8x_9, & x_5x_9 - x_{12}x_{10}, & x_5x_7 - x_{12}x_8, & x_5x_{14} - x_8x_{10}^2, \\
x_5x_{13} - x_8^2x_{10}, & x_5x_{11} - x_{12}^2, & x_4x_8 - x_{14}x_{10}, & x_4x_7 - x_{14}x_9, \\
x_4x_{13} - x_{14}^2, & x_4x_{12} - x_9x_{10}^2, & x_4x_{11} - x_9^2x_{10}, & x_4x_5 - x_{10}^3, \\
x_2x_{10} - x_{11}x_9, & x_2x_8 - x_{11}x_7, & x_2x_{14} - x_7x_9^2, & x_2x_{13} - x_7^2x_9, \\
x_2x_{12} - x_{11}^2, & x_2x_5 - x_{11}x_{12}, & x_2x_4 - x_9^3, & x_1x_{10} - x_{13}x_8, \\
x_1x_9 - x_{13}x_7, & x_1x_{14} - x_{13}^3, & x_1x_{11} - x_7x_8^2, & x_1x_{11} - x_7^2x_8, \\
x_1x_5 - x_8^3, & x_1x_4 - x_{13}x_{14}, & x_1x_2 - x_7^3. &
\end{array}$$

(2) A Gröbner basis for $I_{\mathcal{E}_7}$ with respect to lexicographic ordering with $x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > x_7 > x_8 > x_9 > x_{10}$ is given by the following binomials

$$x_7x_8 - x_9x_{10}, \quad x_6x_9 - x_8x_{10}, \quad x_6x_7 - x_{10}^2, \quad x_4x_{10} - x_8x_9, \quad x_4x_7 - x_9^2, \quad x_4x_6 - x_8^2.$$

(3) The toric ideal $I_{\mathcal{E}_8} = (0)$.

4. Minimal generating sets

In this part, using [6] we show that the Gröbner bases obtained in the previous section are in fact minimal generating sets for each toric ideal. This will be achieved as follows.

Since our semigroups $\mathbb{N}\mathcal{A}$ are pointed, there is a partial order on them given by

$$\mathbf{c} \leq \mathbf{d} \Leftrightarrow \text{there is a } \mathbf{c}' \in \mathbb{N}\mathcal{A} \text{ such that } \mathbf{c} + \mathbf{c}' = \mathbf{d}.$$

As $I_{\mathcal{A}}$ is generated by binomials $x_{\mathbf{a}} - x_{\mathbf{b}}$ with $\pi(x_{\mathbf{a}}) = \pi(x_{\mathbf{b}})$, $x_{\mathbf{a}}$ and $x_{\mathbf{b}}$ will have the same $\mathcal{A}$-*degree*. Recall that for $\mathbf{p} = (p_1, \dots, p_N) \in \mathbb{N}^N$, the $\mathcal{A}$-*degree* of the monomial $x^{\mathbf{p}} := x_1^{p_1} \dots x_N^{p_N}$ is $\deg_{\mathcal{A}}(x^{\mathbf{p}}) = p_1\mathbf{a}_1 + \cdots + p_N\mathbf{a}_N \in \mathbb{N}\mathcal{A}$. A vector $\mathbf{b} \in \mathbb{N}\mathcal{A}$ is called a *Betti* $\mathcal{A}$-degree, if $I_{\mathcal{A}}$ has a minimal generating set containing an element of $\mathcal{A}$-degree $\mathbf{b}$. Since *Betti* $\mathcal{A}$-degrees are independent of the minimal generating sets our Gröbner bases will determine all the candidate vectors $\mathbf{b} \in \mathbb{N}\mathcal{A}$.

For a vector $\mathbf{b} \in \mathbb{N}\mathcal{A}$, $G(\mathbf{b})$ is the graph with vertices the elements of the fiber

$$deg_{\mathcal{A}}^{-1}(\mathbf{b}) = \{x^{\mathbf{p}} \mid deg_{\mathcal{A}}(x^{\mathbf{p}}) = \mathbf{b}\}$$

and edges all the sets $\{x^{\mathbf{p}}, x^{\mathbf{q}}\}$, whenever $x^{\mathbf{p}} - x^{\mathbf{q}} \in I_{\mathcal{A},\mathbf{b}}$, where the ideal $I_{\mathcal{A},\mathbf{b}}$ is defined by $I_{\mathcal{A},\mathbf{b}} = \langle x^{\mathbf{p}} - x^{\mathbf{p}} \mid deg_{\mathcal{A}}(x^{\mathbf{p}}) = deg_{\mathcal{A}}(x^{\mathbf{q}}) \precneqq \mathbf{b}\rangle$.

For each possible *Betti* $\mathcal{A}$-degree $\mathbf{b}$, we consider the complete graph $S_{\mathbf{b}}$ with vertices $G(\mathbf{b})_i$, the connected components of $G(\mathbf{b})$. Let $T_{\mathbf{b}}$ be a spanning tree of $S_{\mathbf{b}}$. Then $\mathcal{F}_{T_{\mathbf{b}}}$ is the collection of binomials $x^{\mathbf{p}} - x^{\mathbf{q}}$ corresponding to edges $\{x^{\mathbf{p}}, x^{\mathbf{q}}\}$ of $T_{\mathbf{b}}$ with $x^{\mathbf{p}} \in G(\mathbf{b})_i$ and $x^{\mathbf{q}} \in G(\mathbf{b})_j$. We will use the following to show the minimality of the generating sets given by the Gröbner bases presented in section 3.

4.1. Theorem. [6, Theorem 2.6]. *$\mathcal{F} = \bigcup_{\mathbf{b} \in \mathbb{N}\mathcal{A}} \mathcal{F}_{T_{\mathbf{b}}}$ is a minimal generating set of $I_{\mathcal{A}}$.*

Notice that if $\mathbf{b}$ is not a *Betti*$\mathcal{A}$-degree, then $\mathcal{F}_{T_{\mathbf{b}}} = \emptyset$ and that the number of possible spanning trees determine the number of different minimal generating sets.

4.1. Even Case $\mathcal{D}_{2m}$. We consider the subset $\mathcal{D}_{2m}$ defined by,

$$\mathcal{D}_{2m} := \{2\mathbf{e}_i, \mathbf{e}_j, 2\mathbf{e}_1, 2\mathbf{e}_n, \mathbf{e}_k + \mathbf{e}_\ell, \mathbf{e}_i + \mathbf{e}_1 + \mathbf{e}_n \mid i, k, \ell \in J,\, j \in J^c \text{ and } k < \ell\},$$

where $J = \{3, 5, \dots, n-1\}$, $J^c = \{2, 4, \dots, n-2\}$ and $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{Z}^n$. Recall that the elements of $\mathcal{D}_{2m}$ are the $\mathcal{D}_{2m}$-degrees of the variables x_i, x_j, x_1, x_n, $x_{k,l}$ and y_i respectively.

By Theorem 3.1, we see that $I_{\mathcal{D}_{2m}}$ is generated by the set $\mathcal{G}_{\mathcal{D}_{2m}}$

$x_{i,k}x_{j,\ell} - x_{i,j}x_{k,\ell}$	$x_{i,\ell}x_{j,k} - x_{i,j}x_{k,\ell}$	$i < j < k < \ell \in J$
$x_{i,j}x_{i,k} - x_i x_{j,k}$	$x_j x_{i,k} - x_{i,j}x_{j,k}$	$i < j < k \in J$
$x_k x_{i,j} - x_{i,k}x_{j,k}$	$x_{j,k}y_i - x_{i,j}y_k$	$i < j < k \in J$
$x_{i,k}y_j - x_{i,j}y_k$		$i < j < k \in J$
$x_i x_j - x_{i,j}^2$	$x_j y_i - x_{i,j}y_j$	$i < j \in J$
$x_{i,j}y_i - x_i y_j$	$x_{i,j}x_1 x_n - y_i y_j$	$i < j \in J$
$x_i x_1 x_n - y_i^2$		$i \in J.$

Therefore, possible *Betti* $\mathcal{D}_{2m}$-degrees are

$$\begin{array}{ll} \mathbf{b_1} = 2\mathbf{e}_i + 2\mathbf{e}_j, & \mathbf{b_2} = \mathbf{e}_i + \mathbf{e}_j + 2\mathbf{e}_1 + 2\mathbf{e}_n, \\ \mathbf{b_3} = 2\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_1 + \mathbf{e}_n, & \mathbf{b_4} = \mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_1 + \mathbf{e}_n, \\ \mathbf{b_5} = 2\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k, & \mathbf{b_6} = \mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_k, \\ \mathbf{b_7} = \mathbf{e}_i + \mathbf{e}_j + 2\mathbf{e}_k, & \mathbf{b_8} = \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_1 + \mathbf{e}_n, \\ \mathbf{b_9} = 2\mathbf{e}_i + 2\mathbf{e}_1 + 2\mathbf{e}_n, & \mathbf{b_{10}} = \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_\ell \end{array}$$

Next we prove that these binomials constitute a minimal generating set for $I_{\mathcal{D}_{2m}}$.

4.2. Proposition. *The set $\mathcal{G}_{\mathcal{D}_{2m}}$ is a minimal generating set of $I_{\mathcal{D}_{2m}}$.*

Proof. Since there is no binomial in $I_{\mathcal{D}_{2m},\mathbf{b_1}}$, $G(\mathbf{b_1})$ consists of two connected components $\{x_i x_j\}$ and $\{x_{i,j}^2\}$. Similarly, $G(\mathbf{b_2})$ has $\{x_{i,j}x_1x_n\}$ and $\{y_iy_j\}$, $G(\mathbf{b_3})$ has $\{x_{i,j}y_i\}$ and $\{x_iy_j\}$, $G(\mathbf{b_4})$ has $\{x_jy_i\}$ and $\{x_{i,j}y_j\}$, $G(\mathbf{b_5})$ has $\{x_{i,j}x_{i,k}\}$ and $\{x_ix_{j,k}\}$, $G(\mathbf{b_6})$ has $\{x_jx_{i,k}\}$ and $\{x_{i,j}x_{j,k}\}$, $G(\mathbf{b_7})$ has $\{x_kx_{i,j}\}$ and $\{x_{i,k}x_{j,k}\}$, $G(\mathbf{b_9})$ has $\{x_ix_1x_n\}$ and $\{y_i^2\}$ as its connected components.

By Corollary 2.10 in [6], these graphs determine all indispensable binomials of $I_{\mathcal{D}_{2m}}$. Since these binomials are indispensable, they must belong to any minimal generating set. Let us find the other binomials needed to obtain a minimal generating set for $I_{\mathcal{D}_{2m}}$.

$G(\mathbf{b_8})$ and $G(\mathbf{b_{10}})$ have three connected components: $\{x_{i,j}y_k\} \cup \{x_{j,k}y_i\} \cup \{x_{i,k}y_j\}$ and $\{x_{i,j}x_{k,\ell}\} \cup \{x_{i,k}x_{j,\ell}\} \cup \{x_{i,\ell}x_{j,k}\}$, respectively. Since each connected component of these graphs is a singleton, the complete graphs $S_{\mathbf{b_8}}$ and $S_{\mathbf{b_{10}}}$ are triangles obtained by joining connected components of $G(\mathbf{b_8})$ and $G(\mathbf{b_{10}})$, respectively. Thus, spanning trees of these complete graphs can be obtained by deleting one edge from the triangle.

Therefore, in a minimal generating set only one of the following three binomial couples may appear corresponding to $G(\mathbf{b_8})$;

$x_{i,j}y_k - x_{j,k}y_i$ and $x_{i,j}y_k - x_{i,k}y_j$, or
$x_{j,k}y_i - x_{i,j}y_k$ and $x_{j,k}y_i - x_{i,k}y_j$, or
$x_{i,k}y_j - x_{i,j}y_k$ and $x_{i,k}y_j - x_{j,k}y_i$

and similarly for $G(\mathbf{b_{10}})$;

$x_{i,j}x_{k,\ell} - x_{i,k}x_{j,\ell}$ and $x_{i,j}x_{k,\ell} - x_{i,\ell}x_{j,k}$, or
$x_{i,k}x_{j,\ell} - x_{i,j}x_{k,\ell}$ and $x_{i,k}x_{j,\ell} - x_{i,\ell}x_{j,k}$, or
$x_{i,\ell}x_{j,k} - x_{i,j}x_{k,\ell}$ and $x_{i,\ell}x_{j,k} - x_{i,k}x_{j,\ell}$.

Hence, there are many different minimal generating sets for the toric ideal $I_{\mathcal{D}_{2m}}$, and in particular the set $\mathcal{G}_{\mathcal{D}_{2m}}$ is a minimal generating set of $I_{\mathcal{D}_{2m}}$. □

4.2. Odd Case $\mathcal{D}_{2m+1}$. In this case, we consider the set $\mathcal{D}_{2m+1} \subset \mathbb{Z}^n$ given by

$$\{2\mathbf{e}_i, \mathbf{e}_j, 4\mathbf{e}_1, 4\mathbf{e}_n, \mathbf{e}_k + \mathbf{e}_\ell, \mathbf{e}_1 + \mathbf{e}_n, \mathbf{e}_i + 2\mathbf{e}_1, \mathbf{e}_i + 2\mathbf{e}_n, \mathbf{e}_i + 3\mathbf{e}_1 + \mathbf{e}_n, \mathbf{e}_i + \mathbf{e}_1 + 3\mathbf{e}_n$$
$$\mid i, k, \ell \in J,\, j \in J^c \text{ and } k < \ell\},$$

where $J = \{2, 4, \ldots, n-1\}$, $J^c = \{3, 5, \ldots, n-2\}$ and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{Z}^n$. Again, the $\mathcal{D}_{2m+1}$-degrees of the variables are exactly the elements of $\mathcal{D}_{2m+1}$ as before.

By Theorem 3.3, we see that $I_{\mathcal{D}_{2m+1}}$ is generated by the set $\mathcal{G}_{\mathcal{D}_{2m+1}}$ of binomials

$x_{i,k}x_{j,\ell} - x_{i,j}x_{k,\ell}$	$x_{i,\ell}x_{j,k} - x_{i,j}x_{k,\ell}$	$i < j < k < \ell \in J$
$x_{j,k}x_{i,n-1} - x_{i,j}x_{k,n-1}$	$x_{i,k}x_{j,n-1} - x_{i,j}x_{k,n-1}$	$i < j < k \in J$
$x_{j,k}x_{i,n} - x_{i,j}x_{k,n}$	$x_{i,k}x_{j,n} - x_{i,j}x_{k,n}$	$i < j < k \in J$
$x_jx_{i,k} - x_{i,j}x_{j,k}$	$x_{i,j}x_{i,k} - x_ix_{j,k}$	$i < j < k \in J$
$x_kx_{i,j} - x_{i,k}x_{j,k}$		$i < j < k \in J$
$x_ix_j - x_{i,j}^2$	$x_{i,j}x_1 - x_{i,n-1}x_{j,n-1}$	$i < j \in J$
$x_{i,j}x_n - x_{i,n}x_{j,n}$	$x_{i,j}x_{i,n-1} - x_ix_{j,n-1}$	$i < j \in J$
$x_{i,j}x_{i,n} - x_ix_{j,n}$	$x_jx_{i,n-1} - x_{i,j}x_{j,n-1}$	$i < j \in J$
$x_jx_{i,n} - x_{i,j}x_{j,n}$	$x_{j,n-1}x_{i,n} - x_{i,n-1}x_{j,n}$	$i < j \in J$
$x_{i,n-1}x_{j,n} - x_{1,n}^2x_{i,j}$		$i < j \in J$
$x_ix_1 - x_{i,n-1}^2$	$x_ix_n - x_{i,n}^2$	$i \in J$
$x_{i,n-1}x_{i,n} - x_{1,n}^2x_i$	$x_{i,n}x_1 - x_{1,n}^2x_{i,n-1}$	$i \in J$
$y_{i,1} - x_{1,n}x_{i,1}$	$y_{i,n} - x_{1,n}x_{i,n}$	$i \in J$
$x_{i,n-1}x_n - x_{1,n}^2x_{i,n}$	$x_1x_n - x_{1,n}^4$	$i \in J$

Therefore, possible Betti $\mathcal{D}_{2m+1}$-degrees are

$$\begin{array}{ll}
\mathbf{b_1} = 2\mathbf{e}_i + 2\mathbf{e}_j, & \mathbf{b_2} = 4\mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_j, \\
\mathbf{b_3} = \mathbf{e}_i + \mathbf{e}_j + 4\mathbf{e}_n, & \mathbf{b_4} = 2\mathbf{e}_1 + 2\mathbf{e}_i + \mathbf{e}_j, \\
\mathbf{b_5} = 2\mathbf{e}_i + \mathbf{e}_j + 2\mathbf{e}_n, & \mathbf{b_6} = 2\mathbf{e}_1 + \mathbf{e}_i + 2\mathbf{e}_j, \\
\mathbf{b_7} = \mathbf{e}_i + 2\mathbf{e}_j + 2\mathbf{e}_n, & \mathbf{b_8} = 2\mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_j + 2\mathbf{e}_n, \\
\mathbf{b_9} = 2\mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k, & \mathbf{b_{10}} = \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k + 2\mathbf{e}_n, \\
\mathbf{b_{11}} = \mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_k, & \mathbf{b_{12}} = 2\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k, \\
\mathbf{b_{13}} = \mathbf{e}_i + \mathbf{e}_j + 2\mathbf{e}_k, & \mathbf{b_{14}} = \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_\ell, \\
\mathbf{b_{15}} = 4\mathbf{e}_1 + 2\mathbf{e}_i, & \mathbf{b_{16}} = 2\mathbf{e}_i + 4\mathbf{e}_n, \\
\mathbf{b_{17}} = 2\mathbf{e}_1 + 2\mathbf{e}_i + 2\mathbf{e}_n, & \mathbf{b_{18}} = 4\mathbf{e}_1 + \mathbf{e}_i + 2\mathbf{e}_n, \\
\mathbf{b_{19}} = 3\mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_n, & \mathbf{b_{20}} = \mathbf{e}_1 + \mathbf{e}_i + 3\mathbf{e}_n \\
\mathbf{b_{21}} = 2\mathbf{e}_1 + \mathbf{e}_i + 4\mathbf{e}_n, & \mathbf{b_{22}} = 4\mathbf{e}_1 + 4\mathbf{e}_n.
\end{array}$$

Next we prove that these binomials constitute a minimal generating set for $I_{\mathcal{D}_{2m+1}}$.

4.3. Proposition. *The set $\mathcal{G}_{\mathcal{D}_{2m+1}}$ is a minimal generating set of $I_{\mathcal{D}_{2m+1}}$.*

Proof. There is no binomial in $I_{\mathcal{D}_{2m+1},\mathbf{b_1}}$. Thus, the graph $G(\mathbf{b_1})$ consists of two connected components $\{x_ix_j\}$ and $\{x_{i,j}^2\}$. Similarly, $G(\mathbf{b_2})$ has $\{x_{i,j}x_1\}$ and $\{x_{i,n-1}x_{j,n-1}\}$, $G(\mathbf{b_3})$ has $\{x_{i,j}x_n\}$ and $\{x_{i,n}x_{j,n}\}$, $G(\mathbf{b_4})$ has $\{x_{i,j}x_{i,n-1}\}$ and $\{x_ix_{j,n-1}\}$, $G(\mathbf{b_5})$ has $\{x_{i,j}x_{i,n}\}$ and $\{x_ix_{j,n}\}$, $G(\mathbf{b_6})$ has $\{x_jx_{i,n-1}\}$ and $\{x_{i,j}x_{j,n-1}\}$, $G(\mathbf{b_7})$ has $\{x_jx_{i,n}\}$ and $\{x_{i,j}x_{j,n}\}$, $G(\mathbf{b_{11}})$ has $\{x_jx_{i,k}\}$ and $\{x_{i,j}x_{j,k}\}$, $G(\mathbf{b_{12}})$ has $\{x_{i,j}x_{i,k}\}$ and $\{x_ix_{j,k}\}$, $G(\mathbf{b_{13}})$ has $\{x_kx_{i,j}\}$ and $\{x_{i,k}x_{j,k}\}$, $G(\mathbf{b_{15}})$ has $\{x_i, x_1\}$ and $\{x_{i,n-1}^2\}$, $G(\mathbf{b_{16}})$ has $\{x_i, x_n\}$ and $\{x_{i,n}^2\}$, $G(\mathbf{b_{17}})$ has $\{x_{i,n-1}x_{i,n}\}$ and $\{x_{1,n}^2x_i\}$, $G(\mathbf{b_{18}})$ has $\{x_{i,n}x_1\}$ and $\{x_{1,n}^2x_{i,n-1}\}$, $G(\mathbf{b_{19}})$ has $\{y_{i,1}\}$ and $\{x_{1,n}x_{i,1}\}$, $G(\mathbf{b_{20}})$ has $\{y_{i,n}\}$ and $\{x_{1,n}x_{i,n}\}$, $G(\mathbf{b_{21}})$ has $\{x_{i,n-1}x_n\}$ and $\{x_{1,n}^2x_{i,n}\}$, and finally $G(\mathbf{b_{22}})$ has $\{x_1x_n\}$ and $\{x_{1,n}^4\}$ as its connected components.

Indispensable binomials of $I_{\mathcal{D}_{2m+1}}$ are all determined by these graphs by Corollary 2.10 in [6] and hence, corresponding binomials belong to any minimal generating set.

The other graphs $G(\mathbf{b_8})$, $G(\mathbf{b_9})$, $G(\mathbf{b_{10}})$ and $G(\mathbf{b_{14}})$ have three connected components:

$$\{x_{i,n-1}x_{j,n}\} \cup \{x_{j,n-1}x_{i,n}\} \cup \{x_{1,n}^2x_{i,j}\}, \{x_{j,k}x_{i,n-1}\} \cup \{x_{i,j}x_{k,n-1}\} \cup \{x_{i,k}x_{j,n-1}\},$$

$$\{x_{j,k}x_{i,n}\} \cup \{x_{i,j}x_{k,n}\} \cup \{x_{i,k}x_{j,n}\} \text{ and } \{x_{i,k}x_{j,\ell}\} \cup \{x_{i,j}x_{k,\ell}\} \cup \{x_{i,\ell}x_{j,k}\}$$

respectively. Each connected component of these graphs is a singleton. Therefore, the complete graphs are triangles obtained by joining the connected components of the graphs $G(\mathbf{b_8})$, $G(\mathbf{b_9})$, $G(\mathbf{b_{10}})$ and $G(\mathbf{b_{14}})$, respectively. Thus, we obtain the spanning trees by deleting one edge from each triangle.

Therefore, in a minimal generating set only one of the following three binomial couples may appear corresponding to $G(\mathbf{b_8})$;

$$\begin{gathered} x_{i,n-1}x_{j,n} - x_{1,n}^2x_{i,j} \text{ and } x_{i,n-1}x_{j,n} - x_{j,n-1}x_{i,n}, \\ x_{1,n}^2x_{i,j} - x_{i,n-1}x_{j,n} \text{ and } x_{1,n}^2x_{i,j} - x_{j,n-1}x_{i,n}, \\ x_{j,n-1}x_{i,n} - x_{i,n-1}x_{j,n} \text{ and } x_{j,n-1}x_{i,n} - x_{1,n}^2x_{i,j} \end{gathered}$$

and the same is true for the following couples corresponding to $G(\mathbf{b_9})$;

$$\begin{gathered} x_{j,k}x_{i,n-1} - x_{i,j}x_{k,n-1} \text{ and } x_{j,k}x_{i,n-1} - x_{i,k}x_{j,n-1}, \\ x_{i,j}x_{k,n-1} - x_{j,k}x_{i,n-1} \text{ and } x_{i,j}x_{k,n-1} - x_{i,k}x_{j,n-1}, \\ x_{i,k}x_{j,n-1} - x_{j,k}x_{i,n-1} \text{ and } x_{i,k}x_{j,n-1} - x_{i,j}x_{k,n-1} \end{gathered}$$

and similarly for $G(\mathbf{b_{10}})$;

$$\begin{gathered} x_{j,k}x_{i,n} - x_{i,j}x_{k,n} \text{ and } x_{j,k}x_{i,n} - x_{i,k}x_{j,n}, \\ x_{i,j}x_{k,n} - x_{j,k}x_{i,n} \text{ and } x_{i,j}x_{k,n} - x_{i,k}x_{j,n}, \\ x_{i,k}x_{j,n} - x_{j,k}x_{i,n} \text{ and } x_{i,k}x_{j,n} - x_{i,j}x_{k,n} \end{gathered}$$

and for $G(\mathbf{b_{14}})$;

$$\begin{gathered} x_{i,k}x_{j,\ell} - x_{i,j}x_{k,\ell} \text{ and } x_{i,k}x_{j,\ell} - x_{i,\ell}x_{j,k}, \\ x_{i,j}x_{k,\ell} - x_{i,k}x_{j,\ell} \text{ and } x_{i,j}x_{k,\ell} - x_{i,\ell}x_{j,k}, \\ x_{i,\ell}x_{j,k} - x_{i,j}x_{k,\ell} \text{ and } x_{i,\ell}x_{j,k} - x_{i,k}x_{j,\ell}. \end{gathered}$$

These discussions show that there are many minimal generating sets for $I_{\mathcal{D}_{2m+1}}$ and in particular, the set $\mathcal{G}_{\mathcal{D}_{2m+1}}$ is a minimal generating set of $I_{\mathcal{D}_{2m+1}}$. □

4.3. E_n**-type.** In this case, it is easy to check that the Gröbner basis given in Theorem 3.5 constitutes a minimal generating set for each $n = 6, 7, 8$. Indeed, there is nothing to prove for the case of $n = 8$, as the corresponding toric ideal is trivial. In the case of $n = 7$, the corresponding toric ideal is generated minimally by the 6 binomials given in Theorem 3.5 (2) as we explain now. Let $\mathbf{b}$ be the $\mathcal{E}_7$-degree of a binomial given in Theorem 3.5 (2). Since the graph $G(\mathbf{b})$ has two connected components, the complete graph $S_{\mathbf{b}}$ (and its spanning tree $T_{\mathbf{b}}$) is a line segment and thus $\mathcal{F}_{T_b}$ is a singleton. As the connected components of $G(\mathbf{b})$ are singletons, $\mathcal{F}_{T_b}$ *must* consist of the binomial we have started with. This means that the binomial is *indispensable*, i.e. appears in any minimal generating set. Therefore the toric ideal has a unique minimal generating set provided by Theorem 3.5 (2).

As for the case of $n = 6$, we have a generating set given in Theorem 3.5 (1) consisting of 35 binomials. Let $\mathbf{b} = 2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_5$ which is the $\mathcal{E}_6$-degree of the binomial $x_{11}x_{13} - x_7x_8x_9$. The graph $G(\mathbf{b})$ has two connected components $\{x_{11}x_{13}\}$ and $\{x_7x_8x_9, x_7^2x_{10}\}$. As before the complete graph $S_{\mathbf{b}}$ (and its spanning tree $T_{\mathbf{b}}$) is a line segment and thus $\mathcal{F}_{T_b}$ is a singleton but it changes according to which monomial we choose from the second component of $G(\mathbf{b})$. So, $\mathcal{F}_{T_b}$ is either $\{x_{11}x_{13} - x_7x_8x_9\}$ or $\{x_{11}x_{13} - x_7^2x_{10}\}$. We have the same situation for the following degrees:

$$\begin{gathered} \mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_4 + \mathbf{e}_5, \\ \mathbf{b} = 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4 + 2\mathbf{e}_5, \\ \mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_4 + 2\mathbf{e}_5. \end{gathered}$$

It is a standard procedure to check that the other 31 binomials given in Theorem 3.5 (1) are indispensable, so there are 8 different minimal generating sets for the toric ideal including the one provided by Theorem 3.5 (1).

5. What about A_n-type?

There are two ways to study the question of whether or not toric ideals of these configurations have squarefree initial ideals. The first one is to produce an example with no squarefree initial ideal using computer programs. In order to achieve this goal one has to find all possible initial ideals for a fixed configuration. The toric ideal corresponding to A_2 is generated by a binomial with a squarefree monomial. One can compute 29 different initial ideals for the toric ideal of A_3 and obtain the unique squarefree one generated by 6 monomials by using e.g. Gfan [12]. As long as n gets larger values listing all the possible initial ideals (or regular triangulations of the corresponding convex polytope) using computer programs becomes problematic. In the second way, one has to determine the correct term order with respect to which the initial ideal is generated by squarefree monomials by heuristic/experimental methods. For the toric ideal of A_4 the lexicographic ordering with $x_{14} > x_{12} > x_{10} > x_9 > x_7 > x_4 > x_8 > x_6 > x_5 > x_3 > x_{11} > x_1 > x_2$ gives a Gröbner basis consisting of 54 binomials with a squarefree initial ideal. Similarly, the toric ideal of A_5 has a squarefree initial ideal generated by 105 monomials which are obtained as the initial terms with respect to the lexicographic ordering with $x_{19} > x_{18} > x_{17} > x_{11} > x_{10} > x_3 > x_{16} > x_{13} > x_7 > x_{15} > x_{14} > x_{12} > x_8 > x_5 > x_9 > x_4 > x_6 > x_2 > x_1$. However, for larger values of n, proving the existence of squarefree initial ideals is difficult as well. This is due to the fact that there is no general formula for the vector configuration as in the case of D-type, although one can compute them one by one with e.g. CoCoA using the algorithm described in [21].

Acknowledgments

The authors would like to thank M. Tosun for increasing their interest on toric varieties of rational surface singularities and A. Jensen for correspondences about the possibility to use Gfan to enumerate all initial ideals. They also thank the anonymous referee for useful suggestions which improved the presentation of the paper.

References

[1] S. Altınok and M. Tosun, *Generators for semigroup of Lipman*, Bull. Braz. Math. Soc. 39 (1) (2008), 123-135.

[2] A. Aramova, J. Herzog, T. Hibi, *Finite lattices and lexicographic Gröbner bases*, European J. Combin. 21 (2000), 431-439.

[3] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. 88 (1966), 129-136.

[4] N. Bourbaki, Groupes et algébres de Lie, Ch. IV, V, VI, Hermann, Paris, 1968.

[5] W. Bruns and T. Römer. *h-vectors of Gorenstein polytopes*, J. Comb. Theory, Ser. A 114 (1) (2007), 65-76.

[6] H. Charalambous, A. Katsabekis and A. Thoma, *Minimal systems of binomial generators and the indispensable complex of a toric Ideal*, Proc. of Amer. Math. Soc., 135 (2007), 3443-3451.

[7] CoCoA: a system for doing Computations in Commutative Algebra, Available at http://cocoa.dima.unige.it.

[8] P. Dueck, S. Hoşten, B. Sturmfels, *Normal toric ideals of low codimension*, Journal of Pure and Applied Algebra 213 (8) (2009), 1636-1641.

[9] P. Du Val, *On isolated singularities which do not affect the conditions of adjunction*, Part I, Math. Proc. Cambridge Philos. Soc. 30 (1934), 453-465.

[10] H.Grauert, *Über Modifikationen und exzeptionnelle analytische Mengen*, Math.Ann. 146 (1962), 331-368.
[11] C. Haase and A. Paffenholz, *Quadratic Gröbner bases for smooth* 3×3 *transportation polytopes*, J. Algebraic Combinatorics 30 (4) (2009), 477-489.
[12] A. Jensen, Gfan, a software system for Gröbner fans and tropical varieties, Available at http://home.imf.au.dk/jensen/software/gfan/gfan.html.
[13] J. Jonsson, V. Welker. *A spherical initial ideal forPfaffians*, Illinois J. Math. 51 (4) (2007), 1397-1407.
[14] T. Kitamura, *Gröbner bases associated with positive roots and Catalan numbers*, Osaka J. Math. 42 (2), 421-433 (2005).
[15] J. Lipman, *Rational singularities with applications to algebraic surfaces and unique factorization*, Publ. Math. IHES, 36 (1) (1969), 195-279.
[16] H. Ohsugi and T. Hibi, *Quadratic initial ideals of root systems*, Proc. Amer. Math. Soc. 130 (7) (2002), 1913-1922.
[17] H. Ohsugi and T. Hibi, *Toric ideals arising from contingency tables*, Ramanujan Mathematical Society Lecture Notes Series 2 (2006), 91-115.
[18] E. O'Shea and R. Thomas, *Toric initial ideals of* Δ*-normal configurations: Cohen-Macaulayness and degree bounds*, Journal of Algebraic Combinatorics 21 (3) (2005), 247-268.

[19] A. Simis and R. H. Villarreal, *Constraints for the normality of monomial subrings and birationality*, Proc. Amer. Math. Soc. 131 (7) (2003), 2043-2048.
[20] B. Sturmfels, Gröbner Bases and Convex Polytopes, Univ. Lecture Ser. Vol. 8, Amer. Math. Soc., Providence, RI (1996).
[21] M. Şahin, *Hilbert basis of the Lipman semigroup*, C. R. Acad. Sci. Paris, Ser. I 348 (23-24) (2010), 1311-1314.
[22] M. Tosun, *Tyurina components and rational cycles for rational singularities*, Turkish J. Math. 23 (1999), no. 3, 361-374.
[23] O. Zariski, *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface*, Ann. of Math. 76 (1962), 560-615.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1071 – 1077

Affine singular control systems on Lie groups

Memet KULE*†

Abstract

The purpose of this paper is to show that an affine singular control system S on a connected Lie group G leads to two subsystems: An affine control system on a homogeneous space G/H and an algebraic-differential control system on H of G, where H is some closed subgroup of G.

Keywords: Singular Control System, Homogeneous Space, Algebraic Differential Equations.

2000 AMS Classification: 93B05, 93B27, 93C10

Received 11/04/2014 : Accepted 26/06/2014 Doi : 10.15672/HJMS.2015449662

1. INTRODUCTION

Let G denote a connected Lie group with Lie algebra $L(G)$ (the set of right invariant vector fields on G). Let us denote by $Af(G)$ the affine group on G. An affine singular control system S on G is a family of differential equations

$$E_{g(t)}\left(\dot{g}(t)\right) = F(g(t)) + \sum_{j=1}^{d} u_j(t) F^j(g(t)), \qquad g(t) \in G, \tag{1.1}$$

where $u \in U$ is the class of unrestricted piecewise constant admissible controls with values on $\mathbb{R}^d$, i.e., the set

$$U = \left\{ u : [0, T_u] \to \mathbb{R}^d \mid u \text{ is a piecewise constant function} \right\}.$$

Here, the vector fields $F, F^1, ..., F^d$ belong to the affine algebra $af(G)$ and E is a non-invertible derivation on $L(G)$. The operator $E_g : T_gG \to T_gG$ is defined by $E_g = (l_g)_* \circ E \circ (l_{g^{-1}})_*$, where

$$\left(l_{g^{-1}}\right)_* : T_gG \to T_eG, \qquad E : T_eG \to T_eG, \qquad (l_g)_* : T_eG \to T_gG.$$

*Department of Mathematics, Faculty of Arts and Sciences,University of 7 Aralık, 79000 Kilis, Turkey, Email: memetkule@kilis.edu.tr
†Corresponding Author.

The singular control system on Euclidean spaces was introduced by Dai [5]. The system has been well developed on Lie groups, see [3, 4]. Thus, there exist the basic ingredients to start with the study of affine singular control systems on Lie groups.

Throughout this paper H, which is the subgroup of G will be assumed to be closed, because in this case the quotient set G/H is a homogeneous space. We also assume that the vector fields $F, F^1, \ldots, F^d$ are projectable on the homogeneous space G/H leads to a decomposition of (1.1) in two systems, one on G/H and the one other on H. The algebraic-differential subsystem plays a crucial role in the understanding of the trajectories for affine singular control systems on Lie groups. Actually, the solvability of (1.1) depends just on when we are able to solve (3.1). Furthermore, we establish a special solution of (3.1) and hence the solution of (1.1).

This paper is organized as follows. In the next section we introduce the notion of an affine control system on a connected Lie group G. In Section 3, vector fields of the affine singular control system on homogeneous space are introduced, and we obtain the decomposition for the affine singular control system S on G, as well as the solution of the decomposition (3.1).

2. Affine Control Systems

In this section, the definition of affine vector fields are recalled. More details can found in [2, 8, 7].

Let G denote a connected Lie group of dimension n with Lie algebra $L(G)$. The affine group $Af(G)$ of G is the semidirect product of $Aut(G)$ and G, i.e., $Af(G) = Aut(G) \times_s G$. The semidirect product consists of all pairs $(\phi, g) \in Af(G)$, with the group structure given by

$$(\phi, g_1) \cdot (\psi, g_2) = (\phi \circ \psi, g_1 \phi(g_2)),$$

that (Id, e) is the group identity and that $\left(\phi^{-1}, \phi^{-1}\left(g^{-1}\right)\right)$ is the inverse of (ϕ, g). Then, the mapping $g \to (Id, g)$ embeds G into $Af(G)$ and $\phi \to (\phi, e)$ embeds $Aut(G)$ into $Af(G)$. Therefore, G and $Aut(G)$ are subgroups of $Af(G)$. There is a natural action

$$Af(G) \times G \to G$$

defined by

$$(\phi, g_1) \cdot g_2 \to g_1 \phi(g_2),$$

where $(\phi, g_1) \in Af(G)$ and $g_2 \in G$. This action is transitive. Indeed, if it is taken $g_2 = e$, then $(\phi, g_1) \cdot e = g_1$ since $\phi(g_2) = e$.

Denote by $AutL(G)$ the automorphism group of $L(G)$ and whose Lie algebra is $DerL(G)$, the Lie algebra of derivations of $L(G)$. If G is simply connected, then $Aut(G)$ and $AutL(G)$ are isomorphic. In fact, there is an isomorphism Φ which assigns to each automorphism ϕ of G its differential $d\phi \mid_{Id}$ at the identity. Any automorphism ϕ of $L(G)$ extends to an automorphism of G, therefore, Φ is indeed an isomorphism between $Aut(G)$ and $AutL(G)$. Thus, in this case, the Lie algebra of $Aut(G)$ is $DerL(G)$.

The Lie bracket in $af(G)$ is given by

$$\left[\left(\mathcal{D}^1, Y^1\right), \left(\mathcal{D}^2, Y^2\right)\right] = \left(\left[\mathcal{D}^1, \mathcal{D}^2\right], \mathcal{D}^1 Y^2 - \mathcal{D}^2 Y^1 + \left[Y^1, Y^2\right]\right),$$

where the first coordinate in the bracket is that of $DerL(G)$, while the second is that of $L(G)$ and $\mathcal{D}X$ denotes the derived action of $DerL(G)$ on $L(G)$. The Lie algebra $af(G)$ of $Af(G)$ is the semidirect product $DerL(G) \times_s L(G)$. An affine vector field F on G can be exclusively separated decomposed into a sum

$$F = \mathcal{D} + Y,$$

where $\mathcal{D} \in DerL(G)$ and $Y \in L(G)$. Thus, an affine control system on G is determined by the dynamic parametrized by $u \in U$,

$$\dot{g}(t) = (\mathcal{D} + Y)(g(t)) + \sum_{j=1}^{d} u_j(t) \left(\mathcal{D}^j + Y^j\right)(g(t)), \qquad g(t) \in G,$$

where right invariant vector fields $Y, Y^1, ..., Y^d \in L(G)$ and $\mathcal{D}, \mathcal{D}^1, ..., \mathcal{D}^d \in DerL(G)$.

As usual, for any $g \in G$, denote by r_g the right translation on G by g; that is, $r_g(x) = xg$ for all x in G. l_g will denote the left translation by g; that is, $l_g(x) = gx$. We recall that $L(G)$ is isomorphic to the tangent space T_eG of G at the identity element e. Thus, a right invariant vector field Y on G is determined by its value at e. In particular, $Y(g) = (r_g)_*Y(e)$ and its flow is given by $Y(g(t)) = r_g(Y(e(t)))$, where $(r_g)_*$ is derivative of r_g.

Let $\mathcal{X}$ be an infinitesimal automorphism of the Lie group G, that is, the flow $(\mathcal{X}_t)_{t\in\mathbb{R}}$ induced by the vector field $\mathcal{X}$ is a one-parameter subgroup of $Aut(G)$. Then, $\mathcal{X}$ induces a derivation $\mathcal{D} = -ad_{\mathcal{X}}$ on $L(G)$ for $\mathcal{D} \in DerL(G)$. This condition on ad means

$$\mathcal{D}Y = -[\mathcal{X}, Y]$$

for $\forall Y \in L(G)$ and verifies $\mathcal{X}(e) = 0$.

3. Affine Singular Control Systems

Throughout this section, we can always assume that G is simply connected and Π_*Y is one-to-one.

Let G denote a Lie group and let H denote a closed Lie subgroup of G with Lie algebra $L(H)$. For closed subgroup H of G, $G/H = \{gH : g \in G\}$ denotes the homogeneous space of left cosets of H, and we denote by Π the natural projection of G onto G/H. In order to any right invariant vector field $Y \in L(G)$, Y projects to Π_*Y on G/H, will be induced to as a well-defined invariant vector field on G/H. Furthermore, $\Pi_*L(G) = \{\Pi_*Y;\ Y \in L(G)\}$ is a Lie algebra and Π_* is a Lie algebra morphism from $L(G)$ onto $\Pi_*L(G)$. Also the projection Π_*Y of $Y \in L(G)$ vanishes at the point H iff $Y \in L(H)$.

We consider an affine singular control system S with derivation $E \in Der(L(G))$ and vector field $\mathcal{X}$ induced by a derivation $\mathcal{D} \in Der(L(G))$. Now, we wish to show the existence of a vector field Π-related to $\mathcal{X}$ on G/H. There exists a vector field π-related to $\mathcal{X}$ on G/H such that

$$\Pi(\mathcal{X}(g(t)x(t))) = \Pi(\mathcal{X}(g(t)))$$

for $\forall g \in G, \forall x \in H$ and $\forall t \in \mathbb{R}$. On the other hand, the corresponding flows on G/H are related by

$$\Pi\left(\mathcal{X}(g(t)x(t))\right) = \Pi\left(\mathcal{X}(g(t))\mathcal{X}(x(t))\right) = \Pi\left(\mathcal{X}(g(t))\right)\mathcal{X}(x(t))H,$$

where $\mathcal{X}(x(t))$ is the one-parameter subgroup in H. Because of the existence of the projection, the subgroup H is invariant under the flow of $\mathcal{X}$; thus, $\mathcal{X}$ is tangent to H.

Now, let H be connected. Because of the elements of H, which are products of exponentials, the invariance of H under $\mathcal{X}$ writes

$$\forall Y \in L(H),\ \forall t \in \mathbb{R} \qquad \mathcal{X}_t(\exp Y) = \exp(e^{t\mathcal{D}}Y) \in H,$$

or equivalently as

$$\forall Y \in L(H),\ \forall t \in \mathbb{R} \qquad e^{t\mathcal{D}}Y \in L(H).$$

Finally, its Lie algebra $L(H)$ is invariant under $\mathcal{D}$.

Under the above assumptions, the projection of $\mathcal{X}$ onto G/H will be denoted by $\Pi_*\mathcal{X}$.

Now, we take an affine vector field $F = \mathcal{X} + Y$ on G. This decomposition is chosen in order to ensure that the projection $\Pi_* Y$ of Y onto G/H is well defined. If $\Pi_* \mathcal{X}$ exists, then F is Π-related to a vector field on G/H. It follows that $\Pi_* F = \Pi_* \mathcal{X} + \Pi_* Y$ will stand for the projection of F onto G/H. Then, there exists an affine control system on G

$$\dot{g}(t) = \Pi\left(F(g(t))\right) + \sum_{j=1}^{d} u_j(t)\, \Pi\left(F^j(g(t))\right), \quad g(t) \in G,$$

which projects down onto G/H.

Now, it follows that $(\Pi_* E)^{-1} D \in Der(L(G)/L(H))$ since $E, D \in Der(L(G))$ and $L(H)$-invariant. Let us denote by $\Pi_*\left((\Pi_* E)^{-1} D\right) \in Der(L(G))$ such that its restriction to $L(G)/L(H)$ coincide with $(\Pi_* E)^{-1} D$. Thus, $\Pi_* (\Pi_* E)^{-1} \mathcal{X} = \Pi_* \mathcal{X}$ on G/H. On the other hand, we define $\Pi_*\left((\Pi_* E)^{-1} Y\right)$ as the only invariant vector fields determined by $(\Pi_* E)^{-1} Y(e) \in L(G)/L(H)$. Thus, the mapping $E_g : T_g G \to T_g G$ is invertible on the homogeneous space G/H for any $g \in G$. In particular, we can consider the affine control system $\Pi(S)$ on G/H in the following way:

$$\begin{aligned} \dot{y}(t) \quad = \quad & \left(E_{y(t)}\right)^{-1} \circ \Pi\left(\mathcal{X}(y(t))\right) + \left(E_{y(t)}\right)^{-1} \circ \Pi\left(Y(y(t))\right) + \\ & \left(E_{y(t)}\right)^{-1} \circ \sum_{j=1}^{d} u_j(t)\, \Pi\left(\mathcal{X}^j(y(t))\right) + \left(E_{y(t)}\right)^{-1} \circ \sum_{j=1}^{d} u_j(t)\, \Pi\left(Y^j(y(t))\right), \end{aligned}$$

where $y(t) \in G/H$ is an integral curve of the projected affine control system on the homogeneous space G/H. Also $\dot{y}(t)$ has a well-defined solution for each piecewise admissible control u and any initial condition in G.

3.1. Theorem. *Let G be a connected Lie group with Lie algebra $L(G)$ and assume that the connected Lie subgroup H of G with Lie algebra $L(H)$ is closed. The curve $g(t)$ is solution of the affine singular control system S for the initial condition $y(0) = y \in G/H$ associated to the control $u \in U$. Then, there exists a one parameter group $x(t)$ of the closed subgroup H which together satisfies the algebraic-differential equation*

$$\begin{aligned} E_{g(t)}\left(y(t)\,\dot{x}(t)\right) \quad = \quad & \left(l_{y(t)}\right)_* \left(\mathcal{X}_{L(H)}(x(t)) + \sum_{j=1}^{d} u_j(t)\, \mathcal{X}^j_{L(H)}(x(t))\right) \\ & + \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right), \end{aligned} \tag{3.1}$$

where $\mathcal{X}_{L(H)}, \mathcal{X}^1_{L(H)}, \ldots, \mathcal{X}^d_{L(H)}$ are infinitesimal automorphisms of the Lie subgroup H and $Y_{L(H)}, Y^1_{L(H)}, ..., Y^d_{L(H)} \in L(H)$.

Proof. Assume there exists a solution $g(t)$ of the affine singular control system S with control u and initial condition $y(0) = y$. Then, for almost every t, there exists a curve $x(t) \in H$, with $x(0) = e$, where e is the identity on G, such that

$$\begin{aligned} g(t) &= y(t)\, x(t) \\ \dot{y(t)\, x(t)} &= \left(l_{y(t)}\right)_* \dot{x}(t) + \left(r_{x(t)}\right)_* \dot{y}(t) \end{aligned}$$

Applying $E_{g(t)}$ on both sides, equation takes form,

$$E_{g(t)}\left(\dot{g}(t)\right) = E_{g(t)}\left(y(t)\,\dot{x}(t)\right) + E_{g(t)}\left(\dot{y}(t) x(t)\right).$$

Hence, we get

$$\begin{aligned}
&\mathcal{X}(g(t)) + Y(g(t)) + \sum_{j=1}^{d} u_j(t)\,\mathcal{X}^j(g(t)) + \sum_{j=1}^{d} u_j(t)\,Y^j(g(t)) \\
= \;& E_{g(t)}\left(y(t)\,x\dot{(t)}\right) + \\
&\left(r_{x(t)}\right)_*\left(\Pi(\mathcal{X})(y(t)) + \Pi(Y)(y(t)) + \sum_{j=1}^{d} u_j(t)\,\Pi\left(\mathcal{X}^j\right)(y(t)) + \sum_{j=1}^{d} u_j(t)\,\Pi\left(Y^j\right)(y(t))\right).
\end{aligned}$$

Since $Y, Y^1, ..., Y^d$ are elements of the Lie algebra $L(G)$, we can project this dynamic on any homogeneous space of G. In particular,

$$\left(r_{x(t)}\right)_*\left(\Pi(Y)(y(t)) + \sum_{j=1}^{d} u_j(t)\,\Pi\left(Y^j\right)(y(t))\right) = \Pi(Y)(g(t)) + \sum_{j=1}^{d} u_j(t)\,\Pi\left(Y^j\right)(g(t)).$$

Thus, it follows that

$$\begin{aligned}
&E_{g(t)}\left(y(t)\,x\dot{(t)}\right) = \mathcal{X}(g(t)) - \left(r_{x(t)}\right)_*\Pi(\mathcal{X})(y(t)) + \\
&\sum_{j=1}^{d} u_j(t)\,\mathcal{X}^j(g(t)) - \left(r_{x(t)}\right)_*\sum_{j=1}^{d} u_j(t)\,\Pi\left(\mathcal{X}^j\right)(y(t)) \\
&+ Y(g(t)) - \Pi(Y)(g(t)) + \sum_{j=1}^{d} u_j(t)\,Y^j(g(t)) - \sum_{j=1}^{d} u_j(t)\,\Pi\left(Y^j\right)(g(t))
\end{aligned}$$

On the other hand, $\mathcal{X}_t \in Aut(G)$ for any real number t, and therefore,

$$\mathcal{X}(g(t)) = \mathcal{X}(y(t)\,x(t)) = \mathcal{X}(y(t))\,\mathcal{X}(x(t)).$$

By taking a derivative of the product $\mathcal{X}(g(t))$ at time t, we obtain

$$\mathcal{X}(y(t)\,x(t)) = \left(r_{x(t)}\right)_*\mathcal{X}(y(t)) + \left(l_{y(t)}\right)_*\mathcal{X}(x(t)).$$

By construction for each $t \in \mathbb{R}:\ \mathcal{X}(y(t)) = \Pi(\mathcal{X}(y(t)))$ and $\mathcal{X}(x(t)) = \Pi(\mathcal{X}(x(t)))\,x(t) = x(t)$. Thus, we conclude that

$$\begin{aligned}
E_{g(t)}\left(y(t)\,x\dot{(t)}\right) &= \left(l_{y(t)}\right)_*\left(\mathcal{X}_{L(H)}(x(t)) + \sum_{j=1}^{d} u_j(t)\,\mathcal{X}^j_{L(H)}(x(t))\right) \\
&\quad + \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\,Y^j_{L(H)}(g(t))\right),
\end{aligned}$$

which completes the proof. □

3.2. Theorem. *Under the conditions of theorem3.1, if the derivation E is nilpotent, then the solution of (3.1) is given by*

$$\begin{aligned}
x\dot{(t)} &= -\sum_{i=0}^{k-1} E^i_{x(t)} \circ \left(l_{y(t)^{-1}}\right)_* \circ \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\,Y^j_{L(H)}(g(t))\right) \\
&\quad - \sum_{i=0}^{k-1}\sum_{j=1}^{d} u_j(t)\,E^i_{x(t)} \circ \mathcal{X}^j_{L(H)}(x(t)).
\end{aligned}$$

Proof. Suppose E is nilpotent whose nilpotent index is denoted by k. Let $x(t) \in H$ be such that $x(0) = e$. Taking the left hand side term of (3.1):

$$E_{g(t)}\left(y(t)\,x\dot{(t)}\right) = \left(l_{g(t)}\right)_* \circ E \circ \left(l_{g(t)^{-1}}\right)_* \circ \left(l_{y(t)}\right)_*\ x\dot{(t)} = \left(l_{g(t)}\right)_* \circ E \circ \left(l_{x(t)^{-1}}\right)_*\ x\dot{(t)}$$

because $\left(l_{g(t)}\right)_* = \left(l_{y(t)}\right)_* \circ \left(l_{x(t)}\right)_*$. Otherwise, we have $\dot{x}(t) = \mathfrak{X}_{L(H)}(x(t))$ where the vector field $\mathfrak{X}_{L(H)}$ is induced by a derivation $\mathcal{D} \in Der(L(H))$ and applying $\left(l_{g(t)^{-1}}\right)_*$ on both sides of (3.1),

$$E \circ \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) = \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) + \left(l_{x(t)^{-1}}\right)_* \circ \left(\sum_{j=1}^{d} u_j(t)\, \mathfrak{X}^j_{L(H)}(x(t))\right)$$
$$+ \left(l_{g(t)^{-1}}\right)_* \circ \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right).$$

If $k = 1$, the algebraic-differential equation (3.1) becomes

$$\dot{x}(t) = -\sum_{j=1}^{d} u_j(t)\, \mathfrak{X}^j_{L(H)}(x(t)) - \left(l_{y(t)^{-1}}\right)_* \circ \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right).$$

Now, let $k > 1$. Then, left multiplying both sides by E, we obtain the following equations:

$$E^2 \circ \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) = E \circ \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) + E \circ \left(l_{x(t)^{-1}}\right)_* \circ \left(\sum_{j=1}^{d} u_j(t)\, \mathfrak{X}^j_{L(H)}(x(t))\right)$$
$$+ E \circ \left(l_{g(t)^{-1}}\right)_* \circ \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right)$$
$$\cdots$$
$$E^k \circ \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) = E^{k-1} \circ \left(l_{x(t)^{-1}}\right)_* \dot{x}(t) + E^{k-1} \circ \left(l_{x(t)^{-1}}\right)_* \circ \left(\sum_{j=1}^{d} u_j(t)\, \mathfrak{X}^j_{L(H)}(x(t))\right)$$
$$+ E^{k-1} \circ \left(l_{g(t)^{-1}}\right)_* \circ \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right).$$

From the addition of these equations and the fact $E^k = 0$, $E^{k-1} \neq 0$, we have

$$\dot{x}(t) = -\sum_{i=0}^{k-1} \left(l_{x(t)}\right)_* \circ E^i \circ \left(l_{g(t)^{-1}}\right)_* \circ \left(Y_{L(H)}(g(t)) + \sum_{j=1}^{d} u_j(t)\, Y^j_{L(H)}(g(t))\right)$$
$$-\sum_{i=0}^{k-1} \sum_{j=1}^{d} \left(l_{x(t)}\right)_* \circ E^i \circ \left(l_{x(t)^{-1}}\right)_* \circ \left(u_j(t)\, \mathfrak{X}^j_{L(H)}(x(t))\right),$$

which proves our claim. □

References

[1] Ayala, V. and Tirao, J. *Linear control systems on Lie groups and Controllability*, Proceedings of Symposia in Pure Mathematics, Vol 64, AMS, 47-64, 1999.

[2] Ayala, V. and San Martin, L. *Controllability properties of a class of control systems on Lie groups*, Lectures Notes in Control and Information Science **1**, 258, 83-92, 2001.

[3] Ayala, V. and Kliemann, W. *A decomposition Theorem for Singular Control Systems on Lie Groups*, An International Journal Computer & Mathematics with applications **45**, 635-636, 2003.

[4] Ayala, V. Rodriguez, J. C. Tribuzy, I. A. and Wagner C. *Solutions of Singular Control Systems on Lie Groups*, Journal of Dynamical and Control Systems, Vol. 18, No. 3, 323-338, 2012.

[5] Dai, L. *Singular Control Systems* (Springer, 1989).

[6] Jurdjevic, V. *Geometric Control Theory* (Cambridge University Press, Cambridge, 1997).

[7] Kara, A and Kule, M. *Controllability of Affine Control Systems on Carnot Groups*, International Journal of Contemporary Mathematical Sciences **5**, 2167-2172, 2010.

[8] Kara, A. and San Martin, L. *Controllability of Affine Control System for the Generalized Heisenberg Lie Groups*, International Journal of Pure and Applied Mathematics **29**, 1-6, 2006.

[9] Varadarajan, V. *Lie groups, Lie algebras and their representations* (Prentice Hall, Inc. 1974).

[10] Yip, E.L. and Sincovec, R.F. *Solvability, Controllability and Observability of continuous descriptor systems*, I.E.E.E. Transactions of Automatic Control **26**, 702-707, 1981.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1079 – 1085

On a new reproducing kernel Hilbert space and a boundary value problem for harmonic functions

Alem Memić*

Abstract

In this paper we continue to develop a theory on a new reproducing kernel Hilbert space related to the decomposition theorem for harmonic functions on a domain of the form $\Omega\backslash K$, where Ω is an open subset of $\mathbb{R}^n$ and K a compact subset of Ω.

Keywords: harmonic Bergman space, decomposition theorem, harmonically extendable set, integral operator, boundary value problem.

2000 AMS Classification: 31B05, 46E22, 35J25.

Received 15/03/2014 : Accepted 19/06/2014 Doi : 10.15672/HJMS.2015449659

1. Introduction

There is a lot of papers about reproducing kernel Hilbert spaces since [1]. This theory found her place also in the area of applied mathematics (see [4]). There are also results about reproducing kernel Hilbert spaces in the framework of real harmonic functions (see [2]) and in the framework of harmonic Bergman spaces (see [3]). In [3] there are explicit formulas for reproducing kernels in the case of a unit ball and a half space. There is no explicit formula for the general case of a reproducing kernels for a harmonic Bergman space on arbitrary domain. In [5] we introduced a new spaces $\mathcal{A}^p(\Omega\backslash K)$ of harmonic functions on $\Omega\backslash K$, where Ω is an open subset of $\mathbb{R}^n$ and K is a compact subset of Ω. For these spaces we introduced a new norm and a new inner product (in the case $p = 2$). Then we obtained a new reproducing kernel for the space $\mathcal{A}^2(\Omega\backslash K)$ and found a relation to the standard reproducing kernel on harmonic Bergman space.
This paper is a continuation of [5]. First of all, for an arbitrary nonempty open set E of $\Omega\backslash K$ we introduce a new space $\mathcal{A}^p(E)$ and we consider the problem of equalness of $\mathcal{A}^p(E)$ and $b^p(E)$ and find it's connection to the harmonic extendability. Then we consider the problem of equivalence of norms on the space $\mathcal{A}^p(\Omega\backslash K)$. In some cases norms under consideration are equivalent, so we restrict ourselves to those that are equivalent and find some useful properties. For the standard L^2 inner product on $\mathcal{A}^2(\Omega\backslash K)$ we obtain a new reproducing kernel $K_{\Omega\backslash K}$ on $\mathcal{A}^2(\Omega\backslash K)$ and prove that this kernel is actually a projection

*Department of Mathematics University of Sarajevo, Bosnia and Herzegovina
Email: alem.memic@pmf.unsa.ba

of a kernel $R_{\Omega\backslash K}$ on $\mathcal{A}^2(\Omega\backslash K)$. After that, we introduce a new integral operator on $L^2(\Omega\backslash K)$ related to the reproducing kernel $S_{\Omega\backslash K}$ and obtain some useful properties. In final, a new kind of a boundary value problem related to the space $\mathcal{A}^p(\Omega\backslash K)$ is introduced in the last section. This new boundary value problem is a new type of a boundary value problem for harmonic functions on domains of the form $\Omega\backslash K$. On annular regions we show that this problem has a unique solution. A general case remains open.

2. Preliminaries

Let $n \geq 2$, Ω an open subset of $\mathbb{R}^n$ and K a compact subset of Ω. If u is a harmonic function on $\Omega\backslash K$, there exists functions v and w such that $u = v + w$ on $\Omega\backslash K$, where v is harmonic on Ω and w is harmonic on $\mathbb{R}^n\backslash K$. If we impose condition on w that $\lim_{|x|\to\infty} w(x) = 0$ in the case $n > 2$, or $\lim_{|x|\to\infty} w(x) - \alpha \log|x| = 0$ (for some constant α) in the case $n = 2$, then the decomposition $u = v + w$ is unique. The proof of this can be found in [3]. Let $1 \leq p < \infty$. If E is a nonempty open subset of $\mathbb{R}^n$, we denote by $b^p(E)$ a set of all functions from $L^p(E)$ that are harmonic on E. This is a Banach space called harmonic Bergman space. More on these spaces can be found in [3]. In [5] we introduced a space $\mathcal{A}^p(\Omega\backslash K)$ of all functions $u \in b^p(\Omega\backslash K)$ such that $u = v + w$ on $\Omega\backslash K$, where $v \in b^p(\Omega)$ and $w \in b^p(\mathbb{R}^n\backslash K)$. In [5] we proved that

$$\mathcal{A}^p(\Omega\backslash K) = b^p(\Omega)|_{\Omega\backslash K} \oplus b^p(\mathbb{R}^n\backslash K)|_{\Omega\backslash K}.$$

This is the motivation for the following definition.

2.1. Definition. Let $1 \leq p < \infty$, Ω an open subset of $\mathbb{R}^n$ and K a compact subset of Ω. Let E be an arbitrary nonempty open subset of $\Omega\backslash K$. We define

$$\mathcal{A}^p(E) = b^p(\Omega)|_E \oplus b^p(\mathbb{R}^n\backslash K)|_E.$$

2.2. Remark. We should use notation $\mathcal{A}^p_{\Omega,K}(E)$ instead of $\mathcal{A}^p(E)$ because the previous definition depends also on Ω and K, not just of E. We will continue to use notation $\mathcal{A}^p(E)$ because Ω and K will be seen from the context.

2.3. Lemma. *For every open set E in $\Omega\backslash K$ it holds*

$$\mathcal{A}^p(\Omega\backslash K)|_E = \mathcal{A}^p(E).$$

Proof. Let $u \in \mathcal{A}^p(E)$. There are $v \in b^p(\Omega)$ and $w \in b^p(\mathbb{R}^n\backslash K)$ such that $u = v + w$ on E. Obviously $v+w$ is harmonic on $\Omega\backslash K$. Let $U = v+w$ on $\Omega\backslash K$. We have $U \in \mathcal{A}^p(\Omega\backslash K)$ and $u = U|_E$, so $u \in \mathcal{A}^p(\Omega\backslash K)|_E$. The other direction is obvious. □

In [5] we introduced a problem to find all (n, p, Ω, K) such that $\mathcal{A}^p(\Omega\backslash K) = b^p(\Omega\backslash K)$. Here we introduce an analogous problem, to see when $\mathcal{A}^p(E) = b^p(E)$ for some open set E in $\Omega\backslash K$. We now prove the following theorem.

2.4. Theorem. *Let E be a nonempty open subset of $\Omega\backslash K$. Then $\mathcal{A}^p(E) = b^p(E)$ if and only if $\mathcal{A}^p(\Omega\backslash K) = b^p(\Omega\backslash K)$ and $b^p(\Omega\backslash K)|_E = b^p(E)$.*

Proof. Suppose $\mathcal{A}^p(E) = b^p(E)$. By previous lemma we have $\mathcal{A}^p(\Omega\backslash K)|_E = b^p(E)$. Let $u \in b^p(\Omega\backslash K)$. Then $u|_E \in b^p(E) = \mathcal{A}^p(\Omega\backslash K)|_E$, so there is $\widetilde{u} \in \mathcal{A}^p(\Omega\backslash K)$ such that $u|_E = \widetilde{u}|_E$. This and the fact that u and $\widetilde{u}$ are harmonic on $\Omega\backslash K$, implies $u = \widetilde{u}$ on $\Omega\backslash K$. So, $u = \widetilde{u} \in \mathcal{A}^p(\Omega\backslash K)$. We conclude that $\mathcal{A}^p(\Omega\backslash K) = b^p(\Omega\backslash K)$. From this we get $b^p(E) = \mathcal{A}^p(\Omega\backslash K)|_E = b^p(\Omega\backslash K)|_E$, so one direction of the theorem is proved.
Suppose now that $\mathcal{A}^p(\Omega\backslash K) = b^p(\Omega\backslash K)$ and $b^p(\Omega\backslash K)|_E = b^p(E)$. We have $\mathcal{A}^p(E) = \mathcal{A}^p(\Omega\backslash K)|_E = b^p(\Omega\backslash K)|_E = b^p(E)$, so the other direction of the theorem also holds, and the proof is finished. □

This theorem is a motivation for the following definition.

2.5. Definition. Let $n \geq 2$ and $1 \leq p < \infty$. Let Ω be an open subset of $\mathbb{R}^n$ and K a compact subset of Ω. We say that a nonempty open subset E of $\Omega \backslash K$ is harmonic p-extendable to $\Omega \backslash K$ if $b^p(\Omega \backslash K)|_E = b^p(E)$.

So, the last theorem says that $\mathcal{A}^p(E) = b^p(E)$ if and only if E is a harmonic p-extendable to $\Omega \backslash K$ and $\mathcal{A}^p(\Omega \backslash K) = b^p(\Omega \backslash K)$.

2.6. Corollary. *If $\mathcal{A}^p(\Omega \backslash K) \neq b^p(\Omega \backslash K)$, then $\mathcal{A}^p(E) \neq b^p(E)$ for every nonempty open subset E of $\Omega \backslash K$.*

2.7. Corollary. *If $\mathcal{A}^p(\Omega \backslash K) = b^p(\Omega \backslash K)$, then $\mathcal{A}^p(E) = b^p(E)$ if and only if E is harmonic p-extendable to $\Omega \backslash K$.*

It would be interesting to characterize all harmonic p-extendable sets E to $\Omega \backslash K$.

3. Equivalence of norms

In [5] we proved the following lemma.

3.1. Lemma. *Let $1 \leq p < \infty$ and $u \in \mathcal{A}^p(\Omega \backslash K)$ is arbitrarily chosen. Then*

$$\|u\|_{b^p(\Omega \backslash K)} \leq 2^{\frac{p-1}{p}} \|u\|_{\mathcal{A}^p(\Omega \backslash K)}.$$

From this lemma we could ask: Is there a $C > 0$ such that $\|u\|_{\mathcal{A}^p(\Omega \backslash K)} \leq C\|u\|_{b^p(\Omega \backslash K)}$ for every $u \in \mathcal{A}^p(\Omega \backslash K)$?

3.2. Remark. If $\Omega = \mathbb{R}^n$ and K an arbitrary compact set of $\mathbb{R}^n$, then these norms are equal because $u = v + w$, where $v = 0$ on $\mathbb{R}^n$. Also, in the case when $K = \{a\}$, where $a \in \Omega$, we have $u = v + w$, where $v \in b^p(\Omega)$, $w \in b^p(\mathbb{R}^n \backslash \{a\}) = \{0\}$. So, $w = 0$ on $\mathbb{R}^n \backslash \{a\}$ and $\|u\|_{\mathcal{A}^p(\Omega \backslash \{a\})} = \|v\|_{b^p(\Omega)} = \|v\|_{b^p(\Omega \backslash \{a\})} = \|u\|_{b^p(\Omega \backslash \{a\})}$. In both cases we have $C = 1$. A general case remains open.

In this section we will consider the case of (n, p, Ω, K) such that

$$\|u\|_{\mathcal{A}^p(\Omega \backslash K)} \leq C\|u\|_{b^p(\Omega \backslash K)}$$

for every $u \in \mathcal{A}^p(\Omega \backslash K)$. This condition with the previous lemma is equivalent that $\|\cdot\|_{\mathcal{A}^p(\Omega \backslash K)}$ and $\|\cdot\|_{b^p(\Omega \backslash K)}$ are equivalent. So, without further assumption, we suppose that this equivalence of norms is satisfied in the rest of this section.

3.3. Theorem. *If $\|\cdot\|_{\mathcal{A}^p(\Omega \backslash K)}$ and $\|\cdot\|_{b^p(\Omega \backslash K)}$ are equivalent, then $\mathcal{A}^p(\Omega \backslash K)$ is a closed subspace of $b^p(\Omega \backslash K)$.*

Proof. We proved in [5] that $\mathcal{A}^p(\Omega \backslash K)$ is a Banach space with respect to $\|\cdot\|_{\mathcal{A}^p(\Omega \backslash K)}$. If these norms are equivalent, then $\mathcal{A}^p(\Omega \backslash K)$ is a Banach space with respect to $\|\cdot\|_{b^p(\Omega \backslash K)}$. Since $b^p(\Omega \backslash K)$ is a Banach space with respect to $\|\cdot\|_{b^p(\Omega \backslash K)}$ and $\mathcal{A}^p(\Omega \backslash K)$ is a Banach space with respect to $\|\cdot\|_{b^p(\Omega \backslash K)}$, this implies that $\mathcal{A}^p(\Omega \backslash K)$ is a closed subspace of $\|\cdot\|_{b^p(\Omega \backslash K)}$ and the proof is finished. □

In [5] we proved the following theorem

3.4. Theorem. *Suppose $x \in \Omega \backslash K$. Then*

$$|u(x)| \leq \frac{2^{\frac{p-1}{p}} \|u\|_{\mathcal{A}^p(\Omega \backslash K)}}{V(B)^{1/p} d(x, \partial(\Omega \backslash K))^{n/p}}$$

for every $u \in \mathcal{A}^p(\Omega \backslash K)$.

If we impose condition that the norms $\|\cdot\|_{\mathcal{A}^p(\Omega\backslash K)}$ and $\|\cdot\|_{b^p(\Omega\backslash K)}$ are equivalent, then we have

$$|u(x)| \leq \frac{2^{\frac{p-1}{p}} C\|u\|_{b^p(\Omega\backslash K)}}{V(B)^{1/p}\, d(x, \partial(\Omega\backslash K))^{n/p}}$$

In the case $p = 2$ this means that point evaluation is a bounded linear functional on the Hilbert space $\mathcal{A}^2(\Omega\backslash K)$ with respect to $\|\cdot\|_{b^2(\Omega\backslash K)}$. This implies that $\mathcal{A}^2(\Omega\backslash K)$ is a reproducing kernel Hilbert space. If $x \in \Omega\backslash K$ is arbitrarily chosen, there is a $K_{\Omega\backslash K}(x,\cdot) \in \mathcal{A}^2(\Omega\backslash K)$ such that

$$u(x) = \langle u, K_{\Omega\backslash K}(x,\cdot)\rangle$$

for all $u \in \mathcal{A}^2(\Omega\backslash K)$ with respect to inner product from $b^2(\Omega\backslash K)$.
Because $\mathcal{A}^2(\Omega\backslash K)$ is a closed subspace of $b^2(\Omega\backslash K)$ and $b^2(\Omega\backslash K)$ is a closed subspace of $L^2(\Omega\backslash K)$, we have that $\mathcal{A}^2(\Omega\backslash K)$ is a closed subspace of the Hilbert space $L^2(\Omega\backslash K)$, which implies that there is a unique orthogonal projection of $L^2(\Omega\backslash K)$ onto $\mathcal{A}^2(\Omega\backslash K)$. We denote this projection by $P_{\Omega\backslash K}$. Let $R_{\Omega\backslash K}$ be a reproducing kernel for $b^2(\Omega\backslash K)$. So,

$$u(x) = \langle u, R_{\Omega\backslash K}(x,\cdot)\rangle$$

for every $u \in b^2(\Omega\backslash K)$. If we use the fact that $\mathcal{A}^2(\Omega\backslash K) \subseteq b^2(\Omega\backslash K)$, we get that $K_{\Omega\backslash K}$ is a projection of $R_{\Omega\backslash K}$ to $\mathcal{A}^2(\Omega\backslash K)$.

3.5. Theorem. *If $x \in \Omega\backslash K$, then*

$$P_{\Omega\backslash K}[u](x) = \int_{\Omega\backslash K} u(y)\, K_{\Omega\backslash K}(x,y)\, dy$$

for all $u \in L^2(\Omega\backslash K)$.

Proof. Let $x \in \Omega\backslash K$ and $u \in L^2(\Omega\backslash K)$. Then

$$\begin{aligned} P_{\Omega\backslash K}[u](x) &= \langle P_{\Omega\backslash K}[u], K_{\Omega\backslash K}(x,\cdot)\rangle \\ &= \langle u, K_{\Omega\backslash K}(x,\cdot)\rangle \\ &= \int_{\Omega\backslash K} u(y)\, K_{\Omega\backslash K}(x,y)\, dy, \end{aligned}$$

where the first equality follows from the reproducing property of $K_{\Omega\backslash K}(x,\cdot)$, the second equality holds because $P_{\Omega\backslash K}$ is a self-adjoint projection onto a subspace containing $K_{\Omega\backslash K}(x,\cdot)$, and the third equality follows from the definition of the inner product and the part 1. of the following theorem. □

3.6. Theorem. *The reproducing kernel $K_{\Omega\backslash K}$ has the following properties:*
1. $K_{\Omega\backslash K}$ is real valued.
2. If (u_m) is an orthonormal basis of $\mathcal{A}^2(\Omega\backslash K)$ with respect to $\|\cdot\|_{b^2(\Omega\backslash K)}$, then

$$K_{\Omega\backslash K}(x,y) = \sum_{m=1}^{\infty} \overline{u_m(x)}\, u_m(y)$$

for all $x, y \in \Omega\backslash K$, where the convergence is pointwise.
3. $K_{\Omega\backslash K}(x,y) = K_{\Omega\backslash K}(y,x)$ for all $x, y \in \Omega\backslash K$.
4.

$$\|K_{\Omega\backslash K}(x,\cdot)\|^2_{b^2(\Omega\backslash K)} = K_{\Omega\backslash K}(x,x)$$

Proof. 1. Let u be a real valued function from $\mathcal{A}^2(\Omega\backslash K)$. Then we have

$$0 = Im\,(u\,(x)) = Im\left(\int_{\Omega\backslash K} u\,(y)\,\overline{K_{\Omega\backslash K}\,(x,y)}dy\right) = -\int_{\Omega\backslash K} u\,(y)\,Im\left(K_{\Omega\backslash K}\,(x,y)\right)dy$$

If we take $u = Im\left(K_{\Omega\backslash K}\,(x,\cdot)\right)$ then we obtain $\int_{\Omega\backslash K}\left(Im\left(K_{\Omega\backslash K}\,(x,y)\right)\right)^2 dy = 0$, which implies $Im\left(K_{\Omega\backslash K}\right) \equiv 0$, so $K_{\Omega\backslash K}$ is real valued.
2. Let $(u_m\,(x))$ be any orthonormal basis of $\mathcal{A}^2(\Omega\backslash K)$. It exists because of the separability of this space with respect to $\|\cdot\|_{b^2(\Omega\backslash K)}$ (norms are equivalent). By standard Hilbert space theory

$$K_{\Omega\backslash K}\,(x,\cdot) = \sum_{m=1}^{\infty}\langle K_{\Omega\backslash K}\,(x,\cdot)\,,u_m\rangle = \sum_{m=1}^{\infty}\overline{u_m\,(x)}u_m,$$

where the infinite sum converges in the norm from $b^2(\Omega\backslash K)$ restricted to $\mathcal{A}^2(\Omega\backslash K)$. Since point evaluation is a continuous linear functional on $\mathcal{A}^2(\Omega\backslash K)$, the equation above implies that 2. holds.
3. This part follows immidiately from 1. and 2.
4. Let $x \in \Omega\backslash K$. Then $\|K_{\Omega\backslash K}\,(x,\cdot)\|^2_{b^2(\Omega\backslash K)} = \langle K_{\Omega\backslash K}\,(x,\cdot)\,, K_{\Omega\backslash K}\,(x,\cdot)\rangle = K_{\Omega\backslash K}\,(x,x)$, where the second equality follows from the reproducing property of $K_{\Omega\backslash K}\,(x,\cdot)$.

□

3.7. Remark. In [5], for $x \in \Omega\backslash K$ we introduced a reproducing kernel $S_{\Omega\backslash K}\,(x,\cdot)$ for a Hilbert space $\mathcal{A}^2(\Omega\backslash K)$ with respect to $\|\cdot\|_{\mathcal{A}^2(\Omega\backslash K)}$ as a consequence of a boundedness of a linear functional $u \mapsto u\,(x)$ on $\mathcal{A}^2(\Omega\backslash K)$. It is shown in [5] that for $x \in \Omega\backslash K$, $S_{\Omega\backslash K}\,(x,\cdot) = R_\Omega\,(x,\cdot) + R_{\mathbb{R}^n\backslash K}\,(x,\cdot)$, where $R_\Omega\,(x,\cdot)$ and $R_{\mathbb{R}^n\backslash K}\,(x,\cdot)$ are reproducing kernels for $b^2(\Omega)$ and $b^2(\mathbb{R}^n\backslash K)$, respectively, obtained as a consequence of boundedness of a linear functional $u \mapsto u\,(x)$ on these spaces. It would be interesting to see connection between $K_{\Omega\backslash K}$ and $S_{\Omega\backslash K}$.

3.8. Remark. Notations $K_{\Omega\backslash K}$ and $S_{\Omega\backslash K}$ are not good in the sense that in reality these kernels depend on Ω and K, not just on $\Omega\backslash K$. We will use these notations because they are easier to write and we can see what are Ω and K from the context.

4. Integral operators

4.1. Definition. For $u \in L^2(\Omega\backslash K)$ we define $M_{\Omega\backslash K}[u]$ by

$$M_{\Omega\backslash K}[u]\,(x) = \int_{\Omega\backslash K} u\,(y)\,S_{\Omega\backslash K}\,(x,y)\,dy$$

for all $x \in \Omega\backslash K$.

4.2. Lemma. *If*

$$\int_{\Omega\backslash K}\int_{\Omega\backslash K}|S_{\Omega\backslash K}\,(x,y)\,|^2 dxdy < \infty,$$

then $M_{\Omega\backslash K}$ is a bounded linear operator on $L^2(\Omega\backslash K)$.

Proof. Linearity is obvious. A boundedness is an immidiate consequence of a Schwartz inequality. □

4.3. Remark. Condition on $S_{\Omega\backslash K}$ in the previous lemma is trivially satisfied in the case $\Omega = \mathbb{R}^n$ and $K = \{a\}$ for any $a \in \mathbb{R}^n$. It would be interesting to characterize all Ω and K such that this condition is satisfied. We can consider also the question on conditions on Ω and K that imply boundedness of $M_{\Omega\backslash K}$ on $L^2(\Omega\backslash K)$.

4.4. Lemma. *$M_{\Omega\backslash K}[u] = u$ for all $u \in \mathcal{A}^2(\Omega\backslash K)$ if and only if $K_{\Omega\backslash K}(x,\cdot) = S_{\Omega\backslash K}(x,\cdot)$ for every $x \in \Omega\backslash K$.*

Proof. " $\Longrightarrow$ ". If $M_{\Omega\backslash K}[u] = u$ for all $u \in \mathcal{A}^2(\Omega\backslash K)$ then

$$\int_{\Omega\backslash K} u(y)\, S_{\Omega\backslash K}(x,y)\, dy = \int_{\Omega\backslash K} u(y)\, K_{\Omega\backslash K}(x,y)\, dy,$$

for all $x \in \Omega\backslash K$. This implies that $K_{\Omega\backslash K}(x,\cdot) - S_{\Omega\backslash K}(x,\cdot)$ belongs to an orthogonal complement of $\mathcal{A}^2(\Omega\backslash K)$ and to the space $\mathcal{A}^2(\Omega\backslash K)$ itself. So, it belongs to their intersection and this is a zero set. From this we conclude that $K_{\Omega\backslash K}(x,\cdot) = S_{\Omega\backslash K}(x,\cdot)$.
" $\Longleftarrow$ ". This direction follows immidiately from the reproducing property of $K_{\Omega\backslash K}$. $\square$

From the fact that $S_{\Omega\backslash K}(x,y) = R_\Omega(x,y) + R_{\mathbb{R}^n\backslash K}(x,y)$ for all $x, y \in \Omega\backslash K$ (see [5]), we obtain

$$M_{\Omega\backslash K}[u](x) = \int_{\Omega\backslash K} u(y)\, R_\Omega(x,y)\, dy + \int_{\Omega\backslash K} u(y)\, R_{\mathbb{R}^n\backslash K}(x,y)\, dy.$$

5. A new type of a boundary value problem

Let Ω be an open subset of $\mathbb{R}^n$ and K a compact subset of Ω. Suppose f is a continuous function on $\partial\Omega$ and g a continuous function on ∂K. Let us consider the following problem. Problem: Can we find a harmonic function u on $\Omega\backslash K$ that is continuous on $\overline{\Omega\backslash K}$ and that has a decomposition $u = v + w$ on $\Omega\backslash K$, where v is a solution to a Dirichlet problem of Ω with boundary data f, and w is a solution to a Dirichlet problem of $\mathbb{R}^n\backslash K$ with boundary data g? Here v and w are from the decomposition theorem for harmonic functions that we consider in this paper. We will call this problem an (Ω, K) boundary value problem with boudary data f and g.

5.1. Definition. For an (Ω, K) boundary value problem we say it is solvable if for every continuous function f on $\partial\Omega$ and every continuous function g on ∂K there is a solution to the (Ω, K) boundary value problem with boundary data f and g.

5.2. Theorem. *Let $n > 2$, $0 < r_0 < r_1$. Consider an annular region $\mathbb{A} = \Omega\backslash K$, where $\Omega = \{x \in \mathbb{R}^n, |x| < r_1\}$ and $K = \{x \in \mathbb{R}^n, |x| \le r_0\}$. Then an (Ω, K) boundary value problem is solvable with a unique solution.*

Proof. We will use the following lemma which is a Theorem 4.11 in [3].

5.3. Lemma. *Suppose $f \in C(S)$. Then there is a unique function u harmonic on B^* and continuous on $\overline{B}^*$ such that $u|_S = f$. Moreover, $u = P_e[f]$ on $B^*\backslash\{\infty\}$.*

If we modify the proof of this lemma we can prove an analogous theorem for arbitrary ball (see exercise 8 in the same chapter). Let us consider now an (Ω, K) boundary value problem for an annular region $\Omega\backslash K$. Let f and g be a continuous functions on $\partial\Omega = r_1 S$ and $\partial K = r_0 S$, respectively, where S is a unit sphere. In this case we obtain a unique solution v to a Dirichlet problem for Ω with boundary data f and a unique solution w to a Dirichlet problem for $\mathbb{R}^n\backslash K$ with boundary data g. By the previous lemma w is harmonic at infinity and in the case $n > 2$ this is equivalent to the fact that a limit of $w(x)$ is zero when $|x| \longrightarrow \infty$. Let $u = v + w$. Then u is a harmonic function on $\Omega\backslash K$ and a condition at infinity of w is satisfied in the decomposition theorem for harmonic functions. Continuity of v on $\overline{\Omega}$ and w on $\overline{\mathbb{R}^n \backslash K}$ imply continuity of u on $\overline{\Omega \backslash K}$. We conclude that in the case of an annular region in $\mathbb{R}^n$, where $n > 2$, (Ω, K) boundary value problem is solvable with a unique solution, so the proof is finished. □

In general we don't have a solution to (Ω, K)-boundary value problem because the Dirichlet problem is not solvable for an arbitrary open set. If Ω is a bounded open set and if there is a solution to the Dirichlet problem (here we suppose that the boundary data is a continuous function), then this solution is unique, which is a consequence of a maximum principle for harmonic functions. There are unbounded open sets where we still have a unique solution to a Dirichlet problem, as it is the case for a half space (see chapter 7 in [3]), but in general if a Dirichlet problem is solvable for unbounded regions, we cannot conclude that it is unique because maximum principle for harmonic functions is not satisfied for unbounded regions (see [3]).

5.4. Remark. Let $1 \leq p < \infty$. If $u = v + w$ is a solution to the (Ω, K) boundary value problem and if $v \in L^p(\Omega)$, $w \in L^p(\mathbb{R}^n \backslash K)$, then $u \in \mathcal{A}^p(\Omega \backslash K)$. It would be interesting to consider the space $\mathcal{A}^p(\Omega \backslash K)$ in the framework of this (Ω, K) boundary value problem for harmonic functions.

5.5. Remark. We could apply these results also in the case of parabolic partial differential equations because there is an analogous decomposition theorem in that case also. (see [6]).

References

[1] Aronszajn, N. *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68**, 337-404, 1950.

[2] Auchmuty, G. *Reproducing kernels for Hilbert spaces of real harmonic functions*, SIAM. J. Math. Anal. **41**(5), 1994-2009, 2009.

[3] Axler, S. and Bourdon, P. and W. Ramey, W. *Harmonic function theory*, Springer-Verlag, New York, 2011.

[4] Thomas Hofmann, T. and Schölkopf, B. and Smola, A.J. *Kernel methods in machine learning*, Ann. Statist. **36**(3), 1171-1220, 2008.

[5] Memić, A. *A reproducing kernel for a Hilbert space related to harmonic Bergman space on a domain outside compact set*, Turk. J. Math. **38**, 311-317, 2014.

[6] Watson, N. A., *A decomposition theorem for solutions of parabolic equations*, Ann. Acad. Sci. Fenn. Math. **25**, 151-160, 2001.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1087 – 1097

Inversion Laplace transform for integrodifferential parabolic equation with purely nonlocal conditions

Ahcene Merad*†, Abdelfatah Bouziani‡ and Cenap Ozel §

Abstract

In this paper we prove the existence, uniqueness, and continuous dependence upon the data of solution to integrodifferential parabolic equation with purely nonlocal integral conditions. The proofs are based on a priori estimates and Laplace transform method. Finally, we obtain a solution using a numerical technique which is called Stehfest algorithm by inverting the Laplace transform.

Keywords: Integrodifferential parabolic equation, Approximate solution, Nonlocal purely integral conditions, Stehfest Algorithm.

2000 AMS Classification: 44A10, 34A12, 35L10

Received 07/03/2012 : Accepted 09/09/2014 Doi : 10.15672/HJMS.2015449667

1. Introduction

In this paper we are concerned with the following parabolic integrodifferential equation

$$\frac{\partial v}{\partial t}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \int_0^t a(t-s)\, v(x,s)\, ds,\ 0 < x < 1,\ 0 < t \leq T, \tag{1.1}$$

subject to the initial condition

$$v(x,0) = \Phi(x),\ 0 < x < 1, \tag{1.2}$$

*Department of Mathematics, Larbi Ben M'hidi University, 04000, Algeria, and Department of Mathematics, University of Jijel, 18000 Jijel, Algeria , Email: merad_ahcene@yahoo.fr

†Corresponding Author.

‡Department of Mathematics, Larbi Ben M'hidi University,04000, Algeria , Email: author.two@mail.com

§Department of Mathematics, Dokuz Eylul University, Izmir, Turkey, Email: cenap.ozel@gmail.com

and the purely nonlocal (integral) conditions

$$\begin{aligned} \int_0^1 v(x,t)\,dx &= r(t)\,,\ 0<t\leq T, \\ \int_0^1 xv(x,t)\,dx &= q(t)\,,\ 0<t\leq T, \end{aligned} \tag{1.3}$$

where v is an unknown function, r, q, and $\Phi(x)$ are given functions supposed to be sufficiently regular, a is suitably defined function satisfying certain some conditions that will be specified later and T is a positive constant number.

Some problems from modern physics and science can be described in terms of partial differential equations with nonlocal conditions. For instance, the nonlocal term of our problem (i.e $\int_0^t a(t-s)\,v(x,s)\,ds$) appears in the modeling of the quasi-static flexure of a thermo-elastic rod [10, 12]. First this problem with the more general second-order parabolic equation or a $2m$-parabolic equation has been studied by the second author using the energy-integral methods and the Rothe method in [10, 12, 14] and [28] respectively. For other models we refer to [7, 12, 13, 15], [16]-[19],[20]-[27], [29]-[34]. The problem $(1.1)-(1.3)$ is studied by using the Rothe method in [21]. On the other hand Ang in [2] considered a one-dimensional heat equation with nonlocal integral conditions and applied the Laplace transform to the problem. Then he used some numerical techniques to obtain a numerical solution of the inverse Laplace transform.

Recently the various types of the partial differential equations with nonlocal conditions have been studied by [3], [4] and [5], [6] and [8], [9].

This paper is organized as follows. In Section 2, we introduce some certain function spaces what we need in this work, and also give a reduction of our problem to another equivalent problem with the homogeneous integral conditions. In Section 3, we establish the existence of the solution by the Laplace transform method. In Section 4, we deal with a priori estimate which gives the uniqueness and continuous dependence upon the given data.

2. Statement of the Problem and Notations

Since integral conditions are not homogenous, it is convenient to convert the problem $(1.1)-(1.3)$ to an equivalent problem with the homogenous integral conditions. For this reason, we introduce a new function $u(x,t)$ representing the deviation of the function $v(x,t)$ as

$$u(x,t) = v(x,t) - w(x,t)\,,\ 0<x<1,\ 0<t\leq T, \tag{2.1}$$

where

$$w(x,t) = 6\,(2q(t) - r(t))\,x - 2\,(3q(t) - 2r(t))\,. \tag{2.2}$$

The problem $(1.1)-(1.3)$ with non-homogenous integral conditions (1.3) can be equivalently reduced to the problem of finding a function u satisfying

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \int_0^t a(t-s)\,u(x,s)\,ds, 0<x<1,\ 0<t\leq T, \tag{2.3}$$

$$u(x,0) = \varphi(x)\,,\ 0<x<1,$$

$$\int_0^1 u(x,t)\,dx = 0,\ 0<t\le T, \tag{2.4}$$
$$\int_0^1 xu(x,t)\,dx = 0,\ 0<t\le T,$$

where

$$\varphi(x) = \Phi(x) - w(x,0).$$

The solution of problem (1.1) − (1.3) will be obtained by the relation (2.1) and (2.2). Let H be the Hilbert space with the norm $\|.\|_H$ and $L^2(0,1)$ be the space of all the square integrable functions on the interval $(0,1)$. Now we are ready to introduce some appropriate function spaces what we need in this work.

2.1. Definition. (i) We denote by $L^2(0,T;H)$ the set of all measurable functions $u(.,t)$ from $(0,T)$ into H equipped with the norm

$$\|u\|_{L^2(0,T;H)} = \left(\int_0^T \|u(.,t)\|_H^2\,dt\right)^{1/2} < \infty. \tag{2.5}$$

(ii) The space $C(0,T;H)$ is the set of all continuous functions $u(.,t) : (0,T) \longrightarrow H$ equipped with the norm

$$\|u\|_{C(0,T;H)} = \max_{0\le t\le T}\|u(.,t)\|_H < \infty.$$

We denote by $C_0(0,1)$ the space of all continuous functions with a compact support in $(0,1)$. Since such functions are Lebesgue integrable with respect to x, we can define a bilinear form on $C_0(0,1)$ given by

$$(u,w) = \int_0^1 J_x^m u.J_x^m w dx,\ m\ge 1, \tag{2.6}$$

where

$$J_x^m u = \int_0^x \frac{(x-\zeta)^{m-1}}{(m-1)!}u(\zeta,t)\,d\zeta; \quad \text{for } m\ge 1. \tag{2.7}$$

We know that the bilinear form (2.6) is a scalar product on $C_0(0,1)$ but $C_0(0,1)$ is not a complete space.

2.2. Definition. Denote by $B_2^m(0,1)$, the completion of $C_0(0,1)$ for the scalar product (2.6), which is denoted by $(.,.)_{B_2^m(0,1)}$, introduced in [11]. By the norm of a function u from $B_2^m(0,1)$, $m\ge 1$, we understand the nonnegative number:

$$\|u\|_{B_2^m(0,1)} = \left(\int_0^1 (J_x^m u)^2\,dx\right)^{1/2} = \|J_x^m u\|, \text{ for } m\ge 1. \tag{2.8}$$

From [11] we have the following lemma.

2.3. Lemma. *For all $m\in\mathbb{Z}^+$ the following inequality*

$$\|u\|^2_{B_2^m(0,1)} \le \frac{1}{2}\|u\|^2_{B_2^{m-1}(0,1)} \tag{2.9}$$

holds.

2.4. Corollary. *For all $m \in \mathbb{Z}^+$ we have the elementary inequality*

$$\|u\|^2_{B_2^m(0,1)} \leq \left(\frac{1}{2}\right)^m \|u\|^2_{L^2(0,1)} . \tag{2.10}$$

2.5. Definition. We denote by $L^2(0,T;B_2^m(0,1))$ the space of functions which are square integrable in the Bochner sense with the scalar product

$$(u,w)_{L^2(0,T;B_2^m(0,1))} = \int_0^T (u(.,t), w(.,t))_{B_2^m(0,1)}\, dt. \tag{2.11}$$

Since the space $B_2^m(0,1)$ is a Hilbert space, it can be shown that $L^2(0,T;B_2^m(0,1))$ is also a Hilbert space. The set of all continuous functions in $[0,T]$ equipped with the norm

$$\sup_{0\leq t\leq T} \|u(.,t)\|_{B_2^m(0,1)}$$

will be denoted by $C(0,T;B_2^m(0,1))$.

