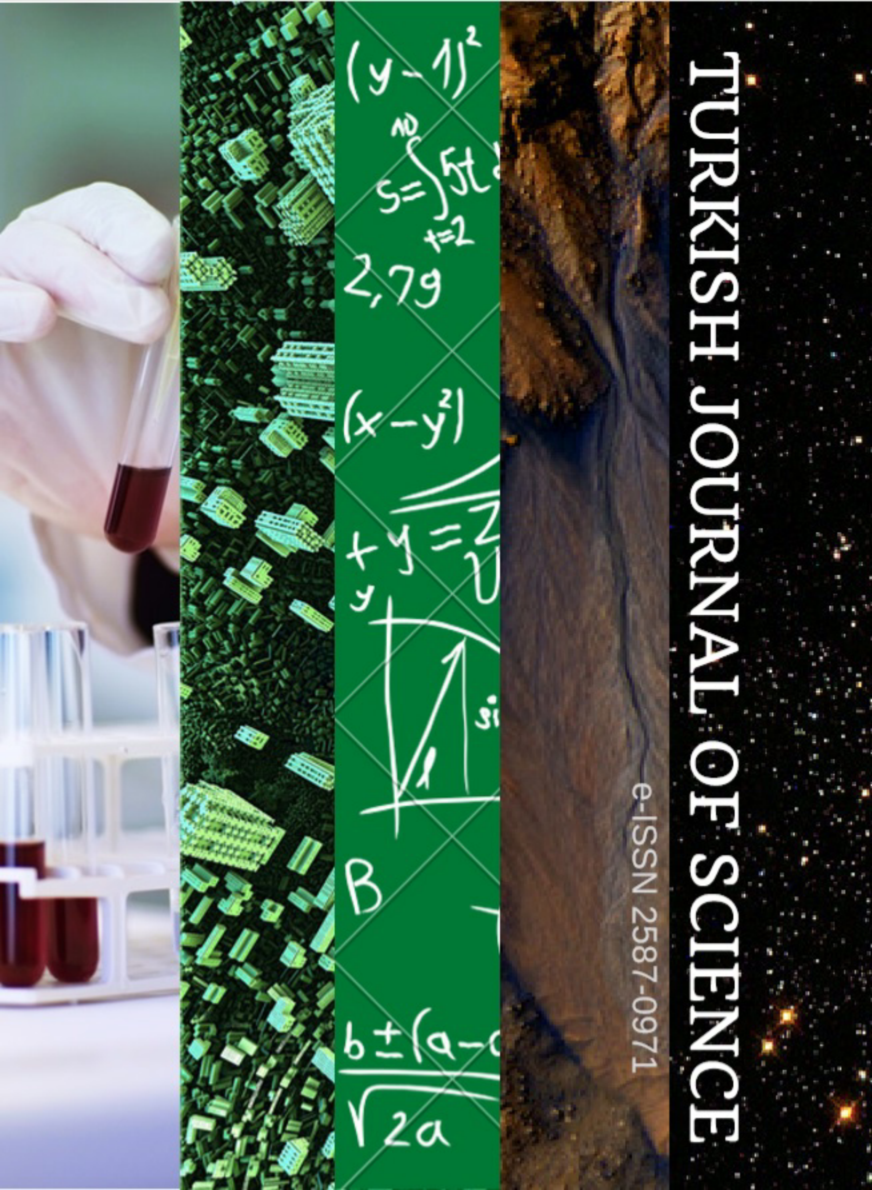
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Smarandache Curves of Spatial Quaternionic Bertrand Curve According to Frenet Frame

Süleyman Şenyurt[a], Ceyda Cevahir Yıldız[b], Yasin Altun[c]

[a]*Department of Mathematics, Faculty of Arts and Sciences, Ordu University, 52200, Ordu/Turkey*
[b]*Department of Mathematics, Institute of Science Ordu University, 52200, Ordu/Turkey*
[c]*Department of Mathematics, Institute of Science Ordu University, 52200, Ordu/Turkey*

Abstract. In this study, Frenet vectors of spatial quaternionic Bertrand curve pair were taken as the position vector. The obtained Smarandache curves from position vector were defined. Frenet vectors, the curvature and torsion of this curve were calculated. This later the Frenet apparatus were expressed in terms of Frenet apparatus of the spatial quaternionic Bertrand curve pair. Example related to the subject was found and their drawings were done with Maple program.

1. Introduction

Quaternion was first introduced by the Irish mathematician William Rowan Hamilton in 1843 in the form of generalized complex numbers. Each quaternion is accompanied by four units $\{1, e_1, e_2, e_3\}$, [9]. In 1987, Bharathi, K. and Nagaraj, M.'s "Quaternion Valued Function of a Real Variable Serret-Frenet Formulae" named article have shed light to many studies related to quaternions. In recent years, many studies have been done on quaternions. These studies are found in [2–4, 6–9, 13, 14, 16]. Many studies have been done on special curves in differential geometry. Studies on one of these, the Bertrand curve, are see in [17, 18]. Some studies of Smarandache curves are available in [1, 9–13, 15].

2. Preliminaries

A real quaternion is defined with q of the form
$\mathbf{Q} = \{q | q = d + ae_1 + be_2 + ce_3, d, a, b, c \in \mathbb{R}, e_1, e_2, e_3 \in \mathbb{R}^3\}$ such that

$$\begin{aligned} e_1^2 &= e_2^2 = e_3^2 = -1\,, \qquad e_1 \times e_2 = -e_2 \times e_1 = e_3, \\ e_1 &\times e_3 = -e_3 \times e_1 = e_2\,, \quad e_2 \times e_3 = -e_3 \times e_2 = e_1. \end{aligned}$$

Corresponding author: SŞ mail address: senyurtsuleyman52@gmail.com ORCID: https://orcid.org/0000-0003-1097-5541, CCY ORCID: https://orcid.org/0000-0002-9498-4930,YA ORCID: https://orcid.org/0000-0002-6977-4958
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We put $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$. Then a quaternion q can rewrite as

$$q = S_q + V_q$$

where S_q and V_q are the scalar part and vectorial part of q, respectively, [5]. For $q_1 = S_{q_1} + V_{q_1}$, $q_2 = S_{q_2} + V_{q_2}$ quaternions, quaternionic summation, multiplication and conjugate operations are, respectively

$$\begin{aligned}
&q_1 + q_2 = S_{q_1} + V_{q_1} + S_{q_2} + V_{q_2} = S_{q_1+q_2} + V_{q_1+q_2}\\
&q_1 \times q_2 = S_{q_1}S_{q_2} - \langle V_{q_1}, V_{q_2}\rangle + S_{q_1}V_{q_2} + S_{q_2}V_{q_1} + V_{q_1} \wedge V_{q_2}\\
&\bar{q} = S_{q_1} - V_{q_1}
\end{aligned}$$

These expression the symmetric real-valued, non-degenerate, bilinear form as follows,

$$\langle , \rangle|_{\mathbf{Q}} : \mathbf{Q} \times \mathbf{Q} \to \mathbb{R},\ \langle q_1, q_2\rangle|_{\mathbf{Q}} = \frac{1}{2}(q_1 \times \bar{q_2} + q_2 \times \bar{q_1}).$$

It is called the quaternionic inner product, [5]. Then the norm of q is

$$N(q) = \sqrt{\langle q, q\rangle|_{\mathbf{Q}}} = \sqrt{q \times \bar{q}},$$

A spatial quaternion set define that $\mathbf{Q}_H = \{q \in \mathbf{Q} | q + \bar{q} = 0\}$, [2]. Let $I = [0, 1]$ be an interval in the real line $\mathbf{R}$ and $s \in I$ be the are-length parameter along the smooth curve, [7]

$$\gamma : [0, 1] \to \mathbf{Q}_H,\ \gamma(s) = \sum_{i=1}^{3} \gamma_i(s)e_i,\ \ (1 \le i \le 3). \tag{1}$$

The tangent vector $\gamma'(s) = t(s)$ has unit length N(t(s))=1 for alls, [2]. Let γ be a differentiable spatial quaternions curve with arc-length parameter s and $\{t(s), n_1(s), n_2(s)\}$ be the Frenet frame of γ at the point $\gamma(s)$, [6],

$$t(s) = \gamma'(s),\quad n_1(s) = \frac{\gamma''(s)}{N(\gamma''(s))},\quad n_2(s) = t(s) \times n_1(s), \tag{2}$$

Let $\{t(s), n_1(s), n_2(s)\}$ be the Frenet frame of $\gamma(s)$. Then Frenet formulae, curvature and the torsion are given by [6]

$$\begin{aligned}
t'(s) &= k(s)n_1(s)\,,\\
n_1{}'(s) &= -k(s)t(s) + r(s)n_2(s)\,,\\
n_2{}'(s) &= -r(s)n_1(s)
\end{aligned} \tag{3}$$

where $t(s)$, $n_1(s)$ and $n_2(s)$ are the unit tangent, the unit principal normal and the unit binormal vector of a quaternionic curve, respectively, [2, 8].

Let $\{k(s), r(s)\}$ be the curvatures of $\gamma(s)$. Then curvature and the torsion are given by[6]

$$\begin{aligned}
k_{\beta_1} &= \frac{N(\beta' \times \beta'')}{N(\beta')^3}\\
r_{\beta_1} &= \frac{\langle \beta' \times \beta'', \beta'''\rangle|_Q}{\left(N(\beta' \times \beta'')\right)^2}.
\end{aligned} \tag{4}$$

Definition 2.1. *Let $\alpha : I \to \mathbf{Q}_H$ unit speed and $\alpha^* : I \to \mathbf{Q}_H$ differentiable two spatial quaternionic curves. If the principal normal vector n_1 of the curve α is linearly dependent on the principal vector n_1^* of the curve α^*, then the pair (α, α^*) is defined to be quaternionic Bertrand curves pair, [7].*

If the curve α^* is Bertrand partner of α and n_1 principal vector of α, then we may write that

$$\alpha^*(s) = \alpha(s) + \lambda n_1(s)\,, \quad \lambda = constant. \tag{5}$$

Theorem 2.2. *Let (α, α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. The relations between the Frenet frames $\{t, n_1, n_2\}$ and $\{t^*, n_1^*, n_2^*\}$ are as follows*

$$\begin{aligned} t^*(s) &= \cos\theta t + \sin\theta n_2, \\ {n_1}^*(s) &= n_1, \\ {n_2}^*(s) &= -\sin\theta t + \cos\theta n_2 \end{aligned} \tag{6}$$

where $\angle(t, t^*) = \theta$, [7].

Theorem 2.3. *Let (α, α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. For the curvatures and the torsions of the Bertrand curves pair (α, α^*) we have*

$$\begin{aligned} k^*(s)\frac{ds^*}{ds} &= \cos\theta k - \sin\theta r\,, \\ r^*(s)\frac{ds^*}{ds} &= \sin\theta k + \cos\theta r, \ [7]. \end{aligned} \tag{7}$$

3. Smarandache Curves of Spatial Quaternionic Bertrand Curve according to Frenet Frame

Frenet vectors of a curve are taken as position vector and a regular curve is defined with this vector. This curve is called as Smarandache curve, [15]. In this study, (α^*, α) will be defined as quaternionic Bertrand curve pair. Curve α^* will be taken as main curve and the other curve α will be taken as Bertrand partner curve of curve α^*. Frenet vectors of α^* curve taken from the curve pair will be taken as position vector. Smarandache curves' Frenet apparatus defined by these position vectors will be calculated. The resulting Frenet apparatus will be expressed Frenet aparatus denominated belonging to α^* curve by using connecting equation between Bertrand curve pair Frenet apparatus.

Definition 3.1. *Let (α, α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. If Frenet frame of curve α^* is shown with $\{t^*, {n_1}^*, {n_2}^*\}$,*

$$\beta_1(s) = \frac{1}{\sqrt{2}}(t^* + {n_1}^*) \tag{8}$$

regular curve drawn by vectors t^* and ${n_1}^*$ is called spatial quaternionic Smarandache curve β_1.

Theorem 3.2. *Frenet vectors of Smarandache curve β_1 are given as follows;*

$$\begin{aligned} t_{\beta_1(s)} &= \frac{-k^*t^* + k^*n_1^* + r^*n_2^*}{\sqrt{2k^{*2} + r^{*2}}}\,, \quad n_{1\beta_1}(s) = \frac{\omega_1 t^* + \phi_1 n_1^* + \sigma_1 n_2^*}{\sqrt{{\omega_1}^2 + {\phi_1}^2 + {\sigma_1}^2}}\,, \\ n_{2\beta_1}(s) &= \frac{(k^*\sigma_1 - r^*\phi_1)t^* + (k^*\sigma_1 + r^*\omega_1)n_1^* + (-k^*\phi_1 - k^*\omega_1)n_2^*}{\sqrt{({\omega_1}^2 + {\phi_1}^2 + \sigma_1^2)(2k^{*2} + r^{*2})}}. \end{aligned} \tag{9}$$

Herein, the coefficients are

$$\begin{aligned} \omega_1 &= -k^{*2}(2k^{*2} + r^{*2}) - r^*(r^*k^{*'} - k^*r^{*'}), \\ \phi_1 &= -k^{*2}(2k^{*2} + 3r^{*2}) - r^*(r^{*3} - r^*k^{*'} + k^*r^{*'}), \\ \sigma_1 &= k^*r^*(2k^{*2} + r^{*2}) - 2k^*(r^*k^{*'} - k^*r^{*'}). \end{aligned} \tag{10}$$

Proof. If derivative according to s_{β_1} arc parameter of curve $\beta_1(s)$ is taken, $t_{\beta_1}(s)$ and $t'_{\beta_1}(s)$ are given, respectively

$$t_{\beta_1(s)} = \frac{-k^*t^* + k^*n_1^* + r^*n_2^*}{\sqrt{2k^{*2} + r^{*2}}}, \quad t'_{\beta_1}(s) = \frac{\sqrt{2}\left(\omega_1 t^* + \phi_1 n_1^* + \sigma_1 n_2^*\right)}{\left(2k^{*2} + r^{*2}\right)^2}. \tag{11}$$

Herein, the coefficients are as seen in (10). From equation (2), principal vector $n_{1\beta_1}$ and binormal vector $n_{2\beta_1}$ are found as in (9). □

Theorem 3.3. *Curvature and torsion belonging to Smarandache curve β_1 are, respectively*

$$k_{\beta_1} = \frac{\sqrt{2({\omega_1}^2 + {\phi_1}^2 + {\sigma_1}^2)}}{\left(2k^{*2} + r^{*2}\right)^2}, r_{\beta_1} = \frac{\sqrt{2}\left(x_1\eta_1 + y_1\theta_1 + z_1\rho_1\right)}{{x_1}^2 + {y_1}^2 + {z_1}^2} \tag{12}$$

where coefficients are

$$\begin{aligned} \eta_1 &= k^{*3} + k^*(r^{*2} - 3k^{*'}) - k^{*''}, \quad \theta_1 = -k^{*3} - k^*(r^{*2} + 3k^{*'}) - 3r^*r^{*'} + k^{*''}, \\ \rho_1 &= -k^{*2}r^* - r^{*3} + 2r^*k^{*'} + k^*r^{*'} + r^{*''}, \\ x_1 &= r^*(2k^{*2} + r^{*2}) + k^*r^{*'} - k^{*'}r^*, \quad y_1 = k^{*'}r^* - k^*r^{*'}, \quad z_1 = 2k^{*3} + k^*r^{*2}. \end{aligned} \tag{13}$$

Proof. First, second and third derivatives of curve β_1 are, respectively

$${\beta_1}' = \frac{-k^*t^* + k^*{n_1}^* + r^*{n_2}^*}{\sqrt{2}}$$

$${\beta_1}'' = \frac{-(k^{*2} + k^{*'})t^* + (k^{*'} - k^{*2} - r^{*2}){n_1}^* + (k^*r^* + r^{*'}){n_2}^*}{\sqrt{2}},$$

$${\beta_1}''' = \frac{\eta_1 t^* + \theta_1 {n_1}^* + \rho_1 {n_2}^*}{\sqrt{2}}$$

where the coefficients are as seen in (13). From (4) equation, curvatures are found as in (12). □

Corollary 3.4. *Let (α, α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of Frenet vectors of Smarandache curve β_1 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$\begin{aligned} t_{\beta_1}(s) &= \frac{-kt + (\cos\theta - \sin\theta)n_1 + rn_2}{\sqrt{k^2 + r^2 + (\cos\theta k - \sin\theta r)^2}}, \quad n_{1\beta_1}(s) = \frac{\bar{\omega}_1 t + \bar{\phi}_1 n_1 + \bar{\sigma}_1 n_2}{\sqrt{\bar{\omega}_1^2 + \bar{\phi}_1^2 + \bar{\sigma}_1^2}}, \\ n_{2\beta_1}(s) &= \frac{((k\cos\theta - r\sin\theta)\bar{\sigma}_1 - r\bar{\phi}_1)t + (k\bar{\sigma}_1 + r\bar{\omega}_1)n_1}{\sqrt{\left(k^2 + r^2 + (\cos\theta k - \sin\theta)^2\right)(\bar{\omega}_1^2 + \bar{\phi}_1^2 + \bar{\sigma}_1^2)}} \\ &\quad - \frac{(k\bar{\phi}_1 + (k\cos\theta - r\sin\theta)\bar{\omega}_1)n_2}{\sqrt{\left(k^2 + r^2 + (\cos\theta k - \sin\theta)^2\right)(\bar{\omega}_1^2 + \bar{\phi}_1^2 + \bar{\sigma}_1^2)}}. \end{aligned} \tag{14}$$

Herein, coefficients are

$$
\begin{aligned}
\bar{\omega}_1 &= (-k^{'} - k^2\cos\theta + kr\sin\theta)(k^2 + r^2 + (k\cos\theta - r\sin\theta)^2) \\
&\quad + k(k^2 + r^2 + (k\cos\theta - r\sin\theta)^{'}, \\
\bar{\phi}_1 &= (-k^2 - r^2 + k'\cos\theta - r'\sin\theta)(k^2 + r^2 + (k\cos\theta - r\sin\theta)^2) \\
&\quad - (k\cos\theta - r\sin\theta)(k^2 + r^2 + (k\cos\theta - r\sin\theta)^2)^{'}, \\
\bar{\sigma}_1 &= (kr\cos\theta - r^2\sin\theta + r')(k^2 + r^2 + (k\cos\theta - r\sin\theta)^2) \\
&\quad - r(k^2 + r^2 + (k\cos\theta - r\sin\theta)^2)^{'}.
\end{aligned}
$$

Proof. If expression (6) instead of t^* and ${n_1}^*$ in curve β_1 is written, we have

$$\beta_1(s) = \frac{1}{\sqrt{2}}\Big(\cos\theta t(s) + n_1(s) + \sin\theta n_2(s)\Big).$$

If equations (6) and (7) into equation (9) and (25) are written, the proof is completed. □

Corollary 3.5. *Let (α, α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of curvatures of Smarandache curve β_1 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$k_{\beta_1} = \frac{\sqrt{{\bar{\omega}_1}^2 + \bar{\phi}_1^2 + \bar{\sigma}_1^2}}{\Big(k^2 + r^2 + (\cos\theta k - \sin\theta)^2\Big)^{\frac{3}{2}}}, \quad r_{\beta_1} = \sqrt{2}\frac{\bar{x}_1\bar{\eta}_1 + \bar{y}_1\bar{\theta}_1 + \bar{z}_1\bar{\rho}_1}{\bar{x}_1^2 + \bar{y}_1^2 + \bar{z}_1^2}. \tag{15}$$

Herein, coefficients are

$$
\begin{aligned}
\bar{\eta}_1 &= (-k' - k^2\cos\theta + kr\sin\theta)^{'} - k(-k^2 - r^2 + (k\cos\theta - r\sin\theta)^{'}), \\
\bar{\theta}_1 &= k(-k' - k^2\cos\theta + kr\sin\theta) + (-k^2 - r^2 + (k\cos\theta - r\sin\theta)^{'})^{'} \\
&\quad - r(kr\cos\theta - r^2\sin\theta + r^{'}), \\
\bar{\rho}_1 &= r(-k^2 - r^2 + (k\cos\theta - r\sin\theta)^{'}) + (kr\cos\theta - r^2\sin\theta + r^{'})^{'}, \\
\bar{x}_1 &= (k\cos\theta - r\sin\theta)(kr\cos\theta - r^2\sin\theta + r^{'}) - r(-k^2 - r^2 + (k\cos\theta - r\sin\theta)^{'}), \\
\bar{y}_1 &= k(kr\cos\theta - r^2\sin\theta + r^{'}) + r(-k^{'} - k^2\cos\theta + kr\sin\theta), \\
\bar{z}_1 &= -\Big(k(-k^2 - r^2 + (k\cos\theta - r\sin\theta)^{'}) + k(k\cos\theta - r\sin\theta) \\
&\quad .(-k^{'} - k^2\cos\theta + kr\sin\theta)\Big).
\end{aligned}
$$

Proof. If equations (6) and (7) into equation (12) and (13) are written, the proof is completed. □

Definition 3.6. *Let (α, α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. If Frenet frame of curve α^* is shown with $\{t^*, {n_1}^*, {n_2}^*\}$,*

$$\beta_2(s) = \frac{(n_1^* + {n_2}^*)}{\sqrt{2}} \tag{16}$$

regular curve drawn by vectors ${n_1}^*$ and ${n_2}^*$ is called spatial quaternionic Smarandache curve β_2.

