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CONTENTS

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Multiplicative (generalized)-derivations and left ideals in semiprime rings

Asma Ali*, Basudeb Dhara†‡, Shahoor Khan§ and Farhat Ali¶

Abstract

Let R be a semiprime ring with center $Z(R)$. A mapping $F : R \to R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation if there exists a map $f : R \to R$ (not necessarily a derivation nor an additive map) such that $F(xy) = F(x)y + xf(y)$ holds for all $x, y \in R$. The objective of the present paper is to study the following identities: (i) $F(x)F(y) \pm [x, y] \in Z(R)$, (ii) $F(x)F(y) \pm x \circ y \in Z(R)$, (iii) $F([x, y]) \pm [x, y] \in Z(R)$, (iv) $F(x \circ y) \pm (x \circ y) \in Z(R)$, (v) $F([x, y]) \pm [F(x), y] \in Z(R)$, (vi) $F(x \circ y) \pm (F(x) \circ y) \in Z(R)$, (vii) $[F(x), y] \pm [G(y), x] \in Z(R)$, (viii) $F([x, y]) \pm [F(x), F(y)] = 0$, (ix) $F(x \circ y) \pm (F(x) \circ F(y)) = 0$, (x) $F(xy) \pm [x, y] \in Z(R)$ and (xi) $F(xy) \pm x \circ y \in Z(R)$ for all x, y in some appropriate subset of R, where $G : R \to R$ is a multiplicative (generalized)-derivation associated with the map $g : R \to R$.

Keywords: Semiprime ring, left ideal, derivation, multiplicative derivation, generalized derivation, multiplicative (generalized)-derivation

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1. Introduction

Throughout the paper R will denote an associative ring with center $Z(R)$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ and is called semiprime if for any $a \in R$, $aRa = \{0\}$ implies that $a = 0$. We shall write for any pair of elements $x, y \in R$ the commutator $[x, y] = xy - yx$ and skew-commutator $x \circ y = xy + yx$. We will frequently use the basic commutator and skew-commutator identities: (i) $[xy, z] = x[y, z] + [x, z]y$, $[x, yz] = y[x, z] + [x, y]z$ and (ii) $x \circ yz = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$, $xy \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$ for all $x, y, z \in R$. Let S be a nonempty subset of R. A map $F : R \to R$ is called centralizing on S if $[F(x), x] \in Z(R)$ for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$ for all $x \in S$. The first well-known result on commuting maps is Posner's second theorem in [15]. This theorem states that the existence of a nonzero commuting derivation on a prime ring R implies R to be commutative. By a derivation, we mean an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. The concept of derivation was extended to generalized derivation in [6] by Brešar. An additive mapping $g : R \to R$ is said to be a generalized derivation if there exists a derivation $d : R \to R$ such that $g(xy) = g(x)y + xd(y)$ holds for all $x, y \in R$. In [13], Hvala gave the algebraic study of generalized derivation in prime rings. Obviously every derivation is a generalized derivation of R.

Many papers in literature have investigated the commutativity of prime and semiprime rings satisfying certain functional identities involving derivations or generalized derivations (see [1], [3], [4], [5], [9], [10], [11], [16], [17]).

In [5], Ashraf and Rehman proved that if R is a prime ring with a nonzero ideal I of R and d is a derivation of R such that either $d(xy) - xy \in Z(R)$ for all $x, y \in I$ or $d(xy) + xy \in Z(R)$ for all $x, y \in I$, then R is commutative. Recently, Ashraf et al. [3] have studied the situations replacing derivation d with a generalized derivation F. More precisely, they proved that the prime ring R must be commutative, if R satisfies any one of the following conditions : (i) $F(xy) - xy \in Z(R)$ for all $x, y \in I$, (ii) $F(xy) + xy \in Z(R)$ for all $x, y \in I$, (iii) $F(xy) - yx \in Z(R)$ for all $x, y \in I$, (iv) $F(xy) + yx \in Z(R)$ for all $x, y \in I$, (v) $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$, (vi) $F(x)F(y) + xy \in Z(R)$ for all $x, y \in I$; where F is a generalized derivation of R associated with a nonzero derivation d and I is a nonzero two-sided ideal of R.

On the other hand, in [9], Daif and Bell proved that if R is a semiprime ring with a nonzero ideal K and d is a derivation of R such that $d([x, y]) = \pm[x, y]$ for all $x, y \in K$, then K is a central ideal. In particular, if $K = R$, then R is commutative. Recently, Quadri et al. [16] generalized this result replacing derivation d with a generalized derivation in a prime ring R. More precisely, they proved the following:

Let R be a prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that any one of the following holds : (i) $F([x, y]) = [x, y]$ for all $x, y \in I$; (ii) $F([x, y]) = -[x, y]$ for all $x, y \in I$; (iii) $F(x \circ y) = (x \circ y)$ for all $x, y \in I$; (iv) $F(x \circ y) = -(x \circ y)$ for all $x, y \in I$; then R is commutative.

Recently in [11], Dhara proved the following result: Let R be a semiprime ring, I be a nonzero ideal of R and F be a generalized derivation of R with associated derivation d satisfying $F([x, y]) \pm [x, y] = 0$ or $F(x \circ y) \pm (x \circ y) = 0$ for all $x, y \in I$, then R must contain a nonzero central ideal, provided $d(I) \neq (0)$. In case R is prime satisfying $F([x, y]) \pm [x, y] \in Z(R)$ or $F(x \circ y) \pm (x \circ y) \in Z(R)$ for all $x, y \in I$, then R must be commutative, provided $d(Z) \neq (0)$.

In this line of investigation, recently, Asma et al. [1] have studied the following situations: (i) $F(xy) \in Z(R)$, (ii) $F([x, y]) = 0$, (iii) $(F(xy) \pm yx) \in Z(R)$ and (iv)

$(F(xy) \pm [x, y]) \in Z(R)$; for all x, y in some nonzero left ideal of semiprime ring R, where F is a generalized derivation of R.

Recently, Dhara and Ali [10] studied the above mentioned results of Ashraf et al. [3] in semiprime rings replacing two-sided ideal I with left sided ideal λ and generalized derivation with multiplicative (generalized)-derivation.

Let us introduce the background of investigation about multiplicative (generalized)-derivation. A mapping $D : R \to R$ which satisfies $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$ is called a multiplicative derivation of R. Of course these mappings are not additive. To the best of my knowledge, the concept of multiplicative derivations appeared for the first time in the work of Daif [7]. Then the complete description of those maps was given by Goldmann and Šemrl in [12].

Further, Daif and Tammam-El-Sayiad [8] extended the notion of multiplicative derivation to multiplicative generalized derivation as follows: a mapping $F : R \to R$ is called a multiplicative generalized derivation if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In [10], Dhara and Ali make a slight generalization of Daif and Tammam-El-Sayiad's definition of multiplicative generalized derivation by considering d as any map. In [10], the authors defined that a mapping $F : R \to R$ (not necessarily additive) is said to be multiplicative (generalized)-derivation if $F(xy) = F(x)y + xf(y)$ holds for all $x, y \in R$, where f is any mapping (not necessarily a derivation nor an additive map). For examples of such maps we refer to [10]. Moreover, multiplicative (generalized)-derivation with $f = 0$ covers the notion of multiplicative centralizers (not necessarily additive). Obviously, every generalized derivation is a multiplicative (generalized)-derivation on R.

In this line of investigation, it is more interesting to study the identities replacing generalized derivation with multiplicative (generalized)-derivation. In the present paper, our main object is to investigate the cases when a multiplicative (generalized)-derivations F and G satisfies the identities: (i) $F(x)F(y) \pm [x, y] \in Z(R)$, (ii) $F(x)F(y) \pm x \circ y \in Z(R)$, (iii) $F([x, y]) \pm [x, y] \in Z(R)$, (iv) $F(x \circ y) \pm (x \circ y) \in Z(R)$, (v) $F([x, y]) \pm [F(x), y] \in Z(R)$, (vi) $F(x \circ y) \pm (F(x) \circ y) \in Z(R)$, (vii) $[F(x), y] \pm [G(y), x] \in Z(R)$, (viii) $F([x, y]) \pm [F(x), F(y)] = 0$, (ix) $F(x \circ y) \pm (F(x) \circ F(y)) = 0$, (x) $F(xy) \pm [x, y] \in Z(R)$ and (xi) $F(xy) \pm x \circ y \in Z(R)$ for all x, y in some appropriate subset of R.

2. Main Results

2.1. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x)F(y) \pm [x, y] \in Z(R)$ for all $x, y \in \lambda$, then $\lambda[\lambda, \lambda] = (0)$ and $\lambda[f(x), x] = (0)$ for all $x \in \lambda$.

Proof. First we consider the case

$$F(x)F(y) + [x, y] \in Z(R) \quad \text{for all} \quad x, y \in \lambda. \tag{2.1}$$

Substituting yz for y in (2.1), we have

$$\begin{aligned} &F(x)F(yz) + [x, yz] = F(x)F(y)z + F(x)yf(z) + y[x, z] + [x, y]z \\ &= (F(x)F(y)z + [x, y])z + y[x, z] + F(x)yf(z) \in Z(R) \quad \text{for all} \quad x, y, z \in \lambda. \end{aligned} \tag{2.2}$$

Commuting both sides with z in (2.2) and using (2.1), we obtain

$$[F(x)yf(z), z] + [y[x, z], z] = 0 \quad \text{for all} \quad x, y, z \in \lambda. \tag{2.3}$$

Putting $x = xz$ in the above relation, we get

$$[F(x)zyf(z), z] + [xf(z)yf(z), z] + [y[x, z], z]z = 0 \quad \text{for all} \quad x, y, z \in \lambda. \tag{2.4}$$

Replacing y by zy in (2.3), we obtain

(2.5) $\quad [F(x)zyf(z),z]+z[y[x,z],z]=0 \text{ for all } x,y,z\in\lambda.$

Subtracting (2.5) from (2.4), we get

(2.6) $\quad [xf(z)yf(z),z]+[[y[x,z],z],z]=0 \text{ for all } x,y,z\in\lambda.$

Putting $x=xz$, the above relation yields that

(2.7) $\quad [xzf(z)yf(z),z]+[[y[x,z],z],z]z=0 \text{ for all } x,y,z\in\lambda.$

Right multiplying (2.6) by z and then subtracting it from (2.7), we get

(2.8) $\quad x[[f(z)yf(z),z],z]=0 \text{ for all } x,y,z\in\lambda.$

Now we substitute $f(z)yf(z)x$ for x in (2.8), to get

$$\begin{aligned}0&=[f(z)yf(z)x[f(z)yf(z),z],z]\\&=f(z)yf(z)[x[f(z)yf(z),z],z]+[f(z)yf(z),z]x[f(z)yf(z),z]\\&\quad\text{for all } x,y,z\in\lambda.\end{aligned}\tag{2.9}$$

Using (2.8), it reduces to

(2.10) $\quad [f(z)yf(z),z]x[f(z)yf(z),z]=0 \text{ for all } x,y,z\in\lambda.$

Since λ is a left ideal of R, it follows that $x[f(z)yf(z),z]Rx[f(z)yf(z),z]=(0)$ for all $x,y,z\in\lambda$. Since R is semiprime, we have

(2.11) $\quad x[f(z)yf(z),z]=0 \text{ for all } x,y,z\in\lambda,$

that is,

(2.12) $\quad x(f(z)yf(z)z-zf(z)yf(z))=0 \text{ for all } x,y,z\in\lambda.$

Replacing y by $yf(z)u$ in (2.12), we obtain

(2.13) $\quad x(f(z)yf(z)uf(z)z-zf(z)yf(z)uf(z))=0 \text{ for all } u,x,y,z\in\lambda.$

Using (2.12), this can be written as

(2.14) $\quad x(f(z)yzf(z)uf(z)-f(z)yf(z)zuf(z))=0 \text{ for all } u,x,y,z\in\lambda,$

which gives

(2.15) $\quad xf(z)y[f(z),z]uf(z)=0 \text{ for all } u,x,y,z\in\lambda.$

This implies that $x[f(z),z]y[f(z),z]u[f(z),z]=0$ for all $u,x,y,z\in\lambda$ and so $(\lambda[f(z),z])^3=(0)$ for all $z\in\lambda$. Since a semiprime ring contains no nonzero nilpotent left ideals (see [2]), it follows that $\lambda[f(z),z]=(0)$ for all $z\in\lambda$.

Now replacing y by yz in (2.3), we get

(2.16) $\quad [F(x)yzf(z),z]+[yz[x,z],z]=0 \text{ for all } x,y,z\in\lambda.$

Right multiplying (2.3) by z and then subtracting from (2.16), we get

(2.17) $\quad [F(x)y[f(z),z],z]+[y[x,z]_2,z]=0 \text{ for all } x,y,z\in\lambda.$

By using $\lambda[f(z),z]=(0)$ for all $z\in\lambda$, (2.17) yields $[y[x,z]_2,z]=0$ for all $x,y,z\in\lambda$. Substituting y by xy, we obtain $0=[xy[x,z]_2,z]=x[y[x,z]_2,z]+[x,z]y[x,z]_2=[x,z]y[x,z]_2$ and hence $y[x,z]_2Ry[x,z]_2=(0)$ for all $x,y,z\in\lambda$. Since R is semiprime ring, $\lambda[x,z]_2=(0)$ for all $x,z\in\lambda$. Linearizing the last relation with respect to z, we have $(0)=\lambda[[x,u],v]+\lambda[[x,v],u]$ for all $x,u,v\in\lambda$. Now we put $u=uv$ and get $(0)=\lambda([[x,u],v]v+[u[x,v],v])+\lambda([[x,v],u]v+u[[x,v],v])=\lambda[u[x,v],v]$ for all $x,u,v\in\lambda$. Now we put $u=xu$ in this last relation and then get $(0)=\lambda[xu[x,v],v]=\lambda x[u[x,v],v]+\lambda[x,v]u[x,v]=\lambda[x,v]u[x,v]$ for all $x,u,v\in\lambda$. Thus $\lambda[x,v]R\lambda[x,v]=(0)$ for all $x,v\in\lambda$. Since R is semiprime, it yields $\lambda[\lambda,\lambda]=(0)$, as desired.

Similarly we can prove the result for the case $F(x)F(y) - [x,y] \in Z(R)$ for all $x, y \in \lambda$. $\square$

2.2. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x)F(y) \pm (x \circ y) \in Z(R)$ for all $x, y \in \lambda$, then $\lambda[\lambda, \lambda] = (0)$ and $\lambda[f(x), x] = (0)$ for all $x \in \lambda$.

Proof. First we consider that

$$(2.18) \quad F(x)F(y) - (x \circ y) \in Z(R) \text{ for all } x, y \in \lambda.$$

Substituting yz for y in (2.18), we have

$$(2.19) \quad \begin{aligned} &F(x)F(yz) - (x \circ yz) = F(x)F(y)z + F(x)yf(z) - (x \circ y)z + y[x,z] \\ &= (F(x)F(y) - x \circ y)z + y[x,z] + F(x)yf(z) \in Z(R) \text{ for all } x, y, z \in \lambda. \end{aligned}$$

Commuting both sides with z in (2.19) and using (2.18), we obtain

$$(2.20) \quad [F(x)yf(z), z] + [y[x,z], z] = 0 \text{ for all } x, y, z \in \lambda.$$

This is same as (2.3) in Theorem 2.1. Then by same argument of Theorem 2.1, we conclude the result.

Similarly, we can prove the result for the case $F(x)F(y) + (x \circ y) \in Z(R)$ for all $x, y \in \lambda$. $\square$

2.3. Corollary. Let R be a semiprime ring and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If R satisfies any one of the following conditions:

(1) $F(x)F(y) \pm [x,y] \in Z(R)$ for all $x, y \in R$;
(2) $F(x)F(y) \pm (x \circ y) \in Z(R)$ for all $x, y \in R$;

then R must be commutative.

Note that the map $G(r) = F(r) \pm r$ for all $r \in R$ is a multiplicative (generalized)-derivation of R.

2.4. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x,y]) \pm [x,y] = 0$ for all $x, y \in \lambda$, then $\lambda[f(x), x] = (0)$ for all $x \in \lambda$.

Proof. By hypothesis, we have

$$(2.21) \quad G([x,y]) = 0 \text{ for all } x, y \in \lambda.$$

Replacing y by yx in (2.21) and using (2.21), we obtain

$$(2.22) \quad 0 = G([x,yx]) = G([x,y]x) = G([x,y])x + [x,y]f(x) = [x,y]f(x) \text{ for all } x, y \in \lambda.$$

This gives that

$$(2.23) \quad [x,y]f(x) = 0 \text{ for all } x, y \in \lambda.$$

Substituting $f(x)y$ for y in (2.23), we get

$$(2.24) \quad [x, f(x)]yf(x) = 0 \text{ for all } x, y \in \lambda.$$

Replace y by yx in (2.24), to get

(2.25) $\quad [x, f(x)]yxf(x) = 0 \quad \text{for all} \quad x, y \in \lambda.$

Right multiplying (2.24) by x and then subtracting from (2.25), we obtain

(2.26) $\quad [x, f(x)]y[f(x), x] = 0 \quad \text{for all} \quad x, y \in \lambda.$

This implies that $\lambda[f(x), x]R\lambda[f(x), x] = (0)$ for all $x \in \lambda$. Hence the semiprimeness of R forces that $\lambda[f(x), x] = (0)$ for all $x \in \lambda$. □

2.5. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x \circ y) \pm (x \circ y) = 0$ for all $x, y \in \lambda$, then $\lambda[f(x), x] = (0)$ for all $x \in \lambda$.

Proof. By hypothesis, we have

(2.27) $\quad G(x \circ y) = 0 \quad \text{for all} \quad x, y \in \lambda.$

Replacing y by yx in (2.27) and using (2.27), we obtain

(2.28)
$$0 = G(x \circ yx) = G((x \circ y)x) = G(x \circ y)x + (x \circ y)f(x) = (x \circ y)f(x) \quad \text{for all} \quad x, y \in \lambda.$$

This implies that

(2.29) $\quad (x \circ y)f(x) = 0 \quad \text{for all} \quad x, y \in \lambda.$

Substituting $f(x)y$ for y in (2.29) and using (2.29), we obtain

(2.30) $\quad 0 = (x \circ f(x)y)f(x) = f(x)(x \circ y)f(x) + [x, f(x)]yf(x) \quad \text{for all} \quad x, y \in \lambda.$

This implies that

(2.31) $\quad [x, f(x)]yf(x) = 0 \quad \text{for all} \quad x, y \in \lambda.$

Replace y by yx in (2.31), to get

(2.32) $\quad [x, f(x)]yxf(x) = 0 \quad \text{for all} \quad x, y \in \lambda.$

Right multiplying (2.31) by x and then subtracting from (2.32), we obtain

(2.33) $\quad [x, f(x)]y[f(x), x] = 0 \quad \text{for all} \quad x, y \in \lambda.$

Since λ is a left ideal of R, it follows that $\lambda[f(x), x]R\lambda[f(x), x] = (0)$ for all $x \in \lambda$. Semiprimeness of R yields that $\lambda[f(x), x] = (0)$ for all $x \in \lambda$. □

2.6. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x, y]) \pm [x, y] \in Z(R)$ for all $x, y \in \lambda$, then one of the following holds:

(1) $\lambda[f(x), x] = (0)$ for all $x \in \lambda$;
(2) $\lambda[\lambda, f(Z)] = (0)$.

Proof. By hypothesis, we have $G([x, y]) \in Z(R)$ for all $x, y \in \lambda$. If $G([x, y]) = 0$ for all $x, y \in \lambda$, then by Theorem 2.4, $\lambda[f(x), x] = (0)$ for all $x \in \lambda$, as desired. Assume that there exist some $x, y \in \lambda$ such that $0 \neq G([x, y]) \in Z(R)$. This gives $Z(R) \neq (0)$. Let $z \in Z(R)$. Replacing y by yz in our hypothesis, we have

(2.34) $\quad G([x, y]z) = G([x, y])z + [x, y]f(z) = G([x, y])z + [x, y]f(z) \in Z(R),$

which implies $[x, y]f(z) \in Z(R)$ for all $x, y \in \lambda$. Thus $0 = [[x, y]f(z), r]$ for all $x, y \in \lambda$ and $r \in R$. Replacing x with yx, we get $0 = [[yx, y]f(z), r] = [y[x, y]f(z), r] = [y, r][x, y]f(z)$, Since $[x, y]f(z) \in Z(R)$ for all $x, y \in \lambda$. Replacing r with sr, we get $0 = [y, sr][x, y]f(z) = s[y, r][x, y]f(z) + [y, s]r[x, y]f(z) = [y, s]r[x, y]f(z)$ for all $x, y \in \lambda$ and $r, s \in R$ and hence

$(0) = [y, x]f(z)R[x, y]f(z)$ for all $x, y \in \lambda$. Since R is semiprime, above relation yields $0 = [x, y]f(z)$ for all $x, y \in \lambda$. Replacing y with $f(z)y$, we obtain $0 = [x, f(z)y]f(z) = f(z)[x, y]f(z) + [x, f(z)]yf(z) = [x, f(z)]yf(z)$and hence $(0) = y[x, f(z)]Ry[x, f(z)]$ for all $x, y \in \lambda$. Semiprimeness of R yields $\lambda[\lambda, f(Z)] = (0)$. □

2.7. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x \circ y) \pm (x \circ y) \in Z(R)$ for all $x, y \in \lambda$, then one of the following holds:

(1) $\lambda[f(x), x] = (0)$ for all $x \in \lambda$;

(2) $\lambda[\lambda, f(Z)] = (0)$.

Proof. By hypothesis, we have $G(x \circ y) \in Z(R)$ for all $x, y \in \lambda$. If $G(x \circ y) = 0$ for all $x, y \in \lambda$, then by Theorem 2.5, $\lambda[f(x), x] = (0)$ for all $x \in \lambda$, as desired. Assume that there exist some $x, y \in \lambda$ such that $0 \neq G(x \circ y) \in Z(R)$. This gives $Z(R) \neq (0)$. Let $z \in Z(R)$. Substituting yz for y in our hypothesis, we have

$$G(x \circ yz) = G(x \circ y)z + (x \circ y)f(z) = (x \circ y)f(z) \in Z(R). \tag{2.35}$$

This implies that $(x \circ y)f(z) \in Z(R)$ for all $x, y \in \lambda$ and hence

$$[(x \circ y)f(z), r] = 0 \quad \text{for all} \quad x, y \in \lambda, \quad \text{for all} \quad r \in R. \tag{2.36}$$

Replacing x by yx in (2.36) and then using the fact that $(x \circ y)f(z) \in Z(R)$ for all $x, y \in \lambda$, we get

$$0 = [y(x \circ y)f(z), r] = [y, r](x \circ y)f(z) \quad \text{for all} \quad x, y \in \lambda, \tag{2.37}$$

that is

$$[y, r](x \circ y)f(z) = 0 \quad \text{for all} \quad x, y \in \lambda, \quad \text{for all} \quad r \in R. \tag{2.38}$$

Substituting sx for x in (2.38) and using $(x \circ y)f(z) \in Z(R)$ for all $x, y \in \lambda$, we obtain

$$\begin{aligned} 0 &= [y, r](sx \circ y)f(z) = [y, r]s(x \circ y)f(z) - [y, r][s, y]xf(z) \\ &= [y, r](x \circ y)f(z)s + [r, y][s, y]xf(z) \quad \text{for all} \quad x, y \in \lambda, \quad \text{for all} \quad r, s \in R. \end{aligned} \tag{2.39}$$

Using (2.38), the above relation yields that

$$[r, y][s, y]xf(z) = 0 \quad \text{for all} \quad x, y \in \lambda, \quad \text{for all} \quad r, s \in R. \tag{2.40}$$

Replacing r with rt and using (2.40) we have

$$[r, y]t[s, y]xf(z) = 0 \quad \text{for all} \quad x, y \in \lambda, \quad \text{for all} \quad r, s, t \in R. \tag{2.41}$$

In the same manner, replacing s with sp, we obtain

$$[r, y]t[s, y]pxf(z) = 0 \quad \text{for all} \quad x, y \in \lambda, \quad \text{for all} \quad r, s, t, p \in R. \tag{2.42}$$

Now replacing x with xy and right multiplying (2.42) by y respectively, and then subtract one from another to get

$$[r, y]t[s, y]px[f(z), y] = 0 \quad \text{for all} \quad x, y \in \lambda, \quad \text{for all} \quad r, s, t, p \in R. \tag{2.43}$$

In particular, we have

$$x[f(z), y]Rx[f(z), y]Rx[f(z), y] = (0) \quad \text{for all} \quad x, y \in \lambda, \tag{2.44}$$

that is $(x[f(z), y]R)^3 = (0)$ for all $x, y \in \lambda$. Since a semiprime ring contains no nonzero nilpotent left ideals (see [2]), it follows that $x[f(z), y]R = (0)$, that is $x[f(z), y] = 0$ for all $x, y \in \lambda$ and $z \in Z(R)$. Thus we have $\lambda[\lambda, f(Z)] = (0)$. □

2.8. Corollary. Let R be a semiprime ring and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x, y]) \pm [x, y] \in Z(R)$ for all $x, y \in R$ or $F(x \circ y) \pm (x \circ y) \in Z(R)$ for all $x, y \in R$, then either f is commuting on R or $f : Z(R) \to Z(R)$.

2.9. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x,y]) \pm [F(x),y] \in Z(R)$ for all $x,y \in \lambda$, then one of the following holds:

(1) $\lambda[f(x),x] = (0)$ for all $x \in \lambda$;
(2) $\lambda[\lambda, f(Z)] = (0)$.

Proof. By our hypothesis, we have

$$F([x,y]) \pm [F(x),y] = 0 \quad \text{for all} \quad x,y \in \lambda. \tag{2.45}$$

Then replacing y by yx in (2.45), we get

$$\begin{aligned} 0 &= F([x,yx]) \pm [F(x),yx] = F([x,y]x) \pm ([F(x),y]x + y[F(x),x]) \\ &= F([x,y])x + [x,y]f(x) \pm ([F(x),y]x + y[F(x),x]) \\ &\quad \text{for all} \quad x,y \in \lambda. \end{aligned} \tag{2.46}$$

Using (2.45) in the above relation, we obtain

$$[x,y]f(x) \pm y[F(x),x] = 0 \quad \text{for all} \quad x,y \in \lambda. \tag{2.47}$$

Substituting $f(x)y$ for y in (2.47), we get

$$f(x)[x,y]f(x) + [x,f(x)]yf(x) \pm f(x)y[F(x),x] = 0 \quad \text{for all} \quad x,y \in \lambda. \tag{2.48}$$

Left multiplying (2.47) by $f(x)$ and then comparing with (2.48), we get

$$[x,f(x)]yf(x) = 0 \quad \text{for all} \quad x,y \in \lambda. \tag{2.49}$$

Then by similar argument as in the proof of Theorem 2.4, we have $\lambda[f(x),x] = (0)$ for all $x \in \lambda$.
Next, we assume that there exist some $x,y \in \lambda$ such that $0 \neq F([x,y]) \pm [F(x),y] \in Z(R)$. This implies that $Z(R) \neq (0)$. Let $z \in Z(R)$. Substituting y by yz in our hypothesis, we have

$$\begin{aligned} &F([x,y]z) \pm [F(x),y]z = F([x,y])z + [x,y]f(z) \pm [F(x),y]z \\ &= (F([x,y]) \pm [F(x),y])z + [x,y]f(z) \in Z(R), \end{aligned} \tag{2.50}$$

which implies that $[x,y]f(z) \in Z(R)$ for all $x,y \in \lambda$. Then by the same argument as in the proof of Theorem 2.6, we conclude that $\lambda[\lambda, f(Z)] = (0)$. □

2.10. Theorem. Let R be a semiprime ring, λ be a nonzero left ideal of R and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x \circ y) \pm (F(x) \circ y) \in Z(R)$ for all $x,y \in \lambda$, then one of the following holds:

(1) $\lambda[f(x),x] = (0)$ for all $x \in \lambda$;
(2) $\lambda[\lambda, f(Z)] = (0)$.

Proof. By hypothesis, we have

$$F(x \circ y) \pm (F(x) \circ y) = 0 \quad \text{for all} \quad x,y \in \lambda. \tag{2.51}$$

Then replacing y by yx in (2.51), we have

$$\begin{aligned} 0 &= F(x \circ yx) \pm (F(x) \circ yx) = F((x \circ y)x) \pm ((F(x) \circ y)x - y[F(x),x]) \\ &= F(x \circ y)x + (x \circ y)f(x) \pm ((F(x) \circ y)x - y[F(x),x]) \quad \text{for all} \quad x,y \in \lambda. \end{aligned} \tag{2.52}$$

Using (2.51) in the above relation, we get

$$(x \circ y)f(x) \mp y[F(x),x] = 0 \quad \text{for all} \quad x,y \in \lambda. \tag{2.53}$$

Substituting $f(x)y$ for y in (2.53), we have

$$f(x)(x \circ y)f(x) + [x,f(x)]yf(x) \mp f(x)y[F(x),x] = 0 \quad \text{for all} \quad x,y \in \lambda. \tag{2.54}$$

Left multiplying (2.53) by $f(x)$ and then subtracting from (2.54), we obtain

$$[x,f(x)]yf(x) = 0 \quad \text{for all} \quad x,y \in \lambda. \tag{2.55}$$

Then by similar argument of Theorem 2.4, $\lambda[f(x),x]=(0)$ for all $x\in\lambda$.
Next, assume that there exist some $x,y\in\lambda$ such that $0\neq F(x\circ y)\pm(F(x)\circ y)\in Z(R)$. This gives $Z(R)\neq(0)$. Let $z\in Z(R)$. Substituting yz for y in our hypothesis, we have

$$\begin{aligned}&F((x\circ y)z)\pm(F(x)\circ y)z=F(x\circ y)z+(x\circ y)f(z)\pm(F(x)\circ y)z\\&=(F(x\circ y)\pm F(x)\circ y)z+(x\circ y)f(z)\in Z(R).\end{aligned}\tag{2.56}$$

This implies that $(x\circ y)f(z)\in Z(R)$ for all $x,y\in\lambda$ and hence

$$[(x\circ y)f(z),r]=0 \text{ for all } x,y\in\lambda, \text{ for all } r\in R.\tag{2.57}$$

Then by the same argument as in the proof of Theorem 2.7, we get $\lambda[\lambda,f(Z)]=(0)$, as desired. □

2.11. Corollary. Let R be a semiprime ring and $F:R\to R$ be a multiplicative (generalized)-derivation associated with the map $f:R\to R$. If $F([x,y])\pm[F(x),y]\in Z(R)$ for all $x,y\in R$ or $F(x\circ y)\pm(F(x)\circ y)\in Z(R)$ for all $x,y\in R$, then either f is commuting on R or $f:Z(R)\to Z(R)$.

2.12. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F,G:R\to R$ are multiplicative (generalized)-derivations associated with the maps $f,g:R\to R$. If $[F(x),y]\pm[G(y),x]\in Z(R)$ for all $x,y\in\lambda$, then one of the following holds:

(1) $\lambda[g(x),x]=(0)$ for all $x\in\lambda$;
(2) $\lambda[\lambda,g(Z)]=(0)$.

Proof. By hypothesis, we have $[F(x),y]\pm[G(y),x]\in Z(R)$ for all $x,y\in\lambda$. If

$$[F(x),y]\pm[G(y),x]=0 \text{ for all } x,y\in\lambda,\tag{2.58}$$

then replacing y by yx in (2.58), we get

$$\begin{aligned}&0=[F(x),yx]\pm[G(yx),x]=[F(x),y]x+y[F(x),x]\pm([G(y),x]x+[yg(x),x])\\&=([F(x),y]\pm[G(y),x])x+y[F(x),x]\pm[yg(x),x]\\&\quad\text{for all } x,y\in\lambda.\end{aligned}\tag{2.59}$$

Using (2.58) in the above relation, we obtain

$$y[F(x),x]\pm[yg(x),x]=0 \text{ for all } x,y\in\lambda.\tag{2.60}$$

Substituting $g(x)y$ for y in (2.60), we get

$$g(x)y[F(x),x]\pm g(x)[yg(x),x]\pm[g(x),x]yg(x)=0 \text{ for all } x,y\in\lambda.\tag{2.61}$$

Left multiplying (2.60) by $g(x)$ and then comparing with (2.61), we get

$$[g(x),x]yg(x)=0 \text{ for all } x,y\in\lambda.\tag{2.62}$$

This is the same as (2.24) in Theorem 2.4, we obtain $\lambda[g(x),x]=(0)$.
Next, we assume that there exist some $x,y\in\lambda$ such that $0\neq[F(x),y]\pm[G(y),x]\in Z(R)$. This implies that $Z(R)\neq(0)$. Let $z\in Z(R)$. Substituting y by yz in our hypothesis, we have

$$\begin{aligned}&[F(x),yz]\pm[G(yz),x]=[F(x),y]z\pm[G(y),x]z\\&+[yg(z),x]=([F(x),y]\pm[G(y),x])z\pm[yg(z),x]\in Z(R),\end{aligned}\tag{2.63}$$

For any $r\in R$, this implies that

$$[[yg(z),x],r]=0 \text{ for all } x,y\in\lambda.\tag{2.64}$$

Replacing y by wy in the above expression and using it, we get

$$[w,r][yg(z),x]=[w,x][yg(z),r]+[[w,x],r]yg(z)=0 \text{ for all } x,y,w\in\lambda, \text{ for all } r\in R.\tag{2.65}$$

Taking $x = w$ in (2.65), we obtain

(2.66) $\quad [w, r][yg(z), w] = 0$ for all $y, w \in \lambda$, for all $r \in R$.

Replacing r by $yg(z)r$ in the above relation, we get

(2.67) $\quad [yg(z), w]r[yg(z), w] = 0$ for all $y, w \in \lambda$, for all $r \in R$.

Semiprimeness of R yields that

(2.68) $\quad [yg(z), w] = 0$ for all $y, w \in \lambda$.

Substituting $g(z)y$ for y in (2.68), we obtain

(2.69) $\quad [g(z)yg(z), w] = 0$ for all $y, w \in \lambda$.

This implies that

(2.70) $\quad g(z)yg(z)w - wg(z)yg(z) = 0$ for all $y, w \in \lambda$.

Replacing y by $yg(z)x$ in the above expression, we have

(2.71) $\quad g(z)yg(z)xg(z)w - wg(z)yg(z)xg(z) = 0$ for all $x, y, w \in \lambda$.

Using (2.70), we get

(2.72) $\quad g(z)y[g(z), x]wg(z) = 0$ for all $x, y, w \in \lambda$.

This implies that $(\lambda[\lambda, g(z)])^3 = (0)$ for any $z \in Z(R)$. Since a semiprime ring contains no nonzero nilpotent left ideals (see [2]), it follows that $\lambda[\lambda, g(z)] = (0)$. □

Using the similar arguments and taking $G = F$ or $G = -F$ in Theorem 2.12, one can prove the following theorem:

2.13. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ are multiplicative (generalized)-derivations associated with the maps $f : R \to R$. If $[F(x), y] \pm [F(y), x] \in Z(R)$ for all $x, y \in \lambda$, then one of the following holds:

(1) $\lambda[f(x), x] = (0)$ for all $x \in \lambda$;
(2) $\lambda[\lambda, f(Z)] = (0)$.

2.14. Corollary. Let R be a semiprime ring and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $[F(x), y] \pm [F(y), x] \in Z(R)$ for all $x, y \in R$, then either f is commuting on R or $f : Z(R) \to Z(R)$.

2.15. Theorem. Let R be a semiprime ring with $Z(R) \neq (0)$, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x, y]) \pm [F(x), F(y)] = 0$ for all $x, y \in \lambda$, then $\lambda[\lambda, f(Z)] = (0)$.

Proof. Suppose that

(2.73) $\quad F([x, y]) \pm [F(x), F(y)] = 0$ for all $x, y \in \lambda$.

Since $Z(R) \neq (0)$, replacing y by yz in (2.73), where $z \in Z(R)$, we get

$$\begin{aligned}
0 &= F([x, yz]) \pm [F(x), F(yz)] = F([x, y]z) \pm ([F(x), y]z + y[F(x), f(z)]) \\
&\quad +[F(x), y]f(z) = F([x, y])z + [x, y]f(z) \pm ([F(x), f(y)]z + y[F(x), f(z)]) \\
&\quad +[F(x), y]f(z) = [x, y]f(z) + y[F(x), f(z)] + [F(x), y]f(z) \\
&\quad \text{for all } x, y \in \lambda.
\end{aligned} \tag{2.74}$$

Using (2.73) in the above relation, we obtain

(2.75) $\quad [x, y]f(z) \pm y[F(x), f(z)] + [F(x), y]f(z) = 0$ for all $x, y \in \lambda$.

Replacing ry for y in (2.75), we get

$$\begin{aligned}
&r[x, y]f(z) + [x, r]yf(z) \pm ry[F(x), f(z)] + r[F(x), y]f(z) + [F(x), r]yf(z) = 0 \\
&\quad \text{for all } x, y \in \lambda, \text{ for all } r \in R.
\end{aligned} \tag{2.76}$$

Left multiplying (2.75) by r and then subtracting from (2.76), we get

(2.77) $\quad [x,r]yf(z) \pm [F(x),r]yf(z) = 0$ for all $x, y \in \lambda$, for all $r \in R$.

Replacing x by xz in (2.77), where $z \in Z(R)$, we have

(2.78)
$$z[x,r]yf(z) \pm z[F(x),r]yf(z) + [xf(z),r]yf(z) = 0 \text{ for all } x, y \in \lambda, \text{ for all } r \in R.$$

Using (2.77), we get

(2.79) $\quad [xf(z),r]yf(z) = 0$ for all $x, y \in \lambda$, for all $r \in R$.

Replacing r by sr in the above relation and using it, we get

(2.80) $\quad [xf(z),s]ryf(z) = 0$ for all $x, y \in \lambda$, for all $r \in R$.

Substituting y by ty in (2.80), we obtain

(2.81) $\quad [xf(z),s]rtyf(z) = 0$ for all $x, y \in \lambda$, for all $r, t \in R$

Right multiplying (2.80) by t and then subtracting from (2.81), we get

(2.82) $\quad [xf(z),s]r[yf(z),t] = 0$ for all $x, y \in \lambda$, for all $r, s, t \in R$.

Semiprimeness of R yields that $[xf(z),r] = 0$ for all $x \in \lambda$ and $r \in R$. Replacing x by $f(z)x$ in the above relation, we get

(2.83) $\quad [f(z)xf(z),r] = 0$ for all $x \in \lambda$, for all $r \in R$,

that is

(2.84) $\quad f(z)xf(z)r - rf(z)xf(z) = 0$ for all $x \in \lambda$, for all $r \in R$.

Replacing x by $xf(z)y$ in (2.84), we obtain

(2.85) $\quad f(z)xf(z)yf(z)r - rf(z)xf(z)yf(z) = 0$ for all $x, y \in \lambda$, for all $r \in R$.

Using (2.84) in the above relation, we get

(2.86) $\quad f(z)xrf(z)yf(z) - f(z)xf(z)ryf(z) = 0$ for all $x, y \in \lambda$, for all $r \in R$.

We find that $f(z)x[f(z),r]yf(z) = 0$ for all $x, y \in \lambda$, $r \in R$. Which implies that $(\lambda[\lambda, f(z)])^3 = (0)$ for any $z \in Z(R)$. Since a semiprime ring contains no nonzero nilpotent left ideals (see [2]), we obtain $\lambda[\lambda, f(z)] = (0)$ for any $z \in Z(R)$. □

2.16. Theorem. Let R be a semiprime ring with $Z(R) \neq (0)$, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(x \circ y) \pm (F(x) \circ F(y)) = 0$ for all $x, y \in \lambda$, then $\lambda[\lambda, f(Z)] = (0)$.

Proof. By hypothesis, we have

(2.87) $\quad F(x \circ y) \pm F(x) \circ F(y) = 0$ for all $x, y \in \lambda$.

Since $Z(R) \neq (0)$. Let $z \in Z(R)$. Replacing y by yz in (2.87), we have

(2.88)
$$\begin{aligned} 0 = F(x \circ yz) \pm F(x) \circ F(yz) = F((x \circ y)z) \pm (F(x) \circ y)z + (F(x) \circ y)f(z) \\ -y[F(x), f(z)] = (x \circ y)f(z) \pm ((F(x) \circ y)f(z) - y[F(x), f(z)]) \text{ for all } x, y \in \lambda. \end{aligned}$$

Using (2.87) in the above relation, we get

(2.89) $\quad (x \circ y)f(z) \mp [F(x),y]f(z) = 0$ for all $x, y \in \lambda$.

Substituting ry for y in (2.89), we obtain

(2.90) $\quad r(x \circ y)f(z) + [x,r]yf(z) \mp r[F(x),y]f(z) + [F(x),r]yf(z) = 0$ for all $x, y \in \lambda$.

Left multiplying (2.89) by r and then subtracting from (2.90), we get

(2.91) $\quad [x,r]yf(z) \mp [F(x),r]yf(z) = 0$ for all $x,y \in \lambda$.

Arguing in the similar manner as in the proof of Theorem 2.15, we get the result. □

2.17. Corollary. Let R be a semiprime ring with $Z(R) \neq (0)$ and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F([x,y]) \pm [F(x), F(y)] = 0$ or $F(x \circ y) \pm (F(x) \circ F(y)) = 0$ for all $x,y \in R$, then $f : Z(R) \to Z(R)$.

2.18. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(xy) \pm [x,y] \in Z(R)$ for all $x,y \in \lambda$, then $\lambda \subseteq Z(R)$ for all $x \in \lambda$ and $F(xy) \in Z(R)$ for all $x,y \in \lambda$.

Proof. By hypothesis, we have

(2.92) $\quad F(xy) \pm [x,y] = G(xy) \mp yx \in Z(R)$

for all $x,y \in \lambda$. By [10, Theorem 2.11], we obtain that $x[x,\lambda] \subseteq Z(R)$ for all $x \in \lambda$. Replacing y with xy in (2.92) and then using the fact $x[x,\lambda] \subseteq Z(R)$ for all $x \in \lambda$, we get $F(x^2y) \in Z(R)$ for all $x,y \in \lambda$. Now we put $x = x^2$ in (2.92) and then obtain

(2.93) $\quad F(x^2y) \pm x[x,y] \pm [x,y]x \in Z(R)$ for all $x,y \in \lambda$.

This implies $[x,y]x \in Z(R)$ for all $x,y \in \lambda$. Therefore we can write that $x[y,x] - [y,x]x \in Z(R)$ for all $x \in \lambda$, that gives $[y,x]_3 = [[[y,x],x],x] = 0$ for all $x,y \in \lambda$. Then by [14, Theorem 2], we get $\lambda \subseteq Z(R)$. Thus our hypothesis reduces to $F(xy) \in Z(R)$ for all $x,y \in \lambda$. □

2.19. Theorem. Let R be a semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If $F(xy) \pm (x \circ y) \in Z(R)$ for all $x,y \in \lambda$, then $\lambda \subseteq Z(R)$ and $F(xy) \in Z(R)$ for all $x,y \in \lambda$.

Proof. By hypothesis, we have

(2.94) $\quad F(xy) \pm (x \circ y) = G(xy) \pm yx \in Z(R)$

for all $x,y \in \lambda$. By [10, Theorem 2.11], we obtain that $x[x,\lambda] \subseteq Z(R)$ for all $x \in \lambda$. Now replacing y with xy in (2.94) and then using the fact $x[x,\lambda] \subseteq Z(R)$ for all $x \in \lambda$, we get $F(x^2y) \pm 2xyx \in Z(R)$ for all $x,y \in \lambda$. Now we put $x = x^2$ in (2.94) and then obtain

(2.95) $\quad F(x^2y) \pm (x^2 \circ y) \in Z(R)$

that is

(2.96) $\quad F(x^2y) \pm (2xyx + x[x,y] + [y,x]x) \in Z(R)$ for all $x,y \in \lambda$.

This implies $[x,y]x \in Z(R)$ for all $x,y \in \lambda$. Therefore we can write that $x[y,x] - [y,x]x \in Z(R)$ for all $x \in \lambda$, which gives $[y,x]_3 = [[[y,x],x],x] = 0$ for all $x,y \in \lambda$. Then by [14, Theorem 2], we get $\lambda \subseteq Z(R)$. Thus our hypothesis gives $F(xy) \in Z(R)$ for all $x,y \in \lambda$. □

2.20. Corollary. Let R be a semiprime ring and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $f : R \to R$. If

(1) $F(xy) \pm [x,y] \in Z(R)$ for all $x,y \in R$;
(2) $F(xy) \pm (x \circ y) \in Z(R)$ for all $x,y \in R$;

then R is commutative.

3. Examples

The following examples demonstrate that the restrictions in the hypothesis of the results are not superfluous.

3.1. Example. Consider $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of all integers. Since $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$, so R is not semiprime ring. We define maps $F, f : R \to R$, by $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $f \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then F is a multiplicative (generalized)-derivation associated with the map f.

It is very easy to verify that R satisfies (i) $F(x)F(y) \pm [x, y] \in Z(R)$; (ii) $F(x)F(y) \pm (x \circ y) \in Z(R)$, (iii) $F(xy) \pm [x, y] \in Z(R)$; (iv) $F(xy) \pm (x \circ y) \in Z(R)$; Since R is not commutative, the hypothesis of semiprimeness in Corollary 2.3 and Corollary 2.20 can not be omitted.

3.2. Example. Consider $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}$. Note that R is not a semiprime ring. Define maps $F, f : R \to R$ by $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ and $f \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^2 & a^2 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. Then it is verified that F is a multiplicative (generalized)-derivation associated with the map f. It is easy to see that $F([x, y]) \pm [x, y] \in Z(R)$ and $F(x \circ y) \pm (x \circ y) \in Z(R)$ for all $x, y \in R$. But neither f is commuting on R nor $f : Z(R) \to Z(R)$. Hence R to be semiprime in the hypothesis of Corollary 2.8 is essential.

3.3. Example. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{S} \right\}$, where S is any ring. Note that R is not a semiprime ring. Define maps F and $f : R \to R$ by $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $f \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then F is a multiplicative generalized derivation associated with the map f. It is easy to see that (i) $[F(x), y] \pm [F(y), x] \in Z(R)$ and (ii)$F([x, y]) \pm [F(x), y] = 0$ or $F(x \circ y) \pm (F(x) \circ y) = 0$ for all $x, y \in R$. But neither f is commuting nor $f : Z(R) \to Z(R)$. Hence R to be semiprime in the hypothesis of Corollary 2.11 and Corollary 2.14 are essential.

Moreover, it satisfies $F([x, y]) \pm [F(x), F(y)] = 0$ or $F(x \circ y) \pm (F(x) \circ F(y)) = 0$ for all $x, y \in R$. But f does not map $Z(R)$ to $Z(R)$. Hence R to be semiprime in the hypothesis of Corollary 2.17 is essential.

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M-Cofaithful modules and correspondences of closed submodules with coclosed submodules

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Abstract

In this paper we introduce and investigate M-cofaithful modules. A module $N \in \sigma[M]$ is called M-cofaithful if for every $o \neq f \in Hom_R(N, X)$ with $X \in \sigma[M]$, $Hom_R(X, M)f \neq 0$. We show that if N is an M-cofaithful weak supplemented module and $Hom_R(N, M)$ a noetherian S-module, then there exists an order-preserving correspondence between the cocolsed R-submodules of N and the closed S-submodules of $Hom_R(N, M)$, where $S = End_R(M)$. Some applications are: (1) the connection between M's being a lifting module and $End_R(M)$'s being an extending ring; (2) the equality between the hollow dimension of a quasi-injective coretractable module M and the uniform dimension of $End_R(M)$.

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1. Introduction

Throughout this paper, R will denote an arbitrary associative ring with identity, M and N unitary right R-modules with $U = Hom_R(N, M)$ the set of R-homomorphisms of N in M and $S = End_R(M)$ the ring of all R-endomorphisms of M; U is then a left S-module. By $\sigma[M]$ we mean the full subcategory of Mod-R whose objects are submodules of M-generated modules.

Following [5], a module $N \in \sigma[M]$ is said to be *M-faithful* if for every $0 \neq f \in Hom_R(X, N)$ with $X \in \sigma[M]$, $fHom_R(M, X) \neq 0$. When M is itself M-faithful, M is called a *self-faithful* module. Self-faithful modules have been studied by some authors

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(see, for example, [5, 6, 7, 8]). It is of obvious interest to investigate the dual notion of M-faithful modules. We call a right R-module $N \in \sigma[M]$ *M-cofaithful* if for every $0 \neq f \in Hom_R(N, X)$ with $X \in \sigma[M]$, $Hom_R(X, M)f \neq 0$. When M is itself M-cofaithful, M is called a *self-cofaithful* module. Example of self-cofaithful modules is quasi-injective coretractable modules (Theorem 3.1). In this paper, we investigate M-cofaithful modules.

It is known that there exists a correspondence between the closed submodules of a suitably restricted module and the closed one-side ideals of its endomorphism ring. Such a correspondence is known to hold for semisimple modules, for free modules (see [2]), and for nonsingular modules M when $End_R(E(M))$ is the maximal right quotient ring of $End_R(M)$ (see [13]), hence in particular, for nonsingular retractable modules (see [9]). Some properties of the endomorphism rings of modules, such as being Baer, extending, etc., were then obtained by means of the above lattice isomorphism. Zelmanowitz showed in [12, Theorem 1.2] that when N is an M-faithful R-module, then there exists an order-preserving correspondence between the closed R-submodules of N and the closed S-submodules of $Hom_R(M, N)$, where $S = End_R(M)$. In this paper, we give conditions under which there exists a correspondence between the coclosed R-submodules of an M-cofaithful module N and the closed S-submodules of $Hom_R(N, M)$.

In section 2, we characterize M-cofaithful modules (Proposition 2.1) and study some properties of M-cofaithful modules. For an M-cofaithful module N, we show that $u.dim({}_SU) = h.dim(N_R)$, where $U = Hom_R(N, M)$ (Theorem 2.12). We show that there is a correspondence between the coclosed R-submodules of an M-cofaithful weak supplemented module N and the closed S-submodules of $Hom_R(N, M)$ whenever $Hom_R(N, M)$ is a noetherian S-module. (Theorem 2.13). This result is used in proving that if $Hom_R(N, M)$ is a noetherian S-module, then an M-cofaithful M-cogenerated amply supplemented module N is a lifting right R-module if and only if $Hom_R(N, M)$ is a left extending S-module, where $S = End_R(M)$ (Theorem 2.15). In section 3, we show that M-coretractability characterizes M-cofaithfulness for some important families of modules and conclude that if either (i) M is an amply supplemented quasi-injective coretractable module and S is noetherian, or (ii) M is an amply supplemented $\sum$-self-cogenerator module and S is noetherian, then:

(a) There exist mutually inverse lattice correspondences between the coclosed submodules of M and the closed left ideals of $S = End_R(M)$.

(b) M is a lifting module if and only if S is a left extending ring.

We will use the notation $N \leq_e M$ to indicate that N is essential in M (i.e., $N \cap L \neq 0\ \forall 0 \neq L \leq M$); $N \ll M$ means that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). For $K \leq N_R$ and $A \leq {}_SU$ we denote:
$An(K) = \{f \in Hom_R(N, M) \mid f(K) = 0\} (\simeq Hom_R(N/K, M))$,
$Ke(A) = \bigcap\{Keg \mid g \in A\}$.

A submodule N of M is called a *closed* submodule of M if it is not contained as a proper essential submodule of any other submodule of M. We recall that L is a *cosmall submodule of K in M* (denoted by $L \overset{cs}{\hookrightarrow} K$ in M) if $K/L \ll M/L$. Recall that a submodule L of M is called *coclosed* if L has no proper cosmall submodule (denoted by $L \overset{cc}{\hookrightarrow} M$). A *coclosure* of a submodule L of M (denoted by $\widetilde{L}$) is a cosmall submodule of L in M which is also a coclosed submodule of M.

If N and L are submodules of the module M, then N is called a *supplement* (*weak supplement*) of L, if $N + L = M$ and $N \cap L \ll N$ ($N \cap L \ll M$). M is called *supplemented* (*weakly supplemented*) if each of its submodules has a supplement (weak supplement) in M. M is called *amply supplemented*, if for all submodules N and L of M with $N + L = M$, N contains a supplement of L in M.

2. M-Cofaithful Modules

A module $N \in \sigma[M]$ is called *M-cofaithful* if for every $0 \neq f \in Hom_R(N, X)$ with $X \in \sigma[M]$, $Hom_R(X, M)f \neq 0$.

2.1. Proposition. *An R-module N is M-cofaithful if and only if $Hom_R(N, Ke(Hom_R(X, M))) = 0$ for every $X \in \sigma[M]$.*

Proof. Let $h : N \to Ke(Hom_R(X, M))$ be a nonzero homomorphism. Composing with the natural inclusion map $i : Ke(Hom_R(X, M)) \to X$ we get a nonzero homomorphism $g : N \to X$ such that $\operatorname{Im} g \subseteq Ke(Hom_R(X, M))$. Then for every $f : X \to M$, $\operatorname{Im} g \subseteq Ke(Hom_R(X, M)) \subseteq \ker f$. Thus $fg = 0$ which is a contradiction.

Conversely, let $\forall X \in \sigma[M]$, $Hom_R(N, Ke(Hom_R(X, M))) = 0$ and $0 \neq g : N \to X$ be a nonzero homomorphism. If $Hom_R(X, M)g = 0$, then $\operatorname{Im} g \subseteq Ke(Hom_R(X, M))$. This gives a nonzero homomorphism $h : N \to Ke(Hom_R(X, M))$ which is a contradiction. $\square$

2.2. Proposition. *If N is an M-cofaithful module, then $Hom_R(N, \frac{KeAn(K)}{K}) = 0$ for every $K \leq N$.*

Proof. It is a direct consequence of Proposition 2.1, because $\frac{KeAn(K)}{K} = Ke(Hom_R(\frac{N}{K}, M))$. $\square$

2.3. Proposition. *Let M be an R-module. If M is a cogenerator in $\sigma[M]$, then every $N \in \sigma[M]$ is M-cofaithful.*

Proof. Suppose that M is a cogenerator in $\sigma[M]$. Then for every $X \in \sigma[M]$, X is M-cogenerated. Thus $Ke(Hom_R(X, M)) = 0$. So $Hom_R(N, Ke(Hom_R(X, M))) = 0$ for every $N \in \sigma[M]$. Hence every $N \in \sigma[M]$ is M-cofaithful. $\square$

2.4. Proposition. *Let M be an R-module. Then every generator in $\sigma[M]$ is an M-cofaithful module if and only if every R-module in $\sigma[M]$ is an M-cofaithful module.*

Proof. Let every generator in $\sigma[M]$ is an M-cofaithful module. Suppose that $N \in \sigma[M]$ and $0 \neq f \in Hom_R(N, X)$ is given with $X \in \sigma[M]$. Then there is a generator F and an epimorphism $g : F \to N$. Since F is M-cofaithful, there exists $h \in Hom_R(X, M)$ with $hfg \neq 0$. Thus $hf \neq 0$ and this proves that N is M-cofaithful. The converse is clear. $\square$

2.5. Proposition. *Let $\{N_\alpha \mid \alpha \in I\}$ be a family of M-cofaithful modules. Then $N = \oplus_{\alpha \in I} N_\alpha$ is M-cofaithful.*

Proof. Let $0 \neq f \in Hom_R(N, X)$ for $X \in \sigma[M]$. Since N_α is M-cofaithful for any $\alpha \in I$, hence there exists $h_\alpha : X \to M$ such that $h_\alpha f i_\alpha \neq 0$, where $i_\alpha : N_\alpha \to N$ is the natural injection map. Then $h_\alpha f \neq 0$ and so N is M-cofaithful. $\square$

2.6. Proposition. *Let N be an M-cofaithful R-module. Then every supplement submodule of N is M-cofaithful.*

Proof. Let K be a supplement submodule of N and $0 \neq g \in Hom_R(K, X)$ for $X \in \sigma[M]$. Then there exists $L \leq N$ such that $K + L = N$ and $K \cap L \ll K$. Put $X' = g(K \cap L) \ll X$ and let g' denote the composition $N \overset{\pi}{\to} (K + L)/K \cong K/(K \cap L) \overset{g}{\to} X/X'$. Then $0 \neq g' : N \to X/X'$. By assumption, there exists $0 \neq h \in Hom_R(X/X', M)$ with $hg' \neq 0$. Then $hg \neq 0$ and so $Hom_R(X, M)g \neq 0$ because $h\pi' \neq 0$, where $\pi' : X \to X/X'$ denotes the natural map. $\square$

2.7. Corollary. *Let N be an M-cofaithful R-module. Then:*

(i) *Every direct summand of N is M-cofaithful.*

(ii) *Every weak supplement coclosed submodule of N is M-cofaithful.*

Proof. (i) By Proposition 2.6.

(ii) Since every weak supplement coclosed submodule is a supplement submodule, it follows by Proposition 2.6. $\square$

2.8. Lemma. *Let N be an M-cofaithful R-module. Then for every proper submodule K of N, $K \overset{cs}{\hookrightarrow} KeAn(K)$ in N and $KeAn(K) \lneq N$; in particular, $Hom_R(N/K, M) \neq 0$.*

Proof. If $K \lneq N$ and $\pi : N \to N/K$ is the natural epimorphism, then $Hom_R(N/K, M)\pi \neq 0$ since $N/K \in \sigma[M]$. Thus $KeAn(K) \lneq N$. Let $K \leq L \lneq N$, then $KeAn(K) + L \leq KeAn(L) \lneq N$. Therefore $K \overset{cs}{\hookrightarrow} KeAn(K)$ in N. $\square$

2.9. Proposition. *Assume that N is an M-cofaithful R-module. Let $K \leq N$ and L be a weak supplement coclosed submodule of N such that $L \subseteq KeAn(K)$. Then $L \subseteq K$. In particular, if K is a weak supplement coclosed submodule of N, then K is the unique coclosure of $KeAn(K)$ in N.*

Proof. Let $K \leq N$ and L be a weak supplement coclosed submodule of N such that $L \subseteq KeAn(K)$. Suppose that g denotes the composition $L \overset{\subseteq}{\to} KeAn(K) \overset{\pi}{\to} \frac{KeAn(K)}{K}$. Then by Proposition 2.2 and Corollary 2.7, $g = 0$, and so $L \subseteq K$. $\square$

2.10. Proposition. *Let N be an M-cofaithful R-module. Then the following conditions hold:*

(1) *For every finitely generated S-submodule $A \leq {}_SU$, $A \leq_e Hom_R(N/Ke(A), M)$ (equivalently, $A \leq_e AnKe(A)$).*

(2) *Let $L \leq K \leq N$. If $An(K) \leq_e An(L)$, then $L \overset{cs}{\hookrightarrow} K$ in N. The converse holds if $Hom_R(N, M)$ is a noetherian S-module.*

(3) *Let $A \leq B \leq {}_SU$ and $Hom_R(N, M)$ be a noetherian S-module. Then $A \leq_e B$ if and only if $Ke(B) \overset{cs}{\hookrightarrow} Ke(A)$ in N.*

Proof. (1) Let $0 \neq f \in Hom_R(N/Ke(A), M)$. Set $A = Sg_1 + Sg_2 + ... + Sg_k$ with $g_i \in Hom_R(N, M)$. Then $Ke(A) = \bigcap_{i \leq k} Keg_i$. Let $P = \{(f(n + Ke(A)), (\prod_{i=1}^k g_i)(n + Ke(A))) \mid n \in N\}$ and let $\bar{i}_1 : M^{(k)} \to M \oplus M^{(k)} \to \frac{M \oplus M^{(k)}}{P}$ and $\bar{i}_2 : M \to M \oplus M^{(k)} \to \frac{M \oplus M^{(k)}}{P}$ be the canonical maps. We have the following commutative diagram:

$$\begin{array}{ccccc} 0 \longrightarrow & N/Ke(A) & \overset{\prod_{i=1}^k g_i}{\longrightarrow} & M^{(k)} \\ & \downarrow f & & \downarrow -\bar{i}_1 \\ & M & \overset{\bar{i}_2}{\longrightarrow} & (M \oplus M^{(k)})/P. \end{array}$$

Then $0 \neq \bar{i_2} f = -\bar{i_1}(\prod_{i=1}^k g_i) : N/Ke(A) \to (M \oplus M^{(k)})/P$. By hypothesis, there exists $h \in Hom_R(\frac{M \oplus M^{(k)}}{P}, M)$ with $h\bar{i_2}f \neq 0$. We may consider $h(-\bar{i_1})$ as $\sum_{i=1}^k s_i$ for some $s_i \in S$. Thus $0 \neq h\bar{i_2}f = h(-\bar{i_1})(\prod_{i=1}^k g_i) = \sum_{i=1}^k s_i g_i \in A$. Therefore $A \leq_e Hom_R(N/Ke(A), M)$.

(2) Let $An(K) \leq_e An(L)$ for $L \leq K \leq N$. Suppose that $K/L + X/L = N/L$, where $L \leq X \leq N$. If $X \neq N$, then by hypothesis, there exists $0 \neq f \in U$ with $f(X) = 0$. Thus $f(L) = 0$ and so $0 \neq f \in An(L)$. As $An(K) \leq_e An(L)$, there exists $g \in S$ such that $0 \neq gf \in An(K)$. Hence $gf(N) = gf(K + X) = 0$, which is a contradiction. Therefore $L \overset{cs}{\hookrightarrow} K$ in N. Conversely, assume that $L \overset{cs}{\hookrightarrow} K$ in N and let $0 \neq A \leq An(L)$. Then $L \leq Ke(A) \lneq N$ and so $K + Ke(A) \lneq N$. Thus $0 \neq An(K + Ke(A)) = An(K) \cap AnKe(A)$. But $A \leq_e AnKe(A)$ from (1), so $An(K) \cap A \neq 0$. Therefore $An(K) \leq_e An(L)$.

(3) It is clear that A is essential in B if and only if $AnKe(A)$ is essential in $AnKe(B)$, by (1) (because A and B are finitely generated, so (1) can be applied). By using (2), the claimed property holds. $\square$

Recall that a module M is said to have *uniform* (*or Goldie*) *dimension* n, denoted by $u.dim(M) = n$ for some $n \in \mathbb{N}$, if $sup\{k \in \mathbb{N} \mid M$ contains k independent submodules $\} = n$ [4]. A module M is said to have *hollow dimension* n, denoting this by $h.dim(M) = n$ for some $n \in \mathbb{N}$, if $sup\{k \in \mathbb{N} \mid M$ has k coindependent submodules $\} = n$ [3].

2.11. Lemma. *Let $N \in \sigma[M]$ be a nonzero R-module and $K, L \leq N$. If $K + L = N$, then $An(K \cap L) = An(K) + An(L)$.*

Proof. It follows from [1, Lemma 4.9]. □

2.12. Theorem. *Let N be an M-cofaithful module. Then $u.dim({}_SU) = h.dim(N_R)$.*

Proof. Assume first that $Sf_1, Sf_2, ..., Sf_n$ is an independent family of submodules of ${}_SU$ and $0 \neq f_i \in {}_SU$ for all $1 \leq i \leq n$. Since $Sf_i \cap Sf_j = 0$ for any $i \neq j$, and $Sf_i \leq_e AnKe(Sf_i)$ for all $1 \leq i \leq n$, $AnKe(Sf_i) \cap AnKe(Sf_j) = 0$. Thus $An(Ke(Sf_i) + Ke(Sf_j)) = 0$. Since N is M-cofaithful, $Ke(Sf_i) + Ke(Sf_j) = N$. By Lemma 2.11, $An(Ke(Sf_i) \cap Ke(Sf_j)) = AnKe(Sf_i) + AnKe(Sf_j)$ for each $i \neq j$. Let $i, j, k \in \{1, 2, ..., n\}$ be distinct. Since $Sf_i \cap (Sf_j + Sf_k) = 0$ and $Sf_i \cap (Sf_j + Sf_k) \leq_e AnKe(Sf_i) \cap (AnKe(Sf_j) + AnKe(Sf_k))$, hence $0 = AnKe(Sf_i) \cap (AnKe(Sf_j) + AnKe(Sf_k)) = AnKe(Sf_i) \cap An(Ke(Sf_j) \cap Ke(Sf_k)) = An(Ke(Sf_i) + (Ke(Sf_j) \cap Ke(Sf_k)))$. Therefore $Ke(Sf_i) + (Ke(Sf_j) \cap Ke(Sf_k)) = N$. It is easy to see by induction that for every $1 \leq i \leq n$, $Ke(Sf_i) + (\bigcap_{j \neq i} Ke(Sf_j)) = N$. Hence $\{Ke(Sf_i), ..., Ke(Sf_n)\}$ is coindependent. Thus $u.dim({}_SU) \leq h.dim(N_R)$. On the other hand, from [1, Proposition 4.10], $u.dim({}_SU) \geq h.dim(N_R)$ and the proof is completed. □

2.13. Theorem. *Assume that N is an M-cofaithful weak supplemented module and $Hom_R(N, M)$ is a noetherian S-module. Then for every $A \leq^c {}_SU = Hom_R(N, M)$, $Ke(A)$ has a unique coclosure $\widetilde{Ke(A)}$ in N and the maps $K \to An(K)$ and $A \to \widetilde{Ke(A)}$ determine mutually inverse correspondences between the coclosed R-submodules of N and the closed S-submodules of $U = Hom_R(N, M)$.*

Proof. Let $K \overset{cc}{\hookrightarrow} N$ and $An(K) \leq_e A \leq {}_SU$. By Zorn's Lemma, we may assume that A is closed in ${}_SU$. From Proposition 2.10, $Ke(A) \overset{cs}{\hookrightarrow} KeAn(K)$ in N. By Proposition 2.9, $K \overset{cs}{\hookrightarrow} Ke(A)$ in N. Hence $A \subseteq AnKe(A) \subseteq An(K)$. Thus $A = An(K)$; that is, $An(K) \leq^c {}_SU$. Also, $K = \widetilde{KeAn}(K)$.

Assume that $A \leq^c {}_SU$. We show that $Ke(A)$ has a unique coclosure in N. Let $K \overset{cc}{\hookrightarrow} N$ and $K \overset{cs}{\hookrightarrow} Ke(A)$ in N. By using Proposition 2.10, $A \leq_e AnKe(A) \leq_e An(K)$, and so $A = An(K)$. Thus $Ke(A) = KeAn(K)$. Therefore K is a unique coclosure of $Ke(A)$ (by Proposition 2.9). So $A = An(K) = An(\widetilde{Ke(A)})$. □

2.14. Corollary. *Let N be an M-cofaithful module and $Hom_R(N, M)$ be a noetherian S-module. Then, $\widetilde{Ke(A)} = Ke(A)$ for every $A \leq^c {}_SU$ if and only if every $K \overset{cc}{\hookrightarrow} N$ is M-cogenerated.*

Proof. Assume that for every $A \leq^c {}_SU$, $\widetilde{Ke(A)} = Ke(A)$ and let $K \overset{cc}{\hookrightarrow} N$. Then $An(K) \leq^c {}_SU$. From Theorem 2.13 and hypothesis, $K = \widetilde{KeAn}(K) = KeAn(K)$. Thus K is M-cogenerated. Conversely, suppose that every $K \overset{cc}{\hookrightarrow} N$ is M-cogenerated and $A \leq^c {}_SU$. By Theorem 2.13, $A = An(\widetilde{Ke(A)})$. On the other hand, by hypothesis, $\widetilde{Ke(A)} = KeAn(\widetilde{Ke(A)})$. Therefore $\widetilde{Ke(A)} = Ke(A)$. □

Recall that an R-module M is an *extending* module if for every submodule K of M there exists a direct summand L of M such that $K \leq_e L$, or equivalently, every closed submodule of M is a direct summand. A left extending ring is a ring which is a extending module over itself. Dually, a module M is called a *lifting* module if, every submodule N of M can be written in the form $N = K \oplus D$ where K is a direct summand of M and $D \ll M$. By [10, 4.8], M is lifting if and only if it is amply supplemented and its coclosed submodules are direct summands.

2.15. Theorem. *Let N be an M-cofaithful module and $Hom_R(N, M)$ be a noetherian S-module. If N_R is a lifting module, then ${}_SU$ is an extending module; and the converse holds when N is M-cogenerated and amply supplemented.*

Proof. Let N be a lifting module and let $A \leq^c {}_SU$. Then, by Theorem 2.13, $N = \widetilde{Ke(A)} \oplus D$ for some $D \leq N$. Thus $U = An(\widetilde{Ke(A)}) \oplus An(D) = A \oplus An(D)$. Conversely, suppose that N is M-cogenerated and amply supplemented and let ${}_SU$ be an extending module and $K \overset{cc}{\hookrightarrow} N$. Then, by Theorem 2.13 again, $U = An(K) \oplus B$ for some $B \leq {}_SU$. Thus $0 = Ke(U) = KeAn(K) \cap Ke(B) = K \cap Ke(B)$. On the other hand, $N = K + Ke(B)$ since if $K + Ke(B) \lneq N$, then $0 \neq An(K + Ke(B)) = An(K) \cap AnKe(B)$, whence $An(K) \cap B \neq 0$, which is a contradiction. Therefore $N = K \oplus Ke(B)$ and so N is lifting. □

3. Applications to coretractable modules

Recall that an R-module N is called *M-coretractable* if, for any proper submodule K of N, there exists a nonzero homomorphism $f : N \to M$ with $f(K) = 0$, that is, $Hom_R(N/K, M) \neq 0$. An R-module M is called *coretractable* if M is itself M-coretractable [3]. By Lemma 2.8, every M-cofaithful module N is M-coretractable.

3.1. Theorem. *Let M be a quasi-injective R-module. Then $N \in \sigma[M]$ is M-cofaithful if and only if N is M-coretractable.*

Proof. By Lemma 2.8, every M-cofaithful module N is M-coretractable. Conversely, suppose that N is M-coretractable. It suffices to show that for every $X \in \sigma[M]$, $Hom_R(N, Ke(Hom_R(X, M))) = 0$. Assume that there exists a nonzero homomorphism $f : N \to Ke(Hom_R(X, M))$. Then $0 \neq if : N \to X$, where $i : Ke(Hom_R(X, M)) \to X$ is the inclusion map. Since $if \neq 0$, there exists $Z = \mathrm{Im}\,(if) \neq 0$. Now, $Hom_R(Z, M) \neq 0$ because Z is a quotient of N and N is M-coretractable. But every homomorphism $g : Z \to M$ can be extended to a homomorphism $h : X \to M$ because M is quasi-injective and $X \in \sigma[M]$ (by [11, 16.3]). Since $Z \subseteq Ke(Hom_R(X, M)) \subseteq X$, $h(Z) = 0$, which is a contradiction. □

3.2. Corollary. *Let $S = End_R(M)$ be a noetherian ring and M an amply supplemented module with one of the following properties:*

(i) M is a quasi-injective coretractable module;

(ii) M is a $\sum$-self-cogenerator module (that is, any direct sum of copies of M is a self-cogenerator). Then:

(a) There exist mutually inverse lattice correspondences between the coclosed submodules of M and the closed left ideals of $S = End_R(M)$.

(b) M is a lifting module if and only if S is a left extending ring.

Proof. By combining Theorems 2.13, 2.15, 3.1, and in the special case when $N = M$ and $U = S$. □

3.3. Corollary. *Let M be an R-module. If any of the following conditions is satisfied, then the hollow dimension of M is equal to n if and only if the right uniform dimension of S is n:*

(*i*) M *is a quasi-injective coretractable module.*
(*ii*) M *is a* $\sum$*-self-cogenerator.*

Proof. Using Theorems 2.12 and 3.1 in the special case when $N = M$ and $U = S$. □

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On the fine spectrum of the generalized difference operator defined by a double sequential band matrix over the sequence space ℓ_p, $(1 < p < \infty)$

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Abstract

The main purpose of this paper is to determine the fine spectrum with respect to the Goldberg's classification of the operator $B(\widetilde{r}, \widetilde{s})$ defined by a double sequential band matrix over the sequence space ℓ_p, where $1 < p < \infty$. These results are more general than the spectrum of the generalized difference operator $B(r, s)$ over ℓ_p of Bilgiç and Furkan [Nonlinear Anal. **68**(3)(2008), 499–506].

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1. Introduction

Let X and Y be Banach spaces, and $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx, \ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^* f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Given an operator $T \in B(X)$, the set

$$\rho(T) := \{\lambda \in \mathbb{C} : T_\lambda = \lambda I - T \text{ is a bijection}\}$$

is called the *resolvent set* of T and its complement with respect to the complex plain

$$\sigma(T) := \mathbb{C} \backslash \rho(T)$$

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is called the *spectrum* of T. By the closed graph theorem, the inverse operator

$$R(\lambda;T) := (\lambda I - T)^{-1},\ (\lambda \in \rho(T)) \tag{1.1}$$

is always bounded and is usually called *resolvent operator* of T at λ.

2. Subdivisions of the spectrum

In this section, we give the definitions of the parts point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

2.1. The point spectrum, continuous spectrum and residual spectrum. Associated with each complex number λ is the perturbed operator $T_\lambda = \lambda I - T$, defined on the same domain $D(T)$ as T. The *spectrum* $\sigma(T,X)$ consist of those $\lambda \in \mathbb{C}$ for which T_λ is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T,X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\lambda \mapsto T_\lambda^{-1}$. The name *resolvent* is appropriate, since T_λ^{-1} helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1} y$ provided T_λ^{-1} exists. More important, the investigation of properties of T_λ^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ's in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ's the domain of T_λ^{-1} is dense in X, to name just a few aspects. A *regular value* λ of T is a complex number such that T_λ^{-1} exists and bounded and whose domain is dense in X. For our investigation of T, T_λ and T_λ^{-1}, we need some basic concepts in spectral theory which are given as follows (see [30, pp. 370-371]):

The *resolvent set* $\rho(T,X)$ of T is the set of all regular values λ of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:

The *point (discrete) spectrum* $\sigma_p(T,X)$ is the set such that T_λ^{-1} does not exist. An $\lambda \in \sigma_p(T,X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T,X)$ is the set such that T_λ^{-1} exists and is unbounded and the domain of T_λ^{-1} is dense in X.

The *residual spectrum* $\sigma_r(T,X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but the domain of T_λ^{-1} is not dense in X.

Therefore, these three subspectras form a disjoint subdivisions

$$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X). \tag{2.1}$$

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T,X) = \sigma_r(T,X) = \emptyset$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite dimensional case.

2.2. The approximate point spectrum, defect spectrum and compression spectrum. In this subsection, following Appell et al. [9], we define the three more subdivisions of the spectrum called as the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(T,X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - T\} \tag{2.2}$$

the *approximate point spectrum of* T. Moreover, the subspectrum

$$\sigma_\delta(T, X) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\} \tag{2.3}$$

is called *defect spectrum* of T.

The two subspectra given by (2.2) and (2.3) form a (not necessarily disjoint) subdivisions

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_\delta(T, X)$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T, X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - T)} \neq X\}$$

which is often called *compression spectrum* in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_{co}(T, X)$$

of the spectrum. Clearly, $\sigma_p(T, X) \subseteq \sigma_{ap}(T, X)$ and $\sigma_{co}(T, X) \subseteq \sigma_\delta(T, X)$. Moreover, comparing these subspectra with those in (2.1) we note that

$$\begin{aligned} \sigma_r(T, X) &= \sigma_{co}(T, X) \backslash \sigma_p(T, X), \\ \sigma_c(T, X) &= \sigma(T, X) \backslash [\sigma_p(T, X) \cup \sigma_{co}(T, X)]. \end{aligned}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

2.1. Proposition. [9, Proposition 1.3, p. 28] *Spectra and subspectra of an operator* $T \in B(X)$ *and its adjoint* $T^* \in B(X^*)$ *are related by the following relations:*

(a) $\sigma(T^*, X^*) = \sigma(T, X)$.
(b) $\sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T, X)$.
(c) $\sigma_{ap}(T^*, X^*) = \sigma_\delta(T, X)$.
(d) $\sigma_\delta(T^*, X^*) = \sigma_{ap}(T, X)$.
(e) $\sigma_p(T^*, X^*) = \sigma_{co}(T, X)$.
(f) $\sigma_{co}(T^*, X^*) \supseteq \sigma_p(T, X)$.
(g) $\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_p(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{ap}(T^*, X^*)$.

The relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The equality (g) implies, in particular, that $\sigma(T, X) = \sigma_{ap}(T, X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (see [9]).

2.3. Goldberg's classification of spectrum. If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

(A) $R(T) = X$.
(B) $R(T) \neq \overline{R(T)} = X$.
(C) $\overline{R(T)} \neq X$.

and

(1) T^{-1} exists and is continuous.
(2) T^{-1} exists but is discontinuous.
(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3. If an operator is in state C_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous (see [22]).

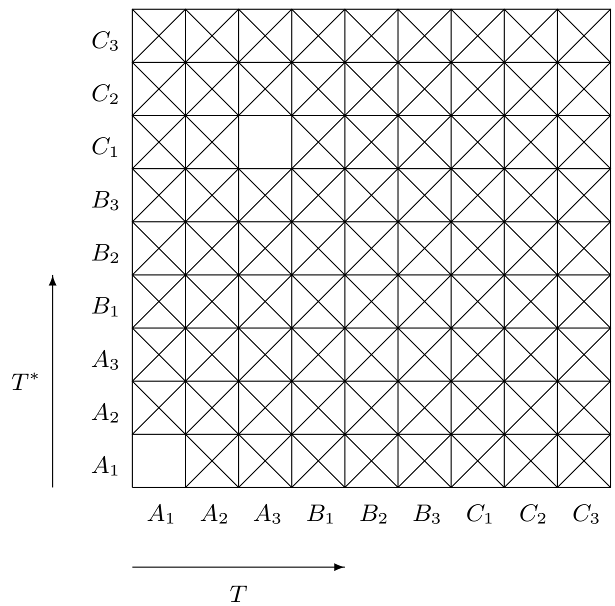

Table 1.1: State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X

If λ is a complex number such that $T_\lambda = \lambda I - T \in A_1$ or $T_\lambda = \lambda I - T \in B_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$ comprise the spectrum of T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T. That is, $\sigma(T, X)$ can be divided into the subsets $A_2\sigma(T, X) = \emptyset$, $A_3\sigma(T, X)$, $B_2\sigma(T, X)$, $B_3\sigma(T, X)$, $C_1\sigma(T, X)$, $C_2\sigma(T, X)$, $C_3\sigma(T, X)$. For example, if $T_\lambda = \lambda I - T$ is in a given state, C_2 (say), then we write $\lambda \in C_2\sigma(T, X)$.

By the definitions given above, we can illustrate the subdivisions (2.1) in the following table:

		1	2	3
		T_λ^{-1} exists and is bounded	T_λ^{-1} exists and is unbounded	T_λ^{-1} does not exist
A	$R(\lambda I - T) = X$	$\lambda \in \rho(T,X)$	–	$\lambda \in \sigma_p(T,X)$ $\lambda \in \sigma_{ap}(T,X)$
B	$\overline{R(\lambda I - T)} = X$	$\lambda \in \rho(T,X)$	$\lambda \in \sigma_c(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$	$\lambda \in \sigma_p(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$
C	$\overline{R(\lambda I - T)} \neq X$	$\lambda \in \sigma_r(T,X)$ $\lambda \in \sigma_\delta(T,X)$ $\lambda \in \sigma_{co}(T,X)$	$\lambda \in \sigma_r(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$ $\lambda \in \sigma_{co}(T,X)$	$\lambda \in \sigma_p(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$ $\lambda \in \sigma_{co}(T,X)$

Table 1.2: Subdivisions of spectrum of a linear operator

Observe that the case in the first row and second column cannot occur in a Banach space X, by the closed graph theorem. If we are not in the third column, i.e., if λ is not an eigenvalue of T, we may always consider the resolvent operator T_λ^{-1} (on a possibly "thin" domain of definition) as "algebraic" inverse of $\lambda I - T$.

From now on, we should note that the index p has different meanings in the notation of the spaces ℓ_p, ℓ_p^* and the point spectrums $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$, $\sigma_p[B(\widetilde{r},\widetilde{s})^*,\ell_p^*]$ which occur in theorems given in Section 3.

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}_1}$ of all complex sequences which contains ϕ, the set of all finitely non–zero sequences, where $\mathbb{N}_1$ denotes the set of positive integers. We write ℓ_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences which are the Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k\in\mathbb{N}}|x_k|$ and $\|x\|_{bv} = \sum_{k=0}^\infty |x_k - x_{k+1}|$ while ϕ is not a Banach space with respect to any norm, respectively, where $\mathbb{N} = \{0,1,2,\dots\}$. Also by ℓ_p, we denote the space of all p–absolutely summable sequences which is a Banach space with the norm $\|x\|_p = \left(\sum_{k=0}^\infty |x_k|^p\right)^{1/p}$, where $1 \le p < \infty$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n,k \in \mathbb{N}$, and write

$$(Ax)_n = \sum_k a_{nk}x_k \;; \quad (n \in \mathbb{N},\ x \in D_{00}(A)), \tag{2.4}$$

where $D_{00}(A)$ denotes the subspace of w consisting of $x \in w$ for which the sum exists as a finite sum. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞ and we shall use the convention that any term with negative subscript is equal to naught. More generally if μ is a normed sequence space, we can write $D_\mu(A)$ for the $x \in w$ for which the sum in (2.4) converges in the norm of μ. We write

$$(\lambda : \mu) = \{A : \lambda \subseteq D_\mu(A)\}$$

for the space of those matrices which send the whole of the sequence space λ into μ in this sense.

We give a short survey concerning with the spectrum and the fine spectrum of the linear operators defined by some particular triangle matrices over certain sequence spaces. The fine spectrum of the Cesàro operator of order one on the sequence space ℓ_p studied by Gonzàlez [23], where $1 < p < \infty$. Also, weighted mean matrices of operators on ℓ_p investigated by Cartlidge [15]. The spectrum of the Cesàro operator of order one on the sequence spaces bv_0 and bv investigated by Okutoyi [32, 33]. The spectrum and fine spectrum of the Rhally operators on the sequence spaces c_0, c and ℓ_p examined by Yıldırım [41, 42, 43, 44]. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c studied by Altay and Başar [5]. The same authors also worked the fine spectrum of the generalized difference operator $B(r,s)$ over c_0 and c, in [6].

The fine spectra of Δ over ℓ_1 and bv studied by Kayaduman and Furkan [29]. The fine spectra of the difference operator Δ over the sequence spaces ℓ_p and bv_p studied by Akhmedov and Başar [2, 3], where bv_p is the space of p-bounded variation sequences and introduced by Başar and Altay [10] with $1 \leq p < \infty$. The fine spectrum of $B(r,s,t)$ over the sequence spaces c_0 and c studied by Furkan et al. [20]. de Malafosse [31] studied the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r, where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by $\|x\|_{s_r} = \sup_{k\in\mathbb{N}} \frac{|x_k|}{r^k}$, $(r > 0)$. Altay and Karakuş [7] determined the fine spectrum of the Zweier matrix which is a band matrix as an operator over the sequence spaces ℓ_1 and bv. Farés and de Malafosse [19] studied the spectra of the difference operator on the sequence spaces $\ell_p(\alpha)$, where (α_n) denotes the sequence of positive reals and $\ell_p(\alpha)$ is the Banach space of all sequences $x = (x_n)$ normed by $\|x\|_{\ell_p(\alpha)} = \left[\sum_{n=1}^{\infty} (|x_n|/\alpha_n)^p\right]^{1/p}$ with $p \geq 1$. The fine spectrum of the operator $B(r,s)$ over ℓ_p and bv_p studied by Bilgiç and Furkan [11]. Besides, the fine spectrum with respect to the Goldberg's classification of the operator $B(r,s,t)$ defined by a triple band matrix over the sequence spaces ℓ_p and bv_p with $1 < p < \infty$ studied by Furkan et al. [21]. In 2010, Srivastava and Kumar [36] determined the spectra and the fine spectra of generalized difference operator Δ_ν on ℓ_1, where Δ_ν is defined by $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$ for all $n \in \mathbb{N}$, under certain conditions on the sequence $\nu = (\nu_n)$ and they generalized these results by the generalized difference operator Δ_{uv} defined by $\Delta_{uv}x = (u_n x_n + v_{n-1}x_{n-1})_{n\in\mathbb{N}}$ for all $n \in \mathbb{N}$, (see [38]).

Recently, Altun [8] have obtained the fine spectra of the Toeplitz operators, which are represented by upper and lower triangular n-band infinite matrices, over the sequence spaces c_0 and c. Later, Karaisa [26, 25] have determined the fine spectrum of the generalized difference operator $A(\widetilde{r},\widetilde{s})$, defined as an upper triangular double-band matrix with the convergent sequences $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ having certain properties, over the sequence space ℓ_p, where $(1 \leq p < \infty)$.

Quite recently, Akhmedov and El-Shabrawy [4], and El-Shabrawy [28] have obtained the fine spectrum of the generalized difference operator $\Delta_{a,b}$, defined as a double band matrix with the convergent sequences $\widetilde{a} = (a_k)$ and $\widetilde{b} = (b_k)$ having certain properties, over the sequence spaces c and c_0. Karaisa and Başar [13, 14, 27] have determined the fine spectrum of the upper triangular triple band matrix $A(r,s,t)$ over some sequence spaces. Yeşilkayagil and Başar [40] have computed the fine spectrum with respect to Goldberg's classification of the operator defined by the lambda matrix over the sequence spaces c_0 and c. Finally, Dündar and Başar [18] have studied the fine spectrum of the matrix operator Δ^+ defined by an upper triangle double band matrix acting on the sequence space c_0 with respect to the Goldberg's classification. At this stage, the following table may be useful:

$\sigma(A,\lambda)$	$\sigma_p(A,\lambda)$	$\sigma_c(A,\lambda)$	$\sigma_r(A,\lambda)$	refer to:
$\sigma(C_1^p,c)$	-	-	-	[39]
$\sigma(W,c)$	-	-	-	[35]
$\sigma(C_1,c_0)$	-	-	-	[34]
$\sigma(C_1,c_0)$	$\sigma_p(C_1,c_0)$	$\sigma_c(C_1,c_0)$	$\sigma_r(C_1,c_0)$	[1]
$\sigma(C_1,bv)$	-	-	-	[33]
$\sigma(C_1^p,c_0)$	-	-	-	[16]
$\sigma(\Delta,s_r)$	-	-	-	[31]
$\sigma(\Delta,c_0)$	-	-	-	[31]
$\sigma(\Delta,c)$	-	-	-	[31]
$\sigma(\Delta^{(1)},c)$	$\sigma_p(\Delta^{(1)},c)$	$\sigma_c(\Delta^{(1)},c)$	$\sigma_r(\Delta^{(1)},c)$	[5]
$\sigma(\Delta^{(1)},c_0)$	$\sigma_p(\Delta^{(1)},c_0)$	$\sigma_c(\Delta^{(1)},c_0)$	$\sigma_r(\Delta^{(1)},c_0)$	[5]
$\sigma(B(r,s),\ell_p)$	$\sigma_p(B(r,s),\ell_p)$	$\sigma_c(B(r,s),\ell_p)$	$\sigma_r(B(r,s),\ell_p)$	[12]
$\sigma(B(r,s),bv_p)$	$\sigma_p(B(r,s),bv_p)$	$\sigma_c(B(r,s),bv_p)$	$\sigma_r(B(r,s),bv_p)$	[12]
$\sigma(B(r,s,t),\ell_p)$	$\sigma_p(B(r,s,t),\ell_p)$	$\sigma_c(B(r,s,t),\ell_p)$	$\sigma_r(B(r,s,t),\ell_p)$	[21]
$\sigma(B(r,s,t),bv_p)$	$\sigma_p(B(r,s,t),bv_p)$	$\sigma_c(B(r,s,t),bv_p)$	$\sigma_r(B(r,s,t),bv_p)$	[21]
$\sigma(\Delta_{a,b},c)$	$\sigma_p(\Delta_{a,b},c)$	$\sigma_c(\Delta_{a,b},c)$	$\sigma_r(\Delta_{a,b},c)$	[4]
$\sigma(\Delta_\nu,\ell_1)$	$\sigma_p(\Delta_\nu,\ell_1)$	$\sigma_c(\Delta_\nu,\ell_1)$	$\sigma_r(\Delta_\nu,\ell_1)$	[36]
$\sigma(\Delta^2_{uv},c_0)$	$\sigma_p(\Delta^2_{uv},c_0)$	$\sigma_c(\Delta^2_{uv},c_0)$	$\sigma_r(\Delta^2_{uv},c_0)$	[37]
$\sigma(\Delta_{uv},\ell_1)$	$\sigma_p(\Delta_{uv},\ell_1)$	$\sigma_c(\Delta_{uv},\ell_1)$	$\sigma_r(\Delta_{uv},\ell_1)$	[38]
$\sigma(\Lambda,c_0)$	$\sigma_p(\Lambda,c_0)$	$\sigma_c(\Lambda,c_0)$	$\sigma_r(\Lambda,c_0)$	[40]
$\sigma(\Delta^+,c_0)$	$\sigma_p(\Delta^+,c_0)$	$\sigma_c(\Delta^+,c_0)$	$\sigma_r(\Delta^+,c_0)$	[18]

Table 1.3: Spectrum and fine spectrum of some triangle matrices in certain sequence spaces.

In this paper, we study the fine spectrum of the generalized difference operator $B(\widetilde{r},\widetilde{s})$ defined by a double sequential band matrix acting on the sequence space ℓ_p with respect to the Goldberg's classification, where $1<p<\infty$. Additionally, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $B(\widetilde{r},\widetilde{s})$ over the space ℓ_p.

We quote some lemmas which are needed in proving the theorems given in Section 3.

2.2. Lemma. [17, p. 253, Theorem 34.16] *The matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T\in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

2.3. Lemma. [17, p. 245, Theorem 34.3] *The matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T\in B(\ell_\infty)$ from ℓ_∞ to itself if and only if the supremum of ℓ_1 norms of the rows of A is bounded.*

2.4. Lemma. [17, p. 254, Theorem 34.18] *Let $1<p<\infty$ and $A\in(\ell_\infty:\ell_\infty)\cap(\ell_1:\ell_1)$. Then, $A\in(\ell_p:\ell_p)$.*

Let $\widetilde{r}=(r_k)$ and $\widetilde{s}=(s_k)$ be sequences whose entries either constants or distinct none-zero real numbers satisfying the following conditions:

$$\lim_{k\to\infty} r_k = r,$$
$$\lim_{k\to\infty} s_k = s\neq 0$$
$$|r_k-r|\neq|s|.$$

Then, we define the sequential generalized difference matrix $B(\widetilde{r},\widetilde{s})$ by

$$B(\widetilde{r},\widetilde{s}) = \begin{bmatrix} r_0 & 0 & 0 & 0 & \dots \\ s_0 & r_1 & 0 & 0 & \dots \\ 0 & s_1 & r_2 & 0 & \dots \\ 0 & 0 & s_2 & r_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore, we introduce the operator $B(\widetilde{r},\widetilde{s})$ from ℓ_p to itself by

$$B(\widetilde{r},\widetilde{s})x = (r_k x_k + s_{k-1}x_{k-1})_{k=0}^{\infty} \quad \text{with} \quad x_{-1} = 0, \quad \text{where} \quad x = (x_k) \in \ell_p.$$

3. The Fine spectrum of the Operator $B(\widetilde{r},\widetilde{s})$ on the Operator sequence space ℓ_p

3.1. Theorem. *The operator $B(\widetilde{r},\widetilde{s}) : \ell_p \to \ell_p$ is a bounded linear operator and*

$$(|r_k|^p + |s_k|^p)^{1/p} \le \|B(\widetilde{r},\widetilde{s}\|_p \le \|\widetilde{s}\|_\infty + \|\widetilde{r}\|_\infty. \tag{3.1}$$

Proof. Since the linearity of the operator $B(\widetilde{r},\widetilde{s})$ does not hard, we omit the detail.

Now we prove that (3.1) holds for the operator $B(\widetilde{r},\widetilde{s})$ on the space ℓ_p. It is trivial that $B(\widetilde{r},\widetilde{s})e^{(k)} = (0,0,\dots,r_k,s_k,0,0,\dots)$ for $e^{(k)} \in \ell_p$. Therefore, we have

$$\frac{\|B(\widetilde{r},\widetilde{s})e^{(k)}\|_p}{\|e^{(k)}\|_p} = (|r_k|^p + |s_k|^p)^{1/p}$$

which implies that

$$(|r_k|^p + |s_k|^p)^{1/p} \le \|B(\widetilde{r},\widetilde{s})\|_p. \tag{3.2}$$

Let $x = (x_k) \in \ell_p$, where $p > 1$. Then, since $(s_{k-1}x_{k-1}), (r_k x_k) \in \ell_p$ it is easy to see by Minkowski's inequality that

$$\begin{aligned} \|B(\widetilde{r},\widetilde{s})x\|_p &= \left(\sum_{k=0}^{\infty} |s_{k-1}x_{k-1} + r_k x_k|^p\right)^{1/p} \\ &\le \left(\sum_{k=0}^{\infty} |s_{k-1}x_{k-1}|^p\right)^{1/p} + \left(\sum_{k=0}^{\infty} |r_k x_k|^p\right)^{1/p} \\ &\le (\|\widetilde{s}\|_\infty + \|\widetilde{r}\|_\infty)\|x\|_p \end{aligned}$$

which leads us to the the result that

$$\|B(\widetilde{r},\widetilde{s})\|_p \le \|\widetilde{s}\|_\infty + \|\widetilde{r}\|_\infty. \tag{3.3}$$

Therefore, by combining the inequalities in (3.2) and (3.3) we have (3.1), as desired. □

3.2. Theorem. *Let $\mathcal{A} = \{\alpha \in \mathbb{C} : |r - \alpha| \le |s|\}$ and $\mathcal{B} = \{r_k : k \in \mathbb{N}, |r - r_k| > |s|\}$. Then, the set $\mathcal{B}$ is finite and $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p] = \mathcal{A} \cup \mathcal{B}$.*

Proof. We firstly prove that

$$\sigma[B(\widetilde{r},\widetilde{s}),\ell_p] \subseteq \mathcal{A} \cup \mathcal{B} \tag{3.4}$$

which is equivalent to show that $\alpha \in \mathbb{C}$ such that $|r - \alpha| > |s|$ and $\alpha \ne r_k$ for all $k \in \mathbb{N}$ implies $\alpha \notin \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$. It is easy to see that $\mathcal{B}$ is finite and $\{r_k \in \mathbb{C} : k \in \mathbb{N}\} \subseteq \mathcal{A} \cup \mathcal{B}$.

It is immediate that $B(\widetilde{r},\widetilde{s})-\alpha I$ is a triangle and so has an inverse. Let $y=(y_k)\in\ell_1$. Then, by solving the equation

$$\begin{aligned}[B(\widetilde{r},\widetilde{s})-\alpha I]x &= \begin{bmatrix} r_0-\alpha & 0 & 0 & \dots \\ s_0 & r_1-\alpha & 0 & \dots \\ 0 & s_1 & r_2-\alpha & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} (r_0-\alpha)x_0 \\ s_0x_0+(r_1-\alpha)x_1 \\ s_1x_1+(r_2-\alpha)x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}\end{aligned}$$

for $x=(x_k)$ in terms of y, we obtain

$$\begin{aligned}x_0 &= \frac{y_0}{r_0-\alpha}, \\ x_1 &= \frac{y_1}{r_1-\alpha}+\frac{-s_0y_0}{(r_1-\alpha)(r_0-\alpha)}, \\ x_2 &= \frac{y_2}{r_2-\alpha}+\frac{-s_1y_1}{(r_2-\alpha)(r_1-\alpha)}+\frac{s_0s_1y_0}{(r_2-\alpha)(r_1-\alpha)(r_0-\alpha)}, \\ &\ \vdots \\ x_k &= \frac{(-1)^k s_0s_1s_2\cdots s_{k-1}y_0}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)\cdots(r_k-\alpha)}+\cdots-\frac{s_{k-1}y_{k-1}}{(r_k-\alpha)(r_{k-1}-\alpha)}+\frac{y_k}{r_k-\alpha}, \\ &\ \vdots\end{aligned}$$

Therefore, we obtain $B=(b_{nk})=[B(\widetilde{r},\widetilde{s})-\alpha I]^{-1}$ as follows:

$$(b_{nk})=\begin{bmatrix} \frac{1}{r_0-\alpha} & 0 & 0 & \dots \\ \frac{-s_0}{(r_1-\alpha)(r_0-\alpha)} & \frac{1}{r_1-\alpha} & 0 & \dots \\ \frac{s_0s_1}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)} & \frac{-s_1}{(r_2-\alpha)(r_1-\alpha)} & \frac{1}{r_2-\alpha} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then, $\sum_k |x_k| \le \sum_k S^k|y_k|$, where

$$\begin{aligned}S^k &= \left|\frac{1}{r_k-\alpha}\right|+\left|\frac{s_k}{(r_k-\alpha)(r_{k+1}-\alpha)}\right|+\left|\frac{s_ks_{k+1}}{(r_k-\alpha)(r_{k+1}-\alpha)(r_{k+2}-\alpha)}\right|+\cdots. \\ S_n^k &= \left|\frac{1}{r_k-\alpha}\right|+\left|\frac{s_k}{(r_k-\alpha)(r_{k+1}-\alpha)}\right|+\left|\frac{s_ks_{k+1}}{(r_k-\alpha)(r_{k+1}-\alpha)(r_{k+2}-\alpha)}\right|+\cdots \\ &\quad+\left|\frac{s_ks_{k+1}\cdots s_{n+k}}{(r_k-\alpha)(r_{k+1}-\alpha)(r_{k+2}-\alpha)\cdots(r_{k+n+1}-\alpha)}\right| \quad \text{for all } k,n\in\mathbb{N}.\end{aligned}$$

Then, since

$$S_n=\lim_{k\to\infty}S_n^k=\left|\frac{1}{r-\alpha}\right|+\left|\frac{s}{(r-\alpha)^2}\right|+\left|\frac{s^2}{(r-\alpha)^3}\right|+\cdots+\left|\frac{s^{n+1}}{(r-\alpha)^{n+2}}\right|,$$

we have

$$S=\lim_{n\to\infty}S_n=\left|\frac{1}{r-\alpha}\right|\left(1+\left|\frac{s}{r-\alpha}\right|+\left|\frac{s}{r-\alpha}\right|^2+\cdots\right)<\infty, \tag{3.5}$$

since $|r-\alpha|>|s|$. Then we have

$$\lim_{n\to\infty}\lim_{k\to\infty}S_n^k=\lim_{k\to\infty}\lim_{n\to\infty}S_n^k=S$$

and $(S^k)_k \in c$. Thus,

$$\sum_k |x_k| \leq \sum_k S^k |y_k| \leq \|(S^k)\|_\infty \sum_k |y_k| < \infty,$$

since $y \in \ell_1$. This shows that $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \in (\ell_1 : \ell_1)$.

Suppose that $y = (y_k) \in \ell_\infty$. By solving the equation $[B(\widetilde{r},\widetilde{s}) - \alpha I]x = y$, for $x = (x_k)$ in terms of y, we get

$$|x_k| \leq S_k \left(\sup_{k \in \mathbb{N}} |y_k| \right),$$

where;

$$\begin{aligned} S_k \quad &= \quad \left| \frac{1}{r_k - \alpha} \right| + \left| \frac{s_{k-1}}{(r_{k-1} - \alpha)(r_k - \alpha)} \right| + \left| \frac{s_{k-1} s_{k-2}}{(r_{k-2} - \alpha)(r_{k-1} - \alpha)(r_k - \alpha)} \right| \\ &\quad + \cdots + \left| \frac{s_0 s_1 \ldots s_{k-1}}{(r_0 - \alpha)(r_1 - \alpha) \cdots (r_k - \alpha)} \right|. \end{aligned}$$

Now, we prove that $(S_k) \in \ell_\infty$. Since $\lim_{k\to\infty} |s_k/(r_k - \alpha)| = |s/(r - \alpha)| = p < 1$, then there exists $k_0 \in \mathbb{N}$ such that $|s_k/(r_k - \alpha)| < p_0$ with $p_0 < 1$, for all $k \geq k_0 + 1$,

$$\begin{aligned} S_k \quad &= \quad \frac{1}{|r_k - \alpha|} \left[1 + \left| \frac{s_{k-1}}{r_{k-1} - \alpha} \right| + \left| \frac{s_{k-1} s_{k-2}}{(r_{k-1} - \alpha)(r_{k-2} - \alpha)} \right| \right. \\ &\quad \left. + \cdots + \left| \frac{s_{k-1} s_{k-2} \ldots s_{k_0+1} s_{k_0} \ldots s_0}{(r_{k-1} - \alpha)(r_{k-2} - \alpha) \cdots (r_{k_0+1} - \alpha)(r_{k_0} - \alpha) \cdots (r_0 - \alpha)} \right| \right] \\ &\leq \quad \frac{1}{|r_k - \alpha|} \left[1 + p_0 + p_0^2 + \cdots + p_0^{k-k_0} + p_0^{k-k_0} \frac{|s_{k_0-1}|}{|r_{k_0-1} - \alpha|} \right. \\ &\quad \left. + \cdots + p_0^{k-k_0} \left| \frac{s_{k_0-1} s_{k_0-2} \ldots s_0}{(r_{k_0-1} - \alpha)(r_{k_0-2} - \alpha) \cdots (r_0 - \alpha)} \right| \right]. \end{aligned}$$

Therefore;

$$S_k \leq \frac{1}{|r_k - \alpha|} \left(1 + p_0 + p_0^2 + \cdots p_0^{k-k_0} + p_0^{k-k_0} M k_0 \right),$$

where

$$Mk_0 = 1 + \left| \frac{s_{k_0-1}}{r_{k_0-1} - \alpha} \right| + \left| \frac{s_{k_0-1} s_{k_0-2}}{(r_{k_0-1} - \alpha)(r_{k_0-2} - \alpha)} \right| + \cdots + \left| \frac{s_{k_0-1} s_{k_0-2} \ldots s_0}{(r_{k_0-1} - \alpha)(r_{k_0-2} - \alpha) \cdots (r_0 - \alpha)} \right|.$$

Then, $Mk_0 \geq 1$ and so

$$S_k \leq \frac{Mk_0}{|r_k - \alpha|} \left(1 + p_0 + p_0^2 + \cdots + p_0^{k-k_0} \right).$$

But there exists $k_1 \in \mathbb{N}$ and a real number p_1 such that $\frac{1}{|r_k - \alpha|} < p_1$ for all $k \geq k_1$. Then, $S_k \leq (Mp_1k_0)/(1-p_0)$ for all $k > max\{k_0, k_1\}$. Hence, $\sup_{k\in\mathbb{N}} S_k < \infty$. This shows that $\|x\|_\infty \leq \|(S_k)\|_\infty \|y\|_\infty < \infty$ which means $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \in (\ell_\infty : \ell_\infty)$. By Lemma 2.4, we have

$$[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \in (\ell_p : \ell_p) \quad \text{for} \quad |r - \alpha| > |s| \quad \text{and} \quad \alpha \neq r_k. \tag{3.6}$$

Hence,

$$\sigma[B(\widetilde{r},\widetilde{s}), \ell_p] \subseteq \mathcal{A} \cup \mathcal{B}. \tag{3.7}$$

Now we show that $\mathcal{A} \cup \mathcal{B} \subseteq \sigma[B(\widetilde{r},\widetilde{s}), \ell_p]$. We assume that $\alpha \neq r_k$ for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ with $|r - \alpha| \leq |s|$. Clearly, $B(\widetilde{r},\widetilde{s}) - \alpha I$ is a triangle and so, $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1}$ exists. For $e^{(0)} = (1, 0, 0, \ldots) \in \ell_p$, $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} e^{(0)} = S^0 \notin \ell_p$, and so $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \notin$

$B(\ell_p)$. Then, $\alpha \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$. Case $r_k = \alpha$ for some k. We then have either $\alpha = r$ or $\alpha = r_k \neq r$ for some k. We have

$$\begin{aligned}[B(\widetilde{r},\widetilde{s}) - r_k I]x &= \begin{bmatrix} r_0 - r_k & 0 & 0 & \cdots \\ s_0 & r_1 - r_k & 0 & \cdots \\ 0 & s_1 & r_2 - r_k & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} (r_0 - r_k)x_0 \\ s_0x_0 + (r_1 - r_k)x_1 \\ s_1x_1 + (r_2 - r_k)x_2 \\ \vdots \\ s_{k-2}x_{k-2} + (r_{k-1} - r_k)x_{k-1} \\ s_{k-1}x_{k-1} + (r_k - r_k)x_k \\ s_kx_k + (r_{k+1} - r_k)x_{k+1} \\ \vdots \end{bmatrix}.\end{aligned}$$

Let $\alpha = r_k = r$ for all k and solving the equation $[B(\widetilde{r},\widetilde{s}) - \alpha I]x = \theta$ we obtain $x_0 = x_1 = x_2 = \cdots = 0$ which shows that $B(\widetilde{r},\widetilde{s}) - \alpha I$ is one to one but its range $R[B(\widetilde{r},\widetilde{s}) - \alpha I] = \{y = (y_k) \in \omega : y \in \ell_p,\ y_1 = 0\}$ is not dense in ℓ_p and $\alpha = r \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$. Now let $\alpha = r_k$ for some k. Then the equation $[B(\widetilde{r},\widetilde{s}) - \alpha I]x = \theta$ yields

$$x_0 = x_1 = x_2 = \cdots = x_{k-1} = 0 \ \text{ and } \ x_n = \frac{s_{n-1}}{r_k - r_n}x_{n-1} \ \text{ for all } \ n \geq k+1.$$

This shows that $B(\widetilde{r},\widetilde{s}) - \alpha I$ is not injective for $\alpha = r_k$ such that $|\alpha - r| > |s|$. Therefore $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1}$ does not exist. So $r_k \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$ for all $k \in \mathbb{N}$. Thus,

$$\mathcal{A} \cup \mathcal{B} \subseteq \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]. \tag{3.8}$$

Combining the inclusions (3.7) and (3.8), we get $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p] = \mathcal{A} \cup \mathcal{B}$.

This completes the proof. □

Throughout the paper, by $\mathcal{C}$ and $\mathcal{SD}$ we denote the set of constant sequences and the set of sequences of distinct none-zero real numbers, respectively.

3.3. Theorem. $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p] = \begin{cases} \emptyset & , \quad \widetilde{r},\widetilde{s} \in \mathcal{C}, \\ \mathcal{B} & , \quad \widetilde{r},\widetilde{s} \in \mathcal{SD}, \end{cases}$

Proof. We prove the theorem by dividing into two parts.

Part 1. Assume that $\widetilde{r},\widetilde{s} \in \mathcal{C}$. Consider $B(\widetilde{r},\widetilde{s})x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in ℓ_p. Then, by solving the system of linear equations

$$\begin{aligned} rx_0 &= \alpha x_0 \\ sx_0 + rx_1 &= \alpha x_1 \\ sx_1 + rx_2 &= \alpha x_2 \\ &\vdots \\ sx_{k-1} + rx_k &= \alpha x_k \\ &\vdots \end{aligned}$$

Case $\alpha = r$. Let x_{n_0} is the first non zero entry of the sequence $x = (x_n)$ and $\alpha = r$, then we get $sx_{n_0} + rx_{n_0+1} = \alpha x_{n_0+1}$ which implies $x_{n_0} = 0$ which contradicts the assumption $x_{n_0} \neq 0$. Hence, the equation $B(\widetilde{r},\widetilde{s})x = \alpha x$ has no solution $x \neq \theta$.

Part 2. Assume that $\widetilde{r},\widetilde{s} \in \mathcal{SD}$. Then, by solving the equation $B(\widetilde{r},\widetilde{s})x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in ℓ_p we obtain $(r_0 - \alpha)x_0 = 0$ and $(r_{k+1} - \alpha)x_{k+1} + s_kx_k = 0$ for all

$k \in \mathbb{N}$. Hence, for all $\alpha \notin \{r_k : k \in \mathbb{N}\}$, we have $x_k = 0$ for all $k \in \mathbb{N}$, which contradicts our assumption. So, $\alpha \notin \sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$. This shows that $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p] \subseteq \{r_k : k \in \mathbb{N}\}\backslash\{r\}$. Now, we prove that

$$\alpha \in \sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p] \text{ if and only if } \alpha \in \mathcal{B}.$$

Let $\alpha \in \sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$. We consider the case $\alpha = r_0$ and $\alpha = r_k$ for some $k \geq 1$. Then, by solving the equation $B(\widetilde{r},\widetilde{s})x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in ℓ_p with $\alpha = r_0$

$$x_k = \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0 - r_k)(r_0 - r_{k-1})(r_0 - r_{k-2}) \cdots (r_0 - r_1)} x_0 \quad \text{for all} \quad k \geq 1$$

which can expressed by the recursion relation

$$x_k = \frac{s_{k-1}}{r_0 - r_k} x_{k-1} \quad \text{for all} \quad k \in \mathbb{N}_1.$$

Therefore, since

$$\lim_{k\to\infty} \left|\frac{x_k}{x_{k-1}}\right|^p = \lim_{k\to\infty} \left|\frac{s_{k-1}}{r_k - r_0}\right|^p = \left|\frac{s}{r - r_0}\right|^p \leq 1,$$

But $\left|\frac{s}{r-r_0}\right|^p \neq 1$. Then $\alpha = r_0 \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$.
If we choose $\alpha = r_k \neq r$ for all $k \in \mathbb{N}_1$, then we get $x_0 = x_1 = x_2 = \cdots = x_{k-1} = 0$ and

$$x_{n+1} = \frac{s_n s_{n-1} s_{n-2} \ldots s_k}{(r_k - r_{n+1})(r_k - r_n)(r_k - r_{n-1}) \cdots (r_k - r_{k+1})} x_k \quad \text{for all} \quad n \geq k$$

which can also be expressed by the recursion relation

$$x_{n+1} = \frac{s_n}{r_k - r_{n+1}} x_n \quad \text{for all} \quad n \geq k.$$

Therefore, we have

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|^p = \lim_{n\to\infty} \left|\frac{s_n}{r_{n+1} - r_k}\right|^p = \left|\frac{s}{r - r_k}\right|^p \leq 1.$$

But $\left|\frac{s}{r-r_k}\right| \neq 1$. Then $\alpha = r_k \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$. Thus $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p] \subseteq \mathcal{B}$.

Conversely, let $\alpha \in \mathcal{B}$. Then, there exists $k \in \mathbb{N}$, $\alpha = r_k \neq r$ and

$$\lim_{n\to\infty} \left|\frac{s_n}{r_{n+1} - r_k}\right| = \left|\frac{s}{r - r_k}\right| < 1,$$

so we have $x \in \ell_p$. Thus $\mathcal{B} \subseteq \sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$. This completes the proof. $\square$

3.4. Theorem. $\sigma_p[B(\widetilde{r},\widetilde{s})^*,\ell_p^*] = \begin{cases} \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} & , \quad \widetilde{r},\widetilde{s} \in \mathcal{C}, \\ \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\} \cup \mathcal{B} & , \quad \widetilde{r},\widetilde{s} \in \mathcal{SD}. \end{cases}$

Proof. By solving the equation $B(\widetilde{r},\widetilde{s})^* f = \alpha f$ for $\theta \neq f \in \ell_p^* \cong \ell_q$, we derive the system of linear equations

$$\begin{array}{rcl} r_0 f_0 + s_0 f_1 & = & \alpha f_0 \\ r_1 f_1 + s_1 f_2 & = & \alpha f_1 \\ r_2 f_2 + s_2 f_3 & = & \alpha f_2 \\ & \vdots & \\ r_{k-1} f_{k-1} + s_{k-1} f_k & = & \alpha f_{k-1} \\ & \vdots & \end{array}$$

This gives $f_k=\left(\frac{\alpha-r_{k-1}}{s_{k-1}}\right)f_{k-1}$ for all $k\geq 1$. Therefore, we have

$$|f_k|=\left|\frac{\alpha-r_{k-1}}{s_{k-1}}\right||f_{k-1}| \text{ for all } k\in\mathbb{N}_1. \tag{3.9}$$

We also prove this theorem by dividing into two parts.

Part 1. Assume that $\widetilde{r},\widetilde{s}\in\mathcal{C}$ with $r_k=r$ and $s_k=s$ for all $k\in\mathbb{N}$. Using (3.9), we get

$$f_k=\left(\frac{\alpha-r}{s}\right)^k f_0 \text{ for all } k\in\mathbb{N}_1.$$

Then, since

$$\lim_{k\to\infty}\left|\frac{f_{k+1}}{f_k}\right|^q=\left|\frac{\alpha-r}{s}\right|^q<1 \text{ provided } \left|\frac{r-\alpha}{s}\right|<1$$

the series $\sum_{k=1}^{\infty}|f_k|^q=\sum_{k=1}^{\infty}|(\alpha-r)/s|^{q(k-1)}|f_0|$ converges by the ratio test, i.e., $f\in\ell_q$.

If $\alpha\in\mathbb{C}$ with $|\alpha-r|=|s|$, then the ratio test fails. But, since $\lim_{k\to\infty}|f_k|=|f_0|\neq 0$ the series $\sum_{k=0}^{\infty}|f_k|^q$ is divergent. This means that $f\in\ell_q$ if and only if $f_0\neq 0$ and $|r-\alpha|<|s|$. Hence, $\sigma_p[B(\widetilde{r},\widetilde{s})^*,\ell_p^*]=\{\alpha\in\mathbb{C}:|r-\alpha|<|s|\}$.

Part 2. Let $\widetilde{r},\widetilde{s}\in\mathcal{SD}$. It is clear that for all $k\in\mathbb{N}$, the vector $f=(f_0,f_1,\ldots\ldots,f_k,0,0,\ldots)$ is an eigenvector of the operator $B(\widetilde{r},\widetilde{s})^*$ corresponding to the eigenvalue $\alpha=r_k$, where $f_0\neq 0$ and $f_n=\left(\frac{\alpha-r_{n-1}}{s_{n-1}}\right)f_{n-1}$ for all $k\in\{1,2,3,\ldots,n\}$. Thus $\mathcal{B}\subseteq\sigma_p[B(\widetilde{r},\widetilde{s})^*,\ell_p^*]$. If $|r-\alpha|<|s|$ and $\alpha=r_k$, by taking into account (3.9), since

$$\lim_{k\to\infty}\left|\frac{f_k}{f_{k-1}}\right|^q=\lim_{k\to\infty}\left|\frac{\alpha-r_{k-1}}{s_{k-1}}\right|^q=\left|\frac{r-\alpha}{s}\right|^q<1,$$

the ratio test gives that $f\in\ell_q$. If $\alpha\in\mathbb{C}$ with $|r-\alpha|=|s|$, the ratio test fails. But one can easily find a decreasing sequence of positive real numbers $f=(f_k)\in\ell_q$ such that $\lim_{k\to\infty}(|f_k/f_{k-1}|)=1$, for example $f=(f_k)=(1/k^2)$. Hence, $|r-\alpha|\leq s$ implies $f\in\ell_q$.