2.6. Corollary. *The following imbedding $L^2(0,1) \longrightarrow B_2^m(0,1)$ is continuous for $m \geq 1$.*

By Lemma 1.3.19 in [25], we have the following result.

2.7. Lemma (Gronwall Lemma)**.** *Let $f_1(t)$, $f_2(t) \geq 0$ be two integrable functions on $[0,T]$, let us suppose that $f_2(t)$ is nondecreasing. If we have*

$$f_1(\tau) \leq f_2(\tau) + c\int_0^\tau f_1(t)\, dt, \ \forall \tau \in [0,T], \tag{2.12}$$

where $c \in \mathbb{R}^+$ then we have

$$f_1(t) \leq f_2(t)\exp(ct), \ \forall t \in [0,T]. \tag{2.13}$$

.

3. Existence of the Solution.

The Laplace transform method is an efficient way to solve many ordinary and partial differential equations. But the main difficulty with the Laplace transform method is in the inverting the Laplace domain solution into the real domain. In this section we will carry out the Laplace transform techniques to find solutions of the partial differential equations.

Suppose that $v(x,t)$ is defined and is of exponential order for $t \geq 0$ i.e. there exists A, $\gamma > 0$ and $t_0 > 0$ such that $|v(x,t)| \leq A\exp(\gamma t)$ for $t \geq t_0$. Then the Laplace transform $V(x,s)$ exists and it is given by

$$V(x,s) = \{v(x,t); t \longrightarrow s\} = \int_0^\infty v(x,t)\exp(-st)\, dt, \tag{3.1}$$

where s is a positive real parameter. Applying the Laplace transform on both sides of (1.1), we have

$$(s - A(s))V(x,s) - \frac{d^2}{dx^2}V(x,s) = s\Phi(x), \tag{3.2}$$

where $G(x,s) = \{g(x,t); t \longrightarrow s\}$. Similarly, we have

$$\begin{aligned} \int_0^1 V(x,s)\, dx &= R(s), \\ \int_0^1 xV(x,s)\, dx &= Q(s), \end{aligned} \tag{3.3}$$

where

$$R(s) = \{r(t); t \longrightarrow s\},$$
$$Q(s) = \{q(t); t \longrightarrow s\}.$$

Now we have three distinguished cases:

Case 1. $s - A(s) > 0$.

Case 2. $s - A(s) < 0$.

Case 3. $s - A(s) = 0$.

Here we consider only Case 2 and 3, because Case 1 can be dealt as like in [2]. For $(s - A(s)) = 0$, we have

$$\frac{d^2}{dx^2} V(x,s) = -s\Phi(x). \tag{3.4}$$

The general solution for Case 3 is given by

$$V(x,s) = -\int_0^x \int_0^y [s\Phi(x)]\, dzdy + C_1(s)\, x + C_2(s). \tag{3.5}$$

Putting the integral conditions (3.3) in (3.5) we get

$$\begin{aligned} &\frac{1}{2} C_1(s) + C_2(s) \\ = &\int_0^1 \int_0^x \int_0^y [s\Phi(x)]\, dzdy + R(s), \\ &\frac{1}{3} C_1(s) + \frac{1}{2} C_2(s) \\ = &\int_0^1 \int_0^x \int_0^y x\, [s\Phi(x)]\, dzdy + Q(s), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} C_1(s) &= 12 \int_0^1 \int_0^x \int_0^y x\, [s\Phi(x)]\, dzdy - \\ & 6 \int_0^1 \int_0^x \int_0^y [s\Phi(x)]\, dzdy + \\ & 12Q(s) - 6R(s), \\ C_2(s) &= 4 \int_0^1 \int_0^x \int_0^y [s\Phi(x)]\, dzdy - \\ & 6 \int_0^1 \int_0^x \int_0^y x\, [s\Phi(x)]\, dzdy - \\ & 6Q(s) + 4R(s). \end{aligned} \tag{3.7}$$

For Case 2, that is, when $(s - A(s)) < 0$, using the method of variation of parameters, we have the general solution as

$$\begin{aligned} V(x,s) = &\frac{1}{\sqrt{A(s) - s^2}} \int_0^x (s\Phi(x)) \cdot \\ & \sin\left(\sqrt{A(s) - s}\right)(x - \tau)\, d\tau + d_1(s) \cos \sqrt{(A(s) - s)}x + \\ & d_2(s) \sin \sqrt{(A(s) - s)}x. \end{aligned} \tag{3.8}$$

From the integral conditions (3.3) we get

$$
\begin{aligned}
(3.9)\qquad & d_1(s)\int_0^1 \cos\sqrt{(A(s)-s)}x dx + d_2(s)\int_0^1 \sin\sqrt{(A(s)-s)}x dx \\
= \; & R(s) - \frac{1}{\sqrt{A(s)-s^2}}\int_0^1\int_0^x (s\Phi(x))\cdot \\
& \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\, d\tau dx, \\
& d_1(s)\int_0^1 x\cos\sqrt{(A(s)-s)}x dx + d_2(s)\int_0^1 x\sin\sqrt{(A(s)-s)}x dx \\
= \; & Q(s) - \frac{1}{\sqrt{A(s)-s}}\int_0^1\int_0^x x(s\Phi(x))\cdot \\
& \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\, d\tau dx.
\end{aligned}
$$

Thus d_1, d_2 are given by

$$
(3.10)\qquad \begin{pmatrix} d_1(s) \\ d_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \cdot \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix},
$$

where

$$
\begin{aligned}
(3.11)\qquad a_{11}(s) &= \int_0^1 \cos\sqrt{(A(s)-s)}x dx, \\
a_{12}(s) &= \int_0^1 \sin\sqrt{(A(s)-s)}x dx, \\
a_{21}(s) &= \int_0^1 x\cos\sqrt{(A(s)-s)}x dx, \\
a_{22}(s) &= \int_0^1 x\sin\sqrt{(A(s)-s)}x dx, \\
b_1(s) &= R(s) - \frac{1}{\sqrt{A(s)-s}}\int_0^1\int_0^x (s\Phi(x))\cdot \\
&\quad \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\, d\tau dx, \\
b_2(s) &= Q(s) - \frac{1}{\sqrt{A(s)-s}}\int_0^1\int_0^x x(s\Phi(x))\cdot \\
&\quad \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\, d\tau dx.
\end{aligned}
$$

If it is not possible to calculate the integrals directly, then we can calculate them numerically. So we can approximate them similarly as done in [2]. If the Laplace inversion is possibly computed directly for (3.5) and (3.8), then we reach the solution explicitly. Otherwise we have to use the suitable approximate technique to get numerical solution, therefore we need the numerical inversion of the Laplace transform. Considering $A(s) - s = k(s)$ and using Gauss's formula given in [1] we have the following appoximations of the integrals

$$\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \cos \sqrt{k(s)} x dx \simeq \frac{1}{2}\sum_{i=1}^{N} w_i \begin{pmatrix} 1 \\ \frac{1}{2}[x_i+1] \end{pmatrix} \cos\left(\sqrt{k(s)}\frac{1}{2}[x_i+1]\right), \tag{3.12}$$

$$\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin \sqrt{k(s)} x dx \simeq \frac{1}{2}\sum_{i=1}^{N} w_i \begin{pmatrix} 1 \\ \frac{1}{2}[x_i+1] \end{pmatrix} \sin\left(\sqrt{k(s)}\frac{1}{2}[x_i+1]\right),$$

$$\int_0^x (s\Phi(x)) \sin\left(\sqrt{k(s)}\right)(x-\tau)\, d\tau \simeq \frac{x}{2}\sum_{i=1}^{N} w_i \left[s\Phi\left(\frac{x}{2}[x_i+1]\right)\right] \sin\left(\sqrt{k(s)}\left[x-\frac{x}{2}[x_i+1]\right]\right),$$

$$\int_0^1 \left[[s\Phi(\tau)]\int_\tau^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin\left(\sqrt{k(s)}\right)(x-\tau)\, dx\right] d\tau \simeq \frac{1}{2}\sum_{i=1}^{N} w_i \left[s\Phi\left(\frac{1}{2}[x_i+1]\right)\right] \left(\frac{1-\frac{1}{2}[x_i+1]}{2}\right)\sum_{i=1}^{N} w_j \begin{pmatrix} 1 \\ \frac{1-\frac{1}{2}[x_i+1]}{2}x_j+\frac{1-\frac{1}{2}[x_i+1]}{2} \end{pmatrix} \cdot \sin\left(\sqrt{k(s)}\left[\frac{1-\frac{1}{2}[x_i+1]}{2}x_j+\frac{1+\frac{1}{2}[x_i+1]}{2}-\frac{1}{2}(x_i+1)\right]\right),$$

where x_i and w_i are the abscissa and weights defined as

$$x_i : i^{th} \text{ zero of } P_n(x), \ \omega_i \ = 2/\left(1-x_i^2\right)\left[P_n^{'}(x)\right]^2.$$

Their tabulated values can be found in [1] for different values of N.

3.1. A numerical inversion of a Laplace transform. Sometimes an analytical inversion of the Laplace domain solution is difficult to obtain. Therefore, a numerical inversion method has to be required. An important comparison of four frequently used numerical Laplace inversion algorithms is given by H. Hassanzadeh et al in [24]. Here we use the Stehfest algorithm [34] that is easy to implement. This numerical technique was first introduced by Graver [22] and then its algorithm is improved by [34]. The Stehfest algorithm approximates the time domain solution as

$$v(x,t) \approx \frac{\ln 2}{t}\sum_{n=1}^{2m}\beta_n V\left(x; \frac{n\ln 2}{t}\right), \tag{3.13}$$

where m is a positive integer,

$$\beta_n = (-1)^{n+m}\sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)} \frac{k^m(2k)!}{(m-k)!k!(k-1)!(n-k)!(2k-n)!}, \tag{3.14}$$

and $[q]$ is the integer part of the real number q.

4. A Numerical Example

In this section we perform some results of numerical computations using the Laplace transform method proposed in the previous section. This technique can be carried out to solve the problem defined by the problem $(1.1)-(1.3)$. The method is easily applicable via Matlab 7.9.3 program. So we can give the following example.

4.1. Example. We take the integrodifferential equation

$$\begin{aligned}
\frac{\partial v}{\partial t}(x,t)-\frac{\partial^2 v}{\partial x^2}(x,t) &= \int_0^t \exp(t-s)u(x,s)\,ds, 0<x<1,\ 0<t\le T,\\
v(x,0) &= \sin x,\ 0<x<1,\\
\int_0^1 v(x,t)\,dx &= 0,\ 0<t\le T,\\
\int_0^1 xv(x,t)\,dx &= 0,\ 0<t\le T.
\end{aligned}$$

In this case the exact solution is given by

$$v(x,t)=\exp(-t).\cos t.\sin x,\ 0<x<1,\ 0<t\le T.$$

The method of solution is easily implemented on the computer, and numerical results obtained by $N=8$ in (3.12) and $m=5$ in (3.13). Now we can compare the exact solution with numerical solution. For $t=0.10$ and $x\in[0.10,0.90]$, we calculate v numerically using the proposed method of solution and compare it with the exact solution as in Table 1.

The relative error computed by the formula $\frac{v\ numerical-v\ exact}{v\ exact}$.

x	0.10	0.30	0.50	0.70	0.90
v exact	0.0898817	0.2660619	0.4316350	0.5800001	0.7052425
v numerical	0.0898818	0.2660623	0.4316355	0.5800058	0.7052395
relativ error	$-0,0000058$	$0,0000017$	$0,0000012$	$0,0000099$	$-0,0000043$

Table1

5. Uniqueness and Continuous Dependence of the Solution.

First we establish a priori estimate, then the uniqueness and continuous dependence of the solution with respect to the given data are immediately obtained.

5.1. Theorem. *If $u(x,t)$ is a solution of the Problem $(2.3)-(2.4)$, then we have the following inequalities*

$$\begin{aligned}
&\|u(.,\tau)\|^2_{L^2(0,1)} \le c_1\left(\|\varphi\|^2_{L^2(0,1)}\right) \text{ and} \\
&\left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} \le c_2\left(\|\varphi\|^2_{L^2(0,1)}\right),
\end{aligned} \tag{5.1}$$

where $c_1=\exp(a_0T)$, $c_2=\frac{\exp(a_0T)}{1-a_0}$, $1<a(x,t)<a_0$, *and* $0\le\tau\le T$.

Proof. If we take the scalar product of the both side of equation (2.3) by u, and integrate over $(0,\tau)$, then we have

$$\int_0^\tau \left(\frac{\partial u(.,t)}{\partial t}, u\right)_{B_2^1(0,1)} dt - \int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, u\right)_{B_2^1(0,1)} dt = \int_0^\tau \left(\int_0^t a(t-s)\,u(x,s)\,ds, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt. \tag{5.2}$$

Integrating by parts on the left-hand side of (5.2) we obtain

$$\frac{1}{2}\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)} + \frac{1}{2}\|u(.,\tau)\|^2_{L^2(0,1)} - \frac{1}{2}\|\varphi\|^2_{L^2(0,1)} = \int_0^\tau \left(\int_0^t a(t-s)\,u(x,s)\,ds, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt. \tag{5.3}$$

By the Cauchy inequality, the right-hand side of (5.3) is bounded by

$$\frac{a_0}{2}\int_0^t \|u(x,s)\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} ds + \frac{a_0}{2}\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)}. \tag{5.4}$$

Substitution of (5.4) into (5.3) yields

$$(1-a_0)\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} + \|u(.,\tau)\|^2_{L^2(0,1)} \leq \|\varphi\|^2_{L^2(0,1)} + \frac{a_0}{2}\int_0^t \|u(x,s)\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} ds. \tag{5.5}$$

By the **Gronwall Lemma** we have

$$(1-a_0)\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} + \|u(.,\tau)\|^2_{L^2(0,1)} \leq \exp(a_0 T)\left(\|\varphi\|^2_{L^2(0,1)}\right). \tag{5.6}$$

From (5.6), we obtain the estimates (5.1). □

5.2. Corollary. *If Problem* $(2.3)-(2.4)$ *has a solution, then this solution is unique and depends continuously on* φ.

References

[1] M. Abramowitz and I. A. Stegun, *Hand book of Mathematical Functions*, Dover, New York, 1972.

[2] W. T. Ang, *A Method of Solution for the One-Dimentional Heat Equation Subject to Non-local Conditions*, Southeast Asian Bulletin of Mathematics **26** (2002), 185-191.

[3] A. Ashyralyev and F.S.O. Tetikoglu, *FDM for elliptic equations with Bitsadze-Samarskii-Dirichlet conditions*, Abstract & Applied Analysis, 1-22, 2013. Article ID 454831.

[4] A. Ashyralyev and O. Yildirim, *A note on the second order of accuracy stable difference schemes for the nonlocal boundary value hyperbolic problem*, Abstract & Applied Analysis, pages 1 - 29, 2012. Article ID 846582.

[5] G. Avalishvili, M. Avalishvili, and D. Gordeziani, *On integral nonlocal boundary value problems for some partial differential equations*, Bulletion of the Georegian National Academy of Sciences, 5(1):31-37, 2011.

[6] G. Avalishvili, M. Avalishvili, and D. Gordeziani, *On a nonlocal problem with integral boundary conditions for a multidimensional elliptic equation*, Applied Mathematics Letters, 24(4):566-571, 2011.

[7] S. A. Beïlin, *Existence of solutions for one-dimentional wave nonlocal conditions*, Electron. J. Differential Equations **2001** (2001) , no. 76, 1-8.

[8] G. Berikelashvili and N. Khomeriki, *On the convergence of difference schemes for one nonlocal boundary-value problem*, Lithuanian Mathematical Journal, 52(4):353-362, 2012.

[9] G. Berikelashvili and N. Khomeriki, *On a numerical solution of one nonlocal boundary-value problem with mixed Dirichlet-Neumann conditions*, Lithuanian Mathematical Journal, 53(4):367-380, 2013.

[10] A. Bouziani, *Problèmes mixtes avec conditions intégrales pour quelques équations aux dérivées partielles*, Ph.D. thesis, Constantine University, (1996).

[11] A. Bouziani, *Mixed problem with boundary integral conditions for a certain parabolic equation,* J. Appl. Math. Stochastic Anal. **09** ,no. 3, 323-330 (1996).

[12] A. Bouziani, *Solution forte d'un problème mixte avec une condition non locale pour une classe d'équations hyperboliques [Strong solution of a mixed problem with a nonlocal condition for a class of hyperbolic equations]*, Acad. Roy. Belg. Bull. Cl. Sci. **8** , 53-70 (1997) .

[13] A. Bouziani, *Strong solution to an hyperbolic evolution problem with nonlocal boundary conditions,* Maghreb Math. Rev., 9 , no. 1-2, 71–84 (2000).

[14] A. Bouziani, *On the quasi static flexur of thermoelastic rod, Communication in Applied Analysis for Theory and Application,* Vol. 6, no.4,pp. 549-568, (2002).

[15] A. Bouziani, *Initial-boundary value problem with nonlocal condition for a viscosity equation*, Int. J. Math. & Math. Sci. **30** , no. 6, 327-338.(2002) .

[16] A. Bouziani, *On the solvabiliy of parabolic and hyperbolic problems with a boundary integral condition*, Internat. J. Math. & Math. Sci., **31,** 435-447.(2002) .

[17] A. Bouziani, *On a class of nonclassical hyperbolic equations with nonlocal conditions*, J. Appl. Math. Stochastic Anal. **15** ,no. 2, 136-153.(2002)

[18] A. Bouziani, *Mixed problem with only integral boundary conditions for an hyperbolic equation,* Internat. J. Math. & Math. Sci., 26, 1279-1291 (2004).

[19] A. Bouziani and N. Benouar, *Problème mixte avec conditions intégrales pour une classe d'équations hyperboliques,* Bull. Belg. Math. Soc. 3 , 137-145 (1996).

[20] A. Bouziani and N. Benouar, *Sur un problème mixte avec uniquement des conditions aux limites intégrales pour une classe d'équations paraboliques*, Maghreb Mathematical Review, vol. 9, no. 1-2, pp. 55-70, (2000).

[21] A. Bouziani and R. Mechri, *The Rothe Method to a Parabolic Integro-differential Equation with a Nonclassical Boundary Conditions,*Int. Jour. of Stochastic Analysis, Article ID 519684/ 16 page, doi: 10.1155/519684/(2010).

[22] D. P. Graver, *Observing stochastic processes and aproximate transform inversion*, Oper. Res. 14, 444-459.(1966).

[23] D. G. Gordeziani and G. A. Avalishvili, *Solution of nonlocal problems for one-dimensional oscillations of a medium*, Mat. Model. 12, no. 1, 94–103 (2000).

[24] H. Hassanzadeh and M. Pooladi-Darvish, *Comparision of different numerical Laplace inversion methods for engineering applications*, Appl. Math. Comp. 189 1966-1981(2007).

[25] J.Kacŭr, *Method of Rothe in Evolution Equations*, Teubner-Texte zur Mathematik, vol. 80, BSB B. G. Teubner Verlagsgesellschaft, Leipzig,(1985).

[26] S.Mesloub and A. Bouziani,*On a class of singular hyperbolic equation with a weighted integral condition,* Int. J. Math. Math. Sci. 22 , no. 3, 511–519 (1999).

[27] S.Mesloub and A. Bouziani, *Mixed problem with integral conditions for a certain class of hyperbolic equations*, Journal of Applied Mathematics, Vol. 1 , no. 3, 107-116(2001).

[28] N. Merazga and A. Bouziani, *Rothe time-discretization method for a nonlocal problem arising in thermoelasticity,* Journal of Applied Mathematics and Stochastic Analysis, Vol. 2005, no. 1, pp. 13-28, (2005) .

[29] A. Merad, *Adomian Decomposition Method for Solution of Parabolic Equation to Nonlocal Conditions,* Int. Journal of Contemp. Math. Sciences, Vol 6, no. 29-32, 1491-1496(2011).

[30] A. Merad and A. L. Marhoune, *Strong Solution for a High Order Boundary Value Problem with Integral Condition,* Turk. Jour. Math., Vol. 37, no.3, 1-9(2013) .

[31] L. S. Pul'kina, *A non-local problem with integral conditions for hyperbolic equations*, Electron. J. Differential Equations 1999 , no. 45, 1–6(1999).

[32] L. S. Pul'kina, *On the solvability in L2 of a nonlocal problem with integral conditions for a hyperbolic equation*, Differ. Equ. 36 , no. 2, 316–318.(2000)

[33] L. S. Pul'kina, *A mixed problem with integral condition for the hyperbolic equation*, Matematicheskie Zametki, vol. 74, no. 3, , pp. 435–445.(2003) .

[34] H. Stehfest, *Numerical Inversion of the Laplace Transform*, Comm. ACM **13**, 47-49.(1970)

[35] A. D. Shruti, *Numerical Solution for Nonlocal Sobolev-type Differential Equations*, Electronic Journal of Differential Equations, Conf. 19 , pp. 75-83.(2010)

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1099 – 1108

Signed degree sequences in signed multipartite graphs

S. Pirzada* and T. A. Naikoo†

Abstract

A signed k-partite graph (signed multipartite graph) is a k-partite graph in which each edge is assigned a positive or a negative sign. If $G(V_1, V_2, \cdots, V_k)$ is a signed k-partite graph with $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $1 \leq i \leq k$, the signed degree of v_{ij} is $sdeg(v_{ij}) = d_{ij} = d_{ij}^+ - d_{ij}^-$, where $1 \leq i \leq k$, $1 \leq j \leq n_i$ and $d_{ij}^+(d_{ij}^-)$ is the number of positive (negative) edges incident with v_{ij}. The sequences $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, are called the signed degree sequences of $G(V_1, V_2, \cdots, V_k)$. The set of distinct signed degrees of the vertices in a signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ is called its signed degree set. In this paper, we characterize signed degree sequences of signed k-partite graphs. Also, we give the existence of signed k-partite graphs with given signed degree sets.

Keywords: Signed graphs, signed multipartite graph, signed degree, signed set.

2000 AMS Classification: 05C22.

Received 17/09/2011 : Accepted 24/06/2014 Doi : 10.15672/HJMS.2015449661

1. Introduction

A signed graph is a graph in which each edge is assigned a positive or a negative sign. The concept of signed graphs is given by Harary [3]. Let G be a signed graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The signed degree of v_i is $sdeg(v_i) = d_i = d_i^+ - d_i^-$, where $1 \leq i \leq n$ and $d_i^+(d_i^-)$ is the number of positive(negative) edges incident with v_i. A signed degree sequence $\sigma = [d_1, d_2, \cdots, d_n]$ of a signed graph G is formed by listing the vertex signed degrees in non-increasing order. An integral sequence is s-graphical if it is the signed degree sequence of a signed graph. Also, a non-zero sequence $\sigma = [d_1, d_2, \cdots, d_n]$ is a standard sequence if σ is non-increasing, $\sum_{i=1}^{n} d_i$ is even, $d_1 > 0$, each $|d_i| < n$ and

*Department of Mathematics, University of Kashmir, Srinagar, Kashmir, India
Email: `sdpirzada@yahoo.co.in; pirzadasd@kashmiruniversity.ac.in`
†Department of Mathematics, Islamia College for Science and Commerce, Srinagar, Kashmir, India
Email:`tariqnaikoo@yahoo.co.in`

$|d_1| \geq |d_n|$.

The following result, due to Charttrand et al. [1], gives a necessary and sufficient condition for an integral sequence to be s-graphical, and this is similar to Hakimi's result for degree sequences in graphs [2].

Theorem 1. A standard integral sequence $\sigma = [d_1, d_2, \cdots, d_n]$ is s-graphical if and only if

$$\sigma' = [d_2 - 1, d_3 - 1, \cdots, d_{d_1+s+1} - 1, d_{d_1+s+2}, \cdots, d_{n-s}, d_{n-s+1} + 1, \cdots, d_n + 1]$$

is s-graphical for some s, $0 \leq s \leq \frac{n-1-d_1}{2}$.

The next result [12] provides a good candidate for parameter s in Theorem 1.

Theorem 2. A standard integral sequence $\sigma = [d_1, d_2, \cdots, d_n]$ is s-graphical if and only if

$$\sigma'_m = [d_2 - 1, d_3 - 1, \cdots, d_{d_1+m+1} - 1, d_{d_1+m+2}, \cdots, d_{n-m}, d_{n-m+1} + 1, \cdots, d_n + 1]$$

is s-graphical, where m is the maximum non-negative integer such that $d_{d_1+m+1} > d_{n-m+1}$.

The set of distinct signed degrees of the vertices in a signed graph G is called its signed degree set. In [6], it is proved that every set of positive (negative) integers is the signed degree set of some connected signed graph and the smallest possible order for such a signed graph is also determined. Hoffman and Jordan [4] have shown that the degree sequences of signed graphs can be characterized by a system of linear inequalities. The set of all n-tuples satisfying this system of linear inequalities is a polytope P_n. In [5], Jordan et al. have proved that P_n is the convex hull of the set of degree sequences of signed graphs of order n. We can find more results on signed degrees in [4,5].

A signed bipartite graph is a bipartite graph in which each edge is assigned a positive or a negative sign. Let $G(U, V)$ be a signed bipartite graph with $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$. Then signed degree of u_i is $sdeg(u_i) = d_i = d_i^+ - d_i^-$, where $1 \leq i \leq p$ and $d_i^+(d_i^-)$ is the number of positive (negative) edges incident with u_i and signed degree of v_j is $sdeg(v_j) = e_j = e_j^+ - e_j^-$, where $1 \leq j \leq q$ and $e_j^+(e_j^-)$ is the number of positive (negative) edges incident with v_j. The sequences $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ are called the signed degree sequences of the signed bipartite graph $G(U, V)$. Two sequences $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ are standard sequences if α is non-zero and non-increasing, $|d_1| \geq |d_p|$, $\sum_{i=1}^{p} d_i = \sum_{j=1}^{q} e_j$, $d_1 > 0$, each $|d_i| \leq q$, each $|e_j| \leq p$ and $|e_j| \leq |d_1|$.

The following result due to Pirzada et al. [8], gives necessary and sufficient conditions for two sequences of integers to be the signed degree sequences of some signed bipartite graph. .

Theorem 3. Let $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ be standard sequences. Then, α and β are the signed degree sequences of a signed bipartite graph if and only if there exist integers r and s with $d_1 = r - s$ and $0 \leq s \leq \frac{q-d_1}{2}$ such that α' and β' are the signed degree sequences of a signed bipartite graph, where α' is obtained from α by deleting d_1 and β' is obtained from β by reducing r greatest entries of β by 1 each and adding s least entries of β by 1 each.

The set of distinct signed degrees of the vertices in a signed bipartite graph $G(U, V)$ is called its signed degree set. The work for signed degree sets in signed bipartite graphs can be found in [7]. Also the work on signed degrees in signed tripartite graphs can be found in [10, 11].

2. Signed degree sequences in signed k-partite graphs

A signed k-partite graph (signed multipartite graph) is a k-partite graph in which each edge is assigned a positive or a negative sign. Let $G(V_1, V_2, \cdots, V_k)$ be a signed k-partite graph with $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $1 \le i \le k$. The signed degree of v_{ij} is $sdeg(v_{ij}) = d_{ij} = d_{ij}^+ - d_{ij}^-$, where $1 \le i \le k$, $1 \le j \le n_i$ and d_{ij}^+ (d_{ij}^-) is the number of positive (negative) edges incident with v_{ij}. The sequences $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \le i \le k$, are called the signed degree sequences of $G(V_1, V_2, \cdots, V_k)$. Also the sequences $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \le i \le k$, of integers are s-graphical if $\alpha_i' s$ are the signed degree sequences of some signed k-partite graph. Denote a positive edge xy by xy^+ and a negative edge xy by xy^-. Several results on signed degree sequences in signed multipartite graphs can be found in [9]. We start with the following observation.

Theorem 4. Let $G(V_1, V_2, \cdots, V_k)$ be a signed k-partite graph with $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $1 \le i \le k$ and having q edges. Then

$$p = \sum_{i=1}^{k} \sum_{j=1}^{n_i} s\deg(v_{ij}) \equiv 2q(mod\ 4),$$

and the number of positive edges and negative edges of $G(V_1, V_2, \cdots, V_k)$ are respectively $\frac{2q+p}{4}$ and $\frac{2q-p}{4}$.

Proof. Let v_{ij} ($1 \le i \le k$, $1 \le j \le n_i$) be incident with d_{ij}^+ positive edges and d_{ij}^- negative edges so that

$$sdeg(v_{ij}) = d_{ij}^+ - d_{ij}^- \text{ while } deg(v_{ij}) = d_{ij}^+ + d_{ij}^-.$$

Obviously, $\sum_{i=1}^{k} \sum_{j=1}^{n_i} \deg(v_{ij}) = 2q$.

Let $G(V_1, V_2, \cdots, V_k)$ have g positive edges and h negative edges. Then $q = g + h$,

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} d_{ij}^+ = 2g \text{ and } \sum_{i=1}^{k} \sum_{j=1}^{n_i} d_{ij}^- = 2h.$$

Further,

$$\begin{aligned}\sum_{i=1}^{k} \sum_{j=1}^{n_i} s\deg(v_{ij}) &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (d_{ij}^+ - d_{ij}^-) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} d_{ij}^+ - \sum_{i=1}^{k} \sum_{j=1}^{n_i} d_{ij}^- \\ &= 2g - 2h.\end{aligned}$$

Hence,

$$\begin{aligned}p = \sum_{i=1}^{k} \sum_{j=1}^{n_i} s\deg(v_{ij}) &\equiv 2g - 2h \\ &= 2(q-h) - 2h \\ &= 2q - 4h,\end{aligned}$$

so that $p \equiv 2q(mod\ 4)$. Again, from $g + h = q$ and $2g - 2h = p$, we have $g = \frac{2q+p}{4}$ and $h = \frac{2q-p}{4}$. $\square$

Corollary 5. A necessary condition for the k sequences $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, of integers to be s-graphical is that $\sum_{i=1}^{k}\sum_{j=1}^{n_i} d_{ij}$ is even.

A zero sequence is a finite sequence each term of which is 0. Clearly, every k finite zero sequences are the signed degree sequences of a signed k-partite graph. If $\beta = [a_1, a_2, \cdots, a_n]$ is a sequence of integers, then the negative of β is the sequence $\beta = [-a_1, -a_2, \cdots, -a_n]$.

The next result follows by interchanging positive edges with negative edges.

Theorem 6. The sequences $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, are the signed degree sequences of some signed k-partite graph if and only if $-\alpha_i = [-d_{i1}, -d_{i2}, \cdots, -d_{in_i}]$ are the signed degree sequences of some signed k-partite graph.

Assume without loss of generality, that a non-zero sequence $\beta = [a_1, a_2, \cdots, a_n]$ is non-increasing and $|a_1| \geq |a_n|$, for we can always replace β by $-\beta$ if necessary. The k sequences of integers $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, are said to be standard sequences if α_1 is non-zero and non-increasing, $\sum_{i=1}^{k}\sum_{j=1}^{n_i} d_{ij}$ is even, $d_{11} > 0$, each $|d_{ij}| \leq \sum_{r=1, r\neq i}^{k} n_r$, $1 \leq i \leq k$, $1 \leq j \leq n_i$, $|d_{11}| \geq \left|d_{1n_1}\right|$ and $|d_{11}| \geq |d_{ij}|$ for each $2 \leq i \leq k$,$1 \leq j \leq n_i$.

A complete signed k-partite graph is a complete k-partite graph in which each edge is assigned a positive or a negative sign. The following result provides a useful recursive test whether the k sequences of integers form the signed degree sequences of some complete signed k-partite graph.

Theorem 7. Let $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, be standard sequences and let $r = \frac{1}{2}\left(d_{11} + \sum_{j=2}^{k} n_j\right)$. Let α_1' be obtained from α_1 by deleting d_{11} and $\alpha_2', \alpha_3', \cdots, \alpha_k'$ be obtained from $\alpha_2, \alpha_3, \cdots, \alpha_k$ by reducing r greatest entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each and adding remaining entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each. Then α_i are the signed degree sequences of some complete signed k-partite graph if and only if α_i' are also signed degree sequences of some complete signed k-partite graph, $1 \leq i \leq k$.

Proof. Let $G'(V_1', V_2', \cdots, V_k')$ be a complete signed k-partite graph with signed degree sequences α_i', $1 \leq i \leq k$. Let $V_1' = \{v_{12}, v_{13}, \cdots, v_{1n_1}\}$ and $V_i' = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $2 \leq i \leq k$. Then a complete signed k-partite graph with signed degree sequences α_i , $1 \leq i \leq k$, can be obtained by adding a vertex v_{11} to V_1' so that there are r positive edges from v_{11} to those r vertices of $V_2', V_3', \cdots, V_k'$, whose signed degrees were reduced by 1 in going from α_i to α_i', and there are negative edges from v_{11} to the remaining vertices of $V_2', V_3', \cdots, V_k'$, whose signed degrees were increased by 1 in going from α_i to α_i'. Note that the signed degree of v_{11} is $r - \left(\sum_{j=2}^{k} n_j - r\right) = 2r - \sum_{j=2}^{k} n_j = d_{11}$.

Conversely, let α_i , $1 \leq i \leq k$, be the signed degree sequences of a complete signed k-partite graph. Let the vertex sets of the complete signed k-partite graph be $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$ such that $sdeg(v_{ij}) = d_{ij}$, $1 \leq i \leq k$, $1 \leq j \leq n_i$.

Among all the complete signed k-partite graphs with α_i, $1 \leq i \leq k$, as the signed degree sequences, let $G(V_1, V_2, \cdots, V_k)$ be one with the property that the sum S of the signed degrees of the vertices of $V_2, V_3, \cdots, V_k$ joined to v_{11} by positive edges is maximum. Let d_{11}^+ and d_{11}^- be respectively the number of positive edges and the number of negative edges incident with v_{11}. Then $sdeg(v_{11}) = d_{11} = d_{11}^+ - d_{11}^-$, $deg(v_{11}) = d_{11}^+ + d_{11}^- = \sum_{j=2}^{k} n_j$, and hence $d_{11}^+ = \frac{1}{2}\left(d_{11} + \sum_{j=2}^{k} n_j\right) = r$. Let U be the set of r

vertices of $V_2, V_2, \cdots, V_k$ with highest signed degrees and let $W = \cup_{j=2}^{k} V_j - U$. We claim that v_{11} must be joined by positive edges to the vertices of U. If this is not true, then there exist vertices $v_{gh} \in U$ and $v_{ij} \in W$ such that the edge $v_{11}v_{gh}$ is negative and the edge $v_{11}v_{ij}$ is positive. Since $sdeg\,(v_{gh}) \geq sdeg\,(v_{ij})$, there exist vertices v_{mn} and v_{pq} such that the edge $v_{gh}v_{mn}$ is positive and the edge $v_{ij}v_{pq}$ is negative. If the edge $v_{gh}v_{pq}$ is positive, then change the signs of the edges $v_{11}v_{gh}$, $v_{gh}v_{pq}$, $v_{pq}v_{ij}$, $v_{ij}v_{11}$ so that the edges $v_{11}v_{gh}$ and $v_{pq}v_{ij}$ are positive and the edges $v_{11}v_{ij}$and $v_{gh}v_{pq}$ are negative. But if the edge $v_{gh}v_{pq}$ is negative, then $sdeg\,(v_{gh}) < sdeg\,(v_{ij})$, which is a contradiction. The case when $v_{mn} = v_{pq}$ follows by the same argument as in above.

Hence we obtain a complete signed k-partite graph with signed degree sequences α_i, $1 \leq i \leq k$, in which the sum of the signed degrees of the vertices of $V_2, V_3, \cdots, V_k$ joined to v_{11} by positive edges exceeds S, a contradiction.

Thus we may assume that v_{11} is joined by positive edges to the vertices of U and by negative edges to the vertices of W. So $G(V_1, V_2, \cdots, V_k) - v_{11}$ is a complete signed k-partite graph with α'_i, $1 \leq i \leq k$, as the signed degree sequences. $\square$

Theorem 7 provides an algorithm of checking whether the standard sequences α_i, $1 \leq i \leq k$, are the signed degree sequences, and for constructing a corresponding complete signed k-partite graph. Suppose $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, be the standard signed degree sequences of a complete signed k-partite graph with parts $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$. Deleting d_{11} and reducing $r = \frac{1}{2}\left(d_{11} + \sum_{j=2}^{k} n_j\right)$ greatest entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each and adding remaining entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each to form $\alpha'_2, \alpha'_3, \cdots, \alpha'_k$. Then edges are defined by $v_{11}\ v_{ij}^{+}$ if $d'_{ij}s$ are reduced by 1 and $v_{11}v_{ij}^{-}$ if $d'_{ij}s$ are increased by 1. For $-\alpha_i$, $1 \leq i \leq k$, edges are defined by $v_{11}v_{ij}^{-}$ if $d'_{ij}s$ are reduced by 1 and $v_{11}v_{ij}^{+}$ if $d'_{ij}s$ are increased by 1. If the conditions of standard sequences do not hold, then we delete d_{i1} for that i for which the conditions of standard sequences get satisfied. If this method is applied recursively, then a complete signed k-partite graph with signed degree sequences α_i, $1 \leq i \leq k$, is constructed.

The next result gives necessary and sufficient conditions for the k sequences of integers to be the signed degree sequences of some signed k-partite graph.

Theorem 8. Let $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, be standard sequences. Then α_i , $1 \leq i \leq k$, are the signed degree sequences of a signed k-partite graph if and only if there exist integers r and s with $d_{11} = r - s$ and $0 \leq s \leq \frac{1}{2}\left(\sum_{j=2}^{k} n_j - d_{11}\right)$ such that α'_i are the signed degree sequences of a signed k-partite graph, where α'_1 is obtained from α_1 by deleting d_{11} and $\alpha'_2, \alpha'_3, \cdots, \alpha'_k$ are obtained from $\alpha_2, \alpha_3, \cdots, \alpha_k$ by reducing r greatest entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each and adding s least entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each.

Proof. Let r and s be integers with $d_{11} = r - s$ and $\leq s \leq \frac{1}{2}\left(\sum_{j=2}^{k} n_j - d_{11}\right)$ such that α'_i, $1 \leq i \leq k$, are the signed degree sequences of a signed k-partite graph $G'(V'_1, V'_2, \prime, V'_k)$.

Let $V'_1 = \{v_{12}, v_{13}, \cdots, v_{1n_1}\}$ and $V'_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $2 \leq i \leq k$. Let U be the set of r vertices of $V'_2, V'_3, \cdots, V'_k$ with highest signed degrees, W be the set of s vertices of $V'_2, V'_3, \cdots, V'_k$ with least signed degrees and let $Z = \cup_{j=2}^{k} V'_j - U - W$. Then a signed k-partite graph with signed degree sequences α_i, $1 \leq i \leq k$, can be obtained by adding a vertex v_{11} to V'_1 so that there are r positive edges from v_{11} to the vertices of U and s negative edges from v_{11} to the vertices of W. Note that the signed degree of v_{11} is $r - s = d_{11}$.

Conversely, let α_i, $1 \leq i \leq k$, be the signed degree sequences of a signed k-partite graph. Let the vertex sets of the signed k-partite graph be $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$ such that $sdeg(v_{ij}) = d_{ij}$, $1 \leq i \leq k$, $1 \leq j \leq n_i$.

Among all the signed k-partite graphs with α_i, $1 \leq i \leq k$, as the signed degree sequences, let $G(V_1, V_2, \cdots, V_k)$ be one with the property that the sum S of the signed degrees of the vertices of $V_2, V_3, \cdots, V_k$ joined to v_{11} by positive edges is maximum. Let $d_{11}^+ = r$ and $d_{11}^- = s$ be respectively the number of positive edges and the number of negative edges incident with v_{11}. Then $sdeg(v_{11}) = d_{11} = d_{11}^+ - d_{11}^- = r - s$ and $deg(v_{11}) = d_{11}^+ + d_{11}^- = r + s \leq \sum_{j=2}^k n_j$, and hence $0 \leq s \leq \frac{1}{2}\left(\sum_{j=2}^k n_j - d_{11}\right)$. Let U be the set of r vertices of $V_2, V_3, \cdots, V_k$ with highest signed degrees and let $W = \cup_{j=2}^k V_j - U$.

We claim that v_{11} must be joined by positive edges to the vertices of U. If this is not true, then there exist vertices $v_{gh} \in U$ and $v_{mn} \in W$ such that the edge $v_{11}v_{mn}$ is positive and either (i) $v_{11}v_{gh}$ is a negative edge or (ii) v_{11} and v_{gh} are not adjacent in $G(V_1, V_2, \cdots, V_k)$. As $sdeg(v_{gh}) \geq sdeg(v_{mn})$, that is $d_{gh} \geq d_{mn}$, therefore we consider only (i) and then (ii) is similar to (i).

We note that if there exists a vertex v_{pq} ($\neq v_{11}$) such that $v_{pq}v_{gh}$ is a positive edge and $v_{pq}v_{mn}$ is a negative edge, then change the signs of these edges so that $v_{11}v_{gh}$ and $v_{pq}v_{mn}$ are positive, and $v_{11}v_{mn}$ and $v_{pq}v_{gh}$ are negative. Hence we obtain a signed k-partite graph with signed degree sequences α_i, $1 \leq i \leq k$, in which the sum of the signed degrees of the vertices of $V_2, V_3, \cdots, V_k$ joined to v_{11} by positive edges exceeds S, a contradiction. So assume that no such vertex v_{pq} exists.

Now, suppose that v_{gh} is not incident to any positive edge. Since $sdeg(v_{gh}) \geq sdeg(v_{mn})$, that is $d_{gh} \geq d_{mn}$, then there exist at least two vertices v_{pq} and v_{lt} (both distinct from v_{11}) such that $v_{pq}v_{mn}$ and $v_{lt}v_{mn}$ are negative edges and both v_{pq} and v_{lt} are not adjacent to v_{gh}. Then by changing the edges so that $v_{11}v_{gh}$ is a positive edge, and $v_{11}v_{mn}$,$v_{gh}v_{pq}$, $v_{gh}v_{lt}$ are negative edges, we again get a contradiction. Hence v_{gh} is incident to at least one positive edge.

We claim that there exists at least one vertex v_{yz} such that $v_{yz}v_{gh}$ is a positive edge and v_{yz} is not adjacent to v_{mn}. Suppose on contrary that whenever v_{gh} is joined to a vertex by a positive edge, then v_{mn} is also joined to this vertex by a positive edge. Since sdeg$(v_{gh}) \geq sdeg(v_{mn})$, that is $d_{gh} \geq d_{mn}$, then again we have the same situation as above, from which we get a contradiction. Thus there exists a vertex v_{yz} such that $v_{yz}v_{gh}$ is a positive edge and v_{yz} is not adjacent to v_{mn}. Similarly, it can be shown that there exists a vertex v_{pq} such that $v_{pq}v_{mn}$ is a negative edge and v_{pq} is not adjacent to v_{gh}. By changing the edges so that $v_{11}v_{gh}$,$v_{mn}v_{yz}$ are positive edges, and $v_{11}v_{mn}$, $v_{gh}v_{pq}$ are negative edges, we again get a contradiction. Hence v_{11} is joined by positive edges to the vertex of U.

In a similar way, it can be shown that v_{11} is joined by negative edge to the s vertices of V_2,V_3,$\cdots$, V_k with least signed degrees.

Hence $G(V_1, V_2, \cdots, V_k) - v_{11}$ is a signed k-partite graph with α_i', $1 \leq i \leq k$, as the signed degree sequences. $\square$

Theorem 8 also provides an algorithm for determining whether or not the standard sequences α_i, $1 \leq i \leq k$, are the signed degree sequences, and for constructing a corresponding signed k-partite graph. Suppose $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, be the standard signed degrees sequences of a signed k-partite graph with parts $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$. Let $d_{11} = r - s$ and $0 \leq s \leq \frac{1}{2}\left(\sum_{j=2}^k n_j - d_{11}\right)$. Deleting d_{11} and reducing r greatest entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each and adding s least entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each to form $\alpha_2', \alpha_3', \cdots, \alpha_k'$. Then edges are defined by $v_{11}v_{ij}^+$ if $d_{ij}'s$ are reduced by 1 ; $v_{11}v_{1j}^-$ if d_{ij}' s are increased by 1, and v_{11} and $v_{ij}$are not adjacent if d_{ij}' s are unchanged. For α_i, edges are defined by $v_{11}v_{ij}^-$ if d_{ij}' s are reduced by 1; $v_{11}v_{ij}^+$ if d_{ij}' s are increased by

1, and v_{11} and v_{ij} are not adjacent if d'_{ij} s are unchanged. If the conditions of standard sequences do not hold, then we delete d_{i1} for that i for which the conditions of standard sequences get satisfied. If this method is applied recursively, then a signed k-partite graph with signed degree sequences α_i, $1 \le i \le k$, is constructed.

3. Signed degree sets in signed k-partite graphs

Let $G(V_1, V_2, \cdots, V_k)$ be a signed k-partite graph with $X \subseteq V_i, Y \subseteq V_j$ $(i \neq j)$. If each vertex of X is joined to every vertex of Y by a positive (negative) edge, then it is denoted by $X \oplus Y (X \ominus Y)$.

The set S of distinct signed degrees of the vertices in a signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ is called its signed degree set. Also, a signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ is said to be connected if each vertex $v_i \in V_i$; is connected to every vertex $v_j \in V_j$.

The following result shows that every set of positive integers is a signed degree set of some connected signed k-partite graph.

Theorem 9. Let $d_1, d_2, \cdots, d_t$ be positive integers. Then there exists a connected signed k-partite graph with signed degree set

$$S = \{d_1, \sum_{i=1}^{2} d_i, \cdots, \sum_{i=1}^{t} d_i\}.$$

Proof. We consider the following two cases. (i) k even, (ii) k odd.
Case (i). Let $k = 2m$, where $m \ge 1$. Construct a signed k-partite graph $G(V_1, V_2, \cdots, V_{2m})$ as follows.
Let

$$\begin{aligned}
&V_1 = P_1 \cup Q_1 \cup R_1 \cup S_1 \cup X_1 \cup X_1' \cup X_1'' \cup X_2 \cup X_2' \cup X_2'' \cup \cdots \cup X_{t-1} \cup X_{t-1}' \cup X_{t-1}'',\\
&V_2 = P_2 \cup Q_2 \cup R_2 \cup S_2 \cup Y_1 \cup Y_1' \cup Y_2 \cup Y_2' \cup \cdots \cup Y_{t-1} \cup Y_{t-1}',\\
&V_3 = P_3 \cup Q_3,\\
&\vdots\\
&V_{2m-1} = P_{2m-1} \cup Q_{2m-1},\\
&V_{2m} = P_{2m} \cup Q_{2m},
\end{aligned}$$

where
(a) $P_1, Q_1, R_1, S_1, X_1, X_1', X_1'', X_2, X_2', X_2'', \cdots, X_{t-1}, X_{t-1}', X_{t-1}''$ are pairwise disjoint,
(b) $P_2, Q_2, R_2, S_2, Y_1, Y_1', Y_2, Y_2', \cdots, Y_{t-1}, Y_{t-1}'$ are pairwise disjoint,
(c) For all i, $P_i \cap Q_i = \phi$, $3 \le i \le 2m$ and $|P_i| = |Q_i| = d_1$, $1 \le i \le 2m$; $|R_i| = |S_i| = d_1$, $1 \le i \le 2$; $|X_i| = |X_i'| = |Y_i| = |Y_i'| = d_1$, $1 \le i \le t-1$; $|X_i''| = d_2 + d_3 + \cdots + d_{i+1}$, $1 \le i \le t-1$.

For all i, let $P_i \oplus Q_{i+1}$, $1 \le i \le 2m-1$; $Q_i \oplus P_{i+1}$, $1 \le i \le 2m-1$; $Q_1 \oplus R_2, R_1 \oplus Q_2, R_1 \oplus S_2, S_1 \oplus R_2, X_1 \oplus S_2, X_1' \oplus R_2, X_i \oplus Y_i'$, $1 \le i \le t-1$; $X_i' \oplus Y_i$, $1 \le i \le t-1$; $X_i'' \oplus Y_i'$, $1 \le i \le t-1$; $X_i \oplus Y_{i-1}'$, $2 \le i \le t-1$; $X_i' \oplus Y_{i-1}$, $2 \le i \le t-1$;
for all even i, $P_i \ominus P_{i+1}$, $2 \le i \le 2m-2$; $Q_i \ominus Q_{i+1}$, $2 \le i \le 2m-2$;
and for all i, $Q_1 \ominus Q_2, R_1 \ominus R_2, X_1 \ominus R_2, X_1' \ominus S_2, X_i \ominus Y_{i-1}$, $2 \le i \le t-1$; $X_i' \ominus Y_{i-1}'$, $2 \le i \le t-1$.
Then the signed degrees of the vertices of $G(V_1, V_2, \cdots, V_{2m})$ are as follows.

$sdeg(p_1) = |Q_2| - 0 = d_1$ for all $p_1 \in P_1$;
for even i, $2 \leq i \leq 2m-2$
$sdeg(p_i) = |Q_{i-1}| + |Q_{i+1}| - |P_{i+1}| = d_1 + d_1 - d_1 = d_1$, for all $p_i \in P_i$;
for odd i, $3 \leq i \leq 2m-1$
$sdeg(p_i) = |Q_{i-1}| + |Q_{i+1}| - |P_{i-1}| = d_1 + d_1 - d_1 = d_1$, for all $p_i \in P_i$,
$sdeg(p_{2m}) = |Q_{2m-1}| - 0 = d_1$, for all $p_{2m} \in P_{2m}$;
$sdeg(q_1) = |P_2| + |R_2| - |Q_2| = d_1 + d_1 - d_1 = d_1$, for all $q_1 \in Q_1$;
$sdeg(q_2) = |P_1| + |R_1| + |P_3| - (|Q_1| + |Q_3|) = d_1 + d_1 + d_1 - (d_1 + d_1) = d_1$, for all $q_2 \in Q_2$;
for odd i, $3 \leq i \leq 2m-1$
$sdeg(q_i) = |P_{i-1}| + |P_{i+1}| - |Q_{i-1}| = d_1 + d_1 - d_1 = d_1$, for all $q_i \in Q_i$;
for even i, $4 \leq i \leq 2m-2$
$sdeg(q_i) = |_{i-1}| + |P_{i+1}| - |Q_{i+1}| = d_1 + d_1 - d_1 = d_1$, for all $q_i \in Q_i$, $sdeg(q_{2m}) = |P_{2m-1}| - 0 = d_1$, for all $q_{2m} \in Q_{2m}$, $sdeg(r_1) = |Q_2| + |S_2| - |R_2| = d_1 + d_1 - d_1 = d_1$, for all $r_1 \in R_1$,
$sdeg(s_1) = |R_2| - 0 = d_1$, for all $s_1 \in S_1$,
$sdeg(r_2) = |Q_1| + |S_1| + |X_1'| - (|R_1| + |X_1|) = d_1 + d_1 + d_1 - (d_1 + d_1) = d_1$, for all $r_2 \in R_2$,
$sdeg(s_2) = |R_1| + |X_1| - |X_1'| = d_1 + d_1 - d_1 = d_1$, for all $s_2 \in S_2$,
$sdeg(x_1) = |S_2| + |Y_1'| - |R_2| = d_1 + d_1 - d_1 = d_1$, for all $x_1 \in X_1$,
$sdeg(x_1') = |R_2| + |Y_1| - |S_2| = d_1 + d_1 - d_1 = d_1$, for all $x_1' \in X_1'$,
$sdeg(x_1'') = |Y_1'| - 0 = d_1$, for all $x_1'' \in X_1''$;
for $2 \leq i \leq t-1$
$sdeg(x_i) = |Y_{i-1}'| + |Y_i'| - |Y_{i-1}| = d_1 + d_1 - d_1 = d_1$, for all $x_i \in X_i$;
for $2 \leq i \leq t-1$
$sdeg(x_i') = |Y_{i-1}| + |Y_i| - |Y_{i-1}'| = d_1 + d_1 - d_1 = d_1$, for all $x_i' \in X_i'$;
for $2 \leq i \leq t-1$
$sdeg(x_i'') = |Y_i'| - 0 = d_1$, for all $x_i'' \in X_i''$;
for $1 \leq i \leq t-2$
$sdeg(y_i) = |X_i'| + |X_{i+1}'| - |X_{i+1}| = d_1 + d_1 - d_1 = d_1$, for all $y_i \in Y_i$
$sdeg(y_{t-1}) = |X_{t-1}'| - 0 = d_1$, for all $y_{t-1} \in Y_{t-1}$;
for $1 \leq i \leq t-2$
$sdeg(y_i') = |X_i| + |X_i''| + |X_{i+1}| - |X_{i+1}'| = d_1 + d_2 + d_3 + \cdots + d_{i+1} + d_1 - d_1 = \sum_{j=1}^{i+1} d_j$, for all $y_i' \in Y_i'$,
and $sdeg(y_{t-1}') = |X_{t-1}| + |X_{t-1}''| = d_1 + d_2 + d_3 + \cdots + d_t = \sum_{j=1}^{t} d_j$, for all $y_{t-1}' \in Y_{t-1}'$.
Therefore signed degree set of $G(V_1, V_2, \prime, V_{2m})$ is $S = \{d_1, \sum_{i=1}^{2} d_i, \prime, \sum_{i=1}^{t} d_i\}$.
Case (ii). Let $k = 2m+1$, where $m \geq 1$. This follows by using the construction as in case (i), and taking another partite set $V_{2m+1} = P_{2m+1} \cup Q_{2m+1}$ with $P_{2m+1} \cap Q_{2m+1} = \phi, |P_{2m+1}| = |Q_{2m+1}| = d_1, P_{2m} \oplus Q_{2m+1}, Q_{2m} \oplus P_{2m+1}, P_{2m+1} \oplus P_1, P_{2m+1} \oplus R_2, Q_{2m+1} \oplus S_1, Q_1 \oplus S_2$ and $P_{2m} \ominus P_{2m+1}, Q_{2m} \ominus Q_{2m+1}, P_{2m+1} \ominus Q_1, P_1 \ominus R_2$, $S_1 \ominus S_2$.
Clearly, by construction , the above signed k-partite graphs are connected. Hence the result follows. □

By interchanging positive edges with negative edges in Theorem 9, we obtain the following result.

Corollary 10. Every set of negative integers is a signed degree set of some connected signed k-partite graph.

Finally, we have the following result.

Theorem 11. Every set of integers is a signed degree set of some connected signed k-partite graph.

Proof. Let S be a set of integers. Then we have the following five cases.

Case (i). S is a set of positive (negative) integers. Then the result follows by Theorem 9 (Corollary 10).

Case (ii). $S = \{0\}$. Then a signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ with $V_i = \{v_i, v_i'\}$ for all i, $1 \le i \le k$, in which $v_i v_{i+1}', v_i' v_{i+1}$ for all i, $1 \le i \le k-1$, are positive edges and $v_i v_{i+1}, v_i' v_{i+1}'$ for all i, $1 \le i \le k-1$, are negative edges has signed degree set S.

Case (iii). S is a set of non-negative (non-positive) integers. Let $S = S' \cup \{0\}$, where S' be a set of positive(negative) integers. Then by Theorem 9(Corollary 10), there is a connected signed k-partite graph $G'(V_1', V_2', \cdots, V_k')$ with signed degree set S'. Construct a new signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ as follows.

Let $V_1 = V_1' \cup \{x_1\} \cup \{y_1\}$, $V_2 = V_2' \cup \{x_2\} \cup \{y_2\}$, $V_3 = V_3'$, $\cdots$, $V_k = V_k'$, with $V_1' \cap \{x_1\} = \phi, V_1' \cap \{y_1\} = \phi, \{x_1\} \cap \{y_1\} = \phi, V_2' \cap \{x_2\} = \phi, V_2' \cap \{y_2\} = \phi, \{x_2\} \cap \{y_2\} = \phi$. Let $v_1' x_2, x_1 v_2', y_1 y_2$ be positive edges, $v_1' y_2, x_1 x_2, y_1 v_2'$ be negative edges, where $v_1' \in V_1', v_2' \in V_2'$ and let there be all the edges of $G'(V_1', V_2', \cdots, V_k')$. Then $G(V_1, V_2, \cdots, V_k)$ has signed degree set S. We note that addition of the positive edges $v_1' x_2, x_1 v_2', y_1 y_2$ and negative edges $v_1' y_2, x_1 x_2, y_1 v_2'$ do not effect the signed degrees of the vertices of $G'(V_1', V_2', \cdots, V_k')$, and the vertices x_1, y_1, x_2, y_2 have signed degrees zero each.

Case (iv). S is a set of non-zero integers. Let $S = S' \cup S''$, where S' and S'' are sets of positive and negative integers respectively. Then by Theorem 9 (Corollary 10), there are connected signed k-partite graphs $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$ with signed degree sets S' and S'' respectively. Suppose $G_1'(V_{11}', V_{21}', \cdots, V_{k1}')$ and $G_2''(V_{12}'', V_{22}'', \cdots, V_{k2}'')$ are the copies of $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$ with signed degree sets S' and S'' respectively. Construct a new signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ as follows.

Let

$$V_1 = V_1' \cup V_{11}' \cup V_1'' \cup V_{12}'',$$

$$V_2 = V_2' \cup V_{21}' \cup V_2'' \cup V_{22}'',$$

$$V_3 = V_3' \cup V_{31}' \cup V_3'' \cup V_{32}'',$$

$$\vdots$$

$$V_k = V_k' \cup V_{k1}' \cup V_k'' \cup V_{k2}'',$$

with $V_i' \cap V_{i1}' = \phi, V_i' \cap V_i'' = \phi, V_i' \cap V_{i2}'' = \phi, V_{i1}' \cap V_i'' = \phi, V_{i1}' \cap V_{i2}'' = \phi, V_i'' \cap V_{i2}'' = \phi$. Let $v_1' v_{22}'', v_{11}' v_2''$ be positive edges, $v_1' v_2', v_{11}' v_{22}''$ be negative edges, where $v_1' \in V_1', v_{11}' \in V_{11}', v_2'' \in V_2'', v_{22}'' \in V_{22}''$ and let there be all the edges of $G'(V_1', V_2', \cdots, V_k'), G_1'(V_{11}', V_{21}', \cdots, V_{k1}'), G''(V_1'', V_2'', \cdots, V_k'')$ and $G_2''(V_{12}'', V_{22}'', \cdots, V_{k2}'')$. Then $G(V_1, V_2, \cdots, V_k)$ has signed degree set S.

We note that addition of the positive edges $v_1' v_{22}'', v_{11}' v_2''$ and negative edges $v_1' v_2'', v_{11}' v_{22}''$ do not effect the signed degrees of the vertices of $G'(V_1', V_2', \cdots, V_k')$, $G_1'(V_{11}', V_{21}', \cdots, V_{k1}')$, $G''(V_1'', V_2'', \cdots, V_k'')$ and $G_2''(V_{12}'', V_{22}'', \cdots, V_{k2}'')$.

Case (v). S is the set of all integers. Let $S = S' \cup S'' \cup \{0\}$, where S' and S'' are sets of positive and negative integers respectively. Then by Theorem 9(Corollary 10), there exist connected signed k-partite graphs $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$ with signed degree sets S' and S'' respectively. Construct a new signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ as follows.

Let

$$V_1 = V_1' \cup V_1'' \cup \{x\},$$
$$V_2 = V_2' \cup V_2'' \cup \{y\},$$
$$V_3 = V_3' \cup V_3'',$$
$$\vdots$$
$$V_k = V_k' \cup V_k'',$$

with $V_i' \cap V_i'' = \phi, V_1' \cap \{x\} = \phi, V_1'' \cap \{x\} = \phi, V_2' \cap \{y\} = \phi, V_2'' \cap \{y\} = \phi$. Let $v_1'v_2'', v_1''y, xv_2'$ be positive edges, $v_1'y, v_1''v_2', xv_2''$ be negative edges, where $v_1' \in V_1', v_1'' \in V_1'', v_2' \in V_2', v_2'' \in V_2''$, and let there be all the edges of $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$. Therefore $G(V_1, V_2, \cdots, V_k)$ has signed degree set S. We note that addition of the positive edges $v_1'v_2'', v_1''y, xv_2'$ and negative edges $v_1'y, v_1''v_2', xv_2''$ do not effect the signed degrees of the vertices of $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$, and the vertices x and y have signed degrees zero each.

Clearly, by construction, all the signed k-partite graphs are connected. This proves the result. □

References

[1] G. Charttrand, H. Gavlas, F. Harary, M. Schultz, *On signed degrees in signed graphs*, Czech. Math. J., **44** (1994) 677-690.

[2] S. L. Hakimi, *On the realizability of a set of integers as degrees of the vertices of a graph*, SIAM J. Appl. Math., **10** (1962) 496-506.

[3] F. Harary, *On the notion of balance in a signed graph*, Michigan Math. J., **2**(1953) 143-146.

[4] D. Hoffman, H. Jordan, *Signed graph factors and degree sequences*, J. Graph Theory, **52** (2006) 27-36.

[5] H. Jordan, R. McBride, S. Tipnis, *The convex hull of degree sequences of signed graphs*, Discrete Math., **309** (2009) 5841-5848.

[6] S. Pirzada, T. A. Naikoo, F. A. Dar, *Signed degree sets in signed graphs*, Czech. Math. J., **57, 3** (2007) 843-848.

[7] S. Pirzada, T. A. Naikoo, F. A. Dar, *A note on signed degree sets in signed bipartite graphs*, Applicable Analysis and Discrete Math., **2, 1** (2008) 114-117.

[8] S. Pirzada, T. A. Naikoo, S. Pirzada, *Signed degree sequences in signed bipartite graphs*, AKCE International J. Graphs and Comb., **4, 2** (2007) 1-12.

[9] S. Pirzada, *Signed degree sequences in signed graphs*, Journal of Combinatorics, Information and System Sciences, **37**, **2-4** (2012) 333-358.

[10] S. Pirzada, F. A. Dar, *Signed degree sequences in signed tripartite graphs*, J. Korean Society of Industrial and Applied Mathematics, **11, 2** (2007) 9-14.

[11] S. Pirzada, F. A. Dar, *Signed degree sets in signed tripartite graphs*, Math. Vesnik, **59** (2007) 121-124.

[12] J. H. Yan, K. W. Lie, D. Kuo, G. J. Chang, *Signed degree sequences of signed graphs*, J. Graph Theory, **26** (1997) 111-117.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1109 – 1124

On admissibility and inadmissibility of estimators after selection under reflected gamma loss function

Mehran Naghizadeh Qomi* , Nader Nematollahi† and Ahmad Parsian‡

Abstract

Let Π_1 and Π_2 denote two gamma populations with common known shape parameter $\alpha > 0$ and unknown scale parameters θ_1 and θ_2, respectively. Let X_1 and X_2 be two independent random variables from Π_1 and Π_2, and $X_{(1)} \leq X_{(2)}$ denote the ordered statistics of X_1 and X_2. Suppose the population corresponding to the largest $X_{(2)}$ or the smallest $X_{(1)}$ observation is selected. This paper concerns on the admissible estimation of the scale parameters θ_M and θ_J of the selected population under reflected gamma loss function. We provide sufficient conditions for the inadmissibility of invariant estimators of θ_M and θ_J. The admissibility and inadmissibility of estimators in the class of linear estimators of the form $cX_{(2)}$ and $dX_{(1)}$ are discussed. We apply our results on k-Records, censored data and extend to a subclass of exponential family.

Keywords: Admissibility; Estimation after selection; Inadmissibility; Invariant estimators; Gamma distribution; Reflected gamma loss function; k-Records data; Type-II censoring.

2000 AMS Classification: 62F07, 62F10, 62C15.

Received 20/03/2011 : Accepted 17/05/2014 Doi : 10.15672/HJMS.2015449655

*Department of Statistics, University of Mazandaran, Babolsar, Iran
Email: m.naghizadeh@umz.ac.ir
†Department of Statistics, Allameh Tabataba'i University, Tehran, Iran
Email:nematollahi@atu.ac.ir Corresponding Author.
‡School of Mathematics, Statistics and Computer Science, University of Tehran, Tehran, Iran
Email:ahmad_p@khayam.ut.ac.ir

1. Introduction

Estimation of the parameter(s) of the selected population is an important estimation problem and arises in various practical problems. For example, we wish to select the treatment with the highest cure rate and then estimate the actual probability of success with this treatment, see Tappin [30]. A car manufacturer, who has selected the most reliable model of engine for his cars, would like to know the reliability of the selected engine during actual use, see Kumar and Kar [12]. A textile designer chooses the best quality cloth from k available varieties for his usage. Naturally, he would be interested in estimating the durability of the best cloth that he has selected, see Gangopadhyay and Kumar [10].

The problem of estimation after selection is related to ranking and selection methodology. Readers may refer to Sarkadi [28], Dahiya [9], Sackrowitz and Cahn [26,27], Misra and Singh [17], Kumar and Kar [12], Balakrishnan et al. [5] and Kumar et al. [14].

During the past three decades, a lot of work has been done on estimation after selection from Gamma populations. Some of the main results are as follows: For positive integer value shape parameter, Vellaisamy and Sharma [35] derived the UMVUE of the scale parameter of the larger selected population and obtained estimators which are admissible (or inadmissible) within a subclass of equivariant estimators under the Squared Error Loss (SEL) function. Some of their results were extended to real valued shape parameter by Vellaisamy and Sharma [36]. Later, Vellaisamy [33] obtained estimators which dominates natural estimators under the SEL function. Vellaisamy [34] showed that the UMVUE of the selected scale parameters are inadmissible under the SEL function. Misra et al. [18,19] extended the admissibility and inadmissibility results of Vellaisamy and Sharma [35] to the case of known and arbitrary shape parameter. Motamed-Shariati and Nematollahi [20] derived the minimax estimator of the scale parameter of the larger selected population under the Scale Invariant Squared Error Loss (SISEL) function. Nematollahi and Motamed-Shariati [22] dealt with estimating the scale parameter of the selected gamma population under the entropy loss function and extended their results to a subclass of exponential family.

Let X_1 and X_2 be two independent random variables from populations Π_1 and Π_2, respectively, where Π_i has probability density function (pdf)

$$f(x|\theta_i,\alpha) = \frac{1}{\theta_i^{\alpha}\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\frac{x}{\theta_i}}, \quad x>0, \quad \alpha>0, \quad \theta_i>0, \quad i=1,2, \tag{1.1}$$

where α is the common known shape parameter and θ_1, θ_2 are unknown scale parameters. Let $X_{(1)} = \min(X_1, X_2)$ and $X_{(2)} = \max(X_1, X_2)$. For selecting the population corresponding to the larger (or smaller) θ_i's, we use the natural selection rule and select the population corresponding to the $X_{(2)}$ (or $X_{(1)}$). Therefore the scale parameter associated with the larger and smaller selected population are given by

$$\theta_M = \begin{cases} \theta_1 & X_1 \geq X_2 \\ \theta_2 & X_1 < X_2 \end{cases} \quad \textit{and} \quad \theta_J = \begin{cases} \theta_2 & X_1 \geq X_2 \\ \theta_1 & X_1 < X_2. \end{cases}$$

In literature the estimation of the selected gamma scale parameters θ_M and θ_J considered under SEL, SISEL and entropy loss functions, which are either symmetric or asymmetric and unbounded. In some estimation problems, the use of unbounded loss function may be inappropriate. For example in estimating the mean life θ of a component, the amount of loss for estimating θ by an estimator is essentially bounded.

For estimation of the parameter of the selected population under a bounded loss function, see Naghizadeh Qomi et al. [21]. They estimate the mean of the selected

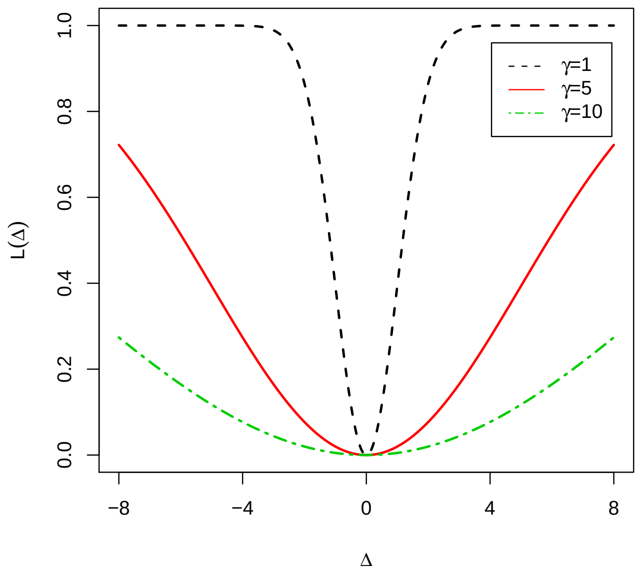


Figure 1. Plot of the RNL function for $k = 1$ and certain values of γ

normal population under Reflected Normal Loss (RNL) function given by

$$L(\Delta) = k\left\{1 - e^{-\frac{\Delta^2}{2\gamma^2}}\right\},$$

where $\Delta = \delta - \theta$, $k > 0$ is the maximum loss and $\gamma > 0$ is a shape parameter. The RNL function is a symmetric and bounded function of Δ (see Figure 1).

Although the RNL function is bounded, but it is symmetric and give the same penalty for underestimation and overestimation. Also, it is appropriate for location parameter θ. In some estimation problem, underestimation may be more serious than overestimation or vise versa. For example, in estimating the average life of the components of an aircraft, overestimation is usually more serious than underestimation. In such cases, for estimation the average life, which is a multiple of a scale parameter, an asymmetric bounded scale invariant loss function is appropriate to use.

In this paper, we discuss the estimation of the scale parameter of the selected gamma population under Reflected Gamma Loss (RGL) function. The RGL function is a simple transformation of the gamma density and introduced by Spiring and Yeung [29] in response to the criticisms of the SISEL function and is defined by

$$L(\theta, \delta) = k\left\{1 - \left(\frac{\delta}{\theta}\right)^{\gamma} e^{-\gamma(\frac{\delta}{\theta} - 1)}\right\} = k\left\{1 - e^{-\gamma\left(\frac{\delta}{\theta} - \ln\frac{\delta}{\theta} - 1\right)}\right\} \tag{1.2}$$

where $k > 0$ is the maximum loss and $\gamma > 0$ is a shape parameter. The RGL function is a bounded and asymmetric function of δ but not convex in δ and is essentially a gamma density flipped upside down, whence its name (see Figure 2). This loss is scale invariant, which is appropriate for estimating scale parameter θ, and it penalizes heavily underestimation. Towhidi and Behboodian [31,32] used this loss in some problem of scale

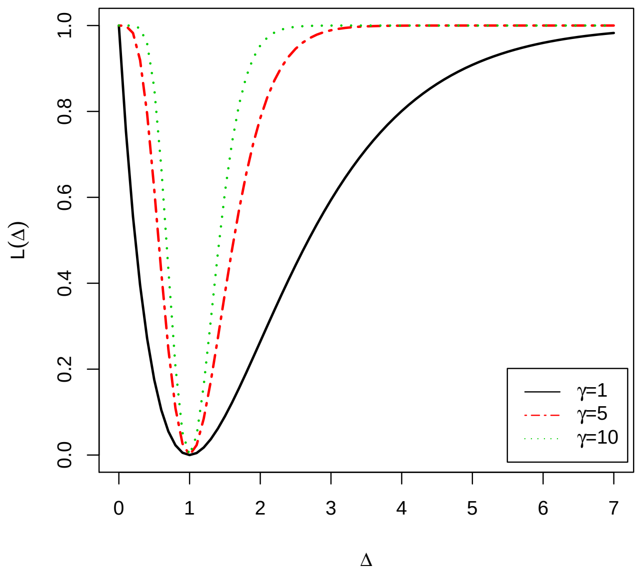


Figure 2. Plot of the RGL function for $\Delta = \frac{\delta}{\theta}$, $k = 1$ and certain values of γ

parameter estimation. Clearly the value of $k > 0$ does not have any influence on our results, therefore without loss of generality, we shall take $k = 1$ in the rest of the paper.

Since the RGL function is bounded, so by a result of Basu [6], Uniformly Minimum Risk Unbiased estimator of any unknown parameter does not exist under the RGL function. We are interested in estimation of the random parameters θ_M and θ_J of the selected gamma population under the RGL function and we concentrate our attention on admissible and inadmissible estimators of θ_M and θ_J. To this end, in section 2, we employ the technique of Brewster and Zidek [7] for finding dominating estimators for some intended scale and permutation invariant estimators. In section 3, we discuss the admissibility of invariant estimators of the form $cX_{(2)}$ and $dX_{(1)}$ for estimating θ_M and θ_J, respectively. In section 4, applications on k-records, censored data and extension of the problem to a subclass of exponential family are considered.

2. Sufficient Conditions for Inadmissibility

Let X_1 and X_2 be two independent random variables, where $X_i,\ i = 1, 2$ has pdf (1.1). In estimation of unknown random parameters θ_M and θ_J under the RGL function, the problem is invariant under the scale and permutation groups of transformations. Therefore, it is natural to consider only those estimators which are permutation and scale invariant, i.e, estimators satisfying $\delta(X_1, X_2) = \delta(X_2, X_1)$ and $\delta(cX_1, cX_2) = c\delta(X_1, X_2),\ \forall c > 0$. For this purpose, consider the following classes of invariant estimators

$$D_U = \{\delta_\psi : \delta_\psi(X_1, X_2) = X_{(2)}\psi(Y)\}, \tag{2.1}$$

and

$$D_L = \{\delta_\varphi : \delta_\varphi(X_1, X_2) = X_{(1)}\varphi(T)\}, \tag{2.2}$$

for θ_M and θ_J respectively, where $Y = \frac{X_{(1)}}{X_{(2)}}$, $T = \frac{1}{Y}$ and ψ and φ are some real valued functions defined on $(0,1]$ and $[1,\infty)$, respectively. In this section, we will employ the technique of Brewster and Zidek [7] to derive dominating estimators to show the inadmissibility of some scale and permutation invariant estimators of θ_M and θ_J, under the RGL function. As a consequence, we show that several of the proposed estimators are inadmissible and present improved estimators for those.

The following lemma will be useful in deriving the improved estimators on estimating θ_M and θ_J.

2.1. Lemma *Let* $Y = \frac{X_{(1)}}{X_{(2)}}$, $T = \frac{1}{Y}$, $\mu = \frac{\max(\theta_1,\theta_2)}{\min(\theta_1,\theta_2)}$ *and* ψ *and* φ *are real valued functions defined on* $(0,1]$ *and* $[1,\infty)$, *respectively. Define the functions* $\eta_{y,\psi}(\mu)$ *and* $\xi_{t,\varphi}(\mu)$ *as*

$$\eta_{y,\psi}(\mu) = (2\alpha+\gamma)\,\frac{\left[\frac{\mu}{(1+\gamma\psi(y))\mu+y}\right]^{2\alpha+\gamma+1} + \frac{1}{\mu^{\gamma+1}}\left[\frac{\mu}{1+\gamma\psi(y)+\mu y}\right]^{2\alpha+\gamma+1}}{\left[\frac{\mu}{(1+\gamma\psi(y))\mu+y}\right]^{2\alpha+\gamma} + \frac{1}{\mu^{\gamma}}\left[\frac{\mu}{1+\gamma\psi(y)+\mu y}\right]^{2\alpha+\gamma}},\quad 0<y\le 1,\ \mu\ge 1,$$

and

$$\xi_{t,\varphi}(\mu) = (2\alpha+\gamma)\,\frac{\left[\frac{\mu}{(1+\gamma\varphi(t))\mu+t}\right]^{2\alpha+\gamma+1} + \frac{1}{\mu^{\gamma+1}}\left[\frac{\mu}{1+\gamma\varphi(t)+\mu t}\right]^{2\alpha+\gamma+1}}{\left[\frac{\mu}{(1+\gamma\varphi(t))\mu+t}\right]^{2\alpha+\gamma} + \frac{1}{\mu^{\gamma}}\left[\frac{\mu}{1+\gamma\varphi(t)+\mu t}\right]^{2\alpha+\gamma}},\quad t\ge 1,\ \mu\ge 1.$$

(i) *For* $y \in (0,1]$, *the conditional pdf of* $S = \frac{X_{(2)}}{\theta_M}$ *given* $Y = y$ *is*

$$f_{S|Y=y}(s) = \frac{y^{\alpha-1}s^{2\alpha-1}}{\Gamma^2(\alpha)f_Y(y)}\left[\mu^{-\alpha}e^{-(\frac{y}{\mu}+1)s} + \mu^{\alpha}e^{-(1+\mu y)s}\right],\quad s>0,$$

where $f_Y(y)$ *denotes the pdf of* Y.

(ii) *For* $t \in [1,\infty)$, *the conditional pdf of* $U = \frac{X_{(1)}}{\theta_J}$ *given* $T = t$ *is*

$$f_{U|T=t}(u) = \frac{t^{\alpha-1}u^{2\alpha-1}}{\Gamma^2(\alpha)f_T(t)}\left[\mu^{-\alpha}e^{-(\frac{t}{\mu}+1)u} + \mu^{\alpha}e^{-(1+\mu t)u}\right],\quad u>0,$$

where $f_T(t)$ *denotes the pdf of* T.

(iii) *For* $y \in (0,1]$

$$\sup_{\mu\ge 1}\eta_{y,\psi}(\mu) = \frac{2\alpha+\gamma}{1+\gamma\psi(y)} = \frac{1}{\psi^\star(y)}. \tag{2.3}$$

(iv) *For* $t \in [1,\infty]$

$$\sup_{\mu\ge 1}\xi_{t,\varphi}(\mu) = \frac{2\alpha+\gamma}{1+\gamma\varphi(t)} = \frac{1}{\varphi^\star(t)}. \tag{2.4}$$

Proof. **(i),(ii)** For a proof, see Lemma 16(i) and 16(ii) of Misra et al. [18].

(iii) Note that

$$\lim_{\mu\uparrow\infty}\eta_{y,\psi}(\mu) = \frac{2\alpha+\gamma}{1+\gamma\psi(y)}.$$

So, we need to show that $\eta_{y,\psi}(\mu) \leq \frac{2\alpha+\gamma}{1+\gamma\psi(y)}$. But this inequality is equivalent to:

$$[1+\gamma\psi(y)]\eta_{y,\psi}(\mu) \leq (2\alpha+\gamma)$$

$$\begin{aligned}
\Leftrightarrow \quad & [1+\gamma\psi(y)]\mu^{\gamma+1}\big\{[1+\gamma\psi(y)]\mu+y\big\}^{-(2\alpha+\gamma+1)} \\
& \qquad +[1+\gamma\psi(y)]\big\{1+\gamma\psi(y)+y\mu\big\}^{-(2\alpha+\gamma+1)} \\
& \quad \leq \mu^{\gamma}\big\{[1+\gamma\psi(y)]\mu+y\big\}^{-(2\alpha+\gamma)}+\big\{1+\gamma\psi(y)+y\mu\big\}^{-(2\alpha+\gamma)} \\
\Leftrightarrow \quad & \big\{[1+\gamma\psi(y)]\mu+y\big\}^{-(2\alpha+\gamma+1)}\Big\{\mu^{\gamma+1}[1+\gamma\psi(y)]-\mu^{\gamma}\big\{[1+\gamma\psi(y)]\mu+y\big\}\Big\} \\
& + \quad \big\{1+\gamma\psi(y)+y\mu\big\}^{-(2\alpha+\gamma+1)}\Big\{1+\gamma\psi(y)-[1+\gamma\psi(y)+y\mu]\Big\} \leq 0 \\
\Leftrightarrow \quad & -y\mu^{\gamma}\big\{[1+\gamma\psi(y)]\mu+y\big\}^{-(2\alpha+\gamma+1)}-y\mu\big\{1+\gamma\psi(y)+y\mu\big\}^{-(2\alpha+\gamma+1)} \leq 0
\end{aligned}$$

which is always true for $\mu \geq 1$ and $y \in (0,1]$. So, the result follows.

(iv) Similar to the proof of (**iii**).

The following theorem provides a sufficient condition for invariant estimators $\delta_\psi(X_1, X_2) \in D_U$ to be inadmissible under the RGL function.

2.2. Theorem *Let $\delta_\psi(X_1,X_2) \in D_U$ be an invariant estimator of θ_M, $\psi_{11}(y)$ be any function defined on $(0,1]$ such that $P_\theta\big(\psi(Y) < \psi_{11}(Y) \leq \psi^{\star}(Y)\big) > 0$, $\forall\theta = (\theta_1,\theta_2) \in (0,\infty)\times(0,\infty) = \Re^2_+$. Then under the RGL function, the invariant estimator δ_ψ is inadmissible for estimating θ_M, and is dominated by $\delta_{\psi_1}(X_1,X_2) = X_{(2)}\psi_1(Y)$, where*

$$\psi_1(Y) \quad = \quad \begin{cases} \psi_{11}(Y) & \psi(Y) < \psi_{11}(Y) \leq \psi^{\star}(Y) \\ \psi(Y) & o.w. \end{cases}$$

Proof. For $\mu \geq 1$, the risk difference of δ_ψ and δ_{ψ_1} is

$$\begin{aligned}
\Delta(\mu) &= R(\theta_M,\delta_\psi)-R(\theta_M,\delta_{\psi_1}) \\
&= E_\theta\left[e^{-\gamma\left(\frac{X_{(2)}\psi_1(Y)}{\theta_M}\,-\,\ln\frac{X_{(2)}\psi_1(Y)}{\theta_M}\,-1\right)}\right]-E_\theta\left[e^{-\gamma\left(\frac{X_{(2)}\psi(Y)}{\theta_M}\,-\,\ln\frac{X_{(2)}\psi(Y)}{\theta_M}\,-1\right)}\right] \\
&= e^{\gamma}E_\theta\left[e^{-\gamma\left(S\psi_1(Y)-\ln S\psi_1(Y)\right)}-e^{-\gamma\left(S\psi(Y)-\ln S\psi(Y)\right)}\right] \\
&= e^{\gamma}E_\theta[D_\theta(Y)],
\end{aligned}$$

where

$$D_\theta(y) = E_\theta\left[e^{-\gamma\left(S\psi_1(y)-\ln S\psi_1(y)\right)}-e^{-\gamma\left(S\psi(y)-\ln S\psi(y)\right)}|Y=y\right], \qquad y \in (0,1].$$

Using the fact that $e^a-e^b \geq (a-b)e^b$, $\forall a,b \in \Re$, we have

$$\begin{aligned}
D_\theta(y) \quad &\geq \quad \gamma E_\theta\Big\{\big[(S\psi(y)-\ln S\psi(y))-(S\psi_1(y)-\ln S\psi_1(y))\big] \\
& \qquad\qquad \times e^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\Big\} \\
&= \quad \gamma(\psi(y)-\psi_1(y))\; E_\theta\big\{e^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\big\}
\end{aligned}$$

$$\times\left[\frac{\ln\psi_1(y)-\ln\psi(y)}{\psi(y)-\psi_1(y)}+\frac{E_\theta\{Se^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\}}{E_\theta\{e^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\}}\right]. \tag{2.5}$$

Let $K(y,\mu)=E_\theta\{e^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\}$, then from Lemma 2.1(i), we have

$$\begin{aligned} K(y,\mu) &= [\psi(y)]^\gamma\int_0^\infty s^\gamma e^{-\gamma\psi(y)s}f_{S|Y=y}(s)ds \\ &= \frac{[\psi(y)]^\gamma y^{\alpha-1}\Gamma(2\alpha+\gamma)\mu^\alpha}{\Gamma^2(\alpha)f_Y(y)} \\ &\quad\times\left[\frac{\mu^\gamma}{[(1+\gamma\psi(y))\mu+y]^{2\alpha+\gamma}}+\frac{1}{[1+\gamma\psi(y)+y\mu]^{2\alpha+\gamma}}\right] \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &E_\theta\{Se^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\} \\ &\quad= [\psi(y)]^\gamma\int_0^\infty s^{\gamma+1}e^{-\gamma\psi(y)s}f_{S|Y=y}(s)ds \\ &\quad= \frac{[\psi(y)]^\gamma y^{\alpha-1}\Gamma(2\alpha+\gamma+1)\mu^\alpha}{\Gamma^2(\alpha)f_Y(y)} \\ &\qquad\times\left[\frac{\mu^{\gamma+1}}{[(1+\gamma\psi(y))\mu+y]^{2\alpha+\gamma+1}}+\frac{1}{[1+\gamma\psi(y)+y\mu]^{2\alpha+\gamma+1}}\right]. \end{aligned} \tag{2.7}$$

Now, substituting (2.6) and (2.7) in (2.5), we have

$$D_\theta(y)\geq\gamma K(y,\mu)(\psi(y)-\psi_1(y))\left[\frac{\ln\frac{\psi_1(y)}{\psi(y)}}{\psi(y)-\psi_1(y)}+\eta_{y,\psi}(\mu)\right],$$

where $\eta_{y,\psi}(\mu)$ is defined in Lemma 2.1. Clearly, if the condition $\psi(y)<\psi_{11}(y)\leq\psi^\star(y)$ does not hold, then $D_\theta(y)=0,\ \forall\theta\in\Re_+^2$ and $\forall y\in(0,1]$. For $\psi(y)<\psi_{11}(y)\leq\psi^\star(y)$, using (2.3) and the inequality $\ln a>1-\frac{1}{a},\ \forall a>0$, we have

$$D_\theta(y)\geq\gamma K(y,\mu)(\psi(y)-\psi_{11}(y))\left[\frac{\ln\frac{\psi_{11}(y)}{\psi(y)}}{\psi(y)-\psi_{11}(y)}+\frac{1}{\psi^\star(y)}\right]>0,\quad\forall\theta\in\Re_+^2 \textit{ and } \forall y\in(0,1].$$

Since $P_\theta(\psi(Y)<\psi_{11}(Y)\leq\psi^\star(Y))>0,\ \forall\theta\in\Re_+^2$, it follows that $\Delta(\mu)>0,\ \forall\theta\in\Re_+^2$.

2.3. Remark *In Theorem 2.2 we have the condition $\psi(Y)<\psi_{11}(Y)\leq\psi^\star(Y)$ with positive probability. So, $P_\theta(\psi(Y)<\psi^\star(Y))=P_\theta(\psi(Y)<\frac{1}{2\alpha})>0$. Therefore, a necessary condition on the function $\psi(Y)$ for Theorem 2.2 to actually offer an improved estimator (i.e., $\psi_1(Y)$ is different than $\psi(Y)$ with positive probability) is that $P_\theta(\psi(Y)<\frac{1}{2\alpha})>0$.*

The following Corollary is an immediate consequence of the Theorem 2.2.

2.4. Corollary *Let $\delta_\psi(X_1,X_2)\in D_U$ be an invariant estimator of θ_M such that $P_\theta(\psi(Y)<\frac{1}{2\alpha})>0$. If for some function $\psi^{\star\star}(Y)$, $P_\theta\big(\psi^{\star\star}(Y)\leq\psi(Y)<\frac{1+\gamma\psi^{\star\star}(Y)}{2\alpha+\gamma}\big)>0$, $\forall\theta\in\Re_+^2$, then under the RGL function, the invariant estimator δ_ψ is inadmissible for estimating θ_M, and is dominated by $\delta_{\psi_1^\star}(X_1,X_2)=X_{(2)}\psi_1^\star(Y)$, where*

$$\psi_1^\star(Y)=\begin{cases}\frac{1+\gamma\psi^{\star\star}(Y)}{2\alpha+\gamma} & \psi^{\star\star}(Y)\leq\psi(Y)<\frac{1+\gamma\psi^{\star\star}(Y)}{2\alpha+\gamma}\\ \psi(Y) & o.w.\end{cases}$$

Proof. Use Theorem 2.2 with $\psi_{11}(Y)=\frac{1+\gamma\psi^{\star\star}(Y)}{2\alpha+\gamma}\leq\psi^\star(Y)$.

2.5. Corollary *Let $\delta_\psi(X_1,X_2) \in D_U$ be an invariant estimator of θ_M such that $P_\theta(\psi(Y) < \frac{1}{2\alpha}) > 0$. Define*

$$\psi_1(Y) = \begin{cases} \frac{1+\gamma\psi(Y)}{2\alpha+\gamma} & \psi(Y) < \frac{1}{2\alpha} \\ \psi(Y) & o.w. \end{cases}$$

Then the estimator $\delta_{\psi_1}(X_1,X_2) = X_{(2)}\psi_1(Y)$ dominates $\delta_\psi(X_1,X_2)$.

Proof. Apply Corollary 2.4 with $\psi^{\star\star}(Y) = \psi(Y)$.

2.6. Remark *Consider the following mixed estimators of θ_M*

$$\begin{aligned} \delta_{p,\psi}(X_1,X_2) &= pX_{(2)} + (1-p)X_{(1)} \\ &= X_{(2)}[p + (1-p)Y] \end{aligned}$$

where $p \geq 0$. Following the Corollary 2.4 and taking $\psi^{\star\star}(y) = \frac{y}{2\alpha}$ for $\alpha > \frac{1}{2}$, this estimator is inadmissible and is dominated by

$$\delta^\star_{p,\psi}(X_1,X_2) = \begin{cases} \frac{2\alpha X_{(2)}+\gamma X_{(1)}}{2\alpha(2\alpha+\gamma)} & \frac{Y}{2\alpha} < p + (1-p)Y < \frac{2\alpha+\gamma Y}{2\alpha(2\alpha+\gamma)} \\ \delta_{p,\psi}(X_1,X_2) & o.w. \end{cases}$$

Also, using Corollary 2.5 one can get another improved estimator of $\delta_{p,\psi}(X_1,X_2)$, which is given by

$$\delta_{p,\psi_1}(X_1,X_2) = \begin{cases} \frac{X_{(2)}+\gamma[pX_{(2)}+(1-p)X_{(1)}]}{2\alpha+\gamma} & p + (1-p)Y < \frac{1}{2\alpha} \\ \delta_{p,\psi}(X_1,X_2) & o.w. \end{cases}$$

The following Theorem gives a sufficient condition for inadmissibility of invariant estimators δ_φ in D_L under the RGL function.

2.7. Theorem *Let $\delta_\varphi(X_1,X_2) \in D_L$ be an invariant estimator of θ_J, $\varphi_{11}(t)$ be any function defined on $[1,\infty)$ such that $P_\theta\big(\varphi(T) < \varphi_{11}(T) \leq \varphi^\star(T)\big) > 0$, $\forall\theta \in \Re^2_+$. Then under the RGL function, the invariant estimator δ_φ is inadmissible for estimating θ_J, and is dominated by $\delta_{\varphi_1}(X_1,X_2) = X_{(1)}\varphi_1(T)$, where*

$$\varphi_1(T) = \begin{cases} \varphi_{11}(T) & \varphi(T) < \varphi_{11}(T) \leq \varphi^\star(T) \\ \varphi(T) & o.w. \end{cases}$$

Proof. The proof is similar to the proof of Theorem 2.2 by replacing $Y, \psi, \psi^\star, \psi_1$ and ψ_{11} by $T, \varphi, \varphi^\star, \varphi_1$ and φ_{11}, respectively.

2.8. Remark *In Theorem 2.7 we have the condition $\varphi(T) < \varphi_{11}(T) \leq \varphi^\star(T)$ with positive probability. So, $P_\theta(\varphi(T) < \varphi^\star(T)) = P_\theta(\varphi(T) < \frac{1}{2\alpha}) > 0$. Therefore, a necessary condition on the function $\varphi(T)$ for Theorem 2.7 to actually offer an improved estimator (i.e., $\varphi_1(T)$ is different than $\varphi(T)$ with positive probability) is that $P_\theta(\varphi(T) < \frac{1}{2\alpha}) > 0$.*

The following Corollary is an immediate consequence of the Theorem 2.7.