Theorem 3.7. *The Frenet vectors of Smarandache curve β_2 are given as follows:*

$$
\begin{aligned}
t_{\beta_2}(s) &= \frac{-kt^* - rn_1^* + rn_2^*)}{\sqrt{2r_2^* + k_2^*}}, \quad n_{1\beta_2}(s) = \frac{\omega_2 t^* + \phi_2 n_1^* + \sigma_2 n_2^*}{\sqrt{\omega_2{}^2 + \phi_2{}^2 + \sigma_2{}^2}}, \\
n_{2\beta_2}(s) &= \frac{-r^*(\sigma_2 + \phi_2)t^* + (r^*\omega_2 + k^*\sigma_2)n_1^* + (-k^*\phi_2 + r^*\omega_2)n_2^*}{\sqrt{(\omega_2{}^2 + \phi_2{}^2 + \sigma_2^2)(2r^{*2} + k^{*2})}}.
\end{aligned}
\tag{17}
$$

Herein, the coefficients are

$$
\begin{aligned}
\omega_2 &= 2r^{*2}(-k^{*'} + r^*r^*) + k^*r^*(k^{*2} + 2r^{*'}), \\
\phi_2 &= k^*(-k^{*3} - r^{*'}k^* + r^*k^{*'}) - r^{*2}(3k^{*2} + 2r^{*2}), \\
\sigma_2 &= k^{*2}(r^{*'} - r^{*2}) - r^*(2r^{*3} + k^*k^{*'}).
\end{aligned}
\tag{18}
$$

Proof. If derivative is taken according to s_{β_2} arc parameter of curve $\beta_2(s)$, $t_{\beta_2}(s)$ and $t'_{\beta_2}(s)$ are given, respectively

$$
t_{\beta_2(s)} = \frac{-k^*t^* - r^*n_1^* + r^*n_2^*}{\sqrt{2k^{*2} + r^{*2}}}, \quad t'_{\beta_1}(s) = \frac{\sqrt{2}\left(\omega_2 t^* + \phi_2 n_1^* + \sigma_2 n_2^*\right)}{\left(2k^{*2} + r^{*2}\right)^2}.
$$

Herein, the coefficients are as seen in (18). From equation (2), principal vector $n_{1\beta_2}$ and binormal vector $n_{2\beta_2}$ are found as in (17). □

Theorem 3.8. *Curvature and torsion belonging to Smarandache curve β_2 are, respectively*

$$
k_{\beta_2} = \sqrt{2}\frac{\sqrt{\omega_2{}^2 + \phi_2{}^2 + \sigma_2{}^2}}{\left(k^{*2} + 2r^{*2}\right)^2}, \quad r_{\beta_2} = \sqrt{2}\frac{x_2\eta_2 + y_2\theta_2 + z_2\rho_2}{x_2{}^2 + y_2{}^2 + z_2{}^2}
\tag{19}
$$

where coefficients are

$$
\begin{aligned}
\eta_2 &= -r^{*3}k^* + k^{*3} + k^{*'}r^* + 2k^*r^{*'} - k^{*''}, \\
\theta_2 &= r^{*3} - r^*k^{*2} - 3k^*k^{*'} + 3r^{*2}r^{*'} - r^{*'}, \\
\rho_2 &= r^{*3} + r^*k^{*2} - 3r^*r^{*'} - r^*r^{*''}, \\
x_2 &= r^*(2r^{*2} + k^{*2}), \quad y_2 = k^*r^{*'} - r^*k^{*'}, \quad z_2 = k^*(k^{*2} + 2r^{*2} + r^{*'}) - r^*k^{*'}.
\end{aligned}
\tag{20}
$$

Proof. First, second and third derivatives of curve β_2 are, respectively

$$
\beta_2{}' = \frac{-k^*t^* - r^*n_1{}^* + r^*n_2{}^*}{\sqrt{2}}
$$

$$
\beta_2'' = \frac{(-k^* + r^*k^*)t^* - (k^{*2} - k^{*2} + r^{*2} + r^{*'})n_1{}^* + (r^{*'} - r^{*2})n_2{}^*}{\sqrt{2}},
$$

$$
\beta_2''' = \frac{\eta_2 t^* + \theta_2 n_1{}^* + \rho_2 n_2{}^*}{\sqrt{2}}
$$

where the coefficients are as seen in (20). From (4) equation, curvatures are found as in (19). □

Corollary 3.9. *Let (α,α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of Frenet vectors of Smarandache curve β_2 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$\begin{aligned}
t_{\beta_2}(s) &= \frac{-kt-(\sin\theta+\cos\theta r)n_1+rn_2}{\sqrt{k^2+r^2+(\sin\theta+\cos\theta r)^2}}, \quad n_{1\beta_2}(s)=\frac{\bar{\omega}_1 t+\bar{\phi}_1 n_1+\bar{\sigma}_1 n_2}{\sqrt{\bar{\omega}_1^2+\bar{\phi}_1^2+\bar{\sigma}_1^2}},\\
n_{2\beta_2}(s) &= \frac{-(\bar{\sigma}_2(k\sin\theta+r\cos\theta)+r\bar{\phi}_2)t+(k\bar{\sigma}_2+r\bar{\omega}_2)n_1}{\sqrt{\left(k^2+r^2+(k\sin\theta+r\cos\theta)^2\right)(\bar{\omega}_2^2+\bar{\phi}_2^2+\bar{\sigma}_2^2)}}\\
&\quad+\frac{(-k\bar{\phi}_2+\left(k\sin\theta+r\cos\theta)\bar{\omega}_2\right)n_2}{\sqrt{\left(k^2+r^2+(k\sin\theta+r\cos\theta)^2\right)(\bar{\omega}_2^2+\bar{\phi}_2^2+\bar{\sigma}_2^2)}}.
\end{aligned} \tag{21}$$

Herein, the coefficients are

$$\begin{aligned}
\bar{\omega}_2 &= (-k'+k(k\sin\theta+r\cos\theta))(k^2+r^2+(k\sin\theta+r\cos\theta)^2)\\
&\quad+k(k^2+r^2+(k\sin\theta+r\cos\theta)^2)',\\
\bar{\phi}_2 &= (-k^2-r^2-(k'\sin\theta-r'\cos\theta))(k^2+r^2+(k\sin\theta+r\cos\theta)^2)\\
&\quad+(k\sin\theta+r\cos\theta)(k^2+r^2+(k\sin\theta+r\cos\theta)^2)',\\
\bar{\sigma}_2 &= (-r(k\sin\theta+r\cos\theta)+r')(k^2+r^2+(k\sin\theta+r\cos\theta)^2)\\
&\quad-r(k^2+r^2+(k\sin\theta+r\cos\theta)^2)'.
\end{aligned}$$

Proof. If expression (6) instead of $n_1{}^*$ and $n_2{}^*$ in curve β_2 is written, we have

$$\beta_2(s)=\frac{1}{\sqrt{2}}\Big(-\sin\theta t+n_1+\cos\theta n_2\Big).$$

If equations (6) and (7) into equation (17) and (18) are written, the proof is completed. □

Corollary 3.10. *Let (α,α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of curvatures of Smarandache curve β_2 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$k_{\beta_2}=\sqrt{2}\frac{\sqrt{\bar{\omega}_2{}^2+\bar{\phi}_2^2+\bar{\sigma}_2^2}}{\left(k^2+r^2+\left(\sin\theta k+\cos\theta r\right)^2\right)^{\frac{3}{2}}}, \quad r_{\beta_2}=\sqrt{2}\frac{\bar{x}_2\bar{\eta}_2+\bar{y}_2\bar{\theta}_2+\bar{z}_2\bar{\rho}_2}{\bar{x}_2^2+\bar{y}_2^2+\bar{z}_2^2}. \tag{22}$$

Herein, the coefficients are

$$\begin{aligned}
\bar{\eta}_2 &= \Big(-k'+k(k\sin\theta+r\cos\theta)\Big)'+k\Big(k^2+r^2-(k'\sin\theta+r'\cos\theta)\Big),\\
\bar{\theta}_2 &= -kk'+(k^2+r^2)(k\sin\theta+r\cos\theta)-(2kk'+2rr'+(k''\sin\theta+r''\cos\theta))-rr',\\
\bar{\rho}_2 &= r\Big(-k'+k(k\sin\theta+r\cos\theta)\Big)+\Big(-r(k\sin\theta+r\cos\theta)+r'\Big),\\
\bar{x}_2 &= (k\sin\theta+r\cos\theta)(r(\sin\theta+r\cos\theta)-r')+r\Big(k^2+r^2-(k'\sin\theta+r'\cos\theta)\Big),\\
\bar{y}_2 &= k(-r(k\sin\theta+r\cos\theta)+r')+r\Big(-k'+k(k\sin\theta+r\cos\theta)\Big),\\
\bar{z}_2 &= k\Big(k^2+r^2-(k'\sin\theta+r'\cos\theta\Big)+(k\sin\theta+r\cos\theta)\Big(-k'+k(k\sin\theta+r\cos\theta)\Big).
\end{aligned}$$

Proof. If equations (6) and (7) into equation (19) and (20) are written, the proof is completed. □

Definition 3.11. *Let (α, α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. If Frenet frame of α^* curve is shown with $\{t^*, {n_1}^*, {n_2}^*\}$,*

$$\beta_3(s) = \frac{(t^* + n_2^*)}{\sqrt{2}} \tag{23}$$

regular curve drawn by vectors t^* and n_2^* is called spatial quaternionic Smarandache curve β_3.

Theorem 3.12. *Frenet vectors of Smarandache curve β_3 are given as follows:*

$$t_{\beta_3}(s) = {n_1}^*, \quad n_{1\beta_3}(s) = \frac{-k^* t^* + r^* {n_2}^*}{k^{*2} + r^{*2}}, \quad n_{2\beta_3}(s) = \frac{r^* t^* + k^* {n_2}^*}{\sqrt{k^{*2} + r^{*2}}}. \tag{24}$$

Proof. If derivative is taken according to s_{β_3} arc parameter of curve $\beta_3(s)$, $t_{\beta_3}(s)$ and $t'_{\beta_3}(s)$ are given, respectively

$$t_{\beta_3(s)} = \frac{(k^* - r^*)n_1^*}{\sqrt{2k^{*2} + r^{*2}}}, \quad {t_{\beta_3}}'(s) = \frac{\sqrt{2}\left(-k^* t^* + r^* n_2^*\right)}{k^* - r^*}.$$

From equation (2), principal vector $n_{1\beta_3}$ and binormal vector $n_{2\beta_3}$ are found as in (24). □

Theorem 3.13. *Curvature and torsion belonging to Smarandache curve β_3 are, respectively*

$$k_{\beta_3} = \frac{\sqrt{2(k^{*2} + r^{*2})}}{k^* - r^*}, \quad r_{\beta_3} = \sqrt{2}\frac{x_3\eta_3 + z_3\varphi_3}{{\eta_3}^2 + {\varphi_3}^2} \tag{25}$$

where coefficients are

$$\eta_3 = -3k^* k^{*'} + 2k^* r^{*'} + k^{*'} r^*, \quad \theta_3 = -k^{*3} + r^* k^{*2} - k^* r^{*2} + r^{*3} + k^{*''} - r^{*''},$$

$$\varphi_3 = k^* r^{*'} + 2k^{*'} r^* - 3r^* r^{*'}, \quad x_3 = r^*(k^* - r^*)^2, \quad z_3 = k^*(k^* - r^*)^2. \tag{26}$$

Proof. First, second and third derivatives of curve β_3 are, respectively

$${\beta_3}' = \frac{(k^* - r^*){n_1}^*}{\sqrt{2}}$$

$${\beta_3}'' = \frac{(-k^{*2} + k^* r^*)t^* + (k^{*'} - r^{*'}){n_1}^* + (k^* r^* - r^{*2}){n_2}^*}{\sqrt{2}},$$

$${\beta_3}''' = \frac{\eta_3 t^* + \theta_3 {n_1}^* + \rho_3 {n_2}^*}{\sqrt{2}}.$$

From (4) equation, curvatures are found as in (25). □

Corollary 3.14. *Let (α, α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of Frenet vectors of Smarandache curve β_3 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$t_{\beta_3}(s) = n_1, \quad n_{1\beta_3}(s) = \frac{\bar{\omega}_3 t + \bar{\phi}_3 n_1 + \bar{\sigma}_3 n_2}{\sqrt{{\bar{\omega}_3}^2 + {\bar{\phi}_3}^2 + {\bar{\sigma}_3}^2}},$$

$$n_{2\beta_3}(s) = \frac{(\bar{\omega}_3 t - \bar{\sigma}_3 n_2)[k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta)]}{\sqrt{(k^2 - r^2 - (k^2 - r^2)\sin 2\theta)({\bar{\omega}_3}^2 + {\bar{\phi}_3}^2 + {\bar{\sigma}_3}^2)}}. \tag{27}$$

Herein, coefficients are

$$\begin{aligned}
\bar{\omega_3} &= -k\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)\sqrt{k^2-r^2-(k^2-r^2)^2\sin 2\theta},\\
\bar{\phi_3} &= \Big(k'(\cos\theta-\sin\theta)-r'(\cos\theta+\sin\theta)\Big)\sqrt{k^2-r^2-(k^2-r^2)^2\sin 2\theta}\\
&\quad -\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)\Big(\sqrt{k^2-r^2-(k^2-r^2)^2\sin 2\theta}\Big)',\\
\bar{\sigma_3} &= \Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)\sqrt{k^2-r^2-(k^2-r^2)^2\sin 2\theta}.
\end{aligned}$$

Proof. If expression (6) instead of t^* and ${n_2}^*$ in curve β_3 is written, we have

$$\beta_3(s)=\frac{1}{\sqrt{2}}\Big((\cos\theta-\sin\theta)t+(\sin\theta+\cos\theta)n_2\Big).$$

If equations (6) and (7) into equation (21) and (22) are written, the proof is completed. □

Corollary 3.15. *Let (α,α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of curvatures of Smarandache curve β_3 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$k_{\beta_3}=\sqrt{2}\frac{\sqrt{\bar{\omega_3}^2+\bar{\phi_3}^2+\bar{\sigma_3}^2}}{k^2-r^2-(k^2-r^2)\sin 2\theta},\quad r_{\beta_3}=\sqrt{2}\frac{\bar{x_3}\bar{\eta_3}+\bar{z_3}\bar{\varphi_3}}{\bar{\eta_3}^2+\bar{\varphi_3}^2}.$$

Herein, the coefficients are

$$\begin{aligned}
\bar{\eta_3} &= -k'\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)-2k\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)',\\
\bar{\theta_3} &= -k^2\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)-r^2\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)\\
&\quad +\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)'',\\
\bar{\varphi_3} &= r'\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)+2r\Big(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)\Big)',\\
\bar{x_3} &= k(k(\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta)))^2,\quad \bar{z_3}=r((\cos\theta-\sin\theta)-r(\cos\theta+\sin\theta))^2.
\end{aligned}$$

Proof. If equations (6) and (7) into equation (25) and (26) are written, the proof is completed. □

Definition 3.16. *Let (α,α^*) be a quaternionic Bertrand pair curves in $\mathbf{Q}_H$. If Frenet frame of α^* curve is shown with $\{t^*,{n_1}^*,{n_2}^*\}$,*

$$\beta_4(s)=\frac{(t^*+n_1^*+{n_2}^*)}{\sqrt{2}} \tag{28}$$

regular curve drawn by vectors t^*, ${n_1}^*$ and ${n_2}^*$ is called spatial quaternionic Smarandache curve β_4.

Theorem 3.17. *Frenet vectors of Smarandache curve β_4 are given as follows:*

$$\begin{aligned}
t_{\beta_4}(s) &= \frac{k^*t^*+(k^*-r^*){n_1}^*+r^*{n_2}^*}{\sqrt{2(k^*+r^*-k^*r^*)}},\quad n_{1\beta_4}(s)=\frac{\omega_4t^*+\phi_4{n_1}^*+\sigma_4{n_2}^*}{\sqrt{{\omega_4}^2+{\phi_4}^2+{\sigma_4}^2}},\\
n_{2\beta_4}(s) &= \frac{((k^*-r^*)\sigma_4-r^*\phi_4)t^*+(r^*\omega_4+k^*\sigma_4){n_1}^*}{\sqrt{(2k^{*2}+2r^{*2}-2k^*r^*)({\omega_4}^2+{\phi_4}^2+{\sigma_4}^2)}}\\
&\quad -\frac{(k^*\phi_4+(k^*-r^*)\omega_4){n_2}^*}{\sqrt{(2k^{*2}+2r^{*2}-2k^*r^*)({\omega_4}^2+{\phi_4}^2+{\sigma_4}^2)}}.
\end{aligned} \tag{29}$$

Herein, the coefficients are

$$\begin{aligned}
\omega_4 &= k^{*2}(-2k^{*2} - 4r^{*2} + 4r^*k^* - k^{*2}r^{*'}) + k^*r^*(k^{*'} + 2r^{*2} + 2r^{*'}) - 2k^{*'}r^{*2},\\
\phi_4 &= k^{*2}(-2k^{*2} - 4r^{*2} + 2k^*r^* - r^{*'}) + r^{*2}(-2r^{*2} + 2k^*r^* + k^{*'}) + k^*r^*(k^{*'} - r^{*'}),\\
\sigma_4 &= 2k^{*2}(k^*r^* - 2r^{*2} + r^{*'}) + r^{*2}(4k^*r^* - 2r^{*2} + k^{*'}) - k^*r^*(r^{*'} + 2k^{*'}).
\end{aligned}$$

Proof. If derivative is taken according to s_{β_4} arc parameter of curve $\beta_4(s)$, $t_{\beta_4}(s)$ and $t'_{\beta_4}(s)$ are given, respectively

$$\begin{aligned}
t_{\beta_4(s)} &= \frac{-k^*t^* + (k^* - r^*)n_1^* + r^*n_2^*}{2(\sqrt{k^{*2} + r^{*2} - k^*r^*})^2}\\
t'_{\beta_4}(s) &= \frac{\sqrt{3}\left(\omega_4 t^* + \phi_4 n_1^* + \sigma_4 n_2^*\right)}{4\left(2k^{*2} + r^{*2}\right)^2}.
\end{aligned}$$

Herein, the coefficients are as seen in (30). From equation (2), principal vector $n_{1\beta_4}$ and binormal vector $n_{2\beta_4}$ are found as in (29). □

Theorem 3.18. *Curvature and torsion belonging to Smarandache curve β_4 are, respectively*

$$k_{\beta_4} = \frac{\sqrt{3}}{4}\frac{\sqrt{\omega_4^2 + \phi_4^2 + \sigma_4^2}}{(k^{*2} + r^{*2} - k^*r^*)^2}, \quad r_{\beta_4} = \frac{\sqrt{3}\left[\eta_4 x_4 + \theta_4 y_4 + \rho_4 z_4\right]}{x_4{}^2 + y_4{}^2 + z_4{}^2} \tag{30}$$

where coefficients are

$$\begin{aligned}
\eta_4 &= k^{*'}r^* - k^{*''} - 3k^*k^{*'} + 2k^*r^{*'} + k^{*3} + k^*r^{*2},\\
\theta_4 &= r^{*3} - k^{*3} - 3(k^*k^{*'} + r^*r^{*'}) - (-k^{*''} + r^{*''}) + k^*r^*(k^* - r^*),\\
\rho_4 &= r^{*''} - k^{*2}r^* - 3r^*r^{*'} - r^{*3} + 2r^*k^{*'} + k^*r^{*'},\\
x_4 &= 2k^*r^*(k^* - r^*) + k^*r^{*'} - r^*k^{*'} + 2r^{*3},\\
y_4 &= k^*r^{*'} - r^*k^{*'}, \quad z_4 = 2k^{*3} + k^*r^{*'} + 2k^*r^{*2} - 2k^{*2}r^* - k^{*'}r^*.
\end{aligned} \tag{31}$$