Conversely, we have to show that $f\in\ell_q$ implies $|r-\alpha|\leq s$. If the condition $|r-\alpha|\leq|s|$ does not hold, then $|r-\alpha|>|s|$ which implies that $\sum_{k=0}^{\infty}|f_k|^q$ is divergent. This means that $f\in\ell_q$ if and only if $f_0\neq 0$ and $|r-\alpha|\leq|s|$. Hence,

$$\sigma_p[B(\widetilde{r},\widetilde{s})^*,\ell_p^*]=\{\alpha\in\mathbb{C}:|r-\alpha|\leq|s|\}\cup\mathcal{B}.$$

This completes the proof. ☐

3.5. Lemma. [22, p. 59] *T has a dense range if and only if T^* is one to one.*

3.6. Lemma. [22, p. 60] *The adjoint operator T^* of T is onto if and only if T is a bounded operator.*

3.7. Theorem. $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]=\begin{cases}\{\alpha\in\mathbb{C}:|r-\alpha|<|s|\} & , \quad \widetilde{r},\widetilde{s}\in\mathcal{C},\\ \{\alpha\in\mathbb{C}:|r-\alpha|\leq|s|\} & , \quad \widetilde{r},\widetilde{s}\in\mathcal{SD}.\end{cases}$

Proof. We prove the theorem by dividing into two parts.

Part 1. Let $\widetilde{r},\widetilde{s}\in\mathcal{C}$. We show that the operator $B(\widetilde{r},\widetilde{s})-\alpha I$ has an inverse and $\overline{R(B(\widetilde{r},\widetilde{s})-\alpha I)}\neq\ell_p$ for α satisfying $|r-\alpha|<|s|$. For $\alpha\neq r$ $B(\widetilde{r},\widetilde{s})-\alpha I$ is triangle so has an inverse. For $\alpha=r$, the operator $B(\widetilde{r},\widetilde{s})-\alpha I$ is one to one by Theorem 3.3. So it has a inverse. By Theorem 3.4, the operator $[B(\widetilde{r},\widetilde{s})-\alpha I)]^*=B(\widetilde{r},\widetilde{s})^*-\alpha I$ is not one to one for $\alpha\in\mathbb{C}$ such that $|r-\alpha|<|s|$. Hence the range of the operator $B(\widetilde{r},\widetilde{s})-\alpha I$ is not dense in ℓ_p by Lemma 3.5. So, $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]=\{\alpha\in\mathbb{C}:|r-\alpha|<|s|\}$.

Part 2. Let $\widetilde{r},\widetilde{s}\in\mathcal{SD}$ with $r_k\longrightarrow r$ and $s_k\longrightarrow s$ as $k\longrightarrow\infty$ for $\alpha\in\mathbb{C}$ such that $|r-\alpha|\leq|s|$. Then, the operator $B(\widetilde{r},\widetilde{s})-\alpha I$ is triangle with $\alpha\neq r_k$ for all $k\in\mathbb{N}$. So, the operator $B(\widetilde{r},\widetilde{s})-\alpha I$ has an inverse. By Theorem 3.3 the operator $B(\widetilde{r},\widetilde{s})-\alpha I$ is one to one for $\alpha = r_k$ for all $k \in \mathbb{N}$. Thus, $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1}$ exists. But by Theorem 3.4, $[B(\widetilde{r},\widetilde{s}) - \alpha I]^* = B(\widetilde{r},\widetilde{s})^* - \alpha I$ is not one to one with $\alpha \in \mathbb{C}$ such that $|r-\alpha| \leq |s|$. Hence, the range of the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is not dense in ℓ_p, by Lemma 3.5. So, $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p] = \{\alpha \in \mathbb{C} : |r-\alpha| \leq |s|\}$.

This completes the proof. □

3.8. Theorem. $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p] = \begin{cases} \{\alpha \in \mathbb{C} : |r-\alpha| = |s|\} & , \quad \widetilde{r},\widetilde{s} \in \mathcal{C}, \\ \emptyset & , \quad \widetilde{r},\widetilde{s} \in \mathcal{SD}. \end{cases}$

Proof. We prove the theorem by dividing into two parts.

Part 1. Let $\widetilde{r},\widetilde{s} \in \mathcal{C}$ for $\alpha \in \mathbb{C}$ such that $|r-\alpha| = |s|$. Since $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$ is the disjoint union of the parts $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$, $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$ and $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p]$, we must have $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p] = \{\alpha \in \mathbb{C} : |r-\alpha| = |s|\}$.

Part 2. Let $\widetilde{r},\widetilde{s} \in \mathcal{SD}$. It is known that $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$, $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$ and $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p]$ are mutually disjoint sets and their union is $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$. Therefore, it is immediate from Theorems 3.2, 3.3 and 3.7 that $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p] = \sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p] \cup \sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$ and hence $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p] = \emptyset$.

This completes the proof. □

3.9. Theorem. *When $|r-\alpha| > |s|$ for $\alpha \neq r_k$, $[B(\widetilde{r},\widetilde{s}) - \alpha I] \in A_1$.*

Proof. We show that the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is bijective and has a continuous inverse for $\alpha \in \mathbb{C}$ such that $|r-\alpha| > |s|$. Since $\alpha \neq r_k$, then $B(\widetilde{r},\widetilde{s}) - \alpha I$ is a triangle. So, it has an inverse. The inverse of the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is continuous for $\alpha \in \mathbb{C}$ such that $|r-\alpha| > |s|$, by equation (3.6). Thus for every $y \in \ell_p$, we can find that $x \in \ell_p$ such that

$$[B(\widetilde{r},\widetilde{s}) - \alpha I]x = y, \quad \text{since} \quad [B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \in (\ell_p : \ell_p).$$

This shows that the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is onto and so $B(\widetilde{r},\widetilde{s}) - \alpha I \in A_1$. □

3.10. Theorem. *Let $\widetilde{r},\widetilde{s} \in \mathcal{C}$ with $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$. Then, $r \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]C_1$.*

Proof. We have $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p] = \{\alpha \in \mathbb{C} : |r-\alpha| < |s|\}$, by Theorem 3.7. Clearly, $r \in \sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$. It is sufficient to show that the operator $[B(\widetilde{r},\widetilde{s}) - rI]^{-1}$ is continuous. By Lemma 3.6, it is enough to show that $[B(\widetilde{r},\widetilde{s}) - Ir]^*$ is onto and for given $y = (y_k) \in \ell_p^* = \ell_q$, we have to find $x = (x_k) \in \ell_q$ such that $[B(\widetilde{r},\widetilde{s}) - Ir]^* x = y$. Solving the system of linear equations

$$\begin{array}{rcl} s_0 x_1 & = & y_0 \\ s_1 x_2 & = & y_1 \\ s_2 x_3 & = & y_2 \\ & \vdots & \\ s_{k-1} x_k & = & y_{k-1} \\ & \vdots & \end{array}$$

one can easily observe that $sx_k = y_{k-1}$ for all $k \geq 1$ which implies that $(x_k) \in \ell_q$, since $y = (y_k) \in \ell_q$. This shows that $[B(\widetilde{r},\widetilde{s}) - Ir]^*$ is onto. Hence, $r \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]C_1$. □

3.11. Theorem. *Let $\widetilde{r},\widetilde{s} \in \mathcal{C}$ with $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$ and $\alpha \in \sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$ for all $r \neq \alpha$. Then, $\alpha \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]C_2$.*

Proof. It is sufficient to show that the operator $[B(\widetilde{r},\widetilde{s}) - I\alpha]^{-1}$ is discontinuous for $r \neq \alpha$ and $\alpha \in \sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$. It is obvious that the operator $[B(\widetilde{r},\widetilde{s}) - I\alpha]^{-1}$ is discontinuous for $r \neq \alpha$ and $\alpha \in \mathbb{C}$ such that $|r-\alpha| < |s|$ with $r_k \neq \alpha$, by (3.5). □

3.12. Theorem. *If* $\widetilde{r},\widetilde{s}\in\mathcal{SD}$ *and* $\alpha\in\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$, *then* $\alpha\in\sigma[B(\widetilde{r},\widetilde{s}),\ell_p]C_2$.

Proof. It is sufficient to show that the operator $[B(\widetilde{r},\widetilde{s})-I\alpha]^{-1}$ is discontinuous for $\alpha\in\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$. By (3.5), the operator $[B(\widetilde{r},\widetilde{s})-I\alpha]^{-1}$ is discontinuous for $r_k\neq\alpha$ and $\alpha\in\mathbb{C}$ with $|r-\alpha|\leq|s|$. □

3.13. Theorem. *Let* $\widetilde{r},\widetilde{s}\in\mathcal{C}$ *with* $r_k=r$, $s_k=s$ *for all* $k\in\mathbb{N}$. *Then, the following statements hold:*

(i) $\sigma_{ap}[B(\widetilde{r},\widetilde{s}),\ell_p]=\mathcal{A}\setminus\{r\}$.
(ii) $\sigma_{\delta}[B(\widetilde{r},\widetilde{s}),\ell_p]=\mathcal{A}$.
(iii) $\sigma_{co}[B(\widetilde{r},\widetilde{s}),\ell_p]=\mathcal{A}^{\circ}$.

Proof. (i) From Table 1.2, we get

$$\sigma_{ap}[B(\widetilde{r},\widetilde{s}),\ell_p]=\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]\setminus\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_1.$$

We have by Theorem 3.10 and Theorem 3.2 that

$$\sigma_{ap}[B(\widetilde{r},\widetilde{s}),\ell_p]=(\mathcal{A}\cup\mathcal{B})\setminus\{r\}=\mathcal{A}\setminus\{r\}.$$

(ii) Since the following equality

$$\sigma_{\delta}[B(\widetilde{r},\widetilde{s}),\ell_p]=\sigma[B(\widetilde{r},\widetilde{s}),\ell_p]\setminus\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]A_3$$

holds from Table 1.2, we derive by Theorem 3.2 and Theorem 3.3 that $\sigma_{\delta}[B(\widetilde{r},\widetilde{s}),\ell_p]=\mathcal{A}$.

(iii) From Table 1.2, we have

$$\sigma_{\delta}[B(\widetilde{r},\widetilde{s}),\ell_p]=\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_1\cup\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_2\cup\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_3$$

and since $\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_3=\emptyset$ by Theorem 3.3 it is immediate that $\sigma_{co}[B(\widetilde{r},\widetilde{s}),\ell_p]=\sigma_r\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]$. Therefore, we obtain by Theorem 3.11 that $\sigma_{co}[B(\widetilde{r},\widetilde{s}),\ell_p]=\mathcal{A}^{\circ}$. □

3.14. Theorem. *Let* $\widetilde{r},\widetilde{s}\in\mathcal{SD}$. *Then*

$$\sigma_{ap}[B(\widetilde{r},\widetilde{s}),\ell_p]=\sigma_{\delta}[B(\widetilde{r},\widetilde{s}),\ell_p]=\sigma_{co}[B(\widetilde{r},\widetilde{s}),\ell_p]=\mathcal{A}\cup\mathcal{B}.$$

Proof. We have by Theorem 3.4 and Part (e) of Proposition 2.1 that

$$\sigma_p[B^*(\widetilde{r},\widetilde{s}),\ell_p^*]=\sigma_{co}[B(\widetilde{r},\widetilde{s}),\ell_p]=\{\alpha\in\mathbb{C}\ :\ |r-\alpha|\leq|s|\}.$$

Furthermore, because of $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]=\{r_k\}$ by Theorem 3.3 and the subdivisions in Goldberg's classification are disjoint, we must have

$$\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]A_3=\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]B_3=\emptyset.$$

Hence, $\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_3=\{r_k\}$. Additionally, since $\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_1=\emptyset$ by Theorem 3.7 and Theorem 3.12, we have

$$\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]=\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_2\cup\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_3.$$

Therefore, we derive from Table 1.2 that

$$\begin{aligned}
\sigma_{ap}[B(\widetilde{r},\widetilde{s}),\ell_p] &= \sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]\setminus\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_1=\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]\\
\sigma_{\delta}[B(\widetilde{r},\widetilde{s}),\ell_p] &= \sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]\setminus\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]A_3=\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]\\
\sigma_{\delta}[B(\widetilde{r},\widetilde{s}),\ell_p] &= \sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_2\cup\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right]C_3=\sigma\left[B(\widetilde{r},\widetilde{s}),\ell_p\right].
\end{aligned}$$

□

4. Conclusion

In the present work, as a natural continuation of Akhmedov and El-Shabrawy [4] and, Srivastava and Kumar [38], we have determined the spectrum and the fine spectrum of the double sequential band matrix $B(\widetilde{r}, \widetilde{s})$ on the space ℓ_p. Many researchers determine the spectrum and fine spectrum of a matrix operator in some sequence spaces. In addition to this, we add the definition of some new divisions of spectrum called as approximate point spectrum, defect spectrum and compression spectrum of the matrix operator and give the related results for the matrix operator $B(\widetilde{r}, \widetilde{s})$ on the space ℓ_p which is a new development for this type works giving the fine spectrum of a matrix operator on a sequence space with respect to the Goldberg's classification.

Finally, we should note that in the case $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$ since the operator $B(\widetilde{r}, \widetilde{s})$ defined by a double sequential band matrix reduces to the operator $B(r, s)$ defined by the generalized difference matrix our results are more general and more comprehensive than the corresponding results obtained by Furkan et al. [12] and Bilgiç and Furkan [11], respectively. We record from now on that our next paper will be devoted to the investigation of the fine spectrum of the matrix operator $B(\widetilde{r}, \widetilde{s})$ on the space bv_p.

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Morita theory for group corings

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Abstract

Using the theory of group corings, we study (graded) Morita contexts associated to a comodule over a group coring, which generalize and unify some classical morita contexts. Some applications of our theory are also discussed.

Keywords: group coring, Morita context, cofree.

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1. Introduction

An A-coring is a coalgebra in the monoidal category of A-bimodules over an arbitrary ring A. The concept was introduced by M. Sweedler [14]. In 2000, Takeuchi pointed out that to each entwining structure (A, C, ψ) over a commutative ring k, which was introduced by T. Brzezinski and S. Majid [2], there corresponds an A-coring structure on $\mathcal{C} := A \otimes_k C$. This motivated the revival of the theory of corings and comodules and Brzezinski's paper [3] was the engine behind the revival of the theory of corings and comodules over corings. Many examples of classical categories in noncommutative algebra are special cases of comodules over corings. Let us mention a few of them: the category of a descent datum of a ring extension, graded modules, Hopf modules, Long dimodules, Yetter-Drinfeld modules, Doi-Koppinen modules or entwined modules, and several other categories studied earlier by Hopf algebraists.

One of the important observations is that coring theory provides an elegant approach to descent theory and Galois theory. A systematic study of coring has been carried out in [1, 3, 6, 7, 15]. As the generalization of coring, Caenepeel, Janssen and Wang [5] introduced the group coring and developed Galois theory for group corings.

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It is well-known that the Morita context plays a important role in the theory of Hopf algebras. The first Morita context was constructed by Chase and Sweedler [9], which was generalized by Doi [12]. Morita contexts similar to the one of Doi were studied by Cohen, Fischman and Montgomery in [11]. As the generalization of both contexts, Caenepeel, Vercruysse and Wang associate different types of Morita contexts to a coring with a fixed grouplike element, which was generalized by Caenepeel, Janssen and Wang to group coring with a grouplike family [5]. Without the assumption of a coring with a fixed grouplike element, Caenepeel, De Groot and Vercruysse associated a Morita context to a comodule over a coring in [8]. Morita theory for group corings with fixed grouplike family is a remarkable tool to discuss Hopf-Galois extensions. In order to further discuss coalgebra-Galois extensions, we need to generalize the Morita context for group corings. Naturally, it occurs to us to how to develop (graded) Morita context associated to a comodule over a group coring. This is the motivation of this paper.

The paper is organized as follows.

In Section 2, we recall some basic definitions such as group corings, comodules over a group coring and graded Morita contexts. In Section 3, we associate a Morita context to a comodule over a group coring. In Section 4, we will discuss the graded Morita contexts and their relationship. Some applications of our theory are discussed in Section 5.

2. Preliminaries

Throughout this paper, let G be a group with unit e, and A a ring with unit 1_A, and M an A-module. We will often need collections of A-modules isomorphic to M and indexed by G. We will consider these modules as isomorphic, but distinct. Let $M \times \{\alpha\}$ be the module with index α. We then have isomorphisms

$$\mu_\alpha : M \to M \times \{\alpha\},\ \mu_\alpha(m) = (m, \alpha).$$

We can then write $M \times \{\alpha\} = \mu_\alpha(M)$. μ can be considered as a dummy variable, and we will also use the symbols $\nu, \kappa, \cdots$. We will identify M and $M \times \{e\}$ using μ_e.

2.1. Group Corings. Let A be an algebra. Recall from [5] that a G-group A-coring (or shortly a G-A-coring) $\underline{C}$ is a family $\{C_\alpha\}_{\alpha\in G}$ of A-bimodules together with a family of A-bimodule maps

$$\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes_A C_\beta,\ \varepsilon : C_e \to A$$

such that the following conditions hold:

$$(\Delta_{\alpha,\beta} \otimes_A id) \circ \Delta_{\alpha\beta,\gamma} = (id \otimes_A \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma},$$
$$(id \otimes_A \varepsilon) \circ \Delta_{\alpha,e} = id = (\varepsilon \otimes_A id) \circ \Delta_{e,\alpha}$$

for all $\alpha, \beta, \gamma \in G$.

For a G-A-coring $\underline{C}$, we also use the following Sweedler-type notation for the comultiplication maps $\Delta_{\alpha,\beta}$:

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes_A c_{(2,\beta)}$$

for all $c \in C_{\alpha\beta}$.

A morphism between two G-A-corings $\underline{C}$ and $\underline{D}$ consists of a family of A-bimodule maps $f = \{f_\alpha : C_\alpha \to D_\alpha\}_{\alpha\in G}$ such that

$$(f_\alpha \otimes_A f_\beta) \circ \Delta_{\alpha,\beta} = \Delta_{\alpha,\beta} \circ f_{\alpha\beta}, \quad \varepsilon \circ f_e = \varepsilon.$$

Over a G-A-coring $\underline{C}$, we can define two different types of comodules. A right $\underline{C}$-comodule is a right A-module M together with a family of right A-linear maps $\rho^M = \{\rho^M_\alpha : M \to M \otimes_A C_\alpha\}_{\alpha\in G}$ such that

$$(id \otimes_A \Delta_{\alpha,\beta}) \circ \rho^M_{\alpha\beta} = (\rho^M_\alpha \otimes_A id) \circ \rho_\beta,\ (id \otimes_A \varepsilon) \circ \rho^M_e = id.$$

We use the following Sweedler-type notation:

$$\rho_\alpha^M(m) = m_{[0,\alpha]} \otimes_A m_{[1,\alpha]}$$

for all $m \in M_\alpha$.

A morphism of right $\underline{C}$-comodules is a right A-linear map $f : M \to N$ satisfying the condition

$$(f \otimes_A id) \circ \rho_\alpha^M = \rho_\alpha^N \circ f$$

for all $\alpha \in G$. Let $\mathcal{M}^{\underline{C}}$ denote the category of right $\underline{C}$-comodules.

Similarly, we can define the left $\underline{C}$-comodule and the category ${}^{\underline{C}}\mathcal{M}$ of all left $\underline{C}$-comodules. We use the following Sweedler-type notation for the left $\underline{C}$-comodule structure maps ${}^M\rho_\alpha$:

$${}^M\rho_\alpha(m) = m_{[-1,\alpha]} \otimes_A m_{[0,\alpha]}$$

for all $m \in M_\alpha$.

A right G-$\underline{C}$-comodule $\underline{M}$ is a family of right A-modules $\{M_\alpha\}_{\alpha\in G}$(meaning that each M_α is right A-module), together with a family of right A-linear maps $\rho = \{\rho_{\alpha,\beta}\}_{\alpha,\beta\in G}$, where $\rho_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes_A C_\beta$, such that the following conditions hold:

$$(id \otimes_A \Delta_{\beta,\gamma}) \circ \rho_{\alpha,\beta\gamma} = (\rho_{\alpha,\beta} \otimes_A id) \circ \rho_{\alpha\beta,\gamma},\ (id \otimes_A \varepsilon) \circ \rho_{\alpha,e} = id$$

for all $\alpha, \beta, \gamma \in G$.

We use the following standard notation:

$$\rho_{\alpha,\beta}(m) = m_{[0,\alpha]} \otimes_A m_{[1,\beta]}$$

for $m \in M_{\alpha\beta}$.

A morphism between two right G-$\underline{C}$-comodules $\underline{M} = \{M_\alpha\}_{\alpha\in G}$ and $\underline{N} = \{N_\alpha\}_{\alpha\in G}$ is a family of right A-linear maps $f = \{f_\alpha : M_\alpha \to N_\alpha\}_{\alpha\in G}$ such that

$$(f_\alpha \otimes_A id) \circ \rho_{\alpha,\beta} = \rho_{\alpha,\beta} \circ f_{\alpha\beta}.$$

The category of right G-$\underline{C}$-comodules will be denoted by $\mathcal{M}^{G,\underline{C}}$.

Let $\underline{C}$ be a G-A-coring. A family $g = (g_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} C_\alpha$ is called grouplike, if $\Delta_{\alpha,\beta}(g_{\alpha\beta}) = g_\alpha \otimes_A g_\beta$ and $\varepsilon(g_e) = 1$ for all $\alpha, \beta \in G$.

Let $\underline{C}$ be a G-A-coring with a fixed grouplike family $g = (g_\alpha)_{\alpha\in G}$. Then A can be endowed with a structure of right $\underline{C}$-comodule via the coaction maps

$$\rho_\alpha : A \to A \otimes_A C_\alpha,\ \rho_\alpha(a) = 1_A \otimes_A g_\alpha \cdot a.$$

For $M \in \mathcal{M}^{\underline{C}}$, we define

$$M^{co\underline{C}} = \{m \in M | \rho_\alpha(m) = m \otimes_A g_\alpha, \forall\alpha \in G\}.$$

In particular,

$$A^{co\underline{C}} = \{a \in A | a \cdot g_\alpha = g_\alpha \cdot a, \forall\alpha \in G\}.$$

Let $A \otimes_B A$ be the canonical Sweedler coring associated to the ring morphism $B \to A$ with its comultiplication and counit given by the formulas

$$\Delta(a \otimes_B b) = (a \otimes_B 1_A) \otimes_A (1_A \otimes_B b),\ \varepsilon(a \otimes_B b) = ab.$$

2.2. Graded Rings and Modules. Let A be a ring and $\mathcal{R} = \bigoplus_{\alpha\in G} \mathcal{R}_\alpha$ a G-graded ring. Suppose that we have a ring morphism $i : A \to \mathcal{R}_e$. Then we call $\mathcal{R}$ a G-graded A-ring. Every $\mathcal{R}_\alpha$ is then an A-bimodule and the decomposition of $\mathcal{R}$ is a decomposition of A-bimodules. The category of G-graded right $\mathcal{R}$-modules will be denoted by $\mathcal{M}_{\mathcal{R}}^G$.

Let $\underline{C}$ be a G-A-coring. For every $\alpha \in G$, $\mathcal{R}_\alpha = {}_A\mathrm{HOM}(C_{\alpha^{-1}}, A)$ is an A-bimodule via

$$(a \cdot f_\alpha \cdot b)(c) = f_\alpha(c \cdot a)b$$

for all $f_\alpha \in \mathcal{R}_\alpha$, $a, b \in A$ and $c \in C_{\alpha^{-1}}$. Take $f_\alpha \in \mathcal{R}_\alpha, g_\beta \in \mathcal{R}_\beta$ and define $f_\alpha \star g_\beta \in \mathcal{R}_{\alpha\beta}$ by the following formula:

$$(f_\alpha \star g_\beta)(c) = g_\beta(c_{(1,\beta^{-1})} \cdot f_\alpha(c_{(2,\alpha^{-1})}))$$

for all $c \in C_{(\alpha\beta)^{-1}}$. This defines maps $m_{\alpha,\beta} : \mathcal{R}_\alpha \otimes_A \mathcal{R}_\beta \to \mathcal{R}_{\alpha\beta}$, which make $\mathcal{R} = \oplus_{\alpha\in G}\mathcal{R}_\alpha$ into a G-graded ring with the unit ε. Define $i : A \to \mathcal{R}_e, i(a)(c) = \varepsilon(c)a$ is a ring homomorphism, which make $\mathcal{R} = \bigoplus_{\alpha\in G}\mathcal{R}_\alpha$ be a G-graded A-ring, called the (left) dual (graded) ring of the group coring $\underline{C}$. We will also write $\mathcal{R} = {}^*\underline{C}$.

2.3. Graded Morita Contexts. Let $\mathcal{R}$ be a G-graded ring, and $M, N \in \mathcal{M}^G_{\mathcal{R}}$. A right $\mathcal{R}$-linear map $f : M \to N$ is called homogeneous of degree σ, if $f(M_\alpha) \subset M_{\sigma\alpha}$ for all $\alpha \in G$. $\mathrm{HOM}_{\mathcal{R}}(M, N)_\sigma$ denotes the additive group of all right $\mathcal{R}$-module maps of degree σ.

Let S and $\mathcal{R}$ be G-graded rings. A G-graded Morita context connecting S and $\mathcal{R}$ is a Morita context $(S, \mathcal{R}, P, Q, \varphi, \psi)$ with the following additional structure: P and Q are graded bimodules, and the maps

$$\varphi : P \otimes_{\mathcal{R}} Q \to S,\ \psi : Q \otimes_S P \to \mathcal{R}$$

are homogeneous of degree e.

Given two graded Morita contexts $(S, \mathcal{R}, P, Q, \varphi, \psi)$ and $(\tilde{S}.\tilde{\mathcal{R}}, \tilde{P}, \tilde{Q}, \tilde{\varphi}, \tilde{\psi})$, if there exist two graded ring morphism $\Phi : S \to \tilde{S}$, $\Psi : \mathcal{R} \to \tilde{\mathcal{R}}$ and two graded bimodule morphism $\Theta : Q \to \tilde{Q}$, $\Xi : P \to \tilde{P}$ such that the following two diagrams

$$\begin{tikzcd} P \otimes_{\mathcal{R}} Q \arrow[r, "\varphi"] \arrow[d, "\Xi\otimes\Theta"'] & S \arrow[d, "\Phi"] \\ \tilde{P} \otimes_{\tilde{\mathcal{R}}} \tilde{Q} \arrow[r, "\tilde{\varphi}"] & \tilde{S} \end{tikzcd} \qquad \begin{tikzcd} Q \otimes_S P \arrow[r, "\psi"] \arrow[d, "\Theta\otimes\Xi"'] & \mathcal{R} \arrow[d, "\Psi"] \\ \tilde{Q} \otimes_{\tilde{S}} \tilde{P} \arrow[r, "\tilde{\psi}"] & \tilde{\mathcal{R}} \end{tikzcd}$$

are communicative, then we say a quadruple $\widetilde{\Upsilon} = (\Phi, \Psi, \Theta, \Xi)$ a morphism from $(S, \mathcal{R}, P, Q, \varphi, \psi)$ to $(\tilde{S}, \tilde{\mathcal{R}}, \tilde{P}, \tilde{Q}, \tilde{\varphi}, \tilde{\psi})$

Let P be a G-graded right $\mathcal{R}$-module. Then $S = \mathrm{END}_{\mathcal{R}}(P)$ is a G-graded ring, and $Q = \mathrm{HOM}_{\mathcal{R}}(P, R) \in {}_R\mathcal{M}^G_S$ with structure

$$(r \cdot q \cdot s)(p) = rq(s(p))$$

for all $r \in \mathcal{R}$, $s \in S$, $q \in Q$ and $p \in P$. The connecting maps are the following

$$\varphi : P \otimes_{\mathcal{R}} Q \to S,\ \varphi(p \otimes_{\mathcal{R}} q)(p') = pq(p'),$$

$$\psi : Q \otimes_S P \to R,\ \psi(q \otimes_S p) = q(p).$$

Then $(S, \mathcal{R}, P, Q, \varphi, \psi)$ is a graded Morita context.

2.4. Cofree Group Corings. A G-A-coring $\underline{C} = \{C_\alpha\}_{\alpha\in G}$ is called cofree, if there exist A-bimodule isomorphisms $\gamma_\alpha : C_e \to C_\alpha$ such that

$$\Delta_{\alpha,\beta}(\gamma_{\alpha\beta}(c)) = \gamma_\alpha(c_{(1,e)}) \otimes_A \gamma_\beta(c_{(2,e)})$$

for all $c \in C_e$. If $\underline{C}$ is a cofree group coring, then, for every $\alpha \in G$, we have A-bimodule isomorphisms

$$\gamma_{\alpha^{-1}} : C_e \to C_{\alpha^{-1}},\ {}^*\gamma_{\alpha^{-1}} : \mathcal{R}_\alpha \to \mathcal{R}_e,$$

and

$$\chi_\alpha = ({}^*\gamma_{\alpha^{-1}})^{-1} : \mathcal{R}_e \to \mathcal{R}_\alpha.$$

From Proposition 4.6 in [5], the left dual $\mathcal{R} = {}^*\underline{C}$ is the group ring $\mathcal{R}_e[G]$.

Let $\underline{C} = C_e\langle G\rangle$ be a cofree G-A-coring and M be a right $\underline{C}$-comodule. Recall from [10] that we call that M is a cofree $\underline{C}$-comodule, if $(id \otimes_A \gamma_\alpha) \circ \rho^M_e = \rho^M_\alpha$.

2.1. Example. If $\underline{C} = C_e\langle G\rangle$ is a cofree G-A-coring and $g = (g_\alpha)_{\alpha\in G}$ a grouplike family of $\underline{C}$ such that $g_\alpha = \gamma_\alpha(g_e)$. Then A can be endowed with a structure of right $\underline{C}$-comodule via the coaction maps

$$\rho_\alpha^A : A \to A \otimes_A C_\alpha,\ \rho_\alpha^A(a) = 1_A \otimes_A g_\alpha \cdot a.$$

For all $a \in A$, we have

$$(id \otimes_A \gamma_\alpha) \circ \rho_e^A(a) = 1_A \otimes_A \gamma_\alpha(g_e \cdot a) = 1_A \otimes_A g_\alpha \cdot a = \rho_\alpha^A(a),$$

this shows that A is a cofree $\underline{C}$-module.

2.5. Group Entwining Structures. Let $\underline{C} = \{C_\alpha\}_{\alpha\in G}$ be a G-coalgebra and A an algebra. We say that the G-coalgebra $\underline{C}$ and the algebra A are G-entwined, if there is a family of linear maps $\psi = \{\psi_\alpha : C_\alpha \otimes A \to A \otimes C_\alpha\}_{\alpha\in G}$ such that

- $(ab)_{\psi_\alpha} \otimes c^{\psi_\alpha} = a_{\psi_\alpha} b_{\psi'_\alpha} \otimes c^{\psi_\alpha\psi'_\alpha}$,
- $1_{A\psi_\alpha} \otimes c^{\psi_\alpha} = 1_A \otimes c$, for any $c \in C_\alpha$,
- $a_{\psi_{\alpha\beta}} \otimes c^{\psi_{\alpha\beta}}{}_{(1,\alpha)} \otimes c^{\psi_{\alpha\beta}}{}_{(2,\beta)} = a_{\psi_\beta\psi_\alpha} \otimes c_{(1,\alpha)}{}^{\psi_\alpha} \otimes c_{(2,\beta)}{}^{\psi_\beta}$,
- $a_{\psi_e}\varepsilon(c^{\psi_e}) = a\varepsilon(c)$, for any $c \in C_e$ and $a \in A$.

where, we set $\psi_\alpha(c\otimes a) = a_{\psi_\alpha} \otimes c^{\psi_\alpha} = a_{\psi'_\alpha} \otimes c^{\psi'_\alpha} = \cdots$, for $a \in A$ and $c \in C_\alpha$. The triple $(A, \underline{C}, \psi)$ is called a right and right G-entwining structure and is denoted by $(A, \underline{C})_{G-\psi}$.

Given a right-right G-entwining structure $(A, \underline{C})_{G-\psi}$, then $\mathcal{U}_A^{\underline{C}}(\psi)$ is the category of right $(A, \underline{C})_\psi$. The object of $\mathcal{U}_A^{\underline{C}}(\psi)$ are right $\underline{C}$-comodules (M, ρ_α^M) which is also A-module such that

$$\rho_\alpha^M(m \cdot a) = m_{[0,\alpha]} \cdot a_{\psi_\alpha} \otimes m_{[1,\alpha]}{}^{\psi_\alpha}$$

for all $m \in M$ and $a \in A$. Morphisms in $\mathcal{U}_A^{\underline{C}}(\psi)$ are right $\underline{C}$-comodule and right A-module maps and let $\mathcal{U}_A^{G,\underline{C}}(\psi)$ be the category of right $(A, \underline{C})_{G-\psi}$ of which the objects are right G-$\underline{C}$-comodules $(\underline{M}, \rho_{\alpha,\beta}^M)$ which is also right A-module, i.e., each M_α is right A-module, such that

$$\rho_{\alpha,\beta}^M(m \cdot a) = m_{[0,\alpha]} \cdot a_{\psi_\beta} \otimes m_{[1,\beta]}{}^{\psi_\beta}$$

for all $m \in M_{\alpha\beta}$ and $a \in A$. Morphisms in $\mathcal{U}_A^{G,\underline{C}}(\psi)$ are right G-$\underline{C}$-comodule and right A-module maps.

2.6. Group Coalgebra Galois Extensions. Let $\underline{C}$ be a G-coalgebra and A an algebra. Let A be a right $\underline{C}$-comodule. Let

$$B = A^{co\underline{C}} = \{a \in A | \rho_\alpha^A(ab) = a\rho_\alpha^A(b), \forall b \in A, \alpha \in G\}.$$

We say that A is a right G-$\underline{C}$-Galois extension of B, if the canonical left A-module right G-$\underline{C}$-comodule map $can = \{can_\alpha : A \otimes_B A \to A \otimes C_\alpha\}$, by $a \otimes_B b \mapsto ab_{[0,\alpha]} \otimes b_{[1,\alpha]}$ for all $a, b \in A$ is bijective, i.e., every map can_α is bijective for al $\alpha \in G$.

3. Morita Context associated to a Comodule over a Group Coring

Let $\underline{C}$ be a G-A-coring, and $M \in {}^{\underline{C}}\mathcal{M}$. We can associate a Morita context to M. The context will connect $T = {}^{\underline{C}}\mathrm{END}(M)^{op}$ and ${}^*\underline{C} = \mathcal{R}$.

For every $\alpha \in G$, $Q_\alpha = {}_A\mathrm{HOM}(C_{\alpha^{-1}}, M) \in {}_R\mathcal{M}_T$ is a left A-module with

$$(a \cdot f_\alpha)(c) = f_\alpha(c \cdot a)$$

for all $f_\alpha \in Q_\alpha$, $a \in A$ and $c \in C_{\alpha^{-1}}$. Let

$$Q = \{q \in \bigoplus_{\alpha\in G} Q_\alpha | q_{\alpha\beta}(c)_{[-1,\beta^{-1}]} \otimes_A q_{\alpha\beta}(c)_{[0,\beta^{-1}]} = c_{(1,\beta^{-1})} \otimes_A q_\alpha(c_{(2,\alpha^{-1})}), \forall c \in C_{\beta^{-1}\alpha^{-1}}\}.$$

3.1. Lemma. *With the notation as above, ${}^*M = {}_A\mathrm{HOM}(M, A) \in {}_T\mathcal{M}_{\mathcal{R}}$ and $Q \in {}_{\mathcal{R}}\mathcal{M}_T$.*

Proof. Let $\zeta \in {}^*M$, $f_\alpha \in \mathcal{R}_\alpha$, $t \in T$, $q_\beta \in Q_\beta$ and $m \in M$. We define the bimodule structure on ${}^*M$ as follows:

$$(\zeta \cdot f_\alpha)(m) = f_\alpha(m_{[-1,\alpha^{-1}]} \cdot \zeta(m_{[0,\alpha^{-1}]})) \text{ and } t \cdot \zeta = \zeta \circ t.$$

For all $g_\beta \in \mathcal{R}_\beta$, we have

$$\begin{aligned}
&((\zeta \cdot f_\alpha) \cdot g_\beta)(m)\\
&= g_\beta(m_{[-1,\beta^{-1}]} \cdot f_\alpha(m_{[0,\beta^{-1}][-1,\alpha^{-1}]} \cdot \zeta(m_{[0,\beta^{-1}][0,\alpha^{-1}]})))\\
&= g_\beta(m_{[-1,\beta^{-1}\alpha^{-1}](1,\beta^{-1})} \cdot f_\alpha(m_{[-1,\beta^{-1}\alpha^{-1}](2,\alpha^{-1})} \cdot \zeta(m_{[0,\beta^{-1}\alpha^{-1}]})))\\
&= (f_\alpha \star g_\beta)(m_{[-1,\beta^{-1}\alpha^{-1}]} \cdot \zeta(m_{[0,\beta^{-1}\alpha^{-1}]}))\\
&= (\zeta \cdot (f_\alpha \star g_\beta))(m).
\end{aligned}$$

This shows that ${}^*M$ is a G-graded right $\mathcal{R}$-module. Let us show that the two actions commute. Indeed, we compute

$$\begin{aligned}
(t \cdot (\zeta \cdot f_\alpha))(m) =&(\zeta \cdot f_\alpha)(t(m))\\
=&f_\alpha(t(m)_{[-1,\alpha^{-1}]} \cdot \zeta(t(m)_{[0,\alpha^{-1}]}))\\
=&f_\alpha(m_{[-1,\alpha^{-1}]} \cdot \zeta(t(m_{[0,\alpha^{-1}]})))\\
=&(t \cdot \zeta) \cdot f_\alpha.
\end{aligned}$$

The bimodule structure on Q is defined by

$$(f_\alpha \cdot q_\beta)(c) = q_\beta(c_{(1,\beta^{-1})} \cdot f_\alpha(c_{(2,\alpha^{-1})}))$$

for all $c \in C_{\beta^{-1}\alpha^{-1}}$ and $q_\beta \cdot t = t \circ q_\beta$. □

3.2. Lemma. *With the notation as above, we have well-defined bimodule maps*

$$\mu : Q \otimes_T {}^*M \to \mathcal{R},\ \mu((q \otimes_T \zeta) = \sum_{\alpha \in G} \zeta \circ q_\alpha,$$

$$\tau : {}^*M \otimes_{\mathcal{R}} Q \to T,\ \tau(\zeta \otimes_{\mathcal{R}} q)(m) = \sum_{\alpha \in G} q_\alpha(m_{[-1,\alpha^{-1}]} \cdot \zeta(m_{[0,\alpha^{-1}]})).$$

3.3. Theorem. *With the notation as above, we have a Morita context $(T, \mathcal{R}, {}^*M, Q, \tau, \mu)$.*

Proof. Here we only check that, for $\zeta, \zeta' \in {}^*M$, $q, q' \in Q$ and $m \in M$,

$$q' \cdot \tau(\zeta \otimes_{\mathcal{R}} q) = \mu(q' \otimes_T \zeta) \cdot q,\ \zeta \cdot \mu(q \otimes_T \zeta') = \tau(\zeta \otimes_{\mathcal{R}} q) \cdot \zeta' \tag{3.1}$$

hold. Indeed, for all $c \in C_{(\alpha\gamma)^{-1}}$, we compute

$$\begin{aligned}
(q'_{\alpha\gamma} \cdot \tau(\zeta \otimes_{\mathcal{R}} q))(c) &= \tau(\zeta \otimes_{\mathcal{R}} q)(q'_{\alpha\gamma}(c))\\
&= \sum_{\beta \in G} q_\beta(q'_{\alpha\gamma}(c)_{[-1,\beta^{-1}]} \cdot \zeta(q'_{\alpha\gamma}(c)_{[0,\beta^{-1}]}))\\
&= \sum_{\beta \in G} q_\beta(c_{(1,\beta^{-1})} \cdot \zeta(q'_{\alpha\gamma\beta^{-1}}(c_{(2,\beta\gamma^{-1}\alpha^{-1})})))\\
&= \sum_{\beta \in G} ((\zeta \circ q'_{\alpha\gamma\beta^{-1}}) \cdot q_\beta)(c)\\
&= (\mu(q'_{\alpha\gamma} \otimes_T \zeta) \cdot q)(c).
\end{aligned}$$

Thus we show that the first identity in (3.1) holds. The other identity can be checked similarly. □

Next, we want to make an application of Theorem 3.3 in order to get a new Morita context.

Let M be a right $\underline{C}$-comodule. Assume that M is finitely generated and projective with the finite dual basis $\{e_i, e_i^*\}$ or $\{e'_i, e'^*_i\}$. M^* $=\mathrm{Hom}_A(M, A)$ can be viewed as a left A-module via $(a\cdot f)(m) = af(m)$. Then M^* is a left $\underline{C}$-comodule with the coaction maps

$$^{M^*}\rho_\alpha : M^* \to C_\alpha \otimes_A M^*,\ \ ^{M^*}\rho_\alpha(f) = \sum_i f(e_{i[0,\alpha]})\cdot e_{i[1,\alpha]} \otimes_A e_i^*$$

Indeed, we compute

$$\begin{aligned}(id \otimes^{M^*} \rho_\beta)\circ^{M^*} \rho_\alpha(f) &= (id \otimes^{M^*} \rho_\beta)(\sum_i f(e_{i[0,\alpha]})\cdot e_{i[1,\alpha]} \otimes_A e_i^*)\\ &= \sum_{i,j} f(e_{i[0,\alpha]})\cdot e_{i[1,\alpha]} \otimes_A e_i^*(e'_{j[0,\beta]})\cdot e'_{j[1,\beta]} \otimes_A e'^*_j\\ &= \sum_{i,j} f(e_{i[0,\alpha]})\cdot e_{i[1,\alpha]}\cdot e_i^*(e'_{j[0,\beta]}) \otimes_A e'_{j[1,\beta]} \otimes_A e'^*_j\\ &= \sum_i f(e_{i[0,\beta][0,\alpha]]})\cdot e_{i[0,\beta][1,\alpha]} \otimes_A e_{i[1,\beta]} \otimes_A e_i^*\\ &= \sum_i f(e_{i[0,\alpha\beta]})\cdot e_{i[1,\alpha\beta](1,\alpha)} \otimes_A e_{i[1,\alpha\beta](2,\beta)} \otimes_A e_i^*.\end{aligned}$$

This shows that $^{M^*}\rho = \{^{M^*}\rho_\alpha\}_{\alpha\in G}$ is $\underline{C}$-colinear.

3.4. Lemma. *Let M be a right $\underline{C}$-comodule. Assume that M is finitely generated and projective. Then*

$$^{\underline{C}}\mathrm{END}(M^*)^{op} \cong \mathrm{END}^{\underline{C}}(M).$$

Proof. Let $\{e_i, e_i^*\}$ be the dual basis of M. We construct the desired maps as follows:

$$\Phi :^{\underline{C}} \mathrm{END}(M^*)^{op} \to \mathrm{END}^{\underline{C}}(M),\ \Phi(f)(m) = \sum_i e_i \cdot f(e_i^*)(m)$$

and

$$\Psi : \mathrm{END}^{\underline{C}}(M) \to^{\underline{C}} \mathrm{END}(M^*)^{op},\ \Psi(f)(g)(m) = g(f(m)).$$

The other verifications are straightforward. □

From Lemma 3.4 and Theorem 3.3, we have the following result.

3.5. Corollary. *Let M be a right $\underline{C}$-comodule. Assume that M is finitely generated and projective. We obtain a Morita context*

$$(\mathrm{END}^{\underline{C}}(M), \mathcal{R}, M, Q =^{\underline{C}} \mathrm{HOM}(\underline{C}, M^*), \tau, \mu)$$

with $M \in_T \mathcal{M}_\mathcal{R}$ by

$$m\cdot f_\alpha = m_{[0,\alpha^{-1}]}\cdot f_\alpha(m_{[1,\alpha^{-1}]}) \text{ and } t\cdot m = t(m)$$

for all $m \in M$, $f \in \mathcal{R}_\alpha$, $t \in T$, and $Q \in_R \mathcal{M}_T$ by

$$(f_\alpha \cdot q_\beta)(c) = q_\beta(c_{(1,\beta^{-1})}\cdot f_\alpha(c_{(2,\alpha^{-1})}))$$

for all $c \in C_{\beta^{-1}\alpha^{-1}}$ and $q_\beta \in Q_\beta$ and $(q_\beta\cdot t)(c') = q_\beta(c')\circ t$ for all $c' \in C_{\beta^{-1}}$, and

$$\mu : Q \otimes_T M \to R,\ \mu(q \otimes_T m)_\alpha(c) = q_\alpha(c)(m), \forall c \in C_{\alpha^{-1}}$$

$$\tau : M \otimes_R Q \to T,\ \tau(m \otimes_R q)(m') = \sum_{\alpha\in G} m_{[0,\alpha^{-1}]}\cdot(q_\alpha(m_{[0,\alpha^{-1}]})(m')).$$

3.6. Example. Let $\underline{C}$ be a G-A-coring with a grouplike family $g = (g_\alpha)_{\alpha\in G}$. Then A is a right $\underline{C}$-comodule via

$$\rho_\alpha^A : A \to A \otimes_A C_\alpha,\ \rho_\alpha^A(a) = 1_A \otimes_A g_\alpha \cdot a.$$

By [10], $T = \mathrm{END}^{\underline{C}}(A)$ is nothing but the $A^{co\underline{C}}$. Since $A^* \cong A$, we have

$$Q = \{q \in \mathcal{R} | q_{\alpha\beta}(c) g_{\beta^{-1}} = c_{(1,\beta^{-1})} \cdot q_\alpha(c_{(2,\alpha^{-1})}), \forall c \in C_{\beta^{-1}\alpha^{-1}}\}.$$

Applying Corollary 3.5, we have a Morita context as in [5].

4. Graded Morita Context associated to a Comodule over a Group Coring

In this section, we assume that $M \in \mathcal{M}^{\underline{C}}$ is finitely generated and projective with the dual basis $\{e_i, e_i^*\}$. We say that a G-A-coring $\underline{C}$ is left homogeneously finite, if each C_α is finitely generated and projective as a left A-module. For $M \in \mathcal{M}^{\underline{C}}$, it follows that $\{\mu_\alpha(M)\}_{\alpha\in G} \in \mathcal{M}^{G,\underline{C}}$ with the coaction maps

$$\rho_{\alpha,\beta} : \mu_{\alpha\beta}(M) \to \mu_\alpha(M) \otimes_A C_\beta,\ \rho_{\alpha,\beta}(\mu_{\alpha\beta}(m)) = \mu_\alpha(m_{[0,\beta]}) \otimes_A m_{[1,\beta]}.$$

From Proposition 4.1 in [5], we then obtain that

$$M\{G\} = \bigoplus_{\alpha\in G} \mu_\alpha(M) \in \mathcal{M}_{\mathcal{R}}^G.$$

The right $\mathcal{R}$-action is defined by the following formula,

$$\mu_\alpha(m) \cdot f_\beta = \mu_{\alpha\beta}(m_{[0,\beta^{-1}]}) \cdot f_\beta(m_{[1,\beta^{-1}]})$$

for all $f_\beta \in \mathcal{R}_\beta$ and $m \in M$.

Next, we will compute the graded Morita context associated to the graded right $\mathcal{R}$-module $M\{G\}$. Consider the ring

$$S = \{\underline{f} = (f_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} \mathrm{END}_A(M) | f_\alpha(m)_{[0,\beta^{-1}]} \otimes_A f_\alpha(m)_{[1,\beta^{-1}]} = f_{\alpha\beta}(m_{[0,\beta^{-1}]}) \otimes_A m_{[1,\beta^{-1}]}\}.$$

Observe that we have a ring monomorphism

$$i : T \to S,\ i(f) = \underline{f} = (f)_\alpha.$$

On S, we have the following right G-action:

$$\underline{f}^\sigma = \underline{f} \cdot \sigma = (f_{\sigma\alpha})_{\alpha\in G}.$$

Indeed, if $\underline{f} \in S$, we have $\underline{f} \cdot \sigma \in S$, since

$$f_{\sigma\alpha}(m)_{[0,\beta^{-1}]} \otimes_A f_{\sigma\alpha}(m)_{[1,\beta^{-1}]} = f_{\sigma\alpha\beta}(m_{[0,\beta^{-1}]}) \otimes_A m_{[1,\beta^{-1}]},$$

Now, we consider the twisted group ring $G * S = \bigoplus_{\alpha\in G} \mu_\alpha S$ with multiplication

$$\mu_\alpha \underline{f} \mu_\beta \underline{g} = \mu_{\alpha\beta}((\underline{f} \cdot \beta)\underline{g}).$$

4.1. Proposition. *If G-A-coring $\underline{C}$ is left homogeneously finite, We then have a graded ring isomorphism*

$$\Omega : \mathrm{END}_{\mathcal{R}}(M\{G\}) \to G * S.$$

Proof. For each $\sigma \in G$, we construct a map by

$$\Omega_\sigma : \mathrm{END}_{\mathcal{R}}(M\{G\})_\alpha \to \mu_\sigma S,\ \Omega_\sigma(h) = \mu_\sigma \underline{f}$$

with $f_\alpha(m) = \mu^{-1}_{\sigma\alpha}(h(\mu_\alpha(m)))$. Since h is right $\mathcal{R}$-linear, we have, for all $m \in M$ and $g \in \mathcal{R}_\beta$ that

$$\begin{aligned} h(\mu_\alpha(m) \cdot g_\beta) &= h(\mu_{\alpha\beta}(m_{[0,\beta^{-1}]}) \cdot g_\beta(m_{[1,\beta^{-1}]})) \\ &= \mu_{\sigma\alpha\beta}(f_{\alpha\beta}(m_{[0,\beta^{-1}]})) \cdot g_\beta(m_{[1,\beta^{-1}]})). \end{aligned}$$

Since

$$\begin{aligned} h(\mu_\alpha(m) \cdot g_\beta) &= h(\mu_\alpha(m)) \cdot g_\beta \\ &= \mu_{\sigma\alpha}(f_\alpha(m)) \cdot g_\beta \\ &= \mu_{\sigma\alpha\beta}((f_\alpha(m)_{[0,\beta^{-1}]}) \cdot g_\beta(f_\alpha(m)_{[1,\beta^{-1}]})), \end{aligned}$$

it follows that

$$(f_\alpha(m)_{[0,\beta^{-1}]}) \cdot g_\beta(f_\alpha(m)_{[1,\beta^{-1}]}) = f_{\alpha\beta}(m_{[0,\beta^{-1}]}) \cdot g_\beta(m_{[1,\beta^{-1}]}).$$

Since $\underline{C}$ is left homogeneously finite (also see Lemma 4.2 in [5]), we have

$$(f_\alpha(m)_{[0,\beta^{-1}]}) \otimes_A f_\alpha(m)_{[1,\beta^{-1}]}) = f_{\alpha\beta}(m_{[0,\beta^{-1}]}) \otimes_A m_{[1,\beta^{-1}]}.$$

This means $\underline{f} \in S$. Next, we define a map

$$\Upsilon_\sigma : \mu_\sigma S \to \mathrm{END}_{\mathcal{R}}(M\{G\})_\alpha,\ \Upsilon_\sigma(\underline{f}) = h$$

where h satisfies $h(\mu_\alpha(m)) = \mu_{\sigma\alpha}(f_\alpha(m))$. It is straightforward to check that Υ_σ and Ω_σ are mutually inverses. It is routine to check that

$$\Omega = \bigoplus_{\alpha \in G} \Omega_\alpha : \mathrm{END}_{\mathcal{R}}(M\{G\}) \to G * S$$

preserves the multiplication and the unit. □

Our next aim is to describe $\mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R})$. Consider

$$\begin{aligned} Q = \{\underline{q} = (q_\alpha)_{\alpha\in G} \in \prod_{\alpha \in G} {}_A\mathrm{HOM}(C_{\alpha^{-1}}, M^*)| \\ c_{(1,\beta^{-1})} \otimes_A q_\alpha(c_{(2,\alpha^{-1})}) = q_{\alpha\beta}(c)(e_{i[0,\beta^{-1}]}) \cdot e_{i[1,\beta^{-1}]} \otimes_A e_i^*,\ c \in C_{(\alpha\beta)^{-1}}\}. \end{aligned}$$

4.2. Lemma. *If $f_\gamma \in \mathcal{R}_\gamma$ and $\underline{q} \in Q$, then*

$$f_\gamma \cdot \underline{q} = (f_\gamma \cdot q_{\gamma^{-1}\alpha})_{\alpha \in G} \in Q.$$

Proof. For all $c \in C_{(\alpha\beta)^{-1}}$ and $m \in M$, we have

$$\begin{aligned} & c_{(1,\beta^{-1})} \otimes_A (f_\gamma \cdot q_{\gamma^{-1}\alpha})(c_{(2,\alpha^{-1})}) \\ &= c_{(1,\beta^{-1})} \otimes_A q_{\gamma^{-1}\alpha}(c_{(2,\alpha^{-1}\gamma)} \cdot f_\gamma(c_{(3,\gamma^{-1})})) \\ &= (c_{(1,\beta^{-1}\alpha^{-1}\gamma)} \cdot f_\gamma(c_{(2,\gamma^{-1})}))_{(1,\beta^{-1})} \otimes_A q_{\gamma^{-1}\alpha}((c_{(1,\beta^{-1}\alpha^{-1}\gamma)} \cdot f_\gamma(c_{(2,\gamma^{-1})}))_{(2,\alpha^{-1}\gamma)}) \\ &= \sum_i q_{\gamma^{-1}\alpha\beta}(c_{(1,\beta^{-1}\alpha^{-1}\gamma)} \cdot f_\gamma(c_{(2,\gamma^{-1})}))(e_{i[0,\beta^{-1}]}) \cdot e_{i[1,\beta^{-1}]} \otimes_A e_i^* \\ &= (f_\gamma \cdot q_{\gamma^{-1}\alpha\beta})(c)(e_{i[0,\beta^{-1}]}) \cdot e_{i[1,\beta^{-1}]} \otimes_A e_i^*. \end{aligned}$$

□

4.3. Lemma. *If $\underline{q} \in Q$ and $\underline{f} \in S$, then $\underline{q} \cdot \underline{f} = (q_\alpha \cdot f_\alpha)_{\alpha \in G} \in Q$, where*

$$(q_\alpha \cdot f_\alpha)(c) = q_\alpha(c) \circ f_\alpha$$

for all $c \in C_{\alpha^{-1}}$.

4.4. Lemma.

$$QG = \bigoplus_{\alpha \in G} \omega_\alpha(Q) \in {}_{\mathcal{R}}\mathcal{M}^G_{G*S}$$

with bimodule structures defined as follows: for all $f \in \mathcal{R}_\beta$, $\underline{q} \in Q$ and $\underline{b} \in S$,

$$f_\beta \cdot \omega_\alpha(\underline{q}) = \omega_{\beta\alpha}(f_\beta \cdot \underline{q}),\ \ \omega_\alpha(\underline{q}) \cdot \mu_\tau \underline{b} = \omega_{\alpha\tau}(\underline{q} \cdot (\underline{b} \cdot (\alpha\tau)^{-1})).$$

4.5. Proposition. *If G-A coring $\underline{C}$ is left homogeneously finite, and $M \in \mathcal{M}^{\underline{C}}$ is finitely generated and projective as a right A-module. We then have an isomorphism of graded bimodules*

$$\Psi : \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) \to QG$$

Proof. For each $\sigma \in G$, we construct a map by

$$\Psi_\sigma : \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R})_\sigma \to \omega_\sigma(Q),\ \ \Psi_\sigma(g) = \omega_\sigma(\underline{q})$$

with $q_\alpha(c)(m) = g(\mu_{\sigma^{-1}\alpha}(m))(c)$ for all $c \in C_{\alpha^{-1}}$ and $m \in M$. Take $\beta \in G$ and $f_\beta \in \mathcal{R}_\beta$. Since g is right $\mathcal{R}$-linear, we have, for all $m \in M$ that

$$\begin{aligned} g(\mu_{\sigma^{-1}\alpha}(m) \cdot f_\beta) &= g(\mu_{\sigma^{-1}\alpha\beta}(m_{[0,\beta^{-1}]}) \cdot f_\beta(m_{[1,\beta^{-1}]})) \\ &= g(\mu_{\sigma^{-1}\alpha\beta}(m_{[0,\beta^{-1}]})) \cdot f_\beta(m_{[1,\beta^{-1}]}). \end{aligned}$$

Notice that

$$g(\mu_{\sigma^{-1}\alpha}(m) \cdot f_\beta) = g(\mu_{\sigma^{-1}\alpha}(m)) \cdot f_\beta.$$

Thus, for all $c \in C_{(\alpha\beta)^{-1}}$, we have

$$\begin{aligned} &(g(\mu_{\sigma^{-1}\alpha\beta}(m_{[0,\beta^{-1}]})) \cdot f_\beta(m_{[1,\beta^{-1}]}))(c) \\ &= (g(\mu_{\sigma^{-1}\alpha\beta}(m_{[0,\beta^{-1}]})))(c) f_\beta(m_{[1,\beta^{-1}]}) \\ &= q_{\alpha\beta}(c)(m_{[0,\beta^{-1}]}) f_\beta(m_{[1,\beta^{-1}]}) \end{aligned}$$

and

$$(g(\mu_{\sigma^{-1}\alpha}(m)) \cdot f_\beta)(c) = f_\beta(c_{(1,\beta^{-1})} \cdot (q_\alpha(c_{(2,\alpha^{-1})}))(m)),$$

it follows that

$$f(q_{\alpha\beta}(c)(m_{[0,\beta^{-1}]}) \cdot m_{[1,\beta^{-1}]}) = f(c_{(1,\beta^{-1})} \cdot g(\mu_{\sigma^{-1}\alpha}(m))(c_{(2,\alpha^{-1})})).$$

Since $\underline{C}$ is left homogeneously finite, we have

$$q_{\alpha\beta}(c)(m_{[0,\beta^{-1}]}) \cdot m_{[1,\beta^{-1}]} = c_{(1,\beta^{-1})} \cdot (q_\alpha(c_{(2,\alpha^{-1})}))(m).$$

Using the above equation and by M being finitely generated and projective (also see Lemma 4.2 in [5]), we have

$$c_{(1,\beta^{-1})} \otimes_A q_\alpha(c_{(2,\alpha^{-1})}) = q_{\alpha\beta}(c)(e_{i[0,\beta^{-1}]}) \cdot e_{i[1,\beta^{-1}]} \otimes_A e_i^*.$$

This means $\underline{q} \in Q$. Next, we define a map

$$\Phi_\sigma : \omega_\sigma(Q) \to \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R})_\sigma,\ \ \Phi_\sigma(\omega_\sigma(\underline{q})) = g$$

where g satisfies $g(\mu_{\sigma^{-1}\alpha}(m))(c) = q_\alpha(c)(m)$ for all $c \in C_{\alpha^{-1}}$ with $\alpha \in G$. It is straightforward to check that Ψ_σ and Φ_σ are mutually inverse. It is routine to check that the bijection

$$\Psi = \bigoplus_{\alpha \in G} \Psi_\alpha : \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) \to QG$$

preserves the bimodule structure. □

Now, we will achieve the main goal in this section.

4.6. Theorem. *If G-A coring $\underline{C}$ is left homogeneously finite, and $M \in \mathcal{M}^{\underline{C}}$ is finitely generated and projective as a right A-module. Consider the graded Morita context* $(\mathrm{END}_{\mathcal{R}}(M\{G\}), \mathcal{R}, M\{G\}, \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}), \mathcal{R}, \varphi, \psi)$ *associated to the graded $\mathcal{R}$-module $M\{G\}$. Using the isomorphism Ω and Ψ from Proposition 4.1 and 4.5, we find an isomorphic graded Morita context $\mathbb{GM} = (G * S, \mathcal{R}, M\{G\}, QG, \omega', \nu')$ with connecting map ω' and ν' given by the formulas*

$$\omega' : M\{G\} \otimes_{\mathcal{R}} QG \to G * S,$$
$$\omega'(\mu_\alpha(m) \otimes_{\mathcal{R}} \omega_\sigma(\underline{q})) = \mu_{\alpha\sigma}(\{f_{\sigma\beta}\}_{\beta\in G}),$$
$$f_{\sigma\beta}(m') = m_{[0,(\sigma\beta)^{-1}]} \cdot q_{\sigma\beta}(m_{[1,(\sigma\beta)^{-1}]})(m')$$
$$\nu' : QG \otimes_{G*S} M\{G\} \to \mathcal{R},$$
$$\nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(m))(c) = q_{\sigma\alpha}(c)(m),\ \forall c \in C_{(\sigma\alpha)^{-1}}.$$

Proof. It is routine to check that the following two diagrams are commutative

$$\begin{tikzcd} M\{G\} \otimes_{\mathcal{R}} \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) \arrow[r] \arrow[d, "id\otimes_{\mathcal{R}}\Psi"'] & \mathrm{END}_{\mathcal{R}}(M\{G\}) \arrow[d, "\Omega"] \\ M\{G\} \otimes_{\mathcal{R}} QG \arrow[r, "\omega'"] & G * S \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) \otimes_{\mathrm{END}_{\mathcal{R}}(M\{G\})} M\{G\} \arrow[r] \arrow[d, "\Psi\otimes_{\mathcal{R}} id"'] & \mathcal{R} \arrow[d, "="] \\ QG \otimes_{G*S} M\{G\} \arrow[r, "\nu'"] & \mathcal{R} \end{tikzcd}$$

This ends the proof. □

4.7. Remark. Let $(\underline{C}, \underline{x})$ be a G-A-coring with a fixed grouplike family $\underline{x} = (x_\alpha)_{\alpha\in G}$. The Morita context in Theorem 4.6 is just the Morita context studied in [5].

Let C_e be an A-coring and M a C_e-comodule such that M is finitely generated and projective as right A-module. Recall from [8] that we have a Morita context

$$\mathbb{M}_e = (T = \mathrm{END}^{C_e}(M), \mathcal{R}_e, M, Q_e = {}^{C_e}\mathrm{HOM}(C_e, M^*), \tau_e, \mu_e)$$

with $M \in {}_T\mathcal{M}_{\mathcal{R}_e}$ by

$$m \cdot f_e = m_{[0,e]} \cdot f_e(m_{[1,e]}) \text{ and } t \cdot m = t(m)$$

for all $m \in M$, $f_e \in \mathcal{R}_e$, $t \in T$, and $Q_e \in {}_{\mathcal{R}_e}\mathcal{M}_T$ by

$$(f_e \cdot q_e)(c) = q_e(c_{(1,e)} \cdot f_e(c_{(2,e)}))$$

for all $c \in C_e$ and $q_e \in Q_e$ and $(q_e \cdot t)(c') = q_e(c') \circ t$ for all $c' \in C_e$, and

$$\mu_e : Q_e \otimes_T M \to \mathcal{R}_e,\ \mu_e(q_e \otimes_T m)(c) = q_e(c)(m), \forall c \in C_e$$
$$\tau_e : M \otimes_{\mathcal{R}_e} Q_e \to T,\ \tau_e(m \otimes_{\mathcal{R}_e} q_e)(m') = m_{[0,e]} \cdot (q_e(m_{[1,e]})(m')).$$

4.8. Proposition. *Let $\mathbb{M}_e$ be the Morita context defined as above. Consider the group rings $T[G]$ and $\mathcal{R}_e[G]$. Then $M[G] = \bigoplus_{\sigma\in G} M\mu_\sigma \in_{T[G]} \mathcal{M}^G_{\mathcal{R}_e[G]}$ and $Q_e[G] = \bigoplus_{\sigma\in G} Q_e\mu_\sigma \in_{\mathcal{R}_e[G]} \mathcal{M}^G_{T[G]}$ with*

$$f\mu_\sigma \cdot m\mu_\alpha \cdot r_e\mu_\beta = (f \cdot m \cdot r_e)\mu_{\sigma\alpha\beta},\ r_e\mu_\beta \cdot q_e\mu_\alpha \cdot f\mu_\sigma = (r_e \cdot q_e \cdot f)\mu_{\beta\alpha\sigma}$$

for all $\sigma, \alpha, \beta \in G$, $f \in T$, $r_e \in \mathcal{R}_e$, $m \in M$ and $q_e \in Q_e$. We have well-defined maps

$$\mu : Q_e[G] \otimes_{T[G]} M[G] \to \mathcal{R}_e[G],\ \mu(q_e\mu_\sigma \otimes_{T[G]} m\mu_\alpha) = \mu_e(q_e \otimes_T m)\mu_{\sigma\alpha},$$
$$\tau : M[G] \otimes_{\mathcal{R}_e[G]} Q_e[G] \to T[G],\ \tau(m\mu_\sigma \otimes_{\mathcal{R}_e[G]} q_e\mu_\alpha) = \tau_e(m \otimes_{\mathcal{R}_e} q_e)\mu_{\sigma\alpha}.$$

Then $\mathbb{M}_e[G] = (T[G], \mathcal{R}_e[G], M[G], Q_e[G], \tau, \mu)$ *is a graded Morita context.*

4.9. Lemma. *Let* $\underline{C}$ *be a cofree group coring and* M *a cofree* $\underline{C}$*-comodule such that* M *is finitely generated and projective. Then* $i : T \to S$ *is isomorphism, and* $\mathrm{END}_{\mathcal{R}}(M\{G\}) \cong G * S$ *is isomorphic as a graded ring to the group ring* $T[G]$.

Proof. It suffices to show that i is surjective. For $\underline{f} \in S$, we have that

$$\begin{aligned} f_\alpha(m)_{[0,e]} \otimes_A \gamma_{\beta^{-1}}(f_\alpha(m)_{[1,e]}) &= f_\alpha(m)_{[0,\beta^{-1}]} \otimes_A f_\alpha(m)_{[1,\beta^{-1}]} \\ &= f_{\alpha\beta}(m_{[0,\beta^{-1}]}) \otimes_A m_{[1,\beta^{-1}]} \\ &= f_{\alpha\beta}(m_{[0,e]}) \otimes_A \gamma_{\beta^{-1}}(m_{[1,e]}). \end{aligned}$$

Applying $id \otimes_A \varepsilon \circ \gamma^{-1}_{\beta^{-1}}$, we have $f_\alpha(m) = f_{\alpha\beta}(m)$, hence $f_e = f_\beta$ for all $\beta \in G$, and $\underline{f} = i(f_e)$. □

4.10. Proposition. *Let* $\underline{C}$ *be a cofree group coring and* M *a cofree* $\underline{C}$*-comodule. Then we have an isomorphism of* G*-graded* $(G * S, \mathcal{R})$*-bimodules*

$$\vartheta : M\{G\} \to M[G],\ \vartheta(\mu_\alpha(m)) = m\mu_\alpha.$$

Proof. Straightforward. □

4.11. Lemma. *Let* $\underline{C}$ *be a cofree group coring and* M *a cofree* $\underline{C}$*-comodule such that* M *is finitely generated and projective. Then* $Q \cong Q_e$. *Consequently* $\mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) \cong Q_e[G]$.

Proof. Let us take $\underline{q} = \{q_\alpha\}_{\alpha \in G} \in Q$. Then for all $\alpha, \beta \in G$ and $c \in C_e$, we have

$$\begin{aligned} &\gamma_{\beta^{-1}}(c_{(1,e)}) \otimes_A q_\alpha(\gamma_{\alpha^{-1}}(c_{(2,e)})) \\ =&\gamma_{(\alpha\beta)^{-1}}(c)_{(1,\beta^{-1})} \otimes_A q_\alpha(\gamma_{(\alpha\beta)^{-1}}(c)_{(2,\alpha^{-1})}) \\ =&q_{\alpha\beta}(\gamma_{(\alpha\beta)^{-1}}(c))(e_{i[0,e]}) \cdot \gamma_{\beta^{-1}}(e_{i[1,e]}) \otimes_A e_i^* \end{aligned}$$

Taking $\alpha = \beta = e$, we find that $q_e \in Q_e$. For all $m \in M$, it follows that

$$\gamma_{\beta^{-1}}(c_{(1,e)}) \cdot q_\alpha(\gamma_{\alpha^{-1}}(c_{(2,e)}))(m) = q_{\alpha\beta}(\gamma_{(\alpha\beta)^{-1}}(c))(m_{[0,e]}) \cdot \gamma_{\beta^{-1}}(m_{[1,e]}).$$

Applying $\gamma^{-1}_{\beta^{-1}}$ to both sides of the equation above, we have

$$c_{(1,e)} \cdot q_\alpha(\gamma_{\alpha^{-1}}(c_{(2,e)}))(m) = q_{\alpha\beta}(\gamma_{(\alpha\beta)^{-1}}(c))(m_{[0,e]}) \cdot m_{[1,e]}.$$

Applying ε to both sides, we find that

$$q_\alpha(\gamma_{\alpha^{-1}}(c))(m) = q_{\alpha\beta}(\gamma_{(\alpha\beta)^{-1}}(c))(m),$$

and

$$q_e(c) = q_\beta(\gamma_{\beta^{-1}}(c)).$$

Hence, we have $q_\beta = q_e \circ \gamma^{-1}_{\beta^{-1}}$. These arguments show that the map

$$j : Q_e \to Q,\ j(q) = (\sigma_\alpha(q))_{\alpha \in \pi}$$

is a well-defined isomorphism. □

4.12. Theorem. *Let* $\underline{C}$ *be a cofree group coring and* M *a cofree* $\underline{C}$*-comodule such that* M *is finitely generated and projective. Then the graded Morita contexts* $\mathbb{GM}$ *and* $\mathbb{M}_e[G]$ *are isomorphic.*

Proof. Let $\Xi : G * S \to T[G]$ be the isomorphism in Lemma 4.9. We will show that the diagram

$$\begin{tikzcd} M\{G\} \otimes_{\mathcal{R}} QG \arrow[r, "\omega'"] \arrow[d, "\vartheta\otimes j^{-1}G"'] & G * S \arrow[d, "\Xi"] \\ M[G] \otimes_{\mathcal{R}_e[G]} Q_e[G] \arrow[r, "\varphi"] & T[G] \end{tikzcd}$$

commutes. Indeed, for $\alpha, \sigma \in G$, $a \in A$ and $\underline{q} \in Q$, we have

$$\begin{aligned}(\Xi \circ \omega')(\mu_\alpha(m) \otimes \omega_\sigma(\underline{q})) =& \Xi(\mu_{\alpha\sigma}((f_{\sigma\beta})_{\beta\in G})) \\ =& f_\sigma \mu_{\alpha\sigma}\end{aligned}$$

where $f_{\sigma\beta}(m') = m_{[0,(\sigma\beta)^{-1}]} \cdot q_{\sigma\beta}(m_{[1,(\sigma\beta)^{-1}]})(m')$, and

$$\begin{aligned}&(\varphi \circ (\vartheta \otimes j^{-1}G))(\mu_\alpha(m) \otimes \omega_\sigma(\underline{q})) \\ &= \varphi(m\mu_\alpha \otimes q_e\mu_\sigma) \\ &= \varphi_e(m \otimes q_e)\mu_{\alpha\sigma},\end{aligned}$$

for all $m' \in M$, since

$$\begin{aligned}f_\sigma(m') =& m_{[0,\sigma^{-1}]} \cdot q_\sigma(m_{[1,\sigma^{-1}]})(m') \\ =& m_{[0,e]} \cdot q_\sigma(\gamma_{\sigma^{-1}}(m_{[1,e]}))(m') \\ =& m_{[0,e]} \cdot q_e(m_{[1,e]})(m') \\ =& \varphi_e(m \otimes q_e)(m'),\end{aligned}$$

it follows that $\Xi \circ \omega = \varphi \circ (\vartheta \otimes j^{-1}G)$. Let

$$\Gamma : \mathcal{R}_e[G] \to \mathcal{R},\ \Gamma(f\mu_\alpha) = f \circ \gamma^{-1}_{\alpha^{-1}}$$

be the isomorphism from Proposition 4.6 in [5]. We will show that the diagram

$$\begin{tikzcd} QG \otimes_{G*S} M\{G\} \arrow[r, "\nu'"] \arrow[d, "j^{-1}G\otimes\vartheta"'] & \mathcal{R} \arrow[d, "\Gamma^{-1}"] \\ Q_e[G] \otimes_{T[G]} M[G] \arrow[r, "\psi"] & R_e[G] \end{tikzcd}$$

commutes. Take $\sigma, \alpha \in G$, $\underline{q} \in Q$ and $a \in A$,

$$\begin{aligned}&(\Gamma \circ \psi \circ (j^{-1}G \otimes \vartheta))(\omega_\sigma(\underline{q}) \otimes \mu_\alpha(m)) \\ &= (\Gamma \circ \psi)(q_e\mu_\sigma \otimes m\mu_\alpha) \\ &= \Gamma(q_e(-)(m)\mu_{\sigma\alpha}) \\ &= q_e(-)(m) \circ \gamma^{-1}_{(\sigma\alpha)^{-1}}\end{aligned}$$

and

$$\nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(m))(-) = q_{\sigma\alpha}(-)(m).$$

For $\gamma_{(\sigma\alpha)^{-1}}(c) \in C_{(\sigma\alpha)^{-1}}$, we compute that

$$\begin{aligned}(q_e(-)(m) \circ \gamma^{-1}_{(\sigma\alpha)^{-1}})(\gamma_{(\sigma\alpha)^{-1}}(c)) =& q_e((\gamma^{-1}_{(\sigma\alpha)^{-1}} \circ \gamma_{(\sigma\alpha)^{-1}})(c))(m) \\ =& q_{\sigma\alpha}(\gamma_{(\sigma\alpha)^{-1}}(c))(m) \\ =& \nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(m))(\gamma_{(\sigma\alpha)^{-1}}(c)).\end{aligned}$$

Thus we have $\psi \circ (j^{-1}G \otimes \vartheta) = \Gamma^{-1} \circ \nu'$. $\square$

5. Applications

In this section, we will give the application of our theory to G-entwining structure.

Given a G-entwining structure $(A, \underline{C})_{G-\psi}$, then we have a G-A-coring $\{A \otimes C_\alpha\}_{\alpha\in G}$ arising from $(A, \underline{C})_{G-\psi}$. First observe that

$$\mathcal{R}_\alpha = {}_A\mathrm{HOM}(A \otimes C_{\alpha^{-1}}, A) \cong \mathrm{HOM}(C_{\alpha^{-1}}, A)$$

as spaces. This graded ring structure on $\mathcal{R}$ induces a graded ring structure on $\bigoplus_{\alpha\in G}\mathrm{HOM}(C_{\alpha^{-1}}, A)$, and this graded ring is denoted by $\sharp(\underline{C}, A)$. The product is given by the formula

$$(f_\alpha \sharp g_\beta)(c) = f_\alpha(c_{(2,\alpha^{-1})})_{\psi_{\beta^{-1}}} g_\beta(c_{(1,\beta^{-1})}{}^{\psi_{\beta^{-1}}})$$

for all $f_\alpha \in \mathrm{HOM}(C_{\alpha^{-1}}, A)$, $g_\beta \in \mathrm{HOM}(C_{\beta^{-1}}, A)$ and $c \in C_{(\alpha\beta)^{-1}}$.

Fix a grouplike family $x = \{x_\alpha\}_{\alpha\in G}$ of $\underline{C}$, then we have that $A \in \mathcal{U}_A^{\underline{C}}(\psi)$ with right $\underline{C}$-coaction $\rho_\alpha^A(a) = a_{\psi_\alpha} \otimes x_\alpha{}^{\psi_\alpha}$.

Generally, let A be a right $\underline{C}$-comodule. Suppose that A is an object of $\mathcal{U}_A^{\underline{C}}(\psi)$ with the structure maps m_A and $\rho^A = \{\rho_\alpha^A\}$. Then we have $\rho_\alpha^A(ab) = a_{[0,\alpha]}b_{\psi_\alpha} \otimes a_{[1,\alpha]}{}^{\psi_\alpha}$. Specially, the coaction can be written as $\rho_\alpha^A(b) = 1_{A[0,\alpha]}b_{\psi_\alpha} \otimes 1_{A[1,\alpha]}{}^{\psi_\alpha}$. The ring of coinvariants is

$$B = \{a \in A | 1_{A[0,\alpha]}a_{\psi_\alpha} \otimes 1_{A[1,\alpha]}{}^{\psi_\alpha} = a1_{A[0,\alpha]} \otimes 1_{A[1,\alpha]}, \forall \alpha \in G\}.$$

Q and S can be described as follows:

$$\begin{aligned} Q = &\{\underline{q} = (q_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} \sharp(C_{\alpha^{-1}}, A)_\alpha | \\ &q_\alpha(c_{(2,\alpha^{-1})})_{\psi_{\beta^{-1}}} \otimes c_{(1,\beta^{-1})}{}^{\psi_{\beta^{-1}}} = q_{\alpha\beta}(c)1_{A[0,\beta^{-1}]} \otimes 1_{A[1,\beta^{-1}]}, c \in C_{(\alpha\beta)^{-1}}\}\end{aligned}$$

and

$$\begin{aligned} S = \{\underline{b} = &(b_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} A | \\ &1_{A[0,\beta^{-1}]}b_{\alpha\psi_{\beta^{-1}}} \otimes 1_{A[0,\beta^{-1}]}{}^{\psi_{\beta^{-1}}} = 1_{A[0,\beta^{-1}]}b_{\alpha\beta} \otimes 1_{A[1,\beta^{-1}]}\}.\end{aligned}$$

Then we have the twisted group ring $G * S = \bigoplus_{\alpha\in G}\mu_\alpha S$ with the multiplication given by $\mu_\alpha\underline{b}\mu_\beta\underline{c} = \mu_{\alpha\beta}(\underline{b}^\beta\underline{c})$, where $\underline{b}^\beta = (b_{\beta\alpha})_{\alpha\in G}$.

From Theorem 4.6, we have the following result.

5.1. Theorem. *With the notation as above, we have a graded Morita context* $\mathbb{GM} = (G * S, \sharp(\underline{C}, A), A\{G\}, QG, \omega', \nu')$ *with connecting map* ω' *and* ν' *given by the formulas*

$$\omega' : A\{G\} \otimes_{\sharp(\underline{C},A)} QG \to G * S,$$

$$\omega'(\mu_\alpha(a) \otimes_{\sharp(\underline{C},A)} \omega_\sigma(\underline{q})) = \mu_{\alpha\sigma}(1_{A[0,(\sigma\beta)^{-1}]}a_{\psi_{(\sigma\beta)^{-1}}}q_{\sigma\beta}(1_{A[1,(\sigma\beta)^{-1}]}{}^{(\sigma\beta)^{-1}})),$$

$$\nu' : QG \otimes_{G*S} A\{G\} \to \sharp(\underline{C}, A),$$

$$\nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(a))(c) = q_{\sigma\alpha}(c)a, \forall c \in C_{(\sigma\alpha)^{-1}}.$$

As was stated above, if we fix a grouplike family $x = \{x_\alpha\}_{\alpha\in G}$ of $\underline{C}$, then we have that $A \in \mathcal{U}_A^{\underline{C}}(\psi)$ with right G-$\underline{C}$-coaction $\rho_\alpha^A(a) = a_{\psi_\alpha} \otimes x_\alpha{}^{\psi_\alpha}$. In particular, it follows that $\rho_\alpha^A(1_A) = 1_A \otimes x_\alpha$. Then B, S and Q have the following forms:

$$B = \{a \in A | a_{\psi_\alpha} \otimes x_\alpha{}^{\psi_\alpha} = a \otimes x_\alpha, \forall \alpha \in G\},$$

$$\begin{aligned} Q = &\{\underline{q} = (q_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} \sharp(C_{\alpha^{-1}}, A)_\alpha | \\ &q_\alpha(c_{(2,\alpha^{-1})})_{\psi_{\beta^{-1}}} \otimes c_{(1,\beta^{-1})}{}^{\psi_{\beta^{-1}}} = q_{\alpha\beta}(c) \otimes x_{\beta^{-1}}, c \in C_{(\alpha\beta)^{-1}}\}\end{aligned}$$

and

$$S = \{\underline{b} = (b_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} A | b_{\alpha\psi_{\beta^{-1}}} \otimes x_{\beta^{-1}}{}^{\psi_{\beta^{-1}}} = b_{\alpha\beta} \otimes x_{\beta^{-1}}\}.$$

From Theorem 5.1, we have a graded Morita context $\mathbb{GM} = (G * S, \sharp(\underline{C}, A), A\{G\}, QG, \omega', \nu')$ with connecting map ω' and ν' given by the formulas

$$\omega' : A\{G\} \otimes_{\sharp(\underline{C},A)} QG \to G * S,$$

$$\omega'(\mu_\alpha(a) \otimes_{\sharp(\underline{C},A)} \omega_\sigma(\underline{q})) = \mu_{\alpha\sigma}(a_{\psi_{(\sigma\beta)^{-1}}} q_{\sigma\beta}(x_{(\sigma\beta)^{-1}}{}^{(\sigma\beta)^{-1}})),$$

$$\nu' : QG \otimes_{G*S} A\{G\} \to \sharp(\underline{C}, A),$$

$$\nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(a))(c) = q_{\sigma\alpha}(c)a, \forall c \in C_{(\sigma\alpha)^{-1}}.$$

Furthermore, if G is a trivial group, then $B = S$ and the graded Morita context $\mathbb{GM} = (G*S, \sharp(\underline{C}, A), A\{G\}, QG, \omega', \nu')$ recovers to the Morita context in the sense of [7, Section 4].

In order to proceed the further discussion, we need the following result [10].

5.2. Proposition. *Let A and E be rings, and $\underline{C}$ a G-A-coring, and M both a $\underline{C}$-comodule and a (E, A)-bimodule such that the comodule maps ρ_α are left E-linear. Then we have a pair of adjoint functors (F, U):*

$$F : \mathcal{M}_E \to \mathcal{M}^{G,\underline{C}}, \ \underline{F(N)} = \{\mu_\alpha(N \otimes_E M)\}_{\alpha\in G}.$$

The coaction maps are

$$\rho_{\alpha,\beta} : \mu_{\alpha\beta}(N \otimes_E M) \to \mu_\alpha(N \otimes_E M) \otimes_A C_\beta,$$

$$\rho_{\alpha,\beta}(\mu_{\alpha\beta}(n \otimes_E m)) = \mu_\alpha(n \otimes_E m_{[0,\beta]}) \otimes_A m_{[1,\beta]}.$$

For $\underline{X} \in \mathcal{M}^{G,\underline{C}}$, define U as follows:

$$U : \mathcal{M}^{G,\underline{C}} \to \mathcal{M}_E, \ U_2(\underline{X}) = \mathrm{HOM}^{G,\underline{C}}(\mu_\alpha(M), \underline{X}).$$

Next, we apply Proposition 5.2 to the particular G-A-coring $\{A \otimes C_\alpha\}_{\alpha\in G}$ arising from $(A, \underline{C})_{G-\psi}$. Under the assumption that A is an object of $\mathcal{U}^{\underline{C}}_A(\psi)$ with the structure maps m_A and $\rho^A = \{\rho^A_\alpha\}$, we have a special pair of adjoint functors $(\widetilde{F}, \widetilde{U})$:

$$\widetilde{F} : \mathcal{M}_B \to \mathcal{U}^{G,\underline{C}}_A(\psi), \ \underline{\widetilde{F}(N)} = \{\mu_\alpha(N \otimes_B A)\}_{\alpha\in G}.$$

The coaction maps are

$$\rho_{\alpha,\beta} : \mu_{\alpha\beta}(N \otimes_B A) \to \mu_\alpha(N \otimes_B A) \otimes_A C_\beta,$$

$$\rho_{\alpha,\beta}(\mu_{\alpha\beta}(n \otimes_B a)) = \mu_\alpha(n \otimes_B 1_{A[0,\beta]} a_{\psi_\beta}) \otimes_A 1_{A[1,\beta]}{}^{\psi_\beta}.$$

For $\underline{X} \in \mathcal{U}^{G,\underline{C}}_A(\psi)$, define $\widetilde{U}$ as follows:

$$\widetilde{U} : \mathcal{U}^{G,\underline{C}}_A(\psi) \to \mathcal{M}_B, \ \widetilde{U}(\underline{X}) = \mathrm{HOM}^{\underline{C}}_A(\mu_\alpha(A), \underline{X}).$$

For $\underline{M} \in \mathcal{U}^{G,\underline{C}}_A$, we define

$$\underline{M}^{co} = \{m = (m_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} M_\alpha | m_{[0,\alpha]} \otimes m_{[1,\beta]} = m \cdot 1_{A[0,\beta]} \otimes 1_{A[1,\beta]}\}.$$

Then we have

5.3. Lemma. *There exists an isomorphism*

$$\mathrm{HOM}^{\underline{C}}_A(\mu_\alpha(A), \underline{M}) \cong \underline{M}^{co}$$

as right B-modules.

Proof. For any $\underline{f} \in \mathrm{HOM}_A^{\underline{C}}(\mu_\alpha(A), \underline{M})$, since f is a right $\underline{C}$-comodule, then we have

$$\rho^M_{\alpha\beta}(f_{\alpha\beta}(\mu_{\alpha\beta}(1_A))) = f_\alpha(\mu_\alpha(1_{A[0,\beta]})) \otimes 1_{A[1,\beta]}.$$

Set $m = (m_\alpha)_{\alpha\in G}$, where $m_\alpha = f_\alpha(\mu_\alpha(1_A))$. Straightforward calculation can show that $\underline{m} \in \underline{M}^{co}$. Thus we define a map

$$\widehat{\Phi} : \mathrm{HOM}_A^{\underline{C}}(\mu_\alpha(A), \underline{M}) \to \underline{M}^{co},\ \widehat{\Phi}(\underline{f}) = (f_\alpha(\mu_\alpha(1_A)))_{\alpha\in G}.$$

Take $\underline{m} \in \underline{M}^{co}$, we define a map

$$\widehat{\Psi} : \underline{M}^{co} \to \mathrm{HOM}_A^{\underline{C}}(\mu_\alpha(A), \underline{M}),\ \widehat{\Psi}(m)_\alpha(\mu_\alpha(a)) = m_\alpha \cdot a.$$

It follow easily that $\widehat{\Phi}$ and $\widehat{\Psi}$ are both B-linear and mutually inverses. □

From Lemma 5.3 and what was discussed above, we have a pair of adjoint functors $(\widetilde{F}, \widetilde{U})$:

$$\widetilde{F} : \mathcal{M}_B \to \mathcal{U}_A^{G,\underline{C}}(\psi),\ \underline{\widetilde{F}(N)} = \{\mu_\alpha(N \otimes_B A)\}_{\alpha\in G}.$$

$$\widetilde{U} : \mathcal{U}_A^{G,\underline{C}}(\psi) \to \mathcal{M}_B,\ \widetilde{U}(\underline{X}) = \underline{X}^{co}.$$

By the discussion as above, and [7, Theorem 9.2], we can achieve the main goal in this section.

5.4. Theorem. *Let $(A, \underline{C})$ be a G-entwined structure. Suppose that $A \in \mathcal{U}_A^{\underline{C}}(\psi)$. Consider the map*

$$can : (A \otimes_B A)\langle G\rangle \to A \otimes \underline{C},\ can_\alpha(a \otimes_B b) = a1_{A[0,\alpha]}b_{\psi_\beta} \otimes {1_{A[1,\alpha]}}^{\psi_\beta}.$$

Then the following statements are equivalent:

(1) *can is an isomorphism of group corings, and A is faithfully flat as a left B-module,*
(2) *${}^*can$ is an isomorphism of graded rings and A is a left B-progenerator,*
(3) *The graded Morita context $\mathbb{GM} = (G * S, \sharp(\underline{C}, A), A\{G\}, QG, \omega', \nu')$ is strict,*
(4) *$(\widetilde{F}, \widetilde{U})$ is an equivalence of categories.*

As the end of this paper, we discuss the $(\underline{H}, A)$-Hopf module for an $\underline{H}$-comodule algebra A over a Hopf G-coalgebra $\underline{H}$.

Let $\underline{H} = (\{H_\alpha, m_\alpha, 1_\alpha, \Delta, \varepsilon\})$ be a Hopf G-coalgebra in the sense of [16] and A an algebra. We recall from [16] that a right $\underline{H}$-comodule algebra is a right $\underline{H}$-comodule $(A, \rho^A = \{\rho^A_\alpha\})$, such that the following conditions are satisfied:

- $\rho^A_\alpha(ab) = a_{[0,\alpha]}b_{[0,\alpha]} \otimes a_{[1,\alpha]}b_{[1,\alpha]}$ for all $a, b \in A$ and $\alpha \in G$,
- $\rho^A_\alpha(1_A) = 1_A \otimes 1_\alpha$ for all $\alpha \in G$.

Given an $\underline{H}$-comodule algebra A, we have a G-entwined structure $\psi_\alpha : H_\alpha \otimes A \to A \otimes H_\alpha$, $\psi_\alpha(h \otimes a) = a_{[0,\alpha]} \otimes ha_{[1,\alpha]}$. We call a special $(A, \underline{C})_\psi$-module a (right-right) $(\underline{H}, A)$-Hopf module and denote the category of $(\underline{H}, A)$-Hopf modules by $\mathcal{U}_A^{\underline{H}}$. It is easy to see that $A \in \mathcal{U}_A^{\underline{H}}$. Let us take the grouplike family $\{1_\alpha\}_{\alpha\in G}$. Then we have a graded Morita context $\mathbb{GM} = (G * S, \sharp(\underline{H}, A), A\{G\}, QG, \omega', \nu')$ with connecting map ω' and ν' given by the formulas

$$\omega' : A\{G\} \otimes_{\sharp(\underline{H},A)} QG \to G * S,$$

$$\omega'(\mu_\alpha(a) \otimes_{\sharp(\underline{H},A)} \omega_\sigma(\underline{q})) = \mu_{\alpha\sigma}(a_{[0,(\sigma\beta)^{-1}]}q_{\sigma\beta}(a_{[1,(\sigma\beta)^{-1}]})),$$

$$\nu' : QG \otimes_{G*S} A\{G\} \to \sharp(\underline{H}, A),$$

$$\nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(a))(c) = q_{\sigma\alpha}(c)a, \forall c \in C_{(\sigma\alpha)^{-1}},$$

where

$$Q = \{\underline{q} = (q_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} \sharp(H_{\alpha^{-1}}, A)_\alpha | \\ q_\alpha(h_{(2,\alpha^{-1})})_{[0,\beta^{-1}]} \otimes h_{(1,\beta^{-1})} q_\alpha(h_{(2,\alpha^{-1})})_{[1,\beta^{-1}]} = q_{\alpha\beta}(h) \otimes 1_{\beta^{-1}}, h \in H_{(\alpha\beta)^{-1}}\}$$

and

$$S = \{\underline{b} = (b_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} A | b_{\alpha[0,\beta^{-1}]} \otimes b_{\alpha[1,\beta^{-1}]} = b_{\alpha\beta} \otimes 1_{\beta^{-1}}\}.$$

5.5. Remark. If π is a trivial group, then $S = A^{coH}$ and

$$Q = \{q \in \sharp(H, A) | q(h_{(2)})_{[0]} \otimes h_{(1)} q(h)_{[1]} = q(h) \otimes 1_H, h \in H\}.$$

Hence, the graded Morita context $\mathbb{GM} = (G * S, \sharp(\underline{H}, A), A\{G\}, QG, \omega', \nu')$ is just the Morita context of Doi in [12].

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On strongly and nicely almost ω_1-$p^{\omega+n}$-projective Abelian p-groups

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Abstract

We define the classes of *strongly almost* ω_1-$p^{\omega+n}$-*projective* abelian p-groups and *nicely almost* ω_1-$p^{\omega+n}$-*projective* abelian p-groups as well as we study their crucial properties. Our results support those obtained by us in Hacettepe J. Math. Stat. (2014) and Korean J. Math. (2014).

Keywords: almost Σ-cyclic groups, almost $p^{\omega+n}$-projective groups, almost ω_1-$p^{\omega+n}$-projective groups, strongly almost ω_1-$p^{\omega+n}$-projective groups, nicely almost ω_1-$p^{\omega+n}$-projective groups.

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1. Introduction and Terminology

Let all groups into consideration be p-primary abelian, where p is a fixed prime integer, written additively as it is customary. As usual, for some ordinal $\alpha \geq 0$ and a group G, we state the α-*th Ulm subgroup* $p^\alpha G$, consisting of all elements of G with height $\geq \alpha$, inductively as follows: $p^0 G = G$, $pG = \{pg \mid g \in G\}$, $p^\alpha G = p(p^{\alpha-1}G)$ if $\alpha - 1$ exists (so α is non-limit) and $p^\alpha G = \cap_{\beta<\alpha} p^\beta G$ if $\alpha - 1$ does not exist (so α is limit). The group G is called p^α-*bounded* if $p^\alpha G = \{0\}$; note that these groups are necessarily reduced. We shall say that G is *separable* if it is p^ω-bounded. Most of the important unexplained here notations and notions will follow mainly those from [9].

In their seminal work [12], Hill and Ullery have given the following critical concept.

• The reduced group G is called *almost totally projective* if it has a collection $\mathcal{C}$ consisting of nice subgroups of G satisfying the following three conditions:

(1) $\{0\} \in \mathcal{C}$;

(2) $\mathcal{C}$ is closed with respect to ascending unions, i.e., if $H_i \in \mathcal{C}$ with $H_i \subseteq H_j$ whenever $i \leq j$ $(i, j \in I)$ then $\cup_{i \in I} H_i \in \mathcal{C}$;

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(3) If K is a countable subgroup of G, then there is $L \in \mathcal{C}$ (that is, a nice subgroup L of G) such that $K \subseteq L$ and L is countable.

This concept generalizes the notion of *an almost direct sum of cyclic groups*, defined in [11], hereafter abbreviated as *almost Σ-cyclic*. Actually separable almost totally projective groups are almost Σ-cyclic. Moreover, the direct sum of a divisible group and an almost totally projective group is called *almost simply presented*. It readily follows that a group is almost simply presented if and only if its reduced part is almost simply presented as well as that the direct sum of almost simply presented groups is again an almost simply presented group.

Extending the meaning of almost Σ-cyclic groups, the current author defines in [2] (see also [4], [5] and [7]) the following:

• The group G is said to be *almost $p^{\omega+n}$-projective* if there is $B \leq G[p^n]$ such that G/B is almost Σ-cyclic.

Observe that when $n = 0$ we obtain almost Σ-cyclic groups, i.e., the almost p^{ω}-projective groups. Moreover, note that P is of necessity nice in G because G/P is separable.

• If there exists a countable subgroup $C \leq G$ of a group G with the property that G/C is almost $p^{\omega+n}$-projective, then we will say that G is *almost ω_1-$p^{\omega+n}$-projective* – see [7]. Note that by Theorem 2.15 of [7] the subgroup C can be taken to be nice in G.

The following two notions were stated in [4].

• A group G is said to be *almost weak $p^{\omega\cdot 2+n}$-projective* if there is an almost $p^{\omega+n}$-projective subgroup $H \leq G$ such that G/H is almost Σ-cyclic.

• A group G is said to be *almost ω_1-weak $p^{\omega\cdot 2+n}$-projective* if there is a countable subgroup $K \leq G$ such that G/K is almost weak $p^{\omega\cdot 2+n}$-projective.

On the other hand, in [2] it was formulated the following:

• The group G is said to be *almost n-simply presented* if there is $H \leq G[p^n]$ such that G/H is almost simply presented.

If G/H is almost totally projective, then we will say that G is *almost n-totally projective.*

In case that H is nice in G, we give

• The group G is called *nicely almost n-simply presented* if there exists a p^n-bounded nice subgroup $N \leq G$ with G/N almost simply presented.

On the other hand these groups could be termed as *strongly almost n-simply presented* and *strongly almost n-totally projective*, respectively.

Apparently almost $p^{\omega+n}$-projective groups are nicely almost n-totally projective.

We will now state our new machinery like this:

1.1 Definition. A group G is said to be *strongly almost ω_1-$p^{\omega+n}$-projective* if it contains a p^n-bounded nice subgroup P such that G/P is the sum of a countable group and an almost Σ-cyclic group.

1.2 Definition. A group G is said to be *nicely almost ω_1-$p^{\omega+n}$-projective* if it contains a nice subgroup X such that X is almost $p^{\omega+n}$-projective and G/X is countable.

The goal of the present paper is to give a comprehensive study of these two concept. The work is organized as follows: In the next section, we establish our basic results which are stated in two different subsections. In the final section, we list some interesting left-open questions.

And so, we come to

2. Main Results

We distribute the chief results into two subsections. We start with

2.1. Strongly Almost ω_1-$p^{\omega+n}$-Projective p-Groups. We begin here with two useful necessary and sufficient conditions when a group is strongly almost ω_1-$p^{\omega+n}$-projective. First, we need two more preliminaries.

The following can be seen in [2].

2.1. Lemma. *If C is a countable subgroup of a group A such that A/C is almost Σ-cyclic, then A is the sum of a countable group and an almost Σ-cyclic group.*

The following somewhat extends the corresponding result from [13] (see [8] and [2], too).

2.2. Proposition. *The group A is almost simply presented with countable $p^\omega A$ if and only if A is the sum of a countable group and an almost Σ-cyclic group.*

Proof. **"Necessity"**. In conjunction with [12], one may write that $A/p^\omega A$ is almost Σ-cyclic. We furthermore appeal to Lemma 2.1 to get the desired decomposition of the group A.

"Sufficiency". Write $A = C+S$, where C is countable and S is almost Σ-cyclic. Since $C \cap S \subseteq S$ is countable, there is a nice countable subgroup K of S such that $C \cap S \subseteq K$. Therefore, $A/K = [(C+K)/K] \oplus [S/K]$. But $p^\omega(S/K) = (p^\omega S + K)/K = \{0\}$, so that $p^\omega(A/K) = p^\omega((C+K)/K)$ is countable because it is obvious that the same is $(C+K)/K$. Thus $p^\omega A/(p^\omega A \cap K) \cong (p^\omega A + K)/K \subseteq p^\omega(A/K)$ is countable, whence so is $p^\omega A$ as asserted, since $p^\omega A \cap K$ is countable. ■

The last can be slightly extended to the following one:

2.3. Lemma. *Suppose G is a group. Then the following are equivalent:*

(1) G is almost simply presented with countable $p^\omega G$;

(2) $G/p^\omega G$ is almost Σ-cyclic such that $p^\omega G$ is countable;

(3) G is the sum of a countable group and an almost Σ-cyclic group.

Proof. The implication (1) $\Rightarrow$ (2) follows from [12]. The implication (2) $\Rightarrow$ (3) follows from Lemma 2.1. The implication (3) $\Rightarrow$ (1) follows from Proposition 2.2. ■

As a helpful consequence we derive:

2.4. Corollary. *(a) A subgroup of the sum of a countable group and an almost Σ-cyclic group is again the sum of a countable group and an almost Σ-cyclic group.*

(b) If G is the sum of a countable group and an almost Σ-cyclic group, then for each $\alpha \geq \omega$ the quotient $G/p^\alpha G$ is also the sum of a countable group and an almost Σ-cyclic group.

Proof. (a) By the usage of Lemma 2.3 (2), let $A \leq G$ where $G/p^\omega G$ is almost Σ-cyclic and $p^\omega G$ is countable. Thus

$$A/(A \cap p^\omega G) \cong (A + p^\omega G)/p^\omega G \subseteq G/p^\omega G$$

is almost Σ-cyclic with the aid of [1]. But $A \cap p^\omega G \leq p^\omega G$ is countable. Hence we employ Lemma 2.1 to deduce the desired claim.

(b) By virtue of Lemma 2.3 (2) we have that $G/p^\omega G$ is almost Σ-cyclic and $p^\omega G$ is countable. Consequently,

$$G/p^\omega G \cong (G/p^\alpha G)/(p^\omega G/p^\alpha G) = (G/p^\alpha G)/(p^\omega(G/p^\alpha G))$$

is almost Σ-cyclic with $p^\omega(G/p^\alpha G) = p^\omega G/p^\alpha G$ being countable. Again an application of point (2) in Lemma 2.3 gives the wanted claim. ■

The next assertion gives two new necessary and sufficient conditions when a group is strongly almost ω_1-$p^{\omega+n}$-projective.

2.5. Proposition. *(a) A group G is strongly almost ω_1-$p^{\omega+n}$-projective if and only if there exists a p^n-bounded nice subgroup $N \leq G$ such that $p^\omega(G/N)$ is countable and $G/(N + p^\omega G)$ is almost Σ-cyclic.*

(b) A group G is strongly almost ω_1-$p^{\omega+n}$-projective if and only if there exists a countable subgroup K and a p^n-bounded nice subgroup N such that $G/(K + N)$ is almost Σ-cyclic.

Proof. (a) "$\Rightarrow$". By definition G/P is the sum of a countable group and an almost Σ-cyclic group for some nice subgroup P of G which is bounded by p^n. Since G/P is almost simply presented in conjunction with Proposition 2.2 (see [2] as well), we deduce that

$$(G/P)/p^\omega(G/P) = (G/P)/((p^\omega G + P)/P) \cong G/(p^\omega G + P)$$

is almost Σ-cyclic, as stated. That $p^\omega(G/P)$ is countable again follows directly from Proposition 2.2.

"$\Leftarrow$". Since $G/(N + p^\omega G) \cong [G/N]/[(N + p^\omega G)/N]$ is almost Σ-cyclic with countable quotient $(N + p^\omega G)/N = p^\omega(G/N)$, Lemma 2.1 leads us to this that G/N is the sum of a countable group and an almost Σ-cyclic subgroup, as expected.

(b) "$\Rightarrow$". Write $G/P = (A/P) + (B/P)$, where the first term A/P is countable and the second term B/P is almost Σ-cyclic for some $A, B \leq G$ and some nice subgroup P of G with $p^n P = \{0\}$. Since $(A/P) \cap (B/P) \subseteq B/P$ is countable, there is a countable nice subgroup C/P of B/P for some $C \leq B$ such that $(A/P) \cap (B/P) \subseteq C/P$. In accordance to [7], the factor-group $(B/P)/(C/P) \cong B/C$ is always almost Σ-cyclic. We also may write twice $A = K_1 + P$ and $C = K_2 + P$, where both K_1 and K_2 are countable groups. Furthermore, one can decompose

$$(G/P)/(C/P) = [((A/P) + (C/P))/(C/P)] \oplus [(B/P)/(C/P)].$$

Since $(G/P)/(C/P) \cong G/C$ as the first term of the above decomposition is isomorphic to $(A+C)/C$ while the second one is isomorphic to B/C, it is routinely seen that $G/(K+P) = G/(A+C) \cong B/C$ is almost Σ-cyclic for some countable subgroup $K = K_1 + K_2$, as required.

"$\Leftarrow$". Suppose that $G/(K+N)$ is almost Σ-cyclic, for some countable subgroup $K \leq G$ and some bounded by p^n nice subgroup $N \leq G$. Observing that $G/(K+N) \cong (G/N)/((K+N)/N)$, where $(K+N)/N \cong K/(K \cap N)$ is obviously countable, Lemma 2.1 allows us to conclude that G/N is the sum of a countable group and an almost Σ-cyclic group, as required in Definition 1.1. ■

2.6. Corollary. *If G is strongly almost ω_1-$p^{\omega+n}$-projective, then so are both $p^\alpha G$ and $G/p^\alpha G$ for every ordinal α.*

Proof. Assume that $G/P = (L/P) + (S/P)$, where $L, S \leq G$ and L/P is countable while S/P is an almost Σ-cyclic group, for some nice subgroup $P \leq G$ with $p^n P = \{0\}$. Thus, with Corollary 2.4 at hand, all of

$$G/P \supseteq (p^\alpha G + P)/P \cong p^\alpha G/(p^\alpha G \cap P)$$

are also sums of countable groups and almost Σ-cyclic groups, where $p^\alpha G \cap P$ is nice in $p^\alpha G$, as needed.

Concerning the second half-part, it follows directly from Corollary 2.4, because the isomorphism sequence

$$(G/p^\alpha G)/((P + p^\alpha G)/p^\alpha G) \cong G/(P + p^\alpha G) \cong (G/P)/((P + p^\alpha G)/P).$$

holds. ■

2.7. Corollary. *If G is a group such that $p^{\omega+n}G = \{0\}$, then G is strongly almost ω_1-$p^{\omega+n}$-projective if and only if G is almost $p^{\omega+n}$-projective.*

Proof. In accordance with Proposition 2.5, the quotient $G/(N + p^\omega G)$ is almost Σ-cyclic for some $N \leq G[p^n]$. Thus $p^n(N + p^\omega G) = \{0\}$ and the claim follows at once by definition. ■

We are now ready to proceed by proving one of our basic results, which reduces the investigation of strongly almost ω_1-$p^{\omega+n}$-projective groups to groups of lengths not exceeding $\omega + n$.

2.8. Theorem. *For every $n \geq 1$ the group G is strongly almost ω_1-$p^{\omega+n}$-projective if and only if*

(1) $p^{\omega+n}G$ is countable;

(2) $G/p^{\omega+n}G$ is almost $p^{\omega+n}$-projective.

Proof. "$\Rightarrow$". According to Proposition 2.5, one may write that $p^\omega G/(p^\omega G \cap N)$ is countable for some p^n-bounded nice subgroup N of G. Thus $p^\omega G = p^\omega G \cap N + C$ where $C \leq p^\omega G$ is countable. Furthermore, $p^{\omega+n}G = p^n C$ is countable, so that clause (1) follows.

Next, point (2) follows directly from Corollary 2.6.

"$\Leftarrow$". Suppose that $P \leq G$ such that $p^{\omega+n}G \subseteq P$, $p^n P \subseteq p^{\omega+n}G$ (thereby $P/p^{\omega+n}G$ is p^n-bounded) and G/P is Σ-cyclic. Let Y be a maximal p^n-bounded summand of $p^\omega G$; so there is a decomposition $p^\omega G = X \oplus Y$ and thus the inclusions $X \subseteq p^\omega G \subseteq P$ hold. We may assume without loss of generality that X is countable; in fact, $p^{\omega+n}G = p^n X$

is countable and so we can decompose $X = K \oplus T$ where K is countable and T is p^n-bounded (whence T is a p^n-bounded summand of $p^\omega G$ and thereby $T \subseteq Y$; then even $T = T \cap Y \subseteq X \cap Y = \{0\}$ and $X = K$ - in any case $p^\omega G = K \oplus (T \oplus Y)$ where $T \oplus Y$ is p^n-bounded). That is why $p^\omega G = K \oplus Y$ with a countable summand K, as desired. An other verification of this fact is like this: Note that $X[p] = (p^{\omega+n}G)[p] = (p^n X)[p]$, and hence $X[p]$ is countable. So X will be countable, provided that it is reduced.

Let us now H be a $p^{\omega+n}$-high subgroup of G containing Y (thus H is maximal with respect to $H \cap p^{\omega+n}G = \{0\}$). We next assert that $(G/p^{\omega+n}G)[p^n] = (X \oplus H[p^n])/p^{\omega+n}G$. To this aim, given $v \in G$ with $p^n v \in p^{\omega+n}G$, it suffices to prove that $v \in X \oplus H[p^n]$. If $x \in X$ is chosen such that $p^n x = p^n v$, then replacing v by $v - x$, we may assume that $p^n v = 0$. Since $G[p] = (p^{\omega+n}G)[p] \oplus H[p] = X[p] \oplus H[p]$ and H is pure in G, it easily follows that $G[p^n] = X[p^n] \oplus H[p^n]$. Therefore, $v = x' + h$ where $x' \in X[p^n]$ and $h \in H[p^n]$ as required. Moreover, $X \cap H = \{0\}$ because as noted above $X[p] = (p^{\omega+n}G)[p]$, which substantiates our assertion. Furthermore, by what we have just shown above, $P/p^{\omega+n}G \subseteq (G/p^{\omega+n}G)[p^n]$ implies that $P \subseteq X \oplus H[p^n]$. Note also the fact from above that $X \leq P$. Let $L = P \cap H[p^n] \subseteq H[p^n] \subseteq G[p^n]$; so $p^n L = \{0\}$. Clearly, the inclusion $L \subseteq H$ forces that $L \cap p^{\omega+n}G = \{0\}$. Likewise, $P \subseteq X \oplus H[p^n]$ yields that $P = X + (P \cap H[p^n]) = X + L$; indeed the modular law applies to get that $P = (X \oplus H[p^n]) \cap P = X + P \cap H[p^n]$ as stated. Consequently, we conclude that $P = p^\omega G + P = p^\omega G + L$. Thus $G/P = G/(p^\omega G + L)$ is Σ-cyclic.

We next will show that L is nice in G. Since $L \cap p^{\omega+n}G = \{0\}$, it readily follows via some technical efforts that $L \cap p^\omega G$ is nice in $p^\omega G$ and so nice in G. But $L + p^\omega G = P$ is also nice in G because $G/(p^\omega G + L)$ is separable, and these two conditions together imply that L is nice in G, as wanted (see, e.g., Section 79, Exercise 10 of [9]).

Furthermore, we claim that $p^\omega(G/L) = (p^\omega G + L)/L = P/L$ is countable. In fact, $P/L = P/(P \cap H[p^n]) \cong (P + H[p^n])/H[p^n] = (p^\omega G + H[p^n])/H[p^n] \cong p^\omega G/(p^\omega G \cap H[p^n])$. But $p^\omega G = X \oplus Y$ and since $Y \subseteq H$, one may have in view of the modular law that $p^\omega G \cap H = (X \oplus Y) \cap H = (X \cap H) \oplus Y = Y$. We therefore establish that $P/L \cong (X \oplus Y)/Y[p^n] \cong X \oplus (Y/Y[p^n]) \cong X \oplus p^n Y = X$, because $p^n Y = \{0\}$. As noticed above, X is countable, so that $p^\omega(G/L)$ is really countable as claimed. Finally, Proposition 2.5 (a) allows us to infer that G is strongly ω_1-$p^{\omega+n}$-projective, as required. ■

As a direct consequence, we obtain the following:

2.9. Corollary. *The group G is strongly almost n-simply presented with countable $p^{\omega+n}G$ if and only if G is strongly almost ω_1-$p^{\omega+n}$-projective.*

Proof. Concerning the necessity, in conjunction with [2], the quotient $G/p^{\omega+n}G$ is almost $p^{\omega+n}$-projective. We next apply Theorem 2.8 to get the desired assertion.

As for the sufficiency, it follows immediately from either Proposition 2.2 or Lemma 2.3 accomplished with Theorem 2.8. ■

Another immediate consequence is the following one:

2.10. Corollary. *Strongly almost n-simply presented groups are almost ω_1-$p^{\omega+n}$-projective if and only if they are strongly almost ω_1-$p^{\omega+n}$-projective.*

Proof. The "if" part being elementary, we concentrate on the "and only if" part. To this aim, owing to [7], the Ulm subgroup $p^{\omega+n}G$ has to be countable. On the other hand, according to [2], the factor-group $G/p^{\omega+n}G$ must be almost $p^{\omega+n}$-projective. We therefore with Theorem 2.8 at hand deduce that G is strongly almost ω_1-$p^{\omega+n}$-projective groups, as claimed. ■

2.11. Proposition. *The countable direct sum of strongly almost ω_1-$p^{\omega+n}$-projective groups is again a strongly almost ω_1-$p^{\omega+n}$-projective group.*

Proof. Write $G = \oplus_{i\in I}G_i$, where all summands G_i are strongly almost ω_1-$p^{\omega+n}$-projective groups, and $|I| = \aleph_0$. Thus, in view of Theorem 2.8, $p^{\omega+n}G = \oplus_{i\in I}p^{\omega+n}G_i$ remains countable. On the other vein, in virtue of [7] along with Theorem 2.8, the quotient $G/p^{\omega+n}G \cong \oplus_{i\in I}G_i/p^{\omega+n}G_i$ remains almost $p^{\omega+n}$-projective. We finally again take into account Theorem 2.8 to get the wanted assertion that G is a strongly almost ω_1-$p^{\omega+n}$-projective group. ∎

2.12. Proposition. *Let $G = H \oplus K$, where K is a countable subgroup of a group G. Then G is a strongly almost ω_1-$p^{\omega+n}$-projective group if and only if H is a strongly almost ω_1-$p^{\omega+n}$-projective group.*

Proof. The "if" part follows directly from Proposition 2.11.

To treat the "and only if" part, since by Theorem 2.8 the group $p^{\omega+n}G$ is countable, it follows at once that so is its subgroup $p^{\omega+n}H$. Moreover, the direct decomposition $G/p^{\omega+n}G \cong (H/p^{\omega+n}H) \oplus (K/p^{\omega+n}K)$ implies with the aid of [7] that $H/p^{\omega+n}H$ is almost $p^{\omega+n}$-projective, because by virtue of Theorem 2.8 the same is $G/p^{\omega+n}G$. We consequently may now employ once again Theorem 2.8 to obtain that H is strongly almost ω_1-$p^{\omega+n}$-projective, as expected. ∎

2.13. Proposition. *(i) Suppose $H \leq G$ with G/H finite. If H is strongly almost ω_1-$p^{\omega+n}$-projective, then G is strongly almost ω_1-$p^{\omega+n}$-projective.*

(ii) Suppose $F \leq G$ is finite. If G is strongly almost ω_1-$p^{\omega+n}$-projective, then G/F is strongly almost ω_1-$p^{\omega+n}$-projective.

Proof. (i) Write $G = H + F$ where $F \leq G$ is finite. By definition, let $H/P = (C/P) + (S/P)$, where the first term is countable while the second one is almost Σ-cyclic, for some p^n-bounded nice subgroup P of H. It follows that $G/P = [(C/P)+((F+P)/P)]+(S/P)$, where the first term remain countable. Owing to [2] or [3] it follows that P is nice in $H + F = G$, as required.