2.9. Corollary *Let $\delta_\varphi(X_1,X_2) \in D_L$ be an invariant estimator of θ_J such that $P_\theta(\varphi(T) < \frac{1}{2\alpha}) > 0$. If for some function $\varphi^{\star\star}(T)$, $P_\theta\big(\varphi^{\star\star}(t) \leq \varphi(T) < \frac{1+\gamma\varphi^{\star\star}(t)}{2\alpha+\gamma}\big) >$*

0, $\forall \theta \in \Re_+^2$, *then under the RGL function, the invariant estimator* δ_φ *is inadmissible for estimating* θ_J*, and is dominated by* $\delta_{\varphi_1^\star}(X_1, X_2) = X_{(1)}\varphi_1^\star(T)$*, where*

$$\varphi_1^\star(T) = \begin{cases} \frac{1+\gamma\varphi^{\star\star}(T)}{2\alpha+\gamma} & \varphi^{\star\star}(T) \leq \varphi(T) < \frac{1+\gamma\varphi^{\star\star}(T)}{2\alpha+\gamma} \\ \varphi(T) & o.w. \end{cases}$$

Proof. Use Theorem 2.7 with $\varphi_{11}(T) = \frac{1+\gamma\varphi^{\star\star}(T)}{2\alpha+\gamma} \leq \varphi^\star(T)$.

2.10. Corollary *Let* $\delta_\varphi(X_1, X_2) \in D_L$ *be an invariant estimator of* θ_J *such that* $P_\theta(\varphi(T) < \frac{1}{2\alpha}) > 0$. *Define*

$$\varphi_1(T) = \begin{cases} \frac{1+\gamma\varphi(T)}{2\alpha+\gamma} & \varphi(T) < \frac{1}{2\alpha} \\ \varphi(T) & o.w. \end{cases}$$

Then the estimator $\delta_{\varphi_1}(X_1, X_2) = X_{(2)}\varphi_1(T)$ *dominates* $\delta_\varphi(X_1, X_2)$.

Proof. Apply Corollary 2.9 with $\varphi^{\star\star}(T) = \varphi(T)$.

2.11. Remark *Consider the following mixed estimators of* θ_J

$$\begin{aligned} \delta_{p,\varphi}(X_1, X_2) &= pX_{(1)} + (1-p)X_{(2)} \\ &= X_{(1)}[1 + (1-p)(T-1)] \end{aligned}$$

where $p \geq 0$. *Following the Corollary 2.9 and taking* $\varphi^{\star\star}(t) = 1$ *for* $\alpha < \frac{1}{2}$, *this estimator is inadmissible and is dominated by*

$$\delta_{p,\varphi}^\star(X_1, X_2) = \begin{cases} \frac{1+\gamma}{2\alpha+\gamma} X_{(1)} & 1 \leq p + (1-p)T < \frac{1+\gamma}{2\alpha+\gamma} \\ \delta_{p,\varphi}(X_1, X_2) & o.w. \end{cases}$$

Also, using Corollary 2.10 we get another improved estimator of $\delta_{p,\varphi}(X_1, X_2)$*, which is given by*

$$\delta_{p,\varphi_1}(X_1, X_2) = \begin{cases} \frac{X_{(1)}+\gamma[pX_{(1)}+(1-p)X_{(2)}]}{2\alpha+\gamma} & p + (1-p)T < \frac{1}{2\alpha} \\ \delta_{p,\varphi}(X_1, X_2) & o.w. \end{cases}$$

3. Discussion on Admissible Estimators

An important problem in estimation of θ_M and θ_J in the family of scale distributions, is to determine admissible estimators of the form $cX_{(2)}$ and $dX_{(1)}$ in the class of scale invariant estimators of the form

(3.1) $$D_{1U} = \{\delta_{1c} : \delta_{1c}(X_1, X_2) = cX_{(2)},\ c > 0\}$$

and

(3.2) $$D_{1L} = \{\delta_{2d} : \delta_{2d}(X_1, X_2) = dX_{(1)},\ d > 0\},$$

respectively. In this section, we discuss the admissibility of δ_{1c} and δ_{2d} within the subclass D_{1U} and D_{1L}, respectively under the RGL function.

The following lemma will be useful in characterization of admissible estimators of θ_M and θ_J within the subclasses D_{1U} and D_{1L}, respectively, under the RGL function.

Table 1. Values of $c^\star(1,\gamma)$ for $\alpha = 1,2,4$ and certain values of γ

	γ					
α	0.25	0.5	0.75	1	5	10
1	0.6693	0.6714	0.6732	0.6747	0.6843	0.6874
2	0.3643	0.3650	0.3655	0.3660	0.3702	0.3721
4	0.1965	0.1966	0.1968	0.1969	0.1983	0.1992

3.1. Lemma *Let X_1 and X_2 be two independent random variables such that X_i, $i = 1,2$ has pdf (1.1) and $X_{(1)} \le X_{(2)}$ be the ordered statistics of X_1 and X_2.*
(i) *If $S = \frac{X_{(2)}}{\theta_M}$, then the pdf of S is*

$$f_S(s) = \left[F_\alpha(\mu s) + F_\alpha\left(\frac{s}{\mu}\right)\right] f_\alpha(s), \quad s > 0, \tag{3.3}$$

where $\mu = \frac{\max(\theta_1,\theta_2)}{\min(\theta_1,\theta_2)} \ge 1$, F_α and f_α denote the cumulative distribution function (cdf) and the pdf of Gamma$(\alpha,1)$-distribution, respectively.
(ii) *If $U = \frac{X_{(1)}}{\theta_J}$, then the pdf of U is given by*

$$f_U(u) = \left[2 - F_\alpha(\mu u) - F_\alpha\left(\frac{u}{\mu}\right)\right] f_\alpha(u), \quad u > 0. \tag{3.4}$$

Proof. For a proof, see Lemma 7(i) and 7(ii) of Misra et al. [18].

3.1. Admissibility of δ_{1c}. For deriving admissible estimators of θ_M in the class of invariant estimators (3.1), we find values of c that minimizes the risk function $\delta_{1c} = cX_{(2)}$ which is

$$\begin{aligned} R(\theta_M, \delta_{1c}) &= 1 - E\left[e^{-\gamma(\frac{cX_{(2)}}{\theta_M} - \ln\frac{cX_{(2)}}{\theta_M} - 1)}\right] \\ &= 1 - E\left[e^{-\gamma(cS - \ln cS - 1)}\right], \end{aligned} \tag{3.5}$$

where $S = \frac{X_{(2)}}{\theta_M}$. The risk function (3.5) is not necessarily convex, but has a unique minimum w.r.t. c. Figure 3 shows the graph of the risk function as a function of c for certain values of μ, $\gamma = 1$ and $\alpha = 1,2,4$. It seems that the minimum point c, which depends on the values of μ and γ, of the risk function is near to α^{-1} when μ gets larger and larger. Therefore $R(\theta_M, cX_{(2)})$ will be minimized at the point c given by $\frac{\partial R(\theta_M,\delta_{1c})}{\partial c} = 0$ which reduces to

$$E\left[\left(S - \frac{1}{c(\mu,\gamma)}\right) e^{-\gamma(c(\mu,\gamma)S - \ln(c(\mu,\gamma)S) - 1)}\right] = 0 \tag{3.6}$$

The behavior of $c(\mu,\gamma)$ can not be determined analytically. The graph of $c(\mu,\gamma)$ as a function of $\mu \ge 1$ for $\alpha = 1,2,4$ and certain values of γ are shown in Figure 4. It is seen from Figure 4 (and also from numerical solution of equation (3.6)) that for fixed γ, $c(\mu,\gamma)$ increases as μ increases and $c(\mu,\gamma) \to \alpha^{-1}$ as $\mu \to \infty$. So

$$\inf_{\mu\ge1} c(\mu,\gamma) = c(1,\gamma) = c^\star \quad \textit{and} \quad \sup_{\mu\ge1} c(\mu,\gamma) = \lim_{\mu\to\infty} c(\mu,\gamma) = \alpha^{-1}.$$

where $c^\star = c^\star(1,\gamma)$ is the root of equation (3.6) with $\mu = 1$. Table 1 shows the root $c^\star = c^\star(1,\gamma)$ for certain values of $\gamma > 0$. Therefore for each $c \in [c^\star, \alpha^{-1}]$ there is a μ for which $R(\theta_M, \delta_{1c})$ is minimum, which implies that for $c \in [c^\star, \alpha^{-1}]$, δ_{1c} is admissible in the class of estimators (3.1). So, we have the following conjecture.

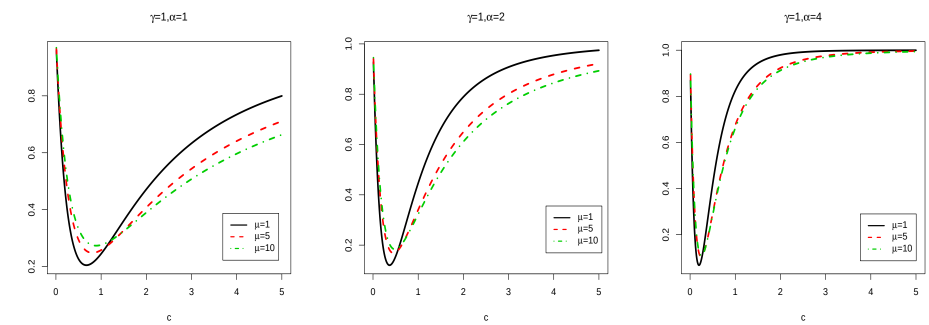


Figure 3. Plots of risk function for $\gamma = 1,\ \alpha = 1, 2, 4$ and certain values of μ

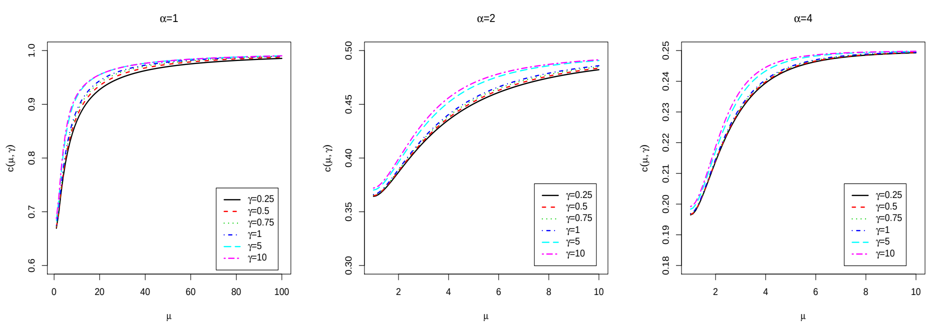


Figure 4. Graph of $c(\mu, \gamma)$ for $\alpha = 1, 2, 4$ and certain values of γ

3.2. Conjecture *Let $c^\star$ be the root of equation (3.6) with $\mu = 1$. Then, under the RGL function, the estimators $\delta_{1c}(X_1, X_2) = cX_{(2)}$ are admissible within the subclass D_{1U} of invariant estimators of θ_M, if and only if $c \in [c^\star, \alpha^{-1}]$.*

3.3. Remark *From Corollary 2.5 the estimator $\delta_{1c}(X_1, X_2) = cX_{(2)}$ for $c < \frac{1}{2\alpha}$ is inadmissible and is dominated by $\delta_1(X_1, X_2) = \frac{1+\gamma c}{2\alpha+\gamma} X_{(2)}$. Note that from Table 1, $c^\star > \frac{1}{2\alpha}$ for certain values of γ, which satisfy the condition of Conjecture 3.2.*

3.4. Remark *Based on the Conjecture 3.2, the natural and generalized Bayes estimator $\frac{X_{(2)}}{\alpha}$ of θ_M, which is the analog of the maximum likelihood and best scale invariant estimators of θ_2, is admissible within the subclass D_{1U} of invariant estimators of θ_M.*

3.2. Admissibility of δ_{2c}. Similarly, the risk function of $\delta_{2d} = dX_{(1)}$ as an estimator of θ_J has a unique minimum w.r.t. d, and can be yield from $\frac{\partial R(\theta_J, dX_{(1)})}{\partial d} = 0$ which is

$$E\left[\left(U - \frac{1}{d(\mu, \gamma)}\right) e^{-\gamma(d(\mu,\gamma)U - \ln(d(\mu,\gamma)U) - 1)}\right] = 0. \tag{3.7}$$

For $\mu = 1$, the root $d^\star = d^\star(1, \gamma)$ of this equation are summarized in Table 2 for the values $\alpha = 1, 2, 4$ and for certain values of γ. Note that we are able to prove analytically that the root $d^\star(1, \gamma)$ for $\alpha = 1$ and arbitrary $\gamma > 0$ is always equal to 2 (see the Appendix).

Table 2. Values of $d^\star(1,\gamma)$ for $\alpha = 1, 2, 4$ and certain values of γ

	γ					
α	0.25	0.5	0.75	1	5	10
1	2	2	2	2	2	2
2	0.7982	0.7967	0.7954	0.7944	0.7870	0.7845
4	0.3437	0.3433	0.3430	0.3427	0.3400	0.3386

The graph of $d(\mu,\gamma)$ as a function of $\mu \geq 1$ (and also numerical solution of equation (3.7)) shows that for fixed $\gamma > 0$, $d(\mu,\gamma)$ decreases as μ increases and $d(\mu,\gamma) \to \alpha^{-1}$ as $\mu \to \infty$. So

$$\inf_{\mu\geq 1} d(\mu,\gamma) = \lim_{\mu\to\infty} d(\mu,\gamma) = \alpha^{-1} \quad \textit{and} \quad \sup_{\mu\geq 1} d(\mu,\gamma) = d(1,\gamma) = d^\star$$

Therefore, we conjecture that the estimators $\delta_{2d}(X_1, X_2) = dX_{(1)}$ are admissible within the subclass D_{1L} of invariant estimators of θ_J, under the RGL function, if and only if $d \in [\alpha^{-1}, d^\star]$.

3.5. Remark *Let* $X_{i1}, X_{i2}, \ldots, X_{in},\ i = 1, 2,$ *be a pair of independent random samples from* $\Pi_i,\ i = 1, 2,$ *and* Π_i *has p.d.f.* (1.1). *Then* $T_i(\boldsymbol{X}_i) = \sum_{i=1}^{n} X_{ij},\ i = 1, 2,$ *is complete sufficient statistic for* θ_i *and has gamma distribution with parameters* $(n\alpha, \theta_i)$, *respectively, where* $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{in})$. *Therefore, the results of Sections 2-3 hold true upon replacing* α *by* $n\alpha$ *and* X_i *by* $T_i(\boldsymbol{X}_i),\ i = 1, 2,$ *in this case.*

4. Applications and Extensions

In this section, we apply the results of Sections 2 and 3 to k-records and Type-II censored data and extend these results to a subclass of exponential family.

4.1. Estimation After Selection Based on k-Record Data. In statistical inference, a rich literature has developed on record data since Chandler [8] formulated the theory of records. Let $X_{i1}, X_{i2}, \ldots, X_{in},\ i = 1, 2$, be a pair of independent random samples from negative exponential populations Π_1, Π_2 with Π_i having the associated pdf

$$f(x|\theta_i) = \frac{1}{\theta_i}\ e^{-\frac{x}{\theta_i}}, \quad \theta_i > 0, \quad i = 1, 2, \tag{4.1}$$

where θ_1, θ_2 are unknown scale parameters. Let $R^i_{m(k)}$ be upper k-records of i-th sample, $i = 1, 2$. It is easy to verify that the mth k-Records, $R^i_{m(k)}$, has a $\text{Gamma}\big(m, \frac{\theta_i}{k}\big)$-distribution and $kR^i_{m(k)}$ has a $\text{Gamma}\big(m, \theta_i\big)$-distribution, see Arnold et al. [4], Nevzorov [23], Ahmadi et al. [1] and Ahmadi et al. [2,3] and references therein. Let $R^{(1)}_{m(k)} \leq R^{(2)}_{m(k)}$ represent the ordered statistics of $R^1_{m(k)}$ and $R^2_{m(k)}$. Suppose the population corresponding to largest $R^{(2)}_{m(k)}$ (or the smallest $R^{(1)}_{m(k)}$) observation is selected. The problems that we are interested here are the estimation of the following random parameters:

$$\theta^m_M = \begin{cases} \theta_1 & R^1_{m(k)} \geq R^2_{m(k)} \\ \theta_2 & R^1_{m(k)} < R^2_{m(k)} \end{cases} \quad \textit{and} \quad \theta^m_J = \begin{cases} \theta_2 & R^1_{m(k)} \geq R^2_{m(k)} \\ \theta_1 & R^1_{m(k)} < R^2_{m(k)}. \end{cases}$$

Since $kR^i_{m(k)}$ has $\text{Gamma}\big(m, \theta_i\big)$-distribution, therefore the results of Sections 2-3, except Remark 2.11, hold for this case upon replacing α by m and X_i by $kR^i_{m(k)},\ i = 1, 2$.

4.2. Estimation after Selection using Type-II Censored Data. The most common censoring scheme which is of importance in the field of reliability and life-testing, is Type-II censoring. In this scheme, after starting the life-testing experiment with n items, the experiment continues until a pre-specified number of failures , say $r(\leq n)$ occur. For more details about this scheme, see Lawless [16].

Let $X_{i1}, X_{i2}, \ldots, X_{in},\ i = 1, 2$, be a pair of independent random samples from negative exponential populations with pdf (4.1). It is easy to show that in this scheme $T_i = \sum_{j=1}^{r} X_{i(j)} + (n-r)X_{i(r)},\ i = 1, 2$, has a Gamma$(r, \theta_i)$-distribution, see Lehmann and Romano [15]. Now, Suppose $T_{(1)} = \min(T_1, T_2)$ and $T_{(2)} = \max(T_1, T_2)$ and the population corresponding to the largest $T_{(2)}$ (or smallest $T_{(1)}$) is selected. We are interested in estimation of the random parameters

$$\theta_M = \begin{cases} \theta_1 & T_1 \geq T_2 \\ \theta_2 & T_1 < T_2 \end{cases} \quad and \quad \theta_J = \begin{cases} \theta_2 & T_1 \geq T_2 \\ \theta_1 & T_1 < T_2. \end{cases}$$

Since $T_i,\ i = 1, 2$, has Gamma$\big(r, \theta_i\big)$-distribution, therefore the results of Sections 2-3, except Remark 2.11, hold true upon replacing α by r and X_i by $T_i,\ i = 1, 2$, in this case.

4.3. Extension to a Subclass of Exponential Family. Let $\mathbf{X}_i = (X_{i1}, X_{i2}, \cdots, X_{in})$, $i = 1, 2$, be a random sample of size n from the ith population $\Pi_i,\ i = 1, 2$, with the joint scale probability density function

$$f(\mathbf{x}_i, \tau_i) = \frac{1}{\tau_i^n} f\big(\frac{\mathbf{x}_i}{\tau_i}\big), \quad i = 1, 2,$$

where $\mathbf{x}_i = (x_{i1}, \cdots, x_{in})$. In some cases the above model reduces to

$$f(\mathbf{x}_i, \theta_i) = C(\mathbf{x}_i, n)\theta_i^{-\gamma} e^{-T_i(\mathbf{x}_i)/\theta_i}, \qquad i = 1, 2, \tag{4.2}$$

where $C(\mathbf{x}_i, n)$ is a function of $\mathbf{x}_i$ and n, $\theta_i = \tau_i^r$ for some $r > 0, \gamma$ is a function of n and $T_i(\mathbf{X}_i)$ is a complete sufficient statistic for θ_i with $Gamma(\gamma, \theta_i)$-distribution. For example $Exponential(\beta_i)$, $Gamma(\nu, \beta_i)$, $Inverse\ Gaussian(\infty, \lambda_i)$, $Normal(0, \sigma_i^2)$, $Weibull(\eta_i, \beta)$, $Rayleigh(\beta_i)$, $Generalized\ Gamma(\alpha, \lambda_i, p_i)$, $Generalized\ laplace(\lambda_i, k)$ belong to the family of distributions (4.2), see Parsian and Nematollahi [24] and references therein.

Since $T_i = T_i(\mathbf{X}_i), i = 1, 2$, has a $Gamma(\gamma, \theta_i)$-distribution, therefore we can extend the results of Sections 2-3 to the subclass of exponential family (4.2) by replacing α and X_i by γ and $T_i(\mathbf{X}_i)$, respectively.

The results of Section 2-3 can also be extended to the family of transformed chi-square distributions which is introduced by Rahman and Gupta [25] and includes Pareto and beta distributions. For details see Jafari Jozani et al. [11].

5. Appendix

In this section, we show analytically that the root $d^\star(1, \gamma)$ for $\alpha = 1$ and arbitrary $\gamma > 0$ is always equal to 2. To see this, note that the pdf of U given in (3.4), for $\mu = \alpha = 1$, reduces to

$$f_U(u) = 2e^{-2u}, \quad u > 0.$$

Therefore $R(\theta_J, dX_{(1)})$ will be minimized at the point d given by $\frac{\partial R(\theta_J, \delta_{2d})}{\partial d} = 0$ which reduces to (3.7) and for $\mu = 1$ can be written as

$$\int_0^\infty \left(u - \frac{1}{d(1,\gamma)}\right) e^{-\gamma(d(1,\gamma)u - \ln(d(1,\gamma)u) - 1)} f_U(u) d(u) = 0$$

and with simple computations is equivalent to

$$2(ed(1,\gamma))^{\gamma}\left(\frac{1}{\gamma d(1,\gamma)+2}\right)^{\gamma+1}\left\{\frac{\Gamma(\gamma+2)}{\gamma d(1,\gamma)+2}-\frac{\Gamma(\gamma+1)}{d(1,\gamma)}\right\}=0.$$

The root of the above equation is simply equal to $d^{\star}(1,\gamma)=2$.

Acknowledgments
The authors are grateful to the editor and two anonymous referees for making helpful comments and suggestions on an earlier version of this paper. Nader Nematollahi's research supported by a grant of the Research Council of Allameh Tabataba'i University and Ahmad Parsian's research supported by a grant of the Research Council of the University of Tehran.

References

[1] Ahmadi, J., Dostparast, M., Parsian, A. *Estimation and prediction in a two-parameter exponential distribution based on $k-$Record values under LINEX loss function*, Communications in Statistics: Theory and Methods **34**, 795-805, 2005.

[2] Ahmadi, J., Jafari Jozani, J., Marchand, E., Parsian, A. *Prediction of k-records from a general class of distributions under balanced type loss functions*, Metrika **70**, 19-33, 2009a.

[3] Ahmadi, J., Jafari Jozani, J., Marchand, E., Parsian, A. (2009b). *Bayes estimation based on k-records from a general class of distributions under balanced type loss functions,* Jr. Statist. Plan. Inf. **139**, 1180-1189, 2009b.

[4] Arnold. B.C., Balakrishnan, N., Nagaraja, H.N. *Records*, (John Wiley and Sons, New York, 1998)

[5] Balakrishnan. N., Kannan, N., Nagaraja, H. N. *Advances in ranking and selection, multiple comparisions and reliability,* (Birkhauser, Boston, 2005).

[6] Basu, D. *A note on the theory of unbiassed estimation,* Ann. Math. Statist. **26**(2), 345-348, 1955.

[7] Brewster, J.F. and Zidek, Z.V. *Improving on invariant estimators,* Ann. Statist. **2**, 21-28, 1974.

[8] Chandler, K.N. *The distribution and frequency of record values,* J. Roy. Statist. Soc. Ser. B **1**, 220-228, 1952.

[9] Dahiya, R. C. *Estimation of the mean of the selected population,* Jr. Amer. Statist. Assoc. **69**, 226-230, 1974.

[10] Gangopadhyay, A. K., Kumar, S. *Estimation average worth of the selected subset from two-parameter exponential populations,* Communications in Statistics: Theory and Methods **34**, 2257-2267, 2005.

[11] Jafari Jozani, M., Nematollahi, N., Shafiee, K. *An admissible minimax estimator of a bounded scale-parameter in a subclass of the exponential family under scale-invariant squared-error loss,* Statist. Probab. Lett. **60**, 437-444, 2002.

[12] Kumar, S. and Kar, A. *Estimation quantiles of a selected exponential population,* Statist. Probab. Lett. **52**, 9-19, 2001.

[13] Kumar, S., Gangopadhyay, A. K. *Estimating parameters of a selected pareto population,* Statist. Metho. **2**, 121-130, 2005.

[14] Kumar, S., Mahapatra, A.K. and Vellaisamy, P. *Reliability estimation of the selected exponential populations,* Statist. Probab. Lett. **79**, 1372-1377, 2005.

[15] Lehmann, E.L, Romano, J.P. *Testing Statistical Hypotheses*, (Springer, 2005).

[16] Lawless, J.F. *Statistical models and methods for lifetime data*, (John Wiley and Sons, New York, 2003).

[17] Misra, N., Singh, G. N. *On the umvue for estimating the parameter of the selected exponential population,* Jr. Ind. Statist. Assoc. **31**, 61-69, 1993.

[18] Misra, N., van der Meulen, E.C. and Branden, K.V. *On estimating the scale parameter of the selected gamma population under the scale invariant squared error loss function,* J. Comput. Appl. Math. **186**, 268-282, 2006a.

[19] Misra, N., van der Meulen, E.C. and Branden, K.V. *On some inadmissibility results for the scale parameters of selected gamma populations,* J. Statist. Plann. Infer. **136**, 2340-2351, 2006b.
[20] Motamed-Shariati, F. and Nematollahi, N. *Minimax estimation of the scale parameter of the selected gamma population with arbitrary known shape parameter,* Applied Mathematical Sciences **3**(51), 2543-2552, 2009.
[21] Naghizadeh Qomi, M., Nematollahi, N. and Parsian, A. *Estimation after selection under reflected normal loss function,* Communications in Statistics: Theory and Methods **41**, 1040-1051, 2012.
[22] Nematollahi, N. and Motamed-Shariati, F. *Estimation of the scale parameter of the selected gamma population under the entropy loss function,* Communications in Statistics: Theory and Methods **38**, 208-221, 2009.
[23] Nevzorov, V. *Records: Mathematical Theory. In: Translation of Mathematical Monographs,* vol. 194. Amer. Math. Soc. Providence, RI. USA, 2001.
[24] Parsian, A., Nematollahi, N. *Estimation of scale parameter under entropy loss function,* J. Statist. Plann. Infer. **52**, 77-91, 1996.
[25] Rahman, M.S., Gupta, R.P. *Family of transformed chi-square distributions,* Communications in Statistics: Theory and Methods **22**, 135-146, 1993.
[26] Sackrowitz, H., Cahn, E. S. *Estimation of the mean of a selected negative exponential population,* Jr. Roy. Statist. Soc. Ser. B. **46**, 242-249, 1984.
[27] Sackrowitz, H. and Cahn, E.S. *Evaluating the chosen population: a Bayes and minimax approach,* IMS Lecture Notes-Monograph Sereis. Adaptive Statistical Procedures and Related Topics **8**, 386-395, ed. John Van Ryzin, 1987.
[28] Sarkadi, K. *Estimation after selection,* Studia Scientarium Mathematicarum Hungarica **2**, 341-350, 1967.
[29] Spiring, F.A. and Yeung, A.S. *A general class of loss functions with industrial applications,* Journal of Quality Technology **30**, 152-162, 1998.
[30] Tappin, L. *Unbiased estimation of the parameter of a selected binomial population,* Communications in Statistics: Theory and Methods **21**(4), 1067-1083, 1992.
[31] Towhidi, M. and Behboodian, J. *Estimation of a scale parameter under a reflected gamma loss functions,* Iranian Journal of Science **10**, 256-269, 1999.
[32] Towhidi, M. and Behboodian, J. *Minimax estimation of a bounded parameter under some bounded loss functions,* Far East J. Theo. Stat. **6**(1), 39-48, 2002.
[33] Vellaisamy, P. *Inadmissibility results for the selected scale parameter,* Ann. Statist. **20**, 2183-2191, 1992.
[34] Vellaisamy, P. *A note on the estimation of the selected scale parameters,* J. Statist. Plann. Infer. **55**, 39-46, 1996.
[35] Vellaisamy, P. and Sharma, D. *Estimation the mean of the selected gamma population,* Communications in Statistics: Theory and Methods **17**(8), 2797-2918, 1988.
[36] Vellaisamy, P. and Sharma, D. *A note on the estimation of the mean of the selected gamma population,* Communications in Statistics: Theory and Methods **18**(2), 555-560, 1989.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1125 – 1132

s-pure extensions of locally compact abelian groups

H.Sahleh* and A.A. Alijani†

Abstract

A subgroup H of a locally compact abelian (LCA) group G is called s-pure if $\overline{H \bigcap nG} = H$ for every positive integer n. A proper short exact sequence $0 \to A \overset{\phi}{\to} B \to C \to 0$ in the category of LCA groups is said to be *s-pure* if $\phi(A)$ is an s-pure subgroup of G. We establish conditions under which the s-pure exact sequences split and determine those LCA groups which are s-pure injective. We also gives a necessary condition for an LCA group to be s-pure projective in $\pounds$.

Keywords: s-pure injective ;s-pure projective;s-pure extension.

2000 AMS Classification: 22B05.

Received 12/11/2013 : Accepted 21/06/2014 Doi : 10.15672/HJMS.2015449660

Intr

All groups considered in this paper are Hausdorff topological abelian groups and they will be written additively. For a group G and a positive integer n, we denote by nG, the subgroup of G defined by $nG = \{nx : x \in G\}$ and $G[n]$, the subgroup of G defined by $G[n] = \{x \in G; nx = 0\}$. In a multiplicative group, we will use G^n instead of nG and define $G^n = \{x^n : x \in G\}$. Let $\pounds$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. A morphism is called proper if it is open onto its image, and a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is said to be an extension of A by C if ϕ and ψ are proper morphism. We let $Ext(C, A)$ denote the group of extensions of A by C [6]. Let $\overline{S}$ denotes the closure of $S \subseteq G$. We say that a closed subgroup H of an LCA group G is s-pure if $\overline{H \bigcap nG} = H$ for every positive integer n. A subgroup H of a group G is said to be pure if $H \bigcap nG = nH$ for every positive integer n [3]. A pure subgroup need not be s-pure and vice versa (Example 1.9). In Section 1, we show that an s-pure subgroup is pure if and only if it is densely divisible (Lemma 1.10). An LCA group G is said to be pure simple if G contains no nontrivial closed pure subgroup [1]. Armacost [1] has determined the pure simple LCA group G. Also, Armacost has determined the LCA group G such that every closed subgroup of G is pure [1]. We say that an LCA group G is s-pure simple if G contains no nonzero s-pure

*Department of Mathematics, University of Guilan, P.O.BOX 1914, Rasht-Iran
Email: sahleh@guilan.ac.ir
†Department of Mathematics, University of Guilan
Email:taleshalijan@phd.guilan.ac.ir

subgroup. We say that a LCA group G is s-pure full if every closed subgroup of G is s-pure. We show that a LCA group G is s-pure full if and only if it is divisible (Theorem 1.15). Also, we show that a compact group G is s-pure simple if and only if it is totally disconnected(Theorem 1.16). A proper short exact sequence $0 \to A \xrightarrow{\phi} B \to C \to 0$ in $\pounds$ is said to be s-pure if $\phi(A)$ is s-pure in B. In section 2, we study s-pure exact sequence in $\pounds$. In [4], Fulp studied pure injective and pure projective in $\pounds$. In section 3, we study s-pure injective and s-pure projective in $\pounds$. An LCA group G is an s-pure injective group in $\pounds$ if and only if $G \cong R^n \bigoplus (R/Z)^\sigma$ (Theorem 3.2). If G is an s-pure projective group in $\pounds$ then $G \cong R^n \bigoplus G'$ where G' is a discrete torsion-free, non divisible group (Theorem 3.4).

The additive topological group of real numbers is denoted by R, Q is the group of rationals with the discrete topology and Z is the group of integers. Also, $Z(n)$ is the cyclic group of order n and $Z(p^\infty)$ denotes the quasicyclic group. For any group G, G_0 is the identity component of G, tG is the maximal torsion subgroup of G and 1_G is the identity map $G \to G$. An element $g \in G$ is called compact if the smallest closed subgroup which its contains is compact [8, Definition 9.9]. We denote by bG, the subgroup of all compact elements of G. If $\{G_i\}_{i\in I}$ is a family of groups in $\pounds$, then we denote their direct product by $\prod_{i\in I} G_i$. If all the G_i are equal, we will write G^I instead of $\prod_{i\in I} G_i$. For any group G and H, $Hom(G,H)$ is the group of all continuous homomorphisms from G to H, endowed with the compact-open topology. The dual group of G is $\hat{G} = Hom(G, R/Z)$ and $(\hat{G}, S)$ denotes the annihilator of $S \subseteq G$ in $\hat{G}$. For a group G, we define $G^{(1)} = \bigcap_{n=1}^{\infty} nG$.

1. s-pure subgroups

Let $G \in \pounds$. In this section, we introduce the concept and study some properties of an s-pure subgroup of G.

1.1. Definition. *A closed subgroup H of a group G is called s-pure if $\overline{H \bigcap nG} = H$ for every positive integer n.*

1.2. Note.

(a) A closed divisible subgroup of a group is s-pure.
(b) A closed subgroup of a divisible group is s-pure.

1.3. Remark. *Let $G \in \pounds$. Then G has two trivial subgroups,$\{0\}$ and G. Clearly, $\{0\}$ is s-pure. But G need not be an s-pure in itself.*

Recall that a group G is said to be densely divisible if it has a dense divisible subgroup.

1.4. Lemma. *A group G is densely divisible if and only if $\overline{nG} = G$ for every positive integer n.*

Proof. See [2, 4.16(a)].

1.5. Corollary. *Let $G \in \pounds$. Then, G is s-pure in itself if and only if G is densely divisible.*

Proof. It is clear by Lemma 1.4.

1.6. Lemma. *Let $G \in \pounds$. Then, $\overline{G^{(1)}}$ is an s-pure subgroup of G.*

Proof. It is clear that $G^{(1)} \subseteq \overline{G^{(1)}} \bigcap mG$ for every positive integer m. So, $\overline{G^{(1)}} \subseteq \overline{\overline{G^{(1)}} \bigcap mG} \subseteq \overline{G^{(1)}}$ for all m. Hence, $\overline{G^{(1)}}$ is an s-pure subgroup.

1.7. Remark. *Let $G \in \pounds$ and H be an s-pure subgroup of G. Then, $H \subseteq \overline{nG}$ for all positive integers n. Hence, $H \subseteq \bigcap_{n=1}^{\infty} \overline{nG} = (G, t\hat{G})$ [8].*

Now, we present an example of a LCA group G and a closed subgroup H of G such that $H \subseteq (G, t\hat{G})$, but H is not an s-pure subgroup.

1.8. Example. *Let S^1 be the (multiplicative) circle group of unitary complex numbers and σ any infinite cardinal number. Let G be the subgroup of $(S^1)^{\sigma}$ consisting of all (x_ι) such that $x_\iota = \pm 1$ for all but a finite number of ι. Let K be the subgroup of G consisting of all (x_ι) such that $x_\iota = 1$ for all but a finite number of ι. By [8, section 24.44(a)], G is a locally compact abelian group, and $\hat{G}$ is torsion-free. Let $H = \{(x)_\iota, (y)_\iota\}$ where $x_\iota = 1$ and $y_\iota = -1$ for $\iota \neq \iota_1, ..., \iota_m$ and $x_\iota = y_\iota = 0$ for $\iota = \iota_1, ..., \iota_m$. Then, H is a closed subgroup of G, and $H \subseteq G = (G, t\hat{G})$. Now, suppose that n is even. Then, $\overline{H \bigcap G^n} = \overline{H \bigcap K} = \{(x)_\iota\}$. Hence, H is not s-pure.*

Recall that a subgroup H of a group G is called pure if $nH = H \bigcap nG$ for every positive integer n[3]. A pure subgroup need not be s-pure, and an s-pure subgroup need not be pure.

1.9. Example. *Since R is divisible, so the subgroup Z of R is s-pure. But it is not a pure subgroup. Let p be a prime and $G = \prod_{n=1}^{\infty} Z(p^n)$, with discrete topology. Then, tG is a pure subgroup of G. Since $(1, 0, 0, ...) \in tG$ and $(1, 0, 0, ...) \notin p(tG)$, so it is not s-pure.*

1.10. Lemma. *A pure subgroup is s-pure if and only if it is densely divisible.*

Proof. Let H be a pure subgroup of G. If H is an s-pure subgroup, then $\overline{nH} = H$ for every positive integer n. So, by Lemma 1.4, H is densely divisible. Conversely, let H be a densely divisible, pure subgroup of G. Then, $\overline{H \bigcap nG} = \overline{nH}$ for every positive integer n. By Lemma 1.4, $\overline{nH} = H$ for all n. So, $\overline{H \bigcap nG} = H$ for all n. Hence, H is an s-pure subgroup in G.

Let G be a group in $\pounds$. Then G is called s-pure simple if G contains no nonzero s-pure subgroups. Similarly, G is called s-pure full if every closed subgroup of G is s-pure.

1.11. Lemma. *Let G_1 and G_2 be two groups in $\pounds$. If $G_1 \times G_2$ is s-pure full, then G_1 and G_2 are s-pure full.*

Proof. Let G_1,$G_2 \in \pounds$ and H be a closed subgroup of G_1. Then, $H \times G_2$ is a closed subgroup of $G_1 \times G_2$. So, $\overline{(H \times G_2) \bigcap (nG_1 \times nG_2)} = H \times G_2$ for any positive integer n. Hence, $\overline{(H \bigcap nG_1) \times (G_2 \bigcap nG_2)} = H \times G_2$. Therefore, $\pi_1(\overline{(H \bigcap nG_1) \times (nG_2)}) = \pi_1(H \times G_2)$ where π_1 is the first projection map of $G_1 \times G_2$ onto G_1. Consequently, $\overline{H \bigcap nG_1} = H$. Similarly, it can be show that G_2 is s-pure full.

1.12. Remark. *Recall that a discrete group is densely divisible if and only if it is divisible.*

1.13. Remark. *Let G be a densely divisible group and H a closed subgroup of G. Since $(\hat{G}, H)$ is a subgroup of $\hat{G}$ and $\hat{G}$ is torsion-free, so G/H is densely divisible.*

1.14. Remark. *Let G be a densely divisible group and H an open, pure subgroup of G. An easy calculation shows that H is divisible.*

1.15. Theorem. *Let $G \in \pounds$. Then, G is s-pure full if and only if G is divisible.*

Proof. Let G be an s-pure full group in $\pounds$. By [8, Theorem 24.30], $G \cong R^n \bigoplus G'$, where G' is an LCA group which contains a compact open subgroup. By Lemma 1.11, G' is s-pure full. So, by Corollary 1.5, G' is densely divisible. By Remark 1.13 , G'/bG' is densely divisible. On the other hand, G'/bG' is discrete and torsion-free (see the proof of Theorem 2.7 [9]). Hence, by Remark 1.12, G'/bG' is divisible. By Remark 1.14, bG' is divisible. Consequently, the short exact sequence $0 \to bG' \to G' \to G'/bG' \to 0$ splits. Hence, $G' \cong bG' \bigoplus G'/bG'$ and G' is divisible. Therefore, G is divisible. The converse is clear by Note 1.2.b.

1.16. Theorem. *A compact group G is an s-pure simple group if and only if it is totally disconnected.*

Proof. Let G be a compact group. If G is an s-pure simple group, then by Note 1.2(a), $G_0 = 0$ because G_0 is a closed divisible subgroup of G. So G is totally disconnected. Conversely, Let G be a compact, totally disconnected group and H an s-pure subgroup of G. By Remark 1.7, $H \subseteq (G, t\hat{G})$. Since $\hat{G}$ is a discrete and a torsion group, so $t\hat{G} = \hat{G}$. Hence, $H = 0$.

2. s-pure exact sequence

In this section, we introduce the concept and study some properties of s-pure extensions in $\pounds$.

2.1. Definition. *An extension $0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow C \longrightarrow 0$ in $\pounds$ is called s-pure if $\phi(A)$ is s-pure in B.*

2.2. Remark. *Let A be a divisible group in $\pounds$ and $E : 0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow C \longrightarrow 0$ an extension in $\pounds$. Then $\phi(A)$ is a closed divisible subgroup of B. So, by Note 1.2(a),E is an s-pure extension.*

2.3. Lemma. *Let A, C be groups in $\pounds$. Then the extension $0 \to A \to A \bigoplus C \to C \to 0$ is an s-pure extension if and only if A is densely divisible.*

Proof. The extension $0 \to A \to A \bigoplus C \to C \to 0$ is pure. Hence, by Lemma 1.10, it is s-pure if and only if A is densely divisible.

2.4. Remark. *Lemma 2.3 shows that the set of all s-pure extensions of A by C need not be a subgroup of $Ext(C, A)$.*

The dual of an extension $E : 0 \to A \to B \to C \to 0$ is defined by $\hat{E} : 0 \to \hat{C} \to \hat{B} \to \hat{A} \to 0$. The following example shows that the dual of an s-pure extension need not be s-pure.

2.5. Example *There exists a non splitting extension*

$$E : 0 \to Z(p^{\infty}) \to B \to C \to 0$$

of $Z(p^{\infty})$ *with compact group* C *which is not torsion-free [2, Example 6.4]. By Note 1.2(a),* E *is s-pure. Since* $\widehat{Z(p^{\infty})}$ *is torsion-free, so* $\hat{E}$ *is pure. By Lemma 1.10,* $\hat{E}$ *is s-pure if and only if* $\hat{C}$ *is densely divisible. But* C *is compact. So,* $\hat{C}$ *is discrete. Hence,* $\hat{E}$ *is s-pure if and only if* $\hat{C}$ *is a discrete divisible group. Consequently,* $\hat{E}$ *is s-pure if and only if* C *is a compact torsion-free group. Since* C *is not torsion-free, it follows that* $\hat{E}$ *is not s-pure.*

Recall that two extensions $0 \to A \xrightarrow{\phi_1} B \xrightarrow{\psi_1} C \to 0$ *and* $0 \to A \xrightarrow{\phi_2} X \xrightarrow{\psi_2} C \to 0$ *is said to be equivalent if there is a topological isomorphism* $\beta : B \to X$ *such that the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi_1} & B & \xrightarrow{\psi_1} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A & \xrightarrow{\phi_2} & X & \xrightarrow{\psi_2} & C & \longrightarrow & 0 \end{array}$$

is commutative.

2.6. Lemma *An extension equivalent to an s-pure extension is s-pure.*

Proof. Suppose that

$$E_1 : 0 \to A \xrightarrow{\phi_1} B \to C \to 0, E_2 : 0 \to A \xrightarrow{\phi_2} X \to C \to 0$$

be two equivalent extension such that E_1 *is s-pure. Then there is a topological isomorphism* $\beta : B \to X$ *such that* $\beta\phi_1 = \phi_2$ *. Since* E_1 *is s-pure,* $\phi_1(A) = \overline{\phi_1(A) \bigcap nB}$*. Then* $\beta\phi_1(A) = \beta(\overline{\phi_1(A) \bigcap nB})$*. So,* $\phi_2(A) = \overline{\phi_2(A) \bigcap nX}$*. Hence,* E_2 *is s-pure.*

2.7. Corollary. *If the s-pure extension* $0 \to A \to B \to C \to 0$ *splits, Then* A *is densely divisible.*

Proof. Let $0 \to A \to B \to C \to 0$ *be a split, s-pure extension. Then, it is equivalent to* $0 \to A \to A \bigoplus C \to C \to 0$*. So,* $0 \to A \to A \bigoplus C \to C \to 0$ *is s-pure. Hence, by Lemma 2.3,* A *is densely divisible.*

2.8. Remark. *The converse of Corollary 2.7 may not hold. Consider Example 2.5.*

We will now show that a pullback or pushout of an s-pure extension need not be s-pure. For more on a pullback and a pushout of an extension in £, see [6].

2.9. Example *Let* α *be the map* $\alpha : Z \to Z : n \longmapsto 2n$*. Consider the s-pure extension* $E : 0 \to Z_2 \to R/Z \to R/Z \to 0$ *which is the dual of* $0 \to Z \xrightarrow{\alpha} Z \to Z_2 \to 0$*. Let* $f : Q \to R/Z$ *be any continuous homomorphism. Since* Q *is torsion-free, so the standard pullback of* E *is pure, but not s-pure by Lemma 1.10 because* Z_2 *is not densely divisible. Now consider the s-pure extension* $E' : 0 \to Z \to Q \to Q/Z \to 0$ *. Then the map* α

induces a pushout diagram

$$\begin{array}{ccccccccc} E' : 0 & \longrightarrow & Z & \longrightarrow & Q & \longrightarrow & Q/Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle 1_{Q/Z}} & & \\ \alpha E' : 0 & \longrightarrow & Z & \xrightarrow{\mu} & (Z \bigoplus Q)/H & \longrightarrow & Q/Z & \longrightarrow & 0 \end{array}$$

Where $H = \{(2n, -n); n \in Z\}$ *and* $\mu : n \longmapsto (n, 0) + H$. *If* $\alpha E'$ *is s-pure, then* $\mu(Z) \subseteq 2((Z \bigoplus Q)/H)$ *which is a contradiction.*

3. s-pure injectives and s-pure projectives

In this section, we define the concept of s-pure injective and s-pure projective in £ and express some of their properties .

3.1. Definition *Let* G *be a group in £. We call* G *an s-pure injective group in £ if for every s-pure exact sequence*

$$0 \to A \xrightarrow{\phi} B \to C \to 0$$

and continuous homomorphism $f : A \to G$, *there is a continuous homomorphism* $\bar{f} : B \longrightarrow G$ *such that* $\bar{f}\phi = f$. *Similarly, we call* G *an s-pure projective group in £ if for every s-pure exact sequence*

$$0 \to A \to B \xrightarrow{\psi} C \to 0$$

and continuous homomorphism $f : G \to C$, *there is a continuous homomorphism* $\bar{f} : G \to B$ *such that* $\psi \bar{f} = f$.

3.2. Theorem *Let* $G \in$ £. *The following statements are equivalent:*

(1) *G is an s-pure injective in £.*
(2) $G \cong R^n \bigoplus (\frac{R}{Z})^\sigma$ *where* σ *is a cardinal number.*

Proof. $1 \Longrightarrow 2$: *Let* G *be an s-pure injective in £. For a group* X *in £, consider the s-pure extension*

$$E : 0 \to G \xrightarrow{\phi} B \longrightarrow X \to 0$$

Then there is a continuous homomorphism $\bar{\phi} : B \to G$ *such that* $\bar{\phi}\phi = 1_G$. *Consequently,* E *splits. In particular, the s-pure extension* $0 \to G \longrightarrow G^* \to G^*/G \to 0$ *splits where* G^* *is the minimal divisible extension of* G. *Hence,* G *is divisible. So, by Remark 2.2, every extension of* G *by* X *is an s-pure extension. On the other hand, every s-pure extension of* G *by* X *splits. Hence,* $Ext(X, G) = 0$. *By [10, Theorem 3.2],* $G \cong R^n \bigoplus (R/Z)^\sigma$.

$2 \Longrightarrow 1$: *It is clear.*

Recall that a discrete group G *is called reduced if it has no nontrivial divisible subgroup.*

3.3. Lemma Q *is not an s-pure projective group.*

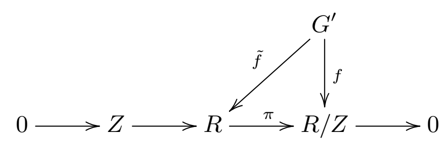
Proof. *Consider the s-pure exact sequence* $0 \to Z \longrightarrow R \xrightarrow{\pi} R/Z \to 0$ *where* π *is the natural mapping. Assume that* Q *is an s-pure projective group and* $f \in Hom(Q, R/Z)$. *Then, there is* $\bar{f} \in Hom(Q, R)$ *such that* $\pi \bar{f} = f$. *Hence,* $\pi^* : Hom(Q, R) \to Hom(Q, R/Z)$ *is surjective. Now consider the following exact sequence*

$$0 \to Hom(Q, Z) \to Hom(Q, R) \xrightarrow{\pi_*} Hom(Q, R/Z) \to Ext(Q, Z) \to Ext(Q, R)$$

Since Q is divisible and Z is reduced, so $Hom(Q, Z) = 0$. Hence, π^* is one to one. This shows that π^* is an isomorphism. On the other hand, $Ext(Q, R) = 0$. Consequently, $Ext(Q, Z) = 0$ which is a contradiction.

3.4. Theorem *Let $G \in \pounds$. If G is an s-pure projective in $\pounds$, then $G \cong R^n \bigoplus G'$ where G' is a discrete torsion-free , reduced group.*

Proof. *It is known that an LCA group G can be written as $G \cong R^n \bigoplus G'$ where G' contains a compact open subgroup [8, Theorem 24.30]. An easy calculation shows that if G is an s-pure projective group, then G' is an s-pure projective in $\pounds$. Let $f \in Hom(G', \frac{R}{Z})$. Then there exists a continuous homomorphism $\tilde{f} : G' \to R$ such that the following diagram is commutative:*



Consider the following exact sequence

$$0 \to Hom(G', Z) \to Hom(G', R) \xrightarrow{\pi_*} Hom(G', R/Z) \to Ext(G', Z) \to 0$$

Since π_* is surjective, so $Ext(G', Z) = 0$. Let K be a compact open subgroup of G'. Then the inclusion map $i : K \to G'$ induces the surjective homomorphism $i_* : Ext(G', Z) \to Ext(K, Z)$. So, $Ext(K, Z) = 0$. Hence, $Ext(R/Z, \hat{K}) = 0$. By [7, Proposition 2.17], $\hat{K} = 0$. So, $K = 0$. Hence, G' is discrete. If G' contains a subgroup of the form $Z(n)$, then $Z(n)$ is a nontrivial compact open subgroup of G' which is a contradiction. So G' is torsion-free. Suppose G' has a nontrivial divisible subgroup. Then G' has a direct summand $H \cong Q$. But then H is s-pure projective, contradicting Lemma 3.3. Therefore, G' is reduced.

Acknowledgment

The authors sincerely thank the referee for his helpful comments and suggestions.

References

[1] Armacost, D. L. On pure subgroups of LCA groups, Trans. Amer. Math. Soc. 45, 414-418, 1974.

[2] Armacost, D. L. The structure of locally compact abelian groups (Marcel Dekker, Inc.,New York,1981).

[3] Fuchs, L. Infinite Abelian Groups, Vol. I, (Academic Press, New York, 1970).

[4] Fulp, R. O. Homological study of purity in locally compact groups, Proc. London Math. Soc. 21, 501-512, 1970.

[5] Fulp, R. O. Splitting locally compact abelian groups, Michigan Math. J. 19, 47-55, 1972.

[6] Fulp, R. O. and Griffith, P. Extensions of locally compact abelian groups I, Trans. Amer. Math. Soc. 154, 341-356, 1971.

[7] Fulp, R. O. and Griffith, P. Extensions of locally compact abelian groups II, Trans. Amer. Math. Soc. 154, 357-363, 1971.

[8] Hewitt, E. and Ross, K. Abstract Harmonic Analysis, Vol I, Second Edition,(Springer-Verlag, Berlin, 1979).

[9] Loth, P. Pure extensions of locally compact abelian groups, Rend. Sem. Mat. Univ. Padova. 116, 31-40, 2006.

[10] Moskowitz, M. Homological algebra in locally compact abelian groups, Trans. Amer. Math. Soc. 127, 361-404, 1967.

[11] Robertson, L. C. Connectivity, divisibility and torsion, Trans. Amer. Math. Soc. 128, 482-505, 1967.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1133 – 1145

Some properties of soft θ-topology

O. R. SAYED * and A. M. KHALIL †

Abstract

For dealing with uncertainties researchers introduced the concept of soft sets. Georgiou et al. [10] defined several basic notions on soft θ-topology and they studied many properties of them. This paper continues the study of the theory of soft θ-topological spaces and presents for this theory new definitions, characterizations, and results concerning soft θ-boundary, soft θ-exterior, soft θ-generalized closed sets, soft Λ-sets, and soft strongly pu-θ-continuity.

2000 AMS Classification: 54A05,54C08,54C10,54D10.

Keywords: soft set; soft θ-topology; soft θ-generalized closed set; soft Λ-set.

Received 27/08/2013 : Accepted 15/04/2014 Doi : 10.15672 /HJMS.2015449437

1. Introduction

In 1999, Molodtsov [20] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. Also, he applied this theory to several directions (see, for example, [21-23]). The soft set theory has been applied to many different fields (see, for example, [1-2], [4-5], [7-8], [13-18], [24], [26], [28], [30]). Later, few researches (see, for example, [3], [6], [11-12], [19], [25], [27], [29]) introduced and studied the notion of soft topological spaces. Recently, in 2013, D. N. Georgiou, A. C. Megaritis, and V. I. Petropoulos [10] initiated the study of soft θ-topology. They proved that the family of all soft θ-open sets defines a soft topology on X. Consequently, they defined some basic notions of soft θ-topological spaces such as soft θ-interior point, soft θ-closure set, and soft θ-continuity and established some of their properties. This paper continues the study of the theory of soft θ-topology. It is organized as follows . The first section is the introduction. In section 2 known basic notions and results concerning the theory of soft sets, soft topological spaces and soft θ-topological spaces are given. In section 3 the notions of soft θ-boundary and soft θ-exterior are defined and some of their properties are studied. Also, some other characterizations of soft θ-closure and soft θ-interior are

*Department of Mathematics, Faculty of Science, Assiut University Assiut, 71516, Egypt Email: o_r_sayed@yahoo.com; o_sayed@aun.edu.eg

†Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt Email:a_ m_ mohmed1@yahoo.com; a.khalil@azhar.edu.eg

given. In section 4 the basic properties of soft θ-generalized closed sets, soft θ-generalized open sets, and soft Λ-sets are introduced. Finally, in section 5, the basic properties of soft strongly pu-θ-continuity are introduced and studied.

2. preliminaries

2.1. Definition. [20]. Let X be an initial universe set, $P(X)$ the power set of X, that is the set of all subsets of X, and A a set of parameters. A pair (F, A), where F is a map from A to $P(X)$, is called a soft set over X.
In what follows by $SS(X, A)$ we denote the family of all soft sets (F, A) over X.

2.2. Definition. [20]. Let $(F, A), (G, A) \in SS(X, A)$. We say that the pair (F, A) is a soft subset of (G, A) if $F(p) \subseteq G(p)$, for every $p \in A$. Symbolically, we write $(F, A) \sqsubseteq (G, A)$. Also, we say that the pairs (F, A) and (G, A) are soft equal if $(F, A) \sqsubseteq (G, A)$ and $(G, A) \sqsubseteq (F, A)$. Symbolically, we write $(F, A) = (G, A)$.

2.3. Definition. [20]. Let I be an arbitrary index set and $\{(F_i, A) : i \in I\} \subseteq SS(X, A)$. Then

(1) The soft union of these soft sets is the soft set $(F, A) \in SS(X, A)$, where the map $F : A \to P(X)$ is defined as follows: $F(p) = \cup\{F_i(p) : i \in I\}$, for every $p \in A$. Symbolically, we write $(F, A) = \sqcup\{(F_i, A) : i \in I\}$.

(2) The soft intersection of these soft sets is the soft set $(F, A) \in SS(X, A)$, where the map $F : A \to P(X)$ is defined as follows: $F(p) = \cap\{F_i(p) : i \in I\}$, for every $p \in A$. Symbolically, we write $(F, A) = \sqcap\{(F_i, A) : i \in I\}$.

2.4. Definition. [29]. Let $(F, A) \in SS(X, A)$. The soft complement of (F, A) is the soft set $(H, A) \in SS(X, A)$, where the map $H : A \to P(X)$ defined as follows: $H(p) = X \backslash F(p)$, for every $p \in A$. Symbolically, we write $(H, A) = (F, A)^c$. Obviously, $(F, A)^c = (F^c, A)$ [10]. For two given subsets $(M, A), (N, A) \in SS(X, A)$ [27], we have

(i) $((M, A) \sqcup (N, A))^c = (M, A)^c \sqcap (N, A)^c$;

(ii) $((M, A) \sqcap (N, A))^c = (M, A)^c \sqcup (N, A)^c$.

2.5. Definition. [20]. The soft set $(F, A) \in SS(X, A)$, where $F(p) = \phi$, for every $p \in A$ is called the A-null soft set of $SS(X, A)$ and denoted by $\mathbf{0}_A$. The soft set $(F, A) \in SS(X, A)$, where $F(p) = X$, for every $p \in A$ is called the A-absolute soft set of $SS(X, A)$ and denoted by $\mathbf{1}_A$.

2.6. Definition. [29]. The soft set $(F, A) \in SS(X, A)$ is called a soft point in X, denoted by e_F, if for the element $e \in A, F(e) \neq \mathbf{0}_A$ and $F(e') = \mathbf{0}_A$ for all $e' \in A \backslash \{e\}$. The set of all soft points of X is denoted by $\mathbf{SP}(X)$. The soft point e_F is said to be in the soft set (G, A), denoted by $e_F \widetilde{\in} (G, A)$, if for the element $e \in A$ and $F(e) \subseteq G(e)$.

2.7. Definition. [29]. Let $SS(X, A)$ and $SS(Y, B)$ be families of soft sets. Let $u : X \to Y$ and $p : A \to B$ be mappings. Then the mapping $f_{pu} : SS(X, A) \to SS(Y, B)$ is defined as:

(1) The image of $(F, A) \in SS(X, A)$ under f_{pu} is the soft set $f_{pu}(F, A) = (f_{pu}(F), B)$ in $SS(Y, B)$ such that

$$f_{pu}(F)(y) = \begin{cases} \bigcup\limits_{x \in p^{-1}(y)} u(F(x)), & p^{-1}(y) \neq \phi \\ \phi, & \text{otherwise} \end{cases}$$

for all $y \in B$.

(2) The inverse image of $(G, B) \in SS(Y, B)$ under f_{pu} is the soft set $f_{pu}^{-1}(G, B) = (f_{pu}^{-1}(G), A)$ in $SS(X, A)$ such that $f_{pu}^{-1}(G)(x) = u^{-1}(G(p(x)))$ for all $x \in A$.

2.8. Proposition. [9]. *Let $(F,A),(F_1,A) \in SS(X,A)$ and $(G,B),(G_1,B) \in SS(Y,B)$. The following statements are true:*

(1) If $(F,A) \sqsubseteq (F_1,A)$, then $f_{pu}(F,A) \sqsubseteq f_{pu}(F_1,A)$.
(2) If $(G,B) \sqsubseteq (G_1,B)$, then $f_{pu}^{-1}(G,B) \sqsubseteq f_{pu}^{-1}(G_1,B)$.
(3) $(F,A) \sqsubseteq f_{pu}^{-1}(f_{pu}(F,A))$.
(4) $f_{pu}(f_{pu}^{-1}(G,B)) \sqsubseteq (G,B)$.
(5) $f_{pu}^{-1}((G,B)^c) = (f_{pu}^{-1}(G,B))^c$.
(6) $f_{pu}((F,A) \sqcup (F_1,A)) = f_{pu}(F,A) \sqcup f_{pu}(F_1,A)$.
(7) $f_{pu}((F,A) \sqcap (F_1,A)) \sqsubseteq f_{pu}(F,A) \sqcap f_{pu}(F_1,A)$.
(8) $f_{pu}^{-1}((G,B) \sqcup (G_1,B)) = f_{pu}^{-1}(G,B) \sqcup f_{pu}^{-1}(G_1,B)$.
(9) $f_{pu}^{-1}((G,B) \sqcap (G_1,B)) = f_{pu}^{-1}(G,B) \sqcap f_{pu}^{-1}(G_1,B)$.

2.9. Definition. [29]. Let X be an initial universe set, A a set of parameters, and $\widetilde{\tau} \subseteq SS(X,A)$. We say that the family $\widetilde{\tau}$ defines a soft topology on X if the following axioms are true:

(1) $\mathbf{0}_A, \mathbf{1}_A \in \widetilde{\tau}$.
(2) If $(G,A),(H,A) \in \widetilde{\tau}$, then $(G,A) \sqcap (H,A) \in \widetilde{\tau}$.
(3) If $(G_i,A) \in \widetilde{\tau}$ for every $i \in I$, then $\sqcup\{(G_i,A) : i \in I\} \in \widetilde{\tau}$.

The triplet $(X,\widetilde{\tau},A)$ is called a soft topological space. The members of $\widetilde{\tau}$ are called soft open sets in X. Also, a soft set (F,A) is called soft closed if the complement $(F,A)^c$ belongs to $\widetilde{\tau}$. The family of all soft closed sets is denoted by $\widetilde{\tau}^c$.

2.10. Definition. Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(F,A) \in SS(X,A)$.

(1) The soft closure of (F,A) [27] is the soft set

$$Cl_S(F,A) = \sqcap\{(S,A) : (S,A) \in \widetilde{\tau}^c, (F,A) \sqsubseteq (S,A)\}.$$

(2) The soft interior of (F,A) [29] is the soft set

$$Int_S(F,A) = \sqcup\{(S,A) : (S,A) \in \widetilde{\tau}, (S,A) \sqsubseteq (F,A)\}.$$

2.11. Definition. [29]. A soft set (G,A) in a soft topological space $(X,\widetilde{\tau},A)$ is called a soft neighborhood (briefly: nbd) of a soft point $e_F \in \mathbf{SP}(X)$ if there exists a soft open set (H,A) such that $e_F \widetilde{\in} (H,A) \sqsubseteq (G,A)$. The soft neighborhood system of a soft point e_F, denoted by $N_{\widetilde{\tau}}(e_F)$, is the family of all of its soft neighborhoods.

2.12. Definition. [3]. Let $(X,\widetilde{\tau},A)$ be a soft topological space.

(1) A subcollection B of $\widetilde{\tau}$ is called a base for $\widetilde{\tau}$ if every member of $\widetilde{\tau}$ can be expressed as a union of members of B.

(2) A subcollection S of $\widetilde{\tau}$ is said to be a subbase for $\widetilde{\tau}$ if the family of all finite intersections of members of S forms a base for $\widetilde{\tau}$.

2.13. Definition. [29]. Let $(X,\widetilde{\tau},A)$ and $(Y,\widetilde{\tau^*},B)$ be two soft topological spaces, $u : X \to Y$ and $p : A \to B$ be mappings, and $e_F \in \mathbf{SP}(X)$.

(1) The map $f_{pu} : SS(X,A) \to SS(Y,B)$ is soft pu-continuous at $e_F \in \mathbf{SP}(X)$ if for each $(G,B) \in N_{\widetilde{\tau^*}}(f_{pu}(e_F))$, there exists $(H,A) \in N_{\widetilde{\tau}}(e_F)$ such that $f_{pu}(H,A) \sqsubseteq (G,B)$.

(2) The map $f_{pu} : SS(X,A) \to SS(Y,B)$ is soft pu-continuous on X if f_{pu} is soft pu-continuous at each soft point in X.

2.14. Definition. [12]. Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(F,A) \in SS(X,A)$.

(1) (F,A) is said to be a soft generalized closed set in $(X,\widetilde{\tau},A)$ if $Cl_S(F,A) \sqsubseteq (G,A)$ whenever $(F,A) \sqsubseteq (G,A)$ and $(G,A) \in \widetilde{\tau}$. The set of all soft generalized closed sets of X is denoted by $(GC)_S(X)$.

(2) (F, A) is said to be a soft generalized open set in $(X, \widetilde{\tau}, A)$ if $(F, A)^c$ is a soft generalized closed set. The set of all soft generalized open sets of X is denoted by $(GO)_S(X)$.

2.15. Definition. [10]. Let $(X, \widetilde{\tau}, A)$ be a soft topological space. The soft θ-interior of a soft subset $(F, A) \in SS(X, A)$ is the soft union of all soft open sets over X whose soft closures are soft contained in (F, A), and is denoted by $Int_S^{\theta}(F, A)$. The soft subset (F, A) is called soft θ-open if $Int_S^{\theta}(F, A) = (F, A)$. The complement of a soft θ-open set is called soft θ-closed. Alternatively, a soft set (F, A) of X is called soft θ-closed set if $Cl_S^{\theta}(F, A) = (F, A)$, where $Cl_S^{\theta}(F, A)$ is the soft θ-closure of (F, A) and is defined to be the soft intersection of all soft closed soft subsets of X whose soft interiors contain $(F, A)^c$ [10, Proposition 5.18 (3) and Definitions 5.10 and 5.11]. We observe that $Cl_S^{\theta}(F, A) = (Int_S^{\theta}(F, A)^c)^c$ [10, Corollary 5.17 (1)]. The family of all soft θ-open sets forms a soft topology on X, denoted by $\widetilde{\tau}_{\theta}$, and is called soft θ-topology. The set of all soft θ-closed sets over X is denoted by $\widetilde{\tau}_{\theta}^{c}$.

2.16. Definition. [10]. Let $(X, \widetilde{\tau}, A)$ and $(Y, \widetilde{\tau^*}, B)$ be two soft topological spaces, $u : X \to Y$ and $p : A \to B$ be mappings, and $e_F \in \mathbf{SP}(X)$.

(1) The map $f_{pu} : SS(X, A) \to SS(Y, B)$ is soft pu-θ-continuous at e_F if for each $(G, B) \in N_{\widetilde{\tau^*}}(f_{pu}(e_F))$, there exists $(H, A) \in N_{\widetilde{\tau}}(e_F)$ such that $f_{pu}(Cl_S(H, A)) \sqsubseteq Cl_S(G, B)$.

(2) The map $f_{pu} : SS(X, A) \to SS(Y, B)$ is soft pu-θ-continuous on X if f_{pu} is soft pu-θ-continuous at each soft point in X.

3. Soft θ-boundary and soft θ-exterior

3.1. Definition. Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(F, A) \in SS(X, A)$. The soft θ-boundary of soft set (F, A) over X is denoted by $Bd_S^{\theta}(F, A)$ and is defined as $Bd_S^{\theta}(F, A) = Cl_S^{\theta}(F, A) \sqcap Cl_S^{\theta}(F^c, A)$.

3.2. Remark. From the above definition it follows directly that the soft sets (F, A) and (F^c, A) have same soft θ-boundary.

3.3. Proposition. *Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(F, A), (G, A) \in SS(X, A)$. Then:*

(1) $Int_S^{\theta}(\mathbf{0}_A) = \mathbf{0}_A$ *and* $Int_S^{\theta}(\mathbf{1}_A) = \mathbf{1}_A$;

(2) $Int_S^{\theta}(F, A) \sqsubseteq (F, A)$;

(3) $Int_S^{\theta}(Int_S^{\theta}(F, A)) \sqsubseteq Int_S^{\theta}(F, A)$;

(4) $(F, A) \sqsubseteq (G, A)$ *implies* $Int_S^{\theta}(F, A) \sqsubseteq Int_S^{\theta}(G, A)$;

(5) $Int_S^{\theta}(F, A) \sqcap Int_S^{\theta}(G, A) = Int_S^{\theta}((F, A) \sqcap (G, A))$;

(6) $Int_S^{\theta}(F, A) \sqcup Int_S^{\theta}(G, A) \sqsubseteq Int_S^{\theta}((F, A) \sqcup (G, A))$.

Proof. Obvious. ∎

The following example shows that the equalities do not hold in Proposition 3.3 (3) and (6).

3.4. Example. (1) Let $X = \{h_1, h_2\}, A = \{e_1, e_2\}$ and $\widetilde{\tau} = \{\mathbf{0}_A, \mathbf{1}_A, (F_1, A), (F_2, A), (F_3, A), (F_4, A)\}$, where $(F_1, A) = \{(e_1, X), (e_2, \{h_2\})\}$, $(F_2, A) = \{(e_1, \{h_1\}), (e_2, \phi)\}$, $(F_3, A) = \{(e_1, \{h_2\}), (e_2, \phi)\}$, and $(F_4, A) = \{(e_1, X), (e_2, \phi)\}$. Then $\widetilde{\tau}$ defines a soft topology on X. Let $(F, A) = \{(e_1, \{h_1\}), (e_2, X)\}$. One observe that $Int_S^{\theta}(Int_S^{\theta}(F, A)) \sqsubseteq Int_S^{\theta}(F, A)$ and $Int_S^{\theta}(F, A) \neq Int_S^{\theta}(Int_S^{\theta}(F, A))$.

(2) Let $X=\{h_1,h_2,h_3\}$, $A=\{e_1,e_2\}$ and $\widetilde{\tau}=\{\mathbf{0}_A,\mathbf{1}_A,(F_1,A),(F_2,A)\}$, where $(F_1,A)=\{(e_1,\{h_1\}),(e_2,\{h_1,h_2\})\}$, and $(F_2,A)=\{(e_1,\{h_2,h_3\}),(e_2,\{h_3\})\}$. Then $\widetilde{\tau}$ defines a soft topology on X. Suppose that $(F,A)=\{(e_1,\{h_1,h_3\}),(e_2,\{h_1,h_2\})\}$, and $(G,A)=\{(e_1,\{h_2\}),(e_2,\{h_3\})\}$. One can deduce that $Int^{\theta}_{S}(F,A)\sqcup Int^{\theta}_{S}(G,A)\sqsubset Int^{\theta}_{S}((F,A)\sqcup(G,A))$ and $Int^{\theta}_{S}((F,A)\sqcup(G,A))\neq Int^{\theta}_{S}(F,A)\sqcup Int^{\theta}_{S}(G,A)$.

3.5. Proposition. *Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(H,A),(M,A)\in SS(X,A)$. Then:*

(1) $Cl^{\theta}_{S}(\mathbf{0}_A)=\mathbf{0}_A$ *and* $Cl^{\theta}_{S}(\mathbf{1}_A)=\mathbf{1}_A$;
(2) $(H,A)\sqsubseteq Cl^{\theta}_{S}(H,A)$;
(3) $Cl^{\theta}_{S}(H,A)\sqsubseteq Cl^{\theta}_{S}(Cl^{\theta}_{S}(H,A))$;
(4) $(H,A)\sqsubseteq(M,A)$ *implies* $Cl^{\theta}_{S}(H,A)\sqsubseteq Cl^{\theta}_{S}(M,A)$;
(5) $Cl^{\theta}_{S}((H,A)\sqcup(M,A))=Cl^{\theta}_{S}(H,A)\sqcup Cl^{\theta}_{S}(M,A)$;
(6) $Cl^{\theta}_{S}((H,A)\sqcap(M,A))\sqsubseteq Cl^{\theta}_{S}(H,A)\sqcap Cl^{\theta}_{S}(M,A)$.

Proof. (1), (2) and (4) are obvious.
(3) Follows from [10, Proposition 5.13 (3)].
(5) Follows from (2) above and [10, Proposition 5.13 (2)].
(6) Follows from (4) above. ∎

The following example shows that the equalities do not hold in Proposition 3.5 (3) and (6).

3.6. Example. (1) The soft topological space $(X,\widetilde{\tau},A)$ is the same as in Example 3.4 (1). Let $(R,A)=(F,A)^c$. We have $Cl^{\theta}_{S}(Cl^{\theta}_{S}(R,A))=\{(e_1,X),(e_2,X)\}\neq Cl^{\theta}_{S}(R,A)=\{(e_1,\{h_2\}),(e_2,X)\}$.
(2) The soft topological space $(X,\widetilde{\tau},A)$ is the same as in Example 3.4 (2). Suppose that $(H,A)=(F,A)^c$ and $(M,A)=(G,A)^c$. So $Cl^{\theta}_{S}((H,A)\sqcap(M,A))=\mathbf{0}_A\sqsubset Cl^{\theta}_{S}(H,A)\sqcap Cl^{\theta}_{S}(M,A)=\{(e_1,\{h_2,h_3\}),(e_2,\{h_3\})\}$. Therefore $Cl^{\theta}_{S}(H,A)\sqcap Cl^{\theta}_{S}(M,A)\neq Cl^{\theta}_{S}((H,A)\sqcap(M,A))$.

3.7. Proposition. *Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(F,A)\in SS(X,A)$. Then the following statements are true.*

(1) $Bd^{\theta}_{S}(F,A)=Cl^{\theta}_{S}(F,A)\backslash Int^{\theta}_{S}(F,A)$.
(2) $Bd^{\theta}_{S}(F,A)\sqcap Int^{\theta}_{S}(F,A)=\mathbf{0}_A$.
(3) $(F,A)\sqcup Bd^{\theta}_{S}(F,A)=Cl^{\theta}_{S}(F,A)$.
(4) $Bd^{\theta}_{S}(F,A)\notin\widetilde{\tau}^{c}_{\theta}$.

Proof. (1), (2) and (3) are obvious.
(4) Let $(X,\widetilde{\tau},A)$ be a soft topological space, where $X=\{h_1,h_2,h_3\}$, $A=\{e_1,e_2\}$ and $\widetilde{\tau}=\{\mathbf{0}_A,\mathbf{1}_A,\{(e_1,\{h_1\}),(e_2,\{h_1\})\},\{(e_1,\{h_2\}),(e_2,\{h_2\})\},\{(e_1,\{h_1,h_2\}),(e_2,\{h_1,h_2\})\}\}$. Then $Bd^{\theta}_{S}(\{(e_1,X),(e_2,\{h_1,h_3\})\})=\{(e_1,\{h_2,h_3\}),(e_2,\{h_2,h_3\})\}\notin\widetilde{\tau}^{c}_{\theta}$. ∎

3.8. Theorem. *Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(F,A)\in SS(X,A)$. Then $Bd^{\theta}_{S}(F,A)=\mathbf{0}_A$ if and only if (F,A) is soft θ-closed and soft θ-open.*

Proof. Obvious. ∎

3.9. Theorem. *Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(F,A)\in SS(X,A)$. Then*

(1) (F,A) *is soft θ-open if and only if* $(F,A)\sqcap Bd^{\theta}_{S}(F,A)=\mathbf{0}_A$.
(2) (F,A) *is soft θ-closed if and only if* $Bd^{\theta}_{S}(F,A)\sqsubseteq(F,A)$.

Proof. (1) *Necessity.* Follows from Proposition 3.7 (2).
Sufficiency. Follows from [29, Proposition 3.6 (1)].
(2) *Necessity.* Obvious.
Sufficiency. Follows from (1) above. ∎

3.10. Proposition. *Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(F, A) \in SS(X, A)$. Then the following statements are true.*
(1) $(F, A) \backslash Bd_S^{\theta}(F, A) = Int_S^{\theta}(F, A)$.
(2) If (F, A) is soft θ-closed, then $(F, A) \backslash Int_S^{\theta}(F, A) = Bd_S^{\theta}(F, A)$.

Proof. (1) Obvious.
(2) Follows from Proposition 3.7 (1). ∎

3.11. Definition. Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(F, A) \in SS(X, A)$. The soft θ-exterior of (F, A) over X is denoted by $Ext_S^{\theta}(F, A)$ and is defined as $Ext_S^{\theta}(F, A) = Int_S^{\theta}(F, A)^c$.

3.12. Proposition. *Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(F, A) \in SS(X, A)$. Then the following statements are true.*
(1) $Ext_S^{\theta}(\mathbf{0}_A) = \mathbf{1}_A$ *and* $Ext_S^{\theta}(\mathbf{1}_A) = \mathbf{0}_A$.
(2) $Ext_S^{\theta}((F, A) \sqcup (G, A)) = Ext_S^{\theta}(F, A) \sqcap Ext_S^{\theta}(G, A)$.
(3) $Ext_S^{\theta}(F, A) \sqcup Ext_S^{\theta}(G, A) \sqsubseteq Ext_S^{\theta}((F, A) \sqcap (G, A))$.
(4) $Ext_S^{\theta}((Ext_S^{\theta}(F, A))^c) \sqsubseteq Ext_S^{\theta}(F, A)$.
(5) $Ext_S^{\theta}(F, A) \notin \widetilde{\tau}_{\theta}$.

Proof. (1), (2), (3) and (4) are obvious.
(5) See Example 3.13. ∎

The following example shows that the equalities do not hold in Proposition 3.12 (3) and (4).

3.13. Example. In Example 3.4 (1), we have $Ext_S^{\theta}((Ext_S^{\theta}(F_3, A))^c) \neq Ext_S^{\theta}(F_3, A)$ and $Ext_S^{\theta}(F_3, A) \notin \widetilde{\tau}_{\theta}$. In Example 3.4 (2), we obtain $Ext_S^{\theta}(F, A) \sqcup Ext_S^{\theta}(G, A) \sqsubset Ext_S^{\theta}((F, A) \sqcap (G, A))$ and $Ext_S^{\theta}((F, A) \sqcap (G, A)) \neq Ext_S^{\theta}(F, A) \sqcup Ext_S^{\theta}(G, A)$.

4. Basic properties of soft θ-generalized closed sets

4.1. Definition. Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(F, A) \in SS(X, A)$. (F, A) is said to be a soft θ-generalized closed set in $(X, \widetilde{\tau}, A)$ if $Cl_S^{\theta}(F, A) \sqsubseteq (G, A)$ whenever $(F, A) \sqsubseteq (G, A)$ and $(G, A) \in \widetilde{\tau}$. The set of all soft θ-generalized closed sets over X is denoted by $(GC)_S^{\theta}(X)$.

4.2. Proposition. *Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(F, A) \in SS(X, A)$. Then the following statement are true.*
(1) If $(F, A) \in \widetilde{\tau}_{\theta}^{c}$, then $(F, A) \in (GC)_S^{\theta}$;
(2) If $(F, A) \in (GC)_S^{\theta}$, then $(F, A) \in (GC)_S$.

Proof. (1) Obvious.
(2) Follows from [10, Definition 5.11]. ∎

The converses of (1) and (2) in Proposition 4.2 are not true as illustrated by the following examples.

4.3. Example. Let $(X,\widetilde{\tau},A)$ be the soft topological space of Example 3.4 (2) and Example 3.6 (2). Since $(F_2,A)\in\widetilde{\tau}$, $(H,A)\sqsubseteq(F_2,A)$ and $Cl_S^{\theta}(H,A)\sqsubseteq(F_2,A)$, we have $(H,A)\in(GC)_S^{\theta}$. But $(H,A)\notin\widetilde{\tau}_{\theta}^{c}$.

4.4. Example. Let $X=\{h_1,h_2\}, A=\{e_1,e_2\}$ and $\widetilde{\tau}=\{\mathbf{0}_A,\mathbf{1}_A,(F_1,A),(F_2,A),(F_3,A)\}$ where $(F_1,A)=\{(e_1,X),(e_2,\{h_2\})\}$, $(F_2,A)=\{(e_1,\{h_1\}),(e_2,X)\}$, and $(F_3,A)=\{(e_1,\{h_1\}),(e_2,\{h_2\})\}$. Then $(X,\widetilde{\tau},A)$ is a soft topological space over X. We have $(H_2,A)=(F_2,A)^c$ is a soft closed set and hence soft generalized-closed. But $(H_2,A)\notin(GC)_S^{\theta}$.

4.5. Proposition. *Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(F_1,A),(F_2,A)\in SS(X,A)$. If $(F_1,A),(F_2,A)\in(GC)_S^{\theta}$, then $(F_1,A)\sqcup(F_2,A)\in(GC)_S^{\theta}$.*

Proof. Follows from Proposition 3.5 (5). ∎

4.6. Corollary. *Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(F,A),(G,A)\in SS(X,A)$. Then the following statement are true.*

(1) If $(F,A)\in\widetilde{\tau}_{\theta}^{c}$ and $(G,A)\in(GC)_S^{\theta}$, then $(F,A)\sqcup(G,A)\in(GC)_S^{\theta}$.

(2) If $(F,A)\in(GC)_S^{\theta}$ and $(G,A)\in(GC)_S$, then $(F,A)\sqcup(G,A)\in(GC)_S$.

Proof. (1) Follows from Proposition 4.2 (1) and Proposition 4.5.

(2) Follows from Proposition 4.2 (2) and [12, Theorem 3.5]. ∎

4.7. Proposition. *Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(F,A)\in SS(X,A)$. Then the following statement are true.*

(1) If $(F,A)\in\widetilde{\tau}$ and $(F,A)\in(GC)_S^{\theta}$, then $(F,A)\in\widetilde{\tau}_{\theta}^{c}$.

(2) If $\widetilde{\tau}=\widetilde{\tau}_{\theta}^{c}$, then every soft subset of X is in $(GC)_S^{\theta}$.

Proof. Clear. ∎

4.8. Proposition. *Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(G,A)\in SS(X,A)$. Then $(G,A)\in(GC)_S^{\theta}$ if and only if the only soft closed soft subset of $Cl_S^{\theta}(G,A)\backslash(G,A)$ is $\mathbf{0}_A$.*

Proof. Necessity. Let $(F,A)\in\widetilde{\tau}^{c}$ such that $(F,A)\sqsubseteq Cl_S^{\theta}(G,A)\backslash(G,A)=Cl_S^{\theta}(G,A)\sqcap(G,A)^{c}$ which implies that $(F,A)\sqsubseteq Cl_S^{\theta}(G,A),(F,A)\sqsubseteq(G,A)^{c}$. Thus $(G,A)\sqsubseteq(F,A)^{c}$. Since $(G,A)\in(GC)_S^{\theta}$ and $(F,A)^{c}\in\widetilde{\tau}$, we have $Cl_S^{\theta}(G,A)\sqsubseteq(F,A)^{c}$ or $(F,A)\sqsubseteq(Cl_S^{\theta}(G,A))^{c}$. Since $(F,A)\sqsubseteq Cl_S^{\theta}(G,A)$, we have $(F,A)\sqsubseteq(Cl_S^{\theta}(G,A))^{c}\sqcap Cl_S^{\theta}(G,A)=\mathbf{0}_A$. This shows that $(F,A)=\mathbf{0}_A$.

Sufficiency. Suppose that $(G,A)\sqsubseteq(U,A)$ and that $(U,A)\in\widetilde{\tau}$. If $Cl_S^{\theta}(G,A)\not\sqsubseteq(U,A)$, then $Cl_S^{\theta}(G,A)\sqcap(U,A)^{c}$ is a non-A-null soft closed soft subset of $Cl_S^{\theta}(G,A)\backslash(G,A)$, a contradiction. Therefore $Cl_S^{\theta}(G,A)\sqsubseteq(U,A)$ and $(G,A)\in(GC)_S^{\theta}$. ∎

4.9. Corollary. *Let $(X,\widetilde{\tau},A)$ be a soft topological space, $(F,A)\in SS(X,A)$ and $(F,A)\in(GC)_S^{\theta}$. Then $(F,A)\in\widetilde{\tau}_{\theta}^{c}$ if and only if $Cl_S^{\theta}(F,A)\backslash(F,A)\in\widetilde{\tau}^{c}$.*

4.10. Proposition. *Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(F,A)\in SS(X,A)$. Then $(F,A)\in(GC)_S^{\theta}$ if and only if $(F,A)\sqcup(Cl_S^{\theta}(F,A))^{c}\in(GC)_S^{\theta}$.*

Proof. Follows from Proposition 4.8. ∎

4.11. Lemma. *Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(F,A)\in SS(X,A)$. If $(F,A)\in\widetilde{\tau}_{\theta}^{c}$, then $(F,A)\in\widetilde{\tau}^{c}$.*

The converse of Lemma 4.11 is not true in general as illustrated by the following example.

4.12. Example. Let $X = \{h_1, h_2\}, A = \{e_1, e_2\}$ and $\widetilde{\tau} = \{\mathbf{0}_A, \mathbf{1}_A, (F_1, A)\}$ is a soft topology over X, where $(F_1, A) = \{(e_1, X), (e_2, \{h_2\})\}$. We observe that $(H_1, A) = (F_1, A)^c \in \widetilde{\tau}^{c}$. But $(H_1, A) \notin \widetilde{\tau}_{\theta}^{c}$.

4.13. Proposition. *Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(G, A) \in SS(X, A)$. Then $(G, A) \in (GC)_S^{\theta}$ if and only if $(G, A) = (F, A) \backslash (H, A)$, where $(F, A) \in \widetilde{\tau}_{\theta}^{c}$ and the only soft closed soft subset of (H, A) is $\mathbf{0}_A$.*

Proof. Necessity. Follows from Proposition 4.8.
Sufficiency. Follows from Lemma 4.11. ∎

4.14. Definition. Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(G, A) \in SS(X, A)$. (G, A) is said to be a soft θ-generalized open set in $(X, \widetilde{\tau}, A)$ if $(G, A)^{c}$ is soft θ-generalized closed. The set of all soft θ-generalized open sets over X is denoted by $(GO)_S^{\theta}(X)$.

4.15. Proposition. *Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(G, A), (F, A) \in SS(X, A)$. Then $(G, A) \in (GO)_S^{\theta}$ if and only if $(F, A) \sqsubseteq Int_S^{\theta}(G, A)$ whenever $(F, A) \sqsubseteq (G, A)$ and $(F, A) \in \widetilde{\tau}^{c}$.*

Proof. Obvious. ∎

As a direct consequence of Proposition 4.2 we have

4.16. Proposition. *Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(G, A) \in SS(X, A)$. Then*

(1) If $(G, A) \in \widetilde{\tau}_{\theta}$, then $(G, A) \in (GO)_S^{\theta}$;
(2) If $(G, A) \in (GO)_S^{\theta}$, then $(G, A) \in (GO)_S$.

The converses of (1) and (2) in Proposition 4.16 are not true as illustrated by the following examples.

4.17. Example. Let $(X, \widetilde{\tau}, A)$ be the soft topological space of Example 3.4 (2) and Example 3.6 (2). Since $(R_2, A) \in \widetilde{\tau}^{c}$, $(R_2, A) \sqsubseteq (F, A)$ and $(R_2, A) \sqsubseteq Int_S^{\theta}(F, A)$, we have $(F, A) \in (GO)_S^{\theta}$. But $(F, A) \notin \widetilde{\tau}_{\theta}$.

4.18. Example. The soft topological space $(X, \widetilde{\tau}, A)$ is the same as in Example 4.4. We observe that $(F_2, A) \in (GO)_S$. But $(F_2, A) \notin (GO)_S^{\theta}$.

4.19. Proposition. *Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(G_1, A), (G_2, A) \in SS(X, A)$. If $(G_1, A), (G_2, A) \in (GO)_S^{\theta}$, then $(G_1, A) \sqcap (G_2, A) \in (GO)_S^{\theta}$.*

Proof. Follows from Proposition 3.3 (5). ∎

4.20. Corollary. *Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(F, A), (G, A) \in SS(X, A)$.*

(1) If $(F, A) \in \widetilde{\tau}_{\theta}$ and $(G, A) \in (GO)_S^{\theta}$, then $(F, A) \sqcap (G, A) \in (GO)_S^{\theta}$.
(2) If $(F, A) \in (GO)_S^{\theta}$ and $(G, A) \in (GO)_S$, then $(F, A) \sqcap (G, A) \in (GO)_S$.

Proof. (1) Follows from Proposition 4.16 (1) and Proposition 4.19 .
(2) Follows from Proposition 4.16 (2) and [12, Theorem 4.5]. ∎

4.21. Proposition. *Let $(X, \widetilde{\tau}, A)$ be a soft topological space and $(G, A) \in SS(X, A)$. Then $(G, A) \in (GO)_S^{\theta}$ if and only if $(U, A) = \mathbf{1}_A$ whenever $(U, A) \in \widetilde{\tau}$ and $Int_S^{\theta}(G, A) \sqcup (G, A)^{c} \sqsubseteq (U, A)$.*

Proof. Necessity. Follows from Proposition 4.8.
Sufficiency. Obvious. ∎

4.22. Proposition. *Let* $(X, \widetilde{\tau}, A)$ *be a soft topological space and* $(G, A) \in SS(X, A)$. *Then* $(G, A) \in (GC)^{\theta}_{S}$ *if and only if* $Cl^{\theta}_{S}(G, A) \backslash (G, A) \in (GO)^{\theta}_{S}$.

Proof. Necessity. Follows from Propositions 4.8 and 4.15.
Sufficiency. Suppose that $(U, A) \in \widetilde{\tau}$ such that $(G, A) \sqsubseteq (U, A)$ or $(U, A)^{c} \sqsubseteq (G, A)^{c}$. Now, $Cl^{\theta}_{S}(G, A) \sqcap (U, A)^{c} \sqsubseteq Cl^{\theta}_{S}(G, A) \sqcap (G, A)^{c} = Cl^{\theta}_{S}(G, A) \backslash (G, A)$ and since $Cl^{\theta}_{S}(G, A) \sqcap (U, A)^{c} \in \widetilde{\tau}^{c}$ and $Cl^{\theta}_{S}(G, A) \backslash (G, A) \in (GO)^{\theta}_{S}$, it follows that $Cl^{\theta}_{S}(G, A) \sqcap (U, A)^{c} \sqsubseteq Int^{\theta}_{S}(Cl^{\theta}_{S}(G, A) \backslash (G, A)) = \mathbf{0}_A$. Therefore $Cl^{\theta}_{S}(G, A) \sqcap (U, A)^{c} = \mathbf{0}_A$ or $Cl^{\theta}_{S}(G, A) \sqsubseteq (U, A)$. Hence $(G, A) \in (GC)^{\theta}_{S}$. ∎

4.23. Proposition. *Let* $(X, \widetilde{\tau}, A)$ *be a soft topological space and* $(F, A), (G, A) \in SS(X, A)$. *If* $(G, A) \in \widetilde{\tau}^{c}$ *and* $(G, A) \in (GO)^{\theta}_{S}$, *then* $(G, A) \in \widetilde{\tau}_{\theta}$.

Proof. Obvious. ∎

4.24. Definition. A soft set (F, A) in a soft topological space $(X, \widetilde{\tau}, A)$ is said to be soft Λ-set if $(F, A) = (F, A)^{\Lambda}$, where $(F, A)^{\Lambda} = \sqcap\{(G, A) \in \widetilde{\tau} : (F, A) \sqsubseteq (G, A)\}$.

4.25. Proposition. *Let* $(X, \widetilde{\tau}, A)$ *be a soft topological space and* $(F, A), (H, A), (F_i, A) \in SS(X, A), i \in I$. *Then the following statements are true.*
(1) $(F, A) \sqsubseteq (F, A)^{\Lambda}$.
(2) If $(F, A) \sqsubseteq (H, A)$, *then* $(F, A)^{\Lambda} \sqsubseteq (H, A)^{\Lambda}$.
(3) $((F, A)^{\Lambda})^{\Lambda} = (F, A)^{\Lambda}$.
(4) $(\sqcap_{i \in I} (F_i, A))^{\Lambda} \sqsubseteq \sqcap_{i \in I} (F_i, A)^{\Lambda}$.
(5) $(\sqcup_{i \in I} (F_i, A))^{\Lambda} = \sqcup_{i \in I} (F_i, A)^{\Lambda}$.

Proof. Clear. ∎

The following example shows that the equality does not hold in Proposition 4.25 (4).

4.26. Example. Let us consider the soft topological space $(X, \widetilde{\tau}, A)$ over X in Example 3.4 (2). One can deduce that $(F, A)^{\Lambda} \sqcap (G, A)^{\Lambda} \neq ((F, A) \sqcap (G, A))^{\Lambda}$.

4.27. Proposition. *Let* $(X, \widetilde{\tau}, A)$ *be a soft topological space. Then the following statements are true.*
(1) $\mathbf{0}_A$ *and* $\mathbf{1}_A$ *are soft* Λ*-sets.*
(2) Every soft union of soft Λ*-sets is a soft* Λ*-set.*
(3) Every soft intersection of soft Λ*-sets is a soft* Λ*-set.*

Proof. Follows from Proposition 4.25. ∎

4.28. Proposition. *Let* $(X, \widetilde{\tau}, A)$ *be a soft topological space and* $(F, A) \in SS(X, A)$. *Then* $(F, A) \in (GC)^{\theta}_{S}$ *if and only if* $Cl^{\theta}_{S}(F, A) \sqsubseteq (F, A)^{\Lambda}$.

Proof. Clear. ∎

4.29. Proposition. *Let* $(X, \widetilde{\tau}, A)$ *be a soft topological space and* $(F, A) \in SS(X, A)$. *Let* (F, A) *be a soft* Λ*-set. Then* $(F, A) \in (GC)^{\theta}_{S}$ *if and only if* $(F, A) \in \widetilde{\tau}^{c}_{\theta}$.

Proof. Necessity. Follows from Proposition 4.28.
Sufficiency. Follows from the fact that every soft θ-closed set is soft θ-generalized closed (Proposition 4.2(1)). ∎

4.30. Proposition. *Let $(X,\widetilde{\tau},A)$ be a soft topological space and $(F,A)\in SS(X,A)$. If $(F,A)^{\Lambda}\in (GC)^{\theta}_{S}$, then $(F,A)\in (GC)^{\theta}_{S}$.*

Proof. Clear. ∎

5. Soft strongly pu-θ-continuity

In this section, we introduce the notion of soft strongly pu-θ-continuity of functions induced by two mappings $u:X\to Y$ and $p:A\to B$ on soft topological spaces $(X,\widetilde{\tau},A)$ and $(Y,\widetilde{\tau^*},B)$.