Proof. First, second and third derivatives of curve β_4 are, respectively

$$\beta_4{}' = \frac{-k^*t^* + (k^* - r^*)n_1{}^* + r^*n_2{}^*}{\sqrt{3}}$$

$$\beta_4'' = \frac{(-k^{*'} - k^{*2} + k^*r^*)t^* - (k^{*2} - k^{*'} + r^{*'} + r^{*2})n_1{}^* + (k^*r^* - r^{*2} + r^{*'})n_2{}^*}{\sqrt{3}},$$

$$\beta_4''' = \frac{\eta_4 t^* + \theta_4 n_1{}^* + \rho_4 n_2{}^*}{\sqrt{2}}$$

where the coefficients are as seen in (31). From (4) equation, curvatures are found as in (30). □

Corollary 3.19. *Let (α, α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of Frenet vectors of*

Smarandache curve β_4 in terms of Frenet apparatus of Bertrand partner curve are as follows:

$$
\begin{aligned}
t_{\beta_4}(s) &= \frac{1}{\sqrt{2}} \frac{-kt + \big(k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta)\big) + rn_2}{\sqrt{(-2\cos\theta kr - \cos\theta\sin\theta k^2 + \cos\theta\sin\theta r^2 + k^2 + kr + r^2)}},\\
n_{1\beta_4}(s) &= \frac{\bar{\omega}_4 t + \bar{\phi}_4 n_1 + \bar{\sigma}_4 n_2}{\sqrt{\bar{\omega}_4{}^2 + \bar{\phi}_4{}^2 + \bar{\sigma}_4{}^2}},\\
n_{2\beta_4}(s) &= \frac{\big((k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta))\bar{\sigma}_4 - r\bar{\phi}_4\big)t}{\sqrt{k^2 + r^2 + [k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta)]^2(\bar{\omega}_4{}^2 + \bar{\phi}_4{}^2 + \bar{\sigma}_4{}^2)}}\\
&\quad + \frac{(k\bar{\sigma}_4 + r\bar{\omega}_4)n_1}{\sqrt{k^2 + r^2 + [k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta)]^2(\bar{\omega}_4{}^2 + \bar{\phi}_4{}^2 + \bar{\sigma}_4{}^2)}}\\
&\quad + \frac{-(k\bar{\phi}_4 + (k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta))\bar{\omega}_4)n_2}{\sqrt{k^2 + r^2 + [k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta)]^2(\bar{\omega}_4{}^2 + \bar{\phi}_4{}^2 + \bar{\sigma}_4{}^2)}}.
\end{aligned}
\tag{32}
$$

Herein, the coefficients are

$$
\begin{aligned}
\bar{\omega}_4 &= \big(-k' - k(k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta))\big).\big(k^2 + r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta))^2\big)\\
&\quad + k\Big((k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta))^2\Big)',\\
\bar{\phi}_4 &= \big(-k^2 - r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta))'\big)\\
&\quad .\big(k^2 + r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta))^2\big)\\
&\quad - (k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta)).\big(k^2 + r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta))^2\big)',\\
\bar{\sigma}_4 &= (r(k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta)) + r').\big(k^2 + r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta))^2\big)\\
&\quad - r\big(k^2 + r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta))^2\big)'.
\end{aligned}
$$

Proof. If expression (6) instead of t^*, $n_1{}^*$ and $n_2{}^*$ in curve β_4 is written, we have

$$\beta_4 = \frac{1}{\sqrt{3}}\big((\cos\theta - \sin\theta)t + n_1 + (\sin\theta + \cos\theta)n_2\big).$$

If equations (6) and (7) into (29) and (30) equations are written, the proof is completed. □

Corollary 3.20. *Let (α, α^*) be a spatial quaternionic Bertrand curve pair in $\mathbf{Q}_H$. The expressions of curvatures of Smarandache curve β_4 in terms of Frenet apparatus of Bertrand partner curve are as follows:*

$$
\begin{aligned}
k_{\beta_4} &= \frac{\sqrt{3}\sqrt{\bar{\omega}_4{}^2 + \bar{\phi}_4{}^2 + \bar{\sigma}_4{}^2}}{\big((k(\cos\theta - \sin\theta) - r(\cos\theta + \sin\theta))^2 + k^2 + r^2\big)^{\frac{3}{2}}},\\
r_{\beta_4} &= \sqrt{2}\frac{\bar{x}_4\bar{\eta}_4 + \bar{y}_4\bar{\Theta}_4 + \bar{z}_4\bar{\rho}_4}{\bar{x}_4^2 + \bar{y}_4^2 + \bar{z}_4^2}.
\end{aligned}
\tag{33}
$$

Herein, the coefficients are

$$
\begin{aligned}
\bar{\eta_4} &= \Big(-k' - k(k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))\Big)' \\
&\quad -k\Big(-k^2 - r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))'\Big), \\
\bar{\theta_4} &= k\Big(-k' - k(k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))'\Big) - r\Big((rk - r^2)(\cos\theta - \sin\theta) + r'\Big) \\
&\quad +\Big(-k^2 - r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))'\Big)', \\
\bar{\rho_4} &= r\Big(-k^2 - r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))'\Big) + \Big((rk - r^2)(\cos\theta - \sin\theta) + r'\Big)', \\
\bar{x_4} &= \Big(k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta)\Big).\Big(r + (k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta)) + r'\Big) \\
&\quad -r\Big(-k^2 - r^2 + (k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))'\Big), \\
\bar{y_4} &= k\Big(r(k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta)) + r'\Big) + r\Big(-k' - (k^2 + kr)(\cos\theta - \sin\theta)\Big), \\
\bar{z_4} &= k\Big(-k^2 - r^2 + [k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta)]'\Big) \\
&\quad \Big((k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))\Big).\Big(-k' - k(k(\cos\theta - \sin\theta) - r(\cos\theta - \sin\theta))\Big).
\end{aligned}
$$

Proof. If equations (6) and (7) into equation (30) and (31) are written, the proof is completed. □

Example. Let be spatial quaternionic curve

$$\alpha(s) = \Big(\frac{\sqrt{2}}{2}\cos\big(\frac{\sqrt{5}}{5}s\big) + \frac{\sqrt{2}}{2}\sin\big(\frac{\sqrt{5}}{5}s\big), -\frac{2\sqrt{5}}{5}s, \frac{-\sqrt{2}}{2}\cos(\frac{\sqrt{5}}{5}s) + \frac{\sqrt{2}}{2}\sin(\frac{\sqrt{5}}{5}s)\Big)$$

and if taken as $\lambda = 1$, Bertrand partner belonging to this curve,

$$\alpha^*(s) = \Big(0, \frac{-2\sqrt{5}}{5}s, 0\Big).$$

In terms of definition, we obtain special Smarandache curves β_1, β_2, β_3 and β_4 according to Frenet frame of spatial quaternionic curve, (Figure 1).

$$
\begin{aligned}
\beta_1(s) &= \Big(-\frac{1}{2}\cos\Big(\frac{\sqrt{5}}{5}s\Big) - \frac{1}{2}\sin\Big(\frac{\sqrt{5}}{5}s\Big), -\frac{\sqrt{2}}{2}, \frac{1}{2}\cos\Big(\frac{\sqrt{5}}{5}s\Big) - \frac{1}{2}\sin\Big(\frac{\sqrt{5}}{5}s\Big)\Big), \\
\beta_2(s) &= \Big(-\cos\Big(\frac{\sqrt{5}}{5}s\Big), 0, -\sin\Big(\frac{\sqrt{5}}{5}s\Big)\Big), \\
\beta_3(s) &= \Big(\frac{1}{2}\sin\Big(\frac{\sqrt{5}}{5}s\Big) - \frac{1}{2}\cos\Big(\frac{\sqrt{5}}{5}s\Big), -\frac{\sqrt{2}}{2}, -\frac{1}{2}\cos\Big(\frac{\sqrt{5}}{5}s\Big) - \frac{1}{2}\sin\Big(\frac{\sqrt{5}}{5}s\Big)\Big), \\
\beta_4(s) &= \Big(-\frac{\sqrt{6}}{3}\cos\Big(\frac{\sqrt{5}}{5}s\Big), -\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}\sin\Big(\frac{\sqrt{5}}{5}s\Big)\Big)
\end{aligned}
$$

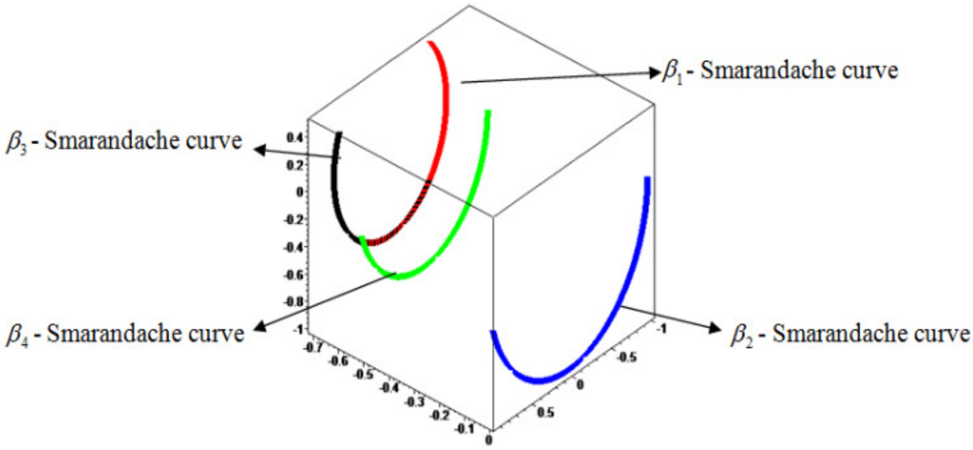

Figure 1: Smarandache Curves of Quaternionic Bertrand Curve

4. Conclusion

In this study, We have calculated the Smarandache curves of the Bertrand curve pairs. To put it simply, we derived curves from a curve according to a method. We found the Frenet frames and curvatures of these curves, which we call Smarandache curves. Finally, we found these results depending on the Frenet frames of the Bertrand curve pair. We saw that we could switch between Frenet frames. It is possible to examine whether these obtained curves are included in special curves.

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Calculation of the differential equations and harmonicity of the involute curve according to unit Darboux vector with a new method

Süleyman Şenyurt [a], Osman Çakır[b]

[a]*Faculty of Arts and Sciences, Department of Mathematics, Ordu University, 52200, Ordu/Turkey*
[b]*Institute of Science, Department of Mathematics, Ordu University, 52200, Ordu/Turkey*

Abstract. In this study we first write the characterizations of involute of a curve by means of the unit Darboux vector of the involute curve. Then we make use of the Frenet formulas obtained by O. Çakır and S. Şenyurt to explain the characterizations of involute of a curve by means of Frenet apparatus of the main curve. Finally we examined the helix as an example.

1. Introduction and Preliminaries

To state a correlation between the invariants of a curve and characterizations of the curve in Euclidean space and non-Euclidean spaces and then to interpret it from the language of geometry has been the focus of interest for many researchers. Some curves are well-known by their explorers such as involute and evolute curves,[2]. Afterwards, many studies have been conducted in Euclidean and non-Euclidean spaces closely related to involute curves, [3, 4]. Later it has been revealed that curves can be classified, [5, 6, 8]. In this paper, we first take a regular curve, that is, a main curve, then write the characterizations of the involute curve by means of Frenet apparatus of the main curve. This work is one of the applications of [1] by which looking from such a point of view that we make the complex calculations more elementary. Eventually we put the example which support our assumption.

Now we may look at the main concepts related to the curve theory. Frenet vector fields can be expressed by means of covariant derivative of these vectors and this relation is known as Frenet formulas, see [9]

$$T' = \vartheta\kappa N, \qquad N' = -\vartheta\kappa T + \vartheta\tau B\,, \qquad B' = -\vartheta\tau N\,. \tag{1}$$

Frenet vectors $T\,,N\,,B$ form a Frenet frame and every Frenet frame moves along an instantaneous rotation axis which is called a Darboux vector and given by, see [9]

$$W = \tau T + \kappa B. \tag{2}$$

Corresponding author: OC, email adress: osmancakir75@hotmail.com ORCID:https://orcid.org/0000-0002-2664-5232, SS ORCID:https://orcid.org/0000-0003-1097-5541
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When we denote the angle between W and B by ϕ, the Darboux vector can be expressed as a unit Darboux vector C given by, see [10]

$$C = sin\phi T + cos\phi B\,,\;\; sin\phi = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\,,\;\; cos\phi = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\,. \tag{3}$$

Definition 1.1. *Let α and β be two differentiable curves. If the tangent vector of α is perpendicular to the tangent vector of β, then we call β as the involute of α. According to this definition, following parametrization can be given*

$$\beta(s) = \alpha(s) + \lambda(s)T(s)\,,\quad \lambda(s) = c - s\,,\quad c \in \mathbb{R}. \tag{4}$$

When β is the involute of α, we have $d(\alpha(s), \beta(s)) = \mid c - s \mid$, $\forall s \in I$ and $c = const$. The relationship between the Frenet apparatus of the curves α and β is given by

$$T_\beta = N\,,\qquad N_\beta = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2+\tau^2}}\,,\qquad B_\beta = \frac{\tau T + \kappa B}{\sqrt{\kappa^2+\tau^2}}\,,\qquad \kappa_\beta = \frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\,,\qquad \tau_\beta = \frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\,. \tag{5}$$

By this definition, Darboux vector of the curve β is given by, see [9]

$$W_\beta = \tau_\beta T_\beta + \kappa_\beta B_\beta. \tag{6}$$

There is still another way to express Darboux vector named as unit Darboux vector in [10]

$$C_\beta = sin\phi_\beta T_\beta + cos\phi_\beta B_\beta\,,\quad sin\phi_\beta = \frac{\tau_\beta}{\sqrt{\kappa_\beta^2+\tau_\beta^2}}\,,\;\; cos\phi_\beta = \frac{\kappa_\beta}{\sqrt{\kappa_\beta^2+\tau_\beta^2}}\,. \tag{7}$$

with the angle ϕ_β between the vectors W_β and B_β. It is also worth noting the relation here is that, see [11]

$$\begin{aligned} sin\phi_\beta &= \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\,,\qquad cos\phi_\beta = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\,, \\ \phi_\beta' &= \Big(\frac{\phi'}{\sqrt{\phi'^2+\kappa^2+\tau^2}}\Big)'\,\frac{\sqrt{\phi'^2+\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,. \end{aligned} \tag{8}$$

This leads us the following relation, see [11]

$$C_\beta = \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}N + \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}C\,. \tag{9}$$

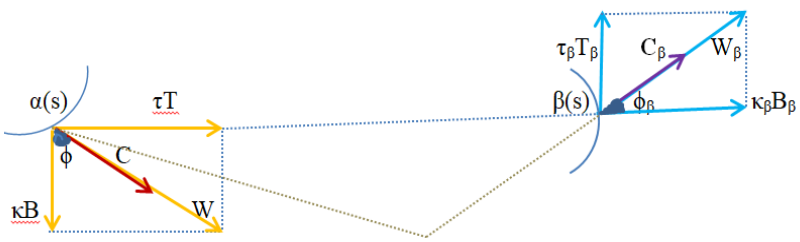

Figure 1: Unit Darboux vectors of the curves α and β.

Definition 1.2. *Let α be the unit speed curve, then the mean curvature vector field H along the curve α is defined as,* see [7]

$$H = D_{\alpha'}\alpha' = \kappa N \tag{10}$$

where D is the Levi-Civita connection. According to this definition the mapping

$$\Delta : \chi^{\perp}(\alpha(I)) \to \chi(\alpha(I)), \qquad \Delta H = -D_T^2 H \tag{11}$$

is called a Laplace operator. Let us denote the normal bundle of a curve $\alpha = \alpha(s)$ by $\chi^{\perp}(\alpha(s))$. Then the normal connection $D^{\perp}$ is given as

$$D_T^{\perp} : \chi^{\perp}(\alpha(I)) \to \chi^{\perp}(\alpha(I)), \qquad D_T^{\perp}X = D_T X - \left\langle D_T X, T\right\rangle T \tag{12}$$

and the normal Laplace operator $\Delta^{\perp}$ is given by the following mapping

$$\Delta_T^{\perp}X = -D_T^{\perp}D_T^{\perp}X, \;\; \forall X \in \chi^{\perp}(\alpha(I)). \tag{13}$$

Theorem 1.3. *Let α be the unit speed curve and H, W be the mean curvature and Darboux vector along the curve α, respectively. Then we have the following propositions,* see [8]

a) $\Delta C = 0$ then α is a biharmonic curve.

b) $\Delta C = \mu C,\ \lambda, \mu \in \mathbb{R}$, then α is a 1-type harmonic curve.

c) $\Delta^{\perp}C^{\perp} = 0$ then α is a weak biharmonic curve.

d) $\Delta^{\perp}C^{\perp} = \mu C^{\perp},\ \lambda, \mu \in \mathbb{R}$, then α is a 1-type harmonic curve.

Theorem 1.4. *Let α be a differentiable curve with unit Darboux vector C, then the differential equation characterizing α according to unit Darboux vector is given as,* see [8]

$$D_T^3 C + \lambda_1 D_T^2 C + \lambda_2 D_T C + \lambda_3 C = 0 \tag{14}$$

with the coefficients λ_1, λ_2, λ_3

$$\lambda_1 = -\Big(\frac{\phi''}{\phi'} + \frac{(\phi'\vartheta \parallel W \parallel)'}{\vartheta \parallel W \parallel \phi'}\Big), \qquad \lambda_2 = (\vartheta \parallel W \parallel)^2 + (\phi')^2 - \Big(\frac{\phi''}{\phi'}\Big)' + \frac{(\phi'\vartheta \parallel W \parallel)'}{\vartheta \parallel W \parallel (\phi')^2}\phi'',$$

$$\lambda_3 = ((\phi')^2)' - \frac{(\phi'\vartheta \parallel W \parallel)'}{\vartheta \parallel W \parallel}\phi'.$$

Theorem 1.5. *Let α be a differentiable curve with unit normal Darboux vector $C^{\perp}$, then the differential equation characterizing α according to unit normal Darboux vector is given as,* see [8]

$$\lambda_2 D_T^{\perp} D_T^{\perp} C^{\perp} + \lambda_1 D_T^{\perp} C^{\perp} + \lambda_0 C^{\perp} = 0 \tag{15}$$

with the coefficients λ_0, λ_1, λ_2

$\lambda_0 = \phi' sin\phi(\phi' sin\phi\vartheta\tau - (\vartheta\tau cos\phi)') + \vartheta\tau cos\phi(\vartheta^2\tau^2 cos\phi + (\phi' sin\phi)')$,

$\lambda_1 = cos\phi(\phi' sin\phi\vartheta\tau - (\vartheta\tau cos\phi)')$,

$\lambda_2 = \vartheta\tau cos^2\phi$.

Theorem 1.6. [1] Let β be the involute of a unit speed curve α. Then the Frenet formulas for the curve β with respect to Levi-Civita connection D and normal Levi-Civita connection $D^{\perp}$ are given, respectively, as

$$D_N T = \kappa N, \quad D_N N = -\kappa T + \tau B, \quad D_N B = -\tau N\,, \tag{16}$$

$$D^{\perp}_N T = 0, \quad D^{\perp}_N B = 0\,. \tag{17}$$

2. Calculation of the differential equations and harmonicity of the involute curve according to unit Darboux vector with a new method

When we say α, unless we stated otherwise, we mean a unit speed curve in Euclidean 3-space with the Frenet apparatus of T, N, B, κ, τ and when we mention β, it stands for the involute of the curve α in the same space with the Frenet apparatus of $T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_\beta$ and $\vartheta = \| \frac{d}{ds}\beta(s) \|$. Throughout the paper we use C to denote the unit Darboux vector of α and C_β to express the unit Darboux vector of β respectively.