(ii) With Theorem 2.8 in hand, we know that $p^{\omega+n}G$ is countable and $G/p^{\omega+n}G$ is almost Σ-cyclic. But F being finite is nice in G, so that $p^{\omega+n}(G/F) = (p^{\omega+n}G+F)/F \cong p^{\omega+n}G/(F \cap p^{\omega+n}G)$ is also countable. Moreover,

$$(G/F)/p^{\omega+n}(G/F) \cong G/(p^{\omega+n}G + F) \cong (G/p^{\omega+n}G)/((p^{\omega+n}G + F)/p^{\omega+n}G).$$

Since $(p^{\omega+n}G + F)/p^{\omega+n}G) \cong F/(p^{\omega+n}G \cap F)$ is finite, we refer to [2] or to [7] to obtain that $(G/p^{\omega+n}G)/((p^{\omega+n}G + F)/p^{\omega+n}G)$ is almost Σ-cyclic, and hence so is $(G/F)/p^{\omega+n}(G/F)$ thus getting the wanted claim. ∎

2.2. Nicely Almost ω_1-$p^{\omega+n}$-Projective p-Groups.

2.14. Proposition. *If G is nicely almost ω_1-$p^{\omega+n}$-projective, then so is $p^\alpha G$ for any ordinal α.*

Proof. Letting G/X be countable for some nice subgroup $X \leq G$ such that X is almost $p^{\omega+n}$-projective, one sees that $p^\alpha(G/X) = (p^\alpha G + X)/X \cong p^\alpha G/(p^\alpha G \cap X)$ remains also countable. Besides, in accordance to [9], $p^\alpha G \cap X$ is nice in $p^\alpha G$ as well as $p^\alpha G \cap X \subseteq X$ is almost $p^{\omega+n}$-projective by application of [7]. Thus Definition 1.2 is satisfied, as required. ∎

2.15. Proposition. *The countable direct sum of nicely almost ω_1-$p^{\omega+n}$-projective groups is again a nicely almost ω_1-$p^{\omega+n}$-projective group.*

Proof. Write $G = \oplus_{i\in I} G_i$, where all summands G_i are strongly almost ω_1-$p^{\omega+n}$-projective groups, and $|I| = \aleph_0$. By definition, for each index $i \in I$, there is a nice subgroup $X_i \leq G_i$ such that G_i/X_i is countable and X_i is almost $p^{\omega+n}$=projective. Setting $X = \oplus_{i\in I} X_i$, one can see that X is nice in G, and X is almost $p^{\omega+n}$-projective by [7]. Moreover, $G/X \cong \oplus_{i\in I} G_i/X_i$ is countable, so that Definition 1.2 is applicable to obtain that G is a nicely almost ω_1-$p^{\omega+n}$-projective group, as promised. ■

2.16. Proposition. *(i) Suppose $H \leq G$ with G/H finite. If H is nicely almost ω_1-$p^{\omega+n}$-projective, then G is nicely almost ω_1-$p^{\omega+n}$-projective.*

(ii) Suppose $F \leq G$ is finite. If G is nicely almost ω_1-$p^{\omega+n}$-projective, then G/F is nicely almost ω_1-$p^{\omega+n}$-projective.

Proof. (i) Letting H/X be countable for some nice subgroup $X \leq H$ such that X is almost $p^{\omega+n}$-projective and writing $G = H + F$, where F is a finite subgroup of G, we observe that $G/X = (H/X) + ((F + X)/X)$ is countable. By [2] or [3], we have that X is nice in $H + F = G$, as required.

(ii) Let G/X be countable for some nice subgroup $X \leq G$ such that X is almost $p^{\omega+n}$-projective. Thus, as being its epimorphic image, $G/(X+F) \cong (G/F)/((X+F)/F)$ remains countable as well. But, in virtue of [2] or [3], $X + F$ is nice in G whence by [9] the factor-group $(X + F)/F$ is nice in G/F. Finally, [7] enables us that $(X + F)/F \cong X/(F \cap X)$ is almost $p^{\omega+n}$-projective, because $F \cap X$ is finite. Consequently, Definition 1.2 leads us to G/F is nicely almost ω_1-$p^{\omega+n}$-projective, as claimed. ■

3. Open Problems

We state here two problems of interest.

Problem 1. Does it follow that strongly almost ω_1-$p^{\omega+n}$-projective groups are nicely almost ω_1-$p^{\omega+n}$-projective?

Removing off the word "almost" this is true (see [3]). However, the same proof does not work directly in the current case, because the two definitions are almost identical.

Problem 2. Decide whether or not nicely almost ω_1-$p^{\omega+n}$-projective groups are nicely almost n-simply presented.

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Periodicity and solutions for some systems of nonlinear rational difference equations

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Abstract

In this paper, we investigated the periodic nature and the form of the solutions of nonlinear difference equations systems of order three

$$x_{n+1}=\frac{x_{n-2}y_{n-1}}{y_n\left(\pm 1\pm x_{n-2}y_{n-1}\right)},\ y_{n+1}=\frac{y_{n-2}x_{n-1}}{x_n\left(\pm 1\pm y_{n-2}x_{n-1}\right)},$$

with initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are nonzero real numbers.

Keywords: difference equations, recursive sequences, stability, periodic solution, solution of difference equation, system of difference equations.

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1. Introduction

In this paper we deal with the existence of solutions and the periodicity character of the following systems of rational difference equations with order three

$$x_{n+1}=\frac{x_{n-2}y_{n-1}}{y_n\left(\pm 1\pm x_{n-2}y_{n-1}\right)},\ y_{n+1}=\frac{y_{n-2}x_{n-1}}{x_n\left(\pm 1\pm y_{n-2}x_{n-1}\right)},$$

with initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are nonzero real numbers.

In recent years, rational difference equations have attracted the attention of many researchers for varied reasons. On the one hand, they provide examples of nonlinear equations which are, in some cases, treatable but whose dynamics present some new features with respect to the linear case. On the other hand, rational equations frequently appear in some biological models, and, hence, their study is of interest also due to their applications. A good example of both facts is Ricatti difference equations; the richness of the dynamics of Ricatti equations is very well-known (see, e.g., [10, 29]), and a particular case of these equations provides the classical Beverton-Holt model on the dynamics of

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exploited fish populations [5]. Obviously, higher-order rational difference equations and systems of rational equations have also been widely studied but still have many aspects to be investigated. The reader can find in the following books [1, 29, 30], and the works cited therein, many results, applications, and open problems on higher-order equations and rational systems.

There are many papers related to the difference equations systems for example, The global asymptotic behavior of the positive solutions of the rational difference system

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}},$$

has been studied by Camouzis et al. in [6].

The periodicity of the positive solutions of the rational difference equations systems

$$\begin{aligned} x_{n+1} &= \frac{m}{y_n}, \ y_{n+1} = \frac{p y_n}{x_{n-1} y_{n-1}}, \\ x_{n+1} &= \frac{1}{z_n}, \ y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}}, \ z_{n+1} = \frac{1}{x_{n-1}}, \end{aligned}$$

has been obtained by Cinar in [7-8].
In [9] Clark and Kulenovic investigated the global asymptotic stability

$$x_{n+1} = \frac{x_n}{a + c y_n}, \quad y_{n+1} = \frac{y_n}{b + d x_n}.$$

Elsayed [14] has got the solutions of the following systems of the difference equations

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 + x_{n-1} y_n}, \ y_{n+1} = \frac{y_{n-1}}{\mp 1 + y_{n-1} x_n}.$$

Grove et al. [23] has studied existence and behavior of solution of the rational system

$$x_{n+1} = \frac{a}{x_n} + \frac{b}{y_n}, \ y_{n+1} = \frac{c}{x_n} + \frac{d}{y_n}.$$

The behavior of positive solutions of the following system

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1} y_n}, \ y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1} x_n}.$$

has been studied by Kurbanli et al. [31].
Özban [32] has investigated the positive solution of the system of rational difference equations

$$x_{n+1} = \frac{a}{y_{n-3}}, \ y_{n+1} = \frac{b y_{n-3}}{x_{n-q} y_{n-q}}.$$

Also, Touafek et al. [36] studied the periodicity and gave the form of the solutions of the following systems

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)}, \ y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}.$$

In [37] Yalçınkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, \ y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}.$$

Similar to difference equations and nonlinear systems of rational difference equations were investigated see [1]-[43].

1.1. Definition. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2. The First System: $x_{n+1}=\frac{x_{n-2}y_{n-1}}{y_n(1+x_{n-2}y_{n-1})}$, $y_{n+1}=\frac{y_{n-2}x_{n-1}}{x_n(1+y_{n-2}x_{n-1})}$

In this section, we investigate the solutions of the system of two difference equations

$$x_{n+1}=\frac{x_{n-2}y_{n-1}}{y_n\left(1+x_{n-2}y_{n-1}\right)},\ y_{n+1}=\frac{y_{n-2}x_{n-1}}{x_n\left(1+y_{n-2}x_{n-1}\right)}, \tag{1}$$

where $n\in\mathbb{N}_0$ and the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are arbitrary nonzero real numbers.

2.1. Theorem. *Assume that $\{x_n,y_n\}$ are solutions of system (1). Then for $n=0,1,2,...$, we see that all solutions of system (1) are given by the following formula*

$$\begin{aligned}
x_{4n-2} &= \frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)},\\
x_{4n-1} &= \frac{x_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})},\\
x_{4n} &= \frac{x_0^{n+1}y_0^n}{x_{-2}^n y_{-2}^n}\prod_{i=0}^{n-1}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)},\\
x_{4n+1} &= \frac{y_{-1}x_{-2}^{n+1}y_{-2}^n}{x_0^n y_0^{n+1}(1+x_{-2}y_{-1})}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})},
\end{aligned}$$

and

$$\begin{aligned}
y_{4n-2} &= \frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)},\\
y_{4n-1} &= \frac{y_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n}\prod_{i=0}^{n-1}\frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})},\\
y_{4n} &= \frac{x_0^n y_0^{n+1}}{x_{-2}^n y_{-2}^n}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)},\\
y_{4n+1} &= \frac{x_{-1}x_{-2}^n y_{-2}^{n+1}}{x_0^{n+1}y_0^n(1+x_{-1}y_{-2})}\prod_{i=0}^{n-1}\frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}.
\end{aligned}$$

Proof. For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. that is,

$$\begin{aligned}
x_{4n-6} &= \frac{x_0^{n-1}y_0^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-2}}\prod_{i=0}^{n-2}\frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)},\\
x_{4n-5} &= \frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})},\\
x_{4n-4} &= \frac{x_0^n y_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)},\\
x_{4n-3} &= \frac{y_{-1}x_{-2}^n y_{-2}^{n-1}}{x_0^{n-1}y_0^n(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})},
\end{aligned}$$

$$
\begin{aligned}
y_{4n-6} &= \frac{x_0^{n-1}y_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-2}} \prod_{i=0}^{n-2} \frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}, \\
y_{4n-5} &= \frac{y_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}, \\
y_{4n-4} &= \frac{x_0^{n-1}y_0^{n}}{x_{-2}^{n-1}y_{-2}^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}, \\
y_{4n-3} &= \frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n}}{x_0^{n}y_0^{n-1}(1+x_{-1}y_{-2})} \prod_{i=0}^{n-2} \frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}.
\end{aligned}
$$

Now it follows from Eq.(1) that

$$
\begin{aligned}
x_{4n-2} &= \frac{x_{4n-5}y_{4n-4}}{y_{4n-3}\left(1+x_{4n-5}y_{4n-4}\right)} \\
&= \frac{\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\left(\frac{x_0^{n-1}y_0^{n}}{x_{-2}^{n-1}y_{-2}^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}\right)}{\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n}}{x_0^{n}y_0^{n-1}(1+x_{-1}y_{-2})} \prod_{i=0}^{n-2} \frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}\right)\left(1+\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\left(\frac{x_0^{n-1}y_0^{n}}{x_{-2}^{n-1}y_{-2}^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}\right)\right)} \\
&= \frac{\left(x_{-1}y_0 \prod_{i=0}^{n-2}(1+(2i)x_{-1}y_0)\right)\left(\prod_{i=0}^{n-2}\frac{1}{(1+(2i+2)x_{-1}y_0)}\right)}{\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n}}{x_0^{n}y_0^{n-1}(1+x_{-1}y_{-2})} \prod_{i=0}^{n-2} \frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}\right)\left(1+\left(x_{-1}y_0 \prod_{i=0}^{n-2}(1+(2i)x_{-1}y_0)\right)\left(\prod_{i=0}^{n-2}\frac{1}{(1+(2i+2)x_{-1}y_0)}\right)\right)} \\
&= \frac{\left(\frac{x_{-1}y_0}{(1+(2n-2)x_{-1}y_0)}\right)}{\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n}}{x_0^{n}y_0^{n-1}(1+x_{-1}y_{-2})} \prod_{i=0}^{n-2} \frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}\right)\left(1+\left(\frac{x_{-1}y_0}{(1+(2n-2)x_{-1}y_0)}\right)\right)} \\
&= \frac{x_0^{n}y_0^{n-1}(1+x_{-1}y_{-2})\left(\frac{x_{-1}y_0}{1+(2n-2)x_{-1}y_0}\right)}{x_{-1}x_{-2}^{n-1}y_{-2}^{n}\left(1+\frac{x_{-1}y_0}{1+(2n-2)x_{-1}y_0}\right)} \prod_{i=0}^{n-2} \frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)} \\
&= \frac{x_0^{n}y_0^{n}(1+x_{-1}y_{-2})}{x_{-2}^{n-1}y_{-2}^{n}\left(1+(2n-2)x_{-1}y_0+x_{-1}y_0\right)} \prod_{i=0}^{n-2} \frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{x_0^{n}y_0^{n}(1+x_{-1}y_{-2})}{x_{-2}^{n-1}y_{-2}^{n}\left(1+(2n-1)x_{-1}y_0\right)} \prod_{i=0}^{n-2} \frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)} \\
&= \frac{x_0^{n}y_0^{n}}{x_{-2}^{n-1}y_{-2}^{n}} \prod_{i=0}^{n-2} \frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)},
\end{aligned}
$$

$$
\begin{aligned}
y_{4n-2} &= \frac{y_{4n-5}x_{4n-4}}{x_{4n-3}\left(1+y_{4n-5}x_{4n-4}\right)} \\
&= \frac{\left(\frac{y_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}\right)\left(\frac{x_0^ny_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)}{\left(\frac{y_{-1}x_{-2}^ny_{-2}^{n-1}}{x_0^{n-1}y_0^n(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\left(1+\left(\frac{y_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}\right)\left(\frac{x_0^ny_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\right)} \\
&= \frac{\left(x_0y_{-1}\prod_{i=0}^{n-2}(1+(2i)x_0y_{-1})\right)\left(\prod_{i=0}^{n-2}\frac{1}{(1+(2i+2)x_0y_{-1})}\right)}{\left(\frac{y_{-1}x_{-2}^ny_{-2}^{n-1}}{x_0^{n-1}y_0^n(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\left(1+\left(x_0y_{-1}\prod_{i=0}^{n-2}(1+(2i)x_0y_{-1})\right)\left(\prod_{i=0}^{n-2}\frac{1}{(1+(2i+2)x_0y_{-1})}\right)\right)} \\
&= \frac{\left(\frac{x_0y_{-1}}{(1+(2n-2)x_0y_{-1})}\right)}{\left(\frac{y_{-1}x_{-2}^ny_{-2}^{n-1}}{x_0^{n-1}y_0^n(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\left(1+\left(\frac{x_0y_{-1}}{(1+(2n-2)x_0y_{-1})}\right)\right)} \\
&= \frac{x_0y_{-1}x_0^{n-1}y_0^n(1+x_{-2}y_{-1})}{y_{-1}x_{-2}^ny_{-2}^{n-1}\left(1+(2n-1)x_0y_{-1}\right)}\prod_{i=0}^{n-2}\frac{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)} \\
&= \frac{x_0^ny_0^n}{x_{-2}^ny_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}.
\end{aligned}
$$

Also, we see from Eq.(1) that

$$
\begin{aligned}
x_{4n-1} &= \frac{x_{4n-4}y_{4n-3}}{y_{4n-2}\left(1+x_{4n-4}y_{4n-3}\right)} \\
&= \frac{\left(\frac{x_0^n y_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n}}{x_0^n y_0^{n-1}(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}\right)}{\left(\frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}\right)\left(\begin{array}{c}1+\left(\frac{x_0^n y_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\\ \left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n}}{x_0^n y_0^{n-1}(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}\right)\end{array}\right)} \\
&= \frac{\left(\prod_{i=0}^{n-2}(1+(2i+1)x_{-1}y_{-2})\right)\left(\frac{x_{-1}y_{-2}}{(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{1}{(1+(2i+3)x_{-1}y_{-2})}\right)}{\left(\frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}\right)\left(1+\left({}_{i=0}^{n-2}(1+(2i+1)x_{-1}y_{-2})\right)\left(\frac{x_{-1}y_{-2}}{(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{1}{(1+(2i+3)x_{-1}y_{-2})}\right)\right)} \\
&= \frac{\left(\frac{x_{-1}y_{-2}}{(1+(2n-1)x_{-1}y_{-2})}\right)}{\left(\frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}\right)\left(1+\left(\frac{x_{-1}y_{-2}}{(1+(2n-1)x_{-1}y_{-2})}\right)\right)} \\
&= \frac{x_{-1}y_{-2}}{\left(\frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}\right)\left(1+(2n-1)x_{-1}y_{-2}+x_{-1}y_{-2}\right)} \\
&= \frac{x_{-2}^n y_{-2}^{n-1}x_{-1}y_{-2}}{x_0^n y_0^n\left(1+(2n)x_{-1}y_{-2}\right)}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})} \\
&= \frac{x_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})},
\end{aligned}
$$

and

$$\begin{aligned}
y_{4n-1} &= \frac{y_{4n-4}x_{4n-3}}{x_{4n-2}\left(1+y_{4n-4}x_{4n-3}\right)} \\
&= \frac{\left(\frac{x_0^{n-1}y_0^n}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}\right)\left(\frac{y_{-1}x_{-2}^n y_{-2}^{n-1}}{x_0^{n-1}y_0^n(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)}{\left(\frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\left(1+\left(\frac{x_0^{n-1}y_0^n}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}\right)\left(\frac{y_{-1}x_{-2}^n y_{-2}^{n-1}}{x_0^{n-1}y_0^n(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\right)} \\
&= \frac{\left(\prod_{i=0}^{n-2}(1+(2i+1)x_{-2}y_{-1})\right)\left(\frac{y_{-1}x_{-2}}{(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{1}{(1+(2i+3)x_{-2}y_{-1})}\right)}{\left(\frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\left(1+\left(\prod_{i=0}^{n-2}(1+(2i+1)x_{-2}y_{-1})\right)\left(\frac{y_{-1}x_{-2}}{(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{1}{(1+(2i+3)x_{-2}y_{-1})}\right)\right)} \\
&= \frac{\left(\frac{y_{-1}x_{-2}}{(1+(2n-1)x_{-2}y_{-1})}\right)}{\left(\frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\left(1+\left(\frac{y_{-1}x_{-2}}{(1+(2n-1)x_{-2}y_{-1})}\right)\right)} \\
&= \frac{y_{-2}^n x_{-2}^{n-1}y_{-1}x_{-2}}{x_0^n y_0^n\left(1+(2n-1)x_{-2}y_{-1}+y_{-1}x_{-2}\right)}\prod_{i=0}^{n-1}\frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})} \\
&= \frac{y_{-1}y_{-2}^n x_{-2}^n}{x_0^n y_0^n}\prod_{i=0}^{n-1}\frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}.
\end{aligned}$$

Also, we can prove the other relations. The proof is complete. ∎

The following Theorems can be proved similarly:

2.2. Theorem. *Assume that $\{x_n, y_n\}$ are solutions of the system*

$$x_{n+1}=\frac{x_{n-2}y_{n-1}}{y_n\left(1-x_{n-2}y_{n-1}\right)},\ y_{n+1}=\frac{y_{n-2}x_{n-1}}{x_n\left(1-y_{n-2}x_{n-1}\right)}.$$

Then for $n = 0, 1, 2, \ldots,$

$$\begin{aligned}
x_{4n-2} &= \frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(1-(2i)x_{-2}y_{-1})(1-(2i+1)x_{-1}y_{-2})}{(1-(2i)x_0y_{-1})(1-(2i+1)x_{-1}y_0)},\\
x_{4n-1} &= \frac{x_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(1-(2i+1)x_0y_{-1})(1-(2i)x_{-1}y_0)}{(1-(2i+1)x_{-2}y_{-1})(1-(2i+2)x_{-1}y_{-2})},\\
x_{4n} &= \frac{x_0^{n+1} y_0^n}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(1-(2i+2)x_{-2}y_{-1})(1-(2i+1)x_{-1}y_{-2})}{(1-(2i+2)x_0y_{-1})(1-(2i+1)x_{-1}y_0)},\\
x_{4n+1} &= \frac{y_{-1}x_{-2}^{n+1} y_{-2}^n}{x_0^n y_0^{n+1}(1-x_{-2}y_{-1})} \prod_{i=0}^{n-1} \frac{(1-(2i+1)x_0y_{-1})(1-(2i+2)x_{-1}y_0)}{(1-(2i+3)x_{-2}y_{-1})(1-(2i+2)x_{-1}y_{-2})},
\end{aligned}$$

and

$$\begin{aligned}
y_{4n-2} &= \frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(1-(2i+1)x_{-2}y_{-1})(1-(2i)x_{-1}y_{-2})}{(1-(2i+1)x_0y_{-1})(1-(2i)x_{-1}y_0)},\\
y_{4n-1} &= \frac{y_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(1-(2i)x_0y_{-1})(1-(2i+1)x_{-1}y_0)}{(1-(2i+2)x_{-2}y_{-1})(1-(2i+1)x_{-1}y_{-2})},\\
y_{4n} &= \frac{x_0^n y_0^{n+1}}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(1-(2i+1)x_{-2}y_{-1})(1-(2i+2)x_{-1}y_{-2})}{(1-(2i+1)x_0y_{-1})(1-(2i+2)x_{-1}y_0)},\\
y_{4n+1} &= \frac{x_{-1}x_{-2}^n y_{-2}^{n+1}}{x_0^{n+1} y_0^n(1-x_{-1}y_{-2})} \prod_{i=0}^{n-1} \frac{(1-(2i+2)x_0y_{-1})(1-(2i+1)x_{-1}y_0)}{(1-(2i+2)x_{-2}y_{-1})(1-(2i+3)x_{-1}y_{-2})}.
\end{aligned}$$

2.3. Theorem. *Let* $\{x_n, y_n\}$ *are solutions of the following system*

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n(-1+x_{n-2}y_{n-1})},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n(-1-y_{n-2}x_{n-1})}.$$

Then for $n = 0, 1, 2, \ldots,$

$$\begin{aligned}
x_{4n-2} &= \frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(-1+(2i)x_{-2}y_{-1})(-1-(2i+1)x_{-1}y_{-2})}{(-1-(2i)x_0y_{-1})(-1+(2i+1)x_{-1}y_0)},\\
x_{4n-1} &= \frac{x_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(-1-(2i+1)x_0y_{-1})(-1+(2i)x_{-1}y_0)}{(-1+(2i+1)x_{-2}y_{-1})(-1-(2i+2)x_{-1}y_{-2})},\\
x_{4n} &= \frac{x_0^{n+1} y_0^n}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(-1+(2i+2)x_{-2}y_{-1})(-1-(2i+1)x_{-1}y_{-2})}{(-1-(2i+2)x_0y_{-1})(-1+(2i+1)x_{-1}y_0)},\\
x_{4n+1} &= \frac{y_{-1}x_{-2}^{n+1} y_{-2}^n}{x_0^n y_0^{n+1}(-1+x_{-2}y_{-1})} \prod_{i=0}^{n-1} \frac{(-1-(2i+1)x_0y_{-1})(-1+(2i+2)x_{-1}y_0)}{(-1+(2i+3)x_{-2}y_{-1})(-1-(2i+2)x_{-1}y_{-2})},
\end{aligned}$$

$$y_{4n-2} = \frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(-1+(2i+1)x_{-2}y_{-1})(-1-(2i)x_{-1}y_{-2})}{(-1-(2i+1)x_0y_{-1})(-1+(2i)x_{-1}y_0)},$$

$$y_{4n-1} = \frac{y_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(-1-(2i)x_0y_{-1})(-1+(2i+1)x_{-1}y_0)}{(-1+(2i+2)x_{-2}y_{-1})(-1-(2i+1)x_{-1}y_{-2})},$$

$$y_{4n} = \frac{x_0^n y_0^{n+1}}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(-1+(2i+1)x_{-2}y_{-1})(-1-(2i+2)x_{-1}y_{-2})}{(-1-(2i+1)x_0y_{-1})(-1+(2i+2)x_{-1}y_0)},$$

$$y_{4n+1} = \frac{x_{-1}x_{-2}^n y_{-2}^{n+1}}{x_0^{n+1} y_0^n(-1-x_{-1}y_{-2})} \prod_{i=0}^{n-1} \frac{(-1-(2i+2)x_0y_{-1})(-1+(2i+1)x_{-1}y_0)}{(-1+(2i+2)x_{-2}y_{-1})(-1-(2i+3)x_{-1}y_{-2})}.$$

2.4. Theorem. *The solutions of the following system of difference equations*

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(-1-x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(-1+y_{n-2}x_{n-1}\right)}.$$

are given by the following formula for $n = 0, 1, 2, ...,$

$$x_{4n-2} = \frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(-1-(2i)x_{-2}y_{-1})(-1+(2i+1)x_{-1}y_{-2})}{(-1+(2i)x_0y_{-1})(-1-(2i+1)x_{-1}y_0)},$$

$$x_{4n-1} = \frac{x_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(-1+(2i+1)x_0y_{-1})(-1-(2i)x_{-1}y_0)}{(-1-(2i+1)x_{-2}y_{-1})(-1+(2i+2)x_{-1}y_{-2})},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(-1-(2i+2)x_{-2}y_{-1})(-1+(2i+1)x_{-1}y_{-2})}{(-1+(2i+2)x_0y_{-1})(-1-(2i+1)x_{-1}y_0)},$$

$$x_{4n+1} = \frac{y_{-1}x_{-2}^{n+1} y_{-2}^n}{x_0^n y_0^{n+1}(-1-x_{-2}y_{-1})} \prod_{i=0}^{n-1} \frac{(-1+(2i+1)x_0y_{-1})(-1-(2i+2)x_{-1}y_0)}{(-1-(2i+3)x_{-2}y_{-1})(-1+(2i+2)x_{-1}y_{-2})},$$

$$y_{4n-2} = \frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(-1-(2i+1)x_{-2}y_{-1})(-1+(2i)x_{-1}y_{-2})}{(-1+(2i+1)x_0y_{-1})(-1-(2i)x_{-1}y_0)},$$

$$y_{4n-1} = \frac{y_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(-1+(2i)x_0y_{-1})(-1-(2i+1)x_{-1}y_0)}{(-1-(2i+2)x_{-2}y_{-1})(-1+(2i+1)x_{-1}y_{-2})},$$

$$y_{4n} = \frac{x_0^n y_0^{n+1}}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(-1-(2i+1)x_{-2}y_{-1})(-1+(2i+2)x_{-1}y_{-2})}{(-1+(2i+1)x_0y_{-1})(-1-(2i+2)x_{-1}y_0)},$$

$$y_{4n+1} = \frac{x_{-1}x_{-2}^n y_{-2}^{n+1}}{x_0^{n+1} y_0^n(-1+x_{-1}y_{-2})} \prod_{i=0}^{n-1} \frac{(-1+(2i+2)x_0y_{-1})(-1-(2i+1)x_{-1}y_0)}{(-1-(2i+2)x_{-2}y_{-1})(-1+(2i+3)x_{-1}y_{-2})}.$$

2.5. Example. For confirming the results of this section, we consider numerical example for the difference system (1) with the initial conditions $x_{-2} = 3,\ x_{-1} = 5, x_0 = -4,\ y_{-2} = 2,\ y_{-1} = 6$ *and* $y_0 = 7$. (See Fig. 1).

3. The Second System: $x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n(-1+x_{n-2}y_{n-1})},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n(1+y_{n-2}x_{n-1})}$

In this section, we obtain the form of the solutions of the system of two difference equations

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(-1+x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(1+y_{n-2}x_{n-1}\right)}, \tag{2}$$

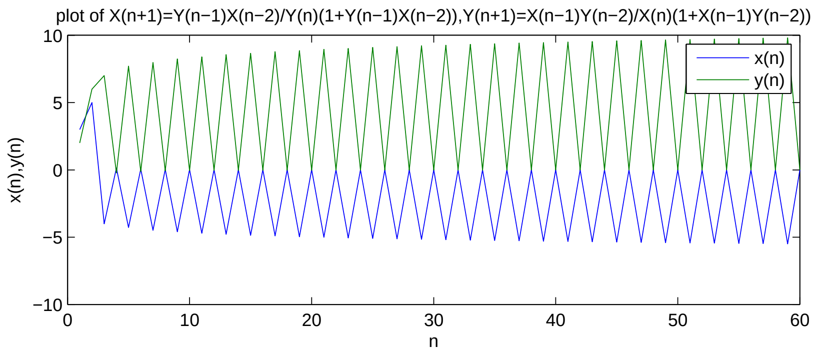

Figure 1

where $n \in \mathbb{N}_0$ and the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are arbitrary non zero real numbers with $x_{-1}y_0,\ x_{-2}y_{-1} \neq 1, \dfrac{1}{2}$, and $x_0y_{-1},\ x_{-1}y_{-2} \neq \pm 1$.

The following theorem is devoted to the expression of the form of the solutions of system (2).

3.1. Theorem. *Let $\{x_n, y_n\}_{n=-2}^{+\infty}$ be solutions of system (2). Then $\{x_n\}_{n=-2}^{+\infty}$ and $\{y_n\}_{n=-2}^{+\infty}$ are given by the formulae for $n = 0, 1, 2, ...$,*

$$\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}},\\
x_{8n-1} &= \frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}},\\
x_{8n} &= \frac{x_0^{2n+1}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}},
\end{aligned}$$

$$\begin{aligned}
x_{8n+1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n+1}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n+1}},\\
x_{8n+2} &= \frac{x_0^{2n+1}y_0^{2n+1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n+1}},\\
x_{8n+3} &= -\frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n+1}},\\
x_{8n+4} &= -\frac{x_0^{2n+2}y_0^{2n+1}(-1+2x_{-2}y_{-1})^{n+1}(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+x_{-1}y_0)^{2n+1}},\\
x_{8n+5} &= \frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+2}(-1+2x_{-1}y_0)^{n+1}(-1+x_0y_{-1})^n(1+x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+2}(-1+x_{-2}y_{-1})^{2n+2}},
\end{aligned}$$

$$
\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n},\\
y_{8n-1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n},\\
y_{8n} &= \frac{x_0^{2n}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n},\\
y_{8n+1} &= \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n+1}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^{n+1}},\\
y_{8n+2} &= \frac{x_0^{2n+1}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n}x_{-2}^{2n+1}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^{n+1}},\\
y_{8n+3} &= \frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+x_{-1}y_0)^{2n+1}}{x_0^{2n+1}y_0^{2n+1}(-1+2x_{-2}y_{-1})^{n+1}(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^{n+1}},\\
y_{8n+4} &= -\frac{x_0^{2n+1}y_0^{2n+2}(-1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+2x_{-1}y_0)^{n+1}(-1+x_0y_{-1})^n(1+x_0y_{-1})^{n+1}},\\
y_{8n+5} &= -\frac{x_{-1}y_{-2}^{2n+2}x_{-2}^{2n+1}(-1+x_{-1}y_0)^{2n+1}}{x_0^{2n+2}y_0^{2n+1}(-1+2x_{-2}y_{-1})^{n+1}(-1+x_{-1}y_{-2})^{n+1}(1+x_{-1}y_{-2})^{n+1}}.
\end{aligned}
$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. that is,

$$
\begin{aligned}
x_{8n-10} &= \frac{x_0^{2n-2}y_0^{2n-2}(-1+2x_{-2}y_{-1})^{n-1}(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^{n-1}}{y_{-2}^{2n-2}x_{-2}^{2n-3}(-1+x_{-1}y_0)^{2n-2}},\\
x_{8n-9} &= \frac{x_{-1}y_{-2}^{2n-2}x_{-2}^{2n-2}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n-1}}{x_0^{2n-2}y_0^{2n-2}(-1+x_{-2}y_{-1})^{2n-2}},\\
x_{8n-8} &= \frac{x_0^{2n-1}y_0^{2n-2}(-1+2x_{-2}y_{-1})^{n-1}(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^{n-1}}{y_{-2}^{2n-2}x_{-2}^{2n-2}(-1+x_{-1}y_0)^{2n-2}},
\end{aligned}
$$

$$
\begin{aligned}
x_{8n-7} &= \frac{y_{-1}y_{-2}^{2n-2}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n-1}}{x_0^{2n-2}y_0^{2n-1}(-1+x_{-2}y_{-1})^{2n-1}},\\
x_{8n-6} &= \frac{x_0^{2n-1}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n-1}(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^{n}}{y_{-2}^{2n-1}x_{-2}^{2n-2}(-1+x_{-1}y_0)^{2n-1}},\\
x_{8n-5} &= -\frac{x_{-1}y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}}{x_0^{2n-1}y_0^{2n-1}(-1+x_{-2}y_{-1})^{2n-1}},\\
x_{8n-4} &= -\frac{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^{n}}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}},\\
x_{8n-3} &= \frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^{n}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}},
\end{aligned}
$$

and

$$
\begin{aligned}
y_{8n-10} &= \frac{x_0^{2n-2}y_0^{2n-2}(-1+x_{-2}y_{-1})^{2n-2}}{y_{-2}^{2n-3}x_{-2}^{2n-2}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n-1}},\\
y_{8n-9} &= \frac{y_{-1}y_{-2}^{2n-2}x_{-2}^{2n-2}(-1+x_{-1}y_0)^{2n-2}}{x_0^{2n-2}y_0^{2n-2}(-1+2x_{-2}y_{-1})^{n-1}(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^{n-1}},\\
y_{8n-8} &= \frac{x_0^{2n-2}y_0^{2n-1}(-1+x_{-2}y_{-1})^{2n-2}}{y_{-2}^{2n-2}x_{-2}^{2n-2}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n-1}},\\
y_{8n-7} &= \frac{x_{-1}y_{-2}^{2n-1}x_{-2}^{2n-2}(-1+x_{-1}y_0)^{2n-2}}{x_0^{2n-1}y_0^{2n-2}(-1+2x_{-2}y_{-1})^{n-1}(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^{n}},\\
y_{8n-6} &= \frac{x_0^{2n-1}y_0^{2n-1}(-1+x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-2}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}},\\
y_{8n-5} &= \frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n-1}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^{n}},\\
y_{8n-4} &= -\frac{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}},\\
y_{8n-3} &= -\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n}(1+x_{-1}y_{-2})^{n}}.
\end{aligned}
$$

Now it follows from Eq.(2) that

$$
x_{8n-2} = \frac{x_{8n-5}y_{8n-4}}{y_{8n-3}\left(-1+x_{8n-5}y_{8n-4}\right)}
$$

$$
\begin{aligned}
&= \frac{\left(-\dfrac{x_{-1}y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}}{x_0^{2n-1}y_0^{2n-1}(-1+x_{-2}y_{-1})^{2n-1}}\right)\left(-\dfrac{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}}\right)}{\left(-\dfrac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n}(1+x_{-1}y_{-2})^{n}}\right)\left(-1+\left(-\dfrac{x_{-1}y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}}{x_0^{2n-1}y_0^{2n-1}(-1+x_{-2}y_{-1})^{2n-1}}\right)\left(-\dfrac{x_0^{2n-1}y_0^{2n}(1-x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}}\right)\right)}\\
&= \frac{\left(\dfrac{x_{-1}y_0}{(-1+2x_{-1}y_0)}\right)}{\left(\dfrac{-x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n}(1+x_{-1}y_{-2})^{n}}\right)\left(-1+\dfrac{x_{-1}y_0}{-1+2x_{-1}y_0}\right)}\\
&= -\frac{x_{-1}y_0x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n}(1+x_{-1}y_{-2})^{n}}{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}\left(1-2x_{-1}y_0+x_{-1}y_0\right)}\\
&= \frac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n}(1+x_{-1}y_{-2})^{n}}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}},
\end{aligned}
$$

$$\begin{aligned}
y_{8n-2} &= \frac{y_{8n-5}x_{8n-4}}{x_{8n-3}\left(1+y_{8n-5}x_{8n-4}\right)} \\
&= \frac{\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n-1}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}\right)\left(-\frac{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}\right)}{\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)\left(1+\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n-1}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}\right)\left(-\frac{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}\right)\right)} \\
&= \frac{-x_0y_{-1}}{\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)(1-x_0y_{-1})} \\
&= \frac{x_0y_{-1}x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n\,(-1+x_0y_{-1})} \\
&= \frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}.
\end{aligned}$$

Also, we see from Eq.(2) that

$$\begin{aligned}
x_{8n-1} &= \frac{x_{8n-4}y_{8n-3}}{y_{8n-2}\left(-1+x_{8n-4}y_{8n-3}\right)} \\
&= \frac{\left(-\frac{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}\right)\left(-\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)}{\left(\frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}\right)\left(-1+\left(-\frac{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}\right)\left(-\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)\right)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\left(\frac{x_{-1}y_{-2}}{(-1+x_{-1}y_{-2})}\right)}{\left(\frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}\right)\left(-1+\frac{x_{-1}y_{-2}}{(-1+x_{-1}y_{-2})}\right)} \\
&= \frac{x_{-1}y_{-2}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}\,(1-x_{-1}y_{-2}+x_{-1}y_{-2})} \\
&= \frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}},
\end{aligned}$$

and

$$
\begin{aligned}
y_{8n-1} &= \frac{y_{8n-4}x_{8n-3}}{x_{8n-2}\left(1+y_{8n-4}x_{8n-3}\right)} \\
&= \frac{\begin{pmatrix} -\dfrac{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n} \end{pmatrix} \begin{pmatrix} \dfrac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}} \end{pmatrix}}{\begin{pmatrix} \dfrac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}} \end{pmatrix} \begin{pmatrix} 1+\begin{pmatrix} -\dfrac{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n} \end{pmatrix} \\ \begin{pmatrix} \dfrac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}} \end{pmatrix} \end{pmatrix}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\left(\dfrac{-x_{-2}y_{-1}}{(-1+x_{-2}y_{-1})}\right)}{\left(\dfrac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}}\right)\left(1+\left(\dfrac{-x_{-2}y_{-1}}{(-1+x_{-2}y_{-1})}\right)\right)} \\
&= \frac{-x_{-2}y_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n\left(-1+x_{-2}y_{-1}-x_{-2}y_{-1}\right)} \\
&= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}.
\end{aligned}
$$

We get from Eq.(2) that

$$
\begin{aligned}
x_{8n} &= \frac{x_{8n-3}y_{8n-2}}{y_{8n-1}\left(-1+x_{8n-3}y_{8n-2}\right)}\\
&= \frac{\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)\left(\frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}\right)}{\left(\frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)\left(-1+\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)\left(\frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}\right)\right)}\\
&= \frac{\left(\frac{x_0y_{-1}}{(-1+x_0y_{-1})}\right)}{\left(\frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)\left(-1+\frac{x_0y_{-1}}{-1+x_0y_{-1}}\right)}\\
&= \frac{x_0y_{-1}x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}\left(1-x_0y_{-1}+x_0y_{-1}\right)}\\
&= \frac{x_0^{2n+1}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}},
\end{aligned}
$$

and

$$
\begin{aligned}
y_{8n} &= \frac{y_{8n-3}x_{8n-2}}{x_{8n-1}\left(1+y_{8n-3}x_{8n-2}\right)}\\
&= \frac{\left(-\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)\left(\frac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}}\right)}{\left(\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)\left(1+\left(-\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)\left(\frac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}}\right)\right)}\\
&= \frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}\left(-\frac{x_{-1}y_0}{(-1+x_{-1}y_0)}\right)}{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n\left(1-\frac{x_{-1}y_0}{-1+x_{-1}y_0}\right)}\\
&= \frac{-x_{-1}y_0x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n\left(-1+x_{-1}y_0-x_{-1}y_0\right)}\\
&= \frac{x_0^{2n}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}.
\end{aligned}
$$

Also, we can prove the other relations. This completes the proof. ∎

Here, we consider the following systems and the proof of the theorems are similarly to above theorem and so, left to the reader.

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(-1-x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(1-y_{n-2}x_{n-1}\right)}.3 \tag{3.1}$$

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(1+x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(-1-y_{n-2}x_{n-1}\right)}.4 \tag{3.2}$$

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(1-x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(-1+y_{n-2}x_{n-1}\right)}.5 \tag{3.3}$$

The following theorems is devoted to the expressions of the form of the solutions of systems (3), (4), (5).

3.2. Theorem. *Let $\{x_n, y_n\}_{n=-2}^{+\infty}$ be solutions of system (3) and $x_{-1}y_0,\ x_{-2}y_{-1} \neq -1, -\frac{1}{2}$, and $x_0y_{-1},\ x_{-1}y_{-2} \neq \pm 1$. Then for $n = 0, 1, 2, \ldots$,*

$$\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(1+2x_{-2}y_{-1})^n(1-x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(1+x_{-1}y_0)^{2n}},\\
x_{8n-1} &= \frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(1+2x_{-1}y_0)^n(1-x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(1+x_{-2}y_{-1})^{2n}},\\
x_{8n} &= \frac{x_0^{2n+1}y_0^{2n}(1+2x_{-2}y_{-1})^n(1-x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}},\\
x_{8n+1} &= -\frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(1-x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}},\\
x_{8n+2} &= -\frac{x_0^{2n+1}y_0^{2n+1}(1+2x_{-2}y_{-1})^n(1-x_{-1}y_{-2})^{n+1}(1+x_{-1}y_{-2})^n}{y_{-2}^{2n+1}x_{-2}^{2n}(1+x_{-1}y_0)^{2n+1}},\\
x_{8n+3} &= \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(1+x_0y_{-1})^n(1-x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}},\\
x_{8n+4} &= -\frac{x_0^{2n+2}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(1+x_{-1}y_{-2})^n(1-x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}},\\
x_{8n+5} &= -\frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+2}(1+2x_{-1}y_0)^{n+1}(1+x_0y_{-1})^n(1-x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+2}(1+x_{-2}y_{-1})^{2n+2}},
\end{aligned}$$

and

$$
\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(1+2x_{-1}y_0)^n(1-x_0y_{-1})^n(1+x_0y_{-1})^n},\\
y_{8n-1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(1+2x_{-2}y_{-1})^n(1-x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n},\\
y_{8n} &= \frac{x_0^{2n}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(1+2x_{-1}y_0)^n(1-x_0y_{-1})^n(1+x_0y_{-1})^n},\\
y_{8n+1} &= \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}}{x_0^{2n+1}y_0^{2n}(1+2x_{-2}y_{-1})^n(1+x_{-1}y_{-2})^n(1-x_{-1}y_{-2})^{n+1}},\\
y_{8n+2} &= -\frac{x_0^{2n+1}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(1+x_0y_{-1})^n(1-x_0y_{-1})^{n+1}},\\
y_{8n+3} &= \frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}}{x_0^{2n+1}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(1+x_{-1}y_{-2})^n(1-x_{-1}y_{-2})^{n+1}},\\
y_{8n+4} &= -\frac{x_0^{2n+1}y_0^{2n+2}(1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(1+2x_{-1}y_0)^{n+1}(1+x_0y_{-1})^n(1-x_0y_{-1})^{n+1}},\\
y_{8n+5} &= \frac{x_{-1}y_{-2}^{2n+2}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}}{x_0^{2n+2}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(1+x_{-1}y_{-2})^{n+1}(1-x_{-1}y_{-2})^{n+1}}.
\end{aligned}
$$

3.3. Theorem. *Assume that $\{x_n, y_n\}$ are solutions of system (4) with $x_{-1}y_0,\ x_{-2}y_{-1} \neq -1, -\frac{1}{2}$, and $x_0y_{-1},\ x_{-1}y_{-2} \neq \pm 1$. Then for $n = 0, 1, 2, ...$,*

$$
\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(1+x_{-1}y_0)^{2n}},\\
x_{8n-1} &= \frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(1+x_{-2}y_{-1})^{2n}},\\
x_{8n} &= \frac{x_0^{2n+1}y_0^{2n}(1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}},\\
x_{8n+1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^n}{x_0^{2n}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}},\\
x_{8n+2} &= \frac{x_0^{2n+1}y_0^{2n+1}(1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1}x_{-2}^{2n}(1+x_{-1}y_0)^{2n+1}},\\
x_{8n+3} &= -\frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}},\\
x_{8n+4} &= \frac{x_0^{2n+2}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}},\\
x_{8n+5} &= -\frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+2}(1+2x_{-1}y_0)^{n+1}(-1+x_0y_{-1})^n(-1-x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+2}(1+x_{-2}y_{-1})^{2n+2}},
\end{aligned}
$$

and

$$\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^n},\\
y_{8n-1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^n},\\
y_{8n} &= \frac{x_0^{2n}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^n},\\
y_{8n+1} &= \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}}{x_0^{2n+1}y_0^{2n}(1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^{n+1}},\\
y_{8n+2} &= \frac{x_0^{2n+1}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^{n+1}},\\
y_{8n+3} &= -\frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}}{x_0^{2n+1}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^{n+1}},\\
y_{8n+4} &= \frac{x_0^{2n+1}y_0^{2n+2}(1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(1+2x_{-1}y_0)^{n+1}(-1+x_0y_{-1})^n(-1-x_0y_{-1})^{n+1}},\\
y_{8n+5} &= -\frac{x_{-1}y_{-2}^{2n+2}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}}{x_0^{2n+2}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(-1+x_{-1}y_{-2})^{n+1}(-1-x_{-1}y_{-2})^{n+1}}.
\end{aligned}$$

3.4. Theorem. *Suppose that* $\{x_n, y_n\}$ *are solutions of system (5) such that* $x_{-1}y_0$, $x_{-2}y_{-1} \neq 1, \frac{1}{2}$, *and* x_0y_{-1}, $x_{-1}y_{-2} \neq \pm 1$. *Then for* $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(-1+x_{-1}y_{-2})^{2n}}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+2x_0y_{-1})^n(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^n},\\
x_{8n-1} &= \frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_0y_{-1})^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-1}y_{-2})^n(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^n},\\
x_{8n} &= \frac{x_0^{2n+1}y_0^{2n}(-1+x_{-1}y_{-2})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(-1+2x_0y_{-1})^n(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^n},\\
x_{8n+1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n+1}(-1+x_0y_{-1})^{2n}}{x_0^{2n}y_0^{2n+1}(-1+2x_{-1}y_{-2})^n(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^{n+1}},\\
x_{8n+2} &= \frac{x_0^{2n+1}y_0^{2n+1}(-1+x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n}(-1+2x_0y_{-1})^n(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^{n+1}},\\
x_{8n+3} &= \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+x_0y_{-1})^{2n+1}}{x_0^{2n+1}y_0^{2n+1}(-1+2x_{-1}y_{-2})^{n+1}(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^{n+1}},\\
x_{8n+4} &= -\frac{x_0^{2n+2}y_0^{2n+1}(-1+x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+2x_0y_{-1})^{n+1}(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^{n+1}},\\
x_{8n+5} &= -\frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+2}(-1+x_0y_{-1})^{2n+1}}{x_0^{2n+1}y_0^{2n+2}(-1+2x_{-1}y_{-2})^{n+1}(-1+x_{-2}y_{-1})^{n+1}(1+x_{-2}y_{-1})^{n+1}},
\end{aligned}$$

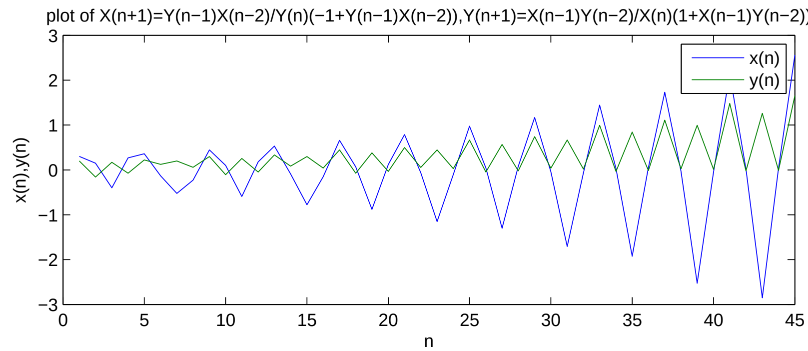

Figure 2

and

$$\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n},\\
y_{8n-1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n},\\
y_{8n} &= \frac{x_0^{2n}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n},\\
y_{8n+1} &= \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n+1}y_0^{2n}(-1+2x_{-2}y_{-1})^n(1+x_{-1}y_{-2})^n(-1+x_{-1}y_{-2})^{n+1}},\\
y_{8n+2} &= -\frac{x_0^{2n+1}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n}x_{-2}^{2n+1}(-1+2x_{-1}y_0)^n(1+x_0y_{-1})^n(-1+x_0y_{-1})^{n+1}},\\
y_{8n+3} &= -\frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+x_{-1}y_0)^{2n+1}}{x_0^{2n+1}y_0^{2n+1}(-1+2x_{-2}y_{-1})^{n+1}(1+x_{-1}y_{-2})^n(-1+x_{-1}y_{-2})^{n+1}},\\
y_{8n+4} &= \frac{x_0^{2n+1}y_0^{2n+2}(-1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+2x_{-1}y_0)^{n+1}(1+x_0y_{-1})^n(-1+x_0y_{-1})^{n+1}},\\
y_{8n+5} &= \frac{x_{-1}y_{-2}^{2n+2}x_{-2}^{2n+1}(-1+x_{-1}y_0)^{2n+1}}{x_0^{2n+2}y_0^{2n+1}(-1+2x_{-2}y_{-1})^{n+1}(1+x_{-1}y_{-2})^{n+1}(-1+x_{-1}y_{-2})^{n+1}}.
\end{aligned}$$

3.5. Example. We consider interesting numerical example for the difference system (2) with the initial conditions $x_{-2} = 0.3,\ x_{-1} = 0.15,\ x_0 = -0.4,\ y_{-2} = 0.2,\ y_{-1} = -0.16$ *and* $y_0 = 0.17$. (See Fig. 2).

4. The Third System: $x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n(1+x_{n-2}y_{n-1})}$, $y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n(-1+y_{n-2}x_{n-1})}$

In this section, we get the solutions of the system of the difference equations

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(1+x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(-1+y_{n-2}x_{n-1}\right)}, \tag{6}$$

where $n = 0, 1, 2, ...$ and the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are arbitrary nonzero real numbers with $x_{-1}y_0,\ x_{-2}y_{-1} \neq \pm 1$, and $x_0y_{-1},\ x_{-1}y_{-2} \neq 1, \frac{1}{2}$.

The following Theorems can be proved similarly as the previous section.

4.1. Theorem. *If* $\{x_n, y_n\}$ *are solutions of difference equation system (6). Then for* $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n} y_0^{2n} (-1 + x_{-1}y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n-1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0y_{-1})^n}, \\
x_{8n-1} &= \frac{x_{-1} y_{-2}^{2n} x_{-2}^{2n} (-1 + x_0y_{-1})^{2n}}{x_0^{2n} y_0^{2n} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 + 2x_{-1}y_{-2})^n}, \\
x_{8n} &= \frac{x_0^{2n+1} y_0^{2n} (-1 + x_{-1}y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0y_{-1})^n}, \\
x_{8n+1} &= \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n+1} (-1 + x_0y_{-1})^{2n}}{x_0^{2n} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^n},
\end{aligned}$$

$$\begin{aligned}
x_{8n+2} &= \frac{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 + 2x_0y_{-1})^n}, \\
x_{8n+3} &= \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (-1 + x_0y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^{n+1}}, \\
x_{8n+4} &= -\frac{x_0^{2n+2} y_0^{2n+1} (-1 + x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 + 2x_0y_{-1})^{n+1}}, \\
x_{8n+5} &= -\frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+2} (-1 + x_0y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+2} (-1 + x_{-2}y_{-1})^{n+1} (1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^{n+1}},
\end{aligned}$$

and

$$\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n} y_0^{2n} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 + 2x_{-1}y_{-2})^n}{y_{-2}^{2n-1} x_{-2}^{2n} (-1 + x_0y_{-1})^{2n}}, \\
y_{8n-1} &= \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0y_{-1})^n}{x_0^{2n} y_0^{2n} (-1 + x_{-1}y_{-2})^{2n}}, \\
y_{8n} &= \frac{x_0^{2n} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 + 2x_{-1}y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n} (-1 + x_0y_{-1})^{2n}}, \\
y_{8n+1} &= \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0y_{-1})^n}{x_0^{2n+1} y_0^{2n} (-1 + x_{-1}y_{-2})^{2n+1}}, \\
y_{8n+2} &= \frac{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n+1} (-1 + x_0y_{-1})^{2n+1}}, \\
y_{8n+3} &= -\frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 + 2x_0y_{-1})^n}{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-1}y_{-2})^{2n+1}}, \\
y_{8n+4} &= -\frac{x_0^{2n+1} y_0^{2n+2} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (-1 + x_0y_{-1})^{2n+1}}, \\
y_{8n+5} &= \frac{x_{-1} y_{-2}^{2n+2} x_{-2}^{2n+1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 + 2x_0y_{-1})^{n+1}}{x_0^{2n+2} y_0^{2n+1} (-1 + x_{-1}y_{-2})^{2n+2}}.
\end{aligned}$$

4.2. Theorem. *If* $\{x_n, y_n\}$ *are solutions of the following difference equation system*

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(-1 - x_{n-2}y_{n-1}\right)}, \quad y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(1 + y_{n-2}x_{n-1}\right)},$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 *are arbitrary nonzero real numbers with* $x_{-1}y_0,\ x_{-2}y_{-1} \neq \pm 1,$ *and* $x_0y_{-1},\ x_{-1}y_{-2} \neq -1, -\frac{1}{2}$. *Then for* $n =$

$0, 1, 2, ...,$

$$\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(1+x_{-1}y_{-2})^{2n}}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^n(-1-2x_0y_{-1})^n},\\
x_{8n-1} &= \frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(1+x_0y_{-1})^{2n}}{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^n(-1-2x_{-1}y_{-2})^n},\\
x_{8n} &= \frac{x_0^{2n+1}y_0^{2n}(1+x_{-1}y_{-2})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^n(-1-2x_0y_{-1})^n},\\
x_{8n+1} &= -\frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n+1}(1+x_0y_{-1})^{2n}}{x_0^{2n}y_0^{2n+1}(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^{n+1}(-1-2x_{-1}y_{-2})^n},\\
x_{8n+2} &= -\frac{x_0^{2n+1}y_0^{2n+1}(1+x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n}(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^{n+1}(-1-2x_0y_{-1})^n},\\
x_{8n+3} &= -\frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(1+x_0y_{-1})^{2n+1}}{x_0^{2n+1}y_0^{2n+1}(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^{n+1}(-1-2x_{-1}y_{-2})^{n+1}},\\
x_{8n+4} &= \frac{x_0^{2n+2}y_0^{2n+1}(1+x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^{n+1}(-1-2x_0y_{-1})^{n+1}},\\
x_{8n+5} &= -\frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+2}(1+x_0y_{-1})^{2n+1}}{x_0^{2n+1}y_0^{2n+2}(-1+x_{-2}y_{-1})^{n+1}(1+x_{-2}y_{-1})^{n+1}(-1-2x_{-1}y_{-2})^{n+1}},
\end{aligned}$$

and

$$\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^n(-1-2x_{-1}y_{-2})^n}{y_{-2}^{2n-1}x_{-2}^{2n}(1+x_0y_{-1})^{2n}},\\
y_{8n-1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^n(-1-2x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(1+x_{-1}y_{-2})^{2n}},\\
y_{8n} &= \frac{x_0^{2n}y_0^{2n+1}(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^n(-1-2x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n}(1+x_0y_{-1})^{2n}},\\
y_{8n+1} &= \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n}(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^n(-1-2x_0y_{-1})^n}{x_0^{2n+1}y_0^{2n}(1+x_{-1}y_{-2})^{2n+1}},\\
y_{8n+2} &= -\frac{x_0^{2n+1}y_0^{2n+1}(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^{n+1}(-1-2x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n+1}(1+x_0y_{-1})^{2n+1}},\\
y_{8n+3} &= \frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^{n+1}(-1-2x_0y_{-1})^n}{x_0^{2n+1}y_0^{2n+1}(1+x_{-1}y_{-2})^{2n+1}},\\
y_{8n+4} &= \frac{x_0^{2n+1}y_0^{2n+2}(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^{n+1}(-1-2x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(1+x_0y_{-1})^{2n+1}},\\
y_{8n+5} &= -\frac{x_{-1}y_{-2}^{2n+2}x_{-2}^{2n+1}(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^{n+1}(-1-2x_0y_{-1})^{n+1}}{x_0^{2n+2}y_0^{2n+1}(1+x_{-1}y_{-2})^{2n+2}}.
\end{aligned}$$

4.3. Theorem. *If* $\{x_n, y_n\}$ *are solutions of the difference equations system*

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\,(1-x_{n-2}y_{n-1})},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\,(-1-y_{n-2}x_{n-1})},$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 *are arbitrary nonzero real numbers with* $x_{-1}y_0,\ x_{-2}y_{-1} \neq \pm 1,$ *and* $x_0y_{-1},\ x_{-1}y_{-2} \neq -1, -\dfrac{1}{2}$. *Then for* $n =$

$0, 1, 2, \ldots,$

$$\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n} y_0^{2n} (1+x_{-1}y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n-1} (1-x_{-1}y_0)^n (1+x_{-1}y_0)^n (1+2x_0y_{-1})^n}, \\
x_{8n-1} &= \frac{x_{-1} y_{-2}^{2n} x_{-2}^{2n} (1+x_0y_{-1})^{2n}}{x_0^{2n} y_0^{2n} (1-x_{-2}y_{-1})^n (1+x_{-2}y_{-1})^n (1+2x_{-1}y_{-2})^n}, \\
x_{8n} &= \frac{x_0^{2n+1} y_0^{2n} (1+x_{-1}y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n} (1-x_{-1}y_0)^n (1+x_{-1}y_0)^n (1+2x_0y_{-1})^n}, \\
x_{8n+1} &= \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n+1} (1+x_0y_{-1})^{2n}}{x_0^{2n} y_0^{2n+1} (1-x_{-2}y_{-1})^{n+1} (1+x_{-2}y_{-1})^n (1+2x_{-1}y_{-2})^n}, \\
x_{8n+2} &= -\frac{x_0^{2n+1} y_0^{2n+1} (1+x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n} (1+x_{-1}y_0)^n (1-x_{-1}y_0)^{n+1} (1+2x_0y_{-1})^n}, \\
x_{8n+3} &= \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (1+x_0y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+1} (1+x_{-2}y_{-1})^n (1-x_{-2}y_{-1})^{n+1} (1+2x_{-1}y_{-2})^{n+1}}, \\
x_{8n+4} &= -\frac{x_0^{2n+2} y_0^{2n+1} (1+x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (1+x_{-1}y_0)^n (1-x_{-1}y_0)^{n+1} (1+2x_0y_{-1})^{n+1}}, \\
x_{8n+5} &= \frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+2} (1+x_0y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+2} (1+x_{-2}y_{-1})^{n+1} (1-x_{-2}y_{-1})^{n+1} (1+2x_{-1}y_{-2})^{n+1}},
\end{aligned}$$

and

$$\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n} y_0^{2n} (1+x_{-2}y_{-1})^n (1-x_{-2}y_{-1})^n (1+2x_{-1}y_{-2})^n}{y_{-2}^{2n-1} x_{-2}^{2n} (1+x_0y_{-1})^{2n}}, \\
y_{8n-1} &= \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n} (1+x_{-1}y_0)^n (1-x_{-1}y_0)^n (1+2x_0y_{-1})^n}{x_0^{2n} y_0^{2n} (1+x_{-1}y_{-2})^{2n}}, \\
y_{8n} &= \frac{x_0^{2n} y_0^{2n+1} (1+x_{-2}y_{-1})^n (1-x_{-2}y_{-1})^n (1+2x_{-1}y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n} (1+x_0y_{-1})^{2n}}, \\
y_{8n+1} &= -\frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n} (1+x_{-1}y_0)^n (1-x_{-1}y_0)^n (1+2x_0y_{-1})^n}{x_0^{2n+1} y_0^{2n} (1+x_{-1}y_{-2})^{2n+1}}, \\
y_{8n+2} &= -\frac{x_0^{2n+1} y_0^{2n+1} (1+x_{-2}y_{-1})^n (1-x_{-2}y_{-1})^{n+1} (1+2x_{-1}y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n+1} (1+x_0y_{-1})^{2n+1}}, \\
y_{8n+3} &= \frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (1+x_{-1}y_0)^n (1-x_{-1}y_0)^{n+1} (1+2x_0y_{-1})^n}{x_0^{2n+1} y_0^{2n+1} (1+x_{-1}y_{-2})^{2n+1}}, \\
y_{8n+4} &= -\frac{x_0^{2n+1} y_0^{2n+2} (1+x_{-2}y_{-1})^n (1-x_{-2}y_{-1})^{n+1} (1+2x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (1+x_0y_{-1})^{2n+1}}, \\
y_{8n+5} &= -\frac{x_{-1} y_{-2}^{2n+2} x_{-2}^{2n+1} (1+x_{-1}y_0)^n (1-x_{-1}y_0)^{n+1} (1+2x_0y_{-1})^{n+1}}{x_0^{2n+2} y_0^{2n+1} (1+x_{-1}y_{-2})^{2n+2}}.
\end{aligned}$$

4.4. Theorem. *Assume that* $\{x_n, y_n\}$ *are solutions of the following system with the initial conditions* x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} *and* y_0 *are arbitrary nonzero real numbers with* $x_{-1}y_0$, $x_{-2}y_{-1} \neq \pm 1$, *and* x_0y_{-1}, $x_{-1}y_{-2} \neq 1, \frac{1}{2}$

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(-1 + x_{n-2}y_{n-1}\right)}, \quad y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(1 - y_{n-2}x_{n-1}\right)}.$$

Then for $n = 0, 1, 2, \ldots,$

$$\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n} y_0^{2n} (-1 + x_{-1}y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n-1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0y_{-1})^n}, \\
x_{8n-1} &= \frac{x_{-1} y_{-2}^{2n} x_{-2}^{2n} (-1 + x_0y_{-1})^{2n}}{x_0^{2n} y_0^{2n} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 + 2x_{-1}y_{-2})^n}, \\
x_{8n} &= \frac{x_0^{2n+1} y_0^{2n} (-1 + x_{-1}y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0y_{-1})^n}, \\
x_{8n+1} &= \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n+1} (-1 + x_0y_{-1})^{2n}}{x_0^{2n} y_0^{2n+1} (-1 + x_{-2}y_{-1})^{n+1} (1 + x_{-2}y_{-1})^n (-1 + 2x_{-1}y_{-2})^n}, \\
x_{8n+2} &= \frac{x_0^{2n+1} y_0^{2n+1} (1 - x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n} (-1 + x_{-1}y_0)^{n+1} (1 + x_{-1}y_0)^n (-1 + 2x_0y_{-1})^n}, \\
x_{8n+3} &= \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (1 - x_0y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-2}y_{-1})^{n+1} (1 + x_{-2}y_{-1})^n (-1 + 2x_{-1}y_{-2})^{n+1}}, \\
x_{8n+4} &= \frac{x_0^{2n+2} y_0^{2n+1} (1 - x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (1 + x_{-1}y_0)^n (-1 + x_{-1}y_0)^{n+1} (-1 + 2x_0y_{-1})^{n+1}}, \\
x_{8n+5} &= -\frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+2} (1 - x_0y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+2} (-1 + x_{-2}y_{-1})^{n+1} (1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^{n+1}},
\end{aligned}$$

and

$$\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n} y_0^{2n} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 + 2x_{-1}y_{-2})^n}{y_{-2}^{2n-1} x_{-2}^{2n} (1 - x_0y_{-1})^{2n}}, \\
y_{8n-1} &= \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0y_{-1})^n}{x_0^{2n} y_0^{2n} (1 - x_{-1}y_{-2})^{2n}}, \\
y_{8n} &= \frac{x_0^{2n} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 + 2x_{-1}y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n} (1 - x_0y_{-1})^{2n}}, \\
y_{8n+1} &= \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0y_{-1})^n}{x_0^{2n+1} y_0^{2n} (1 - x_{-1}y_{-2})^{2n+1}}, \\
y_{8n+2} &= \frac{x_0^{2n+1} y_0^{2n+1} (1 + x_{-2}y_{-1})^n (-1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n+1} (1 - x_0y_{-1})^{2n+1}}, \\
y_{8n+3} &= -\frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (1 + x_{-1}y_0)^n (-1 + x_{-1}y_0)^{n+1} (-1 + 2x_0y_{-1})^n}{x_0^{2n+1} y_0^{2n+1} (1 - x_{-1}y_{-2})^{2n+1}}, \\
y_{8n+4} &= -\frac{x_0^{2n+1} y_0^{2n+2} (1 + x_{-2}y_{-1})^n (-1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (1 - x_0y_{-1})^{2n+1}}, \\
y_{8n+5} &= \frac{x_{-1} y_{-2}^{2n+2} x_{-2}^{2n+1} (1 + x_{-1}y_0)^n (-1 + x_{-1}y_0)^{n+1} (-1 + 2x_0y_{-1})^{n+1}}{x_0^{2n+2} y_0^{2n+1} (1 - x_{-1}y_{-2})^{2n+2}}.
\end{aligned}$$

5. The Fourth System: $x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n(1+x_{n-2}y_{n-1})}$, $y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n(1-y_{n-2}x_{n-1})}$

In this section, we get the form of the solutions of the system of the difference equations

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\,(1 + x_{n-2}y_{n-1})}, \quad y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\,(1 - y_{n-2}x_{n-1})}, \tag{7}$$

where $n = 0, 1, 2, \ldots$ and the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are arbitrary nonzero real numbers with $x_{-1}y_{-2},\ x_0y_{-1} \neq 1,\ x_{-2}y_{-1},\ x_{-1}y_0 \neq -1$.

5.1. Theorem. *If $\{x_n, y_n\}$ are solutions of difference equation system (7). Then for* $n = 0, 1, 2, ...,$

$$\begin{aligned} x_{4n-2} &= \frac{x_0^n y_0^n (1 - x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(1 + x_{-1}y_0)^n}, \quad x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n (1 - x_0 y_{-1})^n}{x_0^n y_0^n (1 + x_{-2}y_{-1})^n}, \\ x_{4n} &= \frac{x_0^{n+1} y_0^n (1 - x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n (1 + x_{-1}y_0)^n}, \quad x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1}(1 - x_0 y_{-1})^n}{x_0^n y_0^{n+1}(1 + x_{-2}y_{-1})^{n+1}}, \end{aligned}$$

and

$$\begin{aligned} y_{4n-2} &= \frac{x_0^n y_0^n (1 + x_{-2}y_{-1})^n}{y_{-2}^{n-1} x_{-2}^n (1 - x_0 y_{-1})^n}, \quad y_{4n-1} = \frac{y_{-1}y_{-2}^n x_{-2}^n (1 + x_{-1}y_0)^n}{x_0^n y_0^n (1 - x_{-1}y_{-2})^n}, \\ y_{4n} &= \frac{x_0^n y_0^{n+1} (1 + x_{-2}y_{-1})^n}{y_{-2}^n x_{-2}^n (1 - x_0 y_{-1})^n}, \quad y_{4n+1} = \frac{x_{-1}y_{-2}^{n+1} x_{-2}^n (1 + x_{-1}y_0)^n}{x_0^{n+1} y_0^n (1 - x_{-1}y_{-2})^{n+1}}. \end{aligned}$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 1$. that is,

$$\begin{aligned} x_{4n-6} &= \frac{x_0^{n-1} y_0^{n-1} (1 - x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1} x_{-2}^{n-2}(1 + x_{-1}y_0)^{n-1}}, \quad x_{4n-5} = \frac{x_{-1}y_{-2}^{n-1} x_{-2}^{n-1} (1 - x_0 y_{-1})^{n-1}}{x_0^{n-1} y_0^{n-1} (1 + x_{-2}y_{-1})^{n-1}}, \\ x_{4n-4} &= \frac{x_0^n y_0^{n-1} (1 - x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1} x_{-2}^{n-1} (1 + x_{-1}y_0)^{n-1}}, \quad x_{4n-3} = \frac{y_{-1}y_{-2}^{n-1} x_{-2}^{n}(1 - x_0 y_{-1})^{n-1}}{x_0^{n-1} y_0^{n}(1 + x_{-2}y_{-1})^{n}}, \end{aligned}$$

$$\begin{aligned} y_{4n-6} &= \frac{x_0^{n-1} y_0^{n-1} (1 + x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-2} x_{-2}^{n-1} (1 - x_0 y_{-1})^{n-1}}, \quad y_{4n-5} = \frac{y_{-1}y_{-2}^{n-1} x_{-2}^{n-1} (1 + x_{-1}y_0)^{n-1}}{x_0^{n-1} y_0^{n-1} (1 - x_{-1}y_{-2})^{n-1}}, \\ y_{4n-4} &= \frac{x_0^{n-1} y_0^{n} (1 + x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-1} x_{-2}^{n-1} (1 - x_0 y_{-1})^{n-1}}, \quad y_{4n-3} = \frac{x_{-1}y_{-2}^{n} x_{-2}^{n-1} (1 + x_{-1}y_0)^{n-1}}{x_0^{n} y_0^{n-1} (1 - x_{-1}y_{-2})^{n}}. \end{aligned}$$

It follows from Eq.(7) that

$$\begin{aligned} x_{4n-2} &= \frac{x_{4n-5}y_{4n-4}}{y_{4n-3}\left(+x_{4n-5}y_{4n-4}\right)} \\ &= \frac{\left(\dfrac{x_{-1}y_{-2}^{n-1}x_{-2}^{n-1}(1 - x_0 y_{-1})^{n-1}}{x_0^{n-1}y_0^{n-1}(1 + x_{-2}y_{-1})^{n-1}}\right)\left(\dfrac{x_0^{n-1}y_0^{n}(1 + x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1 - x_0 y_{-1})^{n-1}}\right)}{\left(\dfrac{x_{-1}y_{-2}^{n}x_{-2}^{n-1}(1 + x_{-1}y_0)^{n-1}}{x_0^{n}y_0^{n-1}(1 - x_{-1}y_{-2})^{n}}\right)\left(1 + \left(\dfrac{x_{-1}y_{-2}^{n-1}x_{-2}^{n-1}(1 - x_0 y_{-1})^{n-1}}{x_0^{n-1}y_0^{n-1}(1 + x_{-2}y_{-1})^{n-1}}\right)\left(\dfrac{x_0^{n-1}y_0^{n}(1 + x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1 - x_0 y_{-1})^{n-1}}\right)\right)} \\ &= \frac{x_{-1}y_0}{\left(\dfrac{x_{-1}y_{-2}^{n}x_{-2}^{n-1}(1 + x_{-1}y_0)^{n-1}}{x_0^{n}y_0^{n-1}(1 - x_{-1}y_{-2})^{n}}\right)(1 + x_{-1}y_0)} \\ &= \frac{x_{-1}y_0 x_0^n y_0^{n-1}(1 - x_{-1}y_{-2})^n}{x_{-1}y_{-2}^n x_{-2}^{n-1}(1 + x_{-1}y_0)^{n-1}(1 + x_{-1}y_0)} = \frac{x_0^n y_0^n (1 - x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(1 + x_{-1}y_0)^n}, \end{aligned}$$

$$\begin{aligned}
y_{4n-2} &= \frac{y_{4n-5}x_{4n-4}}{x_{4n-3}\left(1-y_{4n-5}x_{4n-4}\right)}\\
&= \frac{\left(\frac{y_{-1}y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}{x_0^{n-1}y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}\right)\left(\frac{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}\right)}{\left(\frac{y_{-1}y_{-2}^{n-1}x_{-2}^{n}(1-x_0y_{-1})^{n-1}}{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})^{n}}\right)\left(1-\left(\frac{y_{-1}y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}{x_0^{n-1}y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}\right)\left(\frac{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}\right)\right)}\\
&= \frac{x_0y_{-1}x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})^{n}}{y_{-1}y_{-2}^{n-1}x_{-2}^{n}(1-x_0y_{-1})^{n-1}\left(1-x_0y_{-1}\right)} = \frac{x_0^n y_0^n(1+x_{-2}y_{-1})^n}{y_{-2}^{n-1}x_{-2}^{n}(1-x_0y_{-1})^n}.
\end{aligned}$$

Also, we see from Eq.(7) that

$$\begin{aligned}
x_{4n-1} &= \frac{x_{4n-4}y_{4n-3}}{y_{4n-2}\left(1+x_{4n-4}y_{4n-3}\right)}\\
&= \frac{\left(\frac{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}\right)\left(\frac{x_{-1}y_{-2}^{n}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n}}\right)}{\left(\frac{x_0^n y_0^n(1+x_{-2}y_{-1})^n}{y_{-2}^{n-1}x_{-2}^{n}(1-x_0y_{-1})^n}\right)\left(1+\left(\frac{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}\right)\left(\frac{x_{-1}y_{-2}^{n}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n}}\right)\right)}\\
&= \frac{\left(\frac{x_{-1}y_{-2}}{(1-x_{-1}y_{-2})}\right)}{\left(\frac{x_0^n y_0^n(1+x_{-2}y_{-1})^n}{y_{-2}^{n-1}x_{-2}^{n}(1-x_0y_{-1})^n}\right)\left(1+\frac{x_{-1}y_{-2}}{(1-x_{-1}y_{-2})}\right)}\\
&= \frac{y_{-2}^{n-1}x_{-2}^{n}(1-x_0y_{-1})^n x_{-1}y_{-2}}{x_0^n y_0^n(1+x_{-2}y_{-1})^n\left(1-x_{-1}y_{-2}+x_{-1}y_{-2}\right)} = \frac{x_{-1}y_{-2}^{n}x_{-2}^{n}(1-x_0y_{-1})^n}{x_0^n y_0^n(1+x_{-2}y_{-1})^n},
\end{aligned}$$

and

$$\begin{aligned}
y_{4n-1} &= \frac{y_{4n-4}x_{4n-3}}{x_{4n-2}\left(1-y_{4n-4}x_{4n-3}\right)}\\
&= \frac{\left(\frac{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1-x_0y_{-1})^{n-1}}\right)\left(\frac{y_{-1}y_{-2}^{n-1}x_{-2}^{n}(1-x_0y_{-1})^{n-1}}{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})^{n}}\right)}{\left(\frac{x_0^n y_0^n(1-x_{-1}y_{-2})^n}{y_{-2}^{n}x_{-2}^{n-1}(1+x_{-1}y_0)^n}\right)\left(1-\left(\frac{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1-x_0y_{-1})^{n-1}}\right)\left(\frac{y_{-1}y_{-2}^{n-1}x_{-2}^{n}(1-x_0y_{-1})^{n-1}}{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})^{n}}\right)\right)}\\
&= \frac{\left(\frac{y_{-1}x_{-2}}{(1+x_{-2}y_{-1})}\right)}{\left(\frac{x_0^n y_0^n(1-x_{-1}y_{-2})^n}{y_{-2}^{n}x_{-2}^{n-1}(1+x_{-1}y_0)^n}\right)\left(1-\left(\frac{y_{-1}x_{-2}}{(1+x_{-2}y_{-1})}\right)\right)}\\
&= \frac{y_{-2}^{n}x_{-2}^{n-1}(1+x_{-1}y_0)^n y_{-1}x_{-2}}{x_0^n y_0^n(1-x_{-1}y_{-2})^n\left(1+x_{-2}y_{-1}-x_{-2}y_{-1}\right)} = \frac{y_{-1}y_{-2}^{n}x_{-2}^{n}(1+x_{-1}y_0)^n}{x_0^n y_0^n(1-x_{-1}y_{-2})^n}.
\end{aligned}$$

Also, the other relations can be proved similarly. This completes the proof. ∎

We consider the following systems and the proof of the theorems are similarly to above theorem and so, left to the reader.

$$(5.1)\qquad x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(1-x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(1+y_{n-2}x_{n-1}\right)}.8$$

$$(5.2)\qquad x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(-1+x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(-1+y_{n-2}x_{n-1}\right)}.9$$

$$(5.3)\qquad x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(-1-x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(-1-y_{n-2}x_{n-1}\right)}.10$$

The following theorems is devoted to the expressions of the form of the solutions of systems (8), (9), (10).

5.2. Theorem. *Let $\{x_n, y_n\}_{n=-2}^{+\infty}$ be solutions of system (8) and $x_{-1}y_{-2}$, $x_0y_{-1} \neq -1$, $x_{-2}y_{-1}$, $x_{-1}y_0 \neq 1$. Then for $n = 0,1,2,...$,*

$$x_{4n-2} = \frac{x_0^n y_0^n(1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(1-x_{-1}y_0)^n},\quad x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n(1+x_0y_{-1})^n}{x_0^n y_0^n(1-x_{-2}y_{-1})^n},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n(1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n(1-x_{-1}y_0)^n},\quad x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1}(1+x_0y_{-1})^n}{x_0^n y_0^{n+1}(1-x_{-2}y_{-1})^{n+1}},$$

$$y_{4n-2} = \frac{x_0^n y_0^n(1-x_{-2}y_{-1})^n}{y_{-2}^{n-1} x_{-2}^n(1+x_0y_{-1})^n},\quad y_{4n-1} = \frac{y_{-1}y_{-2}^n x_{-2}^n(1-x_{-1}y_0)^n}{x_0^n y_0^n(1+x_{-1}y_{-2})^n},$$

$$y_{4n} = \frac{x_0^n y_0^{n+1}(1-x_{-2}y_{-1})^n}{y_{-2}^n x_{-2}^n(1+x_0y_{-1})^n},\quad y_{4n+1} = \frac{x_{-1}y_{-2}^{n+1} x_{-2}^n(1-x_{-1}y_0)^n}{x_0^{n+1} y_0^n(1+x_{-1}y_{-2})^{n+1}}.$$

5.3. Theorem. *Assume that $\{x_n, y_n\}$ are solutions of system (9) with $x_{-1}y_{-2}$, x_0y_{-1}, $x_{-2}y_{-1}$, $x_{-1}y_0 \neq 1$. Then for $n = 0,1,2,...$,*

$$x_{4n-2} = \frac{x_0^n y_0^n(-1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(-1+x_{-1}y_0)^n},\quad x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n(-1+x_0y_{-1})^n}{x_0^n y_0^n(-1+x_{-2}y_{-1})^n},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n(-1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n(-1+x_{-1}y_0)^n},\quad x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1}(-1+x_0y_{-1})^n}{x_0^n y_0^{n+1}(-1+x_{-2}y_{-1})^{n+1}},$$

$$y_{4n-2} = \frac{x_0^n y_0^n(-1+x_{-2}y_{-1})^n}{y_{-2}^{n-1} x_{-2}^n(-1+x_0y_{-1})^n},\quad y_{4n-1} = \frac{y_{-1}y_{-2}^n x_{-2}^n(-1+x_{-1}y_0)^n}{x_0^n y_0^n(-1+x_{-1}y_{-2})^n},$$

$$y_{4n} = \frac{x_0^n y_0^{n+1}(-1+x_{-2}y_{-1})^n}{y_{-2}^n x_{-2}^n(-1+x_0y_{-1})^n},\quad y_{4n+1} = \frac{x_{-1}y_{-2}^{n+1} x_{-2}^n(-1+x_{-1}y_0)^n}{x_0^{n+1} y_0^n(-1+x_{-1}y_{-2})^{n+1}}.$$

5.4. Theorem. *Suppose that $\{x_n, y_n\}$ are solutions of system (10) such that $x_{-1}y_{-2}$, x_0y_{-1}, $x_{-2}y_{-1}$, $x_{-1}y_0 \neq -1$. Then for $n = 0,1,2,...$,*

$$x_{4n-2} = \frac{x_0^n y_0^n(-1-x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(-1-x_{-1}y_0)^n},\quad x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n(-1-x_0y_{-1})^n}{x_0^n y_0^n(-1-x_{-2}y_{-1})^n},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n(-1-x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n(-1-x_{-1}y_0)^n},\quad x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1}(-1-x_0y_{-1})^n}{x_0^n y_0^{n+1}(-1-x_{-2}y_{-1})^{n+1}},$$

$$y_{4n-2} = \frac{x_0^n y_0^n(-1-x_{-2}y_{-1})^n}{y_{-2}^{n-1} x_{-2}^n(-1-x_0y_{-1})^n},\quad y_{4n-1} = \frac{y_{-1}y_{-2}^n x_{-2}^n(-1-x_{-1}y_0)^n}{x_0^n y_0^n(-1-x_{-1}y_{-2})^n},$$

$$y_{4n} = \frac{x_0^n y_0^{n+1}(-1-x_{-2}y_{-1})^n}{y_{-2}^n x_{-2}^n(-1-x_0y_{-1})^n},\quad y_{4n+1} = \frac{x_{-1}y_{-2}^{n+1} x_{-2}^n(-1-x_{-1}y_0)^n}{x_0^{n+1} y_0^n(-1-x_{-1}y_{-2})^{n+1}}.$$

5.5. Lemma. *The solution of system (7) is unbounded except in the following case.*

5.6. Theorem. *System (7) has a periodic solution of period four iff* $y_{-2}=-y_0,\ x_{-2}=-x_0$ *and it will be taken the following form* $\{x_n\}=\left\{x_{-2},x_{-1},x_0,\frac{y_{-1}x_{-2}}{y_0(1+x_{-2}y_{-1})},x_{-2},x_{-1},x_0,...\right\}$, $\{y_n\}=\left\{y_{-2},y_{-1},y_0,\frac{x_{-1}y_{-2}}{x_0(1+y_{-2}x_{-1})},y_{-2},y_{-1},y_0,,...\right\}$.

Proof. First suppose that there exists a prime period four solution

$$\begin{aligned}\{x_n\} &= \left\{x_{-2},x_{-1},x_0,\tfrac{y_{-1}x_{-2}}{y_0(1+x_{-2}y_{-1})},x_{-2},x_{-1},x_0,...\right\},\\ \{y_n\} &= \left\{y_{-2},y_{-1},y_0,\tfrac{x_{-1}y_{-2}}{x_0(1+y_{-2}x_{-1})},y_{-2},y_{-1},y_0,,...\right\},\end{aligned}$$

of system (7), we see from the form of the solution of system (7) that

$$\begin{aligned}x_{4n-2} &= x_{-2}=\frac{x_0^n y_0^n(1-x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(1+x_{-1}y_0)^n},\ x_{4n-1}=x_{-1}=\frac{x_{-1}y_{-2}^n x_{-2}^n(1-x_0y_{-1})^n}{x_0^n y_0^n(1+x_{-2}y_{-1})^n},\\ x_{4n} &= x_0=\frac{x_0^{n+1}y_0^n(1-x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n(1+x_{-1}y_0)^n},\ x_{4n+1}=\tfrac{y_{-1}x_{-2}}{y_0(1+x_{-2}y_{-1})}=\tfrac{y_{-1}y_{-2}^n x_{-2}^{n+1}(1-x_0y_{-1})^n}{x_0^n y_0^{n+1}(1+x_{-2}y_{-1})^{n+1}},\\ y_{4n-2} &= y_{-2}=\frac{x_0^n y_0^n(1+x_{-2}y_{-1})^n}{y_{-2}^{n-1}x_{-2}^n(1-x_0y_{-1})^n},\ y_{4n-1}=y_{-1}=\frac{y_{-1}y_{-2}^n x_{-2}^n(1+x_{-1}y_0)^n}{x_0^n y_0^n(1-x_{-1}y_{-2})^n},\\ y_{4n} &= y_0=\frac{x_0^n y_0^{n+1}(1+x_{-2}y_{-1})^n}{y_{-2}^n x_{-2}^n(1-x_0y_{-1})^n},\ y_{4n+1}=\tfrac{x_{-1}y_{-2}}{x_0(1+y_{-2}x_{-1})}=\tfrac{x_{-1}y_{-2}^{n+1}x_{-2}^n(1+x_{-1}y_0)^n}{x_0^{n+1}y_0^n(1-x_{-1}y_{-2})^{n+1}}.\end{aligned}$$

Then we get

$$y_{-2}=-y_0,\ x_{-2}=-x_0.$$

Second assume that $y_{-2}=-y_0,\ x_{-2}=-x_0$. Then we see from the form of the solution of system (7) that

$$\begin{aligned}x_{4n-2} &= x_{-2},\ x_{4n-1}=x_{-1},\ \ x_{4n}=x_0,\ x_{4n+1}=\tfrac{y_{-1}x_{-2}}{y_0(1+x_{-2}y_{-1})},\\ y_{4n-2} &= y_{-2},\ \ \ y_{4n-1}=y_{-1},\ \ y_{4n}=y_0,\ \ \ y_{4n+1}=\tfrac{x_{-1}y_{-2}}{x_0(1+y_{-2}x_{-1})}.\end{aligned}$$

Thus we have a periodic solution of period four and the proof is complete. ∎

Also, we can prove the following Theorems:

5.7. Lemma. *The solutions of all systems (8), (9) and (10) are unbounded except in the following cases.*

5.8. Theorem. *System (8) has a periodic solution of period four iff* $y_{-2}=-y_0,\ x_{-2}=-x_0$ *and it will be taken the following form* $\{x_n\}=\left\{x_{-2},x_{-1},x_0,\frac{y_{-1}x_{-2}}{y_0(1-x_{-2}y_{-1})},x_{-2},x_{-1},x_0,...\right\}$, $\{y_n\}=\left\{y_{-2},y_{-1},y_0,\frac{x_{-1}y_{-2}}{x_0(1-y_{-2}x_{-1})},y_{-2},y_{-1},y_0,,...\right\}$.

5.9. Theorem. *All Solutions of the difference equations system (9) are periodic solution with period four iff* $y_{-2}=y_0,\ x_{-2}=x_0$ *and it will be taken the following form* $\{x_n\}=\left\{x_{-2},x_{-1},x_0,\frac{y_{-1}x_{-2}}{y_0(-1+x_{-2}y_{-1})},x_{-2},...\right\}$, $\{y_n\}=\left\{y_{-2},y_{-1},y_0,\frac{x_{-1}y_{-2}}{x_0(-1+y_{-2}x_{-1})},y_{-2},,...\right\}$.

5.10. Theorem. *If* $\{x_n\}$, $\{y_n\}$ *are solutions of system (10), then* $\{x_n\}$, $\{y_n\}$ *are periodic solutions of period four iff* $y_{-2}=y_0,\ x_{-2}=x_0$ *and it will be in the following form* $\{x_n\}=\left\{x_{-2},x_{-1},x_0,\frac{y_{-1}x_{-2}}{y_0(-1-x_{-2}y_{-1})},x_{-2},...\right\}$, $\{y_n\}=\left\{y_{-2},y_{-1},y_0,\frac{x_{-1}y_{-2}}{x_0(-1-y_{-2}x_{-1})},y_{-2},,...\right\}$.

5.11. Example. We consider interesting numerical example for the difference system (7) with the initial conditions $x_{-2}=0.3,\ x_{-1}=0.15,\ x_0=-0.4,\ y_{-2}=0.2,\ y_{-1}=-0.16$ *and* $y_0=0.17$. See Figure (3).

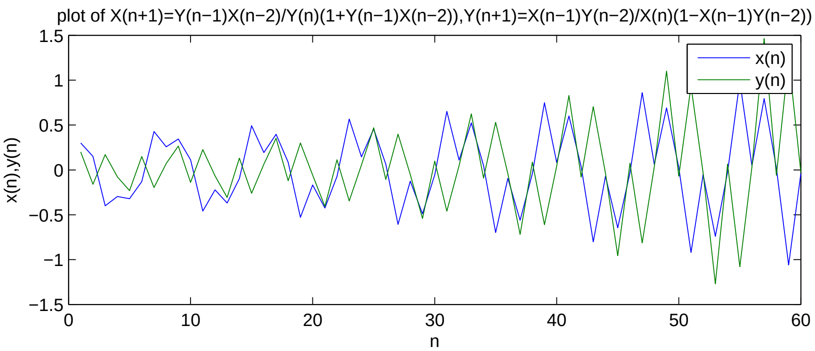

Figure 3

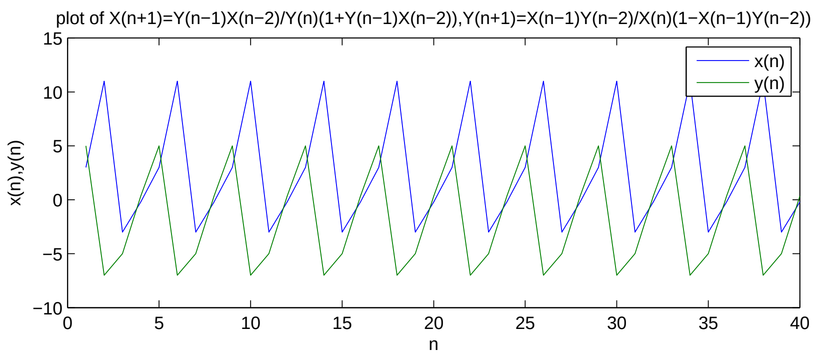

Figure 4

5.12. Example. See Figure (4) when we take system (7) with the initial conditions $x_{-2} = 3,\ x_{-} = 11,\ x_0 = -3,\ y_{-2} = 5,\ y_{-1} = -7$ *and* $y_0 = -5$.

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On M_1- and M_3-properties in the setting of ordered topological spaces

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Abstract

In 1961, J. G. Ceder [3] introduced and studied classes of topological spaces called M_i-spaces ($i = 1, 2, 3$) and established that *metrizable* $\Rightarrow M_1 \Rightarrow M_2 \Rightarrow M_3$. He then asked whether these implications are reversible. Gruenhage [5] and Junnila [8] independently showed that $M_3 \Rightarrow M_2$. In this paper, we investigate the M_1- and M_3- properties in the setting of ordered topological spaces. Among other results, we show that if $(X, \mathcal{T}, \leq)$ is an M_1 ordered topological C- and I-space then the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1. Here, $\mathcal{T}^\natural := \{U \in \mathcal{T} \,|\, U \text{ is an upper set}\}$ and $\mathcal{T}^\flat := \{L \in \mathcal{T} \,|\, L \text{ is a lower set}\}$.

Keywords: C-space, I-space, closure-preserving, (pairwise) M_1, (pairwise) stratifiable

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1. Introduction

It is a well-known fact that σ-locally finite collections are σ-closure-preserving (see [5] or [15]). Thus the characterization of metrizable spaces by Bing-Nagata-Smirnov [15, Theorem 23.9] in terms of σ-locally finite bases motivated Ceder to study spaces with σ-closure preserving bases. In his paper [3], Ceder gave examples of non-metrizable M_1-spaces and got researchers on their feet by asking whether the implications $M_1 \Rightarrow M_2 \Rightarrow M_3$ are reversible. See the definitions of these concepts at the bottom of the preliminaries section below. Many researchers have worked on this problem and have produced a number of partial results but, as far as we know, no general solution yet. In 1966, C. J. R. Borges [1] reviewed Ceder's work on M_3-spaces and improved some of his results, and he generally illustrated the importance of M_3-spaces and thus renamed them stratifiable spaces. In 1973, following Ceder's efforts [3, Theorem 7.6, p. 117], F. G. Slaughter, Jr established that if f is a closed continuous mapping from a metric space X onto a topological space Y then Y is an M_1-space [14].

2. Preliminaries

Following Priestley [13], we denote the intersection of all lower sets containing a subset S of an ordered set X by $d(S)$. Dually, the intersection of all upper sets containing S is denoted by $i(S)$. Then we say that an ordered topological space $(X, \mathcal{T}, \leq)$ is a C-space if $d(F)$ and $i(F)$ are closed whenever F is a closed subset of X. Similarly, $(X, \mathcal{T}, \leq)$ is called an I-space if $d(G)$ and $i(G)$ are open whenever G is an open subset of X. A collection $\mathcal{B}$ of subsets of a topological space $(X, \mathcal{T})$ is said to be $\mathcal{T}$-closure-preserving if for each subcollection $\mathcal{B}' \subseteq \mathcal{B}$, we have $\bigcup_{B\in\mathcal{B}'} \overline{B} = \overline{\bigcup_{B\in\mathcal{B}'} B}$.

For brevity, we are going to refer to bitopological spaces as bispaces. A bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be $\mathcal{T}_1$-*regular with respect to* $\mathcal{T}_2$ [§] if and only if for each point $x \in X$ and each $\mathcal{T}_1$-closed set F with $x \notin F$, there are a $\mathcal{T}_1$-open set U and a $\mathcal{T}_2$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Similarly, a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be $\mathcal{T}_2$-*regular with respect to* $\mathcal{T}_1$ if and only if for each point $x \in X$ and each $\mathcal{T}_2$-closed set F with $x \notin F$, there are a $\mathcal{T}_2$-open set U and a $\mathcal{T}_1$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. We say that a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pairwise regular* if and only if it is both $\mathcal{T}_1$-regular with respect to $\mathcal{T}_2$ and $\mathcal{T}_2$-regular with respect to $\mathcal{T}_1$. We define $\mathcal{T}^\natural$ and $\mathcal{T}^\flat$ like this: $\mathcal{T}^\natural := \{U \in \mathcal{T} \mid U \text{ is an upper set}\}$ and $\mathcal{T}^\flat := \{L \in \mathcal{T} \mid L \text{ is a lower set}\}$.

Let $\mathcal{J}$ be the Euclidean topology on the unit interval $[0, 1]$, carrying its usual order. A bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pairwise completely regular* if and only if for each $x \in X$ and each $\mathcal{T}_1$-closed set F with $x \notin F$, there exists a bicontinuous function $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to ([0, 1], \mathcal{J}^\natural, \mathcal{J}^\flat)$ such that $f(x) = 1$ and $f(F) = \{0\}$; and for each $\mathcal{T}_2$-closed set Q with $x \notin Q$, there exists a bicontinuous function $g : (X, \mathcal{T}_1, \mathcal{T}_2) \to ([0, 1], \mathcal{J}^\natural, \mathcal{J}^\flat)$ such that $g(x) = 0$ and $g(Q) = \{1\}$ (see [9]).

Furthermore, recall that a topological space X is called an M_1-*space* if it is regular and has a σ-closure preserving base. In a bispace setting we follow Gutierrez and Romaguera [6] and say that a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is $\mathcal{T}_1$-M_1 *with respect to* $\mathcal{T}_2$ if and only if it is $\mathcal{T}_1$-regular with respect to $\mathcal{T}_2$ and there exists a base of $\mathcal{T}_1$ which is $\mathcal{T}_2$-σ-closure preserving. A $\mathcal{T}_2$-M_1 *with respect to* $\mathcal{T}_1$ bispace is defined similarly. Then a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise* M_1 if and only if it is both $\mathcal{T}_1$-M_1 with respect to $\mathcal{T}_2$ and $\mathcal{T}_2$-M_1 with respect to $\mathcal{T}_1$.

[§]Alternatively, some authors say that in the bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$, the topology $\mathcal{T}_1$ is *regular with respect to* $\mathcal{T}_2$ whenever the condition given above holds.

We also need the notion of stratifiability. A bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise semi-stratifiable* if and only if for each $\mathcal{T}_i$-closed set $F \subseteq X$ there exists a sequence of $\mathcal{T}_j$-open sets $(F_n)_{n\in\mathbb{N}}$ satisfying the following two conditions ($i, j \in \{1, 2\}$ and $i \neq j$): (i) If $F \subseteq K$ then $F_n \subseteq K_n$ for all $n \in \mathbb{N}$; (ii) $F = \bigcap_{n=1}^{\infty} F_n$. If, in addition, we also have (iii) $F = \bigcap_{n=1}^{\infty} cl_{\mathcal{T}_i} F_n$, then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise stratifiable*. It has been established that a bispace is pairwise M_3 if and only if it is pairwise stratifiable [6, Proposition 1(b)]. Hence the terms pairwise stratifiable and pairwise M_3 shall be used exchangeably below.

3. Closure-Preserving Collections

In this section we prove some facts about closure-preserving collections which are interesting in their own right, and we will apply them in the next section. As usual, $\overline{A}$ and $cl_{\mathcal{T}^\natural} A$ denote the closure of A in $(X, \mathcal{T})$, and in $\mathcal{T}^\natural$ respectively.

1. Lemma. If $(X, \mathcal{T}, \leq)$ is an ordered topological C-space and $A \subseteq X$ then $cl_{\mathcal{T}^\natural} A = d(\overline{A}) = d(\overline{d(A)})$.

Proof. Let A be a subset of an ordered topological C-space $(X, \mathcal{T}, \leq)$. Then $d(\overline{A})$ is closed. Since $A \subseteq \overline{A} \subseteq d(\overline{A})$, $A \subseteq d(\overline{A})$. Then we have

$$d(\overline{A}) \subseteq d(\overline{d(A)}) \subseteq cl_{\mathcal{T}^\natural} (cl_{\mathcal{T}^\natural}(cl_{\mathcal{T}^\natural}(A))) = cl_{\mathcal{T}^\natural}(A) \subseteq cl_{\mathcal{T}^\natural}(d(\overline{A})) = d(\overline{A}),$$

the last equality because $d(\overline{A})$ is a closed lower set given that X is a C-space. Therefore the result holds. □

A similar argument proves the following:

2. Lemma. If $(X, \mathcal{T}, \leq)$ is an ordered topological C-space and $A \subseteq X$ then $cl_{\mathcal{T}^\flat} A = i(\overline{A}) = i(\overline{i(A)})$. □

1. Proposition. If $(X, \mathcal{T}, \leq)$ is an ordered topological C- and I-space and $\mathcal{B}$ is an open and closure-preserving collection in $(X, \mathcal{T})$ then $\mathcal{B}_d = \{d(B) \mid B \in \mathcal{B}\}$ is an open collection in $(X, \mathcal{T}^\flat)$ which is closure-preserving in $(X, \mathcal{T}^\natural)$.

Proof. Suppose $(X, \mathcal{T}, \leq)$ is an ordered topological C- and I-space. Let $\mathcal{B}$ be an open and closure-preserving collection in $(X, \mathcal{T})$. Since X is an I-space, $d(B)$ is an open lower set for each $B \in \mathcal{B}$. Hence $\mathcal{B}_d$ is open in $(X, \mathcal{T}^\flat)$. It remains to show that $\mathcal{B}_d$ is closure-preserving in $(X, \mathcal{T}^\natural)$. Note that the operator d commutes with set union. Let $\mathcal{B}' \subseteq \mathcal{B}$. Then by Lemma 1, we get

$$cl_{\mathcal{T}^\natural} \left(\bigcup_{B\in\mathcal{B}'} d(B) \right) = d \left(\overline{\bigcup_{B\in\mathcal{B}'} d(B)} \right) = d \left(\overline{d(\bigcup_{B\in\mathcal{B}'} B)} \right) = d \left(\overline{\bigcup_{B\in\mathcal{B}'} B} \right)$$

$$= d \left(\bigcup_{B\in\mathcal{B}'} \overline{B} \right) = \bigcup_{B\in\mathcal{B}'} d(\overline{B}) = \bigcup_{B\in\mathcal{B}'} d(\overline{d(B)}) = \bigcup_{B\in\mathcal{B}'} cl_{\mathcal{T}^\natural} d(B). \text{ So,}$$

$cl_{\mathcal{T}^\natural}(\bigcup_{B\in\mathcal{B}'} d(B)) = \bigcup_{B\in\mathcal{B}'} cl_{\mathcal{T}^\natural} d(B)$. Hence $\mathcal{B}_d$ is closure-preserving in $(X, \mathcal{T}^\natural)$. □

Similarly, the following result emerges.

2. Proposition. If $(X, \mathcal{T}, \leq)$ is an ordered topological C- and I-space and $\mathcal{B}$ is an open and closure-preserving collection in $(X, \mathcal{T})$ then $\mathcal{B}_i = \{i(B) \mid B \in \mathcal{B}\}$ is an open collection in $(X, \mathcal{T}^\natural)$ which is closure-preserving in $(X, \mathcal{T}^\flat)$. □

4. On Pairwise M_1- versus Pairwise M_3- (Stratifiable) Bispaces

In 1986, A. Gutierrez and S. Romaguera [6] introduced the concepts of pairwise M_i-spaces into the theory of bispaces as a generalization of Ceder's M_i-spaces (i=1,2,3). We recall the following nice result.

3. Proposition. ([10]) If $(X, \mathcal{T}, \leq)$ is a stratifiable ordered topological C-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise stratifiable. □

The reader is referred to [7] for the definition and basic properties of monotonically normal spaces. For these in the bispace setting, see [12]. It is known that a (bi) space is (pairwise) stratifiable if and only if it is (pairwise) semi-stratifiable and (pairwise) monotonically normal. K. Li and F. Lin showed that one can relax the assumption of the above proposition and obtain:

4. Proposition. ([11]) If $(X, \mathcal{T}, \leq)$ is a monotonically normal ordered topological C-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise monotonically normal.

We are now ready to present the following observation.

1. Theorem. If $(X, \mathcal{T}, \leq)$ is an M_1 ordered topological C- and I-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1.

Proof. Let $(X, \mathcal{T}, \leq)$ be an M_1 ordered topological C- and I-space. We first show that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise regular. Let $x \in X$, and $F \subseteq X$ be a closed lower set such that $x \notin F$. Then $G := X \setminus F$ is a open neighbourhood of x. Since $(X, \mathcal{T}, \leq)$ is M_1, it is regular. Hence there exists an open neighbourhood H of x such that $\overline{H} \subseteq G$. Since X is an I-space, the upper set $U = i(H)$ is an open neighbourhood of x. By Lemma 2, we have

$$\overline{U} = \overline{i(H)} \subseteq i(\overline{i(H)}) = i(\overline{H}) \subseteq i(G) = G.$$

Since X is a C-space, $i(\overline{H})$ is closed and hence $V := X \setminus i(\overline{H})$ is an open lower set containing F and $U \cap V = \emptyset$. Thus the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is $\mathcal{T}^\natural$-regular with respect to $\mathcal{T}^\flat$. Similarly, one can easily show that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is $\mathcal{T}^\flat$-regular with respect to $\mathcal{T}^\natural$ and hence pairwise regular. Since $(X, \mathcal{T}, \leq)$ is an M_1-space, $\mathcal{T}$ has a σ-closure-preserving base, say $\mathcal{B}$. Let $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ where each $\mathcal{B}_n$ is a $\mathcal{T}$-closure-preserving subcollection of $\mathcal{B}$. Now we need to produce σ-closure-preserving bases for $\mathcal{T}^\natural$ and $\mathcal{T}^\flat$. Let $\mathcal{D}_n = \{d(B) \mid B \in \mathcal{B}_n\}$ and put $\mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$. Then $\mathcal{D}$ is a base for $\mathcal{T}^\flat$ which is, by Proposition 1, σ-closure-preserving in $\mathcal{T}^\natural$. Similarly, let $\mathcal{I}_n = \{i(B) \mid B \in \mathcal{B}_n\}$ and $\mathcal{I} = \bigcup_{n \in \mathbb{N}} \mathcal{I}_n$. Then $\mathcal{I}$ is a base for $\mathcal{T}^\natural$ which is, by Proposition 2, σ-closure-preserving in $\mathcal{T}^\flat$. Hence the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is $\mathcal{T}^\natural$-M_1 with respect to $\mathcal{T}^\flat$ and $\mathcal{T}^\flat$-M_1 with respect to $\mathcal{T}^\natural$. Therefore $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1. □.

Since every pairwise M_1-bispace is pairwise stratifiable [6], we get:

1. Corollary. If $(X, \mathcal{T}, \leq)$ is an M_1 ordered topological C- and I-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise stratifiable. □

Finally, we briefly turn our minds to the following result involving countability. Recall that a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is doubly first countable if both topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are first countable (see for instance J. Deak [4]).

5. Proposition. ([10]) If $(X, \mathcal{T}, \leq)$ is a first countable ordered topological I-space then $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is doubly first countable. □

Since any metric space is first countable and stratifiable, the following fact follows immediately and it fits in here.

2. Corollary. If $(X, \mathcal{T}, \leq)$ is a metrizable ordered topological C- and I-space then $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1 (and thus pairwise stratifiable) and doubly first countable. $\square$

Remark. As mentioned in the introduction above, F. G. Slaughter, Jr showed that if f is a closed continuous mapping from a metric space X onto the space Y, then Y is an M_1-space [14, Theorem 6]. It is therefore natural to wonder whether, in the same vein, the assumption of the above theorem can be relaxed without destroying the theorem in the sense that the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1 whenever $(X, \mathcal{T}, \leq)$ is an M_1 ordered topological C-space.

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Some identities and recurrences relations for the q-Bernoulli and q-Euler polynomials

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Abstract

In this article we prove some relations between two-variable q-Bernoulli polynomials and two-variable q-Euler polynomials. By using the equality $e_q(z)\,E_q(-z) = 1$, we give an identity for the two-variable q-Genocchi polynomials. Also, we obtain an identity for the two-variable q-Bernoulli polynomials. Furthermore, we prove two theorems which are analogues of the q-extension Srivastava-Pinter additional theorem.

Keywords: Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi polynomials, generating functions, generalized Bernoulli polynomials, generalized Genocchi polynomials, q-Bernoulli polynomials, q-Euler polynomials, q-Genocchi polynomials.

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1. Introduction Definition and Notation

The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are usually defined by means of the following generating functions;

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi$$

and

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi,$$

respectively. The corresponding Bernoulli numbers B_n and Euler numbers E_n are given by

$$B_n := B_n(0) = (-1)^n B_n(1) = \left(2^{1-n}-1\right)^{-1} B_n\left(\frac{1}{2}\right)$$

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and

$$E_n := 2^n E_n\left(\frac{1}{2}\right), \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

respectively.

Many mathematicians investigated these polynomials in ([2]-[17]). They proved some theorems and gave some interesting recurrences relations. Firstly, Carlitz in [2] gave q-Bernoulli polynomials.

In this work we give some recurrences relations and properties for two-variable q-Bernoulli polynomials and q-Euler polynomials.

Throughout this paper, we make use of the following notations; $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the set of complex numbers and $q \in \mathbb{C}$ with $|q| < 1$. The q-basic numbers and q-factorials are defined ([2], [7]-[15]) by

$$\begin{aligned} [a]_q &= \frac{1-q^a}{1-q} = 1 + q + \ldots + q^{a-1}, \quad (q \neq 1), \\ [n]_q! &= [n]_q [n-1]_q \ldots [2]_q [1]_q, \end{aligned}$$

respectively, where $[0]_q! = 1$ and $n \in \mathbb{N}, a \in \mathbb{C}$.

The q-binomial formula is defined ([8], [14]) by

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{n-k} y^k,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient (or Gaussian binomial coefficient) given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} = \frac{[n]_q!}{[n-k]_q! [k]_q!}.$$

The q-exponential functions are given ([1], [8], [12], [13]) by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, \quad |z| < \frac{1}{|1-q|}$$

and

$$E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1 + (1-q) q^k z\right), \quad 0 < |q| < 1, \quad z \in \mathbb{C}.$$

From the last equations, we can easliy see that $e_q(z) E_q(-z) = 1$.

The Jack-derivative D_q is defined ([7], [10], [13], [14]) by

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, \quad 0 \neq z \in \mathbb{C}.$$

The derivative of the product of two functions and the derivative of the division of two functions are given by the following equations in [7], respectively.

$$\begin{aligned} D_q\left(\frac{f(z)}{g(z)}\right) &= \frac{g(qz) D_q f(z) - f(qz) D_q g(z)}{g(z) g(qz)}, \\ D_q(f(z) g(z)) &= f(qz) D_q g(z) + g(z) D_q f(z). \end{aligned} \tag{1.1}$$

Carlitz was the first to extend the classical Bernoulli numbers and polynomials, Euler numbers and polynomials ([2], [3]). Cheon in [5] gave explicit expansions for the classical Bernoulli polynomials and the classical Euler polynomials. Srivastava et al [16] proved some formulae for the Bernoulli polynomials and the Euler polynomials. Also, they gave the addition-formulae between the Bernoulli polynomials and the Euler polynomials. There are numerous recent investigations on the q-Bernoulli polynomials and q-Euler polynomials by many mathematicians, including as Cenkci et al [4], Choi et al [6], Kim ([8], [9]), Kim et al [10], Luo [11], Luo and Srivastava [12], Srivastava et al ([16], [17]), Tremblay et al [18] and Mahmudov ([13], [14]).

Mahmudov defined and studied properties of the following generalized q-Bernoulli polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ of order α and q-Euler polynomials $\mathfrak{E}_{n,q}^{(\alpha)}(x,y)$ of order α as follows ([13], [14]).

Let $q \in \mathbb{C}, \alpha \in \mathbb{N}$ and $0 < |q| < 1$. The q-Bernoulli numbers $\mathfrak{B}_{n,q}^{(\alpha)}$ and polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t)-1}\right)^{\alpha}, \qquad |t| < 2\pi \tag{1.2}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(xt) E_q(yt), \qquad |t| < 2\pi. \tag{1.3}$$

The q-Euler numbers $\mathfrak{E}_{n,q}^{(\alpha)}$ and polynomials $\mathfrak{E}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t)+1}\right)^{\alpha}, \qquad |t| < \pi \tag{1.4}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t)+1}\right)^{\alpha} e_q(xt) E_q(yt), \qquad |t| < \pi. \tag{1.5}$$

The q-Genocchi numbers $\mathfrak{G}_{n,q}^{(\alpha)}$ and polynomials $\mathfrak{G}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t)+1}\right)^{\alpha}, \qquad |t| < \pi$$

and

$$\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t)+1}\right)^{\alpha} e_q(xt) E_q(yt), \qquad |t| < \pi.$$

It is obvious that

$$\begin{aligned} \lim_{q \to 1^-} \mathfrak{B}_{n,q}^{(\alpha)}(x,y) &= B_n^{(\alpha)}(x+y), \\ \lim_{q \to 1^-} \mathfrak{E}_{n,q}^{(\alpha)}(x,y) &= E_n^{(\alpha)}(x+y), \\ \lim_{q \to 1^-} \mathfrak{G}_{n,q}^{(\alpha)}(x,y) &= G_n^{(\alpha)}(x+y) \end{aligned}$$

and

$$\begin{aligned} D_{q,x}^{(\alpha)} \mathfrak{B}_{n,q}(x,y) &= [n]_q \mathfrak{B}_{n-1,q}^{(\alpha)}(x,y), \quad D_{q,y} \mathfrak{B}_{n,q}^{(\alpha)}(x,y) = [n]_q \mathfrak{B}_{n-1,q}^{(\alpha)}(x,qy), \\ D_{q,t} e_q(xt) &= x e_q(xt), \quad D_{q,t} E_q(yt) = y E_q(qyt). \end{aligned}$$

2. Main Theorems

In this section, we give some relations for q-Bernoulli polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ and q-Euler polynomials $\mathfrak{E}_{n,q}^{(\alpha)}(x,y)$. By applying the derivative operator to q-Bernoulli polynomials and q-Euler polynomials, we have recurrences relations for these polynomials.

2.1. Proposition. *The generalized q-Bernoulli polynomials satisfy the following relation.*

$$\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \mathfrak{B}_{n-l,q}^{(\alpha)}(x,y)-\mathfrak{B}_{n,q}^{(\alpha)}(x,y)=[n]_q\,\mathfrak{B}_{n-1,q}^{(\alpha-1)}(x,y)\,. \tag{2.1}$$

Proof. From (1.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} &= \left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(xt)\,E_q(yt)\\ \sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}(e_q(t)-1) &= t\left(\frac{t}{e_q(t)-1}\right)^{\alpha-1} e_q(xt)\,E_q(yt)\\ \sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} &= t\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}.\end{aligned}$$

By using Cauchy product and comparing the coefficient of $\frac{t^n}{[n]_q!}$ we have (2.1). ∎

The following equations can be obtained easily from (1.2)-(1.5).

$$\mathfrak{B}_{n,q}^{(\alpha-\beta)}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha-\beta)}(0,0)\,(x+y)_q^{n-k}\,, \tag{2.2}$$

$$\mathfrak{B}_{n,q}^{(\alpha-\beta)}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)}(x,0)\,\mathfrak{B}_{n-k,q}^{(-\beta)}(0,y)\,, \tag{2.3}$$

$$(x+y)_q^{n} = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{n-k,q}^{(\alpha)}(x,y)\,\mathfrak{E}_{k,q}^{(-\alpha)}(0,0)\,, \tag{2.4}$$

$$2\mathfrak{E}_{n,q}^{(\alpha-1)}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{n-k,q}^{(\alpha)}(x,y)+\mathfrak{E}_{n,q}^{(\alpha)}(x,y)\,, \tag{2.5}$$

where $\alpha,\beta\in\mathbb{N}$.