5.1. Definition. Let $(X,\widetilde{\tau},A)$ and $(Y,\widetilde{\tau^*},B)$ be two soft topological spaces, $u:X\to Y$ and $p:A\to B$ be mappings, and $e_F\in \mathbf{SP}(X)$.

(1) The map $f_{pu}:SS(X,A)\to SS(Y,B)$ is soft strongly pu-θ-continuous at e_F if for each $(G,B)\in N_{\widetilde{\tau^*}}(f_{pu}(e_F))$, there exists $(H,A)\in N_{\widetilde{\tau}}(e_F)$ such that $f_{pu}(Cl_S(H,A))\sqsubseteq (G,B)$.

(2) The map $f_{pu}:SS(X,A)\to SS(Y,B)$ is soft strongly pu-θ-continuous on X if f_{pu} is soft strongly pu-θ-continuous at each soft point in X.

5.2. Proposition. *Let $(X,\widetilde{\tau},A)$ and $(Y,\widetilde{\tau^*},B)$ be two soft topological spaces. Then the following statements are equivalent.*

(1) The map $f_{pu}:SS(X,A)\to SS(Y,B)$ is soft strongly pu-θ-continuous;

(2) For each $(G,B)\in\widetilde{\tau^*}$, $f_{pu}^{-1}(G,B)\in\widetilde{\tau}_{\theta}$;

(3) For each $(H,B)\in(\widetilde{\tau^*})^{c}$, $f_{pu}^{-1}(H,B)\in\widetilde{\tau}_{\theta}^{c}$.

Proof. Similar to the proof of [29, Theorem 6.3]. ∎

5.3. Proposition. *Let $(X,\widetilde{\tau},A)$ and $(Y,\widetilde{\tau^*},B)$ be two soft topological spaces, $u:X\to Y$, $p:A\to B$ and $f_{pu}:SS(X,A)\to SS(Y,B)$ be mappings. Then the following statements are true.*

(1) If f_{pu} is soft strongly pu-θ-continuous, then f_{pu} is soft pu-continuous.

(2) If f_{pu} is soft strongly pu-θ-continuous, then f_{pu} is soft pu-θ-continuous.

Proof. (1) Obvious.

(2) Follows from (1) and [10, Proposition 5.26]. ∎

The converses of (1) and (2) in Proposition 5.3 are not true as illustrated by the following example.

5.4. Example. Let $X=\{h_1,h_2,h_3\}, Y=\{m_1,m_2,m_3\}, A=\{e_1,e_2\}$, and $B=\{u_1,u_2\}$. We consider the soft topology $\widetilde{\tau}=\{\mathbf{0}_A,\mathbf{1}_A,\{(e_1,\{h_3\}),(e_2,\{h_1,h_2\})\},\{(e_1,\phi),(e_2,\{h_3\})\},\{(e_1,\{h_3\}),(e_2,X)\}\}$ over X and the soft topology $\widetilde{\tau^*}=\{\mathbf{0}_B,\mathbf{1}_B,\{(u_1,\{m_1\}),(u_2,\{m_3\})\},\{(u_1,\{m_1,m_2\}),(u_2,\{m_3\})\}\}$ over Y. Let $u:X\to Y$ be the map such that $u(h_1)=u(h_2)=m_1$ and $u(h_3)=m_3$ and $p:A\to B$ be the map such that $p(e_1)=u_2$ and $p(e_2)=u_1$. Then, the map $f_{pu}:SS(X,A)\to SS(Y,B)$ is both soft pu-continuous and soft pu-θ-continuous but it is not soft strongly pu-θ-continuous.

5.5. Proposition. *Let $(X,\widetilde{\tau},A)$ and $(Y,\widetilde{\tau^*},B)$ be two soft topological spaces and S^* be a soft subbase of $\widetilde{\tau^*}$. A map $f_{pu}:SS(X,A)\to SS(Y,B)$ is soft strongly pu-θ-continuous if and only if for each $(G,B)\in\mathrm{S}^*, f_{pu}^{-1}(G,B)\in\widetilde{\tau}_{\theta}$.*

Proof. Necessity. Follows from Proposition 5.2.

Sufficiency. Follows from [10, Proposition 5.7] and Proposition 5.2. ∎

5.6. Proposition. *Let* $(X, \widetilde{\tau}, A)$ *and* $(Y, \widetilde{\tau^*}, B)$ *be two soft topological spaces. Then the following statements are equivalent.*
(1) The map $f_{pu} : SS(X,A) \to SS(Y,B)$ *is soft strongly pu-θ-continuous;*
(2) For each $(F,A) \in SS(X,A), f_{pu}(Cl_S^{\theta}(F,A)) \sqsubseteq Cl_S(f_{pu}(F,A))$*;*
(3) For each $(G,B) \in SS(Y,B), Cl_S^{\theta}(f_{pu}^{-1}(G,B)) \sqsubseteq f_{pu}^{-1}(Cl_S(G,B))$*.*

Proof. (1)⇒(2) Follows from Proposition 5.2 (3).

(2)⇒(3) This is trivial.

(3)⇒(1) Let $e_F \in \mathbf{SP}(X)$ and $(M,B) \in N_{\widetilde{\tau^*}}(f_{pu}(e_F))$. Since $(M,B)^c \in (\widetilde{\tau^*})^c$, we have $Cl_S^{\theta}(f_{pu}^{-1}(M,B)^c) \sqsubseteq f_{pu}^{-1}(Cl_S(M,B)^c) = f_{pu}^{-1}(M,B)^c$. Therefore $f_{pu}^{-1}(M,B)^c = (f_{pu}^{-1}(M,B))^c \in \widetilde{\tau}_{\theta}^{c}$ and so $f_{pu}^{-1}(M,B) \in \widetilde{\tau}_{\theta}$. Moreover, $e_F \widetilde{\in} f_{pu}^{-1}(M,B)$. There exists $(U,A) \in N_{\widetilde{\tau}}(e_F)$ such that $Cl_S(U,A) \sqsubseteq f_{pu}^{-1}(M,B)$. Therefore $f_{pu}(Cl_S(U,A)) \sqsubseteq (M,B)$. Hence f_{pu} is soft strongly pu-θ-continuous. ∎

5.7. Definition. A soft set (F,A) in a soft topological space $(X, \widetilde{\tau}, A)$ is called a soft θ-neighborhood of a soft point $e_F \in \mathbf{SP}(X)$ if there exists a soft open set (G,A) such that $e_F \widetilde{\in} (G,A) \sqsubseteq Cl_S(G,A) \sqsubseteq (F,A)$. The soft θ-neighborhood system of a soft point e_F, denoted by $N_{\widetilde{\tau}_{\theta}}(e_F)$, is the family of all its soft θ-neighborhoods.

Note that a soft θ-neighborhood is not necessarily a soft neighborhood in the soft θ-topology.

5.8. Proposition. *The soft θ-neighborhood system* $N_{\widetilde{\tau}_{\theta}}(e_F)$ *at* e_F *in a soft topological space* $(X, \widetilde{\tau}, A)$ *has the following properties:*
(1) If $(F,A) \in N_{\widetilde{\tau}_{\theta}}(e_F)$*, then* $e_F \widetilde{\in} (F,A)$*.*
(2) If $(F,A) \in N_{\widetilde{\tau}_{\theta}}(e_F)$ *and* $(F,A) \sqsubseteq (G,A)$*, then* $(G,A) \in N_{\widetilde{\tau}_{\theta}}(e_F)$*.*
(3) If $(F,A), (G,A) \in N_{\widetilde{\tau}_{\theta}}(e_F)$*, then* $(F,A) \sqcap (G,A) \in N_{\widetilde{\tau}_{\theta}}(e_F)$*.*
(4) If $(F,A) \in N_{\widetilde{\tau}_{\theta}}(e_F)$*, then there is a* $(H,A) \in N_{\widetilde{\tau}_{\theta}}(e_F)$ *such that* $(F,A) \in N_{\tau_{\theta}}(e'_M)$ *for each* $e'_M \widetilde{\in} (H,A)$*.*

Proof. Similar to the proof of [29, Theorem 4.10]. ∎

The main results can be paraphrased as follows: soft pu-θ-continuity corresponds to f_{pu}^{-1} (soft θ-neighborhood)= soft θ-neighborhood and strong pu-θ-continuity corresponds to f_{pu}^{-1} (soft neighborhood)= soft θ-neighborhood.

5.9. Proposition. *Let* $(X, \widetilde{\tau}, A)$ *and* $(Y, \widetilde{\tau^*}, B)$ *be two soft topological spaces,* $u : X \to Y$ *and* $p : A \to B$ *be mappings. Then the following statements are equivalent.*
(1) f_{pu} *is soft pu-θ-continuous;*
(2) For each $e_F \in \mathbf{SP}(X)$ *and* $(H,B) \in N_{\widetilde{\tau^*}}(f_{pu}(e_F)), f_{pu}^{-1}(H,B) \in N_{\widetilde{\tau}_{\theta}}(e_F)$*.*

Proof. (1)⇒(2) Follows from Proposition 2.8 (2) and (3).

(2)⇒(1) Follows from Propositions 5.8 (2) and 2.8 (1) and (4). ∎

5.10. Proposition. *Let* $(X, \widetilde{\tau}, A)$ *and* $(Y, \widetilde{\tau^*}, B)$ *be two soft topological spaces,* $u : X \to Y$ *and* $p : A \to B$ *be mappings. Then the following statements are equivalent.*

(1) f_{pu} *is soft strongly pu-θ-continuous;*

(2) For each $e_F \in \mathbf{SP}(X)$ *and* $(H,B) \in N_{\widetilde{\tau^*}}(f_{pu}(e_F)), f_{pu}^{-1}(H,B) \in N_{\widetilde{\tau}_{\theta}}(e_F)$*.*

Proof. Similar to the proof of Proposition 5.9. ∎

Acknowledgement. The authors are grateful to the referees for their valuable comments and suggestions.

References

[1] Aktas, H. and Cağman, N. Soft sets and soft groups, *Inf. Sci.*, **177** (2007), 2726-2735.

[2] Ali, M. I. Feng, F. Liu, X. Min, W. K. and Shabir, M. On some new operations in soft set theory, *Comput. Math. Appl.*, **57** (2009), 1547-1553.

[3] A. Aygünoğlu and H. Aygün, Some notes on soft topological spaces, *Neural. Comput. Appl.*, 21 (1) (2012), 113-119.

[4] Cağman, N. and Enginoglu, S. Soft set theory and uni-int decision making, *Eur. J. Oper. Res.*, **207** (2010), 848-855.

[5] Cağman, N. and Enginoglu, S. Soft matrix theory and its decision making, *Comput. Math. Appl.*, **59** (2010), 3308-3314.

[6] Cağman, N. Karatas, S. and Enginoglu, S. Soft topology, *Comput. Math. Appl.*, **62** (2011), 351-358.

[7] Chen, D. Tsang, E. C. C. Yeung, D. S. and Wang, X. The parameterization reduction of soft sets and its applications, *Comput. Math. Appl.*, **49** (2005), 757-763.

[8] Feng, F. Jun, Y. B. Zhao, X. Z. Soft semirings, *Comput. Math. Appl.*, **56** (2008), 2621-2628.

[9] Georgiou, D. N. and Megaritis, A. C. Soft Set Theory and Topology, *Appl. Gen. Topol.*, **15** (1) (2014), 93-109.

[10] Georgiou, D. N. Megaritis, A. C. and Petropoulos, V. I. On soft topological spaces, *Appl. Math. Inf. Sci.*, **7** (2013), 1889-1901.

[11] Hussain, S. and Ahmad, B. Some properties of soft topological spaces, *Comput. Math. Appl.*, **62** (2011), 4058-4067.

[12] Kannan, K. Soft generalized closed sets in soft topological spaces, *J. Theo. Appl. Inf. Techn.*, **37** (2012), 17-21.

[13] Kazanci, O. Yilmaz, S. and Yamak, S. Soft Sets and Soft BCH-Algebras, *Hacet. J. Math. Stat.*, **39** (2010), 205-217.

[14] Kharal, A. and Ahmad, B. Mappings of soft classes, *New. Math. Nat. Comput.*, **7** (3) (2011), 471-481.

[15] Maji, P. K. Roy, A. R. and Biswas, R. An application of soft sets in a decision making problem, *Comput. Math. Appl.*, **44** (2002), 1077-1083.

[16] Maji, P. K. Biswas, R. and Roy, A. R. Soft set theory, *Comput. Math. Appl.*, **45** (2003), 555-562.

[17] Maji, P. K. Biswas, R. and Roy, A. R., Fuzzy soft sets, *J. Fuzzy Math.*, **9** (2001), 589-602.

[18] Majumdar, P. and Samanta, S. K. Similarity measure of soft sets, *New. Math. Nat. Comput.*, **4** (2008), 1-12.

[19] Min, W. K. A note on soft topological spaces, *Comput. Math. Appl.*, **62** (2011), 3524-3528.

[20] Molodtsov, D. A. Soft set theory-first results, *Comput. Math. Appl.*, **37** (1999), 19-31.

[21] Molodtsov, D. A. The description of a dependence with the help of soft sets, *J. Comput. Sys. Sc. Int.*, **40** (2001), 977-984.

[22] Molodtsov, D. A. The theory of soft sets (in Russian), URSS Publishers, Moscow, (2004).

[23] Molodtsov, D. A. Leonov, V. Y. and Kovkov, D. V. Soft sets technique and its application, *Nech. Siste. Myagkie Vychisleniya*, **1** (2006), 8-39.

[24] Pei, D. and Miao, D. From soft sets to information systems, In: Hu, X. Liu, Q. Skowron, A. Lin, T. Y. Yager, R. R. Zhang, B. ,eds., *Proceedings of Granular Computing*, IEEE, **2** (2005), 617-621.

[25] Peyghan, E. Samadi, B. Tayebi, A. About soft topological spaces, *J. New Results in Sci.*, (2) (2013), 60-75.

[26] Zhu, P. and Qiaoyanwen, Operations on soft sets revisited, *J. Applied Math.*, **2013** (2013), 7 pages. http://dx.doi.org/10.1155/2013/105752.

[27] Shabir M. and Naz, M. On soft topological spaces, *Comput. Math. Appl.*, **61** (2011), 1786-1799.

[28] Shao, Y. C. Qin, K. The lattice structure of the soft groups, *Procedia Engineering*, **15** (2011), 3621-3625.

[29] Zorlutuna, I. Akdag, M. Min, W. K. and Atmaca, S. Remarks on soft topological spaces, *Ann. Fuzzy Math. Inform.*, **3** (2012), 171-185.

[30] Zou, Y. and Xiao, Z. Data analysis approaches of soft sets under incomplete information, *Knowl. Base. Syst.*, **21** (2008), 941-945.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1147 – 1156

Some results on σ-ideal of σ-prime ring

Selin Türkmen* and Neşet Aydın†

Abstract

Let R be a σ-prime ring with characteristic not 2, $Z(R)$ be the center of R, I be a nonzero σ-ideal of R, $\alpha, \beta : R \to R$ be two automorphisms, d be a nonzero (α, β)-derivation of R and h be a nonzero derivation of R. In the present paper, it is shown that (i) If $d(I) \subset C_{\alpha,\beta}$ and β commutes with σ then R is commutative. (ii) Let α and β commute with σ. If $a \in I \cap S_\sigma(R)$ and $[d(I), a]_{\alpha,\beta} \subset C_{\alpha,\beta}$ then $a \in Z(R)$. (iii) Let α, β and h commute with σ. If $dh(I) \subset C_{\alpha,\beta}$ and $h(I) \subset I$ then R is commutative.

Keywords: σ-prime ring, σ-ideal, (α, β)-derivation

2000 AMS Classification: Primary 16N60; Secondary 16W25, 16U80

Received 27/03/2014 : Accepted 28/07/2014 Doi : 10.15672/HJMS.2015449665

1. Introduction

Let R be an associative ring with center $Z(R)$. R is said to be 2-torsion free if whenever $2x = 0$ with $x \in R$, then $x = 0$. Recall that a ring R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An involution σ of a ring R is an additive mapping satisfying $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2(x) = x$ for all $x, y \in R$. A ring R equipped with an involution σ is said to be σ-prime if $aRb = aR\sigma(b) = 0$ implies $a = 0$ or $b = 0$. Note that every prime ring which has an involution σ is a σ-prime but the converse is in generally not true. An example, due to Shuliang [8], if R^0 denotes the opposite ring of a prime ring R, then $R \times R^0$ equipped with the exchange involution σ_{ex}, defined by $\sigma_{ex}(x, y) = (y, x)$, is σ_{ex}-prime but not prime. An additive subgroup I of R is said to be an ideal of R if $xr, rx \in I$ for all $x \in I$ and $r \in R$. An ideal I which satisfies $\sigma(I) = I$ is called a σ- ideal of R. An example, due to Rehman [8], Set $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. We define a map $\sigma : R \to R$ as follows: $\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$. It is easy to check that $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$ is a

*Çanakkale Onsekiz Mart University, Dept. Math. Çanakkale - Turkey
Email: selinvurkac@gmail.com
†Çanakkale Onsekiz Mart University, Dept. Math. Çanakkale - Turkey
Email: neseta@comu.edu.tr

σ-ideal of R. Note that an ideal I of a ring R may be not a σ-ideal. Let $R = \mathbb{Z} \times \mathbb{Z}$. Consider a map $\sigma : R \to R$ defined by $\sigma((a,b)) = (b,a)$ for all $(a,b) \in R$. For an ideal $I = \mathbb{Z} \times \{0\}$ of R, I is not a σ-ideal of R since $\sigma(I) = \{0\} \times \mathbb{Z} \neq I$. $S_\sigma(R)$ will denote the set of symmetric and skew symmetric elements of R. i.e. $S_\sigma(R) = \{x \in R \mid \sigma(x) = \pm x\}$. As usual the commutator $xy - yx$ will be denoted by $[x,y] = xy - yx$. An additive mapping $h : R \to R$ is called a derivation if $h(xy) = h(x)y + xh(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ is given by $I_a(x) = [a,x]$ is a derivation which is said to be an inner derivation which is determined by a. Let α and β be two maps of R. Set $C_{\alpha,\beta} = \{c \in R \mid c\alpha(r) = \beta(r)c \text{ for all } r \in R\}$ and known as (α,β)-center of R. In particular, $C_{1,1} = Z(R)$ is the center of R, where $1 : R \to R$ is identity map. As usual the (α,β)-commutator $a\alpha(b) - \beta(b)a$ will be denoted by $[a,b]_{\alpha,\beta} = a\alpha(b) - \beta(b)a$. An additive mapping $d : R \to R$ is called an (α,β)-derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ is given by $I_a(x) = [a,x]_{\alpha,\beta}$ is an (α,β)-inner derivation which is determined by a.

Many studies have been objected the relationship between commutativity of a ring and the act of derivations defined on this ring. These results have been generalized by many authors in several ways. Herstein [2] proved that if R is a prime ring of characteristic not 2, d is a nonzero derivation of R and $a \in R$ such that $[a, d(R)] = 0$ then $a \in Z(R)$. N. Aydın and K. Kaya [1] proved that if R is a prime ring of characteristic not 2, I is a nonzero right ideal of R, σ and τ are two automorphisms of R, $d : R \to R$ is a nonzero (σ,τ)-derivations of R and $a \in R$ such that (i) $d(I) \subset Z(R)$ then R is commutative. (ii) $[d(R), a]_{\sigma,\tau} \subset C_{\alpha,\beta}$ then $a \in Z(R)$. In [5], this result was extended to on a σ-ideal of a σ-prime ring by L. Oukhtite and S. Salhi. On the other hand, Posner [7] proved that if R is a prime ring of characteristic not 2 and d_1, d_2 are derivations of R such that the composition $d_1 d_2$ is also a derivation; then one at least of d_1, d_2 is zero. K. Kaya [3] proved that if R is a prime ring of characteristic not 2, I is a nonzero ideal of R, σ and τ are two automorphisms of R, $d_1 : R \to R$ is a nonzero (σ,τ)-derivations of R and d_2 is a nonzero derivation of R such that $d_1 d_2(I) \subset C_{\sigma,\tau}$ then R is commutative. In [4], Posner's result was extended to a nonzero σ-ideal of a σ-prime ring by L. Oukhtite and S. Salhi. Motivated by these results, we follow this line of investigation.

In this paper, our main goal is to extend these results on a σ-ideal of a σ-prime ring.

Throughout the present paper, R is a σ-prime ring, $Z(R)$ is the center of R and α, β are two automorphisms of R. We use the following basic commutator identities:

$$[x, yz] = y[x,z] + [x,y]z$$
$$[xy, z] = x[y,z] + [x,z]y$$
$$[xy, z]_{\alpha,\beta} = x[y,z]_{\alpha,\beta} + [x, \beta(z)]y = x[y, \alpha(z)] + [x,z]_{\alpha,\beta} y$$
$$[x, yz]_{\alpha,\beta} = \beta(y)[x,z]_{\alpha,\beta} + [x,y]_{\alpha,\beta}\alpha(z)$$
$$\left[[x,y]_{\alpha,\beta}, z\right]_{\alpha,\beta} = \left[[x,z]_{\alpha,\beta}, y\right]_{\alpha,\beta} + [x,[y,z]]_{\alpha,\beta}$$

The material in this work is a part of first author's PH. Dissertation which is supervised by Prof. Dr. Neşet Aydın.

2. Results

For the proof of our theorems, we give the following known Lemmas.

2.1. Lemma. [6, Theorem 2.2] *Let I be a nonzero σ-ideal of σ-prime ring R. If a, b in R are such that $aIb = 0 = aI\sigma(b)$ then $a = 0$ or $b = 0$.*

2.2. Lemma. [5, Lemma 4] *Let R be a σ-prime ring with characteristic not two, d be a derivation of R satisfying $d\sigma = \pm\sigma d$ and I be a nonzero σ-ideal of R. If $d^2(I) = 0$ then $d = 0$.*

2.3. Lemma. *Let I be a nonzero σ-ideal of R and $a \in R$. If $Ia = 0$ (or $aI = 0$) then $a = 0$.*

Proof. Since I is a σ-ideal, we know that $IR \subset I$. By hypothesis, we have $IRa \subset Ia = 0$. Thus, we get $IRa = 0$. Moreover, since I is invariant under σ, we have $\sigma(I)Ra = 0$. It follows that

$$IRa = \sigma(I)Ra = 0$$

Using σ-primeness of R, we get

$$a = 0$$

Similarly, using $RI \subset I$, one can show that if $aI = 0$ then $a = 0$. □

2.4. Lemma. *Let $a, b \in R$.*

i) *If $b, ab \in C_{\alpha,\beta}$ and a (or b) $\in S_\sigma(R)$ then $a \in Z(R)$ or $b = 0$.*
ii) *If $a, ab \in C_{\alpha,\beta}$ and a (or b) $\in S_\sigma(R)$ then $a = 0$ or $b \in Z(R)$.*

Proof. *i)* By the hypothesis, we have $[ab, r]_{\alpha,\beta} = 0$ for all $r \in R$. Expanding this equation by using $b \in C_{\alpha,\beta}$, holding for all $r \in R$

$$\begin{aligned} 0 = [ab, r]_{\alpha,\beta} &= a[b, r]_{\alpha,\beta} + [a, \beta(r)]b \\ &= [a, \beta(r)]b \end{aligned}$$

Since $b \in C_{\alpha,\beta}$, we get

$$[a, R]Rb = 0 \tag{2.1}$$

In the event of $a \in S_\sigma(R)$, we derive $\sigma([a, R])Rb = 0$. Using the last obtained equation together with (2.1), we yield

$$[a, R]Rb = \sigma([a, R])Rb = 0$$

Applying the σ-primeness of R, we have

$$a \in Z(R) \text{ or } b = 0$$

In case of $b \in S_\sigma(R)$, from (2.1), we get $[a, R]R\sigma(b) = 0$. Using the last obtained equation together with (2.1), we find

$$[a, R]Rb = [a, R]R\sigma(b) = 0$$

Applying the σ-primeness of R,

$$a \in Z(R) \text{ or } b = 0$$

is obtained.

ii) Since $ab \in C_{\alpha,\beta}$, we have $[ab, r]_{\alpha,\beta} = 0$ for all $r \in R$. Expanding this equation by using $a \in C_{\alpha,\beta}$, holding for all $r \in R$

$$\begin{aligned} 0 = [ab, r]_{\alpha,\beta} &= a[b, \alpha(r)] + [a, r]_{\alpha,\beta}b \\ &= a[b, \alpha(r)] \end{aligned}$$

Since $a \in C_{\alpha,\beta}$,

$$aR[b, R] = 0$$

is obtained. After here, it is similar as above. □

2.5. Lemma. *Let I be a nonzero σ-ideal of R and h be a nonzero derivation of R. If $h(I) \subset Z(R)$ then R is commutative.*

Proof. For any $x, y \in I$ and $r \in R$, using hypothesis,

$$\begin{aligned} 0 &= [r, h(xy)] = [r, h(x)y + xh(y)] \\ &= h(x)[r, y] + [r, h(x)]y + x[r, h(y)] + [r, x]h(y) \\ &= h(x)[r, y] + [r, x]h(y) \end{aligned}$$

And so,

$$h(x)[r, y] + [r, x]h(y) = 0, \ \forall x, y \in I, r \in R$$

is obtained. In the last equality, x is taken instead of r and we obtain $h(x)[x, y] = 0$ for all $x, y \in I$. Substituting y by zy where $z \in I$, it holds that

$$(2.2) \qquad h(x)I[x, y] = 0, \ \forall x, y \in I$$

It is supposed that $x \in I \cap S_\sigma(R)$. In (2.2), replacing y with $\sigma(y)$, we get $h(x)I\sigma([x, y]) = 0$ for all $y \in I$. According to Lemma 2.1, it is derived that

$$(2.3) \qquad h(x) = 0 \text{ or } x \in Z(R), \ \forall x \in I \cap S_\sigma(R)$$

Assume that $x \in I$. In this case, $x - \sigma(x) \in I \cap S_\sigma(R)$. So, from (2.3), we have $h(x - \sigma(x)) = 0$ or $x - \sigma(x) \in Z(R)$ for all $x \in I$. We set $A = \{x \in I \mid h(x - \sigma(x)) = 0\}$ and $B = \{x \in I \mid x - \sigma(x) \in Z(R)\}$. It is clear that A and B are additive subgroups of I such that $I = A \cup B$. But, a group can not be an union of two of its proper subgroups. Therefore, it is implied $I = A$ or $I = B$. In the former case, $h(x) = h(\sigma(x))$ for all $x \in I$. In (2.2), replacing y by $\sigma(y)$ and x by $\sigma(x)$, we have $h(x)I\sigma([x, y]) = 0$ for all $x, y \in I$. And so,

$$h(x)I[x, y] = h(x)I\sigma([x, y]) = 0, \ \forall x, y \in I$$

is obtained. By Lemma 2.1, get $h(x) = 0$ or $x \in Z(R)$ for all $x \in I$. In the latter case, $x - \sigma(x) \in Z(R)$ for all $x \in I$. This means $[x, r] = [\sigma(x), r]$ for all $x \in I, r \in R$. In (2.2), taking $\sigma(y)$ instead of y, we get $h(x)I\sigma([x, y]) = 0$ for all $x, y \in I$. And so,

$$h(x)I[x, y] = h(x)I\sigma([x, y]) = 0, \ \forall x, y \in I$$

is derived. According to Lemma 2.1, we have $h(x) = 0$ or $x \in Z(R)$ for all $x \in I$. So, both the cases yield either

$$h(x) = 0 \text{ or } x \in Z(R), \ \forall x \in I$$

Now, we set $K = \{x \in I \mid h(x) = 0\}$ and $L = \{x \in I \mid x \in Z(R)\}$. Each of K and L is an additive subgroup of I. Moreover, I is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $I = K$ or $I = L$. In the former case, $h(I) = 0$. So, we have $h = 0$. But, h is a nonzero derivation of R. So, from the latter case, we get $I \subseteq Z(R)$. Therefore, R is commutative. □

2.6. Lemma. *Let I be a nonzero σ-ideal of R, d be a (α, β)-derivation of R and $a \in R$. If $ad(I) = \sigma(a)d(I) = 0$ and β commutes with σ (or $d(I)a = d(I)\sigma(a) = 0$ and α commutes with σ) then $a = 0$ or $d = 0$.*

Proof. For any $x \in I$ and $r \in R$, using $ad(I) = 0$, we get

$$\begin{aligned} 0 &= ad(xr) = ad(x)\alpha(r) + a\beta(x)d(r) \\ &= a\beta(x)d(r) \end{aligned}$$

It becomes

$$a\beta(I)d(r) = 0, \ \forall r \in R$$

Similarly, using $\sigma(a)\,d(I)=0$, we derive

$$\sigma(a)\,\beta(I)\,d(r)=0,\ \forall r\in R$$

And so,

$$a\beta(I)\,d(r)=\sigma(a)\,\beta(I)\,d(r)=0,\ \forall r\in R$$

is obtained. Since β commutes with σ, $\beta(I)$ is a nonzero σ-ideal of R. Therefore, according to Lemma 2.1, we have

$$a=0 \text{ or } d=0$$

Let us consider $d(I)\,a=d(I)\,\sigma(a)=0$ and α commutes with σ. Since $\alpha(I)$ is a nonzero σ-ideal of R, one can show that $a=0$ or $d=0$ similarly as above. □

2.7. Lemma. *Let I be a nonzero σ-ideal of R and d be a (α,β)-derivation of R. If $d(I)=0$ and α (or β) commutes with σ then $d=0$.*

Proof. By hypothesis, it holds that for all $x\in I$ and $r\in R$

$$\begin{aligned}0&=d(rx)=d(r)\,\alpha(x)+\beta(r)\,d(x)\\&=d(r)\,\alpha(x)\end{aligned}$$

Thus, we get

$$d(r)\,\alpha(I)=0,\ \forall r\in R$$

Since α commutes with σ, $\alpha(I)$ is a nonzero σ-ideal of R. Therefore, by Lemma 2.3, we have $d=0$.

Suppose that β commutes with σ. For any $x\in I$ and $r\in R$, from the hypothesis, we get

$$\begin{aligned}0&=d(xr)=d(x)\,\alpha(r)+\beta(x)\,d(r)\\&=\beta(x)\,d(r)\end{aligned}$$

So, it yields that

$$\beta(I)\,d(r)=0,\ \forall r\in R$$

Since β commutes with σ, $\beta(I)$ is a nonzero σ-ideal of R. Therefore, by Lemma 2.3, we have $d=0$. □

2.8. Theorem. *Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R and d be a nonzero (α,β)-derivation of R such that β commutes with σ. If $d(I)\subset C_{\alpha,\beta}$ then R is commutative.*

Proof. By hypothesis, $d\left(x^2\right)=d(x)\,\alpha(x)+\beta(x)\,d(x)\in C_{\alpha,\beta}$ for all $x\in I$. Using $d(x)\in C_{\alpha,\beta}$, we get $2\beta(x)\,d(x)\in C_{\alpha,\beta}$. Since $char R\neq 2$, we obtain $\beta(x)\,d(x)\in C_{\alpha,\beta}$ which means $[\beta(x)\,d(x),r]_{\alpha,\beta}=0$ for all $r\in R, x\in I$. Expanding this equation by using $d(x)\in C_{\alpha,\beta}$, we arrive

$$\begin{aligned}0&=[\beta(x)\,d(x),r]_{\alpha,\beta}=\beta(x)\,[d(x),r]_{\alpha,\beta}+\beta([x,r])\,d(x)\\&=\beta([x,r])\,d(x)\end{aligned}$$

Since $d(x)\in C_{\alpha,\beta}$, it follows that

$$\beta([x,r])\,Rd(x)=0,\ \forall x\in I, r\in R \tag{2.4}$$

Assume that $x\in I\cap S_\sigma(R)$. In (2.4) taking $\sigma(r)$ instead of r and using the fact that β commutes with σ, we have $\sigma(\beta([x,r]))\,Rd(x)=0$ for all $x\in I, r\in R$. Since R is σ-prime, we derive

$$x\in Z(R) \text{ or } d(x)=0,\ \forall x\in I\cap S_\sigma(R)$$

Assume that $x \in I$. In this case, $x - \sigma(x) \in I \cap S_\sigma(R)$. Therefore, we have $x - \sigma(x) \in Z(R)$ or $d(x - \sigma(x)) = 0$ for all $x \in I$. Set $A = \{x \in I \mid d(x - \sigma(x)) = 0\}$ and $B = \{x \in I \mid x - \sigma(x) \in Z(R)\}$. It is clear that A and B are additive subgroups of I such that $I = A \cup B$. But, a group can not be an union of two of its proper subgroups. Therefore, we yield either $I = A$ or $I = B$. In the former case, $d(x) = d(\sigma(x))$ for all $x \in I$. In (2.4) substituting x by $\sigma(x)$ and r by $\sigma(r)$ and using the fact that β commutes with σ, we have $\sigma(\beta([x,r]))Rd(x) = 0$ for all $x \in I, r \in R$. Since R is σ-prime, we arrive $x \in Z(R)$ or $d(x) = 0$ for all $x \in I$. In the latter case, $x - \sigma(x) \in Z(R)$ for all $x \in I$. This means, $[x,r] = [\sigma(x), r]$ for all $r \in R$. In (2.4), replacing r by $\sigma(r)$ and using the fact that β commutes with σ, we get $\sigma(\beta([x,r]))Rd(x) = 0$ for all $x \in I, r \in R$. Since R is σ-prime, we have $x \in Z(R)$ or $d(x) = 0$ for all $x \in I$. As a result, both the cases yield either

$$x \in Z(R) \text{ or } d(x) = 0, \ \forall x \in I$$

Now, we set $K = \{x \in I \mid d(x) = 0\}$ and $L = \{x \in I \mid x \in Z(R)\}$. Each of K and L is an additive subgroup of I. Moreover, I is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two of its proper subgroups, hence $I = K$ or $I = L$. In the former case, $d(I) = 0$. Since β commutes with σ, by Lemma 2.7, we obtain $d = 0$. But, d is a nonzero (α, β)-derivation of R, then I must be contained in $Z(R)$. So, R is commutative. □

2.9. Lemma. *Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R, d be a (α, β)-derivation of R such that β commutes with σ and h be a derivation of R satisfying $h\sigma = \pm\sigma h$. If $dh(I) = 0$ and $h(I) \subset I$ then $d = 0$ or $h = 0$.*

Proof. By hypothesis, it holds that for all $x, y \in I$

$$\begin{aligned} 0 &= dh(xy) \\ &= dh(x)\alpha(y) + \beta(h(x))d(y) + d(x)\alpha(h(y)) + \beta(x)dh(y) \\ &= \beta(h(x))d(y) + d(x)\alpha(h(y)) \end{aligned}$$

And so,

$$\beta(h(x))d(y) + d(x)\alpha(h(y)) = 0, \ \forall x, y \in I$$

Since $h(I) \subset I$, we take $h(x)$ instead of x. Using the hypothesis, we get

$$\beta\left(h^2(x)\right)d(I) = 0, \ \forall x \in I$$

Moreover, replacing x by $\sigma(x)$ in the above obtained relation and using the fact that β commute with σ and $h\sigma = \pm\sigma h$, we derive

$$\sigma\left(\beta\left(h^2(x)\right)\right)d(I) = 0, \ \forall x \in I$$

And so,

$$\beta\left(h^2(x)\right)d(I) = \sigma\left(\beta\left(h^2(x)\right)\right)d(I) = 0, \ \forall x \in I$$

Since β commutes with σ, by Lemma 2.6, we yield either $h^2(I) = 0$ or $d = 0$. Since $h\sigma = \pm\sigma h$, by Lemma 2.2, we have $h = 0$ or $d = 0$. □

2.10. Lemma. *Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R, d be a nonzero (α, β)-derivation of R such that β commutes with σ. If $a \in I \cap S_\sigma(R)$ and $[d(I), a]_{\alpha,\beta} = 0$ then $a \in Z(R)$.*

Proof. For any $x, y \in I$, from the hypothesis, we have $[d([x,y]),a]_{\alpha,\beta} = 0$. Since $d([x,y]) = [d(x),y]_{\alpha,\beta} - [d(y),x]_{\alpha,\beta}$, we get

$$\left[[d(y),x]_{\alpha,\beta},a\right]_{\alpha,\beta} = \left[[d(x),y]_{\alpha,\beta},a\right]_{\alpha,\beta}, \ \forall x,y \in I$$

In the above obtained relation, applying $\left[[a,b]_{\alpha,\beta},c\right]_{\alpha,\beta} = \left[[a,c]_{\alpha,\beta},b\right]_{\alpha,\beta} + [a,[b,c]]_{\alpha,\beta}$ for all $a,b,c \in R$ and using the hypothesis, it becomes

$$\begin{aligned}\left[[d(y),x]_{\alpha,\beta},a\right]_{\alpha,\beta} &= \left[[d(x),y]_{\alpha,\beta},a\right]_{\alpha,\beta} \\ &= \left[[d(x),a]_{\alpha,\beta},y\right]_{\alpha,\beta} + [d(x),[y,a]]_{\alpha,\beta} \\ &= [d(x),[y,a]]_{\alpha,\beta}\end{aligned}$$

And so,

$$\left[[d(y),x]_{\alpha,\beta},a\right]_{\alpha,\beta} = [d(x),[y,a]]_{\alpha,\beta}, \ \forall x,y \in I$$

is obtained. In the last equation, substituting x by a and using the hypothesis, we yield

$$[d(a),[y,a]]_{\alpha,\beta} = 0, \ \forall y \in I$$

The mapping $I_{d(a)} : R \to R$ is given by $I_{d(a)}(r) = [d(a),r]_{\alpha,\beta}$ is a (α,β)-derivation which is determined by $d(a)$ and $I_a : R \to R$ is given by $I_a(r) = [r,a]$ is a derivation which is determined by a. So, we derive

$$\left(I_{d(a)}I_a\right)(I) = 0$$

Since $a \in I \cap S_\sigma(R)$, we have $I_a\sigma = \pm\sigma I_a$. Therefore, by Lemma 2.9, we have

$$d(a) \in C_{\alpha,\beta} \text{ or } a \in Z(R)$$

Assume that $a \notin Z(R)$ which means that $d(a) \in C_{\alpha,\beta}$. From the hypothesis, we get $d([x,a]) = [d(x),a]_{\alpha,\beta} - [d(a),x]_{\alpha,\beta} = 0$ for all $x \in I$. That is,

$$d([I,a]) = 0 \tag{2.5}$$

On the other hand, by hypothesis, we have $[d(xy),a]_{\alpha,\beta} = 0$ for $x,y \in I$. Expanding this equation, it becomes $d(x)\alpha([y,a]) + \beta([x,a])d(y) = 0$ for all $x,y \in I$. Taking $[x,a]$ instead of x and using (2.5), we derive $\beta([[x,a],a])d(I) = 0$ for all $x \in I$. In this equation, replacing x by $\sigma(x)$ and using the fact that β commutes with σ, we obtain $\sigma(\beta[[x,a],a])d(I) = 0$ for all $x \in I$. And so, we yield

$$\beta([[x,a],a])d(I) = \sigma(\beta([[x,a],a]))d(I) = 0, \ \forall x \in I$$

Since β commutes with σ, by Lemma 2.6, it implies that $d = 0$ or $[[x,a],a] = 0$ for all $x \in I$. That is, $d = 0$ or $I_a^2(I) = 0$. Since $a \in I \cap S_\sigma(R)$, we have $I_a\sigma = \pm\sigma I_a$. So, by Lemma 2.9, we have $d = 0$. This is a contradiction which completes the proof. □

2.11. Theorem. *Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R, d be a nonzero (α,β)-derivation of R such that α,β commute with σ. If $a \in I \cap S_\sigma(R)$ and $[d(I),a]_{\alpha,\beta} \subset C_{\alpha,\beta}$ then $a \in Z(R)$.*

Proof. By hypothesis, $\left[d(a^2),a\right]_{\alpha,\beta} \in C_{\alpha,\beta}$. Expanding this, it becomes

$$\begin{aligned}\left[d(a^2),a\right]_{\alpha,\beta} &= [d(a)\alpha(a) + \beta(a)d(a),a]_{\alpha,\beta} \\ &= d(a)\alpha[a,a] + [d(a),a]_{\alpha,\beta}\alpha(a) + \beta(a)[d(a),a]_{\alpha,\beta} \\ &\quad + \beta([a,a])d(a) \\ &= [d(a),a]_{\alpha,\beta}\alpha(a) + \beta(a)[d(a),a]_{\alpha,\beta} \in C_{\alpha,\beta}\end{aligned}$$

And so,

$$[d(a), a]_{\alpha,\beta}\, \alpha(a) + \beta(a)\, [d(a), a]_{\alpha,\beta} \in C_{\alpha,\beta}$$

is obtained. In the above obtained relation, using $[d(a), a]_{\alpha,\beta} \in C_{\alpha,\beta}$, we have $2\beta(a)\,[d(a), a]_{\alpha,\beta} \in C_{\alpha,\beta}$. Since $char R \neq 2$, we get

$$\beta(a)\,[d(a), a]_{\alpha,\beta} \in C_{\alpha,\beta} \tag{2.6}$$

Since $a \in I \cap S_\sigma(R)$, it is clear that $\beta(a) \in S_\sigma(R)$. Using the hypothesis together with (2.6), according to Lemma 2.4 (i), we yield either

$$a \in Z(R) \text{ or } [d(a), a]_{\alpha,\beta} = 0$$

Assume that $a \notin Z(R)$ which means $[d(a), a]_{\alpha,\beta} = 0$. On the other hand, by hypothesis, it holds that $[d([a,x]), a]_{\alpha,\beta} \in C_{\alpha,\beta}$. So,

$$[d([a,x]), a]_{\alpha,\beta} = \left[[d(a), x]_{\alpha,\beta}, a\right]_{\alpha,\beta} - \left[[d(x), a]_{\alpha,\beta}, a\right]_{\alpha,\beta} \in C_{\alpha,\beta}$$

is obtained. Using the hypothesis, we have

$$\left[[d(a), x]_{\alpha,\beta}, a\right]_{\alpha,\beta} \in C_{\alpha,\beta},\ \forall x \in I$$

Replacing x by ax and using $[d(a), a]_{\alpha,\beta} = 0$, it becomes

$$\beta(a)\left[[d(a), x]_{\alpha,\beta}, a\right]_{\alpha,\beta} \in C_{\alpha,\beta},\ \forall x \in I$$

We know that $\beta(a) \in S_\sigma(R)$ and $\left[[d(a), x]_{\alpha,\beta}, a\right]_{\alpha,\beta} \in C_{\alpha,\beta}$. Therefore, by Lemma 2.4 (i), we derive $\left[[d(a), x]_{\alpha,\beta}, a\right]_{\alpha,\beta} = 0$ for all $x \in I$. Applying the identity $\left[[a,b]_{\alpha,\beta}, c\right]_{\alpha,\beta} = \left[[a,c]_{\alpha,\beta}, b\right]_{\alpha,\beta} + [a,[b,c]]_{\alpha,\beta}$ for all $a, b, c \in R$ and using the assumption, we arrive

$$[d(a), [x,a]]_{\alpha,\beta} = 0,\ \forall x \in I$$

The mapping $I_{d(a)} : R \to R$ is given by $I_{d(a)}(r) = [d(a), r]_{\alpha,\beta}$ is a (α,β)-derivation which is determined by $d(a)$ and $I_a : R \to R$ is given by $I_a(r) = [r, a]$ is a derivation which is determined by a. So,

$$\left(I_{d(a)} I_a\right)(I) = 0$$

is obtained. Since $a \in I \cap S_\sigma(R)$, we have $I_a \sigma = \pm \sigma I_a$. According to Lemma 2.9, we yield either

$$I_{d(a)} = 0 \text{ or } I_a = 0$$

which means $d(a) \in C_{\alpha,\beta}$. On the other hand, by hypothesis, we have $[d(ax), a]_{\alpha,\beta} \in C_{\alpha,\beta}$ for all $x \in I$. So, we get

$$d(a)\,\alpha([x,a]) + \beta(a)\,[d(x), a]_{\alpha,\beta} \in C_{\alpha,\beta}, \forall x \in I \tag{2.7}$$

Commuting (2.7) with a, it follows that

$$\begin{aligned}
0 &= \left[d(a)\,\alpha([x,a]) + \beta(a)\,[d(x), a]_{\alpha,\beta}, a\right]_{\alpha,\beta} \\
&= [d(a)\,\alpha([x,a]), a]_{\alpha,\beta} + \left[\beta(a)\,[d(x), a]_{\alpha,\beta}, a\right]_{\alpha,\beta} \\
&= d(a)\,\alpha([[x,a], a]) + [d(a), a]_{\alpha,\beta}\, \alpha([x,a]) \\
&\quad + \beta(a)\left[[d(x), a]_{\alpha,\beta}, a\right]_{\alpha,\beta} + \beta([a,a])\,[d(x), a]_{\alpha,\beta} \\
&= d(a)\,\alpha([[x,a], a]) + \beta(a)\left[[d(x), a]_{\alpha,\beta}, a\right]_{\alpha,\beta}
\end{aligned}$$

And so, it becomes

$$d(a)\,\alpha([[x,a],a]) + \beta(a)\left[[d(x),a]_{\alpha,\beta},a\right]_{\alpha,\beta} = 0,\ \forall x \in I$$

Using $[d(x),a]_{\alpha,\beta} \in C_{\alpha,\beta}$, we have $d(a)\,\alpha([[x,a],a]) = 0$ for all $x \in I$. Since $d(a) \in C_{\alpha,\beta}$,

$$d(a)\,R\alpha([[x,a],a]) = 0,\ \forall x \in I$$

is obtained. In the above obtained relation, taking $\sigma(x)$ instead of x and using the fact that α commutes with σ, we derive

$$d(a)\,R\sigma(\alpha([[x,a],a])) = 0,\ \forall x \in I$$

And so, we yield

$$d(a)\,R\alpha([[x,a],a]) = d(a)\,R\sigma(\alpha([[x,a],a])) = 0,\ \forall x \in I$$

Since R is σ-prime, we get $d(a) = 0$ or $[[x,a],a] = 0$ for all $x \in I$. That is, $d(a) = 0$ or $I_a^2(I) = 0$. Since $I_a\sigma = \pm\sigma I_a$, by Lemma 2.9, we have $d(a) = 0$. In (2.7), using $d(a) = 0$, it becomes

$$\beta(a)\,[d(x),a]_{\alpha,\beta} \in C_{\alpha,\beta}, \forall x \in I$$

We know that $\beta(a) \in S_\sigma(R)$ and $[d(x),a]_{\alpha,\beta} \in C_{\alpha,\beta}$ from the hypothesis. Therefore, according to Lemma 2.4 (i), we have $[d(x),a]_{\alpha,\beta} = 0$ for all $x \in I$. Since $a \in I \cap S_\sigma(R)$ and β commutes with σ, by Lemma 2.10, we derive $a \in Z(R)$. This is a contradiction which completes the proof. □

2.12. Theorem. *Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R, d be a nonzero (α,β)-derivation of R such that α and β commute with σ and h be a nonzero derivation of R which commutes with σ. If $dh(I) \subset C_{\alpha,\beta}$ and $h(I) \subset I$ then R is commutative.*

Proof. For any $x, y \in I$, from the hypothesis, we have $dh([x,y]) \in C_{\alpha,\beta}$. Expanding this identity, it follows that

$$\begin{aligned} dh([x,y]) &= d([h(x),y] + [x,h(y)]) \\ &= [(dh)(x),y]_{\alpha,\beta} - [d(y),h(x)]_{\alpha,\beta} + [d(x),h(y)]_{\alpha,\beta} \\ &\quad - [(dh)(y),x]_{\alpha,\beta} \\ &= [d(x),h(y)]_{\alpha,\beta} - [d(y),h(x)]_{\alpha,\beta} \in C_{\alpha,\beta} \end{aligned}$$

And it becomes

$$[d(x),h(y)]_{\alpha,\beta} - [d(y),h(x)]_{\alpha,\beta} \in C_{\alpha,\beta},\ \forall x,y \in I$$

Since $h(I) \subset I$, we replace y by $h(y)$. So, we arrive $\left[d(x),h^2(y)\right]_{\alpha,\beta} \in C_{\alpha,\beta}$ for all $x, y \in I$. That is,

$$\left[d(I),h^2(I)\right]_{\alpha,\beta} \subset C_{\alpha,\beta}$$

Using the fact that $h(I) \subset I$ and h commutes with σ, we assure $h^2(I) \subset I \cap S_\sigma(R)$. In additional, we know that from the hypothesis α and β commute with σ. Thereby, according to Theorem 2.11, it yields $h^2(I) \subset Z(R)$. So, for all $x, y \in I$

$$\begin{aligned} h^2([x,y]) &= h([h(x),y] + [x,h(y)]) \\ &= \left[h^2(x),y\right] + 2[h(x),h(y)] + \left[x,h^2(y)\right] \\ &= 2[h(x),h(y)] \in Z(R) \end{aligned}$$

is obtained. Since $char R \neq 2$, we have $[h(x),h(y)] \in Z(R)$ for all $x, y \in I$. Thus,

$$[h(I),h(I)] \subset Z(R)$$

Using $h(I) \subset I \cap S_\sigma(R)$, by Theorem 2.11, we derive $h(I) \subset Z(R)$. According to Lemma 2.5, it implies that R is commutative. □

Acknowledgement. The first author thanks to TÜBİTAK (The Scientific and Technical Research Council of Turkey) for financial support of BİDEB 2211 National Research Fellowship.

References

[1] Aydın N., Kaya K.: Some Generalizations in Prime Rings with (σ, τ)-Derivation, *Doğa-Tr. J. Mathematics*, vol 16 (1992) $169 - 176$.

[2] Herstein I. N. : A Note on Derivation II, Canad. Math. Bull., 22, 4, (1979) $509 - 511$.

[3] Kaya K. : (σ, τ)-Türevli Asal Halkalar Üzerine, Doğa-Tr. J. Mathematics, (1988) $42 - 45$.

[4] Oukhtite L., Salhi S.: On Commutativity of σ-Prime Rings, Glasnik Matematicki, vol. 41 no. 1 (2006) $57 - 64$.

[5] Oukhtite L., Salhi S.: Derivations and Commutativity of σ-Prime Rings, Int. J. Contemp. Sci., vol. 1 no. 9 (2006) $439 - 448$.

[6] Oukhtite L., Salhi S.: σ-Prime Rings with a special kind of automorphism, Int. J. Contemp. Math. Sci., vol. 2 no. 3 (2007) $127 - 133$.

[7] Posner E. : Derivations in Prime Rings, Proc. Amer. Math. Soc., 8, (1957).

[8] Shuliang H.: Some Generalizations in Certain Classes of Rings with Involution, Bol. Soc. Paran. Mat., 29, 1, (2011) $9 - 16$.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1157 – 1162

An asymptotic criterion for third-order dynamic equations with positive and negative coefficients

Nadide Utku* , Tongxing Li† and M. Tamer Şenel‡

Abstract

We establish a criterion for the asymptotic properties of all bounded solutions to a class of third-order linear dynamic equations with positive and negative coefficients. New theorem improves and complements the related results reported in the literature. An example is provided to illustrate the main results.

Keywords: asymptotic behavior, third-order dynamic equation, linear equation, positive and negative coefficients, time scale.

Received 20/05/2014 : Accepted 19/08/2014 Doi : 10.15672/HJMS.2015449666

1. Introduction

In this paper, we deal with the asymptotic behavior of all bounded solutions to a class of third-order linear dynamic equations with positive and negative coefficients

$$\left(rx^{\Delta\Delta}\right)^{\Delta}(t) + B(t)x(\beta(t)) - C(t)x(\gamma(t)) = 0, \tag{1.1}$$

where $t_0 \in \mathbb{T}$ and $t \in [t_0, \infty)_{\mathbb{T}}$. Throughout the paper, we always assume that the following hypotheses are satisfied:

(h1) $r \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$, $B, C \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, [0,\infty))$, and

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty; \tag{1.2}$$

(h2) $\beta, \gamma \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$ are strictly increasing functions such that $\lim_{t\to\infty} \beta(t) = \lim_{t\to\infty} \gamma(t) = \infty$;

(h3) $\delta := \gamma^{-1} \circ \beta \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$ is strictly increasing with $\delta([t_0,\infty)_{\mathbb{T}}) = [\delta(t_0),\infty)_{\mathbb{T}}$ and $\delta(t) < t$, the notation γ^{-1} stands for the inverse of the function γ;

*Institute of Sciences, Erciyes University, 38039, Kayseri, Turkey
Email: nadideutku@gmail.com
†Department of Mathematics, Linyi University, Linyi, Shandong 276005, P. R. China
Email:litongx2007@163.com
‡Department of Mathematics, Faculty of Sciences, Erciyes University, 38039, Kayseri, Turkey
Email:senel@erciyes.edu.tr

(h4) $D \in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, where $D(t) := B(t) - \delta^{\Delta}(t)C(\delta(t))$.

A solution of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Hilger [11] initiated the theory of time scales (which unifies continuous and discrete analysis). Agarwal et al. [3] and Bohner and Peterson [5] summarize and organize much of the time scale calculus and advances in dynamic equations on time scales.

In recent years, there has been an increasing interest in obtaining sufficient conditions for the oscillatory and asymptotic behavior of solutions to various classes of differential and dynamic equations on time scales. We refer the reader to [1,2,4,6–10,12–27] and the references cited therein. For the study of asymptotic properties of third-order dynamic equations, Agarwal et al. [1] and Erbe et al. [8] established Hille and Nehari type criteria for third-order dynamic equations

$$(a(rx^{\Delta})^{\Delta})^{\Delta}(t) + p(t)x(\tau(t)) = 0$$

and

$$x^{\Delta^3}(t) + p(t)x(t) = 0,$$

respectively. Assuming that γ is a quotient of odd positive integers, Agarwal et al. [4], Hassan [10], and Li et al. [21] studied a third-order nonlinear delay dynamic equation

$$(a((rx^{\Delta})^{\Delta})^{\gamma})^{\Delta}(t) + f(t, x(\tau(t))) = 0.$$

Şenel [26] examined a third-order dynamic equation

$$(a(rx^{\Delta})^{\Delta})^{\Delta}(t) + p(t, x(t), x^{\Delta}(t)) + F(t, x(t)) = 0.$$

Grace et al. [9] considered a third-order neutral delay dynamic equation

$$(r(t)(x(t) - a(t)x(\tau(t)))^{\Delta\Delta})^{\Delta} + p(t)x^{\gamma}(\delta(t)) = 0.$$

So far, there are few results regarding the oscillation of dynamic equations with positive and negative coefficients. Karpuz and Öcalan [14] investigated a first-order delay dynamic equation

$$x^{\Delta}(t) + B(t)x(\beta(t)) - C(t)x(\gamma(t)) = 0.$$

Chen et al. [7] considered a second-order nonlinear dynamic equation

$$(rx^{\Delta})^{\Delta}(t) + p(t)f(x(\xi(t))) - q(t)h(x(\delta(t))) = 0.$$

Karpuz and Öcalan [16] and Karpuz et al. [17] studied the first-order neutral delay dynamic equations

$$[x(t) - A(t)x(\alpha(t))]^{\Delta} + B(t)x(\beta(t)) - C(t)x(\gamma(t)) = 0$$

and

$$[x(t) + A(t)x(\alpha(t))]^{\Delta} + B(t)F(x(\beta(t))) - C(t)G(x(\gamma(t))) = \varphi(t),$$

respectively. Karpuz et al. [19] obtained some necessary and sufficient conditions which guarantee that every solution y of a neutral differential equation

$$(y(t) - p(t)y(r(t)))^{(n)} + q(t)G(y(g(t))) - u(t)H(y(h(t))) = f(t)$$

is either oscillatory or satisfies $\lim_{t\to\infty} y(t) = 0$.

In the real world, one can predict dynamic behavior of solutions of third-order partial differential equations by using the qualitative behavior of the third-order differential equations; see, for instance, Agarwal et al. [2]. In order to develop oscillation theory of third-order dynamic equations with positive and negative coefficients, we present an asymptotic test for equation (1.1) in the next section. As usual, all functional equalities and inequalities considered in the paper are assumed to hold for all t large enough.

2. Main results

In what follows, the notation δ^{-1} stands for the inverse of the function δ and

$$z(t) := x(t) - \int_t^\infty \int_v^\infty \frac{1}{r(u)} \int_{\delta(u)}^u \frac{B(\delta^{-1}(s))}{\delta^\Delta(\delta^{-1}(s))} x(\gamma(s))\Delta s \Delta u \Delta v$$

for $t \in [t_0, \infty)_\mathbb{T}$.

2.1. Theorem. *Assume that conditions (h1)–(h4) are satisfied and let*

$$\lim_{t\to\infty} \int_t^\infty [\sigma(v) - t] F(v)\Delta v < \infty, \tag{2.1}$$

where

$$F(v) := \frac{1}{r(v)} \int_{\delta(v)}^v \frac{B(\delta^{-1}(s))}{\delta^\Delta(\delta^{-1}(s))} \Delta s.$$

Then every bounded solution x of (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists (finite).*

Proof. Without loss of generality, we may assume that x is a bounded eventually positive solution of (1.1). Then there exists a $t_1 \in [t_0, \infty)_\mathbb{T}$ such that $x(t) > 0$, $x(\beta(t)) > 0$, and $x(\gamma(t)) > 0$ for all $t \in [t_1, \infty)_\mathbb{T}$. Differentiation of z yields

$$z^\Delta(t) = x^\Delta(t) + \int_t^\infty \frac{1}{r(u)} \int_{\delta(u)}^u \frac{B(\delta^{-1}(s))}{\delta^\Delta(\delta^{-1}(s))} x(\gamma(s))\Delta s \Delta u$$

and

$$z^{\Delta\Delta}(t) = x^{\Delta\Delta}(t) - \frac{1}{r(t)} \int_{\delta(t)}^t \frac{B(\delta^{-1}(s))}{\delta^\Delta(\delta^{-1}(s))} x(\gamma(s))\Delta s.$$

Writing the latter equality in the form

$$r(t)z^{\Delta\Delta}(t) = r(t)x^{\Delta\Delta}(t) - \int_{\delta(t)}^t \frac{B(\delta^{-1}(s))}{\delta^\Delta(\delta^{-1}(s))} x(\gamma(s))\Delta s.$$

Using (1.1) and [5, Theorem 1.93], we deduce that

$$\begin{aligned}
(rz^{\Delta\Delta})^\Delta(t) &= (rx^{\Delta\Delta})^\Delta(t) - \frac{B(\delta^{-1}(t))}{\delta^\Delta(\delta^{-1}(t))} x(\gamma(t)) + B(t)x(\beta(t)) \\
&= -B(t)x(\beta(t)) + C(t)x(\gamma(t)) - \frac{B(\delta^{-1}(t))}{\delta^\Delta(\delta^{-1}(t))} x(\gamma(t)) + B(t)x(\beta(t)) \\
&= C(t)x(\gamma(t)) - \frac{B(\delta^{-1}(t))}{\delta^\Delta(\delta^{-1}(t))} x(\gamma(t)) \\
&= -\left(\frac{B(\delta^{-1}(t))}{\delta^\Delta(\delta^{-1}(t))} - C(t) \right) x(\gamma(t)).
\end{aligned}$$

Then, we obtain

$$(rz^{\Delta\Delta})^\Delta(t) = -\frac{D(\delta^{-1}(t))}{\delta^\Delta(\delta^{-1}(t))} x(\gamma(t)) < 0, \tag{2.2}$$

which implies that $rz^{\Delta\Delta}$ is decreasing, and thus the sign of $z^{\Delta\Delta}$ is fixed. Next, we assert that there exists a $t_2 \in [t_1, \infty)_\mathbb{T}$ such that $z^{\Delta\Delta}(t) > 0$ for $t \in [t_2, \infty)_\mathbb{T}$. If $z^{\Delta\Delta} < 0$, then there exist a $t_3 \in [t_1, \infty)_\mathbb{T}$ and a constant $M > 0$ such that, for $t \in [t_3, \infty)_\mathbb{T}$,

$$z^{\Delta\Delta}(t) \leq -\frac{M}{r(t)} < 0.$$

Integrating the latter inequality from t_3 to t, we obtain

$$z^{\Delta}(t) \leq z^{\Delta}(t_3) - M \int_{t_3}^{t} \frac{\Delta s}{r(s)}.$$

Letting $t \to \infty$ and using condition (1.2), we have $\lim_{t\to\infty} z^{\Delta}(t) = -\infty$. It follows from inequalities $z^{\Delta\Delta} < 0$ and $z^{\Delta} < 0$ that

$$\lim_{t\to\infty} z(t) = -\infty,$$

which contradicts the fact that z is bounded. Hence, there exists a $t_4 \in [t_1, \infty)_{\mathbb{T}}$ such that, for $t \in [t_4, \infty)_{\mathbb{T}}$,

$$z(t) > 0, \quad z^{\Delta}(t) < 0, \quad z^{\Delta\Delta}(t) > 0, \quad (rz^{\Delta\Delta})^{\Delta}(t) < 0, \tag{2.3}$$

or

$$z(t) < 0, \quad z^{\Delta}(t) < 0, \quad z^{\Delta\Delta}(t) > 0, \quad (rz^{\Delta\Delta})^{\Delta}(t) < 0. \tag{2.4}$$

Assume first that (2.3) holds. Using condition (2.1) and the definition of z, we conclude that there exists a constant $\ell \geq 0$ such that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = \ell$. Assume now that (2.4) holds. Then

$$x(t) \leq \int_{t}^{\infty} \int_{v}^{\infty} \frac{1}{r(u)} \int_{\delta(u)}^{u} \frac{B(\delta^{-1}(s))}{\delta^{\Delta}(\delta^{-1}(s))} x(\gamma(s)) \Delta s \Delta u \Delta v.$$

On the other hand, by virtue of [12, Lemma 2.1],

$$\int_{t}^{\infty} \int_{v}^{\infty} \frac{1}{r(u)} \int_{\delta(u)}^{u} \frac{B(\delta^{-1}(s))}{\delta^{\Delta}(\delta^{-1}(s))} \Delta s \Delta u \Delta v = \int_{t}^{\infty} [\sigma(v) - t] F(v) \Delta v.$$

It follows now from condition (2.1) that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. ∎

2.2. Example. For $t \geq t_0$, consider a third-order differential equation

$$x'''(t) + \frac{b}{t^4} x(t) - \frac{c}{t^5} x(2t) = 0, \tag{2.5}$$

where b and c are positive constants. It is not difficult to verify that all assumptions of Theorem 2.1 are satisfied. Hence, every bounded solution x of (2.5) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

3. Conclusions

Most oscillation and asymptotic results reported in the literature for third-order dynamic equation (1.1) and its particular cases have been obtained in the case where $C(t) = 0$. In this paper, we establish an asymptotic criterion for equation (1.1) under the assumption that $C(t) \geq 0$, which, in a certain sense, improves and complements the related results in the cited papers.

We stress that the study of asymptotic behavior of equation (1.1) in the case $C(t) \geq 0$ brings additional difficulties. The main difficulty one encounters lies in how to obtain inequality such as (2.2). Since $z^{\Delta} < 0$, it is hard to establish criteria which ensure that all bounded solutions of (1.1) are just oscillatory. The question regarding the study of sufficient conditions which guarantee that all bounded solutions of (1.1) tend to zero remains open at the moment.

It is not easy to use the technique exploited in this paper for deriving similar results for the odd-order dynamic equation

$$\left(rx^{\Delta^{n-1}}\right)^{\Delta}(t) + B(t)x(\beta(t)) - C(t)x(\gamma(t)) = 0, \tag{3.1}$$

where $n \geq 3$ is an odd natural number. Therefore, an interesting problem for future research can be formulated as follows.

(P) Is it possible to establish similar asymptotic tests for equation (3.1)?

4. Acknowledgements

The authors express their sincere gratitude to the Editor and referee for the careful reading of the original manuscript and useful comments that helped to improve the presentation of the results and accentuate important details. This research is supported by the AMEP of Linyi University, P. R. China.

References

[1] R. P. Agarwal, M. Bohner, T. Li, and C. Zhang, Hille and Nehari type criteria for third-order delay dynamic equations, J. Difference Equ. Appl. 19 (2013) 1563–1579.

[2] R. P. Agarwal, M. Bohner, T. Li, and C. Zhang, Oscillation of third-order nonlinear delay differential equations, Taiwanese J. Math. 17 (2013) 545–558.

[3] R. P. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, Dynamic equations on time scales: a survey, J. Comput. Appl. Math. 141 (2002) 1–26.

[4] R. P. Agarwal, M. Bohner, S. Tang, T. Li, and C. Zhang, Oscillation and asymptotic behavior of third-order nonlinear retarded dynamic equations, Appl. Math. Comput. 219 (2012) 3600–3609.

[5] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[6] T. Candan, Asymptotic properties of solutions of third-order nonlinear neutral dynamic equations, Adv. Difference Equ. 2014 (2014) 1–10.

[7] D. Chen, P. Qu, and Y. Lan, Oscillation of second order nonlinar dynamic equations with positive and negative coefficients, Adv. Difference Equ. 2013 (2013) 1–18.

[8] L. Erbe, A. Peterson, and S. H. Saker, Hille and Nehari type criteria for third-order dynamic equations, J. Math. Anal. Appl. 329 (2007) 112–131.

[9] S. R. Grace, J. R. Graef, and M. A. El-Beltagy, On the oscillation of third order neutral delay dynamic equations on time scales, Comput. Math. Appl. 63 (2012) 775–782.

[10] T. S. Hassan, Oscillation of third order nonlinear delay dynamic equations on time scales, Math. Comput. Modelling 49 (2009) 1573–1586.

[11] S. Hilger, Analysis on measure chains–a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.

[12] B. Karpuz, Asymptotic behaviour of bounded solutions of a class of higher-order neutral dynamic equations, Appl. Math. Comput. 215 (2009) 2174–2183.

[13] B. Karpuz, Necessary and sufficient conditions on the asymptotic behavior of second-order neutral delay dynamic equations with positive and negative coefficients, Math. Methods Appl. Sci. 37 (2014) 1219–1231.

[14] B. Karpuz and Ö. Öcalan, Oscillation and nonoscillation of first-order dynamic equations with positive and negative coefficients, Dyn. Syst. Appl. 18 (2009) 363–374.

[15] B. Karpuz and Ö. Öcalan, Necessary and sufficient conditions on asymptotic behaviour of solutions of forced neutral delay dynamic equations, Nonlinear Anal. 71 (2009) 3063–3071.

[16] B. Karpuz and Ö. Öcalan, Oscillation and nonoscillation in neutral delay dynamic equations with positive and negative coefficients, Commun. Appl. Anal. 16 (2012) 113–129.

[17] B. Karpuz, Ö. Öcalan, and R. Rath, Necessary and sufficient conditions for the oscillatory and asymptotic behaviour of solutions to neutral delay dynamic equations, Electron. J. Diff. Equ. 2009 (2009) 1–15.

[18] B. Karpuz, Ö. Öcalan, and M. K. Yıldız, Oscillation of higher-order nonlinear delay differential equations with oscillatory coefficients, Turkish J. Math. 33 (2009) 259–263.

[19] B. Karpuz, L. N. Padhy, and R. Rath, Oscillation and asymptotic behaviour of a higher order neutral differential equation with positive and negative coefficients, Electron. J. Diff. Equ. 2008 (2008) 1–15.

[20] T. Li, R. P. Agarwal, and M. Bohner, Some oscillation results for second-order neutral dynamic equations, Hacet. J. Math. Stat. 41 (2012) 715–721.
[21] T. Li, Z. Han, S. Sun, and Y. Zhao, Oscillation results for third order nonlinear delay dynamic equations on time scales, Bull. Malays. Math. Sci. Soc. 34 (2011) 639–648.
[22] T. Li, Z. Han, C. Zhang, and Y. Sun, Oscillation criteria for third-order nonlinear delay dynamic equations on time scales, Bull. Math. Anal. Appl., 3 (2011) 52–60.
[23] S. H. Saker, Oscillation Theory of Dynamic Equations on Time Scales, Lambert Academic Publishing, Germany, 2010.
[24] M. T. Şenel, Oscillation theorems for dynamic equation on time scales, Bull. Math. Anal. Appl. 3 (2011) 101–105.
[25] M. T. Şenel, Kamenev-type oscillation criteria for the second-order nonlinear dynamic equations with damping on time scales, Abstr. Appl. Anal. 2012 (2012) 1–18.
[26] M. T. Şenel, Behavior of solutions of a third-order dynamic equation on time scales, J. Inequal. Appl. 2013 (2013) 1–7.
[27] M. K. Yıldız, B. Karpuz, and Ö. Öcalan, Oscillation of nonlinear neutral delay differential equations of second-order with positive and negative coefficients, Turkish J. Math. 33 (2009) 341–350.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1163 – 1179

Hom-Leibniz superalgebras and hom-Leibniz poisson superalgebras

Chunyue Wang* , Qingcheng Zhang† and Zhu Wei‡

Abstract

This paper aims to characterize Hom-Leibniz superalgebras and Hom-Leibniz Poisson superalgebras, presents the methods to construct these superalgebras. Moreover, derivations and representations of Hom-Leibniz Poisson superalgebras are also investigated.

Keywords: Hom-Leibniz superalgebras, Hom-superdialgebras, Hom-Leibniz Poisson superalgebras, endomorphism.

2000 AMS Classification: 16E40,17B56,17B68,17B70.

Received 01/02/2014 : Accepted 24/07/2014 Doi : 10.15672/HJMS.2015449664

1. Introduction

Leibniz algebras are introduced by Cuvier and Loday [11,17], motivated by the study of algebraic K-theory. Such algebras are a non-antisymmetric version of Lie algebras. Active investigations on Leibniz algebras show that many results of Lie algebras can be extended to Leibniz algebras [1,5-7,18-19]. Leibniz superalgebras, originally were introduced by Dzhumadil'daev in [12], can be seen as a direct generalization of Leibniz algebras. Some theories of superdialgebras and (co)homology of Leibniz superalgebras are investigated [14-16].

During the past decades, there is an increasing interest in exploring some exotic algebraic structures [9-10]. In particular, Casas and Datuashoili considered algebras with brackets [8]. Such algebras are called noncommutative Leibniz Poisson algebras. On the other hand the dual algebraic operads of the classical operads provide some kinds of algebraic structures: Dialgebras, Dendriform algebras and Trialgebras [20].

Recently, Leibniz algebras are generalized to Hom-Leibniz algebras by Makhlouf and Silvestrov in [21]. Some structure theories of Hom-Leibniz algebras are developed [22].

*School of Applied Sciences, Jilin Teachers' Institute of Engineering and Technology, Changchun, China.
Email: wang1chun2yue3@163.com

†School of Mathematics and Statistics, Northeast Normal University, Changchun, China.
Email:zhangqc569@nenu.edu.cn Corresponding Author.

‡School of Mathematics and Statistics, Northeast Normal University, Changchun, China.
Email:weiz986@nenu.edu.cn

Moreover, the dialgebras are also generalized to Hom-dialgebras by Yau in [26], which give rise to Hom-Leibniz algebras. Hom-Lie algebras, Hom-Lie superalgebras and Hom-Lie color algebras have been widely investigated [13,25,3,4,2,27,23-24]. The purpose of this paper is to introduce and study Hom-Leibniz superalgebras and Hom-Leibniz Poisson superalgebras.

The paper is organized as follows. In section 2, we give the definition and some important constructions of Hom-Leibniz superalgebras. In section 3, the notion of Hom-superdialgebras is proposed, the construction of Hom-Leibniz superalgebras is provided. Moreover, we give the definition of representation of Hom-superdialgebras and show that the representation of Hom-superdialgebras gives rise to the representation of Hom-Leibniz superalgebras via a special bracket. In section 4, we introduce the notions of Hom-Leibniz Poisson superalgebras, Hom-associative supertrialgebras and Hom-dendriform superalgebras, furthermore, construct several classes of Hom-Leibniz Poisson superalgebras. Section 5 and Section 6 are devoted to dealing with the derivations and representations of Hom-Leibniz Poisson superalgebras.

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero. All vector spaces and algebras are $\mathbb{Z}_2$-graded over $\mathbb{K}$.

2. Hom-Leibniz Superalgebras

In this section, we introduce the notion of Hom-Leibniz superalgebras, and then give the construction of Hom-Leibniz superalgebras.

2.1. Definition. ([3]) *A Hom-associative superalgebra is a triple $(V, \circ, \alpha)$ consisting of a superspace V, an even bilinear map $\circ : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying*

$$\alpha(x \circ y) = \alpha(x) \circ \alpha(y), \tag{0.1}$$

$$\alpha(x) \circ (y \circ z) = (x \circ y) \circ \alpha(z), \tag{0.2}$$

for all homogeneous elements $x, y, z \in V$.

2.2. Definition. ([3]) *A Hom-Lie superalgebra is a triple $(V, [.,], \alpha)$ consisting of a superspace V, an even bilinear map $[.,.] : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying*

$$\alpha([x, y]) = [\alpha(x), \alpha(y)], \tag{0.3}$$

$$[x, y] = -(-1)^{|x||y|}[y, x], \tag{0.4}$$

$$(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|z||y|}[\alpha(z), [x, y]] + (-1)^{|x||y|}[\alpha(y), [z, x]] = 0, \tag{0.5}$$

for all homogeneous elements $x, y, z \in V$.

2.3. Definition. *A Hom-Leibniz superalgebra is a triple $(V, [.,], \alpha)$ consisting of a superspace V, an even bilinear map $[.,.] : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying*

$$\alpha([x, y]) = [\alpha(x), \alpha(y)], \tag{0.6}$$

$$[[x, y], \alpha(z)] = [\alpha(x), [y, z]] + (-1)^{|y||z|}[[x, z], \alpha(y)], \tag{0.7}$$

for all homogeneous elements $x, y, z \in V$.

Let $(V, [.,.], \alpha)$ and $(V', [.,.]', \alpha')$ be two Hom-Leibniz superalgebras. An even homomorphism $f : V \to V'$ is said to be a morphism of Hom-Leibniz superalgebras if

$$f \circ \alpha = \alpha' \circ f, \quad [f(x), f(y)]' = f([x, y]), \forall x, y \in V. \tag{0.8}$$

2.4. Remark. We recover the classical Leibniz superalgebra when α is an identity map and reduces to a Hom-Leibniz algebra when the part of parity one is trivial. Obviously, a Hom-Lie superalgebra is a Hom-Leibniz superalgebra. While a Hom-Leibniz superalgebra is a Hom-Lie superalgebra if and only if $[x,x]=0$, for all homogeneous element $x \in V$.

Suppose that $(V,[.,.],\alpha)$ is a Hom-Leibniz superalgebra. For any $x \in V$, define $Ad_y \in \mathrm{End}(V)$ by

$$Ad_y(x) = (-1)^{|x||y|}[x,y]. \tag{0.9}$$

Then the Hom-Leibniz superalgebra identity (0.7) is written into

$$Ad_{\alpha(z)}([x,y]) = (-1)^{|x||z|}[\alpha(x), Ad_z(y)] + [Ad_z(x), \alpha(y)], \tag{0.10}$$

or into pure operation form

$$Ad_{\alpha(z)}Ad_y = Ad_{Ad_z(y)} \circ \alpha + (-1)^{|y||z|}Ad_{\alpha(y)} \circ Ad_z. \tag{0.11}$$

The following proposition provides a method to construct a Hom-Leibniz superalgebra by a Leibniz superalgebra and an even endomorphism.

2.5. Proposition. *Let $(V,[.,.])$ be a Leibniz superalgebra and $\alpha : V \to V$ be an even Leibniz superalgebra endomorphism. Then $(V,[.,.]_\alpha,\alpha)$ is a Hom-Leibniz superalgebra, where $[x,y]_\alpha = \alpha([x,y])$.*

Moreover, suppose that $(V',[.,.]')$ is another Leibniz superalgebra and $\alpha' : V' \to V'$ is a Leibniz superalgebra endomorphism. If $f : V \to V'$ is a Leibniz superalgebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$, then

$$f : (V,[.,.]_\alpha,\alpha) \to (V',[.,.]'_{\alpha'},\alpha') \tag{0.12}$$

is a morphism of Hom-Leibniz superalgebras.

Proof. We show that $(V,[.,.]_\alpha,\alpha]$ satisfies the Hom-Leibniz superalgebra identity (0.7). In fact,

$$\begin{aligned}
&[\alpha(x),[y,z]_\alpha]_\alpha + (-1)^{|y||z|}[[x,z]_\alpha,\alpha(y)]_\alpha \\
&= \alpha([\alpha(x),\alpha([y,z])]) + (-1)^{|y||z|}\alpha([\alpha([x,z]),\alpha(y)]) \\
&= \alpha^2([x,[y,z]] + (-1)^{|y||z|}[[x,z],y]) \\
&= \alpha^2([[x,y],z]) \\
&= [[x,y]_\alpha,\alpha(z)]_\alpha
\end{aligned}$$

The second assertion follows from

$$\begin{aligned}
f([x,y]_\alpha) &= f([\alpha(x),\alpha(y)]) \\
&= [f\circ\alpha(x), f\circ\alpha(y)]' \\
&= [\alpha'\circ f(x), \alpha'\circ f(y)]' \\
&= [f(x),f(y)]'_{\alpha'}.
\end{aligned}$$

□

2.6. Example. (3-dimensional Hom-Leibniz superalgebras) Let $A = A_{\bar 0} \oplus A_{\bar 1}$ be a 3-dimensional superspace, where $A_{\bar 0}$ is generated by e_1 and $A_{\bar 1}$ is generated by e_2, e_3 and the nonzero product is given by $[e_2,e_1] = e_2$. For any $a, b \in \mathbb{K}$, we consider the homomorphism $\alpha : A \to A$ defined by $\alpha(e_1) = ae_1, \alpha(e_3) = be_2$. By Proposition 2.5, for any $a \in \mathbb{K}$, there is the corresponding Hom-Leibniz superalgebra $A_\alpha = (A,[.,.]_\alpha,\alpha)$ with the nonzero product $[e_2, e_1]_\alpha = ae_2$. It is not a Leibniz superalgebra when $a \neq 0, 1$.

2.7. Lemma. *Let V be a Hom-Lie superalgebra, then the bracket*

$$[x \otimes y, a \otimes b] = [[x, y], \alpha(a)] \otimes b + (-1)^{|a||x|+|a||y|} a \otimes [[x, y], \alpha(b)]$$

definies a Hom-Leibniz superalgebra structure on the vector superspace $V \otimes V$.

2.8. Definition. *A representation (module) of the Hom-Leibniz superalgebra $(V, [.,.], \alpha)$ is a Hom-supermodule (U, α_U) equipped with two even V-actions (left and right)*

$$[.,.] : U \times V \to U \quad ((u, x) \mapsto [u, x]) \quad \text{and} \quad [.,.] : V \times U \to U \quad ((x, u) \mapsto [x, u])$$

satisfying the following axioms,

(0.13) $[U_\alpha, V_\beta] \subseteq U_{\alpha+\beta}, \forall \alpha, \beta \in \mathbb{Z}_2,$

(0.14) $[V_\alpha, U_\beta] \subseteq U_{\alpha+\beta}, \forall \alpha, \beta \in \mathbb{Z}_2,$

(0.15) $\alpha_U([u, x]) = [\alpha_U(u), \alpha(x)],$

(0.16) $\alpha_U([x, u]) = [\alpha(x), \alpha_U(u)],$

(0.17) $[[u, x], \alpha(y)] = [\alpha_U(u), [x, y]] + (-1)^{|x||y|}[[u, y], \alpha(x)],$

(0.18) $[[x, u], \alpha(y)] = [\alpha(x), [u, y]] + (-1)^{|u||y|}[[x, y], \alpha_U(u)],$

(0.19) $[[x, y], \alpha_U(u)] = [\alpha(x), [y, u]] + (-1)^{|u||y|}[[x, u], \alpha(y)],$

for all homogeneous elements $x, y \in V$ and $u \in U$.

Note that the last two relations imply the following identity

$$[\alpha(x), [u, y]] + (-1)^{|u||y|}[\alpha(x), [y, u]] = 0.$$

3. Hom-Superdialgebras

In this section, we extend in one hand superdialgebras and the Hom-dialgebras introduced in [14] and [26] to Hom-superdialgebras. In the other hand we describe some constructions of Hom-Leibniz superalgebras.

3.1. Definition. ([14]) *A superdialgebra is a triple $(V, \dashv, \vdash)$ consisting of a superspace V, two even bilinear maps $\dashv, \vdash: V \times V \to V$ satisfying*

(0.20) $x \vdash (y \dashv z) = (x \vdash y) \dashv z,$

(0.21) $x \dashv (y \dashv z) = (x \dashv y) \dashv z = x \dashv (y \vdash z),$

(0.22) $x \vdash (y \vdash z) = (x \vdash y) \vdash z = (x \dashv y) \vdash z,$

for all homogeneous elements $x, y, z \in V$.

3.2. Definition. *A Hom-superdialgebra is a tuple $(V, \dashv, \vdash, \alpha)$ consisting of a superspace V, two even bilinear maps $\dashv, \vdash: V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying*

(0.23) $\alpha(x \dashv y) = \alpha(x) \dashv \alpha(y), \quad \alpha(x \vdash y) = \alpha(x) \vdash \alpha(y),$

(0.24) $\alpha(x) \dashv (y \dashv z) = (x \dashv y) \dashv \alpha(z) = \alpha(x) \dashv (y \vdash z),$

(0.25) $\alpha(x) \vdash (y \vdash z) = (x \vdash y) \vdash \alpha(z) = (x \dashv y) \vdash \alpha(z),$

(0.26) $\alpha(x) \vdash (y \dashv z) = (x \vdash y) \dashv \alpha(z),$

for all homogeneous elements $x, y, z \in V$.

3.3. Remark. We recover the classical superdialgebra [14] when α is an identity map and reduces to a Hom-dialgebra [26] when the part of parity one is trivial. Any Hom-associative superalgebra is a Hom-superdialgebra if $a \vdash b = a \dashv b = ab$.

3.4. Proposition. *If $(V_1, \circ_1, \alpha_1)$ and $(V_2, \circ_2, \alpha_2)$ are two Hom-superdialgebras, then the tensor product $V_1 \otimes V_2$ is a Hom-superdialgebra with*

$$\alpha = \alpha_1 \otimes \alpha_2,$$

and

$$(v_1 \otimes v_2) \star (u_1 \otimes u_2) = (-1)^{|v_2||u_1|}(v_1 \star u_1) \otimes (v_2 \star u_2)$$

for all homogeneous elements $v_1, v_2 \in V_1, u_1, u_2 \in V_2$ and $\star = \dashv, \ \vdash$.

3.5. Definition. *Let $(V, \dashv, \vdash, \alpha)$ and $(V', \dashv', \vdash', \alpha')$ be two Hom-superdialgebras. An even homomorphism $f : V \to V'$ is said to be a morphism of Hom-superdialgebras if $f \circ \alpha = \alpha' \circ f$, and $f(x) \dashv' f(y) = f(x \dashv y)$, and $f(x) \vdash' f(y) = f(x \vdash y)$ for any $x, y \in V$.*

3.6. Proposition. *Let $(V, \dashv, \vdash)$ be a superdialgebra and $\alpha : V \to V$ be an even superdialgebra endomorphism. Then $(V, \dashv_\alpha, \vdash_\alpha, \alpha)$ is a Hom-superdialgebra, where $x \dashv_\alpha y = \alpha(x \dashv y)$ and $x \vdash_\alpha y = \alpha(x \vdash y)$.*

Moreover, suppose that $(V', \dashv', \vdash')$ is another superdialgebra and $\alpha' : V' \to V'$ is a superdialgebra endomorphism. If $f : V \to V'$ is a superdialgebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$, then

(0.27) $\quad f : (V, \dashv_\alpha, \vdash_\alpha, \alpha) \to (V', \dashv'_{\alpha'}, \vdash'_{\alpha'}, \alpha')$

is a morphism of Hom-superdialgebras.

Proof. We only need to show that $(V, \dashv_\alpha, \vdash_\alpha, \alpha)$ satisfies the Hom-superdialgebra identity (0.24)-(0.26). Direct calculations show that

$$\begin{aligned} \alpha(x) \dashv_\alpha (y \vdash_\alpha z) &= \alpha(\alpha(x) \dashv \alpha(y \dashv z)) \\ &= \alpha^2(x \dashv (y \dashv z)) \\ &= \alpha^2((x \dashv y) \dashv z) \\ &= (x \dashv_\alpha y) \dashv_\alpha \alpha(z), \end{aligned}$$

and

$$\begin{aligned} \alpha(x) \dashv_\alpha (y \vdash_\alpha z) &= \alpha^2(x \dashv (y \dashv z)) \\ &= \alpha^2(x \dashv (y \vdash z)) \\ &= \alpha(\alpha(x) \dashv (y \vdash_\alpha z)) \\ &= \alpha(x) \dashv_\alpha (y \vdash_\alpha z), \end{aligned}$$

thus (0.24) holds. Similarly, we can prove (0.25) and (0.26).

Setting $\star_\alpha = \dashv_\alpha$ and $\star_\alpha = \vdash_\alpha$. The second assertion follows from

$$f \circ \star_\alpha = f \circ \alpha \circ \star = \alpha' \circ f \circ \star = \alpha' \circ \star' \circ f = \star_{\alpha'} \circ f.$$

□

3.7. Proposition. *Let $(V, \dashv, \vdash, \alpha)$ be a Hom-superdialgebra. Define an even bilinear map $[.,.] : V \times V \to V$ by*

(0.28) $\quad [x, y] = x \dashv y - (-1)^{|x||y|} y \vdash x, \forall x, y \in V.$

Then $(V, [.,.], \alpha)$ is a Hom-Leibniz superalgebra.

Proof. We only need to show that $(V,[.,.],\alpha)$ satisfies the Hom-Leibniz superalgebra identity (0.7). Direct calculations show that

$$\begin{aligned}
&[\alpha(x),[y,z]]+(-1)^{|y||z|}[[x,z],\alpha(y)]\\
&=\alpha(x)\dashv(y\dashv z)-(-1)^{|x||y|+|x||z|}(y\dashv z)\vdash\alpha(x)\\
&-(-1)^{|y||z|}\alpha(x)\dashv(z\vdash y)+(-1)^{|x||y|+|x||z|+|y||z|}(z\vdash y)\vdash\alpha(x)\\
&+(-1)^{|y||z|}(x\dashv z)\dashv\alpha(y)-(-1)^{|x||y|}\alpha(y)\vdash(x\dashv z)\\
&-(-1)^{|x||z|+|y||z|}(z\vdash x)\dashv\alpha(y)+(-1)^{|x||y|+|x||z|}\alpha(y)\vdash(z\vdash x)\\
&=(x\dashv y)\dashv\alpha(z)-(-1)^{|x||z|+|y||z}\alpha(z)\vdash(x\dashv y)\\
&-(-1)^{|x||y|}(y\vdash x)\dashv\alpha(z)+(-1)^{|x||y|+|x|||z|+|y||z|}\alpha(z)\vdash(y\vdash x)\\
&+(-1)^{|y||z|}\{(x\dashv z)\dashv\alpha(y)-\alpha(x)\dashv(z\vdash y)\}\\
&+(-1)^{|x||y|+|x||z|}\{\alpha(y)\vdash(z\vdash x)-(y\dashv z)\vdash\alpha(x)\}\\
&=[x\dashv y,\alpha(z)]-(-1)^{|x||y|}[y\vdash x,\alpha(z)]\\
&=[[x,y],\alpha(z)].
\end{aligned}$$

□

3.8. Proposition. *Let $(V,[.,.],\alpha_1)$ be a Hom-Leibniz superalgebra , $(U,\dashv,\vdash,\alpha_2)$ be a super commutative Hom-superdialgebra and let $g=V\otimes U$. Define the operations $\alpha:g\to g$ and $[.,.]:g^{\otimes 2}\to g$ by*

(0.29) $\alpha=\alpha_1\otimes\alpha_2,$

(0.30) $[x\otimes a,y\otimes b]=(-1)^{|a||y|}[x,y]\otimes(a\vdash b).$

Then $(g,[.,.],\alpha)$ is a Hom-Leibniz superalgebra.

Proof. We only need to show that $(g,[.,.],\alpha)$ satisfies the Hom-Leibniz superalgebra identity (0.7). Direct calculations show that

$$\begin{aligned}
&[\alpha(x\otimes a),[y\otimes b,z\otimes c]]+(-1)^{|y||z|+|y||c|+|b||z|+|b||c|}[[x\otimes a,z\otimes c],\alpha(y\otimes b)]\\
&=[\alpha_1(x)\otimes\alpha_2(a),(-1)^{|b||z|}[y,z]\otimes(b\vdash c)]\\
&+(-1)^{|y||z|+|y||c|+|b||z|+|b||c|+|a||z|}[[x,z]\otimes(a\vdash c),\alpha_1(y)\otimes\alpha_2(b)]\\
&=(-1)^{|a||y|+|a||z|+|b||z|}[\alpha_1(x),[y,z]]\otimes(\alpha_2(a)\vdash(b\vdash c))\\
&+(-1)^{|a||y|+|a||z|+|b||z|+|b||c|+|y||z|}[[x,z],\alpha_1(y)]\otimes((a\vdash c)\vdash\alpha_2(b))\\
&=(-1)^{|a||y|+|a||z|+|b||z|}[\alpha_1(x),[y,z]]\otimes(\alpha_2(a)\vdash(b\vdash c))\\
&+(-1)^{|a||y|+|a||z|+|b||z|+|y||z|}[[x,z],\alpha_1(y)]\otimes(\alpha_2(a)\vdash(b\vdash c))\\
&=(-1)^{|a||y|+|a||z|+|b||z|}\{[\alpha_1(x),[y,z]]\\
&+(-1)^{|y||z|}[[x,z],\alpha_1(y)]\}\otimes(\alpha_2(a)\vdash(b\vdash c))\\
&=(-1)^{|a||y|+|a||z|+|b||z|}[[x,y],\alpha_1(z)]\otimes(\alpha_2(a)\vdash(b\vdash c)).
\end{aligned}$$

and

$$\begin{aligned}
&[[x\otimes a,y\otimes b],\alpha(z\otimes c)]=[(-1)^{|a||y|}[x,y]\otimes(a\vdash b),\alpha_1(z)\otimes\alpha_2(c)]\\
&=(-1)^{|a||y|+|a||z|+|b||z|}[[x,y],\alpha_1(z)]\otimes((a\vdash b)\vdash\alpha_2(c))\\
&=(-1)^{|a||y|+|a||z|+|b||z|}[[x,y],\alpha_1(z)]\otimes(\alpha_2(a)\vdash(b\vdash c)).
\end{aligned}$$

This shows that $(g, [.,.], \alpha)$ is a Hom-Leibniz superalgebra. □

3.9. Definition. *Let $(V, \dashv, \vdash, \alpha)$ be a Hom-superdialgebra and (U, α_U) be a Hom-superspace. The pair (U, α_U) is said to be a V-supermodule if (U, α_U) is a Hom-supermodule equipped with four actions (left and right) of V*

$$V \otimes U \to U \quad (x \otimes u \mapsto x \dashv u \quad or \quad x \vdash u),$$
$$U \otimes V \to U \quad (u \otimes x \mapsto u \dashv x \quad or \quad u \vdash x)$$

satisfying the following axioms

$$\begin{aligned}
\alpha_U(x \dashv u) &= \alpha(x) \dashv \alpha_U(u),\\
\alpha_U(x \vdash u) &= \alpha(x) \vdash \alpha_U(u),\\
\alpha_U(u \dashv x) &= \alpha_U(u) \dashv \alpha(x),\\
\alpha_U(u \vdash x) &= \alpha_U(u) \vdash \alpha(x),\\
\alpha(x) \dashv (y \dashv u) &= (x \dashv y) \dashv \alpha_U(u) \quad = \quad \alpha(x) \dashv (y \vdash u),\\
(x \vdash y) \dashv \alpha_U(u) &= \alpha(x) \vdash (y \dashv u),\\
(x \dashv y) \vdash \alpha_U(u) &= \alpha(x) \vdash (y \vdash u) \quad = \quad (x \vdash y) \vdash \alpha_U(u),\\
\alpha(x) \dashv (u \dashv y) &= (x \dashv u) \dashv \alpha(y) \quad = \quad \alpha(x) \dashv (u \vdash y),\\
(x \vdash u) \dashv \alpha(y) &= \alpha(x) \vdash (u \dashv y),\\
(x \dashv u) \vdash \alpha(y) &= \alpha(x) \vdash (u \vdash y) \quad = \quad (x \vdash u) \vdash \alpha(y),\\
\alpha_U(u) \dashv (x \dashv y) &= (u \dashv x) \dashv \alpha(y) \quad = \quad \alpha_U(u) \dashv (x \vdash y),\\
(u \vdash x) \dashv \alpha(y) &= \alpha_U(u) \vdash (x \dashv y),\\
(u \dashv x) \vdash \alpha(y) &= \alpha_U(u) \vdash (x \vdash y) \quad = \quad (u \vdash x) \vdash \alpha(y),
\end{aligned}$$

for all $x, y \in V$ and $u \in U$.