Theorem 2.1. *Let β be the involute of the curve α. Then the differential equation with respect to connection characterizing the curve β by means of the unit Darboux vector C_β is given as*

$$D^3_{T_\beta} C_\beta + \mu_{\beta 1} D^2_{T_\beta} C_\beta + \mu_{\beta 2} D_{T_\beta} C_\beta + \mu_{\beta 3} C_\beta = 0 \tag{18}$$

with the coefficients $\mu_{\beta 1}$, $\mu_{\beta 2}$, $\mu_{\beta 3}$

$$\mu_{\beta 1} = -\Big(\frac{(\phi_\beta)''}{(\phi_\beta)'} + \frac{((\phi_\beta)'\vartheta \| W_\beta \|)'}{\vartheta \| W_\beta \| (\phi_\beta)'}\Big), \qquad \mu_{\beta 3} = (((\phi_\beta)')^2)' - \frac{((\phi_\beta)'\vartheta \| W_\beta \|)'}{\vartheta \| W_\beta \|}(\phi_\beta)',$$

$$\mu_{\beta 2} = (\vartheta \| W_\beta \|)^2 + ((\phi_\beta)')^2 - \Big(\frac{(\phi_\beta)''}{(\phi_\beta)'}\Big)' + \frac{((\phi_\beta)'\vartheta \| W_\beta \|)'}{\vartheta \| W_\beta \| ((\phi_\beta)')^2}(\phi_\beta)''\,.$$

Proof. *From equ.(3) we have* $C_\beta = sin\phi_\beta T_\beta + cos\phi_\beta B_\beta\cdot$ *Taking the derivative with respect to* T_β *gives us*

$$D_{T_\beta} C_\beta = \phi'_\beta (cos\phi_\beta T_\beta - sin\phi_\beta B_\beta)\cdot \tag{19}$$

From the equalities (3) and (19) we write the equivalents of T_β *and* B_β *as,*

$$T_\beta = sin\phi_\beta C_\beta + \frac{cos\phi_\beta}{(\phi_\beta)'} D_{T_\beta} C_\beta\,,$$

$$B_\beta = cos\phi_\beta C_\beta - \frac{sin\phi_\beta}{(\phi_\beta)'} D_{T_\beta} C_\beta\,\cdot$$

Second derivative of C_β *with respect to* T_β *gives us*

$$D^2_{T_\beta} C_\beta = \frac{(\phi_\beta)''}{(\phi_\beta)'} D_{T_\beta} C_\beta - ((\phi_\beta)')^2 C_\beta + (\phi_\beta)'\vartheta \| W_\beta \| N_\beta\,\cdot$$

From this equality we derive N_β *as,*

$$N_\beta = \frac{1}{\vartheta ((\phi_\beta)')^2 \| W_\beta \|}((\phi_\beta)' D^2_{T_\beta} C_\beta - (\phi_\beta)'' D_{T_\beta} C_\beta + ((\phi_\beta)')^3 C_\beta)\,\cdot$$

After third derivative of C_β we find

$$D^3_{T_\beta}C_\beta = (\frac{(\phi_\beta)''}{(\phi_\beta)'} + \frac{((\phi_\beta)'\vartheta \parallel W_\beta \parallel)'}{\vartheta \parallel W_\beta \parallel (\phi_\beta)'})D^2_{T_\beta}C_\beta + ((\frac{(\phi_\beta)''}{(\phi_\beta)'})' - (\vartheta \parallel W_\beta \parallel)^2 - ((\phi_\beta)')^2 - \frac{((\phi_\beta)'\vartheta \parallel W_\beta \parallel)'}{\vartheta \parallel W_\beta \parallel ((\phi_\beta)')^2}(\phi_\beta)'')D_{T_\beta}C_\beta$$

$$+(\frac{((\phi_\beta)'\vartheta \parallel W_\beta \parallel)'}{\vartheta \parallel W_\beta \parallel}(\phi_\beta)' - (((\phi_\beta)')^2)')C_\beta \cdot$$

It remains only to rearrange the above equality as a linear combinations of $D^3_{T_\beta}C_\beta$, $D^2_{T_\beta}C_\beta$, $D_{T_\beta}C_\beta$ and $C_\beta \cdot$ Then we obtain the required equation which completes the proof. □

Theorem 2.2. *Let α be a differentiable curve with principal normal N, unit Darboux vector C and β be the involute of α. Then the differential equation characterizing the curve β with respect to connection is given as*

$$c_1D^3_NC + \left(3c_1' + \mu_1c_1\right)D^2_NC + \left(3c_1'' + 2\mu_1c_1' + \mu_2c_1\right)D_NC$$

$$+\left(c_1''' + \mu_1c_1'' + \mu_2c_1' + \mu_3c_1\right)C + c_2D^3_NN + \left(3c_2' + \mu_1c_2\right)D^2_NN$$

$$+\left(3c_2'' + 2\mu_1c_2' + \mu_2c_2\right)D_NN + \left(c_2''' + \mu_1c_2'' + \mu_2c_2' + \mu_3c_2\right)N = 0 \tag{20}$$

with the coefficients c_1, c_2, μ_1, μ_2, μ_3

$$c_1 = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}, \quad c_2 = \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}},$$

$$\mu_1 = -\frac{(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})''}{(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'} - \frac{\left((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\sqrt{(\phi')^2+\kappa^2+\tau^2}\right)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}\left(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)'},$$

$$\mu_2 = (\phi')^2+\kappa^2+\tau^2 + ((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})')^2 - (\frac{(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})''}{(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'})'$$

$$+\frac{\left((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\sqrt{(\phi')^2+\kappa^2+\tau^2}\right)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}\left((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\right)^2} \cdot (arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'',$$

$$\mu_3 = \left(((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})')^2\right)'$$

$$-\frac{\left((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\sqrt{(\phi')^2+\kappa^2+\tau^2}\right)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'.$$

Proof. *We can compute the equivalents of coefficients $\mu_{\beta 1}$, $\mu_{\beta 2}$, $\mu_{\beta 3}$ and the angle ϕ_β in the equation (18) by taking equations (5), (8) and (9) into consideration as μ_1, μ_2, μ_3 and the angle ϕ. It follows from the equ.(9) we have*

$$c_1 = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}, \quad c_2 = \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}.$$

Making use of the equalities (5), (8) and (9) again, we can write the equivalents of coefficients $\mu_{\beta 1}$, $\mu_{\beta 2}$, $\mu_{\beta 3}$ and the Darboux vector W_β as

$$W_\beta = \frac{sin\phi\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}T + \frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}N + \frac{cos\phi\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}B.$$

By referring the equalities (8) and (14) we can write that

$$C_\beta = \frac{1}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}(sin\phi\sqrt{\kappa^2+\tau^2}\,T + \phi' N + cos\phi\sqrt{\kappa^2+\tau^2}\,B).$$

Applying the equ.(16) we may write the counterparts of $D_{T_\beta}C_\beta$, $D^2_{T_\beta}C_\beta$, $D^3_{T_\beta}C_\beta$ as in the following form

$$D_{T_\beta}C_\beta = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D_N C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' C + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D_N N + \Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' N,$$

$$\begin{aligned}
D^2_{T_\beta}C_\beta = {} & \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^2_N C + 2\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' D_N C \\
& + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'' C + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^2_N N \\
& + 2\Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' D_N N + \Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'' N, \qquad (21) \\
D^3_{T_\beta}C_\beta = {} & \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^3_N C + 3\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' D^2_N C \\
& + 3\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'' D_N C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)''' C \\
& + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^3_N N + 3\Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)' D^2_N N \\
& + 3\Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'' D_N N + \Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)''' N.
\end{aligned}$$

Finally setting the equivalents of coefficients and derivatives with respect to N into the first equation we get desired result which completes the proof. □

Theorem 2.3. *Let β be the involute of the curve α. Then the differential equation with respect to normal connection characterizing the curve β by means of the unit Darboux vector $C_\beta^\perp$ is given as*

$$\lambda_{\beta 2} D^\perp_{T_\beta} D^\perp_{T_\beta} C^\perp_\beta + \lambda_{\beta 1} D^\perp_{T_\beta} C^\perp_\beta + \lambda_{\beta 0} C^\perp_\beta = 0 \tag{22}$$

with the coefficients $\lambda_{\beta 0}$, $\lambda_{\beta 1}$, $\lambda_{\beta 2}$

$\lambda_{\beta 2} = \vartheta\tau_\beta cos^2\phi_\beta, \quad \lambda_{\beta 1} = cos\phi_\beta\Big(\phi'_\beta sin\phi_\beta \vartheta\tau_\beta - (\vartheta\tau_\beta cos\phi_\beta)'\Big),$

$\lambda_{\beta 0} = \phi'_\beta sin\phi_\beta\Big(\phi'_\beta sin\phi_\beta \vartheta\tau_\beta - (\vartheta\tau_\beta cos\phi_\beta)'\Big) + \vartheta\tau_\beta cos\phi_\beta\Big(\vartheta^2(\tau_\beta)^2 cos\phi_\beta + (\phi'_\beta sin\phi_\beta)'\Big).$

Proof. *From equ. (13) we write the normal component of* C_β *as*

$$C^\perp_\beta = cos\phi_\beta B_\beta \cdot \tag{23}$$

Taking the first and second derivatives of this equality with respect to normal connection gives us,

$$D^\perp_{T_\beta} C^\perp_\beta = -\vartheta\tau_\beta cos\phi_\beta N_\beta - \phi'_\beta sin\phi_\beta B_\beta, \tag{24}$$

$$D^\perp_{T_\beta} D^\perp_{T_\beta} C^\perp_\beta = \Big(\phi'_\beta sin\phi_\beta \vartheta\tau_\beta - (\vartheta\tau_\beta cos\phi_\beta)'\Big)N_\beta - \Big(\vartheta^2(\tau_\beta)^2 cos\phi_\beta + (\phi'_\beta sin\phi_\beta)'\Big)B_\beta. \tag{25}$$

If we extract the vectors N_β *and* B_β *from equ.(23) , (24) we have*

$$B_\beta = \frac{1}{cos\phi_\beta} C^\perp_\beta ,$$

$$N_\beta = \frac{-1}{\vartheta\tau_\beta cos\phi_\beta} D^\perp_{T_\beta} C^\perp_\beta - \frac{\phi'_\beta\, sin\phi_\beta}{\vartheta\tau_\beta cos^2\phi_\beta} C^\perp_\beta \cdot$$

Putting the equivalents of B_β *and* N_β *into the equ.(25) we obtain the desired equation which completes the proof.* □

Theorem 2.4. *Let α be a differentiable curve with principal normal N, unit Darboux vector C and β be the involute of α. Then the differential equation characterizing the curve β with respect to normal connection is given as*

$$\Big(\rho\lambda_2\Big) D^\perp_N D^\perp_N C + \Big(2\rho'\lambda_2 + \rho\lambda_1\Big) D^\perp_N C + \Big(\rho''\lambda_2 + \rho'\lambda_1 + \rho\lambda_0\Big) C = 0 \tag{26}$$

with the coefficients $\rho,\ \lambda_0,\ \lambda_1,\ \lambda_2$

$$\rho = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}, \quad \lambda_2 = \frac{\kappa\tau'-\kappa'\tau}{(\phi')^2+\kappa^2+\tau^2},$$

$$\lambda_1 = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\cdot$$

$$\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}-(\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}})'\Big),$$

$$\lambda_0 = (arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\cdot$$

$$\Big((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}$$

$$-(\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}})'\Big)$$

$$+\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}}\Big((\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)})^2\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}$$

$$+\Big((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'\Big).$$

Proof. From equ.(3) we have $cos\phi = \kappa/\sqrt{\kappa^2+\tau^2}$ *and* $sin\phi = \tau/\sqrt{\kappa^2+\tau^2}$ *it follows from the equalities (8) and (14)*

we figure out that $sin\phi_\beta = \phi'/\sqrt{(\phi')^2+\kappa^2+\tau^2}$, $cos\phi_\beta = \sqrt{\kappa^2+\tau^2}/\sqrt{(\phi')^2+\kappa^2+\tau^2}$. *Then we get,*

$$C_\beta^\perp = \frac{\tau}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}T + \frac{\kappa}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}B.$$

On the other hand we can evaluate the equivalents of coefficients of the equation (22) by using the equalities (5), (8) and (17) as $\lambda_0,\ \lambda_1,\ \lambda_2$. *By the same way we can make use of the equalities (5), (8) and (17)again, in order to write* *the equivalents of derivatives of* $D^{\perp}_{T_\beta}C^{\perp}_{\beta}$ *and* $D^{\perp}_{T_\beta}D^{\perp}_{T_\beta}C^{\perp}_{\beta}$ *with respect to N. It follows that*

$$D^{\perp}_{T_\beta}C^{\perp}_{\beta} = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^{\perp}_{N}C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'C\,,$$

$$D^{\perp}_{T_\beta}D^{\perp}_{T_\beta}C^{\perp}_{\beta} = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^{\perp}_{N}D^{\perp}_{N}C + 2\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'D^{\perp}_{N}C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)''C\,. \quad (27)$$

Setting the equivalents of coefficients of the equation with the aid of equ.(5) and then the derivatives with respect to N into the equation above we get desired result which completes the proof. □

Theorem 2.5. *Let* β *be the involute of a differentiable curve* α *with the unit Darboux vector* C_β*. According to connection, harmonicity (biharmonic or 1-type harmonic) of the curve* β *may not be expressed by means of the Frenet apparatus of the main curve* α*.*

Proof. *From equ.(21), it is obvious that we have the following*

$$D^{2}_{T_\beta}C_{\beta} = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^{2}_{N}C + 2\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'D_{N}C$$

$$+\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)''C + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^{2}_{N}N$$

$$+2\Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'D_{N}N + \Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)''N\cdot$$

Considering the case $\Delta C_\beta = 0$ *or* $\Delta C_\beta = \lambda C_\beta$*, from Theorem 1.3 of a and b we get* $D_N N = 0$ *and* $D_N C = 0$*.*
Hence we cannot decide whether the curve β *is biharmonic or 1-type harmonic.* □

Theorem 2.6. *Let* β *be the involute of a differentiable curve* α *with the normal Darboux vector* $C^{\perp}_{\beta}$*. According to normal connection, harmonicity (weak biharmonic or 1-type harmonic) of the curve* β *may not be expressed by means of the Frenet apparatus of the main curve* α*.*

Proof. *From equ.(27), it is clear that we have the following*

$$D^{\perp}_{T_\beta}D^{\perp}_{T_\beta}C^{\perp}_{\beta} = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^{\perp}_{N}D^{\perp}_{N}C + 2\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'D^{\perp}_{N}C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)''C\cdot$$

Considering the case $\Delta C^{\perp}_{\beta} = 0$ *or* $\Delta C^{\perp}_{\beta} = \lambda C^{\perp}_{\beta}$*, from Theorem 1.3 of c and d we get* $D_N C = 0$*.*
Hence we cannot decide whether the curve β *is weak biharmonic or 1-type harmonic.* □

Example 2.7. *Let a curve* $\alpha(s) = \frac{1}{\sqrt{2}}(coss, sins, s)$ *be given. Then we have an involute of* α*, that is, curve* β*,*

$\beta(s) = \frac{1}{\sqrt{2}}(coss - (c-s)sins,\ sins + (c-s)coss,\ c)$*,* $c \in \mathbb{R}$*. It follows that* $C_\beta = sin\phi_\beta T_\beta + cos\phi_\beta B_\beta$ *with*

$sin\phi_\beta = 0$*,* $cos\phi_\beta = 1$*. By the equ.(9) also we get* $B_\beta = C$*. Hence we obtain,* $D_N C = 0$ *and* $D^{\perp}_{N}C = 0$*.*

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The multiplicity of eigenvalues of a vectorial diffusion equations with discontinuous function inside a finite interval

Abdullah ERGÜN[a]

[a] *Vocational High School of Sivas, Cumhuriyet University*

Abstract. In this study , m-dimensional vectorial diffusion equation with discontinuous function inside a finite interval is considered. Considering the asymptotic representation of the solution of the problem, we have obtained some conclusions about the multiplicity of eigenvalues. We have proved that, under certain conditions on potential matrix, the problem can only have a finite number of eigenvalues with multiplicity m.

sectionIntroduction Consider the m-dimensional vectorial singular diffusion equations

$$-y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 \delta(x)\, y,\ \ x \in (0, \pi) \tag{1}$$

$$y'(0) = \theta \tag{2}$$

$$y'(\pi) = \theta \tag{3}$$

where λ is the spectral parameter ,$y = (y_1, y_2, ...y_m)^T$ is an m-dimensional vector function,

$$\delta(x) = \begin{cases} 1\ \ , & x \in (0, a_1) \\ \alpha^2, & x \in (a_1, a_2) \\ \beta^2, & x \in (a_2, \pi) \end{cases}$$

and $\alpha > 0$, $\alpha \neq 1$,$\beta > 0$, $\beta \neq 1$, $q(x) \in L_2[0, \pi]$, $p(x) \in W_2^1[0, \pi]$, $a_1, a_2 \in (0, \pi)$, $a_1 < a_2$. The potential matrix $(2\lambda p(x) + q(x))$ is an $m \times m$ real symmetric matrix function. θ denotes the m-dimensional zero vector.
Many studies on the theory of second-order differential operators have been studied in [7, 18]. One of the most important of these was made in 1946 by Titchmarsh [20]. In 1984, the studies on the spectral theory of singular differential operators were conducted by Levitan [21]. Many physical phenomena, such as fluid flow and heat dissipation [23], atomic mixing modelling [24] include a diffusion process. Singular differential operators with conditions of discontinuity are often used in mathematical physics, in geophysics and natural sciences. In general, these problems are associated with discontinuous material properties. For example; It is used to in determining the parameters of the electricity line in electronics [22]. Also, it is used to determine geophysical models for the release of the earth [9]. The discontinuity here is the reflection of

Corresponding author: AE mail address: aergun@cumhuriyet.edu.tr ORCID: https://orcid.org/0000-0002-2795-8097
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the shear waves at the base of the earth's crust. In 1999, C. L. Shen and C.T. Shies [5] studied the multiplicity of eigenvalues of the m-dimensional the vectorial Sturm-Liouville problem

$$-y'' + Q(x)\,y = \lambda y\ ,\ \ y(0) = y(1) = \theta$$

where Q is continuous $m \times m$ Jacobi matrix-valued function defined on $0 \le x \le 1$. Q. Kong [4] generalized to the case when $Q$is real symmetric. However, there are no such result for the discontinuous problem (1) – (3).
In this study, firstly we define the characteristic function of the eigenvalues of vectorial problem (1) – (3). Following this, we prove the conclusion that the eigenvalues of the problem coincide with the zeros of characteristic function. Then, we show the asymptotic forms of the solutions and obtain some results about multiplicity of the eigenvalues.