2.2. Theorem. *The generalized q-Bernoulli polynomials satisfy the following recurrence relation.*

$$\begin{aligned}\mathfrak{B}_{n+1,q}(x,y) &= \mathfrak{B}_{n,q}(x,y)+[n+1]_q\left\{qy\mathfrak{B}_{n,q}(qx,y)+qx\mathfrak{B}_{n,q}(x,y)\right\}\\ &\quad-\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}(x,y)\,q^k\mathfrak{B}_{n+1-k,q}(1,0)\,.\end{aligned} \tag{2.6}$$

Proof. In (1.3), for $\alpha=1$, we take the q-Jackson derivative of the generalized q-Bernoulli polynomials $\mathfrak{B}_{n,q}(x,y)$ according to t, then we have

$$\sum_{n=0}^{\infty}D_{q,t}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}=D_{q,t}\left(\frac{te_q(xt)\,E_q(yt)}{e_q(t)-1}\right).$$

By using the equalities (1.1) in the last expression we have

$$\begin{aligned}\sum_{n=0}^{\infty} D_{q,t}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\ = \frac{(e_q(qxt)-1)\,D_{q,t}\left[te_q(xt)\,E_q(yt)\right]-qte_q(qxt)\,E_q(qyt)\,D_{q,t}\left[e_q(t)-1\right]}{(e_q(t)-1)(e_q(qt)-1)},\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{[n+1]_q}\mathfrak{B}_{n+1,q}(x,y)\frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty} qy\mathfrak{B}_{n,q}(qx,qy)+qx\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\ &+\frac{1}{[n+1]_q}\left\{\mathfrak{B}_{n,q}(x,y)+\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k\end{bmatrix}_q \mathfrak{B}_{k,q}(x,y)\,q^k\mathfrak{B}_{n+1-k,q}(1,0)\frac{t^n}{[n]_q!}\right\}.\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{[n]_q!}$ we obtain (2.6). ∎

2.3. Theorem. *The generalized q-Euler polynomials $\mathfrak{E}_{n,q}(x,y)$ satisfy the following relation.*

$$\begin{aligned}\mathfrak{E}_{n+1,q}(x,y) &= [n+1]_q \\ &\times\left\{y\mathfrak{E}_{n,q}(qx,qy)+x\mathfrak{E}_{n,q}(x,y)-\frac{1}{4}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q \mathfrak{E}_{k,q}(x,y)\,q^k\mathfrak{E}_{n-k,q}(1,0)\right\}\end{aligned}$$

Proof. In (1.5), for $\alpha=1$, by using the equalities (1.1), the proof can be obtained. ∎

2.4. Theorem. *There is the following relation.*

(2.7)
$$\mathfrak{B}^{(\alpha)}_{n,q}(x,y)=\frac{m^{-n}}{[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k\end{bmatrix}_q\left\{\left[\mathfrak{B}^{(\alpha)}_{k,q}\left(\frac{1}{m},0\right)-\mathfrak{B}^{(\alpha)}_{k,q}(0,0)\right]\mathfrak{B}_{n+1-k,q}(x,y)\,m^k\right\}.$$

Proof. From (1.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}^{(\alpha)}_{n,q}\frac{t^n}{[n]_q!} &= \left(\frac{t}{e_q(t)-1}\right)^{\alpha}\frac{e_q\left(\frac{t}{m}\right)-1}{\frac{t}{m}}\frac{\frac{t}{m}}{e_q\left(\frac{t}{m}\right)-1} \\ &= \frac{m}{t}\left\{\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q\left(\frac{t}{m}\right)\frac{\frac{t}{m}}{e_q\left(\frac{t}{m}\right)-1}-\left(\frac{t}{e_q(t)-1}\right)^{\alpha}\frac{\frac{t}{m}}{e_q\left(\frac{t}{m}\right)-1}\right\} \\ &= \frac{m}{t}\sum_{n=0}^{\infty}\left[\mathfrak{B}^{(\alpha)}_{n,q}\left(\frac{1}{m},0\right)-\mathfrak{B}^{(\alpha)}_{n,q}(0,0)\right]\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}^{(\alpha)}_{n,q}(0,0)\frac{t^n}{m^n[n]_q!}.\end{aligned}$$

Using the Cauchy product and comparing the coefficient of $\frac{t^n}{[n]_q!}$ we obtain (2.7). ∎

2.5. Theorem. *The generalized q-Euler numbers $\mathfrak{E}^{(\alpha)}_{n,q}(0,0)$ satisfy the following relation.*

$$\mathfrak{E}^{(\alpha)}_{n,q}=\frac{1}{2[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k\end{bmatrix}_q\left\{\left[\mathfrak{E}^{(\alpha)}_{k,q}\left(\frac{1}{m},0\right)+\mathfrak{E}^{(\alpha)}_{k,q}(0,0)\right]\mathfrak{G}_{n+1-k,q}(0,0)\,m^{k-n}\right\}.$$

3. Some Relations Between the q-Bernoulli Polynomials and q-Euler Polynomials

In this section, we prove an interesting relationship between the q-Bernoulli polynomials $\mathfrak{B}^{(\alpha)}_{n,q}(x,y)$ of order α and q-Euler polynomials $\mathfrak{E}^{(\alpha)}_{n,q}(x,y)$ of order α.

3.1. Theorem. *There is the following relation between the q-Euler polynomials and q-Bernoulli polynomials.*

(3.1)
$$\mathfrak{B}^{(\alpha)}_{n,q}(x,y)=\frac{1}{2}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q\left\{\sum_{r=0}^{p}\left[\begin{array}{c} p \\ r \end{array}\right]_q\mathfrak{B}^{(\alpha)}_{r,q}(x,0)\,m^{r-n}+\mathfrak{B}^{(\alpha)}_{n-k,q}(x,0)\,m^{-k}\right\}\mathfrak{E}_{k,q}(0,my).$$

Proof. From (1.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}^{(\alpha)}_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\frac{2}{e_q(t)+1}E_q\left(my\frac{t}{m}\right)\frac{e_q\left(\frac{t}{m}\right)+1}{2}\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q(xt)\\
&=\frac{1}{2}\frac{2}{e_q\left(\frac{t}{m}\right)+1}E_q\left(my\frac{t}{m}\right)e_q\left(\frac{t}{m}\right)\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q(xt)\\
&\quad+\frac{1}{2}\frac{2}{e_q\left(\frac{t}{m}\right)+1}E_q\left(my\frac{t}{m}\right)\left(\frac{t}{e_q\left(\frac{t}{m}\right)-1}\right)^{\alpha}e_q(xt)\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}^{(\alpha)}_{n,q}(x,0)\frac{t^n}{[n]_q!}\\
&\quad+\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}^{(\alpha)}_{n,q}(x,0)\frac{t^n}{[n]_q!}\\
&=\frac{1}{2}\left[\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\right]\\
&\quad\times\left[\sum_{p=0}^{\infty}\sum_{r=0}^{p}\left[\begin{array}{c} p \\ r \end{array}\right]_q\mathfrak{B}^{(\alpha)}_{r,q}(x,0)\,m^{r-p}\frac{t^p}{[p]_q!}+\sum_{p=0}^{\infty}\mathfrak{B}^{(\alpha)}_{p,q}(x,0)\frac{t^p}{[p]_q!}\right].\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{[n]_q!}$ we obtain

$$\mathfrak{B}^{(\alpha)}_{n,q}(x,y)=\frac{1}{2}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q\left\{\sum_{r=0}^{p}\left[\begin{array}{c} p \\ r \end{array}\right]_q mathfrakB^{(\alpha)}_{r,q}(x,0)\,m^{r-n}+\mathfrak{B}^{(\alpha)}_{n-k,q}(x,0)\,m^{-k}\right\}\mathfrak{E}_{k,q}(0,my).$$

∎

3.2. Theorem. *There is the following relation between the q-Bernoulli polynomials and q-Euler polynomials.*

(3.2)
$$\begin{aligned}\mathfrak{E}^{(\alpha)}_{n,q}(x,y)&=\frac{m}{[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q\\
&\quad\times\left\{\sum_{r=0}^{n+1-k}\left[\begin{array}{c} n+1-k \\ r \end{array}\right]_q\mathfrak{E}^{(\alpha)}_{r,q}(x,0)\,m^{r-n-1}-\mathfrak{E}^{(\alpha)}_{n+1-k,q}(x,0)\,m^{-k}\right\}\mathfrak{B}_{k,q}(0,my).\end{aligned}$$

Proof. From (1.5), we write

$$\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}=\frac{\frac{t}{m}}{e_q\left(\frac{t}{m}\right)-1}E_q\left(my\frac{t}{m}\right)\frac{e_q\left(\frac{t}{m}\right)-1}{\frac{t}{m}}\left(\frac{2}{e_q(t)+1}\right)^{\alpha}e_q(xt)$$

$$=\frac{m}{t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}$$

$$-\frac{m}{t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}$$

$$=\frac{m}{t}\sum_{k=0}^{\infty}\mathfrak{B}_{k,q}(0,my)\frac{t^k}{m^k[k]_q!}\left\{\sum_{p=0}^{\infty}\sum_{r=0}^{p}\left[\begin{array}{c}p\\ r\end{array}\right]_q\mathfrak{E}_{r,q}^{(\alpha)}(x,0)\,m^{r-p}-\mathfrak{E}_{r,q}^{(\alpha)}(x,0)\right\}\frac{t^p}{[p]_q!}.$$

Using the Cauchy product and comparing the the coefficient of $\frac{t^n}{[n]_q!}$ we obtain (3.2). ∎

3.3. Corollary. *The following relations holds*

(3.3)

$$\mathfrak{B}_{n,q}^{(\alpha)}=\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\ k\end{array}\right]_q m^{k-n}\left\{\mathfrak{B}_{k,q}^{(\alpha)}\left(\frac{1}{m},0\right)+\mathfrak{B}_{k,q}^{(\alpha)}(0,0)\right\}\mathfrak{E}_{n+1-k,q}(0,0)$$

and

(3.4)

$$\mathfrak{E}_{n,q}^{(\alpha)}=\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\ k\end{array}\right]_q m^{k-n}\left\{\mathfrak{E}_{k,q}^{(\alpha)}\left(\frac{1}{m},0\right)-\mathfrak{E}_{k,q}^{(\alpha)}(0,0)\right\}\mathfrak{B}_{n+1-k,q}^{(\alpha)}(0,0).$$

3.4. Corollary. *From (3.3) and (3.4), we have*

$$\left\{\mathfrak{B}_{k,q}^{(\alpha)}\left(\frac{1}{m},0\right)+\mathfrak{B}_{k,q}^{(\alpha)}(0,0)\right\}\mathfrak{E}_{n+1-k,q}(0,0)\,\mathfrak{E}_{n,q}^{(\alpha)}(0,0)$$
$$=\left\{\mathfrak{E}_{k,q}^{(\alpha)}\left(\frac{1}{m},0\right)-\mathfrak{E}_{k,q}^{(\alpha)}(0,0)\right\}\mathfrak{B}_{n+1-k,q}^{(\alpha)}(0,0)\,\mathfrak{B}_{n,q}^{(\alpha)}(0,0).$$

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Some statistical cluster point theorems

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Abstract

In this paper, we present results related to sets of statistical limit points and cluster points of sequences and their matrix transformations, single and double sequences and stretchings of sequences.

Keywords: Sequences,statistical limit points, statistical cluster points, matrix transformations.

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1. Introduction

In real analysis there are many characterization theorems of the type- if $f : R \longrightarrow R$, then F, the set of discontinuities of f, is a closed set and conversely, if F is any closed subset of R then there exists a function $f : R \longrightarrow R$, whose set of discontinuities is precisely F.

R.C. Buck ([1] , [2] , [3]), in a series of articles considered similar questions concerning subsequential limit points of a given sequence. Pratulananda Das [5], at the suggestion of Brian Thompson, continued the inquiries of Buck [1].Our initial result showed that if (x_n) is a bounded sequence of reals, having L as its set of (subsequential) limit points, and if M, $M \neq \emptyset$, is any closed subset of L, then there is a subsequence (x_{n_k}) of (x_n), whose set of (subsequential) limit points is precisely M. Fortunately Cihan Orhan pointed out to us, that in fact, we had rediscovered a known result, pointing us to Theorem 1.62 II, on page 142 in Cooke [4].

Here we consider statistical cluster point analogues of the result mentioned above. Statistical limit points and statistical cluster points were first considered by Fridy in [6]. Our results are concerned with single sequences as well as with double sequences. Similarities between our results and core theorems (for example [10], [12]) will be apparent

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to the reader. Also we present a result about stretchings of sequences. Miller and Patterson have previously had stretching results.

2. Preliminaries

If K is a subset of the positive integers $\mathbb{N}$, then following Fridy [6], K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n. The natural density of K (see [4] , Chapter 11) is given by $\delta(K) = \lim_{n\to\infty} n^{-1}|K_n|$. In the case that $\delta(K) = 0$ we say that K is thin, and otherwise we say that K is non-thin. We continue following Fridy [6] .

2.1. Definition. We say that a number λ is a statistical limit point of a sequence of reals (x_n) if $\lim_{k\to\infty} x_{n_k} = \lambda$ for some non-thin subsequence (x_{n_k}) of (x_n).

2.2. Definition. Given a sequence of reals (x_n), a stretching of that sequence is any sequence of the form $x_1, x_1, \ldots x_1, x_2, x_2 \ldots, x_2, \ldots, x_n, x_n, \ldots, x_n \ldots$.

2.3. Definition. We say that a number γ is a statistical cluster point of a sequence of reals (x_n) if for every $\epsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \gamma| < \epsilon\}$ is non-thin.

We also consider bounded double sequences $x = (x_{n,k})$ and 4-dimensional bounded regular matrix transformations.

2.4. Definition. A double sequence $x = (x_{n,k})$ of reals is said to be bounded if there exists an M such that $|x_{n,k}| < M$ for all n, k.

2.5. Definition. A double sequence $x = (x_{n,k})$ has Pringsheim limit L (or in what follows, just limit L) denoted by $\lim x_{n,k} = L$, if given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_{n,k} - L| < \epsilon$ whenever $n, k > \mathbb{N}$. Briefly, we say that x is convergent and has limit L.

Let $A = (a_{s,t,n,k})$, denote a four dimensional summability method (see [8]) that maps the double real sequence x into the double real sequence Ax where the s,t-th term of Ax is defined as follows:

$$(Ax)_{s,t} = \sum_{n,k=1,1}^{\infty,\infty} a_{s,t,n,k} x_{n,k}$$

and is called an A-mean. For the above definition and for what follows see Móricz [11].

We say that a double sequence is A-summable to the limit L if the A-means exist for all $s, t = 1, 2, 3 \ldots$ and

$$\lim_{s,t} (Ax)_{s,t} = L.$$

2.6. Definition. The four dimensional real matrix A is said to be bounded regular if every bounded convergent double sequence with real entries x is A-summable to the same limit and the A-means are also bounded.

Finally, a classical theorem, ([8], [13]) characterizes bounded regular four dimensional matrices.

2.7. Theorem. Necessary and sufficient conditions for $A = (a_{s,t,n,k})$ to be bounded regular are

$\lim_{s,t} a_{s,t,n,k} = 0$ for each n and k
$\lim_{s,t} \sum_{n.k=1,1}^{\infty,\infty} a_{s,t,n,k} = 1$
$\lim_{s,t} \sum_{n=1}^{\infty} |a_{s,t,n,k}| = 0$
$\lim_{s,t} \sum_{k=1}^{\infty} |a_{s,t,n,k}| = 0$
$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{s,t,n,k}|$ is convergent; and
there exist positive integers A and B such that $\sum_{n,k>B} |a_{s,t,n,k}| < A$ for each s, t.

3. Results

Our first result is the statistical cluster point analogue of the result (see [4]) mentioned in our introduction.

3.1. Theorem. Suppose $x = (x_n)$ is a bounded sequence and L is the set of limit points of x. If $M \subseteq L$, M is closed and nonempty, there exists a subsequence $y = (y_n)$ of x such that M is the set of statistical cluster points of y.

Proof. Since M is closed and separable, there is a countable subset of M, $\{a_m : m \in \mathbb{N}\}$ such that its closure is M. Now for $m \in \mathbb{N}$, fix a subsequence of (x_n), $(x_{n_{k,m}})_{k=1}^{\infty})$, converging to a_m and contained in $(a_m - \frac{1}{m}, a_m + \frac{1}{m})$.

We construct $y = (y_n)$ as follows:

$y_1, y_3, y_5, \ldots, y_{2j+1}, \ldots$ will be chosen from $(x_{n_{k,1}})$,
$y_2, y_6, y_{10}, \ldots, y_{2(2j+1)}, \ldots$ will be chosen from $(x_{n_{k,2}})$,
$y_4, y_{12}, y_{20}, \ldots, y_{4(2j+1)}, \ldots$ will be chosen from $(x_{n_{k,3}})$,
$\ldots$
$y_{2^{m-1}}, y_{2^{m-1}.3}, y_{2^{m-1}.5}, \ldots, y_{2^{m-1}.(2j+1)}, \ldots$
will be chosen from $(x_{n_{k,m}})$,
$\ldots$

where

$y_1 = x_{n_{1,1}}$, and
$y_2 = x_{n_{k,2}}$ where k is the smallest number so that $n_{k,2} > n_{1,1}$
if $y_1, y_2 \ldots y_{i-1}$ have been chosen, and $i = 2^{m-1}(2j+1)$ we choose $y_i = x_{n_{k,m}}$ so that k is the smallest number such that the index $n_{k,m}$ is bigger than the indices of $y_1, y_2 \ldots y_{i-1}$ in terms of x.

Hence (y_n) is a subsequence of x. Also $(y_{2^{m-1}.(2j+1)})_{j=0}^{\infty}$ has density $\frac{1}{2^m}$ in (y_n) so a_m is a statistical limit point (and cluster point) of (y_n). Also it is clear that every $a \in M$ is a statistical cluster point of (y_n). Likewise for every $a \in R \setminus M$, there is a sufficiently small neighborhood around it that is disjoint from (y_n) (since M is closed), so the set M is precisely the set of statistical cluster points of (y_n). $\square$

3.2. Corollary. If $x = (x_n)$ is bounded and $M \subset L$, M closed and nonempty, where L is the limit point set of (x_n), then there exists a regular summability method A such that M is the set of statistical cluster points of (Ax).

Proof. Suppose that (x_{n_k}) is the subsequence of (x_n) with M as the set of its statistical cluster points from Theorem 3.1. If $A = (a_{km})$ has entries $a_{kn_k} = 1$ for all k and $a_{km} = 0$ otherwise, then $(Ax) = (x_{n_k})$ and the corollary follows. $\square$

Next, we show the analogous result for stretchings of sequences.

3.3. Theorem. Suppose $x = (x_n)$ is a bounded sequence and L is the set of limit points of x. If $M \subseteq L$, M is closed and nonempty, there exists a stretching of x, y such that M is the set of statistical cluster points of y.

Proof. Suppose that (x_{n_k}) is the subsequence of (x_n) with M as the set of its statistical cluster points constructed in the proof of Theorem 3.1. We construct the following stretching of the sequence (x_n):

$x_1, x_2, x_3, \ldots, x_{n_1-1}$ remain as before,
$x_{n_1}, x_{n_1}, x_{n_1}, \ldots, x_{n_1}, \ldots$ will be repeated $2n_2$ times, followed by
$x_{n_1+1}, x_{n_1+2}, x_{n_1+3}, \ldots, x_{n_2-1}$,
$x_{n_2}, x_{n_2}, x_{n_2}, \ldots, x_{n_2}, \ldots$ will be repeated $4n_3$ times, followed by

$x_{n_2+1}, x_{n_2+2}, x_{n_2+3}, \ldots, x_{n_3-1}$
$\ldots$
$x_{n_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}, \ldots, x_{n_{k-1}}, \ldots$ will be repeated $2^{k-1} \cdot n_k$ times, followed by
$x_{n_{k-1}+1}, x_{n_{k-1}+2}, x_{n_{k-1}+3}, \ldots, x_{n_k-1}$,
$x_{n_k}, x_{n_k}, x_{n_k}, \ldots, x_{n_k}, \ldots$ will be repeated $2^k \cdot n_{k+1}$ times, ,
$\ldots$

It is easy to check that the part of the new sequence that was not stretched (the odd numbered rows above) is thin (has density 0). Also since $2 \cdot n_2 < 4 \cdot n_3 < \ldots < 2^{k-1} \cdot n_k < 2^k \cdot n_{k+1} < \ldots$ we see that M is still the set of all statistical cluster points of the stretched part of the new sequence (even rows) and consequently of the whole new sequence (stretching).

□

Now, as mentioned in the introduction, we consider the two-dimensional analogue of our corollary. First we mention that we say l is a limit point of $(x_{n,k})$ if there exist $n_j \to \infty$ and $k_j \to \infty$ such that $\lim_{j\to\infty} x_{n_j,k_j} = l$ and we notice that L the set of limit points of $(x_{n,k})$ is always closed.

3.4. Theorem. If $x = (x_{n,k})$ is a bounded double sequence, and M is a closed nonempty subset of L, the set of limit points of x, then there exists a four dimensional bounded regular matrix transformation A of double sequences such that the set of limit points of Ax is exactly M.

Proof. Without loss of generality, assume that x is contained in the interval $[0, 1]$. We define

$I^1 = [0, \frac{1}{2}]$, $I^2 = [\frac{1}{2}, 1]$;
$I^3 = [0, \frac{1}{4}]$, $I^4 = [\frac{1}{4}, \frac{1}{2}]$, $I^5 = [\frac{1}{2}, \frac{3}{4}]$, $I^6 = [\frac{3}{4}, 1]$;
$I^7 = [0, \frac{1}{8}]$, $I^8 = [\frac{1}{8}, \frac{1}{4}]$, etc.
and so on. . . .

Let (s_i) be the sequence of integers satisfying $I^{s_i} \bigcap M \neq \emptyset$. For each i, pick a $y_i \in I^{s_i} \bigcap M$. Since $y_i \in M \subseteq L$, for each i, there exists $u_i \to \infty$, $v_i \to \infty$, such that

$$|y_i - x_{u_i,v_i}| < \frac{1}{i}.$$

Now we define the required matrix $A = (a_{m,n,u,v})$. For any m, n, if $m + n = i$ ($i = 2, 3, \ldots$), let $a_{m,n,u_i,v_i} = 1$ for ($i = 2, 3, \ldots$), and $a_{m,n,u,v} = 0$ otherwise. It is easy to see that A satisfies the conditions in Theorem 2.7 and that the set of limit points of Ax is exactly M. □

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On the isomorphy of categories of probabilistic limit spaces under t-norms

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Abstract

We show that for the classes of strict t-norms, the categories of probabilistic limit spaces under these t-norms are all isomorphic to each other. The same is true for the categories of probabilistic limit spaces under nilpotent t-norms. To show this, we study the isomorphisms between the categories of probabilistic limit spaces under a t-norm, limit tower spaces and approach limit spaces. Similar results are obtained for probabilistic Cauchy spaces and probabilistic uniform limit spaces.

Keywords: Probabilistic limit space, probabilistic uniform limit space, probabilistic Cauchy space, limit tower space, uniform limit tower space, Cauchy tower space, convergence approach space, approach Cauchy space, approach uniform limit space, t-norm.

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1. Introduction

Probabilistic limit spaces go back to the work of Florescu [3] and a formulation by means of filter convergence was given by Richardson and Kent [15]. These spaces are extensions of probabilistic metric spaces and probabilistic topological spaces as studied by Menger [12], Schweizer and Sklar [16] and Frank [4]. The category of probabilistic limit spaces is a Cartesian closed, extensional and topological category in the sense of [1]. Triangular norms (t-norms for short) were already used in [16] to model a triangular inequality in probabilistic metric spaces and it therefore seems appropriate to include t-norms in the generalizations of such spaces. This was, consequently, done by Nusser [14] who studied various categories of probabilistic spaces under t-norms.

Ultra-approach limit spaces were introduced by Lowen and Lowen [8] under the name convergence approach spaces. The category of these spaces is a Cartesian closed, extensional and topological category and forms a common framework that encompasses metric spaces and classical convergence spaces. The category of ultra-approach limit spaces contains the category of approach spaces [9, 10] (which form a common framework for topological, metric and uniform spaces) as a reflective subcategory.

In order to study the relationship between probabilistic limit spaces and ultra-approach limit spaces, Brock and Kent [2] introduced the category of limit tower spaces. They could show that the category of probabilistic limit spaces (under the minimum t-norm) is isomorphic to the category of ultra-approach limit spaces.

In this paper, we are extending the results of Brock and Kent [2] to probabilistic limit spaces under a t-norm. In order to do so, we generalize the definition of a limit tower space and introduce a certain subclass of these spaces. It turns out that for certain classes of t-norms, all probabilistic limit spaces under these t-norms are isomorphic. Similar results can be shown for probabilistic Cauchy spaces under a t-norm and for probabilistic uniform limit spaces under a t-norm.

We are finally going to introduce the basic concepts that we need later and fix the notation. A *t-norm* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a binary operation on $[0,1]$ which is associative, commutative, non-decreasing in each argument and which has 1 as the unit. A t-norm is called continuous if it is continuous as a mapping from $[0,1] \times [0,1] \longrightarrow [0,1]$. A special class of t-norms is given by *continuous Archimedean t-norms*. These are determined by continuous, strictly decreasing additive generators $S : [0,1] \longrightarrow [0,\infty]$ with $S(1) = 0$ such that

$$\alpha * \beta = S^{(-1)}(S(\alpha) + S(\beta))$$

with the *pseudo-inverse* $S^{(-1)}(u) = \bigvee\{x \in [0,1] \ : \ S(x) > u\} = \begin{cases} v & \text{if } S(v) = u \\ 0 & \text{if } u > S(0) \end{cases}$.
Note that $\bigvee \emptyset = \bigwedge[0,1] = 0$ here.

We further note that the pseudo-inverse $S^{(-1)} : [0,\infty] \longrightarrow [0,1]$ is continuous, surjective and strictly decreasing on $[0, S(0)]$ and that $S(S^{(-1)}(u)) = u$ if $u \leq S(0)$ and that $S^{(-1)}(S(u)) = u$ for all $u \in [0,1]$. Continuous Archimedean t-norms can be separated into two classes.

- $S(0) = \infty$. These are the *strict t-norms*. In this case $S^{(-1)} = S^{-1}$. A typical example is the product t-norm $\alpha * \beta = \alpha\beta$ with additive generator $S(x) = -\ln(x)$ (and $S(0) = \infty$).
- $S(0) < \infty$. These are the *nilpotent t-norms*. Noting that for an additive generator S for a continuous Archimedean t-norm and for all $a > 0$, $\overline{S}(x) = aS(x)$ defines an additive generator for the same t-norm, we can always assume for a nilpotent t-norm that $S(0) = 1$. A typical example for a nilpotent t-norm is the Lukasiewicz t-norm $\alpha * \beta = (\alpha + \beta - 1) \vee 0$ with additive generator $S(x) = 1 - x$.

An example of a non-Archimedean t-norm is the *minimum t-norm* $\alpha * \beta = \alpha \wedge \beta$. For further results on t-norms we refer to Schweizer and Sklar [16] and to [6].

We finally fix some notation. For a set X, we denote $P(X)$ its power set. We denote the set of all filters $\mathbb{F}, \mathbb{G}, \mathbb{H}, ...$ on the set X by $\mathbb{F}(X)$. We order this set by set inclusion and we denote for $x \in X$ the point filter by $[x] = \{F \subseteq X \ : \ x \in F\}$. For a subset A of an ordered set X we write, in case of existence, $\bigvee A$ for its supremum and $\bigwedge A$ for its infimum. If $A = \{\alpha, \beta\}$, then we write $\alpha \wedge \beta = \bigwedge A$ and $\alpha \vee \beta = \bigvee A$. For notions from category theory we refer to [1].

2. Probabilistic limit spaces, limit tower spaces and approach convergence spaces

A *probabilistic limit space under a t-norm* $*$ [14] is a pair $(X, \overline{q})$ of a set X and a non-empty family of mappings $\overline{q} = (q_\lambda : \mathbb{F}(X) \longrightarrow P(X))_{\lambda \in [0,1]}$ that satisfies the following axioms.

(PL1) $x \in q_\alpha([x])$ for all $\alpha \in [0,1]$, $x \in X$;
(PL2) $q_\alpha(\mathbb{F}) \subseteq q_\alpha(\mathbb{G})$, whenever $\mathbb{F} \leq \mathbb{G}$;
(PL3) $q_\beta(\mathbb{F}) \subseteq q_\alpha(\mathbb{F})$ whenever $\alpha \leq \beta$;
(PL4) $q_0(\mathbb{F}) = X$;
(PL5) $x \in q_{\alpha * \beta}(\mathbb{F} \wedge \mathbb{G})$ whenever $x \in q_\alpha(\mathbb{F})$ and $x \in q_\beta(\mathbb{G})$;
(PLLC) $q_\alpha(\mathbb{F}) = \bigcap_{\beta < \alpha} q_\beta(\mathbb{F})$; for all $\alpha, \beta \in [0,1], \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$.

The condition (PLLC) is called *left-continuity*. It is not required in the original definition by Nusser [14], however we will need it later. A mapping $f : X \longrightarrow X'$ between the probabilistic limit spaces under the t-norm $*$, $(X, \overline{q}), (X', \overline{q}')$, is *continuous* if for all $\alpha \in [0,1]$ and all $\mathbb{F} \in \mathbb{F}(X)$ we have $f(q_\alpha(\mathbb{F})) \subseteq q'_\alpha(f(\mathbb{F}))$. The category of all probabilistic limit spaces under the t-norm $*$ with the continuous mappings as morphisms is denoted by $PLIM^*$. It is shown in [14] that $PLIM^*$ is a topological and extensional construct and for $* = \wedge$, $PLIM^\wedge$ is Cartesian closed.

2.1. Lemma. *Let $(X, \overline{q})$ be a probabilistic limit space under the minimum t-norm $\wedge$. Then (PL5) is equivalent to the axiom*

(uPL5) $x \in q_\alpha(\mathbb{F} \wedge \mathbb{G})$ *whenever* $x \in q_\alpha(\mathbb{F})$ *and* $x \in q_\alpha(\mathbb{G})$.

Proof. If (PL5) is true, then we simply choose $\alpha = \beta$. If (uPL5) is true, then for $x \in q_\alpha(\mathbb{F}) \cap q_\beta(\mathbb{G})$ we have, because $\alpha \wedge \beta \leq \alpha, \beta$ and (PL3) that $x \in q_{\alpha\wedge\beta}(\mathbb{F}) \cap q_{\alpha\wedge\beta}(\mathbb{G})$ and hence, by (uPL5), also $x \in q_{\alpha\wedge\beta}(\mathbb{F} \wedge \mathbb{G})$. □

Therefore, probabilistic limit spaces under the minimum t-norm $\wedge$ are (left-continuous) *componentwise probabilistic limit spaces* in the definition of [14].

A *limit tower space* is a pair $(X, \overline{p})$ of a set X and a non-empty family of mappings $\overline{p} = (p_\epsilon : \mathbb{F}(X) \longrightarrow P(X))_{\epsilon\in[0,\infty]}$ that satisfies the following axioms.

(LT1) $x \in p_\epsilon([x])$ for all $\epsilon \in [0, \infty]$, $x \in X$;

(LT2) $p_\epsilon(\mathbb{F}) \subseteq p_\epsilon(\mathbb{G})$, whenever $\mathbb{F} \leq \mathbb{G}$;

(LT3) $p_\delta(\mathbb{F}) \subseteq p_\epsilon(\mathbb{F})$ whenever $\delta \leq \epsilon$;

(LT4) $p_\infty(\mathbb{F}) = X$;

(LT5) $x \in p_{\epsilon+\delta}(\mathbb{F} \wedge \mathbb{G})$ whenever $x \in p_\epsilon(\mathbb{F})$ and $x \in p_\delta(\mathbb{G})$;

(LTLC) $p_\epsilon(\mathbb{F}) = \bigcap_{\epsilon<\delta} p_\delta(\mathbb{F})$, for all $\epsilon, \delta \in [0, \infty], \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$.

The condition (LTLC) is again called *left-continuity*. A mapping $f : X \longrightarrow X'$ between the limit tower spaces $(X, \overline{p})$, $(X', \overline{p}')$ is *continuous* if for all $\epsilon \in [0, \infty]$ and all $\mathbb{F} \in \mathbb{F}(X)$ we have $f(p_\epsilon(\mathbb{F})) \subseteq p'_\epsilon(f(\mathbb{F}))$. The category of all limit tower spaces with the continuous mappings as morphisms is denoted by LTS.

If we replace the axiom (LT5) by the axiom

(uLT5) $x \in p_{\epsilon\vee\delta}(\mathbb{F} \wedge \mathbb{G})$ whenever $x \in p_\epsilon(\mathbb{F})$ and $x \in p_\delta(\mathbb{G})$;

then we speak of a *ultra-limit tower space*. The category of ultra-limit tower spaces with continuous mappings as morphisms is denoted by $uLTS$.

2.2. Lemma. *Let $(X, \overline{p})$ be an ultra-limit tower space. Then (uLT5) is equivalent to the axiom*

(uLT5') $x \in p_\epsilon(\mathbb{F} \wedge \mathbb{G})$ *whenever* $x \in p_\epsilon(\mathbb{F})$ *and* $x \in p_\epsilon(\mathbb{G})$.

Proof. Similar to the proof of Lemma 2.1. □

The preceding Lemma shows that ultra-limit tower spaces are the same as limit tower spaces as originally introduced and studied in [2]. We prefer to rename them in the light of the subsequent sections.

An *approach limit space* [11] is a pair (X, λ) of a set X and a mapping $\lambda : \mathbb{F}(X) \longrightarrow [0, \infty]^X$ that satisfies the following axioms.

(AL1) $\lambda([x])(x) = 0$ for all $x \in X$;

(AL2) $\lambda(\mathbb{G})(x) \leq \lambda(\mathbb{F})(x)$ whenever $\mathbb{F} \leq \mathbb{G}$;

(AL3) $\lambda(\mathbb{F} \wedge \mathbb{G})(x) \leq \lambda(\mathbb{F})(x) + \lambda(\mathbb{G})(x)$.

The value $\lambda(\mathbb{F})(x)$ has the interpretation as the distance that x is away from being a limit point of $\mathbb{F}$ [10]. A mapping $f : X \longrightarrow X'$ between two approach limit spaces (X, λ), (X', λ') is called a *contraction* if for all $\mathbb{F} \in \mathbb{F}(X)$ and all $x \in X$ we have $\lambda'(f(\mathbb{F}))(f(x)) \leq \lambda(\mathbb{F})(x)$

If we replace the axiom (AC3) by the stronger axiom

(uAL3) $\lambda(\mathbb{F} \wedge \mathbb{G})(x) \leq \lambda(\mathbb{F})(x) \vee \lambda(\mathbb{G})(x)$

then we call the pair (X, λ) an *ultra-approach limit space*. Note that these spaces were originally called *convergence approach spaces* and introduced and studied by Lowen and Lowen [8]. What we call here an approach limit space is called *weak convergence approach space* in [11]. Again we prefer to change the names in order to reach consistency with other notations. The category of approach limit spaces with contractions as morphisms is denoted by ALS, the subcategory of ultra-approach limit spaces is denoted by $uALS$. The category $uALS$ is topological, extensional and Cartesian closed [8] and whereas ALS is topological and contains $uALS$ as a bireflective subcategory [11].

3. Isomorphisms between the categories ALS and LTS

The following isomorphism functors between the categories of ultra-approach limit spaces and ultra-limit tower spaces were introduced in [2]. We extend their definition to the categories of approach limit spaces and limit tower spaces.

For $(X, \lambda) \in |uALS|$ we define $\eta\lambda = ((\eta\lambda)_\epsilon)_{\epsilon \in [0,\infty]}$ by

$$x \in (\eta\lambda)_\epsilon(\mathbb{F}) \quad \iff \quad \lambda(\mathbb{F})(x) \leq \epsilon.$$

It is shown in [2] that $\eta : uALS \longrightarrow uLTS$, $(X, \lambda) \longmapsto (X, \eta\lambda)$, $f \longmapsto f$ is a functor.

For $(X, \overline{p}) \in |uLTS|$ we define $\rho\overline{p} : \mathbb{F}(X) \longrightarrow [0, \infty]^X$ by

$$(\rho\overline{p})(\mathbf{F})(x) = \bigwedge \{\epsilon \in [0, \infty] \ : \ x \in p_\epsilon(\mathbb{F})\}.$$

It is shown in [2] that $\rho : uLTS \longrightarrow uALS$, $(X, \overline{p}) \longmapsto (X, \rho\overline{p})$, $f \longmapsto f$ is a functor and that $\eta \circ \rho = id_{uLTS}$ and $\rho \circ \eta = id_{uACP}$. Hence both functors are isomorphism functors and the categories $uALS$ and $uLTS$ are isomorphic. We will show with the next two lemmas that both functors can be extended to the categories ALS and LTS. To this end, we simply use for $(X, \lambda) \in |ALS|$, resp. for $(X, \overline{p}) \in |LTS|$ the same definitions of $\eta\lambda$ and $\rho\overline{p}$, i.e. we extend the domains of the functors η and ρ to ALS and LTS, respectively. We will show that the co-domains then are again the categories LTS and ALS, respectively.

3.1. Lemma. *Let $(X, \lambda) \in |ALS|$. Then $(X, \eta\lambda)$ satisfies the axiom (LT5).*

Proof. Let $x \in (\eta\lambda)_\epsilon(\mathbb{F})$ and $x \in (\eta\lambda)_\delta(\mathbb{G})$. Then $\lambda(\mathbb{F})(x) \leq \epsilon$ and $\lambda(\mathbb{G})(x) \leq \delta$. By (AC3) then $\lambda(\mathbb{F} \wedge \mathbb{G})(x) \leq \epsilon + \delta$, i.e. $x \in (\eta\lambda)_{\epsilon+\delta}(\mathbb{F} \wedge \mathbb{G})$. $\square$

3.2. Lemma. *Let $(X, \overline{p}) \in |LTS|$. Then $(X, \rho\overline{p})$ satisfies the axiom (AC3).*

Proof. Let $\rho\overline{p}(\mathbb{F})(x) = \epsilon$ and $\rho\overline{p}(\mathbb{G})(x) = \delta$. For $\epsilon' > \epsilon$ and $\delta' > \delta$ then $x \in p_{\epsilon'}(\mathbb{F})$ and $x \in p_{\delta'}(\mathbb{G})$. By (LT5) then $x \in p_{\epsilon'+\delta'}(\mathbb{F} \wedge \mathbb{G})$ and hence $\rho\overline{p}(\mathbb{F} \wedge \mathbb{G})(x) \leq \epsilon' + \delta'$. As $\epsilon' > \epsilon$ and $\delta' > \delta$ are arbitrary we conclude $\rho\overline{p}(\mathbb{F} \wedge \mathbb{G})(x) \leq \epsilon + \delta = \rho\overline{p}(\mathbb{F})(x) + \rho\overline{p}(\mathbb{G})(x)$. $\square$

3.3. Theorem. *The categories ALS and LTS are isomorphic.*

4. The case of strict t-norms

Let now $*$ be a strict t-norm with additive generator $S : [0, 1] \longrightarrow [0, \infty]$. Brock and Kent [2] have defined the following isomorphism functors between the categories $PLIM^\wedge$ and $uLTS$. For an ultra-limit tower space $(X, \overline{p})$ we define $(\Phi_S\overline{p})_\alpha = p_{S(\alpha)}$. Then $\Phi_S : uLTS \longrightarrow PLIM^\wedge$, $(X, \overline{p}) \longmapsto (X, \Phi_S\overline{p})$, $f \longmapsto f$ is a functor. For a levelwise probabilistic limit space $(X, \overline{q})$ we define $(\Psi_S\overline{q})_\epsilon = q_{S^{-1}(\epsilon)}$. Then $\Psi_S : PLIM^\wedge \longrightarrow uLTS$, $(X, \overline{q}) \longmapsto (X, \Psi_S\overline{q})$, $f \longmapsto f$ is a functor and $\Phi_S \circ \Psi_S = id_{PLIM^\wedge}$ and $\Psi_S \circ \Phi_S = id_{uLTS}$. Hence both functors are isomorphism functors and $PLIM^\wedge$ and $uLTS$ are isomorphic. We will show with the next two lemmas that these functors can be extended to the categories $PLIM^*$ and LTS, provided that $*$ is the strict t-norm generated by S.

4.1. Lemma. *Let the strict t-norm $*$ have the additive generator S and let $(X, \overline{q}) \in |PLIM^*|$. Then $(X, \Psi_S\overline{q})$ satisfies the axiom (LT5).*

Proof. Let $x \in (\Psi_S\overline{q})_\epsilon(\mathbb{F})$ and $x \in (\Psi_S\overline{q})_\delta(\mathbb{G})$. Then $x \in q_{S^{-1}(\epsilon)}(\mathbb{F})$ and $x \in q_{S^{-1}(\delta)}(\mathbb{G})$ and by (PL5) then $x \in q_{S^{-1}(\epsilon)*S^{-1}(\delta)}(\mathbb{F} \wedge \mathbb{G})$. By the definition of the t-norm $*$ it is easily verified that $S^{-1}(\epsilon) * S^{-1}(\delta) = S^{-1}(\epsilon + \delta)$ and hence $x \in q_{S^{-1}(\epsilon+\delta)}(\mathbb{F} \wedge \mathbb{G})$, which means $x \in (\Psi_S\overline{q})_{\epsilon+\delta}(\mathbb{F} \wedge \mathbb{G})$. $\square$

4.2. Lemma. *Let the strict t-norm $*$ have the additive generator S and let $(X, \overline{p}) \in |LTS|$. Then $(X, \Phi_S\overline{p})$ satisfies the axiom (PL5).*

Proof. Let $x \in (\Phi_S\overline{p})_\alpha(\mathbb{F})$ and $x \in (\Phi_S\overline{p})_\beta(\mathbb{G})$. Then $x \in p_{S(\alpha)}(\mathbb{F})$ and $x \in p_{S(\beta)}(\mathbb{G})$ and hence by (LT5) $x \in p_{S(\alpha)+S(\beta)}(\mathbb{F} \wedge \mathbb{G})$. By definition of the t-norm $*$ we see that $S(\alpha) + S(\beta) = S(\alpha * \beta)$ and hence $x \in p_{S(\alpha*\beta)}(\mathbb{F} \wedge \mathbb{G})$. But this means that $x \in (\Phi_S\overline{p})_{\alpha*\beta}(\mathbb{F} \wedge \mathbb{G})$. $\square$

4.3. Corollary. *For a strict t-norm $*$, the categories $PLIM^*$ and LTS are isomorphic.*

We conclude the following main result of this section.

4.4. Theorem. *For strict t-norms, all categories $PLIM^*$ are isomorphic.*

5. The case of nilpotent t-norms

We are now showing similar results for the class of nilpotent t-norms. To this end, we first introduce a subcategory of LTS.

For $\omega \in (0, \infty]$ we call $(X, \overline{p}) \in |LTS|$ an *ω-limit tower space* if the following strengthening of (LT4) is valid:

(LT4ω) $\quad p_\epsilon(\mathbb{F}) = X$ whenever $\omega \leq \epsilon$.

We see that a limit tower space is the same as an ∞-limit tower space. The subcategory of LTS with objects the ω-limit tower spaces is denoted by LTS_ω. It is not difficult to show that LTS_ω is a bireflective subcategory of LTS.

We consider now a nilpotent t-norm with additive generator S. We will show that $PLIM^*$ and $LTS_{S(0)}$ are isomorphic. To this end, we generalize the two functors of the previous section. For $(X, \overline{q}) \in |PLIM^*|$ we define $(\Psi\overline{q})_\epsilon = q_{S^{(-1)}(\epsilon)}$.

5.1. Lemma. *For $(X, \overline{q}) \in |PLIM^*|$ we have that $(X, \Psi\overline{q}) \in |LTS_{S(0)}|$.*

Proof. (LT1) and (LT2) are easy. For (LT3) we may assume $\epsilon \leq \delta < S(0)$. Then $S^{(-1)}(\epsilon) \geq S^{(-1)}(\delta)$ and hence $(\Psi\overline{q})_\epsilon = q_{S^{(-1)}(\epsilon)} \subseteq q_{S^{(-1)}(\delta)} = (\Psi\overline{q})_\delta$.
For (LT4$_{S(0)}$), let $\epsilon \geq S(0)$. Then $S^{(-1)}(\epsilon) = 0$ and hence $(\Psi\overline{q})_\epsilon(\mathbb{F}) = q_0(\mathbb{F}) = X$.
For (LT5), let $x \in (\Psi\overline{q})_\epsilon(\mathbb{F}) \cap (\Psi\overline{q})_\delta(\mathbb{G})$. If $\epsilon + \delta \geq S(0)$, then there is nothing to prove. If $\epsilon + \delta < S(0)$, then both $\epsilon, \delta < 0$ and hence $S^{(-1)}(\epsilon) * S^{(-1)}(\delta) = S^{(-1)}(\epsilon + \delta)$ and we conclude

$$(\Psi\overline{q})_\epsilon(\mathbb{F}) \cap (\Psi\overline{q})_\delta(\mathbb{G}) = q_{S^{(-1)}(\epsilon)}(\mathbb{F}) \cap q_{S^{(-1)}(\delta)}(\mathbb{G}) \subseteq q_{S^{(-1)}(\epsilon)*S^{(-1)}(\delta)}(\mathbb{F} \wedge \mathbb{G})$$

$$= q_{S^{(-1)}(\epsilon+\delta)}(\mathbb{F} \wedge \mathbb{G}) = (\Psi\overline{q})_{\epsilon+\delta}(\mathbb{F} \wedge \mathbb{G}).$$

We finally show (LTLC). If $\epsilon \geq S(0)$ then for $\delta > \epsilon$ we have $(\Psi\overline{q})_\delta(\mathbb{F}) = X$ and hence $(\Psi\overline{q})_\epsilon(\mathbb{F}) = X = \bigcap_{\delta>\epsilon}(\Psi\overline{q})_\delta(\mathbb{F})$. If $\epsilon < S(0)$ then by continuity and surjectivity of $S^{(-1)}$

and because $S^{(-1)}$ is strictly decreasing on $[0,S(0)]$, for $\beta < S^{(-1)}(\epsilon)$ there is a unique $\delta \in (\epsilon, S(0)]$ such that $\beta = S^{(-1)}(\delta)$. Hence

$$(\Psi\overline{q})_\epsilon(\mathbb{F}) = q_{S^{(-1)}(\epsilon)}(\mathbb{F}) = \bigcap_{\beta<S^{(-1)}(\epsilon)} q_\beta(\mathbb{F}) = \bigcap_{S^{(-1)}(\delta)<S^{(-1)}(\epsilon)} q_{S^{(-1)}(\delta)}(\mathbb{F}).$$

Now for $\epsilon < \delta \leq S(0)$, $S^{(-1)}(\delta) < S^{(-1)}(\epsilon)$ is equivalent to $\epsilon < \delta$ and hence we obtain

$$(\Psi\overline{q})_\epsilon(\mathbb{F}) = \bigcap_{\epsilon<\delta} q_{S^{(-1)}(\delta)}(\mathbb{F}) = \bigcap_{\epsilon<\delta} (\Psi\overline{q})_\delta(\mathbb{F}).$$

□

It follows easily from this that $\Psi : PLIM^* \longrightarrow LTS_{S(0)}$, $(X,\overline{q}) \longmapsto (X,\Psi\overline{q})$, $f \longmapsto f$ is a functor.

For $(X,\overline{p}) \in |LTS_{S(0)}|$ we define now $(\Phi\overline{p})_\alpha = p_{S(\alpha)}$.

5.2. Lemma. *For $(X,\overline{p}) \in |LTS_{S(0)}|$ we have that $(X,\Phi\overline{p}) \in |PLIM^*|$.*

Proof. (PL1) and (PL2) are again easy. (PL3) follows because S is order-reversing. For (PL4) we note that $(\Phi\overline{p})_0(\mathbb{F}) = p_{S(0)}(\mathbb{F}) = X$. For (PL5), we have

$$(\Phi\overline{p})_\alpha(\mathbb{F}) \cap (\Phi\overline{p})_\beta(\mathbb{G}) = p_{S(\alpha)}(\mathbb{F}) \cap p_{S(\beta)}(\mathbb{G}) \subseteq P_{S(\alpha)+S(\beta)}(\mathbb{F} \wedge \mathbb{G}).$$

By definition of the t-norm we have $S(\alpha * \beta) = S(S^{(-1)}(S(\alpha) + S(\beta)))$. We distinguish two cases. If $S(\alpha) + S(\beta) \leq S(0)$, then $S(\alpha * \beta) = S(\alpha) + S(\beta)$. Then

$$(\Phi\overline{p})_\alpha(\mathbb{F}) \cap (\Phi\overline{p})_\beta(\mathbb{G}) \subseteq p_{S(\alpha*\beta)}(\mathbb{F} \wedge \mathbb{G}) = (\Phi\overline{p})_{\alpha*\beta}(\mathbb{F} \wedge \mathbb{G}).$$

If $S(\alpha) + S(\beta) > S(0)$, then $S(\alpha * \beta) = S(0)$ and hence

$$(\Phi\overline{p})_\alpha(\mathbb{F}) \cap (\Phi\overline{p})_\beta(\mathbb{G}) \subseteq p_{S(\alpha)+S(\beta)}(\mathbb{F} \wedge \mathbb{G}) = X$$

$$= p_{S(0)}(\mathbb{F} \wedge \mathbb{G}) = p_{S(\alpha*\beta)}(\mathbb{F} \wedge \mathbb{G}) = (\Phi\overline{p})_{\alpha*\beta}(\mathbb{F} \wedge \mathbb{G}).$$

The axiom (PLLC) finally follows with similar arguments as the proof of (LTLC) in the previous Lemma. □

It follows easily from this that $\Phi : LTS_{S(0)} \longrightarrow PLIM^*$, $(X,\overline{p}) \longmapsto (X,\Phi\overline{p})$, $f \longmapsto f$ is a functor. Now we note that $(\Phi \circ \Psi\overline{q})_\alpha = q_{S^{(-1)}(S(\alpha))} = q_\alpha$. If $\epsilon \leq S(0)$, then $S(S^{(-1)}(\epsilon)) = \epsilon$ and hence $(\Psi \circ \Phi\overline{p})_\epsilon = p_{S(S^{(-1)}(\epsilon))} = p_\epsilon$. If $\epsilon > S(0)$ then trivially $(\Psi \circ \overline{p})_\epsilon = X = p_\epsilon$. Hence both functors, Ψ and Φ are isomorphism functors and we can state the following result.

5.3. Lemma. *$PLIM^*$ and $LTS_{S(0)}$ are isomorphic categories.*

As noted above, for a nilpotent t-norm, we can always assume that $S(0) = 1$ for an additive generator. Hence we obtain the following result.

5.4. Theorem. *For nilpotent t-norms, all categories $PLIM^*$ are isomorphic.*

6. Probabilistic Cauchy spaces, Cauchy tower spaces and approach Cauchy spaces

A *probabilistic Cauchy space under the t-norm* $*$ [14] is a pair $(X, \overline{C})$ of a set X and a non-empty family of subsets of $\mathbb{F}(X)$, $\overline{C} = (C_\alpha)_{\alpha\in[0,1]}$, that satisfies the following axioms.

(PC1) $[x] \in C_\alpha$ for all $x \in X$ and all $\alpha \in [0,1]$;

(PC2) $\mathbb{G} \in C_\alpha$ whenever $\mathbb{F} \in C_\alpha$ and $\mathbb{F} \leq \mathbb{G}$;

(PC3) $C_\beta \subseteq C_\alpha$ whenever $\alpha \leq \beta$;

(PC4) $C_0 = \mathbb{F}(X)$;

(PC5) $\mathbb{F} \wedge \mathbb{G} \in C_{\alpha*\beta}$ whenever $\mathbb{F} \in C_\alpha$, $\mathbb{G} \in C_\beta$ and $\mathbb{F} \vee \mathbb{G}$ exists;

(PCLC) $C_\alpha = \bigcap_{\beta<\alpha} C_\beta$.

A mapping $f : X \longrightarrow X'$ between two probabilistic Cauchy spaces under the t-norm $*$, $(X, \overline{C}), (X, \overline{C}')$, is called *Cauchy-continuous* if for all $\alpha \in [0,1]$ we have $f(C_\alpha) \subseteq C'_\alpha$. The category of probabilistic Cauchy spaces under the t-norm $*$ and Cauchy continuous mappings is denoted by $PChy^*$.

6.1. Lemma. *Let $(X, \overline{C})$ be a probabilistic Cauchy space under the t-norm $\wedge$. Then (PC5) is equivalent to the axiom*

(uPC5) $\mathbb{F} \wedge \mathbb{G} \in C_\alpha$ *whenever* $\mathbb{F} \in C_\alpha$ *and* $\mathbb{G} \in C_\alpha$ *and* $\mathbb{F} \vee \mathbb{G}$ *exists.*

Proof. If (PC5) is true, then we simply choose $\alpha = \beta$. If (uPC5) is true, then for $\mathbb{F} \in C_\alpha$ and $\mathbb{G} \in C_\beta$ we conclude with (PC3) that $\mathbb{F} \in C_{\alpha\wedge\beta}$ and $\mathbb{G} \in C_{\alpha\wedge\beta}$. Therefore, if $\mathbb{F} \vee \mathbb{G}$ exists, by (uPC5) then $\mathbb{F} \wedge \mathbb{G} \in C_{\alpha\wedge\beta}$. □

Therefore, probabilistic Cauchy spaces under the t-norm $\wedge$ are (left-continuous) *componentwise probabilistic Cauchy spaces* in the definition of [14]. The category $PChy^*$ is topological but it is not hereditary and quotients are not productive, not even for $* = \wedge$. However, $PChy^\wedge$ is Cartesian closed, see [14].

A *Cauchy tower space* is a pair $(X, \overline{D})$ of a set X and a non-empty family of subsets of $\mathbb{F}(X)$, $\overline{D} = (D_\epsilon)_{\epsilon\in[0,\infty]}$, that satisfies the following axioms.

(CT1) $[x] \in D_\epsilon$ for all $x \in X$ and all $\epsilon \in [0,\infty]$;

(CT2) $\mathbb{G} \in D_\epsilon$ whenever $\mathbb{F} \in D_\epsilon$ and $\mathbb{F} \leq \mathbb{G}$;

(CT3) $D_\epsilon \subseteq D_\delta$ whenever $\epsilon \leq \delta$;

(CT4) $D_\infty = \mathbb{F}(X)$;

(CT5) $\mathbb{F} \wedge \mathbb{G} \in D_{\epsilon+\delta}$ whenever $\mathbb{F} \in D_\epsilon$, $\mathbb{G} \in D_\delta$ and $\mathbb{F} \vee \mathbb{G}$ exists;

(CTLC) $D_\epsilon = \bigcap_{\epsilon<\delta} C_\delta$.

A mapping $f : X \longrightarrow X'$ between two Cauchy tower spaces, $(X, \overline{D}), (X, \overline{D}')$, is called *Cauchy-continuous* if for all $\epsilon \in [0, \infty]$ we have $f(D_\epsilon) \subseteq D'_\epsilon$. The category of Cauchy tower spaces and Cauchy continuous mappings is denoted by $ChyTS$.

If we replace the axiom (CT5) by the axiom

(uCT5) $\mathbb{F} \wedge \mathbb{G} \in D_{\epsilon\vee\delta}$ whenever $\mathbb{F} \in D_\epsilon$, $\mathbb{G} \in D_\delta$ and $\mathbb{F} \vee \mathbb{G}$ exists;

then we speak of a *ultra-Cauchy tower space*. The category of ultra-Cauchy tower spaces with continuous mappings as morphisms is denoted by $uChyTS$.

6.2. Lemma. *Let $(X, \overline{D})$ be an ultra-Cauchy tower space. Then (uCT5) is equivalent to the axiom*

(uCT5') $\mathbb{F} \wedge \mathbb{G} \in D_\epsilon$ whenever $\mathbb{F} \in D_\epsilon$ and $\mathbb{G} \in D_\epsilon$ and $\mathbb{F} \vee \mathbb{G}$ exists.

Proof. Similar to the proof of Lemma 6.1. □

We note that Cauchy tower spaces are defined in a different way in [13]. We define for $0 < \omega \leq \infty$ an *ω-Cauchy tower space* as a Cauchy tower space that satisfies the following strengthening of (CT4)

(CT4ω) $D_\epsilon = \mathbb{F}(X)$ whenever $\omega \leq \epsilon$.

We denote the subcategory of CTS with objects the ω-Cauchy tower spaces by CTS_ω. It is not difficult to prove that CTS_ω is bireflective in CTS.

An *approach Cauchy space* [11] is a pair (X, γ) of a set X and a mapping $\gamma : \mathbb{F}(X) \longrightarrow [0, \infty]$ that satisfies the following axioms.

(AChy1) $\gamma([x]) = 0$ for all $x \in X$;

(AChy2) $\gamma(\mathbb{G}) \leq \gamma(\mathbb{F})$ whenever $\mathbb{F} \leq \mathbb{G}$;

(AChy3) $\gamma(\mathbb{F} \wedge \mathbb{G}) \leq \gamma(\mathbb{F}) + \gamma(\mathbb{G})$ whenever $\mathbb{F} \vee \mathbb{G}$ exists.

A mapping $f : X \longrightarrow X'$ between two approach Cauchy spaces (X, γ), (X', γ') is called a *Cauchy contraction* if for all $\mathbb{F} \in \mathbb{F}(X)$ we have $\gamma'(f(\mathbb{F})) \leq \gamma(\mathbb{F})$

If we replace the axiom (AChy3) by the stronger axiom

(uAChy3) $\gamma(\mathbb{F} \wedge \mathbb{G}) \leq \gamma(\mathbb{F}) \vee \gamma(\mathbb{G})$ whenever $\mathbb{F} \vee \mathbb{G}$ exists;

then we call the pair (X, γ) an *ultra-approach Cauchy space*.

The category of approach Cauchy spaces with Cauchy contractions as morphisms is denoted by $AChy$, the subcategory of ultra-approach convergence spaces is denoted by $uAChy$. The category $uAChy$ is a bireflective subcategory of $AChy$. $AChy$ is topological and $uAChy$ is also Cartesian closed [11].

We can define isomorphism functors between the categories in a similar way as in the previous section. For $(X, \gamma) \in |AChy|$ we define the Cauchy tower $\sigma\gamma$ by

$$\mathbb{F} \in (\sigma\gamma)_\epsilon \iff \gamma(\mathbb{F}) \leq \epsilon.$$

For $(X, \overline{D}) \in |CTS|$ we define the mapping $\tau\overline{D} : \mathbb{F}(X) \longrightarrow [0, \infty]$ by

$$\tau\overline{D}(\mathbb{F}) = \bigwedge\{\epsilon \in [0, \infty] \ : \ \mathbb{F} \in D_\epsilon\}.$$

The following result is not difficult to prove.

6.3. Lemma. *(1)* $\sigma : AChy \longrightarrow CTS$, $(X, \gamma) \longmapsto (X, \sigma\gamma)$, $f \longmapsto f$ *is a functor.*
(2) $\tau : CTS \longrightarrow AChy$, $(X, \overline{D}) \longmapsto (X, \tau\overline{D})$, $f \longmapsto f$ *is a functor.*
(3) $\sigma \circ \tau = id_{CTS}$ *and* $\tau \circ \sigma = id_{AChy}$.
(4) $\sigma(uAChy) = uCTS$ *and* $\tau(uCTS) = uAChy$.

6.4. Corollary. *The categories* $AChy$ *and* CTS *are isomorphic and the categories* $uAChy$ *and* $uCTS$ *are isomorphic.*

We can also define isomorphism functors between the categories $PChy^*$ and $CTS_{S(0)}$ provided that the t-norm $*$ is continuous Archimedean with additive generator $S : [0, 1] \longrightarrow [0, \infty]$. For $(X, \overline{C}) \in |PChy^*|$ we define $\Gamma_S\overline{C}$ by

$$\mathbb{F} \in (\Gamma_S\overline{C})_\epsilon \quad \iff \quad \mathbb{F} \in C_{S^{(-1)}(\epsilon)},$$

and for $(X, \overline{D}) \in |CTS_{S(0)}|$ we define $\Delta_S\overline{D}$ by

$$\mathbb{F} \in (\Delta_S\overline{D})_\alpha \quad \iff \quad \mathbb{F} \in D_{S(\alpha)}.$$

The following result is then not difficult to prove.

6.5. Lemma. *(1)* $\Gamma_S : PChy^* \longrightarrow CTS_{S(0)}$, $(X, \overline{C}) \longmapsto (X, \Gamma_S\overline{C})$, $f \longmapsto f$ *is a functor.*
(2) $\Delta_S : CTS_{S(0)} \longrightarrow PChy^*$, $(X, \overline{D}) \longmapsto (X, \Delta_S\overline{D})$, $f \longmapsto f$ *is a functor.*
(3) $\Gamma_S \circ \Delta_S = id_{CTS_{S(0)}}$ *and* $\Delta_S \circ \Gamma_S = id_{PChy^*}$.
(4) $\Gamma_S(PChy^\wedge) = uCTS$ *and* $\Delta_S(uCTS) = PChy^\wedge$.

Noting that $CTS_\infty = CTS$ we can state the following results.

6.6. Corollary. *For a strict t-norm* $*$*, the categories* $PChy^*$ *and* CTS *are isomorphic. For a nilpotent t-norm* $*$*, the categories* $PChy^*$ *and* $CTS_{S(0)}$ *are isomorphic. Furthermore, the categories* $PChy^\wedge$ *and* $uCTS$ *are isomorphic.*

6.7. Theorem. *For strict t-norms, all categories* $PChy^*$ *are isomorphic. For nilpotent t-norms all categories* $PChy^*$ *are isomorphic.*

7. Probabilistic uniform limit spaces, uniform limit tower spaces and approach uniform limit spaces

A *probabilistic uniform limit space under the t-norm* $*$ [14] is a pair $(X, \overline{L})$ of a set X and a non-void family of subsets of $\mathbb{F}(X \times X)$, $\overline{L} = (L_\alpha)_{\alpha \in [0,1]}$ that satisfies the following axioms.

(PUL1) $[x] \times [x] \in L_\alpha$ for all $x \in X$ and all $\alpha \in [0,1]$;

(PUL2) $\mathbb{G} \in L_\alpha$ whenever $\mathbb{F} \leq \mathbb{G}$ and $\mathbb{F} \in L_\alpha$;

(PUL3) $L_\alpha \subseteq L_\beta$ whenever $\beta \leq \alpha$;

(PUL4) $L_0 = \mathbb{F}(X \times X)$;

(PUL5) $\mathbb{F} \wedge \mathbb{G} \in L_\alpha$ whenever $\mathbb{F}, \mathbb{G} \in L_\alpha$;

(PUL6) $\mathbb{F}^{-1} \in L_\alpha$ whenever $\mathbb{F} \in L_\alpha$;

(PUL7) $\mathbb{F} \circ \mathbb{G} \in L_{\alpha * \beta}$ whenever $\mathbb{F} \in L_\alpha$, $\mathbb{G} \in L_\beta$ and $\mathbb{F} \circ \mathbb{G}$ exists;

(PULLC) $L_\alpha = \bigcap_{\beta < \alpha} L_\beta$.

A mapping $f : X \longrightarrow X'$ between two probabilistic uniform limit spaces $(X, \overline{L})$ and $(X', \overline{L}')$ is called *uniformly continuous* if $(f \times f)(L_\alpha) \subseteq L'_\alpha$ for all $\alpha \in [0,1]$. The category of all probabilistic uniform limit spaces under the t-norm $*$ with uniformly continuous mappings as morphisms is denoted by $PULIM^*$.

7.1. Lemma. *Let $(X, \overline{L})$ be a probabilistic uniform limit space under the t-norm $\wedge$. Then (PUL7) is equivalent to the axiom*

(uPUL7) $\mathbb{F} \circ \mathbb{G} \in L_\alpha$ *whenever* $\mathbb{F} \in L_\alpha$ *and* $\mathbb{G} \in L_\alpha$ *and* $\mathbb{F} \circ \mathbb{G}$ *exists.*

Proof. Similar to the proof of Lemma 4.1. □

Therefore, probabilistic uniform limit spaces under the t-norm $\wedge$ are (left-continuous) *componentwise probabilistic uniform limit spaces* in the definition of [14]. The category $PULIM^*$ is topological and not hereditary and products of quotients are quotients. $PULIM^\wedge$ is Cartesian closed [14].

A *uniform limit tower space* [7] is a pair $(X, \overline{M})$ of a set X and a non-void family of subsets of $\mathbb{F}(X \times X)$, $\overline{M} = (M_\epsilon)_{\epsilon \in [0,\infty]}$ that satisfies the following axioms.

(ULT1) $[x] \times [x] \in M_\epsilon$ for all $x \in X$ and all $\epsilon \in [0, \infty]$;

(ULT2) $\mathbb{G} \in M_\epsilon$ whenever $\mathbb{F} \leq \mathbb{G}$ and $\mathbb{F} \in M_\epsilon$;

(ULT3) $M_\epsilon \subseteq M_\delta$ whenever $\epsilon \leq \delta$;

(ULT4) $M_\infty = \mathbb{F}(X \times X)$;

(ULT5) $\mathbb{F} \wedge \mathbb{G} \in_\epsilon$ whenever $\mathbb{F}, \mathbb{G} \in M_\epsilon$;

(ULT6) $\mathbb{F}^{-1} \in M_\epsilon$ whenever $\mathbb{F} \in M_\epsilon$;

(ULT7) $\mathbb{F} \circ \mathbb{G} \in M_{\epsilon+\delta}$ whenever $\mathbb{F} \in M_\epsilon$, $\mathbb{G} \in M_\delta$ and $\mathbb{F} \circ \mathbb{G}$ exists;

(ULTLC) $M_\epsilon = \bigcap_{\epsilon < \delta} M_\delta$.

A mapping $f : X \longrightarrow X'$ between two uniform limit tower spaces $(X, \overline{M})$ and $(X', \overline{M}')$ is called *uniformly continuous* if $(f \times f)(M_\epsilon) \subseteq M'_\epsilon$ for all $\epsilon \in [0, \infty]$. The category of all uniform limit tower spaces with uniformly continuous mappings as morphisms is denoted by $ULTS$.

If we replace the axiom (ULT6) by the axiom

(uULT6) $\mathbb{F} \circ \mathbb{G} \in M_{\epsilon \vee \delta}$ whenever $\mathbb{F} \in M_\epsilon$, $\mathbb{G} \in M_\delta$ and $\mathbb{F} \circ \mathbb{G}$ exists;

then we speak of a *ultra-uniform limit tower space*. The category of ultra-uniform limit tower spaces with uniformly continuous mappings as morphisms is denoted by $uULTS$.

7.2. Lemma. *Let $(X, \overline{M})$ be an ultra-uniform limit tower space. Then (uULT6) is equivalent to the axiom*

(uULT6') $\mathbb{F} \circ \mathbb{G} \in M_\epsilon$ whenever $\mathbb{F} \in M_\epsilon$ and $\mathbb{G} \in M_\epsilon$ and $\mathbb{F} \circ \mathbb{G}$ exists.

Proof. Similar to the proof of Lemma 7.1. □

We again define, for $0 < \omega \leq \infty$, an *ω-uniform limit tower space* $(X, \overline{M})$ as a uniform limit tower space that satisfies the following strengthening of the axiom (ULT4):

(ULT4ω) $M_\epsilon = \mathbb{F}(X \times X)$ whenever $\omega \leq \epsilon$.

The subcategory of $ULTS$ with objects the ω-uniform limit tower spaces is denoted by $ULTS_\omega$.

An *approach uniform limit space* [7] is a pair (X, η) of a set X and a mapping $\eta : \mathbb{F}(X \times X) \longrightarrow [0, \infty]$ that satisfies the following axioms.

(AULS1) $\eta([x] \times [x]) = 0$ for all $x \in X$;

(AULS2) $\eta(\mathbb{G}) \leq \eta(\mathbb{F})$ whenever $\mathbb{F} \leq \mathbb{G}$;

(AULS3) $\eta(\mathbb{F} \wedge \mathbb{G}) \leq \eta(\mathbb{F}) \vee \eta(\mathbb{G})$;

(AULS4) $\eta(\mathbb{F}^{-1}) = \eta(\mathbb{F})$;

(AULS5) $\eta(\mathbb{F} \circ \mathbb{G}) \leq \eta(\mathbb{F}) + \eta(\mathbb{G})$ whenever $\mathbb{F} \circ \mathbb{G}$ exists.

A mapping $f : X \longrightarrow X'$ between two approach uniform limit spaces (X, η), (X', η') is called a *uniform contraction* if for all $\mathbb{F} \in \mathbb{F}(X \times)$ we have $\eta'((f \times f)(\mathbb{F})) \leq \eta(\mathbb{F})$

If we replace the axiom (AULS5) by the stronger axiom

(uAULS5) $\eta(\mathbb{F} \circ \mathbb{G}) \leq \eta(\mathbb{F}) \vee \eta(\mathbb{G})$ whenever $\mathbb{F} \circ \mathbb{G}$ exists;

then we call the pair (X, η) an *ultra-approach uniform limit space*.

The category of approach uniform limit spaces with uniform contractions as morphisms is denoted by $AULS$, the subcategory of ultra-approach convergence spaces is denoted by $uAULS$. It is shown in [7] that $uAULS$ is a bireflective subcategory of $AULS$ and that it is a topological construct and is Cartesian closed. It is mentioned that $AULS$ is a topological construct.

We can again define isomorphism functors between these categories. Lee and Windels [7] mention the following. For $(X,\eta) \in |AULS|$ we define the uniform limit tower $\kappa\eta$ by

$$\mathbb{F} \in (\kappa\eta)_\epsilon \quad \Longleftrightarrow \quad \eta(\mathbb{F}) \leq \epsilon.$$

For $(X,\overline{M}) \in |ULTS|$ we define the approach uniform limit $\chi\overline{M} : \mathbb{F}(X \times X) \longrightarrow [0,\infty]$ by

$$\chi\overline{M}(\mathbb{F}) = \bigwedge\{\epsilon \in [0,\infty] \ : \ \mathbb{F} \in M_\epsilon\}.$$

This again gives rise to two isomorphism functors, $\kappa : AULS \longrightarrow ULTS$ and $\chi : ULTS \longrightarrow AULS$ and we obtain the following result.

7.3. Lemma. *(1)* $\kappa : AULS \longrightarrow ULTS,\ (X,\eta) \longmapsto (X,\kappa\eta),\ f \longmapsto f$ *is a functor.*
(2) $\chi : ULTS \longrightarrow AULS,\ (X,\overline{M}) \longmapsto (X,\chi\overline{M}),\ f \longmapsto f$ *is a functor.*
(3) $\kappa \circ \chi = id_{ULTS}$ *and* $\chi \circ \kappa = id_{AULS}$.
(4) $\kappa(uAULS) = uULTS$ *and* $\chi(uULTS) = uAULS$.

We obtain as a corollary the following theorem.

7.4. Theorem. *The categories $AULS$ and $ULTS$ are isomorphic and the categories $uAULS$ and $uULTS$ are isomorphic.*

Now, once again let the continuous Archimedean t-norm $*$ have the additive generator S. For $(X,\overline{L}) \in |PULIM^*|$ we define the $S(0)$-uniform limit tower $\Omega_S\overline{L}$ by

$$\mathbb{F} \in (\Omega_S\overline{L})_\epsilon \quad \Longleftrightarrow \quad \mathbb{F} \in L_{S^{(-1)}(\epsilon)}$$

and for $(X\overline{M}) \in |ULTS_{S(0)}|$ we define the probabilistic uniform limit structure $\Lambda_S\overline{M}$ by

$$\mathbb{F} \in (\Lambda_S\overline{M})_\alpha \quad \Longleftrightarrow \quad \mathbb{F} \in M_{S(\alpha)}.$$

This gives rise to two isomorphism functors and we can prove the following result.

7.5. Lemma. *(1)* $\Omega_S : PULIM^* \longrightarrow ULTS_{S(0)},\ (X,\overline{L}) \longmapsto (X,\Omega_S\overline{L}),\ f \longmapsto f$ *is a functor.*
(2) $\Lambda_S : ULTS_{S(0)} \longrightarrow PULIM^*,\ (X,\overline{M}) \longmapsto (X,\Lambda_S\overline{M}),\ f \longmapsto f$ *is a functor.*
(3) $\Omega_S \circ \Lambda_S = id_{ULTS_{S(0)}}$ *and* $\Lambda_S \circ \Omega_S = id_{PULIM^*}$.
(4) $\Omega_S(PULIM^\wedge) = uULTS$ *and* $\Lambda_S(uULTS) = PULIM^\wedge$.

Noting again that $ULTS_\infty = ULTS$ we obtain the following results.

7.6. Corollary. *For a strict t-norm $*$, the categories $PULIM^*$ and $ULTS$ are isomorphic. For a nilpotent t-norm $*$, the categories $PULIM^*$ and $ULTS_{S(0)}$ are isomorphic. Furthermore, the categories $PULIM^\wedge$ and $uULTS$ are isomorphic.*

7.7. Theorem. *For strict t-norms, all categories $PULIM^*$ are isomorphic. For nilpotent t-norms all categories $PULIM^*$ are isomorphic.*

8. Conclusions

We showed in this paper, that for certain classes of t-norms, all categories of probabilistic limit spaces under these t-norms are isomorphic. We could show this for the class of strict t-norms and for the class of nilpotent t-norms. This essentially means that it is sufficient to study "prototype spaces", i.e. it would be sufficient to study probabilistic limit spaces under the product t-norm (as a prototype for probabilistic limit spaces under strict t-norms) or probabilistic limit spaces under the Lukasiewics t-norm (as a prototype for probabilistic limit spaces under nilpotent t-norms). The proofs depend on the existence of an additive generator. It would be interesting to know if there are other classes of t-norms for which the categories of probabilistic limit spaces are isomorphic. It shall be further remarked that we considered only left-continuous probabilistic limit spaces. This restriction was used in order to accomodate approach limit spaces. The isomorphism functors between the categories of limit tower spaces and of probabilistic limit spaces, however, also work without imposing the left-continuity condition on the spaces.

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On coefficient estimates for a certain class of starlike functions

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Abstract

The purpose of this paper is to consider coefficient estimates in a class of functions $\mathcal{S}^*(q)$ consisting of analytic functions f normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{U}$ which satisfies the subordination condition that

$$zf'(z)/f(z) \prec q(z), \quad z \in \mathbb{U},$$

where $q(z) = \sqrt{1+z^2} + z$.

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\mathbb{U} = \{z : \ |z| < 1\}$ on the complex plane $\mathbb{C}$. Also, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ comprising of functions f normalized by $f(0) = 0$, $f'(0) = 1$, and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions which are univalent in $\mathbb{U}$. We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$ and $f(z) = g(\omega(z))$. In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(|z| < 1) \subset g(|z| < 1).$$

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Let a function f be analytic univalent in the unit disc $\mathbb{U} = \{z : \ |z| < 1\}$ on the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$, then f maps $\mathbb{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{1.1}$$

It is well known that if an analytic function f satisfies (1.1) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathbb{U}$.

A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. Let f be analytic and univalent in $\mathbb{U}_r = \{z : |z| < r \leq 1\}$. Then f maps $\mathbb{U}_r$ onto a convex domain E if and only if

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{U}_r).$$

If $r = 1$, then the function f is said to be convex in $\mathbb{U}$ (or briefly convex). The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{U}$ will be denoted by $\mathcal{S}^*$ and the set of all functions $f \in \mathcal{A}$ that are convex univalent in $\mathbb{U}$ by $\mathcal{K}$.

1. Definition. [8] Let $\mathcal{S}^*(q)$ denote the class of analytic functions f in the unit disc $\mathbb{U}$ normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition that

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1 + z^2} + z =: q(z), \quad z \in \mathbb{U}, \tag{1.2}$$

where the branch of the square root is chosen to be $q(0) = 1$.

We now mention some geometrical facts of curves defined in the open unit disk. For instance, the function $w(z) = \sqrt{1 + z}$ maps $\mathbb{U}$ onto a set bounded by a Bernoulli lemniscate, and a corresponding class of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec \sqrt{1 + z}$ was considered in [10], while the class generated by the subordination that $zf'(z)/f(z) \prec \sqrt{1 + cz}$ was considered in [1]. This way the known class of k-starlike functions was seen to be connected with certain conic domains. For some recent results for k-starlike functions, we refer to [11]. In recent papers [2, 3, 4, 5, 6],certain function classes were considered and were defined by means of the subordination that $zf'(z)/f(z) \prec \widehat{q}(z)$, where $\widehat{q}(z)$ was not univalent and this obviously made the consideration of certain geometric properties for such classes of functions much more difficult. It may be noted from (1.2) of Definition 1 that the set $q(\mathbb{U})$ lies in the right half-plane and it is not a starlike domain with respect to the origin, see Fig. 1 (below).

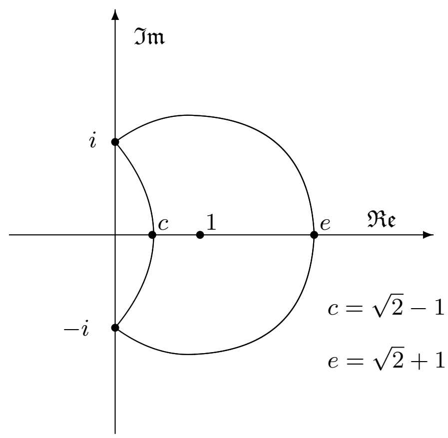

Fig. 1. The boundary of the set $q(\mathbb{U})$.

2. Main result

We first note the following result:

2.1. Lemma. [8] $\mathcal{S}^*(q) \subset \mathcal{S}^*$.

Therefore, if $f \in \mathcal{S}^*(q)$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}), \tag{2.1}$$

then $|a_n| \leq n$.

In this paper, we shall find estimations of first few coefficients of functions f of the form (2.1) belonging to $\mathcal{S}^*(q)$ and also consider the estimations of the familiar functionals like $|a_3 - \lambda a_2^2|$ and $|a_2 a_4 - a_3^2|$.

2.1. Theorem. Let the function f defined by (2.1) belong to the class $\mathcal{S}^*(q)$, then

$$|a_2| \leq 1, \quad |a_3| \leq 3/4, \quad |a_4| \leq 1/2. \tag{2.2}$$

Proof. Since the function f defined by (2.1) belongs to the class $\mathcal{S}^*(q)$, therefore from (1.2), we have

$$zf'(z) - \omega(z) f(z) = f(z)\sqrt{\omega^2(z) + 1}, \tag{2.3}$$

where ω is such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $|z| < 1$.

Let us denote the function $\omega(z)$ by

$$\omega(z) = \sum_{k=1}^{\infty} c_k z^k. \tag{2.4}$$

Then, (2.3) and (2.4) readily give

$$\sqrt{\omega^2(z) + 1} = 1 + \frac{1}{2} c_1^2 z^2 + c_1 c_2 z^3 + \left(c_1 c_3 + \frac{1}{2} c_2^2 - \frac{1}{8} c_1^2 \right) z^4 + \cdots \tag{2.5}$$

and

$$f(z)\sqrt{\omega^2(z) + 1} = z + a_2 z^2 + \left(\frac{1}{2} c_1^2 + a_3 \right) z^3 + \left(c_1 c_2 + \frac{1}{2} c_1^2 a_2 + a_4 \right) z^4 + \cdots . \tag{2.6}$$

Moreover,

(2.7)

$$zf'(z)-\omega(z)f(z) = z+(2a_2-c_1)z^2+(3a_3-c_1a_2-c_2)z^3+(4a_4-c_1a_3-c_2a_2-c_3)z^4+\cdots.$$

Equating now the second, third and fourth coefficients in (2.6) and (2.7), we have

(i) $a_2 = 2a_2 - c_1$,
(ii) $\frac{1}{2}c_1^2 + a_3 = 3a_3 - c_1a_2 - c_2$,
(iii) $c_1c_2 + \frac{1}{2}c_1^2a_2 + a_4 = 4a_4 - c_1a_3 - c_2a_2 - c_3$.

From (i), we get

$$a_2 = c_1. \tag{2.8}$$

It is well known that the coefficients of the bounded function $\omega(z)$ satisfies the inequality that $|c_k| \le 1$, so from (2.8), we have the first inequality that $|a_2| \le 1$. Now, from (ii), we have

$$\begin{aligned} |2a_3| &= \left|\frac{1}{2}c_1^2 + c_1a_2 + c_2\right| \\ &= \left|\frac{1}{2}c_1^2 + c_1^2 + c_2\right| \\ &= \left|c_2 + \frac{3}{2}c_1^2\right|. \end{aligned} \tag{2.9}$$

Using the estimate (see [7]) that if $\omega(z)$ has the form (2.4), then

$$|c_2 - \mu c_1^2| \le \max\{1, |\mu|\}, \quad \text{for all} \quad \mu \in \mathbb{C}, \tag{2.10}$$

and we obtain from (2.9) and (2.10) that

$$|2a_3| \le \frac{3}{2},$$

which gives the second inequality that $|a_3| \le 3/4$. Also, from $(i) - (iii)$, we find that

$$\begin{aligned} |3a_4| &= \left|c_1a_3 + c_2a_2 + c_3 + c_1c_2 + \frac{1}{2}c_1^2a_2\right| \\ &= \left|c_1\left(\frac{3}{4}c_1^2 + \frac{1}{2}c_2\right) + c_2c_1 + c_3 + c_1c_2 + \frac{1}{2}c_1^3\right| \\ &= \left|\frac{5}{4}c_1^3 + \frac{5}{2}c_1c_2 + c_3\right| \\ &= \left|\frac{5}{4}\left(c_1^3 + 2c_1c_2 + c_3\right) - \frac{1}{4}c_3\right| \\ &\le \left|\frac{5}{4}\left(c_1^3 + 2c_1c_2 + c_3\right)\right| + \left|\frac{1}{4}c_3\right| \\ &\le \left|\frac{5}{4}\left(c_1^3 + 2c_1c_2 + c_3\right)\right| + \frac{1}{4}. \end{aligned} \tag{2.11}$$

Next, we establish some properties of c_k involved in (2.4). It is known that the function $p(z)$ given by

$$\frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1z + p_2z^2 + \cdots =: p(z) \tag{2.12}$$

defines a Caratheodory function with the property that $\mathfrak{Re}\{p(z)\} > 0$ in $\mathbb{U}$ and that $|p_k| \le 2$ $(k = 1, 2, 3, \ldots)$. Equating of the coefficients in (2.12) yields that

$$p_2 = 2(c_1^2 + c_2)$$

and

$$p_3 = 2(c_1^3 + 2c_1c_2 + c_3).$$

Hence

(2.13) $\quad |c_1^2 + c_2| \le 1$

and

(2.14) $\quad |c_1^3 + 2c_1c_2 + c_3| \le 1.$

By applying (2.11) and (2.14), we find that

$$\begin{aligned} |3a_4| &\le \left|\frac{5}{4}\left(c_1^3 + 2c_1c_2 + c_3\right)\right| + \frac{1}{4} \\ &\le \frac{5}{4} + \frac{1}{4} \\ &= \frac{3}{2}, \end{aligned}$$

which gives the third inequality that $|a_4| \le 1/2$. □

2.2. Theorem. If the function defined by (2.1) belongs to the class $\mathcal{S}^*(q)$, then

(2.15) $\quad |a_3 - \lambda a_2^2| \le \max\{1/2, |\lambda - 3/4|\} \quad (\lambda \in \mathbb{C}).$

Furthermore, (2.15) is sharp.

Proof. Applying the notations used in the proof of Theorem 2.1, we obtain from (2.8) and (2.9) that

(2.16) $$|a_3 - \lambda a_2^2| = \left|\frac{1}{2}c_2 + \frac{3}{4}c_1^2 - \lambda c_1^2\right| = \left|\frac{1}{2}c_2 - \left(\lambda - \frac{3}{4}\right)c_1^2\right|.$$

In view of (2.10), we have then

$$\begin{aligned} |a_3 - \lambda a_2^2| &= \left|\frac{1}{2}c_2 - \left(\lambda - \frac{3}{4}\right)c_1^2\right| \\ &= \frac{1}{2}\left|c_2 - \left(\frac{4\lambda - 3}{2}\right)c_1^2\right| \\ &\le \frac{1}{2}\max\left\{1, \left|\frac{4\lambda - 3}{2}\right|\right\} \\ &= \max\{1/2, |\lambda - 3/4|\}. \end{aligned}$$

If

(2.17) $$\frac{zf_1'(z)}{f_1(z)} = q(z) = \sqrt{1+z^2} + z, \quad f_1(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then $f_1 \in \mathcal{S}^*(q)$. Moreover, we have from (2.17) that

(2.18) $\quad zf_1'(z) - zf_1(z) = f_1(z)\sqrt{1+z^2}.$

Hence,

$$z+(2b_2-1)z^2+(3b_3-b_2)z^3+\cdots = (z+b_2z^2+b_3z^3+\cdots)(1+z^2/2-z^4/8+z^6/16+\cdots).$$

Equating the coefficients of like powers of z, we obtain the following first few coefficients of the series involved in (2.18):

$$b_2 = 1,\ b_3 = 3/4,\ b_4 = 5/12.$$

Upon integrating (2.17), we can express the function $f_1(z)$ by

$$\begin{aligned} f_1(z) &= z\exp\int_0^z \frac{\sqrt{1+t^2}+t-1}{t}\mathrm{d}t \\ &= \frac{2\sqrt{1+z^2}-2}{z}\exp\left\{z-1+\sqrt{1+z^2}\right\} \\ &= z+z^2+\frac{3}{4}z^3+\frac{5}{12}z^4+\frac{2}{9}z^5+\cdots \quad (z\in\mathbb{U}). \end{aligned} \tag{2.19}$$

For the above function f_1, we have

$$\left|b_3-\lambda b_2^2\right| = |\lambda - 3/4|\,.$$

Next, if a function f_2 is such that

$$\frac{zf_2'(z)}{f_2(z)} = q(z^2) = \sqrt{1+z^4}+z^2, \tag{2.20}$$

then $f_2 \in \mathcal{S}^*(q)$ and

$$\begin{aligned} f_2(z) &= z\exp\int_0^z \frac{\sqrt{1+t^4}+t^2-1}{t}\mathrm{d}t \\ &= z+\frac{1}{2}z^3+\cdots \\ &= z+\sum_{n=2}^{\infty} d_n z^n \quad (z\in\mathbb{U}). \end{aligned} \tag{2.21}$$

Evidently, then for f_2 given by (2.21), we have

$$\left|d_3-\lambda d_2^2\right| = |1/2|.$$

$\square$

Conjecture. *Let $f\in\mathcal{S}^*(q)$ and $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. Then*

$$|a_n| \le |b_n|, \; n=2,3,4,\ldots, \tag{2.22}$$

where the coefficients b_n are those given in (2.17).

From (2.19), we have

$$b_2=1,\; b_3=3/4,\; b_4=5/12,\; b_5=2/9,\ldots,$$

and from (2.22), we get $|a_3|\le 3/4$, as is in Theorem 2.1. Also, for $n=4$, the inequality (2.22) gives $|a_4|\le 5/12 = 0.416\ldots$, while Theorem 2.1 gives $|a_4|\le 5/10$.

The second Hankel determinant for the function $f(z)=z+a_2z^2+a_3z^3+\cdots$ is given by $a_2a_4-a_3^2$. For more details and applications of this determinant, we refer to the recent paper [9]. We next find the second Hankel determinant estimation in the class $\mathcal{S}^*(q)$.

2.3. Theorem. If the function defined by (2.1) belongs to the class $\mathcal{S}^*(q)$, then

$$|a_2a_4-a_3^2| \le 39/48.$$

Proof. Applying the notations used in the proof of Theorem 2.1, we obtain the following relations from (2.8), (2.9) and (2.11):

$$a_2=c_1, \quad a_3=\frac{1}{2}\left(c_2+\frac{3}{2}c_1^2\right)$$

and

$$a_4=\frac{1}{3}\left(\frac{5}{4}c_1^3+\frac{5}{2}c_1c_2+c_3\right),$$

where c_k are coefficients of a Schwarz function. Therefore, we have

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{c_1}{3}\left(\frac{5}{4}c_1^3 + \frac{5}{2}c_1c_2 + c_3\right) - \frac{1}{4}\left(c_2 + \frac{3}{2}c_1^2\right)^2 \\ &= \frac{c_1}{3}\left(c_1^2 + 2c_1c_2 + c_3\right) - \frac{1}{4}\left(c_2 + c_1^2\right)^2 - \frac{1}{12}c_1^2\left(c_2 + c_1^2\right) - \frac{7}{48}c_1^4. \end{aligned}$$

From (2.13) and (2.14), we obtain that

$$\begin{aligned} & |a_2a_4 - a_3^2| \\ = & \left|\frac{c_1}{3}\left(c_1^3 + 2c_1c_2 + c_3\right) - \frac{1}{4}\left(c_2 + c_1^2\right)^2 - \frac{1}{12}c_1^2\left(c_2 + c_1^2\right) - \frac{7}{48}c_1^4\right| \\ \leq & \left|\frac{c_1}{3}\left(c_1^3 + 2c_1c_2 + c_3\right)\right| + \left|\frac{1}{4}\left(c_2 + c_1^2\right)^2\right| + \left|\frac{1}{12}c_1^2\left(c_2 + c_1^2\right)\right| + \left|\frac{7}{48}c_1^4\right| \\ \leq & \frac{1}{3} + \frac{1}{4} + \frac{1}{12} + \frac{7}{48} \\ = & 39/48. \end{aligned}$$

□

Remark. In view of (2.22), it may be observed that the value of the Hankel determinant $|b_2b_4 - b_3^2|$ is $7/48$ for the function f_1 (defined above by (2.17)).

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Applications of n-Gorenstein projective and injective modules

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Abstract

Over a commutative noetherian ring, we introduce a generalization of Gorenstein projective and injective modules, which we call, respectively, n-Gorenstein projective and injective modules. These last two classes of modules give us a new characterization of Gorenstein rings in terms of top local cohomology modules of flat modules. We also utilize the n-Gorenstein injective dimension to study an open question of Takahashi. Furthermore, we prove that a nonzero finite module with finite n-Gorenstein projective dimension satisfies the Auslander-Bridger formula.

Keywords: n-Gorenstein projective module; n-Gorenstein injective module; n-Gorenstein projective dimension; n-Gorenstein injective dimension.

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1. Introduction

Throughout this paper, R is a commutative noetherian ring with identity element, and all R-modules are unital. Also, for any R-module M, $Z(M)$ denotes the set of all zerodivisors of M.

When R is two-sided and noetherian, Auslander and Bridger [1] introduced the G-dimension for finitely generated modules. Several decades later, over a general ring R, Enochs and Jenda in [7] extended this homological dimension to Gorenstein projective dimension for arbitrary (non-finite) modules. Dually, they defined in [7] the Gorenstein injective dimension. The Gorenstein projective, injective dimension of a module is defined in terms of resolutions by Gorenstein projective, injective modules, respectively. Those modules are constructed from some special acyclic complexes. A complex of R-modules $\mathbf{A} = \cdots \to A_{i+1} \to A_i \to A_{i-1} \to A_{i-2} \to \cdots$ is *acyclic* if $\mathrm{H}(\mathbf{A}) = 0$.

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1.1. Definition. (1) An R-module M is said to be *Gorenstein projective*, if there exists an acyclic complex of projective modules $\mathbf{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(P_0 \to P_{-1})$ and such that the complex $\mathrm{Hom}_R(\mathbf{P}, Q)$ is acyclic whenever Q is projective.

(2) An R-module M is said to be *Gorenstein injective*, if there exists an acyclic complex of injective modules $\mathbf{I} = \cdots \to I_1 \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(I_0 \to I_{-1})$ and such that the complex $\mathrm{Hom}_R(E, \mathbf{I})$ is acyclic whenever E is injective.

Inspired by the definitions of Gorenstein projective/injective modules and sesqui-acyclic complexes in [13], the main idea of this paper is to introduce and study two new classes of modules called n-Gorenstein projective/injective modules and related homological dimensions.

The structure of the paper is summarized below. Section 2 is devoted to introducing the concept of n-Gorenstein projective (resp., injective) modules. We will find some useful properties of these classes of modules. Section 3 discusses the n-Gorenstein projective (resp., injective) dimension. We prove that a module of finite n-Gorenstein projective dimension can be approximated by a module, for which the corresponding classical homological dimension is finite. Section 4 consists of three applications. Theorem 4.2 states that if a local ring R admits a nonzero finitely generated R-module M with n-$\mathrm{Gid}_R M$ finite and $\dim_R M = \dim R$, then R is Cohen-Macaulay. This result in fact gives a partial answer to the following question of Takahashi: Is a local ring Cohen-Macaulay if it admits a nonzero finitely generated module of finite Gorenstein injective dimension? Theorem 4.3 can be regarded as the following expansion of a result of Yoshizawa (see [18, Theorem 2.6]). It shows that a complete Cohen-Macaulay local ring R of Krull dimension d is Gorenstein if and only if $\mathrm{H}^d_{\mathfrak{m}}(R)$ is n-Gorenstein injective for some positive integer n. The last Theorem 4.7, is in fact a generalized version of the Auslader-Bridge formula, which is proved by Lars in [4, Theorem 1.4.8]. However the method we use here is somewhat different from theirs.

Let $\mathcal{X}$ be a class of R-modules and M an R-module. Following [10], a *left* $\mathcal{X}$*-resolution* of M is an exact sequence of R-modules in $\mathcal{X}$ of the form $\mathbf{X} = \cdots \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$. Now let $\mathbf{X}$ be any left $\mathcal{X}$-resolution of M. We say that $\mathbf{X}$ is proper if the sequence $\mathrm{Hom}_R(Y, \mathbf{X})$ is exact for all $Y \in \mathbf{X}$. The $\mathcal{X}$*-projective dimension* of M is defined as $\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \mid \mathbf{X}$ is an $\mathcal{X}$-resolution of $M\}$. The *right* $\mathcal{X}$*-resolution*, *co-proper right* $\mathcal{X}$*-resolution* and $\mathcal{X}$*-injective dimension* are defined dually. We write $\mathcal{P}(R)$, $\mathcal{I}(R)$ for the classes of projective, injective R-modules, respectively. Following established conventions, we use abbreviations (pd, id) for the homological dimensions ($\mathcal{P}(R)$-pd, $\mathcal{I}(R)$-id). For each positive integer n, we denote $\mathcal{X}^{\perp_n} := \{M \mid \mathrm{Ext}^i_R(X, M) = 0$ for any $X \in \mathcal{X}$ and $1 \leq i \leq n\}$. Dually, we have the class ${}^{\perp_n}\mathcal{X}$.

2. n-Gorenstein projective and injective modules

We begin with the following

2.1. Definition. (1) Suppose that n is a positive integer, an R-module M is said to be *n-Gorenstein projective*, if there exists an acyclic complex of projective modules $\mathbf{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(P_0 \to P_{-1})$ and such that for any projective module Q the complex $\mathrm{Hom}_R(\mathbf{P}, Q) = \cdots \to P_1^* \to P_0^* \to P_{-1}^* \to P_{-2}^* \to \cdots$ is exact at P_i^* for all $i \geq -n$, where $P_i^* = \mathrm{Hom}_R(P_{-i}, Q)$. The class of n-Gorenstein projective modules is denoted by n-$\mathcal{GP}(R)$.

(2) Suppose that n is a positive integer, an R-module M is said to be *n-Gorenstein injective*, if there exists an acyclic complex of injective modules $\mathbf{I} = \cdots \to I_1 \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(I_0 \to I_{-1})$ and such that for any injective module

E the complex $\mathrm{Hom}_R(E, \mathbf{I}) = \cdots \to I_1^* \to I_0^* \to I_{-1}^* \to I_{-2}^* \to \cdots$ is exact at I_i^* for all $i \geq -n$, where $I_i^* = \mathrm{Hom}_R(E, I_i)$. The class of n-Gorenstein injective modules is denoted by n-$\mathcal{GI}(R)$.

Almost by the definitions we have that Gorenstein projective (resp., injective) modules are n-Gorenstein projective (resp., injective) modules. However, there are n-Gorenstein projective (resp., injective) modules which are not Gorenstein projective (resp., injective)(see Example 2.4 below).

2.2. Proposition. *Suppose that M is an R-module, and m, n are positive integers such that $m < n$, then the following statements hold.*

(1) *M is n-Gorenstein projective if and only if M belongs to the class ${}^{\perp_n}\mathcal{P}(R)$, and admits a co-proper right $\mathcal{P}(R)$-resolution.*
(2) *n-Gorenstein projective modules are m-Gorenstein projective modules.*
(3) *$\mathcal{GP}(R) = \bigcap_{n=1}^{\infty} n$-$\mathcal{GP}(R)$.*
(4) *If M is n-Gorenstein projective, then there is an exact sequence $0 \to M \to P \to G \to 0$ such that P is projective and G is $(n+1)$-Gorenstein projective.*

Proof. (1). Since every module has a left $\mathcal{P}(R)$-resolution, the statement is clear from the definition of n-Gorenstein projective modules.

(2) follows immediately from (1).

(3) is true by [10, Proposition 2.3].

(4). Also by the definition of n-Gorenstein projective modules. □

Next we discuss some connections between n-Gorenstein projective modules and n-Gorenstein injective modules.

2.3. Proposition. *Let M and E be R-modules. Then the following statements hold.*

(1) *If M is a finitely generated n-Gorenstein projective module and E is injective, then $\mathrm{Hom}_R(M, E)$ is n-Gorenstein injective.*
(2) *If R is a complete local ring with the residue field k, M is an artinian n-Gorenstein injective module, then $\mathrm{Hom}_R(M, E(k))$ is n-Gorenstein projective.*
(3) *If M is finitely generated, then M is n-Gorenstein projective module if and only if M is reflexive, $\mathrm{Ext}_R^i(M, R) = 0$ for $1 \leq i \leq n$, and $\mathrm{Ext}_R^i(M^*, R) = 0$ for $i \geq 1$, where $M^* = \mathrm{Hom}_R(M, R)$.*

Proof. (1). Since M is n-Gorenstein projective and finitely generated, there exists an exact sequence $0 \to M \to P_{-1} \to L \to 0$ such that P_{-1} is a finitely generated projective module and such that the exact sequence remains exact after applying $\mathrm{Hom}_R(-, \mathcal{P}(R))$ to it. Hence we obtain that $\mathrm{Ext}_R^1(L, Q) = 0$ for all projective modules Q. Since M is n-Gorenstein projective, there also exists an exact sequence $0 \to M \to P \to G' \to 0$ with

$P \in \mathcal{P}(R)$, $G' \in n$-$\mathcal{GP}(R)$. Consider the following pushout diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & P_{-1} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & L \arrow[r] & 0 \\
 & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

So L is n-Gorenstein projective by Proposition 2.6 and Corollary 2.7 below. Continuing in this way gives a co-proper right $\mathcal{P}(R)$-resolution $\mathbf{P} = 0 \to M \to P_{-1} \to P_{-2} \to \cdots$ with each term finitely generated. Because R is commutative, for each injective R-module I, $\mathrm{Hom}_R(I, \mathrm{Hom}_R(\mathbf{P},\ E)) \cong \mathrm{Hom}_R(I \otimes_R \mathbf{P},\ E) \cong \mathrm{Hom}_R(\mathbf{P},\ \mathrm{Hom}_R(I, E))$. Note that $\mathrm{Hom}_R(I, E)$ is flat by [8, Theorem 3.2.16] and every flat module is a direct limit of finitely generated projective modules. Hence $\mathrm{Hom}_R(\mathbf{P}, E)$ becomes a proper left $\mathcal{I}(R)$-resolution of $\mathrm{Hom}_R(M, E)$ by [9, Lemma 3.1.6]. For each $i \geq 1$, we have $\mathrm{Ext}^i_R(I, \mathrm{Hom}_R(M, E)) \cong \mathrm{Ext}^i_R(\mathrm{Tor}^R_i(I, M), E) \cong \mathrm{Ext}^i_R(M, \mathrm{Hom}_R(I, E))$ by [8, Theorem 3.2.1]. Therefore $\mathrm{Hom}_R(M, E)$ is n-Gorenstein injective by a dual statement of Proposition 2.2(1) and [9, Lemma 3.1.6].

(2) is proved in a fashion similar to [6, Theorem 4.8].

(3) is easy by the definition of n-Gorenstein projective modules.

$\square$

2.4. Example. *Let R be the local artinian ring in* [14]*, there exists a family $\{M_s\}_{s \geq 1}$ of reflexive modules such that*

(1) $\mathrm{Ext}^i_R(M_s, R) = 0$ if and only if $1 \leq i \leq s-1$.

(2) $\mathrm{Ext}^i_R(M_s^*, R) = 0$ for all $i > 0$.

Then the following statements hold for any $s > 1$.

(a) M_s is $(s-1)$-Gorenstein projective but not s-Gorenstein projective.

(b) $\mathrm{Hom}_R(M_s, E(k))$ is $(s-1)$-Gorenstein injective but not s-Gorenstein injective.

Proof. (a) is easy since the reflexive module M_s satisfies conditions (1) and (2).

(b). Since M_s is (s-1)-Gorenstein projective by (a), it is deduced from Proposition 2.3(1) that $\mathrm{Hom}_R(M_s, E(k))$ is (s-1)-Gorenstein injective. Now suppose that $\mathrm{Hom}_R(M_s, E(k))$ is s-Gorenstein injective. Indeed, since artinian local rings are complete and $M_s \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M_s, E(k)), E(k))$, the Matlis duality between noetherian modules and artinian modules implies that M_s must be s-Gorenstein projective. It is impossible. So $\mathrm{Hom}_R(M_s, E(k))$ is not s-Gorenstein injective for $s > 1$. $\square$

2.5. Remark. From the construction of M_s (see [14]), we know that there is an exact sequence $0 \to M_s \to R^2 \to M_{s+1} \to 0$ such that M_{s+1} is s-Gorenstein projective, M_s is (s-1)-Gorenstein projective but not s-Gorenstein projective. This implies that n-$\mathcal{GP}(R)$ is not closed under kernels of epimorphisms. Hence n-$\mathcal{GP}(R)$ is not a projectively resolving class (see [10, 1.1]).

The next proposition provides ways to create n-Gorenstein projective modules.

2.6. Proposition. *n-$\mathcal{GP}(R)$ is closed under direct sums and extensions.*

Proof. By [10, Proposition 1.6], [15, Proposition 7.21] and Proposition 2.2(1), n-$\mathcal{GP}(R)$ is closed under direct sums. It follows from [10, 1.7] that n-$\mathcal{GP}(R)$ is closed under extensions. □

Although n-$\mathcal{GP}(R)$ is not projectively resolving, we may show that the class of n-Gorenstein projective modules is closed under direct summands without using Eilenberg's technique (see [10, Proposition 1.4]). This technique is applied by Holm in [10, Theorem 2.5] to show that the class of Gorenstein projective modules is closed under direct summands.

2.7. Corollary. *n-$\mathcal{GP}(R)$ is closed under direct summands.*

Proof. Let G be an n-Gorenstein projective module and H be a direct summand of G. Since $G \in {}^{\perp_n}\mathcal{P}(R)$, $H \in {}^{\perp_n}\mathcal{P}(R)$. By [12, Theorem 4.6(1)], one sees that H has a co-proper right $\mathcal{P}(R)$-resolution. Now Proposition 2.2(1) gives the result. □

3. n-Gorenstein projective and injective dimensions

In this section, we turn to studying n-Gorenstein projective and injective dimensions. For an R-module M, we use $n\text{-Gpd}_R M$ (resp., $n\text{-Gid}_R M$) to denote the n-Gorenstein projective (resp., injective) dimension of M. Holm in [10, Theorem 2.10] showed that an R-module M with Gorenstein projective dimension n admits such an exact sequence $0 \to K \to G \to M \to 0$ with G Gorenstein projective and $\text{pd}_R K = n-1$. The proof of this result depends on the fact that for an R-module M with Gorenstein projective dimension n every projective resolution of M has its nth syzygy Gorenstein projective. However we don't know weather the same fact is true for modules with finite n-Gorenstein projective dimension. But by showing in a different way we still have the following result which is similar to that of Holm.

3.1. Proposition. *Let M be an R-module with finite n-Gorenstein projective dimension m. Then there exists an exact sequence $0 \to K \to G \to M \to 0$, where G is n-Gorenstein projective and $pd_R K = m-1$.*

Proof. If $m = 0$, our claim clearly holds. If $m = 1$, we have an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ with $G_0, G_1 \in n$-$\mathcal{GP}(R)$. Since G_1 is n-Gorenstein projective, there also exists an exact sequence $0 \to G_1 \to P \to G' \to 0$ with $P \in \mathcal{P}(R)$, $G' \in n$-$\mathcal{GP}(R)$. Consider the following pushout diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G_1 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

According to Proposition 2.6, G is n-Gorenstein projective. Therefore $0 \to P \to G \to M \to 0$ is the desired sequence. Now suppose $m > 1$. We will use induction to show this statement. Since $n\text{-Gpd}_R M = m$, we have such an exact sequence $0 \to K \to G_0 \to M \to 0$ with $G_0 \in n\text{-}\mathcal{GP}(R)$ and $n\text{-Gpd}_F K = m-1$. Hence there exists an exact sequence $0 \to K_1 \to G_1 \to K \to 0$ with $\text{pd}_F K_1 = m-2$ and $G_1 \in n\text{-}\mathcal{GP}(R)$ by induction. Observe that G_1 is n-Gorenstein projective, there is an exact sequence $0 \to G_1 \to P' \to G_2 \to 0$ with $P' \in \mathcal{P}(R)$, $G_2 \in n\text{-}\mathcal{GP}(R)$. Similarly, consider the following pushout diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
\varepsilon_1 : 0 \arrow[r] & K_1 \arrow[r] \arrow[d, equal] & G_1 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
\varepsilon_2 : 0 \arrow[r] & K_1 \arrow[r] & P' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & & G_2 \arrow[r, equal] \arrow[d] & G_2 \arrow[d] & & \\
 & & 0 & 0 & &
\end{tikzcd}$$

Since G_0, $G_2 \in n\text{-}\mathcal{GP}(R)$, $G \in n\text{-}\mathcal{GP}(R)$. Let $K = \text{Ker}(G \to M)$, $0 \to K \to G \to M \to 0$ is the desired sequence. □

A complement to Proposition 2.6 is given below.

3.2. Corollary. *Let $0 \to G' \to G \to M \to 0$ be an exact sequence, where G and G' are n-Gorenstein projective and $\text{Ext}^1_R(M, Q) = 0$ for all projective modules Q. Then M is n-Gorenstein projective.*

If M is n-Gorenstein projective, $\text{Ext}^i_R(M, L) = 0$ for all $i \geq 1$ and all modules L with finite injective dimension. If N is n-Gorenstein injective, $\text{Ext}^i_R(H,\text{N}) = 0$ for all $i \geq 1$ and all modules H with finite projective dimension. Using this fact, we may provide the following results. Their proofs are similar to those in [11, Theorem 2.1, 2.2].

3.3. Corollary. *Let M be an R-module, then:*
(1) *If $pd_R M < \infty$, n-$Gid_R M = id_R M$.*
(2) *If $id_R M < \infty$, n-$Gpd_R M = pd_R M$.*

Using Propositions 2.2 and 3.1, we get the following result which is analogous to [5, Lemma 2.17].

3.4. Proposition. *Let M be an R-module with n-$Gpd_R M < \infty$. There is an exact sequence $0 \to M \to H \to A \to 0$, where A is n-Gorenstein projective and $pd_R H = n$-$Gpd_R M$.*

4. Applications

For an R-module M, the *ith local cohomology module* of M with respect to an ideal I is defined as $\text{H}^i_I(M) = \varinjlim_n \text{Ext}^i_R(R/I^n, M)$. Let $(R, \mathfrak{m}, k)$ be a local ring. We say that R is *Cohen-Macaulay* if $\text{depth}R = \dim R$. R is *Gorenstein* if it has finite self-injective dimension.

A careful reading of the proof in [17, Lemma 1.1], combined with some basic facts about the n-Gorenstein injective dimension, gives the following vanishing result of local cohomology modules.

4.1. Lemma. *Let M be an R-module with $n\text{-Gid}_R M$ finite, and let I be a nonzero ideal of R. Then $\mathrm{H}_I^i(M) = 0$ for all $i > n\text{-Gid}_R M$.*

With the aid of Lemma 4.1, We are now in a position to give one of the main results in this paper, which partially answers a question of Takahashi in [16]. A similar result was proved by Yassemi([17, Theorem 1.3]).

4.2. Theorem. *Let $(R, \mathfrak{m}, k)$ be a local ring. If R admits a nonzero finitely generated R-module M with $n\text{-Gid}_R M$ finite and $\dim_R M = \dim R$, then R is Cohen-Macaulay.*

Proof. Since $n\text{-Gid}_R M$ is finite, by Lemma 4.1, $\mathrm{H}_{\mathfrak{m}}^i(M) = 0$ for all $i > n\text{-Gid}_R M$. On the other hand, we have $\mathrm{H}_{\mathfrak{m}}^{dim_R M}(M) \neq 0$ by [2, Theorem 7.3.2]. Hence $\dim_R M \leqslant n\text{-Gid}_R M$ follows. Also by a dual argument of Proposition 3.4, the finiteness of n-Gorenstein injective dimension of M means that there is an R-module L such that $\mathrm{id}_R L = n\text{-Gid}_R M$. But $\mathrm{id}_R L = \sup\{\mathrm{depth} R_{\mathfrak{p}} - \mathrm{width}_{R_{\mathfrak{p}}} L_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}(L)\}$ by [3, 3.1]. Thus $\dim R = \dim_R M$ $\leq n\text{-Gid}_R M = \mathrm{id}_R L \leq \mathrm{depth} R_{\mathfrak{p}} \leq \dim R_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$. Hence $\mathfrak{p}$ must be the maximal ideal $\mathfrak{m}$. So we get that $\mathrm{depth} R = \dim R$, and R is Cohen-Macaulay. □

Enochs and Jenda in [8, Corollary 9.5.13] proved that a local ring R is Gorenstein if and only if R is Cohen-Macaulay and the top local cohomology module of R, $\mathrm{H}_{\mathfrak{m}}^{dim R}(R)$, is isomorphic to $E(k)$. Recently, Yoshizawa in [18] generalized this result. It says that a complete Cohen-Macaulay local ring R of krull dimension d is Gorenstein if and only if the top local cohomology module $\mathrm{H}_{\mathfrak{m}}^{dim R}(R)$ is Gorenstein injective. Motivated by these established facts, an extended version of this result is presented as follows.

4.3. Theorem. *Let R be a complete Cohen-Macaulay local ring of krull dimension d, then the following statements are equivalent.*

(1) *R is Gorenstein.*
(2) *For any positive integer n, $H_{\mathfrak{m}}^d(F)$ is n-Gorenstein injective for every flat R-module F.*
(3) *For some positive integer n, $H_{\mathfrak{m}}^d(F)$ is n-Gorenstein injective for every flat R-module F.*
(4) *For some positive integer n, $H_{\mathfrak{m}}^d(R)$ is n-Gorenstein injective.*

Proof. (1) ⇒ (2). Since flat modules are direct limits of finitely generated free modules and the local cohomology functors commute with direct limits, the result follows from [18, Theorem 2.6].

(2) ⇒ (3) and (3) ⇒ (4) are trivial.

(4) ⇒ (1). By assumption, $\mathrm{H}_{\mathfrak{m}}^d(R)$ is n-Gorenstein injective for some positive integer n. Since $\mathrm{H}_{\mathfrak{m}}^d(R)$ is artinian, by Proposition 2.3, $n\text{-Gpd}_R(\mathrm{Hom}_R(\mathrm{H}_{\mathfrak{m}}^d(R), E(k))) = 0$. By [8, Theorem 9.5.16], we know that $\mathrm{Hom}_R(\mathrm{H}_{\mathfrak{m}}^d(R), E(k))$ is a dualizing module. Now Corollary 3.3 says that $\mathrm{Hom}_R(\mathrm{H}_{\mathfrak{m}}^d(R), E(k))$ is projective. Note that projective modules over local rings are free. Hence $\mathrm{Hom}_R(\mathrm{H}_{\mathfrak{m}}^d(R), E(k)) \cong R^n$. Thus R is Gorenstein. □

Now it is natural to ask what can we say about R when the top local cohomology module has finite n-Gorenstein injective dimension?

4.4. Corollary. *The following statements are equivalent for a commutative artinian local ring R.*

(1) *R is quasi-Frobenius.*
(2) *For any positive integer n, $H_{\mathfrak{m}}^0(R)$ is n-Gorenstein injective.*

(3) *For some positive integer n, all modules are n-Gorenstein injective.*
(4) *For some positive integer n, all modules are n-Gorenstein projective.*

Proof. (1) $\Leftrightarrow$ (2) is easy directly by Theorem 4.3.

(1) $\Rightarrow$ (4). Because all n-Gorenstein projective modules are Gorenstein projective over Gorenstein rings, it is a consequence of [8, Exercise 10.3.5].

(4) $\Rightarrow$ (1). Since all modules are n-Gorenstein projective, all injective modules are also n-Gorenstein projective. Thus every injective module can be embedded into a projective module. So every injective module is projective. It means that R is quasi-Frobenius.

(1) $\Leftrightarrow$ (3) can be shown dually. □

Finally we aim at investigating some applications of n-Gorenstein projective dimensions among the category of all finitely generated R-modules. For convenience, all modules appeared below are finitely generated.

4.5. Lemma. *Let M be an R-module, and let x be an M- and R-regular element. If M is n-Gorenstein projective, then M/xM is n-Gorenstein projective $R/(x)$-module.*

Proof. Set $\bar{M} = M/xM$ and $\bar{R} = R/(x)$. Since M is n-Gorenstein projective, there is an exact complex of free modules $\mathbf{F} = \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} F_{-1} \xrightarrow{d_{-1}} F_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(F_0 \to F_{-1})$ and such that the complex $\mathbf{F}^* = \cdots \to F_1^* \to F_0^* \to F_{-1}^* \to F_{-2}^* \to \cdots$ is exact at F_i^* for all $i \geq -n$, where $F_i^* = \mathrm{Hom}_R(F_{-i},\, R)$. Let $M_i = \mathrm{Im}d_i$ for each i. Since x is M_i-regular for each i, by [4, Lemma 1.3.4(a)], applying $- \otimes_R \bar{R}$ to the exact complex $\mathbf{F}$ gives an exact complex $\bar{\mathbf{F}} = \cdots \to \bar{F}_1 \to \bar{F}_0 \to \bar{F}_{-1} \to \bar{F}_{-2} \to \cdots$. Again by [4, Lemma 1.3.4(a)], there is a complex $\cdots \to \overline{F_1^*} \to \overline{F_0^*} \to \overline{F_{-1}^*} \to \overline{F_{-2}^*} \to \cdots$, which is exact at $\overline{F_i^*}$ for all $i \geq -n$. It is deduced from [4, Lemma 1.3.4(b)] that $\mathrm{Hom}_{\bar{R}}(\bar{F}_{-i}, \bar{R}) \cong \overline{F_i^*}$. Therefore $\bar{M}$ is n-Gorenstein projective $\bar{R}$-module. □

4.6. Proposition. *If R is a local ring, and suppose that there exists an exact sequence $0 \to M \to N \to H \to 0$ with n-$Gpd_RM \leq pd_RN < \infty$ and H n-Gorenstein projective. Then $depth_RM = depth_RN$.*

Proof. Case 1: depth$R$=0. Since $\mathrm{pd}_RN < \infty$, we infer by the Auslander-Buchsbaum formula that N is projective and $\mathrm{depth}_RN = 0$, and hence M is n-Gorenstein projective. Now assume that $\mathrm{depth}_RM \geq 1$, so there is a regular element x of M. Note that M is n-Gorenstein projective. Applying $\mathrm{Hom}_R(-, R)$ to the exact sequence $0 \to M \xrightarrow{x} M \to \bar{M} \to 0$ yields an exact sequence $0 \to \bar{M}^* \to M^* \xrightarrow{x} M^* \to \mathrm{Ext}^1_R(\bar{M}, R) \to 0$. Applying $\mathrm{Hom}_R(-, R)$ to this exact sequence gives the following exact sequence $0 \to \mathrm{Ext}^1_R(\bar{M}, R)^* \to M^{**} \xrightarrow{x} M^{**}$. As M is reflexive by Proposition 2.3(3), $\mathrm{Ext}^1_R(\bar{M}, R)^* = 0$. We get from the formula $\mathrm{Ass}(\mathrm{Ext}^1_R(\bar{M}, R)^*) = \mathrm{Supp}(\mathrm{Ext}^1_R(\bar{M}, R)) \cap \mathrm{Ass}R$ that $\mathrm{Ext}^1_R(\bar{M}, R) = 0$. Therefore $M^* = xM^*$. It follows from the Nakayama's lemma that $M^* = 0$. Hence $M = 0$, which is a contradiction.