3.10. Proposition. *Let $(V, \dashv, \vdash, \alpha)$ be a Hom-superdialgebra, $(V, [.,.], \alpha)$ be a Hom-Leibniz superalgebra, where $[x, y] = x \dashv y - (-1)^{|x||y|} y \vdash x$ for any $x, y \in V$, and (U, α_U) be a representation of $(V, \dashv, \vdash, \alpha)$. Then (U, α_U) is also a representation of $(V, [.,.], \alpha)$.*

Proof. We just check

$$[[x, y], \alpha_U(u)] = [\alpha(x), [y, u]] + (-1)^{|y||u|}[[x, u], \alpha(y)].$$

Using the axioms of the supermodule of Hom-superdialgebra, we have

$$\begin{aligned}
&[\alpha(x), [y, u]] + (-1)^{|y||u|}[[x, u], \alpha(y)]\\
&= \alpha(x) \dashv (y \dashv u) - (-1)^{|x||y|+|x||u|}((y \dashv u) \vdash \alpha(x))\\
&- (-1)^{|y||u|}\alpha(x) \dashv (u \vdash y) + (-1)^{|x||y|+|x||u|+|y||u|}(u \vdash y) \vdash \alpha(x)\\
&+ (-1)^{|y||u|}(x \dashv u) \dashv \alpha(y) - (-1)^{|x||y|}\alpha(y) \vdash (x \dashv u)\\
&- (-1)^{|x||u|+|y||u|}(u \vdash x) \dashv \alpha(y) + (-1)^{|x||y|+|x||u|}\alpha(y) \vdash (u \vdash x)\\
&= (x \dashv y) \dashv \alpha_U(u) - (-1)^{|x||u|+|y||u|}\alpha_U(u) \vdash (x \dashv y)\\
&- (-1)^{|x||y|}(y \vdash x) \vdash \alpha_U(u) + (-1)^{|x||y|+|y||u|+|x||u|}\alpha_U(u) \vdash (y \vdash x)\\
&= [[x, y], \alpha_U(u)].
\end{aligned}$$

□

4. Hom-Leibniz Poisson Superalgebras

In this section, we introduce the notions of Hom-Leibniz Poisson superalgebras, Hom-associative supertrialgebras and Hom-dendriform superalgebras. Moreover, we construct several classes of Hom-Leibniz Poisson superalgebras.

4.1. Definition. *A Hom-Poisson superalgebra is a tuple $(A, \circ, [.,.], \alpha)$ consisting of a superspace V, two even bilinear maps $\circ, [.,.] : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying the following axioms*

(1) *$(A, \circ, \alpha)$ is a Hom-associative superalgebra ,*
(2) *$(A, [.,.], \alpha)$ is a Hom-Lie superalgebra,*
(3) *the Hom-Leibniz superidentity*

$$[x \circ y, \alpha(z)] = \alpha(x) \circ [y, z] + (-1)^{|y||z|}[x, z] \circ \alpha(y)$$

holds, for all homogeneous elements $x, y, z \in A$.

4.2. Theorem. *Let $(A, \cdot, [.,.])$ be a Poisson superalgebra and $\alpha : A \to A$ be an even Poisson superalgebra endomorphism. Then $(A, \cdot_\alpha, [.,.]_\alpha, \alpha)$ is a Hom-Poisson superalgebra, where $x \cdot_\alpha y = \alpha(x \cdot y)$ and $[x, y]_\alpha = \alpha([x, y])$.*

Proof. It is straightforward. □

This theorem provides a method to construct Hom-Poisson superalgebra by a Poisson superalgebra and an even Poisson superalgebra endomorphism.

4.3. Example. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a 2-dimensional superspace, where $A_{\bar{0}}$ is generated by e_1 and $A_{\bar{1}}$ is generated by e_2 and nonzero products are given by

$$e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_2 = e_1, \quad e_1 \cdot e_2 = e_2 \cdot e_1 = e_2, \quad [e_2, e_2] = 2e_1.$$

For any $a \in \mathbb{K}$, we consider the homomorphism $\alpha : A \to A$ defined by

$$\alpha(e_1) = ae_1, \quad \alpha(e_2) = ae_2.$$

By Theorem 4.2, for any $a \in \mathbb{K}$, there is the corresponding Hom-Poisson superalgebra $A_a = (A, \cdot_\alpha, [.,.]_\alpha, \alpha)$ with the nonzero products

$$e_1 \cdot_\alpha e_1 = ae_1, \quad e_2 \cdot_\alpha e_2 = ae_1, \quad e_1 \cdot_\alpha e_\alpha = ae_2, \quad [e_2, e_2]_\alpha = 2ae_1.$$

It is not a Poisson superalgebra when $a \neq 0, 1$.

4.4. Example. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a 3-dimensional superspace, where $A_{\bar{0}}$ is generated by e_1, e_2 and $A_{\bar{1}}$ is generated by e_3 and the nonzero products are given by

$$e_1 \cdot e_2 = e_1, \quad e_2 \cdot e_2 = e_2, \quad e_3 \cdot e_2 = e_3, \quad [e_1, e_2] = ae_1.$$

For any $a \in \mathbb{K}$, we consider the homomorphism $\alpha : A \to A$ defined by

$$\alpha(e_1) = ae_1, \quad \alpha(e_2) = e_1 + e_2.$$

By Theorem 4.2, for any $a \in \mathbb{K}$, there is the corresponding Hom-Poisson superalgebra $A_\alpha = (A, \cdot_\alpha, [.,.]_\alpha, \alpha)$ with the nonzero products

$$e_1 \cdot_\alpha e_2 = ae_1, \quad e_2 \cdot_\alpha e_2 = e_1 + e_2, \quad [e_1, e_2]_\alpha = ae_1.$$

It is not a Poisson superalgebra when $a \neq 0, 1$.

4.5. Definition. *A Hom-Leibniz Poisson superalgebra is a tuple $(V, \circ, [.,.], \alpha)$ consisting of a superspace V, two even bilinear maps $\circ, [.,.] : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying the following axioms*

(1) *$(V, \circ, \alpha)$ is a Hom-associative superalgebra,*
(2) *$(V, [.,.], \alpha)$ is a Hom-Leibniz superalgebra,*

(3) the Hom-Leibniz superidentity

$$[x \circ y, \alpha(z)] = \alpha(x) \circ [y, z] + (-1)^{|y||z|}[x, z] \circ \alpha(y)$$

holds, for all homogeneous elements $x, y, z \in V$.

4.6. Definition. *Let $(V, \circ, [.,.], \alpha)$ and $(V', \circ', [.,.]', \alpha')$ be two Hom-Leibniz Poisson superalgebras. An even homomorphism $f : V \to V'$ is said to be a morphism of Hom-Leibniz Poisson superalgebras if*

(0.31) $\quad f \circ \alpha = \alpha' \circ f$

(0.32) $\quad f(x) \circ' f(y) = f(x \circ y), \quad [f(x), f(y)]' = f([x, y]), \quad \forall x, y \in V.$

4.7. Remark. Any Hom-Poisson superalgebra is a Hom-Leibniz Poisson superalgebra. Any Hom-Leibniz Poisson superalgebra $(V, \circ, [.,.], \alpha)$ is a Hom-Poisson superalgebra if and only if $[x, y] + (-1)^{|x||y|}[y, x] = 0$ holds, for all homogeneous elements $x, y \in V$. If $\alpha = Id$, then a Hom-Leibniz Poisson superalgebra becomes a Leibniz-Poisson superalgebra. On the other hand, any Hom-associative superalgebra is a Hom-Leibniz Poisson superalgebra with usual bracket $[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$.

4.8. Proposition. *Let $(V, \dashv, \vdash, \alpha)$ be a Hom-superdialgebra and $\circ, [.,.] : V \times V \to V$ be two binary operations on V defined by*

$$x \circ y = x \vdash y, \quad [x, y] = x \dashv y - (-1)^{|x||y|} y \vdash x, \forall x, y \in V.$$

Then $(V, \circ, [.,.], \alpha)$ is a Hom-Leibniz Poisson superalgebra.

Proof. It is obvious that $(V, \circ, \alpha)$ is a Hom-associative superalgebra. Moreover, from Proposition 3.7, it follows that $(V, [.,.], \alpha)$ is a Hom-Leibniz superalgebra. Next we show the remaining Hom-Leibniz superidentity. In fact

$$\begin{aligned}
&\alpha(x) \circ [y, z] + (-1)^{|y||z|}[x, z] \circ \alpha(y) \\
&= \alpha(x) \vdash (y \dashv z) - (-1)^{|y||z|} \alpha(x) \vdash (z \vdash y) \\
&+ (-1)^{|y||z|}(x \dashv z) \vdash \alpha(y) - (-1)^{|x||z|+|y||z|}(z \vdash x) \vdash \alpha(y) \\
&= (x \vdash y) \dashv \alpha(z) - (-1)^{|x||z|+|y||z|} \alpha(z) \vdash (x \vdash y) \\
&= [x \circ y, \alpha(z)].
\end{aligned}$$

□

Taking $\alpha = Id$ in Proposition 4.8, we obtain the following result about Leibniz-Poisson superalgebras.

4.9. Corollary. *Let $(V, \dashv, \vdash)$ be a superdialgebra and $\circ, [.,.] : V \times V \to V$ be two binary operations on V defined by*

$$x \circ y = x \vdash y, \quad [x, y] = x \dashv y - (-1)^{|x||y|} y \vdash x, \forall x, y \in V.$$

Then $(V, \circ, [.,.])$ is a Leibniz-Poisson superalgebra.

4.10. Proposition. *Let $(V, \circ, [.,.])$ be a Leibniz-Poisson superalgebra and $\alpha : V \to V$ be an even Leibniz-Poisson superalgebras endomorphism. Then $(V, \circ_\alpha, [.,.]_\alpha, \alpha)$ is a Hom-Leibniz Poisson superalgebra, where $x \circ_\alpha y = \alpha(x \circ y)$ and $[x, y]_\alpha = \alpha([x, y])$.*

Moreover, suppose that $(V', \circ', [.,.]')$ is another Leibniz superalgebra and $\alpha' : V' \to V'$ is a Leibniz superalgebras endomorphism. If $f : V \to V'$ is a Leibniz superalgebra *morphism that satisfies* $f \circ \alpha = \alpha' \circ f$*, then*

$$f : (V, \circ_\alpha, [.,.]_\alpha, \alpha) \to (V', \circ_{\alpha'}, [.,.]_{\alpha'}, \alpha')$$

is a morphism of Hom-Leibniz superalgebras.

Proof. It is obvious that $(V, \circ_\alpha, \alpha)$ is a Hom-associative superalgebra. Moreover, from Proposition 2.5, we have $(V, [.,.]_\alpha, \alpha)$ is a Hom-Leibniz superalgebra. Next we will show that the Hom-Leibniz superidentity holds. In fact

$$\begin{aligned}
&\alpha(x) \circ_\alpha [y,z]_\alpha + (-1)^{|y||z|}[x,z]_\alpha \circ_\alpha \alpha(y) \\
&= \alpha(\alpha(x) \circ \alpha([y,z])) + (-1)^{|y||z|}\alpha(\alpha([x,z]) \circ \alpha(y)) \\
&= \alpha^2(x \circ [y,z] + (-1)^{|y||z|}[x,z] \circ y) \\
&= \alpha^2[x \circ y, z] \\
&= \alpha([x \circ_\alpha y, \alpha(z)]) \\
&= [x \circ_\alpha y, \alpha(z)]_\alpha .
\end{aligned}$$

By Proposition 2.5, the second assertion is straightforward. □

4.11. Definition. *An Hom-associative supertrialgebra is a quintuple* $(V, \dashv, \vdash, \perp, \alpha)$ *consisting of a superspace* V*, three even bilinear maps* $\dashv, \vdash, \perp : V \times V \to V$ *and an even superspace homomorphism* $\alpha : V \to V$ *satisfying the following axioms*

$$\begin{array}{rclrcl}
\alpha(x \dashv y) &=& \alpha(x) \dashv \alpha(y), & \alpha(x \vdash y) &=& \alpha(x) \vdash \alpha(y), \\
\alpha(x \perp y) &=& \alpha(x) \perp \alpha(y), & (x \dashv y) \dashv \alpha(z) &=& \alpha(x) \dashv (y \dashv z), \\
(x \dashv y) \dashv \alpha(z) &=& \alpha(x) \dashv (y \vdash z), & (x \vdash y) \dashv \alpha(z) &=& \alpha(x) \vdash (y \dashv z), \\
(x \dashv y) \vdash \alpha(z) &=& \alpha(x) \vdash (y \vdash z), & (x \vdash y) \vdash \alpha(z) &=& \alpha(x) \vdash (y \vdash z), \\
(x \dashv y) \dashv \alpha(z) &=& \alpha(x) \dashv (y \perp z), & (x \perp y) \dashv \alpha(z) &=& \alpha(x) \perp (y \dashv z), \\
(x \dashv y) \perp \alpha(z) &=& \alpha(x) \perp (y \vdash z), & (x \vdash y) \perp \alpha(z) &=& \alpha(x) \vdash (y \perp z), \\
(x \perp y) \vdash \alpha(z) &=& \alpha(x) \vdash (y \vdash z), & (x \perp y) \perp \alpha(z) &=& \alpha(x) \perp (y \perp z).
\end{array}$$

4.12. Remark. We recover the classical associative trialgebra when $\alpha = Id$ and the part of parity one is trivial in [14,20]. The associative supertrialgebra is obtained when $\alpha = Id$. Any Hom-associative supertrialgebra gives rise to a Hom-associative superdialgebra by forgetting the operation $\perp$.

4.13. Proposition. *Let* $(V, \dashv, \vdash, \perp, \alpha)$ *be a Hom-associative supertrialgebra and* $\circ, [.,.] : V \times V \to V$ *be two binary operations on* V *defined by*

$$x \circ y = x \perp y, \qquad [x,y] = x \dashv y - (-1)^{|x||y|} y \vdash x, \forall x, y \in V.$$

Then $(V, \circ, [.,.], \alpha)$ *is a Hom-Leibniz Poisson superalgebra.*

Proof. It is obvious that $(V, \circ, \alpha)$ is a Hom-associative superalgebra. Moreover, from Proposition 3.7, we have $(V, [.,.], \alpha)$ is a Hom-Leibniz superalgebra. Next we will show that the remaining Hom-Leibniz superidentity holds. In fact,

$$\begin{aligned}[][\alpha(x), y\circ z] + (-1)^{|y||z|}[x\circ z, \alpha(y)] &= \alpha(x)\perp(y\dashv z)\\ &- (-1)^{|y||z|}\alpha(x)\perp(z\vdash y)\\ &+ (-1)^{|y||z|}(x\dashv z)\perp\alpha(y)\\ &- (-1)^{|x||z|+|y||z|}(z\vdash x)\perp\alpha(y)\\ &= (x\perp y)\dashv\alpha(z) - (-1)^{|x||z|+|y||z|}\alpha(z)\vdash(x\perp y)\\ &= [x\circ y, \alpha(z)].\end{aligned}$$

□

4.14. Definition. [24] *A Hom-dendriform superalgebra is a tuple $(V, \prec, \succ, \alpha)$ consisting of a superspace V, two even bilinear maps $\prec, \succ: V\times V\to V$ and an even superspace homomorphism $\alpha : V\to V$ satisfying the following axioms*

$$\begin{aligned}\alpha(x<y) &= \alpha(x)<\alpha(y),\\ \alpha(x>y) &= \alpha(x)>\alpha(y),\\ (x\prec y)\prec\alpha(z) &= \alpha(x)\prec(y\prec z)+\alpha(x)\prec(y\succ z),\\ (x\succ y)\prec\alpha(z) &= \alpha(x)\succ(y\prec z),\\ (x\prec y)\succ\alpha(z)+(x\succ y)\succ\alpha(z) &= \alpha(x)\succ(y\succ z),\end{aligned}$$

for all homogeneous elements $x, y, z\in V$.

4.15. Lemma. *Let $(V, \prec, \succ, \alpha)$ be a Hom-dendriform superalgebra, define the product on homogeneous elements by*

$$x * y = x\prec y + x\succ y.$$

Then $(V, *, \alpha)$ is a Hom-associative superalgebra.

4.16. Proposition. *Let $(V, \prec, \succ, \alpha)$ be a Hom-dendriform superalgebra. Define the products on homogeneous elements by*

$$x * y = x\prec y + x\succ y, \qquad [x, y] = x * y - (-1)^{|x||y|}y * x.$$

Then $(V, *, [.,.], \alpha)$ is a Hom-Leibniz Poisson superalgebra.

Proof. It is straightforward. □

5. Derivation of Hom-Leibniz Poisson Superalgebras

In this section, we extend the α-derivations of Hom-Lie algebras introduced in [25] to Hom-Leibniz Poisson superalgebras.

Let $(V, \circ, [.,.], \alpha)$ be a Hom-Leibniz Poisson superalgebra, denote by α^k the k-times composition of α, i.e., $\alpha^k = \alpha\circ\alpha\cdot\dots\cdot\circ\alpha$ (k-times). In particular, $\alpha^{-1} = 0, \alpha^0 = Id$, and $\alpha^1 = \alpha$.

5.1. Definition. *For any $k\geq -1$, we call $D\in(\mathrm{EndV})_i$, where $i\in\mathbb{Z}_2$, an α^k-derivation of the Hom-Leibniz Poisson superalgebra $(V, \circ, [.,.], \alpha)$ if*

(0.33) $\alpha\circ D = D\circ\alpha,$

(0.34) $D([x, y]) = [D(x), \alpha^k(y)] + (-1)^{|x||D|}[\alpha^k(x), D(y)],$

(0.35) $D(x\circ y) = D(x)\circ\alpha^k(y) + (-1)^{|x||D|}\alpha^k(x)\circ D(y),$

for all homogeneous elements $x, y \in V$.

We denote by $\mathrm{Der}_{\alpha^k}(V) = \mathrm{Der}_{\alpha^k}(V)_{\bar{0}} \oplus \mathrm{Der}_{\alpha^k}(V)_{\bar{1}}$ the set of α^k-derivations of the Hom-Leibniz Poisson superalgebra $(V, \circ, [.,.], \alpha)$, and $\mathrm{Der}(V) = \oplus_{k\geq -1}\mathrm{Der}_{\alpha^k}(V)$.

For any homogeneous elements $a \in V$, satisfying $\alpha(a) = a$, define $ad_k(a) \in \mathrm{End}(V)$ by

$$ad_k(a)(x) = -(-1)^{|a||x|}[\alpha^k(x), a], \forall x \in V.$$

Notice that $|ad_k(a)| = |a|$.

5.2. Proposition. *Let* $(V, \circ, [.,.], \alpha)$ *be a Hom-Leibniz Poisson superalgebra. Then* $ad_k(a)$ *is an* α^{k+1}*-derivation, which is said to be an inner* α^{k+1}*-derivation.*

Proof. Direct calculations show that

$$\begin{aligned} ad_k(a) \circ \alpha(x) &= -(-1)^{|a||x|}[\alpha^{k+1}(x), a] \\ &= -(-1)^{|a||x|}[\alpha^{k+1}(x), \alpha(a)] \\ &= -(-1)^{|a||x|}\alpha([\alpha^k(x), a] \\ &= \alpha \circ ad_k(a)(x), \end{aligned}$$

and

$$\begin{aligned} ad_k(a)([x,y]) &= -(-1)^{|a||x|+|a||y|}[\alpha^k([x,y]), a] \\ &= -(-1)^{|a||x|+|a||y|}[[\alpha^k(x), \alpha^k(y)], \alpha(a)] \\ &= -(-1)^{|a||x|+|a||y|}[\alpha^{k+1}(x), [\alpha^k(y), a]] - (-1)^{|a||x|}[[\alpha^k(x), a], \alpha^{k+1}(y)] \\ &= (-1)^{|a||x|}[\alpha^{k+1}(x), ad_k(a)(y)] + [ad_k(a)(x), \alpha^{k+1}(y)], \end{aligned}$$

and

$$\begin{aligned} ad_k(a)(x \circ y) &= -(-1)^{|a||x|+|a||y|}[\alpha^k(x \circ y), a] \\ &= -(-1)^{|a||x|+|a||y|}[\alpha^k(x) \circ \alpha^k(y), \alpha(a)] \\ &= -(-1)^{|a||x|+|a||y|}\alpha^{k+1}(x) \circ [\alpha^k(y), a]] - (-1)^{|a||x|}[\alpha^k(x), a] \circ \alpha^{k+1}(y) \\ &= (-1)^{|a||x|}\alpha^{k+1}(x) \circ ad_k(a)(y) + ad_k(a)(x) \circ \alpha^{k+1}(y). \end{aligned}$$

Therefore, $ad_k(a)$ is an α^{k+1}-derivation. □

We denote by $\mathrm{Inn}_{\alpha^k}(V)$ the set of inner α^k-derivations, i.e.,

$$\mathrm{Inn}_{\alpha^k}(V) = \{ad_k(a) | a \in V_{\bar{0}} \cup V_{\bar{1}}, \alpha(a) = a\}.$$

For any $D \in \mathrm{Der}(V)$ and $D' \in \mathrm{Der}(V)$, define their commutator $[D, D']$ as usual:

$$[D, D'] = D \circ D' - (-1)^{|D||D'|}D' \circ D.$$

5.3. Lemma. *For any* $D \in (Der_{\alpha^k}(V))_i$ *and* $D' \in (Der_{\alpha^k}(V))_j$*, then* $[D, D'] \in Der_{\alpha^{k+s}}(V)_{|D|+|D'|}$*, where* $k + s \geq -1$ *and* $(i, j) \in \mathbb{Z}_2^2$*.*

Proof. For any $x, y \in V$, we have

$$\begin{aligned}
[D,D']([x,y]) &= D\circ D'([x,y]) - (-1)^{|D||D'|}D'\circ D([x,y])\\
&= D([D'(x),\alpha^s(y)] + (-1)^{|D'||x|}[\alpha^s(x),D'(y)])\\
&- (-1)^{|D||D'|}D'([D(x),\alpha^k(y)] + (-1)^{|D||x|}[\alpha^k(x),D(y)])\\
&= [DD'(x),\alpha^{k+s}(y)] + (-1)^{|D||D'|+|D||x|}[\alpha^k(D'(x)),D\alpha^s(y)]\\
&+ (-1)^{|D'||x|}[D\alpha^s(x),\alpha^k D'(y)] + (-1)^{|D'||x|+|D||x|}[\alpha^{k+s}(x),DD'(y)]\\
&- (-1)^{|D||D'|}[D'D(x),\alpha^{k+s}(y)] - (-1)^{|D'||x|}[\alpha^s D(x),D'\alpha^k(y)]\\
&- (-1)^{|D||D'|+|D||x|}[D'\alpha^k(x),\alpha^s D(y)]\\
&- (-1)^{|D||D'|+|D||x|+|D'||x|}[\alpha^{k+s}(x),D'D(y)].
\end{aligned}$$

Since D and D' satisfy $D\circ\alpha = \alpha\circ D$ and $D'\circ\alpha = \alpha\circ D'$, we obtain

$$\begin{aligned}
[D,D']([x,y]) &= [DD'(x) - (-1)^{|D||D'|}D'D(x),\alpha^{k+s}(y)]\\
&+ (-1)^{|D||x|+|D'||x|}[\alpha^{k+s}(x),DD'(y) - (-1)^{|D||D'|}D'D(y)]\\
&= [[D,D'](x),\alpha^{k+s}(y)] + (-1)^{|[D,D']||x|}[\alpha^{k+s}(x),[D,D'](y)].
\end{aligned}$$

It is not difficult to show that

$$\begin{aligned}
[D,D'](x\circ y) &= D\circ D'(x\circ y) - (-1)^{|D||D'|}D'\circ D(x\circ y)\\
&= D(D'(x)\circ\alpha^s(y) + (-1)^{|D'||x|}\alpha^s(x)\circ D'(y))\\
&- (-1)^{|D||D'|}D'(D(x)\circ\alpha^k(y) + (-1)^{|D||x|}\alpha^k(x)\circ D(y))\\
&= DD'(x)\circ\alpha^{k+s}(y) + (-1)^{|D||D'|+|D||x|}\alpha^k D'(x)\circ D\alpha^s(y)\\
&+ (-1)^{|D'||x|}D\alpha^s(x)\circ\alpha^k D'(y) + (-1)^{|D'||x|+|D||x|}\alpha^{k+s}(x)\circ DD'(y)\\
&- (-1)^{|D||D'|}D'D(x)\circ\alpha^{k+s}(y) - (-1)^{|D'||x|}\alpha^s D(x)\circ D'\alpha^k(y)\\
&- (-1)^{|D||D'|+|D||x|}D'\alpha^k(x)\circ\alpha^s D(y)\\
&- (-1)^{|D||D'|+|D||x|+|D'||x|}\alpha^{k+s}(x)\circ D'D(y)\\
&= (DD' - (-1)^{|D||D'|}D'D)(x)\circ\alpha^{k+s}(y)\\
&+ (-1)^{|[D,D']||x|}(DD' - (-1)^{|D||D'|}D'D)(y)\\
&= [D,D'](x)\circ\alpha^{k+s}(y) + (-1)^{|[D,D']||x|}\alpha^{k+s}(x)\circ[D,D'](y).
\end{aligned}$$

It is easy to verify that $\alpha\circ[D,D'] = [D,D']\circ\alpha$, which leads to $[D,D'] \in Der_{\alpha^{k+s}}(V)_{|D|+|D'|}$.

5.4. Remark. Obviously, we have

$$\mathrm{Der}_{\alpha^{-1}}(V) = \{D \in \mathrm{End}(V) | D\circ\alpha = \alpha\circ D, D([x,y]) = 0, D(x\circ y) = 0, \forall x,y \in V\}.$$

Thus for any $D, D' \in \mathrm{Der}_{\alpha^{-1}}(V)$, we have $[D,D'] \in \mathrm{Der}_{\alpha^{-1}}(V)$.

5.5. Proposition. *With the above notations,* $\mathrm{Der}(V)$ *is a Hom-Leibniz Poisson superalgebra, in which the bracket is given by* $[D,D'] = DD' - (-1)^{|D||D'|}D'D$ *and an even endomorphism* α' *is defined by* $\alpha'(D) = \alpha\circ D$.

6. Representations of Hom-Leibniz Poisson Superalgebras

Let $(V,\circ,[.,.],\alpha)$ be a Hom-Leibniz Poisson superalgebra, then $(V,\circ,\alpha)$ is a Hom-associative superalgebra and $(V,[.,.],\alpha)$ a Hom-Leibniz superalgebra, so we can study $V-V$-bimodules, and the representation of Hom-Leibniz superalgebras over V.

6.1. Definition. *Let $(V,\circ,[.,.],\alpha)$ be a Hom-Leibniz Poisson superalgebra. A $V-V$-bimodule (M,α_M) is two $\mathbb{K}$-module homomorphisms*

$$[.,.]:V\otimes M\to M,[.,.]:M\otimes V\to M$$

such that the following axioms hold:

$$[V_\alpha,M_\beta]\subseteq M_{\alpha+\beta},\qquad [M_\alpha,V_\beta]\subseteq M_{\alpha+\beta},\qquad \forall\alpha,\beta\in\mathbb{Z}_2,$$

$$\begin{aligned}
\alpha_M([v,m]) &= [\alpha(v),\alpha_M(m)],\\
\alpha_M([m,v]) &= [\alpha_M(m),\alpha(v)],\\
[[v_1,v_2],\alpha_M(m)] &= [\alpha(v_1),[v_2,m]]+(-1)^{|v_2||m|}[[v_1,m],\alpha(v_2)],\\
[[v_1,m],\alpha(v_2)] &= [\alpha(v_1),[m,v_2]]+(-1)^{|v_2||m|}[[v_1,v_2],\alpha_M(m)],\\
[[m,v_1],\alpha(v_2)] &= [\alpha_M(m),[v_1,v_2]]+(-1)^{|v_1||v_2|}[[m,v_2],\alpha(v_1)],\\
[v_1\circ m,\alpha(v_2)] &= \alpha(v_1)\circ[m,v_2]+(-1)^{|m||v_2|}[v_1,v_2]\circ\alpha_M(m),\\
[m\circ v_1,\alpha(v_2)] &= \alpha_M(m)\circ[v_1,v_2]+(-1)^{|v_1||v_2|}[m,v_2]\circ\alpha(v_1),\\
[v_1\circ v_2,\alpha_M(m)] &= \alpha(v_1)\circ[v_2,m]+(-1)^{|v_2||m|}[v_1,m]\circ\alpha(v_2),
\end{aligned}$$

for all homogeneous elements $m\in M,v_1,v_2\in V$.

A representation over V is defined by a $V-V$-bimodule (M,α_M).

6.2. Proposition. *Let $(V_1,\circ_1,[.,.]_1,\alpha_1)$ and $(V_2,\circ_2,[.,.]_2,\alpha_2)$ be Hom-Leibniz Poisson superalgebras and $\varphi:V_1\to V_2$ be a morphism of Hom-Leibniz Poisson superalgebras, then V_2 is a representation over V_1 with respect to the operations*

$$v_1\cdot m=\varphi(v_1)\cdot m,\qquad m\cdot v_1=m\cdot\varphi(v_1),$$

$$[v_1,m]=[\varphi(v_1),m],\quad [m,v_1]=[m,\varphi(v_1)],\quad \forall v_1\in V_1,\quad m\in V_2.$$

Proof. For any $v_1,v_2\in V_1,m\in V_2$, We just check

$$[[v_1,v_2],\alpha_2(m)]=[\alpha_1(v_1),[v_2,m]]+(-1)^{|v_2||m|}[[v_1,m],\alpha_1(v_2)]$$

and

$$[v_1\cdot v_2,\alpha_2(m)]=\alpha_1(v_1)\cdot[v_2,m]+(-1)^{|v_2||m|}[v_1,m]\cdot\alpha_1(v_2).$$

By the definition of the operations, we have

$$\begin{aligned}
[[v_1,v_2],\alpha_2(m)] &= [\varphi([v_1,v_2]),\alpha_2(m)]\\
&= [[\varphi(v_1),\varphi(v_2)],\alpha_2(m)]\\
&= [\alpha_2\varphi(v_1),[\varphi(v_2),m]]+(-1)^{|v_2||m|}[[\varphi(v_1),m],\alpha_2\varphi(v_2)]\\
&= [\varphi\alpha_1(v_1),[\varphi(v_2),m]]+(-1)^{|v_2||m|}[[\varphi(v_1),m],\varphi\alpha_1(v_2)]\\
&= [\alpha_1(v_1),[v_2,m]]+(-1)^{|v_2||m|}[[v_1,m],\alpha_1(v_2)].
\end{aligned}$$

and

$$\begin{aligned}[v_1 \cdot v_2, \alpha_2(m)] &= [\varphi(v_1 \cdot v_2), \alpha_2(m)] \\ &= [\varphi(v_1) \cdot \varphi(v_2), \alpha_2(m)] \\ &= \alpha_2\varphi(v_1) \cdot [\varphi(v_2), m] + (-1)^{|v_2||m|}[\varphi(v_1), m] \cdot \alpha_2\varphi(v_2) \\ &= \varphi\alpha_1(v_1) \cdot [v_2, m] + (-1)^{|v_2||m|}[v_1, m] \cdot \varphi\alpha_1(v_2) \\ &= \alpha_1(v_1) \cdot [v_2, m] + (-1)^{|v_2||m|}[v_1, m] \cdot \alpha_1(v_2).\end{aligned}$$

□

6.3. Proposition. *Let $(V, \circ, [.,.], \alpha)$ be a Hom-Leibniz Poisson superalgebra, then $(\mathrm{End}(V), \alpha')$ can be endowed with a representation over V by means of the operations*

$$\alpha'(f) = \alpha \circ f, \quad (v \cdot f)(a) = v \cdot f(a), \quad (f \cdot v)(a) = (-1)^{|f||a|} f(a) \cdot v,$$

$$[v, f](a) = [v, f(a)], \quad [f, v](a) = (-1)^{|v||a|}[f(a), v],$$

for any $a, v \in V$, $f \in \mathrm{End}(V)$.

Proof. For any $v_1, v_2 \in V, f \in \mathrm{End}(V)$, We just check

$$[[v_1, v_2], \alpha'(f)] = [\alpha(v_1), [v_2, f]] + (-1)^{|v_2||f|}[[v_1, f], \alpha(v_2)]$$

and

$$[v_1 \cdot v_2, \alpha'(f)] = \alpha(v_1) \cdot [v_2, f] + (-1)^{|v_2||f|}[v_1, f] \cdot \alpha(v_2).$$

By the definition of the operations, we have

$$\begin{aligned}[[v_1, v_2], \alpha'(f)](a) &= [[v_1, v_2], \alpha'(f)(a)] \\ &= [[v_1, v_2], \alpha \circ f(a)] \\ &= [\alpha(v_1), [v_2, f(a)]] + (-1)^{|v_2||f|+|v_2||a|}[[v_1, f(a)], \alpha(v_2)] \\ &= [\alpha(v_1), [v_2, f](a)] + (-1)^{|v_2||f|+|v_2||a|}[[v_1, f](a), \alpha(v_2)] \\ &= [\alpha(v_1), [v_2, f]](a) + (-1)^{|v_2||f|}[[v_1, f], \alpha(v_2)](a).\end{aligned}$$

Then $[[v_1, v_2], \alpha'(f)] = [\alpha(v_1), [v_2, f]] + (-1)^{|v_2||f|}[[v_1, f], \alpha(v_2)]$. Since

$$\begin{aligned}[v_1 \cdot v_2, \alpha'(f)](a) &= [v_1 \cdot v_2, \alpha(f(a))] \\ &= \alpha(v_1) \cdot [v_2, f(a)] + (-1)^{|v_2||f|+|v_2||a|}[v_1, f(a)] \cdot \alpha(v_2) \\ &= \alpha(v_1) \cdot [v_2, f](a) + (-1)^{|v_2||f|+|v_2||a|}[v_1, f](a) \cdot \alpha(v_2) \\ &= (\alpha(v_1) \cdot [v_2, f])(a) + (-1)^{|v_2||f|}([v_1, f] \cdot \alpha(v_2))(a).\end{aligned}$$

We obtain $[v_1 \cdot v_2, \alpha'(f)] = \alpha(v_1) \cdot [v_2, f] + (-1)^{|v_2||f|}[v_1, f] \cdot \alpha(v_2)$. □

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 10871057) (No. 11171055), the Natural Science Foundation of Jilin Province (No. 20130101068) and the Fundamental Research Funds for the Central Universities (No. 14QNJJ003).

References

[1] Albeverio, S., Ayupov, Sh. A. and Omirov, B. A. On nilpotent and simple Leibniz algebras, Comm. Alg., 33 (1), 159-172, 2005.

[2] Ammar, F., Ayadi, I., Mabrouk, S. and Makhlouf, A. Quadratic color Hom-Lie algebras, arXiv:1204.5155v1, 2012.

[3] Ammar, F. and Makhlouf, A. Hom-Lie superalgebras and Hom-Lie admissible superalgebras, J. Algebra., 324 (7), 1513-1528, 2010.

[4] Ammar, F., Makhlouf, A. and Saadoui, N. Cohomology of Hom-Lie superalgebras and q-deformed Witt superalgebra, Czechoslovak Mathematical Journal, 63(3), 721-761, 2013.

[5] Ayupov, Sh. A. and Omirov, B. A. On Leibniz algebras, Algebra and Operators Theory, 1-12, 1998.

[6] Ayupov, Sh. A. and Omirov, B. A. On 3-dimensional Leibniz algebras, Uzbek Math. J., 1(6), 9-14, 1999.

[7] Ayupov, Sh. A. and Omirov, B. A. On some classes of nilpotent Leibniz algebras, Siberian Math.J., 42 (1), 18-29, 2001.

[8] Casas, J. M. and Datuashoili, T. Noncommutative Leibniz Poisson algebras, Comm. Alg., 34 (7), 2507-2530, 2006.

[9] Casas, J. M. and Pirashvili, T. Algebras with bracket, Manuscripta Math., 119(1), 1-15, 2006.

[10] Casas, J. M. Homology with trivial coefficients and universal central extnesions of algebras with bracket, Comm. Alg., 35 (8), 2431-2449, 2007.

[11] Cuvier, C. Homologie de Leibniz et homologie de hochschild, C.R.Acad. Sci. Paris Se'r. A-B., 313 (12), 569-572, 1991.

[12] Dzhumadil'daev, A. A. Cohomologies of colour Leibniz algebras: Pre-simplicial approach, Lie Theory and its applications in physics III, preceeding of the third international workshop, 124-135, 1999.

[13] Hartwig, J.T., Larsson, D. and Silvesrov, S.D. Deformations of Lie algebras using σ derivations, J. Algebra., 295 (2), 314-361, 2006.

[14] Liu, D. and Hu, N. Leibniz superalgebras and central extensions, J. Algebra Appl., 5 (6), 765-780, 2006.

[15] Liu, D. and Hu, N. Steinberg Leibniz algebras and superalgebras, J. Algebra., 283 (1), 199-221, 2005.

[16] Liu, D. and Hu, N. Leibniz central extensions on some infinite dimensional Lie algebras, Comm. Alg., 32 (6), 2385-2405, 2004.

[17] Loday, J. L. Une version non commutative des algèbres de Lie: Les algèbres de Leibniz, Enseign. Math., 39 (1) 269-294, 1993.

[18] Loday, J. L. Cup-product for Leibniz cohomogy and dual Leibniz algebras, Math. Scand., 77 (2), 189-196, 1995.

[19] Loday, J. L. and Pirashvili, T. Universal enveloping algebras of Leibniz algebras and (co)homology, Math. Ann., 296 (1), 139-158, 1993.

[20] Loday, J. L. Dialgebra (co)homology with coefficients, in: Dialgebras and Related Operads, Lecture Notes in Math. 1763 (Springer-Verlag, Berlin, 2001), 67-103.

[21] Makhlouf, A. and Silvestrov, S. D. Notes on formal deformations of Hom-associative and Hom-Lie algebras, Forum Math., 22 (4), 715-739, 2010.

[22] Makhlouf, A. and Silvestrov, S. D. Hom-Lie admissible Hom-coalgebras and Hom-hopf algebra, Generalized Lie Theory in Mathematics, Physics and Beyond, Chapter 17, 189-206, 2009.

[23] Makhlouf, A. and Silvestrov, S. D. Hom-algebras and hom-coalgebras, J. Algebra App., 9 (4), 553-589, 2010.

[24] Makhlouf, A. Hom-dendriform Algebras and Rota-Baxter Hom-algebras, Proc. Int. Conf., in Nankai Series in Pure, Applied Mathematics and Theoretical Physics, (Eds. C. Bai, L. Guo and J-L. Loday), Vol 9, (World Scientific, Singapore, 2012), 147-171.

[25] Sheng, Y. Representations of Hom-Lie algebras, Alg Represent Theory, 5 (6), 1081-1098, 2012.

[26] Yau, D. Envelopping algebras of Hom-Lie algebras, J. Gen. Lie Theory Appl., 2(2), 95-108, 2008.
[27] Yuan, L. Hom-Lie color algebra structures, Comm. Alg., 40 (2), 575-592, 2012.

STATISTICS

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1181 – 1195

Asymptotic properties of risks ratios of shrinkage estimators

Hamdaoui Abdenour* and Benmansour Djamel†

Abstract

We study the estimation of the mean θ of a multivariate normal distribution $N_p\left(\theta, \sigma^2 I_p\right)$ in $\mathbb{R}^p$, $\sigma^2$is unknown and estimated by the chi-square variable $S^2 \sim \sigma^2\chi^2_n$. In this work we are interested in studying bounds and limits of risk ratios of shrinkage estimators to the maximum likelihood estimator, when n and p tend to infinity provided that $\lim_{p\to\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$.The risk ratio for this class of estimators has a lower bound $B_m = \frac{c}{1+c}$, when n and p tend to infinity provided that $\lim_{p\to\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$.We give simple conditions for shrinkage minimax estimators, to attain the limiting lower bound B_m. We also show that the risk ratio of James-Stein estimator and those that dominate it, attain this lower bound B_m (in particularly its positive-part version).We graph the corresponding risk ratios for estimators of James-Stein δ_{JS}, its positive part δ^+_{JS}, that of a minimax estimator, and an estimator dominating the James-Stein estimator in the sense of the quadratic risk (polynomial estimators proposed by Tze Fen Li and Hou Wen Kuo [13]) for some values of n and p.

Keywords: James-Stein estimator, multivariate gaussian random variable, non-central chi-square distribution, shrinkage estimator, quadratic risk.

2000 AMS Classification: 62F12, 62C20

Received 11/03/2014 : Accepted 17/10/2014 Doi : 10.15672/HJMS.2014377624

---

*Department of Mathematics, Oran (U.S.T.O)University, 31000, Algeria
Email: abdenour.hamdaoui@yahoo.fr
†Department of Environnement and Ecology, Laboratory of Statistics and Random Modelisations, Tlemcen University,13000, Algeria
Email: dbenmansour@yahoo.fr

1. Introduction

Since the papers of Stein [10],[11] and James and Stein [6], many studies were carried out in the direction of shrinkage estimators, of the mean θ of a multivariate normal distribution $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ in $\mathbb{R}^p$. In these works one estimates the mean θ of a multivariate normal distribution $N_p\left(\theta, \sigma^2 I_p\right)$ in $\mathbb{R}^p$ by shrinkage estimators deduced from the empirical mean estimator, which are better in quadratic loss than the empirical mean estimator. A summary of these proceedings is made by Hoffmann [5] who presents an expository development of Stein estimation in several distribution families. He considered both the point estimation and confidence interval cases. Emphasis is laid on the chronological development. In our work we are interested only in the case where the observation X is Gaussian.

More precisely, if X represents an observation or a sample of multivariate normal distribution $N_p\left(\theta, \sigma^2 I_p\right)$, the aim is to estimate θ by an estimator δ relatively at the quadratic loss function :

$$(1.1) \qquad L(\delta, \theta) = \|\delta - \theta\|_p^2$$

where$\|.\|_p$ is the usual norm in $\mathbb{R}^p$. To this loss we associate its risk function:

$$R(\delta, \theta) = E_\theta\left(L(\delta, \theta)\right).$$

James and Stein [6] introduced a class of estimators improving $\delta_0 = X$, when the dimension of the space of the observations p is $\geqslant 3$, denoted by

$$(1.2) \qquad \delta_{JS} = \left(1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right) X \ ,$$

in the case where σ^2 is unknown where $S^2 \sim \sigma^2 \chi_n^2$ is an estimate of σ^2, independent of X.

Baranchik [1] proposed the positive-part version of the James-Stein estimator, an estimator dominating the James-Stein estimator when $p \geqslant 3$:

$$(1.3) \qquad \delta_{JS}^{+} = \max\left(0, 1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right) X.$$

Robert [9] gives an explicit formula of its quadratic risk. We give a simple demonstration of this domination in Section 4.

Casella and Hwang [4] studied the case where σ^2 is known $\left(\sigma^2 = 1\right)$ and showed that if the limit of the ratio $\frac{\|\theta\|^2}{p}$, when p tends to infinity, is a constant $c > 0$, then

$$\lim_{p \to +\infty} \frac{R(\delta_{JS}(X), \theta)}{R(X, \theta)} = \lim_{p \to +\infty} \frac{R(\delta_{JS}^{+}(X), \theta)}{R(X, \theta)} = \frac{c}{1+c} \quad , \quad c > 0.$$

Li Sun [7] has considered the following ANOVA1 model :

$(X_{ij} \mid \theta_j, \sigma^2) \sim N(\theta_j, \sigma^2) \quad i = 1, ..., n, \quad j = 1, ..., m$ where $E(X_{ij}) = \theta_j$ for the group j and $var(X_{ij}) = \sigma^2$ is unknown. In this case it is clear that the maximum likelihood estimator, denoted by δ_0, has risk $R(\delta_0, \theta) = \dfrac{m\sigma^2}{n}$.

The James-Stein estimators are written in this case

$$\delta_{JS} = (\delta_{JS}^1, \delta_{JS}^2, ..., \delta_{JS}^m)^t$$

with

$$\delta_{JS}^j = \left(1 - \frac{(m-3)S^2}{(N+2)T^2}\right)(X_{ij} - \overline{X}) + \overline{X}, \ j = 1, ..., m$$

and

$$S^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} (X_{ij} - \overline{X}_j)^2 \quad , \quad T^2 = n \sum_{j=1}^{m} (\overline{X}_j - \overline{X})^2 \ ,$$

$$\overline{X}_j = \frac{\sum_{i=1}^{n} X_{ij}}{n} \quad , \ \overline{X} = \frac{\sum_{j=1}^{m} \ \overline{X}_j}{m} \qquad , \quad N = (n-1)m \ .$$

He shows that for any estimator of the form

$$\delta = (\delta_1, ..., \delta_m)^t \quad \text{where} \quad \delta_j = \left[1 - \psi\left(S^2, T^2\right)\right] (\overline{X}_j - \overline{X}) + \ \overline{X}, \ \ j = 1, ..., m \ ,$$

if $\lim\limits_{m \to +\infty} \frac{1}{m} \left(\sum\limits_{j=1}^{m} \left(\theta_j - \overline{\theta}\right)^2 \right) = c$ exists, then $\lim\limits_{m \to +\infty} \dfrac{R(\delta, \theta)}{R\left(\delta_0, \theta\right)} \geqslant \dfrac{c}{c + \frac{\sigma^2}{n}}$ and also $\lim\limits_{m \to +\infty} \dfrac{R(\delta_{JS}, \theta)}{R(\delta_0, \theta)} = \dfrac{c}{c + \frac{\sigma^2}{n}}$. On the other hand $\dfrac{c}{c + \frac{\sigma^2}{n}}$ constitutes a lower bound for the ratio $\lim\limits_{m \to +\infty} \dfrac{R(\delta, \theta)}{R\left(\delta_0, \theta\right)}$ and is equal to $\lim\limits_{m \to +\infty} \dfrac{R(\delta_{JS}, \theta)}{R(\delta_0, \theta)}$.

Li Sun [7] also shows that this bound is attained for a class of estimators defined by

$$\delta = (\delta_1, ..., \delta_m)^t \text{ where } \delta_j = \left[1 - \psi\left(S^2, T^2\right) \frac{S^2}{T^2}\right] (\overline{X}_j - \overline{X}) + \ \overline{X}, \ \ j = 1, ..., m$$

and ψ satisfies certain conditions.

This bound is also attained for any estimator dominating the James-Stein estimator, in particular the positive-part version of the James-Stein estimator.

Finally, we note that if n tends to infinity then the ratio $\dfrac{c}{c + \frac{\sigma^2}{n}}$ tends to 1, and thus the risk of the James-Stein estimator is that of δ_0 (when m and n tend to infinity).

Maruyama [8] considered the following model : $Z \sim N_d\left(\theta, I_d\right)$ and the so-called l_p-norm given by: $\left\|z\right\|_p = \left\{\sum_{i=1}^{i=d} \left|z_i\right|^p\right\}^{\frac{1}{p}}$, $p > 0$.

He also notes:$\left\|z\right\|_p^m = \left\{\sum_{i=1}^{i=d} \left|z_i\right|^p\right\}^{\frac{m}{p}}$. He defined a new class of James-Stein estimators with ‘l_p-norm based shrinkage factor, defined as follows :

$\widehat{\theta}_\phi = (\widehat{\theta}_{1\phi}, \widehat{\theta}_{2\phi}, ..., \widehat{\theta}_{d\phi})$ with: $\widehat{\theta}_{i\phi} = \left(1 - \phi(\left\|z\right\|_p)/ \left\|z\right\|_p^{2-\alpha} \left|z_i\right|^\alpha\right) z_i$ where $0 \leq \alpha < (d-2)/d-1)$, $p > 0$. (Since some components of the estimator can be exactly zero, the choice between a full model and reduced models is possible).

When $d \geq 3$, he establishes minimaxity and sparsity simultaneously, of this class of estimators with ‘l_p-norm based shrinkage factor, under conditions on $\widehat{\theta}_\phi$, and any positive p.

Note that the risk functions of these estimators are calculated relatively to the usual quadratic loss function (1.1).

The calculation of risk ratios in this case, and the conditions on the report of the l_p-norm of θ to the dimension of its space, change completely. Extension of our work to this type of estimators presents technical difficulties.

In our work we consider a different model and we obtain for several classes of shrinkage estimators (in particular the James-Stein estimator and its positive-part) that if $\lim\limits_{p \to +\infty} \dfrac{\left\|\theta\right\|^2}{\sigma^2 p} = c$ then the risk ratios tend to $\dfrac{c}{1+c} < 1$, when n and p tend to infinity.

In the following we denote the general form of a shrinkage estimator as follows:

$$(1.4) \qquad \delta = \left(1 - \psi\left(S^2, \left\|X\right\|^2\right)\right) X.$$

We adopt the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ and independently of the observations X, we observe $S^2 \sim \sigma^2 \chi_n^2$ an estimator of σ^2. Note that $R(X,\theta) = p\sigma^2$ is the risk of the maximum likelihood estimator.

In Section 2, we recall two results obtained in the paper of Benmansour and Hamdaoui [2]. The authors showed, that if $\lim_{p\to\infty} \dfrac{\|\theta\|^2}{p\sigma^2} = c\,(>0)$, then the risk ratio of James-Stein estimator δ_{JS} to the maximum likelihood estimator X, tends to the value $\dfrac{\frac{2}{n+2}+c}{1+c}$ when p tends to infinity and n is fixed. The second result indicates that under the condition $\lim_{p\to\infty} \dfrac{\|\theta\|^2}{p\sigma^2} = c\ (>0)$, the risk ratio of James-Stein estimator δ_{JS} to the maximum likelihood estimator X, tends to the value $\dfrac{c}{1+c}$ when n and p tend simultaneously to infinity. We also get the same results with James-Stein positive-part estimator.

In the first part of Section 3 we show that under condition $\lim\limits_{p\to+\infty} \dfrac{\|\theta\|^2}{\sigma^2 p} = c$, $\lim\limits_{n,p\to+\infty} \dfrac{R(\delta,\theta)}{R(X,\theta)} \geq \dfrac{c}{1+c}$ and we prove by an argument which is different from the one in Benmansour and Hamdaoui [2], that under the same condition $\lim\limits_{p\to+\infty} \dfrac{\|\theta\|^2}{\sigma^2 p} = c$, $\lim\limits_{n,p\to+\infty} \dfrac{R(\delta_{JS},\theta)}{R(X,\theta)} = \dfrac{c}{1+c}$. We deduce that any shrinkage estimator defined in (1.4) dominating the James-Stein estimator also satisfies this property. In the second part of this section, we show that if $\lim\limits_{p\to+\infty} \dfrac{\|\theta\|^2}{\sigma^2 p} = c$, then $\lim\limits_{n,p\to+\infty} \dfrac{R(\delta,\theta)}{R(X,\theta)} \geq \dfrac{c}{1+c}$ on the one hand, and for certain forms of ψ, we show that $\lim\limits_{n,p\to+\infty} \dfrac{R(\delta,\theta)}{R(X,\theta)} = \dfrac{c}{1+c}$.

In Section 4 we consider conditions of minimaxity of an estimator, and show that for certain forms of minimax δ, we have the same result as above.

By taking a class of estimators proposed by Benmansour and Mourid [3] (Proposition 4.4), estimators dominating the James-Stein estimator in the case σ^2 is known, we propose a simple proof of the domination of the James Stein estimator by its positive-part in the case σ^2 is unknown.

Finally, we graph the corresponding risks ratios for estimators of James-Stein δ_{JS}, its positive-part δ_{JS}^{+}, that of a minimax estimator, and an estimator dominating the James-Stein estimator in the sense of the quadratic risk (polynomial estimators proposed by Tze Fen Li and Hou Wen Kuo [13]) for various values of n and p.

2. Preliminaries

We recall that if X is a multivariate Gaussian random $N_p\left(\theta, \sigma^2 I_p\right)$ in $\mathbb{R}^p$, then $U = \dfrac{\|X\|^2}{\sigma^2} \sim \chi_p^2\ (\lambda)$ where $\chi_p^2\ (\lambda)$ denotes the non-central chi-square distribution with p degrees of freedom and non-centrality parameter $\lambda = \dfrac{\|\theta\|^2}{2\sigma^2}$.

In this case, for $\sigma^2 = 1$, Casella and Hwang [4] have shown the inequalities

$$\frac{1}{\left(p-2+\|\theta\|^2\right)} \leq E\left(\frac{1}{\|X\|^2}\right) \leq \frac{p}{(p-2)\left(p+\|\theta\|^2\right)} \quad , p \geq 3$$

that we generalize in the following lemma, in the case σ^2 is unknown.

2.1. Lemma. *Let* $X \sim N_p(\theta, \sigma^2 I_p)$; *if* $p \geq 3$ *then*

$$\text{(2.1)} \qquad \frac{1}{\sigma^2\left(p-2+\frac{\|\theta\|^2}{\sigma^2}\right)} \le E\left(\frac{1}{\|X\|^2}\right) \le \frac{p}{\sigma^2\,(p-2)\left(p+\frac{\|\theta\|^2}{\sigma^2}\right)}$$

Proof. It follows immediately from the inequalities of Casella and Hwang [4], since $\frac{X}{\sigma} \sim N_p\left(\frac{\theta}{\sigma}, I_p\right)$ □

From Robert [9], it is clear that the risk of the James-Stein estimator given in (1.2) is

$$R(\delta_{JS},\theta) = \sigma^2\left\{p - \frac{n}{n+2}\,(p-2)^2\,E\left(\frac{1}{p-2+2K}\right)\right\}$$

with $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ being the Poisson distribution of parameter $\frac{\|\theta\|^2}{2\sigma^2}$.

2.2. Theorem. *If* $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2} = c(>0)$ *, we have*

$$\text{(2.2)} \qquad \lim_{p\to+\infty}\frac{R(\delta_{JS},\theta)}{R\,(X,\theta)} = \frac{c+\frac{2}{n+2}}{c+1}\ .$$

Proof. See Benmansour and Hamdaoui [2]. □

2.3. Corollary. *If* $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2} = c\ (>0)$ *, we have*

$$\text{(2.3)} \qquad \lim_{n,p\to+\infty}\frac{R(\delta_{JS},\theta)}{R\,(X,\theta)} = \frac{c}{c+1}\ .$$

Proof. See Benmansour and Hamdaoui [2]. □

3. Lower bound of shrinkage estimators

To calculate the risk function, we recall a lemma similar to Lemma 2.1 of Li Sun [7].

3.1. Lemma. *Let* $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ *. Then*

$$(a) \quad E\left\{f\left(S^2,\|X\|^2\right)\right\} = E\left\{f\left(\sigma^2\chi^2_n,\sigma^2\chi^2_{p+2K}\right)\right\}$$

$$(b) \quad E\left\{g\left(S^2,\|X\|^2\right)\sum_{j=1}^{p}\theta_j X_j\right\} = 2\sigma^2\ E\left\{K\ g\left(\sigma^2\chi^2_n,\sigma^2\chi^2_{p+2K}\right)\right\}$$

for any functions of two variables such that all expectations of (*a*) *and* (*b*) *exist.*

Proof. Analogous to the proof of Lemma 2.1 of Li Sun [7]. □

In the case of our model, Theorem 2.1 of Li Sun [7] is written as follows:

3.2. Theorem. *The risk of the estimator given in (1.4) is*

$$R(\delta\ ,\theta) = \sigma^2 E\left\{\psi_K^2\ \chi^2_{p+2K} - 2\psi_K\left(\chi^2_{p+2K} - 2K\right) + p\right\}$$

where $\psi_K = \psi\left(\sigma^2\chi^2_n, \sigma^2\chi^2_{p+2K}\right)$ *and* $K \sim P\left(\dfrac{\|\theta\|^2}{2\sigma^2}\right)$.

Furthermore $R(\delta\ ,\theta) \geq B_p(\theta)$ *with*

$$B_p(\theta) = \sigma^2\left\{p - 2 - E\left\{\frac{(p-2)^2}{p-2+2K}\right\}\right\}\ .$$

Proof. Analogous to the proof of Theorem 2.1 of Li Sun [7], using Lemma 2.1. □

We set $b_p(\theta) = \dfrac{B_p(\theta)}{R(\theta, X)}$, then using Lemma 3.1 of Li Sun [7] and the fact that $R(\theta, X) = p\sigma^2$, we have

$$\frac{p-2}{p} - \frac{(p-2)^2}{p^2}\frac{1}{\frac{p-4}{p} + \frac{\|\theta\|^2}{p\sigma^2}} \leq b_p(\theta) \leq \frac{p-2}{p} - \frac{(p-2)^2}{p^2}\frac{1}{\frac{p-2}{p} + \frac{\|\theta\|^2}{p\sigma^2}}\ .$$

It is clear that if $\lim_{p\to\infty}\dfrac{\|\theta\|^2}{p\sigma^2} = c$, then

$$\text{(3.1)}\qquad \lim_{p\to\infty} b_p(\theta) = \frac{c}{1+c}$$

In the case where $\psi(S^2, \|X\|^2) = d\dfrac{S^2}{\|X\|^2}$, we have $\delta_d = \left(1 - d\dfrac{S^2}{\|X\|^2}\right)X$ hence

$$\text{(3.2)}\qquad R(\delta_d, \theta) = \sigma^2\left\{p + n\left[d^2(n+2) - 2d(p-2)\right]E\left(\frac{1}{p-2+2K}\right)\right\}.$$

For $d = \dfrac{(p-2)}{(n+2)}$ we obtain the James-Stein estimator (defined in (1.2)) which minimizes the risk of δ_d whose quadratic risk is

$$\text{(3.3)}\qquad R(\delta_{JS}, \theta) = \sigma^2\left\{p - \frac{n}{n+2}(p-2)^2 E\left(\frac{1}{p-2+2K}\right)\right\}.$$

Next we are interested in the ratios $\dfrac{R(\delta,\theta)}{R(X,\theta)}$ in particular when n and p tend to infinity. Casella and Hwang [4], showed in the case $\sigma^2 = 1$ that if $\lim_{p\to\infty}\dfrac{\|\theta\|^2}{p} = c(c > 0)$ then $\lim_{p\to+\infty}\dfrac{R(\delta_{JS}(X), \theta)}{R(X,\theta)} = \dfrac{c}{1+c}$. Li Sun [7] in his case showed that if $\lim\limits_{p\to+\infty}\frac{\sum_{j=1}^p(\theta_j - \overline{\theta})^2}{p} = c(c > 0)$, then $\lim_{p\to+\infty}\dfrac{R(\delta,\theta)}{R(\delta_0,\theta)} \geq \dfrac{c}{\frac{\sigma^2}{n} + c}$ and also $\lim_{p\to\infty}\dfrac{R(\delta_{JS},\theta)}{R(\delta_0,\theta)} = \dfrac{c}{\frac{\sigma^2}{n} + c}$ and therefore $\lim_{n,p\to\infty}\dfrac{R(\delta_{JS},\theta)}{R(\delta_0,\theta)} = 1$.

We show in our work that if $\lim_{p\to\infty}\dfrac{\|\theta\|^2}{\sigma^2 p} = c$, then $\lim_{n,p\to\infty}\dfrac{R(\delta,\theta)}{R(X,\theta)} \geq \dfrac{c}{1+c}$ on the one hand, and for some forms of δ, we show that $\lim_{n,p\to\infty}\dfrac{R(\delta,\theta)}{R(X,\theta)} = \dfrac{c}{1+c}$.

Thus we ameliorate the result of Li Sun [7], obtaining a limit strictly less than 1.

3.3. Proposition. *If* $\lim_{p\to\infty} \dfrac{\|\theta\|^2}{\sigma^2 p} = c$, *then*

$$\lim_{n,p\to\infty} \frac{R(\delta,\theta)}{R(X,\theta)} \geq \frac{c}{1+c}\ , \tag{3.4}$$

$$\lim_{n,p\to\infty} \frac{R(\delta_{JS},\theta)}{R(X,\theta)} = \frac{c}{1+c}\ . \tag{3.5}$$

Proof. Formula (3.4) follows immediately from Theorem 3.2 and Formula (3.1). Formula (3.5) follows immediately from Corollary 2.3. Indeed Theorem 3.2 implies that $\dfrac{R(\delta_{JS},\theta)}{R(X,\theta)} \geq \dfrac{B_p(\theta)}{R(X,\theta)} = b_p(\theta)$, and from (3.3), Lemma (2.1) and (3.1) we have

$$\frac{\frac{2}{n+2}+c}{1+c} \geq \lim_{p\to\infty} \frac{R(\delta_{JS},\theta)}{R(X,\theta)} \geq \frac{c}{1+c}$$

thus

$$\lim_{n\to\infty} \frac{\frac{2}{n+2}+c}{1+c} \geq \lim_{n,p\to\infty} \frac{R(\delta_{JS},\theta)}{R(X,\theta)} \geq \frac{c}{1+c}$$

hence

$$\lim_{n,p\to\infty} \frac{R(\delta_{JS},\theta)}{R(X,\theta)} = \frac{c}{1+c}\ . \tag{3.6}$$

Thus we find exactly the same limit ratio Casella and Hwang [4], in the case where σ^2 is unknown. □

In the following we study the families of estimators written as follows

$$\delta_\psi = \delta_{JS} + l\psi\left(S^2, \|X\|^2\right) \frac{S^2}{\|X\|^2} X\ ,\ \ l > 0 \tag{3.7}$$

and we give simple conditions on ψ so that the limiting ratio $\lim_{n,p\to\infty} \dfrac{R(\delta_\psi,\theta)}{R(X,\theta)}$ equals $\dfrac{c}{1+c}$, when $\lim_{p\to\infty} \dfrac{\|\theta\|^2}{\sigma^2 p} = c$, where ψ is a measurable function such that $E\left[\psi^2\left(\sigma^2\chi^2_n, \sigma^2\chi^2_p(\lambda)\right)\right] < \infty$.

In this case, the difference of risks denoted by $\Delta_{\psi_{JS}} = R(\delta_\psi,\theta) - R(\delta_{JS},\theta)$ is:

$$\Delta_{\psi_{JS}} = E\left[l^2 \frac{\left(\sigma^2\chi^2_n\right)^2 \psi^2\left(\sigma^2\chi^2_n, \sigma^2\chi^2_p(\lambda)\right)}{\sigma^2\chi^2_p(\lambda)} + 2l\sigma^2\chi^2_n\psi\left(\sigma^2\chi^2_n, \sigma^2\chi^2_p(\lambda)\right)\right]$$

$$-2ldE\left[\frac{\left(\sigma^2\chi^2_n\right)^2 \psi\left(\sigma^2\chi^2_n, \sigma^2\chi^2_p(\lambda)\right)}{\sigma^2\chi^2_p(\lambda}\right] - 4l\lambda E\left[\frac{\sigma^2\chi^2_n(\psi(\sigma^2\chi^2_n, \sigma^2\chi^2_{p+2}(\lambda))}{\chi^2_{p+2}(\lambda)}\right], \tag{3.8}$$

where $\lambda = \dfrac{\|\theta\|^2}{2\sigma^2}$ and $d = \dfrac{p-2}{n+2}$, see (Benmansour and Mourid [3]).

For estimators of the form (3.7), which are not necessarily minimax we give the following two results which are analogous to Theorem 3.2 of Li Sun [7] , with different conditions on ψ and whose risks ratios attain the lower bound B_m.

3.4. Theorem. *Assume that* δ_ψ *is given in (3.7) and that* $\psi\left(S^2, \|X\|^2\right)$ *satisfies*
a) $\left|\psi\left(S^2, \|X\|^2\right)\right| \le g\left(S^2\right)$ *a.s; where* $E\left\{\left(g^2\left(S^2\right)\right)^{1+\gamma}\right\} \le (M(n))^{1+\gamma}$ *for some* $\gamma > 0$.
If $\lim_{p\to\infty} \dfrac{\|\theta\|^2}{p\sigma^2} = c\,(>0)$ *then*

$$\lim_{n,p\to+\infty} \frac{R(\delta_\psi, \theta)}{R(X,\theta)} = \frac{c}{1+c} \tag{3.9}$$

for all l *such that* $l\,(M(n))^{1/2} = O\left(\dfrac{1}{n}\right)$ *in the neighborhood of* $+\infty$. *Note that* l *may depend on* n .

Proof. Relation (3.8) and condition a) give

$$\begin{aligned}\Delta_{\psi_{JS}} \quad &\le \quad E\left[l^2\frac{\left(S^2\right)^2 g^2(S^2)}{\|X\|^2} + 2lS^2g(S^2) + \frac{2ld\left(S^2\right)^2 g(S^2)}{\|X\|^2}\right] \\ &\qquad +4l\lambda E(S^2g(S^2))E\left(\frac{1}{\chi^2_{p+2}(\lambda)}\right)\end{aligned}$$

thus

$$\begin{aligned}\Delta_{\psi_{JS}} \quad \le \quad &\frac{l^2}{\sigma^2(p-2)}\left(E\left[(\sigma^2\chi^2_n)^{2(1+\gamma)/\gamma}\right]\right)^{\gamma/(1+\gamma)} M(n) \\ &+2l\left[E\left(\left(\sigma^2\chi^2_n\right)^2\right)\right]^{1/2}(M(n))^{1/2} \\ &+\frac{2l}{\sigma^2(n+2)}\left[E\left(\left(\sigma^2\chi^2_n\right)^4\right)\right]^{1/2}(M(n))^{1/2} \\ &+4l\lambda\frac{\left[E\left(\left(\sigma^2\chi^2_n\right)^2\right)\right]^{1/2}(M(n))^{1/2}}{p}.\end{aligned}$$

The last inequality follows from Holder inequality, Schwarz inequality, the independence of $\|X\|^2$ and S^2 and that $E\left(\dfrac{1}{\chi^2(p,\lambda)}\right) \le \dfrac{1}{p-2}$. Thus, for n close to infinity we have

$$\begin{aligned}\Delta_{\psi_{JS}} \quad \le \quad &\frac{4\sigma^2 l^2 M(n)}{p-2}\left(\frac{\Gamma\left(\frac{n}{2}+\frac{2}{\gamma}+2\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\gamma/(1+\gamma)} + 2l\sigma^2(M(n))^{1/2}[n(n+2)]^{1/2} \\ &+\frac{2\sigma^2 l(M(n))^{1/2}[(n+6)(n+4)(n+2)n]^{1/2}}{(n+2)} \\ &+\frac{4l\lambda\sigma^2(M(n))^{1/2}[n(n+2)]^{1/2}}{p}\ .\end{aligned}$$

Now from Stirling's formula which expresses that in the neighborhood of $+\infty$, we have: $\Gamma(y+1) \simeq \sqrt{2\pi}y^{y+\frac{1}{2}}e^{-y}$ and the fact that $e^y = \lim_{n\to+\infty}\left(1+\frac{y}{n}\right)^n$, we have

$$\left(\frac{\Gamma\left(\frac{n}{2}+\frac{2}{\gamma}+2\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\gamma/(1+\gamma)} \simeq \left(\frac{n}{2}+\frac{2}{\gamma}+1\right)^2 \tag{3.10}$$

thus

$$\lim_{n,p\to+\infty} \frac{\Delta_{\psi_{JS}}}{R(X,\theta)} \leq \frac{4l^2M(n)}{p(p-2)}\left(\frac{n}{2}+\frac{2}{\gamma}+1\right)^2 + \frac{2l(M(n))^{1/2}[n(n+2)]^{1/2}}{p}$$
$$+\frac{2l(M(n))^{1/2}[(n+6)(n+4)(n+2)n]^{1/2}}{p(n+2)}$$
$$+\frac{4l\lambda(M(n))^{1/2}[n(n+2)]^{1/2}}{p^2}$$

Since $\lim_{p\to\infty} \frac{2\lambda}{p} = c$ and $l(M(n))^{1/2} = O\left(\frac{1}{n}\right)$ we finally obtain

$$\lim_{n,p\to+\infty} \frac{\Delta_{\psi_{JS}}}{R(X,\theta)} = \lim_{n,p\to\infty} \frac{R(\delta_\psi,\theta)}{R(X,\theta)} - \lim_{n,p\to\infty} \frac{R(\delta_{JS},\theta)}{R(X,\theta)} \leq 0$$

and thus from (3.4) and (3.5)

$$\lim_{n,p\to\infty} \frac{R(\delta_\psi,\theta)}{R(X,\theta)} = \frac{c}{1+c}\ ,$$

hence the result. □

3.5. Example. Let $\psi_1\left(S^2,\|X\|^2\right) = \frac{\|X\|^2}{S^2\left(\|X\|^2+1\right)}$. In this case it suffices to take $g\left(S^2\right) = \frac{1}{S^2}$ and to choose $l = 1$.

The following proposition gives the same result as Theorem 3.4 for a particular class of the shrinkage function $\psi\left(S^2,\|X\|^2\right)$. Indeed, we will choose g in L^2 and not in $L^{2(1+\gamma)}$ but with the constraint that $g(S^2)$ is monotone non-increasing.

3.6. Proposition. *Assume that δ_ψ is given in (3.7) and that ψ satisfies:*

a) $\left|\psi\left(S^2,\|X\|^2\right)\right| \leq g(S^2)$ a.s where $g(S^2)$ is monotone non-increasing such that $E\left[g^2(S^2)\right] \leq M(n)$.

If $\lim_{p\to\infty} \frac{\|\theta\|^2}{\sigma^2 p} = c$, then

$$\lim_{n,p\to\infty} \frac{R(\delta_\psi,\theta)}{R(X,\theta)} = \frac{c}{1+c} \tag{3.11}$$

for all l such that $l\left(M(n)\right)^{1/2} = O\left(\frac{1}{n}\right)$ in the neighborhood of $+\infty$ (l may depend on n).

Proof. Analogous to the proof of Theorem 3.4, so we give a brief idea.(3.8) and condition a) give

$$\Delta_{\psi_{JS}} \leq \frac{l^2}{\sigma^2}\frac{E\left[\left(\sigma^2\chi^2_n\right)^2\right]E\left[g^2(\sigma^2\chi^2_n)\right]}{p-2} + 2lE\left[\sigma^2\chi^2_n\right]E\left[g(\sigma^2\chi^2_n)\right]$$
$$+\frac{2l}{\sigma^2(n+2)}E\left[\left(\sigma^2\chi^2_n\right)^2\right]E\left[g(\sigma^2\chi^2_n)\right] + 4l\lambda\frac{E(\sigma^2\chi^2_n)E\left[g(\sigma^2\chi^2_n)\right]}{p}.$$

The last inequality comes from the fact that $E\left(\frac{1}{\chi^2(p,\lambda)}\right) \leq \frac{1}{p-2}$ and that the covariance of two functions, one increasing and the other decreasing, is negative. Thus,

$$\lim_{n,p\to\infty} \frac{\Delta_{\psi_{JS}}}{R(X,\theta)} \leq \lim_{n,p\to\infty} \frac{nl(M(n))^{1/2}}{p}\left(l\frac{(n+2)(M(n))^{1/2}}{(p-2)} + 4 + \frac{4\lambda}{p}\right)$$
$$\leq 0$$

because $\lim_{p\to\infty} \dfrac{2\lambda}{p} = c$, and $l(M(n))^{1/2} = O\left(\dfrac{1}{n}\right)$. We finally obtain

$$\lim_{n,p\to\infty} \frac{\Delta_{\psi_{JS}}}{R(X,\theta)} = \lim_{n,p\to\infty} \frac{R(\delta_\psi,\theta)}{R(X,\theta)} - \lim_{n,p\to\infty} \frac{R(\delta_{JS},\theta)}{R(X,\theta)} \leq 0$$

and from (3.4) and (3.5)

$$\lim_{n,p\to\infty} \frac{R(\delta_\psi,\theta)}{R(X,\theta)} = \frac{c}{1+c} ,$$

hence the result. □

3.7. Example. Let $\psi_1\left(S^2, \|X\|^2\right) = \dfrac{\|X\|^2}{S^2\left(\|X\|^2+1\right)}$, and therefore

$$\delta_{\psi_1} = \delta_{JS} + \frac{1}{\|X\|^2+1}X. \tag{3.12}$$

In this case we simply take $g\left(S^2\right) = \frac{1}{S^2}$ and choose $l = 1$.

4. Minimaxity

Now, we recall a result of Strawderman [12] about the minimaxity of the following class of estimators. Let:

$$\delta_\phi = \left(1 - l\phi\left(S^2, \|X\|^2\right)\frac{S^2}{\|X\|^2}\right)X \ , \ \ l>0 \tag{4.1}$$

4.1. Theorem. *If :*
a) $\phi\left(S^2, \|X\|^2\right)$ *is monotone non-increasing in* S^2 *and non-decreasing in* $\|X\|^2$,
b) $0 \leq \phi\left(S^2, \|X\|^2\right) \leq \dfrac{2(p-2)}{l(n+2)}$, *then* δ_ϕ *is minimax.*

Proof. A simple proof of this result is as follows: For $U = \frac{\|X\|^2}{\sigma^2}$, we have

$$\begin{aligned} R(\delta_\phi,\theta) &= p\sigma^2 + \sigma^2 lE\left[l\frac{\left(S^2\right)^2\phi^2\left(S^2,\sigma^2U\right)}{U} - 2(p-2)\frac{S^2\phi\left(S^2,\sigma^2U\right)}{U}\right] \\ &\quad -\sigma^2 E\left[4l\frac{S^2\partial\phi\left(S^2,\sigma^2U\right)}{\partial U}\right], \end{aligned}$$

by using the equality of Stein [11]. Since $\phi\left(S^2, \|X\|^2\right)$ is non-decreasing in U it suffices to have

$$E\left[l\frac{\left(S^2\right)^2\phi^2\left(S^2,\sigma^2U\right)}{U} - 2(p-2)\frac{S^2\phi\left(S^2,\sigma^2U\right)}{U}\right] \leq 0 .$$

Setting $C_0 = \dfrac{2(p-2)}{l(n+2)}$, we have

$$\begin{aligned} & E\left[l\frac{\left(S^2\right)^2\phi^2\left(S^2,\sigma^2U\right)}{U} - 2(p-2)\frac{S^2\phi\left(S^2,\sigma^2U\right)}{U}\right] \\ = \ & E\left[\frac{\phi\left(S^2,\sigma^2U\right)}{U}S^2\left[lS^2\phi\left(S^2,\sigma^2U\right) - 2(p-2)\right]\right] \end{aligned}$$

$$\leq E\left[\frac{\phi\left(S^2,\sigma^2U\right)}{U}S^2\left[lS^2C_0-2(p-2)\right]\right].$$

Because $\phi\left(S^2,\|X\|^2\right)$ is non-increasing in S^2, therefore in both cases where $\phi\left(S^2,\|X\|^2\right)>C_0$ and $\phi\left(S^2,\|X\|^2\right)\leq C_0$, we have

$$\begin{aligned}&E\left[l\frac{\left(S^2\right)^2\phi^2\left(S^2,\sigma^2U\right)}{U}-2(p-2)\frac{S^2\phi\left(S^2,\sigma^2U\right)}{U}\right]\\ \leq\quad &E\left[\frac{\phi\left(C_0,\sigma^2U\right)}{U}S^2\left[lS^2C_0-2(p-2)\right]\right].\end{aligned}$$

As S^2 and U are independent we obtain

$$\begin{aligned}&E\left[l\frac{\left(S^2\right)^2\phi^2\left(S^2,\sigma^2U\right)}{U}-2(p-2)\frac{S^2\phi\left(S^2,\sigma^2U\right)}{U}\right]\\ \leq\quad &E\left[\frac{\phi\left(C_0,\sigma^2U\right)}{U}\right]E\left[l\left(S^2\right)^2C_0-2(p-2)S^2\right]\\ \leq\quad &0\end{aligned}$$

hence the result. □

Note that this class of minimax estimators admits as lower bound $B_m=\dfrac{c}{1+c}$ (Proposition 3.3) but does not attain it.

Then we have the following proposition which gives a class of minimax estimators whose risks ratios attains the lower bound.

4.2. Proposition. *Assume that δ_ψ is as given in (3.7), i.e.,*

$$\delta_\psi=\delta_{JS}+l\psi\left(S^2,\|X\|^2\right)\frac{S^2}{\|X\|^2}X \tag{4.2}$$

$$=\left(1-\left[\frac{S^2}{\|X\|^2}\left(\frac{p-2}{n+2}-l\psi\left(S^2,\|X\|^2\right)\right)\right]\right)X,\quad l>0.$$

If ψ satisfies the following conditions:

1) $\psi\left(S^2,\|X\|^2\right)$ is monotone non-decreasing in S^2 and non-increasing in $\|X\|^2$.

2) $\left|l\psi\left(S^2,\|X\|^2\right)\right|\leq\dfrac{p-2}{n+2}$,

then $\lim_{p\to\infty}\dfrac{\|\theta\|^2}{\sigma^2p}=c$ implies

$$\lim_{n,p\to\infty}\frac{R(\delta_\psi,\theta)}{R(X,\theta)}=\frac{c}{1+c}$$

for all l such that $\lim_{n\to\infty}l\,(p-2)=0$ (l depends on n).

Proof. It follows immediately from Theorems 3.4 and 4.1. □ □

4.3. Example. Let the estimator

$$\delta_{\psi_2}=\delta_{JS}+l\psi_2\left(S^2,\|X\|^2\right)\frac{S^2}{\|X\|^2}X \tag{4.3}$$

such that $l\psi_2\left(S^2,\|X\|^2\right)=\dfrac{p-2}{n+2}\dfrac{S^2}{S^2+1}\exp\left(-\|X\|^2\right).$

Note that the function ψ_2 satisfies the conditions of Proposition 4.2.

We note that the estimators of the form (4.1) are minimax but do not necessarily dominate the James-Stein estimator under the usual quadratic risk.

A class of estimators dominating the James Stein estimator is given as follows:

Let:

$$\delta_\phi = \delta_{JS} + m\phi\left(S^2, \|X\|^2\right) X \quad m > 0 \tag{4.4}$$

where ϕ is a measurable positive function, such that $E\left[\phi^2\left(S^2, \|X\|^2\right)\right] < \infty$. In this case, the difference of risks denoted by $\Delta_{\phi_{JS}} = R(\delta_\phi, \theta) - R(\delta_{JS}, \theta)$ is:

$$\begin{aligned} \Delta_{\phi_{JS}} &= E\left[m^2\left(\|X\|^2\right)\phi^2\left(S^2, \|X\|^2\right) + 2m\left(\|X\|^2\right)\phi\left(S^2, \|X\|^2\right)\right] \\ &\quad - E\left[2mdS^2\phi\left(S^2, \|X\|^2\right) + 4m\lambda(\phi(S^2, \sigma^2\chi^2_{p+2}(\lambda))\right] \end{aligned}$$

thus

$$\Delta_{\phi_{JS}} \leq E\left[m\phi\left(S^2, \|X\|^2\right)\left[m\|X\|^2\phi\left(S^2, \|X\|^2\right) + 2\|X\|^2 - 2dS^2\right]\right], \tag{4.5}$$

where $d = \dfrac{p-2}{n+2}$.

Then we have the following proposition.

4.4. Proposition. *Estimators given in (4.2) dominate the James-Stein estimator if*

1) $0 \leq \phi\left(S^2, \|X\|^2\right) \leq \dfrac{2}{m}\left(d\dfrac{S^2}{\|X\|^2} - 1\right) I_{\left(\frac{p-2}{n+2}\frac{s^2}{\|X\|^2} - 1 \geq 0\right)}$.

2) If in addition, $\lim_{p\to\infty} \dfrac{\|\theta\|^2}{\sigma^2 p} = c$, *then* $\lim_{n,p\to\infty} \dfrac{R(\delta_\phi, \theta)}{R(X, \theta)} = \dfrac{c}{1+c}$.

Proof. 1) It follows from inequality (4.5). 2) Immediate from (3.4) and (3.5). □

We observe that any estimator dominating the James-Stein estimator satisfies the property 2 of Proposition 4.3. Thus the class of estimators:

$\delta_m = \delta_{JS} + m\phi\left(S^2, \|X\|^2\right) = \delta_{JS} + m\left(\dfrac{p-2}{n+2}\dfrac{S^2}{\|X\|^2} - 1\right) I_{\left(\frac{p-2}{n+2}\frac{s^2}{\|X\|^2} - 1 \geq 0\right)} = \delta_{JS} + m\delta^-_{JS}(S^2, \|X\|^2)X$, dominates the James Stein estimator. And for $m = 1$ we have $\delta_\phi = \delta_{JS} + \delta^-_{JS}(S^2, \|X\|^2)X$, hence $\delta_\phi = \delta^+_{JS}\left(S^2, \|X\|^2\right) X$ dominates $\delta_{JS}\left(S^2, \|X\|^2\right)$ according to Proposition 4.3.

Moreover, its risk is minimal at $\lambda = 0$, relative to the whole family of the class of estimators $\delta_m = \delta_{JS} + m\delta^-_{JS}(S^2, \|X\|^2)X$.

5. Simulation

We recall the form of the estimator introduced by Tze Fen Li and Wen Hou Kuo [13]. Let $X \sim N_p\left(\theta, \sigma^2 I_p\right)$, $Y = \dfrac{X}{\sigma} \sim N_p\left(\dfrac{\theta}{\sigma}, I_p\right)$.

For all r $\left(2 < r < \dfrac{p+2}{2}\right)$, we consider the family of polynomial estimators:

$$\delta_{TZ} = \delta_{JS} + \alpha(S^2)^{\frac{r}{2}} X \|X\|^{-r} \tag{5.1}$$

where

$$\alpha = \frac{(r-2)}{2}\frac{(n+p)}{(n+2)}\frac{\Gamma\left(\frac{n+r}{2}\right)}{\Gamma\left(\frac{n+2r}{2}\right)}\frac{\Gamma\left(\frac{p-r}{2}\right)}{\Gamma\left(\frac{p-2r+2}{2}\right)}.$$

It is known by Tze Fen Li and Wen Hou Kuo [13], that the risk of the estimator δ_{TZ} is

$$R(\delta_{TZ},\theta) = R(\delta_{JS},\theta) + \sigma^2 \frac{2\alpha 2^{\frac{r}{2}}\Gamma\left(\frac{n+r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left[(p-r) - \frac{(p-2)(n+r)}{n+2}\right] E\left(\|Y\|^{-r}\right)$$

$$+\sigma^2\alpha^2 2^r \frac{\Gamma\left(\frac{n+2r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} E\left(\|Y\|^{-2r+2}\right). \tag{5.2}$$

We recall the form of the estimators given in Example 3.7 (3.12), i.e., $\delta_{\psi_1} = \delta_{JS} + \frac{\|X\|^2}{S^2\left(\|X\|^2+1\right)}\frac{S^2}{\|X\|^2}X = \delta_{JS} + \frac{X}{\|X\|^2+1}$, as well as in Example 4.3 (4.3), i.e., $\delta_{\psi_2} = \delta_{JS} + \frac{p-2}{n+2}\frac{S^4}{(S^2+1)\|X\|^2}\exp\left(-\|X\|^2\right)X$, of which we graph their risks ratios as well as those of Tze Fen Li, James-Stein and the positive part- of James-Stein denoted respectively:
$\frac{R(\delta_{\psi_1},\theta)}{R(X,\theta)}, \frac{R(\delta_{\psi_2},\theta)}{R(X,\theta)}, \frac{R(\delta_{TZ},\theta)}{R(X,\theta)}, \frac{R(\delta_{JS},\theta)}{R(X,\theta)}, \frac{R(\delta^+_{JS},\theta)}{R(X,\theta)}$,for various values of n and p.

Fig. 1 Graph of risk ratios $\frac{R(\delta_{\psi_1},\theta)}{R(X,\theta)}, \frac{R(\delta_{\psi_2},\theta)}{R(X,\theta)}, \frac{R(\delta_{TZ},\theta)}{R(X,\theta)}, \frac{R(\delta_{JS},\theta)}{R(X,\theta)}, \frac{R(\delta^+_{JS},\theta)}{R(X,\theta)}$ *as functions of* $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$ *for* $n = 10$ and $p = 4$.

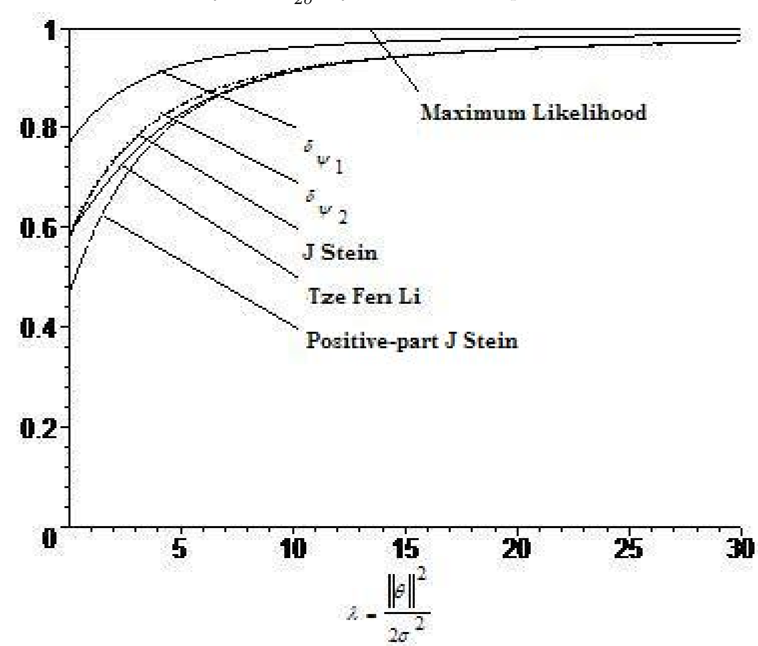
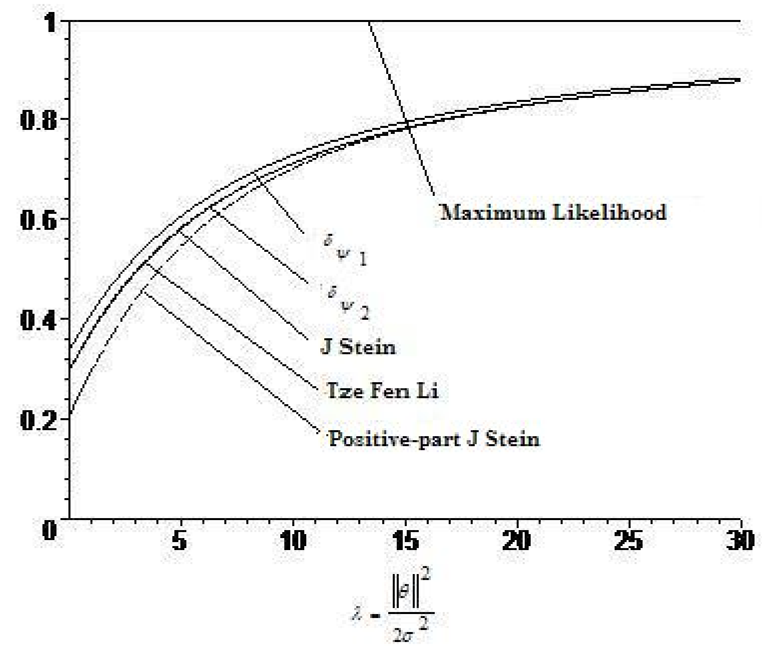


Fig. 2 Graph of risk ratios $\frac{R(\delta_{\psi_1},\theta)}{R(X,\theta)}$, $\frac{R(\delta_{\psi_2},\theta)}{R(X,\theta)}$, $\frac{R(\delta_{TZ},\theta)}{R(X,\theta)}$, $\frac{R(\delta_{JS},\theta)}{R(X,\theta)}$, $\frac{R(\delta^{+}_{JS},\theta)}{R(X,\theta)}$ *as functions of* $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$ *for* $n = 30$ and $p = 8$.



We note that in both graphs, the risk ratios tend to the same limit less than 1 where λ increases as well as n and p.

6. Conclusion

In the case of the estimate of the mean θ of a multivariate gaussian random $N_p(\theta, I_p)$ in $\mathbb{R}^p$, Casella and Hwang [4] showed that if $\lim_{p\to+\infty} \frac{\|\theta\|^2}{p} = c > 0$ then the ratio $\frac{R(\delta_{JS},\theta)}{R(X,\theta)}$ and $\frac{R(\delta^{+}_{JS},\theta)}{R(X,\theta)}$ tend to $\frac{c}{1+c}$. In our work by taking the same model, namely $X \sim N_p(\theta, \sigma^2 I_p)$ with σ^2 unknown, and estimated by the statistic $S^2 \sim \sigma^2\chi^2_n$ independent of X, we have showed that for the shrinkage estimators of the form $\delta = \left(1 - \psi\left(S^2, \|X\|^2\right)\right) X$, we obtain a similar ratio dependent of the sample size n, as soon as $\lim_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c > 0$. Moreover, we obtain a ratio constant less than 1, when n and p tend simultaneously to $+\infty$, without assuming any order relation or functional relation between n and p. We obtained the same result for particular forms of δ, which are not necessarily minimax, and for other forms of δ which are minimax. Finally we concluded that any shrinkage estimator dominating the James-Stein estimator has a risk ratio tending to $\frac{c}{1+c}$ when n and p tend to infinity.

Li Sun [7] was also interested in the case where σ^2 is unknown, but he studied the behaviour of the ratio $\frac{R(\delta,\theta)}{R(X,\theta)}$, $\frac{R(\delta_{JS},\theta)}{R(X,\theta)}$ and $\frac{R(\delta^{+}_{JS},\theta)}{R(X,\theta)}$, when only p tends to infinity.

The simulations in the case of selected examples, show that the asymptotic behaviour of risk ratios are identical and converge to the same limit that is strictly less than 1.

An idea would be to see whether one can obtain similar ratios in the general case of the symmetrical spherical models. Expanding our work to minimax estimators proposed by Maruyama [8] is also an idea that we currently explore.

References

[1] Baranchik, A. J. *Multiple regression and estimation of the mean of a multivariate normal distribution*. Stanford Univ. Technical Report 51 (1964).

[2] Benmansour Djamel and Hamdaoui Abdenour. *Limit of the ratio of risks of James-Stein estimators with unknown variance*. Far East Journal of Theorical Statistics. Volume 36, Number 1, pp 31-53 (2011).

[3] Benmansour, D and Mourid, T. *Etude d'une classe d'estimateurs avec rétrécisseur de la moyenne d'une loi gaussienne*. Annales de l'ISUP volume 51 - Fascicule 1-2 (2007).

[4] Casella, G and Hwang, J.T. *Limit expressions for the risk of James-Stein estimators*. The canadian Journal of Statistics. Vol. 10. 4, Pages 305-309 (1982).

[5] Hoffmann Kurt. *Stein estimation - A review*. Statistical Papers 41, 127-158 (2000).

[6] James, W and Stein, C. *Estimation of quadratique loss*. Proc 4th Berkeley Symp,Math .Statist.Prob., Univ. of California Press, Berkeley.Vol.1, 361-379 (1961).

[7] Li Sun. *Risk Ratio and Minimaxity in Estimating the Multivariate Normal Mean with unknown Variance*. Scandinavian Journal of Statistics, Vol. 22, No. 1,pp. 105-120 (1995).