1. Characteristic function and asymptotics of solutions

Denote $H = L^2(I, C^m)$ the Hilbert space of vector-valued functioons with the scalar product

$$(f,g) = \int_0^{a_1} g_1^* f_1 dx + \int_{a_1}^{a_2} g_2^* f_2 dx + \int_{a_2}^{\pi} g_r^* f_r dx = \int_0^{\pi} g^* f dx$$

where $f = (f_1, f_2, ..., f_m)^T$, $g = (g_1, g_2, ...g_m)^T$ and $f_i, g_i \in L^2(I)$, $f_1(x) = f(x)\big|_{(0,a_1)}$, $f_2(x) = f(x)\big|_{(a_1,a_2)}$ and $f_r(x) = f(x)\big|_{(a_2,\pi)}$. We can define an operator L associated with the problem (1) – (3) on H

$$L : -y'' + [2\lambda p(x) + q(x)]\,y = \lambda^2 \delta(x)\,y\ ,\ \ y \in D(L)$$
$$D(L) = \{y \in H;\ y, y' \in AC[I, C^m]\},\ Ly \in L^2[I, C^m]$$
$$y'(0) = y'(\pi) = \theta.$$

Lemma 1.1. *The operator L is self-adjoint.*

The proof is similar to the scaler case in [12].
We consider the problem on the three intervals $(0, a_1)$, (a_1, a_2) and (a_2, π) respectively, where θ_m denotes $m \times m$ zero matrix and $E_m$denotes $m \times m$ identify matrix. On $(0, a_1)$, the matrix initial value problem

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))\,Y = \lambda^2 \cdot 1 \cdot Y\ ,\ \ x \in (0, a_1) \\ \phi_1(0,\lambda) = E_m\ , \phi_1'(0,\lambda) = \theta_m \end{cases} \tag{4}$$

has a unique solution $\phi_1(x,\lambda)$. What's more, for any fixed $x \in (0, a_1)$, $\phi_1(x,\lambda)$ is an entire matrix function in λ [1],p17. By variation of constants, we have

$$\phi_1(x,\lambda) = \cos\lambda x E_m + \frac{1}{\lambda}\int_0^x \sin\lambda(x-t)\,(2\lambda p(t) + q(t))\,\phi_1(t,\lambda)\,dt. \tag{5}$$

on (a_1, a_2) the matrix value problem

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))\,Y = \lambda^2\alpha^2 Y\ ,\ \ x \in (a_1, a_2) \\ \phi_2(a_1 + 0,\lambda) = \phi_1(a_1 - 0,\lambda) \\ \phi_2'(a_1 + 0,\lambda) = \phi_1'(a_1 - 0,\lambda) \end{cases} \tag{6}$$

has a unique solution $\phi_2(x,\lambda)$. In addition to, for any fixed $x \in (a_1, a_2)$, $\phi_2(x,\lambda)$ is an entire matrix function in λ. By variation of constants, we have

$$\begin{aligned} \varphi(x,\lambda) &= \alpha^+ e^{i\lambda\mu^+(x)} + \alpha^- e^{i\lambda\mu^-(x)} + \alpha^+ \int_0^{a_1} \tfrac{\sin\lambda\left(\mu^+(x)-t\right)}{\lambda} Q(t)\,y(t,\lambda)\,dt \\ &+\alpha^- \int_0^{a_1} \tfrac{\sin\lambda\left(\mu^-(x)-t\right)}{\lambda} Q(t)\,y(t,\lambda)\,dt + \int_{a_1}^{x} \tfrac{\sin\lambda\alpha(x-t)}{\lambda\alpha} Q(t)\,y(t,\lambda)\,dt \end{aligned} \tag{7}$$

where $\mu^{\pm}(x) = \pm\alpha x \mp \alpha a_1 + a_1$, $Q(t) = 2\lambda p(t) + q(t)$,
or

$$\begin{aligned}&\phi_2(x,\lambda) = \cos\lambda\alpha(x-a_1)\,\phi_1(a_1-0)\,E_m + \tfrac{1}{\lambda\alpha}\sin\lambda\alpha(x-a_1)\,\phi_1'(a_1-0)\,E_m\\&+\int_{a_1}^{x}\tfrac{\sin\lambda\alpha(x-t)}{\lambda\alpha}\left(2\lambda p(t)+q(t)\right)\phi_2(t,\lambda)\,dt.\end{aligned} \tag{8}$$

on (a_2,π) the matrix value problem

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))\,Y = \lambda^2\beta^2 Y\ ,\ x\in(a_2,\pi)\\ \phi_3(a_2+0,\lambda) = \phi_2(a_2-0,\lambda)\\ \phi_3'(a_2+0,\lambda) = \phi_2'(a_2-0,\lambda)\end{cases} \tag{9}$$

has a unique solution $\phi_3(x,\lambda)$. In addition to, for any fixed $x\in(a_2,\pi)$, $\phi_3(x,\lambda)$ is an entire matrix function in λ. By variation of constants, we have

$$\begin{aligned}&\phi_3(x,\lambda) = \alpha^+\beta^+e^{i\lambda k^+(x)} + \alpha^-\beta^-e^{i\lambda k^-(x)} + \alpha^+\beta^-e^{i\lambda s^+(x)} + \alpha^-\beta^+e^{i\lambda s^-(x)}\\&+\alpha^+\beta^+\int_0^{a_1}\tfrac{\sin\lambda(k^+(x)-t)}{\lambda}Q(t)\,y(t,\lambda)\,dt + \alpha^+\beta^-\int_0^{a_1}\tfrac{\sin\lambda(s^+(x)-t)}{\lambda}Q(t)\,y(t,\lambda)\,dt\\&+\alpha^-\beta^-\int_0^{a_1}\tfrac{\sin\lambda(k^-(x)-t)}{\lambda}Q(t)\,y(t,\lambda)\,dt + \alpha^-\beta^+\int_{a_1}^{a_2}\tfrac{\sin\lambda(s^-(x)-t)}{\lambda}Q(t)\,y(t,\lambda)\,dt\\&+\tfrac{\beta^+}{\alpha}\int_0^{a_1}\tfrac{\sin\lambda\left(\beta x-\beta a_2+\alpha a_2-\alpha t\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt\\&-\tfrac{\beta^-}{\alpha}\int_0^{a_1}\tfrac{\sin\lambda\left(\beta x-\beta a_2-\alpha a_2+\alpha t\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt + \int_{a_2}^{x}\tfrac{\sin\lambda\beta(x-t)}{\lambda\beta}Q(t)\,y(t,\lambda)\,dt\end{aligned} \tag{10}$$

where $Q(t) = 2\lambda p(t) + q(t)$,$\mu^{\pm}(x) = \pm\alpha x \mp \alpha a_1 + a_1$,$\alpha^{\pm} = \frac{1}{2}\left(1\pm\frac{1}{\alpha}\right)$,$\beta^{\pm} = \frac{1}{2}\left(1\pm\frac{1}{\beta}\right)$, $k^{\pm}(x) = \beta x - \beta a_2 + \mu^{\pm}(a_2)$, $s^{\pm}(x) = -\beta x + \beta a_2 + \mu^{\pm}(a_2)$,
or

$$\begin{aligned}&\phi_3(x,\lambda) = \cos\lambda\beta(x-a_2)\,\phi_2(a_2-0,\lambda)\,E_m + \tfrac{1}{\lambda\beta}\sin\lambda\beta(x-a_2)\,\phi_2'(a_2-0,\lambda)\,E_m\\&+\int_{a_2}^{x}\tfrac{\sin\lambda\beta(x-t)}{\lambda\beta}\left(2\lambda p(t)+q(t)\right)\phi_3(t,\lambda)\,dt\,.\end{aligned} \tag{11}$$

Let

$$\phi(x,\lambda) = \begin{cases}\phi_1(x,\lambda)\,,\ x\in(0,a_1)\\ \phi_2(x,\lambda)\,, x\in(a_1,a_2)\\ \phi_3(x,\lambda)\,, x\in(a_2,\pi)\end{cases}.$$

Then, any solution of the equations (1) satisfying boundary condition (2) can be expressed as

$$y(x,\lambda) = \phi(x,\lambda)\,c_1 = \begin{cases}\phi_1(x,\lambda)\,c_0, & x\in(0,a_1)\\ \phi_2(x,\lambda)\,c_0, & x\in(a_1,a_2)\\ \phi_3(x,\lambda)\,c_0, & x\in(a_2,\pi)\end{cases} \tag{12}$$

where c_1 is an arbitrary m-dimensional constant vector. If λ is an eigenvalue of the problem (1) − (3), then $c_0 \neq \theta$ and $y(x,\lambda)$ satisfies the boundary condition at $x=\pi$, that is,

$$y'(\pi,\lambda) = \phi'(\pi,\lambda)\,c_0 = \phi_3'(\pi,\lambda)\,c_0 = \theta.$$

Thus, we get

$$\det\left(\phi_3'(\pi,\lambda)\right) = 0.$$

Similarly, on (a_2,π), consider the matrix initial value problem

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))\,Y = \lambda^2\beta^2 Y\ ,\ x\in(a_2,\pi)\\ \psi_3(\pi,\lambda) = E_m\ ,\psi_3'(\pi,\lambda) = \theta_m\end{cases}. \tag{13}$$

The problem (13) has a unique solution $\psi_3(x,\lambda)$. Furthermore, for any fixed $x\in(a_2,\pi)$, $\psi_3(x,\lambda)$ is an entire matrix function in λ.

Consider the matrix initial value problem on (a_1, a_2),

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))\,Y = \lambda^2\alpha^2 Y\ ,\ x \in (a_1, a_2) \\ \psi_3(a_2 + 0, \lambda) = \psi_2(a_2 - 0, \lambda) \\ \psi_3'(a_2 + 0, \lambda) = \psi_2'(a_2 - 0, \lambda) \end{cases}. \tag{14}$$

The problem (14) has a unique solution $\psi_2(x, \lambda)$. Furthermore, for any fixed $x \in (a_1, a_2)$, $\psi_2(x, \lambda)$ is an entire matrix function in λ.
Consider the matrix initial value problem on $(0, a_1)$,

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))\,Y = \lambda^2 \cdot 1 \cdot Y\ ,\ x \in (0, a_1) \\ \psi_2(a_1 + 0, \lambda) = \psi_1(a_1 - 0, \lambda) \\ \psi_2'(a_1 + 0, \lambda) = \psi_1'(a_1 - 0, \lambda) \end{cases} \tag{15}$$

The problem (15) has a unique solution $\psi_1(x, \lambda)$. Furthermore, for any fixed $x \in (0, a_1)$, $\psi_1(x, \lambda)$ is an entire matrix function in λ. Let

$$\psi(x, \lambda) = \begin{cases} \psi_1(x, \lambda)\ ,\ x \in (0, a_1) \\ \psi_2(x, \lambda)\,, x \in (a_1, a_2) \\ \psi_3(x, \lambda)\,, x \in (a_2, \pi) \end{cases}.$$

Then, any solution of the equations (1) satisfying boundary condition (3) can be expressed as

$$y(x, \lambda) = \psi(x, \lambda)\, c_2 = \begin{cases} \psi_1(x, \lambda)\, c_1,\ x \in (0, a_1) \\ \psi_2(x, \lambda)\, c_1, x \in (a_1, a_2) \\ \psi_3(x, \lambda)\, c_1, x \in (a_2, \pi) \end{cases} \tag{16}$$

where c_2 is an arbitrary m-dimensional constant vector. If λ is an eigenvalue of the problem (1) − (3), then $c_1 \neq \theta$and $y(x, \lambda)$ satisfies the boundary condition at $x = 0$, that is,

$$y'(0, \lambda) = \psi'(0, \lambda)\, c_1 = \psi_1'(0, \lambda)\, c_1 = \theta$$

Thus, we get

$$\det\left(\psi_1'(0, \lambda)\right) = 0.$$

Let $\Delta_j(\lambda) = W\left(\phi_j(x, \lambda), \psi_j(x, \lambda)\right)$ be the Wronskian of solution matrices $\phi_j(x, \lambda)$and $\psi_j(x, \lambda)$, $j = 1, 2, 3$, that is,

$$\begin{aligned} \Delta_1(\lambda) &= \begin{vmatrix} \phi_1(x, \lambda) & \psi_1(x, \lambda) \\ \phi_1'(x, \lambda) & \psi_1'(x, \lambda) \end{vmatrix}.\ \Delta_2(\lambda) = \begin{vmatrix} \phi_2(x, \lambda) & \psi_2(x, \lambda) \\ \phi_2'(x, \lambda) & \psi_2'(x, \lambda) \end{vmatrix}. \\ \Delta_3(\lambda) &= \begin{vmatrix} \phi_3(x, \lambda) & \psi_3(x, \lambda) \\ \phi_3'(x, \lambda) & \psi_3'(x, \lambda) \end{vmatrix}. \end{aligned} \tag{17}$$

Lemma 1.2. $\Delta_1(\lambda) = \Delta_2(\lambda) = \Delta_3(\lambda)$ *for all* $\lambda \in C$.

Proof. Because the Wronskian of the solution matrices $\phi_j(x, \lambda)$ and $\psi_j(x, \lambda)$ is independent of x ,

$$\Delta_3(\lambda) = \Delta_3(\lambda)|_{x=a_2+0} = \begin{vmatrix} \phi_3(a_2 + 0, \lambda) & \psi_3(a_2 + 0, \lambda) \\ \phi_3'(a_2 + 0, \lambda) & \psi_3'(a_2 + 0, \lambda) \end{vmatrix} = \begin{vmatrix} \phi_2(a_2 - 0, \lambda) & \psi_2(a_2 - 0, \lambda) \\ \phi_2'(a_2 - 0, \lambda) & \psi_2'(a_2 - 0, \lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \phi_2(x, \lambda) & \psi_2(x, \lambda) \\ \phi_2'(x, \lambda) & \psi_2'(x, \lambda) \end{vmatrix}_{x=a_2-0} = \Delta_2(\lambda) = \Delta_2(\lambda)|_{x=a_1+0} = \begin{vmatrix} \phi_2(a_1 + 0, \lambda) & \psi_2(a_1 + 0, \lambda) \\ \phi_2'(a_1 + 0, \lambda) & \psi_2'(a_1 + 0, \lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \phi_1(a_1 - 0, \lambda) & \psi_1(a_1 - 0, \lambda) \\ \phi_1'(a_1 - 0, \lambda) & \psi_1'(a_1 - 0, \lambda) \end{vmatrix} = \begin{vmatrix} \phi_1(x, \lambda) & \psi_1(x, \lambda) \\ \phi_1'(x, \lambda) & \psi_1'(x, \lambda) \end{vmatrix}_{x=a_1-0} = \Delta_1(\lambda)$$

the proof is completed. □

Denote $\Delta(\lambda) = \Delta_1(\lambda) = \Delta_2(\lambda) = \Delta_3(\lambda)$, we have the following lemma.

Lemma 1.3. *λ is an eigenvalue of* (1) − (3) *if any only if* $\Delta(\lambda) = 0$.

Proof. Necessity: Assume that $\lambda_0$is an eigenvalue of (1) − (3). $y(x,\lambda_0)$ is the eigenfunctions corresponding to λ_0, then by (16) we have

$$y(x,\lambda_0) = \phi(x,\lambda_0)\,c_{30} = \begin{cases} \phi_1(x,\lambda_0)\,c_{30}, & x \in (0,a_1) \\ \phi_2(x,\lambda_0)\,c_{30}, & x \in (a_1,a_2) \\ \phi_3(x,\lambda_0)\,c_{30}, & x \in (a_2,\pi) \end{cases} \tag{18}$$

$$y(x,\lambda_0) = \psi(x,\lambda_0)\,c_{40} = \begin{cases} \psi_1(x,\lambda_0)\,c_{40}, & x \in (0,a_1) \\ \psi_2(x,\lambda_0)\,c_{40}, & x \in (a_1,a_2) \\ \psi_3(x,\lambda_0)\,c_{40}, & x \in (a_2,\pi) \end{cases} \tag{19}$$

c_{30}, c_{40} are m-dimensional nonzero constant vector. So from (18) and (19), we have

$$\left.\begin{array}{l} \phi_1(x,\lambda_0)\,c_{30} = \psi_1(x,\lambda_0)\,c_{40} \\ \phi_1'(x,\lambda_0)\,c_{30} = \psi_1'(x,\lambda_0)\,c_{40} \end{array}\right\} x \in (0,a_1).$$

By direct simplification, we get

$$\begin{pmatrix} \phi_1(x,\lambda_0) & -\psi_1(x,\lambda_0) \\ \phi_1'(x,\lambda_0) & -\psi_1'(x,\lambda_0) \end{pmatrix} \cdot \begin{pmatrix} c_{30} \\ c_{40} \end{pmatrix} = \begin{pmatrix} \theta \\ \theta \end{pmatrix}.$$

Because $c_{30}, c_{40} \neq 0$, the coefficient determinant of above linear system of equations

$$\begin{vmatrix} \phi_1(x,\lambda_0) & -\psi_1(x,\lambda_0) \\ \phi_1'(x,\lambda_0) & -\psi_1'(x,\lambda_0) \end{vmatrix} = (-1)^m \begin{vmatrix} \phi_1(x,\lambda_0) & \psi_1(x,\lambda_0) \\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0) \end{vmatrix}$$
$$= (-1)^m \Delta_1(\lambda_0) = \Delta_2(\lambda_0) = \Delta_3(\lambda_0) = \Delta(\lambda_0) = 0$$

Sufficiency:
If $\lambda_0 \in \mathrm{C}$, $\Delta(\lambda_0) = 0$. Then the linear systems of equations

$$\begin{pmatrix} \phi_1(x,\lambda_0) & \psi_1(x,\lambda_0) \\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0) \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \theta \\ \theta \end{pmatrix}, \begin{pmatrix} \phi_2(x,\lambda_0) & \psi_2(x,\lambda_0) \\ \phi_2'(x,\lambda_0) & \psi_2'(x,\lambda_0) \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \theta \\ \theta \end{pmatrix}$$

$$\begin{pmatrix} \phi_3(x,\lambda_0) & \psi_3(x,\lambda_0) \\ \phi_3'(x,\lambda_0) & \psi_3'(x,\lambda_0) \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \theta \\ \theta \end{pmatrix}$$

have nonzero solutions. By a direct computation, we get

$$\left.\begin{array}{l} \phi_1(x,\lambda_0)\,c_0 = -\psi_1(x,\lambda_0)\,c_1 \\ \phi_1'(x,\lambda_0)\,c_0 = -\psi_1'(x,\lambda_0)\,c_1 \end{array}\right\} x \in (0,a_1), \quad \left.\begin{array}{l} \phi_2(x,\lambda_0)\,c_0 = -\psi_2(x,\lambda_0)\,c_1 \\ \phi_2'(x,\lambda_0)\,c_0 = -\psi_2'(x,\lambda_0)\,c_1 \end{array}\right\} x \in (a_1,a_2)$$

and

$$\left.\begin{array}{l} \phi_3(x,\lambda_0)\,c_0 = -\psi_3(x,\lambda_0)\,c_1 \\ \phi_3'(x,\lambda_0)\,c_0 = -\psi_3'(x,\lambda_0)\,c_1 \end{array}\right\} x \in (a_2,\pi).$$

Denote

$$y(x,\lambda_0) = \begin{cases} \phi_1(x,\lambda_0)\,c_0 = -\psi_1(x,\lambda_0)\,c_1, & x \in (0,a_1) \\ \phi_2(x,\lambda_0)\,c_0 = -\psi_2(x,\lambda_0)\,c_1, & x \in (a_1,a_2) \\ \phi_3(x,\lambda_0)\,c_0 = -\psi_3(x,\lambda_0)\,c_1, & x \in (a_2,\pi) \end{cases}.$$

We note that $y(x,\lambda_0)$ satisfies the boundary condition (2), (3). That is, $y(x,\lambda_0)$ is the eigenfunctions corresponding to λ_0. Thus λ_0 is an eigenvalue of the problem (1) − (3).

□

Remark 1.4. *As two especial case*

$$\Delta(\lambda) = \begin{vmatrix} \phi_1(x,\lambda_0) & \psi_1(x,\lambda_0) \\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0) \end{vmatrix}_{x=0} = \begin{vmatrix} E_m & \psi_1(0,\lambda_0) \\ \theta_m & \psi_1'(0,\lambda_0) \end{vmatrix} = \det\left(\psi_1'(0,\lambda)\right)$$

$$\Delta(\lambda) = \begin{vmatrix} \phi_3(x,\lambda_0) & \psi_3(x,\lambda_0) \\ \phi_3'(x,\lambda_0) & \psi_3'(x,\lambda_0) \end{vmatrix}_{x=\pi} = \begin{vmatrix} \phi_3(\pi,\lambda_0) & E_m \\ \phi_3'(\pi,\lambda_0) & \theta_m \end{vmatrix} = (-1)^m \det\left(\phi_3'(\pi,\lambda)\right).$$

Definition 1.5. $\Delta(\lambda)$ *will be called the characteristic function of the eigenvalues of the problem* (1) − (3).

Definition 1.6. *If there is a* $\Delta_1(\lambda)$ *to be* $\Delta(\lambda) = (\lambda-\lambda_0)^m \Delta_1(\lambda)$, *algebraic multiplicity of eigenvalue* λ*is called* m. *The geometric multiplicity of* λ *as an eigenvalue of the problem* (1) − (3) *is defined to be the number of linearly independent solutions of the boundary value problem. If we denote* $2m \times 2m$*matrices*
$A(x,\lambda_0) = \begin{pmatrix} \phi_1(x,\lambda_0) & \psi_1(x,\lambda_0) \\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0) \end{pmatrix}$,$B(x,\lambda_0) = \begin{pmatrix} \phi_2(x,\lambda_0) & \psi_2(x,\lambda_0) \\ \phi_2'(x,\lambda_0) & \psi_2'(x,\lambda_0) \end{pmatrix}$ *and*
$C(x,\lambda_0) = \begin{pmatrix} \phi_3(x,\lambda_0) & \psi_3(x,\lambda_0) \\ \phi_3'(x,\lambda_0) & \psi_3'(x,\lambda_0) \end{pmatrix}$
the rank of matrix $A(x,\lambda_0)$ *as* $R(A(x,\lambda_0))$. *Similarly,* $B(x,\lambda_0)$ *as* $R(B(x,\lambda_0))$ *and* $C(x,\lambda_0)$*as* $R(C(x,\lambda_0))$.

Corollary 1.7. *The geometric multiplicity of* λ_0 *as an eigenvalue of the problem* (1) − (3) *is equal to* $2m - R(A(x,\lambda_0))$ *or* $2m - R(B(x,\lambda_0))$ *or* $2m - R(C(x,\lambda_0))$.