Case 2: depth$R \geq 1$. Because M is a submodule of N, $\mathrm{depth}_RM = 0$ implies that $\mathrm{depth}_RN = 0$. So we may assume that $\mathrm{depth}_RM \geq 1$. Since depth$R \geq 1$, there must be an element $x \in R \backslash Z(M) \cup Z(R)$. Moreover x is H-regular element as H can be embedded into a free module. By Lemma 4.5 and [4, Lemma 1.3.4(a)], tensoring the exact sequence $0 \to M \to N \to H \to 0$ gives an exact sequence $0 \to \bar{M} \to \bar{N} \to \bar{H} \to 0$ such that $\bar{H}$ is n-Gorenstein projective and n-$\mathrm{Gpd}_{\bar{R}}\bar{M} \leq n$-$\mathrm{Gpd}_RM$. Note that $\mathrm{pd}_{\bar{R}}(\bar{N}) = \mathrm{pd}_RN$. We have that n-$\mathrm{Gpd}_{\bar{R}}\bar{M} \leq \mathrm{pd}_{\bar{R}}(\bar{N})$. Hence, as depth$\bar{R}$ = depth$R$ -1, we obtain, by induction on depth$R$, $\mathrm{depth}_{\bar{R}}\bar{M} = \mathrm{depth}_{\bar{R}}\bar{N}$. Therefore $\mathrm{depth}_RM = \mathrm{depth}_RN$. □

4.7. Theorem. *If R is a local ring, M is a nonzero R-module with n-Gpd_RM finite. Then n-$Gpd_RM + depth_RM = depthR$.*

Proof. There exists an exact sequence $0 \to M \to H \to A \to 0$ with n-$\mathrm{Gpd}_R M = \mathrm{pd}_R H$ and A n-Gorenstein projective by Proposition 3.4. Then by Proposition 4.6 we have that $\mathrm{depth}_R M = \mathrm{depth}_R H$. But we know from the Auslander-Buchsbaum formula that $\mathrm{pd}_R H + \mathrm{depth}_R H = \mathrm{depth} R$. Hence we get the result. $\square$

We end this section with the following corollary.

4.8. Corollary. *If R is a Gorenstein local ring and M is a nonzero R-module. Then $Gpd_R M = n$-$Gpd_R M$.*

Proof. Since R is Gorenstein, $\mathrm{Gpd}_R M < \infty$ by [8, Theorem 12.3.1]. Furthermore we have $\mathrm{Gpd}_R M = \mathrm{depth} R - \mathrm{depth}_R M$ by [4, Theorem 1.4.8]. But n-$\mathrm{Gpd}_R M < \mathrm{Gpd}_R M$, it now follows from Theorem 4.7 that n-$\mathrm{Gpd}_R M = \mathrm{depth} R - \mathrm{depth}_R M$. Hence the result is true. $\square$

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Effect of hall current on the MHD fluid flow and heat transfer due to a rotating disk with uniform radial electric field

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Abstract

In this paper the steady Von Kármán flow of incompressible fluid in which the Hall effect exists is analyzed over the infinite rotating disk with additional assumptions: the uniform magnetic field applied normally to the disk and the radial electric field imposed to the disk. Therefore, the stability equations and energy equation have been modified in the presence of Hall effect, uniform magnetic field and radial electric field. The system of equations generated by stability and energy equations has been solved using Chebyshev collocation technique for varying values of Hall parameters, magnetic interaction and radial electric parameters. Accuracy of the method is verified comparing results in the literature. Effects of parameters are depicted graphically and are analyzed.

Keywords: Incompressible Flow, Rotating-Disk flow, Magnetic field, Electric field, Hall effect.

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1. Introduction

Rotating disk flow has been extensively studied in the literature. An interesting problem from both engineering and mathematical point of view has been investigated for the last half of the century using experimental, analytical and numerical means. Rotating disk flows are important in many applications such as turbomachinery, oceanography, computer storage devices, nuclear reactors, lubrication, and so on.

Von Kármán [13] has carried out the pioneering study of fluid flow, triggered further studies, many explanations are initiated on infinite rotating disk. Cochran [7] and Benton [5] have considered by Kármán [13], they investigated the steady motion of an incompressible viscous fluid. The effect of uniform magnetic field on the flow over a rotating infinite disk has been studied by many researchers [8], [12], [19], [20], [22], [23], [24], [25]. Hall effect has been taken into consideration in some of the works in the literature. To the best of our knowledge Attia[2] has initiated in his studies examining Hall effect on the flow over a infinite rotating disk. The study has been followed by Attia & Aboul-Hassan[1], and Siddiqui, Rana & Naseer[19]. The case without Hall effect on the rotating infinite disk has been investigated [4], [6], [8], [9], [12], [14], [15], [16], [19], [20], [21], [22], [23], [24], [25].

Millsaps & Pohlhausen[15] have considered the heat transfer problem on the rotating infinite disk. After their work, the heat transfer on a flat plate was analyzed by Sparrow

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& Gregg[21] for Prandtl numbers. Sparrow & Cess[20], Riley[16], Kumar & Thacker & Watson[14] studied the effects of magnetic field to the heat transfer over a infinite rotating disk. Finally, effects of the uniform radial electric field on the MHD heat and fluid flow due to a rotating disk was investigated by Turkyilmazoglu[22].

In most of the studies, the Hall term is neglected for small or moderate values of the magnetic field in applying Ohm's law in the analysis. When a strong magnetic field is applied, the influence of electromagnetic force is noticeable as stated by Cramer and Pai [8]. Therefore, the Hall current is important and it has a marked effect on the magnitude and direction of the current density and consequently on the magnetic force term.

In this work, following the above approach, steady hydromagnetic flow of viscous, incompressible fluid over rotating infinite disk is examined with the radial electric field taking Hall effect into consideration. An external uniform magnetic field is imposed on the normal direction. The radial electric field is produced by electric potential. In the rotating infinite disk, the magnetic Reynolds number is assumed to be very small. Navier-Stokes equations and energy equation are solved by using Chebyshev collocation method. The effects of Hall parameters, magnetic field and electric field are analyzed.

2. Basic Equations

Let us consider the three-dimensional steady viscous incompressible conducting fluid over infinite rotating disk. The disk is assumed to be rotating about z-axis with a constant angular velocity Ω in the cylindrical coordinates (r, θ, z). An external uniform magnetic field is applied in the z-direction and has a constant magnetic flux density B_0. The magnetic Reynolds number is assumed to be very small, therefore, the effect on the imposed magnetic field is negligible. The disk is taken electrically conducting with $e = (e_r, e_\theta, e_z)$ denoting the electric field, in which $e_\theta = 0$ due to axisymmetric flow assumption, by the work of Kármán [13]. Moreover, the effect of uniform electric field on the disk flow is produced by electrical potential is given by $e_r = -B_0\Omega\gamma r$ [10]. In magnetic field, the electric current can be written by Ohm's law $j = \sigma(e + \mathbf{v} \times B - \beta(j \times B))$ where $j = (j_r, j_\theta, j_z)$ is the current density vector, σ is the electrical conductivity, and the last term defines the Hall effect as β is the Hall factor. The disk flow motion is governed by Maxwell's equation, continuity equation, the Navier-Stokes equations including the Lorentz force as follows

$$\nabla \cdot j = 0, \tag{2.1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{2.2}$$

$$\rho\Big[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathrm{v}\Big] = -\nabla p + \frac{1}{Re}\big[\nabla^2 \mathbf{v}\big] + M_n(j \times B)_i \tag{2.3}$$

Lorentz force terms $M_n(j \times B)_i$ represents the existence of magnetic field in the fluid motion equations. The presence of the force, originating from magnetic field, on the flow of conducting fluids can alter the velocity and pressure characteristics of the flow.

In general Maxwell's equation is defined by

$$\nabla \times e = -\frac{\partial B}{\partial t}. \tag{2.4}$$

In the case of time-independent flow, the equation(2.4) is turned into the equation below,

$$\nabla \times e = 0.$$

Therefore, there is a conservative electric field which arises by electric potential Φ, arriving to $e = -\nabla\Phi$.

Several parameters appearing in equations (2.1-2.3) are defined as follows, ρ is the density, $\mathbf{v} = (u, v, w)$ is the velocity vector, ∇ is the usual gradient operator in cylindrical coordinates, p is the pressure, Re is the Reynolds number characterizing the flow defined by $Re = \frac{\Omega}{\nu}$, ν is the kinematic viscosity of the fluid. Finally M_n is the magnetic interaction parameter, which represents the ratio between the magnetic force and the fluid inertia force. In component form of Maxwell's equation, continuity equation and momentum equations with Lorentz force can be written as

$$\frac{\partial j_r}{\partial r} + \frac{\partial j_z}{\partial z} + \frac{j_r}{r} = 0, \tag{2.5}$$

$$\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{w}{r} = 0, \tag{2.6}$$

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{Re}\left[\nabla^2 u - \frac{u}{r^2}\right] + \frac{M_n}{1+m^2}\left[me_r - u + mv\right], \tag{2.7}$$

$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + 2u = \frac{1}{Re}\left[\nabla^2 v - \frac{v}{r^2}\right] + \frac{M_n}{1+m^2}\left[-e_r - mu - v\right], \tag{2.8}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\left[\nabla^2 w\right], \tag{2.9}$$

where $m = \sigma\beta B_0$ is the Hall parameter. The Hall parameter can take any value. In case of positive values of m, B_0 is upwards and the electrons of the conducting fluid gyrate in the same sense as the rotating disk. On the other hand, when m takes negative values, B_0 is downwards and the electrons gyrate in an opposite sense to the disk.

In equations(2.7-2.8), Lorentz force terms are $j \times B = B_0(j_\theta, -j_r, 0)$, and the components of current density vector are easily derived by Ohm's law as

$$(j_r, j_\theta, j_z) = \frac{\sigma}{1+m^2}\left(e_r + mu + v, me_r - u + mv, (1+m^2)e_z\right).$$

Because of imposing radial electric field in velocity at infinity, the tangential direction velocity is given by $v = \Omega\gamma r$. Furthermore, existence of potential flow due to radial electric field at the edge of the boundary layer implies that pressure gradient in the radial direction is $\frac{\partial p}{\partial r} = \rho\Omega^2\gamma^2 r$ (see Evans[10]). When these are taken into consideration, boundary conditions become

$$\begin{array}{llllll} u = 0, & v = r\Omega, & w = 0, & j_z = 2r\Omega B_0 C\gamma, & \text{at} & z = 0, \\ u \to 0, & v \to r\Omega\gamma, & & & \text{as} & z \to \infty, \end{array} \tag{2.10}$$

where C is the wall conduction ratio of the electrical conductance of the wall to electrical conductivity of the fluid.

The basic flow of incompressible case, which is also called as Von Kármán's steady state flow is well known. The Von Kármán's[13] flow will be considered here, which means that the disk flow is assumed to evolve alongside the boundary layer coordinate $\eta = Re^{1/2}z$, in conformity with the self similarity variables (see Hossain,Hossain& Wilson[11])

$$\begin{array}{l} (u, v, w, p) = (r\Omega F(\eta), r\Omega G(\eta), Re^{-1/2}H(\eta), \rho\Omega^2 P(\eta)), \\ (j_r, j_\theta, j_z) = (B_0 r\Omega J_r(\eta), B_0 r\Omega J_\theta(\eta), B_o\Omega Re^{-1/2}J_z(\eta)), \\ (e_r, e_\theta, e_z) = (B_0 r\Omega E_r(\eta), 0, B_0\Omega Re^{-1/2}E_z(\eta)). \end{array} \tag{2.11}$$

These quantities substitute into the governing equations (2.5-2.9), and also neglect terms of $O(Re^{-1})$, the disk flow quantities are determined from the subsequent equations and boundary conditions(2.10) as

$$2J_r + J_z' = 0, \tag{2.12}$$

$$2F + H' = 0, \tag{2.13}$$

$$F^2 - G^2 + F'H - F'' - \frac{M_n}{1+m^2}\big[-m\gamma - F + mG\big] + \gamma^2 = 0, \tag{2.14}$$

$$2FG + G'H - G'' - \frac{M_n}{1+m^2}\big[\gamma - G - mF\big] = 0, \tag{2.15}$$

$$P' + H'H - H'' = 0, \tag{2.16}$$

$$\begin{array}{lllll} F = 0, & G = 1, & H = 0, & J_z = 2C\gamma & \text{at} \quad \eta = 0, \\ F \to 0, & G \to \gamma & & & \text{as} \quad \eta \to \infty, \end{array} \tag{2.17}$$

a prime denotes derivative with respect to η. The initial and boundary conditions(3.3) show the no-slip boundary conditions of governing equations at the surface of disk and a far field disk flow, respectively.

3. Analysis of the Heat Transfer

Due to the difference in the temperature between the surface of the disk and the ambient fluid, heat transfer takes place. The energy equation, with viscous dissipation and Joule heating depending on the Hall effect, takes the form

$$\begin{aligned} \rho\big[\tfrac{\partial T}{\partial t} \quad &+(\mathbf{v}\cdot\nabla)T\big] = M_\infty^2(\Gamma-1)\big[\tfrac{\partial p}{\partial t}+(\mathbf{v}\cdot\nabla)p\big]+\tfrac{1}{Pr}\tfrac{1}{Re}\big[\nabla^2 T\big] \\ &+\tfrac{\Gamma-1}{Re}M_\infty^2\big[\Phi\big]+M_n(\Gamma-1)M_\infty^2\tfrac{j^2}{\sigma} \end{aligned} \tag{3.1}$$

where T is the temperature of the fluid, $Pr=\frac{\mu c_p}{k}$ is the Prandtl number c_p is the specific heat capacity, μ is the dynamical viscosity and k is thermal conductivity of the fluid. Moreover, Γ is the ratio of the specific heats, M_∞ is the free stream Mach number. Finally, the last two terms in the right-hand-side of Eq.(3.1) represent

$$\Phi=(\frac{\partial u}{\partial z})^2+(\frac{\partial v}{\partial z})^2,$$

the viscous dissipation and

$$\frac{j^2}{\sigma}=\frac{1}{(1+m^2)^2}\big[(e_r+mu+v)^2+(me_r-u+mv)^2+(1+m^2)^2e_z^2\big]$$

Joule heating terms respectively.

Using the Von Kármán [13] assumptions, similarities of (2.11) and also neglecting terms of $O(Re^{-1})$, equation(3.1) becomes

$$\begin{aligned} \tfrac{1}{Pr}T'' \quad &-HT'+M_\infty^2(\Gamma-1)\big[\gamma^2F+F'^2+G'^2\big] \\ &+M_\infty^2(\Gamma-1)\tfrac{M_n}{(1+m^2)^2}\big[(-\gamma+mF+G)^2+(-m\gamma-F+mG)^2\big]=0, \end{aligned} \tag{3.2}$$

and the initial and boundary conditions for the energy equation are

$$\begin{aligned} &T=T_w, && \text{at} \quad \eta=0, \\ &T\to T_\infty, && \text{as} \quad \eta\to\infty, \end{aligned} \tag{3.3}$$

recalling that a prime indicates derivative in term of η. In the last two equations, T_w is the temperature at the surface of the disk, T_∞ is the temperature of the ambient fluid at a large distance from the disk. Introducing the non-dimensional variable $\theta=\frac{T-T_\infty}{T_w-T_\infty}$, the equation(3.2), the initial and boundary conditions(3.3) take the forms

$$\begin{aligned} \tfrac{1}{Pr}\theta'' \quad &-H\theta'+E_c\big[\gamma^2F+F'^2+G'^2\big] \\ &+\tfrac{M_nE_c}{(1+m^2)^2}\big[(-\gamma+mF+G)^2+(-m\gamma-F+mG)^2\big]=0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\theta=0, && \text{at} \quad \eta=0, \\ &\theta\to 1, && \text{as} \quad \eta\to\infty, \end{aligned} \tag{3.5}$$

where $E_c=\frac{M_\infty^2(\Gamma-1)}{T_w-T_\infty}$ is the Eckert number. The heat transfer from the disk surface to the fluid is computed by the application of the Fourier's law, and using transformation for heat term we have

$$\begin{aligned} q \quad &= -k\left(\tfrac{\partial T}{\partial z}\right)_w \\ &= -k(T_w-T_\infty)\sqrt{\tfrac{\Omega}{\nu}}\tfrac{d\theta(0)}{d\eta}, \end{aligned} \tag{3.6}$$

by rephrasing the heat transfer result in terms of the Nusselt number, defined as

$$N_u=\frac{q\sqrt{\frac{\nu}{\Omega}}}{k(T_w-T_\infty)}$$

Therefore the second part of the equation (3.6) turns into

$$(3.7) \qquad N_u = -\frac{d\theta(0)}{d\eta}$$

The action of viscosity in the fluid adjacent to the disk tends to set up tangential shear stress, which opposes the rotation of the disk. There is also a surface shear stress in the radial direction. Consequently, it is necessary to provide a torque at the shaft to maintain a steady rotation. Applying the Newtonian formula, the radial component τ_r and tangential component τ_θ of the shear stress are respectively expressed by

$$(3.8) \qquad \tau_r = \left(\frac{\partial u}{\partial z}\right)_w = r\Omega\sqrt{\frac{\Omega}{\nu}}F'(0)$$

$$(3.9) \qquad \tau_\theta = \left(\frac{\partial v}{\partial z}\right)_w = r\Omega\sqrt{\frac{\Omega}{\nu}}G'(0)$$

Of physical interest is also the magnitude of the constant axial velocity at infinity, given by $H(\infty)$ and the resisting the turning moment (or torque) T_0 on the disk of radius R

$$(3.10) \qquad T_0 = -\int_0^R \mu\left(\frac{\partial v}{\partial z}\right)_w 2\pi r^2 dr = -\frac{\rho\Omega\pi}{2}\sqrt{\Omega\nu}G'(0)$$

In this study, a matrix method called Chebyshev collocation method is presented for numerical solution of the equations (2.12-2.16) and (3.3) under the initial and boundary conditions (2.17) and (3.4) respectively by a truncated Chebyshev series. Using the Chebyshev collocation points, this method transforms the differential-integral equations to a matrix equation which corresponds to a system of linear algebraic equations with unknown Chebyshev coefficients. Therefore, this allows us to use computer for solution of the equations. In addition, the Chebyshev collocation method can be used for differential and integral equations.

4. Results and Discussions

In this section, we numerically solved the system of differential equations (2.12-2.16) under the initial and boundary conditions (2.17). The energy equation (3.3) relating to the initial and boundary conditions (3.4) was calculated using velocity profiles which were given in the previous case. The numerical results were obtained by utilizing Spectral Chebyshev collocation scheme.

In many boundary layer problems different methods have been applied to solve the system of the equations. For example, Sahoo [17], Attia [2], Jasmine & Gajjar [12] and Turkyilmazoglu [22], [25] reached their results using finite-difference method, a special technique, and also Chebyshev collocation method respectively.

In this work we use spectral Chebyshev collocation scheme based on the Chebsyhev polynomials. We briefly summarize the numerical scheme as follows: Nonlinear terms are linearized with the Newton linearization technique in the given equations. Using the Chebyshev collocation points, the linearized equations are transformed to a matrix equations with unknown Chebyshev coefficients and matrix system is solved by decomposition technique.

To verify the accuracy of the numerical scheme, as well as, to validate the code, we compared our results with the outcome of the studies by Sahoo [17] and Turkyilmazoglu [22]. For comparison purpose, the results of Sahoo [17], and Turkyilmazoglu [22] are tabulated in Table 1 and Table 2, which presents a clear evidence for accuracy of the numerical method. Moreover, Figure 1, which demonstrates the velocity profiles of the generalized Von Kármán's flow for the boundary layer over the rotating disk, is given below. This figure has been included in many relevant studies, as well.

Equations (2.13-2.15) under the conditions (2.17) are solved to compute the various velocity profiles in relation with the several Magnetic interaction parameters, Hall parameters and the radial electric parameters, as depicted in Figures (2-7). It is observed that if the radial electric parameter γ becomes larger than unity, the radial velocity profile decreases as the Hall parameter increases, if not, that is, the radial electric parameter γ

M_n	$F'(0)$		$-G'(0)$	
	Present	Sahoo	Present	Sahoo
0.0	0.510232	0.510214	0.615922	0.615909

Table 1. Comparison of the numerical solutions of shear stress coefficients in radial and tangential directions $F'(0)$, $-G'(0)$ respectively.

M_n	P_r	Γ	$H(\infty)$		-$\theta'(0)$	
			Present	Turkyilmazoglu	Present	Turkyilmazoglu
0.5	1.0	0.0	-0.458880064	-0.45888005	0.282655934	0.28265593

Table 2. Comparison of numerical solutions of the vertical velocity, $H(\infty)$ and coefficients of the heat transfer, $-\theta'(0)$.

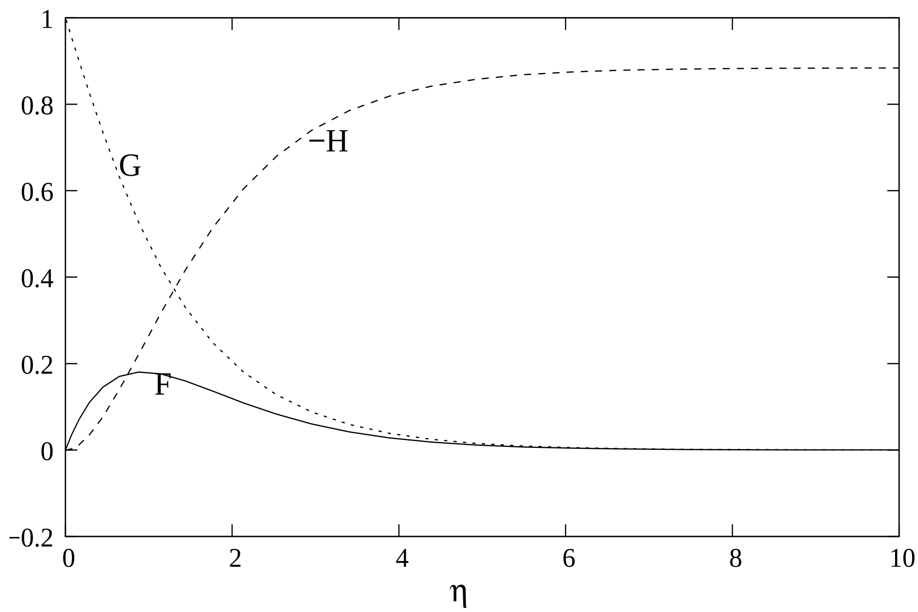
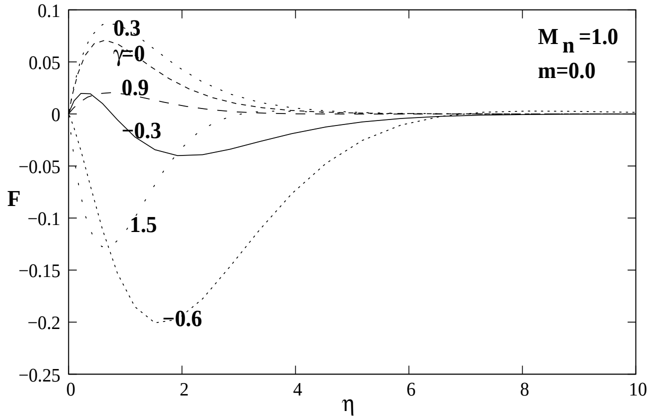


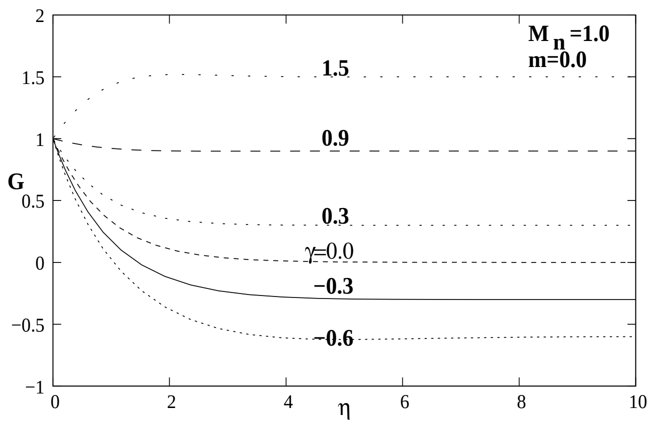
Figure 1. Velocity profiles of the generalized Von Kármán's flow are shown against the coordinate η.

gets less than unity a reverse effect takes place. It should be noted that, in both cases the size of the interval of η decreases as Hall parameter increases in Figures (2(a)-4(a)). Similarly, the size of the interval of ν decreases while a Magnetic interaction parameter increases according to graphs (2(a)-5(a)). These figures delineate that the negative Hall parameter has prominent effect on the radial component of velocity. It is interesting to find out from Figures (2(a)-7(a)) that F reverses its sign for some values of η, which proves that radial reverse flow can occur near the surface. This interesting phenomenon is interpreted as follows: the decelerated fluid particles in the boundary layer do not, in all cases, remain the thin layer which adheres to the disk along the whole wetted length of the surface. In some cases the boundary layer increases its thickness considerably in the downstream direction and the flow in the boundary layer becomes reversed. This causes the decelerated fluid particles to be forced outwards (see Schlicting[18]). Similar effect is observed in figures (2(a)-7(a)) and also in the paper by Turkyilmazoglu [22] for negative radial electric parameters.

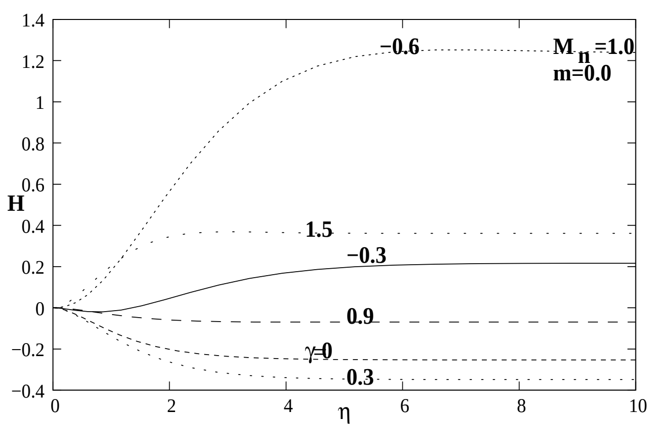
In graphs (2(b)-7(b)), there is no meaningful change in the tangential velocity profile when the Hall parameter or the magnetic interaction parameter increases or decreases.



(a)



(b)



(c)

Figure 2. Velocity profiles of the generalized Von Karman‘s flow are shown for $M_n = 1.0$ and $m = 0.0$ at six different radial electric parameters, respectively in (a) for radial F, in (b) for tangential G, and in (c) for axial H components.

The effect of the Hall parameter on the axial component of the velocity can be visualized as in Figures (2(c)-7(c)). In case of having positive radial electric parameter values, it does

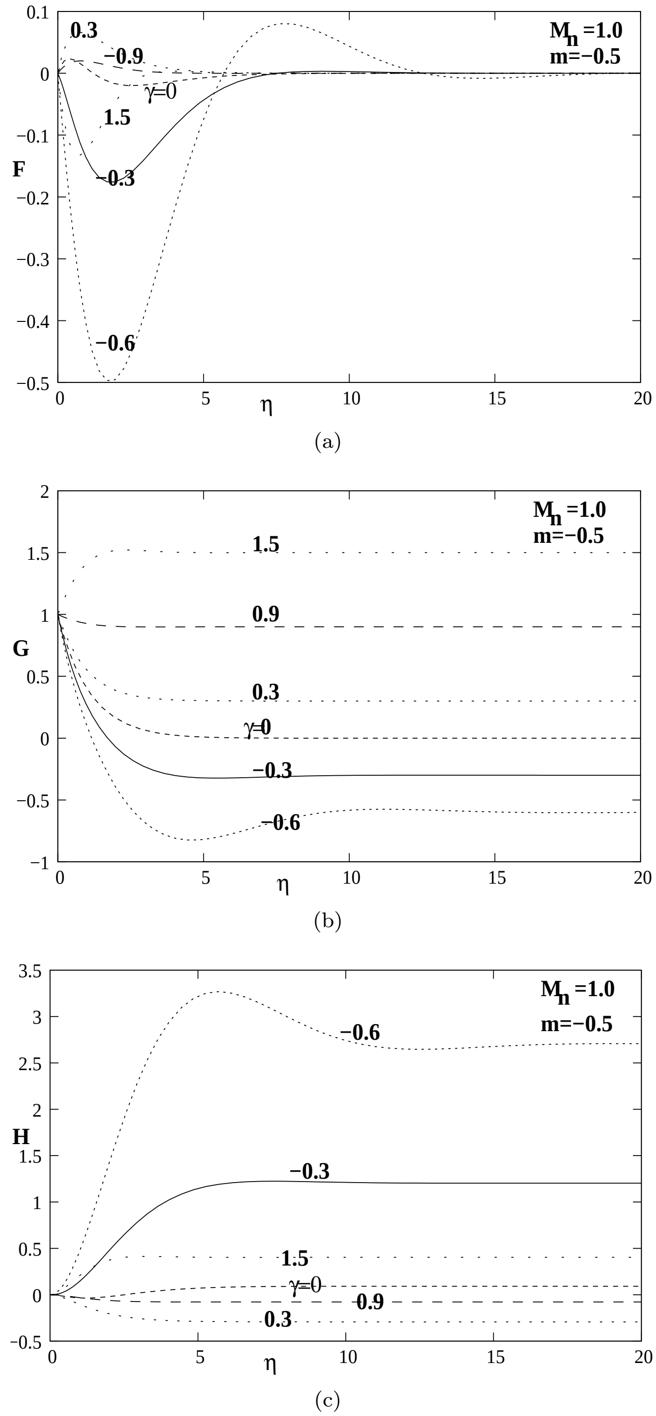
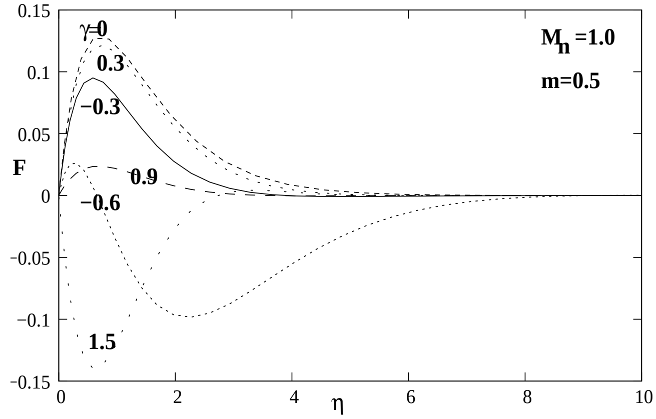
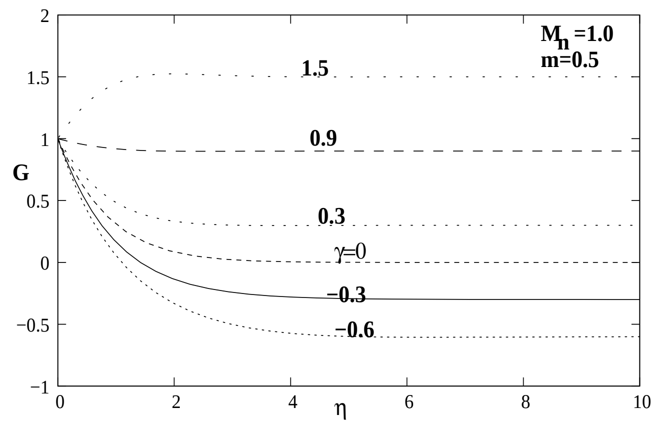


Figure 3. Velocity profiles of the generalized Von Kármán's flow are shown for $M_n = 1.0$ and $m = -0.5$ at six different radial electric parameters, respectively in (a) for radial F, in (b) for tangential G, and in (c) for axial H components.

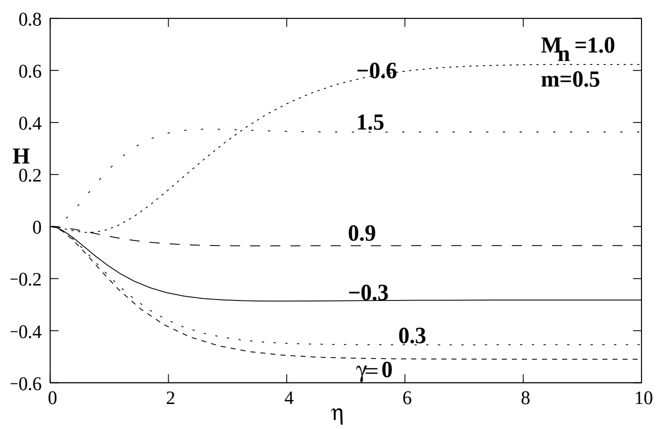
not matter whether the change on the axial velocity profile as Hall parameter increases or decreases. On the other hand, while the radial electric parameter takes negative values, the



(a)



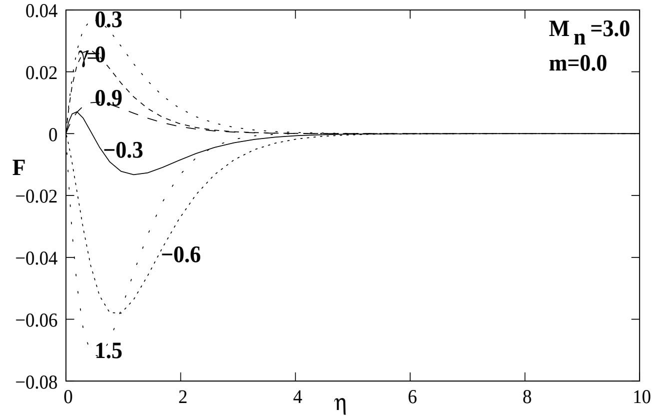
(b)



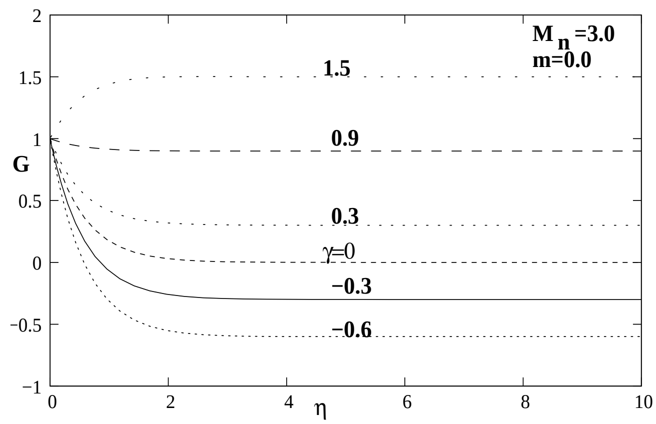
(c)

Figure 4. Velocity profiles of the generalized Von Kármán are shown for $M_n = 1.0$ and $m = 0.5$ at six different radial electric parameters, respectively in (a) for radial F, in (b) for tangential G, and in (c) for axial H components.

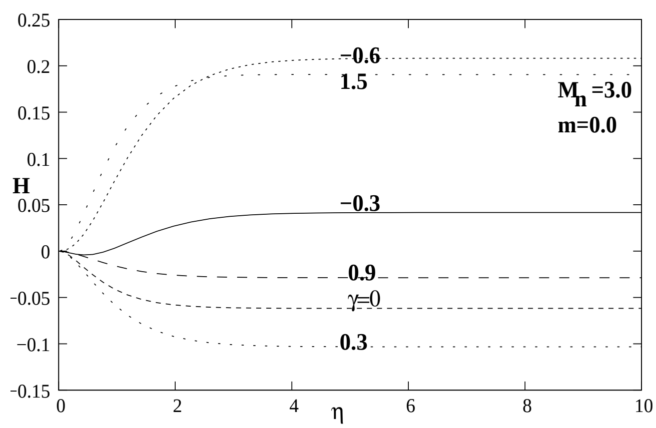
axial component of the velocity profiles decreases as Hall parameter increases. Above all, when the Hall parameter has a small negative value, H may become positive. Meanwhile,



(a)



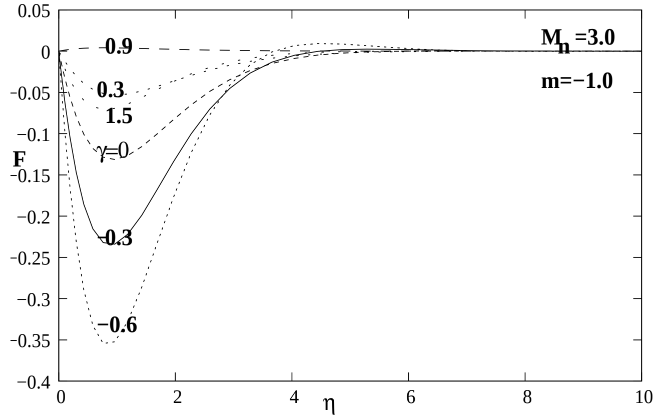
(b)



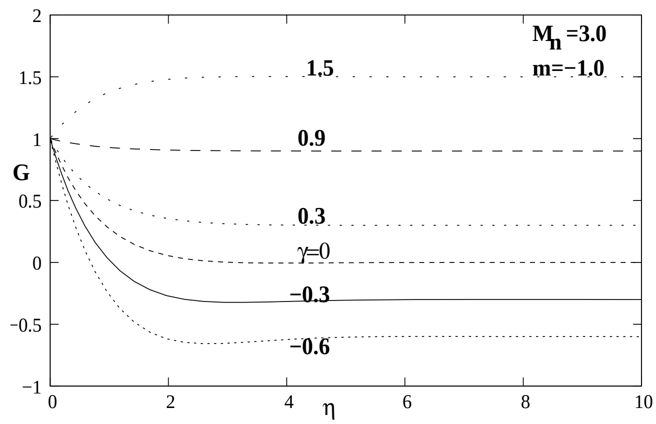
(c)

Figure 5. Velocity profiles of the generalized Von Kármán are shown for $M_n = 3.0$ and $m = 0.0$ at six different radial electric parameters, respectively in (a) for radial F, in (b) for tangential G, and in (c) for axial H components.

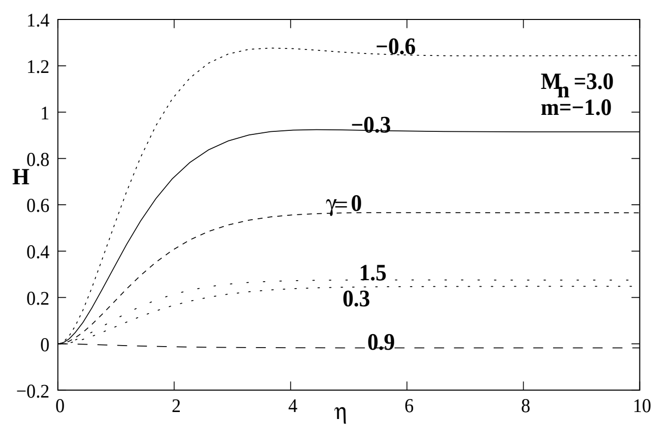
the impacts of magnetic interaction parameter are depicted in graphs(2(c)-7(c)). These



(a)



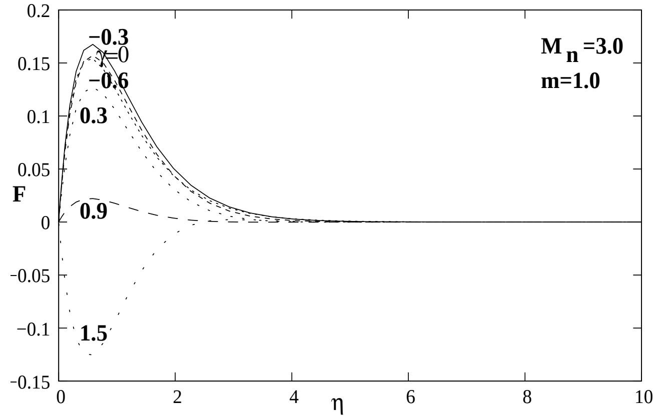
(b)



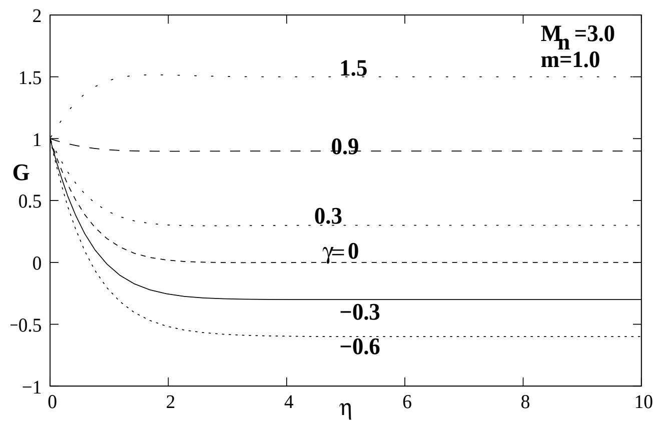
(c)

Figure 6. Velocity profiles of the generalized Von Kármán are shown for $M_n = 3.0$ and $m = -1.0$ at six different radial electric parameters, respectively in (a) for radial F, in (b) for tangential G, and in (c) for axial H components.

graphs demonstrate that the increment in the magnetic interaction parameter causes increment in the axial velocity values depending on positive electric parameters in radial



(a)



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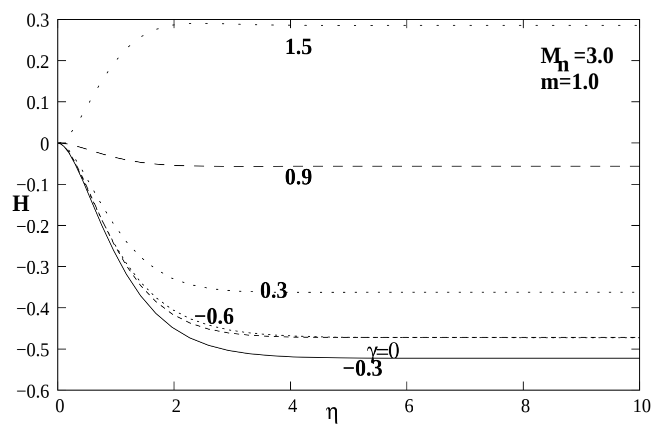


(c)

Figure 7. Velocity profiles of generalized Von Kármán are shown for $M_n = 3.0$ and $m = 1.0$ at six different radial electric parameters, respectively in (a) for F radial, in (b) for G tangential, and in (c) for H axial components.

direction and also depending on negative electric parameters in the same direction. All of these relations can be fairly seen in Table (3-4).

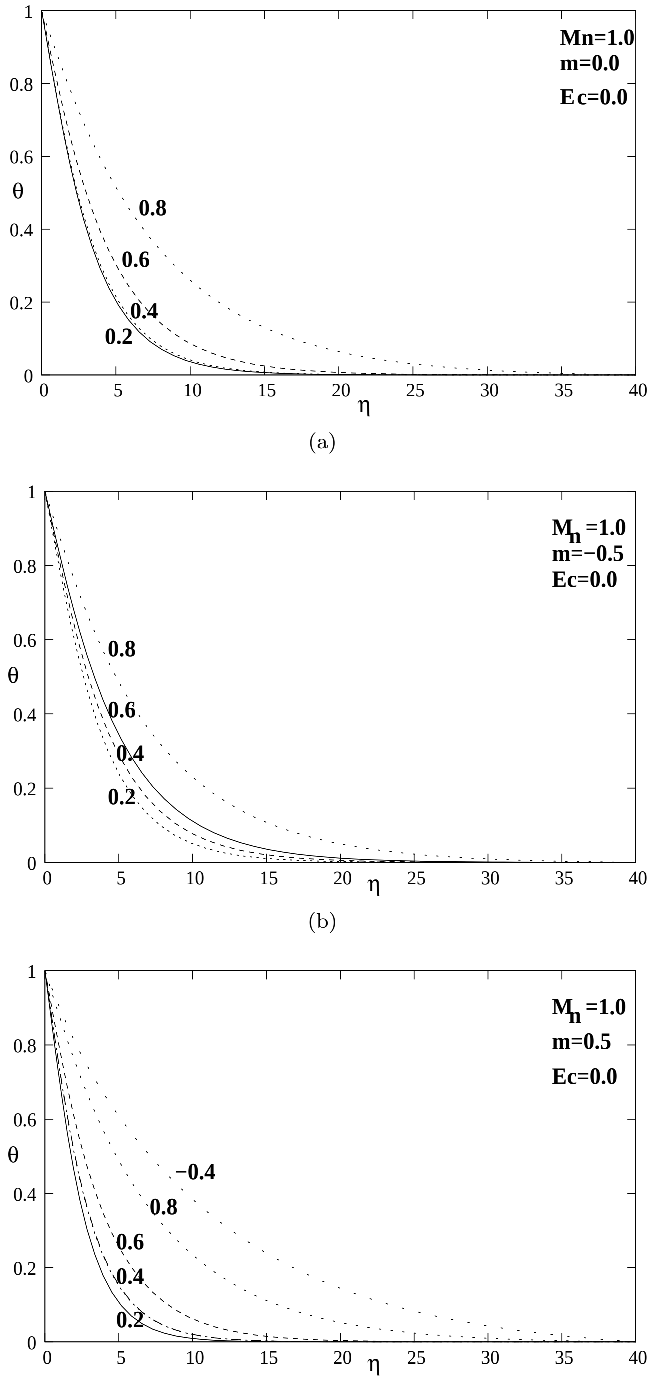
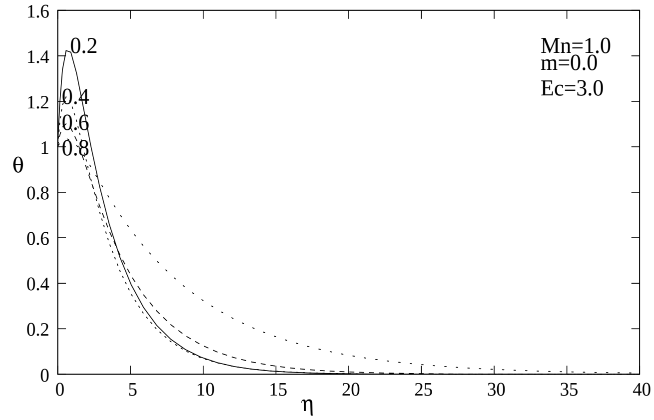
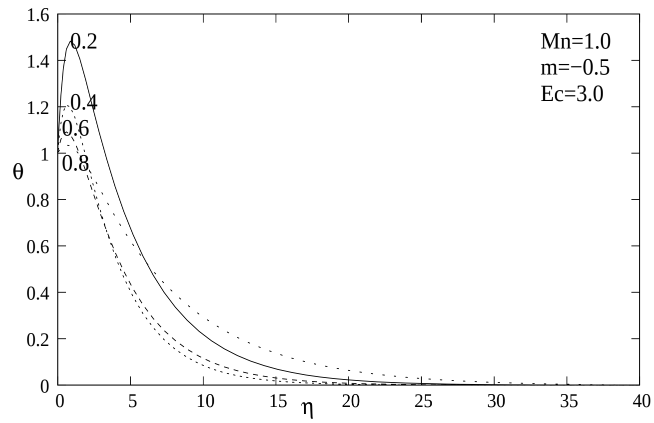


Figure 8. Temperature profile corresponding to heat transfer case is shown for $M_n = 1.0$ and $E_c = 0.0$ at different radial electric parameters respectively in (a) for $m = 0.0$, in (b) for $m = -0.5$, and in (c) for $m = 0.5$.

Temperature profiles, depending on the velocity components, are demonstrated in Figures(8-10), which are evaluated by using the equations (3.4-3.5) for different Eckert



(a)



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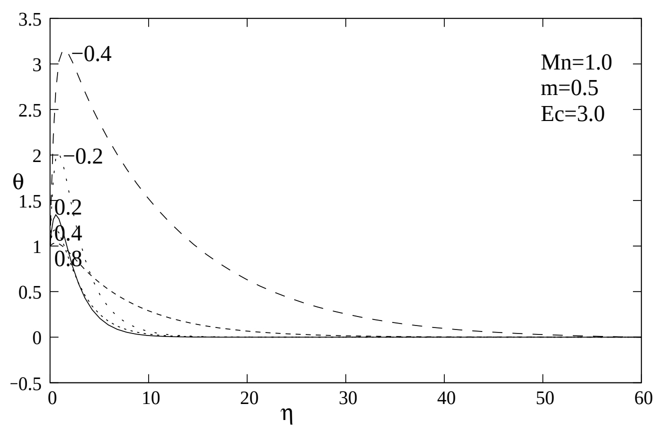

(c)

Figure 9. Temperature profile corresponding to heat transfer case is shown for $M_n = 1.0$ and $E_c = 3.0$ at different radial electric parameters respectively in (a) for $m = 0.0$, in (b) for $m = -0.5$, and in (c) for $m = 0.5$.

numbers, Hall parameters, magnetic interaction parameters with varying electric parameter in the radial direction at the fixed Prandtl number $P_r = 1.0$.

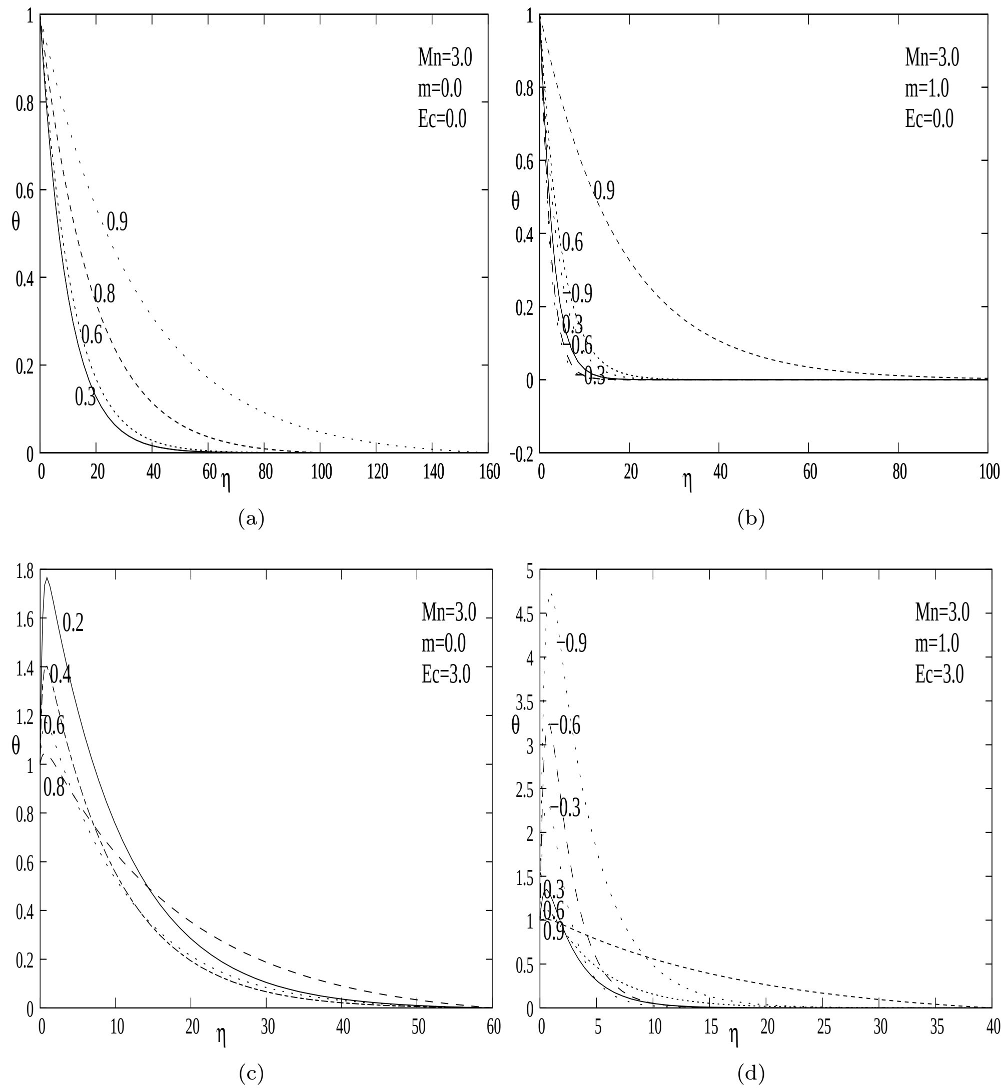

Figure 10. Temperature profile corresponding to heat transfer case is shown for $M_n = 3.0$ at different radial electric parameters respectively in (a) for $m = 0.0$, $E_c = 0.0$, in (b) for $m = 1.0$, $E_c = 0.0$, in (c) for $m = 0.0$, $E_c = 3.0$, and in (d) for $m = 1.0$, $E_c = 3.0$.

The effect of the Hall parameter is emphasized in Figures (8-10). It can be fairly inferred from the figures that for increasing Hall parameters the size of the interval of η shrinks, then this seems to occur for increasing Eckert numbers, as well. The case can also be observed easily for large magnetic interaction parameters. Furthermore, whenever Hall parameter increases, it is more temperature profiles are likely to be present for negative radial electric parameters. Table(5) also confirms the case, that is, the number of the presence of the more temperature profiles increases for the negative radial electric parameters.

It is also apparent from graphs (8) and (10) that when the Eckert number increases temperature profile increases too. Finally, the impacts of magnetic interaction numbers is given for increasing magnetic interaction numbers. As illustrated in Figures (8-10), the size of interval of η extends as magnetic interaction parameter increases.

Variations of the radial shear stress $F'(0)$, tangential shear stress $G'(0)$, the velocity in the radial direction $H(\infty)$ and coefficients of heat transfer $-\theta'(0)$ have been tabulated

M_n	m	γ	$F'(0)$	$G'(0)$	$H(\infty)$
1.0	-0.5	-0.6	-0.546781	-1.314478	2.707162
		-0.3	-0.102931	-1.113456	1.202871
		0.0	0.145142	-0.913193	9.205420E-002
		0.3	0.236486	-0.684928	-0.291948
		0.9	6.565356E-002	-0.111510	-7.689249E-002
		1.5	-0.457895	0.624115	0.403455
	0.0	-0.6	-0.112813	-1.538095	1.236077
		-0.3	0.170415	-1.315308	0.216396
		0.0	0.309257	-1.069053	-0.253314
		0.3	0.328034	-0.790558	-0.348514
		0.9	7.445934E-002	-0.125180	-6.913082E-002
		1.5	-0.489523	0.685647	0.361547
	0.5	-0.6	0.233191	-1.447853	0.621125
		-0.3	0.432856	-1.283261	-0.282036
		0.0	0.495221	-1.062616	-0.509727
		0.3	0.447946	-0.793985	-0.453959
		0.9	8.918274E-002	-0.126886	-7.328618E-002
		1.5	-0.554112	0.696890	0.363279

Table 3. Shear stress coefficients $F'(0)$ and $G'(0)$, vertical velocity $H(\infty)$ are tabulated at some chosen Hall parameters, radial electric parameters for fixed Magnetic interaction number $M_n = 1.0$.

M_n	m	γ	$F'(0)$	$G'(0)$	$H(\infty)$
3.0	-1.0	-0.6	-1.020161	-2.088827	1.243738
		-0.3	-0.640020	-1.649686	0.914910
		0.0	-0.338087	-1.241569	0.565473
		0.3	-0.124081	-0.858124	0.247915
		0.9	1.484372E-002	-0.123473	-1.746047E-002
		1.5	-0.226986	0.645086	0.275293
	0.0	-0.6	-6.191555E-002	-2.760873	0.208113
		-0.3	9.987538E-002	-2.254771	4.168429E-002
		0.0	0.190502	-1.747685	-6.176540E-002
		0.3	0.211255	-1.235386	-0.103263
		0.9	5.157919E-002	-0.180893	-2.865252E-002
		1.5	-0.358583	0.931506	0.190445
	1.0	-0.6	0.795771	-2.175726	-0.472718
		-0.3	0.805884	-1.834793	-0.522486
		0.0	0.734807	-1.462603	-0.472049
		0.3	0.590146	-1.059327	-0.361939
		0.9	0.104282	-0.161279	-5.608166E-002
		1.5	-0.612114	0.854623	0.285565

Table 4. Shear stress coefficients $F'(0)$ and $G'(0)$, vertical velocity $H(\infty)$ are tabulated at some chosen Hall parameters, radial electric parameters for fixed Magnetic interaction number $M_n = 3.0$.

for various radial electric parameter γ for the two different magnetic interaction numbers $M_n = 1.0$, $M_n = 3.0$, and Eckert numbers $E_c = 0.0$, $E_c = 3.0$ respectively in Tables (3-5). Because of increasing the Hall number m, the radial shear stress increases as the radial electric parameter gets less than unity. However, radial shear stress decreases when radial electric parameter gets larger than unity. It is apparent that reverse effect as a Magnetic interaction parameter is getting bigger.

M_n	E_c	m	$\gamma=-0.6$	$\gamma=-0.3$	$\gamma=0.0$	$\gamma=0.3$	$\gamma=0.9$
1.0	0.0	-0.5	-	-	-	0.204831	8.911136E-002
		0.0	-	-	0.193041	0.239180	7.320891E-002
		0.5	-	0.222168	0.310924	0.294007	8.858441E-002
	3.0	-0.5	-	-	-	-1.086214	-2.805792E-002
		0.0	-	-	-2.661671	-1.250502	-2.820800E-002
		0.5	-	-4.106938	-2.312196	-1.162085	-3.069879E-002
3.0	0.0	-1.0	-	-	-	-	5.775309E-002
		0.0	-	-	8.285834E-002	0.107275	6.388499E-002
		1.0	0.319910	0.341921	0.322544	0.269701	7.936438E-002
	3.0	-1.0	-	-	-	-	2.106896E-003
		0.0	-	-	-5.020784	-2.411661	-2.226743E-002
		1.0	-8.848756	-5.836229	-3.493386	-1.762939	-3.213039E-002

Table 5. Heat transfer parameter $-\theta'(0)$ is tabulated at some chosen Hall parameters, radial electric parameters for the two different Magnetic interaction numbers $M_n = 1.0$, $M_n = 3.0$, and Eckert numbers $E_c = 0.0$, $E_c = 3.0$ respectively, and fixed Prandtl number $P_r = 1.0$.

Impact of Hall numbers on tangential shear stress can be deduced from these tables. It can be seen that the tangential shear stress increases in the case of increasing or decreasing Hall parameters values. In the event of the radial electric parameter becomes less than unity, when the magnetic interaction parameter increases the tangential shear stress decreases. On the other hand, if the radial electric parameter gets larger than unity, the effect on the shear stress in tangential direction becomes reversed.

5. Conclusions

The velocity and temperature profiles governing the steady-incompressible boundary layer flow over a rotating disk have been obtained using self-consistent assumptions. The resulting equations have then been solved numerically by using Chebyshev collocation method, and then the behavior of the velocity and temperature profiles are displayed graphically.

The effects of Hall parameter, radial electric parameter, Eckert parameter and magnetic interaction parameter are tabulated. One of the main outcomes of the present study is defining the effect of the Hall parameters on temperature profiles. This has been observed throughout for varying magnetic interaction parameters and radial electric parameters. Although the positive values of Hall parameter reveal the more temperature profiles for negative radial electric parameters, negative Hall parameter reverses the effect.

In this paper the effect of Hall parameter on the rotating disk is studied. Following this, we believe that, it would be interesting to study the effect of the electric field and also Hall parameter on instability mechanisms over rotating disk. For similar works, we refer to Jasmine &Gajjar [12] and Turkyilmazoglu [25]

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Soft hyperrings and their (fuzzy) isomorphism theorems

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Abstract

We introduce the notions of soft hyperrings, idealistic soft hyperrings, soft subhyperrings and soft hyperideals, and discuss some related properties. Moreover, we establish three (fuzzy) isomorphism theorems of soft hyperrings.

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1. Introduction

Soft set theory has been considered as an effective mathematical tool for modeling uncertainties [21]. Different from traditional mathematical tools for dealing with uncertainties, such as probability theory, fuzzy set theory [34] and rough set theory [25], soft set theory is free from the inadequacy of the parametrization tools of these theories [21]. Molodtsov demonstrated that soft set theory has potential applications in many directions, including function smoothness, Riemann integration, measurement theory, game theory and operations research [21]. Also, soft set theory has been applied to forecasting [32], decision making [6, 17, 39], association rules mining [13] and mobile cloud computing [31].

In theoretical aspect of soft sets, after Molodtsov's pioneer work [21], Maji et al. [20] gave further a detailed theoretical study on soft sets. Based on the analysis of several operations on soft sets introduced in [20], Ali et al. [3] proposed some new operations. In [4], Çağman and Enginoğlu defined the soft matrices, which are representative of soft

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sets, and Gong et al. [12] presented the bijective soft sets, which are special soft sets. As an extended concept of bijective soft sets, the exclusive disjunctive soft sets [33] were introduced. Furthermore, Jiang et al. [14] presented an extended soft set theory by using the concepts of description logics to act as the parameters of soft sets. Recently, many researchers studied the algebraic structures of soft sets. Aktaş and Çağman [2] defined soft groups and showed that fuzzy groups can be considered a special case of soft groups. Moreover, some basic properties of soft semirings [11] and soft rings [1] were introduced. Also, Sun et al. [28] presented the soft modules, and Li [19] analyzed the soft lattices. In addition, Jun et al. [15, 16, 26, 38] considered the applications of soft sets in BCK/BCI-algebras, BCH-algebras, WS-algebras and BL-algebras, and considered their related properties.

On the other hand, the theory of algebraic hyperstructures, introduced by Marty in 1934 [23], is a natural generalization of the theory of algebraic structures. It has been applied to many areas [5], such as probabilities, geometry, fuzzy sets, automata, cryptography, combinatorics, and artificial intelligence. Several books on hyperstructure theory have been published [5, 7, 29]. The book [7] was devoted especially to the study of hyperring theory and applications, in which several kinds of hyperrings were introduced and investigated. Krasner hyperring [18], which is a well known type of hyperring, has been studied by many authors. In what follows, by a hyperring we mean a Krasner hyperring. In [8], Davvaz and Salasi defined the notions of normal hyperideal, prime hyperideal, maximal hyperideal, and Jacobson radical of a hyperring and obtained some related results. Furthermore, Davvaz [9] established three isomorphism theorems of hyperrings, and derived the Jordan-Holder theorem for hyperrings. Moreover, Vougiouklis [30] considered the fundamental relation on a hyperring as the smallest equivalence relation so that the quotient is the fundamental ring. In [35], Zhan et al. applied fuzzy sets to hyperrings and introduced the concept of fuzzy hyperideals of a hyperring. By using the normal fuzzy hyperideals of a hyperring, Ma and Zhan [22] derived three fuzzy isomorphism theorems of hyperrings. Also, they considered isomorphism theorems and fuzzy isomorphism theorems of hypermodules [36, 37].

In this paper, we apply the notion of soft sets to hyperrings. Some related notions, such as soft hyperrings, idealistc soft hyperrings, soft subhyperrings, soft hyperideals, are defined, and several basic properties are investigated. Furthermore, we consider the isomorphism of soft hyperrings, and establish three (fuzzy) isomorphism theorems of soft hyperrings.

2. Preliminaries

In this section, we review some notions and results about hyperrings and soft sets. A hypergroupoid $(H, \circ)$ is a non-empty set H together with a hyperoperation $\circ$ defined on H, i.e., a mapping $H \times H \to \mathscr{P}^*(H)$, where $\mathscr{P}^*(H)$ is the set of all non-empty subsets of H. If $x \in H$ and A, B are subsets of H, then $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$. $(H, \circ)$ is called a hypergroup if for all $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and $x \circ H = H \circ x = H$ [27].

2.1. Definition. [18] A hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup, i.e.,
  (a) for every $x, y, z \in R$, $(x + y) + z = x + (y + z)$;
  (b) for every $x, y \in R$, $x + y = y + x$;
  (c) there exists $0 \in R$ such that $0 + x = x$ for all $x \in R$;

(d) for every $x \in R$ there exists a unique element $x' \in R$ such that $0 \in x + x'$ (we shall write $-x$ for x' and we call it the oposite of x);

(e) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

(2) Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$.

(3) The multiplication is distributive with respect to the hyperoperation $+$.

The following elementary facts follow easily from the axioms: $-(-x) = x$ and $-(x + y) = -x - y$ for all $x, y \in R$.

2.2. Example. [18] Let $(R, +, \cdot)$ be a ring and N a normal subgroup of its multiplicative semigroup. Then the multiplicative classes $\overline{x} = x \cdot N (x \in R)$ form a partition of R, and let $\overline{R} = R/N$ be the set of these classes. Define the hyperaddition and the multiplication on $\overline{R}$ by $\overline{x} \oplus \overline{y} = \{\overline{z} | z \in \overline{x} + \overline{y}\}$ and $\overline{x} \odot \overline{y} = \overline{x \cdot y}$. Then $(\overline{R}, \oplus, \odot)$ is a hyperring.

A non-empty subset S of a hyperring $(R, +, \cdot)$ is called a subhyperring of R if $(S, +, \cdot)$ itself is a hyperring. A subhyperring I of R is a left (right) hyperideal of R if $r \cdot a \in I (a \cdot r \in I)$ for all $r \in R$ and $a \in I$. A subhyperring I is called a hyperideal if I is both left and right hyperideal [9].

2.3. Lemma. [9] A non-empty subset I of a hyperring R is a left (right) hyperideal if and only if (1) $a, b \in I$ implies $a - b \subseteq I$; (2) $a \in I, r \in R$ imply $r \cdot a \in I (a \cdot r \in I)$.

A subhyperring I of a hyperring R is normal if and only if $x + I - x \subseteq I$ for all $x \in R$. Let I be a normal hyperideal of a hyperring R, then for all $x, y \in R$, $(I + x) + (I + y) = I + x + y = I + z$ for all $z \in x + y$ and $I + x = I + y$ for all $y \in I + x$. If K and N are hyperideals of a hyperring R with N normal in R, then $K \cap N$ is a normal hyperideal of K, and N is a normal hyperideal of $K + N$ [9].

If I is a normal hyperideal of a hyperring R, then the relation I^* defined by $x \equiv y (mod\ I)$ if and only if $(x - y) \cap I \neq \emptyset$ is an equivalence relation [9]. Let $I^*[x]$ be the equivalence class of the element $x \in R$. Then $I + x = I^*[x]$ for all $x \in R$. On the set of all classes $R/I = \{I^*[x] \mid x \in R\}$, the hyperoperation $\oplus$ and the multiplication $\odot$ are defined by $I^*[x] \oplus I^*[y] = \{I^*[z] \mid z \in I^*[x] + I^*[y]\}$, and $I^*[x] \odot I^*[y] = I^*[x \cdot y]$, respectively. Then $(R/I, \oplus, \odot)$ is a hyperring. For all $I + x, I + y \in R/I$, we have $(I + x) \oplus (I + y) = \{I + z \mid z \in x + y\}$.

Let R_1 and R_2 be two hyperrings. A mapping φ from R_1 into R_2 is called a strong homomorphism if $\varphi(a + b) = \varphi(a) + \varphi(b)$, $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$, and $\varphi(0) = 0$, for all $a, b \in R_1$. A strong homomorphism φ is an isomorphism if φ is one to one and onto. If φ is a strong homomorphism from R_1 into R_2, then the kernel of φ is the set $ker\varphi = \{x \in R_1 \mid \varphi(x) = 0\}$. It is trivial that $ker\varphi$ is a hyperideal of R_1, but in general it is not normal in R_1 [9].

Let U be an initial universe set and E be a set of parameters. $\mathscr{P}(U)$ denotes the power set of U and $A \subseteq E$.

2.4. Definition. [21] A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to \mathscr{P}(U)$.

In fact, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). Please readers see the reference [20] for some examples.

2.5. Definition. [20] For two soft sets (F,A) and (G,B) over U, we say that (F,A) is a soft subset of (G,B), denoted by $(F,A)\widetilde{\subseteq}(G,B)$, if the following conditions hold: (1) $A \subseteq B$; (2) for all $e \in A$, $F(e) \subseteq G(e)$. Two soft sets (F,A) and (G,B) over U are called soft equal if $(F,A)\widetilde{\subseteq}(G,B)$ and $(G,B)\widetilde{\subseteq}(F,A)$.

2.6. Definition. [3, 20] The extended intersection (or union) of two soft sets (F,A) and (G,B) over U is the soft set $(H,C) = (F,A) \cap_{\mathscr{E}} (G,B)$ (or $(F,A)\widetilde{\cup}(G,B)$), where $C = A \cup B$, and for all $e \in C$, if $e \in A - B$, $H(e) = F(e)$; if $e \in B - A$, $H(e) = G(e)$; if $e \in A \cap B$, $H(e) = F(e) \cap G(e)$ (or $F(e) \cup G(e)$).

2.7. Definition. [3] The restricted intersection (or restricted union) of two soft sets (F,A) and (G,B) over U such that $A \cap B \neq \varnothing$, is the soft set $(H,C) = (F,A) \cap_{\mathscr{R}} (G,B)$ (or $(F,A) \cup_{\mathscr{R}} (G,B)$), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$ (or $F(e) \cup G(e)$).

2.8. Definition. [20] If (F,A) and (G,B) are two soft sets over U, then "(F,A) AND (G,B) (or (F,A) OR (G,B))", denoted by $(F,A)\widetilde{\wedge}(G,B)$ (or $(F,A)\widetilde{\vee}(G,B)$), is defined as $(F,A)\widetilde{\wedge}(G,B)$ (or $(F,A)\widetilde{\vee}(G,B)$)$=(H, A \times B)$, where $H(x,y) = F(x) \cap G(y)$ (or $F(x) \cup G(y)$) for all $(x,y) \in A \times B$.

2.9. Definition. [11] Let (F,A) be a soft set. The set $\mathrm{Supp}(F,A) = \{x \in A \mid F(x) \neq \varnothing\}$ is called the support of the soft set (F,A). A soft set (F,A) is non-null if $\mathrm{Supp}(F,A) \neq \varnothing$.

3. (Idealistic) soft hyperrings

In what follows, R denotes a hyperring and A is a nonempty set. A set-valued function $F : A \to \mathscr{P}(R)$ can be defined as $F(x) = \{y \in R \mid (x,y) \in \rho\}$ for all $x \in A$, where ρ is an arbitrary binary relation between an element of A and an element of R, i.e., ρ is a subset of $A \times R$, then (F,A) is a soft set over R.

3.1. Definition. Let (F,A) be a non-null soft set over R. Then (F,A) is called an (idealistic) soft hyperring over R if $F(x)$ is a subhyperring (hyperideal) of R for all $x \in \mathrm{Supp}(F,A)$.

3.2. Example. Suppose that $R = \{0,1,2,3\}$ and define the operations $+$ and $\cdot$ on R as follows:

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	1	2	3
3	0	0	0	0

Then $(R,+,\cdot)$ is a hyperring [10]. Let (F,A) be a soft set over R, where $A = R$ and $F : A \to \mathscr{P}(R)$ is a set-valued function defined by $F(x) = \{0\} \cup \{y \in R \mid x\rho y \Leftrightarrow x + y = \{2\}\}$ for all $x \in A$. Then $F(0) = \{0,2\}$, $F(1) = \{0,3\}$, $F(2) = \{0\}$ and $F(3) = \{0,1\}$ are subhyperrings of R. Hence (F,A) is a soft hyperring over R.

Let $B = R$ and $G : B \to \mathscr{P}(R)$ be a set-valued function defined by $G(x) = \{0,3\} \cup \{y \in R \mid x\rho' y \Leftrightarrow x + y \subseteq \{0,3\}\}$ for all $x \in B$. Then $G(0) = G(3) = \{0,3\}$ and $G(1) = G(2) = \{0,1,2,3\}$ are hyperideals of R. Thus (G,B) is an idealistic soft hyperring over R.

Clearly, every idealistic soft hyperring over R is a soft hyperring over R, but the converse is not true in general. In Example 3.2, the (F, A) is not an idealistic soft hyperring over R since $\{0, 1\}$ and $\{0, 2\}$ are not hyperideals of R.

(F, A) is an (idealistic) soft hyperring over R and $B \subseteq A$. From the Definition 3.1, we have that $(F\,|_B, B)$ is an (idealistic) soft hyperring over R when it is non-null. Next, we give an example to show that (F, A) is not an (idealistic) soft hyperring over R, but there exists a subset B of A such that $(F\,|_B, B)$ is an (idealistic) soft hyperring over R.

3.3. Example. Let $R = \{0, 1, 2, 3\}$ be a set with the hyperoperation $+$ and the multiplication $\cdot$ defined as follows:

$+$	0	1	2	3
0	0	1	2	3
1	1	$\{0,1\}$	3	$\{2,3\}$
2	2	3	0	1
3	3	$\{2,3\}$	1	$\{0,1\}$

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	2
3	0	0	2	2

It follows that $(R, +, \cdot)$ is a hyperring [24]. Let (F, A) be the soft set over R where $A = R$ and $F : A \to \mathscr{P}(R)$ is a set-valued function defined by $F(x) = \{y \in R \mid x\rho y \Leftrightarrow x + y \subseteq \{1, 3\}\}$ for all $x \in A$. Then $F(1) = F(3) = \{0, 2\}$ is a hyperideal of R, but $F(0) = F(2) = \{1, 3\}$ is not a hyperideal of R, and also is not a subhyperring of R since $1 + 3 = \{2, 3\} \nsubseteq \{1, 3\}$. Therefore, (F, A) is not an idealistic soft hyperring over R, and also is not a soft hyperring over R. However, if we take $B = \{1, 3\} \subseteq A$, then $(F\,|_B, B)$ is an idealistic soft hyperring over R. Also, it is a soft hyperring over R.

3.4. Theorem. Let (F, A) and (G, B) be two (idealistic) soft hyperrings over R, then

(1) $(F, A) \cap_{\mathscr{E}} (G, B)$ is an (idealistic) soft hyperring over R if it is non-null;
(2) if $A \cap B \neq \varnothing$, then $(F, A) \cap_{\mathscr{R}} (G, B)$ is an (idealistic) soft hyperring over R whenever it is non-null;
(3) if $A \cap B = \varnothing$, then $(F, A)\widetilde{\cup}(G, B)$ is an (idealistic) soft hyperring over R;
(4) $(F, A)\widetilde{\wedge}(G, B)$ is an (idealistic) soft hyperring over R.

Proof. We only prove (1), and the proofs of (2)-(4) are similar. By Definition 2.6, we have $(H, C) = (F, A) \cap_{\mathscr{E}} (G, B)$. For all $x \in \mathrm{Supp}(H, C)$, if $x \in A - B$, because (F, A) is an (idealistic) soft hyperring over R, we have $H(x) = F(x)$ is a subhyperring (hyperideal) of R; if $x \in B - A$, because (G, B) is an (idealistic) soft hyperring over R, $H(x) = G(x)$ is a subhyperring (hyperideal) of R; if $x \in A \cap B$, $H(x) = F(x) \cap G(x)$ is a subhyperring (hyperideal) of R, since the intersection of any two subhyperrings (hyperideals) of R is also a subhyperring (hyperideal) of R. Therefore, $(H, C) = (F, A) \cap_{\mathscr{E}} (G, B)$ is an (idealistic) soft hyperring over R.

If A and B are not disjoint, Theorem 3.4(3) is not true in general.

3.5. Example. Consider the hyperring R defined in Example 3.3. Let $A = R$ and $F : A \to \mathscr{P}(R)$ be a set-valued function defined by $F(x) = \{0, 1\} \cup \{y \in R \mid x\rho y \Leftrightarrow x + y \subseteq \{2, 3\}\}$ for all $x \in A$. Then $F(0) = F(1) = \{0, 1, 2, 3\}$ and $F(2) = F(3) = \{0, 1\}$ are hyperideals of R. Thus (F, A) is an idealistic soft hyperring over R.

Let $B = \{0, 2\}$ and $G : B \to \mathscr{P}(R)$ be the set-valued function defined by $G(x) = \{y \in R \mid x\rho' y \Leftrightarrow x + y \subseteq \{0, 2\}\}$ for all $x \in B$. Then $G(0) = G(2) = \{0, 2\}$ is a hyperideal of R. Hence, (G, B) is an idealistic soft hyperring over R. However, $(H, C) = (F, A)\widetilde{\cup}(G, B)$ is not an idealistic soft hyperring over R and also is not a soft hyperring, since $H(2) = F(2) \cup G(2) = \{0, 1, 2\}$ is not a subhyperring of R for

$1+2=\{3\} \nsubseteq H(2)$.

3.6. Corollary. Let $(F_i, A_i)_{i\in\Lambda}$ be a non-empty family of (idealistic) soft hyperrings over R, where Λ is an index set, then

(1) $(\cap_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i)$ is an (idealistic) soft hyperring over R if it is non-null;
(2) if $\cap_{i\in\Lambda} A_i \neq \varnothing$, then $(\cap_{\mathscr{R}})_{i\in\Lambda}(F_i, A_i)$ is an (idealistic) soft hyperring over R whenever it is non-null;
(3) if $A_i \cap A_j = \varnothing$ for all $i, j \in \Lambda$ and $i \neq j$, then $\widetilde{\cup}_{i\in\Lambda}(F_i, A_i)$ is an (idealistic) soft hyperring over R;
(4) $\widetilde{\wedge}_{i\in\Lambda}(F_i, A_i)$ is an (idealistic) soft hyperring over R.

3.7. Definition. Let (F, A) be an idealistic soft hyperring over R, then (F, A) is called an identity idealistic soft hyperring over R if $F(x) = \{0\}$ for all $x \in A$; (F, A) is called an absolute idealistic soft hyperring over R if $F(x) = R$ for all $x \in A$.

3.8. Example. Consider the hyperring R defined in Example 3.3. Let $A = R$ and $F : A \to \mathscr{P}(R)$ be the set-valued function defined by $F(x) = \{y \in R \mid x\rho y \Leftrightarrow x{+}y = \{x\}\}$ for all $x \in A$. Then $F(0) = F(1) = F(2) = F(3) = \{0\}$ and so (F, A) is an identity idealistic soft hyperring over R.

Let $B = R$ and $G : B \to \mathscr{P}(R)$ be the set-valued function defined by $G(x) = \{y \in R \mid x\rho' y \Leftrightarrow x + y \subseteq R\}$ for all $x \in B$. Then $G(x) = R$ for all $x \in B$ and so (G, B) is an absolute idealistic soft hyperring over R.

3.9. Theorem. Let φ be a strong homomorphism from hyperring R_1 to hyperring R_2. If (F, A) is a soft hyperring over R_1, then $(\varphi(F), A)$ is a soft hyperring over R_2; if φ is onto and (F, A) is an idealistic soft hyperring over R_1, then $(\varphi(F), A)$ is an idealistic soft hyperring over R_2, where $\varphi(F)(x) = \varphi(F(x))$ for all $x \in A$.

Proof. Clearly, $\mathrm{Supp}(\varphi(F), A) = \mathrm{Supp}(F, A)$. For all $x \in \mathrm{Supp}(\varphi(F), A)$, $\varphi(F)(x) = \varphi(F(x))$. Since (F, A) is a soft hyperring over R_1, it follows that $F(x)$ is a subhyperring of R_1, so $\varphi(F(x))$ is also a subhyperring of R_2. Hence, $(\varphi(F), A)$ is a soft hyperring over R_2. Moreover, for every $x \in \mathrm{Supp}(\varphi(F), A)$, because $F(x)$ is a hyperideal of R_1 and φ is onto, we have that $\varphi(F)(x) = \varphi(F(x))$ is a hyperideal of R_2. Therefore, $(\varphi(F), A)$ is an idealistic soft hyperring over R_2.

3.10. Theorem. Let φ be a strong homomorphism from hyperring R_1 to hyperring R_2, and (F, A) be an idealistic soft hyperring over R_1. If $F(x) = ker\varphi$ for all $x \in A$, then $(\varphi(F), A)$ is an identity idealistic soft hyperring over R_2. If φ is onto and (F, A) is an absolute idealistic soft hyperring over R_1, then $(\varphi(F), A)$ is an absolute idealistic soft hyperring over R_2.

Proof. It is straightforward.

3.11. Definition. Let (F, A) and (G, B) be two soft hyperrings over R. Then (G, B) is called a soft subhyperring (hyperideal) of (F, A) if $B \subseteq A$, and $G(x)$ is a subhyperring (hyperideal) of $F(x)$ for all $x \in \mathrm{Supp}(G, B)$.

3.12. Example. Consider the hyperring R given in Example 3.2. Let $A = R$ and $F : A \to \mathscr{P}(R)$ be the set-valued function defined by $F(x) = \{0, 2\} \cup \{y \in R \mid x\rho y \Leftrightarrow x + y \subseteq \{1, 3\}\}$ for all $x \in A$. Then $F(0) = F(2) = \{0, 1, 2, 3\}$, and $F(1) = F(3) = \{0, 2\}$ are subhyperrings of R. Therefore, (F, A) is a soft hyperring over R.

Let $B = \{1, 2, 3\} \subseteq A$ and $G : B \to \mathscr{P}(R)$ be the set-valued function defined by $G(x) = \{0\} \cup \{y \in R \mid x\rho' y \Leftrightarrow x + y = \{1\}\}$ for all $x \in B$. Then $G(1) = \{0\}$, $G(2) = \{0, 3\}$ and $G(3) = \{0, 2\}$ are hyperideals of $F(1)$, $F(2)$ and $F(3)$, respectively, so (G, B) is a soft hyperideal of (F, A).

3.13. Theorem. Let (F, A) and (G, B) be soft hyperrings over R. For all $x \in \mathrm{Supp}(G, B)$, if $B \subseteq A$ and $G(x) \subseteq F(x)$, then (G, B) is a soft subhyperring of (F, A). Furthermore, if (G, B) is an idealistic soft hyperring over R, then (G, B) is a soft hyperideal of (F, A).

Proof. Straightforward.

3.14. Theorem. Let (F, A) be a soft hyperring over R, and $(G_i, B_i)_{i\in\Lambda}$ be a non-empty family of soft subhyperrings (hyperideals) of (F, A), where Λ is an index set, then

(1) $(\cap_{\mathscr{E}})_{i\in\Lambda}(G_i, B_i)$ is a soft subhyperring (hyperideal) of (F, A) if it is non-null;
(2) if $\cap_{i\in\Lambda} B_i \neq \varnothing$, then $(\cap_{\mathscr{R}})_{i\in\Lambda}(G_i, B_i)$ is a soft subhyperring (hyperideal) of (F, A) whenever it is non-null;
(3) if $B_i \cap B_j = \varnothing$ for all $i, j \in \Lambda$ and $i \neq j$, then $\widetilde{\cup}_{i\in\Lambda}(G_i, B_i)$ is a soft subhyperring (hyperideal)of (F, A);
(4) $\widetilde{\wedge}_{i\in\Lambda}(G_i, B_i)$ is a soft subhyperring (hyperideal) of the soft hyperring $\widetilde{\wedge}_{i\in\Lambda}(F, A)$ if it is non-null.

Proof. We only prove (1), and the proofs of (2)-(4) are similar. By Definition 2.6, we have $(H, C) = (\cap_{\mathscr{E}})_{i\in\Lambda}(G_i, B_i)$, where $C = \bigcup_{i\in\Lambda} B_i$, $H(x) = \bigcap_{i\in\Lambda(x)} G_i(x)$ and $\Lambda(x) = \{i \in \Lambda | x \in B_i\}$, for all $x \in C$. Since $(G_i, B_i)_{i\in\Lambda}$ be a non-empty family of soft subhyperrings (hyperideals) of (F, A), we have that $C = \bigcup_{i\in\Lambda} B_i \subseteq A$, and $H(x) = \bigcap_{i\in\Lambda(x)} G_i(x)$ is a subhyperring (hyperideal) of $F(x)$, for all $x \in \mathrm{Supp}(H, C)$. Therefore, $(H, C) = (\cap_{\mathscr{E}})_{i\in\Lambda}(G_i, B_i)$ is a soft subhyperring (hyperideal) of (F, A).

3.15. Theorem. Let φ be a strong homomorphism from hyperring R_1 to hyperring R_2. If (F, A) is a soft hyperring over R_1, and (G, B) is a soft subhyperring (hyperideal) of (F, A), then $(\varphi(G), B)$ is a soft subhyperring (hyperideal) of $(\varphi(F), A)$.

Proof. From Theorem 3.9, we have that $(\varphi(F), A)$ and $(\varphi(G), B)$ are soft hyperrings over R_2. Clearly, $\mathrm{Supp}(\varphi(G), B) = \mathrm{Supp}(G, B)$. It follows that $B \subseteq A$ and $G(x)$ is a subhyperring of $F(x)$ for all $x \in \mathrm{Supp}(G, B)$, because (G, B) is a soft subhyperring of (F, A). So $\varphi(G)(x) \subseteq \varphi(F)(x)$ for all $x \in \mathrm{Supp}(\varphi(G), B)$. According to Theorem 3.13, $(\varphi(G), B)$ is a soft subhyperring of $(\varphi(F), A)$.

Now, for all $x \in \mathrm{Supp}(\varphi(G), B)$, $r' \in \varphi(F)(x)$, $a' \in \varphi(G)(x)$, there exists $r \in F(x)$, $a \in G(x)$ such that $\varphi(r) = r'$, $\varphi(a) = a'$. Because $G(x)$ is a hyperideal of $F(x)$, we have that $r' \cdot a' = \varphi(r) \cdot \varphi(a) = \varphi(r \cdot a) \in \varphi(G(x)) = \varphi(G)(x)$ and $a' \cdot r' = \varphi(a) \cdot \varphi(r) = \varphi(a \cdot r) \in \varphi(G(x)) = \varphi(G)(x)$. It follows that $\varphi(G)(x)$ is a hyperideal of $\varphi(F)(x)$ for all $x \in \mathrm{Supp}(\varphi(G), B)$. Therefore, $(\varphi(G), B)$ is a soft hyperideal of $(\varphi(F), A)$.

4. Isomorphism theorems of soft hyperrings

In this section, we consider the isomorphism theorems of soft hyperrings. First, we give the notions of soft homomorphism, soft monomorphism, soft epimorphism, and soft isomorphism.

4.1. Definition. Let (F, A) and (G, B) be soft hyperrings over hyperring R_1 and hyperring R_2, respectively, and $\varphi : R_1 \to R_2$ and $\psi : A \to B$ be two mappings. If φ is a strong homomorphism and for all $x \in A$, $\varphi(F(x)) = G(\psi(x))$, then (φ, ψ) is called a soft homomorphism, and (F, A) is soft homomorphic to (G, B), denoted by $(F, A) \sim (G, B)$. If φ is a monomorphism (resp. epimorphism, isomorphism) and ψ is a injective (resp. surjective, bijective) mapping, then (φ, ψ) is called a soft monomorphism (resp. epimorphism, isomorphism), and (F, A) is soft monomorphic (resp. epimorphic, isomorphic) to (G, B). $(F, A) \simeq (G, B)$ is used to denote that (F, A) is soft isomorphic to (G, B).

4.2. Theorem. Let (F, A) and (G, B) be soft hyperrings over hyperring R_1 and hyperring R_2, respectively, and (F, A) be soft epimorphic to (G, B). If (F, A) is an idealistic soft hyperring over R_1, then (G, B) is an idealistic soft hyperring over R_2.

Proof. Suppose that (φ, ψ) is a soft epimorphism from (F, A) to (G, B). For every $x \in \mathrm{Supp}(F, A)$, $F(x)$ is a hyperideal of R_1, by Theorem 3.9, we have that $\varphi(F(x))$ a hyperideal of R_2. For every $y \in \mathrm{Supp}(G, B)$, there exists $x \in A$ such that $\psi(x) = y$, so $G(y) = G(\psi(x)) = \varphi(F(x))$ is a hyperideal of R_2. It follows that (G, B) is an idealistic soft hyperring over R_2.

In what follows, we say that $(F/I, A)$ is a soft hyperring over R/I, which means $(F/I)(x) = F(x)/I$ for all $x \in A$, $I \subseteq F(x)$ for all $x \in \mathrm{Supp}(F, A)$, and $(F/I)(x) = \varnothing$ for $x \in A-$ $\mathrm{Supp}(F, A)$, where (F, A) is a soft hyperring over R, and I is a normal hyperideal of R.

4.3. Theorem. Let I be a normal hyperideal of R, and (F, A) be a soft hyperring over R, then (F, A) is soft epimorphic to $(F/I, A)$.

Proof. Since $I \subseteq F(x)$ for all $x \in \mathrm{Supp}(F, A)$, it follows that $F(x)/I$ is a subhyperring of R/I. So $(F/I, A)$ is a soft hyperring over R/I. Define $\varphi : R \to R/I$ by $\varphi(x) = I^*[x]$, for all $x \in R$, then φ is an epimorphism. Define $\psi : A \to A$ by $\psi(x) = x$ for all $x \in A$. Clearly, ψ is surjective. For all $x \in A$, $\varphi(F(x)) = F(x)/I = F(\psi(x))/I$. Therefore, (φ, ψ) is a soft epimorphism, and (F, A) is soft epimorphic to $(F/I, A)$.

4.4. Theorem. (*First Isomorphism Theorem*) Let (F, A) and (G, B) be soft hyperrings over hyperring R_1 and hyperring R_2, respectively. If (φ, ψ) is a soft epimorphism from (F, A) to (G, B) with kernel I such that I is a normal hyperideal of R_1, then $(F/I, A) \simeq (\varphi(F), A)$. Moreover, if ψ is bijective, then $(F/I, A) \simeq (G, B)$.

Proof. Clearly, $(F/I, A)$ and $(\varphi(F), A)$ are soft hyperrings over R_1/I and R_2, respectively. We define $\varphi' : R_1/I \to R_2$ by $\varphi'(I^*[x]) = \varphi(x)$, for all $x \in R_1$. According to the first isomorphism theorem of hyperrings, φ' is an isomorphism. Define $\psi' : A \to A$ by $\psi'(x) = x$ for all $x \in A$, then ψ' is bijective. Also $\varphi'(F(x)/I) = \varphi(F(x)) = \varphi(F(\psi'(x)))$ for all $x \in A$. It follows that (φ', ψ') is a soft isomorphism, and $(F/I, A) \simeq (\varphi(F), A)$. Moreover, since φ' is an isomorphism, ψ is bijective and for all $x \in A$, $\varphi'(F(x)/I) = \varphi(F(x)) = G(\psi(x))$. So (φ', ψ) is a soft isomorphism, and $(F/I, A) \simeq (G, B)$.

4.5. Theorem. (*Second Isomorphism Theorem*) Let I and K be hyperideals of R, with I normal in R. If (F, A) is a soft hyperring of K, then $(F/(I \cap K), A) \simeq ((I + F)/I, A)$.

Proof. Clearly, $(F/(I \cap K), A)$ and $((I+F)/I, A)$ are soft hyperring over $(K/(I \cap K)$ and $(I+K)/I$, respectively. $\varphi : K \to (I+K)/I$ is defined by $\varphi(x) = I^*[x]$ for all $x \in K$. Then φ is an epimorphism. $\psi : A \to A$ is defined by $\psi(x) = x$ for all $x \in A$. Then ψ is bijective. For all $x \in A$, we have $\varphi(F(x)) = \{I^*[a] \mid a \in F(x)\} = (I + F(x))/I = (I + F(\psi(x)))/I$. For $\{I^*[a] \mid a \in F(x)\} = (I + F(x))/I$, the proof is showed as follows.

Clearly, $\{I^*[a] \mid a \in F(x)\} \subseteq (I + F(x))/I$. For any $I^*[b] \in (I + F(x))/I$, where $b \in I + F(x)$, which implies that there exist $i \in I$ and $k \in F(x)$ such that $b \in i + k$, so $I^*[b] = I + b = I + i + k = I + k = I^*[k] \in \{I^*[a] \mid a \in F(x)\}$.

Therefore, (φ, ψ) is a soft epimorphism from (F, A) to $((I+F)/I, A)$. Since $I \cap K$ is a normal hyperideal of K, if $ker\varphi = I \cap K$, then, by Theorem 4.4, $(F/(I \cap K), A) \simeq ((I + F)/I, A)$. For any $x \in K$, $x \in ker\varphi \Leftrightarrow \varphi(x) = I^*[0] = I \Leftrightarrow I^*[x] = I + x = I \Leftrightarrow x \in I$ (since $x \in K$)$\Leftrightarrow x \in I \cap K$.

4.6. Theorem. (*Third Isomorphism Theorem*) Let K and I be normal hyperideals of R such that $I \subseteq K$. If (F, A) is a soft hyperring over R, and $K \subseteq F(x)$ for all $x \in \mathrm{Supp}(F, A)$, then $((F/I)/(K/I), A) \simeq (F/K, A)$.

Proof. We have that K/I is a normal hyperideal of R/I, because K and I are normal hyperideals of R with $I \subseteq K$. Thus, $(R/I)/(K/I)$ is defined. Since $F(x)$ is a subhyperring of R and $I \subseteq K \subseteq F(x)$ for all $x \in \mathrm{Supp}(F, A)$, $(F(x)/I)/(K/I)$ is defined and is a subhyperring of $(R/I)/(K/I)$. Clearly, $\mathrm{Supp}((F/I)/(K/I), A)= \mathrm{Supp}(F, A)$. It follows that $((F/I)/(K/I), A)$ is a soft hyperring over $(R/I)/(K/I)$. Also, it is easy to obtain that $(F/I, A)$ and $(F/K, A)$ are soft hyperrings over R/I and R/K, respectively. $\varphi : R/I \to R/K$, defined by $\varphi(I^*[x]) = K^*[x]$, is an epimorphism, and $\psi : A \to A$, defined by $\psi(x) = x$ for all $x \in A$, is bijective. Moreover, for all $x \in A$, $\varphi(F(x)/I) = F(x)/K = F(\psi(x))/K$. So (φ, ψ) is a soft epimorphism from $(F/I, A)$ to $(F/K, A)$. By Theorem 4.4, if $kerf = K/I$, then $((F/I)/(K/I), A) \simeq (F/K, A)$. For any $I^*[x] \in R/I$, $I^*[x] \in kerf \Leftrightarrow f(I^*[x]) = K^*[0] = K \Leftrightarrow K^*[x] = K + x = K \Leftrightarrow x \in K \Leftrightarrow I^*[x] \in K/I$.

5. Fuzzy isomorphism theorems of soft hyperrings

In this scetion, we eatablish three fuzzy isomorphism theorems of soft hyperrings. Firstly, we review some related results about fuzzy hyperideal of hyperrings [22, 35].

A fuzzy set μ of a hyperring R is called a fuzzy hyperideal of R if the following conditions hold: (1) $\min\{\mu(x), \mu(y)\} \leq \inf_{z\in x+y} \mu(z)$ for all $x, y \in R$; (2) $\mu(x) \leq \mu(-x)$ for all $x \in R$; (3) $\max\{\mu(x), \mu(y)\} \leq \mu(xy)$ for all $x, y \in R$. A fuzzy hyperideal μ of R is called normal if $\mu(y) \leq \inf_{\alpha\in x+y-x} \mu(\alpha)$ for all $x, y \in R$.

Let μ be a normal fuzzy hyperideal of R. Define the relation on R: $x \equiv y(mod\ \mu)$ if and only if there exists $\alpha \in x - y$ such that $\mu(\alpha) = \mu(0)$, denoted by $x\mu^*y$, and μ^* is an equivalence relation. If $x\mu^*y$, then $\mu(x) = \mu(y)$. Let $\mu^*[x]$ be the equivalence class containing $x \in R$, and R/μ be the set of all equivalence classes, i.e., $R/\mu = \{\mu^*[x] \mid x \in R\}$. Define operations $\oplus$ and $\odot$ in R/μ by $\mu^*[x] \oplus \mu^*[y] = \{\mu^*[z] \mid z \in \mu^*[x] + \mu^*[y]\}$, and $\mu^*[x] \odot \mu^*[y] = \mu^*[x \cdot y]$, respectively. Then $(R/\mu, \oplus, \odot)$ is a hyperring.

Let I be a normal hyperideal of R, and μ be a normal fuzzy hyperideal of R. If μ is restricted to I, then μ is a normal fuzzy hyperideal of I, and I/μ is a normal hyperideal of R/μ. If μ and ν are normal fuzzy hyperideals of R, then $\mu\cap\nu$ is normal fuzzy hyperideals of R.

If X and Y are two non-empty sets, $\varphi : X \to Y$ is a mapping, and μ and ν are the fuzzy sets of X and Y, respectively, then the image $\varphi(\mu)$ of μ is the fuzzy subset of Y defined as follows: for all $y \in Y$, if $\varphi^{-1}(y) \neq \emptyset$, $\varphi(\mu)(y) = \sup\limits_{x\in\varphi^{-1}(y)} \{\mu(x)\}$; otherwise, $\varphi(\mu)(y) = 0$. The inverse image $\varphi^{-1}(\nu)$ of ν is the fuzzy subset of X defined by $\varphi^{-1}(\nu)(x) = \nu(\varphi(x))$ for all $x \in X$.

Let R_1 and R_2 be two hyperrings, and $\varphi : R_1 \to R_2$ be a strong homomorphism. If μ and ν are (normal) fuzzy hyperideals of R_1 and R_2, respectively, then (1) $\varphi(\mu)$ is a (normal) fuzzy hyperideal of R_2; (2) if φ is onto, then $\varphi^{-1}(\nu)$ is a (normal) fuzzy hyperideal of R_1. If μ and ν are normal fuzzy hyperideals of R_1 and R_2, respectively, then (1) if φ is onto, then $\varphi(\varphi^{-1}(\nu)) = \nu$; (2) if μ is a constant on $ker\varphi$, then $\varphi^{-1}(\varphi(\mu)) = \mu$. Let μ be a normal fuzzy hyperideal of R, then $R_\mu = \{x \in R \mid \mu(x) = \mu(0)\}$ is a normal hyperideal of R.

5.1. Theorem. (*First Fuzzy Isomorphism Theorem*) Let (F, A) and (G, B) be soft hyperrings over hyperring R_1 and hyperring R_2, respectively. If (φ, ψ) is a soft epimorphism from (F, A) to (G, B) and μ is a normal fuzzy hyperideal of R_1 with $(R_1)_\mu \supseteq ker\varphi$, then $(F/\mu, A) \simeq (\varphi(F)/\varphi(\mu), A)$, where $(F/\mu)(x) = F(x)/\mu$ for all $x \in A$. Moreover, if ψ is bijective, then $(F/\mu, A) \simeq (G/\varphi(\mu), B)$.

Proof. We obtain that $(F/\mu, A)$ is a soft hyperring over R_1/μ, since (F, A) is soft hyperring over R_1, and μ is a normal fuzzy hyperideal of R_1. For all $x \in \mathrm{Supp}(F, A)$, $\varphi(F(x)) = G(\psi(x))$ is a subhyperring of R_2, so $(\varphi(F)/\varphi(\mu), A)$ is a soft hyperring over $R_2/\varphi(\mu)$. $\varphi' : R_1/\mu \to R_2/\varphi(\mu)$ defined by $\varphi'(\mu^*[x]) = \varphi(\mu)^*[\varphi(x)]$, for all $x \in R_1$, is an isomorphism, by the first fuzzy isomorphism theorem of hyperrings. $\psi' : A \to A$ defined by $\psi'(x) = x$ for all $x \in A$, is bijective. Moreover, $\varphi'(F(x)/\mu) = \{\varphi(\mu)^*[a] \mid a \in \varphi(F(x))\} = \varphi(F(x))/\varphi(\mu) = \varphi(F(\psi'(x)))/\varphi(\mu)$, for all $x \in A$. It follows that (φ', ψ') is a soft isomorphism, and $(F/\mu, A) \simeq (\varphi(F)/\varphi(\mu), A)$.

Moreover, for all $x \in A$, we have that $\varphi'(F(x)/\mu) = \{\varphi(\mu)^*[a] \mid a \in \varphi(F(x))\} = \varphi(F(x))/\varphi(\mu) = G(\psi(x))/\varphi(\mu)$. φ' is an isomorphism, and ψ is bijective. It follows that (φ', ψ) is a soft isomorphism, and $(F/\mu, A) \simeq (G/\varphi(\mu), B)$.

5.2. Theorem. Let (F, A) and (G, B) be soft hyperrings over hyperring R_1 and hyperring R_2 respectively. If (φ, ψ) is a soft epimorphism from (F, A) to (G, B) and ν is a normal fuzzy hyperideal of R_2, then $(F/\varphi^{-1}(\nu), A) \simeq (\varphi(F)/\nu, A)$. Moreover, if ψ is bijective, then $(F/\varphi^{-1}(\nu), A) \simeq (G/\nu, B)$.

Proof. Since ν is a normal fuzzy hyperideal of R_2 and φ is an epimorphism, we have that $\varphi(\varphi^{-1}(\nu)) = \nu$ and $\varphi^{-1}(\nu)$ is a normal fuzzy hyperideal of R_1. Thus, $(F/\varphi^{-1}(\nu), A)$ and $(\varphi(F)/\nu, A)$ are soft hyperrings over hyperrings $R_1/\varphi^{-1}(\nu)$ and R_2/ν, respectively. For any $x \in ker\varphi$, $\varphi(x) = \varphi(0)$. It follows that $\nu(\varphi(x)) = \nu(\varphi(0))$, i.e., $\varphi^{-1}(\nu)(x) = \varphi^{-1}(\nu)(0)$, which implies that $x \in (R_1)_{\varphi^{-1}(\nu)}$. So $(R_1)_{\varphi^{-1}(\nu)} \supseteq ker\varphi$. By Theorem 5.1, we have $(F/\varphi^{-1}(\nu), A) \simeq (\varphi(F)/\nu, A)$. Furthermore, if ψ is bijective, then we have $(F/\varphi^{-1}(\nu), A) \simeq (G/\nu, B)$.

5.3. Theorem. (*Second Fuzzy Isomorphism Theorem*) Let (F, A) be a soft hyperring over R. If μ and ν are two normal fuzzy hyperideals with $\mu(0) = \nu(0)$, then $(F_\mu/(\mu \cap \nu), A) \simeq ((F_\mu + F_\nu)/\nu, A)$.

Proof. We have that $\mu \cap \nu$ and ν are normal fuzzy hyperideal of R_μ and $R_\mu + R_\nu$, respectively. It follows that $R_\mu/(\mu \cap \nu)$ and $(R_\mu + R_\nu)/\nu$ are hyperrings. Since (F, A) is a soft hyperring over R, we can obtain easily that $(F_\mu/(\mu \cap \nu), A)$ and $((F_\mu + F_\nu)/\nu, A)$ are soft hyperrings over $R_\mu/(\mu \cap \nu)$ and $(R_\mu + R_\nu)/\nu$, respectively. $\varphi : R_\mu/(\mu \cap \nu) \to (R_\mu + R_\nu)/\nu$ is defined by $\varphi((\mu \cap \nu)^*[x]) = \nu^*[x]$ for all $x \in R_\mu$. If $(\mu \cap \nu)^*[x] = (\mu \cap \nu)^*[y]$, then $(\mu \cap \nu)(x) = (\mu \cap \nu)(y)$, i.e., $\min\{(\mu(x), \nu(x)\} = \min\{(\mu(y), \nu(y)\}$. Because $x, y \in R_\mu$ and $\mu(0) = \nu(0)$, we have $\mu(x) = \mu(0) = \nu(0)$ and $\mu(y) = \mu(0) = \nu(0)$. So $\nu(x) = \nu(y)$. It follows that $\nu^*(x) = \nu^*(y)$. Thus, φ is well-defined. Moreover, we have

$$\begin{aligned}
&\varphi((\mu \cap \nu)^*[x] \oplus (\mu \cap \nu)^*[y]) = \varphi(\{(\mu \cap \nu)^*[z] \mid z \in (\mu \cap \nu)^*[x] + (\mu \cap \nu)^*[y]\}) \\
&= \{\nu^*[z] \mid z \in (\mu \cap \nu)^*[x] + (\mu \cap \nu)^*[y]\} = \nu^*((\mu \cap \nu)^*[x]) \oplus \nu^*((\mu \cap \nu)^*[y]) \\
&= \varphi((\mu \cap \nu)^*[x]) \oplus \varphi((\mu \cap \nu)^*[y]),
\end{aligned}$$

$$\varphi((\mu\cap\nu)^*[x]\odot(\mu\cap\nu)^*[y])=\varphi((\mu\cap\nu)^*[x\cdot y])=\nu^*[x\cdot y]$$
$$=\nu^*[x]\odot\nu^*[y]=\varphi((\mu\cap\nu)^*[x])\odot\varphi((\mu\cap\nu)^*[y])$$

and $\varphi((\mu\cap\nu)^*[0])=\nu^*[0]=0$. Consequently, φ is a homomorphism.

If $(\mu\cap\nu)^*[x]\neq(\mu\cap\nu)^*[y]$, we have $(\mu\cap\nu)(x)\neq(\mu\cap\nu)(y)$. It follows that $\nu(x)\neq\nu(y)$, so $\nu^*[x]\neq\nu^*[y]$. Hence, φ is a monomorphism. For any $\nu^*[x]\in(R_\mu+R_\nu)/\nu$, where $x\in R_\mu+R_\nu$, which implies that there exist $a\in R_\mu$ and $b\in R_\nu$ such that $x\in a+b$, there is $\alpha\in x-a\subseteq a+b-a\subseteq R_\nu$, i.e., $\nu(\alpha)=\nu(0)$. Hence we have $\nu^*[x]=\nu^*[a]$. So $\varphi((\mu\cap\nu)^*[a])=\nu^*[x]$, and φ is an epimorphism. Thus, φ is an isomorphism.

$\psi:A\to A$ defined by $\psi(x)=x$ for all $x\in A$, is bijective. For all $x\in A$, $\varphi(F_\mu(x)/(\mu\cap\nu))=F_\mu(x)/\nu=(F_\mu+F_\nu)(x)/\nu=(F_\mu+F_\nu)(\psi(x))/\nu$. The proof of $F_\mu(x)/\nu=(F_\mu+F_\nu)(x)/\nu$ is showed as follows.

Clearly, $F_\mu(x)/\nu\subseteq(F_\mu+F_\nu)(x)/\nu$. For all $\nu^*[a]\in(F_\mu+F_\nu)(x)/\nu$, where $a\in(F_\mu+F_\nu)(x)$, which implies that there exist $m\in F_\mu(x)$ and $n\in F_\nu(x)$ such that $a\in m+n$, there is $\alpha\in a-m\subseteq m+n-m\subseteq F_\nu(x)$, i.e., $\nu(\alpha)=\nu(0)$. It follows that $\nu^*[a]=\nu^*[m]\in F_\mu(x)/\nu$.

Therefore, (φ,ψ) is a soft isomorphism and $(F_\mu/\mu\cap\nu,A)\simeq((F_\mu+F_\nu)/\nu,A)$.

5.4. Theorem. (*Third Fuzzy Isomorphism Theorem*) Let (F,A) be a soft hyperring over a hyperring R. If μ and ν are two normal fuzzy hyperideals with $\nu\leq\mu$, $\mu(0)=\nu(0)$ and $F_\mu(x)=R_\mu$ for all $x\in$Supp(F,A), then $((F/\nu)/(F_\mu/\nu),A)\simeq(F/\mu,A)$.

Proof. We can easily deduce that R_μ/ν is a normal hyperideal of R/ν. Because (F,A) be a soft hyperring over R, we have that $(F/\nu,A)$, $((F/\nu)/(F_\mu/\nu),A)$ and $(F/\mu,A)$ are soft hyperrings over R/ν, $(R/\nu)/(R_\mu/\nu)$ and R/μ, respectively. $\varphi:R/\nu\to R/\mu$ is defined by $\varphi(\nu^*[x])=\mu^*[x]$ for all $x\in R$. If $\nu^*[x]=\nu^*[y]$ for all $x,y\in R$, then there exists $\alpha\in x-y$ such that $\nu(\alpha)=\nu(0)$. Because $\nu\leq\mu$ and $\mu(0)=\nu(0)$, we get $\mu(\alpha)\geq\nu(\alpha)=\nu(0)=\mu(0)$, which implies that $\mu(\alpha)=\mu(0)$. So we have $\mu^*[x]=\mu^*[y]$. Thus, φ is well-defined. Clearly, φ is an epimorphism. $\psi:A\to A$ defined by $g(x)=x$ for all $x\in A$, is bijective. For all $x\in A$, $\varphi(F(x)/\nu)=F(x)/\mu=F(\psi(x))/\mu$. Hence, (φ,ψ) is a soft epimorphism from $(F/\nu,A)$ to $(F/\mu,A)$. Moreover, $ker\varphi=\{\nu^*[x]\in R/\nu\mid\varphi(\nu^*[x])=\mu^*[0]\}=\{\nu^*[x]\in R/\nu\mid\mu^*[x]=\mu^*[0]\}=\{\nu^*[x]\in R/\nu\mid\mu(x)=\mu(0)\}=\{\nu^*[x]\in R/\nu\mid x\in R_\mu\}=R_\mu/\nu$. By Theorem 4.4, we have $((F/\nu)/(F_\mu/\nu),A)\simeq(F/\mu,A)$.

6. Conclusions

In this paper, we define soft hyperrings, idealistic soft hyperrings, soft subhyperrings and soft hyperideals, and introduce homomorphism and isomorphism of soft hyperrings. Furthermore, we generalize three (fuzzy) isomorphism theorems of hyperrings to three (fuzzy) isomorphism theorems of soft hyperrings. Based on these results, we will apply the notion of soft sets to other algebraic hyperstructures, and consider some applications of soft hyperrings in decision making problems.

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Oscillation theorems for fractional neutral differential equations

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Abstract

In this paper we study the oscillation of the fractional neutral differential equation

$$D_t^\alpha[a(t)D_t^\alpha(x(t)+p(t)x(\tau(t)))]+q(t)x(\sigma(t))=0,$$

where D_t^α is a modified Riemann-Liouville derivative. The obtained results are based on the new comparison theorems, which enable us to reduce the oscillatory problem of 2α-order fractional differential equation to the oscillation of the first order equation. The results are easily verified.

Keywords: Oscillation; Fractional differential equation; Modified Riemann-Liouville derivative; Comparison theorem

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1. Introduction

In this paper, we shall study the oscillation behavior of a class of fractional neutral differential equations with the form

$$D_t^\alpha[a(t)D_t^\alpha(x(t)+p(t)x(\tau(t)))]+q(t)x(\sigma(t))=0, \quad t\geq t_0>0,\ 0<\alpha<1, \tag{1.1}$$

where D_t^α denotes the modified Riemann-Liouville derivative [1] with respect to the variable t, $q(t)\in C([t_0,+\infty))$, $D_t^\alpha a(t)\in C([t_0,+\infty))$, $D_t^{2\alpha}p(t)\in C([t_0,+\infty))$, and we define $z(t)=x(t)+p(t)x(\tau(t))$. The equation also satisfies that:

(H_1) $a(t)>0,\ q(t)>0,\ 0\leq p(t)\leq p_0<\infty$;
(H_2) $\lim\limits_{t\to+\infty}\tau(t)=+\infty,\ \lim\limits_{t\to+\infty}\sigma(t)=+\infty$;
(H_3) $\tau'(t)\geq\tau_0>0,\ \tau\circ\sigma=\sigma\circ\tau$;
(H_4) $\frac{t}{\tau(t)}\geq l>0$.

In recent years, there has been much research activity concerning the fractional differential equation and many useful achievement have been obtained. Due to the fractional differential equation is more realistic in describing some practical models, it has been used widely in establishing mathematical models in electrochemistry, control, electromagnetic field theories and other natural phenomena and physical problems. Furthermore, it can also provide an excellent instrument for the description of memory and hereditary properties of various materials and processes due to the existence of a “memory" term in the model. Its initial and boundary value problems, stability of solutions, explicit and numerical solutions and many other properties have obtained significant development [2–6]. Particularly, the oscillation of fractional differential equations as a new research field has been received attention, and some interesting results have already been obtained. The relative works we refer to [7–17].

In 2012, Grace *et al.* [7] studied the oscillation theory for fractional differential equations by considering equations of the form

$$D_a^q x+f_1(t,x)=v(t)+f_2(t,x), \lim_{t\to a^+}J_a^{1-q}x(t)=b_1,$$

under the conditions

$$xf_i(t,x)>0 \quad \text{for} \quad i=1,2, x\neq 0, \quad \text{and} \quad t\geq a,$$

and

$$|f_1(t,x)|>p_1(t)|x|^\beta \quad \text{and} \quad |f_2(t,x)|>p_2(t)|x|^\gamma \quad \text{for} \quad x\neq 0, \quad \text{and} \quad t\geq a,$$

where D_a^q denotes the Riemann-Liouville differential operator of order q with $0<q\leq 1$, and the operator J_a^p is the Rieman-Liouville fractional integral operator. The authors obtained some new oscillation criteria by reducing the fractional differential equation to the equivalent Volterra fractional integral equation and by applying inequality technique.

In 2012, Chen *et al.* [8] studied the oscillatory behavior of the following fractional differential equation

$$[r(t)(D_-^\alpha y)^\eta(t)]'-q(t)f(\int_t^\infty(v-t)^{-\alpha}y(v)dv)=0 \quad \text{for} \quad t>0,$$

where $D_-^\alpha y$ denotes the Liouville right-sided fractional derivative of order α with the form

$$(D_-^\alpha y)(t):=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty(v-t)^{-\alpha}y(v)dv \quad \text{for} \quad t\in\mathbb{R}_+:=(0,\infty).$$

By the Riccati transformation technique the authors obtained some sufficient conditions, which guarantee that every solution of the equation is oscillatory.

Using the same method, in 2013, Chen [9] studied oscillatory behavior of the fractional differential equation in the form

$$(D_-^{1+\alpha}y)(t) - p(t)(D_-^{\alpha}y)(t) + q(t)f(\int_t^{\infty}(v-t)^{-\alpha}y(v)dv) = 0 \quad \text{for} \quad t > 0,$$

where $D_-^{\alpha}y$ is the Liouville right-sided fractional derivative of order $\alpha \in (0,1)$ of y.

Zheng [10] considered the oscillation of the nonlinear fractional differential equation with damping term

$$[a(t)(D_-^{\alpha}x(t))^{\gamma}]' + p(t)(D_-^{\alpha}x(t))^{\gamma} - q(t)f(\int_t^{\infty}(\xi-t)^{-\alpha}x(\xi)d\xi) = 0,\ t \in [t_0,\infty),$$

where $D_-^{\alpha}x(t)$ denotes the Liouville right-sided fractional derivative of order α of x. Using a generalized Riccati function and inequality technique, he established some new oscillation criteria.