[8] Maruyama Yuzo. *l_p-norm based James-Stein estimation with minimaxity and sparsity,* Statistics Theory (math.ST) arXiv: 1402.0302.(2014).

[9] Robert Christian. *An explicit Formula for the Risk of the Positive-Part James-Stein Estimator*. The Canadian Journal of Statistics; Vol. 16, No. 2 Pages 161-168 (1988).

[10] Stein, C. *Inadmissibilty of the usual estimator for the mean of a multivariate normal distribution*. Proc 3th Berkeley Symp, Math.Statist.Prob., Univ. of California Press, Berkeley. Vol.1,197-206 (1956).

[11] Stein, C. *Estimation of the mean of a multivariate normal distribution*. Ann. Statis. Vol 9, No. 6, 1135-1151 (1981).

[12] Strawderman, W. E. *Proper bayes minimax estimators of the multivariate normal mean vector for the case of common unknown variances.* Ann of Statis. Vol. 1,No.6, 1189-1194 (1973).

[13] Tze Fen Li and Wen Hou Kuo. *Generalized James-Stein Estimators*, Commun Statist. Thor.Math., 11(20),2249,2257 (1982).

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1197 – 1213

A computational approach for testing equality of coefficients of variation in k normal populations

Fikri GOKPINAR * and Esra GOKPINAR†

Abstract

In this article, a testing procedure based on computational approach testis proposed for the equality of coefficients of variation in k normal populations. We compare this procedure to some of the existing tests; the likelihood ratio, modified Bennett's, score, generalized p-value tests in terms of the estimated type I error rates and powers by using Monte Carlo simulation. Furthermore, applications of these tests are given on a real dataset.

Keywords: Computational approach test, Generalized p-value test, Likelihood ratio test, Modified Bennett's test, Score test

2000 AMS Classification: 62F03, 65C05, 65C60, 62E17.

Received 09/06/2014 : Accepted 09/09/2014 Doi : 10.15672/HJMS.2014317482

1. Introduction

The ratio of population standard deviation to the population mean is called coefficient of variation (CV) of a population which is free from the unit of measurement. CV has a wide range of applications in many physical, biological, medical sciences, etc. For example, in haematology and serology, CV values of the measurement of the blood sample taken from the different laboratories are compared to determine performances of these laboratories [19]. Also the usage of CV includes other applications such as the determination of instrumental precision and the homogeneity of test samples.

A very common problem in applied statistics is to test equality of coefficients of variation. Many methods are developed for this problem. Miller and Karson [14] proposed a test for the equality of coefficients of variation in two normal populations. Doornbos and Dijkstra [6] presented two tests which are called likelihood ratio test and non-central t test for testing equality of coefficients of variation in k normal populations. Likelihood ratio test involves estimators of parameters which can be only obtained from iteration method. Gupta and Ma [11] used a better iteration method called bisection method to

*Gazi University, Department of Statistics, Ankara,Turkey.
Email: fikri@gazi.edu.tr Corresponding author.
†Gazi University, Department of Statistics, Ankara,Turkey.

obtain these estimators of parameters. Bennet [1] proposed a test for k normal populations using transformed sample CV's. Afterwards, this test was modified by Shafer and Sullivan [17]. Gupta and Ma [11] developed score test for the case of k normal populations and compared their test to some of the existing tests. Fung and Tsang [8] compared parametric and nonparametric tests by using simulation studies for the equality of coefficients of variation in k normal populations. All of the test statistics mentioned above have asymptotic chi-squared distributions with k-1 degrees of freedom. Thus, there is no exact statistical test for the equality of coefficients of variation in k normal populations. Approximate methods have been applied to solve a number of problems when conventional methods are difficult to apply or fail to provide exact solutions. Exact distributions of approximate methods are not known and the p-values can be only found by simulation. One of these methods is introduced by Tsui and Weerahandi [21], which is called the generalized p-value approach. Many researchers developed test procedures based on the generalized p-value for hypothesis testing [22, 12, 18, 24, 13]. Liu et al. [13] applied the generalized p-value approach for the equality of coefficients of variation in k normal populations and compared this approach to some of the existing tests; the likelihood ratio, modified Bennett's, score tests.

In this paper, a computational approach test (CAT) which is one of the most popular approximate methods is proposed for the equality of coefficients of variation in k normal populations. The CAT method based on simulation and numerical computations uses the maximum likelihood estimates (MLEs), and does not require the knowledge of any sampling distribution. This approach provides an algorithmic framework based on the Monte-Carlo simulation and numerical computations when a suitable parametric model is assumed for a given dataset [16]. Some papers related to CAT can be given as follows. Chang et al. [3, 5] showed how the CAT can be applied to Poisson and Gamma models for hypothesis testing. Chang and Pal [2] applied CAT to test the equality of two population means when the variances are unknown and arbitrary. Also Chang et al. [4] demonstrated that the CAT is as powerful as the classical F test for one-way ANOVA under homoscedasticity. Gokpinar and Gokpinar [9] modified CAT to test the equality of k population means when the variances are unequal. Also Gokpinar et al. [10] proposed a test based on CAT for the equality of several inverse Gaussian means under heterogeneity.

This article is organized as follows. In Section 2, the likelihood ratio, modified Bennett's, score tests used for the equality of coefficients of variation in k normal populations, are presented. In Section 3, the general CAT procedure and its algorithm are given in detail. Also in the third section the application of the CAT procedure for equality of coefficients of variation in k normal populations is presented. In Section 4, simulation results of estimated type I error rates and powers are obtained by using Monte Carlo studies. In section 5, CAT procedure is applied to a real life dataset. Concluding remarks are summarized in Section 6.

2. Tests for equality of coefficients of variation

In this section, the likelihood ratio, modified Bennett's, score, generalized p-value tests for testing equality of coefficients of variation in k normal populations, are presented. Initially, we need to give some notations and assumptions.

Let $X_{i1}, X_{i2}, \cdots, X_{in_i}$ be a random sample with size n_i from$N\left(\mu_i, \sigma_i^2\right), \; i = 1, \cdots, k$ where μ_i and σ_i^2 are the mean and variance of ith population, respectively and the coefficients of variation for ith population are defined as $R_i = \sigma_i / \mu_i, \; i = 1, ..., k$. The problem of interest involves testing:

$$H_0 : R_1 = R_2 = \ldots = R_k = R \quad \text{against} \quad H_A : \; \exists R_i \neq R_j \quad i = 1, \ldots, k. \tag{2.1}$$

Let $\bar{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i$, $S_i^2 = \sum_{j=1}^{n_i} \left(X_{ij} - \bar{X}_i\right)^2 \Big/ n_i$ and $r_i = S_i \big/ \bar{X}_i$ denote the ith sample mean, variance and coefficients of variation for $i = 1, \cdots, k$, respectively. Let $\bar{x}_i$ and s_i^2denote the ith observed sample mean and variance for $i = 1, \cdots, k$, respectively. Similar to Liu et. al. [13], we assume,

(1) $\mu_i > 0$;
(2) $P\left(\bar{X}_i < 0\right)$,for each of $i = 1, \cdots, k$ is very small.

2.1. *Likelihood ratio test*

Likelihood ratio test is proposed by Doornbos and Dijkstra [6]. The likelihood function under H_0 is given as follows:

$$(2.2) \qquad L_0 = \prod_{i=1}^{k} \left(\frac{1}{\sqrt{2\pi}\mu_i R}\right)^{n_i} \exp\left(-\sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\left(X_{ij} - \mu_i\right)^2}{2\mu_i^2 R^2}\right)$$

Differentiating the Equation (2.2) with respect to μ_i and R yields the following results:

$$(2.3) \qquad \frac{\partial \ln L_0}{\partial R} = -\sum_{i=1}^{k} \frac{n_i}{R} + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\left(X_{ij} - \mu_i\right)^2}{\mu_i^2 R^3} = 0,$$

$$(2.4) \qquad \frac{\partial \ln L_0}{\partial \mu_i} = -\frac{n_i}{\mu_i} + \sum_{j=1}^{n_i} \frac{X_{ij}\left(X_{ij} - \mu_i\right)}{\mu_i^3 R^2} = 0, \quad i = 1, \cdots, k\,.$$

The Equation (2.3) and Equation (2.4) are simplified as follows:

$$(2.5) \qquad \sum_{i=1}^{k} \frac{n_i\left(1 + \sqrt{1 + 4\left(1 + r_i^2\right)R^2}\right)}{2\left(1 + r_i^2\right)} - \sum_{i=1}^{k} n_i = 0$$

$$(2.6) \qquad \mu_i = \frac{2\left(1 + r_i^2\right)\bar{X}_i}{1 + \sqrt{1 + 4\left(1 + r_i^2\right)R^2}}, \quad i = 1, ..., k.$$

As seen from Equation (2.5) and Equation (2.6), the restricted MLEs (RMLEs) of the R and μ_i have no closed forms. Therefore, the numerical method called bisection method, which is proposed by Gupta and Ma [11], could be used for the RMLEs of these parameters. Hence, the likelihood ratio test statistic is given as follows:

$$(2.7) \qquad -2\ln\lambda = \sum_{i=1}^{k} n_i \ln\left(\frac{\hat{\mu}_{i(RML)}^2 \hat{R}_{RML}^2}{S_i^2}\right) \sim \chi_{k-1}^2,$$

where $\hat{\mu}_{i(RML)}$ and $\hat{R}_{RML}$ are the RMLEs of μ_i and R. For a given level α, this test rejects the H_0 in Equation (2.1) if $-2\ln\lambda > \chi_{k-1,\alpha}^2$.

2.2. *Modified Bennett's test*

Shafer and Sullivan [17] modified Bennett's test given as

$$(2.8) \qquad -2\ln\lambda = (N-k)\ln\sum_{i=1}^{k}\left(\frac{d_i}{N-k}\right) - \sum_{i=1}^{k}(n_i - 1)\ln\left(\frac{d_i}{n_i - k}\right) \sim \chi_{k-1}^2.$$

Here $N = \sum_{i=1}^{k} n_i$ and $d_i = n_i r_i^2 \big/ \left(r_i^2 + 1\right)$. For a given level α, this test rejects the null hypothesis in Equation (2.1) if $-2\ln\lambda > \chi_{k-1,\alpha}^2$.

2.3. *Score test*

To test the null hypothesis in Equation (2.1), the explicit value of test statistic is given by

$$(2.9) \qquad S = \left[\frac{\hat{R}_{RML}^2\left(2\hat{R}_{RML}^2 + 1\right)}{2}\right] \sum_{i=1}^{k} \frac{a_i^2}{n_i} \sim \chi_{k-1}^2,$$

where $a_i = \sum_{j=1}^{n_i} \left(X_{ij} - \hat{\mu}_{i(RML)} \right)^2 \Big/ \hat{\mu}_{i(RML)}^2 \hat{R}_{RML}^3 - n_i \Big/ \hat{R}_{RML}$ (see [11] for details). For a given level α, this test rejects the null hypothesis in Equation (2.1) if $-2\ln\lambda > \chi^2_{k-1,\alpha}$.

2.4. *Generalized p-value test*

Tsui and Weerahandi [21] presented the concept of generalized p-value for some statistical testing problems. A parallel test for the equality of coefficients of variation in k normal populations is developed as follows [13]. The null hypothesis in Equation (2.1) can be rewritten as follows:

$$H_{10}: \mu_1/\sigma_1 = \mu_2/\sigma_2 = \ldots = \mu_k/\sigma_k \,. \tag{2.10}$$

Put $C = (\mu_1/\sigma_1, \mu_2/\sigma_2, ..., \mu_k/\sigma_k)'$ and

$$A = \begin{pmatrix} 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix}_{(k-1)\times k} .$$

Thus, the null hypothesis in Equation (2.7) is equivalent to $H_{20} : AC = 0$. The generalized pivotal quantities for μ_i and σ_i are given as follows:

$R_{\mu_i} = \bar{x}_i - \frac{s_i}{S_i}\left(X_i - \mu_i\right)$ and $R_{\sigma_i} = \frac{S_i}{s_i}\sigma_i$

[23]. The generalized pivotal quantity for AC could be written as follows.

$$R_{AC} = AR_c = A\left(\frac{R_{\mu_1}}{R_{\sigma_1}}, ..., \frac{R_{\mu_k}}{R_{\sigma_k}}\right)' .$$

Here $\frac{R_{\mu_i}}{R_{\sigma_i}} = \frac{\bar{x}_i - s_i \big/ S_i\left(\bar{X}_i - \mu_i\right)}{s_i \big/ S_i \sigma_i} = \frac{\bar{x}_i}{\sqrt{n_i}s_i}U_i - \frac{Z_i}{\sqrt{n_i}}$ and $U_i^2 \sim \chi^2_{(n_i-1)},\ Z_i \sim N(0,1),\ i = 1, ..., k$.

The conditional expectation and covariance matrix of R_{AC} for given $(\bar{x}, s)$ can be written as:

$$\mu_R = A\left(E\left(\frac{R_{\mu_1}}{R_{\sigma_1}}(\bar{x}, s)\right), ..., E\left(\frac{R_{\mu_k}}{R_{\sigma_k}}(\bar{x}, s)\right)\right)',$$

$$\Sigma_R = A\,diag\left(Var\left(\frac{R_{\mu_1}}{R_{\sigma_1}}(\bar{x}, s)\right), ..., Var\left(\frac{R_{\mu_k}}{R_{\sigma_k}}(\bar{x}, s)\right)\right)A',$$

where

$$E\left(\frac{R_{\mu_1}}{R_{\sigma_1}}(\bar{x}, s)\right) = \frac{\bar{x}_i}{\sqrt{n_i}s_i}E(U_i)\,,$$

$$Var\left(\frac{R_{\mu_1}}{R_{\sigma_1}}(\bar{x}, s)\right) = \frac{\bar{x}_i^2}{n_i s_i^2}Var(U_i) + \frac{1}{n_i}\,,$$

and

$$E(U_i) = \begin{cases} \sqrt{\frac{2}{\pi}}\frac{(n_i-2)!!}{(n_i-3)!!}, & n_i \geq 3 \text{ and } n_i \text{ is odd} \\ \sqrt{\frac{\pi}{2}}\frac{(n_i-2)!!}{(n_i-3)!!}, & n_i \geq 4 \text{ and } n_i \text{ is even} \end{cases}$$

$$Var(U_i) = n_i - 1 - (E(U_i))^2 .$$

The standardized expression of $R_{AC}$is $D = (\Sigma_R)^{-1/2}(R_{AC} - \mu_R)$and d is the observed value of D for $(\bar{X}, S) = (\bar{x}, s)$. Then $\|D\|^2 = (R_{AC} - \mu_R)' \Sigma_R^{-1}(R_{AC} - \mu_R)$ and the observed value $\|d\|^2$under $H_{20} : AC = 0$ is equal to $\mu_R' \Sigma_R^{-1} \mu_R$. The generalized p-value based on $\|D\|^2$could be given as follows:

$$\begin{aligned} p &= P\left(\|D\|^2 \geq \|d\|^2 / H_{20}\right) \\ &= P\left((R_{AC} - \mu_R)' \Sigma_R^{-1}(R_{AC} - \mu_R) \geq \mu_R' \Sigma_R^{-1} \mu_R\right) \\ &= P\left(\sum_{i=1}^k \frac{[\bar{x}_i(U_i - E(U_i)) - s_i Z_i]^2}{\bar{x}_i^2 Var(U_i) + s_i^2} - \frac{1}{\sum_{j=1}^k n_j s_j^2 / \left(\bar{x}_i^2 Var(U_i) + s_i^2\right)}\right. \\ &\qquad \times \left(\sum_{i=1}^k \frac{[\bar{x}_i(U_i - E(U_i)) - s_i Z_i]\sqrt{n_i} s_i}{\bar{x}_i^2 Var(U_i) + s_i^2}\right)^2 \\ &\qquad \left. \geq \sum_{i=1}^k \frac{(\bar{x}_i E(U_i))^2}{\bar{x}_i^2 Var(U_i) + s_i^2} - \frac{1}{\sum_{j=1}^k n_j s_j^2 / \left(\bar{x}_i^2 Var(U_i) + s_i^2\right)} \left(\sum_{i=1}^k \frac{\bar{x}_i E(U_i)\sqrt{n_i} s_i}{\bar{x}_i^2 Var(U_i) + s_i^2}\right)^2 \right). \end{aligned} \tag{2.11}$$

H_{10} in Equation (2.10) is rejected if $p< \alpha$.

3. The computational approach test

In this section, initially we introduce the general framework of CAT procedure. By using this procedure, we give an algorithm for testing equality of coefficients of variation in k normal populations. The algorithm of CAT based on the paper of Pal et al. [16] can be given as follows:

Let X_1, X_2,...,$X_n$be random sample having a probability density function $f(x/\theta)$, where the functional form of f is assumed to be known and θ=($\theta^{(1)}$,$\theta^{(2)}$) is an unknown vector in parameter space Θ. $\theta^{(1)}$ is the parameter of interest and $\theta^{(2)}$ is the nuisance parameter. The problem of interest is to test $H_0' : \theta^{(1)} = \theta_0^{(1)}$ against a suitable alternative $H_1'^*$. To do this, initially we express H_0' as $H_0'^* : \eta\left(\theta^{(1)}, \theta_0^{(1)}\right) = 0$ where η is a scalar valued function. The general methodology of the proposed CAT for testing $H_0'^* : \eta\left(\theta^{(1)}, \theta_0^{(1)}\right) = 0$ against $H_1'^*$ at a desired level α is given through the following steps.

1. Calculate $\hat{\theta}_{ML} = \left(\hat{\theta}_{ML}^{(1)},\ \hat{\theta}_{ML}^{(2)}\right)$, where $\hat{\theta}_{ML}$is maximum likelihood estimation (MLE) of θ. Thus,$\hat{\eta}_{ML} = \eta\left(\hat{\theta}_{ML}^{(1)},\ \theta_0^{(1)}\right)$ is estimated.

2. Find the MLE of $\theta^{(2)}$ under H_0' or $H_0'^*$ from the data which is called the restricted MLE (RMLE) of $\theta^{(2)}$ and denoted by $\hat{\theta}_{RML}^{(2)}$.

3. Generate artificial sample X_1, X_2,...,X_n from $f\left(x/\theta_0^{(1)},\ \hat{\theta}_{RML}^{(2)}\right)$ a large number of times (say m times). For each of these data, recalculate the MLE of $\theta^{(1)}$, i.e., $\tilde{\theta}_1^{(1)}, \tilde{\theta}_2^{(1)}, ..., \tilde{\theta}_m^{(1)}$ and $\tilde{\eta}_{ML}^{(j)} = \eta\left(\tilde{\theta}_j^{(1)},\ \theta_0^{(1)}\right)$, $\quad j = 1, ..., m$.

4. For testing $H_0'^* : \eta\left(\theta^{(1)}, \theta_0^{(1)}\right) = 0$ versus $H_1'^* : \eta\left(\theta^{(1)}, \theta_0^{(1)}\right) > 0$, calculate the p-value as $p = \#\left(\tilde{\eta}_{ML}^{(j)} > \hat{\eta}_{ML}\right) \Big/ m$. In the case of $p<\alpha$, H_0 is rejected.

Remark 3.1: The success of CAT depends heavily on the selection of η. For this purpose, the choice of η needs a little clarification. According to Chang et. al.[5], CAT works best (in terms of maintaining the desired level and attaining a high power) for two different situations as follows. When we have location parameters which can take values over the real line, we can use the standard quadratic expression for η as it is done in classical one-way ANOVA under normality assumption. When our parameters are nonnegative, and so are the observations, the logarithmic transformation makes the parameters behave like location parameters as it is done in gamma model [5]. Thus, interested parameters take nonnegative values, as we have here, firstly we use the logarithmic transformation of parameters, and after that standard quadratic (or squared) expression of these transformed parameters is used for η, i.e. $\eta = \sum_{i=1}^{k} n_i \left(\log \theta_i^{(1)} - \log \overline{\theta^{(1)}}\right)^2$, where $\overline{\theta^{(1)}} = \sum_{i=1}^{k} \theta_i^{(1)} \Big/ k$.

Remark 3.2: The CAT procedure borrows ideas from the classical likelihood ratio test as well as parametric bootstrap [2]. It is well known that the classical likelihood ratio test is based on MLE under H_0, that is, RMLE. Score test is also based on RMLE. Both the classical likelihood ratio test and Score test use test statistics which are asymptotically distributed as Chi-square under H_0. However, the CAT method uses the idea of replicating data from $f\left(x/\theta_0^{(1)}, \hat{\theta}_{RML}^{(2)}\right)$.

In the rest of this section, a test procedure based on CAT is given for testing equality of coefficients of variation in k normal populations based on algorithm given above.

Initially, the null hypothesis given in Equation (2.1) should be expressed in terms of suitable scalar η based on the criteria given in Remark. Thus, η is defined as shown in Equation (3.1):

$$\eta = \eta\left(R_1, R_2 ..., R_k\right) = \sum_{i=1}^{k} n_i \left(\log\, R_i - \log\, \bar{R}\right)^2, \tag{3.1}$$

where $\bar{R} = \sum_{i=1}^{k} R_i/k$. It is clear that testing H_0 against H_1 is equivalent to testing $H_0^* : \eta = 0$against $H_1^* : \eta > 0$. With the general idea of CAT which is given above, its application for testing equality of coefficients of variation in k normal populations can be given as below:

1.The sample coefficient of variation for the ith group is $r_i = S_i \big/ \bar{X}_i$. Therefore, the $\hat{\eta}_{ML}$ is obtained $\hat{\eta}_{ML} = \sum_{i=1}^{k} n_i \left(\log\, r_i - \log\, \bar{r}\right)^2$ by using these sample coefficient of variation. Here $\bar{r} = \sum_{i=1}^{k} r_i/k$. The observed value of $\hat{\eta}_{ML}$ is $\hat{\eta}_{ML}^*$.

2.Under H_0or H_0^*, the restricted MLEs $\left(\hat{\mu}_{i(RML)},\ \hat{R}_{RML}\right)$ of $(\mu_i,\ R)$ are obtained iteratively from Equation (2.5) and Equation (2.6) by using bisection method given in Gupta and Ma [11].

3.Generate artificial sample $X_{i1}, X_{i2}, \ldots, X_{in_i}$, $1 \leq i \leq k$ i.i.d. from $N\left(\hat{\mu}_{i(RML)}, \hat{\mu}_{i(RML)}^2 \times \hat{R}_{RML}^2\right)$ a large of number of times (say m times). For each of these replicated samples, recalculate the values of $\tilde{\eta}_{ML}^{(j)}$ $(j = 1, ..., m)$.

4. Calculate the p-value as $p = \#\left(\tilde{\eta}_{ML}^{(j)} > \hat{\eta}_{ML}^*\right) \Big/ m$. In the case of $p<\alpha$, H_0 is rejected.

Remark 3.3: By generating artificial sample we are trying to mimic the null distribution of $\hat{\eta}_{ML}$. Thus the cut-off point $\tilde{\eta}_C = \tilde{\eta}_{ML((1-\alpha)m)}$ is an approximation of the true critical value based on the null model. $\hat{\eta}_{ML}$ acts as an automatic test statistic, and helps us make a decision based on the value of $\tilde{\eta}_C$ [3].

4. A simulation study

In this section for testing equality of coefficients of variation in k normal populations, the likelihood ratio test (LRT), modified Bennett's test (MBT), score test (SCT), generalized p-value test (GPT) and CAT are compared according to type I errors and powers for different combinations of parameters $(\mu_i,\ \sigma_i)$ and sample sizes. For this purpose, we consider some cases from smaller to larger sample sizes with different number of groups as k=3, 4, 5, 6, 7. For specified nominal level of α=0.05, 5000 replications are used to calculate the estimated type I error rates and powers of each tests. Also 5000 replications are used to obtain the p values of GPT and CAT.

Firstly, we calculate the type I error rates of tests under null hypothesis for $(\mu_i = 3,\ \sigma_i = 1,\ i = 1, 2, \ldots, k)$ .The numerical results for estimated type I error rates are given as in Table 1 to Table 5.

Table 1. Estimated type I error rates of tests for k=3

n	CAT	GPT	LRT	MBT	SCT
6,6,6	0.058	0.026	0.109	0.070	0.053
6,8,10	0.049	0.032	0.086	0.055	0.044
10,10,10	0.048	0.039	0.073	0.055	0.047
15,15,20	0.050	0.041	0.066	0.051	0.046
20,20,20	0.050	0.042	0.061	0.053	0.050
10,15,20	0.049	0.041	0.068	0.054	0.047
10,20,30	0.046	0.038	0.063	0.051	0.047
30,30,30	0.051	0.048	0.057	0.051	0.050

Table 2. Estimated type I error rates of tests for k=4

n	CAT	GPT	LRT	MBT	SCT
6,6,6,6	0.051	0.019	0.103	0.060	0.056
6,8,10,12	0.054	0.035	0.095	0.063	0.056
10,15,20,25	0.055	0.042	0.073	0.056	0.051
10,10,10,10	0.052	0.035	0.081	0.058	0.054
10,10,15,15	0.053	0.038	0.079	0.057	0.055
20,20,20,20	0.050	0.042	0.061	0.052	0.051
15,15,20,20	0.053	0.045	0.069	0.053	0.057
10,20,20,30	0.051	0.044	0.071	0.057	0.057
30,30,30,30	0.047	0.041	0.055	0.046	0.048

Table 3. Estimated type I error rates of tests for k=5

n	CAT	GPT	LRT	MBT	SCT
6,6,6,6,6	0.051	0.020	0.122	0.065	0.068
6,8,10,12,14	0.051	0.033	0.091	0.058	0.059
10,15,20,25,30	0.049	0.038	0.070	0.051	0.051
10,10,10,10,10	0.050	0.031	0.092	0.058	0.065
20,20,20,20,20	0.052	0.043	0.068	0.053	0.053
15,15,15,20,20	0.053	0.043	0.067	0.054	0.056
10,10,10,15,15	0.052	0.033	0.082	0.056	0.058
10,10,20,30,30	0.053	0.042	0.075	0.055	0.057
30,30,30,30,30	0.055	0.050	0.064	0.056	0.058

Table 4. Estimated type I error rates of tests for k=6

n	CAT	GPT	LRT	MBT	SCT
6,6,6,6,6,6	0.053	0.017	0.127	0.064	0.075
6,8,10,10,12,14	0.052	0.028	0.094	0.057	0.060
10,15,20,20,25,30	0.055	0.041	0.075	0.057	0.055
10,10,10,10,10,10	0.050	0.028	0.091	0.057	0.064
20,20,20,20,20,20	0.055	0.043	0.069	0.055	0.057
15,15,15,20,20,20	0.054	0.042	0.071	0.054	0.059
10,10,10,15,15,15	0.056	0.035	0.087	0.061	0.060
10,10,20,20,30,30	0.049	0.038	0.070	0.050	0.054
30,30,30,30,30,30	0.052	0.046	0.062	0.053	0.053

Table 5. Estimated type I error rates of tests for k=7

n	CAT	GPT	LRT	MBT	SCT
6,6,6,6,6,6	0.052	0.016	0.136	0.066	0.079
6,8,10,10,10,12,14	0.053	0.029	0.099	0.063	0.073
10,15,20,20,20,25,30	0.051	0.041	0.071	0.054	0.064
10,10,10,10,10,10,10	0.052	0.030	0.095	0.058	0.073
20,20,20,20,20,20,20	0.055	0.044	0.074	0.057	0.061
15,15,15,20,20,20,20	0.052	0.038	0.067	0.052	0.056
10,10,10,15,15,15,15	0.053	0.033	0.082	0.058	0.066
10,10,20,20,20,30,30	0.049	0.036	0.069	0.054	0.058
30,30,30,30,30,30,30	0.048	0.044	0.058	0.048	0.055

As seen from Table 1-Table 5, the GPT seems to have lower the estimated type I error rates than nominal level, especially for small sample size. Contrary to GPT, LRT has estimated type I error rates greater than the nominal level. In the case of small sample size, the estimated type I error rates of MBT exceed the nominal level for all k. In the case of small sample size, the estimated type I error rates of SCT get larger than nominal level, especially when k is large. Also, it is observed that the MBT and SCT have the estimated type I error rates close to the nominal level for other cases. However, the CAT seems to have the estimated type I error rates close to nominal level in all cases.

After calculating the type I error rates of five methods, we calculate the estimated powers of the tests for different combinations of parameters and sample sizes.

Table 6. Estimated powers of tests for μ_i=3 (i=1,2,3)

n	R_1, R_2, R_3	CAT	GPT	LRT	MBT	SCT
6,6,6	1/9,1/6,1/6	0.109	0.057	0.207	0.138	0.102
	1/9,2/9,2/9	0.246	0.133	0.387	0.287	0.179
	1/9,1/3,1/3	0.592	0.346	0.718	0.612	0.290
	1/4,1/2,1/2	0.215	0.122	0.336	0.235	0.150
6,8,10	1/9,1/6,1/6	0.144	0.054	0.196	0.124	0.073
	1/9,2/9,2/9	0.326	0.122	0.396	0.275	0.117
	1/9,1/3,1/3	0.718	0.314	0.765	0.625	0.188
	1/4,1/2,1/2	0.267	0.118	0.366	0.252	0.116
10,10,10	1/9,1/6,1/6	0.175	0.132	0.231	0.185	0.137
	1/9,2/9,2/9	0.477	0.359	0.545	0.477	0.290
	1/9,1/3,1/3	0.885	0.797	0.913	0.880	0.617
	1/4,1/2,1/2	0.403	0.319	0.478	0.406	0.265
15,15,20	1/9,1/6,1/6	0.312	0.237	0.341	0.299	0.208
	1/9,2/9,2/9	0.748	0.646	0.764	0.728	0.536
	1/9,1/3,1/3	0.990	0.977	0.990	0.988	0.938
	1/4,1/2,1/2	0.652	0.576	0.685	0.636	0.478
20,20,20	1/9,1/6,1/6	0.382	0.334	0.410	0.382	0.309
	1/9,2/9,2/9	0.848	0.798	0.855	0.838	0.742
	1/9,1/3,1/3	0.999	0.998	0.999	0.999	0.994
	1/4,1/2,1/2	0.781	0.736	0.796	0.767	0.676
30,30,30	1/9,1/6,1/6	0.557	0.514	0.571	0.552	0.484
	1/9,2/9,2/9	0.975	0.963	0.974	0.971	0.948
	1/9,1/3,1/3	1.000	1.000	1.000	1.000	1.000
	1/4,1/2,1/2	0.940	0.925	0.945	0.936	0.900

Table 7. Estimated powers of tests for μ_i=3 (i=1,2,3,4)

n	R_1, R_2, R_3, R_4	CAT	GPT	LRT	MBT	SCT
6,6,6,6	1/9,1/6,1/6,1/6	0.095	0.044	0.197	0.122	0.105
	1/9,2/9,2/9,2/9	0.220	0.093	0.360	0.250	0.160
	1/9,1/3,1/3,1/3	0.543	0.215	0.682	0.545	0.247
	1/4,1/2,1/2,1/2	0.195	0.092	0.330	0.213	0.159
6,8,10,12	1/9,1/6,1/6,1/6	0.147	0.054	0.204	0.125	0.082
	1/9,2/9,2/9,2/9	0.324	0.096	0.377	0.249	0.111
	1/9,1/3,1/3,1/3	0.708	0.218	0.727	0.564	0.159
	1/4,1/2,1/2,1/2	0.253	0.093	0.345	0.214	0.115
10,10,10,10	1/9,1/6,1/6,1/6	0.169	0.118	0.230	0.184	0.139
	1/9,2/9,2/9,2/9	0.450	0.299	0.521	0.444	0.255
	1/9,1/3,1/3,1/3	0.886	0.701	0.906	0.862	0.431
	1/4,1/2,1/2,1/2	0.391	0.272	0.469	0.380	0.238
15,15,20,20	1/9,1/6,1/6,1/6	0.305	0.207	0.323	0.279	0.175
	1/9,2/9,2/9,2/9	0.743	0.576	0.743	0.689	0.394
	1/9,1/3,1/3,1/3	0.994	0.967	0.992	0.990	0.769
	1/4,1/2,1/2,1/2	0.635	0.494	0.666	0.596	0.341
20,20,20,20	1/9,1/6,1/6,1/6	0.365	0.294	0.385	0.355	0.252
	1/9,2/9,2/9,2/9	0.477	0.359	0.545	0.477	0.290
	1/9,1/3,1/3,1/3	1.000	0.999	1.000	1.000	0.981
	1/4,1/2,1/2,1/2	0.769	0.694	0.783	0.747	0.562
30,30,30,30	1/9,1/6,1/6,1/6	0.551	0.476	0.552	0.528	0.412
	1/9,2/9,2/9,2/9	0.977	0.958	0.976	0.973	0.914
	1/9,1/3,1/3,1/3	1.000	1.000	1.000	1.000	1.000
	1/4,1/2,1/2,1/2	0.941	0.917	0.942	0.931	0.850

Table 8. Estimated powers of tests for μ_i=3 (i=1,2,...,5)

n	$R_1,\ R_2,\ R_3,\ R_4,\ R_5$	CAT	GPT	LRT	MBT	SCT
6,6,6,6,6	1/9,1/6,1/6,1/6,1/6	0.093	0.040	0.194	0.121	0.108
	1/9,2/9,2/9,2/9,2/9	0.190	0.076	0.338	0.221	0.161
	1/9,1/3,1/3,1/3,1/3	0.491	0.165	0.639	0.481	0.227
	1/4,1/2,1/2,1/2,1/2	0.186	0.073	0.333	0.205	0.167
6,8,10,12,14	1/9,1/6,1/6,1/6,1/6	0.143	0.047	0.184	0.114	0.075
	1/9,2/9,2/9,2/9,2/9	0.323	0.088	0.361	0.233	0.109
	1/9,1/3,1/3,1/3,1/3	0.695	0.180	0.686	0.509	0.147
	1/4,1/2,1/2,1/2,1/2	0.234	0.083	0.316	0.190	0.101
10,10,10,10,10	1/9,1/6,1/6,1/6,1/6	0.150	0.094	0.225	0.166	0.136
	1/9,2/9,2/9,2/9,2/9	0.416	0.234	0.482	0.401	0.222
	1/9,1/3,1/3,1/3,1/3	0.879	0.609	0.897	0.845	0.358
	1/4,1/2,1/2,1/2,1/2	0.345	0.220	0.425	0.328	0.216
15,15,15,20,20	1/9,1/6,1/6,1/6,1/6	0.256	0.168	0.282	0.235	0.155
	1/9,2/9,2/9,2/9,2/9	0.709	0.495	0.705	0.645	0.330
	1/9,1/3,1/3,1/3,1/3	0.993	0.948	0.991	0.984	0.606
	1/4,1/2,1/2,1/2,1/2	0.588	0.435	0.618	0.540	0.293
20,20,20,20,20	1/9,1/6,1/6,1/6,1/6	0.346	0.261	0.369	0.339	0.234
	1/9,2/9,2/9,2/9,2/9	0.844	0.729	0.840	0.814	0.541
	1/9,1/3,1/3,1/3,1/3	1.000	0.995	0.999	0.999	0.918
	1/4,1/2,1/2,1/2,1/2	0.748	0.644	0.758	0.714	0.481
30,30,30,30,30	1/9,1/6,1/6,1/6,1/6	0.531	0.449	0.535	0.506	0.377
	1/9,2/9,2/9,2/9,2/9	0.974	0.944	0.970	0.965	0.858
	1/9,1/3,1/3,1/3,1/3	1.000	1.000	1.000	1.000	0.999
	1/4,1/2,1/2,1/2,1/2	0.928	0.891	0.930	0.913	0.777

Table 9. Estimated powers of tests for μ_i=3 (i=1,2,...,6)

n	R_1, R_2, R_3, R_4, R_5, R_6	CAT	GPT	LRT	MBT	SCT
6,6,6,6,6,6	1/9,1/6,1/6,1/6,1/6,1/6	0.083	0.036	0.195	0.112	0.118
	1/9,2/9,2/9,2/9,2/9,2/9	0.196	0.067	0.359	0.226	0.176
	1/9,1/3,1/3,1/3,1/3,1/3	0.465	0.139	0.637	0.448	0.237
	1/4,1/2,1/2,1/2,1/2,1/2	0.165	0.058	0.320	0.176	0.156
6,8,10,10,12,14	1/9,1/6,1/6,1/6,1/6,1/6	0.132	0.051	0.196	0.114	0.092
	1/9,2/9,2/9,2/9,2/9,2/9	0.273	0.075	0.331	0.207	0.107
	1/9,1/3,1/3,1/3,1/3,1/3	0.620	0.150	0.641	0.454	0.145
	1/4,1/2,1/2,1/2,1/2,1/2	0.222	0.076	0.305	0.177	0.110
10,10,10,10,10,10	1/9,1/6,1/6,1/6,1/6,1/6	0.147	0.089	0.209	0.159	0.129
	1/9,2/9,2/9,2/9,2/9,2/9	0.402	0.214	0.476	0.384	0.230
	1/9,1/3,1/3,1/3,1/3,1/3	0.863	0.534	0.876	0.810	0.338
	1/4,1/2,1/2,1/2,1/2,1/2	0.329	0.182	0.418	0.310	0.206
15,15,15,20,20,20	1/9,1/6,1/6,1/6,1/6,1/6	0.246	0.153	0.278	0.224	0.151
	1/9,2/9,2/9,2/9,2/9,2/9	0.682	0.449	0.678	0.612	0.305
	1/9,1/3,1/3,1/3,1/3,1/3	0.991	0.923	0.987	0.979	0.513
	1/4,1/2,1/2,1/2,1/2,1/2	0.559	0.395	0.586	0.506	0.271
20,20,20,20,20,20	1/9,1/6,1/6,1/6,1/6,1/6	0.306	0.218	0.330	0.292	0.203
	1/9,2/9,2/9,2/9,2/9,2/9	0.680	0.439	0.671	0.605	0.290
	1/9,1/3,1/3,1/3,1/3,1/3	0.999	0.991	0.998	0.997	0.818
	1/4,1/2,1/2,1/2,1/2,1/2	0.718	0.590	0.727	0.673	0.418
30,30,30,30,30,30	1/9,1/6,1/6,1/6,1/6,1/6	0.507	0.407	0.499	0.471	0.331
	1/9,2/9,2/9,2/9,2/9,2/9	0.971	0.919	0.964	0.955	0.776
	1/9,1/3,1/3,1/3,1/3,1/3	1.000	1.000	1.000	1.000	0.999
	1/4,1/2,1/2,1/2,1/2,1/2	0.922	0.863	0.919	0.894	0.695

Table 10. Estimated powers of tests for μ_i=3 (i=1,2,...,7)

n	R_1, R_2, R_3, R_4, R_5, R_6,R_7	CAT	GPT	LRT	MBT	SCT
6,6,6,6,6,6,6	1/9, 1/6,1/6,1/6,1/6,1/6,1/6	0.084	0.035	0.203	0.113	0.124
	1/9,2/9,2/9,2/9,2/9,2/9,2/9	0.168	0.059	0.339	0.203	0.173
	1/9,1/3,1/3,1/3,1/3,1/3,1/3	0.439	0.120	0.607	0.422	0.226
	1/4,1/2,1/2,1/2,1/2,1/2,1/2	0.157	0.060	0.309	0.172	0.167
6,8,10,10,10,12,14	1/9, 1/6,1/6,1/6,1/6,1/6,1/6	0.119	0.044	0.182	0.109	0.084
	1/9,2/9,2/9,2/9,2/9,2/9,2/9	0.265	0.071	0.323	0.192	0.111
	1/9,1/3,1/3,1/3,1/3,1/3,1/3	0.584	0.134	0.597	0.407	0.141
	1/4,1/2,1/2,1/2,1/2,1/2,1/2	0.200	0.067	0.281	0.151	0.106
10,10,10,10,10,10,10	1/9, 1/6,1/6,1/6,1/6,1/6,1/6	0.134	0.077	0.212	0.151	0.137
	1/9,2/9,2/9,2/9,2/9,2/9,2/9	0.364	0.183	0.440	0.341	0.201
	1/9,1/3,1/3,1/3,1/3,1/3,1/3	0.835	0.462	0.849	0.774	0.312
	1/4,1/2,1/2,1/2,1/2,1/2,1/2	0.302	0.163	0.388	0.283	0.189
15,15,15,20,20,20,20	1/9, 1/6,1/6,1/6,1/6,1/6,1/6	0.239	0.145	0.267	0.214	0.154
	1/9,2/9,2/9,2/9,2/9,2/9,2/9	0.665	0.409	0.645	0.577	0.273
	1/9,1/3,1/3,1/3,1/3,1/3,1/3	0.987	0.877	0.981	0.969	0.455
	1/4,1/2,1/2,1/2,1/2,1/2,1/2	0.539	0.357	0.566	0.470	0.246
20,20,20,20,20,20,20	1/9, 1/6,1/6,1/6,1/6,1/6,1/6	0.303	0.217	0.323	0.284	0.202
	1/9,2/9,2/9,2/9,2/9,2/9,2/9	0.818	0.624	0.799	0.760	0.420
	1/9,1/3,1/3,1/3,1/3,1/3,1/3	0.999	0.986	0.998	0.997	0.713
	1/4,1/2,1/2,1/2,1/2,1/2,1/2	0.705	0.550	0.715	0.648	0.372
30,30,30,30,30,30,30	1/9, 1/6,1/6,1/6,1/6,1/6,1/6	0.493	0.383	0.489	0.458	0.301
	1/9,2/9,2/9,2/9,2/9,2/9,2/9	0.963	0.905	0.954	0.948	0.714
	1/9,1/3,1/3,1/3,1/3,1/3,1/3	1.000	1.000	1.000	1.000	0.993
	1/4,1/2,1/2,1/2,1/2,1/2,1/2	0.904	0.827	0.898	0.875	0.631

The numerical results for estimated powers of the tests are presented as above in Table 6 to Table 10. In most cases, the LRT can be disregarded because of its estimated type I error rates exceeding the nominal level. Although the estimated type I error rates of MBT and SCT are close to each other, the MBT performs better than the SCT in terms of their powers.

MBT performs slightly better than the CAT in terms of powers for small sample size. However the estimated type I error rates of MBT exceed the estimated type I error rates of CAT for in this case. If both of the MBT and CAT are compared for other all cases, the CAT appears to be more powerful than MBT does.

5. An application with real life dataset

In this section, the LRT, MBT, SCT, GPT and CAT are applied for two real life datasets given as follows.

Example 5.1. The first analysis uses data collected by Nairy and Rao [15]. The data related to survival times of patients collected from 4 hospitals, which was a part of the data by given Fleming and Harrington [7]. The data containing failure time of the patients and their summary statistics are presented in Table 11. 5000 replications are used to obtain the p values of GPT and CAT. The obtained test statistics are given in Table 12.

Table 11. Survival time of patients from 4 hospitals and their means, standard deviations and coefficient of variations

Hospitals	Survival time of patients	$\bar{x}_i$	s_i^2	$r_i = s_i/\bar{x}_i$
1	176, 105, 266, 227, 66	168	74.19	0.051
2	24, 5, 155, 54	59.5	57.84	0.128
3	58, 64, 15	45.7	21.82	0.102
4	147, 42, 305, 92, 30, 82, 265, 237, 208, 147	155.5	90.53	0.062

Table 12. The results of tests statistics

Tests	Values of test statistics	p
LRT	1.753	0.625
MBT	1.396	0.707
SCT	2.064	0.559
GPT	-	0.699
CAT	1.621	0.754

The values in Table 12 indicate that the tests do not reject the H_0 given in Equation (2.1) at nominal level 0.05.

Example 5.2. The second analysis uses data collected by Tsou [20]. Table 13 gives the respective numbers of birth in 1978 on Monday, Thursday, and Saturday in the United Kingdom and their means, standard deviations and coefficient of variations. 5000 replications are used to obtain the p values of GPT and CAT. 5000 replications are used to obtain the p values of GPT and CAT.The obtained test statistics are given in Table 14.

Table 13. Numbers of birth in 1978 on Monday, Thursday, and Saturday in the United Kingdom and their means, standard deviations and coefficient of variations

Number of birth on Monday				Number of birth on Thursday				Number of birth on Saturday			
7527	9172	9458	9252	9043	9259	9226	9387	8084	8299	7954	7946
9184	9225	8966	9021	9218	9247	9103	9268	8065	8144	8167	8313
9262	9294	9022	9135	9304	9218	9327	9159	8008	8144	7965	7874
9100	9114	8870	8702	8902	8696	8724	8582	8069	7890	7527	7787
9017	8900	8987	9195	8839	8672	8903	9044	7750	7718	7762	8064
9089	7780	9127	9201	9180	9435	9075	9175	8005	7971	8040	8233
9543	9348	9284	9877	9405	9630	10184	9984	8122	8209	8773	8859
10026	9960	9890	10206	10386	10192	10128	10284	9062	8677	8738	8951
10127	9967	9998	8481	10377	10152	9489	10292	9023	9170	8735	8648
9927	9765	9531	9425	9949	9824	9502	9501	8605	8554	8411	8415
9457	9507	9606	9592	9245	9609	9568	7915	8246	8352	8432	8275
9825	8676	9686	10196	9396	9480	9524	9398	8528	8335	8507	7939
10154	10304	10414	7846	10265	10499	10175	10177	8904	8782	8580	8474
$\bar{x}_1$ =9350.3				$\bar{x}_2$ =9471.5				$\bar{x}_3$ =8309.3			
s_1^2 =376030				s_2^2 =307890				s_3^2 =152300			
$r_1 = s_1/\bar{x}_1$ =0.066				$r_2 = s_2/\bar{x}_2$ =0.056				$r_3 = s_3/\bar{x}_3$ =0.047			

Table 14. The results of tests statistics

Tests	Values of test statistics	p
LRT	5.708	0.058
MBT	5.598	0.061
SCT	5.220	0.074
GPT	-	0.064
CAT	3.014	0.057

The Table 14 shows that all of tests lead to the same conclusion, that is, all of tests do not reject the H_0 given in Equation (2.1) at nominal level 0.05.

6. Conclusion

In this article, we propose the CAT for testing the equality of coefficients of variation in k normal populations. We compare the CAT to some of the existing tests; the LRT, MBT, SCT, GPT. For a different sample sizes and number of groups, we investigate the performance of these tests using Monte Carlo simulation.

It could be observed from the simulation results that for small sample size the LRT approach seems to have the estimated type I error rates exceeding the nominal level and the GPT performs contrary to the LRT that its estimated type I error rates are lower than the nominal level. However, the estimated type I error rates of CAT are generally more conservative than other tests for all the sample size. Therefore, we could mention that the CAT is not affected from the changes in the sample size. Furthermore, according to power comparison results, the CAT appears to be more powerful than the other tests when the differences between coefficients of variation in k normal populations are increased.

Consequently, in respect to our simulation study, even when comparing different number of groups (as k=3, 4, 5, 6, 7), CAT could be suggested as a good alternative for testing the equality of coefficients of variation in k normal populations.

Acknowledgments: The authors are sincerely grateful to anonymous referees for their time and effort in providing very constructive, helpful and valuable comments and suggestions that have led to a substantial improvement in the quality of the paper.

References

[1] Bennett, B. M. *On approximate test for homogeneity of coefficients of variation*, In Contribution to Applied Statistics 169-171, 1976.

[2] Chang, C. H. and Pal, N. *A revisit to the Behren-Fisher problem: comparison of five test methods*, Communications in Statistics-Simulation and Computation **37**, 1064-1085, 2008.

[3] Chang, C. H., Pal, N., Lim, W. K. and Lin, J. J. *A note on comparing several poisson means*, Communications in Statistics-Simulation and Computation **39**, 1605-1627, 2010a.

[4] Chang, C. H., Pal, N., Lim, W. K. and Lin, J. J. *Comparing several population means: a parametric bootstrap method and its comparison with usual ANOVA F test as well as ANOM*, Computational Statistics **25**, 71-95, 2010b.

[5] Chang, C. H., Lin, J. J. and Pal, N. *Testing the equality of several gamma means: a parametric boostrap method with applications*, Computational Statistics **26**, 55-76, 2011.

[6] Doornbos, R. and Dijkstra, J. B. *A multi sample test for the equality of coefficients of variation in normal populations*, Communications in Statistics: Simulation and Computation **12**, 147-158, 1983.

[7] Flemingand, T. R. and Harrington, D. P. *Counting Processes and Survival Analysis* (New York, Wiley, 1991).

[8] Fung, W. K. and Tsang, T. S. *A simulation study comparing tests for the equality of coefficients of variation*, Statistics in Medicine **17**, 2003-2014, 1998.

[9] Gokpinar, E. Y. and Gokpinar, F. *A test based on computational approach for equality of means under unequal variance assumption*, Hacettepe *journal of* Mathematics and Statistics **41**, 605-613, 2012.

[10] Gokpinar, E. Y., Polat, E., Gokpinar, F. and Gunay, S. *A new computational approach for testing equality of inverse gaussian means under heterogeneity*, Hacettepe *journal of* Mathematics and Statistics **42**, 581-590, 2013.

[11] Gupta, R. C. and Ma, S. *Testing the equality of the coefficients of variation in k normal populations*, Communications in Statistics: Theory and Methods **25**, 115-132, 1996.

[12] Krishnamoorthy, K. and Mathew, T. *Inferences on the means of lognormal distributions using generalized p-values and generalized confidence intervals*, Journal of Statistical Planning and Inference **115**, 103-121, 2003.

[13] Liu, X., Xu, X. and Zhao, J. *A new generalized p-value approach for testing equality of coefficients of variation in k normal populations*, Journal of Statistical Computation and Simulation **81**, 1121-1130, 2011.

[14] Miller, E. C. and Karson, M. J. *Testing equality of two coefficients of variation*, Proceedings of the Business and Economics Section, Part I, 278-283, 1977.

[15] Nairy, K. S. and Rao, K. A. *Tests of coefficient of variation in normal population*, Communications in Statistics: Simulations and Computation **32**, 641-661, 2003.

[16] Pal, N., Lim, W. K. and Ling, C. H. *A computational approach to statistical inferences*, Journal of Applied Probability and Statistics **2**, 13-35, 2007.

[17] Shafer, N. G. and Sullivan, J. A. *A simulation study of a test for the equality of the coefficients of variation*, Communications in Statistics: Simulation and Computation **15**, 681-695, 1986.

[18] Tian, L. L. *Inferences on the common coefficient of variation*, Statistics in Medicine **24**, 2213-2220, 2005.

[19] Tsim, Y. L., Yip, S. P., Tsang, K. S., Li, K. F. and Wong, H. F. *Haematogoy and Serology* (In Annual Report, Honk Kong Medical Technology Association Quality Assurance Programme 25-40, 1991).

[20] Tsou, T.S. *A robust Score test for testing several coefficients of variation with unknown underlying disributions*, Communications in Statistics: Theory and Methods **38**, 1350-1360, 2009.

[21] Tsui, K. and Weerahandi, S. *Generalized p-values in significance testing of hypotheses in the presence of nuisance parameters*, Journal of the American Statistical Association **84**, 602-607, 1989.

[22] Weerahandi, S. *ANOVA under unequal error variances*, Biometrica **38**, 330-336, 1995.

[23] Xu X. and Li G., *Fiducial inference in the pivotal family of distributions*, Sci. China Ser. A Math. 49, 410–432, 2006.

[24] Xu, L. and Wang, S. *A new generalized p-value for ANOVA under heteroscedasticity*, Statistics and Probability Letters **78**, 963-969, 2007.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1215 – 1233

Mean square error comparisons of the alternative estimators for the distributed lag models

Berrin Gültay* and Selahattin Kaçıranlar†

Abstract

The finite distributed lag models include highly correlated variables as well as lagged and unlagged values of the same variables. Some problems are faced for this model when applying the ordinary least squares (OLS) method or econometric models such as Almon and Koyck models. The primary aim of this study is to compare the performances of alternative estimators to the OLS estimator defined by combining the Almon estimator with some other estimators according to the mean square error (MSE) criterion. We use Almon [2] data to illustrate our theoretical results.

Keywords: Finite distributed lag model, Almon estimator, Ridge estimator, Liu estimator.

2000 AMS Classification: 62J05, 62J07.

Received 15/12/2013 : Accepted 14/09/2014 Doi : 10.15672/HJMS.2014357556

1. Introduction

Consider the finite distributed lag model,

$$y_t = \beta_0 x_t + \beta_1 x_{t-1} + \cdots + \beta_p x_{t-p} + u_t, \qquad t = p+1, \cdots, T$$

$$= \sum_{i=0}^{p} \beta_i x_{t-i} + u_t \tag{1.1}$$

where u_t are $IN\left(0, \sigma_u^2\right)$. The coeffients β_i are called lag weights. The model in Eq.(1.1) can be written in the matrix notation as

$$y = X\beta + u \tag{1.2}$$

*Faculty of Arts and Sciences, Çanakkale Onsekiz Mart University, Çanakkale, Turkey.
Email: berringultay@comu.edu.tr Corresponding Author.
†Faculty of Arts and Sciences, Çukurova University, Adana, Turkey.
Email:skacir@cu.edu.tr

where

$$y = \begin{bmatrix} y_{p+1} \\ y_{p+2} \\ \vdots \\ y_T \end{bmatrix}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}, X = \begin{bmatrix} x_{p+1} & x_p & \dots & x_1 \\ x_{p+2} & x_{p+1} & \dots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_T & x_{T-1} & \dots & x_{T-p} \end{bmatrix}, u = \begin{bmatrix} u_{p+1} \\ u_{p+2} \\ \vdots \\ u_T \end{bmatrix}.$$

In case of estimating the model (1.1) by OLS, the following problems are encountered:

a) Multicollinearity problem among the explanatory variables may be occured. Because there are p lags of the same variables in the model.

b) The length of the lag, p, isn't known. Even if p is known, if this number is large and amount of the sample is small, it is unable to estimate the parameters.

To overcome these problems, some kind of distributed lag models have been suggested such as Koyck and Almon models (Yurdakul [21]). The most of these estimators require some prior information about the behavior of the β's in (1.1). In general, the two sources of prior information can be classified as nonstochastic and stochastic smoothness prior (Vinod and Ullah, [19]; Gujarati, [5]).

Irving Fisher [4] initially introduced nonstochastic smoothness prior information of the following type:

$$\begin{aligned} \beta_i &= (p+1-i)\alpha \qquad 0 \le i \le p \\ &= 0 \qquad\qquad\qquad i > p \end{aligned} \tag{1.3}$$

where α is any unknown parameter. Substituting (1.3) in (1.1) gives,

$$\begin{aligned} y_t &= \left[\sum_{i=0}^{p} (p+1-i) x_{t-i} \right] \alpha + u_t \\ &= z_t \alpha + u_t \end{aligned} \tag{1.4}$$

Thus the OLS estimate of α can be obtained from (1.4) and then using (1.3), the estimate of β_i can be obtained. A generalization of the linear nonstochastic prior on β_i can be written as

$$\beta_i = \alpha_0 + \alpha_1 i + \alpha_2 i^2 + \ldots + \alpha_r i^r \qquad p \ge r \ge 0 \tag{1.5}$$

which is a polynomial of the r^{th} degree. This structure on lag weights β_i was proposed by Almon [2] and is known as the Almon polynomial lag. Again, substituting (1.5) in (1.1) we can get estimates of the α's and then using (1.5) we can obtain the estimates of β_i . Eq. (1.5) can be written in the matrix notation as

$$\beta = A\alpha \tag{1.6}$$

where β is given before, and

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p & p^2 & \vdots & p^r \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_r \end{bmatrix}$$

are $A : (p+1) \times (r+1)$ matrix and $\alpha : (r+1) \times 1$ vector. The ranks of matrices X and A are assume to be $(p+1) < (T-p)$ and $(r+1) < (p+1)$, respectively. If $r < p$, then the rank of A is $r+1$. We estimate β in (1.2), under the nonstochastic prior information on β is given by (1.6), using Almon estimation method. By substituting (1.6) in (1.2),

$$\begin{aligned} y &= XA\alpha + u \\ &= Z\alpha + u, \qquad u \sim N\left(0, \sigma_u^2\right) \end{aligned} \tag{1.7}$$

is obtained. This model can be called a linear Almon distributed lag model. Then, OLS estimator of α in model (1.7) is

$$\widehat{\alpha}_A = \left(Z'Z\right)^{-1} Z'y = \left(A'X'XA\right)^{-1} A'X'y \ . \tag{1.8}$$

In this case,

$$\widehat{\beta}_A = A\widehat{\alpha}_A \tag{1.9}$$

is the Almon estimator of β. $\widehat{\beta}_A$ is the best linear unbiased estimator (BLUE).

2. Alternative methods

In this section some alternative biased estimators to the Almon estimator are defined for the distributed lag model.

2.1. The Almon-modified ridge estimator. Hoerl and Kennard's ridge regression estimator has been discussed as an alternative approach to resolve problems encountered in due to some disadvantages of Almon estimator (Maddala [14], Vinod and Ullah [19], Chanda and Maddala [3]). Distributed lag estimation seems tractable only when prior information on the lag coefficients is incorporated. Ridge regression introduces yet another representation of such prior information and hence is a possible estimation procedure (Yeo and Trivedi [20]).

The Almon-ridge estimator of α in model (1.7) is

$$\begin{aligned} \widehat{\alpha}_k &= \left(Z'Z + kI\right)^{-1} Z'y \\ &= \left(A'SA + kI\right)^{-1} A'X'y \qquad k > 0 \end{aligned} \tag{2.1}$$

where $S = X'X$. Thus

$$\widehat{\beta}_k = A\widehat{\alpha}_k \tag{2.2}$$

is the Almon-ridge estimator for the model (1.2). However, the ridge estimator and the extension given by Lindley and Smith [12] are not as promising for the distributed lag models (Maddala, [14]). They tried various values of the k. But they are not satisfied the results of some empirical examples with this method. Because the selection of k reveals several problems. Therefore, alternative estimation methods must be considered.

Swindel [16] introduced a modified ridge estimator based on prior information b_0. Almon-modified ridge estimator of α in model (1.7) is defined,

$$\widehat{\alpha}(k, b_0) = \left(Z'Z + kI\right)^{-1}\left(Z'y + kb_0\right). \tag{2.3}$$

As pointed out by Swindel [16], it seems more useful and reasonable in the applications to consider the prior information. To overcome multicollinearity problem, if we take $b_0 = \widehat{\alpha}_k$, (2.3) is reduced to

$$\begin{aligned}\widehat{\alpha}_m(k) &= \left(Z'Z + kI\right)^{-1}\left(Z'y + k\widehat{\alpha}_k\right)\\ &= T_k\widehat{\alpha}_A + k\left(Z'Z + kI\right)^{-1}\widehat{\alpha}_k\\ &= T_k\widehat{\alpha}_A + (I - T_k)\,\widehat{\alpha}_k \qquad (2.4)\end{aligned}$$

where $T_k = (Z'Z + kI)^{-1} Z'Z$. Substituting $\widehat{\alpha}_k$ for b_0, it is expected that $\widehat{\alpha}_m(k)$ has advantage according to the Almon-ridge and Almon estimators. Thus, Almon-modified ridge estimator of β in model (1.2) is $\widehat{\beta}_m(k) = A\widehat{\alpha}_m(k)$. In application b_0 might well be chosen to reflect as well as possible the prior information or restricted on β.

2.2. The Almon-modified Liu estimator. In order to overcome the multicollinearity problem, ridge estimator that we have discussed before is widely used in practice, but selection of k poses some problems. To overcome this problem an estimator is defined by combining Ridge and Stein type estimators in Liu [13]. This estimator was called Liu estimator in Akdeniz and Kaçıranlar [1]. The advantage of Liu estimator over ridge estimator is a linear function of d and therefore selection of d is easier. Liu estimator of β in (1.2) is

$$\begin{aligned}\widehat{\beta}_d &= \left(X'X + I\right)^{-1}\left(X'y + db\right)\\ &= \left(X'X + I\right)^{-1}\left(X'X + dI\right)b, \qquad 0 < d < 1 \qquad (2.5)\end{aligned}$$

where b is the OLS estimator for model (1.2). To overcome multicollinearity problem, if we take $\widehat{\alpha}_A$ instead of b, Almon-Liu estimator of α in model (1.7) is

$$\begin{aligned}\widehat{\alpha}_d &= \left(Z'Z + I\right)^{-1}\left(Z'y + d\widehat{\alpha}_A\right)\\ &= \left(A'SA + I\right)^{-1}\left(A'X'y + d\widehat{\alpha}_A\right) \qquad (2.6)\end{aligned}$$

obtained. This estimator can be given,

$$\begin{aligned}\widehat{\alpha}_d &= \left(Z'Z + I\right)^{-1}\left(Z'Z + dI\right)\widehat{\alpha}_A\\ &= \left(A'SA + I\right)^{-1}\left(A'SA + dI\right)\widehat{\alpha}_A\end{aligned}$$

(2.7) $\qquad = F_d \widehat{\alpha}_A$

where $F_d = (Z'Z + I)^{-1}(Z'Z + dI)$. Thus, the Almon-Liu estimator of β is $\widehat{\beta}_d = A\widehat{\alpha}_d$. Comparison of $\widehat{\alpha}_A$ with $\widehat{\alpha}_d$ and selection of d are given in Kaçıranlar [9].

Li and Yang [11] introduced a modified Liu estimator based on prior information similar to (2.3). Almon-modified Liu estimator of α in model (1.7) is defined,

(2.8) $$\widehat{\alpha}(d, b_0) = \left(Z'Z + I\right)^{-1}\left(Z'Z + dI\right)\widehat{\alpha}_A + (1-d)\left(Z'Z + I\right)^{-1} b_0.$$

To overcome multicollinearity problem, if we take $b_0 = \widehat{\alpha}_d$, (2.8) is reduced to

(2.9) $$\widehat{\alpha}_m(d) = F_d\widehat{\alpha}_A + (I - F_d)\widehat{\alpha}_d.$$

Substituting $\widehat{\alpha}_d$ for b_0, it is expected that Almon-modified Liu estimator has advantage according to the Almon-Liu and the Almon estimators.

3. Matrix mean square error comparisons

Bias and variance of an estimator $\tilde{\beta}$ are measured simultaneously by the MSE matrix,

$$MSE\left(\widetilde{\beta}\right) = E\left[\left(\widetilde{\beta} - \beta\right)\left(\widetilde{\beta} - \beta\right)'\right] = V\left(\widetilde{\beta}\right) + Bias(\widetilde{\beta})Bias(\widetilde{\beta})'$$

where

$$V\left(\widetilde{\beta}\right) = E\left[\left((\widetilde{\beta} - E(\widetilde{\beta})\right)\left((\widetilde{\beta} - E(\widetilde{\beta})\right)'\right]$$

and

$$Bias\left(\widetilde{\beta}\right) = E\left(\widetilde{\beta}\right) - \beta.$$

For a given value of β, $\widetilde{\beta}_2$ is preferred to an alternative estimator, $\widetilde{\beta}_1$, when $MSE\left(\widetilde{\beta}_1\right) - MSE\left(\widetilde{\beta}_2\right)$ is a nonnegative definite ($n.n.d$) matrix. Another criterion measure of goodness of an estimator is

$$smse\left(\widetilde{\beta}\right) = tr\left(V\left(\widetilde{\beta}\right)\right) + \left[Bias(\widetilde{\beta})\right]'\left[Bias(\widetilde{\beta})\right],$$

which is called as the scalar mean squared error ($smse$) value of $\widetilde{\beta}$.

If $MSE\left(\widetilde{\beta}_1\right) - MSE\left(\widetilde{\beta}_2\right)$ is a $n.n.d.$, then $smse\left(\widetilde{\beta}_1\right) - smse\left(\widetilde{\beta}_2\right) \geq 0$. The converse is not generally true (Theobald, [17]).

4. Superiority of the biased estimators under the MSE criterion

Almon-modified ridge and Almon-modified Liu estimators are biased alternatives to the Almon estimator in the presence of multicollinearity. In the following five subsections we compare Almon-modified ridge estimator with the Almon-ridge and Almon estimators. Also, Almon-modified Liu estimator is compared to the Almon-Liu and Almon estimators. In addition to these, Almon-modified ridge and Almon-Liu estimators are compared under the MSE criterion. Canonical form of the estimators will be discussed in order to make these comparisons.

Model (1.7) can be written in canonical form

$$y = W\gamma + u, \qquad u \sim N\left(0, \sigma_u^2\right) \tag{4.1}$$

where $W = ZQ$, $\gamma = Q'\alpha$ and Q is the orthogonal matrix whose columns constitute the eigenvectors of $Z'Z$. Then

$$W'W = Q'Z'ZQ = \Lambda = diag\left(\lambda_1, \lambda_2, \ldots, \lambda_{r+1}\right) \tag{4.2}$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{r+1} > 0$ are ordered eigenvalues of $Z'Z$. For model (4.1), we get the following representations.

Almon estimator is,

$$\widehat{\gamma}_A = \Lambda^{-1}W'y = C_1 y. \tag{4.3}$$

Almon-ridge estimator is,

$$\begin{aligned} \widehat{\gamma}_k &= (\Lambda + kI)^{-1} W'y \\ &= G_k W'y = C_2 y \end{aligned} \tag{4.4}$$

where $G_k = (\Lambda + kI)^{-1}$. Here G_k is the diagonal and symmetric matrix.

Almon-modified ridge estimator is,

$$\begin{aligned} \widehat{\gamma}_m(k) &= (\Lambda + kI)^{-1}\left(W'y + k\widehat{\gamma}_k\right) \\ &= (\Lambda + kI)^{-1}\Lambda\widehat{\gamma}_A + k(\Lambda + kI)^{-1}\widehat{\gamma}_k \\ &= \left[(\Lambda + kI)^{-1} + k(\Lambda + kI)^{-2}\right]W'y \\ &= \left[G_k + kG_k^2\right]W'y = C_3 y. \end{aligned} \tag{4.5}$$

Almon-Liu estimator is,

$$\begin{aligned} \widehat{\gamma}_d &= (\Lambda + I)^{-1}(\Lambda + dI)\Lambda^{-1}W'y \\ &= L_d\Lambda^{-1}W'y = C_4 y \end{aligned} \tag{4.6}$$

where $L_d = (\Lambda + I)^{-1}(\Lambda + dI)$. Here L_d is diagonal and symmetric matrix.

Almon-modified Liu estimator is,

$$\begin{aligned} \widehat{\gamma}_m(d) &= \left[(\Lambda + I)^{-1}(\Lambda + dI)\right]\widehat{\gamma}_A + \left[I - (\Lambda + I)^{-1}(\Lambda + dI)\right]\widehat{\gamma}_d \\ &= L_d\widehat{\gamma}_A + (I - L_d)\widehat{\gamma}_d \\ &= \left(2L_d - L_d^2\right)\widehat{\gamma}_A \\ &= \left(2L_d - L_d^2\right)\Lambda^{-1}W'y = C_5 y. \end{aligned} \tag{4.7}$$

It is evident that the above mentioned estimators are homogeneous linear. For the sake of convenience, we have an important Lemma needed in the following comparisons.

Lemma.(Trenkler, [18]). Let $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ be two homogeneous linear estimators of β such that $D = V\left(\widetilde{\beta}_1\right) - V\left(\widetilde{\beta}_2\right)$ is positive definite (*p.d.*).

If $Bias\left(\widetilde{\beta}_2\right)' D^{-1} Bias\left(\widetilde{\beta}_2\right) < \sigma^2$ then $MSE\left(\widetilde{\beta}_1\right) - MSE\left(\widetilde{\beta}_2\right)$ is *p.d.*.

4.1. The comparison of Almon-modified ridge estimator and Almon estimator. In this section, we will discuss the superiority of Almon-modified ridge estimator over the Almon estimator by the MSE criterion. Also, we want to show that for any $k > 0$, we can always find k so that Almon-modified ridge estimator has less MSE as compared with Almon estimator.

As regards the performance by the variance-covariance matrix, we have the following theorem.

4.1. Theorem. *Let k be fixed and $k > 0$.*

If $b_1' D_1^{-1} b_1 < \sigma_u^2$, then $MSE\left(\widehat{\gamma}_A\right) - MSE\left(\widehat{\gamma}_m(k)\right)$ is p.d.,
where $D_1 = C_1C_1' - C_3C_3'$, $C_1 = \Lambda^{-1}W'$, $C_3 = \left[G_k + kG_k^2\right]W'$ and
$b_1 = Bias\left(\widehat{\gamma}_m(k)\right) = -k^2G_k^2\gamma$.

Proof. Using the estimators $\widehat{\gamma}_A$ and $\widehat{\gamma}_m(k)$ in (4.3) and (4.5), the variance-covariance matrix of unbiased $\widehat{\gamma}_A$ is

$$V\left(\widehat{\gamma}_A\right) = \sigma_u^2\Lambda^{-1} \tag{4.8}$$

and the variance-covariance matrix and bias of $\widehat{\gamma}_m(k)$ are respectively,

$$\begin{aligned} V\left(\widehat{\gamma}_m(k)\right) &= \sigma_u^2\left(G_k + kG_k^2\right)\Lambda\left(G_k + kG_k^2\right) \\ &= \sigma_u^2 G_k\left(I - kG_k\right)\left(I + kG_k\right)^2, \end{aligned} \tag{4.9}$$

$$Bias\left(\widehat{\gamma}_m(k)\right) = -k^2G_k^2\gamma \tag{4.10}$$

obtained. Then using (4.9) and (4.10), MSE matrix of $\widehat{\gamma}_m(k)$ is,

$$MSE\left(\widehat{\gamma}_m(k)\right) = \sigma_u^2 G_k\left(I - kG_k\right)\left(I + kG_k\right)^2 + k^4G_k^2\gamma\gamma'G_k^2. \tag{4.11}$$

Considering the following difference from (4.8) and (4.9), we obtain

$$\begin{aligned} \Delta_1 = V\left(\widehat{\gamma}_A\right) - V\left(\widehat{\gamma}_m(k)\right) &= \sigma_u^2\left(C_1C_1' - C_3C_3'\right) \\ &= \sigma_u^2 k^2 G_k\left[G_k + \Lambda^{-1} + kG_k^2\right]G_k. \end{aligned} \tag{4.12}$$

Since $\left[G_k + \Lambda^{-1} + kG_k^2\right] > 0$, $\Delta_1 > 0$, namely D_1 will be *p.d.* for $k > 0$. By the Lemma, the proof is completed. □

4.2. The comparison of Almon-modified ridge estimator and Almon-ridge estimator. We have already seen in the previous section that Almon-modified ridge estimator is superior to the Almon estimator. Now, the aim is to compare the performance of Almon-modified ridge to the Almon-ridge estimator according to the MSE criterion .

In the following theorem, we have obtained sufficient condition for the Almon-modified ridge estimator to outperform the Almon-ridge estimator in terms of MSE criterion.

4.2. Theorem. *Let k be fixed and $k > 0$.*

If $b_1' D_2^{-1} b_1 < \sigma_u^2$, then $MSE\left(\widehat{\gamma}_k\right) - MSE\left(\widehat{\gamma}_m(k)\right)$ is p.d.,
where $D_2 = C_2C_2' - C_3C_3'$, $C_2 = G_kW'$.

Proof. Using the estimator $\widehat{\gamma}_k$ in (4.4), the variance-covariance matrix of this estimator is,

$$V\left(\widehat{\gamma}_k\right) = \sigma_u^2 G_k \Lambda G_k'$$

$$= \sigma_u^2 \left(I - kG_k\right) G_k \tag{4.13}$$

and bias is,

$$Bias\left(\widehat{\gamma}_k\right) = -kG_k\gamma. \tag{4.14}$$

Then using (4.13) and (4.14), MSE matrix of $\widehat{\gamma}_k$ is,

$$MSE\left(\widehat{\gamma}_k\right) = \sigma_u^2 \left(I - kG_k\right) G_k + k^2 G_k \gamma\gamma' G_k' \tag{4.15}$$

obtained. Then considering the following difference from (4.13) and (4.9) we obtain

$$\Delta_2 = V\left(\widehat{\gamma}_k\right) - V\left(\widehat{\gamma}_m(k)\right) = \sigma_u^2 \left(C_2C_2' - C_3C_3'\right)$$

$$= \sigma_u^2 G_k \Lambda G_k \left(2kG_k + k^2 G_k^2\right). \tag{4.16}$$

Since $\left[2kG_k + k^2G_k^2\right] > 0$, $\Delta_2 > 0$. Then D_2 will be *p.d.* for $k > 0$. By the Lemma, the proof is completed. □

4.3. The comparison of Almon-modified Liu estimator and Almon estimator. Li and Yang [11] compared the modified Liu estimator with OLS, Liu, ridge and modified ridge estimators according to the MSE criterion in linear regression model. Now, our goal is to compare the Almon-modified Liu estimator that we have proposed here, with the Almon estimator for the distributed lag model.

Here we show that Almon-modified Liu estimator outperform to the Almon estimator in terms of MSE criterion by the following theorem.

4.3. Theorem. *Let d be fixed and $0 < d < 1$.*

If $b_2' D_3^{-1} b_2 < \sigma_u^2$, then $MSE\left(\widehat{\gamma}_A\right) - MSE\left(\widehat{\gamma}_m\left(d\right)\right)$ is p.d.
where $D_3 = C_1 C_1' - C_5 C_5'$, $C_5 = \left(2L_d - L_d^2\right)\Lambda^{-1} W'$ and
$b_2 = Bias\left(\widehat{\gamma}_m\left(d\right)\right) = -\left(1-d\right)^2\left(\Lambda + I\right)^{-2}\gamma$.

Proof. Using the estimator $\widehat{\gamma}_m\left(d\right)$ in (4.7), the variance-covariance matrix of this estimator is,

(4.17) $$V\left(\widehat{\gamma}_m\left(d\right)\right) = \sigma_u^2\left[2L_d - L_d^2\right]\Lambda^{-1}\left[2L_d - L_d^2\right]$$

and bias is,

(4.18) $$Bias\left(\widehat{\gamma}_m\left(d\right)\right) = -\left(1-d\right)^2\left(\Lambda + I\right)^{-2}\gamma$$

Then using (4.17) and (4.18), MSE matrix of $\widehat{\gamma}_m\left(d\right)$ is,

(4.19) $$MSE\left(\widehat{\gamma}_m\left(d\right)\right) = \sigma_u^2\left[2L_d - L_d^2\right]\Lambda^{-1}\left[2L_d - L_d^2\right] + \left(1-d\right)^4\left(\Lambda + I\right)^{-2}\gamma\gamma'\left(\Lambda + I\right)^{-2}$$

The variance-covariance matrix of $\widehat{\gamma}_m\left(d\right)$ can be rewrite in the following:

(4.20) $$V\left(\widehat{\gamma}_m\left(d\right)\right) = \left[2L_d - L_d^2\right]^2 V\left(\widehat{\gamma}_A\right).$$

Here matrix $\left[2L_d - L_d^2\right]$ is the diagonal and symmetric matrix. Let B defined as

(4.21) $$B = \left[2L_d - L_d^2\right]^2 = diag\left(b_1, b_2, \ldots, b_p\right).$$

We can see that $V\left(\widehat{\gamma}_m\left(d\right)\right)$ is decreasing due to the factor B in equation (4.20). The $i-th$ element of matrix B in (4.21) is

(4.22) $$b_i = \left[\frac{\lambda_i^2 + 2\lambda_i + 2d - d^2}{\left(\lambda_i + 1\right)^2}\right]^2.$$

From (4.22), we have the conclusions that $\lambda_i^2 + 2\lambda_i + 2d - d^2 > 0$ and $\frac{\lambda_i^2 + 2\lambda_i + 2d - d^2}{(\lambda_i + 1)^2} < 1$ for $0 < d < 1$. Therefore, $0 < b_i < 1$ is ensured for the $i-th$ element of matrix B. Consequently, we obtain $V\left(\widehat{\gamma}_A\right) - V\left(\widehat{\gamma}_m\left(d\right)\right) > 0$, namely, D_3 is *p.d.* for $0 < d < 1$. By the Lemma, the proof is completed. □

4.4. The comparison of Almon-modified Liu estimator and Almon-Liu estimator. Modified Liu estimator has smaller estimated MSE values than Liu, ridge and modified ridge estimators, respectively, in Liu and Yang [11]. In this section, we show that Almon-Liu estimator is better than Almon-modified Liu estimator according to the MSE criterion.

In the following theorem, we have obtained a sufficient condition for the Almon-Liu estimator to be superior to the Almon-modified Liu estimator in terms of MSE criterion.

4.4. Theorem. *Let d be fixed and $0 < d < 1$.*

If $b_3' D_4^{-1} b_3 < \sigma_u^2$, then $MSE\left(\widehat{\gamma}_m(d)\right) - MSE\left(\widehat{\gamma}_d\right)$ is p.d., where $D_4 = C_5C_5' - C_4C_4'$, $C_4 = L_d\Lambda^{-1}W'$, $L_d = (\Lambda + I)^{-1}(\Lambda + dI)$ and $b_3 = Bias\left(\widehat{\gamma}_d\right) = -(1-d)(\Lambda + I)^{-1}\gamma$.

Proof. Using the estimator $\widehat{\gamma}_d$ in (4.6), the variance-covariance matrix and the bias of this estimator are obtained respectively in the following:

$$V\left(\widehat{\gamma}_d\right) = \sigma_u^2 L_d \Lambda^{-1} L_d \tag{4.23}$$

$$Bias\left(\widehat{\gamma}_d\right) = -(1-d)(\Lambda + I)^{-1}\gamma. \tag{4.24}$$

Then using (4.23) and (4.24), MSE matrix of $\widehat{\gamma}_d$ is,

$$MSE\left(\widehat{\gamma}_d\right) = \sigma_u^2 L_d \Lambda^{-1} L_d + (1-d)^2 (\Lambda + I)^{-1}\gamma\gamma'(\Lambda + I)^{-1}. \tag{4.25}$$

Considering the following difference from (4.17) and (4.23), we obtain

$$\begin{aligned}\Delta_3 &= V\left(\widehat{\gamma}_m(d)\right) - V\left(\widehat{\gamma}_d\right) = \sigma_u^2\left(C_5C_5' - C_4C_4'\right)\\ &= \sigma_u^2 L_d\left[\left(I + (1-d)(\Lambda + I)^{-1}\right)\Lambda^{-1}\left(I + (1-d)(\Lambda + I)^{-1}\right) - \Lambda^{-1}\right]L_d\\ &= \sigma_u^2 L_d\left[2(1-d)\Lambda^{-1}(\Lambda + I)^{-1} + (1-d)^2(\Lambda + I)^{-1}\Lambda^{-1}(\Lambda + I)^{-1}\right]L_d.\end{aligned} \tag{4.26}$$

Since the last equation in (4.26) is *p.d.* for $0 < d < 1$, $V\left(\widehat{\gamma}_m(d)\right) - V\left(\widehat{\gamma}_d\right) > 0$. Therefore, $D_4 = C_5C_5' - C_4C_4'$ will be *p.d.* for $0 < d < 1$. By the Lemma, the proof is completed. □

4.5. The comparison of Almon-modified ridge estimator and Almon-Liu estimator. Now, we compare the second order moment matrices of Almon-modified ridge and Almon-Liu estimators. Let now d be fixed for the moment, we may state the following theorem.

4.5. Theorem. *Let d be fixed and $0 < d < 1$.*

a. *If $b_3'\left(C_3C_3' - C_4C_4'\right)^{-1} b_3 < \sigma_u^2$, then $MSE\left(\widehat{\gamma}_m(k)\right) - MSE\left(\widehat{\gamma}_d\right)$ is p.d. for $0 < k < k_j$.*

b. *If $b_1'\left(C_4C_4' - C_3C_3'\right)^{-1} b_1 < \sigma_u^2$, then $MSE\left(\widehat{\gamma}_d\right) - MSE\left(\widehat{\gamma}_m(k)\right)$ is p.d. for $0 < k_j < k$, where $k_j = \frac{\lambda_j(1-d)}{\lambda_j + d}$, $\quad j = 1, 2, \ldots, r+1$ $b_1 = Bias\left(\widehat{\gamma}_m(k)\right)$ and $b_3 = Bias\left(\widehat{\gamma}_d\right)$.*

Proof. Using (4.9) and (4.23), we obtain

$$\begin{aligned}\Delta_3 &= V\left(\widehat{\gamma}_m(k)\right) - V\left(\widehat{\gamma}_d\right) = \sigma_u^2\left(C_3C_3' - C_4C_4'\right)\\ &= \sigma_u^2\left[\left(G_k + kG_k^2\right)\Lambda\left(G_k + kG_k^2\right) - L_d\Lambda^{-1}L_d\right].\end{aligned}$$

Evidently, $C_3C_3' - C_4C_4'$ will be *p.d.* if and only if $\Psi_j > 0$, for all $j = 1, 2, \ldots, r+1$ where

$$\Psi_j = \frac{\lambda_j}{(\lambda_j + k)^2} - \frac{(\lambda_j + d)^2}{\lambda_j (\lambda_j + 1)^2} + \frac{2k\lambda_j}{(\lambda_j + k)^3} + \frac{k^2\lambda_j}{(\lambda_j + k)^4}.$$

For $k > 0$, since $\frac{2k\lambda_j}{(\lambda_j+k)^3}$ and $\frac{k^2\lambda_j}{(\lambda_j+k)^4}$ are positive, a sufficient condition for $C_3C_3' - C_4C_4'$ being *p.d.* is

$$\frac{\lambda_j}{(\lambda_j + k)^2} - \frac{(\lambda_j + d)^2}{\lambda_j (\lambda_j + 1)^2} \tag{4.27}$$

greater than zero. So, this inequality requires than $C_3C_3' - C_4C_4'$ is *p.d.* for $0 < k < k_j$. Similarly, $C_4C_4' - C_3C_3'$ will be *p.d.* for $0 < k_j < k$ (see also Sakallioglu et al. [15]). By the Lemma, the proof is completed. □

Let now k be fixed for the moment and let be $0 < k < 1$. Thus we have the following theorem.

4.6. Theorem. *Let k be fixed and $0 < k < 1$.*

a.*If $b_3' (C_3C_3' - C_4C_4')^{-1} b_3 < \sigma_u^2$, then $MSE(\widehat{\gamma}_m(k)) - MSE(\widehat{\gamma}_d)$ is p.d. for $0 < d < d_j < 1$.*

b.*If $b_1' (C_4C_4' - C_3C_3')^{-1} b_1 < \sigma_u^2$, then $MSE(\widehat{\gamma}_d) - MSE(\widehat{\gamma}_m(k))$ is p.d. for $0 < d_j < d < 1$ where $d_j = 1 - \frac{k(\lambda_j+1)}{\lambda_j+k}$, $\quad j = 1, 2, \ldots, r+1$.*

Proof. From the above theorem's proof, we know that $C_3C_3' - C_4C_4'$ will be *p.d.* if and only if $\Psi_j > 0$, for all $j = 1, 2, \ldots, r+1$. For fixed $k > 0$, (4.27) requires that $C_3C_3' - C_4C_4'$ is *p.d.* for $0 < d < d_j < 1$ and $C_4C_4' - C_3C_3'$ will be *p.d.* for $0 < d_j < d < 1$. By the Lemma, the proof is completed. □

To illustrate our theoretical results, it is easy to use *smse* in practical applications. Therefore, the *smse* formulas for the $\widehat{\gamma}_A, \widehat{\gamma}_k, \widehat{\gamma}_m(k), \widehat{\gamma}_d$ and $\widehat{\gamma}_m(d)$ are given respectively:

$$smse(\widehat{\gamma}_A) = \sigma_u^2 \sum_{i=1}^{r+1} \frac{1}{\lambda_i} \tag{4.28}$$

$$smse(\widehat{\gamma}_k) = \sigma_u^2 \sum_{i=1}^{r+1} \frac{\lambda_i}{(\lambda_i + k)^2} + k^2 \sum_{i=1}^{r+1} \frac{\gamma_i^2}{(\lambda_i + k)^2} \tag{4.29}$$

$$smse(\widehat{\gamma}_m(k)) = \sigma_u^2 \sum_{i=1}^{r+1} \frac{\lambda_i (\lambda_i + 2k)^2}{(\lambda_i + k)^4} + k^4 \sum_{i=1}^{r+1} \frac{\gamma_i^2}{(\lambda_i + k)^4} \tag{4.30}$$

$$smse(\widehat{\gamma}_d) = \sigma_u^2 \sum_{i=1}^{r+1} \frac{(\lambda_i + d)^2}{\lambda_i (\lambda_i + 1)^2} + (1 - d)^2 \sum_{i=1}^{r+1} \frac{\gamma_i^2}{(\lambda_i + 1)^2} \tag{4.31}$$

$$(4.32)\qquad smse\left(\widehat{\gamma}_m\left(d\right)\right)=\sigma_u^2\sum_{i=1}^{r+1}\frac{b_i}{\lambda_i}+\left(1-d\right)^4\sum_{i=1}^{r+1}\frac{\gamma_i^2}{\left(\lambda_i+1\right)^4}$$

where b_i is defined in (4.22). A very important issue in the study of ridge regression is how to find an appropriate biasing parameter k. Hoerl and Kennard [6], [7], Hoerl, Kennard and Baldwin [8] and Lawless and Wang [10] suggested the following ridge parameters, that we can estimate for the model (4.1) respectively;

$$(4.33)\qquad \widehat{k}_{HK}=\frac{\widehat{\sigma}_u^2}{\sum_{i=1}^{r+1}\widehat{\gamma}_i^2}$$

$$(4.34)\qquad \widehat{k}_{HKB}=\frac{\left(r+1\right)\widehat{\sigma}_u^2}{\sum_{i=1}^{r+1}\widehat{\gamma}_i^2}$$

$$(4.35)\qquad \widehat{k}_{LW}=\frac{\left(r+1\right)\widehat{\sigma}_u^2}{\sum_{i=1}^{r+1}\lambda_i\widehat{\gamma}_i^2}$$

where $\widehat{\gamma}$ and $\widehat{\sigma}_u^2$ are the OLS estimates of γ and σ_u^2. On the other hand Liu [13] gave the some estimates of d by analogy with the estimate of k in ridge estimate. Two of these estimates are defined as for the model (4.1):

$$(4.36)\qquad \widehat{d}_{mm}=1-\widehat{\sigma}_u^2\left[\sum_{i=1}^{r+1}\frac{1}{\lambda_i\left(\lambda_i+1\right)}\Bigg/\sum_{i=1}^{r+1}\frac{\widehat{\gamma}_i^2}{\left(\lambda_i+1\right)^2}\right]$$

$$(4.37)\qquad \widehat{d}_{CL}=1-\widehat{\sigma}_u^2\left[\sum_{i=1}^{r+1}\frac{1}{\lambda_i+1}\Bigg/\sum_{i=1}^{r+1}\frac{\lambda_i\widehat{\gamma}_i^2}{\left(\lambda_i+1\right)^2}\right]$$

where $\widehat{\gamma}$ and $\widehat{\sigma}_u^2$ are the OLS estimates of γ and σ_u^2.

5. A numeric example with Almon data

To illustrate our theoretical results we now consider a dataset due to Almon [2]. These data was taken in the years 1953-1967 using quarterly data where independent variable is appropriations and dependent variable is expenditures. Consideration of these data, the following results were obtained. Firstly, the smallest value of SIC was obtained 12.75 if the length of lag is p=8 using "*Schwartz Information Criteria* (SIC) ". Starting from the assumption that the prior information on β_i is fifth degree ($r = 5$) polynomial in (1.5), after testing the significance of the coefficient then, the optimal polynomial degree ($r = 2$) is obtained. Here, in order to obtain the form (1.7), Z matrix is obtained by means of X matrix produced by multiplying matrix A defined earlier. The condition number of Z matrix is 63.5 which imply the existence of highly multicollinearity in the data set. In this case, the results of Almon method that based on the OLS will not be appropriate.

Theoretical comparisons for the alternative estimators to the Almon estimator have been made in terms of the MSE criterion. Also, *smse* formulas have been given for these estimators. Using *smse* is generally the most convenient for applications or simulation studies. Then, we decided that which one is the best estimator for distributed lag models. For this data, we find the following results:

(a) The eigenvalues of $Z'Z$: $(0.0007,\ 0.0634,\ 2.9359)$

(b) The Almon estimates of

$$\alpha : (\widehat{\alpha}_A)' = (0.0962,\ 0.0320,\ -0.0052)$$

$$\widehat{\beta}_A = (A\widehat{\alpha}_A)' = (0.096,\ 0.123,\ 0.140,\ 0.146,\ 0.142,\ 0.127,\ 0.102,\ 0.067,\ 0.021)\,.$$

(c) The estimate of $\sigma^2 : \widehat{\sigma}_u^2 = 0.0164$

The 3×3 matrix Q is the matrix of normalized eigenvectors, Λ is a 3×3 diagonal matrix of eigenvalues of $Z'Z$ such that $Z'Z = Q\Lambda Q'$. Then, $W = ZQ$ and $\gamma = Q'\alpha$ so that, $y = Z\alpha + u = W\gamma + u$, where

$$Q = \begin{bmatrix} -0.2478 & -0.7818 & 0.5722 \\ 0.7934 & 0.1751 & 0.5829 \\ -0.5559 & 0.5985 & 0.5769 \end{bmatrix}$$

and

$$W'W = \Lambda = \begin{bmatrix} 0.0007 & 0 & 0 \\ 0 & 0.0634 & 0 \\ 0 & 0 & 2.9359 \end{bmatrix}$$

In orthogonal coordinates the OLS estimator of the regression coefficients is

$$\widehat{\gamma} = \Lambda^{-1}W'y = [1.2297,\ -1.0754,\ 0.5580]'$$

obtained. Using the equations in (4.33)-(4.35) estimators of k obtained for the evaluate the estimated *smse* values of Almon-ridge and Almon-modified ridge estimators. Then, for the practical purposes various values of k and the corresponding estimated *smse* values of the estimators are shown in Table 1. In Figure 1, the graph of estimated *smse* values of the Almon-ridge and Almon-modified ridge estimators is illustrated for the range of k values that performance of Almon-modified ridge estimator is better than Almon-ridge estimator.

Let us consider the Almon-Liu and Almon-modified Liu estimators various values of d and the corresponding estimated *smse* values of the estimators are shown in Table 2.

Also, the performances of Almon-Liu and Almon-modified Liu estimators are illustrated for the various values of d in Figure 2.

In Table 3, we compared the Almon-modified ridge, Almon-modified Liu and also Almon-Liu estimators and comparisons are shown on the graph for the common values of k and d in Figure 3.

Table 1. Estimated *smse* values of Almon, Almon-ridge and Almon-modified ridge estimators

	$\widehat{smse}\left(\widehat{\gamma}_A\right)$	$\widehat{smse}\left(\widehat{\gamma}_k\right)$	$\widehat{smse}\left(\widehat{\gamma}_m\left(k\right)\right)$
$k = 0$	23.6928	23.6928	23.6928
k_{HK}=0.0055	23.6928	1.7206	2.2609
$k = 0.01$	23.6928	1.6411	1.7840
k_{HKB}=0.0165	23.6928	1.6481	1.6743
$k = 0.02$	23.6928	1.6599	1.6605
$k = 0.03$	23.6928	1.7002	1.6524
$k_{LW} = 0.0498$	23.6928	1.7855	1.6649
$k = 0.1$	23.6928	1.9700	1.7440
$k = 0.2$	23.6928	2.1898	1.9286
$k = 0.3$	23.6928	2.3086	2.0675
$k = 0.4$	23.6928	2.3823	2.1662
$k = 0.5$	23.6928	2.4328	2.2384
$k = 0.6$	23.6928	2.4699	2.2931
$k = 0.7$	23.6928	2.4985	2.3359
$k = 0.8$	23.6928	2.5215	2.3702
$k = 0.9$	23.6928	2.5406	2.3984
$k = 1$	23.6928	2.5569	2.3310
$k = 2$	23.6928	2.6510	2.4949

When we compare Almon, Almon-ridge and Almon-modified ridge estimators, we observe that as k increases, Almon-modified ridge estimator always gives better performance than the other estimators. On the other hand, the performance of Almon-ridge estimator is better than Almon estimator with in the wide range k values. The plot of $\widehat{smse}\left(\widehat{\gamma}_k\right)$ and $\widehat{smse}\left(\widehat{\gamma}_m\left(k\right)\right)$ vs. k in the interval [0,1] has been presented in Fig.1. This figure indicates that $\widehat{smse}\left(\widehat{\gamma}_k\right)$ and $\widehat{smse}\left(\widehat{\gamma}_m\left(k\right)\right)$ increase as k increases. The Almon-modified ridge estimator dominates Almon-ridge estimator when $k > 0.02$. These findings have supported the results in Section 4.1 and 4.2.