Corollary 1.8. $R(A(x,\lambda_0))$, $R(B(x,\lambda_0))$ *or* $R(C(x,\lambda_0))$ *is at least equal to* m, *so the geometric multiplicity of* λ_0 *varies from 1 to m. When the geometric multiplicity of an eigenvalue is m, we say the eigenvalue has maximal (full) multiplicity. In this study, we refer multiplicity as the geometric multiplicity.*
An entire function of non-integer order has an infinite set of zeros. The zeros of an analytic function which does not vanish identically are isolated [3]. $\psi_1'(0,\lambda)$ *and* $\phi_3'(\pi,\lambda)$ *are entire function of order* $\frac{1}{2}$ *matrices. The sums and products of such functions are entire of order not exceeding* $\frac{1}{2}$. *Hence, the determinants of* $\psi_1'(0,\lambda)$ *and* $\phi_3'(\pi,\lambda)$, *that is, the caracteristic functions are also non-integer.*

Eigenvalues for (1)−(3) are real. The boundary value problem (1)−(3)has a countable number of eigenvalues that grow unlimitedly, when those are ordered according to their absolute value.
The norm of a constant matrix as well as the norm of a matrix function A is denoted by $\|A\|$.
$A(x) = \left(a_{ij}\right)_{i,j=1}^{m} : I \to M_{mxm}^{\mathrm{R}}$, for any $x \in I$, the norm of $A(x)$ may be taken as

$$\|A(x)\| = \max_{1\le i\le m} \sum_{j=1}^{m} \left|a_{ij}\right| \tag{20}$$

Let $\lambda = s^2, s = \sigma + i\tau$, $\sigma, \tau \in \mathrm{R}$. We have the following three lemmas.

Lemma 1.9. *When* $|\lambda| \to \infty$, *the following asymptotic formulas hold on*
$0 < x < a_1$,

$$\phi_1(x,\lambda) = \cos(\lambda x) E_m + O\left(|\lambda|^{-1} e^{|\sigma| x}\right) \tag{21}$$

$$\phi_1'(x,\lambda) = -\lambda \sin(\lambda x) E_m + O\left(e^{|\sigma| x}\right) \tag{22}$$

Proof. See [1]. □

Lemma 1.10. *When* $|\lambda| \to \infty$, $\phi_2(x,\lambda)$ *and* $\phi_2'(x,\lambda)$ *have the following asymptotic formulas on* $a_1 < x < a_2$,

$$\phi_2(x,\lambda) = \frac{1}{2}\alpha^+ \exp\left(-i\left(\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right) E_m \left(1 + O\left(\frac{1}{\lambda}\right)\right) \tag{23}$$

$$\phi_2'(x,\lambda) = \frac{1}{2}\alpha^+ (p(x) - \lambda\alpha)\, i \exp\left(-i\left(\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right) E_m + O(1) \tag{24}$$

where $\mu^{\mp}(x) = \mp\alpha x \pm \alpha a_1 + a_1$, $\alpha^{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{\alpha}\right)$.

Proof. Since $\phi_2(x,\lambda)$ is the solution of initial value problem (6) , we have

$$\begin{array}{l}\phi_2(x,\lambda) = \alpha^+ \cos\left[\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right]E_m \\ +\alpha^- \cos\left[\lambda\mu^-(x) + \frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right]E_m + O\left(\frac{1}{\lambda}e^{\sigma\mu^+(x)}\right)\end{array}.$$

We get

$$\begin{array}{l}\phi_2(x,\lambda) = \frac{1}{2}\alpha^+ e^{i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m + \frac{1}{2}\alpha^+ e^{-i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m \\ +\frac{1}{2}\alpha^- e^{i\left[\lambda\mu^-(x)+\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m + \frac{1}{2}\alpha^- e^{-i\left[\lambda\mu^-(x)+\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m + O\left(\frac{1}{\lambda}e^{\sigma\mu^+(x)}\right)\end{array} \tag{25}$$

Let $f(x,\lambda) := O\left(\frac{1}{\lambda}e^{\sigma\mu^+(x)}\right)$ and note that

$$\phi_2(x,\lambda) = \frac{1}{2}\alpha^+ e^{-i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m + (1 + g(x,\lambda)).$$

From a simple computation at equations (25), we get

$$\begin{array}{l}g(x,\lambda) = e^{2i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m + \frac{\alpha-1}{\alpha+1}e^{2i\lambda a_1}E_m + \frac{\alpha-1}{\alpha+1}e^{2i\left[\lambda\alpha(x-a_1)-\frac{v(x)}{\alpha}\right]}E_m \\ +\frac{2e^{i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}}{\alpha^+}f(x,\lambda)E_m.\end{array}$$

Let's examine $g(x,\lambda) = O\left(\frac{1}{\lambda}\right)$ accuracy.

$$\begin{array}{l}\left|g(x,\lambda)\right| \le \left|e^{2i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m\right| + \left|\frac{\alpha-1}{\alpha+1}e^{2i\lambda a_1}E_m\right| + \left|\frac{\alpha-1}{\alpha+1}e^{2i\left[\lambda\alpha(x-a_1)-\frac{v(x)}{\alpha}\right]}E_m\right| \\ +\left|\frac{e^{i\left[\lambda\mu^+(x)-\frac{v(x)}{\alpha}\right]}}{s^+}E_m f(x,\lambda)\right| + \left|\frac{2e^{i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}}{\alpha^+}f(x,\lambda)E_m\right| \\ \le e^{-2\sigma\mu^+(x)}E_m + \left|\frac{s^-}{s^+}\right|e^{-2\sigma a_1}E_m + \left|\frac{s^-}{s^+}\right|e^{-2\sigma\alpha x}E_m + \frac{c}{\lambda}e^{-\sigma\mu^+(x)}e^{\sigma\mu^+(x)}E_m\end{array}$$

Furthermore , $\sigma > \varepsilon|\lambda|$, $\varepsilon > 0$ in D. Thus,$-\sigma < -\varepsilon|\lambda|$ and $e^{-2\sigma\mu^+(x)} < e^{-\varepsilon|\lambda|\mu^+(x)}$
Since $\frac{x}{e^x} \to 0$, $x < ce^{\mu^+(x)}$ $(c > 0)$. Thus, $e^{-2\sigma\mu^+(x)} < \frac{c}{\varepsilon|\lambda|\mu^+(x)}$. We get
$g(x,\lambda) = O\left(\frac{1}{\lambda}\right)$ $\lambda \to \infty$. Hence,

$$\phi_2(x,\lambda) = \frac{1}{2}\alpha^+ \exp\left(-i\left(\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right)\right)E_m\left(1 + O\left(\frac{1}{\lambda}\right)\right), |\lambda| \to \infty.$$

Derivativing both sides of (23) and using the first formula (25), we could get the formula of (24) similarly. □

Lemma 1.11. *When $|\lambda| \to \infty$, $\phi_3(x,\lambda)$ and $\phi_3'(x,\lambda)$ have the following asymptotic formulas on $a_2 < x < \pi$,*

$$\phi_3(x,\lambda) = \frac{1}{2}\beta^+ \exp\left(-i\left(\lambda k^+(x) - \frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right)\right)E_m\left(1 + O\left(\frac{1}{\lambda}\right)\right) \tag{26}$$

$$\phi_3'(x,\lambda) = \frac{1}{2}\beta^+ (p(x) - \lambda\beta)\, i\exp\left(-i\left(\lambda k^+(x) - \frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right)\right)E_m + O(1) \tag{27}$$

where $k^\pm(x) = \pm\beta x \mp \beta a_2 + \mu^+(a_2)$, $s^\pm(x) = \pm\beta x \mp \beta a_2 + \mu^-(a_2)$,$\beta_2^\mp = \frac{1}{2}\left(\alpha_2 \mp \frac{\alpha\beta_2}{\beta}\right)$.

Proof. Since $\phi_3(x,\lambda)$ is the solution of initial value problem (9) , we have

$$\begin{array}{l}\phi_3(x,\lambda) = \beta^+ \cos\left[\lambda k^+(x) - \frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right] + \beta^- \cos\left[\lambda k^-(x) - \frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right] \\ +\beta^- \cos\left[\lambda s^+(x) + \frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right] + \beta^+ \cos\left[\lambda s^-(x) + \frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right] + O\left(\frac{1}{\lambda}e^{\sigma k^+(x)}\right)\end{array}$$

We get

$$\begin{aligned}\phi_3(x,\lambda) &= \tfrac{\beta^+}{2}e^{i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}+\tfrac{\beta^+}{2}e^{-i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}\\ &+\tfrac{\beta^-}{2}e^{i\left[\lambda k^-(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}+\tfrac{\beta^-}{2}e^{-i\left[\lambda k^-(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}\\ &+\tfrac{\beta^-}{2}e^{i\left[\lambda s^+(x)+\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}+\tfrac{\beta^-}{2}e^{-i\left[\lambda s^+(x)+\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}\\ &+\tfrac{\beta^+}{2}e^{i\left[\lambda s^-(x)+\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}+\tfrac{\beta^+}{2}e^{-i\left[\lambda s^-(x)+\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}+O\left(\tfrac{1}{\lambda}e^{\sigma k^+(x)}\right)\end{aligned} \tag{28}$$

Let $f(x,\lambda) := O\left(\frac{1}{\lambda}e^{\sigma k^+(x)}\right)$ and note that

$$\phi_3(x,\lambda) = \frac{\beta^+}{2}e^{-i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m + (1+g(x,\lambda))$$

From a simple calculation at equation (28), we get

$$\begin{aligned}g(x,\lambda) &= e^{2i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m+\tfrac{\beta^-}{\beta^+}e^{2i\left[(\beta\pi-\beta a_2+a_1)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m\\ &+\tfrac{\beta^-}{\beta^+}e^{2i\left[\alpha(a_2-a_1)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m+\tfrac{\beta^-}{\beta^+}e^{2i\mu^+(a_2)}+\tfrac{\beta^-}{\beta^+}e^{2i\beta(\pi-a_2)}\\ &+e^{2ia_1}+e^{2i[\beta\pi-\beta a_2+\alpha a_2-\alpha a_1]}+\frac{e^{i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}}{\beta^+}f(x,\lambda)E_m\end{aligned}$$

Let's examine $g(x,\lambda) = O\left(\frac{1}{\lambda}\right)$ accuracy.

$$\begin{aligned}\left|g(x,\lambda)\right| &\le \left|e^{2i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m\right|+\left|\tfrac{\beta^-}{\beta^+}e^{2i\left[(\beta\pi-\beta a_2+a_1)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m\right|\\ &+\left|\tfrac{\beta^-}{\beta^+}e^{2i\left[\alpha(a_2-a_1)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m\right|+\left|\tfrac{\beta^-}{\beta^+}e^{2i\mu^+(a_2)}E_m\right|+\left|\tfrac{\beta^-}{\beta^+}e^{2i\beta(\pi-a_2)}E_m\right|\\ &+\left|e^{2ia_1}E_m\right|+\left|e^{2i[\beta\pi-\beta a_2+\alpha a_2-\alpha a_1]}E_m\right|+\left|\frac{e^{i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}}{\beta^+}f(x,\lambda)E_m\right|\end{aligned}$$

$$\begin{aligned}&\le e^{-2\sigma k^+(x)}+\left|\tfrac{\beta^-}{\beta^+}\right|e^{-2\sigma k^+(x)}+\left|\tfrac{\beta^-}{\beta^+}\right|e^{-2\sigma a_2}+\left|\tfrac{\beta^-}{\beta^+}\right|e^{-2\sigma a_2}+\left|\tfrac{\beta^-}{\beta^+}\right|e^{-2\sigma\beta x}\\ &+e^{-2\sigma a_1}+e^{-2\sigma k^+(x)}+\tfrac{c}{\lambda}e^{-2\sigma k^+(x)}e^{2\sigma k^+(x)}\end{aligned}$$

In addition to , $\sigma > \varepsilon|\lambda|$, $\varepsilon > 0$ in D . Thus,$-\sigma < -\varepsilon|\lambda|$ and $e^{-2\sigma k^+(x)} < e^{-\varepsilon|\lambda|k^+(x)}$
Since $\frac{x}{e^x} \to 0$, $x < ce^{k^+(x)}$ $(c > 0)$. Thus, $e^{-2\sigma k^+(x)} < \frac{c}{\varepsilon|\lambda|k^+(x)}$. We get
$g(x,\lambda) = O\left(\frac{1}{\lambda}\right)$ $\lambda \to \infty$. Hence,

$$\phi_3(x,\lambda) = \frac{1}{2}\beta^+\exp\left(-i\left(\lambda k^+(x) - \frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right)\right)E_m\left(1+O\left(\frac{1}{\lambda}\right)\right), |\lambda| \to \infty.$$

Derivativing both sides of (26) and using the first formula (28), we could get the formula of (27) similarly. □

2. Multiplicities of eigenvalues of the vectorial problem

In the section, we find the conditions on the potential matrix function $(2\lambda p(x) + q(x))$, under some conditions, the problem (1) − (3) can only have a finite number of eigenvalues with multiplicity m. Where $p(x) \in W_2^1[0,\pi]$ ve $p(x) = \left\{p_{ij}(x)\right\}_{i,j=1}^m$, $q(x) \in L_2[0,\pi]$ and $q(x) = \left\{q_{ij}(x)\right\}_{i,j=1}^m$.

Theorem 2.1. *Let $m \ge 2$. Assume that, for some $i, j \in \{1, 2, ..., m\}$ with $i \ne j$*

either

$$(i)\quad \begin{array}{l} \int_0^{a_1} p_{ij}(x)\,dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} p_{ij}(x)\,dx + \frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\,dx \neq 0 \\ \int_0^{a_1} q_{ij}(x)\,dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} q_{ij}(x)\,dx + \frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} q_{ij}(x)\,dx \neq 0 \end{array} \tag{29}$$

or

$$(ii)\quad \begin{array}{l} \int_0^{a_1}\left[p_{ii}(x)-p_{jj}(x)\right]dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left[p_{ii}(x)-p_{jj}(x)\right]dx + \frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}\left[p_{ii}(x)-p_{jj}(x)\right]dx \neq 0 \\ \int_0^{a_1}\left[q_{ii}(x)-q_{jj}(x)\right]dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left[q_{ii}(x)-q_{jj}(x)\right]dx + \frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}\left[q_{ii}(x)-q_{jj}(x)\right]dx \neq 0 \end{array} \tag{30}$$

where $\alpha^{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{\alpha}\right)$, $\beta^{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{\beta}\right)$*.Then, with finitely many exceptions. The multiplicities of the eigenvalues of the problem* (1) − (3) *are at most* $m-1$*.*

Proof. (*i*) We assume that (29) holds. Suppose, to the contrary, that there exists a sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ whole multiplicities are all m. Obviously, $\lambda_n \to \infty$ as $n \to \infty$. From the equations in (9). Denoting $\phi_3(x,\lambda) = \left\{y_{ij}^+(x)\right\}_{i,j=1}^m$, when $\lambda = \lambda_n$ for $n = 1, 2, ...$, we get

$$\left(y_{ii}^+\right)''(x) + \left(\lambda - \left(2\lambda p_{ii}(x) + q_{ii}(x)\right)\right) y_{ii}^+(x) - \sum_{k\neq i}\left(2\lambda p_{ii}(x) + q_{ii}(x)\right) y_{ki}^+(x) = 0 \tag{31}$$

and

$$\left(y_{ij}^+\right)''(x) + \left(\lambda - \left(2\lambda p_{ii}(x) + q_{ii}(x)\right)\right) y_{ij}^+(x) - \sum_{k\neq j}\left(2\lambda p_{ii}(x) + q_{ii}(x)\right) y_{kj}^+(x) = 0 \tag{32}$$

Multiplying (31) and (32) by $y_{ij}^+(x)$ and $y_{ii}^+(x)$ respectively, then subtructing one fom the other and using (26), nothing that the eigenvalues of the problem are all real, we have

$$\begin{aligned} \left(\left(y_{ii}^+\right)'(x)\, y_{ij}^+(x) - y_{ii}^+(x)\left(y_{ij}^+\right)'(x)\right)' &= \sum_{k\neq i}\left(2\lambda p_{ik}(x) + q_{ik}(x)\right)\left(y_{ki}^+(x)\, y_{ij}^+(x) - y_{ii}^+(x)\, y_{kj}^+(x)\right) \\ = \left(2\lambda p_{ij}(x) + q_{ij}(x)\right)&\left[y_{ij}^+(x)\, y_{ji}^+(x) - y_{ii}^+(x)\, y_{ij}^+(x)\right] \\ &+ \textstyle\sum_{k\neq i,j}\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left(y_{ki}^+(x)\, y_{ij}^+(x) - y_{ii}^+(x)\, y_{kj}^+(x)\right) \\ = -\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)&\left[\frac{(\beta^+)^2}{4}\cos^2\left(\lambda k^+(x) - \frac{1}{\beta}\int_{a_2}^{x} p(t)\,dt\right)\right] + O\left(1 + \frac{1}{\lambda}\right) \end{aligned} \tag{33}$$

similarly, from the equations in (6), denoting $\phi_2(x,\lambda) = \left\{y_{ij}^-(x)\right\}_{i,j=1}^m$, we get

$$\begin{aligned} &\left(\left(y_{ii}^-\right)'(x)\, y_{ij}^-(x) - y_{ii}^-(x)\left(y_{ij}^-\right)'(x)\right)' = \\ &\quad -\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\alpha^+)^2}{4} cos^2\left(\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right] O\left(1 + \frac{1}{\lambda}\right) \end{aligned} \tag{34}$$

similarly, from the equations in (4), denoting $\phi_1(x,\lambda) = \left\{y_{ij}^0(x)\right\}_{i,j=1}^m$, we get

$$\left(\left(y_{ii}^0\right)'(x)\, y_{ij}^0(x) - y_{ii}^0(x)\left(y_{ij}^0\right)'(x)\right)' = -\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\cos^2(\lambda x)\right] + O\left(\frac{1}{\lambda}\right) \tag{35}$$

When λ is an eigenvalue with multiplicity m, we have $\phi_3'(\pi,\lambda) = 0_m$. By integrating both sides of (33) from a_2 to π, for $\lambda_n \to \lambda$ and $n \to \infty$, we obtain

$$\begin{aligned} &-\left(\left(y_{ii}^+\right)'(x)\, y_{ij}^+(x) - y_{ii}^+(x)\left(y_{ij}^+\right)'(x)\right) = \\ &\quad = \int_{a_2}^{\pi}\left[-\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\beta^+)^2}{4}\cos^2\left(\lambda k^+(x) - \frac{1}{\beta}\int_{a_2}^{x} p(t)\,dt\right)\right] + O\left(\frac{1}{\lambda}\right)\right] dx \end{aligned} \tag{36}$$

By integrating both sides of (34) from a_1 to a_2 and appliying the boundary condition

$$\begin{aligned} &-\left(\left(y_{ii}^{-}\right)'(x)\, y_{ij}^{-}(x) - y_{ii}^{-}(x)\left(y_{ij}^{-}\right)'(x)\right) = \\ &= \int_{a_1}^{a_2}\left[-\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\alpha^+)^2}{4}\cos^2\left(\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right] + O\left(\frac{1}{\lambda}\right)\right]dx \end{aligned} \tag{37}$$

By integrating both sides of (35) from 0 to a_1 and applying the boundary condition $\phi_1'(0,\lambda) = 0_m$, we obtain, for $\lambda_n \to \lambda$ and $n \to \infty$,

$$\left(\left(y_{ii}^{0}\right)'(x)\, y_{ij}^{0}(x) - y_{ii}^{0}(x)\left(y_{ij}^{0}\right)'(x)\right) = -\int_0^{a_1}\left[\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\cos^2(\lambda x)\right] + O\left(\frac{1}{\lambda}\right)\right]dx \tag{38}$$

Sum the above (36), (37) and (38), then use the initial conditions at point $x = a_1$ and $x = a_2$, we get

$$\begin{aligned} 0 = &-\int_0^{a_1}\left[\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\cos^2(\lambda x)\right] + O\left(\frac{1}{\lambda}\right)\right]dx \\ &+ \int_{a_1}^{a_2}\left[-\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\alpha^+)^2}{4}\cos^2\left(\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right]\right]dx \\ &+ \int_{a_2}^{\pi}\left[-\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{\left(\beta^+\right)^2}{4}\cos^2\left(\lambda k^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right]\right]dx + O\left(\frac{1}{\lambda}\right) \end{aligned}$$