Han *et al.* [11] considered the oscillation for a class of fractional differential equation

$$\left[r(t)g\left((D_-^{\alpha}y)(t)\right)\right]' - p(t)f\left(\int_t^{\infty}(s-t)^{-\alpha}y(s)ds\right) = 0, \quad \text{for} \quad t > 0,$$

where $0 < \alpha < 1$ is a real number, $D_-^{\alpha}y$ is the Liouville right-sided fractional derivative of order α of y. By generalized Riccati transformation technique, oscillation criteria for the nonlinear fractional differential equation are obtained.

In this paper we focus on the fractional neutral differential equations involving a modified Riemann-Liouville derivative, which is given by Jumarie in [1] (see also in [18–22]). The modified Riemann-Liouville derivative is defined as

$$D_t^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n \le \alpha < n+1,\ n \ge 1. \end{cases}$$

And it has some properties that

$$D_t^{\alpha}t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \tag{1.2}$$

$$D_t^{\alpha}(f(t)g(t)) = g(t)D_t^{\alpha}f(t) + f(t)D_t^{\alpha}g(t), \tag{1.3}$$

$$D_t^{\alpha}f[g(t)] = f_g'[g(t)]D_t^{\alpha}g(t) = D_g^{\alpha}f[g(t)](g'(t))^{\alpha}. \tag{1.4}$$

Due to having these especial properties, it can be more appropriately used in studying the oscillatory behavior of the fractional differential equations.

In [12], Feng *et al.* considered the fractional differential equation involving the derivative of this type in the form

$$D_t^{\alpha}[r(t)\psi(x(t))D_t^{\alpha}x(t)] + q(t)f(x(t)) = e(t),\ t \ge t_0 > 0,\ 0 < \alpha < 1,$$

where $D_t^{\alpha}(\cdot)$ denotes the modified Riemann-Liouville derivative. Based on a transformation of variables and properties of the modified Riemann-Liouville derivative, they transformed the fractional differential equation into a second-order ordinary differential equation. Then by a generalized Riccati transformation, inequalities, and an integration average technique, they established some oscillation criteria for the fractional differential equation.

In [13], Liu *et al.* concerned with oscillation of a class of fractional differential equations under the modified Riemann-Liouville derivative

$$D_t^{\alpha}[a(t)(D_t^{\alpha}(r(t)D_t^{\alpha}x(t)))^{\gamma}] + q(t)f(x(t)) = 0,\ t \ge t_0 > 0,\ 0 < \alpha < 1,$$

where $D_t^\alpha(\cdot)$ denotes the modified Riemann-Liouville derivative and they put some sufficient conditions about the oscillation of the equation.

Although the oscillation of fractional differential equation has been initiated to study by some authors, to the best of our knowledge very little is known in the literature regarding the oscillatory behavior of fractional neutral differential equations up to now.

Regarding the integer case of our equation (1.1), that is, $\alpha = 1$, B. Baculíková *et al.* in their article [23] have studied the second-order neutral differential equation

$$(r(t)[x(t) + p(t)x(\tau(t))]')' + q(t)x(\sigma(t)) = 0. \tag{1.5}$$

By comparison theorem, they established some oscillation criteria for the equation (1.5). They proved that: when $\sigma(t) \le t$, if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^t Q(s)R(\sigma(s))ds > \frac{\tau_0 + p_0}{\tau_0 e},$$

where $Q(t) = \min\{q(t), q(\tau(t))\}$, $R(t) = \int_{t_0}^t \frac{1}{r(s)}ds$, then (1.5) is oscillatory.

Moreover, in article [24], B. Baculíková *et al.* investigated the oscillation for the nonlinear case. They studied the equation in the form

$$(a(t)[z'(t)]^\gamma)' + q(t)x^\beta(\sigma(t)) = 0, \tag{1.6}$$

where $z(t) = x(t) + p(t)x(\tau(t))$. Also by comparison theorem, they established some sufficiently conditions for the oscillation of equation (1.6).

In this paper we will consider the oscillation of fractional neutral differential equation (1.1). Comparing to the method used by Feng and Liu [12,13], we will reduce a fractional differential equations to an integer one by appropriate variable transforms and establish some new comparison theorems and then use them to reduce the problem of the fractional order differential equation to the problem of second-order differential equations. In order to treat the delay or advance term in our equations, in this paper, we establish some new variable transformations so that the variable transformation method in [12, 13] can be applied for more classes of fractional differential equations, such as fractional neutral differential equations and fractional differential equations with delays. We also extend B. Baculíková and J. Džurina's results to the fractional order differential equations.

We organize this article as follows. In the next section, we give a transformation of variables to the fractional differential equation similar to that in the references [12, 13], and provide a new transformation on account of the delay term. So we can translate our fractional neutral differential equation to a second-order neutral differential equation. In Section 3, we first establish some new comparison theorems and then use them to get some sufficient conditions for oscillation of all solutions of (1.1). At the last we provide some examples to show applications of our criteria.

A solution of the equation is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is nonoscillatory. Equation is said to be oscillatory if all its solutions are oscillatory.

2. Some preliminary lemmas

First we will use a variable substitution. Denote $\xi = y(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$, $\xi_i = y(t_i) = \frac{t_i^\alpha}{\Gamma(1+\alpha)}$, $i = 0, 1$, $x(t) = \tilde{x}(\xi)$, $a(t) = \tilde{a}(\xi)$, $p(t) = \tilde{p}(\xi)$, $q(t) = \tilde{q}(\xi)$.

Towards to $\tau(t)$, $\sigma(t)$, we have the next transformations.

2.1. Lemma. *Suppose* (H_3), (H_4) *hold, we define the functions* $\tilde{\tau}(\xi)$, $\tilde{\sigma}(\xi)$ *as the following forms*

$$\tilde{\tau}(\xi) = y(\tau(y^{-1}(\xi))),$$

$$\tilde{\sigma}(\xi) = y(\sigma(y^{-1}(\xi))),$$

then it satisfies

$$x(\tau(t)) = \tilde{x}(\tilde{\tau}(\xi)),\ x(\sigma(t)) = \tilde{x}(\tilde{\sigma}(\xi));$$

and a new condition

$$(H_3'): \quad \tilde{\tau}'(\xi) \geq \tau_0 l^{1-\alpha} = \tilde{\tau}_0 > 0,\ \tilde{\tau} \circ \tilde{\sigma} = \tilde{\sigma} \circ \tilde{\tau}.$$

Proof. From the defines of $\tilde{\tau}$, $\tilde{\sigma}$ we get

$$\tilde{x}(\tilde{\tau}(\xi)) = \tilde{x}(y(\tau(y^{-1}(\xi)))) = \tilde{x}(y(\tau(t))).$$

Due to

$$x(t) = \tilde{x}(\xi) = \tilde{x}(y(t)),$$

substituting t with $\tau(t)$ we get

$$\tilde{x}(y(\tau(t))) = x(\tau(t)).$$

Thus

$$\tilde{x}(\tilde{\tau}(\xi)) = x(\tau(t)).$$

The same is

$$\tilde{x}(\tilde{\sigma}(\xi)) = x(\sigma(t)).$$

On the other hand, from H_3, H_4 and the defines of $\tilde{\tau}$ we get

$$\tilde{\tau}\circ\tilde{\sigma} = y(\tau(y^{-1}(\tilde{\sigma}(\xi)))) = y(\tau(y^{-1}(y(\sigma(y^{-1}(\xi)))))) = y(\tau(\sigma(y^{-1}(\xi)))) = y(\sigma(\tau(y^{-1}(\xi)))) = \tilde{\sigma}\circ\tilde{\tau}.$$

Also we have that,

$$\begin{aligned}
\tilde{\tau}'(\xi) = \tfrac{\partial}{\partial \xi} y(\tau(y^{-1}(\xi))) \quad &= \tfrac{\partial y(\tau(y^{-1}(\xi)))}{\partial \tau(y^{-1}(\xi))} \times \tfrac{\partial \tau(y^{-1}(\xi))}{\partial y^{-1}(\xi)} \times \tfrac{\partial y^{-1}(\xi)}{\partial \xi} \\
&= \tfrac{\partial y(\tau(t))}{\partial \tau(t)} \times \tfrac{\partial \tau(t)}{\partial t} \times \tfrac{\partial y^{-1}(\xi)}{\partial \xi} \\
&\geq \tfrac{\alpha(\tau(t))^{\alpha-1}}{\Gamma(1+\alpha)} \times \tau_0 \times \tfrac{1}{\alpha}(\Gamma(1+\alpha))^{\frac{1}{\alpha}} \xi^{\frac{1}{\alpha}-1} \\
&= \tfrac{\alpha(\tau(t))^{\alpha-1}}{\Gamma(1+\alpha)} \times \tau_0 \times \tfrac{1}{\alpha}(\Gamma(1+\alpha))^{\frac{1}{\alpha}} (\tfrac{t^\alpha}{\Gamma(1+\alpha)})^{\frac{1}{\alpha}-1} \\
&= \tau_0 (\tfrac{t}{\tau(t)})^{1-\alpha} \\
&\geq \tau_0 l^{1-\alpha} = \tilde{\tau}_0.
\end{aligned}$$

The proof is complete.

2.2. Lemma. *If x(t) is a eventually positive solution of* (1.1)*, and a sufficient large* t_1 *such that*

$$(2.1) \qquad R(t) = \int_{t_1}^{t} \frac{1}{a(s)} ds \to +\infty \qquad \text{as } t \to +\infty,$$

then the corresponding function $z(t) = x(t) + p(t)x(\tau(t))$ *satisfies*

$$z(t) > 0, \qquad a(t) D_t^\alpha(z(t)) > 0, \qquad D_t^\alpha[a(t) D_t^\alpha(z(t))] < 0,$$

eventually.

Proof. Let $x(t) = \tilde{x}(\xi)$, where $\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}$. Then from (1.2) we get $D_t^\alpha \xi(t) = 1$, and furthermore by use of (1.4) and Lemma 2.1 we have

$$D_t^\alpha x(t) = D_t^\alpha \tilde{x}(\xi) = \tilde{x}'(\xi) D_t^\alpha \xi(t) = \tilde{x}'(\xi),$$

$$D_t^\alpha x(\tau(t)) = D_t^\alpha \tilde{x}(\tilde{\tau}(\xi)) = (\tilde{x}(\tilde{\tau}(\xi)))' D_t^\alpha \xi(t) = (\tilde{x}(\tilde{\tau}(\xi)))'.$$

Similarly we have $D_t^\alpha a(t) = \tilde{a}'(\xi)$, $D_t^\alpha p(t) = \tilde{p}'(\xi)$, $D_t^\alpha q(t) = \tilde{q}'(\xi)$ and $D_t^\alpha x(\sigma(t)) = (\tilde{x}(\tilde{\sigma}(\xi)))'$. Then we get $D_t^\alpha z(t) = (\tilde{x}(\xi) + \tilde{p}(\xi)\tilde{x}(\tilde{\tau}(\xi)))'$. We define $\tilde{z}(\xi) = \tilde{x}(\xi) + \tilde{p}(\xi)\tilde{x}(\tilde{\tau}(\xi))$, then $D_t^\alpha z(t) = \tilde{z}'(\xi)$. So the equation (1.1) can be transformed into the following form:

$$(\tilde{a}(\xi)\tilde{z}'(\xi))' + \tilde{q}(\xi)\tilde{x}(\tilde{\sigma}(\xi)) = 0, \qquad \xi \geq \xi_0 > 0. \tag{2.2}$$

Since $x(t)$ is an eventually positive solution of (1.1), $\tilde{x}(\xi)$ is an eventually positive solution of (2.2). Hence there exists $\xi_1 > \xi_0$ such that $\tilde{x}(\xi) > 0$ on $[\xi_1, \infty)$. Also we know $\tilde{z}(\xi) > 0$ on $[\xi_1, \infty)$. It follows from (2.2) that

$$(\tilde{a}(\xi)\tilde{z}'(\xi))' = -\tilde{q}(\xi)\tilde{x}(\tilde{\sigma}(\xi)) < 0,$$

holds eventually. Consequently, $\tilde{a}(\xi)\tilde{z}'(\xi)$ is decreasing and thus either $\tilde{z}'(\xi) > 0$ or $\tilde{z}'(\xi) < 0$ eventually. We claim $\tilde{z}'(\xi) > 0$. Otherwise if $\tilde{z}'(\xi) < 0$, then also $\tilde{a}(\xi)\tilde{z}'(\xi) < -c < 0$ and integrating this from ξ_1 to ξ, we have

$$\tilde{z}(\xi) \leq \tilde{z}(\xi_1) - c\int_{\xi_1}^{\xi} \frac{1}{\tilde{a}(s)}ds = \tilde{z}(\xi_1) - c\int_{t_1}^{t} \frac{1}{a(s)}ds \to -\infty \quad \text{as} \quad t \to +\infty.$$

This contradicts the positivity of $\tilde{z}(\xi)$ and the proof is complete.

3. Main results

To simplify our notation, let us denote

$$Q(\xi) = \min\{\tilde{q}(\xi), \tilde{q}(\tilde{\tau}(\xi))\}, \quad Q^*(\xi) = Q(\xi)\int_{\xi_1}^{\xi} \frac{1}{\tilde{a}(s)}ds, \tag{3.1}$$

where ξ_1 is defined in Lemma 2.2.

3.1. Theorem. *If the first order neutral differential inequality*

$$(u(t) + \frac{p_0}{\tilde{\tau}_0}u(\tilde{\tau}(t)))' + Q^*(t)u(\tilde{\sigma}(t)) \leq 0, \qquad t \geq \xi_1 = \frac{t_1^\alpha}{\Gamma(1+\alpha)}, \tag{3.2}$$

where $\tilde{\tau}(t)$ is defined in Lemma 2.1, $Q^*(t)$ is defined in (3.1)*, has no positive solution, then* (1.1) *is oscillatory.*

Proof. Assume to the contrary that there exists a non-oscillatory solution x of equation (1.1). Without loss of generality, we only consider the case when $x(t)$ is eventually positive, since the case when $x(t)$ is eventually negative is similar. Then let $x(t) > 0$ on $[t_1, \infty)$. It is equivalent to $\tilde{x}(\xi) > 0$ on $[\xi_1, \infty)$. Then from (H_1) and (H_3') the corresponding function $\tilde{z}(\xi)$ satisfies

$$\begin{aligned} \tilde{z}(\tilde{\sigma}(\xi)) &= \tilde{x}(\tilde{\sigma}(\xi)) + \tilde{p}(\tilde{\sigma}(\xi))\tilde{x}(\tilde{\tau}(\tilde{\sigma}(\xi))) \\ &\leq \tilde{x}(\tilde{\sigma}(\xi)) + p_0\tilde{x}(\tilde{\sigma}(\tilde{\tau}(\xi))). \end{aligned} \tag{3.3}$$

On the other hand from (2.2) we have

$$0 = (\tilde{a}(\xi)\tilde{z}'(\xi))' + \tilde{q}(\xi)\tilde{x}(\tilde{\sigma}(\xi)), \tag{3.4}$$

which in view of (H_1) and (H_3') yields

$$\begin{aligned} 0 &= \tfrac{p_0}{\tilde{\tau}'(\xi)}(\tilde{a}(\tilde{\tau}(\xi))\tilde{z}'(\tilde{\tau}(\xi)))' + p_0\tilde{q}(\tilde{\tau}(\xi))\tilde{x}(\tilde{\sigma}(\tilde{\tau}(\xi))) \\ &\geq \tfrac{p_0}{\tilde{\tau}_0}(\tilde{a}(\tilde{\tau}(\xi))\tilde{z}'(\tilde{\tau}(\xi)))' + p_0\tilde{q}(\tilde{\tau}(\xi))\tilde{x}(\tilde{\sigma}(\tilde{\tau}(\xi))). \end{aligned} \tag{3.5}$$

Then combining (3.4) and (3.5) we get

$$(\tilde{a}(\xi)\tilde{z}'(\xi))' + \tilde{q}(\xi)\tilde{x}(\tilde{\sigma}(\xi)) + \frac{p_0}{\tilde{\tau}_0}(\tilde{a}(\tilde{\tau}(\xi))\tilde{z}'(\tilde{\tau}(\xi)))' + p_0\tilde{q}(\tilde{\tau}(\xi))\tilde{x}(\tilde{\sigma}(\tilde{\tau}(\xi))) \leq 0. \tag{3.6}$$

Furthermore using (3.1) and (3.3) we obtain

$$(3.7)\qquad (\tilde{a}(\xi)\tilde{z}'(\xi))' + \frac{p_0}{\tilde{\tau}_0}(\tilde{a}(\tilde{\tau}(\xi))\tilde{z}'(\tilde{\tau}(\xi)))' + Q(\xi)\tilde{z}(\tilde{\sigma}(\xi)) \le 0,$$

where $Q(\xi)$ is defined in (3.1). Now we denote $u(\xi) = \tilde{a}(\xi)\tilde{z}'(\xi)$. From Lemma 2.2 we get $u(\xi) > 0$ eventually. Also we have

$$(3.8)\qquad \tilde{z}(\xi) \ge \int_{\xi_1}^{\xi} \frac{\tilde{a}(s)\tilde{z}'(s)}{\tilde{a}(s)}ds \ge \tilde{a}(\xi)\tilde{z}'(\xi)\int_{\xi_1}^{\xi}\frac{1}{\tilde{a}(s)}ds = u(\xi)\int_{\xi_1}^{\xi}\frac{1}{\tilde{a}(s)}ds.$$

Then taking (3.8) into (3.7) we get that $u(\xi)$ is a positive solution of

$$(u(\xi) + \frac{p_0}{\tilde{\tau}_0}u(\tilde{\tau}(\xi)))' + Q^*(\xi)u(\tilde{\sigma}(\xi)) \le 0,$$

which is a contradiction and the proof is complete.

Next, by using the conclusion of Theorem 3.1, we will deduce oscillatory problem of our equation into the problem of first-order nonlinear delay differential equations, and establish some new oscillatory criteria for equation (1.1). We shall discuss both cases when τ is a delayed or advanced argument.

3.2. Theorem. *Assume that $\tau(t) \ge t$ and $\sigma(t) \le t$ is increasing. Assumptions $(H_1)-(H_4)$ hold. Then if the first-order delay differential equation*

$$(3.9)\qquad w'(\xi) + \frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0}Q^*(\xi)w(\tilde{\sigma}(\xi)) = 0$$

is oscillatory, the equation (1.1) *is oscillatory.*

Proof. We assume that $x(t)$ is a positive solution of (1.1) eventually. Then it follows from Lemma 2.2 and the proof of Theorem 3.1 that $u(\xi) = \tilde{a}(\xi)\tilde{z}'(\xi) > 0$ is decreasing eventually and satisfies (3.2). We define

$$(3.10)\qquad w(\xi) = u(\xi) + \frac{p_0}{\tilde{\tau}_0}u(\tilde{\tau}(\xi)).$$

From the definition of $\tilde{\tau}(\xi)$ and $\tau(t) \ge t$, we can easily get that $\tilde{\tau}(\xi) \ge \xi$. Similarly we have $\tilde{\sigma}(\xi) \le \xi$. Then

$$w(\xi) \le u(\xi)(1 + \frac{p_0}{\tilde{\tau}_0}),$$

$$\frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0}w(\xi) \le u(\xi).$$

Substituting this into (3.2), we get that $w(\xi)$ is the positive solution of the delay differential inequality

$$(3.11)\qquad w'(\xi) + \frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0}Q^*(\xi)w(\tilde{\sigma}(\xi)) \le 0.$$

Then from [25, Theorem 1] we know that the equation (3.9) also has a positive solution, which is a contradiction. The proof is complete.

3.3. Theorem. *Assume that $\tau(t) \le t$ and $\sigma(t) \le \tau(t) \le t$. Conditions $(H_1)-(H_4)$ hold. Then if the first-order delay differential equation*

$$(3.12)\qquad w'(\xi) + \frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0}Q^*(\xi)w(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))) = 0$$

is oscillatory, the equation (1.1) *is oscillatory.*

Proof. We assume that $x(t)$ is a positive solution of (1.1) eventually. Then it follows from Lemma 2.2 and the proof of Theorem 3.1 that $u(\xi) = \tilde{a}(\xi)\tilde{z}'(\xi) > 0$ is decreasing eventually and satisfies (3.2). Also from Lemma 2.1 we have

$$\tilde{\sigma}(\xi) \le \tilde{\tau}(\xi) \le \xi.$$

Then it follows from (3.10) that

$$w(\xi) \le u(\tilde{\tau}(\xi))(1 + \frac{p_0}{\tilde{\tau}_0}),$$

which is equivalent to

$$u(\tilde{\sigma}(\xi)) \ge \frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0} w(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))).$$

Substituting this into (3.2), we obtain that $w(\xi)$ is a positive solution of the delay differential inequality

$$w'(\xi) + \frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0} Q^*(\xi) w(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))) \le 0.$$

Then from [25, Theorem 1] we know that the equation (3.12) also has a positive solution, and a contradiction. The proof is complete.

Next we will give some sufficient conditions such that equations (3.9) and (3.12) have only oscillatory solutions.

3.4. Lemma. *Assume that $e(\xi)$ is a positive continuous function on $[\xi_0, \infty)$. If*

$$\liminf_{\xi\to\infty} \int_{\tilde{\sigma}(\xi)}^{\xi} e(s)ds > \frac{1}{e}, \tag{3.13}$$

then the first-order delay differential equation

$$w'(\xi) + e(\xi)w(\tilde{\sigma}(\xi)) = 0 \tag{3.14}$$

is oscillatory.

Proof. From (3.13) we can get that

$$\int_{\xi_0}^{\infty} e(s)ds = +\infty. \tag{3.15}$$

Then assume to the contrary that there exists a positive solution $w(\xi)$ of equation (3.14) on $[\xi_1, \infty)$. Since $w(\xi)$ is decreasing, there exists $\lim_{\xi\to+\infty} w(\xi) = k \ge 0$. If $k > 0$, then integrating (3.14) from t_1 to t. We have

$$w(\xi_1) \ge \int_{\xi_1}^{\xi} e(s)w(\tilde{\sigma}(s))ds \ge k \int_{\xi_1}^{\xi} e(s)ds \to +\infty \qquad \text{as} \quad \xi \to +\infty.$$

This is a contradiction. So we get that $\lim_{\xi\to+\infty} w(\xi) = 0$. But from the Theorem 2.1.1 in [26], the condition (3.13) yields that the equation (3.14) has no positive solution, which is a contradiction. The proof is complete.

3.5. Theorem. *Let $\tau(t) \ge t$ and $\sigma(t) \le t$. Conditions $(H_1) - (H_4)$ hold. If*

$$\liminf_{\xi\to\infty} \int_{\tilde{\sigma}(\xi)}^{\xi} Q^*(s)ds > \frac{\tilde{\tau}_0 + p_0}{\tilde{\tau}_0 e}, \tag{3.16}$$

then (1.1) *is oscillatory.*

Proof. From the condition (3.16) and Lemma 3.4 we get that equation (3.9) is oscillatory. Then from Theorem 3.2 we have equation (1.1) is oscillatory, the proof is complete.

3.6. Theorem. *Let $\sigma(t) \le \tau(t) \le t$ and conditions $(H_1) - (H_4)$ hold. If*

$$\liminf_{\xi\to\infty} \int_{\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))}^{\xi} Q^*(s)ds > \frac{\tilde{\tau}_0 + p_0}{\tilde{\tau}_0 e}, \tag{3.17}$$

then (1.1) *is oscillatory.*

Proof. The proof is similar to the proof of Theorem 3.5.

4. Examples

In this section, we will show the application of our main results.
Example 4.1 Consider the fractional differential equation

$$D_t^{\frac{1}{2}}[\sqrt{t}D_t^{\frac{1}{2}}(x(t) + \frac{1}{t}x(t+3))] + tx(t-5) = 0, \qquad t \in [5, +\infty), \tag{4.1}$$

where $D_t^{\alpha}x(t)$ is the modified Riemann-Liouville differential operator. In (4.1), we set $a(t) = \sqrt{t}$, $p(t) = \frac{1}{t}$, $\tau(t) = t+3$, $q(t) = t$, $\sigma(t) = t-5$. Then using a variable substitution we have

$$\xi = y(t) = \frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}, \quad y^{-1}(\xi) = \Gamma^2(\frac{3}{2})\xi^2, \quad \xi_1 = \frac{\sqrt{5}}{\Gamma(\frac{3}{2})}.$$

And we also have

$$\tilde{a}(\xi) = a(y^{-1}(\xi)) = \Gamma(\frac{3}{2})\xi,$$

$$\tilde{\sigma}(\xi) = y(\sigma(y^{-1}(\xi))) = \frac{(\Gamma^2(\frac{3}{2})\xi^2 - 5)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} = (\xi^2 - \frac{5}{\Gamma^2(\frac{3}{2})})^{\frac{1}{2}},$$

$$\tilde{q}(\xi) = q(y^{-1}(\xi)) = \Gamma^2(\frac{3}{2})\xi^2.$$

Easily we see the equation (4.1) satisfies $(H_1) - (H_4)$, furthermore we have

$$\begin{cases} 0 \le p(t) = \frac{1}{t} \le \frac{1}{5} = p_0, \\ \tau_0 = (t+3)' = 1, \\ \lim\limits_{t\to\infty} \frac{t}{\tau(t)} = \frac{t}{t+3} = l = 1, \\ \tilde{\tau}_0 = \tau_0 l^{1-\frac{1}{2}} = 1. \end{cases}$$

We know $\tilde{q}(\xi)$ is increasing and $\tau(t) > t$, $\tilde{\tau}(\xi) > \xi$, so

$$Q(\xi) = \tilde{q}(\xi) = \Gamma^2(\frac{3}{2})\xi^2,$$

$$\begin{aligned} Q^*(\xi) &= \Gamma^2(\tfrac{3}{2})\xi^2 \textstyle\int_{\xi_1}^{\xi} \frac{1}{\Gamma(\frac{3}{2})s} ds \\ &= \Gamma^2(\tfrac{3}{2})\xi^2 (\tfrac{1}{\Gamma(\frac{3}{2})} \ln \xi - \tfrac{1}{\Gamma(\frac{3}{2})} \ln \xi_1). \end{aligned}$$

Then we have

$$\lim_{\xi\to\infty}\inf\int_{\tilde\sigma(\xi)}^{\xi} Q^*(s)ds$$

$$=\lim_{\xi\to\infty}\inf\int_{\tilde\sigma(\xi)}^{\xi}\Gamma(\tfrac{3}{2})(s^2\ln s-s^2\ln m)ds$$

$$=\lim_{\xi\to\infty}\inf[\tfrac{\Gamma(\frac{3}{2})}{3}(\xi^3\ln\tfrac{\xi}{m}-\tilde\sigma^3(\xi)\ln\tfrac{\tilde\sigma(\xi)}{m})-\tfrac{5}{9\Gamma(\frac{3}{2})}]$$

$$\geq\lim_{\xi\to\infty}\inf[\tfrac{\Gamma(\frac{3}{2})}{3}(\xi^3-\tilde\sigma^3(\xi))\ln\tfrac{\tilde\sigma(\xi)}{m})-\tfrac{5}{9\Gamma(\frac{3}{2})}]$$

$$\geq\lim_{\xi\to\infty}\inf[\tfrac{\Gamma(\frac{3}{2})}{3}(\xi^2-\tilde\sigma^2(\xi))\ln\tfrac{\tilde\sigma(\xi)}{m}-\tfrac{5}{9\Gamma(\frac{3}{2})}]$$

$$=\geq\lim_{\xi\to\infty}\inf[\tfrac{\Gamma(\frac{3}{2})}{3}\tfrac{5}{\Gamma^2(\frac{3}{2})}\ln\tfrac{\tilde\sigma(\xi)}{m}-\tfrac{5}{9\Gamma(\frac{3}{2})}]$$

$$=\infty>\tfrac{1+\frac{1}{5}}{e},$$

where $m=\xi_1=\frac{5^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}$. From Theorem 3.5 we get that (4.1) is oscillatory.

Example 4.2 Consider the fractional differential equation

$$D_t^{\frac{1}{3}}[tD_t^{\frac{1}{3}}(x(t)+2x(\frac{t}{2}))]+tx(\frac{t}{8})=0,\qquad t\in[1,+\infty), \tag{4.2}$$

where $D_t^{\alpha}x(t)$ is the modified Riemann-Liouville differential operator. In (4.2), we set $a(t)=t,\ p(t)=2,\ q(t)=t,\ \tau(t)=\frac{t}{2},\ \sigma(t)=\frac{t}{8}$. Then using a variable substitution we have

$$\xi=y(t)=\frac{t^{\frac{1}{3}}}{\Gamma(\frac{4}{3})},\quad y^{-1}(\xi)=\Gamma^3(\frac{4}{3})\xi^3,\quad \xi_1=\frac{1}{\Gamma(\frac{4}{3})}.$$

Then we get

$$\tilde a(\xi)=a(y^{-1}(\xi))=\Gamma^3(\frac{4}{3})\xi^3,$$

$$\tilde\sigma(\xi)=y(\sigma(y^{-1}(\xi)))=y(\frac{\Gamma^3(\frac{4}{3})\xi^3}{8})=\frac{\xi}{2},$$

$$\tilde q(\xi)=q(y^{-1}(\xi))=\Gamma^3(\frac{4}{3})\xi^3,$$

$$\tilde\tau(\xi)=y(\tau(y^{-1}(\xi)))=y(\frac{\Gamma^3(\frac{4}{3})\xi^3}{2})=\frac{\xi}{2^{\frac{1}{3}}},$$

$$\tilde\tau^{-1}(\tilde\sigma(\xi))=2^{\frac{1}{3}}\tilde\sigma(\xi)=\frac{\xi}{2^{\frac{2}{3}}}.$$

Easily we see the equation (4.2) satisfies $(H_1)-(H_4)$, and

$$\begin{cases}0\leq p(t)=2=p_0,\\ \tau_0=(\frac{t}{2})'=\frac{1}{2},\\ \lim\limits_{t\to\infty}\frac{t}{\tau(t)}=\frac{t}{\frac{t}{2}}=2=l,\\ \tilde\tau_0=\tau_0 l^{1-\frac{1}{3}}=2^{-\frac{1}{3}}.\end{cases}$$

In this time $\tilde q(\xi)$ is increasing and $\tau(t)<t,\ \tilde\tau(\xi)<\xi$, so

$$Q(\xi)=\tilde q(\tilde\sigma(\xi))=\Gamma^3(\frac{4}{3})\frac{\xi^3}{2},$$

$$Q^*(\xi) \quad = \Gamma^3(\tfrac{4}{3})\tfrac{\xi^3}{2}\int_{\xi_1}^{\xi} \tfrac{1}{\Gamma^3(\frac{4}{3})s}ds$$

$$= \tfrac{\xi^3}{2}(\ln\xi - \ln\xi_1).$$

Following from (3.17) we have

$$\lim_{\xi\to\infty}\inf\int_{\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))}^{\xi} Q^*(s)ds$$

$$= \lim_{\xi\to\infty}\inf\int_{\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))}^{\xi} \tfrac{s^3}{2}(\ln s - \ln s_1)ds$$

$$= \lim_{\xi\to\infty}\inf[\tfrac{1}{8}\xi^4(\ln\xi - \tfrac{1}{4} - \ln\xi_1) - \tfrac{1}{8}\cdot\tfrac{1}{2^{\frac{8}{3}}}t^4(\ln\xi - \tfrac{2}{3}\ln 2 - \tfrac{1}{4} - \ln\xi_1)]$$

$$= \infty > \tfrac{2^{-\frac{1}{3}}+2}{2^{-\frac{1}{3}}e}.$$

According to Theorem 3.6 we get that (4.2) is oscillatory.

5. Conclusion

We have established some new oscillation criteria for a fractional neutral differential equation. First we can see, the variable transformation used in ξ is very important, transforms a fractional differential equation into an ordinary differential equation of integer order. Then toward to this differential equation with neutral term, we solve it by the comparison theorem, such that we can judge whether its solutions oscillatory by investigating some first-order delay differential equations. And some classical results can be used easily. Finally, we note that the oscillation for other fractional differential equations possessing the modified Riemann-Liouville derivative can also be used this method.

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Volume 44 (6) (2015), 1489 – 1490

ERRATUM

The figure section of the paper *Simplicial homology groups of certain digital surfaces*, Hacettepe Journal of Mathematics and Statistics, Volume 44(5), (2015), 1011 – 1022 by Emel ÜNVER DEMİR and İsmet KARACA has to be as follows:

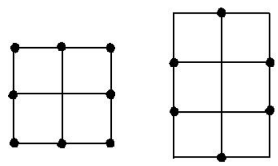

Figure 1. Minimal simple closed curves MSC_4 and MSC_8.

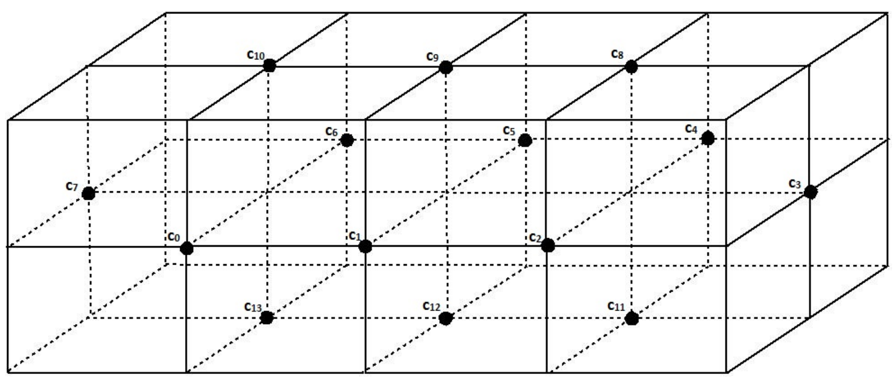

Figure 2. $MSS_{18}\sharp MSS_{18}$

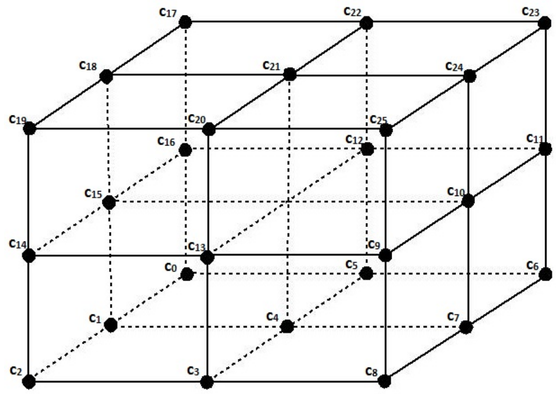

Figure 3. MSS_6

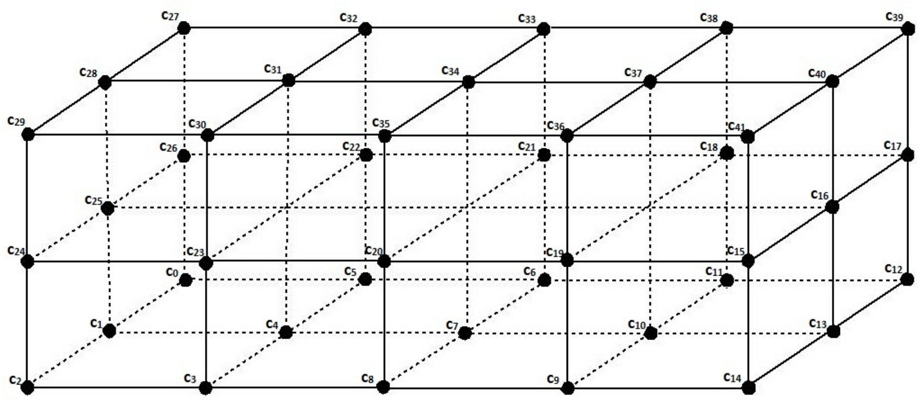

Figure 4. $MSS_6 \sharp MSS_6$

STATISTICS

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A new family of distributions: the Kumaraswamy odd log-logistic, properties and applications

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Abstract

In this paper, a new family of distributions, called the Kumaraswamy odd log-logistic, is proposed and studied. Some mathematical properties are presented and special models are discussed. The asymptotes and shapes are investigated. The family density function is given by a linear combination of exponentiated densities following the same baseline model. We derive a power series for the quantile function, explicit expressions for the moments, quantile and generating functions and order statistics. We provide a bivariate extension of the new family. Its performance is illustrated by means of two real data sets.

Keywords: Generated function; Log-logistic distribution; Maximum likelihood; Moment; Order statistic; Quantile function.

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1. Introduction

Recently, some attempts have been made to define new families of distributions that extend well-known distributions and at the same time provide great flexibility in modelling data in practice. So, several classes by adding one or more parameters to generate new distributions have been proposed in the statistical literature. Some well-known generators are the Marshall-Olkin generated family (MO-G) by Marshall and Olkin (1997), the beta-G by Eugene *et al.* (2002), the Kumaraswamy-G (Kw-G for short) by Cordeiro and de Castro (2011), the McDonald-G (Mc-G) by Alexander *et al.* (2012), the gamma-G by Zografos and Balakrishanan (2012), the transformer (T-X) by Alzaatreh *et al.* (2013), the Weibull-G by Bourguignon *et al.* (2014) and the exponentiated half-logistic family by Cordeiro *et al.* (2014).

Let $G(x;\boldsymbol{\xi})$ be a baseline cumulative distribution function (cdf) and $\boldsymbol{\xi}$ the $p \times 1$ vector of associated parameters. Recently, Gleaton and Lynch (2006) and da Cruz *et al.* (2014) introduced a class of distributions named the *odd log-logistic family* with one extra shape parameter $\alpha > 0$ defined by the cdf

$$H(x) = \frac{G(x;\boldsymbol{\xi})^{\alpha}}{G(x;\boldsymbol{\xi})^{\alpha} + \bar{G}(x;\boldsymbol{\xi})^{\alpha}}, \tag{1.1}$$

where $\bar{G}(x;\boldsymbol{\xi}) = 1 - G(x;\boldsymbol{\xi})$.

Let $r(t)$ be the probability density function (pdf) of a random variable $T \in [c,d]$ for $-\infty \le c < d < \infty$ and let $W[G(x)]$ be a function of the cdf of a random variable X such that $W[G(x)]$ satisfies the following conditions:

$$\begin{cases} (i) & W\left[G(x)\right] \in [c,d], \\ (ii) & W\left[G(x)\right] \text{is differentiable and monotonically non-decreasing, and} \\ (iii) & W\left[G(x)\right] \to c \ \text{ as } \ x \to -\infty \, \text{and} \, W\left[G(x)\right] \to d \ \text{ as } \ x \to \infty. \end{cases} \tag{1.2}$$

Alzaatreh *et al.* (2013) defined the T-X family of distributions by

$$F(x) = \int_{c}^{W[G(x)]} r(t)\, dt, \tag{1.3}$$

where $W[G(x)]$ satisfies the conditions (1.2). The pdf corresponding to (1.3) is given by

$$f(x) = \left\{ \frac{d}{dx} W[G(x)] \right\} r\left\{ W[G(x)] \right\}. \tag{1.4}$$

In this paper, we propose a new wider class of continuous distributions called the *Kumaraswamy odd log-logistic-G* ("KwOLL-G" for short) family by taking $W[G(x)] = \frac{G(x;\boldsymbol{\xi})^{\alpha}}{G(x;\boldsymbol{\xi})^{\alpha}+\bar{G}(x;\boldsymbol{\xi})^{\alpha}}$ and $r(t) = a\, b\, t^{a-1}\,(1-t^{a})^{b-1},\ \ 0 < t < 1$. Its cdf is given by

$$\begin{aligned} F(x) &= \int_{0}^{\frac{G(x;\boldsymbol{\xi})^{\alpha}}{G(x;\boldsymbol{\xi})^{\alpha}+\bar{G}(x;\boldsymbol{\xi})^{\alpha}}} a\, b\, t^{a-1}(1-t^{a})^{b-1} dt \\ &= 1 - \left\{ 1 - \left[\frac{G(x,\boldsymbol{\xi})^{\alpha}}{G(x,\boldsymbol{\xi})^{\alpha} + \bar{G}(x,\boldsymbol{\xi})^{\alpha}} \right]^{a} \right\}^{b}, \end{aligned} \tag{1.5}$$

where $\alpha > 0$, $a > 0$ and $b > 0$ are three extra shape parameters to the baseline cdf $G(x,\boldsymbol{\xi})$. The KwOLL-G family (1.5) includes the *Kumaraswamy generalized* family (Cordeiro and de Castro, 2011), the proportional and reversed hazard rate models, the *odd log-logistic* family (Gleaton and Lynch, 2006 and da Cruz *et al.,* 2014), among others. Some special models of (1.5) are listed in Table 1.

The paper is organized as follows. In Section 2, we provide a physical interpretation of the KwOLL-G family. Four special cases are described in Section 3 with some details. In Section 4, the asymptotes and shapes of the density and hazard rate functions are

Table 1. Some special models.

a	b	α	Reduced distribution
-	-	1	Kumaraswamy generalized family of distributions (Cordeiro and de Castro, 2011)
1	1	-	Odd log-logistic family (Gleaton and Lynch 2006 and da Cruz *et al.* 2014)
1	-	1	Proportional hazard rate model (Gupta *et al.,* 1998)
-	1	1	Proportional reversed hazard rate model (Gupta and Gupta, 2007)
1	1	1	$G(x)$

investigated analytically. Some useful expansions are obtained in Section 5. In Section 6, we derive a power series for the quantile function (qf). In Sections 7 and 8, we obtain the ordinary and incomplete moments and the generating function, respectively. The order statistics are derived in Section 9.
In Section 10, we introduce a bivariate extension of the new family. The estimation of the model parameters by maximum likelihood is performed in Section 11. Two applications to real data illustrate the potentiality of the proposed family in Section 12. Section 13 provides some conclusions.

2. The new family

The pdf corresponding to (1.5) is

$$f(x;a,b,\alpha,\boldsymbol{\xi}) = \frac{a\,b\,\alpha\,g(x,\boldsymbol{\xi})G(x,\boldsymbol{\xi})^{\alpha a-1}\bar{G}(x,\boldsymbol{\xi})^{\alpha-1}}{\left[G(x,\boldsymbol{\xi})^{\alpha}+\bar{G}(x,\boldsymbol{\xi})^{\alpha}\right]^{a+1}}\left\{1-\left[\frac{G(x,\boldsymbol{\xi})^{\alpha}}{G(x,\boldsymbol{\xi})^{\alpha}+\bar{G}(x,\boldsymbol{\xi})^{\alpha}}\right]^{a}\right\}^{b-1}, \tag{2.1}$$

where $g(x;\boldsymbol{\xi}) = dG(x;\boldsymbol{\xi})/dx$. Hereafter, $X \sim$ KwOLL-G$(a,b,\alpha,\boldsymbol{\xi})$ denotes a random variable having the density function (2.1). Further, we sometimes omit the dependence on the vector $\boldsymbol{\xi}$ and write simply $G(x) = G(x;\boldsymbol{\xi})$.

A physical interpretation of the KwOLL-G cdf (for a and b positive integers) is as follows. Equation (1.5) denotes the cdf of the lifetime of a series-parallel system consisting of independent components with the common cdf $H(x)$ given by (1.1). Consider that a system is formed by b independent series subsystems and that each of the subsystems is made up of a independent parallel components. Let $X_{ij} \sim H(x)$, for $1 \le i \le a$ and $1 \le j \le b$, denote the lifetime of the ith component in the jth subsystem and X denotes the lifetime of the entire system. We have

$$Pr(X \le x) = 1-\{1-Pr(X_{11}\le x,\cdots,X_{1a}\le x)\}^{b} = 1-\{1-Pr^{a}(X_{11}\le x)\}^{b},$$

and then X has pdf (2.1).

The hazard rate function (hrf) of X is given by

$$h(x;a,b,\alpha,\boldsymbol{\xi}) = \frac{a\,b\,\alpha\,g(x,\boldsymbol{\xi})\,G(x,\boldsymbol{\xi})^{\alpha a-1}\bar{G}(x,\boldsymbol{\xi})^{\alpha-1}}{\left[G(x,\boldsymbol{\xi})^{\alpha}+\bar{G}(x,\boldsymbol{\xi})^{\alpha}\right]\left\{\left[G(x,\boldsymbol{\xi})^{\alpha}+\bar{G}(x,\boldsymbol{\xi})^{\alpha}\right]^{a}-G(x,\boldsymbol{\xi})^{a\alpha}\right\}}. \tag{2.2}$$

The KwOLL-G family is simulated by inverting $F(x) = u$ in (1.5) as follows: if u has a uniform $U(0,1)$ distribution, the solution of the nonlinear equation

$$x_u = G^{-1}\left(\frac{\left[1-(1-u)^{\frac{1}{b}}\right]^{\frac{1}{a\alpha}}}{\left[1-(1-u)^{\frac{1}{b}}\right]^{\frac{1}{a\alpha}}+\left\{1-\left[1-(1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right\}^{\frac{1}{\alpha}}}\right) \tag{2.3}$$

has the pdf (2.1).

3. Three special cases of the KwOLL-G family

Equation (2.1) will be most tractable when $G(x;\boldsymbol{\xi})$ and $g(x;\boldsymbol{\xi})$ have closed-forms. Now, we provide only three cases of so many distributions which can be special models of the KwOLL-G family.

3.1. The Kumaraswamy odd log-logistic-normal (KwOLLN) distribution. By taking $G(x;\boldsymbol{\xi})$ and $g(x;\boldsymbol{\xi})$ in (2.1) to be the cdf and pdf of the normal $N(\mu,\sigma^2)$ distribution, where $\boldsymbol{\xi}=(\mu,\sigma)^T$, the KwOLLN pdf follows as

$$\begin{aligned} f(x;a,b,\alpha,\mu,\sigma) &= \frac{a\,b\,\alpha\,\phi(\frac{x-\mu}{\sigma})\,\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha\,a-1}\,\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha-1}}{\sigma\,\left\{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}+\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}\right\}^{a+1}} \\ &\quad\times\left\{1-\left[\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}}{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}+\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}}\right]^{a}\right\}^{b-1}, \end{aligned} \tag{3.1}$$

where $x\in\mathbb{R}$, $\mu\in\mathbb{R}$ is a location parameter, $\sigma>0$ is a scale parameter, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. We denote by $X\sim\text{KwOLLN}(a,b,\alpha,\mu,\sigma)$ a random variable with pdf (3.1). For $\mu=0$ and $\sigma=1$, we obtain the standard KwOLLN distribution, and for $a=b=\alpha=1$, it reduces to the normal distribution. For $\alpha=1$, we have the Kumaraswamy normal (KwN) (Cordeiro and de Castro, 2011) distribution. Further, if $\alpha=1$ in addition to $b=1$, it gives the exponentiated-normal (EN) distribution. Plots of the KwOLLN pdf for selected parameter values are displayed in Figure 1.

3.2. The Kumaraswamy odd log-logistic-Weibull (KwOLLW) distribution. By taking $G(x;\boldsymbol{\xi})=1-\mathrm{e}^{-(\beta\mathrm{x})^{\lambda}}$ to be the Weibull distribution with scale parameter $\beta>0$ and shape parameter $\lambda>0$, where $\boldsymbol{\xi}=(\lambda,\beta)^T$, we obtain the KwOLLW pdf (for $x>0$) as

$$\begin{aligned} f(x)=f(x;a,b,\alpha,\lambda,\beta) &= \frac{a\,b\,\alpha\,\lambda\,\beta^{\lambda}\,x^{\lambda-1}\left\{1-\exp\left[-(\beta\,x)^{\lambda}\right]\right\}^{\alpha\,a-1}\left\{\exp\left[-(\beta\,x)^{\lambda}\right]\right\}^{\alpha}}{\left\{\left[1-\exp\left[-(\beta\,x)^{\lambda}\right]\right]^{\alpha}+\left[\exp\left[-(\beta\,x)^{\lambda}\right]\right]^{\alpha}\right\}^{a+1}} \\ &\quad\times\left\{1-\left[\frac{\left\{1-\exp\left[-(\beta\,x)^{\lambda}\right]\right\}^{\alpha}}{\left\{1-\exp\left[-(\beta\,x)^{\lambda}\right]\right\}^{\alpha}+\left\{\exp\left[-(\beta\,x)^{\lambda}\right]\right\}^{\alpha}}\right]^{a}\right\}^{b-1}. \end{aligned} \tag{3.2}$$

The Weibull distribution (with parameters λ and β) is a basic exemplar for $a=b=\alpha=1$. Other special models include the Kumaraswamy Weibull (KwW) (Cordeiro *et al.*, 2010) for $\alpha=1$ and the exponentiated Weibull (EW) (Mudholkar *et al.*, 1995; Mudholkar *et al.*, 1996; Nassar and Eissa, 2003; Nadarajah *et al.*, 2013) and exponentiated exponential (EE) (Gupta and Kundu, 2001) distributions for $b=\alpha=1$ and $b=\alpha=\beta=1$, respectively. Plots of the pdf and hrf of the KwOLLW distribution for selected parameter values are displayed in Figures 2 and 3, respectively. Further, it allows for five major hazard shapes: constant, increasing, decreasing, bathtub and unimodal hazard rates .

(a) (b)

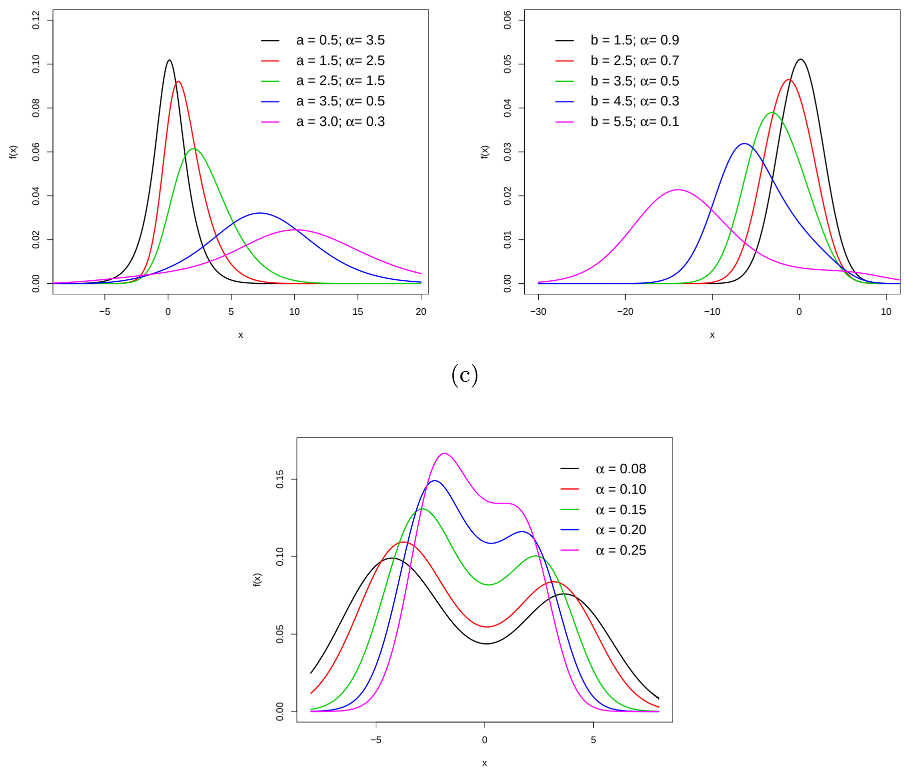


Figure 1. The KwOLLN pdf: (a) For $b = 0.5$, $\mu = 0$ and $\sigma = 3$. (b) For $a = 1.5$, $\mu = 0$ and $\sigma = 3$. (c) For $a = 1.5$, $b = 2.0$, $\mu = 0$ and $\sigma = 1$.

3.3. The Kumaraswamy odd log-logistic-Gumbel (KwOLLGu) distribution. Let $G(x;\boldsymbol{\xi})$ for $x \in \mathbb{R}$ be the Gumbel distribution with parameters (μ, σ), where $\mu \in \mathbb{R}$ is the location parameter and $\sigma > 0$ is the scale parameter, and cdf given by

$$G(x;\boldsymbol{\xi}) = \exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right], \quad x \in \mathbb{R}.$$

Inserting these expressions in equation (2.1) yields the KwOLLGu pdf

$$\begin{aligned} f(x;a,b,\alpha,\mu,\sigma) \quad &= \quad \frac{ab\alpha\exp\{-\frac{x-\mu}{\sigma}-\exp(-\frac{x-\mu}{\sigma})\}\left(\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right)^{\alpha\, a-1}}{\sigma\left\{\left[\exp\{-\exp(-\frac{x-\mu}{\sigma})\}\right]^{\alpha}+\left[1-\exp\{-\exp(-\frac{x-\mu}{\sigma})\}\right]^{\alpha}\right\}^{a+1}} \\ &\times\left\{1-\left[\frac{\left[\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{\alpha}}{\left[\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{\alpha}+\left[1-\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{\alpha}}\right]^{a}\right\}^{b-1} \\ &\times(1-\exp\{-\exp(-\frac{x-\mu}{\sigma})\})^{\alpha-1}, \end{aligned} \tag{3.3}$$

where $x \in \mathbb{R}$. The Kumaraswamy Gumbel (KwGu) (Cordeiro *et al.*, 2010) model corresponds to $\alpha = 1$. The Lehmann type I Gumbel distribution refers to $b = \alpha = 1$. This

(a) (b)

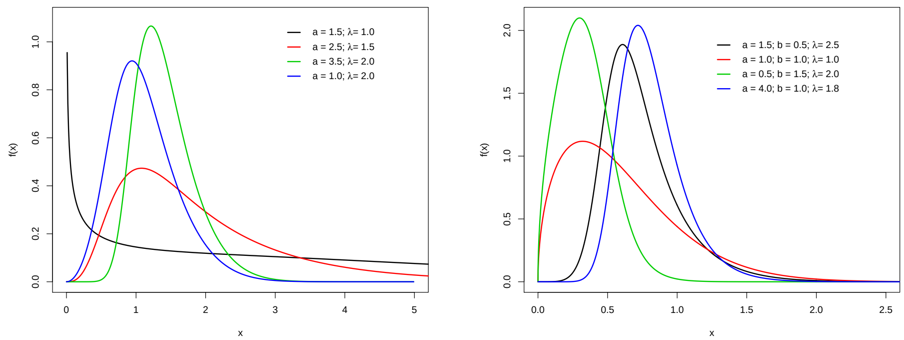


Figure 2. The KwOLLW pdf: (a) For $b = 0.5$, $\alpha = 0.5$ and $\beta = 1$. (b) For $\lambda = 1.5$ and $\beta = 1.5$.

case is usually called the exponentiated Gumbel (EGu) model. Indeed, the EGu cdf is defined by (for $\lambda > 0$)

$$F(x;\lambda,\boldsymbol{\xi}) = 1 - [1 - G(x;\boldsymbol{\xi})]^{\lambda}.$$

Plots of the KwOLLGu pdf for some parameter values are displayed in Figure 4.

4. Asymptotes and Shapes

4.1. Proposition. *The asymptotics of equations (1.5), (2.1) and (2.2) as $x \to 0$ are given by*

$$\begin{aligned} F(x) &\sim b\,G(x)^{a\,\alpha} \quad \text{as G(x)} \to 0, \\ f(x) &\sim a\,b\,\alpha\,g(x)\,G(x)^{a\,\alpha-1} \quad \text{as G(x)} \to 0, \\ h(x) &\sim a\,b\,\alpha\,g(x)\,G(x)^{a\,\alpha-1} \quad \text{as G(x)} \to 0. \end{aligned}$$

4.2. Proposition. *The asymptotics of equations (1.5), (2.1) and (2.2) as $x \to \infty$ are given by*

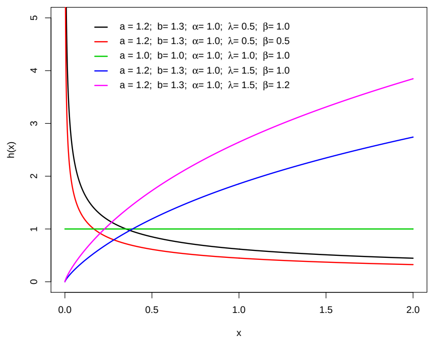
$$\begin{aligned} 1 - F(x) &\sim \left[a\,\alpha\,\bar{G}(x)\right]^{b} \quad \text{as x} \to \infty, \\ f(x) &\sim b\,(a\,\alpha)^{b}\,g(x)\,\bar{G}(x)^{b-1} \quad \text{as x} \to \infty, \\ h(x) &\sim \frac{b\,g(x)}{\bar{G}(x)} \quad \text{as x} \to \infty. \end{aligned}$$

The shapes of the density and hazard rate functions can be described analytically. The critical points of the KwOLL-G pdf are the roots of the equation:

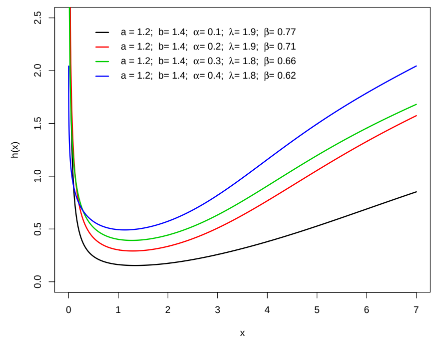
$$\begin{aligned} &\frac{g'(x)}{g(x)} + (a\alpha - 1)\frac{g(x)}{G(x)} + (1-\alpha)\frac{g(x)}{\bar{G}(x)} - \alpha(a+1)g(x)\frac{G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}}{G(x)^{\alpha} + \bar{G}(x)^{\alpha}} \\ = \quad & a(b-1)\alpha g(x)\frac{G(x)^{a\alpha-1}\bar{G}(x)^{\alpha-1}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]\left\{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a} - G(x)^{a\alpha}\right\}}. \end{aligned}$$

(4.1)

(a)



(b)



(c)

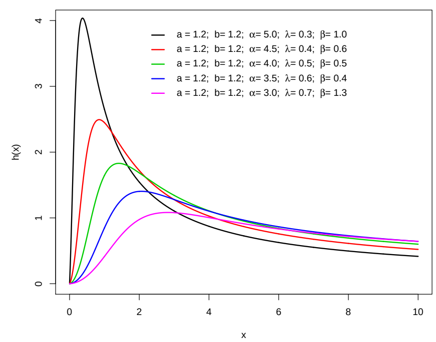


Figure 3. The KwOLLW hrf: (a) Constant, increasing and decreasing hrf. (b) Bathtub hrf. (c) Unimodal hrf.

There may be more than one root to (4.1). Let $\lambda(x) = \frac{d^2 \log[f(x)]}{dx^2}$. Then,

$$
\begin{aligned}
\lambda(x) &= \frac{g''(x)g(x) - g'(x)^2}{g(x)^2} + (a\alpha - 1)\frac{g'(x)G(x) - g(x)^2}{G(x)^2} + (1-\alpha)\frac{g'(x)\bar{G}(x) + g(x)^2}{\bar{G}(x)^2} \\
- &\quad \alpha(a+1)g'(x)\frac{G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}}{G(x)^{\alpha} + \bar{G}(x)^{\alpha}} - \alpha(\alpha-1)(a+1)g(x)^2 \; \frac{G(x)^{\alpha-2} + \bar{G}(x)^{\alpha-2}}{G(x)^{\alpha} + \bar{G}(x)^{\alpha}} \\
+ &\quad (a+1)\left\{\alpha g(x)\frac{G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}}{G(x)^{\alpha} + \bar{G}(x)^{\alpha}}\right\}^2 \\
- &\quad a(b-1)\alpha g'(x)\frac{G(x)^{a\alpha-1}\bar{G}(x)^{\alpha-1}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]\left\{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^a - G(x)^{a\alpha}\right\}} \\
- &\quad a(a\alpha-1)(b-1)\alpha\frac{g(x)^2 G(x)^{a\alpha-2}\bar{G}(x)^{\alpha-1}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]\left\{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^a - G(x)^{a\alpha}\right\}} \\
+ &\quad a(b-1)(\alpha-1)\alpha\frac{g(x)^2 G(x)^{a\alpha-1}\bar{G}(x)^{\alpha-2}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]\left\{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^a - G(x)^{a\alpha}\right\}} \\
+ &\quad a\alpha^2(b-1)\frac{g(x)^2\left[G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}\right]G(x)^{a\alpha-1}\bar{G}(x)^{\alpha-1}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^2\left\{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^a - G(x)^{a\alpha}\right\}} \\
+ &\quad \frac{a^2\alpha^2(b-1)\,g(x)G(x)^{a\alpha-1}\bar{G}(x)^{\alpha-1}\left\{\left[G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}\right]\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a-1}\right\}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]\left\{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^a - G(x)^{a\alpha}\right\}^2} \\
- &\quad \frac{a^2\alpha^2(b-1)\,g(x)G(x)^{2a\alpha-2}\bar{G}(x)^{\alpha-1}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]\left\{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^a - G(x)^{a\alpha}\right\}^2}.
\end{aligned}
$$

(a) (b)

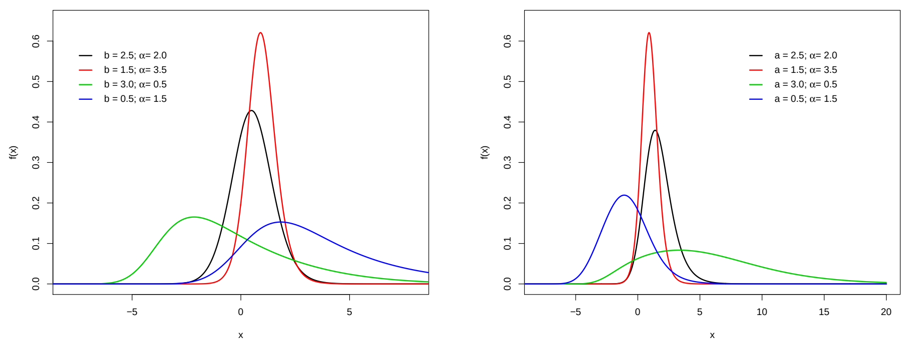


Figure 4. The KwOLLGu pdf: (a) For $a = 1.5$, $\mu = 0$ and $\sigma = 2.5$; (b) For $b = 1.5$, $\mu = 0$ and $\sigma = 2.5$.

If $x = x_0$ is a root of (4.1) then it corresponds to a local maximum (minimum) if $\lambda(x) > 0 (< 0)$ for all $x < x_0$ and $\lambda(x) < 0 (> 0)$ for all $x > x_0$. It yields points of inflexion if either $\lambda(x) > 0$ for all $x \neq x_0$ or $\lambda(x) < 0$ for all $x \neq x_0$.

The critical points of the hrf $h(x)$ are obtained from the equation:

$$
\begin{aligned}
& \frac{g'(x)}{g(x)} + (a\alpha - 1)\frac{g(x)}{G(x)} + (1-\alpha)\frac{g(x)}{\bar{G}(x)} - \alpha g(x)\frac{\left[G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}\right]}{G(x)^{\alpha} + \bar{G}(x)^{\alpha}} \\
= \quad & a\alpha g(x)\frac{\left[G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}\right]\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a-1} - G(x)^{a\alpha-1}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a} - G(x)^{a\alpha}}.
\end{aligned} \tag{4.2}
$$

There may be more than one root to (4.2). Let $\tau(x) = d^2 \log[h(x)]/dx^2$. We have

$$
\begin{aligned}
\tau(x) \;=\; & \frac{g''(x)g(x) - g'(x)^2}{g(x)^2} + (a\alpha - 1)\frac{g'(x)G(x) - g(x)^2}{G(x)^2} + (1-\alpha)\frac{g'(x)\bar{G}(x) + g(x)^2}{\bar{G}(x)^2} \\
- \; & \alpha g'(x)\frac{G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}}{G(x)^{\alpha} + \bar{G}(x)^{\alpha}} - \alpha(\alpha-1)g(x)^2 \;\; \frac{G(x)^{\alpha-2} + \bar{G}(x)^{\alpha-2}}{G(x)^{\alpha} + \bar{G}(x)^{\alpha}} \\
+ \; & \left\{\alpha g(x)\frac{G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}}{G(x)^{\alpha} + \bar{G}(x)^{\alpha}}\right\}^2 \\
- \; & a\alpha g'(x)\frac{\left[G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}\right]\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a-1} - G(x)^{a\,\alpha-1}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a} - G(x)^{a\alpha}} \\
- \; & a\,\alpha(\alpha-1)g(x)^2\frac{\left[G(x)^{\alpha-2} + \bar{G}(x)^{\alpha-2}\right]\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a-1}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a} - G(x)^{a\alpha}} \\
- \; & a\,\alpha^2(a-1)g(x)^2\frac{\left[G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}\right]^2\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a-2}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a} - G(x)^{a\alpha}} \\
+ \; & a\,\alpha(a\,\alpha-1)g(x)^2\frac{G(x)^{a\,\alpha-2}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a} - G(x)^{a\alpha}} \\
+ \; & \left\{a\,\alpha\,g(x)\frac{\left[G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}\right]\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a-1} - G(x)^{a\,\alpha-1}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{a} - G(x)^{a\alpha}}\right\}^2 .
\end{aligned}
$$

If $x = x_0$ is a root of (4.2) then it refers to a local maximum (minimum) if $\tau(x) > 0(< 0)$ for all $x < x_0$ and $\tau(x) < 0(< 0)$ for all $x > x_0$. It gives an inflexion point if either $\tau(x) > 0$ for all $x \neq x_0$ or $\tau(x) < 0$ for all $x \neq x_0$.

5. Some useful expansions

The cdf (1.5) of X admits the expansion

$$
\begin{aligned}
F(x) &= 1 - \sum_{m=0}^{\infty}(-1)^m \binom{b}{m} \; \frac{G(x)^{a\alpha m}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{am}} \\
&= 1 - \sum_{m=0}^{\infty}(-1)^m \binom{b}{m} \; \frac{\sum_{k=0}^{\infty} \delta_{1,k}^{(m)} \, G(x)^k}{\sum_{k=0}^{\infty} \delta_{2,k}^{(m)} \, G(x)^k} \\
&= 1 - \sum_{m=0}^{\infty}(-1)^m \binom{b}{m} \; \sum_{k=0}^{\infty} \beta_k^{(m)} \, G(x)^k,
\end{aligned}
$$

where $\beta_0^{(m)} = \frac{\delta_{1,0}^{(m)}}{\delta_{2,0}^{(m)}}$ and for $k \geq 1$

$$
\beta_k^{(m)} = \frac{1}{\delta_{2,0}^{(m)}}\left(\delta_{1,k}^{(m)} - \frac{1}{\delta_{2,0}^{(m)}}\sum_{r=1}^{k}\delta_{2,k}^{(m)}\,\beta_{k-r}^{(m)}\right), \quad \delta_{1,k}^{(m)} = \sum_{i=k}^{\infty}(-1)^{i+k}\binom{a\alpha m}{i}\binom{i}{k}
$$

and $\delta_{2,k}^{(m)} = h_k(\alpha, a\,m)$ is defined in the Appendix. Then, we can write

$$
F(x) = \sum_{k=0}^{\infty} b_k \, G(x)^k, \tag{5.1}
$$

where

$$
b_0 = 1 - \sum_{m=0}^{\infty}(-1)^m \binom{b}{m} \beta_0^{(m)}, \;\text{ and for } k \geq 1, \; b_k = \sum_{m=0}^{\infty}(-1)^{m+1}\binom{b}{m}\beta_k^{(m)}.
$$

So, the pdf of X can be expressed as an infinite mixture of exponentiated-G ("exp-G") densities

$$f(x) = f(x; a, b, \alpha, \boldsymbol{\xi}) = \sum_{k=0}^{\infty} b_{k+1}\, h_{k+1}(x), \tag{5.2}$$

where $h_{k+1}(x) = (k+1)\, G(x)^k\, g(x)$ denotes the exp-G pdf with power parameter $k+1$. Structural properties of some exp-G distributions were studied by Mudholkar *et al.* (1996), Gupta and Kundu (2001), Nadarajah and Kotz (2006), Nadarajah and Gupta (2007) and Nadarajah *et al.* (2013), among others. So, some mathematical quantities of X can be derived from (5.2) and those exp-G properties. For example, the ordinary and incomplete moments and moment generating function (mgf) of X can be easily obtained from those of the exp-G quantities.

The formulae derived in the next sections can be easily handled in most symbolic computation software platforms such as MAPLE, MATHEMATICA and MATLAB. These platforms have currently the ability to deal with complex expressions. Established closed-form statistical measures can be more efficient than calculating them by numerical integration. The infinity limit in these sums can be substituted by a large positive integer such as twenty or thirty for most applications.

6. Quantile power series

Here, we derive a power series for the qf $x = Q(u) = F^{-1}(u)$ of X by expanding (2.3). First, if $Q_G(u)$ (the baseline qf) does not have an explicit expression, it can usually be expressed as a power series given by

$$Q_G(u) = \sum_{i=0}^{\infty} a_i\, u^i, \tag{6.1}$$

where the coefficients a_i's are suitably chosen real numbers which depend on the parameters of the G distribution. For several important distributions, such as the normal, Student t, gamma and beta distributions, $Q_G(u)$ does not have explicit expressions but it can be expanded as in equation (6.1).

Here and from now on, we use a result by Gradshteyn and Ryzhik (2007, Section 0.314) for a power series raised to a positive integer n (for $n \geq 1$)

$$Q_G(u)^n = \left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i, \tag{6.2}$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) can be obtained from the recurrence equation

$$c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, c_{n,i-m}, \tag{6.3}$$

and $c_{n,0} = a_0^n$. Clearly, the quantity $c_{n,i}$ can be determined numerically in any algebraic or numerical software from the quantities $a_0, \ldots, a_i$.

Second, we derive an expansion for the argument A of $Q_G(\cdot)$ in equation (2.3)

$$A = \frac{\left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{a\alpha}}}{\left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{a\alpha}} + \left\{1 - \left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right\}^{\frac{1}{\alpha}}} = \frac{\sum_{k=0}^{\infty} a_k^*\, u^k}{\sum_{k=0}^{\infty} b_k^*\, u^k},$$

where $a_k^* = \sum_{i=0}^{\infty} (-1)^{i+k} \binom{\frac{1}{\alpha a}}{i} \binom{\frac{i}{b}}{k}$ and $b_k^* = a_k^* + \sum_{i,j=0}^{\infty} (-1)^{i+j+k} \binom{\frac{1}{\alpha}}{i} \binom{\frac{i}{a}}{j} \binom{\frac{j}{b}}{k}$.

The quotient of the two power series is given by

$$A = \sum_{k=0}^{\infty} c_k^* \, u^k, \tag{6.4}$$

where $c_0^* = \frac{a_0^*}{b_0^*}$ and the coefficients c_k^*'s ($k \geq 1$) are determined from the recurrence equation

$$c_k^* = \frac{1}{b_0^*} \left(a_k^* - \frac{1}{b_0^*} \sum_{r=1}^{k} b_r^* \, c_{k-r}^* \right).$$

Then, the qf of the KwOLL-G family can be reduced to

$$Q(u) = Q_G \left(\sum_{k=0}^{\infty} c_k^* \, u^k \right). \tag{6.5}$$

By combining (6.1) and (6.5) gives

$$Q(u) = \sum_{i=0}^{\infty} a_i \left(\sum_{k=0}^{\infty} c_k^* \, u^k \right)^i,$$

and then using (6.2) and (6.3), we have

$$Q(u) = \sum_{k=0}^{\infty} e_k \, u^k, \tag{6.6}$$

where $e_k = \sum_{i=0}^{\infty} a_i \, d_{i,k}$, $d_{i,0} = c_0^{*i}$ and (for $k > 1$)

$$d_{i,k} = (k \, c_0^*)^{-1} \sum_{m=1}^{k} [m(i+1) - k] \, c_m^* \, d_{i,k-m}.$$

Hence, equation (6.6) reveals that the qf of the KwOLL-G family can be expressed as a power series. So, several mathematical quantities of X can be reduced to integrals over $(0,1)$ based on this power series. For the great majority of these quantities, we can adopt twenty terms in this power series.

Let $W(\cdot)$ be any integrable function in the real line. We can write

$$\int_{-\infty}^{\infty} W(x) \, f(x) dx = \int_0^1 W \left(\sum_{k=0}^{\infty} e_k \, u^k \right) du. \tag{6.7}$$

Equations (6.6) and (6.7) are the main results of this section since we can obtain from them various KwOLL-G mathematical properties. In fact, they can follow by using the integral on the hight-hand side for special $W(\cdot)$ functions, which are usually more simple than if they are based on the left-hand integral. For example, a formula for the nth moment of X follows from (6.7) combined with (6.2) and (6.3) as

$$\mu_n' = \int_0^1 \left(\sum_{k=0}^{\infty} e_k \, u^k \right)^n du = \sum_{k=0}^{\infty} \frac{f_{n,k}}{(k+1)},$$

where (for $n \geq 0$) $f_{n,0} = e_0^n$ and, for $k \geq 1$, $f_{n,k} = (k \, e_0)^{-1} \sum_{r=1}^{k} [r \, (n+1) - k] \, e_r \, f_{n,k-r}$.

7. Moments

Let $Y_{k+1}(k \geq 0)$ be a random variable having the pdf $h_{k+1}(x)$. A first formula for the nth moment of X is obtained from (5.2) as

$$E(X^n) = \sum_{k=0}^{\infty} b_{k+1}\, E(Y_{k+1}^n). \tag{7.1}$$

Moments of some exp-G distributions are given by Nadarajah and Kotz (2006), which can be used to obtain $E(X^n)$.

A second formula for $E(X^n)$ can be expressed from (7.1) as

$$E(X^n) = \sum_{k=0}^{\infty} (k+1)\, b_{k+1}\, \tau(n,k), \tag{7.2}$$

where $\tau(n,k) = \int_0^1 Q_G(u)^n\, u^k du$.

The nth incomplete moment of X is determined from (5.2) as

$$m_n(y) = \int_0^y x^n\, f(x) dx = \sum_{k=0}^{\infty} (k+1)\, b_{k+1} \int_0^{G(y)} Q_G(u)^n\, u^k du.$$

Using (6.2), we obtain

$$m_n(y) = \sum_{i,k=0}^{\infty} \frac{(k+1)\, b_{k+1}\, c_{n,i}}{(k+i+1)}\, G(y)^{k+i+1} \tag{7.3}$$

Equations (7.1)-(7.3) are the main results of this section.

8. Generating function

Here, we provide two formulae for the mgf $M(t) = E(\mathrm{e}^{\mathrm{t\,X}})$ of X. Clearly, the first one simply comes from (5.2) as

$$M(t) = \sum_{k=0}^{\infty} b_{k+1}\, M_{k+1}(t), \tag{8.1}$$

where $M_{k+1}(t)$ is the mgf of Y_{k+1}. Hence, $M(t)$ can be determined from the exp-G generating function. A second formula for $M(t)$ can be derived from (5.2) as

$$M(t) = \sum_{i=0}^{\infty} (k+1)\, b_{k+1}\, \rho(t,k), \tag{8.2}$$

where $\rho(t,k) = \int_0^1 \exp[t\, Q_G(u)]\, u^k du$ can be computed numerically for most G distributions.

So, we can obtain the mgfs of several generated distributions from (3.2) directly from equations (8.1) and (8.2).

9. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose that $X_1, \ldots, X_n$ is a random sample from X and let $X_{i:n}$ denote the ith order statistic. From equations (5.1) and (5.2), the pdf of $X_{i:n}$ becomes

$$f_{i:n}(x) = C \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left(\sum_{r=0}^{\infty} (r+1)\, b_{r+1}\;\; G(x)^r\, g(x) \right) \left(\sum_{k=0}^{\infty} b_k\, G(x)^k \right)^{j+i-1},$$

where $C = n!/[(i-1)!\,(n-i)!]$. Using (6.2) and (6.3), we can write

$$\left(\sum_{k=0}^{\infty} b_k\, G(x)^k\right)^{j+i-1} = \sum_{k=0}^{\infty} e_{j+i-1,k}\; G(x)^k,$$

where $e_{j+i-1,0} = b_0^{j+i-1}$ and

$$e_{j+i-1,k} = (k\, b_0)^{-1} \sum_{m=1}^{k} [m(j+i) - k]\, b_m\, e_{j+i-1,k-m}.$$

Hence,

$$f_{i:n}(x) = \sum_{k=0}^{\infty} d_k\, h_{k+1}(x), \tag{9.1}$$

where $d_k = C \sum_{j=0}^{n-i} \sum_{m=0}^{k} b_{m+1}\, e_{j+i-1,k-m}$.

Equation (9.1) gives the pdf of the KwOLL-G order statistics as a linear combination of exp-G densities.

10. A bivariate extension

Here, we construct a bivariate version of the proposed model. The joint cdf of (X_1, X_2) is given by

$$\begin{aligned} F_{X_1,X_2}(x_1,x_2;a,b,\alpha,\boldsymbol{\xi}) &= \int_0^{\frac{G(x_1,x_2;\boldsymbol{\xi})^{\alpha}}{G(x_1,x_2;\boldsymbol{\xi})^{\alpha}+[1-G(x_1,x_2;\boldsymbol{\xi})]^{\alpha}}} a\,b\,t^{a-1}(1-t^{a})^{b-1}dt \\ &= 1-\left\{1-\left[\frac{G(x_1,x_2,\boldsymbol{\xi})^{\alpha}}{G(x_1,x_2,\boldsymbol{\xi})^{\alpha}+[1-G(x_1,x_2,\boldsymbol{\xi})]^{\alpha}}\right]^{a}\right\}^{b}, \end{aligned}$$

where $G(x_1, x_2; \boldsymbol{\xi})$ is a bivariate continuous distribution with marginal cdfs $G_1(x_1; \boldsymbol{\xi})$ and $G_2(x_2; \boldsymbol{\xi})$. This distribution is called the *bivariate Kumaraswamy odd log-logistic* (BKwOLL) family of distributions. The marginal cdfs are given by

$$F_{X_i}(x_i;a,b,\alpha,\boldsymbol{\xi}) = 1-\left\{1-\left[\frac{G_i(x_i,\boldsymbol{\xi})^{\alpha}}{G_i(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}_i(x_i,\boldsymbol{\xi})^{\alpha}}\right]^{a}\right\}^{b}, \quad i=1,2.$$

The joint pdf of (X_1, X_2) can be expressed as $f_{X_1,X_2}(x_1,x_2) = \dfrac{\partial^2 F_{X_1,X_2}(x_1,x_2)}{\partial x_1 \partial x_2}$ and then

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2;a,b,\alpha,\boldsymbol{\xi}) &= \frac{ab\alpha A(x_1,x_2;a,b,\alpha,\boldsymbol{\xi})G(x_1,x_2,\boldsymbol{\xi})^{\alpha a-1}\,[1-G(x_1,x_2,\boldsymbol{\xi})]^{\alpha-1}}{\{G(x_1,x_2,\boldsymbol{\xi})^{\alpha}+[1-G(x_1,x_2,\boldsymbol{\xi})]^{\alpha}\}^{a+1}} \\ &\times \left\{1-\left[\frac{G(x_1,x_2,\boldsymbol{\xi})^{\alpha}}{G(x,y,\boldsymbol{\xi})^{\alpha}+[1-G(x_1,x_2,\boldsymbol{\xi})]^{\alpha}}\right]^{a}\right\}^{b-1}, \end{aligned}$$

where

$$
\begin{aligned}
& A(x_1, x_2; a, b, \alpha, \boldsymbol{\xi}) = g(x_1, x_2; \boldsymbol{\xi}) \\
& + \frac{\partial G(x_1, x_2, \boldsymbol{\xi})}{\partial x_1} \frac{\partial G(x_1, x_2, \boldsymbol{\xi})}{\partial x_2} \left[\frac{a\alpha - 1}{G(x_1, x_2, \boldsymbol{\xi})} + \frac{1-\alpha}{1 - G(x_1, x_2, \boldsymbol{\xi})} \right] \\
- \quad & (a+1)\alpha \frac{\partial G(x_1, x_2, \boldsymbol{\xi})}{\partial x_1} \frac{\partial G(x_1, x_2, \boldsymbol{\xi})}{\partial x_2} \frac{G(x_1, x_2, \boldsymbol{\xi})^{\alpha-1} - [1 - G(x_1, x_2, \boldsymbol{\xi})]^{\alpha-1}}{G(x_1, x_2, \boldsymbol{\xi})^{\alpha} + [1 - G(x_1, x_2, \boldsymbol{\xi})]^{\alpha}} \\
+ \quad & \frac{a\alpha(1-b) G(x_1, x_2, \boldsymbol{\xi})^{a\alpha-1} [1 - G(x_1, x_2, \boldsymbol{\xi})]^{\alpha-1}}{\{G(x_1, x_2, \boldsymbol{\xi})^{\alpha} + [1 - G(x_1, x_2, \boldsymbol{\xi})]^{\alpha}\}} \\
& \times \frac{\frac{\partial G(x_1, x_2, \boldsymbol{\xi})}{\partial x_1} \frac{\partial G(x_1, x_2, \boldsymbol{\xi})}{\partial x_2}}{\{\{G(x_1, x_2, \boldsymbol{\xi})^{\alpha} + [1 - G(x_1, x_2, \boldsymbol{\xi})]^{\alpha}\}^{a} - G(x_1, x_2, \boldsymbol{\xi})^{a\alpha}\}}.
\end{aligned}
$$

The marginal pdfs are given by

$$
f_{X_i}(x_i) = \frac{ab\alpha g_i(x_i, \boldsymbol{\xi}) G_i(x_i, \boldsymbol{\xi})^{\alpha a - 1} \bar{G}_i(x_i, \boldsymbol{\xi})^{\alpha-1}}{\left[G_i(x_i, \boldsymbol{\xi})^{\alpha} + \bar{G}_i(x_i, \boldsymbol{\xi})^{\alpha}\right]^{a+1}} \left\{ 1 - \left[\frac{G_i(x_i, \boldsymbol{\xi})^{\alpha}}{G_i(x_i, \boldsymbol{\xi})^{\alpha} + \bar{G}_i(x_i, \boldsymbol{\xi})^{\alpha}} \right]^{a} \right\}^{b-1}, \quad i = 1, 2.
$$

The conditional cdfs are given by

$$
F_{X_i|X_j}(x_i|x_j) = \frac{1 - \left\{ 1 - \left[\frac{G(x_1, x_2, \boldsymbol{\xi})^{\alpha}}{G(x_1, x_2, \boldsymbol{\xi})^{\alpha} + [1 - G(x_1, x_2, \boldsymbol{\xi})]^{\alpha}} \right]^{a} \right\}^{b}}{1 - \left\{ 1 - \left[\frac{G_j(x_j, \boldsymbol{\xi})^{\alpha}}{G_j(x_j, \boldsymbol{\xi})^{\alpha} + \bar{G}_j(x_j, \boldsymbol{\xi})^{\alpha}} \right]^{a} \right\}^{b}} \text{ for } i, j = 1, 2 \text{ and } i \neq j.
$$

The conditional pdfs are given by

$$
\begin{aligned}
& f_{X_i|X_j}(x_i|x_j) = \frac{A(x_1, x_2; a, b, \alpha, \boldsymbol{\xi}) G(x_1, x_2, \boldsymbol{\xi})^{\alpha a - 1} [1 - G(x_1, x_2, \boldsymbol{\xi})]^{\alpha-1}}{\{G(x_1, x_2, \boldsymbol{\xi})^{\alpha} + [1 - G(x_1, x_2, \boldsymbol{\xi})]^{\alpha}\}^{a+1}} \\
\times \quad & \left\{ 1 - \left[\frac{G(x_1, x_2, \boldsymbol{\xi})^{\alpha}}{G(x, y, \boldsymbol{\xi})^{\alpha} + [1 - G(x_1, x_2, \boldsymbol{\xi})]^{\alpha}} \right]^{a} \right\}^{b-1} \\
& \times \left\{ \frac{g_j(x_j, \boldsymbol{\xi}) G_j(x_j, \boldsymbol{\xi})^{\alpha a - 1} \bar{G}_j(x_j, \boldsymbol{\xi})^{\alpha-1}}{\left[G_j(x_j, \boldsymbol{\xi})^{\alpha} + \bar{G}_j(x_j, \boldsymbol{\xi})^{\alpha}\right]^{a+1}} \left\{ 1 - \left[\frac{G_j(x_j, \boldsymbol{\xi})^{\alpha}}{G_j(x_i, \boldsymbol{\xi})^{\alpha} + \bar{G}_j(x_j, \boldsymbol{\xi})^{\alpha}} \right]^{a} \right\}^{b-1} \right\}^{-1}
\end{aligned}
$$

for $i, j = 1, 2$ and $i \neq j$.

11. Estimation

We determine the maximum likelihood estimates (MLEs) of the parameters of the new family from complete samples only. Let $x_1, \ldots, x_n$ be the observed values from the KwOLL-G distribution with parameters a, b, α and $\boldsymbol{\xi}$. Let $\boldsymbol{\theta} = (a, b, \alpha, \boldsymbol{\xi})^{\top}$ be the $r \times 1$ parameter vector. Then, the total log-likelihood function for $\boldsymbol{\theta}$ is given by

$$
\begin{aligned}
\ell_n(\boldsymbol{\theta}) = {} & n \log[ab\alpha] + \sum_{i=1}^{n} \log[g(x_i; \boldsymbol{\xi})] + (a\alpha - 1) \sum_{i=1}^{n} \log[G(x_i; \boldsymbol{\xi})] \\
& + (\alpha - 1) \sum_{i=1}^{n} \log\left[\bar{G}(x_i; \boldsymbol{\xi})\right] - (a+1) \sum_{i=1}^{n} \log\left\{G(x_i; \boldsymbol{\xi})^{\alpha} + \bar{G}(x_i; \boldsymbol{\xi})^{\alpha}\right\} \\
& + (b-1) \sum_{i=1}^{n} \log\left\{ 1 - \left[\frac{G(x_i, \boldsymbol{\xi})^{\alpha}}{G(x_i, \boldsymbol{\xi})^{\alpha} + \bar{G}(x_i, \boldsymbol{\xi})^{\alpha}} \right]^{a} \right\}. \qquad (11.1)
\end{aligned}
$$

The components of the score function are given by

$$\begin{aligned} U_a(\boldsymbol{\theta}) &= \frac{n}{a} + \sum_{i=1}^{n} \log\left[\frac{G(x_i,\boldsymbol{\xi})^{\alpha}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\right] \\ &+ (1-b)\sum_{i=1}^{n} \frac{\left[\frac{G(x_i,\boldsymbol{\xi})^{\alpha}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\right]^{a} \log\left[\frac{G(x_i,\boldsymbol{\xi})^{\alpha}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\right]}{1-\left[\frac{G(x_i,\boldsymbol{\xi})^{\alpha}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\right]^{a}}, \end{aligned}$$

$$U_b(\boldsymbol{\theta}) = \frac{n}{b} + \sum_{i=1}^{n} \log\left\{1-\left[\frac{G(x_i,\boldsymbol{\xi})^{\alpha}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\right]^{a}\right\},$$

$$\begin{aligned} U_\alpha(\boldsymbol{\theta}) &= \frac{n}{\alpha} + a\sum_{i=1}^{n}\log\left[G(x_i;\boldsymbol{\xi})\right] + \sum_{i=1}^{n}\log\left[\bar{G}(x_i;\boldsymbol{\xi})\right] \\ &-(a+1)\sum_{i=1}^{n}\frac{G(x_i,\boldsymbol{\xi})^{\alpha}\log\left[G(x_i,\boldsymbol{\xi})\right]+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}\log\left[\bar{G}(x_i,\boldsymbol{\xi})\right]}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}} \\ &-a(b-1)\sum_{i=1}^{n}\frac{G(x_i,\boldsymbol{\xi})^{a\alpha}\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}\log\left[\frac{G(x_i,\boldsymbol{\xi})}{\bar{G}(x_i,\boldsymbol{\xi})}\right]}{\left\{\left[G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}\right]^{a}-G(x_i,\boldsymbol{\xi})^{a\alpha}\right\}\left[G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}\right]} \end{aligned}$$

and

$$\begin{aligned} U_{\boldsymbol{\xi}}(\boldsymbol{\theta}) &= \sum_{i=1}^{n}\frac{g^{(\xi)}(x_i,\boldsymbol{\xi})}{g(x_i,\boldsymbol{\xi})} + (a\alpha-1)\sum_{i=1}^{n}\frac{G^{(\xi)}(x_i,\boldsymbol{\xi})}{G(x_i,\boldsymbol{\xi})} + (1-\alpha)\sum_{i=1}^{n}\frac{G^{(\xi)}(x_i,\boldsymbol{\xi})}{\bar{G}(x_i,\boldsymbol{\xi})} \\ &-\alpha(a+1)\sum_{i=1}^{n}G^{(\xi)}(x_i,\boldsymbol{\xi})\frac{G(x_i,\boldsymbol{\xi})^{\alpha-1}-\bar{G}(x_i,\boldsymbol{\xi})^{\alpha-1}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}} \\ &-a\alpha(b-1)\sum_{i=1}^{n}\frac{G^{(\xi)}(x_i,\boldsymbol{\xi})G(x_i,\boldsymbol{\xi})^{a\,\alpha-1}\bar{G}(x_i,\boldsymbol{\xi})^{\alpha-1}}{\left\{\left[G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}\right]^{a}-G(x_i,\boldsymbol{\xi})^{a\alpha}\right\}\left[G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}\right]}. \end{aligned}$$

Numerical maximization of (11.1) is performed by using the RS method (Rigby and Stasinopoulos, 2005) which is available in the gamlss package (R Development Core Team, 2013), SAS (Proc NLMixed) or the Ox program (sub-routine MaxBFGS) (see, Doornik, 2007) or by solving the nonlinear likelihood equations obtained by differentiating (11.1). Setting these equations to zero, $U_a(\boldsymbol{\theta}) = U_b(\boldsymbol{\theta}) = U_\alpha(\boldsymbol{\theta}) = U_{\boldsymbol{\xi}}(\boldsymbol{\theta}) = \mathbf{0}$, and solving them simultaneously yields the MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$.

For interval estimation and hypothesis tests on the parameters in $\boldsymbol{\theta}$, we require the $(p+3)\times(p+3)$ total observed information matrix $\mathbf{J}(\boldsymbol{\theta}) = -\{U_{rs}\}$, where the elements U_{rs} for $r,s = a,b,\alpha,\boldsymbol{\xi}$ are calculated numerically. The estimated multivariate normal $\mathrm{N}_{p+3}(\boldsymbol{\theta},\mathbf{J}(\widehat{\boldsymbol{\theta}})^{-1})$ distribution can be used to construct approximate confidence regions for the parameters in $\widehat{\boldsymbol{\theta}}$. An asymptotic confidence interval (ACI) with significance level γ for each parameter θ_r is given by

$$\mathrm{ACI}(\theta_r, 100(1-\gamma)\%) = (\hat{\theta}_r - z_{\gamma/2}\sqrt{\hat{\kappa}^{\theta_r,\theta_r}}, \hat{\theta}_r + z_{\gamma/2}\sqrt{\hat{\kappa}^{\theta_r,\theta_r}}),$$

where $\hat{\kappa}^{\theta_r,\theta_r}$ is the rth diagonal element of $\mathbf{J}(\boldsymbol{\theta})^{-1}$ estimated at $\widehat{\boldsymbol{\theta}}$ and $z_{\gamma/2}$ is the quantile $1-\gamma/2$ of the standard normal distribution.

We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct likelihood ratio (LR) statistics for testing some sub-models of the KwOLL-G distribution. For example, we may use LR statistics to check if the fit using the KwOLLW distribution is statistically "superior" to the fits using the KwW, EW, EE and Weibull distributions for a given data set. In any case, considering the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T)^T$, tests of hypotheses of the type $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ versus $H_A : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ can be performed using the LR statistic $w = 2\{\ell(\widehat{\boldsymbol{\theta}}) - \ell(\widetilde{\boldsymbol{\theta}})\}$, where $\widehat{\boldsymbol{\theta}}$ and $\widetilde{\boldsymbol{\theta}}$ are the estimates of $\boldsymbol{\theta}$ under H_A and H_0, respectively. Under the null hypothesis H_0, $w \xrightarrow{d} \chi_q^2$, where q is the dimension of the vector $\boldsymbol{\theta}_1$ of interest. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ_q^2 distribution.

12. Applications

We illustrate the importance of the proposed family in two applications to real data. In the last few years, several extensions of the normal and Weibull distributions have been introduced in the literature. For example, Silva *et al.* (2010) studied the beta modified Weibull (BMW) distribution, Cordeiro *et al.* (2012b) proposed the McDonald normal (McN) distribution, Cordeiro *et al.* (2014b) defined the Libby-Novick beta Weibull (LNBW) distribution, Cordeiro *et al.* (2014c) studied the McDonald Weibull (McW) distribution and Cordeiro *et al.* (2014d) introduced the Kummaraswamy modified Weibull (KwMW) distribution.

We compare the fits of the KwOLLN and KwOLLW distributions with those of other known models, namely the McN, beta normal (BN), Kumaraswamy normal (KwN), McW, BMW, KwMW, LNBW, beta Weibull (BW), Kumaraswamy Weibull (KwW) and their baseline distributions themselves, see Alexander *et al.* (2012) and Cordeiro *et al.* (2010) for more details.

12.1. Aarset data. We consider the lifetimes of 50 industrial devices put on life test at time zero presented by Aarset (1987). These data also reported in Mudholkar and Srivastava (1993), Mudholkar *et al.* (1996) and Silva *et al.* (2010) exhibit a bathtub-shaped failure rate property. These authors consider that the data are generated by a Weibull distribution. So, we adopt this distribution as the baseline model for our family.

Table 2 lists the MLEs and their standard errors (in parentheses) of the parameters from the fitted KwOLLW, McW, KwMW, BMW, LNBW, BW, KwW, EW and Weibull models and the values of the statistics: Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC). The computations are performed using the statistical software R. The results indicate that the KwOLLW model has the smallest values of these statistics among the fitted models, and therefore it could be chosen as the best model.

A comparison of the KwOLLW distribution with some of its sub-models using LR statistics is given in Table 3. Clearly, we reject the null hypotheses for the three LR tests in favor of the KwOLLW distribution. In order to assess if the new model is appropriate, Figures 5a and 5b display the histogram of the data and the fitted KwOLLW density function and the densities of some of its sub-models and non-nested models, respectively. Further, Figures 5c and 5d display plots of the empirical and estimated survival functions of the KwOLLW distribution and of some sub-models and non-nested models, respectively. We can conclude that the KwOLLW distribution is a very suitable model to fit to the current data.

We shall apply formal goodness-of-fit tests in order to verify which distribution fits the data better. We consider the Cramér-Von Mises (W^*) and Anderson-Darling (A^*) statistics defined by Chen and Balakrishnan (1995).

The values of these statistics for the fitted models are listed in Table 4. Overall, by comparing the measures of these formal goodness-of-fit tests in Table 4, we conclude that the KwOLLW distribution yields a better fit than the Weibull, EW, KwW, BW and McW distributions and therefore it can be an interesting alternative to these distributions for

Table 2. MLEs and information criteria.

Aarset	λ	β	a	b	α	AIC	CAIC	BIC
KwOLLW	5.4771	0.0203	3.0532	4.9020	0.0514	441.0	442.3	450.5
	(0.0100)	(0.0010)	(1.2819)	(4.5062)	(0.0188)			
KwW	5.5025	0.0165	0.0602	0.2510	1	449.5	450.4	457.2
	(0.0043)	(0.0013)	(0.0205)	(0.0796)	(-)			
EW	4.6978	0.0108	0.1381	1	1	464.3	464.8	470.0
	(0.00002)	(0.0008)	(0.0206)	(-)	(-)			
Weibull	0.9488	0.0222	1	1	1	486.0	486.2	489.8
	(0.1195)	(0.0034)	(-)	(-)	(-)			
	λ	β	a	b	c			
McW	5.4712	0.0202	0.0880	0.0876	0.8457	447.5	448.8	457.0
	(0.0086)	(0.0028)	(0.0195)	(0.0640)	(0.6682)			
BW	5.3386	0.0212	0.0864	0.0731	1	445.7	446.5	453.3
	(0.0146)	(0.0019)	(0.0181)	(0.0306)	(-)			
	λ	β	a_1	b_1	c_1			
LNBW	5.4514	0.0217	0.0838	0.0620	2.1512	447.3	448.7	456.9
	(0.0109)	(0.0041)	(0.0187)	(0.0618)	(14.3571)			
	α_2	λ_2	γ_2	a_2	b_2			
BMW	0.0028	0.0403	1.1337	0.2455	0.1400	453.9	455.2	463.4
	(0.0009)	(0.0125)	(0.2873)	(0.0623)	(0.0671)			
	α_3	λ_3	γ_3	a_3	b_3			
KwMW	0.0038	0.03724	0.9403	0.2654	0.3195	457.7	459.0	467.2
	(0.0020)	(0.0106)	(0.2650)	(0.1058)	(0.1649)			

Table 3. LR tests.

Aarset	Hypotheses	Statistic w	p-value
KwOLLW vs KwW	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	10.55	0.0011
KwOLLW vs EW	$H_0 : b = \alpha = 1$ vs $H_1 : H_0$ is false	27.33	<0.0001
KwOLLW vs Weibull	$H_0 : a = b = \alpha = 1$ vs $H_1 : H_0$ is false	51.00	<0.0001

modeling lifetime data. These results illustrate the importance of the additional shape parameters of the new distribution to analyze real data.

12.2. Respiratory data. Now, we use a real data set to compare the fits of the KwOLLN distribution with those of the McN, BN, KwN and normal distributions. The McN pdf (Cordeiro *et al.*, 2012b) is given by

$$f(x;\mu,\sigma,a,b,c) = \frac{c}{B(a,b)\sigma}\,\phi\left(\sigma^{-1}(x-\mu)\right)\left\{\Phi\left(\sigma^{-1}(x-\mu)\right)\right\}^{ac-1}\left\{1-\Phi\left(\sigma^{-1}(x-\mu)\right)^{c}\right\}^{b-1},$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ is the scale parameter and a, b and c are positive shape parameters.

We consider 630 observations on respiratory rate (Alexander *et al.*, 2012) and a parent normal distribution. These data were taken from a study by the University of São Paulo, ESALQ (Laboratory of Physiology and Post-Harvest Biochemistry), which evaluate the

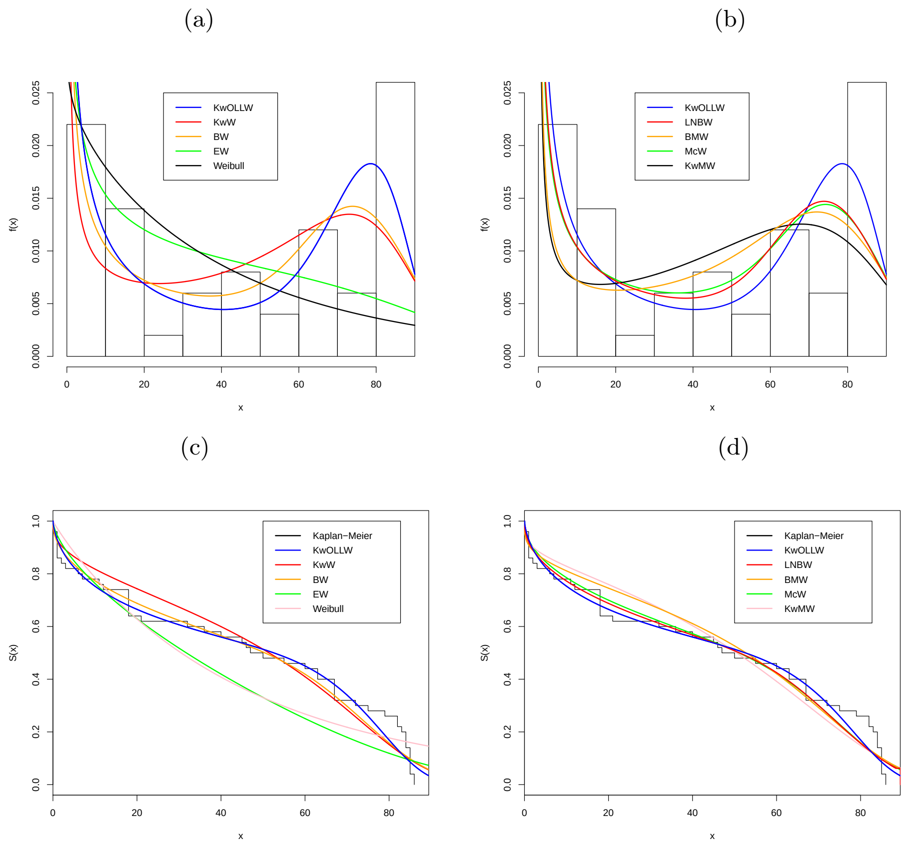


Figure 5. (a) Estimated densities of the KwOLLW, BW, KwW, EW and Weibull models. (b) Estimated densities of the KwOLLW, McW, KwMW, BMW and LNBW models. (c) Empirical and estimated survival functions of the KwOLLWBW, KwW, EW and Weibull. (d) Empirical and estimated survival functions of the KwOLLW, McW, KwMW, BMW and LNBW models.

effects of mechanical damage on banana fruits (genus Musa spp.); see Saavedra del Aguila *et al.* (2010) for more details. The major problem affecting bananas during and after harvest is the susceptibility of the mature fruit to physical damage caused during transport and marketing. The extent of the damage is measured by the respiratory rate.

Initial values for a, b, μ and σ are taken from the fitted KwN model with $\alpha = 1$; see, for example, Cordeiro *et al.* (2012b). The computations are performed using the subroutine NLMIXED in SAS. Table 5 lists the MLEs and their standard errors (in parentheses) of the parameters of the fitted models and the AIC, CAIC and BIC values. The computations are performed using the subroutine NLMixed in SAS. These results indicate that the KwOLLN model has the lowest AIC, CAIC and BIC values among those values of the fitted models, and therefore it could be chosen as the best model.

More information is provided by a visual comparison of the histogram of the data with the fitted densities. In Figure 6, we plot the histogram of the respiratory data and

Table 4. Formal goodness-of-fit tests for Aarset data.

Model	Statistic	
	W^*	A^*
KwOLLW	0.0833	0.7477
KwW	0.1454	1.1324
EW	0.2740	1.8111
Weibull	0.4963	3.0079
BW	0.1041	0.9043
McW	0.1047	0.9056
LNBW	0.1028	0.8972
BMW	0.1677	1.2697
KwMW	0.1912	1.3995

Table 5. MLEs and information criteria.

Respiratory	μ	σ	a	b	α	AIC	CAIC	BIC
KwOLLN	6.5396	113.18	2.2642	0.2778	11.2953	5547.0	5547.1	5569.3
	(2.7772)	(14.6040)	(0.3356)	(0.0147)	(2.3128)			
KwN	-32.7704	29.4031	13.4721	0.4520	1	5775.1	5775.2	5792.9
	(2.5507)	(0.8140)	(1.4283)	(0.0329)	(-)			
Normal	34.3166	27.7500	1	1	1	5979.3	5979.4	5988.2
	(1.1056)	(0.7818)	(-)	(-)	(-)			
	a	b	c	μ	σ	AIC	CAIC	BIC
McN	10021.0	0.4681	4.6369	-186.04	47.9945	5638.3	5638.4	5660.5
	(8.8561)	(0.0305)	(0.6311)	(7.9203)	(1.7718)			
BN	50.9335	0.4135	1	-56.1790	32.2426	5709.9	5710.0	5727.7
	(2.5794)	(0.0296)	-	(2.1684)	(0.9699)			

the fitted KwOLLN, McN, BN, KwN and normal densities. The KwOLLN and McN distributions provide reasonable fits, but it is clear that the KwOLLN model provides a more adequate fit to the histogram and better captures its extreme bathtub shape.

13. Conclusions

We introduce and study a new class of distributions called the Kumaraswamy odd log-logistic-G (KwOLL-G) family, which includes as special cases some classical generators of distributions such as the Kumaraswamy-generalized and exponentiated families. For each baseline G distribution, we define the corresponding KwOLL-G distribution with three additional shape parameters using simple formulas to extend widely-known models such as the normal, Weibull and Gumbel distributions in order to provide more flexibility. Some characteristics of the new family, such as the ordinary moments, generating function and mean deviations, have tractable mathematical properties. The role of the generator parameters is related to the skewness and kurtosis of the new family. We estimate the parameters using maximum likelihood and determine the observed information matrix. Inference on the model parameters is conducted based on likelihood ratio statistics for testing nested models and other formal statistics for non-nested models. Two applications to real data demonstrate the importance of the new family.

(a) (b)

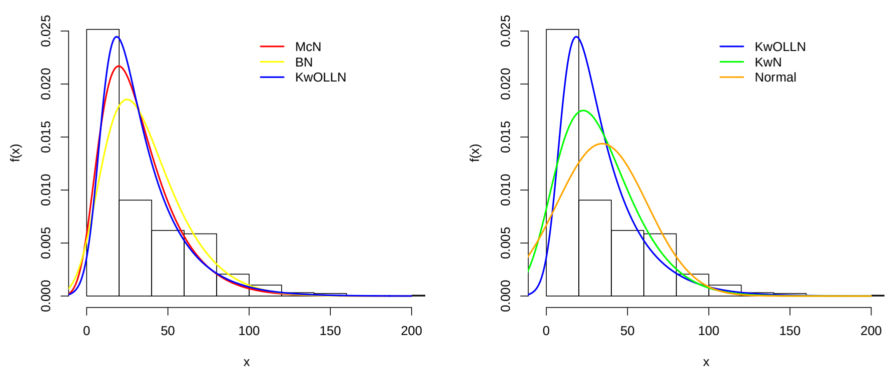


Figure 6. Fitted densities of the KwOLLN, McN and BW models for the respiratory data. (b) Fitted densities of the KwOLLN, KwN and normal models for the respiratory data.

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Appendix A

We present four power series expansions required for the proof of the general result in Section 4. First, for $b > 0$ real non-integer and $0 < u < 1$, we have the binomial expansion

$$(13.1)\qquad (1-u)^a = \sum_{j=0}^{\infty} (-1)^j \binom{a}{j} u^j,$$

where the binomial coefficient is defined for any real.

Second, the following expansion holds for any $\alpha > 0$ real non-integer

$$(13.2)\qquad G(x)^\alpha = \sum_{r=0}^{\infty} s_r(\alpha)\, G(x)^r,$$

where $s_r(\alpha) = \sum_{j=r}^{\infty} (-1)^{r+j} \binom{\alpha}{j} \binom{j}{r}$.

Third, by expanding z^λ in Taylor series, we obtain

$$(13.3)\qquad z^\lambda = \sum_{k=0}^{\infty} (\lambda)_k \,(z-1)^k / k! = \sum_{i=0}^{\infty} f_i\, z^i,$$

where

$$f_i = f_i(\lambda) = \sum_{k=i}^{\infty} \frac{(-1)^{k-i}\,(\lambda)_k}{k!} \binom{k}{i}$$

and $(\lambda)_k = \lambda(\lambda-1)\ldots(\lambda-k+1)$ is the descending factorial.

Fourth, we consider equations (6.2) and (6.3) to obtain an expansion for $[G(x)^a + \bar{G}(x)^a]^c$. We can write from equations (13.1) and (13.2)

$$[G(x)^a + \bar{G}(x)^a] = \sum_{j=0}^{\infty} t_j\, G(x)^j,$$

where $t_j = t_j(a) = s_j(a) + (-1)^j \binom{a}{j}$. Then, using (13.3), we can write

$$[G(x)^a + \bar{G}(x)^a]^c = \sum_{i=0}^{\infty} f_i \left(\sum_{j=0}^{\infty} t_j\, G(x)^j \right)^i,$$

where $f_i = f_i(c)$. Finally, based on equations (6.2) and (6.3), we have

$$[G(x)^a + \bar{G}(x)^a]^c = \sum_{j=0}^{\infty} h_j\, G(x)^j, \tag{13.4}$$

where $h_j = h_j(a, c) = \sum_{i=0}^{\infty} f_i\, m_{i,j}$ and $m_{i,j} = (j\, t_0)^{-1} \sum_{m=1}^{j} [m(j+1) - j]\, t_m\, m_{i,j-m}$ (for $j \geq 1$) and $m_{i,0} = t_0^i$.

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Improved estimation from ranked set sampling

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Abstract

Ranked set sampling is used when the measurement or quantification of units of the variable under study is difficult but the ranking of units of sets of small sizes can be done easily by an inexpensive method. Dell and Clutter (1972) showed that the sample mean based on ranked set sample is more efficient than the sample mean based on simple random sample with replacement sampling procedure for estimation of the population mean. In this paper Dell and Clutter estimator has been improved further by using the ranking variable x as an auxiliary variable when μ_x, the population mean of x is unknown. An empirical investigation based on life data shows all proposed estimators are approximately unbiased and bring gain in efficiency of up to 50 percent.

Keywords: Order statistics, ranked set sampling, relative efficiency, sampling with replacement.

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1. Introduction

Ranked set sampling (RSS) was introduced by McIntyre (1952) to estimate the mean pasture and forage yield. The RSS is used when precise measurement of the variable of interest is difficult or expensive but the variable can be ranked easily without measuring the actual variable by an inexpensive method such as visual perception, judgment and auxiliary information. For example, in estimating the mean height of trees in a forest, the heights of a small sample of two or three trees standing nearby can be ranked easily by visual inspection without measuring them. In estimating the number of bacterial cells per unit volume, we can rearrange two or three test tubes easily in order of concentration using optical instruments without measuring exact values. In a ranked set sampling, instead of selecting a single sample of size m, we select m-sets of samples each of size m. In

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each of the sets all the elements are ranked but only one is measured. Finally, an average of the m-measured units is taken as an estimate of the population mean. Dell and Clutter (1972) proved that the sample mean based on the RSS is unbiased for the population mean regardless of the errors of ranking. The RSS mean is at least as precise as the sample mean of the simple random sampling with replacement (SRSWR) sampling scheme of the same size. Stokes (1980, 1980a, 1988) showed that RSS provides precise estimators for cumulative distribution function, population variance and correlation coefficient.

1.1. Rank set sampling by SRSWR method. First we choose a small number m (set-size) such that one can easily rank the m elements of the population with sufficient accuracy. Then the selection of RSS is as follows: Select a sample of m^2 units from a population U by SRSWR method. Allocate these m^2 units at random into m sets each of size m. Rank all the units in a set with respect to the values of the variable of interest y from 1 (minimum) to m (maximum) by a very inexpensive method such as eye inspection. No actual measurement is done at this stage. After the ranking has been completed, the unit holding rank 1 of the set-1, unit holding rank-2 of the set $2, \ldots,$ and finally the unit holding rank m of the set m is measured accurately by using a suitable instrument. This completes a cycle of the sampling. The process is repeated for r cycles to obtain the desired sample of size $n = mr$ units. Thus in a RSS a total of m^2r units have been drawn from the population but only mr of them are measured and the rest $mr\,(m-1)$ are discarded. These measured mr observations are called "ranked set sample". Since the ordering of a large number of observations is difficult, increase of sample size $n = mr$ is done by increasing the number of cycles r.

It is obvious that the variable used for ranking x (say) e.g. eye estimation, judgment or auxiliary information is expected to have high correlation with the variable of interest y. Stokes (1977) considered ranking as an auxiliary variable. Prasad (1989), Kadilar et al. (2009) and Singh et al. (2014) used the estimation of the population mean μ_y assuming the population mean μ_x is known. In our present paper we have proposed improved methods of estimation of the population mean using the ranking variable as an auxiliary variable when the population mean μ_x is unknown. The proposed estimators fare better than the traditional estimator-sample mean. We also compared the performances of the proposed estimators through simulation studies based on live data collected by Platt et al. (1988), given by Chen et al. (2003). The simulation revealed that all the proposed estimators are approximately unbiased and bring gain in efficiency of up to 50%.

1.2. A fundamental equality. Let $y_{i1|k}, \ldots, y_{ij|k}, \ldots, y_{im|k}$ and $x_{i1|k}, \ldots, x_{ij|k}, \ldots, x_{im|k}$ be the value of the variable of interest y and x of the ith set of elements of the kth cycle, $i = 1, \ldots, m; \quad k = 1, \ldots, r$. Further, let $y_{i(j)|k}$ and $x_{i(j)|k}$ be the smallest jth observation (order statistic) of $y_{i1|k}, \ldots, y_{ij|k}, \ldots, y_{im|k}$ and $x_{i1|k}, \ldots, x_{ij|k}, \ldots, x_{im|k}$ respectively. Here we first assume that y increases with x i.e. $x_{ij|k} > x_{i'j'|k'}$ implies $y_{ij|k} > y_{i'j'|k'}$. Ranking of heights of two and three trees nearby through visual inspection, the eye estimates (x) is expected to provide perfect ranking. Obviously, perfect ranking is not always possible. So, the theory of judgement ranking has been introduced in section 2.6. Let $y_{i1|k}, \ldots, y_{ij|k}, \ldots, y_{im|k}$ be a random sample from a population with cumulative distribution function (cdf) $F(y)$ and probability density function (pdf) $f(y)$. Similarly $x_{i1|k}, \ldots, x_{ij|k}, \ldots, x_{im|k}$ are the random sample from a population with cdf $F(x)$and pdf $f(x)$ respectively. Let the mean and variance of x and y be μ_x, μ_y and

$\sigma_x^2,\ \sigma_y^2$ respectively. Then we have the following equalities following Stokes (1980):

$$
\begin{aligned}
(1.1)\qquad \sum_{j=1}^{m} y_{ij|k} &= \sum_{j=1}^{m} y_{i(j)|k}, \qquad \sum_{j=1}^{m} x_{ij|k} = \sum_{j=1}^{m} x_{i(j)|k} \\
\sum_{j=1}^{m} (x_{ij|k}-\mu_x)^2 &= \sum_{j=1}^{m}, (x_{i(j)|k}-\mu_x)^2, \\
(1.2)\qquad \sum_{j=1}^{m} (y_{ij|k}-\mu_y)^2 &= \sum_{j=1}^{m} (y_{i(j)|k}-\mu_y)^2
\end{aligned}
$$

and

$$(1.3)\qquad \sum_{j=1}^{m} (x_{ij|k}-\mu_x)(y_{ij|k}-\mu_y) = \sum_{j=1}^{m} (x_{i(j)|k}-\mu_x)(y_{i(j)|k}-\mu_y)$$

Let $\mu_{x(j)|m} = E\{x_{i(j)|k}\}$ and $\mu_{y(j)|m} = E\{y_{i(j)|k}\}$ be the mean of the jth order-statistic of random samples of size m of the variables x and y for the cycle k. The order statistics $\mu_{x(j)|m}$ and $\mu_{y(j)|m}$ depend on m but is independent of the set i and the cycle k.