Considering the performance of the other alternative estimators we can see that Almon-modified Liu estimator outperforms to the Almon-Liu and Almon estimator for all values of d satisfying $0 < d < 1$. The plot of $\widehat{smse}\left(\widehat{\gamma}_d\right)$ and $\widehat{smse}\left(\widehat{\gamma}_m\left(d\right)\right)$ has been presented in Fig.2. This figure indicates that $\widehat{smse}\left(\widehat{\gamma}_d\right)$ and $\widehat{smse}\left(\widehat{\gamma}_m\left(d\right)\right)$ increase as

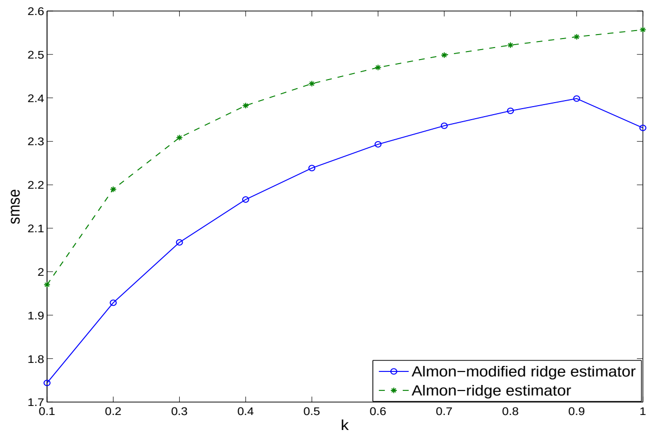


Figure 1. Estimated *smse* of Almon-ridge and Almon-modified ridge estimators versus k

Table 2. Estimated *smse* values of Almon, Almon-Liu and Almon-modified Liu estimators

	$\widehat{smse}\left(\widehat{\gamma}_A\right)$	$\widehat{smse}\left(\widehat{\gamma}_d\right)$	$\widehat{smse}\left(\widehat{\gamma}_m\left(d\right)\right)$
$d = 0$	23.6928	0.1721	0.0161
$d = 0.001$	23.6928	0.1718	0.0160
$d = 0.01$	23.6928	0.1688	0.0156
$d = 0.1$	23.6928	0.1403	0.0119
$d = 0.2$	23.6928	0.1118	0.0089
$d = 0.3$	23.6928	0.0867	0.0069
$d = 0.4$	23.6928	0.0650	0.0056
$d = 0.5$	23.6928	0.0467	0.0049
$d = 0.6$	23.6928	0.0318	0.0045
$d_{CL} = 0.712$	23.6928	0.0192	0.0043
$d = 0.8$	23.6928	0.0122	0.0042
$d = 0.9$	23.6928	0.0076	0.0042

d increases and large value of d Almon-modified Liu estimator dominates the Almon-Liu estimator. On the other hand the increasing of $\widehat{smse}\left(\widehat{\gamma}_m\left(d\right)\right)$ is slowly than the $\widehat{smse}\left(\widehat{\gamma}_d\right)$.

Finally, comparison of the three estimators is illustrated in Figure 3. It can be seen that not only Almon-modified Liu estimator but also Almon-Liu estimator outperforms Almon-modified ridge estimator in Figure 3.

From Table 3 and Figure 3, we can also obtain the following conclusions:

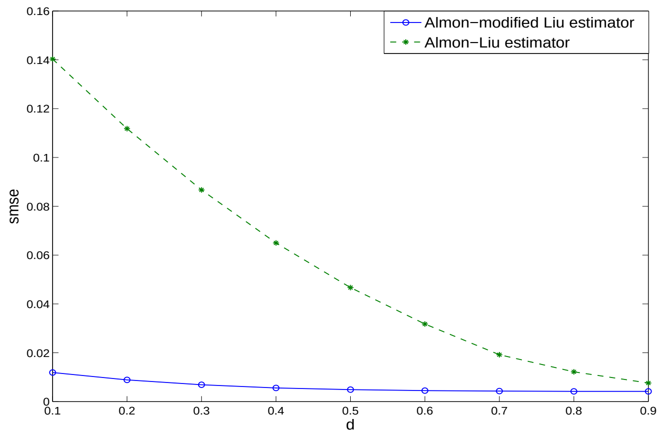


Figure 2. Estimated *smse* of Almon-Liu and Almon-modified Liu estimators versus d

Table 3. Comparisons between Almon-modified ridge, Almon Liu and Almon-modified Liu estimators in *smse* sense

$k = d$	$\widehat{smse}\left(\widehat{\gamma}_d\right)$	$\widehat{smse}\left(\widehat{\gamma}_m\left(k\right)\right)$	$\widehat{smse}\left(\widehat{\gamma}_m\left(d\right)\right)$
0.01	0.1688	1.7840	0.0156
0.1	0.1403	1.7440	0.0119
0.2	0.1118	1.9286	0.0089
0.3	0.0867	2.0675	0.0069
0.4	0.0650	2.1662	0.0056
0.5	0.0467	2.2384	0.0049
0.6	0.0318	2.2931	0.0045
0.7	0.0192	2.3359	0.0043
0.8	0.0122	2.3702	0.0042
0.9	0.0076	2.3984	0.0042

(i) Let $d = 0.1$ be fixed. We get values of k_j by using Theorem 4.5.

$$k_j : 0.8704, 0.3492, 0.0062.$$

Comparing $\widehat{smse}\left(\widehat{\gamma}_{d=0.1}\right) = 0.1403$ with $\widehat{smse}\left(\widehat{\gamma}_m\left(k = 0.005\right)\right) = 2.4019$ for $0 < k < 0.0062$, we see that $\widehat{\gamma}_d$ has a smaller estimated *smse* value than $\widehat{\gamma}_m\left(k\right)$ (see also Figure 3). Comparing $\widehat{smse}\left(\widehat{\gamma}_{d=0.1}\right) = 0.1403$ with $\widehat{smse}\left(\widehat{\gamma}_m\left(k = 0.9\right)\right) = 2.3984$ is obtained for $0 < 0.8704 < k$. Since the sufficient condition in Theorem 4.5.(b) is not satisfied, $\widehat{\gamma}_m\left(k\right)$ does not have estimated *smse* value than $\widehat{\gamma}_d$.

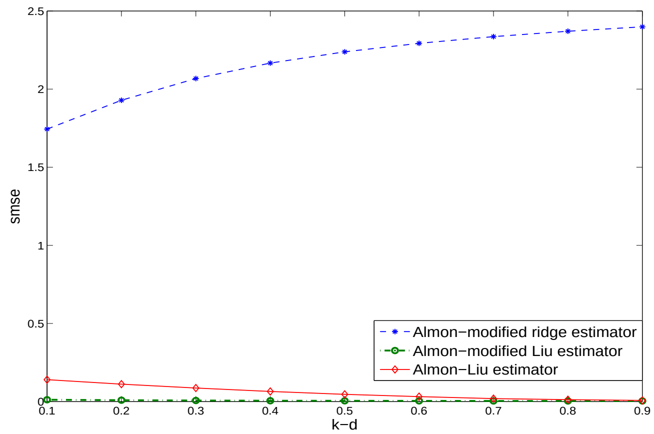


Figure 3. Estimated *smse* of Almon-modified ridge, Almon-Liu, Almon-modified Liu estimators versus $k-d$

(ii) Let $d = 0.9$ be fixed. By using Theorem 4.5 k_j values are obtained as

$$k_j : 0.00765, 0.00658, 0.000077.$$

Comparing $\widehat{smse}\left(\widehat{\gamma}_{d=0.9}\right) = 0.0076$ with $\widehat{smse}\left(\widehat{\gamma}_m\left(k = 0.00007\right)\right) = 23.235$ for $0 < k < 0.000077$, we see that $\widehat{\gamma}_d$ has a smaller estimated *smse* value than $\widehat{\gamma}_m\left(k\right)$ (see also Figure 3). Comparing $\widehat{smse}\left(\widehat{\gamma}_{d=0.9}\right) = 0.0076$ with $\widehat{smse}\left(\widehat{\gamma}_m\left(k = 0.008\right)\right) = 1.8979$ is obtained for $0 < 0.00765 < k$. Since the sufficient condition in Theorem 4.5.(b) is not satisfied, $\widehat{\gamma}_m\left(k\right)$ does not have smaller estimated *smse* value than $\widehat{\gamma}_d$.

(iii) Let $k = 0.2$ be fixed. We get values of d_j by using Theorem 4.6.

$$d_j : 0.749, 0.1926, 0.0028.$$

Comparing $\widehat{smse}\left(\widehat{\gamma}_m\left(k = 0.2\right)\right) = 1.9286$ with $\widehat{smse}\left(\widehat{\gamma}_{d=0.002}\right) = 0.1715$ for $0 < d < 0.0028 < 1$. So $\widehat{\gamma}_d$ has a smaller estimated *smse* value than $\widehat{\gamma}_m\left(k\right)$ as it is indicated in (a) part of the Theorem 4.6 (see also Figure 3). On the other hand, comparing $\widehat{smse}\left(\widehat{\gamma}_m\left(k = 0.2\right)\right) = 1.9286$ with $\widehat{smse}\left(\widehat{\gamma}_{d=0.8}\right) = 0.0122$ is obtained for $0 < 0.749 < d < 1$. Since the sufficient condition in Theorem 4.6.(b) is not satisfied, $\widehat{\gamma}_m\left(k\right)$ does not have smaller estimated *smse* value than $\widehat{\gamma}_d$.

(iv) Let $k = 0.8$ be fixed. By using Theorem 4.6 d_j values are obtained as

$$d_j : 0.1572, 0.0147, 0.0002.$$

Comparing $\widehat{smse}\left(\widehat{\gamma}_m\left(k=0.8\right)\right)=2.3702$ with $\widehat{smse}\left(\widehat{\gamma}_{d=0.0001}\right)=0.1721$ for $0<d<0.0002<1$. So $\widehat{\gamma}_d$ has a smaller estimated $smse$ value than $\widehat{\gamma}_m\left(k\right)$ as it is indicated in (a) part of the Theorem 4.6 (see also Figure 3). Beside this, comparing $\widehat{smse}\left(\widehat{\gamma}_m\left(k=0.8\right)\right)=2.3712$ with $\widehat{smse}\left(\widehat{\gamma}_{d=0.2}\right)=0.1118$ is obtained for $0<0.1572<d<1$. Since the sufficient condition in Theorem 4.6.(b) is not satisfied, $\widehat{\gamma}_m\left(k\right)$ does not have estimated $smse$ value than $\widehat{\gamma}_d$.

6. Conclusions

In this study, we have compared theoretical performances of Almon-ridge $(\widehat{\gamma}_k)$, Almon-modified ridge $(\widehat{\gamma}_m\left(k\right))$, Almon-Liu $(\widehat{\gamma}_d)$, Almon-modified Liu $(\widehat{\gamma}_m\left(d\right))$ estimators to the Almon $(\widehat{\gamma}_A)$ estimator according to the MSE criterion with using some theorems. These alternative estimators showed quite good performance to the Almon estimator. Also, some of the alternative estimators compared with each other. The performances of the estimators depends on biasing parameters k and d. To see more detailed results of the comparisons we plotted estimated $smse$ values of these estimators using k and d values in Figure 1-3.

Liu and Yang [11] showed with the increasing of the levels of multicollinearity, the $smse$ values of ridge, Liu, modified ridge and modified Liu estimators are decreasing in general for the linear regression model. Moreover, they showed that the $smse$ values of these estimators outperformed to the OLS estimator for all cases. Also, for most cases, modified Liu estimator has smaller $smse$ values than those of the Liu, ridge, and modified ridge estimator, respectively. In this study, we find similar results for the distributed lag models. Theoretical results suggested that, for an appropriate value of k and d Almon-modified ridge and Almon-modified Liu estimator give better estimates than the other alternative estimators in terms of MSE criterion for the distributed lag models.

The theoretical section is supported by a numerical example based on widely analyzed Almon [2] dataset. Almon-modified Liu estimator has been showed as the best estimator in distributed lag models.

Acknowledgement

The authors are thankful to anonymous referees for the useful comments and suggestions.

References

[1] Akdeniz, F. and Kaçıranlar, S. *On the almost unbiased generalized Liu estimator and unbiased estimation of the bias and MSE*, Communications in Statistics Theory and Methods **24** (7), 1789-1797, 1995.

[2] Almon, S. *The distributed lag between capital appropriations and expenditures*, Econometrica, 178-196, 1965.

[3] Chanda, A.K. and Maddala, G.S. *Ridge estimators for distributed lag models*, Communications in Statistics Theory and Methods **13** (2), 217-225, 1984.

[4] Fisher, I. *Income in theory and income taxation practice*, Econometrica, **5**, 1-55, 1937.

[5] Gujarati, D. N. *Basic Econometrics*, New York:McGraw-Hill, 1999.

[6] Hoerl, A.E. and Kennard, R.W. *Ridge regression:biased estimation for nonorthogonal problems*, Technometrics, **12**, 55-67, 1970a.

[7] Hoerl, A.E. and Kennard, R.W. *Ridge regression:application for nonorthogonal problems*, Tecnometrics, **12**, 69-82, 1970b.

[8] Hoerl, A.E., Kennard, R.W., Baldwin, K. F. *Ridge regression: some simulation*, Communication in Statistics, **4** (2), 105-123, 1975.

[9] Kaçıranlar, S. *Alternative estimators for distributed lag models*, Journal of Statistical Research, **9** (1), 1-10, 2010.

[10] Lawless, J.F. and Wang, P.A. *Simulation study of ridge and other regression estimators*, Communication in Statistics Theory and Methods, **A5** (4), 307-323, 1976.

[11] Li Y. and Yang H. *A new Liu-type estimator in linear regression model*, Statistical Papers **53** (2), 427-437, 2012.

[12] Lindley, D. V. and Smith, A. F. M. *Bayes estimates for the linear model*, Journal of the Royal Statistical Society, B Series, 1-41, 1972.

[13] Liu, K. *A new class of biased estimate in linear regression*, Communication in Statistics Theory and Methods, **22** (2), 393-402, 1993.

[14] Maddala, G. S. *Ridge estimator for distributed lag models*, NBER Working Paper Series, **69**, 1974.

[15] Sakallıoglu, S., Kaçıranlar S., Akdeniz F. *Mean square error comparisons of some biased regression estimators*, Communication in Statistics Theory and Methods, **30** (2), 347-361, 2001.

[16] Swindel, B.F. *Good ridge estimators based on prior information*, Communication in Statistics Theory and Methods, **A5** (11), 1065-1075, 1976.

[17] Theobald, C.M. *Generalizations of mean square error applied to ridge regression*, J.Roy.Stat.Soc.Ser. **B** (36), 103-106, 1974.

[18] Trenkler G. *Generalized mean squared error comparisions of biased regression estimators*, Communication in Statistics Theory and Methods, **A9** (12), 1247-1259, 1980.

[19] Vinod H. D. and Ullah A. *Recent advances in regression methods*, Marcel Dekker, Inc, 226-238. 1981.

[20] Yeo, S. J. and Trivedi, P. K. *On using ridge-type estimators for a distributed lag model*, Oxford Bulletin of Economics and Statistics, **51** (1), 85-90, 1989.

[21] Yurdakul, F. *The econometrics analysis of relationships between of cotton production and prices: Koyck -Almon Approach*, Journal of Faculty of Economics and Administrative Sciences of Cukurova University, **8** (1), 341-353, 1998.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1235 – 1245

Likelihood and Bayesian estimations for step-stress life test model under Type-I censoring

Ali. A. Ismail*

Abstract

This paper discusses likelihood and Bayesian estimations for partially accelerated step-stress life test model under Type-I censoring assuming Pareto distribution of the second kind. The posterior means and posterior variances are obtained under the squared error loss function using Lindley's approximation procedure. It has been observed that Lindley's method usually provides posterior variances and mean square errors smaller than those of the maximum likelihood estimators. Furthermore, the highest posterior density credible intervals of the model parameters based on Gibbs sampling technique are computed. For illustration, simulation studies and an illustrative example based on a real data set are provided.

Keywords: Reliability, partially accelerated step-stress life test, Bayesian estimation, Gibbs sampling.

2000 AMS Classification: 62N01; 62N05.

Received 30/07/2014 : Accepted 08/10/2014 Doi : 10.15672/HJMS.2014337517

1. Introduction

For most practical tests, it may be not easy to gather data on failure-time of a device under use conditions when this device is a highly reliable. Consequently, such devices should be tested under accelerated (i.e. harsher-than-use) conditions to obtain failures quickly. According to Pathak et al. [33], "the model of acceleration is chosen so that the relationship between the parameters of the failure distribution and the accelerated stress conditions is known. Such relationship is used to extrapolate the accelerated data to the design stress to estimate the life distribution. The tests performed under

*King Saud University, College of Science, Department of Statistics and Operations Research, P. O. BOX 2455, Riyadh 11451, Saudi Arabia,
Cairo University, Faculty of Economics & Political Science, Department of Statistics, Giza 12613, Egypt,
Email: ismail@ksu.edu.sa, ali.ismail@cu.edu.eg

accelerated stress conditions are called fully accelerated life tests (FALT or simply ALT)". Involved persons may refer to "Meeker and Escobar [30] and Nelson [31], which are two comprehensible sources for ALT".

Sometimes, such relationship (the life-stress relationship) may not be known or can't be assumed. So, in this case, ALT can't be applied to predict products' reliability because the cumulative exposure model in this case can't be assumed. Instead, as proposed by DeGroot and Goel [13], "another type of tests called partially accelerated life tests (PALT) is used according to a tampered random variable model".

As Nelson [31] shows, "the stress can be applied in various ways, commonly used method is step-stress. Under step-stress PALT, a test item is first run at use condition and, if it does not fail for a specified time, then it is run at accelerated condition until failure occurs or the test is terminated. Accelerated test stresses involve higher than usual temperature, voltage, pressure, load, humidity, $\ldots$, etc., or some combination of them".

Most of literature performed on PALT discussed non-Bayesianl approaches to make some statistical inferences, for example, see Goel [15], Bhattacharyya and Soejoeti [10], Bai and Chung [6], Bai et al. [7], Attia et al. [5], Abdel-Ghaly et al. [2], Madi [28], Abdel-Ghani [3], Aly and Ismail [4], Ismail and Sarhan [24], Ismail [22], Ismail and Abu-Youssef [23], Ismail [20-21] and Abd-Elfattah et al. [1].

Few of Bayesian researches had been made on PALT. Goel [15] "used the Bayesian approach for estimating the acceleration factor and the parameters in the case of step-stress PALT (SSPALT) with complete sampling for items having exponential and uniform distributions". DeGroot and Goel [13] "investigated the optimal Bayesian design of a PALT in the case of the exponential distribution under complete sampling". Abdel-Ghani [3] "considered the Bayesian approach to estimate the parameters of Weibull distribution in SSPALT with censoring". Ismail [19] "considered the Bayesian approach to estimate the parameters of Gompertz distribution with time-censoring".

In this paper, our objective is to apply a Bayesian analysis of SSPALT considering two-parameter Pareto distribution with Type-I censoring assuming the squared error (SE) loss function. The Bayes estimators (BEs) of the acceleration factor and the distribution parameters are derived and compared with the maximum likelihood estimators (MLEs) counterparts by Monte Carlo simulations.

The rest of this paper is organized as follows. In Section 2, the model and test method are described. Approximate BEs of the parameters under consideration are derived in Section 3. In Section 4, BEs derived in Section 3 are obtained numerically using Lindley's approximation and compared with the MLEs. Also, the highest posterior density credible intervals of the model parameters based on Gibbs sampling technique are presented in Section 3. In Section 4 Monte Carlo simulation study is made for investigating and comparing the methods of ML and Bayes estimators. Section 5 considers an illustrative example with real data set. Finally, a conclusion is presented in Section 6.

2. The model and test method

2.1. The Pareto distribution as a lifetime model. The lifetimes of the test items are assumed to follow two-parameter Pareto distribution of the second kind. Pareto [32]" introduced a distribution (Pareto distribution) as a model for the distribution of income". Many authors, for example, Davis and Feldstein [12], Cohen and Whitten [11], Grimshaw [17] among others "studied its models in several different forms". According to Johnson et al. [25], "Pareto distribution of the second kind also know as Lomax or Pearson's Type VI distribution". Bain and Engelhardt [8] said that "it has been found as a good model in biomedical problems, such as survival time following a heart transplant". Using the

Pareto distribution, Dyer [14] "studied annual wage data of production line workers in a large industrial firm". Lomax [27] "used this distribution in the analysis of business failure data". In addition, Bain and Engelhardt [8] indicated that "the length of wire between flaws also follows a Pareto distribution". Moreover, Howlader and Hossain [18] showed that "since Pareto distribution has a decreasing hazard or failure rate, it has often been used to model incomes and survival times".

The used PDF is expressed by

$$f(t;\theta,\alpha) = \frac{\alpha\theta^{\alpha}}{(\theta+t)^{\alpha+1}}, t \geq 0, \theta > 0, \alpha > 0, \tag{2.1}$$

Its reliability function is given by

$$R(t) = \frac{\theta^{\alpha}}{(\theta+t)^{\alpha}}, \tag{2.2}$$

and its failure-rate function is

$$h(t) = \frac{\alpha}{\theta+t}. \tag{2.3}$$

McCune and McCune [29] indicated that "Pareto distribution has classically been used in economic studies of income, size of cities and firms, service time in queuing systems and so on". Also, according to Davis and Feldstein [12], "it has been used in connection with reliability theory and survival analysis".

2.2. The Test Method. Fundamental Assumptions

(1) Two levels of stress x_1 and x_2 (normal and severe) are applied.
(2) The distribution is Pareto for each stress level.
(3) The total lifetime Y of an item is given by

$$Y = \begin{cases} T, & \text{if } T \leq t \\ \tau + \beta^{-1}(T-\tau), & \text{if } T > \tau, \end{cases} \tag{2.4}$$

where T is the lifetime of an item under normal condition. According to the litrature, "DeGroot Goel [13] proposed this model which is called a tampered random variable (TRV) model". For the tampered random variable models, the readers may also refer to Tang et al. [35].
(4) The failure times y_i; $i = 1, \ldots, n$ are i.i.d. r.v.'s.

Test Process

(1) Each of the n test items is first operate under design stress.
(2) If it does not fail by a pre-specified time τ then it is put on severe condition and run until it fails or the experiment is ended.

The PDF of total lifetime Y of an item under SSPALT is expressed by

$$Y = \begin{cases} 0, & \text{if } y \leq 0 \\ f_1(y) \equiv f_T(t;\theta,\alpha) = \frac{\alpha\theta^{\alpha}}{(\theta+y)^{\alpha+1}}, & \text{if } 0 < y \leq \tau \\ f_2(y) = \frac{\beta\alpha\theta^{\alpha}}{(\theta+\tau+\beta(y-\tau))^{\alpha+1}}, & \text{if } y > \tau, \end{cases} \tag{2.5}$$

where $\theta > 0$ and $\alpha > 0$.

3. Bayesian estimation

3.1. Posterior means and posterior variances. In this section, the SE loss function is used. Under SE loss function, the Bayes estimator of a parameter is its posterior expectation. The Bayes estimators can't be given in explicit forms. Approximate Bayes estimators will be discussed under the assumption of non-informative priors using Lindley's approximation. Basu et al. [9] showed that " in many practical situations, the information about the parameters are available in an independent manner". Thus, here it is assumed that the parameters are independent a priori and let the non-informative prior (NIP) for each parameter be represented by the limiting form of the appropriate natural conjugate prior.

Therefore, the joint NIP of the three parameters can be expressed by

$\pi(\beta,\theta,\alpha)(\beta\theta\alpha)^{-1},\ \beta > 1,\ \theta > 0,\ \alpha > 0.$ (3.1)

The observed values of the total lifetime Y are given by

$$y_{(1)} \leq \ \dots \ \leq \ y_{(nu)} \leq \tau \leq \ y_{(nu+1)} \leq \ \dots \ \leq \ y_{(nu+na)} \leq \eta$$

where n_u is the number of items failed at use condition and n_a is the number of items failed at accelerated condition.

Since the total lifetimes $y_1,\ \dots,\ y_n$ of n items are independent and identically distributed random variables, then the total likelihood function for them is given by

$$L(\underline{y}\,|\beta,\theta,\alpha) = \prod_{i=1}^{n_u}[\tfrac{\alpha\theta^{\alpha}}{(\theta+y_{\{(i)\}})^{\alpha+1}}].\prod_{i=1}^{n_a}[\tfrac{\beta\alpha\theta^{\alpha}}{(\theta+\tau+\beta(y_{\{(i+n_u)\}}-\tau))^{\alpha+1}}].\prod_{i=1}^{n_c}[\tfrac{\theta^{\alpha}}{(\theta+\tau+\beta(\eta-\tau))^{\alpha}}],\quad(3.2)$$

where $n_c = n$ - n_u - n_a.

Forming the product of (3.1) and (3.2), the joint posterior density function of β , θ and α given the data can be written as

$$g(\beta,\theta,\alpha\,\big|\underline{y}) \propto L(\underline{y}\,|\beta,\theta,\alpha).\Pi(\beta,\theta,\alpha)$$
$$\propto \tfrac{\beta^{n_a-1}\theta^{\alpha n-1}\alpha^{n_u+n_a-1}}{(\theta+\tau+\beta(\eta-\tau))^{\alpha n_c}}[\prod_{i=1}^{n_u}\tfrac{1}{(\theta+y_{\{(i)\}})^{\alpha+1}}].[\prod_{i=n_u+1}^{n_u+n_a}\tfrac{1}{(\theta+\tau+\beta(y_{\{(i)\}}-\tau))^{\alpha+1}}].\quad(3.3)$$

According to Lindley [26], "an approximation via an asymptotic expansion of the ratio of two non-tractable integrals is used to obtain approximate Bayes estimators".

Now, let Θ be a set of parameters $\{\Theta_1,\ \Theta_2,\ \dots,\ \ \Theta_m\ \}$, where m is the number of parameters, then the posterior expectation of an arbitrary function $u(\Theta)$ can be asymptotically estimated by

$$\mathrm{E}(u(\Theta)) = \frac{\int_{\Theta} u(\Theta)\pi(\Theta)e^{ln\mathrm{L}(\mathrm{y}|\Theta)}d\Theta}{\int_{\Theta}\pi(\Theta)e^{ln\mathrm{L}(\mathrm{y}|\Theta)}d\Theta}$$
$$\approx [u + (1/2)\sum_{i,j}(u_{ij}^{(2)} + 2u_i^{(1)}\rho_j^{(1)})\sigma_{\mathrm{ij}} + (1/2)\sum_{i,j,k,s}L_{ijk}^{(3)}\sigma_{ij}\sigma_{ks}u_s^{(1)}]\downarrow\hat{\Theta},\quad(3.4)$$

which is the Bayes estimator of $u(\Theta)$ under a squared error loss function, where $\pi(\Theta)$ is the prior distribution of Θ

,$u \equiv u(\Theta)$, $L \equiv L(\Theta)$ is the likelihood function, $\rho \equiv \rho(\Theta) = log\ \pi(\Theta)$, σ_{ij} are the elements of the inverse of the asymptotic Fisher's information matrix of β , θ and α , and

$$\mathrm{u}_i^{(1)} = \frac{\partial u}{\partial \Theta_i}, u_{ij}^{(2)} = \frac{\partial^2 u}{\partial \Theta_i \partial \Theta_j}, \rho_j^{(1)} = \frac{\partial log \pi(\Theta)}{\partial \Theta_j} \text{and} L_{ijk}^{(3)} = \frac{\partial^3 lnL(y|\Theta)}{\partial \Theta_i \partial \Theta_j \partial \Theta_k}$$

According to Green [16], "the linear Bayes estimator in (3.4) is a very good and operational approximation for the ratio of multi-dimension integrals". Also, as pointed out by Sinha [34], "it has led to many useful applications".

Bayesian interval estimators, called credible intervals, for the model parameters are derived from their posterior distributions. We propose the following Markov Chain Monte Carlo (MCMC) method to draw samples from the posterior density function and then to compute the Bayes estimates and the highest posterior density (HPD) credible intervals. We use the Gibbs sampling procedure to compute HPD credible intervals.

3.2. Credible intervals using Gibbs sampling. Assume that the priori are Gamma distributions and that they are independent. Therefore, samples of β , θ and α can be easily generated using any of the gamma generating routines. We use the Gibbs sampling procedure to generate a sample from the posterior density function and then to compute the Bayes estimates and HPD credible intervals. To run the Gibbs sampler algorithm, it is appropriate to start with the approximate BEs. The following algorithm is used for this purpose.

Step 1: Start with an($\theta^{(0)} = \tilde{\theta}$ and $\beta^{(0)} = \tilde{\beta}$) and set I = 1.

Step 2: Generate $\alpha^{(\mathrm{I})}$ from the conditional Gamma distribution $(g(\alpha \left| \theta^{(\mathrm{I}-1)}, \beta^{(\mathrm{I}-1)}, \underline{y}\right.))$

Step 3: Generate $\theta^{(\mathrm{I})}$ from the conditional Gamma distribution $(g(\theta \left| \alpha^{(\mathrm{I}-1)}, \beta^{(\mathrm{I}-1)}, \underline{y}\right.))$

Step 4: Generate $\beta^{(\mathrm{I})}$ from the conditional Gamma distribution $(g(\beta \left| \theta^{(\mathrm{I}-1)}, \alpha^{(\mathrm{I}-1)}, \underline{y}\right.))$

Step 5: Set I = I +1.

Step 6: Repeat steps 2-4 M times and obtain α_i, θ_i and β_i for i=1,..., M.

Step 7: The Bayes MCMC point estimates of β , θ and α with respect to the squared error function are then

$\tilde{\beta} = \tilde{E}(\beta\,|data\,) = \frac{1}{M}\sum_{k=1}^{M} \beta_k$, $\tilde{\theta} = \tilde{E}(\theta\,|data\,) = \frac{1}{M}\sum_{k=1}^{M} \theta_k$ and $\tilde{\alpha} = \tilde{E}(\alpha\,|data\,) = \frac{1}{M}\sum_{k=1}^{M} \alpha_k$.

Step 8: The posterior variances of β , θ and α are

$\tilde{V}(\beta\,|data\,) = \frac{1}{M}\sum_{k=1}^{M} \{\beta_k - \tilde{E}(\beta\,|data\,)\}^2$, $\tilde{V}(\theta\,|data\,) = \frac{1}{M}\sum_{k=1}^{M} \{\theta_k - \tilde{E}(\theta\,|data\,)\}^2$ and $\tilde{V}(\alpha\,|data\,) = \frac{1}{M}\sum_{k=1}^{M} \{\alpha_k - \tilde{E}(\alpha\,|data\,)\}^2$.

Step 9: To compute the credible intervals (CRIs) of ϕ_l $(\phi_1 = \alpha, \phi_2 = \theta \text{and}\, \phi_3 = \beta)$, the quantiles of the sample is usually taken as the endpoints of the intervals. Order $\phi_l^{(1)}, \phi_l^{(2)}, ..., \phi_l^{(M)}, as\, \phi_{l(1)}, \phi_{l(2)}, ..., \phi_{l(M)}$.

Then, the $100(1-2\gamma)\%$ CRIs for ϕ_l become $(\phi_{l(\gamma M)}, \phi_{l((1-\gamma)M)})$.

4. Simulation results and discussion

Simulation results are made for comparing the methods of ML and Bayes estimators, using a SE loss function. The posterior means and posterior variances of the model parameters are obtained suggesting a NIP for each parameter under a SE loss function with time-censored data. Since the BEs of the model parameters can't be found in closed form, approximate BEs are determined numerically using Lindley techniuqe. The performance of the approximate BEs is assessed and compared with the MLEs in Tables 1 and 2 via their variances, MSEs and average confidence interval lengths (CIL) for different settings of true parameter values and sample sizes.

95% confidence intervals of the model parameters are constructed with average CIL presented in Tables 1 and 2. It is shown from the results presented in Tables 1 and 2 that the CRIs obtained under Bayes method (via Gibbs sampling approach) are narrower than those obtained using the ML approach. Also, we observed that the computed coverage probabilities (CP) of the CRIs for each parameter are very close to the nominal level. On the other hand, it was found that these CP using the ML approach are lower than the nominal level in general.

Table 1: Average values of MLEs, BEs, variances and MSEs, when β =2, θ = 0.2, α = 0.5, τ = 3 and η = 7

n	*parameter*	*Method*	*estimate*	*MSE*	*variance*	*CIL*	*CP*
30	β	*ML*	2.4633	0.1569	0.0843	1.1382	92
		Bayes	2.2863	0.1328	0.0669	0.9824	94.1
	θ	*ML*	0.5154	0.0264	0.0376	0.7601	92.3
		Bayes	0.4289	0.0185	0.0115	0.3214	94.3
	α	*ML*	0.7232	0.0172	0.0169	0.5096	93
		Bayes	0.6821	0.0093	0.0081	0.2972	94.4
50	β	*ML*	2.3954	0.1143	0.0548	0.9176	93
		Bayes	2.1627	0.0881	0.0333	0.6302	94.3
	θ	*ML*	0.3653	0.0784	0.0218	0.5788	94
		Bayes	0.3102	0.0113	0.0071	0.2733	94.4
	α	*ML*	0.5563	0.0132	0.0074	0.3372	94.5
		Bayes	0.5382	0.0047	0.0034	0.1765	94.8
75	β	*ML*	2.2233	0.0911	0.0273	0.6477	94.1
		Bayes	2.0793	0.0578	0.0099	0.3140	94.7
	θ	*ML*	0.2977	0.0546	0.0079	0.3484	94.3
		Bayes	0.2286	0.0078	0.0052	0.2261	94.8
	α	*ML*	0.4796	0.0082	0.0025	0.1960	94.6
		Bayes	0.4836	0.0023	0.0015	0.1103	95.1
100	β	*ML*	2.1143	0.0366	0.0057	0.2960	94.3
		Bayes	2.0371	0.0206	0.0046	0.1874	94.9
	θ	*ML*	0.2384	0.0281	0.0047	0.2687	94.4
		Bayes	0.2178	0.0037	0.0016	0.1210	94.9
	α	*ML*	0.4885	0.0026	0.0011	0.1300	94.8
		Bayes	0.4913	0.0014	0.0006	0.0411	95.0

Table 2: Average values of MLEs, BEs, variances and MSEs, when β =3 , θ = 1.5, α = 2, τ = 3 and η = 7

n	*parameter*	*Method*	*estimate*	*MSE*	*variance*	*CIL*	*CP*
30	β	*ML*	3.4943	0.0951	0.0639	0.9909	92.5
		Bayes	3.3511	0.0722	0.0403	0.6820	94.4
	θ	*ML*	1.9411	0.0689	0.0289	0.6664	93
		Bayes	1.7101	0.0522	0.0233	0.5147	94.5
	α	*ML*	2.2677	0.0645	0.0119	0.4276	93.5
		Bayes	2.2153	0.0529	0.0075	0.2655	94.7
50	β	*ML*	3.4461	0.0552	0.0519	0.8930	93.2
		Bayes	3.3289	0.0373	0.0329	0.5918	94.8
	θ	*ML*	1.6533	0.0487	0.0112	0.4149	93.6
		Bayes	1.5790	0.0307	0.0074	0.2413	94.8
	α	*ML*	2.1791	0.0213	0.0043	0.2571	94
		Bayes	2.1342	0.0128	0.0031	0.1643	94.8
75	β	*ML*	3.1731	0.0375	0.0297	0.6756	94.2
		Bayes	3.0944	0.0187	0.0163	0.4101	94.9
	θ	*ML*	1.5832	0.0215	0.0038	0.2416	94.4
		Bayes	1.5346	0.0117	0.0025	0.1386	94.9
	α	*ML*	2.0773	0.0085	0.0020	0.1753	94.5
		Bayes	2.0522	0.0052	0.0016	0.0982	94.9
100	β	*ML*	3.0891	0.0156	0.0112	0.4149	94.6
		Bayes	3.0343	0.0092	0.0074	0.2283	94.9
	θ	*ML*	1.5421	0.0082	0.0022	0.1839	94.7
		Bayes	1.5117	0.0064	0.0014	0.0922	94.9
	α	*ML*	2.0463	0.0036	0.0007	0.1037	94.6
		Bayes	2.0144	0.0028	0.0004	0.0415	95.0

5. Data analysis: A numerical example

To demonstrate the applicability of the methodology introduced in this paper, a numerical example is provided. Pareto model is used to fit the data set. To verify the power of the model, we calculate the Kolmogorov-Smirnov (K-S) distance between the empirical distribution function and the fitted distribution function when the parameters estimates are determined by the maximum likelihood method. The result of K-S test is D=0.0764 with p-value = 0.542. This result obviously shows that the Pareto model provides excellent fit to the data set. So, it can be served successfully for modeling this data set. Assuming Pareto distribution with time-censoring we use n = 76, β =2, θ = 2.5, α = 1.5, τ = 3 and η = 7. The number of failures gained at use and accelerated conditions are n_u=13 and n_a=46, respectively, with censored items n_c=17. The MLEs of the model parameters β,θ and α are respectively 2.09, 2.57 and 1.54, while the BEs are 2.04, 2.53 and 1.52. The MSEs associated with the MLEs of β,θ and α are 0.0241, 0.0207 and 0.0071, respectively, while those associated with the BEs are respectively 0.0156, 0.0111 and 0.0043. In addition, the 95% confidence intervals of β,θ and α using the approaches ML and MCMC are (1.7650, 2.4150), (2.4402, 2.6198), (1.4570, 1.6232) and (1.7976, 2.2824), (2.5040, 2.5760), (1.4467, 1.5933), respectively.

6. Conclusion

In this paper the ML and Bayes estimations of the SSPALT model parameters have been considered. Bayes estimations have been found assuming squared error loss functions and non-informative priors. Lindley approach has been applied to find BEs. It has been seen that the approach acts very well even for small sample sizes. The approach usually provides smaller posterior variances. That is, it gives improved estimates. In the

MCMC approach, it has been noted that the CRIs are shorter than the ML intervals and always include the population parameter values.

Acknowledgment

"This project was supported by King Saud University, Deanship of Scientific Research, College of Science Research Center".

References

[1] Abd-Elfattah, A.M., Hassan, A.S., and Nassr, S.G. *Estimation in step-stress partially accelerated life tests for the Burr type XII distribution using type I censoring*, Stat. Meth. **5** (6), 502-514, 2008.

[2] Abdel-Ghaly, A. A. Attia, A. F., and Abdel-Ghani, M. M. *The Bayesian estimation of Weibull parameters in step partially accelerated life tests with censored data*, Proceedings of the 32stAnnual Conference of Statistics, Computer Science and Operations Research, ISSR, Cairo University, 45-59, 1997.

[3] Abdel-Ghani, M.M. *Investigation of some lifetime models under partially accelerated life tests*, Ph.D. Thesis, Department of Statistics, Faculty of Economics and Political Science, Cairo University, Egypt, 1998.

[4] Aly, H.M. and Ismail, A.A. *Optimum simple time-step stress plans for partially accelerated life testing with censoring*, Far East J.Theor. Stat. **24** (2), 175-200, 2008.

[5] Attia, A.F., Abdel-Ghaly, A.A. and Abdel-Ghani, M.M. *The estimation problem of partially accelerated life tests for Weibull distribution by maximum likelihood method with censored data*, Proceedings of the 31stAnnual Conference of Statistics, Computer Science and Operation Research, ISSR, Cairo University, 128-138, 1996.

[6] Bai, D.S. and Chung, S.W. *Optimal design of partially accelerated life tests for the Exponential distribution under type-I censoring*, IEEE Trans. Reliab. **41** (3), 400-406, 1992.

[7] Bai, D.S., Chung, S.W. and Chun, Y.R. *Optimal design of partially accelerated life tests for the lognormal distribution under type-I censoring*, Reliab. Eng. Syst. Safety **40**, 85-92, 1993.

[8] Bain, L.J. and Engelhardt, M. *Introduction to Probability and Mathematical Statistics* (2nd Edition, PWSKENT Publishing Company, Boston, Mossachusett 1992).

[9] Basu, S., Basu, A.P. and Mukhopadhyay, C. *Bayesian analysis for masked system failure data using non-identical Weibull models*, J. Stat. Plan. Inf. **78**, 255-275, 1999.

[10] Bhattacharyya, G.K. and Soejoeti, Z.A *tampered failure rate model for step-stress accelerated life test*, Commun. Stat.—Theory Methods **18** (5), 1627-1643. 1989.

[11] Cohen, A.C. and Whitten, B. J. *Parameter Estimation in Reliability and Life Span Models* (Marcel Dekker, Inc., New York, 1988).

[12] Davis, H.T. and Feldstein, M.L. *The generalized Pareto law as a model for progressively censored survival data*, Biometrika **66**, 299-306, 1979.

[13] DeGroot, M.H. and Goel, P.K. *Bayesian and optimal design in partially accelerated life testing*, Naval Res. Logist. Quart. **16** (2), 223-235, 1979.

[14] Dyer, D. *The structural probability for the strong Pareto law*, Canad. J. Stat.**9**, 71-77, 1981.

[15] Goel, P.K. Some estimation problems in the study of tampered random variables. Technical Report No. 50, Department of statistics, Carnegie-mellon university, Pittspurgh, Pennsylvania, 1971.

[16] Green, J. *Discussant on D.V. Lindley's (1980) paper on approximate Bayesian methods*, Trabajos Estadistica **31**, 241-243, 1980.

[17] Grimshaw, S.W. *Computing maximum likelihood estimates for the generalized Pareto distribution*, Technometrics **35**, 185-191, 1993.

[18] Howlader, H.A. and Hossain, A.M. *Bayesian survival estimation of Pareto distribution of the second kind based on failure-censored data*, Comput. Stat. and Data Analysis **38**, 301-314. 2002.

[19] Ismail, A.A. *Bayes estimation of Gompertz distribution parameters and acceleration factor under partially accelerated life tests with Type-I censoring*, J. Stat. Comput. Simulat. **80** (11), 1253–1264, 2010.

[20] Ismail, A.A. *Estimating the parameters of Weibull distribution and the acceleration factor from hybrid partially accelerated life test*, Appl. Math. Model. **36**, 2920-2925, 2012.

[21] Ismail, A.A. *Inference in the generalized exponential distribution under partially accelerated tests with progressive Type-II censoring*, Theor. Appl. Fract. Mech. **59**, 49–56, 2012.

[22] Ismail, A.A. On *designing step-stress partially accelerated life tests under failure-censoring scheme*. Proc. IMechE, Part O: J. Risk Reliab. **227** (6), 662–670, 2013.

[23] Ismail, A.A. and Abu-Youssef, S. E. *Parameters estimation under step-stress life test based on censored data from generalized exponential mode*l, J. Test. Eval. **40**, 305-309, 2012.

[24] Ismail, A.A. and Sarhan, A.M. *Optimal planning time-step stress life test using progressively type-II censored exponential data*, Int. Math. Forum **4** (40), 1963–1976, 2009.

[25] Johnson, N.L., Kotz, S. and Balakrishnan, N. *Continuous Univariate Distributions*, Vol. **1**, 2nd edition (Wiley, New York, 1994).

[26] Lindley, D.V. *Approximate Bayesian methods*, Trabajos Estadistica **31**, 223-237, 1980.

[27] Lomax, K.S. *Business failures: another example of the analysis of failure data*, J. Am. Stat. Assoc. **49**, 847-852, 1954.

[28] Madi, M.T. *Bayesian inference for partially accelerated life tests using Gibbs sampling*, Microelectron. Reliab **37** (8), 1165-1168, 1997.

[29] McCune, E.D. and McCune, S.L. *Estimation of the Pareto shape parameter*, Commun. Stat.-Simulat. Comput. **29** (4), 1317-1324, 2000.

[30] Meeker, W.Q. and Escobar, L.A. *Statistical Methods for Reliability Data* (Wiley, New York, 1998).

[31] Nelson, W. *Accelerated Testing-Statistical Models, Test Plans and Data Analysis* (Wiley, New York, 1990).

[32] Pareto, V. *Cours d'Economie politique* (Rouge et Cie, Paris, 1897).

[33] Pathak, P.K., Singh, A.K. and Zimmer, W.J. *Bayes estimation of hazard and acceleration in accelerated testing*, IEEE Trans. Reliab. **40** (5), 615-621, 1991.

[34] Sinha, S.K. *Reliability and Life Testing* (Wiley Eastern Ltd./Halstead Press, 1986).

[35] Tang, Y., Guan, Q., Xu, P., and Xu, H. *Optimum design for type-I step-stress accelerated life tests of two-parameter Weibull distributions*, Commun. Stat.—Theory Methods **41** (21), 3863-3877, 2012.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1247 – 1256

Chain regression-type estimator using multiple auxiliary information in successive sampling

Housila P. Singh[1], Manoj Kumar Srivastava[2], Namita Srivastava[3], Tanveer A. Tarray[1], Virendra Singh[2], and Shivam Dixit[2] *

Abstract

In successive sampling, the use of auxiliary information for estimation of population mean on current occasion is a well explored area. In the present work, the information on an auxiliary variable, which is available on both the occasions, is used along with the information on the study variable from the previous occasion and the current occasion. Consequently, chain regression-type estimator for estimating the population mean are proposed in two occasions successive sampling. The optimal replacement policy is also discussed. We have also given an empirical study along with pictorial representation to examine the merit of the proposed estimator.

2000 AMS Classification: 62DO5.

Keywords: Successive sampling, Chain type ratio estimator, Optimal replacement policy, Rotation pattern, Auxiliary information, Double sampling.

Received 28/02/2014 : Accepted 03/09/2014 Doi : 10.15672/HJMS.2014287479

1. Introduction

When a population is subject to change over time, a survey on a single occasion does not provide information about the nature of change or the rate of change of the characteristics over different occasions and the average value of the characteristic for the most recent occasion or current occasion. To meet these objectives, sampling is done on successive occasions by retaining some units, drawn on the first occasion for its use on the second occasion and replacing the remaining by units drawn on fresh from the current occasion. The related theory and methods are called successive sampling which has drawn considerable attention of survey statisticians. This provides a strong mechanism to produce a reliable estimate of the population mean at the current occasion. In successive sampling over two occasions, the information on the study variable on the first occasion has been utilized as auxiliary information, which provides a strong mechanism to produce

*[1] School of studies in Statistics, Vikram University, Ujjain-456010,(**tanveerstat@gmail.com**)
[2] Institute of Social Sciences, Dr. B. R. A. University, Agra-282002,U.P.-India
[3] Department of Statistics, St. John,s College, Agra

a reliable estimate of the population mean on the current occasion. Some of the reference in this area are Jessen (1942), Yates (1949), Patterson (1950), Tikkiwal (1951), Eckler (1955), Rao and Graham (1964), Singh and Kathuria (1969), Sen (1971, 1972, 1973a, 1973b), Cochran (1977) and Chaturvedi and Tripathi (1983).

Sometimes, the information on auxiliary variables, which are strongly related to the study variable, is available so that their population means are known. The question arises that whether it is possible to utilize the information on the auxiliary variables, which are available on both the occasions, to increase the precision for estimating the population mean on the current occasion. For example in agriculture, the crop infestation due to a pest or disease during a week, in a particular area, may be associated with infestation and ancillary factors such as rainfall, temperature and humidity during the preceding week. Similarly, the yield of a crop during a season in a farm is known to depend to a great extend on the climatic factors, prevailing during the previous season. In biological populations we may be interested to estimate the kill of birds during a season by a hunter in a locality, which is known to be related to the hunter's kill and his disposable income during the previous season. Utilizing the auxiliary information on both the occasions, Feng and Zou (1997), Biradar and Singh (2001), Singh and Priyanka (2007) have proposed a variety of estimators of population mean on the current occasion.

Motivated by Chand's (1975) chain technique, Singh and Priyanka (2008) used the auxiliary information on both the occasions and developed estimators for estimating the population mean on the current occasion in two occasions successive sampling and have discussed their properties.

In the present paper, a chain regression-type difference estimator is proposed for estimating the population mean on the current occasion. Through an empirical investigation the proposed estimator is shown to perform better than Singh and Priyanka (2008) estimator in terms of efficiency. It is noted that higher optimum value of μ (the fraction of the sample taken afresh on the second (current) occasion) is required for the proposed estimator than for Singh and Priyanka estimator when relationship between study variables over two occasions is weak, however, the proposed estimator reports high gain in efficiency. Thus, in case of efficiency is a priority and budget is not a limitation, it is shown that the proposed estimator is superior to Singh and Priyanka (2008) estimator more particularly when relationship between study variables over two occasions is weak.

2. Formulation of Estimator

2.1. Notations and Sampling scheme. Consider a finite population $U = (U_1, U_2, \cdots, U_N)$ with $N(< \infty)$ identifiable units. Let the character under study be denoted by $x(y)$ on the first (second) occasion, respectively. It is assumed that information on an auxiliary variable z is known on the first and second occasions both. We assume that the variable z is closely and positively related with the study variable y . The objective of the present paper is to estimate population mean at the current occasion. For this a sample of size n is drawn from the population on the first occasion by simple random sampling without replacement (SRSWOR) scheme. The observations on z and x are taken for every unit selected in the sample. Out of this sample a subsample of size m is retained (matched subsample) for its use on the second occasion. The y observations are taken on the retained units of the matched subsample on the current occasion. Further, a fresh sample of size $u = n - m = n\mu$ is drawn on the second occasion from the remaining $N - n$ units of the population by simple random sampling without replacement scheme so that total sample size on the second occasion is maintained at n . It is assumed that population is large enough so that finite population correction terms can be ignored. Following notations are used in the present work.

$\bar{X}, \bar{Y}, \bar{Z}$: Population mean of x, y and z respectively.
$\bar{x}_n, \bar{x}_m, \bar{y}_u, \bar{y}_m, \bar{z}_n, \bar{z}_m, \bar{z}_u$: Sample means of the respective variables based on sample sizes shown in suffices
$\rho_{yx}, \rho_{xz}, \rho_{yz}$: Correlation coefficient between the variables given in the subscript.
S_x^2, S_y^2, S_z^2 : Population variance for the variables and .

2.2. Proposed Chain Regression-Type Estimator. Two independent regression-type estimators are suggested for estimating the population mean $\bar{Y}$ on the current occasion. The first estimator is based on sample of size u drawn afresh on the second occasion. The first estimator is a regression estimator defined as

(2.1) $$T_{1u} = \bar{y}_u + b_{yz}(u)(\bar{Z} - \bar{z}_u)$$

where $b_{yz}(u)$ is the sample regression coefficient of y on z based on sample of size u . The second estimator is based on matched subsample of size m which is the common to both the occasions. Motivated by Tripathi and Ahmed (1995) and Ahmad (1998) we define a regression-type estimator based on the sample of size $m = (n\lambda)$ common with both the occasions as ,

(2.2) $$T_{2m} = \bar{y}_m + b_{yx \cdot z}(m)(\bar{x}_n - \bar{x}_m) + b_{yz \cdot x}(m)(\bar{z}_n - \bar{z}_m) + b_{yz}(n)(\bar{Z} - \bar{z}_n)$$

where $b_{yx \cdot z}(m)$ and $b_{yz \cdot x}(m)$ are the sample partial regression coefficients between the variables shown in suffices and based on sample of size m; and $b_{yz}(n)$ is the sample regression coefficient between the variables y and z based on sample of size n. The estimator ($i.e. T_{2m}$) can be also obtained from the equation (9.7.2) in , Sarndal Swensson and Wretman (1992). Combining the estimators T_{1u} and T_{2m}, we have the final estimator of the population mean $\bar{Y}$ as

(2.3) $$T_c = \phi T_{1u} + (1 - \phi) T_{2m}$$

where ϕ is a constant to be determined such that the variance of T_c is minimum.

Adopted the standard techniques given in Cochran (1977, pp.193-194), the variance of the regression–type estimators T_{1u} and T_{2m} to the first degree of approximation (ignoring finite population correction terms) can be easily obtained as

(2.4) $$V(T_{1u}) = (S_y^2/u)(1 - \rho_{yz}^2)$$

and

(2.5) $$V(T_{2m}) = (S_y^2/m)[1 - \rho_{y \cdot xz}^2 + (m/n)(\rho_{y \cdot xz}^2 - \rho_{yz}^2)$$

where

$$\rho_{y \cdot xz}^2 = \frac{(\rho_{yx}^2 + \rho_{yz}^2 - 2\rho_{yz}\rho_{yx}\rho_{xz}}{(1 - \rho_{xz}^2)}$$

Thus the variance of the combined estimator T_c is given by

(2.6) $$V(T_c) = \phi^2 V(T_{1u}) + (1 - \phi)^2 V(T_{2m})$$

which is minimum when

$$\phi = \frac{V(T_{2m})}{V(T_{1u}) + V(T_{2m})} = \phi_{opt}(say)$$

(2.7) $$= \frac{\mu(A + \mu B)}{A + \mu^2 B}$$

where, $\lambda = m/n, \mu = u/n, A = (1 - \rho_{yz}^2), B = \frac{(\rho_{yx} - \rho_{yz}\rho_{xz})^2}{(1 - \rho_{xz}^2)}$.
Here we note that in the expression (2.6), we have not taken the term $\text{cov}(T_{1u}, T_{2m})$

into account because for large population size (i.e. N is very-very large), the term $\text{cov}(T_{1u}, T_{2m})$ is negligible. (i.e. $\lim_{N\to\infty} \text{cov}(T_{1u}, T_{2m}) \to 0$.

Substitution of (2.7) in (2.6) yields the variance of T_c as

$$V(T_c)_{opt} = \frac{V(T_{2u})V(T_{2m})}{V(T_{1u}) + V(T_{2m})}$$

$$= \frac{S_y^2}{n}\frac{A(A+\mu B)}{A+\mu^2 B} \tag{2.8}$$

. Under the assumption $\rho_{xz} = \rho_{yz}$, which has been earlier considered by Cochran (1977), Feng and Zou (1997) and Singh and Priyanka (2008); the expression in (2.8) reduces to

$$V(T_c)_{opt} = \frac{S_y^2}{n}\frac{A(A+\mu B^*)}{A+\mu^2 B^*} \tag{2.9}$$

where

$$B^* = -(\rho_{yx} - \rho_{yz}^2)^2/(1-\rho_{yz}^2)$$

2.3. Comparison of T_c with chain regression-type estimator $T_c^{(1)}$ due to Singh and Priyanka (2008). Using the technique due to Chand (1975), Singh and Priyanka (2008) proposed a chain type regression estimator of population mean on the current occasion by

$$T_c^{(1)} = \phi T_{1u}^{(1)} + (1-\phi)T_{2m}^{(1)} \tag{2.10}$$

with

$$T_{1u}^{(1)} = \bar{y}_u + b_{yz}(u)(\bar{Z} - \bar{z}_u) \tag{2.11}$$

$$T_{2m}^{(1)} = \bar{y}_m^* + b_{yx}(m)(\bar{x}_n^* - \bar{x}_m^*) \tag{2.12}$$

where

$$\bar{y}_m^* = \bar{y}_m + b_{yz}(m)(\bar{Z} - \bar{z}_m),$$
$$\bar{x}_n^* = \bar{x}_n + b_{xz}(n)(\bar{Z} - \bar{z}_n),$$
$$\bar{x}_m^* = \bar{x}_m + b_{xz}(m)(\bar{Z} - \bar{z}_m),$$

The variances of the estimators $T_{1u}^{(1)}$ and $T_{2m}^{(1)}$ to the first degree of approximation (ignoring finite population correction terms) are respectively given by .

$$V(T_{1u}^{(1)}) = \left(\frac{S_y^2}{u}\right)\left(1-\rho_{yz}^2\right)$$

$$V(T_{2m}^{(1)}) = S_y^2\left[\left(\frac{1}{m}\right)(1-\rho_{yz}^2) + \left(\frac{1}{m} - \frac{1}{n}\right)\left\{2\rho_{yz}^2\rho_{yx} - \rho_{yx}^2(1+\rho_{yz}^2)\right\}\right]$$

The variance of $V(T_{2m}^{(1)})$ is derived under the assumption that $\rho_{xz} = \rho_{yz}$ which has been earlier considered by Cochran (1977) and Feng and Zou (1997). Thus the variance of the estimator $T_c^{(1)}$ is given by

$$V(T_c^{(1)}) = \frac{1}{\mu(1-\mu)}[\phi^2(1-\mu)A + (1-\phi)^2\mu(A+\mu B_1)]\frac{S_y^2}{n} \tag{2.13}$$

where $B_1 = 2\rho_{yz}^2\rho_{yx} - \rho_{yx}^2(1+\rho_{yz}^2)$ and $\mu = u/n$ is the fraction of the sample taken a fresh on the second (current) occasion.

The variance of the estimator $T_c^{(1)}$ in (2.13) is minimum for

$$\phi^* = \frac{\mu(A+\mu B_1)}{A+\mu^2 B_1} \tag{2.14}$$

Thus, the resulting variance of $T_c^{(1)}$ is given by

$$\min V(T_c^{(1)}) = \frac{S_y^2}{n}\frac{A(A+\mu B_1)}{A+\mu^2 B_1} \tag{2.15}$$

From (2.9) and (2.15), we have

$$\min V(T_c^{(1)}) - \min V(T_c) = \Big(\frac{S_y^2}{n}\Big)\frac{A\mu(1-\mu)\rho_{yz}^4(1-\rho_{yx})^2}{(A+\mu^2 B_1)(A+\mu^2 B^*)} \tag{2.16}$$

which is always positive.
It follows that the proposed chain regression- type estimator T_c is superior to the chain regression-type estimator $T_c^{(1)}$ due to Singh and Priyanka (2008).

3. Optimum Replacement Policy for T_c

To determine the optimum value of the sample fraction for the required sample to be drawn afresh on the second occasion to estimate population mean $\bar{Y}$ we minimize the minimum variance of the combined estimator in equation (2.9) with respect to μ . The resulting quadratic equation in μ is given by

$$B^*\mu^2 + 2A\mu - A = 0 \tag{3.1}$$

Solving equation (3.1) we get the optimum value for μ

$$\hat{\mu} = \frac{-A \pm \sqrt{A(A+B^*)}}{B^*} \tag{3.2}$$

provided $A(A+B^*) \geq 0$.
Only those value of μ are admissible for which $0 \leq \mu \leq 1$. Otherwise, it is stated that μ does not exist. With this optimum value of μ say μ_0 the minimum $M(T_c)_{opt}$ is given by

$$M(T_c)_{opt} = \frac{S_y^2}{n}\frac{A[A+\mu_0 B^*]}{[A+\mu_0^2 B^*]} \tag{3.3}$$

4. Efficiency Comparison

The proposed estimator T_c is compared with the two estimators namely $\bar{y}_n$, and combined regression-type estimator $\bar{y}_{CD}$. The estimator $\bar{y}_n$ refers to a situation when there is no matching, and , $\bar{y}_{CD} = \psi\bar{y}_u + (1-\psi)\bar{y}_{ld}$, refers to a situation when no auxiliary information is used at any occasion. Here, $\bar{y}_{ld}$ is the regression estimator defined by $\bar{y}_{ld} = \bar{y}_m + b_{yx}(m)(\bar{x}_n - \bar{x}_m)$.
The variance of $\bar{y}_n$ (ignoring fpc terms) is given by

$$V(\bar{y}_n) = \frac{S_y^2}{n} \tag{4.1}$$

and the variance of the estimator $\bar{y}_{CD}$ to the first degree of approximation (ignoring fpc terms) under optimum condition is given by

$$V_{opt}(\bar{y}_C D) = \frac{S_y^2}{n}[1+\sqrt{1-\rho_{yx}^2}] \tag{4.2}$$

The percent relative efficiencies of the proposed estimator T_c and $T_c^{(1)}$ with respect to $\bar{y}_n$ and $\bar{y}_{CD}$ have been calculated for different values of ρ_{yx} and ρ_{yz}

$$E_1(T_c) = \frac{V\bar{y}_n}{V(T_c)_{opt}|_{\mu_0}} \times 100 \qquad and\ E_2(T_c) = \frac{V_{opt}\bar{y}_{CD}}{V(T_c)_{opt}|_{\mu_0}} \times 100$$

$$E_1(T_c^{(1)}) = \frac{V\bar{y}_n}{V(T_c^{(1)})_{opt}|_{\mu_0}} \times 100 \quad and E_2(T_c^{(1)}) = \frac{V_{opt}\bar{y}_{CD}}{V(T_c^{(1)})_{opt}|_{\mu_0}} \times 100 \tag{4.3}$$

Findings are shown in Table 4.1. A pictorial representation of $E_i(T_c)$ and $E_i(T_c^{(1)}), i = 1, 2$ is given in figure 4.1.

Table 1. Relative Efficiencies (%) of T_c and $T_c^{(1)}$ with respect to $\bar{y}_n$ and $\bar{y}_{CD}$

		ρ_{yx}											
		0.3		0.4		0.5		0.6		0.7		0.8	
ρ_{yz}		T_c	$T_c^{(1)}$	T_c	$T_c^{(1)}$	T_c	$T_c^{(1)}$	T_c	$T_c^{(1)}$	T_c	$T_c^{(1)}$	T_c	$T_c^{(1)}$
0.3	μ_0	0.5068	0.5062	0.5154	0.5149	0.5283	0.5279	0.5470	0.5467	0.6152	0.6151	0.6869	0.6869
	E_0	111.39	111.25	113.28	113.17	116.12	116.03	120.22	120.16	135.21	135.18	150.97	150.96
	E_2	108.83	108.69	108.55	108.44	108.34	108.26	108.02	108.14	108.17	108.15	108.39	108.38
0.4	μ_0	0.5035	0.5013	0.5106	0.5089	0.5224	0.5210	0.5400	0.5390	0.6069	0.6065	0.6788	0.6786
	E_1	119.89	119.35	121.58	121.16	124.37	124.05	128.57	128.34	144.50	144.40	161.62	161.57
	E_2	117.30	116.60	116.51	116.10	116.04	115.74	115.72	115.50	115.60	115.52	116.03	116.00
0.6	μ_0	0.5011	0.4829	0.5005	0.4870	0.5061	0.4962	0.5189	0.5118	0.5793	0.5764	0.6507	0.6495
	E_1	156.59	150.92	156.40	152.17	158.17	155.05	162.17	159.93	181.03	118.13	203.35	202.96
	E_2	152.99	147.44	149.87	145.82	147.57	144.66	145.95	143.94	144.83	144.10	146.00	145.72
0.7	μ_0	0.5187	0.4660	0.5040	0.4671	0.5001	0.4741	0.5060	0.4880	0.5574	0.5504	0.6271	0.6240
	E_1	203.32	182.73	197.62	183.19	196.10	185.92	198.41	191.35	218.58	215.83	245.91	244.72
	E_2	198.72	178.52	189.38	175.54	182.96	173.47	178.57	172.22	174.87	172.66	176.55	175.70
0.8	μ_0	0.7526	0.4372	0.5730	0.4345	0.5205	0.4386	0.5016	0.4500	0.5275	0.5092	0.5911	0.5834
	E_1	418.13	242.91	315.37	241.41	288.86	243.64	278.63	250.02	293.02	282.90	328.39	324.10
	E_2	408.50	237.31	302.02	231.33	269.79	227.32	250.78	225.02	234.44	226.32	235.78	232.69

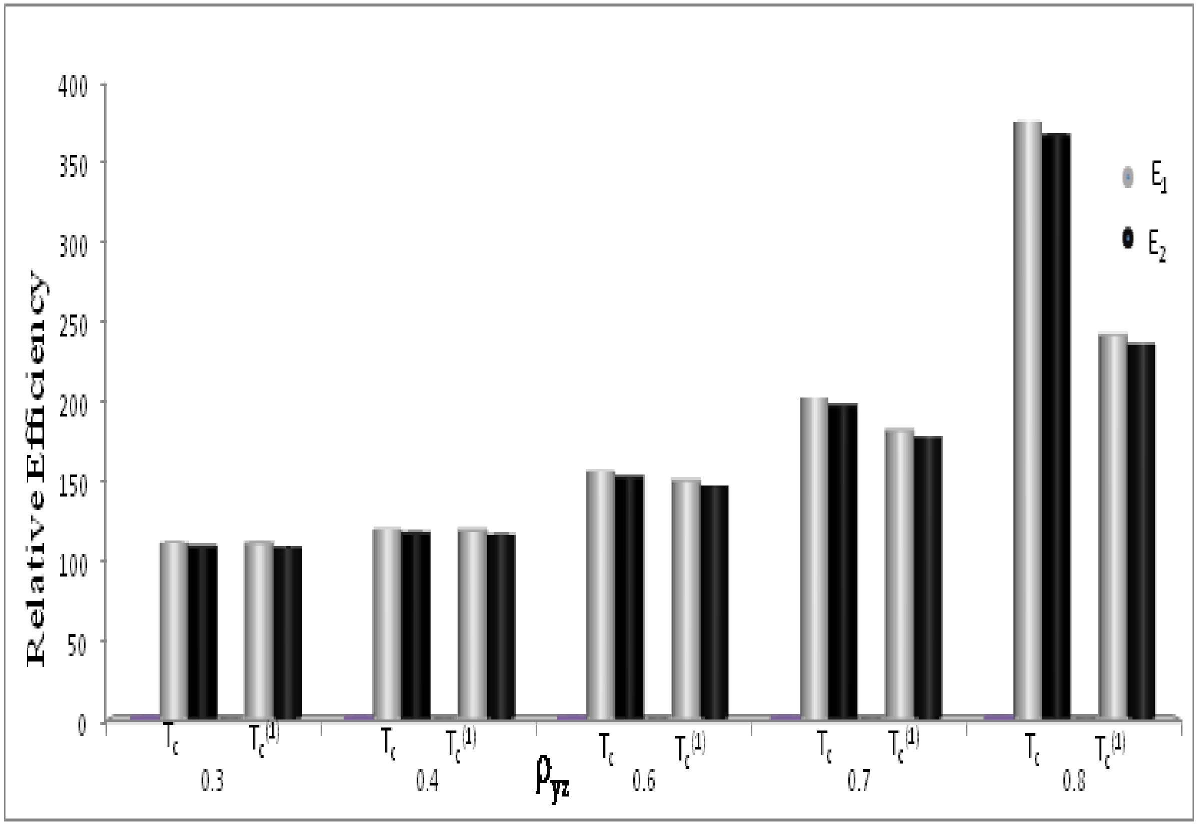


Fig. 4.1 Relative Efficiencies (%) of T_c and $T_c^{(1)}$ with respect to $\bar{y}_n$ and $\bar{y}_{CD}$ when $\rho_{yx} = 0.3$

Remark 4.1 However, if one is able to conduct a well designed simulation study it may throw some more light on the behavior of the suggested estimator in comparison to other existing estimators. Due to authors limitations we have not conducted the simulation study which is one of the criterion to examine the merit of the estimator.

5. Conclusions

The performance of an estimator in successive sampling is generally judged on the basis of relative efficiency and cost of the survey involved in terms of optimum value of μ for using the considered estimator since same is directly associated to the cost of the survey. It is observed from Table 4.1 that the values of $E_1(T_c), E_2(T_c), E_1(T^{(1)})$ and $E_2(T^{(1)})$ are more than 100. Thus, the chain regression type estimators T_c and $T_c^{(1)}$ are better than usual unbiased estimators $\bar{y}_n$ and the estimator $\bar{y}_{CD}$. The proposed estimator utilizes the information on relationship between auxiliary and study variables more efficiently as compared to Singh and Priyanka (2008) estimator. It is further observed from the Table 4.1 that the proposed estimator results into high gain in efficiency at the cost of increased optimum value of μ as compared to that for Singh and Priyanka (2008) estimator particularly when the relationship between study variables over two occasions is weak and between study and auxiliary variables is strong. The price that we pay for using the proposed estimator, in this case, for increased efficiency, is in terms of high cost of survey since more fresh sampling units are required on the current occasion. However, the difference in cost of using proposed and Singh and Priyanka estimators is marginal when the relationship between study variables is strong. Moreover, the proposed estimator continues to be more efficient than Singh and Priyanka (2008) estimator even if it is used with μ which is optimum for Singh and Priyanka estimator. In other words, the proposed estimator continues to be superior to Singh and Priyanka estimator even at a fixed cost. The above observations on the performance of the proposed estimator can easily be seen by considering fixed high value of $\rho_{yx} = 0.8$ and low values of $\rho_{yx} = 0.3$. The proposed estimator results in 72% gain in efficiency over Singh and Priyanka (2008) estimator but with increased cost of the survey that is with increased optimum value of μ about 75%. Further, the proposed estimator continues to report high relative efficiency about 56% at a fixed cost that is when the proposed estimator is used at 44% of an optimum value of μ for Singh and Priyanka estimator. One may thus notice that the proposed estimator addresses the problem of weak relationship between study variables on two occasions and compensates for this situation by allowing for more fresh units on the current occasion while continuing to yield high efficiencies by exploiting strong relationship between study and auxiliary variables. Thus a survey statistician can use the proposed estimator over Singh and Priyanka (2008) estimator in case of strong relationship between study variables over two occasions. However, in case of weak relationship between study variables over two occasions, a survey statistician can use the proposed estimator over Singh and Priyanka (2008) estimator for higher gain in efficiency but with increased cost if efficiency is the priority and budget is not a limitation. Even if the budget is limited, statistician can use the proposed estimator at a fixed cost in terms of optimum value of μ for Singh and Priyanka (2008) estimator for better efficiency. Thus, the proposed estimator is justified.

Acknowledgements. The authors are thankful to the Editor – in- Chief - Professor Cem Kadilar, and to the anonymous learned referees for their valuable suggestions regarding improvement of the paper.

References

[1] Ahmed, M.S. *A note on regression-type estimators using multiple auxiliary information.* Austral. and New Zealand Jour. Statist., **40**(3), 373-376, 1998.

[2] Biradar, R. S., Singh, H. P. *Successive sampling using auxiliary information on both occasions.* Cal. Statist. Assoc. Bull., **51**,243-251, 2001.

[3] Chand, L. *Some ratio-type estimator based on two or more auxiliary variables.* Unpublished Ph.D. dissertation, Iowa State University, Ames, Iowa, 1975.

[4] Chaturvedi, D. K., Tripathi, T.P. . *Estimation of population ratio on two occasions using multivariate auxiliary information.* Jour. Ind. Soc. Agri. Statist., **21**, 113-120, 1983.

[5] Cochran, W. G. *Sampling techniques. 3rd edition.* New York: John Wiley and Sons, 1977.

[6] Eckler, A. R. *Rotation Sampling.* Annals of Math. Statist., **26**,664-685, 1955.

[7] Feng, S., and Zou, G. *Sample rotation method with auxiliary variable.* Comm. Statist., Theory Methods, **26**(6), 1497-1509, 1997.

[8] Jessen, R. J. *Statistical investigation of a sample survey for obtaining farm facts.* In: Iowa Agriculture Experiment Station Road Bulletin. No. 304 lowa, U.S.A: Ames,1-104, 1942.

[9] Kiregyera, B. *Regression type estimators using two auxiliary variable and the model of double sampling from finite population.* Metrika, **31**,215-226, 1984.

[10] Mukerjee, R., Rao, T. J. and Vijayan, K. *Regression type estimator using multiple auxiliary information.* Austral. Jour. Statist. **29**, 244-254, 1987.

[11] Patterson, H. D. *Sampling on successive occasions with partial replacement of units.* Jour. Roy. Statist. Assoc. B, **12**, 241-255, 1950.

[12] Rao, J. N. K. and Graham. J. E. *Rotation design for sampling on repeated occasions,* Jour. Amer. Statist. Assoc., **59**, 492-509, 1964.

[13] C.E., Swensson, B. and Wretman, J. *Model assisted survey sampling,* New York: Springer-verlag, 1992.

[14] Sen, A. R. *Successive sampling with two auxiliary variables,* Sankhya, **33(B)**, 371-378, 1971.

[15] Sen, A. R. *Successive sampling with auxiliary variables,* The Annals of Math. Statist., **43**, 2031-2034, 1972.

[16] Sen, A. R. *Theory and application of sampling on repeated occasions with several auxiliary variables,* Biometrics, **29**, 381-385, 1973a.

[17] Sen, A. R. *Some Theory of Sampling on successive occasion,* Austral. Jour. Statist., **2**, 105-110, 1973b.

[18] Singh, D. *Estimates in successive sampling using multi stage design,* Jour. Amer. Statist. Assoc., **63**, 99-112, 1968.

[19] Singh, D. and Kathuria, O. P. *On two stage successive sampling,* Austral. Jour. Statist., **11(2)**, 59-66, 1969.

[20] Singh, G. N. and Priyanka, K. *On the use of auxiliary information in search of good rotation patterns on successive occasions,* Bull. Statist. and Econo., **1, 07**, 42-60, 2007.

[21] Singh, G. N. and Priyanka, K. *Search of good rotation patterns to improve the precision of estimates at current occasion,* Comm. Statist.-Theory and Methods,**37(3)**, 337-348, 2008.

[22] Sukhatame, P. V., Sukhatame, B. V., Sukhatme, S., Ashok, C. *Sampling theory of surveys with applications,* Ames, lowa: lowa State University Press and New Delhi India: Indi. Soci. Agri. Statist., 1984.

[23] Tikkiwal, B. D. *Theory of successive sampling,* Thesis for diploma, ICAR, New Delhi, 1951.

[24] Tripathi, T.P. and Ahmad, M.S. *A class of estimators for a finite population mean based on multivariate information and general two phase sampling,* Cal. Statist. Assoc. Bull., **45** , 203-218, 1995.

[25] Yates, F. *Sampling method for censuses and surveys,* Charles Griffin and Co., London, 1949.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1257 – 1270

Study of Yadav and Kadilar's improved exponential type ratio estimator of population variance in two-phase sampling

Housila P. Singh , Ragen Goli and Tanveer A. Tarray*

Abstract

This paper presents a double sampling version of Yadav and Kadilar (2013) estimator alongwith its properties under large sample approximation. Cost aspect is also discussed. We have compared the proposed estimator with usual unbiased estimator and usual double sampling ratio estimator and shown that the proposed estimator is better than usual unbiased estimator and other existing estimators under some realistic conditions to two-phase sampling.

Keywords: Auxiliary variable, Bias, Efficiency, Mean squared error, Double sampling.

Received 21/01/2014 : Accepted 18/03/2014 Doi : 10.15672/HJMS.20158612895

1. Introduction

The use of auxiliary information has been dealt with at great length for improving estimators of population parameters in sample surveys. Various estimation procedures in sample surveys need advance knowledge of some auxiliary variable which is then used to increase the precision of estimates. For example, the ratio - type estimator due to Isaki (1983) need the advance knowledge of population variance S_x^2 of the auxiliary variable x. When the population variance $S_x^2$is not known, it is sometimes estimated from a preliminary large sample on which only the auxiliary characteristic x is observed. The value of S_x^2in the estimator is then replaced by its estimate. A smaller second phase sample of the variate under study y is then taken. This technique, known as double sampling or two-phase sampling, is especially appropriate if the x values are easily accessible and much cheaper to collect than the y_i values see. Hidiroglou and Sarandal (1998). The use of double sampling is necessary if the x - value is obtained by performing a non-destructive experiment where as to obtain a y - value of a unit, a destructive experiment has to be performed, see UnniKrishan and Kunte (1995). Double sampling is also an able alternative to simple random sampling when there are expected to be gains from using

*School of Studies in Statistics,Vikram University, Ujjain - 456010 - India
Email:tanveerstat@gmail.com

auxiliary information.

let $U = (U_1, U_2, ...U_N)$ denote the population of N units and let (y, x) be the variate defined on U taking values (y_i, x_i) on $U_i(i = 1, 2, ..., N)$. It is desired to estimate S_y^2 of the study variate y. A simple random sample of size n is drawn without replacement (SRSWOR) from the population U. The usual unbiased estimator of based on SRSWOR is given by :

$$(1.1) \qquad S_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2,$$

where $\bar{y} = \frac{1}{n}\sum_{i}^{n} y_i$ is the sample mean based on n observations.

To improve the usual unbiased estimator s_y^2, using the known population variance S_x^2 of the auxiliary variate x, Isaki (1983) suggested a ratio-type estimator for the population variance S_y^2 as

$$(1.2) \qquad t_l = s_y^2\frac{S_x^2}{s_x^2},$$

where $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$, is an unbiased estimator of the population variance s_x^2 and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the sample mean.

Singh et al. (2011) proposed the exponential ratio estimator for the population variance S_y^2 as

$$(1.3) \qquad t_s = s_y^2 exp(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}).$$

when the population varianceS_x^2 of the auxiliary character x, the usual linear regression estimator for population variance S_x^2 is defined by

$$(1.4) \qquad t_l r = s_y^2 + \hat{\beta}(S_x^2 - s_x^2)$$

where $\hat{\beta} = \frac{s_y^2(\hat{\lambda}_{22} - 1)}{s_x^2(\hat{\lambda}_{04} - 1)}$ is sample regression coefficient,
$\hat{\lambda}_{04} = \frac{\hat{\mu}_{04}}{\hat{\mu}_{02}^2}$, $\hat{\lambda}_{22} = \frac{\hat{\mu}_{22}}{\hat{\mu}_{20}\hat{\mu}_{02}}$,
$\hat{\mu}_{04} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^4$, $\hat{\mu}_{02} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$,
$\hat{\mu}_{20} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^4$, $\hat{\mu}_{22} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2(x_i - \bar{x})^2$.
Motivated by Upadhyaya et al. (2011), Yadav and Kadilar (2013) suggested the following class of estimators of the population variance S_y^2 as

$$(1.5) \qquad t_y = s_y^2 exp[\frac{S_x^2 - s_x^2}{S_x^2 + (\alpha - 1)s_x^2}],$$

where $(\alpha \geq 0)$.

In this paper we have studied the properties of the above estimators $t_1, t_s, t_l r$ and $t_y$in the case of double sampling (i.e. when the population variance S_x^2 of the auxiliary variable x is not known). Cost aspects are also discussed. Numerical illustration is given in support of the present study.

2. Two-phase sampling estimators

When the population variance $S_x^2$of x is not known, a first phase sample of $n_1$is drawn from the population on which only the x-characteristic is measured in order to furnish a good estimate of S_x^2.Then a second phase sample of size n is drawn on which both the variates y and x are measured [see Singh and Ruiz Espejs (2007)].Let $(x_1, x_2, ..., x_{n1})$be the first phase sample drawn by simple random sampling without replacement (SRSWOR) from the given population U and only auxiliary variable x be measured.
Also, let $(y_1, y_2, ..., y_n)$ and $(x_1, x_2, ..., x_n)$,$(n < n_1)$denote respectively, the second phase sample for the study variable y and the auxiliary variable x respectively.
Let us write $\bar{x_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i$, $s^2_{x_1} = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(x_i - \bar{x_1})^2$, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$, $s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$,
Then the two-phase sampling (or double sampling) estimators of population variance S_y^2 are given by

$$t_{ld} = s_y^2[\frac{s^2_{x1}}{s_x^2}], \tag{2.1}$$

$$t_{sd} = s_y^2 exp[\frac{s^2_{x1} - s_x^2}{s^2_{x1} + s_x^2}], \tag{2.2}$$

and

$$t_{yd} = s_y^2 exp[\frac{s^2_{x1} - s_x^2}{s^2_{x1} + (\alpha - 1)s_x^2}]. \tag{2.3}$$

It is to be mentioned that the estimators t_{ld}, t_{sd} and t_{yd} are double sampling versions of Isaki (1983) estimator, Singh et al. (2011) estimator and Yadav and Kadilar (2013) estimator. For $\alpha = 2$ in (8),t_{yd} reduces to the estimator t_{sd}.

3. The first Degree Approximation to the Biases and Variances of the Suggested Estimators.

In order to study the large sample properties of the proposed estimators, we define.
$s_y^2 = S_y^2(1 + \varepsilon_0)$, $s_x^2 = S_x^2(1 + \varepsilon_1)$,$s^2_{x1} = S_x^2(1 + \varepsilon_2)$
such that $E(\varepsilon_0) = E(\varepsilon_1) = E(\varepsilon_2) = 0$
The following two cases will be considered separately.

Case - I : When the second phase sample of size n is a subsample of the first phase of size n_1.

Case - II : When the second phase sample of size n is drawn independently of the first phase sample of size n_1 see Bose (1943)

Case I - When the second phase sample of size n is a subsample of the first phase sample of size n_1 $(n < n_1)$, the expected values are :

$$E(\varepsilon_0^2) = \frac{1}{n}(\lambda_{40}-1), E(\varepsilon_1^2) = \frac{1}{n}(\lambda_{04}-1), E(\varepsilon_0\varepsilon_1) = \frac{1}{n}(\lambda_{22}-1), E(\varepsilon_2^2) = \frac{1}{n_1}(\lambda_{04}-1),$$

$$E(\varepsilon_0\varepsilon_2) = \frac{1}{n_1}(\lambda_{22} - 1), E(\varepsilon_1\varepsilon_2) = \frac{1}{n_1}(\lambda_{40} - 1), \tag{3.1}$$

where $\lambda_{rs} = \frac{\mu_{rs}}{(\mu_{20}^{r/2})(\mu_{02}^{s/2})}$, $\mu_r s = \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{y})^r(x_i - \bar{x})^s$
(r, s) being non-negative integers,

Case II - When the second phase sample of size n is independent of the first phase sample of size n_1 , the expected value are :

$$E(\varepsilon_0^2) = \frac{1}{n}(\lambda_{40} - 1), E(\varepsilon_1^2) = \frac{1}{n}(\lambda_{04} - 1), E(\varepsilon_0\varepsilon_1) = \frac{1}{n}(\lambda_{22} - 1),$$

(3.2) $$E(\varepsilon_2^2) = \frac{1}{n_1}(\lambda_{04} - 1), E(\varepsilon_0\varepsilon_2) = E(\varepsilon_1\varepsilon_2) = 0,$$

Expressing t_{ld}, t_{sd} and t_{yd} in terms of $\varepsilon_i's, (i = 0, 1, 2)$, we have

(3.3) $$t_{ld} = s_y^2(1 + \varepsilon_0)(1 + \varepsilon_1)^{-1}(1 + \varepsilon_2)$$

(3.4) $$t_{sd} = s_y^2(1 + \varepsilon_0)exp[-\frac{(\varepsilon_1 - \varepsilon_2)}{2}(1 + \frac{\varepsilon_1 + \varepsilon_2}{2})^{-1}]$$

(3.5) $$t_{yd} = s_y^2(1 + \varepsilon_0)exp[-\frac{(\varepsilon_1 - \varepsilon_2)}{\alpha}(1 + \frac{(\alpha - 1)\varepsilon_1 + \varepsilon_2}{\alpha})^{-1}]$$

Expanding the right hand side of (11), (12) and (13) multiplying out and neglecting terms of $e's$ having power greater than two we have

$$t_{ld} \cong S_y^2(1 + \varepsilon_0 + \varepsilon_2 - \varepsilon_1 + \varepsilon_0\varepsilon_2 - \varepsilon_0\varepsilon_1 - \varepsilon_1\varepsilon_2 + \varepsilon_1^2)$$

or

(3.6) $$(t_{ld} - S_y^2) \cong S_y^2(\varepsilon_0 + \varepsilon_2 - \varepsilon_1 + \varepsilon_0\varepsilon_2 - \varepsilon_0\varepsilon_1 - \varepsilon_1\varepsilon_2 + \varepsilon_1^2)$$

$$t_{sd} \cong S_y^2[(1 + \varepsilon_0 - \frac{(\varepsilon_1 - \varepsilon_2)}{2} - \frac{(\varepsilon_0\varepsilon_1 - \varepsilon_0\varepsilon_2)}{2} + \frac{(3\varepsilon_1^2 - \varepsilon_2^2 - 2\varepsilon_1\varepsilon_2)}{8}]$$

or

(3.7) $$(t_{sd} - S_y^2) \cong S_y^2[(\varepsilon_0 - \frac{(\varepsilon_1 - \varepsilon_2)}{2} - \frac{(\varepsilon_0\varepsilon_1 - \varepsilon_0\varepsilon_2)}{2} + \frac{(3\varepsilon_1^2 - \varepsilon_2^2 - 2\varepsilon_1\varepsilon_2)}{8}]$$

$$t_{yd} \cong S_y^2[1 + \varepsilon_0 - \frac{(\varepsilon_1 - \varepsilon_2)}{\alpha} - \frac{(\varepsilon_0\varepsilon_1 - \varepsilon_0\varepsilon_2)}{\alpha} + \frac{((2\alpha - 1)\varepsilon_1^2 - \varepsilon_2^2 - \alpha\varepsilon_1\varepsilon_2)}{2\alpha^2}]$$

or

(3.8) $$(t_{yd} - S_y^2) \cong S_y^2[\varepsilon_0 - \frac{(\varepsilon_1 - \varepsilon_2)}{\alpha} - \frac{(\varepsilon_0\varepsilon_1 - \varepsilon_0\varepsilon_2)}{\alpha} + \frac{((2\alpha - 1)\varepsilon_1^2 - \varepsilon_2^2 - 2\varepsilon_1\varepsilon_2)}{2\alpha^2}]$$

Now squaring both sides of (14), (15) and (16) and neglecting terms of $\varepsilon' s$having power greater than two we have

(3.9) $$(t_{ld} - s_y^2) = S_y^4(\varepsilon_0^2 + (\varepsilon_2 - \varepsilon_1)^2 - 2(\varepsilon_0\varepsilon_1 - \varepsilon_0\varepsilon_2)]$$

(3.10) $$(t_{sd} - S_y^2)^2 = S_y^4(\varepsilon_0^2 + \frac{(\varepsilon_1 - \varepsilon_2)^2}{4} - (\varepsilon_0\varepsilon_1 - \varepsilon_0\varepsilon_2)]$$

and

(3.11) $$(t_{yd} - S_y^2)^2 = S_y^4(\varepsilon_0^2 + \frac{(\varepsilon_1 - \varepsilon_2)^2}{\alpha^2} - \frac{(2\varepsilon_0\varepsilon_1 - \varepsilon_0\varepsilon_2)}{\alpha})]$$

Taking expectations of both sides of (14), (15), (16) and (17), (18), (19) and using the results in (9), we get the biases and mean squared errors of $t_{1d}, t_{sd}$and t_{yd} to the first degree of approximation under case-I respectively as

(3.12) $$B(t_{ld})_1 = (\frac{1}{n} - \frac{1}{n_1})(\lambda_{04} - 1)S_y^2(1 - C)$$

(3.13) $$B(t_{sd})_1 = \frac{1}{8}(\frac{1}{n} - \frac{1}{n_1})(\lambda_{04} - 1)S_y^2(3 - 4C)$$

$$(3.14)\quad B(t_{yd})_1 = \frac{(\lambda_{04}-1)}{2\alpha^2}[\frac{1}{n}[2\alpha(1-c)-1] - \frac{1}{n_1}(1+\alpha(1-2c))]$$

$$(3.15)\quad MSE(t_{1d})_1 = S_y^4[\frac{1}{n}(\lambda_{40}-1) + (\frac{1}{n} - \frac{1}{n_1})(\lambda_{04}-1)(1-2c)]$$

$$(3.16)\quad MSE(t_{sd})_1 = S_y^4[\frac{1}{n}(\lambda_{40}-1) + (\frac{1}{n} - \frac{1}{n_1})\frac{1}{4}(\lambda_{04}-1)(1-4c)]$$

$$(3.17)\quad MSE(t_{yd})_1 = S_y^4[\frac{1}{n}(\lambda_{40}-1) + (\frac{1}{n} - \frac{1}{n_1})\frac{1}{\alpha^2}(\lambda_{04}-1)(1-2\alpha c)]$$

where $c = \frac{\lambda_{22}-1}{\lambda_{04}-1}$, and $B(.)_1$ and $MSE(.)_1$stand the bias of (.) under case-I (i.e. when the second phase sample is a subsample of the first phase sample) respectively.
Now taking the expectations of both sides of (14), (15), (16) and (17), (18) and (19) and using results in (10) we get the biases and mean squared errors of the estimators t_{1d}, t_{sd} and t_{yd} to the first degree of approximation under case-II respectively as

$$(3.18)\quad B(t_{ld})_{11} = \frac{S_y^2(\lambda_{04}-1)}{n}(1-c)$$

$$(3.19)\quad B(t_{sd})_{11} = \frac{S_y^2(\lambda_{04}-1)}{8}[\frac{3-4c}{n} - \frac{1}{n_1}]$$

$$(3.20)\quad B(t_{yd})_{11} = \frac{S_y^2(\lambda_{04}-1)}{2\alpha^2}[\frac{(2\alpha-2\alpha c-1)}{n} - \frac{1}{n_1}]$$

$$(3.21)\quad MSE(t_{ld})_{11} = S_y^4[(\frac{1}{n})[(\lambda_{40}-1) + (\lambda_0 4-1)(1-2c)] + \frac{\lambda_{04}-1}{n_1}]$$

$$(3.22)\quad MSE(t_{sd})_{11} = S_y^4[(\frac{1}{n})[(\lambda_{40}-1) + \frac{(\lambda_{04}-1)}{4}(1-4c)] + \frac{\lambda_{04}-1}{4n_1}]$$

$$(3.23)\quad MSE(t_{yd})_{11} = S_y^4[(\frac{1}{n})[(\lambda_{40}-1) + \frac{(\lambda_{04}-1)}{\alpha^2}(1-2\alpha c)] + \frac{\lambda_{04}-1}{\alpha^2 n_1}]$$

where $B(.)_{11}$ and $MSE(.)_{11}$stand the bias of (.) and MSE of (.) under case-II.