By a simple computation, one can see that

$$\begin{aligned} &\int_0^{a_1}\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)dx + \frac{\left(\beta^+\right)^2}{4}\int_{a_2}^{\pi}\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)dx \\ &= -\int_0^{a_1}\left[\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\cos 2\lambda x\right]dx \\ &- \int_{a_1}^{a_2}\left[\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\alpha^+)^2}{4}\cos 2\left(\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right]\right]dx \\ &- \int_{a_2}^{\pi}\left[\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{\left(\beta^+\right)^2}{4}\cos 2\left(\lambda k^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right]\right]dx + O\left(\frac{1}{\lambda}\right) \end{aligned} \tag{39}$$

$$\begin{aligned} &= -2\lambda\int_0^{a_1} p_{ij}(x)\cos(2\lambda x)\,dx - \int_0^{a_1} q_{ij}(x)\cos(2\lambda x)\,dx - 2\lambda\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} p_{ij}(x)\cos 2\lambda\mu^+(x)\cos\frac{2v(x)}{\alpha}dx \\ &-\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} q_{ij}(x)\cos 2\lambda\mu^+(x)\cos\frac{2v(x)}{\alpha}dx - 2\lambda\frac{(\alpha^+)^2}{4}in\int_{a_1}^{a_2} p_{ij}(x)\,2\lambda\mu^+(x)\sin\frac{2v(x)}{\alpha}dx \\ &-\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} q_{ij}(x)\sin 2\lambda\mu^+(x)\sin\frac{2v(x)}{\alpha}dx - 2\lambda\frac{\left(\beta^+\right)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\cos 2\lambda k^+(x)\cos\frac{2t(x)}{\beta}dx \\ &-\frac{\left(\beta^+\right)^2}{4}\int_{a_2}^{\pi} q_{ij}(x)\cos 2\lambda k^+(x)\cos\frac{2t(x)}{\beta}dx - 2\lambda\frac{\left(\beta^+\right)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\sin 2\lambda k^+(x)\sin\frac{2t(x)}{\beta}dx \\ &-\frac{\left(\beta^+\right)^2}{4}\int_{a_2}^{\pi} q_{ij}(x)\sin 2\lambda k^+(x)\sin\frac{2t(x)}{\beta}dx \end{aligned}$$

where $v(x) = \int_{a_1}^{x} p(t)\,dt$, $t(x) = \int_{a_2}^{x} p(t)\,dt$. Then, we obtain, for $\lambda_n \to \infty$ and $n \to \infty$,

$$\begin{aligned} &= -2\int_0^{a_1} p_{ij}(x)\cos(2\lambda x)\,dx - 2\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} p_{ij}(x)\cos 2\lambda\mu^+(x)\cos\frac{2v(x)}{\alpha}dx \\ &-2\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} p_{ij}(x)\,2\lambda\mu^+(x)\sin\frac{2v(x)}{\alpha}dx - 2\frac{\left(\beta^+\right)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\cos 2\lambda k^+(x)\cos\frac{2t(x)}{\beta}dx \\ &-2\frac{\left(\beta^+\right)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\sin 2\lambda k^+(x)\sin\frac{2t(x)}{\beta}dx \end{aligned}$$

By Riemann-Lebesgue Lemma, the right side of (39) approaches 0 as $\lambda_n = \lambda$ and $n \to \infty$. This implies that

$$\begin{aligned} &\int_0^{a_1} p_{ij}(x)\,dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} p_{ij}(x)\,dx + \frac{\left(\beta^+\right)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\,dx = 0 \\ &\int_0^{a_1} q_{ij}(x)\,dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} q_{ij}(x)\,dx + \frac{\left(\beta^+\right)^2}{4}\int_{a_2}^{\pi} q_{ij}(x)\,dx = 0 \end{aligned}$$

We have reached a contradiction. The conclusion for this case is proved.

(*ii*) Next, we assume that

$$\int_0^{a_1}\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)dx=0$$

or
$\int_0^{a_1} s_{ij}(x)\,dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} s_{ij}(x)\,dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} s_{ij}(x)\,dx=0, \forall i\neq j,$
where $s_{ij}(x)=\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)$.
and
$\int_0^{a_1}\left[s_{ii}(x)-s_{jj}(x)\right]dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left[s_{ii}(x)-s_{jj}(x)\right]dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}\left[s_{ii}(x)-s_{jj}(x)\right]dx\neq 0$
without loss of generality, we assume that for $i=1, j=2$

$$\int_0^{a_1}\left[s_{11}(x)-s_{22}(x)\right]dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left[s_{11}(x)-s_{22}(x)\right]dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}\left[s_{11}(x)-s_{22}(x)\right]dx\neq 0$$

$$K=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{bmatrix} K=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{bmatrix}$$

and $y=K\cdot t$. Then, the problem $(1)-(3)$ becomes

$$\left.\begin{array}{l} t''+\left(\lambda^2\delta(x)-R(x)\right)t=0 \\ t'(0)=t'(\pi)=0 \end{array}\right\} \tag{40}$$

where $R(x)=K^{-1}S(x)K$. By making a simple computation, we get

$$R(x)=\begin{bmatrix}\frac{1}{4}(s_{11}+s_{22})+s_{12} & \frac{1}{4}(s_{22}-s_{11}) & * & * & * \\ \frac{1}{4}(s_{22}-s_{11}) & \frac{1}{4}(s_{11}+s_{22})+s_{12} & * & * & * \\ * & * & q_{33} & \cdots & \\ * & * & \vdots & \ddots & \\ * & * & & \cdots & q_{mm}\end{bmatrix}(x)$$

We note that the two poblems $(1)-(3)$and (40)have exactly the same spectral structure. Denote $R(x)=\left\{r_{ij}(x)\right\}_{i,j=1}^{m}$. Since

$$\int_0^{a_1} r_{12}(x)\,dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} r_{12}(x)\,dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} r_{12}(x)\,dx=$$
$$\int_0^{a_1}\left[s_{11}(x)-s_{22}(x)\right]dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left[s_{11}(x)-s_{22}(x)\right]dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}\left[s_{11}(x)-s_{22}(x)\right]dx\neq 0$$

By part (i) , the conclusion of the theorem holds for the problem (40), and hence holds for the problem $(1)-(3)$. □

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A New Family of Odd Generalized Nakagami (Nak-G) Distributions

Ibrahim Abdullahi[a], Obalowu Job[b]

[a] *Yobe State University, Department of Mathematics and Statistics*
[b] *University of Ilorin, Department of Statistics*

Abstract. In this article, we proposed a new family of generalized Nak-G distributions and study some of its statistical properties, such as moments, moment generating function, quantile function, and probability Weighted Moments. The Renyi entropy, expression of distribution order statistic and parameters of the model are estimated by means of maximum likelihood technique. We prove, by providing three applications to real-life data, that Nakagami Exponential (Nak-E) distribution could give a better fit when compared to its competitors.

1. Introduction

There has been recent developments focus on generalized classes of continuous distributions by adding at least one shape parameters to the baseline distribution, studying the properties of these distributions and using these distributions to model data in many applied areas which include engineering, biological studies, environmental sciences and economics. Numerous methods for generating new families of distributions have been proposed [8] many researchers. The beta-generalized family of distribution was developed , Kumaraswamy generated family of distributions [5], Beta-Nakagami distribution [19], Weibull generalized family of distributions [4], Additive weibull generated distributions [12], Kummer beta generalized family of distributions [17], the Exponentiated-G family [6], the Gamma-G (type I) [21], the Gamma-G family (type II) [18], the McDonald-G [1], the Log-Gamma-G [3], A new beta generated Kumaraswamy Marshall-Olkin-G family of distributions with applications [11], Beta Marshall-Olkin-G family [2] and Logistic-G family [20].

The Nakagami distribution is a continuous probability distribution related to gammadistribution with applications in measuring alternation of wireless signal traversing multiple paths. The Nakagami distribution has two parameters; $\lambda \geq 0.5$ is the shape parameter and $\beta >$ is scale parameter. The cumulative distribution function (cdf) is given by

$$F(x;\lambda,\beta) = \int_0^x \frac{2\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} t^{2\lambda-1} \exp\left(\frac{-\lambda}{\beta} t^2\right) dt \tag{1}$$

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probability density function (pdf) is given by

$$f(x;\lambda,\beta)=\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}t^{2\lambda-1}\exp\left(\frac{-\lambda}{\beta}t^{2}\right);\ x>0 \tag{2}$$

It reduces to Rayleigh distribution when $\lambda = 1$ and half normal distribution when $\lambda = 0.5$ The main aim of this study is to develop a new family of generated distributions for the generalized Nakagami distribution and study some of the mathematical and statistical properties of the proposed family of distributions.
This paper is organized as follows: In section 2, the Nakagami (Nak-G) family of distributions was defined. In section 3, a useful linear representation for its probability density function (pdf) was obtained, some mathematical properties and parameter estimators using maximum likelihood estimation are derived. In section 4, the goodness of fit of the distribution using real data was illustrated while section 5, gives the conclusion.

2. Constructions of the Nak-G Distributions

In this section, the probability density function (pdf), cumulative distribution function (cdf), survival function, hazard rate function (hrf), mean remaining lifetime function, order statistic, moment, moment generating function, Renyi and q entropies of Nak-G distributions are derived. We obtain the Nak-G distribution by considering the Nakagami generator applied to the odd ratio $G(x;\eta)/\bar{G}(x;\eta)$ where $G(x;\eta)$ is the cdf of baseline distribution and $\bar{G}(x;\eta)=1-G(x;\eta)$.
Let denote the cdf and pdf of baseline model, η is the parameter vector of the baseline distribution. Based on the family of distributions we define the cdf of Nak-G by replacing x with in equation (1) it become Nak-G distribution.

$$F(x;\lambda,\beta,\eta)=\int_{0}^{\frac{G(x;\eta)}{\bar{G}(x;\eta)}}\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}t^{2\lambda-1}\exp\left(\frac{-\lambda}{\beta}t^{2}\right)dt \tag{3}$$

$$F(x;\lambda,\beta,\eta)=\frac{1}{\Gamma\lambda}\gamma\left(\lambda,\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^{2}\right)$$

$$F(x;\lambda,\beta,\eta)=\gamma_{*}\left(\lambda,\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^{2}\right) \tag{4}$$

Using expansion of incomplete gamma ratio function $\gamma_{*}(a,x)$ in [7] the above equation (4) can be expressed as:

$$\gamma_{*}\left(\lambda,\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^{2}\right)=\sum_{q=0}^{\infty}\frac{(-1)^{q}\left\{\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^{2}\right\}^{\lambda+q}}{(\lambda-1)!q!(\lambda+q)} \tag{5}$$

The pdf of the Nak-G is given by

$$f(x)=\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}g(x;\eta)\frac{[G(x;\eta)]^{2\lambda-1}}{[1-G(x;\eta)]^{2\lambda+1}}\exp\left(-\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^{2}\right);\ x\in\Re \tag{6}$$

A random variable X with pdf in equation (6) is denoted by $X\sim Nak-G(x;\eta)$ the survival function and hazard rate function (hrf) of X are given by:

$$S(x) = 1 - \gamma_* \left(\lambda, \frac{\lambda}{\beta} \left(\frac{G(x;\eta)}{\bar{G}(x;\eta)} \right)^2 \right) \tag{7}$$

and

$$h(x) = \frac{\frac{2\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} g(x;\eta) \frac{[G(x;\eta)]^{2\lambda-1}}{[1-G(x;\eta)]^{2\lambda+1}} \exp\left(-\frac{\lambda}{\beta} \left(\frac{G(x;\eta)}{\bar{G}(x;\eta)} \right)^2 \right)}{1 - \gamma_* \left(\lambda, \frac{\lambda}{\beta} \left(\frac{G(x;\eta)}{\bar{G}(x;\eta)} \right)^2 \right)} \tag{8}$$

2.1. Linear Representation

In this section, we derive some very useful linear representation for the Nak-G density function. Note that.

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \tag{9}$$

Therefore, applying equation (9) to (6)

$$f(x) = \frac{2\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} g(x;\eta) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\lambda}{\beta} \right)^k \frac{[G(x;\eta)]^{2(\lambda+k)-1}}{[1-G(x;\eta)]^{2(\lambda+k)+1}} \tag{10}$$

Consider the binomial expansion theorem

$(1-z)^{-b} = \sum\limits_{j=0}^{\infty} \binom{b+j-1}{j} z^j, |z| < 1, b > 0$ then

$$[1-G(x;\eta)]^{-2(\lambda+k)+1} = \sum_{j=0}^{\infty} \binom{2(\lambda+k)+j}{j} [G(x;\eta)]^j, [2(\lambda+k)+1] > 0 \tag{11}$$

Therefore, applying equation (11) to (10)

$$f(x) = \frac{2\lambda^\lambda \, [2(\lambda+k)+2j(\lambda+k)+j]}{\Gamma(\lambda)\beta^\lambda \, [2(\lambda+k)+j]} \sum_{k,j=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\lambda}{\beta} \right)^k \binom{2(\lambda+k)+j}{j}$$
$$g(x;\eta)[G(x;\eta)]^{2(\lambda+k)+j-1} \tag{12}$$

Also, the pdf equation (12) can be written as

$$f(x) = \sum_{k,j=0}^{\infty} \pi_{k,j} h_{2(\lambda+k)+j}(x) \tag{13}$$

where

$$\pi_{k,j} = \frac{2}{\Gamma(\lambda)} \frac{(-1)^k}{k!} \left(\frac{\lambda}{\beta} \right)^{\lambda+k} \binom{2(\lambda+k)+j}{j}$$

and

$$h_{2(\lambda+k)+j}(x) = (2(\lambda+k)+j) g(x;\eta) [G(x;\eta)]^{2(\lambda+k)+j-1}$$

Equation (13) can be well-defined as an infinite linear combination of exponentiated -G (exp $-G$) densities. Similarly, the cdf of the Nak-G family can also be expressed as a linear combination of exponentiated-G (exp $-G$) cdfs given by

$$F(x) = \sum_{k,j}^{\infty} \pi_{k,j} H_{2(\lambda+k)+j}(x) \tag{14}$$

where $H_{2(\lambda+k)+j}(x) = [G(x;\eta)]^{2(\lambda+k)+j}$ is the cdf of the exp $-G$ family with power parameter.

3. The Nakagami Exponential (NE) Distribution

Our baseline distribution, the Exponential distribution with parameter α has its cdf and pdf given by:

$$G(x;\alpha) = 1 - e^{-\alpha x} \tag{15}$$

$$g(x;\alpha) = \alpha e^{-\alpha x}; \alpha > 0, x > 0 \tag{16}$$

Substituting equation (15) and (16) in (4) and (6) then, the cdf and pdf of NE distribution can be written as

$$F_{NE}(x) = \gamma_{*}\left(\lambda, \frac{\lambda}{\beta}\left(\frac{1 - e^{-\alpha x}}{e^{-\alpha x}}\right)^2\right) \tag{17}$$

$$f_{NE}(x) = \frac{2\lambda^{\lambda} \alpha e^{-\alpha x} \left(1 - e^{-\alpha x}\right)^{2\lambda - 1}}{\Gamma(\lambda)\beta^{\lambda} \left(e^{-\alpha x}\right)^{2\lambda + 1}} e^{-\frac{\lambda}{\beta}\left(\frac{1 - e^{-\alpha x}}{e^{-\alpha x}}\right)^2} \tag{18}$$

3.1. *Investigation of the Proposed (NE) Distribution for PDF*

To show that the proposed distribution is a proper pdf, we proceed to show as follows:

$$\int_0^{\infty} f(x)dx = 1 \tag{19}$$

$$\int_0^{\infty} \frac{2\lambda^{\lambda} \alpha e^{-\alpha x} \left(1 - e^{-\alpha x}\right)^{2\lambda - 1}}{\Gamma(\lambda)\beta^{\lambda} \left(e^{-\alpha x}\right)^{2\lambda + 1}} e^{-\frac{\lambda}{\beta}\left(\frac{1 - e^{-\alpha x}}{e^{-\alpha x}}\right)^2} dx = 1$$

$$y = \frac{\lambda}{\beta}\left(\frac{1 - e^{-\alpha x}}{e^{-\alpha x}}\right)^2 \tag{20}$$

$$\frac{\partial y}{\partial x} = \frac{\lambda}{\beta}\left(\frac{1 - e^{-\alpha x}}{e^{-\alpha x}}\right)\left(\frac{1 - e^{-\alpha x}}{e^{-\alpha x}}\right)$$

$$\partial x = \frac{\beta e^{-2\alpha x}}{2\alpha\lambda(1 - e^{-}\alpha x)} \partial y$$

$$\int_0^{x} f(x)dx \equiv \frac{\lambda^{\lambda - 1}}{\Gamma(\lambda)\beta^{\lambda - 1}} \int_0^{\infty} \frac{(1 - e^{-\alpha})^{2\lambda - 2}}{(e^{-\alpha})^{2\lambda - 2}} e^{-y} \partial y \tag{21}$$

from (20)

$$\left(\frac{y\beta}{\lambda}\right)^{\frac{1}{2}} \tag{22}$$

Therefore, from (21) and (22) we obtained

$$\frac{1}{\Gamma(\lambda)}\int_0^{\infty} y^{\lambda-1}e^{-y}\partial y = 1 \tag{23}$$

Hence Nakagami Exponential Distribution is pdf

3.2. *Expansion for Nakagami Exponential Distribution*

In this part a simple form for the probability density function of NE distribution is derived. Applying equation (9) into (18) we obtained

$$f_{NE}(x) = \frac{2\alpha}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}}\left(1-e^{-\alpha x}\right)^{2(\lambda+k)-1}\left(e^{-\alpha x}\right)^{-2(\lambda+k)} \tag{24}$$

The binomial expansion of $(1-e^{-\alpha x})$ can be expressed as $\sum_{i=0}^{\infty}(-1)^i\binom{2(\lambda+k)-1}{i}e^{-\alpha x}$ Therefore, equation (24) will take the following form

$$f_{NE}(x) = \frac{2\alpha}{\Gamma(\lambda)}\sum_{k,i=0}^{\infty}\frac{(-1)^{k+i}}{k!}\frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}}\binom{2(\lambda+k)-1}{i}\left(e^{-\alpha x}\right)^{i-2(\lambda+k)} \tag{25}$$

Therefore, the NE pdf distribution is reduced to

$$f_{NE}(x) = \frac{2\alpha}{\Gamma(\lambda)}\frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}}\sum_{k,i=0}^{\infty}\omega_{k,i}\left(e^{-\alpha x}\right)^{i-2(\lambda+k)} \tag{26}$$

where $\omega_{k,i} = \frac{(-1)^{k+i}}{k!}\binom{2(\lambda+k)-1}{i}$.

While the cumulative distribution function (cdf), survival function and hazard functions are given respectively by equations (27), (28) and (29).

$$F_{NE}(x) = \gamma_*\left[\lambda, \frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2\right] \tag{27}$$

$$S_{NE}(x) = 1-\gamma_*\left[\lambda, \frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2\right] \tag{28}$$

$$H_{NE}(x) = \frac{\frac{2\alpha}{\Gamma(\lambda)}\frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}}\sum_{k,i=0}^{\infty}\omega_{k,i}\left(e^{-\alpha x}\right)^{i-2(\lambda+k)}}{1-\gamma_*\left[\lambda, \frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2\right]} \tag{29}$$

3.3. *Some Mathematical and Statistical Properties*

In this section, some general mathematical and statistical properties of Nak-G distribution are derived.