The equation (1.1) yields

$$
\begin{aligned}
E\left\{\frac{1}{m}\sum_{j=1}^{m} x_{ij|k}\right\} &= E\left\{\frac{1}{m}\sum_{j=1}^{m} x_{i(j)|k}\right\} \\
(1.4)\qquad \text{i.e.}\quad \mu_x &= \frac{1}{m}\sum_{j=1}^{m} \mu_{x(j)|m}
\end{aligned}
$$

Similarly,

$$(1.5)\qquad \mu_y = \frac{1}{m}\sum_{j=1}^{m} \mu_{y(j)|m}$$

the equation (1.2) yields

$$
\begin{aligned}
\sum_{j=1}^{m} E(x_{ij|k}-\mu_x)^2 &= \sum_{j=1}^{m} E(x_{i(j)|k}-\mu_x)^2 \\
\text{i.e.}\quad m\sigma_x^2 &= \sum_{j=1}^{m} \left\{\sigma^2_{x(j)|m} + (\mu_{x(j)|m}-\mu_x)^2\right\} \\
(\text{where } \sigma^2_{x(j)|m} &= \text{variance of } x_{i(j)|m}) \\
(1.6)\qquad \text{i.e.}\quad \sigma_x^2 &= \frac{1}{m}\sum_{j=1}^{m} \sigma^2_{x(j)|m} + \frac{1}{m}\sum_{j=1}^{m} (\mu_{x(j)|m}-\mu_x)^2
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(1.7)\qquad \sigma_y^2 &= \frac{1}{m}\sum_{j=1}^{m} \sigma^2_{y(j)|m} + \frac{1}{m}\sum_{j=1}^{m} (\mu_{y(j)|m}-\mu_y)^2 \\
(\text{where } \sigma^2_{y(j)|m} &= \text{variance of } y_{i(j)|m})
\end{aligned}
$$

Let us assume that the variables x and y from the same unit are correlated while from the different units are uncorrelated so that

$$(1.8)\qquad Cov(x_{ij|k}, y_{ij|k}) = \mu_{xy} \quad \text{and} \quad Cov(x_{ij|k}, y_{i'j'|k'}) = 0 \quad \text{for} \quad (i,j,k) \neq (i',j',k')$$

1.3. Estimation of the mean. Let $\bar{y}_{[m]|k} = \frac{1}{m}\sum_{i=1}^{m} y_{i(i)|k}$ = arithmetic mean of the m quantified values of the variable y for the cycle k and

$$\hat{\mu}_{y(rss)} = \frac{1}{r}\sum_{k=1}^{r} \bar{y}_{[m]|k} = \frac{1}{n}\sum_{k=1}^{r}\sum_{i=1}^{m} y_{i(i)|k} \tag{1.9}$$

is the mean of $n = mr$ quantified variables based on all the r cycles. The following theorem due to Dell and Clutter (1972)and Kaur et al. (1997) show that the estimator $\hat{\mu}_{y(rss)}$ is unbiased for μ_y and possesses a lower variance than $\hat{\mu}_{y(srs)}$, the sample mean based on an SRSWR sample of the same size n. An unbiased estimator of the variance is also presented here.

1.1. Theorem.

(i) $$E(\hat{\mu}_{y(rss)}) = \mu_y$$

(ii) $$\begin{aligned} V(\hat{\mu}_{y(rss)}) &= \frac{\sigma^2_{y[m]}}{n} \\ &= \frac{1}{n}\left[\sigma_y^2 - \frac{1}{m}\sum_{i=1}^{m}(\mu_{y(j)|m} - \mu_y)^2\right] \\ &\leq \sigma_y^2/n = V(\hat{\mu}_{y(srs)}) \end{aligned}$$

(iii) *An unbiased estimator of the variance of* $V(\hat{\mu}_{y(rss)})$ *is*

$$\hat{V}(\hat{\mu}_{y(rss)}) = \frac{1}{r(r-1)}\sum_{k=1}^{r}(\bar{y}_{[m]|k} - \hat{\mu}_{y(rss)})^2.$$

where $\sigma^2_{y[m]} = \frac{1}{m}\sum_{j=1}^{m}\sigma^2_{y(j)|m}$

1.4. Precision of the rank-set sampling. The relative precision of $\hat{\mu}_{y(rss)}$ compared to $\hat{\mu}_{y(srs)}$, sample mean of an SRSWR sample of size $n = mr$ is

$$RP_{rss/srs} = \frac{V(\hat{\mu}_{y(srs)})}{V(\hat{\mu}_{y(rss)})} = \frac{\sigma_y^2}{\sigma^2_{y[m]}} \tag{1.10}$$

2. Proposed estimator of the population mean

From the ith set of the kth cycle, we construct an estimator for μ_y as follows:

$$\begin{aligned} t_{i|k} &= y_{i(i)|k} - \lambda x_{i(i)|k} + \lambda \bar{x}_{i|k} \quad \text{for} \quad i = 1,..,m \\ &= y_{i(i)|k} - \lambda(x_{i(i)|k} - \bar{x}_{i|k}) \end{aligned} \tag{2.1}$$

where $\bar{x}_{i|k} = \frac{1}{m}\sum_{j=1}^{m} x_{ij|k}$ and λ is a suitably chosen constant to be determined optimally.

The proposed estimator of the population mean μ_y based on the kth cycle is

$$\begin{aligned} t_k &= \frac{1}{m}\sum_{i=1}^{m} t_{i|k} \\ &= \left(\frac{1}{m}\sum_{i=1}^{m} y_{i(i)|k}\right) - \lambda\left(\frac{1}{m}\sum_{i=1}^{m} x_{i(i)|k} - \frac{1}{m}\sum_{i=1}^{m}\bar{x}_{i|k}\right) \end{aligned} \tag{2.2}$$

and the overall estimator for μ_y is

$$\bar{t} = \frac{1}{r}\sum_{k=1}^{r} t_k \tag{2.3}$$

2.1. Mean and variance of $\bar{t}$.

$$\begin{aligned} E(t_{i|k}) &= E(y_{i(i)|k}) - \lambda E(x_{i(i)|k} - \bar{x}_{i|k}) \\ &= \mu_{y(i)|k} - \lambda(\mu_{x(i)|k} - \mu_x) \\ &= \mu_{y(i)} - \lambda(\mu_{x(i)} - \mu_x) \end{aligned} \tag{2.4}$$

(noting $\mu_{y(i)|k} = \mu_{y(i)}$ for every k)
Now using (2.2), we get

$$\begin{aligned} E(t_k) &= \frac{1}{m}\sum_{i=1}^{m}[(\mu_{y(i)} - \lambda\mu_{x(i)}) + \lambda\mu_x] \\ &= \mu_d + \lambda\mu_x \quad (\text{where } \mu_d = \mu_y - \lambda\mu_x) \\ &= \mu_y \end{aligned} \tag{2.5}$$

The variance of t_k is

$$\begin{aligned} V(t_k) &= V\left(\frac{1}{m}\sum_{i=1}^{m} t_{i|k}\right) \\ &= \frac{1}{m^2}\sum_{i=1}^{m} V(t_{i|k}) \end{aligned} \tag{2.6}$$

Now

$$V(t_{i|k}) = V(y_{i(i)|k}) + \lambda^2 V(x_{i(i)|k} - \bar{x}_{i|k}) - 2\lambda Cov(y_{i(i)|k}, x_{i(i)|k} - \bar{x}_{i|k}) \tag{2.7}$$

Further,

$$\begin{aligned} V(x_{i(i)|k} - \bar{x}_{i|k}) &= V(x_{i(i)|k}) + V\left(\bar{x}_{i|k}\right) - 2Cov(x_{i(i)|k,}\bar{x}_{i|k}) \\ &= \sigma^2_{x(i)} + \frac{\sigma^2_x}{m} - \frac{2}{m}\left[V(x_{i(i)|k}) + \sum_{j(\neq i)} Cov(x_{i(i)k}, x_{i(j)|k})\right] \end{aligned} \tag{2.8}$$

$$\begin{aligned} Cov(y_{i(i)|k}, x_{i(i)|k} - \bar{x}_{i|k}) &= Cov(y_{i(i)|k}, x_{i(i)|k}) - Cov\left(y_{i(i)|k}, \frac{1}{m}\sum_{j=1}^{m} x_{i(j)||k}\right) \\ &= \sigma_{xy(i)|k} - \frac{1}{m}\left[Cov(x_{i(i)|k}, y_{i(i)|k}) + \frac{1}{m}\sum_{j(\neq 1)}^{m} Cov(y_{i(i)|k}, x_{i(j)|k})\right] \end{aligned} \tag{2.9}$$

where $\sigma_{xy(i)}$ is the covariance between $x_{i(i)k}$ and $y_{i(i)k}$.
Now substituting(2.8) and (2.9) in (2.7), we get

$$\begin{aligned} V(t_{i|k}) &= \sigma^2_{y(i)} + \lambda^2\left[\sigma^2_{x(i)} + \frac{\sigma^2}{m} - \frac{2}{m}\left\{V(x_{i(i)}) + \sum_{j(\neq i)} Cov(x_{i(i)}, x_{i(j)})\right\}\right] \\ &\quad -2\lambda\left[\sigma_{xy(i)} - \frac{1}{m}\left\{Cov(x_{i(i)}, y_{i(i)}) + \frac{1}{m}\sum_{j(\neq i)}^{m} Cov(y_{i(i)}, x_{i(j)})\right\}\right] \end{aligned}$$

The equation (2.6) yields

$$\begin{aligned}
& V(t_k) \\
= & \frac{1}{m^2}\sum_{i=1}^{m}\sigma^2_{y(i)} + \frac{\lambda^2}{m^2}\left[\sum_{i=1}^{m}\sigma^2_{x(i)} + \sigma^2_x - \frac{2}{m}\right. \\
& \left.\left\{\sum_{i=1}^{m} V(x_{i(i)|k}) + \sum_{i=1}^{m}\sum_{j(\neq i)} Cov(x_{i(i)k}, x_{i(j)|k})\right\}\right] \\
& -2\frac{\lambda}{m^2}\left[\sum_{i=1}^{m}\sigma_{xy(i)} - \frac{1}{m}\right. \\
& \left.\left\{\sum_{i=1}^{m} Cov(x_{i(i)|k}, y_{i(i)k}) + \frac{1}{m}\sum_{i=1}^{m}\sum_{j(\neq i)}^{m} Cov(y_{i(i)|k}, x_{i(j)|k})\right\}\right] \\
= & \frac{1}{m^2}\sum_{i=1}^{m}\sigma^2_{yi} + \frac{\lambda^2}{m^2}\left[\sum_{i=1}^{m}\sigma^2_{x(i)} - \sigma^2_x\right] - 2\frac{\lambda}{m^2}\left[\sum_{i=1}^{m}\sigma_{xy(i)} - \sigma_{xy}\right] \qquad (2.10)
\end{aligned}$$

Further, the equation (2.3) yields the variance of $\bar{t}$ as

$$\begin{aligned}
& V(\bar{t}) \\
= & \frac{1}{r^2}\sum_{k=1}^{r} V(t_k) \\
= & \frac{1}{rm^2}\left[\sum_{i=1}^{m}\sigma^2_{y(i)} + \lambda^2\left(\sum_{i=1}^{m}\sigma^2_{x(i)} - \sigma^2_x\right) - 2\lambda\left(\sum_{i=1}^{m}\sigma_{xy(i)} - \sigma_{xy}\right)\right] \qquad (2.11) \\
= & \frac{1}{n}\left[\frac{1}{m}\sum_{i=1}^{m}\sigma^2_{y(i)} + \lambda^2\frac{1}{m}\sum_{i=1}^{m}\sigma^2_{x(i)} - 2\frac{\lambda}{m}\sum_{i=1}^{m}\sigma_{xy(i)}\right] \qquad (2.12) \\
& -\frac{1}{nm}\left[\lambda^2\sigma^2_x - 2\lambda\sigma_{xy}\right]
\end{aligned}$$

(noting $n = rm$)

Now using (1.6), (1.7) and (1.8),we get

$$\begin{aligned}
V(\bar{t}) = & \frac{1}{n}\left[\left(\sigma^2_y - \frac{1}{m}\sum_{i=1}^{m}\lambda^2_{y(i)}\right) + \lambda^2\left(\sigma^2_x - \frac{1}{m}\sum_{i=1}^{m}\mu^2_{x(i)}\right)\right. \\
& \left.-2\lambda\left(\sigma_{xy} - \frac{1}{m}\sum_{i=1}^{m}\mu_{xy(i)}\right)\right] - \frac{1}{rm^2}[\lambda^2\sigma^2_x - 2\lambda\sigma_{xy}] \\
= & \frac{1}{n}\left[\left(\sigma^2_d - \frac{1}{m}\sum_{i=1}^{m}\mu^2_{d(i)}\right) - \frac{1}{rm}\left(\lambda^2\sigma^2_x - 2\lambda\sigma_{xy}\right)\right] \qquad (2.13)
\end{aligned}$$

The above results are summarized as follows:

2.1. Theorem.

(i) *The estimator $\bar{t}$ is unbiased for μ_y*

(ii) *The variance of $\bar{t}$ is*

$$V(\bar{t}) = \frac{1}{nm}\left[\sum_{i=1}^{m}\sigma_{y(i)}^2 + \lambda^2\left(\sum_{i=1}^{m}\sigma_{x(i)}^2 - \sigma_x^2\right) - 2\lambda\left(\sum_{i=1}^{m}\sigma_{xy(i)} - \sigma_{xy}\right)\right]$$

$$= \frac{1}{rn}\left[\sigma_d^2 - \frac{1}{m}\sum_{i=1}^{m}(\mu_{d(i)} - \mu_d)^2 + \frac{2\lambda\rho\sigma_x\sigma_y - \lambda^2\sigma_x^2}{m}\right]$$

where $\mu_{d(i)} = \mu_{y(i)} - \lambda\mu_{x(i)}$.

(iii) *An unbiased estimator of $V(\bar{t})$ is*

$$\hat{V}(\bar{t}) = \frac{1}{r(r-1)}\sum_{k=1}^{r}(t_k - \bar{t})^2$$

The part (iii) of the Theorem 2.1 follows from the fact that the estimators $t_k(k = 1, \ldots, r)$ are independently identically distributed random variables.

2.2. Optimum value of λ. The optimum value of λ that minimizes $V(\bar{t})$ is obtained from the equation

$$\frac{\partial V(\bar{t})}{\partial \lambda} = 0 \tag{2.14}$$

and it is given by

$$opt\lambda = \lambda_0 = \frac{\sum_{i=1}^{m}\sigma_{xy(i)} - \sigma_{xy}}{\sum_{i=1}^{m}\sigma_{x(i)}^2 - \sigma_x^2} \tag{2.15}$$

$$= \delta\frac{\sqrt{\sum_{i=1}^{m}\sigma_{y(i)}^2}}{\sqrt{\sum_{i=1}^{m}\sigma_{x(i)}^2 - \sigma_x^2}} \tag{2.16}$$

where δ is the correlation coefficient between $\frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{k} y_{i(i)|k}$ and $\frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{k}\left(x_{i(i)|k} - \bar{x}_{i|k}\right)$.

Finally, the variance $\bar{t}_0$, the optimum value of $\bar{t}$ with $\lambda = \lambda_0$ is given by

$$V_0 = (1-\delta^2)\frac{1}{m^2 r}\sum_{i=1}^{m}\sigma_{y(i)}^2$$

$$= (1-\delta^2)\frac{1}{n}\left[\sigma_y^2 - \frac{1}{m}\sum_{i=1}^{m}(\mu_{y(i)} - \mu_y)^2\right] \tag{2.17}$$

2.3. Precision of the proposed optimum estimator $\bar{\mathbf{t}}_0$. The relative precision of $\bar{t}_0$ with respect to the conventional estimator $\hat{\mu}_{y(rss)}$ based on an SRSWR sample mean of size $n = mr$ is given by

$$RP_{t_0|rss} = \frac{V(\hat{\mu}_{y(rss)})}{V(\bar{t}_0)} = \frac{1}{1-\delta^2} \tag{2.18}$$

From the expression (2.18), we note that the modified estimator is more efficient than the conventional RSS estimator $\hat{\mu}_{y(rss)}$ since $\delta^2 \leq 1$.

2.4. Estimator of λ_0. The optimum estimator t_0 cannot be used in practice since the value λ_0 is generally unknown. The following estimators for λ_0 may be used

$$\hat{\lambda}_0 = \frac{\sum_{k=1}^{r} (g_k - \bar{g})(h_k - \bar{h})}{\sum_{k=1}^{r} (h_k - \bar{h})^2} \tag{2.19}$$

and

$$\hat{\lambda}_1 = \frac{\sum_{k=1}^{r}\sum_{i=1}^{m} y_{i(i)|k} x_{i(i)|k} - \left(\sum_{k=1}^{r}\sum_{i=1}^{m} y_{i(i)|k}\right)\left(\sum_{k=1}^{r}\sum_{i=1}^{m} x_{i(i)|k}\right) / (rm)}{\sum_{k=1}^{r}\sum_{i=1}^{m} x^2_{i(i)|k} - \left(\sum_{k=1}^{r}\sum_{i=1}^{m} x_{i(i)|k})\right)^2 / rm} \tag{2.20}$$

where $g_k = \frac{1}{m}\sum_{i=1}^{m} y_{i(i)|k}$, $h_k = \frac{1}{m}\sum_{i=1}^{m}(x_{i(i)|k} - \bar{x}_{i|k})$, $\bar{g} = \frac{1}{r}\sum_{k=1}^{r} g_k$ and $\bar{h} = \frac{1}{r}\sum_{k=1}^{r} h_k$.

2.5. Ratio and difference estimators. Instead of the optimum value of λ_0, one may use the following ratio and difference estimators:

$$\bar{t}_R = \left(\frac{\hat{\mu}_{y(rss)}}{\hat{\mu}_{x(rss)}}\right)\left(\frac{1}{mr}\sum_{k=1}^{r}\sum_{i=1}^{m} \bar{x}_{i|k}\right) \tag{2.21}$$

and

$$\bar{t}_d = \hat{\mu}_{y(rss)} - (\hat{\mu}_{x(rss)} - \bar{x}) \tag{2.22}$$

where $\hat{\mu}_{x(rss)} = \frac{1}{mr}\left(\sum_{k=1}^{r}\sum_{i=1}^{m} x_{i(i)|k}\right)$, $\hat{\mu}_{y(rss)} = \frac{1}{mr}\left(\sum_{k=1}^{r}\sum_{i=1}^{m} y_{i(i)|k}\right)$ and $\bar{x} = \left(\frac{1}{m^2 r}\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m} x_{ij|k}\right)$

For large $n = mr$, the ratio estimator is appropriately unbiased and an approximate estimator of the mean square of $\hat{\mu}_{x(rss)}$ is obtained by using Cochran (1977) as

$$\begin{aligned} M(\bar{t}_R) &\cong \mu_x^2 V\left(\frac{1}{n}\sum_{k=1}^{r}\sum_{i=1}^{m} y_{i(i)|k} - \theta\frac{1}{n}\sum_{k=1}^{r}\sum_{i=1}^{m} x_{i(i)|k}\right) \\ &\cong \frac{\mu_x^2}{n}\Big[\left(\sigma_y^2 - 2\theta\rho_{xy}\sigma_x\sigma_y + \theta^2\sigma_x^2\right) \\ &\quad -\frac{1}{m}\sum_{i=1}^{m}\left\{\left(\mu_{y(i)} - \theta\mu_{x(i)}\right) - (\mu_y - \theta\mu_x)\right\}^2\Big] \end{aligned} \tag{2.23}$$

where $\theta = \frac{\mu_y}{\mu_x}$.

From the expression (2.23), we note that the ratio estimator based on ranked set sample is more precise than the conventional ratio estimator based on the same sample size.

A reasonably good estimator of $M(\bar{t}_R)$ is

$$\hat{M}(\bar{t}_R) \cong \hat{\mu}^2_{x(rss)}\frac{1}{n-1}\sum_{k=1}^{r}\sum_{i=1}^{m}\left(z_{i(i)|k} - \bar{z}\right)^2 \tag{2.24}$$

where $z_{i(i)|k} = y_{i(i)|k} - \hat{\theta}x_{i(i)|k}$, $\bar{z} = \sum_{k=1}^{r}\sum_{i=1}^{m} z_{i(i)|k}/n$ and $\hat{\theta} = \frac{\hat{\mu}_{y<rss>}}{\hat{\mu}_{x(rss)}}$.

It is easy to note that the difference estimator $\bar{t}_d$ is always unbiased and it is more efficient than the conventional difference estimator of the same sample size.

2.6. Judgment ranking. Sometimes ranking may be imperfect. Let $y_{i<j>|k}$ be the smallest jth "judgment order statistic" corresponding to order statistic $x_{i(j)|k}$ in the ith set of the cycle k. In case the judgment ranking is perfect $y_{i<j>|k}$ becomes equal to $y_{i(j)k}$, otherwise if the judgment process is imperfect, we find $y_{i<j>|k} \neq y_{i(j)|k}$. Here we assume that the expectation of $y_{i<j>|k}$ over the judgment process is the true ranking so that $E(y_{i<j>|k}) = y_{i(j)|k}$. In this case we modify the estimators $\hat{\mu}_{y(rss)}, \bar{t}_0,\ \bar{t}_1,\ \bar{t}_R$ and $\bar{t}_d$ by replacing $y_{i(j)|k}$ with $y_{i<j>|k}$. The modified estimators become respectively as follows:

$$
\begin{aligned}
\hat{\mu}_{y<rss>} &= \frac{1}{n}\sum_{k=1}^{r}\sum_{i=1}^{m} y_{i<i>|k} \\
\bar{t}_{<0>} &= \frac{1}{mr}\left[\sum_{k=1}^{r}\sum_{i=1}^{m} y_{i<i>|k} - \lambda_{<0>}\left(\sum_{k=1}^{r}\sum_{i=1}^{m} x_{i(i)|k} - \sum_{k=1}^{r}\sum_{i=1}^{m} \bar{x}_{i|k}\right)\right], \\
\bar{t}_{<1>} &= \frac{1}{mr}\left[\sum_{k=1}^{r}\sum_{i=1}^{m} y_{i<i>|k} - \lambda_{<1>}\left(\sum_{k=1}^{r}\sum_{i=1}^{m} x_{i(i)|k} - \sum_{k=1}^{r}\sum_{i=1}^{m} \bar{x}_{i|k}\right)\right], \\
\bar{t}_{<R>} &= \left(\frac{\hat{\mu}_{y\langle rss\rangle}}{\hat{\mu}_{x(rss)}}\right)\left(\frac{1}{mr}\sum_{k=1}^{r}\sum_{i=1}^{m} \bar{x}_{i|k}\right) \\
&\text{and}
\end{aligned}
$$

$$
\bar{t}_d = \hat{\mu}_{y<rss>} - (\hat{\mu}_{x(rss)} - \bar{x}) \tag{2.25}
$$

where $\hat{\lambda}_{<0>} = \dfrac{\sum\limits_{k=1}^{r}(g_{<k>} - \bar{g}_{<>})(h_k - \bar{h})}{\sum\limits_{k=1}^{r}(h_k - \bar{h})^2}$,

$$
\hat{\lambda}_{<1>} = \frac{\sum\limits_{k=1}^{r}\sum\limits_{i=1}^{m} y_{i<i>|k}x_{i(i)|k} - \left(\sum\limits_{k=1}^{r}\sum\limits_{i=1}^{m} y_{i<i>|k}\right)\left(\sum\limits_{k=1}^{r}\sum\limits_{i=1}^{m} x_{i(i)|k}\right)/\,(rm)}{\sum\limits_{k=1}^{r}\sum\limits_{i=1}^{m} x_{i(i)|k}^2 - \left(\sum\limits_{k=1}^{r}\sum\limits_{i=1}^{m} x_{i(i)|k}\right)^2/\,(rm)},
$$

$g_{<k>} = \dfrac{1}{m}\sum\limits_{i=1}^{m} y_{i<i>|k}, \quad \bar{g}_{<>} = \dfrac{1}{r}\sum\limits_{k=1}^{r} g_{<k>}$ with h_k and $\bar{h}$ as defined in section 2.4.

The modified estimator $\hat{\mu}_{y<rss>}$ remains exactly unbiased for μ_y while the remaining modified estimators based on the judgment order statistics remains approximately unbiased for μ_y.

3. Simulation studies

In the proposed simulation study we consider the tree data set originally collected by Platt et al. (1988) and cited by Chen et al. (2003). The data comprises of diameters in centimetre (cm) at breastheights (x) and entire height (y) in feet of 396 trees. The mean diameter and height of the 396 trees are $\mu_x = 20.9641$ and $\mu_y = 52.6768$ respectively. Treating the 396 trees as a population, initially a sample of m^2 trees is selected by SRSWR sampling procedures. The selection of the sample (cycle) is repeated r times. Since, for this data y does not always increase with x, we have compared performances with the proposed five estimators $\hat{\mu}_{y<rss>}, \bar{t}_{<0>},\ \bar{t}_{<1>},\ \bar{t}_{<R>}$ and $\bar{t}_{<d>}$ based on judgement order statistic. However, as per suggestions from one of the referees, we have considered the following ratio estimator (Kadilar et al., 2009) and regression estimators when the population mean μ_x is known

$$(3.1) \qquad \bar{t}^*_{<R>} = \frac{\hat{\mu}_{y<rss>}}{\hat{\mu}_{x(rss)}}\mu_x \quad \text{and} \quad \bar{t}^*_{<1>} = \hat{\mu}_{y<rss>} - \hat{\lambda}_{<1>}(\hat{\mu}_{x(rss)} - \mu_x)$$

We call the process of selection of m^2 trees and replication r times as an iteration. The iteration is repeated R $= 100,000$ times. Let the values of the $\hat{\mu}_{y<rss>}$, $\bar{t}_{<0>}$, $\bar{t}_{<1>}$, $\bar{t}_{<R>}$, $\bar{t}_{<d>}$, $\bar{t}^*_{<R>}$ and $\bar{t}^*_{<1>}$ based on the q^{th} iteration be denoted by $\hat{\mu}_{y<rss>}(q)$, $\bar{t}_{<0>}(q)$, $\bar{t}_{<1>}(q)$, $\bar{t}_{<R>}(q)$, $\bar{t}_{<d>}(q)$, $\bar{t}^*_{<R>}(q)$ and $\bar{t}^*_{<1>}(q)$ respectively.

The percentage relative biases (RB) and mean square errors (MSE) of the seven estimators are computed by the following formula:

$$(3.2) \qquad RB(\hat{\theta}) = \frac{1}{\mu_y}\left(\frac{1}{R}\sum_{q=1}^{R}\hat{\theta}\,(q) - \mu_y\right) \quad \text{and} \quad MSE(\hat{\theta}) = \frac{1}{R}\sum_{q=1}^{R}(\hat{\theta}(q) - \mu_y)^2$$

where $\mu_y = 52.6768$ and $\hat{\theta} = \hat{\mu}_{y<rss>},\ \bar{t}_{<0>},\ \bar{t}_{<1>},\ \bar{t}_{<R>},\ \bar{t}_{<d>},\ \bar{t}^*_{<R>},\ \bar{t}^*_{<1>}$.

The relative efficiency of the estimator $\hat{\theta}$ compared with the conventional estimator $\mu_{y<rss>}(q)$ is given by

$$(3.3) \qquad RE(\hat{\theta}) = 100 \times MSE(\hat{\mu}_{y<rss>})/MSE(\hat{\theta})\%$$

The values of $RB(\hat{\theta})$ and $RE(\hat{\theta})$ are computed for different combinations of $m(= 3, 4, 6, 10)$ and $r = (3, 6, 8, 9, 12, 15, 18, 20, 36)$. These are presented in the following Table-1 and Table-2. The simulation study shows for unknown, μ_x, the population mean of x, all the proposed estimators are approximately unbiased. The maximum absolute relative bias was 1.25. The minimum standard error (which is approximately $\sqrt{MSE}$) is 3.69 (not shown in the table). The biases of all the estimators are ignorable since the maximum of the ratio of bias of an estimator to its standard error is $0.0034 << 0.1$(see Cochran (1977)). For a given sample size $n(= mr)$ the biases of all the estimators increase with m. As per efficiency, all the proposed estimators are more efficient than the conventional estimator $\hat{\mu}_{y<rss>}$ in all situations considered here. The estimator $\bar{t}_{<1>}$ performed the best, closely followed by $t_{<R>}$ and $t_{<0>}$. The estimator $t_{<d>}$ performed least among the proposed five estimators. The maximum relative efficiency 147.70 was attained by $\bar{t}_{<1>}$ with $m = 10$, $r = 9$ and it attained the minimum 133.66 when $m = 3$, $r = 12$. This shows that the estimator $\bar{t}_{<1>}$ brings gains in efficiency over the conventional estimator $\hat{\mu}_{y<rss>}$ between 33% and 48% for estimating the population mean μ_y. However, in case μ_x, the population mean is known one should use the estimators $t^*_{<R>}$ and $\bar{t}^*_{<1>}$ as they perform much better than all the proposed estimators with respect to bias and mean square errors.

Table-1: Relative Bias of the proposed estimators

Sample size n	m	r	$B\left(\hat{\theta}\right)$ $\hat{\mu}_{y<rss>}$	$\bar{t}_{<0>}$	$\bar{t}_{<1>}$	$\bar{t}_{<R>}$	$\bar{t}_{<d>}$	$\bar{t}^*_{<1>}$	$\bar{t}^*_{<R>}$
36	3	12	0.29	−0.35	0.20	0.21	0.30	−0.24	−0.19
	4	9	0.43	0.38	0.29	0.31	0.42	−0.14	−0.09
	6	6	0.67	0.45	0.52	0.54	0.67	−0.05	0.03
54	3	18	0.26	−0.15	0.20	0.20	0.27	−0.15	−0.11
	6	9	0.66	0.51	0.55	0.56	0.66	0.04	0.10
60	4	15	0.44	0.17	0.33	0.34	0.43	−0.02	0.02
	10	6	1.19	1.14	1.12	1.12	1.19	0.06	0.21
72	3	24	0.30	−0.02	0.24	0.25	0.30	−0.11	−0.07
	4	18	0.42	0.20	0.34	0.35	0.41	−0.00	0.04
	6	12	0.66	0.54	0.58	0.59	0.66	0.07	0.13
80	4	20	0.41	0.21	0.34	0.34	0.41	0.00	0.04
	10	8	1.19	1.18	1.15	1.15	1.19	0.10	0.24
90	3	30	0.30	0.06	0.28	0.28	0.31	−0.08	−0.04
	6	15	0.65	0.57	0.59	0.59	0.65	0.10	0.16
	10	9	1.21	1.20	0.16	1.17	1.21	0.11	0.26
108	3	36	0.32	0.12	0.30	0.30	0.33	−0.05	−0.01
120	6	20	0.65	0.57	0.60	0.60	0.64	0.13	0.19
	10	12	1.21	1.21	1.19	1.19	1.22	0.14	0.28

Table-2: Relative Efficiencies of the proposed estimators

Sample			$E\left(\hat{\theta}\right)$						
size n	m	r	$\hat{\mu}_{y<rss>}$	$\bar{t}_{<0>}$	$\bar{t}_{<1>}$	$\bar{t}_{<R>}$	$\bar{t}_{<d>}$	$\bar{t}^{*}_{<1>}$	$\bar{t}^{*}_{<R>}$
36	3	12	100	119.29	133.66	132.87	115.94	427.32	410.01
	4	9	100	118.63	139.52	138.29	117.99	394.42	380.69
	6	6	100	109.33	145.55	143.67	119.70	356.66	342.76
54	3	18	100	125.59	134.50	133.27	116.05	434.01	409.9
	6	9	100	125.52	146.00	143.67	119.70	363.26	343.26
60	4	15	100	129.13	140.14	138.46	118.05	401.35	379.73
	10	6	100	111.93	147.67	144.93	119.97	320.61	301.68
72	3	24	100	127.54	134.09	132.81	115.94	440.94	412.61
	4	18	100	130.61	139.48	137.83	117.87	402.2	378.78
	6	12	100	132.17	146.41	143.77	119.69	366.35	343.37
80	4	20	100	132.37	140.18	138.39	118.05	403.1	378.85
	10	8	100	124.22	147.28	144.39	119.78	323.22	302.52
90	3	30	100	129.53	134.71	133.33	116.13	441.11	411.54
	6	15	100	135.51	146.19	143.50	119.63	368.25	344.19
	10	9	100	127.61	147.70	144.69	119.86	322.81	302.05
108	3	36	100	130.82	134.74	133.26	116.06	443.02	411.38
120	6	20	100	138.59	145.93	143.15	119.49	366.84	341.58
	10	12	100	133.46	146.66	143.75	119.59	325.82	303.93

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Regular A-optimal spring balance weighing designs with correlated errors

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Abstract

The problems linked with an A-optimal spring balance weighing design with correlated errors are discussed. The topic is focus on the determining the lowest bound of the trace of inverse information matrix in a special class of design matrices. The constructing method of the optimal design, based on the incidence matrices of balanced incomplete block designs, is presented.

Keywords: A-optimal design, balanced incomplete block design, spring balance weighing design.

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1. Introduction

Consider the linear model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}, \tag{1.1}$$

where

(a) $\mathbf{y}$ is an $n \times 1$ random vector of the observations,
(b) $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$, where $\mathbf{\Phi}_{n\times p}(0,1)$ denotes the class of $n \times p$ matrices $\mathbf{X} = (x_{ij})$ of known elements $x_{ij} = 1$ or 0 according as in the ith weighing operation the jth object is placed on the pan or not. Any matrix $\mathbf{X}$ belonging to the class $\mathbf{\Phi}_{n\times p}(0,1)$ is called the design matrix of the spring balance weighing design.
(c) $\mathbf{w}$ is a $p \times 1$ vector of unknown weights of objects,
(d) $\mathbf{e}$ is an $n \times 1$ random vector of errors for that $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2\mathbf{G}$, where $\mathbf{0}_n$ denotes the $n \times 1$ vector with zero elements everywhere, $\mathbf{G}$ is a known positive definite matrix.

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For the estimation of $\mathbf{w}$ we use the normal equations $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\mathbf{w} = \mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{y}$. Any spring balance weighing design is singular or nonsingular, depending on whether $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is singular or nonsingular, respectively. Since $\mathbf{G}$ is a known positive definite matrix then $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is nonsingular if and only if $\mathbf{X}$ has a full column rank. However, if $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, then the generalized least squares estimator of $\mathbf{w}$ is given by $\hat{\mathbf{w}} = \left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{y}$ and $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$.

There are several problems concerning to the optimality criteria of experimental designs. The best general references here are books [14] and [11]. The study results of determining the optimal weighing designs are shown in many papers, see for instance [12]. The standard work on A-, D- and E-optimality is the paper [5]. The deliberation related to A-optimal criterion for $\mathbf{G} = \mathbf{I}_n$ is presented in many papers. In [8] the robustness optimal designs are considered, whereas in [3] the problem of adding additionally weighing operation is presented. For a recent account on the theory of weighing designs, for $\mathbf{G}$ being any positive definite diagonal matrix, we refer the reader to [4].
The problems of determining of the regular D-optimal designs are included in several papers: in [10] some infinite families of D-optimal matrices based on Hadamard matrices are considered, however in [7] the deliberation on D-optimal designs under correlated structure of errors is presented. The construction of optimal design for eight objects is given in [9], while D-optimal weighing designs with autoregressive errors in [6]. Moreover, weighing designs as 2^n factorial designs were presented in [1] and [2].

2. The main result

In this paper, we emphasize a special interest of the existence conditions for A-optimal criterion. For given matrix $\mathbf{G}$, the problem is to determine such matrix $\mathbf{X}$ that $\mathrm{tr}\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ takes the minimal value over all possible matrices in $\mathbf{\Phi}_{n\times p}(0,1)$.

2.1. Definition. For given variance matrix of errors $\sigma^2\mathbf{G}$, any $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ is A-optimal if $\mathrm{tr}\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ is minimal. Moreover, if $\mathrm{tr}\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ attains the lower bound then $\mathbf{X}$ is called regular A-optimal.

It's worth underlining that for given variance matrix of errors $\sigma^2\mathbf{G}$ and in any class $\mathbf{\Phi}_{n\times p}(0,1)$ A-optimal spring balance weighing design exists always, whereas regular A-optimal design may exist.

In order to determine the lower bound of $\mathrm{tr}\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ the following theorems will be required.

2.2. Theorem. *Let* $\mathbf{M}$ *be any positive definite* $p \times p$ *matrix and* $\mathbf{\Pi}$ *be the set of all* $p \times p$ *permutation matrices. The average of* $\mathbf{M}$ *over all elements of* $\mathbf{\Pi}$*, i.e.* $\bar{\mathbf{M}} = \frac{1}{p!}\sum_{\mathbf{P}\in\mathbf{\Pi}}\mathbf{P}^{'}\mathbf{M}\mathbf{P}$ *and*

$$\bar{\mathbf{M}} = \frac{p\,\mathrm{tr}(\mathbf{M}) - \mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p}{p(p-1)}\mathbf{I}_p + \frac{\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p - \mathrm{tr}(\mathbf{M})}{p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'}. \tag{2.1}$$

Besides, $\mathrm{tr}(\mathbf{M}) = \mathrm{tr}(\bar{\mathbf{M}})$ *and* $\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p = \mathbf{1}_p^{'}\bar{\mathbf{M}}\mathbf{1}_p$.

Proof. Let us consider $p!$ elements of the set of all $p \times p$ permutation matrices $\mathbf{\Pi}$. When we put all matrices into $\sum_{\mathbf{P}\in\mathbf{\Pi}} \mathbf{P}^{'}\mathbf{MP}$ an easy computation makes it obvious that

$$\bar{\mathbf{M}} = \frac{1}{p!}\begin{bmatrix} (p-1)!\,\mathrm{tr}(\mathbf{M}) & (p-2)!\,Q(\mathbf{M}) & ... & (p-2)!\,Q(\mathbf{M}) \\ (p-2)!\,Q(\mathbf{M}) & (p-1)!\,\mathrm{tr}(\mathbf{M}) & ... & (p-2)!\,Q(\mathbf{M}) \\ ... & ... & ... & ... \\ (p-2)!\,Q(\mathbf{M}) & (p-2)!\,Q(\mathbf{M}) & ... & (p-1)!\,\mathrm{tr}(\mathbf{M}) \end{bmatrix},$$

where $Q(\mathbf{M})$ denotes the sum of all offdiagonal elements. Because $\mathbf{1}_p^{'}\mathbf{M1}_p = \mathrm{tr}(\mathbf{M}) + Q(\mathbf{M})$ we obtain 2.1. Moreover, the form the matrix $\bar{\mathbf{M}}$ indicates that it has two eigenvalues $\mu_1 = \frac{p\mathrm{tr}(\mathbf{M}) - \mathbf{1}_p^{'}\mathbf{M1}_p}{p(p-1)}$ with the multiplicity $p-1$ and $\mu_2 = \frac{\mathbf{1}_p^{'}\mathbf{M1}_p}{p}$ with the multiplicity 1.

□

2.3. Theorem. *Let t_1 be the eigenvalue with the multiplicity $p-1$, t_2 be the eigenvalue with the multiplicity 1 of any positive definite $p \times p$ matrix $\mathbf{M}$ and let q_1 be the eigenvalue with the multiplicity $p-1$ and q_2 be the eigenvalue with the multiplicity 1 of the matrix $\bar{\mathbf{M}}$. If $(p-1)t_1 + t_2 = (p-1)q_1 + q_2$, $t_1 \le t_2$, $q_1 \le q_2$, $t_1 \le q_1$ then $\mathrm{tr}(\mathbf{M}^{-1}) \ge \mathrm{tr}(\bar{\mathbf{M}}^{-1})$. The equality is satisfied if and only if the eigenvalues of matrices $\mathbf{M}$ and $\bar{\mathbf{M}}$ are the same.*

Proof. $\mathrm{tr}(\mathbf{M}^{-1}) - \mathrm{tr}(\bar{\mathbf{M}}^{-1}) = \frac{p-1}{t_1} + \frac{1}{t_2} - \frac{p-1}{q_1} - \frac{1}{q_2} = \frac{(p-1)t_2q_1q_2 - (p-1)t_1t_2q_2 + t_1q_1q_2 - t_1t_2q_1}{t_1t_2q_1q_2}$. Because $(p-1)q_1 = (p-1)t_1 + t_2 - q_2$ then $\mathrm{tr}(\mathbf{M}^{-1}) - \mathrm{tr}(\bar{\mathbf{M}}^{-1}) = \frac{(t_2-q_2)(t_2q_2-t_1q_1)}{t_1t_2q_1q_2}$. We observe $\frac{t_2}{t_1} \ge 1$, $\frac{q_1}{q_2} \le 1$. Thus $t_2q_2 - t_1q_1 \ge 0$. Finally $\mathrm{tr}(\mathbf{M}^{-1}) \ge \mathrm{tr}(\bar{\mathbf{M}}^{-1})$. It is obvious the equality is satisfied if and only if the eigenvalues of the matrices $\mathbf{M}$ and $\bar{\mathbf{M}}$ are equal. □

To aim at a target determining the regular A-optimal design let us consider the class of all design matrices of the spring balance weighing design $\mathbf{\Phi}_{n\times p}(0,1)$. For positive definite matrix $\mathbf{G}$ and any $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ we take $\mathbf{M} = \mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$. Let $m_1,\ m_2,\ ...\ m_p$, $m_1 \le m_2 \le ... \le m_p$ be the eigenvalus of the matrix $\mathbf{M}^{-1}$. Then $\mathrm{tr}(\mathbf{M}^{-1}) = m_1 + m_2 + ... + m_p \ge pm_1$. The minimum of $\mathrm{tr}(\mathbf{M}^{-1})$ is attained if $m_1 = m_2 = ... = m_p$ and m_1 attains the minimal value. The equality is fulfilled if and only if $\mathbf{M}^{-1}$ is proportional to identity matrix. Such form of the matrix $\mathbf{M} = \mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is not interesting from the point of view of experiment as in each measurement only one object is included. Therefore, let $m_1 = m_2 = ... = m_{p-1} \le m_p$ and $\mathrm{tr}(\mathbf{M}^{-1}) = (p-1)m_1 + m_p$ and its minimum is attained if and only if m_1 and m_p are minimal. So, we consider the matrix $\mathbf{M}$ with two different eigenvalues, only.

Here, we consider the subclass of the spring balance weighing designs in the following form

$$\mathbf{\Omega}_{n\times p}^{\xi}(0,\ 1) =$$

$$\left\{\mathbf{X}:\ \mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,\ 1),\ \mathbf{X1}_p = \xi\mathbf{1}_n,\ \mathbf{X}^{'}\mathbf{1}_n = \tfrac{n\xi}{p}\mathbf{1}_p,\ \tfrac{n\xi}{p} \in \mathrm{N},\ \xi \le p\right\}.$$

Moreover, from now on until the end of the paper we consider $\mathbf{G}$ to be of the form

$$\mathbf{G} = g\left[(1-\rho)\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n^{'}\right],\ \ g > 0,\ \ \frac{-1}{n-1} < \rho < 1. \tag{2.2}$$

Condition on the values of g and ρ is equivalent to the matrix $\mathbf{G}$ being positive definite. When the variance matrix of errors $\sigma^2\mathbf{G}$ is given by the matrix of the form 2.2 then we say that the errors are equally correlated and they have the same variances. Let note, $\mathbf{G}^{-1} = \frac{1}{g(1-\rho)}\left[\mathbf{I}_n - \frac{\rho}{1+\rho(n-1)}\mathbf{1}_n\mathbf{1}_n^{'}\right]$. Next let us consider

$$\mathbf{M} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{1}{g(1-\rho)}\left[\mathbf{X}'\mathbf{X} - \frac{\rho}{1+\rho(n-1)}\mathbf{X}'\mathbf{1}_n\mathbf{1}_n'\mathbf{X}\right].$$

We will denote by s the the number of elements equal to 1 in any row of the design matrix $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,\ 1)$. It is evident that $\mathrm{tr}(\mathbf{M}) = \frac{ns}{g(1-\rho)}\left[1 - \frac{ns\rho}{p(1+\rho(n-1))}\right]$ and $\mathbf{1}_p'\mathbf{M}\mathbf{1}_p = \frac{ns^2}{g(1+\rho(n-1))}$. From the above considerations and Theorem 2.2, eigenvalues of $\bar{\mathbf{M}}$ are $\mu_1 = \frac{ns(p-s)}{p(p-1)g(1-\rho)}$ and $\mu_2 = \frac{ns^2}{pg(1+\rho(n-1))}$. Thus the matrix $\bar{\mathbf{M}}^{-1}$ has also two eigenvalues $\frac{1}{\mu_1}$ with the multiplicity $p-1$ and $\frac{1}{\mu_2}$ with the multiplicity 1. Then $\mathrm{tr}(\bar{\mathbf{M}}^{-1}) = \frac{p-1}{\mu_1} + \frac{1}{\mu_2}$. Furthermore, to determine A-optimal spring balance weighing design, we need to find the smallest value of $\mathrm{tr}(\bar{\mathbf{M}}^{-1})$. The $\mathrm{tr}(\bar{\mathbf{M}}^{-1})$ attains the lowest bound when $\frac{p-1}{\mu_1}$ and $\frac{1}{\mu_2}$ are minimized. We have

$$\mathrm{tr}(\bar{\mathbf{M}}^{-1}) = \frac{pg}{n}\phi(s), \tag{2.3}$$

where $\phi(s) = \frac{(p-1)^2(1-\rho)}{s(p-s)} + \frac{1+\rho(n-1)}{s^2}$, $s = 1, 2, ..., p-1$.

2.4. Theorem. *Let p be even. In any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$*

(i) if $\quad \rho \in \left(\frac{-1}{n-1},\ P_1\right) \quad$ then

$$\mathrm{tr}\left(\mathbf{M}^{-1}\right) \geq \frac{4g}{np}\left(1+\rho(n-1)+(p-1)^2(1-\rho)\right), \tag{2.4}$$

the equality in 2.4 is satisfied if and only if $\mathbf{X}\mathbf{1}_p = \frac{p}{2}\mathbf{1}_n$,

(ii) if $\quad \rho \in (P_a,\ P_{a+1}) \quad$ then

$$\mathrm{tr}\left(\mathbf{M}^{-1}\right) \geq \frac{4pg}{n(p+2a)}\left(\frac{1+\rho(n-1)}{p+2a} + \frac{(p-1)^2(1-\rho)}{p-2a}\right), \tag{2.5}$$

the equality in 2.5 is satisfied if and only if $\mathbf{X}\mathbf{1}_p = \frac{p+2a}{2}\mathbf{1}_n$,

(iii) if $\quad \rho = P_a \quad$ then

$$\mathrm{tr}\left(\mathbf{M}^{-1}\right) \geq \frac{n(p-1)^2\left((p+2a-2)(2a-1)+(p+2a-1)(p-2a+2)\right)}{(p+2a)\left(n(p+2a-1)(p-2a)(p-2a+2)+L(a)\right)} \tag{2.6}$$

the equality in 2.6 is satisfied if and only if $\mathbf{X}\mathbf{1}_p = \frac{p+2a-2}{2}\mathbf{1}_n$ *or* $\mathbf{X}\mathbf{1}_p = \frac{p+2a}{2}\mathbf{1}_n$,

where $P_a = \frac{L(a)}{n(p+2a-1)(p-2a)(p-2a+2)+L(a)}$, $L(a) = (p-1)^2(2a-1)(p+2a-2)(p+2a) - (p+2a-1)(p-2a)(p-2a+2)$, $a = 1, 2, ..., \frac{p-2}{2}$.

Proof. Based on the delibarations given above, we will consider the matrix $\mathbf{M}$ with two eigenvalues. Theorem 2.3 implies $\mathrm{tr}(\mathbf{M}^{-1}) \geq \mathrm{tr}(\bar{\mathbf{M}}^{-1})$. Thus we have to asses the equality 2.3. For given $n,\ p,\ \rho$ and g, 2.3 is the function of s. Furthermore, to determine A-optimal spring balance weighing design, we need to find s for which $\phi(s)$ takes the smallest value. Because $s = 1, 2, ..., p-1$, then we should investigate the sequence $\phi(1),\ \phi(2), ..., \phi(p-1)$. Therefore we study the difference

$$\phi(s) - \phi(s+1) = \frac{(2s+1)(1+\rho(n-1))}{s^2(s+1)^2} + \frac{(p-2s-1)(p-1)^2(1-\rho)}{s(s+1)(p-s-1)(p-s)}. \tag{2.7}$$

For $s=1,2,...,\frac{p-2}{2}$ and any $n,\ p,\ \rho$, we have $\phi(s)\geq\phi(s+1)$. Thus, we investigate the sequence for $s=\frac{p-2}{2}+a,\ a=1,2,...,\frac{p-2}{2}$. We denote $P_a=\frac{L(a)}{n(p+2a-1)(p-2a)(p-2a+2)+L(a)}$, $L(a)=(p-1)^2(2a-1)(p+2a-2)(p+2a)-(p+2a-1)(p-2a)(p-2a+2)$. Next, let us consider the interval $\rho\in\left(\frac{-1}{n-1},\ P_1\right)$. If $s<\frac{p}{2}$ then $\phi(s)\geq\phi(s+1)$, if $s>\frac{p}{2}$, then $\phi(s)\leq\phi(s+1)$. The smallest value of 2.3 is attained if $s=\frac{p}{2}$ and then we obtain (i). Thus, we study $\rho\in(P_a,\ P_{a+1})$. If $s<\frac{p+2a}{2}$, then $\phi(s)\geq\phi(s+1)$. The inequality $s>\frac{p+2a}{2}$ implies $\phi(s)\leq\phi(s+1)$. The smallest value of 2.3 is attained for $s=\frac{p+2a}{2}$, thus (ii). If $\rho=P_a$, then $\phi(s)=\phi(s+1)$ and for $s=\frac{p+2a-2}{2}$ or $s=\frac{p+2a}{2}$, we receive (iii). $\square$

2.5. Theorem. *Let p be even. Any nonsingular spring balance weighing design $\mathbf{X}\in\mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$ is regular A-optimal*

(i) *for fixed $\rho\in\left(\frac{-1}{n-1},\ P_1\right)$ if and only if $\mathbf{X}\mathbf{1}_p=\frac{p}{2}\mathbf{1}_n$,*

(ii) *for fixed $\rho\in(P_a,\ P_{a+1})$ if and only if $\mathbf{X}\mathbf{1}_p=\frac{p+2a}{2}\mathbf{1}_n$,*

(iii) *for fixed $\rho=P_a$ if and only if $\mathbf{X}\mathbf{1}_p=\frac{p+2a-2}{2}\mathbf{1}_n$ or $\mathbf{X}\mathbf{1}_p=\frac{p+2a}{2}\mathbf{1}_n$,*

where $a=1,2,...,\frac{p-2}{2}$.

Proof. Any spring balance weighing design is regular A-optimal if and only if the equalities in 2.4-2.6 hold, i.e. if and only if the design matrix $\mathbf{X}\in\mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ is given as above. $\square$

2.6. Theorem. *Let p be even. Any nonsingular spring balance weighing design $\mathbf{X}\in\mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$ is regular A-optimal*

(i) *for fixed $\rho\in\left(\frac{-1}{n-1},\ P_1\right)$ if and only if*
$$\mathbf{M}=\frac{1}{g(1-\rho)}\left[\frac{np}{4(p-1)}\mathbf{I}_p+\frac{n(p-2)}{4(p-1)}\mathbf{1}_p\mathbf{1}_p^{'}-\frac{\rho n^2}{4(1+\rho(n-1))}\mathbf{1}_p\mathbf{1}_p^{'}\right]$$

(ii) *for fixed $\rho\in(P_a,\ P_{a+1})$ if and only if*
$$\mathbf{M}=\frac{1}{g(1-\rho)}\left[\frac{n(p+2a)(p-2a)}{4p(p-1)}\mathbf{I}_p+\frac{n(p+2a)(p-2a-2)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'}+\phi_a\mathbf{1}_p\mathbf{1}_p^{'}\right],$$

(iii) *for fixed $\rho=P_a$ if and only if*
$$\mathbf{M}=\frac{1}{g(1-\rho)}\left[\frac{n(p+2a)(p-2a)}{4p(p-1)}\mathbf{I}_p+\frac{n(p+2a)(p-2a-2)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'}+\phi_a\mathbf{1}_p\mathbf{1}_p^{'}\right]\quad or$$
$$\mathbf{M}=\frac{1}{g(1-\rho)}\left[\frac{n(p+2a+2)(p-2a-2)}{4p(p-1)}\mathbf{I}_p+\frac{n(p+2a+2)(p-2a-4)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'}+\phi_{a+1}\mathbf{1}_p\mathbf{1}_p^{'}\right]$$

where $\phi_a=\frac{n(p+2a)(4ap(1-\rho)-\rho n(p(p-1)-2a(p+1)))}{4p^2(p-1)(1+\rho(n-1))}$, $a=1,2,...,\frac{p-2}{2}$.

Proof. From Theorem 2.3, we obtain $\mathrm{tr}(\mathbf{M}^{-1})=\mathrm{tr}(\bar{\mathbf{M}}^{-1})$ if and only if the eigenvalues of $\mathbf{M}$ and $\bar{\mathbf{M}}$ are equal. Hence for $\mathbf{G}$ in the form 2.2 and $\mathbf{X}\in\mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ the best design for which minimum of $\mathrm{tr}(\mathbf{M}^{-1})$ is attained if the $\bar{\mathbf{M}}=\mathbf{M}$ one. Thus to prove this Theorem it is worthy to notice that from 2.1 we have $\bar{\mathbf{M}}=\frac{p\mathrm{tr}(\mathbf{M})-\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p}{p(p-1)}\mathbf{I}_p+\frac{\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p-\mathrm{tr}(\mathbf{M})}{p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'}$. Moreover, taking $s=\frac{p+2a}{2}$ we obtain $\frac{p\mathrm{tr}(\mathbf{M})-\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p}{p(p-1)}=\frac{n(p+2a)(p-2a)}{4p(p-1)g(1-\rho)}$ and $\frac{\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p-\mathrm{tr}(\mathbf{M})}{p(p-1)}=\frac{1}{g(1-\rho)}\left(\frac{n(p+2a)(p-2a-2)}{4p(p-1)}+\frac{n(p+2a)(4ap(1-\rho)-\rho n(p(p-1)-2a(p+1)))}{4p^2(p-1)(1+\rho(n-1))}\right)$, thus (ii). For $a=0$ we obtain (i). The above consideration and the condition (iii) of Theorem 2.5 imply formulas given in (iii). $\square$

2.7. Corollary. *In the special case, $g=1$ and $\rho=0$, the Condition (i) of Theorem 2.6 is equivalent to equality given in* [5]*. If additionally, $a=0$ then the condition (ii) of Theorem 2.6 is the same as given in* [5] *one.*

2.8. Theorem. *Let p be odd. In any nonsingular spring balance weighing design* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$

(i) if $\rho \in \left(\frac{-1}{n-1},\ R_1\right)$ then

$$\operatorname{tr}\left(\mathbf{M}^{-1}\right) \geq \frac{4pg}{n(p+1)^2}\left(1+\rho(n-1)+(p^2-1)(1-\rho)\right), \tag{2.8}$$

the equality in 2.8 is satisfied if and only if $\mathbf{X1}_p = \frac{p+1}{2}\mathbf{1}_n$,

(ii) if $\rho \in (R_a,\ R_{a+1})$ then

$$\operatorname{tr}\left(\mathbf{M}^{-1}\right) \geq \frac{4pg}{n(p+2a+1)}\left(\frac{1+\rho(n-1)}{p+2a+1}+\frac{(p-1)^2(1-\rho)}{p-2a-1}\right), \tag{2.9}$$

the equality in 2.9 is satisfied if and only if $\mathbf{X1}_p = \frac{p+2a+1}{2}\mathbf{1}_n$,

(iii) if $\rho = R_a$ then

$$\operatorname{tr}\left(\mathbf{M}^{-1}\right) \geq \frac{4pg(p-1)^2\left(2a(p+2a+1)+(p+2a)(p-2a-1)\right)}{(p+2a-1)\left(n(p+2a)(p-2a+1)(p-2a-1)+N(a)\right)} \tag{2.10}$$

the equality in 2.10 is satisfied if and only if $\mathbf{X1}_p = \frac{p+2a-1}{2}\mathbf{1}_n$ *or* $\mathbf{X1}_p = \frac{p+2a+1}{2}\mathbf{1}_n$,

where $R_a = \frac{N(a)}{n(p+2a)(p-2a+1)(p-2a-1)+N(a)}$, $N(a) = 2(p-1)^2a(p+2a-1)(p+2a+1) - (p+2a)(p-2a+1)(p-2a-1)$, $a = 1,2,...,\frac{p-3}{2}$.

Proof. The proof of Theorem is similar to that given in Theorem 2.4. Since, we will give the most important steps, only. For $s = 1,2,...,\frac{p+1}{2}$, $\phi(s) \geq \phi(s+1)$, for any $n,\ p,\ \rho$. Thus, we investigate the sequence for $s = \frac{p}{2}+a$, $a = 1,2,...,\frac{p-3}{2}$. We denote $R_a = \frac{N(a)}{n(p+2a)(p-2a+1)(p-2a-1)+N(a)}$, $N(a) = 2(p-1)^2a(p+2a-1)(p+2a+1)-(p+2a)(p-2a+1)(p-2a-1)$, $a = 1,2,...,\frac{p-3}{2}$. Next, let us consider the interval $\rho \in \left(\frac{-1}{n-1},\ R_1\right)$. If $s < \frac{p+1}{2}$ then $\phi(s) \geq \phi(s+1)$, if $s > \frac{p+1}{2}$, then $\phi(s) \leq \phi(s+1)$. The smallest value of 2.8 is attained if $s = \frac{p+1}{2}$. When we put $s = \frac{p+1}{2}$ in 2.3 we obtain (i). Now, we study $\rho \in (R_a,\ R_{a+1})$. If $s < \frac{p+2a+1}{2}$, then $\phi(s) \geq \phi(s+1)$. If $s > \frac{p+2a+1}{2}$, then $\phi(s) \leq \phi(s+1)$. The smallest value of 2.3 is attained for $s = \frac{p+2a+1}{2}$, thus (ii). If $\rho = R_a$, then $\phi(s) = \phi(s+1)$ and for $s = \frac{p+2a-1}{2}$ or $s = \frac{p+2a+1}{2}$, we receive (iii). □

2.9. Theorem. *Let p be odd. Any nonsingular spring balance weighing design* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$ *is regular A-optimal*

(i) *for fixed* $\rho \in \left(\frac{-1}{n-1},\ R_1\right)$ *if and only if* $\mathbf{X1}_p = \frac{p+1}{2}\mathbf{1}_n$,
(ii) *for* $\rho \in (R_a,\ R_{a+1})$ *if and only if* $\mathbf{X1}_p = \frac{p+2a+1}{2}\mathbf{1}_n$,
(iii) *for fixed* $\rho = R_a$ *if and only if* $\mathbf{X1}_p = \frac{p+2a-1}{2}\mathbf{1}_n$ *or* $\mathbf{X1}_p = \frac{p+2a+1}{2}\mathbf{1}_n$,

where $a = 1,2,...,\frac{p-3}{2}$.

Proof. According to the investigation given above, a spring balance weighing design is regular A-optimal if and only if the equalities in 2.8-2.10 are satisfied, i.e. if and only if the design matrix $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ is given as in Theorem 2.8. □

2.10. Theorem. *Let p be odd. Any nonsingular spring balance weighing design* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$ *is regular A-optimal*

(i) *for fixed* $\rho \in \left(\frac{-1}{n-1},\ R_1\right)$ *if and only if*

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{n(p+1)}{4p}\mathbf{I}_p + \frac{n(p+1)}{4p}\mathbf{1}_p\mathbf{1}_p^{'} - \frac{\rho n^2(p+1)^2}{4p^2(1+\rho(n-1))}\mathbf{1}_p\mathbf{1}_p^{'}\right]$$

(ii) *for* $\rho \in (R_a,\ R_{a+1})$ *if and only if*

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{n(p+2a+1)(p-2a-1)}{4p(p-1)}\mathbf{I}_p + \frac{n(p+2a+1)(p-2a-1)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'} - \psi_a\mathbf{1}_p\mathbf{1}_p^{'}\right],$$

(iii) *for* $\rho = R_a$ *if and only if*

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{n(p+2a+1)(p-2a-1)}{4p(p-1)}\mathbf{I}_p + \frac{n(p+2a+1)(p-2a-1)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'} - \psi_a\mathbf{1}_p\mathbf{1}_p^{'}\right] \quad or$$

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{n(p+2a+3)(p-2a-3)}{4p(p-1)}\mathbf{I}_p + \frac{n(p+2a+3)(p-2a-3)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'} - \psi_{a+1}\mathbf{1}_p\mathbf{1}_p^{'}\right],$$

where $\psi_a = \frac{n(p+2a+1)(\rho n(p^2-1)-4ap(1-\rho)-2an\rho(p+1))}{4p^2(p-1)(1+\rho(n-1))}$, $a = 1, 2, ..., \frac{p-3}{2}$.

Proof. The proof is similar to given in Theorem 2.6 one. It is sufficient to show that taking $s = \frac{p+2a+1}{2}$ we obtain $\frac{p\mathrm{tr}(\mathbf{M})-\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p}{p(p-1)} = \frac{n(p+2a+1)(p-2a-1)}{4p(p-1)g(1-\rho)}$ and $\frac{\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p-\mathrm{tr}(\mathbf{M})}{p(p-1)} = \frac{1}{g(1-\rho)}\left(\frac{n(p+2a+1)(p-2a-1)}{4p(p-1)} - \frac{n(p+2a+1)(\rho n(p^2-1)-4ap(1-\rho)-2an\rho(p+1)))}{4p^2(p-1)(1+\rho(n-1))}\right)$. Thus (ii). For $a = 0$ we obtain (i). Moreover, the above considerations and the condition (iii) of Theorem 2.9 imply the formulas presented in (iii). □

2.11. Corollary. *In the special case,* $g = 1$ *and* $\rho = 0$*, the Condition (i) of Theorem 2.10 is equivalent to equality given in* [5]*. If additionally,* $a = 0$ *then (ii) of Theorem 2.10 is the same as given in* [5] *one.*

3. Examples

Take into the consideration $\mathbf{X} = \mathbf{N}^{'}$, where $\mathbf{N}$ is the incidence matrix of balanced incomplete block design with the parameters $v,\ b,\ r,\ k,\ \lambda$, see [13]. To simplify the notation it is customary to write v instead of p and b instead of n. It is obvious that we are not able to give the construction of regular A-optimal spring balance weighing design for any combination of p, n and ρ. With the results obtained until now we can establish the following corollaries which indicate the series of the parameters of balanced incomplete block designs. Based on that incidence matrices we form the design matrices of regular A-optimal designs for an appropriate ρ.

3.1. Corollary. *Let* v *be even. If exists the balanced incomplete block design with the parameters* v, $b = v(v-1)$, $r = 0.5(v-1)(v+2a-2)$, $k = 0.5(v+2a-2)$, $\lambda = 0.25(v+2a-2)(v+2a-4)$, $a = 1, 2, ..., \frac{v-2}{2}$, *given by the incidence matrix* $\mathbf{N}$ *then any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{v(v-1)\times v}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}^{'}$ *is regular A-optimal spring balance weighing with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{n-1},\ P_1\right]$ *or* $\rho \in [P_a,\ P_{a+1})$.

3.2. Corollary. *Let* v *be even. If exists the balanced incomplete block design with the parameters* $v = 2(t+1)$, $b = 2(2t+1)$, $r = 2t+1$, $k = t+1$, $\lambda = t$, $t = 1, 2, ...$, *given by incidence matrix* $\mathbf{N}$*, then any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{2(2t+1)\times 2(t+1)}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}^{'}$ *is regular A-optimal spring balance weighing design with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{4t+1},\ \frac{2t^3+5t^2+3t+1}{6t^3+13t^2+6t+1}\right]$.

3.3. Corollary. *Let* v *be even. Any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{b\times v}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}^{'}$*, where* $\mathbf{N}$ *is the incidence matrix of balanced incomplete block design with the parameters* v, $b = \begin{pmatrix} v \\ 0.5(v+2a-2) \end{pmatrix}$, $r = \begin{pmatrix} v-1 \\ 0.5(v+2a-4) \end{pmatrix}$, $k = \frac{v+2a-2}{2}$, $\lambda = \begin{pmatrix} v-2 \\ 0.5(v+2a-6) \end{pmatrix}$, $a = 1, 2, ..., \frac{v-2}{2}$, *is regular A-optimal spring balance weighing design with the variance* *matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{n-1},\ P_1\right]$ *or* $\rho \in [P_a,\ P_{a+1})$*, where* $\binom{\eta}{\tau}$ *denotes binomial coefficient.*

3.4. Corollary. *Let* v *be odd. If exists the balanced incomplete block design with the parameters* $v,\ b = 0.5v(v-1),\ r = 0.25(v-1)(v+2a-1),\ k = 0.5(v+2a-1),\ \lambda = 0.125(v+2a-1)(v+2a-3),\ a = 1, 2, ..., \frac{v-3}{2}$*, given by the incidence matrix* $\mathbf{N}$*, then any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{0.5v(v-1)\times v}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}'$ *is regular A-optimal spring balance weighing design with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{n-1},\ R_1\right]$ *or* $\rho \in [R_a,\ R_{a+1})$.

3.5. Corollary. *Let* v *be odd. If exists the balanced incomplete block design with the parameters* $v = 2t+1,\ b = 2(2t+1),\ r = 2(t+1),\ k = t+1,\ \lambda = t+1,\ t = 2, 3, ...,$ *given by the incidence matrix* $\mathbf{N}$*, then any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{2(2t+1)\times(2t+1)}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}'$ *is regular A-optimal spring balance weighing design with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{4t+1},\ \frac{8t^3+22t^2+15t+3}{16t^3+30t^2+5t-3}\right]$.

3.6. Corollary. *Let* v *be odd. Any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{b\times v}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}'$*, where* $\mathbf{N}$ *is the incidence matrix of balanced incomplete block design with the parameters* $v,\ b = \begin{pmatrix} v \\ 0.5(v+2a-1) \end{pmatrix},\ r = \begin{pmatrix} v-1 \\ 0.5(v+2a-3) \end{pmatrix},\ k = \frac{v+2a-1}{2},\ \lambda = \begin{pmatrix} v-2 \\ 0.5(v+2a-5) \end{pmatrix},$ $a = 1, 2, ..., \frac{v-1}{2}$*, is regular A-optimal spring balance weighing design with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{n-1},\ R_1\right]$ *or* $\rho \in [R_a,\ R_{a+1})$.

3.7. Corollary. *Any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{v\times v}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}'$*, where* $\mathbf{N}$ *is the incidence matrix of balanced incomplete block design with the parameters* $v = b,\ r = k = v-1,\ \lambda = v-2,\ v = 3, 4, ...,$ *is regular A-optimal spring balance weighing design with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left[\frac{v^4-8v^3+24v^2-34v+19}{(v-1)(v^3-7v^2+17v-13)},\ 1\right)$.

3.8. Example. Let $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{30\times 6}(0,1)$ and let for $\mathbf{G}$, $g > 0,\ \rho \in (-0.034,\ 1),\ \xi \le 6$.

(i) If $\rho \in (-0.034,\ 0.170)$ then $\mathbf{X} = \mathbf{N}_1'$,
(ii) if $\rho \in (0.170,\ 0.733)$ then $\mathbf{X} = \mathbf{N}_2'$,
(iii) if $\rho \in (0.733,\ 1)$ then $\mathbf{X} = \mathbf{N}_3'$,
(iv) if $\rho = 0.170$ then $\mathbf{X} = \mathbf{N}_h',\ h = 1, 2$,
(v) if $\rho = 0.733$ then $\mathbf{X} = \mathbf{N}_h',\ h = 2, 3$,

is regular A-optimal spring balance weighing design, where $\mathbf{N}_h,\ h = 1, 2, 3$, is the incidence matrix of the balanced incomplete block design with parameters $v = 6,\ b_1 = 30,\ r_1 = 15,\ k_1 = 3,\ \lambda_1 = 6,\ v = 6,\ b_2 = 30,\ r_2 = 20,\ k_2 = 4,\ \lambda_2 = 12,$ $v = 6,\ b_3 = 30,\ r_3 = 25,\ k_3 = 5,\ \lambda_3 = 20$, respectively.

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A new generalized intuitionistic fuzzy set

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Abstract

A generalized intuitionistic fuzzy set ($GIFS_B$) is proposed. It is shown that Atanassov's intuitionistic fuzzy set, intuitionistic fuzzy sets of root type and intuitionistic fuzzy sets of second type are special cases of this new one. Some important notions, basic algebraic properties of $GIFS_B$, three operators and their relationship are discussed. The algebraic properties include being closed under union, being closed under intersection, being closed under a necessity measure, being closed under a possibility measure and de Morgan type identities.

Keywords: Generalized intuitionistic fuzzy set, Intuitionistic fuzzy set, Intuitionistic fuzzy sets of root type, Intuitionistic fuzzy sets of second type.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [22] whose basic component is only a degree of membership. Atanassov [2] generalized this idea to intuitionistic fuzzy sets (IFS) using a degree of membership and a degree of non-membership, under the constraint that the sum of the two degrees does not exceed one. A fuzzy set can be considered as IFS, since the sum of these grades is one. However, there are different situations when the sum of two degrees is smaller than one, which means that there is a certain ambiguity in the decision of membership or non-membership. For such cases the IFS is an appropriate tool.

A generalized intuitionistic fuzzy set (GIFS) were proposed by Mondal and Samanta [14] under the constraint that the minimum of the two degrees does not exceed half. Following the definition of IFS, Atanassov [3] [4] and Atanassov and Gargov [5] introduced

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interval valued IFSs, IFSs of second type, and temporal IFS. Srinivasan and Palaniappan [19] introduced IFSs of root type.

Some other extensions of the IFSs have also been introduced: IF soft sets due to Maji et al. [12]; IF rough sets due to Samanta and Mondal [18]; rough IFSs due to Rizvi et al. [17].

Some recent applications of IFSs have been: sustainable energy planning in Malaysia (Abdullah and Najib [1]); image fusion (Balasubramaniam and Ananthi [6]); agricultural production planning from a small farm holder perspective (Bharati and Singh [7]); medical diagnosis (Bora et al. [8]); pattern recognition (Chu et al. [9]); reservoir flood control operation (Hashemi et al. [10]); reliability optimization of complex system (Mahapatra and Roy [11]); fault diagnosis using dissolved gas analysis for power transformer (Mani and Jerome [13]); prioritizing the components of SWOT matrix in the Iranian insurance industry (Nikjoo and Saeedpoor [15]); prediction of the best quality of two-wheelers (Pathinathan et al. [16]); study of the decision framework of wind farm project plan selection (Wu et al. [21]).

The aim of this paper is to introduce new generalized IFSs and to derive their properties. The derived properties include: i) if A and B are generalized IFSs then their union and intersection are also generalized IFSs; ii) if A, B and C are generalized IFSs, A is a subset of B and B is a subset of C then A is a subset of C; iii) if A is a generalized IFS then its necessity and possibility measures are also generalized IFSs; iv) if the degree of non-determinacy of an element of a generalized IFS is zero then that for the nth power of the set is also zero; v) if A is a generalized IFS then the necessity measure of the nth power of A is the same as the nth power of the necessity measure of A; vi) if A is a generalized IFS then the possibility measure of the nth power of A is the same as the nth power of the necessity measure of A; vii) if A is a generalized IFS and $m \geq n$ then the mth power of A is a subset of the nth power of A; viii) if A is a generalized IFS and $m \geq n$ then nA is a subset of mA; ix) if A and B are generalized IFSs and A is a subset of B then nA is a subset of nB; x) if A and B are generalized IFSs and A is a subset of B then the nth power of A is a subset of nth power of B; xi) if A and B are generalized IFSs then the nth power of the union of A and B is the same as the union of the nth powers of A and B; xii) if A and B are generalized IFSs then the nth power of the intersection of A and B is the same as the intersection of the nth powers of A and B; xiii) if A and B are generalized IFSs then n times the union of A and B is the same as the union of nA and nB; xiv) if A and B are generalized IFSs then n times the intersection of A and B is the same as the intersection of nA and nB.

2. Preliminaries

In this section, we give some definitions of various types of IFS. We also define triangular norms and triangular conorms. Let X denote a non-empty set.

1. Definition. (Atanassov [2]). An IFS A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote, respectively, the degree of membership and degree of non-membership functions of A, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

2. Definition. (Atanassov [3]). An intuitionistic fuzzy set of second type (IFSST) A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote, respectively, the degree of membership and degree of non-membership functions of A, and $0 \leq [\mu_A(x)]^2 + [\nu_A(x)]^2 \leq 1$ for each $x \in X$.

3. Definition. (Srinivasan and Palaniappan [20]). An intuitionistic fuzzy set of root type (IFSRT) A in X is defined as an object of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x) : x \in X\rangle\},$$

where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote, respectively, the degree of membership and degree of non-membership functions of A, and $0 \leq \frac{1}{2}\sqrt{\mu_A(x)} + \frac{1}{2}\sqrt{\nu_A(x)} \leq 1$ for each $x \in X$.

4. Definition. (Atanassov [4]). A temporal IFS A in X is defined as an object of the form $A(T) = \{(x,t), \mu_A(x,t), \nu_A(x,t) : (x,t) \in E \times T\}$, where the functions $\mu_A(x,t)$ and $\nu_A(x,t)$ denote, respectively, the degree of membership and degree of non-membership functions of A of the element $x \in X$ at the time-moment $t \in T$, $A \subset E$ is a fixed set and $0 \leq \mu_A(x,t) + \nu_A(x,t) \leq 1$ for each $(x,t) \in E \times T$.

5. Definition. A triangular norm is a binary operation on $[0,1]$, i.e., an operator $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following conditions are satisfied:

i) Communicativity: $T(x,y) = T(y,x)$,
ii) Associativity: $T(x, T(y,z)) = T(T(x,y), z)$,
iii) Monotonicity: $T(x,y) \leq T(x,z)$ whenever $y \leq z$,
iv) Boundary condition: $T(x,1) = x$.

6. Definition. A triangular conorm is a binary operation on $[0,1]$, i.e., an operator $S : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following conditions are satisfied:

i) Communicativity: $S(x,y) = S(y,x)$,
ii) Associativity: $S(x, T(y,z)) = S(T(x,y), z)$,
iii) Monotonicity: $S(x,y) \leq S(x,z)$ whenever $y \leq z$,
iv) Boundary condition: $T(x,0) = x$.

Generalized fuzzy intuitionistic metric spaces can be defined based on triangular norms and triangular conorms.

3. New generalized intuitionistic fuzzy sets

7. Definition. Let X denote a non-empty set. Our generalized IFS A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote, respectively, the degree of membership and degree of non-membership functions of A, and $0 \leq \mu_A(x)^\delta + \nu_A(x)^\delta \leq 1$ for each $x \in X$ and $\delta = n$ or $\frac{1}{n}, n = 1, 2, \ldots, N$. The collection of all of our generalized IFSs is denoted by $GIFS_B(\delta, X)$.

One of the geometrical interpretations of the $GIFS_B(\delta, X)$ is shown in Figures 1 and 2. Let X denote a universal set and F a subset in the Euclidean plane with cartesian coordinates. For a $GIFS_B A$, a function f_A from X to F can be constructed such that if $x \in X$ then $p = (\nu_A(x), \mu_A(x)) = f_A(x) \in F$, $0 \leq \mu_A(x)$, $\nu_A(x) \leq 1$.

Let X be a set of ages of men over $[0,75]$. Let A be a set of young men whose ages are between 20 and 30. Define the membership and non membership functions of A as

$$\mu_A(x) = \begin{cases} \left(\dfrac{x-10}{10}\right)^{1/2}, & \text{if } 10 \leq x \leq 20, \\ 1, & \text{if } 20 \leq x \leq 30, \\ \left(\dfrac{40-x}{10}\right)^{1/2}, & \text{if } 30 \leq x \leq 40, \\ 0, & \text{otherwise,} \end{cases}$$

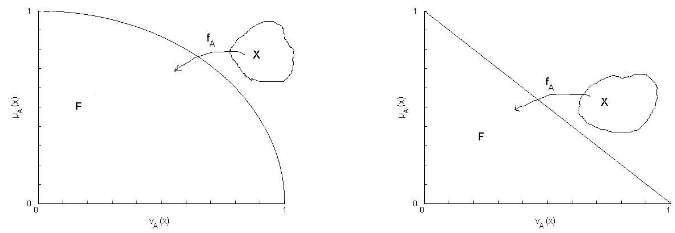

Figure 1. A geometrical interpretation of $GIFS_B$ with $\delta = 1$ and 2.

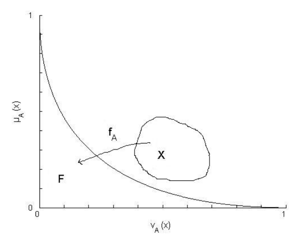

Figure 2. A geometrical interpretation of $GIFS_B$ with $\delta = 0.5$.

and

$$\nu_A(x) = \begin{cases} \left(\dfrac{20-x}{15}\right)^{1/2}, & \text{if } 5 \le x \le 20, \\ 0, & \text{if } 20 \le x \le 30, \\ \left(\dfrac{x-30}{15}\right)^{1/2}, & \text{if } 30 \le x \le 45, \\ 1, & \text{otherwise.} \end{cases}$$

Since $0 \le \mu_A(x)^2 + \nu_A(x)^2 \le 1$, $\forall x \in X$, $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is a $GIFS_B(2)$.

Also, we can define the membership and non membership functions of A as

$$\mu_A(x) = \begin{cases} \left(\dfrac{x-10}{10}\right)^{2}, & \text{if } 10 \le x \le 20, \\ 1, & \text{if } 20 \le x \le 30, \\ \left(\dfrac{40-x}{10}\right)^{2}, & \text{if } 30 \le x \le 40, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\nu_A(x) = \begin{cases} \left(\dfrac{20-x}{15}\right)^2, & \text{if } 5 \le x \le 20, \\ 0, & \text{if } 20 \le x \le 30, \\ \left(\dfrac{x-30}{15}\right)^2, & \text{if } 30 \le x \le 45, \\ 1, & \text{otherwise.} \end{cases}$$

Since $0 \le \mu_A(x)^{0.5} + \nu_A(x)^{0.5} \le 1,\ \forall x \in X,\ A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is a $GIFS_B(0.5)$.

3.1. Remark. It is obvious that for all real numbers $\alpha, \beta \in [0,1]$,

(i) if $0 \le \alpha+\beta \le 1$ and $\delta \ge 1$ then we have $0 \le \alpha^\delta+\beta^\delta \le 1$. With this consideration if $A \in IFS$ then $A \in GIFS_B$.

(ii) if $0 \le \alpha^\delta + \beta^\delta \le 1$ and $\delta \le 1$ then $0 \le \alpha + \beta \le 1$. With this consideration if $A \in GIFS_B$ then $A \in IFS$.

(iii) if $\delta_1 \le \delta_2$ then $\alpha^{\delta_2} \le \alpha^{\delta_1}$ and $\beta^{\delta_2} \le \beta^{\delta_1}$. It follows that $GIFS_B(\delta_1) \subset GIFS_B(\delta_2)$.

3.2. Remark. $GIFS_B(1) = IFS$, $GIFS_B(2) = GIFSST$, and $GIFS_B\left(\frac{1}{2}\right) = GIFSRT$.

8. Definition. Let X denote a non-empty set. Let A and B denote two $GIFS_B$s such that $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$. Define the following relations and operations on A and B:

i. $A \subset B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_A(x)$, $\forall x \in X$,

ii. $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, $\forall x \in X$,

iii. $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \in X\}$,

iv. $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \in X\}$,

v. $A + B = \left\{\left\langle x, \mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta, \nu_A(x)^\delta \nu_B(x)^\delta \right\rangle : x \in X\right\}$, so

$$2A = \left\{\left\langle x, 1 - \left(1 - \mu_A(x)^\delta\right)^2, \nu_A(x)^{2\delta} \right\rangle : x \in X\right\}$$

and

$$nA = \left\{\left\langle x, 1 - \left(1 - \mu_A(x)^\delta\right)^n, \nu_A(x)^{n\delta} \right\rangle : x \in X\right\},$$

vi. $A.B = \left\{\left\langle x, \mu_A(x)^\delta . \mu_B(x)^\delta, \nu_A(x)^\delta + \nu_B(x)^\delta - \nu_A(x)^\delta \nu_B(x)^\delta \right\rangle : x \in X\right\}$, so

$$A^2 = \left\{\left\langle x, \mu_A(x)^{2\delta}, 1 - \left(1 - \nu_A(x)^\delta\right)^2 \right\rangle : x \in X\right\}$$

and

$$A^n = \left\{\left\langle x, \mu_A(x)^{n\delta}, 1 - \left(1 - \nu_A(x)^\delta\right)^n \right\rangle : x \in X\right\},$$

vii. $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$.

Proposition 3.1 For $A, B, C \in GIFS_B$, we have

i. $\overline{\overline{A}} = A$,

ii. $A \subset B,\ B \subset C \Rightarrow A \subset C$.

Proof. The proof is obvious. $\square$

Proposition 3.2 For $A, B \in GIFS_B$, we have

i. $A \cup B \in GIFS_B$,

ii. $A \cap B \in GIFS_B$,

iii. $\delta \geq 1 \Rightarrow A + B \in GIFS_B$, $\delta < 1 \Rightarrow A + B \in IFS$,
iv. $\delta \geq 1 \Rightarrow A.B \in GIFS_B$, $\delta < 1 \Rightarrow A.B \in IFS$.

Proof. (i) Suppose $\max(\mu_A(x), \mu_B(x)) = \mu_A(x)$. Since $\min(\nu_A(x), \nu_B(x)) \leq \nu_A(x)$, we have

$$\begin{aligned} 0 &\leq \mu_{A\cup B}(x)^\delta + \nu_{A\cup B}(x)^\delta \\ &= (\max(\mu_A(x), \mu_B(x)))^\delta + (\min(\nu_A(x), \nu_B(x)))^\delta \\ &= \mu_A(x)^\delta + (\min(\nu_A(x), \nu_B(x)))^\delta \\ &\leq \mu_A(x)^\delta + \nu_A(x)^\delta \leq 1. \end{aligned}$$

Suppose now $\max(\mu_A(x), \mu_B(x)) = \mu_B(x)$. Since $\min(\nu_A(x), \nu_B(x)) \leq \nu_B(x)$, we have

$$\begin{aligned} 0 &\leq (\max(\mu_A(x), \mu_B(x)))^\delta + (\min(\nu_A(x), \nu_B(x)))^\delta \\ &= \mu_B(x)^\delta + (\min(\nu_A(x), \nu_B(x)))^\delta \\ &\leq \mu_B(x)^\delta + \nu_B(x)^\delta \leq 1. \end{aligned}$$

The proof of (i) is complete.
(ii) Proof of (i) is similar.

(iii) Since

$$A + B = \left\{ \left\langle x, \mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta, \nu_A(x)^\delta \nu_B(x)^\delta \right\rangle : x \in X \right\},$$

we have

$$\begin{aligned} &\mu_{A+B}(x)^\delta + \nu_{A+B}(x)^\delta \\ = &\left(\mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta\right)^\delta + \left(\nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta \\ = &\left(\mu_A(x)^\delta \left(1 - \mu_B(x)^\delta\right) + \mu_B(x)^\delta\right)^\delta + \left(\nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta \geq 0 \end{aligned}$$

and

$$\begin{aligned} &\mu_{A+B}(x)^\delta + \nu_{A+B}(x)^\delta \\ = &\left(\mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta\right)^\delta + \left(\nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta \\ \leq &\left(\left(1 - \nu_A(x)^\delta\right) + \left(1 - \nu_B(x)^\delta\right) - \left(1 - \nu_A(x)^\delta\right)\left(1 - \nu_B(x)^\delta\right)\right)^\delta \\ &+ \left(\nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta \\ = &\left(1 - \nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta + \left(\nu_A(x)^\delta \nu_B(x)^\delta\right)^\delta \\ = &(1-u)^\delta + u^\delta, \end{aligned}$$

where $u = \nu_A(x)^\delta \nu_B(x)^\delta$. If $\delta \geq 1$ then $(1-u)^\delta + u^\delta \leq 1$, hence $A + B \in GIFS_B$. If $\delta < 1$ then $(1-u)^\delta + u^\delta \leq 1$, if and only if $\nu_A(x) = 0$ or $\nu_B(x) = 0$. But for any δ, we have

$$\begin{aligned} &\mu_{A+B}(x) + \nu_{A+B}(x) \\ = &\mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta + \nu_A(x)^\delta \nu_B(x)^\delta \\ \leq &\mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta + \left(1 - \mu_A(x)^\delta\right)\left(1 - \mu_B(x)^\delta\right) \\ = &1, \end{aligned}$$

hence $A + B \in IFS$. The proof of (iii) is complete.
(iv). The proof of (iii) is similar. $\square$

9. Definition. The degree of non-determinacy (uncertainty) of an element $x \in X$ to the $GIFS_B A$ is defined by

$$\pi_A(x) = \left(1 - \mu_A(x)^\delta - \nu_A(x)^\delta\right)^{\frac{1}{\delta}}.$$

3.3. Remark. It can be easily shown that $\pi_A(x)^\delta + \mu_A(x)^\delta + \nu_A(x)^\delta = 1$.

10. Definition. For every $GIFS_B A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, we define the modal logic operators, the necessity measure on A and the possibility measure on A, as

$$\Box A = \left\{\left\langle x, \mu_A(x), \left(1 - \mu_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\}$$

and

$$\Diamond A = \left\{\left\langle x, \left(1 - \nu_A(x)^\delta\right)^{\frac{1}{\delta}}, \nu_A(x)\right\rangle : x \in X\right\},$$

respectively.

11. Definition. Let X denote a non-empty finite set. For every $GIFS_B$ as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

two analogues of the topological operators, closure (C) and intersection (I), can be defined on $GIFS_B$s as

$$C(A) = \{\langle x, K, L\rangle : x \in X\}, \quad K = \max_{y \in X} \mu_A(y), \quad L = \min_{y \in X} \nu_A(y)$$

and

$$I(A) = \{\langle x, k, l\rangle : x \in X\}, \quad k = \min_{y \in X} \mu_A(y), \quad l = \max_{y \in X} \nu_A(y).$$

It is obvious that both $C(A)$ and $I(A)$ are $GIFS_B$. These two operators transform a given $GIFS_B$ to a new $GIFS_B$.

12. Definition. Let X denote a non-empty finite set and let A denote a finite $GIFS_B$. The normalization of A denoted by $NORM(A)$ is defined by

$$NORM(A) = \left\{\left\langle x, \frac{\mu_A(x)^\delta}{\sup \mu_A(x)^\delta}, \frac{\nu_A(x)^\delta - \inf \nu_A(x)^\delta}{1 - \inf \nu_A(x)^\delta}\right\rangle : x \in X\right\}.$$

Proposition 3.3 Let $A, B \in GIFS_B$. We have

i. $\Box A \in GIFS_B$,
ii. $\Diamond A \in GIFS_B$,
iii. $\pi_A(x) = 0 \Rightarrow \pi_{A^n}(x) = 0$.

Proof. (i) Follows by noting that

$$\mu_{\Box A}(x)^\delta + \nu_{\Box A}(x)^\delta = \mu_A(x)^\delta + \left(\left(1 - \mu_A(x)^\delta\right)^{\frac{1}{\delta}}\right)^\delta = 1.$$

Proof of (ii) is similar to that of (i). (iii) Since

$$\pi_A(x) = \left(1 - \mu_A(x)^\delta - \nu_A(x)^\delta\right)^{\frac{1}{\delta}},$$

we have

$$\begin{aligned} & \pi_A(x) = 0 \\ \Rightarrow\ & \left(1 - \mu_A(x)^\delta - \nu_A(x)^\delta\right)^{\frac{1}{\delta}} = 0 \\ \Rightarrow\ & \mu_A(x)^\delta + \nu_A(x)^\delta = 1 \\ \Rightarrow\ & \mu_A(x)^\delta = 1 - \nu_A(x)^\delta. \end{aligned}$$

By using this result, we have

$$\begin{aligned} A^n &= \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\mu_A(x)^{n\delta}\right\rangle : x \in X\right\}. \end{aligned}$$

It is now obvious that $\pi_{A^n}(x) = 0$. □

3.4. Proposition. Let A denote a $GIFS_B$ and n any positive real number. Then, the following relations are true at the extreme values of $\mu_A(x)$ and $\nu_A(x)$:

i. $\Box A^n = (\Box A)^n$,
ii. $\Diamond A^n = (\Diamond A)^n$.

Proof. (i) Since

$$A^n = \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\},$$

we have

$$\Box A^n = \left\{\left\langle x, \mu_A(x)^{n\delta}, \left(1-\mu_A(x)^{n\delta^2}\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\}.$$

Also since

$$\Box A = \left\{\left\langle x, \mu_A(x), \left(1-\mu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\},$$

we have

$$\begin{aligned} (\Box A)^n &= \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\left(1-\left(1-\mu_A(x)^{\delta}\right)\right)^n\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\mu_A(x)^{n\delta}\right\rangle : x \in X\right\}. \end{aligned}$$

Assume $\Box A^n = (\Box A)^n$. Consequently, we must have

$$\begin{aligned} &\left(1-\mu_A(x)^{n\delta^2}\right)^{\frac{1}{\delta}} = 1-\mu_A(x)^{n\delta}, \\ &\left(1-\mu_A(x)^{n\delta^2}\right) = \left(1-\mu_A(x)^{n\delta}\right)^{\delta}, \\ &1-u^{\delta} = (1-u)^{\delta}, \qquad u = \left(1-\mu_A(x)^{n\delta}\right). \end{aligned}$$

Hence, (i) is true if and only if $\mu_A(x) = 0$ or 1, $\forall x \in X$.
(ii) We know that

$$A^n = \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\}$$

and

$$\Diamond A^n = \left\{\left\langle x, \left(1-\left(1-\left(1-\nu_A(x)^{\delta}\right)^n\right)^{\delta}\right)^{\frac{1}{\delta}}, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\}.$$

Also

$$\Diamond A = \left\{\left\langle x, \left(1-\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \nu_A(x)\right\rangle : x \in X\right\},$$

so

$$\begin{aligned} (\Diamond A)^n &= \left\{\left\langle x, \left(1-\nu_A(x)^{\delta}\right)^{\frac{n\delta}{\delta}}, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \left(1-\nu_A(x)^{\delta}\right)^n, 1-\left(1-\nu_A(x)^{\delta}\right)^n\right\rangle : x \in X\right\}. \end{aligned}$$

Assume $\diamond A^n = (\diamond A)^n$. Consequently, we must have

$$\left(1-\left(1-\left(1-\nu_A(x)^{\delta}\right)^{n}\right)^{\delta}\right)^{\frac{1}{\delta}} = \left(1-\nu_A(x)^{\delta}\right)^{n},$$
$$\left(1-\left(1-\nu_A(x)^{\delta}\right)^{n}\right)^{\delta} = 1-\left(1-\nu_A(x)^{\delta}\right)^{n\delta},$$
$$(1-u)^{\delta} = 1-u^{\delta}, \qquad u = \left(1-\nu_A(x)^{\delta}\right)^{n}.$$

Hence, (ii) is true if and only if $\nu_A(x) = 0$ or 1, $\forall x \in X$. $\square$

3.5. Proposition. For every $GIFS_B A$, we have

i. $m \geq n \Rightarrow A^m \subset A^n$,
ii. $m \geq n \Rightarrow nA \subset mA$,
iii. $A^n = \overline{n\overline{A}}$,

where m and n are both positive numbers.

Proof. (i) Since

$$A^n = \left\{\left\langle x, \mu_A(x)^{n\delta}, 1-\left(1-\nu_A(x)^{\delta}\right)^{n}\right\rangle : x \in X\right\},$$

we have

$$A^m = \left\{\left\langle x, \mu_A(x)^{m\delta}, 1-\left(1-\nu_A(x)^{\delta}\right)^{m}\right\rangle : x \in X\right\}.$$

Since $m \geq n$, we have $\mu_A(x)^n \geq \mu_A(x)^m$, so $\mu_A(x)^{n\delta} \geq \mu_A(x)^{m\delta}$ and $\mu_{A^n}(x) \geq \mu_{A^m}(x)$. Also since $\nu_A(x) \leq 1$, we have $\left(1-\nu_A(x)^{\delta}\right)^{m} \leq \left(1-\nu_A(x)^{\delta}\right)^{n}$, so

$$1-\left(1-\nu_A(x)^{\delta}\right)^{n} \leq 1-\left(1-\nu_A(x)^{\delta}\right)^{m} \Rightarrow \nu_{A^n}(x) \leq \nu_{A^m}(x),$$

completing the proof. The proof of (ii) is similar to that of (i). The proof of (iii) is immediate. $\square$

3.6. Proposition. Let $A, B \in GIFS_B$. We have

i. $A \subset B \Rightarrow nA \subset nB$,
ii. $A \subset B \Rightarrow A^n \subset B^n$,
iii. $(A \cup B)^n = A^n \cup B^n$,
iv. $(A \cap B)^n = A^n \cap B^n$,
v. $n\,(A \cup B) = nA \cup nB$,
vi. $n\,(A \cap B) = nA \cap nB$.

Proof. (i) Since $A \subset B$, we have $\mu_A(x) \leq \mu_B(x)$ and

$$\mu_A(x)^{\delta} \leq \mu_B(x)^{\delta} \Rightarrow 1-\mu_B(x)^{\delta} \leq 1-\mu_A(x)^{\delta} \Rightarrow \left(1-\mu_B(x)^{\delta}\right)^{n} \leq \left(1-\mu_A(x)^{\delta}\right)^{n},$$

so

$$1-\left(1-\mu_A(x)^{\delta}\right)^{n} \leq 1-\left(1-\mu_B(x)^{\delta}\right)^{n} \Rightarrow \mu_{nA}(x) \leq \mu_{mB}(x).$$

Also since $A \subset B$, we have $\nu_B(x) \leq \nu_A(x)$ and

$$\nu_B(x)^{n\delta} \leq \nu_A(x)^{n\delta} \Rightarrow \nu_{nB}(x) \leq \nu_{nA}(x),$$

completing the proof.
(ii) follows since

$$A \subset B \Rightarrow \overline{B} \subset \overline{A} \Rightarrow n\overline{B} \subset n\overline{A} \Rightarrow \overline{n\overline{A}} \subset \overline{n\overline{B}} \Rightarrow A^n \subset B^n.$$

(iii) follows since

$$A \cup B = \{\langle x, \max\left(\mu_A(x), \mu_B(x)\right), \min\left(\nu_A(x), \nu_B(x)\right)\rangle : x \in X\}$$

and

$$\begin{aligned}
& (A \cup B)^n \\
= \;& \left\{\left\langle x, (\max(\mu_A(x), \mu_B(x)))^{n\delta}, 1 - \left(1 - \min(\nu_A(x), \nu_B(x))^{\delta}\right)^n\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, \max\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1 - \left(1 - \min\left(\nu_A(x)^{\delta}, \nu_B(x)^{\delta}\right)\right)^n\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, \max\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1 - \left(\max\left(1 - \nu_A(x)^{\delta}, 1 - \nu_B(x)^{\delta}\right)\right)^n\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, \max\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1 - \max\left(\left(1 - \nu_A(x)^{\delta}\right)^n, \left(1 - \nu_B(x)^{\delta}\right)^n\right)\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, \max\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), \min\left(1 - \left(1 - \nu_A(x)^{\delta}\right)^n, 1 - \left(1 - \nu_B(x)^{\delta}\right)^n\right)\right\rangle : x \in X\right\} \\
= \;& A^n \cup B^n.
\end{aligned}$$

(iv) follows since

$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \in X\}$$

and

$$\begin{aligned}
& (A \cap B)^n \\
= \;& \left\{\left\langle x, (\min(\mu_A(x), \mu_B(x)))^{n\delta}, 1 - \left(1 - \max(\nu_A(x), \nu_B(x))^{\delta}\right)^n\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, \min\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1 - \left(1 - \max\left(\nu_A(x)^{\delta}, \nu_B(x)^{\delta}\right)\right)^n\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, \min\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1 - \left(\min\left(1 - \nu_A(x)^{\delta}, 1 - \nu_B(x)^{\delta}\right)\right)^n\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, \min\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), 1 - \min\left(\left(1 - \nu_A(x)^{\delta}\right)^n, \left(1 - \nu_B(x)^{\delta}\right)^n\right)\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, \min\left(\mu_A(x)^{n\delta}, \mu_B(x)^{n\delta}\right), \max\left(1 - \left(1 - \nu_A(x)^{\delta}\right)^n, 1 - \left(1 - \nu_B(x)^{\delta}\right)^n\right)\right\rangle : x \in X\right\} \\
= \;& A^n \cap B^n.
\end{aligned}$$

(v) follows since

$$\begin{aligned}
& n(A \cup B) \\
= \;& \left\{\left\langle x, 1 - \left(1 - \max(\mu_A(x), \mu_B(x))^{\delta}\right)^n, \min(\nu_A(x), \nu_B(x))^{n\delta}\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, 1 - \left(1 - \max\left(\mu_A(x)^{\delta}, \mu_B(x)^{\delta}\right)\right)^n, \min\left(\nu_A(x)^{n\delta}, \nu_B(x)^{n\delta}\right)\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, 1 - \left(\min\left(1 - \mu_A(x)^{\delta}, 1 - \mu_B(x)^{\delta}\right)\right)^n, \min\left(\nu_A(x)^{n\delta}, \nu_B(x)^{n\delta}\right)\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, 1 - \min\left(\left(1 - \mu_A(x)^{\delta}\right)^n, \left(1 - \mu_B(x)^{\delta}\right)^n\right), \min\left(\nu_A(x)^{n\delta}, \nu_B(x)^{n\delta}\right)\right\rangle : x \in X\right\} \\
= \;& \left\{\left\langle x, \max\left(1 - \left(1 - \mu_A(x)^{\delta}\right)^n, 1 - \left(1 - \mu_B(x)^{\delta}\right)^n\right), \min\left(\nu_A(x)^{n\delta}, \nu_B(x)^{n\delta}\right)\right\rangle : x \in X\right\} \\
= \;& nA \cup nB.
\end{aligned}$$

The proof of (vi) is similar to that of (v). □

4. The operators $D_\alpha(A)$, $F_{\alpha,\beta}(A)$ and $G_{\alpha,\beta}(A)$

Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ denote a $GIFS_B$.

13. Definition. Let $\alpha \in [0,1]$ and $A \in GIFS_B$. We define the operator of $D_\alpha(A)$ as

$$D_\alpha(A) = \left\{\left\langle x, \left(\mu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^{\delta} + (1-\alpha)\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\}.$$

Clearly, $D_\alpha(A)$ is a $GIFS_B$.

4.1. Theorem. For every $GIFS_B A$ and for every $\alpha, \beta \in [0,1]$, we have

i. $\alpha \le \beta \Rightarrow D_\alpha(A) \subset D_\beta(A)$,
ii. $D_0(A) = \Box A$,
iii. $D_1(A) = \Diamond A$.

Proof. The proof of (i) is immediate.
(ii) We have

$$\begin{aligned} D_0(A) &= \left\{\left\langle x, \left(\mu_A(x)^\delta + 0 \times \pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + (1-0)\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \mu_A(x), \left(\nu_A(x)^\delta + \pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \mu_A(x), \left(1 - \mu_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\} = \Box A, \end{aligned}$$

where the penultimate equality follows since $\pi_A(x)^\delta = 1 - \mu_A(x)^\delta - \nu_A(x)^\delta$. So, (ii) follows.
(iii) We have

$$\begin{aligned} D_1(A) &= \left\{\left\langle x, \left(\mu_A(x)^\delta + 1 \times \pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + (1-1)\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \left(\mu_A(x)^\delta + \pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \nu_A(x)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \left(1 - \nu_A(x)^\delta\right)^{\frac{1}{\delta}}, \nu_A(x)\right\rangle : x \in X\right\} = \Diamond A, \end{aligned}$$

completing the proof. $\Box$

14. Definition. Let $\alpha.\beta \in [0,1]$, where $\alpha + \beta \leq 1$. Let $A \in GIFS_B$. We define the operator of $F_{\alpha,\beta}(A)$ as

$$F_{\alpha,\beta}(A) = \left\{\left\langle x, \left(\mu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\}.$$

4.2. Theorem. For every $GIFS_B A$ and for any $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, we have

i. $F_{\alpha,\beta}(A) \in GIFS_B$,
ii. $0 \leq \gamma \leq \alpha \Rightarrow F_{\gamma,\beta}(A) \subset F_{\alpha,\beta}(A)$,
iii. $0 \leq \gamma \leq \beta \Rightarrow F_{\alpha,\beta}(A) \subset F_{\alpha,\gamma}(A)$,
iv. $D_\alpha(A) = F_{\alpha,1-\alpha}(A)$,
v. $\Box A = F_{0,1}(A)$,
vi. $\Diamond A = F_{1,0}(A)$,
vii. $\overline{F_{\alpha,\beta}\overline{A}} = F_{\beta,\alpha}(A)$.

Proof. (i) follows since

$$\begin{aligned} & \mu_{F_{\alpha,\beta}(A)}(x)^\delta + \nu_{F_{\alpha,\beta}(A)}(x)^\delta \\ =\ & \left[\left(\mu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right]^\delta + \left[\left(\nu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right]^\delta \\ =\ & \mu_A(x)^\delta + \nu_A(x)^\delta + \pi_A(x)^\delta(\alpha+\beta) \\ \leq\ & \mu_A(x)^\delta + \nu_A(x)^\delta + \pi_A(x)^\delta = 1. \end{aligned}$$

The proofs of (ii) and (iii) are immediate.
(iv) follows since

$$\begin{aligned} & F_{\alpha,1-\alpha}(A) \\ =\ & \left\{\left\langle x, \left(\mu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + (1-\alpha)\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\} \\ =\ & D_\alpha(A). \end{aligned}$$

(v) follows by Theorem 4.1 after noting that $D_0(A) = F_{0,1}(A)$ and $D_1(A) = F_{1,0}(A)$ from (iv).
(vi) follows by Theorem 4.1 after noting that $D_0(A) = F_{0,1}(A)$ and $D_1(A) = F_{1,0}(A)$ from (iv).
(vii) since

$$F_{\beta,\alpha}(A) = \left\{\left\langle x, \left(\mu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\}$$

and

$$F_{\alpha,\beta}\left(\overline{A}\right) = \left\{\left\langle x, \left(\nu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\mu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\},$$

we have

$$\overline{F_{\alpha,\beta}\left(\overline{A}\right)} = \left\{\left\langle x, \left(\mu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right\rangle : x \in X\right\}$$

and $\overline{F_{\alpha,\beta}\left(\overline{A}\right)} = F_{\beta,\alpha}(A)$. □

15. Definition. Let $\alpha, \beta \in [0,1]$ and $A \in GIFS_B$. We define the operator of $G_{\alpha,\beta}(A)$ as

$$G_{\alpha,\beta}(A) = \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\}.$$

4.3. Theorem. For every $GIFS_B A$, and for any real numbers $\alpha, \beta, \gamma \in [0,1]$, we have

i. $G_{\alpha,\beta}(A) \in GIFS_B$,
ii. $\alpha \le \gamma \Rightarrow G_{\alpha,\beta}(A) \subset G_{\gamma,\beta}(A)$,
iii. $\beta \le \gamma \Rightarrow G_{\alpha,\beta}(A) \supset G_{\alpha,\gamma}(A)$,
iv. $\tau \in [0,1] \Rightarrow G_{\alpha,\beta}\left(G_{\gamma,\tau}(A)\right) = G_{\alpha\gamma,\beta\tau}(A) = G_{\gamma,\delta}\left(G_{\alpha,\beta}(A)\right)$,
v. $G_{\alpha,\beta}\left(C(A)\right) = C\left(G_{\alpha,\beta}(A)\right)$,
vi. $G_{\alpha,\beta}\left(I(A)\right) = I\left(G_{\alpha,\beta}(A)\right)$,
vii. $\overline{G_{\alpha,\beta}\left(\overline{A}\right)} = G_{\beta,\alpha}(A)$.

Proof. (i) follows since

$$G_{\alpha,\beta}(A) = \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\}$$

and

$$\begin{aligned}
\mu_{G_{\alpha,\beta}(A)}(x)^\delta + \nu_{G_{\alpha,\beta}(A)}(x)^\delta &= \left(\alpha^{\frac{1}{\delta}}\mu_A(x)\right)^\delta + \left(\beta^{\frac{1}{\delta}}\nu_A(x)\right)^\delta \\
&= \alpha\mu_A(x)^\delta + \beta\nu_A(x)^\delta \\
&\le \mu_A(x)^\delta + \nu_A(x)^\delta \le 1.
\end{aligned}$$

(ii) We have

$$G_{\alpha,\beta}(A) = \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\}$$

and

$$G_{\gamma,\beta}(A) = \left\{\left\langle x, \gamma^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\}.$$

Since $\alpha \le \gamma$, we have $\alpha^{\frac{1}{\delta}} \le \gamma^{\frac{1}{\delta}}$ and so $\alpha^{\frac{1}{\delta}}\mu_A(x) \le \gamma^{\frac{1}{\delta}}\mu_A(x)$, completing the proof of (ii).
The proof of (iii) is similar to that of (ii).
(iv) We have

$$G_{\gamma,\tau}(A) = \left\{\left\langle x, \gamma^{\frac{1}{\delta}}\mu_A(x), \tau^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\},$$

$$\begin{aligned} G_{\alpha,\beta}\left(G_{\gamma,\tau}(A)\right) &= \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\gamma^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\tau^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, (\alpha\gamma)^{\frac{1}{\delta}}\mu_A(x), (\beta\tau)^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\} \\ &= G_{\alpha\gamma,\beta\tau}(A) \end{aligned}$$

and

$$\begin{aligned} G_{\gamma,\tau}\left(G_{\alpha,\beta}(A)\right) &= \left\{\left\langle x, \gamma^{\frac{1}{\delta}}\alpha^{\frac{1}{\delta}}\mu_A(x), \tau^{\frac{1}{\delta}}\beta^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, (\gamma\alpha)^{\frac{1}{\delta}}\mu_A(x), (\tau\beta)^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, (\alpha\gamma)^{\frac{1}{\delta}}\mu_A(x), (\beta\tau)^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\} \\ &= G_{\alpha\gamma,\beta\tau}(A), \end{aligned}$$

so

$$G_{\alpha,\beta}\left(G_{\gamma,\tau}(A)\right) = G_{\alpha\gamma,\beta\tau}(A) = G_{\gamma,\tau}\left(G_{\alpha,\beta}(A)\right).$$

(v) follows since

$$C(A) = \left\{\left\langle x, \max_{y\in X}\mu_A(y), \min_{y\in X}\nu_A(y)\right\rangle : x \in X\right\}$$

and

$$\begin{aligned} G_{\alpha,\beta}\left(C(A)\right) &= \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\max_{y\in X}\mu_A(y), \beta^{\frac{1}{\delta}}\min_{y\in X}\nu_A(y)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \max_{y\in X}\alpha^{\frac{1}{\delta}}\mu_A(y), \min_{y\in X}\beta^{\frac{1}{\delta}}\nu_A(y)\right\rangle : x \in X\right\} \\ &= C\left(G_{\alpha,\beta}(A)\right). \end{aligned}$$

(vi) follows since

$$I(A) = \left\{\left\langle x, \min_{y\in X}\mu_A(y), \max_{y\in X}\nu_A(y)\right\rangle : x \in X\right\}$$

and

$$\begin{aligned} G_{\alpha,\beta}\left(I(A)\right) &= \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\min_{y\in X}\mu_A(y), \beta^{\frac{1}{\delta}}\max_{y\in X}\nu_A(y)\right\rangle : x \in X\right\} \\ &= \left\{\left\langle x, \min_{y\in X}\alpha^{\frac{1}{\delta}}\mu_A(y), \max_{y\in X}\beta^{\frac{1}{\delta}}\nu_A(y)\right\rangle : x \in X\right\} \\ &= I\left(G_{\alpha,\beta}(A)\right), \end{aligned}$$

where $\alpha, \beta \in [0,1]$.
(vii) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ denote a $GIFS_B$. Then,

$$\begin{aligned} &\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}, \\ &G_{\beta,\alpha}(A) = \left\{\left\langle x, \beta^{\frac{1}{\delta}}\mu_A(x), \alpha^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\}, \\ &G_{\alpha,\beta}\left(\overline{A}\right) = \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\nu_A(x), \beta^{\frac{1}{\delta}}\mu_A(x)\right\rangle : x \in X\right\}, \\ &\overline{G_{\alpha,\beta}\left(\overline{A}\right)} = \left\{\left\langle x, \beta^{\frac{1}{\delta}}\mu_A(x), \alpha^{\frac{1}{\delta}}\nu_A(x)\right\rangle : x \in X\right\}, \end{aligned}$$

and so $\overline{G_{\alpha,\beta}\left(\overline{A}\right)} = G_{\beta,\alpha}(A)$. □

5. Conclusions

We have introduced a new generalized IFS ($GIFS_B$) as an extension to the IFS. The basic algebraic properties of $GIFS_B$ have been presented. Some operators on $GIFS_B$ are defined and their relationship have been proved. A list of open problems is as follows: i) define the generalized fuzzy intuitionistic number, norms, distances, metrics, metric spaces, etc for the generalized IFS and study of their properties; ii) develop statistical and probabilistic tools for the generalized IFS; iii) construct an axiomatic system for the generalized IFS; iv) develop efficient algorithms and computer software for the construction of degrees of membership and nonmembership of a given generalized IFS; v) define and study the properties of generalized IF boolean algebras; vi) develop information and entropy measures corresponding to generalized IFSs; vii) develop preference theory and utility theory for the generalized IFS; viii) compare with other generalizations of the IFS.

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The modified beta Weibull distribution

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Abstract

A new five-parameter model called the modified beta Weibull probability distribution is being introduced in this paper. This model turns out to be quite flexible for analyzing positive data and has bathtub and upside down bathtub hazard rate function.

Our main objectives are to obtain representations of certain statistical functions and to estimate the parameters of the proposed distribution. As an application, the probability density function is utilized to model two actual data sets. The new distribution is shown to provide a better fit than related distributions.

The proposed distribution may serve as a viable alternative to other distributions available in the literature for modeling positive data arising in various fields of scientific investigation such as reliability theory, hydrology, medicine, meteorology, survival analysis and engineering.

Keywords: Weibull distribution, Modified beta distribution, Goodness–of–fit statistics, Lifetime data.

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1. Introduction

The Weibull distribution is a very popular distribution named after Waloddi Weibull, a Swedish physicist. He used it in 1939 to analyze the breaking strength of materials. Ever since, it has been widely used for analyzing lifetime data. However, this distribution does not have a bathtub or upside–down bathtub shaped hazard rate function, that is why it cannot be utilized to model the life time of certain systems. To overcome this shortcoming, several generalizations of the classical Weibull distribution have been discussed by different authors in recent years. Many authors introduced flexible distributions for modeling complex data and obtaining a better fit. Extensions of Weibull distribution arise in different areas of research as discussed for instance in

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[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 20, 21, 24] and [27]. Many extended Weibull models have an upside–down bath tub shaped hazard rate, which is the case of the extensions discussed by [4], [14], [18] and [25], among others.
Adding parameters to an existing distribution enables one to obtain classes of more flexible distributions. Nadarajah *et al.* [17] introduced an interesting method for adding three new parameters to an existing distribution. The new distribution provides more flexibility to model various types of data. The baseline distribution has the cdf $G(x)$, then the new distribution is

$$F(x) = \frac{1}{B(a\,,\,b)} \int_0^{\left\{\frac{cG(x)}{(c-1)G(x)+1}\right\}} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x\,. \tag{1.1}$$

The Modified beta Weibull probability density function obtained from (1.1) can be expressed in the following form:

$$f(x) = \frac{c^a g(x)\{G(x)\}^{a-1}\{1-G(x)\}^{b-1}}{B(a,b)\{1-(1-c)G(x)\}^{a+b}}. \tag{1.2}$$

The cdf and pdf of Weibull distribution are defined as follows:

$$G(x) = 1 - \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^k}, \quad \lambda > 0\,,\, k > 0\,,\, x > 0 \tag{1.3}$$

and

$$g(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^k}. \tag{1.4}$$

We further generalize this model by applying the modified beta technique [17], which results in what we are referring to as the modified beta Weibull (MBW) distribution. The cdf, survival function, pdf and hazard rate function of the modified beta Weibull distribution, for which $G(x)$ is the baseline function, are respectively given by

$$F(x) = \frac{1}{B(a,b)}\, B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}}; a, b\right), \tag{1.5}$$

$$S(x) = 1 - \frac{1}{B(a,b)}\, B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}}; a, b\right), \tag{1.6}$$

$$\begin{aligned} f(x) &= c^a\, k\,(x)^{-1+k}\left(\mathrm{e}^{-\lambda^{-k}x^k}\right)^b\left(1-\mathrm{e}^{-\lambda^{-k}x^k}\right)^{-1+a} \\ &\times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k}x^k}\right)\right\}^{-a-b}}{\lambda^k\, B(a,b)} \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} h(x) &= k\, c^a\, x^{-1+k}\,\lambda^{-k}\left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^b\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-1+a} \\ &\times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-a-b}}{B(a,b) - B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}}; a, b\right)}\,,\; x > 0\,, \end{aligned} \tag{1.8}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, $Re(a) > 0$, $Re(b) > 0$ and $B(z;a,b) = \int_0^z t^{a-1}(1-t)^{b-1}dt$.

Also $\lambda > 0$, $k > 0$, $a > 0$, $b > 0$, $c > 0$. Equations (1.5) to (1.8) can be easily evaluated numerically using computational packages such as *Mathematica*, *Maple*, *MATLAB* and *R*. The following *Mathematica* code can be used for integration purposes: Integrate[f(x),{x, 0, Infinity}]. Further, Figure 1 shows the correctness of the defined cdf.

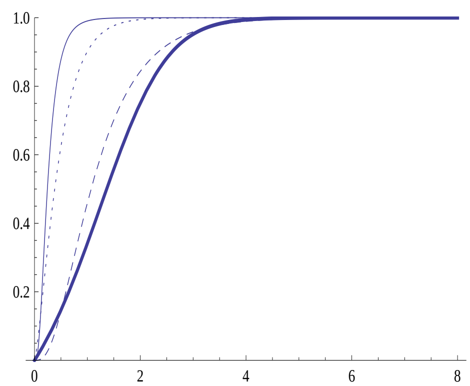

Figure 1. The MBW cdf. $\lambda = 0.8$, $k = 1.6$, $a = 1.4$, $b = 1.5$, $c = 0.8$, (dotted line), $\lambda = 4.8$, $k = 2.6$, $a = 3.4$, $b = 2.5$, $c = 1.8$, (dashed line), $\lambda = 10$, $k = 6$, $a = 4$, $b = 5$, $c = 5$, (solid line), $\lambda = 1$, $k = 1.2$, $a = 1$, $b = 2$, $c = 0.1$, (thick line).