4. Optimum choice of the scalar $'\alpha'$

Case - I The $MSE(t_{yd})_1$ at (25)is minimized for

$$(4.1)\quad \alpha = \frac{1}{c} = \alpha_{opt}(say)$$

Substitution (32) in (8) yields the asymptotically optimum estimator (AOE) of S_y^2 as

$$(4.2)\quad t_{yd(0)} = s_y^2 exp[\frac{c(s_{x1}^2 - s_x^2)}{cs_{x1}^2 + (1-c)s_x^2}]$$

The value of $'c'$ can be guessed quite accurately from the past data or experience gathered in due course of time see Yadav and Kadilar (2013, p. 148). In case c is not known, it is worth advisable to replace c by its consistent estimate $\hat{c} = \frac{(\hat{\lambda}_{22}-1)}{\hat{\lambda}_{04}-1}$ based on sample data at hand, where $\hat{\lambda}_{22}$ and $\hat{\lambda}_{04}$ are same as defined earlier. Thus replacing $'c'$ by its estimate $'\hat{c}'$in (33) , we get an estimator of S_y^2 based on estimated optimum as

$$(4.3)\quad \hat{t}_{yd(0)} = s_y^2 exp[\frac{\hat{c}(s_{x1}^2 - s_x^2)}{\hat{c}s_{x1}^2 + (1-\hat{c}s_x^2)}]$$

It can be shown to the first degree of approximation that

$$(4.4)\qquad MSE(t_{yd(0)})_1 = MSE(\hat{t}_{yd(0)})_1 = \frac{s_y^4}{n}[(\lambda_{40}-1) - (\frac{n_1-1}{n_1})\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}]$$

which equals to the approximate variance / MSE of the regression estimator

$$t_{lrd} = s_y^2 + \frac{s_y^2((\hat{\lambda})_{22}-1)}{s_x^2((\hat{\lambda})_{04}-1)}(s_{x1}^2 - s_x^2)$$

Thus the proposed $\hat{t}_{yd(0)}$ is an alternative to the regression estimator t_{lrd} It is well known under SRSWOR that to the first degree of approximation (ignoring fpc term) that

$$(4.5)\qquad V(s_y^2) = MSE(s_y^2) = \frac{1}{n}S_y^4(\lambda_{40}-1)$$

From (23), (24), (35) and (36) we have

$$(4.6)\qquad MSE(s_y^2) - MSE(\hat{t}_{yd(0)}) = (\frac{1}{n} - \frac{1}{n_1})S_y^4\frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)} \geq 0$$

$$(4.7)\qquad MSE(t_{ld}) - MSE(\hat{t}_{yd(0)}) = (\frac{1}{n} - \frac{1}{n_1})S_y^4(\lambda_{04}-1(1-c)^2 \geq 0$$

$$(4.8)\qquad MSE(t_{sd}) - MSE(\hat{t}_{yd(0)}) = (\frac{1}{n} - \frac{1}{n_1})S_y^4\frac{(\lambda_{04}-1)}{4}(1-2c)^2 \geq 0$$

It follows from (37), (38) and (39) that the proposed estimator $\hat{t}_{d(0)}$ is more efficient than the usual unbiased estimator S_y^2, t_{ld} and t_{sd}. Thus the proposed estimator $\hat{t}_{yd(0)}$ is an appropriate choice among the estimator S_y^2, t_{ld}, t_{sd} and $\hat{t}_{yd(0)}$ to be used in practice.
case - II: The $MSE(t_{yd})_{11}$ at (31) is minimized for

$$(4.9)\qquad \alpha = \frac{n+n_1}{n_1 c} = \alpha^*_{opt}$$

Substitution of (40) is (8) yields the asymptotically optimum estimator (AOE) under case-II as

$$(4.10)\qquad t^*_{yd(0)} = s_y^2 exp[\frac{c(s_{x1}^2 - s_x^2)}{cs_{x1}^2 + (\delta - c)s_x^2}]$$

where $\delta = (n+n_1)/n_1$
if c is not known, then we replace c by its consistent estimate $\hat{c}$. thus the estimator based on estimated optimum value $\hat{c}$ of c is given by

$$(4.11)\qquad (\hat{t})^*_{yd(0)} = s_y^2 exp[\frac{(\hat{c})(s_{x1}^2 - s_x^2)}{(\hat{c})s_{x1}^2 + (\delta - (\hat{c}))s_x^2)}]$$

To the first degree of approximation (ignoring fpc terms), it can be shown that

$$(4.12)\qquad MSE(t^*_{yd(0)}) = \frac{S_y^4}{n}[(\lambda_{40}-1) - \frac{n_1}{(n+n_1)}(\lambda_{04}-1)c^2]$$

From (29), (30), (36) and (43), we have

$$(4.13)\qquad MSE(s_y^2) - MSE(\hat{t}^*_{yd(0)}) = \frac{n_1}{n(n+n_1)}S_y^4(\lambda_{04}-1)c^2 \geq 0$$

$$(4.14)\qquad MSE(t_{ld})_{11} - MSE(\hat{t}^*_{yd(0)}) = \frac{S_y^4(\lambda_{04}-1n+n_1(1-c)^2)}{n(n+n_1)} \geq 0$$

$$(4.15)\quad MSE(t_{sd})_{11} - MSE(\hat{t}^*_{yd(0)}) = \frac{S_y^4(\lambda_{04}-1)(n+n_1-2n_1c^2)}{4nn_1(n+n_1)} \geq 0$$

Thus the proposed estimator $\hat{t}^*_{yd(0)}$ is more efficient than the usual unbiased estimator s_y^2, t_{ld} and t_{sd} under case - II.
From (35) and (43), we have

$$(4.16)\quad [MSE(t^*_{yd(0)})_1 - MSE(t^*_{yd(0)})_{11}] = \frac{ns_y^4(\lambda_{04}-1)c^2)}{n_1(n+n_1)} \geq 0$$

which shows that the proposed estimator $t_{yd(0)}$ under case -I is better than the proposed estimator $t^*_{yd(0)}$ under case - II.

5. EFFICIENCY COMPARISON OF THE PROPOSED ESTIMATOR WHEN THE SCALAR ıαJ DOES NOT COINCIDE EXACTLY WITH ITS OPTIMUM VALUE.

In this section we compare the proposed estimator $t_{yd}$with the estimators s_y^2, t_ld, t_sd under case - I and II.
Case - I: From (25) and (36) we have

$$(5.1)\quad MSE(s_y^2) - MSE(t_{yd})_1 = (\frac{1}{n} - \frac{1}{n_1})s_y^4\frac{1}{\alpha^2}(2\alpha c - 1)$$

which is positive if

$$2\alpha c - 1 > 0$$

i.e. if

$$(5.2)\quad \alpha > \frac{1}{2c}$$

From (23) and (25) we have

$$MSE(t_{ld})_1 - MSE(t_{yd})_1 = (\frac{1}{n} - \frac{1}{n_1})s_y^4(\lambda_{04}-1)[1 - 2c - \frac{1}{\alpha^2} + \frac{2c}{\alpha}]$$

which is positive if $[(1 - \frac{1}{\alpha^2}) - 2c(1 - \frac{1}{\alpha})] > 0$
i.e. if

$$(5.3)\quad either min.[1, \frac{1}{(2c-1)}] < \alpha < max.[1, \frac{1}{(2c-1)}], c > \frac{1}{2}$$

or

$$(5.4)\quad \alpha > 1, 0 \leq c \leq \frac{1}{2}$$

Further from (24) and (25) we have

$$MSE(t_{sd})_1 - MSE(t_{yd})_1 = (\frac{1}{n} - \frac{1}{n_1})s_y^4(\lambda_{04}-1)(\frac{1}{2} - \frac{1}{\alpha})(\frac{1}{2} + \frac{1}{\alpha} - 2c)$$

which is greater than ' zero' if

$$(\frac{1}{2} - \frac{1}{\alpha})(\frac{1}{2} + \frac{1}{\alpha} - 2c)$$

i.e. if

$$eithermin.[2, \frac{2}{4c-1}] < \alpha < max.[2, \frac{2}{4c-1}] \tag{5.5}$$

or

$$\alpha > 2, 0 \leq c \leq \frac{1}{4}$$

Thus we established the following theorem.
Theorem - 5.1 The proposed estimator t_{yd} in case-I is more efficient than :
(i) the usual unbiased estimators_y^2 if

$$\alpha > \frac{1}{2c}$$

(ii)the Isaki (1983) double sampling ratio estimator t_{ld} if

$$eithermin.[1, \frac{1}{(2c-1)}] < \alpha < max.[1, \frac{1}{(2c-1)}], c > \frac{1}{2}$$

or

$$\alpha > 1, 0 \leq c \leq \frac{1}{2} \tag{5.6}$$

(iii)the double sampling version of Singh et al (2011) estimator t_{sd} if

$$eithermin.[2, \frac{2}{4c-1}] < \alpha < max.[2, \frac{2}{4c-1}]$$

or

$$\alpha > 2, 0 \leq c \leq \frac{1}{4}$$

Case II-From (31) and (36) we have

$$MSE(s_y^2) - MSE(t_y d)_{11} = -s_y^4(\lambda_{04} - 1)\frac{1}{\alpha^2}[\frac{1}{n}(1 - 2\alpha c) + \frac{1}{n_1}]$$

which is positive if

$$[\frac{1}{n}(1 - 2\alpha c) + \frac{1}{n_1}] \leq 0 \tag{5.7}$$

i.e. if $\alpha > \frac{\delta}{2c}$,
where $\delta = \frac{(n + n_1)}{n_1}$.
From (29) and (31) we have

$$MSE(t_{ld})_{11} - MSE(t_{yd})_{11} = S_y^4(\lambda_{04} - 1)[\frac{1}{n}(1 - 2c) + \frac{1}{n_1} - \frac{(1 - 2\alpha c)}{n\alpha^2} - \frac{1}{\alpha^2 n_1}]$$

which is positive if

$$[(\frac{1}{n} + \frac{1}{n_1})(1 - \frac{1}{\alpha^2}) - \frac{2}{n}(1 - \frac{1}{\alpha})c] > 0$$

i.e. if

either $1 < \alpha < \frac{\delta}{2c-\delta}$ or $\frac{\delta}{(2c-\delta)} < \alpha < 1$
or equivalently,

$$(5.8) \qquad min.[1, \frac{\delta}{(2c-\delta)}] < \alpha < max.[1, \frac{\delta}{2c-\delta}].$$

Also the difference

$$(5.9) \qquad [MSE(t_{ld})_{11} - MSE(t_{yd})_{11}] \qquad ispositiveif \qquad \alpha > 1, c < \frac{\delta}{2}$$

From (30) and (31) we have

$$[MSE(t_{sd})_{11} - MSE(t_{yd})_{11}] = S_y^4(\lambda_{04}-1)[\frac{1}{n}[\frac{1-4c}{4} - \frac{1-2\alpha c}{\alpha^2}] + (\frac{1}{4} - \frac{1}{\alpha^2})\frac{1}{n_1}]$$

$$= S_y^4(\lambda_{04}-1)(1-\frac{2}{\alpha})[\frac{\delta}{4}(1+\frac{2}{\alpha} - c]$$

which is positive if
either $2 < \alpha < \frac{2\delta}{4c-\delta}$ or $\frac{2\delta}{(4c-\delta)} < \alpha < 2$
or equivalently,

$$(5.10) \qquad min.[2, \frac{2\delta}{(4c-\delta)}] < \alpha < max. - [2, \frac{2\delta}{4c-\delta}].$$

Now established the following theorem.
Theorem - 5.2 The proposed estimator t_{yd} under case II is more efficient than :
(i) the usual unbiased estimator s_y^2 if

$$\alpha > \frac{\delta}{2c}$$

(ii)the Isaki's (1983) ratio type double (two phase) sampling estimator t_{sd} if
either $[min.1, \frac{\delta}{(2c-\delta)}] < \alpha < max.[1, \frac{\delta}{2c-\delta}]$.
(iii) the Singh et al.'s (2011) estimator t_{sd} if
either $[2, \frac{2\delta}{4c-\delta}] < \alpha < max.[2, \frac{2\delta}{4c-\delta}]$

6. Comparison with single phase sampling

In this section following Singh and Ruiz Espejo (2007) the comparisons between double and Single-phase sampling have been made for fixed cost. We shall consider the cases separately.
Case - I - In this case we consider the following cost function:

$$(6.1) \qquad c^* = nc_1 + n_1c_2$$

where c^* equals the total cost of the survey and (c_1, c_2) are the costs per unit of collecting information on the study variate y and the auxiliary variate x respectively.
In this case, we express the minimum MSE of t_{yd}(or the MSE of $\hat{t}_{yd(0)}$) as

$$(6.2) \qquad M_y = \frac{M_{y1}}{n} + \frac{M_{y2}}{n_1}$$

(6.3) $$M_{y1} = [(\lambda_{40}-1) - \frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)} = (\lambda_{40}-1)(1-\rho^{*2})S_y^4$$

(6.4) $$M_{y2} = (\frac{(\lambda_{40}-1)^2}{(\lambda_{04}-1)} = (\lambda_{40}-1)\rho^{*2})S_y^4$$

where $\rho^* = \dfrac{(\lambda_{22}-1)}{\sqrt{(\lambda_{22}-1)(\lambda_{04}-1)}}$

The optimum values of n and n_1 for fixed cost c^* , which minimizes the mean squared error M_y is given by

(6.5) $$n_{yopt} = \frac{C^*\sqrt{\frac{M_{y1}}{c_1}}}{\sqrt{M_{y1}}c_1 + \sqrt{M_{y2}c_2}} \qquad n_{y1opt} = \frac{C^*\sqrt{\frac{M_{y2}}{c_2}}}{\sqrt{M_{y1}}c_1 + \sqrt{M_{y2}c_2}}$$

The mean squared error of $\hat{y}_{yd(0)}$corresponding to optimal double sampling estimator is

$$MSE_{opt}(t_{yd})_1 = (\frac{1}{c^*})(\sqrt{c_1 M_{y1}} + \sqrt{c_2 M_{y2}})^2$$

(6.6) $$(\frac{S_y^4}{c^*})(\lambda_{40}-1)(\sqrt{c_1(1-\rho^{*2}+\rho^*\sqrt{c_2})^2}$$

Case - II In case II, we assume that x is measured on y on $n^*{=}n+n_1$ units and$y$ units. Motivated by Srivastava (1970) we shall consider a simple cost function:

(6.7) $$c^* = c_1 n + c_2^* n^*$$

where c_1 andc_2^* denote costs per unit of observing the study variate y and the auxiliary variate$x$ values respectively. The expression of mean squared error of$\hat{t}_{yd(0)}$ (under case II) can now be written as

(6.8) $$M_y^* = \frac{M_{y1}}{n} + \frac{M_{y2}}{n^*},$$

where $n^* = n + n_1$

To obtain the optimum allocation of sample between phases for a fixed cost c^*, we minimize equation (65) with the condition (64). It is easily obtained that this minimum is attained for

(6.9) $$\frac{n}{n^*} = (\frac{M_{y1}c_2^*}{M_{y2}c_1})^{1/2} = \frac{c_2^*(1-\rho^{*2})}{c_1\rho^{*2}}^{1/2}$$

Thus the minimum MSE corresponding to these optimum values of n and n_1 are given by

(6.10) $$MSE_{opt}(\hat{t}_{yd(0)})_{11} = [\frac{S_y^4(\lambda_{40}-1)}{c^*}][\sqrt{(1-\rho^{*2})c_1} + \rho^*\sqrt{c_2^*}]^2$$

Had all the resources been diverted towards the study variate y only, then we would have optimum sample size as given below

(6.11) $$n^{**} = \frac{c^*}{c_1}$$

Thus the variance of the usual unbiased estimator s_y^2 for a given fixed cost $c$in case of large population is given by

(6.12) $$MSE_{opt}(s_y^2) = (\frac{c_1}{c^*})s_y^4(\lambda_{40}-1)$$

Case - I : From (63) and (69), the suggested double sampling strategy would be profitable if

$$MSE_{opt}(\hat{t}_{yd(0)}) < MSE_{opt}(S_y^2)$$

i.e. if

$$\frac{c_2}{c_1} < \frac{(1-\sqrt{1-\rho^{*2}})^2}{\rho^{*2}}$$

Thus we established the following theorem.

Theorem 6.1 The suggested double sampling strategy $\hat{t}_{yd(0)}$ would be more efficient than the strategy s_y^2 as long as

$$\frac{c_2}{c_1} < \frac{(1-\sqrt{1-\rho^{*2}})^2}{\rho^{*2}}$$

Case-II From (67) and (69) it is observed that the double sampling estimator $\hat{t}_{yd(0)}$ is better than the sample mean square s_y^2 for the same fixed cost, if

$$MSE(\hat{t}_{yd(0)})_{11} < MSE_{opt}(s_y^2)$$

i.e. if

$$\rho^{*2} > \frac{4c_1c_2^*}{(c_1+c_2^*)^2}$$

7. Empirical Study

The appropriateness of the proposed estimator has been examined with the help of the four data sets, given in $Table 1$ earlier considered by Subramani and Kumarapandiyan (2012).

We have computed the percent relative efficiencies of the estimators $s_y^2, t_{ld}, t_{sd}$and $\hat{t}_{yd(0)}$ with respect to the usual unbiased estimator s_y^2 by using the following formulae:

$$(i)PRE(t_{ld}, s_y^2)_1 = \frac{(\frac{1}{n})((\lambda_{40})-1)}{[(\frac{1}{n})((\lambda_{40})-1)+(\frac{1}{n}-\frac{1}{n_1})(\lambda_{04}-1)(1-2c)]}\times 100$$

$$(ii)PRE(t_{sd}, s_y^2)_1 = \frac{(\frac{1}{n})((\lambda_{40})-1)}{[(\frac{1}{n})((\lambda_{40})-1)+(\frac{1}{n}-\frac{1}{n_1})(\frac{1}{4})(\lambda_{04}-1)(1-4c)]}\times 100$$

$$(iii)PRE(\hat{t}_{yd(0)}, s_y^2)_1 = \frac{((\lambda_{40})-1)}{[((\lambda_{40})-1)-\frac{n_1-n}{n_1}c^2(\lambda_{04}-1)]}\times 100$$

$$(iv)PRE(t_{1d}, s_y^2)_{11} = \frac{(\frac{1}{n})((\lambda_{40})-1)}{[(\frac{1}{n})[(\lambda_{40}-1)+(\lambda_{04}-1)(1-2c)+(\frac{1}{n_1})(\lambda_{04}-1)]}\times 100$$

$$(v)PRE(t_{sd}, s_y^2)_{11} = \frac{(\frac{1}{n})((\lambda_{40})-1)}{[(\frac{1}{n})[(\lambda_{40}-1)+\frac{(\lambda_{04}-1)}{4}(1-4c)+(\frac{1}{n_1})\frac{(\lambda_{04}-1)}{4}]}\times 100$$

$$(vi) PRE(\hat{t}^*_{yd(0)}, s_y^2)_{11} = \frac{((\lambda_{40}) - 1)}{[((\lambda_{40}) - 1) + \frac{n_1}{n + n_1} c^2(\lambda_{04} - 1)]} \times 100$$

Findings are shown in Table 2.
It is observed from Table 2 that the performance of the proposed estimator $\hat{t}_{yd(0)}(\hat{t}^*_{yd(0)})$ is more efficient than the estimators s_y^2 ,t_{ld} and t_{sd} . The percent relative efficiency of the proposed estimator $\hat{t}_{yd(0)}$ (under case I) is larger than the proposed estimator $(\hat{t}^*_{yd(0)})$.
Table 3, exhibits the range of α in which the proposed class of estimators $\hat{t}_{yd(0)}$ is more efficient than the usual unbiased estimator s_y^2 , Isaki (1983) ratio type estimator t_{id} in double sampling and the estimator t_{sd} which is double sampling version of Singh et al.'s (2011) exponential type estimator.

8. Conclusion

We have suggested an improved exponential ratio estimator for estimating the population variance in two phase sampling. It has been shown theoretically and numerically that the proposed estimator is better than the existing estimators in literature, the usual sample variance, traditional ratio estimator due to Isaki (1983), Yadav and Kadilar (2013) and Singh et al. (2011) exponential ratio estimator in the sense of having lesser mean square error. We have also given the range α of along with its optimum value for the proposed estimator to be more efficient than other competitors. Hence, the proposed estimator is recommended for its practical use for estimating the population variance when the auxiliary information is available. For the sake of completeness we have also discussed the cost aspect.

Acknowledgements
The authors are thankful to the Editor – in- Chief - Professor Cem- Kadilar, and to the anonymous learned referees for their valuable suggestions regarding improvement of the paper.

References

[1] Hidiroglou M.A and Sarandal C.E. (1988): Use of auxiliary information for two phase sampling . Survey Meth., 873-878.
[2] Isaki C. (1983): Variance estimation using auxiliary information. Jour. Amer. Statist. Assoc., 78,117-123.
[3] Shrivastav S.K. (1970): A two phase sampling estimator in sample surveys. Aust. Jour. Statist. 12 (1),23-27.
[4] Singh H.P and Ruiz- Espejo, M. (2007). Double sampling ratio product estimator of a finite population mean in sampling surveys. Jour. Appl. Statist. 34 (1), 71-85.
[5] Singh R., Chauhan P., Swan N. and Smarandache F. (2011): Improved exponential estimator for population variance using two auxiliary variables. Ital. Jour. Pur. Appl. Math. 28,101-108.
[6] Subramani J. and Kumarapandiyan G. (2012): Variance estimation using median of the auxiliary variable. Inter. Jour. Prob. Statist. 1(3), 36-40.
[7] Unnikrishan N.K. and Kunte S. (1995). Optimality of an analogue of Basu's estimator under a double sampling design. Sankhya, 57,103-111.
[8] Upadhyaya L., Singh H.P., Chatterjee S. and Yadav R. (2011): Improved ratio and product exponential type estimators. Jour. Statist. Theo. Pract. 5(2), 285-302.
[9] Yadav S.H. and Kadilar C. (2013): Improved exponential type ratio estimator of population variance. Rev. Colom. Estad., 36, 145-152.

Table 1. Parameters of the population

Parameters	Population 1	Population 2	Population 3	Population 4
N	103	103	80	49
$\bar{Y}$	626.2123	62.6212	51.8264	116.1633
$\bar{X}$	557.1909	556.5541	11.2646	98.6765
ρ	0.9936	0.7298	0.9413	0.6904
s_y	913.5488	91.3549	18.3569	98.8286
c_y	1.4588	1.4588	0.3542	0.8508
s_x	818.1117	610.1643	8.4563	102.9709
c_x	1.4683	1.0963	0.7507	1.0435
λ_{04}	37.3216	17.8738	2.8664	5.9878
λ_{40}	37.1279	37.1279	2.2667	4.9245
λ_{22}	37.2055	17.2220	2.2209	4.6977
c	0.9969	0.9635	0.7748	0.7846

Table 2. Percent relative efficiencies (PREs) of different estimators of population variance S_y^2 with respect to the unbiased estimator s_y^2 .

Estimator				PRE(., s_y^2)				
				Population				
	I	I	II	II	III	III	IV	IV
	Case I n_1=60 n=40	Case II n_1=60 n=40	Case I n_1=60 n=40	Case II n_1=60 n=40	Case I n_1=30 n=20	Case II n_1=30 n=20	Case I n_1=25 n=20	Case II n_1=25 n=20
s_y^2	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
t_{ld}	149.90	99.38	116.92	96.71	130.34	63.62	112.90	69.78
t_{sd}	133.41	199.80	112.54	127.65	128.62	153.73	112.00	140.10
$\hat{t}_{yd(0)}$	149.91	-	116.92	-	134.11	-	114.13	-
$\hat{t}^*_{yd(0)}$	-	249.73	-	135.22	-	184.23	-	152.45

Table 3. Range of α for t_{yd} to be more efficient than different estimators of the population variance S_y^2 .

Estimator				Population				
	I	I	II	II	III	III	IV	IV
	Case I n_1=60 n=40	Case II n_1=60 n=40	Case I n_1=60 n=40	Case II n_1=60 n=40	Case I n_1=30 n=20	Case II n_1=30 n=20	Case I n_1=25 n=20	Case II n_1=25 n=20
s_y^2	$\alpha > 0.50$	$\alpha > 0.84$	$\alpha > 0.52$	$\alpha > 0.87$	$\alpha > 0.65$	$\alpha > 1.08$	$\alpha > 0.64$	$\alpha > 1.15$
t_{ld}	$\alpha\varepsilon(1.00, 1.01)$	$\alpha\varepsilon(1.00, 1.68)$	$\alpha\varepsilon(1.00, 1.08)$	$\alpha\varepsilon(1.00, 1.79)$	$\alpha\varepsilon(1.00, 1.83)$	$\alpha\varepsilon(1.00, 3.03)$	$\alpha\varepsilon(1.00, 1.76)$	$\alpha\varepsilon(1.00, 3.17)$
t_{sd}	$\alpha\varepsilon(0.67, 2.01)$	$\alpha\varepsilon(1.44, 2.00)$	$\alpha\varepsilon(0.70, 2.00)$	$\alpha\varepsilon(1.52, 2.00)$	$\alpha\varepsilon(0.95, 2.00)$	$\alpha\varepsilon(2.00, 2.32)$	$\alpha\varepsilon(0.94, 2.00)$	$\alpha\varepsilon(2.00, 2.68)$
common range of α for t_{yd} to be more efficient s_y, t_{ld}, t_{sd} $\hat{t}_{yd(0)}$	$\alpha\varepsilon(1.00, 1.01)$	$\alpha\varepsilon(1.43, 1.68)$	$\alpha\varepsilon(1.00, 1.08)$	$\alpha\varepsilon(1.52, 1.79)$	$\alpha\varepsilon(1.00, 1.83)$	$\alpha\varepsilon(2.00, 3.03)$	$\alpha\varepsilon(1.00, 1.76)$	$\alpha\varepsilon(2.00, 2.68)$
Optimum value of α	1.003	1.672	1.038	1.73	1.291	2.152	1.275	2.295

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1271 – 1281

An optimization model for designing acceptance sampling plan based on cumulative count of conforming run length using minimum angle method

Ahmad Ahmadi Yazdi* and Mohammad Saber Fallah Nezhad†

Abstract

The purpose of this article is to present an acceptance sampling plan based on cumulative count of conforming using minimum angle method. In this plan, if the number of inspected items until r_{th} defective items is greater than an upper control threshold then lot is accepted and if it is less than a lower control threshold then the lot is rejected and if it is between control thresholds, process of inspecting the items continues. To design this model, we considered some important concepts like number of inspected items until r_{th} nonconforming item in inspection, first and second type of error, average number inspected (ANI), AQL and LQL. Also derivative of (ANI) function in point AQL is used for optimization. The objective function of this model was constructed based on minimum angle method. Also a comparison study is carried out to evaluate the performance of proposed methodology in 50 different data sets.

Keywords: Quality control, Markovian model, Conforming run length, Acceptance sampling plan, Minimum angle method.

2000 AMS Classification: 08A99

Received 26/05/2014 : Accepted 01/10/2014 Doi : 10.15672/HJMS.2014327483

*Ph.D Student of Industrial Engineering, Isfahan University of Technology, Email: Ahmad_ahmadi_yazdi@yahoo.com Corresponding Author.
†Associate Professor of Industrial Engineering, Yazd University, Email:Fallahnezhad@yazd.ac.ir

1. Introduction

Acceptance sampling plan is a statistical quality control technique. In such plans, a sample is taken from a lot and the lot will either be rejected or accepted or inspection continues upon the results of the sample taken. The purpose of acceptance sampling plan is to determine the quality level of an incoming lot or the end production and also ensure that the quality level of the lot satisfies the predetermined requirement. Many types of acceptance sampling plans have been proposed. One approach to design acceptance sampling plans is minimum angle method (Fallahnezhad [7]). In this research, a new acceptance sampling plan is developed based on minimum angle method using cumulative conforming run length. This idea is based on the concept of cumulative conforming control charts. Design of cumulative conforming control charts is a favorable issue for many authors. Cumulative conforming control charts (CCC-charts) usually are constructed by using geometric and negative binomial variables (Chan et al. [5]). Calvin [7] presented a control chart by using run-length of successive conforming items. Goh [12] presented a method to control the production with low-nonconformity by (CCC-charts). Lai [15] proposed a discrete time renewal event process when a success is preceded by a failure and introduced modified CCC-chart. Also he calculated ANI (average number inspected) and other indicators for this modified chart. Some authors also refer CCC-charts as CRL-type (conforming run length) control charts or SCRL (sum of CRL) chart (Wu et al. [17]). A CCC-chart which is based on number of inspected items until detection of r_{th} defective item is called CCC_r-charts. Calvin [4], Goh [12], Xie and Goh [18] and many other authors have applied CCC_1-charts. Chan et al. [5] denoted that CCC_r-chart is more reliable than CCC_1-chart but it takes more time and inspection items than CCC_1-charts for detecting change in fraction of non-conforming. He also presented a two-stage decision procedure for monitoring processes with low fraction of nonconforming and introduced $CCC_1 + \gamma$ chart for this purpose and presented an economical model for minimizing total cost of the system. Di Bucchianico et al. [6] presented a case study for monitoring the packing process in coffee production based on choosing optimal value of r when using CCC_r-charts. Aslo Bourke [2] has applied the concept of conforming run length in designing the acceptance sampling plans. In this research, we used Markov model in designing the sampling plan based on the concept of conforming run length. An absorbing Markov model is developed for this sampling system (Bowling et al. [3]). In this subject, Fallahnezhad et al. [9] developed a Markov model based on sum of run-lengths of successive conforming items. Fallahnezhad and Niaki [11] proposed a sampling plan using Markov model based on control threshold policy. They considered the run-lengths of successive conforming items as a measure for process performance. Fallahnezhad et al.[10] proposed an economical model for sampling based on decision tree. Fallahnezhad and Hosseininasab [8] proposed a one stage economical acceptance sampling model based on the control threshold policy. In our sampling plan we used the concept of minimum angle method that its purpose is to reach ideal OC curve in order to decrease the risk of sampling plan. Bush et al. [1] analyzed the sampling systems by comparing operation characteristic (OC) curve against the ideal OC curve. His study was a motivation for constructing the concept of minimum angle method. Soundararajan and Christina [16] proposed a method for the selection of optimal single stage sampling plans based on the minimum angle method. They were first authors who used minimum angle method for designing a sampling plan. But little studies have been done on designing a sampling plan based on minimum angle method. Soundararajan and Christina [16] used the tangent of angle between the lines that joins $[AQL, P_a(AQL)]$ to $[LQL, P_a(LQL)]$ in order to reach ideal OC curve. $P_a(AQL)$ is the probability of acceptance when the percentage of the defective items of the lot is AQL. This angle (θ) is denoted in Figure 1 It is obvious that minimizing (θ) is favorable because the OC curve approaches to ideal OC curve. $\tan(\theta)$ is obtained as follows,

$$\tan(\theta) = \frac{LQL - AQL}{P_a(AQL) - P_a(LQL)}$$

Since (θ) should be minimized, thus the value of $\tan(\theta)$ should be minimized also since $LQL - AQL$ is constant thus the value of $[P_a(AQL) - P_a(LQL)]$ should be maximized. In this paper, a nonlinear model for acceptance sampling plans by developing a Markov

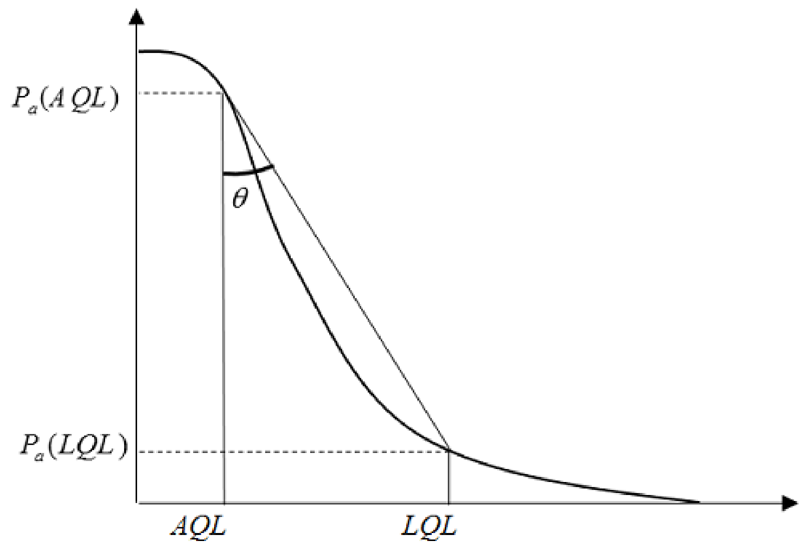


Figure 1. Tangent angle minimizing using AQL, LQL [16]

model is presented. To design this model, we considered some important concepts like number of conforming items until r_{th} nonconforming item in inspection, first and second type of error, average number inspected (ANI), AQL and LQL. Also derivative of (ANI) function in point AQL is used for optimization. The objective function of this model was constructed based on minimum angle method. The model has been solved for 4 scenarios in the cases $r = 1$ or $r = 2$ or $r = 3$ by using visual basic 6 in Microsoft excel 2013. Then the optimal solutions have been collected and analyzed in order to determine which one of these sampling plans is more desirable in practical environment. The rest of the paper is organized as follows. We present the model in Section 2. A case study is solved in Section 3. Section 4 provides a sensitivity analysis for illustrating the effect of different parameters on the objective function. In section 5, a comparison study is carried out in 50 different data sets.

2. Model Development

The purpose of this model is to develop an optimization model for determining the optimum value of thresholds of an acceptance sampling design. This acceptance sampling design is based on run length of conforming items. Assume that in an acceptance sampling plan, Y is defined as the number of inspected items until detecting r_{tk} nonconforming item. It is obvious that Y follows negative binomial distribution.

The decision making method is as follows,

If $Y \geq U$ then the lot is accepted and if $Y \leq L$ then the lot is rejected. If $U > Y > L$ then inspection of the items continues where U is an upper control threshold and L is a lower control threshold. Thus states of the decision making method are as follows,
State 1: $U > Y > L$, continue inspecting.
State 2: $Y \geq U$, the lot is accepted.
State 3: $Y \leq L$, the lot is rejected.
If p_{kl} denotes the probability of going from state k to state l then transition probabilities are obtained as follows, [7]

$$p_{11} = P\{U > Y > L\}, p_{12} = P\{Y \geq U\}, p_{13} = P\{Y \leq L\} \tag{2.1}$$

where $P(Y|r,p) = \binom{i-1}{r-1}(1-p)^{i-r}p^r$; for $i = r, r+1, ...$ is the negative binomial distribution and p denotes the proportion of nonconforming items in the lot.
Fallahnezhad [7] proposed a new optimization model for designing sampling plans based on minimum angle method and run length of inspected items with considering minimum angle method and average number of inspection (ANI) in the optimization model. He tried to solve his model by search procedure just for $r = 1$ (r is number of nonconforming items in inspection process). In the proposed model, we try to optimize some important criteria of sampling plans simultaneously. The objective function is constructed using minimum angle method which optimizes the producer risk and consumer risk simultaneously. Also the constraints of average number inspected (ANI) and first derivative of ANI function and risks are included in the model. Then we tried to solve the proposed model by search method for $r = 1, 2, 3$ with considering all mentioned concepts.
The transition probability matrix is as follows (Fallahnezhad [7]),

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} \overset{1}{p_{11}} & \overset{2}{p_{12}} & \overset{3}{p_{13}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

States 2 and 3 are absorbing state and state 1 is transient. The transition probability matrix should be rewritten in the following form in order to calculate long run probabilities of absorption:

$$\begin{bmatrix} A & O \\ R & Q \end{bmatrix} \tag{2.3}$$

where Q is transition probability matrix among non-absorbing states and R is the matrix containing probabilities of going from non-absorbing states to absorbing states and A is an identity matrix and O is matrix of zeros. Thus following matrix is obtained (Fallahnezhad [7]),

$$\begin{array}{c} 2 \\ 3 \\ 1 \end{array} \begin{bmatrix} \overset{2}{1} & \overset{3}{0} & \overset{1}{0} \\ 0 & 1 & 0 \\ p_{12} & p_{13} & p_{11} \end{bmatrix} \tag{2.4}$$

The fundamental matrix M can be determined as follows (Bowling et al [3]):

$$M = m_{11}(p) = (I - Q)^{-1} = \frac{1}{1 - p_{11}} = \frac{1}{1 - P\{U > Y > L\}} \tag{2.5}$$

where I is the identity matrix. The value $m_{11}(p)$ denotes the expected number of visiting the transient state 1 until absorption occurs. The absorption probability matrix, F is calculated as follows (Bowling et al. [3]):

$$F = M \times R = 1 \begin{bmatrix} \overset{2}{f_{12}(p)} & \overset{3}{f_{13}(p)} \end{bmatrix} = 1 \begin{bmatrix} \overset{2}{\frac{p_{12}}{1-p_{12}}} & \overset{3}{\frac{p_{13}}{1-p_{13}}} \end{bmatrix} \tag{2.6}$$

where $f_{12}(p)$ and $f_{13}(p)$ denote the probabilities of accepting and rejecting the lot, respectively.
The objective function of this model is written by using minimum angle method. In this approach, our goal is to maximize the value of $\{P_a(AQL) - P_a(LQL)\}$ where $P_a(LQL)$ and $P_a(AQL)$ are the probabilities of accepting the lot when the proportion of nonconforming items in the lot is respectively LQL and AQL. It is obvious that $1 - P_a(AQL)$ is risk of producer thus maximizing $P_a(AQL)$ is favorable. Also $P_a(LQL)$ is the risk of consumer thus minimizing $P_a(LQL)$ is favorable. Consequently maximizing $\{P_a(AQL) - P_a(LQL)\}$ for a sampling system would be desired. The values of $P_a(LQL)$ and $P_a(AQL)$ are determined as follows,

$$p = AQL \rightarrow P_a(AQL) = f_{12}(AQL) = \frac{P\{U \leq Y\}}{1 - P\{U > Y > L\}} \tag{2.7}$$

$$p = LQL \rightarrow P_a(LQL) = f_{12}(LQL) = \frac{P\{U \leq Y\}}{1 - P\{U > Y > L\}} \tag{2.8}$$

The objective function in minimum angle method is as follows, (Fallahnezhad [7])

$$Z = \underset{L,U}{Max}\,\{P_a(AQL) - P_a(LQL)\} \tag{2.9}$$

An important performance measure of sampling plans is the average number inspected (ANI) . Since sampling and inspecting has cost, therefore designs with minimum ANI are preferred. Therefore we try to consider the ANI in constraint of optimization model so that its value does not get more that a control threshold. These constraints are written for both cases of acceptable and unacceptable lots where the proportion of nonconforming items in lot is equal to AQL and LOL, respectively. This constraint is written based on the value of $m_{11}(p)$. As mentioned, $m_{11}(p)$ is the expected number of times that the transient state 1 is visited until absorption occurs, since in each visit to transient state, the average number of inspections is $\frac{r}{p}$ which is the mean value of negative binomial distribution, consequently the value of ANI is given by $\frac{r}{p}m_{11}(p)$. Now these constraints are obtained for both cases of acceptable lot $(p = AQL)$ and unacceptable lot $(p = LQL)$ respectively,

$$ANI(AQL) \leq W \tag{2.10}$$

$$ANI(LQL) \leq M \tag{2.11}$$

where W and M are upper control limits for these constraints and,

$$ANI(AQL) = \frac{r}{AQL}m_{11}(AQL) \tag{2.12}$$

$$ANI(LQL) = \frac{r}{LQL}m_{11}(LQL) \tag{2.13}$$

It is very important that acceptance sampling plans satisfy the constraints of first and second type errors. These two types of errors are important performance measure of acceptance sampling plans. First type error probability is the probability of rejecting an acceptable lot and Second type error probability is the probability of accepting an unacceptable lot. So we have included these two concepts as the constraints of optimization model.
Thus we added following constraints to the optimization model for both cases of acceptable lot $(p = AQL)$ and unacceptable lot $(p = LQL)$ respectively,

$$P_a(AQL) \geq 1 - \alpha \tag{2.14}$$

(2.15) $P_a(LQL) \leq \beta$

where α is the value of first type error probability and β value of second type error probability. According to the ANI graph, when the percentage of the defectives in lot is equal to the AQL, the ideal is that the derivation of the function at this point be equal to zero, or in other words, reaches its minimum value. We try to consider this concept as a constraint and examine its impact on the optimal solution of the model. The first derivative of ANI function is written as follows (Chen [13]),

$$ANI_p(p) = \frac{\partial}{\partial p} \frac{r}{k(p)} = \frac{-rk_p(p)}{k^2(p)} \tag{2.16}$$

where

$$\begin{aligned} &k(p) = p\{1 - [F(U-1\,|r,p) - F(L\,|r,p)]\} \\ &k_p(p) = 1 - F(U-1\,|r,p) - F(L\,|r,p) + (L-1)f(L-1\,|r,p) - Uf(U\,|r,p) \end{aligned} \tag{2.17}$$

We considered upper and lower limits for derivative of ANI when the percentage of the defective in lot is equal to AQL in order to apply this constraint in the model. Since AQL is an important parameter in decision making about the lot thus this value is selected as reference value in constraint of ANI derivative. It is obvious that lower limit is negative and upper limit is positive. As much as the interval of these limits would be tighter then it will be closer to zero that is more favorable for us. This constraint is obtained as follows,

(2.18) $\lambda_1 \leq ANI_p(AQL) \leq \lambda_2$

where λ_1 and λ_2 are lower and upper limits for the first derivation of ANI function, respectively. Now the optimization problem can be defined as follows,

$$\begin{aligned} &\underset{L,U}{Max}\, Z \\ &s.t. \\ &ANI(AQL) \leq W \\ &ANI(LQL) \leq M \\ &P_a(AQL) \geq 1-\alpha \\ &P_a(LQL) \leq \beta \\ &\lambda_1 \leq ANI_p(AQL) \leq \lambda_2 \end{aligned} \tag{2.19}$$

Optimal values of L, U, r can be determined by solving above nonlinear optimization problem using search procedures or other optimization tools. The parameters like W, M, α, β, $\lambda_1, \lambda_2, AQL, LQL$ are predetermined for solving the model in order to reach the optimal values of L, U, r. The advantage of this sampling system is to consider most important critical factors affecting on performance of sampling methods in an optimization model which optimizes them simultaneously.

3. Case Study

A case study is solved using Visual basic codes in Microsoft excel 2013 in order to demonstrate the application of the proposed methodology in designing acceptance sampling models. The following example is intended to provide illustrations about application of the model in a juice factory. The quality engineer tries to design an acceptance sampling plan for accepting or rejecting an incoming lot received from suppliers. The values of AQL and LQL and other important parameters are specified as required quality standards by both sides (consumer and producer).

This case is solved and the values in intervals $L = [0, 20]$ and $U = [1, 90]$ and $r = [1, 2, 3]$ are searched for optimal solution in each scenario, while L and U are integer. In the other words, we restricted our search space in order to reach optimal value of L and U

Table 1. Optimal solution of case study

r	L	U	Z	$ANI(AQL)$	$ANI(LQL)$	$ANI_p(AQL)$	$P_a(AQL)$	$P_a(LQL)$
2	3	35	0.94	83.45	37.59	52.44	0.98011	0.00574

Table 2. Input Parameters of Different scenarios

Scenarios	M	W	λ_2	λ_1	β	α	LQL	AQL
1	70	80	250	-250	0.2	0.15	0.3	0.06
2	100	130	200	-200	0.2	0.1	0.2	0.04
3	60	70	400	-400	0.1	0.05	0.2	0.05
4	50	105	100	-100	0.1	0.05	0.2	0.05

Table 3. Number of feasible solutions for each scenario

Scenarios	$r=1$	$r=2$	$r=3$
Scenario 1	10	48	11
Scenario 2	1	33	28
Scenario 3	0	6	0
Scenario 4	0	3	6

and r. It is observed that optimal solution lies in the specified intervals in all considered practical cases. Thus first the feasible value of L and U will be determined and the optimal solution which maximizes the objective function is determined among them. It is assumed that $AQL = 0.05$, $LQL = 0.2$, $\lambda_1 = -80$, $\lambda_2 = 80$, $M = 50$, $W = 90$, $\alpha = 0.05$, $\beta = 0.1$. We solved the proposed model with these input parameters. The results show that there are just 3 feasible solutions in the solution space. Table 1 shows the optimal solutions. It is obvious that the result of the proposed model is applicable in any production environment.
In the cases that required sample size is limited then we can easily consider this limitation in the constraints of the model. It is observed that ANI of proposed method is large for $r = 3, 4, ...$ but when small sample size is an important criterion, we may apply $r = 1, 2$ for sampling system. It is obvious that optimal solution of optimization model for $r = 1$ or $r = 2$ with tighter intervals for ANI function would result in smaller values for required number of inspected items.

4. Sensitivity Analysis

In this section, a sensitivity analysis is done for illustrating the effect of different parameters on the results of the model. This model was solved in several scenarios with different assumptions. Table 2 shows the input parameters of different scenarios.
Each scenario is solved in the cases, $r = 1$, $r = 2$ and $r = 3$ and the number of feasible solutions are summarized in Table 3.
As can be seen in Table 2, the number of feasible solutions for each scenario is not the same in cases, $r = 1$, $r = 2$ and $r = 3$. For example, case $r = 1$ will not have any feasible solutions in Scenario 3 and 4. Also case $r = 3$ will not have any feasible solutions in Scenario 3. Table 4 shows the optimal solution of the model for each scenario.

Table 4. Optimal solution for each scenario

Scenarios	r	L	U	Z	$ANI(AQL)$	$ANI(LQL)$	$ANI_p(AQL)$	$p_a(AQL)$	$p_a(LQL)$
Scenario 1	3	4	34	0.98	78.31	28.30	-200.67	0.99014	0.00099
Scenario 2	3	5	58	0.99	131.42	73.79	190.35	0.9946	0.00095
Scenario 3	2	3	29	0.92	69.43	36.60	-286.97	0.96	0.04
Scenario 4	3	7	46	0.97	103.70	46.28	14.74	0.98011	0.00574

Table 5. Input Parameters

Scenarios	M	W	λ_2	λ_1	β	α	LQL	AQL
1	70	80	250	0	0.2	0.15	0.3	0.06
2	100	130	200	0	0.2	0.1	0.2	0.04

Table 6. Optimal Solution

Scenarios	r	L	U	Z	$ANI(AQL)$	$ANI(LQL)$	$ANI_p(AQL)$	$p_a(AQL)$	$p_a(LQL)$
Scenario 1	2	1	28	0.97	68.82	30.80	49.60	0.978592	0.00199
Scenario 2	3	5	57	0.99	129	73.77	85.93	0.9947	0.00115

According to Table 4, the case $r = 3$ will be optimal in most of the scenarios and case $r = 2$ will be optimal in scenario 3. Since we saw that the model could not find any feasible solution in case $r = 3$ for scenarios 3 thus this result was justified. Also the case $r = 1$ has not been optimal in any of the scenarios. So we can say that the case $r = 3$ is suitable for practical real world problems. But since we have not investigated the cases with the values of $r > 3$, this is suggested as future studies but in general, it seems that the value of $r > 4$ need so much more inspections and may not be feasible as can be seen in Table 3, where the number of feasible solution has decreased significantly by changing $r = 2$ to $r = 3$.
The first derivative of ANI function is included in the model to minimize the number of inspected items. It is needed to analyze the effect of lower limit and upper limit for first derivative of ANI function in order to investigate the behavior of optimal solution by changing them. It is obvious that when the first derivative of a convex function at a point is zero then that point is minimum value of a convex function. Thus considering negative and positive bounds for first derivative is logical which results in finding near optimal solution. We used this concept for monitoring the ANI value by calculating its first derivative. Then we defined an interval for the first derivative of $ANI(ANI_p(AQL))$. We defined two scenarios for $\lambda_1 = 0$ and $\lambda_2 > 0$ inorder to check the effects of λ_1 and λ_2. Table 5 shows the input parameters and Table 6 shows the optimal solutions.

The results shows that when we consider $\lambda_1 = 0$ and $\lambda_2 > 0$, then the variations of objective function is negligible. In this state, a better optimal solution is obtained according to the values of $ANI(AQL)$, $ANI(LQL)$ and $ANI_p(AQL)$.

5. Comparison Study

After constructing proposed method optimization model, it is very beneficial to compare this new model with traditional single stage sampling method. For illustrating the effect of different data sets on the results of the proposed model and discussion about the application of the model in the different practical environments, we carried out a simulation study with 50 different random data sets. Then we compared the proposed model withtraditional single stage sampling methodassuming the same constraints. It is tried to search all feasible points of solution space in order to obtain general optimal values for L, U, r. The optimization model for traditional single stage sampling method is as follows;

$$\begin{aligned} & Z' = \underset{n,c}{Max}\{P_a(AQL) - P_a(LQL)\} \\ & s.t. \\ & P_a(AQL) \geq 1 - \alpha \\ & P_a(LQL) \leq \beta \end{aligned} \tag{5.1}$$

where $P_a(p)$ denotes the probability of accepting the lot which is obtained by cumulative function of binomial distribution as follows;

$$P_a(p) = \sum_{x=0}^{c} \binom{n}{x} p^x (1-p)^{n-x} \tag{5.2}$$

It is obvious that the constraints regarding first derivation of ANI function, $ANI(AQL)$, and have not been considered in the optimization model because ANI in the traditional single stage sampling method is fixed $(ANI = n)$.

50 different scenarios of parameters are randomly generated by uniform distribution. The results are summarized in Table 7. According to Table 7, proposed method has better value of objective function in 28% of cases but proposed model is worse than traditional method in 14% of cases and for the rest of the cases, the objective function of these two methods are equal.

The results shows that since proposed model has more constraints than the traditional single stage sampling method but it has better value for objective function in 28% of cases and both methods have equal objective function in 58% of cases. Also in most of cases, $ANI(LQL)$ in the proposed model is less value than the sample size, n in the traditional method but $ANI(AQL)$ of proposed model is often more than sample size, n in the traditional method. Thus we can assume tighter intervals for constraint regarding $ANI(AQL)$ in order to decrease the average number of inspected items. In general, the results show the advantages of proposed methodology over existing methods and this model can be efficiently applied in practical environment.

6. Conclusion

In this paper, we proposed a general nonlinear model for acceptance sampling based on cumulative count of conforming using minimum angle method. Number of inspected items until r_{th} defective items was selected as criteria for decision making. We presented our model using Markov model and derivative of ANI (average number inspected) in AQL point to ensure that ANI chart behavior is in desired level. It's ideal that the derivative of ANI in AQL point to be equal zero in order to ensure that ANI is minimized. This approach is suitable when our plan for accepting or rejecting a lot is based on number of inspected items until r_{th} nonconforming item. Also it is tried that constraint of first and second type of errors to be included in the model simultaneously. We concluded that the case $r = 3$ which denotes the method of sampling until the third defective item is suitable for practical real world problems. But since we have not investigated the cases with the values of $r > 3$, thus this is suggested as future studies but in general, it seems that the value of $r > 4$ needs so much more inspections and it may not be feasible. As can be seen in Table 3, the number of feasible solution has decreased significantly by changing $r = 2$ to $r = 3$. For analyzing the behavior of proposed model in different data sets, we solved the model for 50 different random scenarios and also we compared

Table 7. Proposed method VS. Traditional single sampling

Scenarios	Input parameters								Proposed Model						Traditional Single Sampling Method		
	AQL	LQL	W	M	λ_1	λ_2	$1-\alpha$	β	L	U	r	ANI(AQL)	ANI(LQL)	Z	n	c	Z'
1	0.04	0.27	241	99	-400	190	0.7	0.2	2	53	3	121	87	0.99	88	10	0.99
2	0.04	0.14	289	66	-311	218	0.72	0.11	12	64	3	141	65	0.95	90	6	0.91
3	0.04	0.31	237	130	-217	156	0.89	0.14	1	52	3	118	110	0.99	79	10	0.99
4	0.03	0.23	181	63	-357	443	0.82	0.25	1	60	2	145	61	0.99	90	8	0.99
5	0.03	0.31	128	68	-372	219	0.87	0.21	0	33	1	84	10	0.99	88	10	0.99
6	0.04	0.12	187	57	-246	464	0.79	0.14	16	61	3	130	57	0.92	90	6	0.85
7	0.03	0.16	216	86	-454	453	0.72	0.16	8	88	3	196	74	0.90	90	6	0.98
8	0.03	0.23	274	120	-319	179	0.80	0.21	3	67	3	151	92	0.99	90	9	0.99
9	0.02	0.15	135	78	-72	223	0.83	0.14	0	49	1	126	41	0.99	90	6	0.98
10	0.04	0.28	278	120	-90	352	0.73	0.11	2	62	3	140	76	0.93	88	10	0.99
11	0.05	0.16	135	113	0	266	0.76	0.17	7	52	3	117	87	0.99	90	8	0.94
12	0.03	0.27	286	112	-231	260	0.79	0.16	2	72	3	162	90	0.96	90	10	0.99
13	0.04	0.25	152	127	-324	161	0.74	0.22	2	59	3	133	109	0.99	90	10	0.99
14	0.03	0.11	284	57	-425	470	0.92	0.23	9	69	2	158	54	0.99	90	5	0.90
15	0.04	0.28	249	57	-46	331	0.88	0.23	3	56	3	127	50	0.93	86	10	0.99
16	0.03	0.21	122	86	-459	214	0.78	0.10	1	49	2	119	85	0.99	90	8	0.99
17	0.04	0.24	255	79	-494	154	0.90	0.12	1	46	3	105	71	0.99	71	10	0.99
18	0.04	0.27	128	124	0	438	0.70	0.13	2	51	2	116	85	0.99	84	10	0.99
19	0.05	0.21	276	67	-440	408	0.89	0.11	5	53	3	121	71	0.99	90	9	0.98
20	0.03	0.18	205	103	-236	182	0.85	0.22	5	73	3	164	85	0.99	90	7	0.98
21	0.04	0.2	170	61	-277	478	0.78	0.18	6	61	3	137	64	0.99	90	8	0.99
22	0.04	0.32	203	143	-355	410	0.88	0.12	1	56	3	127	97	0.99	79	10	0.99
23	0.04	0.33	194	104	-307	151	0.72	0.14	1	48	3	109	55	0.99	73	10	0.99
24	0.04	0.18	237	133	-268	221	0.71	0.12	5	61	3	138	96	0.99	90	8	0.97
25	0.04	0.15	261	57	-72	419	0.92	0.16	11	61	3	136	81	0.99	90	7	0.93
26	0.05	0.29	178	141	-436	265	0.82	0.23	2	45	3	103	105	0.95	78	10	0.99
27	0.02	0.1	169	103	-267	357	0.87	0.11	0	53	1	137	93	0.99	90	4	0.91
28	0.03	0.23	211	84	0	447	0.78	0.21	4	68	3	153	60	0.91	90	9	0.99
29	0.03	0.26	115	74	-358	354	0.85	0.23	0	33	3	84	14	0.99	90	10	0.99
30	0.05	0.28	191	66	-93	166	0.83	0.13	3	49	3	111	50	0.92	82	10	0.99
31	0.03	0.14	128	83	-256	177	0.88	0.15	0	39	3	99	49	0.99	90	6	0.94
32	0.02	0.31	195	66	-108	493	0.81	0.19	1	72	1	174	29	0.89	90	10	0.99
33	0.04	0.28	282	104	-457	161	0.91	0.22	2	48	1	109	74	0.99	81	10	0.99
34	0.03	0.34	239	94	-156	270	0.80	0.12	1	83	2	186	79	0.99	83	10	0.99
35	0.02	0.14	176	78	-177	335	0.79	0.22	0	52	3	135	52	0.99	90	5	0.97
36	0.03	0.26	233	135	-456	231	0.76	0.16	2	63	3	142	95	0.94	90	10	0.99
37	0.05	0.19	205	78	-257	369	0.91	0.22	6	49	3	111	66	0.99	90	9	0.96
38	0.03	0.15	130	110	-192	402	0.85	0.13	3	53	3	127	82	0.98	90	6	0.95
39	0.03	0.25	112	68	-368	228	0.74	017	0	29	1	74	15	0.98	90	10	0.99
40	0.03	0.31	122	92	-161	250	0.87	0.21	0	34	1	86	10	0.92	85	10	0.99
41	0.03	0.20	157	112	-289	369	0.79	0.15	4	67	1	151	107	0.91	90	8	0.99
42	0.05	0.22	158	60	-234	332	0.73	0.22	5	50	2	114	52	0.99	90	10	0.99
43	0.04	0.15	181	84	-113	334	0.79	0.14	9	60	3	134	257	0.99	90	7	0.94
44	0.02	0.30	221	91	-470	349	0.74	0.2	0	72	2	174	74	0.96	90	10	0.99
45	0.03	0.34	249	147	-243	124	0.75	0.2	1	64	3	144	72	0.99	76	10	0.99
46	0.05	0.26	115	147	-156	256	0.90	0.18	2	47	3	108	97	0.99	82	10	0.99
47	0.04	0.12	164	76	-199	441	0.91	0.24	14	71	3	156	72	0.94	90	6	0.88
48	0.04	0.25	205	145	-52	262	0.76	0.22	2	53	3	121	11	0.99	89	10	0.99
49	0.04	0.22	124	103	-297	497	0.72	0.1	1	45	2	110	72	0.99	90	9	0.99
50	0.04	0.15	226	92	-131	464	0.82	0.2	8	62	3	140	81	0.97	90	7	0.95

the results with traditional single sampling method. The results show that the proposed model has better performance.

References

[1] Bush, N. and Leonard, E. J. and Marchan, JR, M. Q. M. *A method of discrimination for single and double sampling OC curves utilizing the tangent at the point of inflection*, Chemical Corps Engineering Agency, USA, 1953.

[2] Bourke PD. *a shift in fraction nonconforming using run-length control charts with* 100% *inspection*, of Quality Technology **23**, 225-238, 1991.

[3] Bowling, SR. and Khasawneh, MT. and Kaewkuekool, S. Cho. BR. *A Markovian approach to determining optimum process target levels for a multi-stage serial production system*, European Journal of Operational Research **159**, 636-650, 2004.

[4] Calvin, TW. *Quality control techniques for zero-defects*, IEEE Transactions on Components, Hybrids, and Manufacturing Technology **6**, 323-328, 1983.

[5] Chan, L.Y., Lin, D.K.J., Xie, M. and Goh, T.N. *Cumulative probability control charts for geometric and exponential process characteristics*, International Journal of Production Research **40** (1), 133-150, 2002.

[6] Di Bucchianico, A., Mooiweer, G.D. and Moonen, E.J.G. *Monitoring Infrequent Failures of High-volume Production Processes*, Quality and Reliability Engineering International **21** (5), 521-528, 2005.

[7] Fallahnezhad M.S. *A new Markov Chain Based Acceptance Sampling Policy via the Minimum Angle Method*, Journal of Operations Research **3** (1), 104-111, 2012.

[8] Fallahnezhad M.S. and Hosseini Nasab H. *Designing a Single Stage Acceptance Sampling Plan based on the control Threshold policy*, International Journal of Industrial Engineering and Production Research **22** (3), 143-150, 2011.

[9] Fallahnezhad M.S., Niaki, S.T.A. and Abooie, M.H. *A New Acceptance Sampling Plan Based on Cumulative Sums of Conforming Run-Lengths*, Journal of Industrial and Systems Engineering **4** (4), 256-264, 2011.

[10] Fallahnezhad M.S., Niaki, S.T.A. and Vahdat Zad, M.A. *A New Acceptance Sampling Design Using Bayesian Modeling and Backwards Induction*, International Journal of Engineering, Islamic Republic of Iran **25** (1), 45-54, 2012.

[11] Fallahnezhad M.S. and Niaki, S.T.A. *A New Acceptance Sampling Policy Based on Number of Successive Conforming Items*, Communications in Statistics-Theory and Methods **42** (8), 1542-1552, 2013.

[12] Goh T.N. *A charting technique for control of low-nonconformity production*, International Journal of Quality and Reliability Management **5**, 53-62, 1987.

[13] Jung-Tai, C. *Design of cumulative count of conforming charts for high yield processes based on average number of items inspected*, International Journal of Quality and Reliability Management **30** (9), 942-957, 2013.

[14] Chan, L.Y., Lai, C.D., Xie, M. and Goh, T.N. *A two-stage decision procedure for monitoring processes with low fraction nonconforming*, European Journal of Operational Research **150**, 420-436, 2003.

[15] Lai, C.D. *The Recurrent Event Process of a Success Preceded by a Failure and its Application*, Journal of Applied Mathematics and Decision Sciences **1** (2), 101-117, 1997.

[16] Soundararajan V. and Christina A.L. *Selection of single sampling variables plans based on the minimum angle*, Journal of Applied Statistics **24** (2), 207-218, 1997.

[17] Wu, Z., Zhang, X. and Yeo, S.H. *Design of the sum-of-conforming-run-length control charts*, Model Assisted Statistics and Applications **132** (1), 187-196, 2001.

[18] Xie, M. and Goh, T.N. *Some procedures for decision making in controlling high yield processes*, Quality and Reliability Engineering International **8**, 355-360, 1992.

Hacettepe Journal of Mathematics and Statistics
Volume 44 (5) (2015), 1283 – 1291

Gamma-admissibility of generalized Bayes estimators under LINEX loss function in a non-regular family of distributions

SH. Moradi Zahraie*

Abstract

Consider an estimation problem in a non-regular family of distributions under the LINEX loss function. Reviewing the admissibility of estimators under a vague prior information leads to the concept of gamma-admissibility. The purpose of this article is to give a sufficient conditions for a generalized Bayes estimator of a parametric function to be gamma-admissible. Some examples are given.

Keywords: Gamma-admissibility, Generalized Bayes estimator, LINEX loss function, Non-regular distribution, Vague prior information.

2000 AMS Classification: 62C15, 62F15.

Received 30/07/2014 : Accepted 15/10/2014 Doi : 10.15672/HJMS.2014347545

1. INTRODUCTION

Admissibility of estimator is an important problem in statistical decision theory; Consequently, this problem has been considered by many authors under various types of loss functions both in an exponential and in a non-regular family of distributions. For example under squared error loss function (Karlin [5], Ghosh & Meeden [3], Ralescu & Ralescu [10], Sinha & Gupta [13], Hoffmann [4], Pulskamp & Ralescu [9], Kim [6] and Kim & Meeden [7]), under entropy loss function (Sanjari Farsipour [11, 12]) and under LINEX loss function (Tanaka [14, 15, 16]) and squared-log error loss function (Zakerzadeh & Moradi Zahraie [18]).

In Bayesian statistical inference arbitrariness of a unique prior distribution is a permanent question. Robust Bayesian inference deals with the problem of expressing uncertainty of the prior information. A gamma-admissible approach is used which allows to take into account vague prior information on the distribution of the unknown parameter θ. The uncertainty about a prior is assumed by introducing a class Γ of priors. If prior information is scarce, the class Γ under consideration is large and a decision is close to a admissible decision. In the extreme case when no information is available the

*Department of Statistics, Yazd University, Yazd, Iran
Email: sh_moradi_zahraie@stu.yazd.ac.ir

Γ-admissible setup is equivalent to the usual admissible setup. If, on the other hand, the statistician has an exactly prior information and the class Γ contains a single prior, then the Γ-admissible decision is an usual Bayes decision. So it is a middle ground between the subjective Bayes setup and full admissible. See Berger [1] for useful references on robust Bayesian analysis.

Eichenauer-Herrmann [2] gained a sufficient conditions for an estimator of the form $(aX + b)/(cX + d)$ to be Γ-admissible under the squared error loss in a one-parameter exponential family.

The most popular convex and symmetric loss function is the squared error loss function which is widely used in decision theory due to its simple mathematical properties. However in some cases, it does not represent the true loss structure. This loss function is symmetric in nature i.e. it gives equal weightage to both over and under estimation. In real life, we encounter many situations where over-estimation may be more serious than under-estimation or vice versa. As an example, in construction an underestimate of the peak water level is usually much more serious than an overestimation.

The LINEX loss function was initially introduced by Varian [17] in the context of real estate assessment; estimation under this loss from the Bayesian perspective was studied by Zellner [19]. Subsequently, it became a workhorse in the literature on asymmetric loss. For an estimator δ of estimand $h(\theta)$, it is given by

$$L(\delta, h(\theta)) = b\left\{e^{c(\delta-h(\theta))} - c(\delta - h(\theta)) - 1\right\}, \tag{1.1}$$

where $c \neq 0$ and $b > 0$. If we define $\nabla := \delta - h(\theta)$, then $L(\nabla) = b\left\{e^{c\nabla} - c\nabla - 1\right\}$.

Some properties of the loss (1.1) are as follows:

(i) The constant b serves to scale this loss and without loss of generality we can assume that it is equal 1.

(ii) The constant c determines the shape of the loss; For $c > 0$ this loss function is quite asymmetric about 0 with overestimation being more costly than underestimation. As $|\nabla| \to \infty$, $L(\nabla)$ increases almost exponentially when $\nabla > 0$ and almost linearly when $\nabla < 0$. For $c < 0$, the linearity-exponentially phenomenon is reversed.

(iii) For $|c| \to 0$, this loss is almost symmetric and not far from a squared error loss function; In fact since $e^{c\nabla} \approx 1 + c\nabla + c^2\nabla^2/2$, thus $L(\nabla) \approx c^2\nabla^2/2$.

(iv) It is everywhere differentiable and its derivatives are continuous.

1.1. Remark. Linear-exponential where the name LINEX is justified by the fact that is this loss function rises approximately linearly on one side of zero and approximately exponentially on the other side.

A full discussion of the properties of this loss, may be found in Zellner [19] and Parsian & Kirmani [8].

In this paper we consider the Γ-admissibility of generalized Bayes estimators in a non-regular family of distributions under the loss (1.1) where class Γ consists of all distributions which are compatible with the vague prior information. To this end, in Section 2, we state some preliminary definitions and results. In Section 3, main theorem will obtain. Finally, in Section 4, we give an application of the Γ-admissibility in proof the Γ-minimaxity of estimators. Some examples are given.

2. Preliminaries

2.1. Definition of Γ-admissibility. In the present paper it is assumed that vague prior density on the distribution of the unknown parameter θ is available. Let Π denote the set of all priors, i.e. Borel probability measures on the parameter interval Θ and Γ be a non-empty subset of Π. Suppose that the available vague prior information can be described by the set Γ, in the sense that Γ contains all prior which are compatible with the vague prior information.

Eichenauer-Herrmann [2] has defined the Γ-admissibility of an estimator as follows.

2.1. Definition. An estimator δ^* is called Γ-admissible, if

$$r(\pi,\delta) \le r(\pi,\delta^*), \qquad \pi \in \Gamma,$$

for some estimator δ implies that

$$r(\pi,\delta) = r(\pi,\delta^*), \qquad \pi \in \Gamma,$$

where $r(\pi,\delta)$ is the Bayes risk of δ.

2.2. Remark. From Definition 2.1, it is obvious that
- A Π-admissible estimator is admissible.
- A $\{\pi\}$-admissible estimator is simply a Bayes strategy with respect to the prior π.
- In general neither Γ-admissibility implies admissibility nor admissibility implies Γ-admissibility.

Hence, the available results on admissibility cannot be applied in order to prove the Γ-admissibility of an estimator. Consequently, it is necessary to study the problem of Γ-admissibility of estimators.

2.2. A non-regular family of distributions. Let X be a random variable whose probability density function with respect to some σ-finite measure μ is given by

$$f_X(x;\theta) = \begin{cases} q(\theta)r(x), & \underline{\theta} < x < \theta \\ 0, & \text{otherwise} \end{cases}$$

where $\theta \in \Theta =: (\underline{\theta}, \bar{\theta})$ and Θ is a nondegenerate interval (possibly infinite) on the real line. Also $r(x)$ is a positive μ-measurable function of x and

$$q^{-1}(\theta) = \int_{\underline{\theta}}^{\theta} r(x)d\mu(x) < \infty$$

for $\theta \in \Theta$. This family is known as *a non-regular family of distributions*.

Suppose $\pi(\theta)$ be a prior (possibly improper) by its Lebesgue density $p_\pi(\theta)$ over Θ which is positive and continuous. Let $h(\theta)$ be a continuous function to be estimated from Θ to $\mathbb{R}$ and the loss to be (1.1). The generalized Bayes estimator of $h(\theta)$ with respect to $\pi(\theta)$ is given by $\delta_\pi(X)$, where

$$\delta_\pi(x) = -\frac{1}{c}\ln\left\{\frac{\int_x^{\bar{\theta}} e^{-ch(\theta)}q(\theta)p_\pi(\theta)d\theta}{\int_x^{\bar{\theta}} q(\theta)p_\pi(\theta)d\theta}\right\} \tag{2.1}$$

for $\underline{\theta} < x < \bar{\theta}$, provided that the integrals in (2.1) exist and are finite.

3. Main results

In this section, main results will obtain.

For some real number λ_0 let $a, b : [\lambda_0, \infty) \mapsto \Theta$ be continuously differentiable functions with $a(\lambda_0) < b(\lambda_0)$, where a and b are supposed to be strictly decreasing and strictly increasing, respectively. For $\lambda \ge \lambda_0$ a prior π_λ is defined by its Lebesgue density p_{π_λ} of the form

$$p_{\pi_\lambda}(\theta) := \left(\int_{a(\lambda)}^{b(\lambda)} p_\pi(t)dt\right)^{-1} I_{[a(\lambda),b(\lambda)]}(\theta)p_\pi(\theta).$$

Throughout this paper, we restrict estimators to the class

$$\Delta := \{\delta | (\text{A1}) \text{ and } (\text{A2}) \text{ are satisfied}\},$$

where

(A1) $E_\theta[|\delta(X)|] < \infty$ and $E_\theta\left[e^{a\delta(X)}\right] < \infty$ for all $\theta \in \Theta$,

(A2) $\int_{a(\lambda)}^{b(\lambda)} E_\theta\left[|\delta(X) - h(\theta)|\right] p_\pi(\theta)d\theta < \infty$ and $\int_{a(\lambda)}^{b(\lambda)} E_\theta\left[e^{a(\delta(X)-h(\theta))}\right] p_\pi(\theta)d\theta$ for $\lambda \geq \lambda_0$ and all θ which $\underline{\theta} < a(\lambda) < \theta < b(\lambda) < \bar{\theta}$.

3.1. Remark. In the statistical game (Γ, Δ, r), a Γ-admissible estimator is an admissible strategy of the second player.

The next lemma is essential to obtain our results.

3.2. Lemma. *Let $S(\theta)$ be a continuous and non-negative function over $\Theta = (\underline{\theta}, \bar{\theta})$. Let $G(\lambda) := \int_{a(\lambda)}^{b(\lambda)} S(\theta)d\theta$. Suppose that there exists a positive function $R(\theta)$ such that*

$$G(\lambda) \leq 4\left(\min\left\{R(b(\lambda))b'(\lambda), -R(a(\lambda))a'(\lambda)\right\}\right)^{-\frac{1}{2}}\left(G'(\lambda)\right)^{\frac{1}{2}}$$

for $\lambda \geq \lambda_0$. If

$$\int_{\lambda_0}^{\infty} \min\left\{R(b(\lambda))b'(\lambda), -R(a(\lambda))a'(\lambda)\right\} d\lambda = \infty,$$

then $S(\theta) = 0$ for a.a. $\theta \in \Theta$.

Proof. See Eichenauer-Herrmann [2]. □

Now, the main result of the present paper can be stated.

3.3. Theorem. *Suppose that $\delta_\pi \in \Delta$ and put*

$$K(x, \theta) := \int_x^\theta \left\{e^{-c\delta_\pi(x)} - e^{-ch(t)}\right\} q(t)p_\pi(t)dt,$$

and

$$\gamma(\theta) := \frac{e^{ch(\theta)}}{p_\pi(\theta)q(\theta)} \int_{\underline{\theta}}^{\theta} r(x)e^{c\delta_\pi(x)}K^2(x, \theta)d\mu(x).$$

If $\pi_\lambda \in \Gamma$ for all $\lambda \geq \lambda_0$ and

$$\int_{\lambda_0}^{\infty} \min\{\gamma^{-1}(b(\lambda))b'(\lambda), -\gamma^{-1}(a(\lambda))a'(\lambda)\}d\lambda = \infty, \tag{3.1}$$

then $\delta_\pi(X)$ is Γ-admissible under the loss (1.1).

Proof. Let $\delta \in \Delta$ be an estimator such that $r(\pi, \delta) \leq r(\pi, \delta_\pi)$ for every prior $\pi \in \Gamma$. Since $\pi_\lambda \in \Gamma$ for $\lambda \geq \lambda_0$, we must have

$$\begin{aligned} 0 &\leq \left(\int_{a(\lambda)}^{b(\lambda)} p_\pi(t)dt\right)\{r(\pi_\lambda, \delta_\pi) - r(\pi_\lambda, \delta)\} \\ &= \int_{a(\lambda)}^{b(\lambda)} E_\theta\left[L(\delta_\pi, h(\theta)) - L(\delta, h(\theta))\right] p_\pi(\theta)d\theta \end{aligned}$$

for all $\theta \in \Theta$. From Condition (A1), we see that it is equivalent to

$$\begin{aligned} 0 &\leq \int_{a(\lambda)}^{b(\lambda)} E_\theta\left[\left\{e^{\frac{c\delta(X)}{2}} - e^{\frac{c\delta_\pi(X)}{2}}\right\}^2\right] e^{-ch(\theta)}p_\pi(\theta)d\theta \\ &\leq \int_{a(\lambda)}^{b(\lambda)} E_\theta\left[c\{\delta(X) - \delta_\pi(X)\}\right] p_\pi(\theta)d\theta \\ &- 2\int_{a(\lambda)}^{b(\lambda)} E_\theta\left[e^{-ch(\theta)}e^{\frac{c\delta_\pi(X)}{2}}\left\{e^{\frac{c\delta(X)}{2}} - e^{\frac{c\delta_\pi(X)}{2}}\right\}\right] p_\pi(\theta)d\theta. \end{aligned}$$

An application of the Fubini's theorem gives

$$\begin{aligned}
0 \quad &\le \quad \int_{a(\lambda)}^{b(\lambda)} \int_{\underline{\theta}}^{\theta} \left\{ e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right\}^2 r(x) d\mu(x) e^{-ch(\theta)} p_\pi(\theta) q(\theta) d\theta \\
&\le \quad \int_{\underline{\theta}}^{b(\lambda)} \int_{x}^{b(\lambda)} \{c(\delta(x) - \delta_\pi(x))\} r(x) q(\theta) p_\pi(\theta) d\theta d\mu(x) \\
&- \quad 2\int_{\underline{\theta}}^{b(\lambda)} \int_{x}^{b(\lambda)} e^{-ch(\theta)} e^{\frac{c\delta_\pi(x)}{2}} \left\{ e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right\} r(x) q(\theta) p_\pi(\theta) d\theta d\mu(x) \\
&- \quad \int_{\underline{\theta}}^{a(\lambda)} \int_{x}^{a(\lambda)} \{c(\delta(x) - \delta_\pi(x))\} r(x) q(\theta) p_\pi(\theta) d\theta d\mu(x) \\
&+ \quad 2\int_{\underline{\theta}}^{a(\lambda)} \int_{x}^{a(\lambda)} e^{-ch(\theta)} e^{\frac{c\delta_\pi(x)}{2}} \left\{ e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right\} r(x) q(\theta) p_\pi(\theta) d\theta d\mu(x)
\end{aligned} \tag{3.2}$$

which is guaranteed by Condition (A2).

Using the inequality $x - y \le e^{-y}(e^x - e^y)$ for all x and y, the first term of the right-hand side in (3.2) is less than

$$2\int_{\underline{\theta}}^{b(\lambda)} \int_{x}^{b(\lambda)} e^{-\frac{c\delta_\pi(x)}{2}} \left\{ e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right\} r(x) q(\theta) p_\pi(\theta) d\theta d\mu(x).$$

By Schwartz inequality, sum of the first and the second terms of the right-hand side in (3.2) is less than

$$2\left\{ \int_{\underline{\theta}}^{b(\lambda)} \left(e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right)^2 r(x) d\mu(x) \right\}^{\frac{1}{2}} \left\{ \int_{\underline{\theta}}^{b(\lambda)} e^{c\delta_\pi(x)} K^2(x, b(\lambda)) r(x) d\mu(x) \right\}^{\frac{1}{2}}.$$

Hence, if we define

$$T(\theta) := \int_{\underline{\theta}}^{\theta} \left\{ e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right\}^2 r(x) d\mu(x),$$

and

$$M(\theta) := T(\theta) e^{-cb(\theta)} q(\theta) p_\pi(\theta),$$

then Equation (3.2) implies

$$\begin{aligned}
0 \quad &\le \quad \int_{a(\lambda)}^{b(\lambda)} T(\theta) e^{-ch(\theta)} q(\theta) p_\pi(\theta) d\theta \\
&\le \quad 2\left\{ T(b(\lambda)) e^{-ch(b(\lambda))} q(b(\lambda)) p_\pi(b(\lambda)) b'(\lambda) \right\}^{\frac{1}{2}} \left\{ \gamma^{-1}(b(\lambda)) b'(\lambda) \right\}^{-\frac{1}{2}} \\
&+ \quad 2\left\{ -T(a(\lambda)) e^{-ch(a(\lambda))} q(a(\lambda)) p_\pi(a(\lambda)) a'(\lambda) \right\}^{\frac{1}{2}} \left\{ -\gamma^{-1}(a(\lambda)) a'(\lambda) \right\}^{-\frac{1}{2}} \\
&\le \quad 4\left(\min\{\gamma^{-1}(b(\lambda)) b'(\lambda), -\gamma^{-1}(a(\lambda)) a'(\lambda)\} \right)^{-\frac{1}{2}} \times \left(M(b(\lambda)) b'(\lambda) - M(a(\lambda)) a'(\lambda) \right)^{\frac{1}{2}}
\end{aligned} \tag{3.3}$$

for $\lambda \ge \lambda_0$, where the definition of the function $\gamma(\theta)$ has been used. Now a continuous, differentiable and increasing function $H : [\lambda_0, \infty] \to \mathbb{R}$ is defined by

$$H(\lambda) := \int_{a(\lambda)}^{b(\lambda)} T(\theta) e^{-ch(\theta)} q(\theta) p_\pi(\theta) d\theta.$$

So (3.3) can be written in the form

$$H(\lambda) \leq 4\left(\min\{\gamma^{-1}(b(\lambda))b'(\lambda), -\gamma^{-1}(a(\lambda))a'(\lambda)\}\right)^{-\frac{1}{2}}(H'(\lambda))^{\frac{1}{2}}$$

for $\lambda \geq \lambda_0$. Therefore, from Lemma 3.2 we obtain $T(\theta) = 0$ for $a.a.\theta \in \Theta$, and consequently from (A1), we have $\delta(x) = \delta_\pi(x)$ $a.e.\mu$. This completes the proof. □

3.4. Remark. $K(x,\theta)$ can expressed as

$$K(x,\theta) = \frac{1}{\int_x^{\bar{\theta}} q(u)p_\pi(u)du}\int_x^\theta \int_\theta^{\bar{\theta}} \left\{e^{-ah(s)} - e^{-ah(t)}\right\} q(s)p_\pi(s)q(t)p_\pi(t)dsdt,$$

by (2.1) and the symmetry of the integrand.

3.5. Example. As Example 1 in [18], suppose that $X_1, ..., X_n$ are i.i.d. random variables according to an exponential distribution whose probability distribution function is given by

$$f(x;\theta) = \begin{cases} e^{x-\theta}, & x < \theta \\ 0, & x > \theta \end{cases}$$

where $\theta(\in \mathbb{R})$ is unknown. $X = X_{(n)}$ is sufficient for θ and its probability distribution function is given by

$$f_X(x;\theta) = \begin{cases} ne^{n(x-\theta)}, & x < \theta \\ 0, & x > \theta \end{cases}$$

The generalized Bayes estimator of $h(\theta) = \theta$ with respect to the Lebesgue prior is given by

$$\delta_\pi(X) = X + \frac{1}{c}\ln\frac{n+c}{n},$$

if $n + c > 0$. A direct calculation gives

$$K(x,\theta) = \frac{1}{n+c}e^{-n\theta}\left(e^{-c\theta} - e^{-cx}\right),$$

and

$$\gamma(\theta) = \frac{2c^2}{n(n+c)^2(n-c)}.$$

Let class Γ_0 consists of all priors with mean 0, i.e., $\Gamma_0 := \{\pi \in \Pi | \int_\Theta \theta p_\pi(\theta)d\theta = 0\}$. Define functions a and b by $a(\lambda) = -\lambda$ and $b(\lambda) = \lambda$ for $\lambda \geq \lambda_0 > 0$, i.e., the prior π_λ is the uniform distribution on the interval $[-\lambda, \lambda]$. Hence, $\pi_\lambda \in \Gamma_0$ for all $\lambda \geq \lambda_0$. Since (3.1) is satisfied, Theorem 3.3 implies that $\delta_\pi(X)$ is Γ_0-admissible under the loss (1.1).

3.6. Remark. It is difficult to express $\gamma(\theta)$ explicitly and it can have a complicated form, so to apply Theorem 3.3, we have to seek the suitable upper bound of $\gamma(\theta)$. For the case when $h(\theta)$ is bounded, we can get the next corollary.

3.7. Corollary. *Suppose that $h(\theta)$ is bounded and $\delta_\pi \in \Delta$. Put*

$$\tilde{K}(x,\theta) := \frac{\int_\theta^{\bar{\theta}} q(s)p_\pi(s)ds \int_x^\theta q(t)p_\pi(t)dt}{\int_x^{\bar{\theta}} q(u)p_\pi(u)du},$$

and

$$\tilde{\gamma}(\theta) := \frac{1}{p_\pi(\theta)q(\theta)}\int_{\underline{\theta}}^\theta r(x)\tilde{K}^2(x,\theta)d\mu(x).$$

If $\pi_\lambda \in \Gamma$ *for all* $\lambda \geq \lambda_0$ *and*

$$\int_{\lambda_0}^{\infty} \min\{\tilde{\gamma}^{-1}(b(\lambda))b'(\lambda), -\tilde{\gamma}^{-1}(a(\lambda))a'(\lambda)\}d\lambda = \infty,$$

then $\delta_\pi(X)$ *is* Γ*-admissible under the loss (1.1).*

Proof. It can be shown that there exist constants $\underline{C}$ and $\bar{C}$ such that $\underline{C} < e^{c\delta_\pi(x)} < \bar{C}$ for all $x \in (\underline{\theta}, \bar{\theta})$. Further, since $h(\theta)$ is bounded, there exists a constant C such that $|K(x,\theta)| \leq C\tilde{K}(x,\theta)$ for all $(x,\theta) \in \{(x,\theta)|\underline{\theta} < x < \theta < \bar{\theta}\}$. This completes the proof by Theorem 3.3. □

3.8. Example. As Example 2 in [18], suppose that $X_1, ..., X_n$ are i.i.d. random variables according to a uniform distribution over the interval $(0,\theta)$ where $\theta(\in \mathbb{R}^+)$ is unknown. Then the probability distribution function of the sufficient statistic $X = X_{(n)}$ is given by

$$f_X(x;\theta) = \begin{cases} \frac{n}{\theta^n}x^{n-1}, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases}$$

Let $h(\theta) = P_\theta(X_1 \leq 1) = \frac{1}{\theta}I_{\{1<\theta\}}(\theta) + I_{\{\theta<1\}}(\theta)$, where $I_A(\theta)$ is the indicator function of the set A. Then the generalized Bayes estimator of $h(\theta)$ with respect to $\pi(\theta)$ by its density $p_\pi(\theta) = 1/\theta$ is given by $\delta_\pi(X)$, where

$$\delta_\pi(x) = \begin{cases} -\frac{1}{c}\ln\left\{e^{-c}(1-x^n) + n\int_0^x y^{n-1}e^{-c\frac{y}{x}}dy\right\}, & 0 < x < 1 \\ -\frac{1}{c}\ln\left\{n\int_0^1 y^{n-1}e^{-c\frac{y}{x}}dy\right\}, & 1 < x \end{cases}$$

We can easily obtain

$$\tilde{K}(x,\theta) = \frac{1}{n\theta^n}\left\{1 - \left(\frac{x}{\theta}\right)^n\right\},$$

and

$$\tilde{\gamma}(\theta) = \frac{\theta}{3n^2}.$$

Let $\Gamma_m := \{\pi \in \Pi | \int_\Theta \theta p_\pi(\theta)d\theta = m\}$, i.e., Γ_m consists of all priors with mean m. Define functions a and b by $a(\lambda) = m\ln(\lambda)/(\lambda-1)$ and $b(\lambda) = \lambda a(\lambda)$ for $\lambda \geq \lambda_0 > 1$. Since

$$\int_\Theta \theta p_{\pi_\lambda}(\theta)d\theta = \left(\int_{a(\lambda)}^{b(\lambda)} \frac{1}{t}dt\right)^{-1}(b(\lambda) - a(\lambda)) = m$$

for all $\lambda \geq \lambda_0$, so that $\pi_\lambda \in \Gamma_m$. A short calculation yields

$$a'(\lambda) = m\frac{\lambda - 1 - \lambda\ln(\lambda)}{\lambda(\lambda-1)^2} < 0,$$

and

$$b'(\lambda) = m\frac{\lambda - 1 - \ln(\lambda)}{(\lambda-1)^2} > 0,$$

for $\lambda \geq \lambda_0$. Because of $\lambda - 1 - \ln(\lambda) < \lambda\ln(\lambda) - \lambda + 1$ for $\lambda \geq \lambda_0$ and $\lim_{\lambda\to\infty} b(\lambda) = \infty$, one obtains

$$\begin{aligned} \int_{\lambda_0}^{\infty} \min\{\tilde{\gamma}^{-1}(b(\lambda))b'(\lambda), -\tilde{\gamma}^{-1}(a(\lambda))a'(\lambda)\}d\lambda &= (3n^2)\int_{\lambda_0}^{\infty} \min\left\{\frac{b'(\lambda)}{b(\lambda)}, -\frac{a'(\lambda)}{a(\lambda)}\right\}d\lambda \\ &= (3n^2)\int_{\lambda_0}^{\infty} \frac{b'(\lambda)}{b(\lambda)}d\lambda = \infty \end{aligned}$$

which implies, according to Corollary 3.7 that $\delta_\pi(X)$ is Γ_m-admissible under the loss (1.1).

3.9. Remark. Typically all the result in this paper go through with some modifications for the density

$$f_X(x,\theta) = \begin{cases} q(\theta)r(x), & \theta < x < \bar{\theta} \\ 0, & \text{otherwise} \end{cases}$$

where $\theta \in \Theta = (\underline{\theta}, \bar{\theta})$ is unknown.

4. An application

In the presence of vague prior information frequently the Γ-minimax approach is used as underlying principle. In this section, we provide the definition of the Γ-minimaxity of an estimator and then express the relation between this concept and the Γ-admissibility. Finally, we give an example.

4.1. Definition. A Γ-minimax estimator is a minimax strategy of the second player in the statistical game (Γ, Δ, r); δ^* is called a Γ-minimax estimator, if

$$\sup_{\pi\in\Gamma} r(\pi,\delta^*) = \inf_{\delta\in\Delta} \sup_{\pi\in\Gamma} r(\pi,\delta),$$

where $r(\pi,\delta)$ is the Bayes risk of δ.

4.2. Definition. A Γ-minimax estimator δ^* is said to be unique, if

$$r(\pi,\delta) = r(\pi,\delta^*), \qquad \pi \in \Gamma,$$

for any other Γ-minimax estimator δ.

4.3. Remark.
- From Definition 4.2, it is obvious that a unique Γ-minimax estimator is Γ-admissible.
- If a Γ-admissible estimator δ is an equalizer on Γ, i.e., $r(.,\delta)$ is constant on Γ, then δ is a unique Γ-minimax estimator.

4.4. Example. In Example 3.5, we have $E_\theta[X] = \theta - (1/n)$ and $E_\theta[e^{cX}] = (n/(n+c))e^{c\theta}$. Thus, the risk function of δ_π is equal to

$$R(\delta_\pi,\theta) = bE_\theta\left[e^{c(\delta_\pi-\theta)} - c(\delta_\pi-\theta) - 1\right] = b\left\{\frac{c}{n} - \ln\left(\frac{n+c}{n}\right)\right\}.$$

So, δ_π is an equalizer on Γ_0, since its risk function is constant. Hence, $\delta_\pi(X)$ is the unique Γ_0-minimax estimator for θ.

References

[1] Berger, JO. *An overview of robust Bayesian analysis*, Test **3**, 5-124, 1994.

[2] Eichenauer-Herrmann, J. *Gamma-admissibility of estimators in the one-parameter exponential family*, Metrika **39**, 199-208, 1992.

[3] Ghosh, G. and Meeden, M. *Admissibility of linear estimators in the one parameter exponential family*, Annals of Mathematical Statistics **5**, 772-778, 1977.

[4] Hoffmann, K. *Admissibility and inadmissibility of estimators in the one-parameter exponential family*, Statistics **16**, 327-349, 1985.

[5] Karlin, S.*Admissibility for estimation with quadratic loss*, Annals of Mathematical Statistics **29**, 406-436, 1958.

[6] Kim, B.H. *Admissibility of generalized Bayes estimators in an one parameter non-regular family*, Metrika **41**, 99-108, 1994.

[7] Kim, B.H. and Meeden, G. *Admissible estimation in an one parameter nonregular family of absolutely continuous distributions*, Communications in Statistics-Theory and Methods **23**, 2993-3001, 1994.

[8] Parsian, A. and Kirmani, SNUA. *Estimation under LINEX loss function. In: Ullah A, Wan ATK, Chaturvedi A (eds) Handbook of applied econometrics and statistical inference. Marcel Dekker*, NewYork, 53-76, 2002.

[9] Pulskamp, R.J. and Ralescu, D.A. *A general class of nonlinear admissible estimators in the one-parameter exponential case*, Journal of Statistical Planning and Inference **28**, 383-390, 1991.

[10] Ralescu, D. and Ralescu, S. *A class of nonlinear admissible estimators in the one-parameter exponential family*, Annals of Mathematical Statistics **9**, 177-183, 1981.

[11] Sajari Farsipour, N. *Admissibility of estimators in the non-regular family under entropy loss function*, Statistical Papers **44**, 249-256, 2003.

[12] Sajari Farsipour, N. *Admissible estimation in an one parameter nonregular family of absolutely continuous distributions*, Statistical Papers **48**, 337-345, 2007.

[13] Sinha, B.K. and Gupta, A.D. *Admissibility of generalized Bayes and Pitman estimates in the nonregular family*, Communications in Statistics-Theory and Methods **13**, 1709-1721, 1984.

[14] Tanaka, H. *Sufficient conditions for the admissibility under LINEX loss function in regular case*, Communications in Statistics-Theory and Methods **39**, 1477-1489, 2010.

[15] Tanaka, H. *Sufficient conditions for the admissibility under LINEX loss function in non-regular case*, Statistics **45**, 199-208, 2011.

[16] Tanaka, H. *Admissibility under the LINEX loss function in non-regular case*, Scientiae Mathematicae Japonicae **75**, 351-358, 2012.

[17] Varian, H.R. *A Bayesian approach to real estate assessment. In: Fienberg S. E., Zellner A, eds. Studies in Bayesian Econometrics and Statistics.*, Amsterdam: North-Holland, 195-208, 1975.

[18] Zakerzadeh, H., and Moradi Zahraie, S.H. *Admissibility in non-regular family under squared-log error loss*, Metrika, **78**, 227-236, 2015.

[19] Zellner, A. *Bayesian estimation and prediction using asymmetric loss functions*, Journal of the American Statistical Association **81**, 446-451, 1986.

ADVICE TO AUTHORS

Hacettepe Journal of Mathematics and Statistics publishes short to medium length research papers and occasional survey articles written in English. All papers are refereed to international standards.

Address for Correspondence

Editorial Office,
Hacettepe Journal of Mathematics and Statistics,
Hacettepe University,
Faculty of Science,
Department of Mathematics,
06532 Beytepe,
Ankara,
Turkey.

E-mail: hjms@hacettepe.edu.tr
Tel: Editors: + 90 312 297 7898
+ 90 312 297 7850 / 122
Associate Editor: + 90 312 297 7880
Fax : + 90 312 299 2017
Web Page : http//www.hjms.hacettepe.edu.tr

Advise to Authors : The style of articles should be lucid but concise. The text should be preceded by a short descriptive title and an informative abstract of not more than 100 words. Keywords or phrases and the 2010 AMS Classification should also be included.The main body of the text should be divided into numbered sections with appropriate headings. Items should be numbered in the form 2.4. Lemma, 2.5. Definition. These items should be referred to in the text using the form Lemma 2.4., Definition 2.5. Figures and tables should be incorporated in the text. A numbered caption should be placed above them. References should be punctuated according to the following examples, be listed in alphabetical order according to in the (first) author' s surname, be numbered consecutively and referred to in the text by the same number enclosed in square brackets. Only recognized abbreviations of the names of journals should be used.

[1] Banaschewski, B. *Extensions of topological spaces*, Canad. Math. Bull. **7** (1), 1–22, 1964.

[2] Ehrig, H. and Herrlich, H. *The construct PRO of projection spaces: its internal structure*, in: Categorical methods in Computer Science, Lecture Notes in Computer Science **393** (Springer-Verlag, Berlin, 1989), 286–293.

[3] Hurvich, C. M. and Tsai, C. L. *Regression and time series model selection in small samples*, Biometrika **76** (2), 297–307, 1989.

[4] Papoulis, A. *Probability random variables and stochastic process* (McGraw-Hill, 1965).

INSTRUCTIONS FOR AUTHORS

Preparation of Manuscripts : Manuscripts will be typeset using the LATEX typesetting system. Authors should prepare the article using the HJMS style before submission by e-mail. The style bundle contains the styles for the article and the bibliography as well as example of formatted document as pdf file. The authors should send their paper directly to hjms@hacettepe.edu.tr with the suggestion of area editor and potential reviewers for their submission. The editorial office may not use these suggestions, but this may help to speed up the selection of appropriate reviewers.

Copyright : No manuscript should be submitted which has previously been published, or which has been simultaneously submitted for publication elsewhere. The copyright in a published article rests solely with the Faculty of Science of Hacettepe University, and the paper may not be reproduced in whole in part by any means whatsoever without prior written permission.

Notice : The content of a paper published in this Journal is the sole responsibility of the author or authors, and its publication does not imply the concurrence of the Editors or the Publisher.