Note that on making use of the identity

$$(1 - z)^{-\tau} = \sum_{n=0}^{\infty} \frac{\Gamma(\tau + n)}{\Gamma(\tau)\, n!}\, z^n\,, \quad |z| < 1\,, \tau > 0\,, \tag{1.9}$$

one has the following series representations of the pdf specified by (1.7)

$$f(x) = \frac{c^a\, x^{-1+k} k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b) n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n) m!} \times \left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b}. \tag{1.10}$$

Moreover, the first derivative of $h(x)$, which is used to study the shapes of hazard rate functions as explained in [13] is

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dx}}\,h(x) = {} & c^{1+a}\,k^2\,x^{-1+k}\,\lambda^{-1-k}\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-1+a}\left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^b \\
& \times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-a-b}}{\left\{c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right\}^2\left\{B(a,b)-B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}};a,b\right)\right\}^2} \\
& \times \left\{\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}}\right\}^{1+a} \\
& \times \left\{1-\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}}\right\}^{-1+b} \\
& + k^2x^{-2+2k}\lambda^{-2k}(a+b)(1-c)c^a\left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^{1+b}\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-1+a} \\
& \times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-1-a-b}}{B(a,b)-B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}};a,b\right)} \\
& + c^ak^2\lambda^{-2k}(a-1)x^{-2+2k}\left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^{1+b}\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-2+a} \\
& \times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-a-b}}{B(a,b)-B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}};a,b\right)} \\
& - b\,c^a\left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^b\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-1+a}k^2x^{-2+2k}\lambda^{-2k} \\
& \times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-a-b}}{B(a,b)-B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}};a,b\right)} \\
& + c^a(k-1)k\,\lambda^{-k}x^{-2+k}\left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^b\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-1+a} \\
& \times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-a-b}}{B(a,b)-B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}};a,b\right)}.
\end{aligned}
$$

Fig. $2,3$ and 4 plots some MBW curves for different choices of parameters for pdf and hazard rate function. Figures 2 and 3 indicate how the new parameters a, b and c affect the MBW density. These graphs illustrate the versatility of the MBW distribution. As can be seen from left panel of Figure 2 that a is a scale parameter and from the right panel of Figure 2 and left panel of Figure 3 that b and c are shape parameters. Similarly

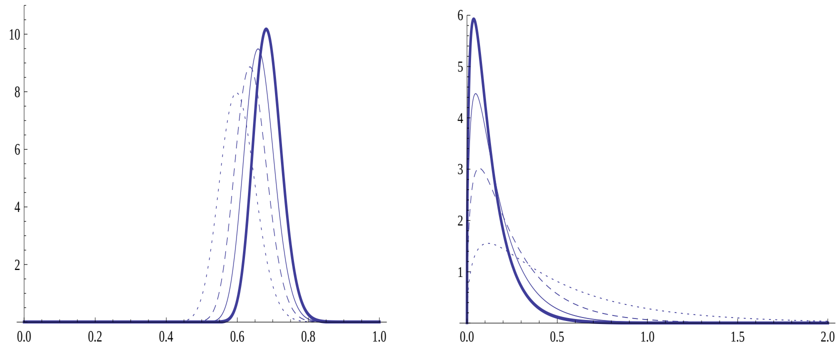

Figure 2. The MBW pdf. Left panel: $\lambda = 0.5$, $k = 3.5$, $b = 2.8$, $c = 2.1$ and $a = 30$ (dotted line) $a = 50$ (dashed line), $a = 70$ (solid line), $a = 100$ (thick line). Right panel: $\lambda = 0.5$, $k = 1$, $a = 1.5$, $c = 1.5$ and $b = 1$ (dotted line) $b = 2$ (dashed line), $b = 3$ (solid line), $b = 4$ (thick line).

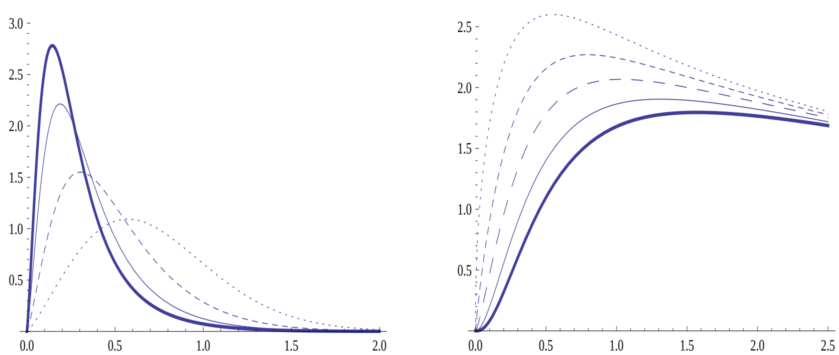

Figure 3. Left panel: The MBW pdf. $\lambda = 0.8$, $k = 1.6$, $a = 1.4$, $b = 1.5$ and $c = 0.8$ (dotted line), $c = 2$ (dashed line), $c = 4$ (solid line), $c = 6$ (thick line). Right panel: The MBW hazard rate function. $\lambda = 1.7$, $k = 1.2$, $b = 1.5$, $c = 3.5$ and $a = 1.2$ (dotted line) $a = 1.6$ (short dashes), $a = 2$ (long dashes), $a = 2.5$ (solid line), $a = 3$ (thick line).

right panel of Figure 3 and left and right panels of Figure 4 represent bathtub shaped and upside down bathtub shaped hazard rate function.

The rest of the paper is organized as follows. Representations of certain statistical functions are provided in Section 2. The parameter estimation technique described in Section 3 is utilized in connection with the modeling of two actual data sets originating from the engineering and biological sciences in Section 4, where the new model is compared with several related distributions.

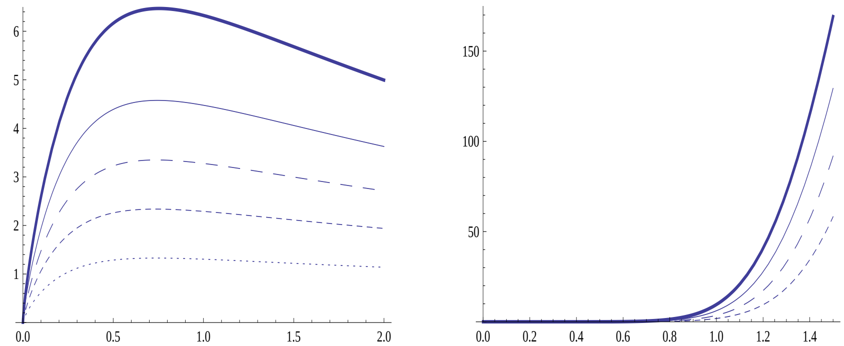

Figure 4. Left panel: The MBW hazard rate function. $\lambda = 1.7$, $k = 1.2$, $a = 1.5$, $c = 3.5$ and $b = 1$ (dotted line), $b = 1.5$ (dashed line), $b = 1.9$ (long dashes), $b = 2.3$ (solid line), $b = 2.8$ (thick line). Right panel: The MBW hazard rate function. $\lambda = 2$, $k = 4$, $a = 2$, $b = 1.5$ and $c = 1.6$ (dashed line), $c = 2$ (long dashes), $c = 2.4$ (solid line), $c = 2.8$ (thick line).

2. Statistical Functions of the MBW Distribution

Here, we derive computable representations of some statistical functions associated with the MBW distribution whose probability density function can be represented by (1.10). The resulting expressions can be evaluated exactly or numerically with symbolic computational packages such as *Mathematica*, *MATLAB* or *Maple*. In numerical applications, infinite sum can be truncated whenever convergence is observed.

2.1. Moments. We now derive closed form representations of the positive, negative and factorial moments of a MBW random variable. The r^{th} raw moment of the MBW distribution is

$$E(X^r) = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \times \int_0^{\infty} x^r\, x^{-1+k} \left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b} \mathrm{d}x\,. \tag{2.1}$$

Which gives

$$E(X^r) = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \times \frac{\left((b+m)\lambda^{-k}\right)^{-\frac{k+r}{k}} \Gamma\left(\frac{k+r}{k}\right)}{k}\,. \tag{2.2}$$

The h^{th} order negative moment can readily be determined by replacing r with $-h$ in (2.1):

$$E(X^{-h}) = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \times \int_0^{\infty} x^{-h}\, x^{-1+k} \left(\mathrm{e}^{-x^k/\lambda^k}\right)^{m+b} \mathrm{d}x$$

Which gives,

$$E(X^{-h}) = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!}$$

$$\times \frac{\left((b+m)\lambda^{-k}\right)^{-1+\frac{h}{k}} \Gamma\left(1-\frac{h}{k}\right)}{k}\,. \tag{2.3}$$

The factorial moments of X are

$$E(X(X-1)(X-2)\cdots(X-\gamma+1)) \equiv \sum_{m=0}^{\gamma-1} \phi_m (-1)^j\, E(X^{\gamma-m})\,, \tag{2.4}$$

where $E(X^{\gamma-m})$ can be evaluated by replacing r by $\gamma-m$ in (2.1).

2.2. Moment Generating Function. The moment generating function of the MBW distribution whose density function is specified by (1.10) will be derived here. First, we consider a result developed in [23]:

$$\int_0^{\infty} x^{\eta-1} \mathrm{e}^{-\theta\, x^k} \mathrm{e}^{s\, x} \mathrm{d}x = \frac{(2\pi)^{1-(q+p)/2} q^{1/2} p^{\eta-1/2}}{(-s)^{\eta}}$$

$$\times\, G^{q,p}_{p,q}\left(\left(-\frac{p}{s}\right)^p \left(\frac{\theta}{q}\right)^q \,\middle|\, \begin{array}{ll} 1-\frac{i+\eta}{p}, & i=0,1,....,p-1 \\ j/q, & j=0,1,....,q-1 \end{array} \right), \tag{2.5}$$

where $\Re(\eta), \Re(\theta), \Re(s) < 0$ and k is rational number such that $k = p/q$, where p and $q \neq 0$ are integers.

The moment generating function of the MBW distribution whose density function is specified by (1.10) is

$$M(t) = \frac{c^a}{B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b\,+n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!}$$

$$\times \int_0^{\infty} x^{k-1} \mathrm{e}^{-(\lambda^{-k}\,(m+b))x^k}\, \mathrm{e}^{t\,x}\, \mathrm{d}x$$

On replacing η with k, θ with $\lambda^{-k}\,(m+b))$ and s with t. In the integrand of integral and making use of (2.5), we have the following representation of the moment generating function when $k = p/q$:

$$M(t) = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!}$$

$$\times \frac{(2\pi)^{1-(q+p)/2}\, q^{1/2} p^{\,k-1/2}}{(-t)^k}$$

$$\times\, G^{q,p}_{p,q}\left(\left(\frac{-p}{t}\right)^p \left(\frac{\lambda^{-k}\,(m+b)}{q}\right)^q \,\middle|\, \begin{array}{ll} 1-\frac{i+k}{p}\,, & i=0,1,\ldots,p-1 \\ j/q\,, & j=0,1,\ldots,q-1 \end{array} \right) \tag{2.6}$$

2.3. Entropy. Entropy is a concept encountered in Physics and Engineering. An extension of Shannon's entropy for the continuous case can be defined as follows:

$$H(f) = -\int_0^{\infty} f(x)\log(f(x))\,\mathrm{d}x\,. \tag{2.7}$$

Combining (1.10) with (2.7), one has the following representation:

$$
\begin{aligned}
H(f) = &-\frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!}\,\frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \\
&\times \log\left(\frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!}\,\frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \right) \\
&\times \int_0^{\infty} x^{-1+k} \left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b} \mathrm{d}x \\
&- \frac{c^a k(-1+k)}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b\,+\,n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!} \\
&\times \int_0^{\infty} x^{-1+k} \left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b} \log(x)\,\mathrm{d}x \\
&+ \frac{c^a k\,(m+b)}{\lambda^{2k} B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b\,+\,n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!} \\
&\times \int_0^{\infty} x^{-1+2k} \left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b} \mathrm{d}x\,.
\end{aligned}
$$

$$
\begin{aligned}
H(f) = &-\frac{c^a}{(m+b)B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!}\,\frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \\
&\times \log\left(\frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!}\,\frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \right) \\
&- \frac{c^a k(-1+k)}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b\,+\,n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!} \\
&\times \int_0^{\infty} (x)^{-1+k} \left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b} \log(x)\,\mathrm{d}x \\
&+ \frac{c^a}{(m+b)B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b\,+\,n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!}\,.
\end{aligned}
\tag{2.8}
$$

Note that, the integral on the right–hand side of (2.8) can be evaluated by numerical integration.

2.4. Mean Residue Life Function. The mean residue life function is defined as

$$
\begin{aligned}
K(x) &= \frac{1}{S(x)} \int_x^{\infty} (y-x)\,f(y)\,\mathrm{d}y \\
&= \frac{1}{S(x)} \int_x^{\infty} y\,f(y)\,\mathrm{d}y - x \\
&= \frac{1}{S(x)} \left[E(Y) - \int_0^{x} y\,f(y)\,\mathrm{d}y\right] - x\,,
\end{aligned}
$$

where $f(y)$, $S(x)$ and $E(Y)$ are as given in (1.10), (1.6) and (2.2), respectively and

$$\begin{aligned}\int_0^x y f(y)\mathrm{d}y = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} & \frac{\Gamma(a+b\,+\,n)\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)n!m!} \\ \times & \int_0^x y^k \left(\mathrm{e}^{-\,\lambda^{-k}\,y^k}\right)^{m+b} \mathrm{d}y\,. \\ = & \frac{c^a\,k\,\lambda^{-k}}{B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b\,+\,n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!} \\ \times & \int_0^x \mathrm{e}^{-(m+b)\,\lambda^{-k}\,y^k}\,y^k\,\mathrm{d}y \\ = & \frac{c^a\,k\,\lambda^{-k}}{B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b\,+\,n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!} \\ \times & \int_0^x y^k\,G_{0,1}^{1,0}\left((m+b)\,\lambda^{-k}\,y^{p/q}\,\middle|\, \begin{matrix} - \\ 0 \end{matrix}\right)\mathrm{d}y\,, \end{aligned} \tag{2.9}$$

where $\mathrm{e}^{-g(x)} = G_{0,1}^{1,0}\left(g(x)\,\middle|\,\begin{matrix} - \\ 0\end{matrix}\right)$, $k = p/q$, $p \geq 1$, $q \geq 1$ are natural co-prime numbers and

$$\begin{aligned}&\int_0^x y^t\,G_{0,1}^{1,0}\left((m+b)\,\beta\,y^{p/q}\,\middle|\,\begin{matrix} - \\ 0\end{matrix}\right)\mathrm{d}y \\ &= \frac{q\,x^{p\,(t+1)}}{p(2\,\pi)^{(q-1)/2}}\,G_{p,p+q}^{q,p}\left(\frac{((m+b)\,\lambda^{-k})^q\,x^p}{q^q}\,\middle|\,\begin{matrix}\frac{-t}{p}\,,\,\frac{1-t}{p}\,,\ldots,\,\frac{p-t-1}{p}\,,\,- \\ 0\,,\,\frac{-t-1}{p}\,,\,\frac{t}{p}\,,\ldots,\,\frac{p-t-2}{p}\end{matrix}\right).\end{aligned} \tag{2.10}$$

Equation (2.10) is obtained by making use of Equation (13) of [5].

2.5. Mean Deviation. The mean deviation about the mean is defined by

$$\begin{aligned}\delta(X) &= \int_0^\infty |x - E(X)|\,f(x)\,\mathrm{d}x \\ &= \int_0^{E(X)} (E(X)-x)\;f(x)\,\mathrm{d}x + \int_{E(X)}^\infty (x-E(X))\;f(x)\,\mathrm{d}x\,.\end{aligned}$$

where $E(X)$ can be evaluated by letting $r = 1$ in (2.2). The mean deviation can easily be evaluated by numerical integration.

3. Parameter Estimation

In this section, we will make use of the MBW, Transmuted–Weibull(TW) [1], Kumaraswamy modified Weibull (KwMW) [9], Extended Weibull (ExtW) [21], Exponential–Weibull (EW) [5], Gamma–Weibull (GW) [22], Generalized modified Weibull (GMW) [4], Modified Weibull (MW) [15], Generalized gamma (GG) [26], Two parameter Weibull (Weibull) and Two parameter gamma (Gamma) distributions to model two well–known real data sets, namely the ‘Carbon fibres’ [19] and the ‘Cancer patients’ [16] data sets. The parameters of the MBW distribution can be estimated from the loglikelihood of the samples in conjunction with the *NMaximize* command in the symbolic computational package *Mathematica*. Additionally, three goodness-of-fit measures are proposed to compare the density estimates.

3.1. Maximum Likelihood Estimation. In order to estimate the parameters of the proposed MBW density function as defined in Equation (1.7), the loglikelihood of the sample is maximized with respect to the parameters. Given the data x_i, $i = 1, \ldots, n$, the loglikelihood function is

$$\begin{aligned}\ell(\lambda,\, k,\, a,\, b,\, c) = n\, \{a\, \log(c) + \log(k) -\, k\, \log(\lambda) - \log(\mathrm{B}(a,b))\}\\ + (k-1)\sum_{i=1}^{n} \log(x_i) + b \sum_{i=1}^{n} \log\left(\mathrm{e}^{-x_i{}^k/\lambda^k}\right)\\ + (a-1)\sum_{i=1}^{n} \log\left(1-\mathrm{e}^{-x_i{}^k/\lambda^k}\right)\\ - (a+b)\sum_{i=1}^{n} \log\left\{1-(1-c)\left(1-\mathrm{e}^{-x_i{}^k/\lambda^k}\right)\right\} \qquad (3.1)\end{aligned}$$

where $f(x)$ is as given in (1.7). The associated nonlinear loglikehood system $\frac{\partial \ell(\theta)}{\partial \theta} = 0$ for MLE estimator derivation reads as follows:

$$\begin{aligned}\frac{\partial \ell(\theta)}{\partial \lambda} &= -\frac{k\, n}{\lambda} + b\sum_{i=1}^{n} k\, \lambda^{-1-k}\, x_i^k + (a-1)\sum_{i=1}^{n} -\frac{\mathrm{e}^{-\lambda^{-k}\, x_i^k}\, k\, \lambda^{-1-k}\, x_i^k}{1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}}\\ &\quad - (a+b)\sum_{i=1}^{n} \frac{(1-c)\,\mathrm{e}^{-\lambda^{-k}\, x_i^k}\, k\, \lambda^{-1-k}\, x_i^k}{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}\right)} = 0\\ \frac{\partial \ell(\theta)}{\partial k} &= n\left\{\frac{1}{k} - \log(\lambda)\right\} + \sum_{i=1}^{n} \log(x_i)\\ &\quad + b\sum_{i=1}^{n}\left\{\lambda^{-k}\, \log(\lambda)\, x_i^k - \lambda^{-k} \log(x_i)\, x_i^k\right\}\\ &\quad - (a-1)\sum_{i=1}^{n} \frac{\mathrm{e}^{-\lambda^{-k} x_i^k}\, \{\lambda^{-k} \log(\lambda)\, x_i^k - \lambda^{-k} \log(x_i)\, x_i^k\}}{1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}}\\ &\quad - (a+b)\sum_{i=1}^{n} \frac{(1-c)\mathrm{e}^{-\lambda^{-k} x_i^k}\, \{\lambda^{-k} \log(\lambda)\, x_i^k - \lambda^{-k} \log(x_i)\, x_i^k\}}{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k} x_i^k}\right)} = 0\\ \frac{\partial \ell(\theta)}{\partial a} &= n\{\log(c) - \psi^{(0)}(a) + \psi^{(0)}(a+b)\} + \sum_{i=1}^{n} \log\left(1-\mathrm{e}^{-\lambda^{-k} x_i^k}\right)\\ &\quad - \sum_{i=1}^{n} \log\left\{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}\right)\right\} = 0\\ \frac{\partial \ell(\theta)}{\partial b} &= n\,\{-\psi^{(0)}(b) + \psi^{(0)}(a+b)\}\\ &\quad + \sum_{i=1}^{n} \log\left(\mathrm{e}^{-\lambda^{-k}\, x_i^k}\right) - \sum_{i=1}^{n} \log\left\{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k} x_i^k}\right)\right\} = 0\\ \frac{\partial \ell(\theta)}{\partial c} &= \frac{a\, n}{c} - (a+b)\sum_{i=1}^{n} \frac{1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}}{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}\right)} = 0\,. \qquad (3.2)\end{aligned}$$

Where $\psi^{(0)}(\cdot)$ is the polygamma function. The above equations cannot be solved analytically and statistical software can be used to solve them numerically.

3.2. Goodness-of-Fit Statistics. To verify the goodness-of-fit of certain statistical models, some goodness-of-fit statistics shall be used. They are computed using the symbolic computation package *Mathematica*. The following goodness-of-fit statistics are considered: the Anderson-Darling, Cramér-von Mises and Akaike Information Criterion (AIC) statistics for comparison purposes. The Anderson-Darling and Cramér-von Mises statistics are widely utilized to determine how closely a specific distribution whose associated cumulative distribution function denoted by cdf($\cdot$) fits the empirical distribution associated with a given data set. Upper tail percentiles of the asymptotic distributions of Anderson-Darling and Cramér-von Mises statistics were tabulated in [19]. The distribution having the better fit will be the one whose goodness-of-fit statistic is the smallest.

4. Empirical illustrations

In this section, we present two applications where the MBW model is compared with other related models, namely Transmuted–Weibull(TW) [1], Kumaraswamy modified Weibull (KwMW) [9] Extended Weibull (ExtW) [21], Exponential–Weibull (EW) [5], Gamma–Weibull (GW) [22], Generalized modified Weibull (GMW) [4], Modified Weibull (MW) [15], Generalized gamma (GG) [26], Two parameter Weibull (Weibull) and Two parameter gamma (Gamma) distributions. We make use of two data sets: first, the Carbon fibres data set [19] and, secondly, the Cancer patients data set [16].

- The classical gamma (Gamma) distribution with density function

$$f(x) = \frac{x^{\xi-1}\,\mathrm{e}^{-x/\phi}}{\phi^{\xi}\,\Gamma(\xi)}, \quad x>0,\, \phi,\, \xi>0\,.$$

- The classical Weibull (Weibull) distribution with density function

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\mathrm{e}^{-(x/\lambda)^{k}}, \quad x>0,\, k,\, \lambda>0\,.$$

- The generalize gamma (GG) distribution [26] with density function

$$f(x) = \frac{k\,\lambda^{-\xi}\,x^{\xi-1}\,\mathrm{e}^{-\lambda^{-k}\,x^{k}}}{\Gamma(\xi/k)}, \quad x>0,\, \xi,\, k,\, \lambda>0\,.$$

- The modified Weibull (MW) distribution [15] with density function

$$f(x) = \alpha\,x^{\gamma-1}(\gamma+\lambda x)\,\mathrm{e}^{\left(\lambda x-\ \alpha x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)}, \quad x>0,\, \gamma,\, \alpha>0,\, \lambda\geq 0.$$

- The generalized modified Weibull (GMW) distribution [4] with density function

$$f(x) = \varphi\,\alpha\,x^{\gamma-1}\,(\gamma+\lambda x)\,\mathrm{e}^{\left(\lambda x-\alpha x^{\gamma}\,\mathrm{e}^{\lambda x}\right)}\left\{1-\mathrm{e}^{\left(-\alpha x^{\gamma}\,\mathrm{e}^{\lambda x}\right)}\right\}^{\varphi-1}, \quad x>0,\, \gamma,\, \alpha,\, \varphi>0,\, \lambda\geq 0.$$

- The gamma–Weibull distribution [22] with density function

$$f(x) = \frac{k\,\lambda^{-k-\xi}\,x^{\xi+k-1}\mathrm{e}^{-\lambda^{-k}\,x^{k}}}{\Gamma(1+\xi/k)}, \quad x>0,\, \xi+k>0,\, \lambda>0\,.$$

- The exponential–Weibull (EW) distribution [5] with density function

$$f(x) = \left(\lambda+\beta\,k\,x^{k-1}\right)\,\mathrm{e}^{-\lambda\,x-\beta\,x^{k}}, \quad x>0,\, \lambda,\, \beta,\, k>0\,.$$

- The Transmuted–Weibull(TW) [1] with density function

$$f(x) = \frac{\eta\mathrm{e}^{-\left(\frac{x}{\sigma}\right)^{\eta}}\left(\frac{x}{\sigma}\right)^{\eta-1}\left\{2\lambda\,\mathrm{e}^{-\left(\frac{x}{\sigma}\right)^{\eta}}-\lambda+1\right\}}{\sigma}, \quad x>0,\, \sigma,\, \eta,\, \lambda>0\,.$$

- The extended Weibull (ExtW) distribution [21] with density function

$$f(x) = a\,(c + b\,x)\,x^{-2+b}\,\mathrm{e}^{-c/x - a x^b \mathrm{e}^{-c/x}}, \quad x > 0,\ a,\ b > 0,\ c \geq 0\,.$$

- The Kumaraswamy modified Weibull (KwMW) distribution [9] with density function

$$f(x) = a\,b\,\alpha\,x^{\gamma-1}\,(\gamma + \lambda\,x)\,\mathrm{e}^{\left(\lambda\,x - \alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)}\left\{1 - \mathrm{e}^{\left(-\alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)}\right\}^{a-1}$$
$$\times\left[1 - \left\{1 - \mathrm{e}^{\left(-\alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)}\right\}^{a}\right]^{b-1}, \quad x > 0,\ a,\ b,\ \alpha,\ \gamma > 0,\ \lambda \geq 0\,.$$

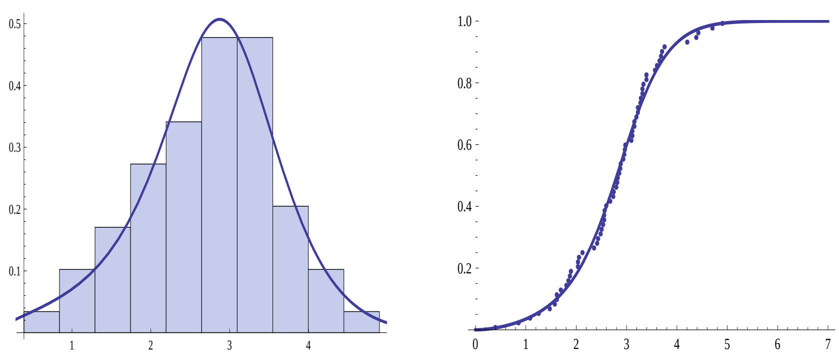

Figure 5. Left panel: The MBW density estimate superimposed on the histogram for Carbon fibres data . Right panel: The MBW cdf estimates and empirical cdf.

Note that: The empirical cdf can be plotted using the following code in mathematica.
ListPlot[Table[{data[[i]], i/n-1/(2n)},{i, 1,n}]].

4.1. The Carbon Fibres Data Set. The first data set represents the uncensored real data set on the breaking stress of carbon fibres (in Gba) as reported in [5]. The data are (n = 66): 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53.

4.2. The Cancer Patients Data Set. The second data set represents the remission times (in months) of a random sample of 128 bladder cancer patients as reported in [16]. The data are 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90 , 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

The pdf and cdf estimates of the MBW distribution are plotted in Figures 5 and 6 for the Carbon fibres and Cancer patients data, respectively. The estimated hazard

Table 1. Estimates of the Parameters, Goodness-of-Fit Statistics and Loglikelihood for the Carbon Fibres Data

Distributions	Estimates				
Gamma $(\xi\,,\,\phi)$	7.48803	0.368528			
Weibull $(k\,,\,\lambda)$	3.4412	47.0505			
GG $(k\,,\,\lambda\,,\,\xi)$	4.0735	3.34592	3.09225		
MW$(\alpha\,,\gamma\,,\lambda)$	0.021813	2.709212	0.248518		
GMW$(\varphi,\,\alpha\,,\gamma\,,\lambda\,)$	5.49894	0.436399	0.148117	0.516284	
GW $(k\,,\,\xi\,,\,\lambda)$	3.4412	1.6×10^{-7}	3.06226		
EW $(k\,,\,\lambda\,,\,\beta)$	3.73666	0.0170948	0.01401		
TW $(\eta\,,\sigma\,,\lambda\,)$	3.441197	3.745584	1		
ExtW $(\,a\,,\,b\,,\,c)$	16.1979	1×10^{-7}	8.05671		
KwMW$(\alpha\,,\gamma\,,\lambda\,,a\,,b)$	0.14981	1.7994	0.49987	0.64975	0.17111
MBW $(\lambda,\,k,\,a\,,b\,,c)$	1.65934	2.23218	0.78685	0.55408	0.07248
Distributions		A_0^*	W_0^*	AIC	$\ell(\hat{\Theta})$
Gamma $(\xi\,,\,\phi)$		1.32674	0.248153	186.335	-91.1675
Weibull $(k\,,\,\lambda)$		0.491678	0.0843011	**176.135**	-86.0676
GG $(k\,,\,\lambda\,,\,\xi)$		0.487573	0.0811144	177.835	-85.9175
MW$(\alpha\,,\gamma\,,\lambda)$		0.485662	0.0793299	177.727	-85.8636
GMW$(\varphi,\,\alpha\,,\gamma\,,\lambda\,)$		0.385439	0.0627953	178.746	-85.3731
GW $(k\,,\,\xi\,,\,\lambda)$		0.491678	0.0843011	178.135	-86.0676
EW $(k\,,\,\lambda\,,\,\beta)$		0.403649	0.06479	177.044	-85.5218
TW $(\eta\,,\sigma\,,\lambda\,)$		0.491678	0.0843011	178.135	-86.0676
ExtW $(\,a\,,\,b\,,\,c)$		2.26745	0.416152	207.47	-100.735
KwMW$(\alpha\,,\gamma\,,\lambda\,,a\,,b)$		1.29338	0.213215	185.980	-87.9902
MBW $(\lambda,\,k,\,a\,,b\,,c)$		**0.24516**	**0.034375**	179.226	**-84.613**

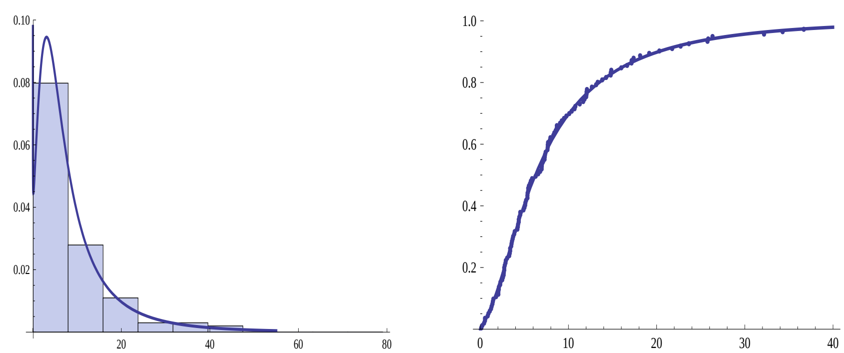

Figure 6. Left panel: The MBW density estimate superimposed on the histogram for Cancer patients data . Right panel: The MBW cdf estimates and empirical cdf.

rate function of MBW distribution are plotted in Figure 7. It can be seen that both shapes of hazard rate function, for carbon fibers and cancer patients data sets are like bathtub shaped hazard rate function. The estimates of the parameters and the values of AIC, Anderson-Darling and Cramér-von Mises goodness–of–fit statistics are given in Tables 1 and 2 for the Carbon fibres and Cancer patients data, respectively. It is seen that the proposed MBW model provides the best fit for both data sets when considering Anderson-Darling and Cramér-von Mises goodness–of–fit statistics and is a competitive model when considering AIC.

Table 2. Estimates of the Parameters, Goodness-of-Fit Statistics and Loglikelihood for the Cancer Patients Data

Distributions	Estimates				
Gamma (ξ , ϕ)	1.17251	7.98766			
Weibull (k , λ)	1.04783	10.651			
GG (k , λ , ξ)	0.520095	0.595104	1.94927		
MW(α , γ , λ)	0.093887	1.047834	3.6×10^{-11}		
GMW(φ, α , γ , λ)	2.796005	0.453691	0.654409	5.8×10^{-13}	
GW (k , ξ , λ)	0.520095	1.42917	0.595104		
EW (k , λ , β)	1.04783	1×10^{-7}	0.093887		
TW (η , σ , λ)	1.133310	14.61979	0.744922		
ExtW (a , b , c)	1.9621	1×10^{-21}	3.74383		
KwMW(α , γ , λ , a , b)	0.639622	0.381865	0.029602	0.375	0.322843
MBW (λ , k, a , b , c)	0.32113	0.52381	1.29997	0.41823	0.053809
Distributions		A_0^*	W_0^*	AIC	$\ell(\hat{\Theta})$
Gamma (ξ , ϕ)		0.77625	0.136063	830.736	-413.368
Weibull (k , λ)		0.963452	0.154303	832.174	-414.087
GG (k , λ , ξ)		0.300873	0.04526	**827.708**	-410.854
MW(α , γ , λ)		0.963452	0.154303	834.174	-414.087
GMW(φ, α , γ , λ)		0.271984	0.04050	829.36	-410.68
GW (k , ξ , λ)		0.300873	0.045261	**827.708**	-410.854
EW (k , λ , β)		0.963452	0.154303	834.174	-414.087
TW (η , σ , λ)		0.563397	0.0882597	829.916	-411.958
ExtW (a , b , c)		13.3317	2.49818	1034.9	-514.498
KwMW(α , γ , λ , a , b)		18.8864	3.68568	979.652	-484.826
MBW (λ , k, a , b , c)		**0.076133**	**0.0119393**	828.612	**-409.306**

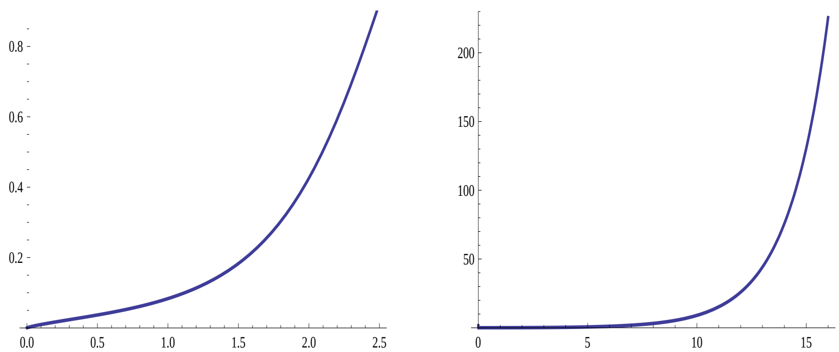

Figure 7. Left panel: The estimated MBW hazard rate function for carbon data . Right panel: The estimated MBW hazard rate function for cancer patients data .

5. Discussion

There has been a growing interest among statisticians and applied researchers in constructing flexible lifetime models in order to improve the modeling of survival data. As a result, significant progress has been made towards the generalization of some well–known lifetime models, which have been successfully applied to problems arising in several areas of research. In particular, several authors have proposed new distributions which are based on the traditional Weibull model. In this paper, we introduce a five–parameter distribution which is obtained by applying the modified beta technique to the Weibull model. Interestingly, our proposed model has bathtub and up side down bathtub shaped hazard rate function.We studied some of its statistical properties. We also provided computable representations of the positive and negative moments, the factorial moments, the moment generating function, the mean residue life function, the mean deviation and the associated Shannon's entropy. The proposed distribution was applied to two data sets and shown to provide a better fit than other related models. The distributional results developed in this article should find numerous applications in the physical and biological sciences, reliability theory, hydrology, medicine, meteorology, engineering and survival analysis.

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A group sequential test of circular data using the von Mises distribution

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Abstract

In this study, the group sequential test is suggested for the mean direction parameter of the von Mises distribution when the concentration parameter is known and unknown. An application of the proposed test is illustrated by using a medical data of the patients, who were complained about internal rotation angles of the shoulder and treated in a rehabilitation and physical therapy center in Eskisehir, Turkey. It is shown that the results of the study demonstrate that the group sequential test can provide a great advantage not only for linear data but also for circular data in terms of sample size.

Keywords: Sequential test, Circular data, Von Mises distribution, Mean direction

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1. Introduction

Circular data often arise in many scientific disciplines like meteorology, geography, biology, geology and medicine etc. As an example, meteorological events are periodical, that's why it is convenient to analyze them by using directional methods. It is shown that the distribution of the wind direction can be approximated by a specific circular model.

Ecologists consider the prevailing wind direction as an important factor in many studies including those of which involve pollutant transport. In Geology, geologists study paleocurrents to find out about the direction of flow of rivers in the past [16] and analyze paleomagnetic directions of the earth's magnetic pole to investigate the phenomenon of pole-reversal as well as in support of the hypothesis of continental drift. In Biology, biologists who study bird-migrations record the flight directions of just-released birds as they disappear over the horizon. Batschelet [2] presented a number of noteworthy applications of circular statistics in Biology. Also, any periodic phenomenon which is known and may be a day, a month or a year, can be represented on a circle by aggregating the necessary data of several individuals or periods if the circumferences corresponds to this period. Examples include arrival times of patients to a hospital over the day, or the time of patients at a hospital in the day. As a last example, the circle may represent the 365 days in the year and could be plotted the occurrence of crash accidents in a specific roadway junction to see if they are uniformly distributed over the different seasons of the year [8].

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Circular data take values on the circumference of a circle and they form the angles in the range $(0^o, 360^o)$ or $(0, 2\pi)$ radians [7]. The circular probability distributions are used to fit the distribution of circular data. The von Mises distribution is the most common probability distribution for circular data. A comprehensive discussion of circular statistics as well as examples of the applications and general properties of the von Mises distribution can be found in [11] and [8].

There are many practical situations in which it is desirable to update the decision with every incoming observation, by sequentially, either in the temporal or in the spatial mode of collecting the circular data.

As an example of using a sequential test for circular data, observations on the imbalanced directions of individually produced wheels can provide for the information of whether the procedure is under control.

Gadsden & Kanji [5] developed a sequential probability ratio test (SPRT) of Wald [17] for the mean direction (μ_{VM}) of the von Mises distribution with a known and an unknown concentration parameter (κ). Gadsden & Kanji [6] represents the applications of SPRT for circular data.

The sample size is a predetermined fixed value in fixed sample size test procedure. In practice, this test cause, the practitioner, to spend more resources such as money and time. When the sequential tests are used, these difficulties can be removed. The test begins with a single observation value and stops when there is sufficient data for statistical comparison and for making a decision on the hypothesis. Thus, it leads to a great saving in the sample size [17].

However, in some cases, when a new data is obtained, testing the data by grouping is an easier way than applying SPRT. A test which is performed sequentially by grouping data is called a group sequential test (GST). Various group sequential testing procedures have been proposed to achieve the desired levels of type I error. Pocock [14], O'Brien & Fleming [12] and Lan & DeMets [10] were among the first scholars to develop group sequential test. A great part of the progress of group sequential tests are reviewed in detail by Jennison & Turnbull [9].

Group sequential tests are widely used in medicine. On the other hand, medical events are convenient to be analyzed using directional methods since many of them are periodical. The occurrences of deaths caused by some disease in several times of year is a typical example for circular data observations. However, none of these studies consider group sequential test for von Mises distribution. In this study, a group sequential test is suggested for the mean direction of the von Mises distribution with known and unknown concentration parameter.

This article is organized as follows: The von Mises distribution and the sequential probability ratio test ($SPRT$) are briefly reviewed in the second and the third sections, respectively. In the fourth section, Pocock's group sequential test is described for the mean of the normal distribution. In the fifth section, it is indicated that Pocock's group sequential test can be used for the mean direction of the von Mises distribution. An application of medical data and conclusions are given in the sixth and the seventh sections, respectively.

2. The Von Mises Distribution

The von Mises distribution is a symmetric distribution which is the most important model for unimodal samples of circular data and it plays the same role in circular statistical inference as the normal distribution on the line.

If a circular random variable θ has a von Mises distribution ($\theta \sim VM(\mu, \kappa)$), its probability density function (pdf) is given by

$$f(\theta; \mu, \kappa) = \frac{1}{2\pi I_o(\kappa)} e^{\kappa cos(\theta-\mu)} \quad , 0 \leq \theta < 2\pi \tag{2.1}$$

where $\kappa \geq 0$ and $0 \leq \mu < 2\pi$. Here, $I_o(\kappa)$ is a particular function of κ and it denotes the modified Bessel function of the first kind and order zero, and is defined by

$$I_o(\kappa) = \frac{1}{2\pi} \int_0^{2\pi} e^{\kappa cos\theta} d\theta = \sum_{r=0}^{\infty} (\frac{1}{r!})^2 (\frac{\kappa}{2})^{2r} \tag{2.2}$$

This function has the effect of scaling the distribution.

For sufficiently large κ, the von Mises distribution is related to the normal distribution. If $\kappa \to \infty$ and $\xi = \kappa^{1/2}(\theta - \mu)$, ξ is approximately distributed as standard normal distribution ($N(0,1)$) [11], [8].

Several properties of the von Mises distribution are similar to those of the normal distribution. For instance, it is completely determined by two parameters. The parameter μ is the mean direction. The von Mises density is unimodal and symmetrical about the mean direction μ. The mode of the distribution is at $\theta = \mu$ and antimode is at $\theta = \mu + \pi$. The parameter κ is the concentration parameter which measures the concentration around the mean direction. As κ approaches zero, the von Mises pdf approaches a uniform distribution and as κ increases, the distribution increasingly concentrated at μ. Due to these properties, the concentration parameter is similar to the variance of a normal distribution.

By giving a random sample $\theta_1, \theta_2, ..., \theta_n$ from $VM(\mu, \kappa)$, the log-likelihood function is given by

$$\text{(2.3)} \qquad logL(\mu, \kappa; \theta_1, \theta_2, ..., \theta_n) = n[log2\pi + \kappa\bar{R}cos(\bar{\theta} - \mu) - logI_o(\kappa)].$$

Then the maximum likelihood estimate $\hat{\mu}$ of μ is

$$\text{(2.4)} \qquad \hat{\mu} = \bar{\theta}$$

where

$$\text{(2.5)} \qquad \bar{\theta} = \begin{cases} tan^{-1}(\frac{\sum_{i=1}^{n} sin\theta_i}{\sum_{i=1}^{n} cos\theta_i}), & \sum_{i=1}^{n} cos\theta_i \geq 0 \\ \\ tan^{-1}(\frac{\sum_{i=1}^{n} sin\theta_i}{\sum_{i=1}^{n} cos\theta_i}) + \pi, & \sum_{i=1}^{n} cos\theta_i < 0. \end{cases}$$

Differentiating (2.3) with respect to κ gives

$$\text{(2.6)} \qquad \frac{logL(\mu, \kappa; \theta_1, \theta_2, ...\theta_n)}{\partial\kappa} = n\{\bar{R}cos(\bar{\theta} - \mu) - A(\kappa)\}$$

where $A(\kappa) = I_1(\kappa)/I_o(\kappa)$ is the ratio of two modified Bessel functions and $I_1(\kappa)$ is the imaginary Bessel function of order one. The maximum likelihood estimate $\hat{\kappa}$ of κ is the solution of

$$\text{(2.7)} \qquad A(\hat{\kappa}) = \bar{R}$$

i.e.

$$\text{(2.8)} \qquad \hat{\kappa} = A^{-1}(\bar{R})$$

where $\bar{R}$ is the mean resultant length of the sample and is given by;

$$\text{(2.9)} \qquad \bar{R} = \sqrt{\left(\frac{1}{n}\sum_{i=1}^{n} cos\theta_i\right)^2 + \left(\frac{1}{n}\sum_{i=1}^{n} sin\theta_i\right)^2}.$$

Values of functions A and A^{-1} are taken from the tables, such as Mardia and Jupp (2000, p. 362-363) and Fisher (1993, p. 224-225). A reasonable approximation to the solution of (2.8) can, also, be obtained by

$$\text{(2.10)} \qquad \hat{\kappa} = \begin{cases} 2\bar{R} + \bar{R}^3 + 5\bar{R}^5/6, & \bar{R} < 0.53 \\ -0.4 + 1.39\bar{R} + 0.43/(1 - \bar{R}), & 0.53 \leq \bar{R} < 0.85 \\ 1/(\bar{R}^3 - 4\bar{R}^2 + 3\bar{R}), & \bar{R} \geq 0.85 \end{cases}$$

[4, 11].

3. Sequential Probability Ratio Test for the Mean Direction

Let θ be a von Mises distributed random variable with a mean direction μ_0 and a concentration parameter. For testing $H_0 : \mu = \mu_0$ against $H_1 : \mu = \mu_1$, sequential probability ratio test is defined as follows; If the values of θ random variable is defined as $\theta_1, \theta_2, ..., \theta_n$, likelihood ratio is defined as,

$$(3.1) \qquad L_n = \prod_{i=1}^{n} \frac{f(\theta_i;\mu_1)}{f(\theta_i;\mu_0)} = \frac{\frac{1}{[2\pi I_0(\kappa)]^n} e^{\kappa \sum_{i=1}^{n} cos(\theta_i-\mu_1)}}{\frac{1}{[2\pi I_0(\kappa)]^n} e^{\kappa \sum_{i=1}^{n} cos(\theta_i-\mu_0)}}.$$

Then by taking logarithm and simplifying, (3.1) can be written as;

$$(3.2) \qquad lnL_n = \sum_{i=1}^{n} Z_i = 2\kappa \sum_{i=1}^{n} sin(\theta_i - v_1) sin(-v_2)$$

where $v_1 = \frac{\mu_0+\mu_1}{2}$ and $v_2 = \frac{\mu_0-\mu_1}{2}$.

At each stage of the test process, the value of $\sum_{i=1}^{n} Z_i$ is computed and compared with lnA and lnB critical values which depend on type-1(α) and type-2(β) errors. A and B values are computed as $A = \frac{1-\beta}{\alpha}$, $B = \frac{\beta}{1-\alpha}$. Then, one of the following decision is made.

(1) If $\sum_{i=1}^{n} Z_i \leq lnB$, the process is terminated with the acceptance of H_0.
(2) If $\sum_{i=1}^{n} Z_i \geq lnA$, the process is terminated with the rejection of H_0.
(3) If $lnB < \sum_{i=1}^{n} Z_i < lnA$, the experiment is continued by taking an additional observation.

[17].

When μ is the test parameter for the von Mises distribution, the approximate formula for the operating characteristic (OC) function $P(\mu)$ is given by;

$$(3.3) \qquad P(\mu) = \frac{A^h - 1}{A^h - B^h}$$

where $h = \frac{sin(\mu - v_1)}{sinv_2}$ [5, 6].

In linear data, acceptance probabilities are computed for the various values of h. Apart from the linear data, minimum and maximum values of operating characteristic function are obtained in circular data. Differentiating OC function with respect to μ, it is obtained that $\mu = 90^0 + v_1$ and $\mu = 270^0 + v_1$, and these can be shown to be a minimum and maximum, respectively.

An approximation to the average sample number function ASN(μ), which is the expected number of observations, is given by;

$$(3.4) \qquad ASN(\mu) = \frac{P(\mu)lnB + [1 - P(\mu)]lnA}{2A(\kappa)sinv_1 sinv_2}.$$

It is possible to compute maximum and minimum values of the average sample number in circular data. Therefore, the average sample numbers, which are obtained when H_0 or H_1 is true in linear data, are computed for the maximum and minimum values in circular data. Differentiating the average sample number with respect to v_2 and setting that equal to zero gives;

$$(3.5) \qquad ASN(\mu)_{min} = \frac{P(\mu)lnB + [1 - P(\mu)]lnA}{2A(\kappa)sinv_1}.$$

Since a minimum can be obtained in only one turning point, the ends of the range of v_2 will give the maximum. This leads to the point 0^0 and it gives

$$(3.6) \qquad ASN(\mu)_{max} = \infty$$

[5, 13].

4. Pocock's Group Sequential Test

The basic concepts of Pocock's group sequential test in one sample are described as follows. Consider K groups (stages) of normally distributed observations with an unknown mean μ and a known variance σ^2, where in group k,$k = 1, 2, ..., K$ and $n_1 = n_2 = ...n_K = n$ observations are obtained. It is planned as a test of the null hypothesis $H_0 : \mu = \mu_0$ against the two sided alternative $H_1 : \mu \neq \mu_0$. Let $\bar{x}_j$ denote the mean response of the sample in the jth group of n observations. In the jth stage, the normal score Z_j is given by

$$Z_j = \sqrt{n}(\bar{x}_j - \mu_0)/\sqrt{\sigma^2}. \tag{4.1}$$

The cumulative normal score

$$S_k = \textstyle\sum_{j=1}^{k} Z_j \quad , k = 1, 2, ...K \tag{4.2}$$

is the usual statistic for testing the hypothesis of the mean at type-I error probability α. Z_j is $N(0, 1)$ and $N(\Delta, 1)$ distributed, under H_0 and H_1 respectively. Where Δ is given as

$$\Delta = E(Z_j) = \sqrt{n}(\mu_1 - \mu_0)/\sqrt{\sigma^2} \tag{4.3}$$

[1, 9]. Formally the test process is as follows:

(1) After group $k = 1, 2, ..., K - 1$
  If $|S_k| \geq z_p(K, \alpha)\sqrt{k}$, stop, reject H_0
  otherwise, continue to group $k + 1$

(2) After group K
  If $|S_K| \geq z_p(K, \alpha)\sqrt{K}$, stop, reject H_0
  otherwise, stop, accept H_0.

Where $z_p(K, \alpha)$ is the Pocock's critical value as in Table 1. The sample size per group is obtained as

$$n = \Delta^2 \left(\frac{\sqrt{\sigma^2}}{\mu_1 - \mu_2} \right)^2 \tag{4.4}$$

where Δ is the value of noncentrality parameter and it can be determined by a given value of $1 - \beta$. The maximum sample size is $n_{max} = nK$. If $K = 1$ is taken as fixed sample size design (4.4) becomes the familiar sample size for a normal response. The average sample number, under H_1 is $ASN = n\bar{K}^*$, where $\bar{K}^*$ is the average number of stages.

$z_p(K, \alpha)$, Δ and $\bar{K}^*$ values are given in Table 1 for $k = 1, 2, .., 5$, $\alpha = 0,05$, $1 - \beta = 0,95$. More complete tabulations of various values can be found in [14] and [9].

Table 1. Pocock's Critical Values $z_p(K, \alpha)$, Δ and $\bar{K}^*$ for $k = 1, 2, ..., 5$, $\alpha = 0,05$, $1 - \beta = 0,95$

k	$z_p(K, \alpha)$					Δ	$\bar{K}^*$
1	1,645					3,290	1
2	1,875	1,875				2,445	1,282
3	1,993	1,993	1,993			2,035	1,656
4	2,067	2,067	2,067	2,067		1,782	2,056
5	2,122	2,122	2,122	2,122	2,122	1,605	2,460

When the variance σ^2 is unknown, group sequential t-test is used. Test procedure is the same as the one with known σ^2. Since σ^2 is unknown, the pooled sample variance is estimated of n observations and is used for σ^2 in (4.1). Furthermore, sample size per group can not be calculated with (4.4) in group sequential t-test. Thus, the researcher supposed that each group contains n observations, in this case [9].

5. Group Sequential Test for the Mean Direction of the Von Mises Distribution

In this section, it is shown that Pocock's group sequential test can be used for the mean direction of the von Mises distribution both for known κ and unknown κ cases.

It is assumed that $\theta_1, \vartheta_2, ..., \vartheta_n$ is a random sample from a von Mises distribution $VM(\mu, \kappa)$. Let the concentration parameter be known as $\kappa = \kappa_0(\kappa_0 \geq 2)$. Then, the population mean resultant length of a von Mises distribution is ρ. The hypothesis to be tested is $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. From (2.3), the score statistic is defined as

$$\left.\frac{\partial logL(\mu, \kappa; \theta_1, \theta_2, ..., \theta_n)}{\partial \mu}\right|_{\mu=\mu_0} = n\kappa\bar{R}sin(\bar{\theta} - \mu_0). \tag{5.1}$$

[3]. Under H_0, the score statistic is equal to

$$\sqrt{n\kappa_0\rho}sin(\bar{\theta} - \mu_0) \tag{5.2}$$

and it has approximately the distribution $N(0, 1)$, for large n. The circular standard error of the mean direction for the von Mises distribution is

$$\sigma_{VM} = \frac{1}{\sqrt{n\kappa_0\rho}}. \tag{5.3}$$

Thus, the test statistic for the score test is given by

$$Z_{VM} = \frac{sin(\bar{\theta} - \mu_0)}{\sigma_{VM}} \tag{5.4}$$

[4]. Let $z_{\alpha/2}$ indicates the upper $100(\alpha/2)\%$ point and z_α indicates $100(\alpha)\%$ point of the standard normal distribution. Then the test of $H_0 : \mu = \mu_0$ against the alternatives are at the $100\alpha\%$ level are follows:

(1) When $H_1 : \mu \neq \mu_0$: if $|Z_{VM}| > z_{\alpha/2}$, then reject H_0.
(2) When $H_1 : \mu < \mu_0$: if $\mu_0 - \pi < \bar{\theta} < \mu_0$ and $Z_{VM} < -z_\alpha$, then reject H_0.
(3) When $H_1 : \mu > \mu_0$: if $\bar{\theta} < \mu_0 + \pi$ and $Z_{VM} > -z_\alpha$, then reject H_0.

In the sense of the information given above, the group sequential test statistic for the von Mises distribution can be defined as:

$$S_{VMk} = \textstyle\sum_{j=1}^{k} Z_{VMj}, \quad k = 1, ..., K \tag{5.5}$$

where

$$Z_{VMj} = \sqrt{n\kappa_0\rho}sin(\bar{\theta}_j - \mu_0) \tag{5.6}$$

where $\bar{\theta}_j$ is computed from the data of n observations for the jth group. For $K = 1$, the test statistic (5.5) transforms into fixed sample test in the von Mises distribution. Therefore, since Z_{VMj} has approximately the distribution $N(0, 1)$ under H_0, the group sequential test can be used for testing the mean direction of the von Mises distribution with the known concentration parameter. The test statistic S_{VMk} is compared with $z_p(K, \alpha)$ as follows:

After group $k = 1, 2, ..., K - 1$
For $H_1 : \mu \neq \mu_0$, if $|S_{VMk}| \geq z_p(K, \alpha)\sqrt{k}$, stop, reject H_0
For $H_1 : \mu < \mu_0$ and $\mu_0 - \pi < \bar{\theta}_k < \mu_0$, if $|S_{VMk}| < -z_p(K, \alpha)\sqrt{k}$, stop, reject H_0
For $H_1 : \mu > \mu_0$ and $\bar{\theta}_k < \mu_0 + \pi$, if $|S_{VMk}| > -z_p(K, \alpha)\sqrt{k}$, stop, reject H_0
otherwise continue to group $k + 1$
After group K
For $H_1 : \mu \neq \mu_0$, if $|S_{VMk}| \geq z_p(K, \alpha)\sqrt{K}$, stop, reject H_0
For $H_1 : \mu < \mu_0$ and $\mu_0 - \pi < \bar{\theta}_k < \mu_0$, if $|S_{VMk}| < -z_p(K, \alpha)\sqrt{K}$, stop, reject H_0
For $H_1 : \mu > \mu_0$ and $\bar{\theta}_k < \mu_0 + \pi$, if $|S_{VMk}| > -z_p(K, \alpha)\sqrt{K}$, stop, reject H_0
otherwise stop, accept H_0.

For this test, the group size n_{VM} is obtained from the expected value of the test statistic (5.6) under H_1;

(5.7) $\Delta = E(Z_{VMj|H_1}) = \sqrt{n_{VM}\rho\kappa} sin(\mu_1 - \mu_0).$

Therefore, the value of n_{VM} for this test is

(5.8) $n_{VM} = \Delta^2 \dfrac{1}{[sin(\mu_1 - \mu_0)]^2 \kappa\rho}.$

The maximum sample size can be defined as

(5.9) $n_{max} = n_{VM} N$

and the average sample number is

(5.10) $ASN_{VM} = n_{VM} \bar{K}^*.$

Now, let the concentration parameter κ be unknown, and then the test statistic for the score test can be defined as

(5.11) $Z_{VM} = \dfrac{sin(\bar{\theta} - \mu_0)}{\hat{\sigma}_{VM}}$

where

(5.12) $\hat{\sigma}_{VM} = \dfrac{1}{\sqrt{n\hat{\kappa}\bar{R}}}.$

Therefore, Z_{VM} is approximately distributed as $N(0, 1)$ under H_0. This approximation is satisfactory for the values of estimated concentration parameter ($\hat{\kappa}$) and sample size (n) which are given in Table 2 [4, 11].

Table 2. $\hat{\kappa}$ and n values for the test

$\hat{\kappa}$	n
$0,4 \leq \hat{\kappa} < 1$	$n \geq 25$
$1,0 \leq \hat{\kappa} < 1,5$	$n \geq 15$
$1,5 \leq \hat{\kappa} < 2,0$	$n \geq 10$
$\hat{\kappa} \geq 2,0$	All n

Then, as for group sequential test statistic, it can be defined as

(5.13) $Z_{VMj} = \sqrt{n\bar{R}_j\hat{\kappa}_j} sin(\bar{\theta}_j - \mu_0)$

where $\bar{\theta}_j$,$\bar{R}_j$ and $\hat{\kappa}_j$ values are computed from the data of n observations for the jth group. The test proceeds as in the same way of known κ. Since κ is unknown, group size can not be calculated in (5.8). Therefore, group size is supposed by researchers.

To give an instance for the application of real-life data on wind directions, the following example compares the group sequential test for the von Mises distribution with known κ, with fixed sample test and SPRT.

Example 5.1: Wind directions, in Anadolu University Airport Eskisehir, are measured sequentially (hourly) in university's weather station. For this data set, κ is known as $\kappa = 4, 58$ (corresponding $\rho = 0, 88263$) and $\alpha = \beta = 0, 05$ is supposed and the hypothesis is tested $H_0 : \mu = 141^0$ against $H_1 : \mu = 130^0$. Table 3 gives the maximum sample sizes and the expected sample sizes for the fixed sample, the sequential probability ratio, and the group sequential test.

Other examples can be presented that have the same general principle with different choices of α,β,μ_0,μ_1 and κ. Pocock [15] compared those tests for the mean of the normal distribution and showed that GST is more advantageous than the fixed sample test and SPRT in terms of sample size; in addition , Bacanlı & Demirhan [1] proposed the group sequential test for the mean of the inverse Gaussian distribution, in a similar way and showed that this test is more advantageous than the others.

Table 3. Comparison of the Fixed Sample, Sequential Probability Ratio and Group Sequential Tests for $\kappa = 4,58$, $\alpha = \beta = 0,05$, $H_0 : \mu = 141^0$, $H_1 : \mu = 130^0$ (von Mises response with known κ)

Tests		**Maximum Sample Size**		**Average Sample Number**	
Fixed Sample Test		73,545		73,545	
				minimum	maximum
SPRT		∞		2,380	∞
		Group sizes	n_{max}		
Group Sequential Test	$K=2$	40,618	81,236	52,072	
	$K=3$	28,138	84,413	46,596	
	$K=4$	21,576	86,305	44,361	
	$K=5$	17,503	87,515	43,057	

Thus, it is seen that these results are, also, valid for circular normal distribution that is known as Von-mises distribution.

6. Application to Medical Data

In this section, the group sequential test is applied to a medical circular data set. The medical data were collected from sequentially patients who was male and female and between the age of 44 and 75 in *Eskisehir Private Fizyomer Rehabilitation and Physical Therapy Center* between the years of 2010 and 2013. These patients were admitted to the center with complaints of pain in their shoulders. After the physical examination, some problems were detected in patients such as shoulder joint motions are painful and, also, partially restrictive and so on. Then, the range of motion the shoulder joints of patients were measured. These measurements include active and passive angular values for *flexion, extension, abduction, internal rotation* and *external rotation* variables. After the patients were diagnosed with the *adhesive capsulitis of shoulder* (also known as *the frozen shoulder*), 30 sessions of physical therapy and rehabilitation were applied to them and the range of motion of the shoulder joints were measured again. After the therapy, it is aimed that the patients will reach a complete joint range of motion in all of the shoulder motions. In this study, the group sequential test is applied for the *internal rotation (passive)* variable which is obtained after the therapy in the data set. In anatomy, *internal rotation* (also known as *medial rotation*) is a term that refers to the rotation towards the center of the body [18] and the term *passive* means that the patient moves with an external support or assistance.

It is theorized that a healthy, "perfect" shoulder should have 90 degrees of internal rotation [19]. Therefore, the group sequential test is applied for $H_0 : \mu = 90^0$ against the alternative $H_1 : \mu = 80^0$ with $\alpha = \beta = 0,05$ and $K = 4$. The concentration parameter is unknown, so the group sizes are supposed as $n = 5$. GST results are given in Table 4.

Table 4. Group Sequential Test Results for (passive) Internal Rotation Data Set

j	1	2	3	4
n	5	5	5	5
$\bar{\theta}_j$	84,133	85,031	86,012	85,031
$\hat{\kappa}_j$	11,486	27,181	47,768	27,181
$\bar{R}_j$	0,978	0,991	0,995	0,991
Z_{VMj}	-0,766	-1,005	-1,072	-1,005
S_{VMk}	-0,766	-1,771	-2,843	-3,848
$Z_p(4; 0,05)\sqrt{k}$	2,067	2,923	3,580	4,134
Decision	Continue	Continue	Continue	Accept H_0

When Table 4 results are examined, it can be seen that, in stage 4;

$$S_{VM4} = 3,848 > -Z_P(4; 0,05)\sqrt{4} = -2,067(2) = -4,134$$

hence we stop and accept H_0.

Therefore, researchers can apply the group sequential test for predetermined α, β, N and n values.

7. Discussion and Conclusions

As in many scientific fields, the most common probability distribution in medical applications of circular data is the von Mises distribution. However, the group sequential tests are often used in medical researches which the data is collected sequentially. Therefore, the group sequential test for the mean of the distributed von Mises data is proposed in this study.

In medical studies, a significant amount of the collected data is in the form of circular. In the literature, there are fixed sample and sequential probability ratio tests for circular data. However, in medical studies, the use of these tests is very difficult in terms of obtaining required sample sizes. The reason of this is that, when SPRT is used in the studies in which the data is collected sequentially, the expected sample size and the maximum sample size are infinite (see Table 3). Therefore, these values cannot be predetermined before the test. In this study, the group sequential test have been proposed for circular data. An application of this test for a medical data set (shoulder internal rotation angles) is carried out and it is shown that the advantages of the test are also valid for circular data.

In GST, researchers can determine required maximum sample size and expected sample size values for their hypotheses, determined α and β probabilities and K values. In this respect, using GST provides a great advantage. GST was generated for linear data in the literature. In this study, GST is defined for circular data and it is indicated that GST can be used for the mean of the von Mises distribution which is frequently encountered in medical studies.

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The Marshall–Olkin exponential Weibull distribution

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Abstract

A new four-parameter model called the Marshall–Olkin exponential–Weibull probability distribution is being introduced in this paper, generalizing a number of known lifetime distributions. This model turns out to be quite flexible for analyzing positive data. The hazard rate functions of the new model can be increasing and bathtub shaped. Our main objectives are to obtain representations of certain associated statistical functions, to estimate the parameters of the proposed distribution and to discuss its modality. As an application, the probability density function is utilized to model two actual data sets. The new distribution is shown to provide a better fit than related distributions as measured by the Anderson–Darling and Cramér–von Mises goodness–of–fit statistics. The proposed distribution may serve as a viable alternative to other distributions available in the literature for modeling positive data arising in various fields of scientific investigation such as reliability theory, hydrology, medicine, meteorology, survival analysis and engineering.

Keywords: Marshall–Olkin exponential–Weibull distribution, goodness–of–fit statistics, moments, median, mode, unimodal distribution, quantile function, Fox–Wright ${}_p\Psi_q$ function, Goyal–Laddha generalized Hurwitz–Lerch zeta function.

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1. Introduction

The Weibull distribution is a popular life time distribution model in reliability engineering. However, this distribution does not have a bathtub or upside–down bathtub shaped hazard rate function, which is why it cannot be utilized to model the life time of certain systems. To overcome this shortcoming, several generalizations of the classical Weibull distribution have been discussed by different authors in recent years. Many authors introduced flexible distributions for modeling complex data and obtaining a better fit. Extensions of the Weibull distribution arise in different areas of research as is often pointed out in the literature, see for instance [2] and the references therein. Various extended Weibull models have an upside–down bathtub shaped hazard rate, which is the case for the extensions discussed by [11] and [20], among others.

Adding parameters to an existing distribution enables one to obtain classes of more flexible distributions. Marshall and Olkin [9] introduced a method for adding a new parameter to an existing distribution, which results in improved flexibility to model different types of data. They consider the so–called *baseline distribution* having *cumulative distribution function* (CDF) G_b, with the associated *probability density function* (PDF) $g_b(x)$, being the Radon-Nikodým derivative of the CDF G_b with respect to the ordinary Lebesgue measure. Then, the associated Marshall–Olkin extended distribution CDF F is given by

$$F(x) = \frac{G_b(x)}{G_b(x) + \alpha \overline{G}_b(x)}\,,$$

where $\overline{G}_b = 1 - G_b$ stands for the survival function of the baseline CDF G_b. Accordingly, *via* the baseline PDF g_b the Marshall–Olkin PDF becomes

$$f(x) = \frac{\alpha\, g_b(x)}{\big(G_b(x) + \alpha \overline{G}_b(x)\big)^2}\,.$$

Recently Cordeiro *et al.* [2] introduced a type of exponential–Weibull distribution by considering the baseline CDF [¶]

$$G_b(x) = \Big(1 - e^{-\lambda\, x - \beta\, x^k}\Big) \cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad \lambda > 0\,,\, \beta > 0\,,\, k > 0\,, \tag{1.1}$$

with the associated PDF

$$g_b(x) = \Big(\lambda + \beta\, k\, x^{k-1}\Big)\; e^{-\lambda\, x - \beta\, x^k} \cdot \boldsymbol{I}_{(0,\infty)}(x)\,. \tag{1.2}$$

Now, we generalize the model (1.1) by Cordeiro *et al.* by applying the Marshall–Olkin technique, which results in what we are referring to as the Marshall–Olkin Exponential–Weibull (MOEW) distribution. Another implementation of the Marshall–Olkin technique was recently considered by Saboor and Pogány, see [20].

Let $\theta = (\lambda, \beta, k, \alpha)$ be a vector parameter having positive coordinates. The random variable (rv) ξ defined on a fixed probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ possesses the *Marshall–Olkin exponential–Weibull distribution* when its CDF and PDF are respectively given by

$$F(x) = \frac{1 - e^{-\left(\lambda\, x + \beta\, x^k\right)}}{1 - (1-\alpha)\, e^{-\left(\lambda\, x + \beta\, x^k\right)}} \cdot \boldsymbol{I}_{(0,\infty)}(x)\,, \tag{1.3}$$

$$f(x) = \frac{\alpha\, \left(\lambda + \beta\, k\, x^{k-1}\right)\, e^{-\lambda\, x - \beta\, x^k}}{\left(1 - (1-\alpha)\, e^{-\left(\lambda\, x + \beta\, x^k\right)}\right)^2} \cdot \boldsymbol{I}_{(0,\infty)}(x)\,, \qquad \lambda, \beta, k, \alpha > 0\,; \tag{1.4}$$

[¶]In this paper, $\boldsymbol{I}_A(x)$ denotes the indicator function of the set A.

and we write $\xi \sim \mathrm{MOEW}(\theta)$ with $\theta = (\lambda, \beta, k, \alpha)$ to indicate that the rv ξ follows this distribution.

One of the main reasons for introducing the MOEW distribution is the following. Consider a sequence of random variables (X_n), $n \in \mathbb{N}$ with IID elements from $G(x)$ distribution, the rv N which possesses geometric distribution with parameter $\alpha \in [0, 1]$, that is with probability mass function $\alpha(1-\alpha)^{n-1}$ for $n \in \mathbb{N}$, and $m_N = \min\{X_1, X_2, \cdots, X_N\}$. Then

$$\mathsf{P}\{m_N < x\} = 1 - \sum_{n \geq 1} \mathsf{P}\{m_N \geq x \,|\, N = n\}\, \mathsf{P}\{N = n\} = \frac{G(x)}{G(x) + \alpha\, \overline{G}(x)}.$$

Graphical illustrations of the effect of the parameter α, considered on the whole set $\mathbb{R}_+$ are included in Section 2. Representations of certain statistical functions are provided in Section 3. The parameter estimation technique described in Section 4 is utilized in Section 5 in connection with the modeling of two actual data sets originating from the engineering and biological sciences, where the new model is compared with several related distributions.

2. Graphical Presentations of the MOEW Distribution

Graphs of the PDF (1.4) and the hazard rate function (2.1) are presented in this section for certain values of the parameters.

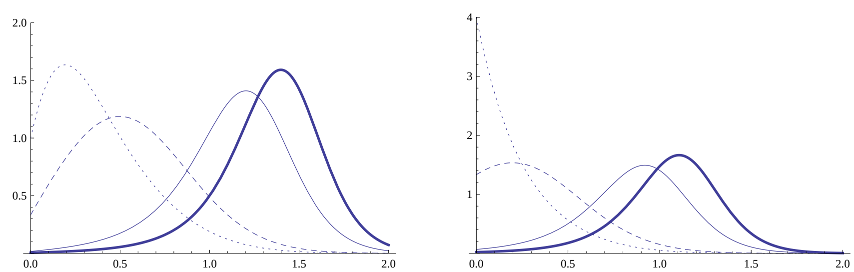


Figure 1. The MOEW PDF. Left panel: $\lambda = 0.5$, $\beta = 2.1$, $k = 2$, and $\alpha = 0.5$ (dotted line) $\alpha = 1.5$ (dashed line), $\alpha = 30$ (solid line), $\alpha = 100$ (thick line). Right panel: $\lambda = 2$, $k = 2$, $\beta = 2.1$ and $\alpha = 0.5$ (dotted line) $\alpha = 1.5$ (dashed line), $\alpha = 30$ (solid line), $\alpha = 100$ (thick line).

Figures 1 and 2 illustrate how the additional parameter α affect the $\mathrm{MOEW}(\theta)$ density (1.4). The graphs illustrate the versatility of the MOEW distribution and indicate that the new parameter α has a noticeable effect on the skewness and kurtosis. Both Figures 1 and 2 suggest that the parameter α acts somewhat as a location parameter. The left and right panels of Figure 3 indicate that the hazard rate function

$$h(x) = \frac{\lambda + \beta\, k\, x^{k-1}}{1 - (1-\alpha)\, e^{-\left(\lambda\, x + \beta\, x^k\right)}} \cdot \boldsymbol{I}_{(0,\infty)}(x) \tag{2.1}$$

can be increasing or bathtub shaped for certain values of the parameters.

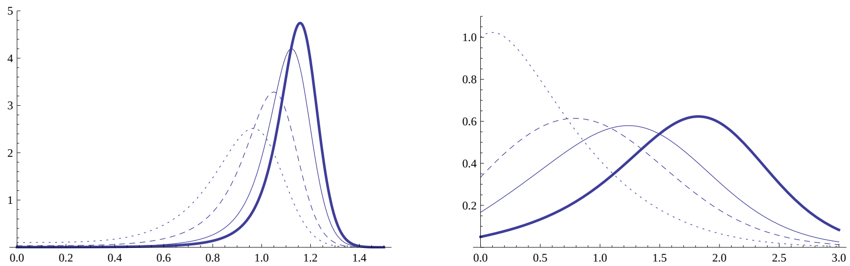


Figure 2. The MOEW PDF. Left panel: $\lambda = 0.5$, $k = 5$, $\beta = 2.1$, and $\alpha = 5$ (dotted line), $\alpha = 15$ (dashed line), $\alpha = 50$ (solid line), $\alpha = 100$ (thick line). Right panel: $\lambda = 0.5$, $\beta = 0.5$, $k = 2$, and $\alpha = 0.5$ (dotted line) $\alpha = 1.5$ (dashed line), $\alpha = 3$ (solid line), $\alpha = 10$ (thick line).

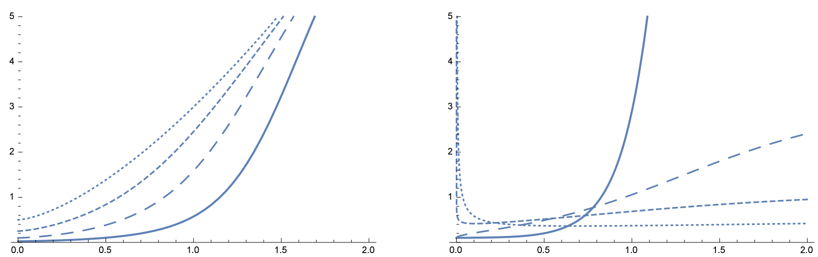


Figure 3. The MOEW hazard rate function. Left panel: $\lambda = 0.5$, $\beta = 1$, $k = 2.5$ and $\alpha = 1$ (dotted line), $\alpha = 2$ (short dashes), $\alpha = 5$ (long dashes), $\alpha = 20$ (solid line). Right panel: $\lambda = 0.5$, $\beta = 1$, $\alpha = 5$, and $k = 0.2$ (dotted line) $k = 0.8$ (small dashed line), $k = 1.5$ (long dashed line), $k = 5$ (solid line).

3. Special Cases

We point out some special cases of the MOEW$(\lambda, \beta, k, \alpha)$ distribution which are obtained by specifying some of its parameters values. For example, the MOEW$(\lambda, \beta, k, 1)$ corresponds to the *exponential–Weibull* distribution [2], the MOEW$(\lambda, \beta, 2, 1)$ is the *modified Rayleigh* distribution, the MOEW$(\lambda, \beta, 1, 1)$ turns out to be the *modified exponential* distribution and finally the MOEW$(0, \beta, k, 1)$ stands for the classical two-parameter Weibull distribution. If $k = 1$ and $k = 2$ in addition to $\alpha = 1$ and $\lambda = 0$, it coincides with the exponential and Rayleigh distributions, respectively.

4. Moments, Quantile Function, Modality Analysis and Mixture representation of the MOEW Distribution

In this section, we derive computable representations of some general order moments associated with the MOEW(θ) distribution having the PDF specified by (1.4). The Fox–Wright generalized hypergeometric ${}_1\Psi_0$, function has been used to obtain the series representations; in the case $k = 1$, the Goyal–Laddha generalized Hurwitz–Lerch Zeta function provides a closed form for the general order moments; in this case, the MOEW distribution is close to the classical Gamma distribution. The resulting expressions can be evaluated exactly or numerically with symbolic computational packages such as *Mathematica*, *MATLAB* or *Maple*. In numerical applications, infinite sum can be truncated whenever convergence is observed.

4.1. Moments. Before concentrating on the derivation of the r^{th} raw moment of the MOEW(θ) distribution, we introduce the Fox-Wright function ${}_p\Psi_q$, which is a generalization of the familiar generalized hypergeometric function ${}_pF_q$, with $p \in \mathbb{N}_0$ numerator parameters $a_1, \cdots, a_p \in \mathbb{C}$ and $q \in \mathbb{N}_0$ denominator parameters $b_1, \cdots, b_q \in \mathbb{C} \setminus \mathbb{Z}_0^-$, defined by

$$ {}_p\Psi_q \left[\begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p) \\ \\ (b_1, B_1), \cdots, (b_q, B_q) \end{array} \middle| z \right] = \sum_{n \geq 0} \frac{\Gamma(a_1 + A_1 n) \cdots \Gamma(a_p + A_p n)}{\Gamma(b_1 + B_1 n) \cdots \Gamma(b_q + B_q n)} \frac{z^n}{n!}, $$

where the empty products are conventionally taken to be equal 1, while

$$ A_j > 0,\, j = \overline{1,p};\; B_k > 0,\, k = \overline{1,q}; \quad \Delta = 1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j \geq 0\,, $$

(see, for instance [5, p. 56]). Convergence will occur for suitably bounded values of $|z|$ such that

$$ |z| < \nabla := \left(\prod_{j=1}^p A_j^{-A_j} \right) \cdot \left(\prod_{j=1}^q B_j^{B_j} \right). $$

We now derive closed form representations of the real order moments of a r.v. $\xi \sim$ MOEW(θ). First, we expand the denominator of the PDF (1.4) into a power series in $\exp\{-\left(\lambda\, x + \beta\, x^k\right)\}$. Then, interchanging the integral and the sum, we have

$$ \begin{aligned} \mathsf{E}\,\xi^r &= \alpha \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{n!} \int_0^\infty x^r \left(\lambda + \beta\, k\, x^{k-1} \right) e^{-(n+1)\,\lambda\, x - (n+1)\,\beta\, x^k}\, \mathrm{d}x \\ &= \alpha\,\lambda \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{n!} \int_0^\infty x^r\, e^{-(n+1)\,\lambda\, x - (n+1)\,\beta\, x^k}\, \mathrm{d}x \\ &\quad + \alpha\,\beta\, k \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{n!} \int_0^\infty x^{r+k-1}\, e^{-(n+1)\,\lambda\, x - (n+1)\,\beta\, x^k}\, \mathrm{d}x\,, \end{aligned} $$

where the Pochhammer symbol $(a)_b := \Gamma(a+b)/\Gamma(a)$, $\min(a, a+b) > 0$, and conventionally $(0)_0 = 1$. The r^{th} moment is a linear combination of integrals $\mathcal{I}(\omega)$ (considered already for a similar purpose by Nadarajah and Kotz in [12, Eq. (2.1)]) where

$$ \mathcal{I}(\omega) = \int_0^\infty x^{\kappa-1}\, e^{-(\mu x + a x^\eta)}\, \mathrm{d}x\,, \qquad \omega = (\kappa, \mu, a, \eta) > 0\,. $$

The following representation of this integral for general parameter values was obtained by Pogány and Saxena in [16, p. 515, Corollary 1.1]:

$$ \mathcal{I}(\omega) = \begin{cases} \mu^{-\kappa}\, {}_1\Psi_0 \left[\begin{array}{c} (\kappa, \eta) \\ \text{---} \end{array} \middle| -\dfrac{a}{\mu^\eta} \right] & 0 < \eta < 1 \\ \dfrac{\Gamma(\kappa)}{(\mu + a)^\kappa} & \eta = 1 \\ \dfrac{1}{\eta a^{\kappa/\eta}}\, {}_1\Psi_0 \left[\begin{array}{c} \left(\dfrac{\kappa}{\eta}, \dfrac{1}{\eta} \right) \\ \text{---} \end{array} \middle| -\dfrac{\mu}{a^{1/\eta}} \right] & \eta > 1 \end{cases}. $$

Thus, for all $k \in (0,1)$, we have

$$
\begin{aligned}
\mathsf{E}\,\xi^r &= \alpha\,\lambda \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{n!}\,\mathcal{I}(r+1,(n+1)\lambda,(n+1)\beta,k) \\
&\qquad + \alpha\,\beta\,k \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{n!}\,\mathcal{I}(r+k-1,(n+1)\lambda,(n+1)\beta,k) \\
&= \frac{\alpha}{\lambda^r} \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{r+1}\,n!}\,{}_1\Psi_0\left[\begin{array}{c|c} (r+1,k) & -\dfrac{\lambda\beta^{-k}}{(n+1)^{k-1}} \\ \text{———} & \end{array}\right] \\
&\qquad + \frac{\alpha\,\beta\,k}{\lambda^{r+k}} \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{r+k}\,n!}\,{}_1\Psi_0\left[\begin{array}{c|c} (r+k,k) & -\dfrac{\lambda\beta^{-k}}{(n+1)^{k-1}} \\ \text{———} & \end{array}\right]. \qquad (4.1)
\end{aligned}
$$

When $k = 1$, we have

$$
\mathsf{E}\,\xi^r = \frac{\alpha\,\Gamma(r+1)}{(\lambda+\beta)^r} \sum_{n\geq 0} \frac{(2)_n}{n!}\,\frac{(1-\alpha)^n}{(n+1)^{r+1}}\,. \tag{4.2}
$$

Now, consider the *Goyal–Laddha generalized Hurwitz–Lerch Zeta function* [4, p. 100, Eq. (1.5)] defined by the series

$$
\Phi^*_\mu(z,s,a) = \sum_{n\geq 0} \frac{(\mu)_n}{n!}\,\frac{z^n}{(n+a)^s}\,, \tag{4.3}
$$

where $\mu \in \mathbb{C}$; $a \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z|<1$; $\Re(s-\mu) > 1$ for $|z| = 1$. Applying (4.3) to the moment expression (4.2) for all $\alpha \in (0,2)$, while for $\alpha \in \{0,2\}$, $r > 2$, we obtain

$$
\mathsf{E}\,\xi^r = \frac{\alpha\,\Gamma(r+1)}{(\lambda+\beta)^r}\,\Phi^*_2(1-\alpha, r+1, 1)\,.
$$

The remaining values of the parameter $k > 1$ lead to the expected value

$$
\begin{aligned}
\mathsf{E}\,\xi^r &= \frac{\alpha\,\lambda}{k\beta^{\frac{r+1}{k}}} \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{\frac{r+1}{k}}\,n!}\,{}_1\Psi_0\left[\begin{array}{c|c} \left(\dfrac{r+1}{k},\dfrac{1}{k}\right) & -\dfrac{(n+1)^{1-\frac{1}{k}}\,\lambda}{\beta^{\frac{1}{k}}} \\ \text{———} & \end{array}\right] \\
&\qquad + \frac{\alpha}{\beta^{\frac{r}{k}}} \sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{\frac{r}{k}+1}\,n!}\,{}_1\Psi_0\left[\begin{array}{c|c} \left(\dfrac{r}{k}+1,\dfrac{1}{k}\right) & -\dfrac{\lambda}{(n+1)^{k-1}\beta^k} \\ \text{———} & \end{array}\right].
\end{aligned}
$$

Thus, the following result:

4.1. Theorem. *Let the rv* $\xi \sim \mathrm{MOEW}(\theta), \theta = (\lambda, \beta, k, \alpha) > 0$. *Then, for all* $r > -1$, *we have*

$$
(4.4)\qquad \mathsf{E}\,\xi^r = \begin{cases}
\dfrac{\alpha}{\lambda^r} \displaystyle\sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{r+1}\,n!}\, {}_1\Psi_0\left[\begin{array}{c|c} (r+1,k) & \dfrac{-\lambda\beta^{-k}}{(n+1)^{k-1}} \\ \text{———} & \end{array}\right] & \\
\quad + \dfrac{\alpha\,\beta\,k}{\lambda^{r+k}} \displaystyle\sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{r+k}\,n!}\, {}_1\Psi_0\left[\begin{array}{c|c} (r+k,k) & \dfrac{-\lambda\beta^{-k}}{(n+1)^{k-1}} \\ \text{———} & \end{array}\right] & 0<k<1 \\[3ex]
\dfrac{\alpha\,\Gamma(r+1)}{(\lambda+\beta)^r}\, \Phi_2^*(1-\alpha, r+1, 1) & k=1 \\[3ex]
\dfrac{\alpha\,\lambda}{k\beta^{\frac{r+1}{k}}} \displaystyle\sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{\frac{r+1}{k}}\,n!}\, {}_1\Psi_0\left[\begin{array}{c|c} \left(\frac{r+1}{k}, \frac{1}{k}\right) & \dfrac{-\lambda\beta^{-\frac{1}{k}}}{(n+1)^{\frac{1}{k}-1}} \\ \text{———} & \end{array}\right] & \\
\quad + \dfrac{\alpha}{\beta^{\frac{r}{k}}} \displaystyle\sum_{n\geq 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)^{\frac{r}{k}+1}\,n!}\, {}_1\Psi_0\left[\begin{array}{c|c} \left(\frac{r}{k}+1, \frac{1}{k}\right) & \dfrac{-\lambda\beta^{-\frac{1}{k}}}{(n+1)^{\frac{1}{k}-1}} \\ \text{———} & \end{array}\right] & k>1
\end{cases},
$$

where in the case $k = 1$, *the additional conditions* $\alpha \in (0, 2)$, *or when* $\alpha \in \{0, 2\}$, $r > 2$, *have to be satisfied.*

Proof. It only remains to verify the convergence conditions of the Fox–Wright series which depend only on the parameter k. Note that, when $k \in (0, 1)$, $\Delta = 1 - k > 0$, so that both series in (4.1) converge. So does the Goyal–Laddha function when $k = 1$. Finally, when $k > 1$, the value $\Delta = 1 - \frac{1}{k} > 0$ ensures that the moment $\mathsf{E}\,\xi^r$ is finite for any $r > -1$. □

4.2. Remark. For certain integer and rational values of the parameter k, we can make use of a representation of the Fox–Wright ${}_1\Psi_0$ in terms of generalized hypergeometric ${}_pF_q$ functions, which is discussed in detail in [10]. By their [10, Eq. (3.3)], for all positive rational $A = \frac{m}{M}$, one has

$$
{}_1\Psi_0\left[\begin{array}{c|c} (a, \frac{m}{M}) & z \\ \text{——} & \end{array}\right] = \Gamma(a) + \sum_{j=1}^{M} \frac{\Gamma(a + \frac{m}{M}j)\, z^j}{j!} \times {}_{m+1}F_M\left[\begin{array}{c|c} 1, \frac{j}{M}+\frac{a}{m}, \cdots, \frac{j}{M}+\frac{a+m-1}{m} & \dfrac{m^m\, z^M}{M^M} \\ \text{———————} & \end{array}\right],
$$

where ${}_pF_q$ stands for the generalized hypergeometric function which is a built–in *Mathematica* function specified by

```
HypergeometricPFQ[{a_1, ..., a_p},{b_1, ..., b_q},z].
```

The same authors also transform Fox–Wright Ψ functions into Meijer G–functions for rational arguments. Referring to [10, Eq. (5.1)], one has

$$
{}_1\Psi_0\left[\begin{array}{c|c} (a, \frac{m}{M}) & z \\ \text{——} & \end{array}\right] = \frac{2\sqrt{M}\, m^a}{\Gamma(a)\, \sqrt{m}\, \pi^{\frac{M+m-1}{2}}} \times G^{M,m}_{m,M}\left(\frac{m^m\,(-z)^M}{M^M} \,\middle|\, \begin{array}{c} 1-\frac{a}{m}, \cdots, 1-\frac{a+m-1}{m} \\ 0, \frac{1}{M}, \cdots, \frac{M-1}{M} \end{array} \right).
$$

The G–function in *Mathematica* code reads

```
MeijerG[{{a_1,...,a_n},{a_{n+1},...,a_p}},{{b_1,...,b_m},{b_{m+1},...,b_q}},z].
```

See, for example, the monographs [8, Ch. V] and [5] for an introduction to the G–function. ■

The factorial moments of order $N \in \mathbb{N}$ for a r.v. ξ are

$$\Phi_N = \mathsf{E}(\xi(\xi-1)(\xi-2)\cdots(\xi-N+1)) = \frac{\mathrm{d}^N\left(\mathsf{E}\, t^\xi\right)}{\mathrm{d}t^N}\bigg|_{t=1}.$$

By virtue of the Viète–Girard formulae for expanding $\xi(\xi-1)(\xi-2)\cdots(\xi-N+1)$, we obtain

$$\Phi_N = \sum_{r=1}^{N} (-1)^{N-r} \left\{ \sum_{1\le \ell_1 < \cdots < \ell_r \le N-1} \ell_1 \cdots \ell_r \right\} \mathsf{E}\xi^r\,,$$

where the second sum represents elementary symmetric polynomials:

$$e_r = e_r(\ell_1, \cdots, \ell_r) = \sum_{1\le \ell_1 < \cdots < \ell_r \le N-1} \ell_1 \cdots \ell_r, \qquad r = \overline{0, N-1}.$$

This in conjunction with the positive integer r^{th} order moment expression given in formula (4.4) provides an exact series representation for the fractional order moments.

4.2. Quantile Function. The next statistical function being considered is the *quantile function* $\mathscr{Q}_\xi$ for the rv $\xi \sim \mathrm{MOEW}(\theta)$. The rv ξ possesses the CDF $F(x)$ given by (1.3) and its quantile function is

$$\mathscr{Q}_\xi(p) = \inf\{x \in \mathbb{R}\colon p \le F(x)\}\,, \qquad p \in (0,1)\,;$$

it consists of the generalized inverse of the CDF for a fixed probability p. A closed form is given in the next theorem for the MOEW distribution.

4.3. Theorem. *Let $\xi \sim \mathrm{MOEW}(\theta)$, $\theta = (\lambda, \beta, k, \alpha)$ with parameter space $\theta \in \mathbb{R}_+^4$. For all $p \in (0,1)$, the quantile function of ξ is*

$$\mathscr{Q}_\xi(p) = \ln\Big(\frac{1-(1-\alpha)p}{1-p}\Big)^{\frac{1}{\lambda}} \Big\{1 + \sum_{n\ge 1} \binom{kn}{n-1} \frac{w^n}{n!}\Big\}\,, \tag{4.5}$$

where

$$w = \left(-\frac{1}{\lambda}\right)^k \left[\ln \frac{1-p}{1-(1-\alpha)p}\right]^{k\beta}.$$

Moreover, for $k>1$ we have

$$\mathscr{Q}_\xi(p) = \ln\Big(\frac{1-(1-\alpha)p}{1-p}\Big)^{\frac{1}{\lambda}} \cdot \left\{1 + w \cdot {}_1\Psi_2\Big[\begin{matrix} (k+1,k) \\ (k+1,k-1),\,(2,1) \end{matrix} \Big| w\Big]\right\}. \tag{4.6}$$

Proof. The quantile function is the solution of $F(x) = p$ in x. Thus, for $p \in (0,1)$ fixed, one has

$$\beta x^k + \lambda x + \ln \frac{1-p}{1-(1-\alpha)p} = 0\,,$$

which is equivalent to

$$1 - t + wt^k = 0; \qquad t = -\frac{\lambda}{c}; \qquad c = \ln\frac{1-p}{1-(1-\alpha)p}\,. \tag{4.7}$$

Applying the Bürmann–Lagrange series expansion [17, p. 153, p. 348, **211.**] for the three-term equation (4.7), we obtain

$$t = 1 + \sum_{n\ge 1} \binom{kn}{n-1} \frac{w^n}{n!}\,,$$

which leads to the solution (4.5).

Further, assuming $k>1$, transforming and writing the generalized binomial coefficient in (4.7) in terms of gamma function, that is,

$$\binom{a}{\ell} = \frac{\Gamma(a+1)}{\Gamma(a-\ell+1)\,\ell!}, \qquad \ell \in \mathbb{N},$$

we have

$$\begin{aligned} t &= 1 + \sum_{n\ge 1} \binom{kn}{n-1} \frac{w^n}{n!} = 1 + \sum_{n\ge 0} \binom{kn+k}{n} \frac{w^{n+1}}{(n+1)!} \\ &= 1 + w \sum_{n\ge 0} \frac{\Gamma(k+1+kn)}{\Gamma(k+1+(k-1)n)\,\Gamma(2+n)} \, \frac{w^n}{n!}\,. \end{aligned}$$

Since $\Delta = 1+k-1+2-k=2$ and all coefficients of the running indices are positive, we recognize the sum as the appropriate converging Fox–Wright generalized ${}_1\Psi_2$ function as stated in (4.6). $\square$

The distribution of ξ being absolutely continuous, the corresponding *median* turns out to be $\mathfrak{m}_\xi = \mathscr{Q}_\xi(\frac{1}{2})$. Therefore we have

4.4. Corollary. *Under the assumptions made in the* Theorem 3.3 *we have*

$$\mathfrak{m}_\xi = \frac{1}{\lambda}\,\ln(1+\alpha)\cdot\left(1+\sum_{n\ge 1}\binom{kn}{n-1}\frac{w^n}{n!}\right), \tag{4.8}$$

where

$$w = \frac{(-1)^{k(\beta+1)}}{\lambda^k}\,\big[\,\ln(1+\alpha)\,\big]^{k\beta}\,.$$

Accordingly, for $k>1$ we have

$$\mathfrak{m}_\xi = \frac{1}{\lambda}\,\ln(1+\alpha)\cdot\left\{1 + w\cdot {}_1\Psi_2\Big[\begin{matrix}(k+1,k)\\ (k+1,k-1),\,(2,1)\end{matrix}\,\Big|\, w\Big]\right\}. \tag{4.9}$$

Finally, we point out that Theorem 4.1 yields the characteristic function $\phi_\xi(t) = \mathsf{E}\, e^{\mathrm{i}t\xi}$ *via* the well–known Maclaurin series expansion $\phi_\xi(t) = \sum_{n\ge 0}(\mathrm{i}t)^n\,\mathsf{E}\,\xi^n/n!$. Further, the moment generating function $M_\xi(t) = \phi_\xi(-\mathrm{i}t)$, while the hazard rate function $h(x)$ and the survival function $\overline{F}(x) = 1-F(x)$ can be expressed in obvious ways in terms of the PDF and the CDF of the rv $\xi \sim \mathrm{MOEW}(\theta)$.

4.3. Modality Analysis. To close this section, we carry out a modality analysis for the $\mathrm{MOEW}(\theta)$ distribution.

Let us recall that in the case of continuous distributions having PDF f, the argument value x_0 belonging to its support $\mathrm{supp}(f) := \{x\colon f(x) > 0\}$ for which $f(x_0) = \max$, is called the *mode* (*peak*)[‖]. The PDF can attain local maximum at several values from $\mathrm{supp}(f)$; the distributions with a single mode are *unimodal*. The following theorem gives certain sufficient conditions for the unimodality of a $\mathrm{MOEW}(\theta)$ distribution for different cases.

4.5. Theorem. *Let $\xi \sim \mathrm{MOEW}(\theta)$, $\theta = (\lambda,\beta,k,\alpha)$ where $(\lambda,\beta,k) \in \mathbb{R}^3_+$, $\alpha \in (0,1]$. Then*

(i) $k \in (0,1]$. *No mode.*

[‖]Let us mention that there are other definitions of the modality in terms of the related CDF or the characteristic function or its Laplace–Stieltjes transform [22]

(ii) $k \in (1,2)$. *The rv* $\xi \sim \mathrm{MOEW}(\theta)$ *is unimodal with* $x_0 \in (0,1)$*, when*

$$\beta k[(\beta-1)k + 2\lambda + 1] + \lambda^2 > 0\,.$$

(iii) $k = 2$. *No mode exists when* $\lambda \geq \sqrt{2\beta}$. *For* $\lambda < \sqrt{2\beta}$ *the distribution is unimodal with the peak at* $x_0 \in (0, x^*)$*, where*

$$x^* = \frac{\sqrt{2\beta}-\lambda}{2\beta}\,.$$

(iv) $k \in (2,4)$. *The rv* $\xi \sim \mathrm{MOEW}(\theta)$ *is unimodal with* $x_0 \in (0,1)$*, when*

$$(\beta k + \lambda)^2 < (4-k)\beta\,.$$

Proof. For MOEW distribution $\mathrm{supp}(f) = \mathbb{R}_+$. As for the peak value of the PDF (1.4), we consider its logarithmic derivative

$$\begin{aligned}\frac{\partial \ln f(x)}{\partial x} &= \frac{\beta k(k-1)x^{k-2}}{\lambda + \beta k x^{k-1}} - \left(\lambda + \beta k x^{k-1}\right)\\ &\quad - \frac{2(1-\alpha)(\lambda + \beta k x^{k-1})}{1-(1-\alpha)e^{-\lambda x - \beta k x^k}}\, e^{-\lambda x - \beta k x^k}\,.\end{aligned} \tag{4.10}$$

The case **(i)**, when $k \in (0,1)$ is obvious, since

$$f'(x) = f(x)\,\frac{\partial \ln f(x)}{\partial x} < 0, \qquad x > 0\,,$$

that is, $f(x)$ monotonically decreases from $f(0^+) = +\infty$ to zero. The case $k=1$ is actually generated by the exponential baseline distribution with parameter $\lambda + \beta$, see (1.1). In all those cases no mode exists.

As for the case **(ii)**, when $k \in (1,2)$, we consider the first two terms on the right-hand-side expression

$$\begin{aligned}h_k(x) &= \frac{\beta k(k-1)x^{k-2}}{\lambda + \beta k x^{k-1}} - \left(\lambda + \beta k x^{k-1}\right)\\ &= -\frac{\beta^2 k^2 x^{2k-2} + 2\lambda\beta k x^{k-1} - \beta k(k-1)x^{k-2} + \lambda^2}{\lambda + \beta k x^{k-1}} =: \frac{-q_k(x)}{\lambda + \beta k x^{k-1}}\,,\end{aligned}$$

say. For $\alpha \in (0,1]$ the third term in (4.10) is negative for all $x > 0$. Since $q_k(0^+) = -\infty$, but $q_k(1) = \beta k[(\beta-1)k + 2\lambda + 1] + \lambda^2 > 0$ and

$$q_k'(x) = \beta k(k-1)x^{k-3}(2\beta k x^k + 2\lambda x + 2 - k) > 0$$

exactly one sign change occurs inside $(0,1)$, so $x_0 \in (0,1)$.

Consider now

$$q_2(x) = 4\beta^2 x^2 + 4\lambda\beta x - 2\beta + \lambda^2 = 0\,.$$

The roots of $q_2(x) = 0$ are

$$x_1 = -\frac{\sqrt{2\beta}+\lambda}{2\beta} < 0\,, \qquad x^* = \frac{\sqrt{2\beta}-\lambda}{2\beta}\,.$$

The solution $x^* > 0$ for $\sqrt{2\beta} - \lambda > 0$, which confirms the assertion **(iii)**.

Finally, for $k > 2$, $q_k(0^+) = \lambda^2 > 0$ and $q_k(1) = (\beta k + \lambda)^2 - \beta k(k-1)$ should be negative. However,

$$(\beta k + \lambda)^2 - \beta k(k-1) < (\beta k + \lambda)^2 - (4-k)\beta\,,$$

which can take negative values for $k \in (2,4)$. For $k \geq 4$, the last estimate becomes redundant. $\square$

4.6. Remark. Obviously, the modality analysis in the cases $\alpha \in (0,1], k > 4$ and $\alpha > 1$ requires another approach to be solved, since in the latter case the third right-hand-side addend in (4.10) becomes

$$\frac{2(\alpha-1)(\lambda+\beta k x^{k-1})}{1+(\alpha-1)e^{-\lambda x-\beta k x^k}}\, e^{-\lambda x-\beta k x^k} > 0\,. \qquad \blacksquare$$

We now show that the density (1.4) can be expressed as a mixture of EW densities. Using the identity

$$(1-z)^{-\tau} = \sum_{n=0}^{\infty} \frac{(\tau)_n}{n!}\, z^n\,, \qquad |z|<1\,,\, \tau>0\,,$$

one has the following mixture representation for the density function (1.4):

$$f(x) = \alpha \sum_{n\ge 0} \frac{(2)_n\,(1-\alpha)^n}{(n+1)!}\, g_{n+1}(x),$$

where $g_{n+1}(x)$ denotes the PDF of the EW model with parameters $\lambda^\star = (n+1)\lambda$, $\beta^\star = (n+1)\beta$ and k. Thus, the MOEW density function is a mixture of EW densities.

5. Parameter Estimation

This section provides a system of equations that can be utilized to determine the maximum likelihood estimates of the parameters of the MOEW distribution. Additionally, two goodness-of-fit measures are proposed to compare the density estimates.

5.1. Maximum Likelihood Estimation. In order to estimate the parameters of the proposed MOEW density function as defined in Equation (6), the loglikelihood of the sample is maximized with respect to the parameters. Given the data $\boldsymbol{x} = (x_1, \cdots, x_n)$, the loglikelihood function is

$$\begin{aligned}\ell(\theta) &= n\,\log\alpha + \sum_{i=1}^n \log\left(\lambda+\beta\,k\,x_i^{k-1}\right) - \sum_{i=1}^n \left(\lambda\,x_i + \beta\,x_i^k\right)\\ &\quad - \sum_{i=1}^n \log\left(\left(1-(1-\alpha)\,e^{-\left(\lambda\,x_i+\beta\,x_i^k\right)}\right)^2\right)\,,\end{aligned}$$

where $f(x)$ is as given in (1.4). The associated nonlinear loglikelihood system $\frac{\partial \ell(\theta)}{\partial\theta} = 0$ for MLE's is

$$\begin{aligned}\frac{\partial\ell(\theta)}{\partial\lambda} &= -\sum_{i=1}^n x_i - \sum_{i=1}^n \frac{2e^{-\lambda x_i-\beta x_i^k}(1-\alpha)x_i}{1-e^{-\lambda x_i-\beta x_i^k}(1-\alpha)} + \sum_{i=1}^n \frac{1}{\lambda+k\beta x_i^{-1+k}} = 0\\ \frac{\partial\ell(\theta)}{\partial\beta} &= -\sum_{i=1}^n x_i^k - \sum_{i=1}^n \frac{2e^{-\lambda x_i-\beta x_i^k}(1-\alpha)x_i^k}{1-e^{-\lambda x_i-\beta x_i^k}(1-\alpha)} + \sum_{i=1}^n \frac{k x_i^{-1+k}}{\lambda+k\beta x_i^{-1+k}} = 0\\ \frac{\partial\ell(\theta)}{\partial k} &= -\beta\sum_{i=1}^n x_i^k\,\log x_i - \sum_{i=1}^n \frac{2e^{-\lambda x_i-\beta x_i^k}(1-\alpha)\beta x_i^k\,\log x_i}{1-e^{-\lambda x_i-\beta x_i^k}(1-\alpha)}\\ &\quad + \sum_{i=1}^n \frac{\beta x_i^{-1+k}+k\beta x_i^{-1+k}\,\log x_i}{\lambda+k\beta x_i^{-1+k}} = 0\\ \frac{\partial\ell(\theta)}{\partial\alpha} &= \frac{n}{\alpha} - \sum_{i=1}^n \frac{2e^{-\lambda x_i-\beta x_i^k}}{1-e^{-\lambda x_i-\beta x_i^k}(1-\alpha)} = 0\,.\end{aligned}$$

Solving these equations simultaneously yields the maximum likelihood estimates (MLEs) of the four parameters. Numerical iterative techniques are then necessary to estimate the model parameters. It is possible to determine the global maximum of the log-likelihood by taking different initial values for the parameters. However, we observed that the MLEs for this model are not very sensitive to the initial estimates. For interval estimation on the model parameters, we require the Fisher information matrix; however in this article we leave this routine calculation to the interested reader.

5.2. Goodness-of-Fit Statistics. The Anderson-Darling and the Cramér-von Mises statistics are widely utilized to determine how closely a specific distribution whose associated cumulative distribution function fits the empirical distribution associated with a given data set. These statistics are

$$A_0^* = -\left(\frac{9}{4n^2}+\frac{3}{4n}+1\right)\left\{n+\frac{1}{n}\sum_{j=1}^{n}(2j-1)\,\log\left(z_j\,(1-z_{n-j+1})\right)\right\}$$

$$W_0^* = \left(\frac{1}{2n}+1\right)\left\{\sum_{j=1}^{n}\left(z_j-\frac{2j-1}{2n}\right)^2+\frac{1}{12n}\right\},$$

respectively, where $z_j = F(y_j)$, the y_j values being the *ordered* observations. The smaller these statistics are, the better the fit. Upper tail percentiles of the asymptotic distributions of these goodness–of–fit statistics were tabulated in [13].

6. Applications

Now, we will make use of the MOEW, beta transmuted Weibull (BTW) [14], Kumaraswamy modified Weibull (KMW) [3], extended Weibull (ExtW) [15], exponential–Weibull (EW) [2], gamma–Weibull (GW) [18] **, generalized gamma (GG) [21], two parameter Weibull (Weibull) and two parameter gamma (Gamma) distributions to model two well–known real data sets, namely the 'Carbon fibres' [13] and the 'Cancer patients' [6] data sets. The parameters of the MOEW distribution can be estimated from the log-likelihood of the samples in conjunction with the *NMaximize* command in the symbolic computational package *Mathematica*. More specifically, the models being considered are:

- The classical gamma distribution with PDF

$$f(x) = \frac{x^{\xi-1}\,e^{-x/\phi}}{\phi^{\xi}\,\Gamma(\xi)}\cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad \phi,\,\xi>0\,.$$

- The classical Weibull distribution with PDF

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}\cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad k,\lambda>0\,.$$

- The generalized gamma (GG) distribution [21] with PDF

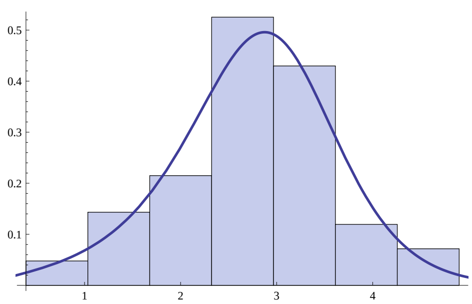
$$f(x) = \frac{k\,\lambda^{-\xi}\,x^{\xi-1}\,e^{-\lambda^{-k}\,x^k}}{\Gamma(\xi/k)}\cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad k,\,\lambda>0\,.$$

- The gamma–Weibull (GW) distribution [18] with PDF

$$f(x) = \frac{k\,\lambda^{-k-\xi}\,x^{\xi+k-1}e^{-\lambda^{-k}\,x^k}}{\Gamma(1+\xi/k)}\cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad \xi+k,\,\lambda>0\,.$$

**It is worth mentioning that following another approach, that is, renormalizing the product of the gamma and the Weibull distribution's PDF, Leipnik and Pearce [7] introduced a five–parameter gamma–Weibull distribution; for further results on this type of investigations consult also [12] and [16]. In turn, the independently introduced, different type of PDF proposed by Provost *et al.* [18] is actually a specific case of Leipnik–Pearce type gamma-Weibull distribution. Fortunately, the both turn out to be good candidates for various applications.

- The gamma exponentiated exponential (GEE) distribution [19] with PDF
$$f(x) = \frac{\lambda\,\alpha^\delta}{\Gamma(\delta)}\,e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}\left(-\log\left(1-e^{-\lambda x}\right)\right)^{\delta-1}\cdot \boldsymbol{I}_{(0,\infty)}(x)\,,$$
where $\lambda,\,\alpha,\,\delta>0\,.$
- The exponential–Weibull (EW) distribution [1] with PDF
$$f(x) = \left(\lambda+\beta\,k\,x^{k-1}\right)\,e^{-\lambda\,x-\beta\,x^k}\cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad \lambda,\,\beta,\,k>0\,.$$
- The extended Weibull (ExtW) distribution [15] with PDF
$$f(x) = a\,(c+b\,x)\,x^{-2+b}\,e^{-c/x-ax^b e^{-c/x}}\cdot \boldsymbol{I}_{(0,\infty)}(x), \qquad a,\,b,\,c\geq 0\,.$$
- The Kumaraswamy modified Weibull (KMW) distribution [3] with PDF
$$f(x) = a\,b\,\alpha\,x^{\gamma-1}(\gamma+\lambda\,x)\exp\left(\lambda\,x-\alpha\,x^\gamma\,e^{\lambda\,x}\right)\left(1-\exp\left(-\alpha\,x^\gamma\,e^{\lambda\,x}\right)\right)^{a-1}$$
$$\cdot\,\left(1-\left(1-\exp\left(-\alpha\,x^\gamma\,e^{\lambda\,x}\right)\right)^a\right)^{b-1}\cdot \boldsymbol{I}_{(0,\infty)}(x)$$
where $a,\,b,\,\alpha,\gamma>0\,,\lambda\geq 0$.
- The beta transmuted Weibull (BTW) distribution [14] with PDF
$$f(x) = \frac{\alpha\beta x^{\beta-1}}{\mathrm{B}(a,b)}\,e^{-\alpha\,x^\beta}(1-\lambda+2\,\lambda\,e^{-\alpha\,x^\beta})(1-e^{-\alpha\,x^\beta})^{a-1}(1+\lambda e^{-\alpha x^\beta})^{a-1}$$
$$\cdot\,(1-(1-e^{-\alpha\,x^\beta})(1+\lambda\,e^{-\alpha\,x^\beta}))^{b-1}\cdot \boldsymbol{I}_{(0,\infty)}(x)$$
where $a,\,b,\,\alpha,\,\beta>0,\ |\lambda|\leq 1.$



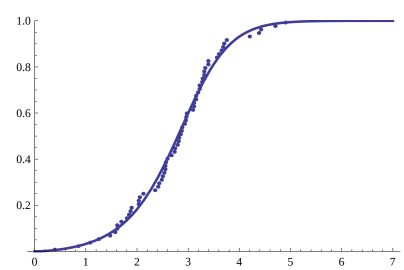


Figure 4. The Carbon fibres data fitted using the maximum likelihood approach; Left panel: The MOEW PDF estimate superimposed on the histogram for Carbon fibres data. Right panel: The MOEW CDF estimate and empirical CDF.

6.1. The Carbon Fibres Data Set. We shall consider the uncensored real data set on the breaking stress of carbon fibres (in Gba) as reported in [13]. The data are ($n = 66$):

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53.

Table 1. Estimates of the Parameters and Goodness-of-Fit Statistics for the Carbon Fibres Data

Distributions	Estimates					A_0^*	W_0^*
Gamma(ξ, ϕ)	7.48803	0.36853				1.32674	0.24815
Weibull(k, λ)	3.44120	47.0505				0.49168	0.08430
GG(k, λ, ξ)	4.07350	3.34592	3.09225			0.48757	0.08111
GW(k, ξ, λ)	3.44120	$1.6 * 10^{-7}$	3.06226			0.49168	0.08430
GEE$(\lambda, \alpha, \delta)$	0.26555	10.0365	7.23658			1.43415	0.26682
EW(k, λ, β)	3.73666	0.01710	0.01402			0.40365	0.06479
ExtW(a, b, c)	16.1979	$1 * 10^{-7}$	8.05671			2.26745	0.41615
KMW$(\alpha, \gamma, \lambda, a, b)$	0.14981	1.79940	0.49987	0.64975	0.17111	1.29338	0.21322
BTW$(\alpha, \beta, \lambda, a, b)$	0.00395	3.49999	0.99982	0.95052	2.39533	0.51603	0.09143
MOEW$(\lambda, \beta, k, \alpha)$	1.62267	$1 * 10^{-6}$	0.61610	25.3808		**0.2565**	**0.0374**

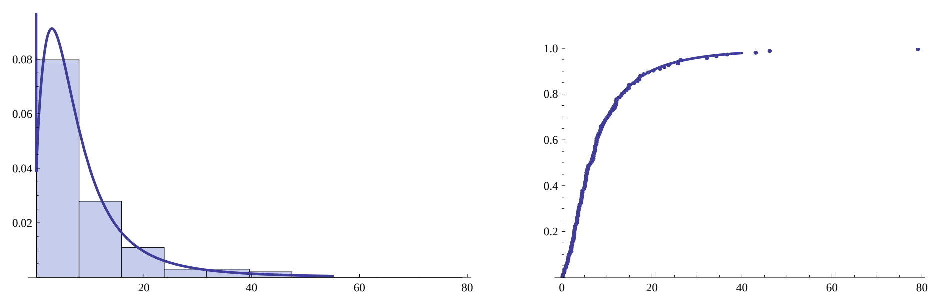


Figure 5. The Cancer Patients data fitted using the maximum likelihood approach; Left panel: The MOEW PDF estimate superimposed on the histogram for Cancer patients data. Right panel: The MOEW CDF estimate and empirical CDF.

6.2. The Cancer Patients Data Set. The second data set represents the remission times (in months) of a random sample of 128 bladder cancer patients as reported in [6]. The data are

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90 , 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

The PDF and CDF estimates of the MOEW distribution are plotted in Figures 4 and 5 for the Carbon fibres and Cancer patients data, respectively. The estimates of the parameters and the values of the Anderson-Darling and Cramér-von Mises goodness–of–fit statistics are given in Tables 1 and 2. It is seen that the proposed MOEW model provides the best fit for the both data sets.

To compare MOEW model with its sub-model EW, the likelihood-ratio (LR) test is applied to both data sets. The LR in this case is $L^* = L_0(k, \lambda, \beta)/L_a(k, \lambda, \beta)$, where L_0 and L_a are the likelihood values for the EW and MOEW distributions, respectively. The LR statistic $-2 \log L^*$ follows a chi-square distribution (asymptotically) with 1 degrees

Table 2. Estimates of the Parameters and Goodness-of-Fit Statistics for the Cancer Patients Data

Distributions	Estimates					A_0^*	W_0^*
Gamma(ξ,ϕ)	1.17251	7.98766				0.77625	0.13606
Weibull(k,λ)	1.04783	10.6510				0.96345	0.15430
GG(k,λ,ξ)	0.52010	0.59510	1.94927			0.30087	0.04526
GW(k,ξ,λ)	0.52001	1.42917	0.59510			0.30087	0.04526
GEE(λ,α,δ)	0.12117	1.21795	1.00156			0.71819	0.12840
EW(k,λ,β)	1.04780	$1*10^{-7}$	0.09389			0.96345	0.15430
ExtW(a,b,c)	1.96210	$1*10^{-21}$	3.74383			13.3317	2.49818
KMW$(\alpha,\gamma,\lambda,a,b)$	0.63962	0.38186	0.02960	0.37500	0.32284	18.8864	3.68568
BTW$(\alpha,\beta,\lambda,a,b)$	0.21333	0.99990	0.97623	1.52665	0.32699	0.16057	0.02367
MOEW(λ,β,k,α)	0.12080	0.01234	10.9988	$1*10^{6}$		**0.09052**	**0.0141**

of freedom. For the first data set $-2\log L^* = 1.613$ with a p-value of 0.2041 whereas for the second data set $-2\log L^* = 9.344$ with a p-value of 0.0022. Both values of the LR statistics suggest that in both cases the MOEW model performs significantly better when compared with its sub-model EW.

7. Discussion

There has been a growing interest among statisticians and applied researchers in constructing flexible lifetime models in order to improve the modeling of survival data. As a result, significant progress has been made towards generalizing some well–known lifetime models, which have been successfully applied to problems arising in several areas of research. In particular, several authors proposed new distributions that are based on the traditional Weibull model. In this paper, we introduce a four–parameter distribution which is obtained by applying the Marshall–Olkin technique to the exponential Weibull model. We studied some of its mathematical and statistical properties. We also provided computable representations of the moments of order $r > -1$, the factorial moments and the quantile function. Also the unimodality analysis was performed for suitable subdomains of the parameter space of the MOEW(θ) distribution.

The proposed distribution was utilized to model two data sets; it was shown to provide a better fit than several other related models, including some with more parameters. The distributional results developed in this article should find numerous applications in reliability theory, hydrology, medicine, meteorology, survival analysis and engineering.

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