3.4. Moment and Moment Generating Function

In this subsection, the r^{th} moment and moment generating function for Nak-G distribution will be derived. The rth moment of random variable can be obtained from pdf equation in (13) as follows;

$$\mu_r' = \int_0^\infty x^r f(x)\partial x = \pi_{k,j} \sum_{k,j=0}^{\infty} x^r h_{2(\lambda+k)+j}(x)\partial x$$

therefore,

$$\mu_r' = \pi_{k,j} I_{r,2(\lambda+k)+j}, r = 1,2,3,\ldots \tag{30}$$

where

$$I_{r,2(\lambda+k)+j} = \sum_{k,j=0}^{\infty} x^r h_{2(\lambda+k)+j}(x)\partial x$$

The mean and variance of Nak-G distribution are obtained, respectively as follows

$$E(x) = \pi_{k,j} I_{r,2(\lambda+k)+j} \tag{31}$$

where,

$$I_{r,2(\lambda+k)+j} = \sum_{k,j=0}^{\infty} x h_{2(\lambda+k)+j}(x)\partial x$$

and

$$Var(x) = \pi_{k,j} I_{2,2(\lambda+k)+j} - \left[\pi_{k,j} I_{1,2(\lambda+k)+j}\right]^2 \tag{32}$$

, where

$$I_{2,2(\lambda+k)+j} = \sum_{k,j=0}^{\infty} x^2 h_{2(\lambda+k)+j}(x)\partial x \tag{33}$$

From equation (30) the measures of skewness γ_1 and kurtosis γ_2 of Nak-G distribution can be expressed as follows

$$\gamma_1 = \frac{\mu_3' - 3\mu_2'\mu_1'\mu_1'^3}{(\mu_2' - \mu_1'^2)^{\frac{3}{2}}}, \tag{34}$$

$$\gamma_2 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}{(\mu_2' - \mu_1'^2)^2} \tag{35}$$

Furthermore, the moment generating function can be obtained by using pdf equation (13) as follows

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r \mu_r'}{r!} = \sum_{r=0}^{\infty} \frac{t^r \pi_{k,j} I_{r,2(\lambda+k)+j}}{r!} \tag{36}$$

3.4.1. Moment for Nakagami Exponential Distribution

moment can be obtained by using pdf in equation (26) as follows

$$E(X^r) = \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\alpha}{\Gamma(\lambda)\beta^{\lambda+k}} \int_0^{\infty} x^r e^{-\alpha x[i-2(\lambda+k)]} \partial x \tag{37}$$

Let

$$u = \alpha x[i - 2(\lambda + k)] \Rightarrow \frac{\partial x}{\partial y} = \alpha[i - 2(\lambda + k)]$$

$$\frac{\partial u}{\alpha[i - 2(\lambda + k)]} = \partial x$$

$$E(X^r) = \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\alpha}{\Gamma(\lambda)\beta^{\lambda+k}} \int_0^{\infty} \frac{u^r}{\alpha^r[i - 2(\lambda + k)]^r} e^{-u} \frac{\partial u}{\alpha[i - 2(\lambda + k)]}$$

$$= \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\alpha}{\Gamma(\lambda)\beta^{\lambda+k}} \int_0^{\infty} \frac{u^r}{\alpha^{r+1}[i - 2(\lambda + k)]^{r+1}} e^{-u} \partial u$$

$$E(X^r) = \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\alpha\Gamma(r + 1)}{\Gamma(\lambda)\beta^{\lambda+k}\alpha^{r+1}[i - 2(\lambda + k)]^{r+1}};\ r = 1, 2, 3, \ldots \tag{38}$$

The mean and variance of NE distribution are obtained, respectively as follows

$$E(X) = \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\Gamma(2)}{\Gamma(\lambda)\beta^{\lambda+k}\alpha[i - 2(\lambda + k)]^2} \tag{39}$$

$$Var(x) = \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{4\lambda^{\lambda+k}\Gamma(2)}{\Gamma(\lambda)\beta^{\lambda+k}\alpha^2[i - 2(\lambda + k)]^3} - \left[\sum_{r=0}^{\infty} \omega_{k,j} \frac{2\lambda^{\lambda+k}\Gamma(2)}{\Gamma(\lambda)\beta^{\lambda+k}\alpha[i - 2(\lambda + k)]^2}\right]^2 \tag{40}$$

Furthermore, the moment generating function can be obtained by using pdf in equation (26) as follows

$$M_X(t) = E(e^{tX}) = \frac{2\alpha}{\Gamma(\lambda)} \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}} \int_0^{\infty} e^{-x[\alpha[i-2(\lambda+k)]-t]} \partial x$$

Therefore, the moment generating function of NE distribution takes the following form

$$M_X(t) = \frac{2\alpha}{\Gamma(\lambda)} \sum_{k,i=0}^{\infty} \omega_{k,j} \frac{\lambda^{\lambda+k}}{\beta^{\lambda+k}[\alpha[i - 2(\lambda + k)] - t]}. \tag{41}$$

3.5. Probability Weighted Moments

[10] stated that for a random variable X, the Probability Weighted Moments (pwm) is given by:

$$\varphi_{s,r} = E\left[X^s F(x)^r\right] = \int_{-\infty}^{\infty} x^s F(x)^r f(x)\partial x \tag{42}$$

we formally define PWM of Nak-G by means of equation (4) and (13)

$$\varphi_{s,r} = \int_0^{\infty} x^s \gamma_* \left[\left(\lambda, \frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^2\right)\right]^r \sum_{k,j=0}^{\infty} \pi_{k,j,i,b} h_{2(\lambda+k)+j}(x)\partial x \tag{43}$$

$$\varphi_{s,r} = \int_0^{\infty} x^s \rho_{k,j,i,b} h_{2[\lambda(r+1)+i+k]+b+j}(x)\partial x \tag{44}$$

where,

$$\rho_{k,j,i,b} = \frac{\sum_{k,j,i,b=0}^{\infty} c_{r,i}\binom{2(\lambda r+i)+b-1}{b}\left(\frac{\lambda}{\beta}\right)^{\lambda r+1}\pi_{k,j}}{[\Gamma(\lambda)]^r}$$

3.6. Measures of Uncertainty

In this subsection, Renyi entropy will be mentioned as an important measure of uncertainty. The Rényi entropy of a random variable X is defined mathematically as follows:

$$I_R(\sigma) = \frac{1}{1-\sigma} log\left(\int_0^{\infty} f^{\sigma}(x)\partial x\right)$$

Where $\sigma > 0$ and $\sigma \neq 1$. Based on $f(x)$ of any distribution. From equation (18)

$$f_{NE}^{\sigma}(x) = \frac{2^{\sigma}\left(\lambda^{\lambda}\right)^{\sigma}\alpha^{\sigma}e^{-\sigma\alpha x}\left(1-e^{-\alpha x}\right)^{\sigma(2\lambda-1)}}{\left(\Gamma(\lambda)\right)^{\sigma}\beta^{\sigma\lambda}\left(e^{-\alpha x}\right)^{\sigma(2\lambda+1)}} e^{-\sigma\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2} \tag{45}$$

Since the power series for the following exponential function can be expressed as

$$e^{-\sigma\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\sigma\frac{\lambda}{\beta}\right)^i\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^{2i}$$

Therefore equation (45) can be expressed as

$$f_{NE}^{\sigma}(x) = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{2\alpha}{\Gamma(\lambda)}\right)^{\sigma}\left(\frac{\lambda}{\beta}\right)^{\lambda\sigma+i}\sigma^i\frac{\left(1-e^{-\alpha x}\right)^{\sigma(2\lambda-1)+2i}}{\left(e^{-\alpha x}\right)^{2\sigma\left(\lambda+\frac{i}{\sigma}\right)}} \tag{46}$$

therefore, (46) is reduced to

$$f_{NE}^{\sigma}(x) = \sum_{i.j=0}^{\infty}\tau_{i,j}e^{\alpha x[2(\sigma\lambda+i)-j]} \tag{47}$$

where

$$\tau_{i,j} = \frac{(-1)^{i+j}}{i!}\binom{\sigma(2\lambda-1)2i}{j}\left(\frac{2\sigma}{\Gamma(\lambda)}\right)^{\sigma}\left(\frac{\lambda}{\beta}\right)^{\lambda\sigma+i}\sigma^{i}$$

since

$$\int_0^\infty f_{NE}^{\sigma}(x)\partial x = \int_0^\infty \sum_{i.j=0}^{\infty} \tau_{i,j} e^{\alpha x[2(\sigma\lambda+i)-j]}\partial x;$$

$$\int_0^\infty f_{NE}^{\sigma}(x)\partial x = \sum_{i.j=0}^{\infty} \frac{\tau_{i,j}}{\alpha[j-2(\sigma\lambda+i)]} \tag{48}$$

therefore, $I_R(\sigma)$ reduces to

$$I_R(\sigma) = \frac{1}{1-\sigma} log\left[\sum_{i.j=0}^{\infty} \frac{\tau_{i,j}}{\alpha[j-2(\sigma\lambda+i)]}\right] \tag{49}$$

3.7. Distribution of Order Statistic

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. denote the order statistics of a random sample, $X_1, X_2, \ldots, X_n$ from a Nak-G distribution with cdf equation (6) and pdf equation (5). Then the pdf of $X_{(j)}$ is given by

$$f_{x_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\sum_{z=0}^{n-j}(-1)^z\binom{n-j}{z} f_X(x)\,[F_X(x)]^{z+j-1} \tag{50}$$

$$f_{x_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\sum_{z=0}^{n-j}(-1)^z\binom{n-j}{z}\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}} g(x;\eta)\frac{[G(x;\eta)]^{2\lambda-1}}{[1-G(x;\eta)]^{2\lambda+1}}\exp\left(-\frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^2\right)$$
$$\left[\gamma_*\left(\lambda, \frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^2\right)\right]^{z+j-1}$$

3.8. The Asymptotic Properties

We study the asymptotic behavior of NE distribution with a view to influential its performance limit as $x \to \infty$ is 0 and the limit as $x \to 0$ is 0.

Proof:

These can be achieved as follows by taking the limiting behavior of the NE density function in equation (18).

$$\lim_{x\to\infty} f_{NE}(x) = \lim_{x\to\infty}\left[\frac{2\lambda^{\lambda}\alpha e^{-\alpha x}\,(1-e^{-\alpha x})^{2\lambda-1}}{\Gamma(\lambda)\beta^{\lambda}\,(e^{-\alpha x})^{2\lambda+1}} e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2}\right] = 0$$

$$\lim_{x\to 0} f_{NE}(x) = \lim_{x\to 0}\left[\frac{2\lambda^{\lambda}\alpha e^{-\alpha x}\,(1-e^{-\alpha x})^{2\lambda-1}}{\Gamma(\lambda)\beta^{\lambda}\,(e^{-\alpha x})^{2\lambda+1}} e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2}\right] = 0$$

Then $f_{NE}(x)$ has at least one mode.

3.9. Quantile Function

Quantile functions are normally used to describe a probability distribution, simulations and statistical application. Simulation techniques utilize quantile function to create simulated random variables for standard and new continuous distributions. In general, it is given as: $Q(u) = F^{-1}(u)$.
U $Uniform(0, 1)$. That is U follows a uniform distribution.
By considering equation (4) quantile function (qf) X is obtained as follows:

$$u = F(x; \lambda\beta\eta) = \frac{1}{\Gamma\lambda}\gamma\left(\lambda, \frac{\lambda}{\beta}\left(\frac{G(x;\eta)}{\bar{G}(x;\eta)}\right)^2\right) \tag{51}$$

$$x = \frac{-1}{\alpha} ln\left\langle 1 - \left\{\frac{\left[\frac{\beta}{\lambda}\gamma^{-1}(\lambda, u\Gamma(\lambda))\right]}{1 + \left[\frac{\beta}{\lambda}\gamma^{-1}(\lambda, u\Gamma(\lambda))\right]}\right\}^{\frac{1}{2}}\right\rangle \tag{52}$$

3.10. Shape of the Crucial Functions

The shapes of the density and hazard function of the Nak-G family can be defined analytically. The critical points of the Nak-G density function equation (6) are the roots of the resulting equation:

$$\frac{g'(x;\eta)}{g(x;\eta)} + \frac{(2\lambda-1)g(x;\eta)}{G(x;\eta)} + \frac{(2\lambda+1)g(x;\eta)}{\bar{G}(x;\eta)} - \frac{2\lambda g(x;\eta)}{\beta\left[\bar{G}(x;\eta)\right]^3} = 0 \tag{53}$$

The critical points of Nak-G hazard function obtained in equation (8) are obtained from the following equation:

$$\frac{g'(x;\eta)}{g(x;\eta)} + \frac{(2\lambda-1)g(x;\eta)}{G(x;\eta)} + \frac{(2\lambda+1)g(x;\eta)}{\bar{G}(x;\eta)} - \frac{2\lambda g(x;\eta)}{\beta\left[\bar{G}(x;\eta)\right]^3} + e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2}\frac{2\lambda^{\lambda}\alpha e^{-\alpha x}\left(1-e^{-\alpha x}\right)^{2\lambda-1}}{\Gamma(\lambda)\beta^{\lambda}\left(e^{-\alpha x}\right)^{2\lambda+1}}$$

$$\left\langle \Gamma(\lambda) - \gamma\left(\lambda, \frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2\right)\right\rangle = 0$$

By using R software, we can examine equations (53) and (54) to determine the local maximums and minimums and inflexion points.

3.11. Maximum Likelihood Estimation

This subsection, deals with the ML estimators of the unknown parameters for the Nak-G family of distributions based on complete samples of size n. Let $X_1, X_2, \ldots, X_n$ be observed values from the Nak-G family with set of parameter $\Theta = (\lambda, \beta, \eta)$. The log-likelihood function for parameter vector $\Theta = (\lambda, \beta, \eta)$ is obtained from equation (6) as follows

$$\ell(\Theta) = nln2 + n\lambda ln\lambda - nln\Gamma(\lambda) - n\lambda ln\beta + \sum_{i=0}^{\infty} ln\left[g(x;\eta)\right] + (2\lambda-1)\cdot\sum_{i=0}^{\infty} ln\left[G(x;\eta)\right] - (2\lambda+1)\sum_{i=0}^{\infty} ln\left[1-G(x;\eta)\right] - \frac{\lambda}{\beta}\sum_{i=0}^{\infty} ln\left[W(x;\eta)\right]^2 \tag{54}$$

where $W(x;\eta) = \frac{G(x;\eta)}{1-G(x;\eta)}$
The components of the score function $U(\Theta) = \left(U_\lambda, U_\beta, U_\eta\right)$ are given by

$$U_\lambda = nln\lambda - n - n\Psi(\lambda) - nln\beta + 2\sum_{i=0}^{\infty} ln\left[G(x;\eta)\right] - 2\sum_{i=0}^{\infty} ln\left[1 - G(x;\eta)\right] - \frac{1}{\beta}\sum_{i=0}^{\infty} ln\left[W(x;\eta)\right] \tag{55}$$

$$U_\beta = n\frac{\lambda}{\beta} + \frac{\lambda \sum\limits_{i=0}^{\infty} ln\left[W(x;\eta)\right]^2}{\beta^2} \tag{56}$$

$$U_\eta = \sum_{i=0}^{\infty} \frac{\partial g(x;\eta)/\partial\eta}{g(x;\eta)} + (2\lambda - 1)\sum_{i=0}^{\infty} \frac{\partial g(x;\eta)/\partial\eta}{G(x;\eta)} + (2\lambda + 1)\sum_{i=0}^{\infty} \frac{\partial g(x;\eta)/\partial\eta}{1 - G(x;\eta)} - \frac{2\lambda}{\beta} \cdot \sum_{i=0}^{\infty} W(x;\eta)w(x;\eta) \tag{57}$$

Setting $U_\lambda, U_\beta, U_\eta$ equate to zero and solving the equations simultaneously result to the ML estimates $\hat{\Theta} = (\hat{\lambda}, \hat{\beta}, \hat{\eta})$ of $\Theta = (\lambda, \beta, \eta)^\tau$.
These estimates can not be solved algebraically and statistical software can be used to solve them numerically via iterative technique.

4. Result and Discussion

The first real life data set was obtained on the breaking stress of carbon fibres of 50 mm length (GPa). The data has been formerly used by [15] and [16]. The data is as follows: 0.39, 0.85, 1.08 ,1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87 ,1.89, 2.03, 2.03 ,2.05, 2.12 ,2.35 ,2.41, 2.43, 2.48, 2.50, 2.53 ,2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87 ,2.88, 2.93, 2.95 ,2.96 ,2.97, 3.09, 3.11 ,3.11, 3.15, 3.15 ,3.19 ,3.22, 3.22, 3.27, 3.28, 3.31 ,3.31, 3.33 ,3.39, 3.39, 3.56 ,3.60 ,3.65, 3.68 ,3.70 ,3.75 ,4.20 ,4.38, 4.42, 4.70, 4.90

Table 1: MLEs and Goodness-of-fit measures for First Data Set

Model	MLE	ℓ	AIC	BIC	CAIC
Nak-Exp	$\lambda = 1.2778$ β=2.1709 α=0.2964	-85.88033	177.7607	187.3296	184.3296
GOG-Exp	$\lambda = 0.6170$ $\beta = 4.0054$ $\alpha = 0.4087$	-85.92746	177.8549	187.4239	184.4239
Wei-Exp	$\lambda = 0.7704$ $\beta = 2.4675$ $\alpha = 0.2389$	-85.97049	177.941	187.5099	184.5099
Kum-Exp	$\lambda = 5.13720$ $\beta = 10.23005$ $\alpha = 0.33140$	-88.10031	182.2006	191.7696	188.7696
Beta-Exp	$\lambda = 8.19864$ $\beta = 4.98148$ $\alpha = 0.37362$	-91.78444	189.5689	199.1378	196.1378
Exp.	$\alpha = 0.36235$	-132.9944	267.9887	271.1785	270.1785
Gamma-Exp	$\lambda = 0.337$ $\beta = 1.141$ $\alpha = 11.458$	-127.4033	260.8066	270.3756	267.3756

The second data set represents the times of failures and running times for sample of devices from an eld-tracking study of a larger system. The data set has been previously studied by [13] and [14]. The data set has thirty (30) observations and they are as follows: 2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00,3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66 The third real life data set [9] corresponds to fifty two ordered annual maximum antecedent rainfall

Table 2: MLEs and Goodness-of-fit measures for Second Data Set

Model	MLE	ℓ	AIC	BIC	CAIC
Nak-Exp	λ=0.35257 β=6.96172 α=0.54568	-38.92824	83.85647	88.06007	91.06007
Wei-Exp	λ=0.13324 β = 0.56404 α = 1.60267	-39.07062	84.14124	88.34483	91.34483
Exp.	α = 0.5648	-47.13504	96.27007	97.67128	98.67128

measurements in mm from Maple 264.9, 314.1, 364.6, 379.8, 419.3, 457.4, 459.4, 460, 490.3, 490.6, 502.2, 525.2, 526.8, 528.6, 528.6, 537.7, 539.6, 540.8, 551.0, 573.5, 579.2, 588.2, 588.7, 589.7, 592.1, 592.8, 600.8, 604.4, 608.4, 609.8, 619.2,626.4, 629.4, 636.4, 645.2, 657.6, 663.5, 664.9, 671.7, 673.0, 682.6, 689.8, 698, 698.6, 698.8, 703.2, 755.9, 786, 787.2, 798.6, 850.4, 895.1.

Table 3: MLEs and Goodness-of-fit measures for Third Data Set

Model	MLE	ℓ	AIC	BIC	CAIC
Nak-Exp	λ= 2.5182774 β=0.2482101 α=0.0006204	-329.275	664.5501	670.4037	673.4037
Wei-Exp	λ=1.6478687 β=1.5943460 α=0.0008296	-351.8995	709.799	715.6527	718.6527
Ext.-Burr III	a=12.0863 b=15.3622 α=0.5868 λ=15.4776 s=11.8405	-339.5244	689.0488	698.805	703.805

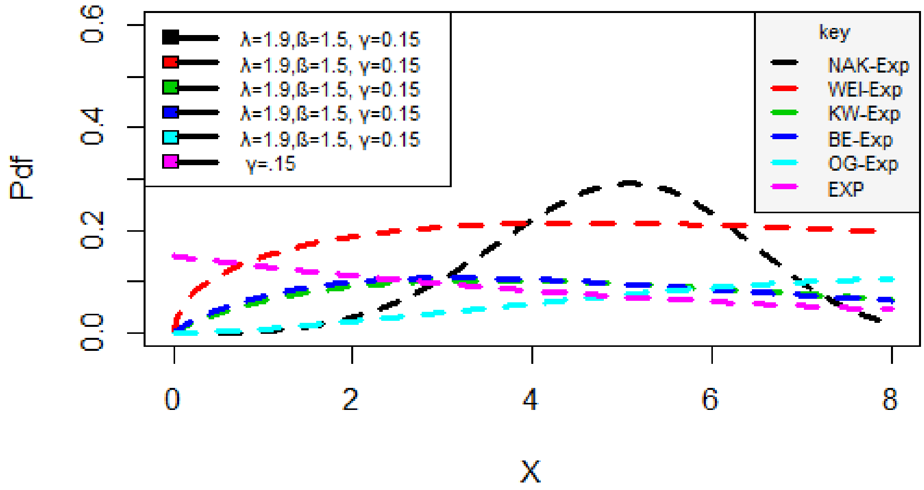

Figure 1: Graph of the Six Distributions Nak-Exp Wei-Exp, KW-Exp, BE-Exp, OG-Exp and Exp (λ= 1.9 (shape parameter) and β, γ= 1.5, 0.15 (scale parameters))

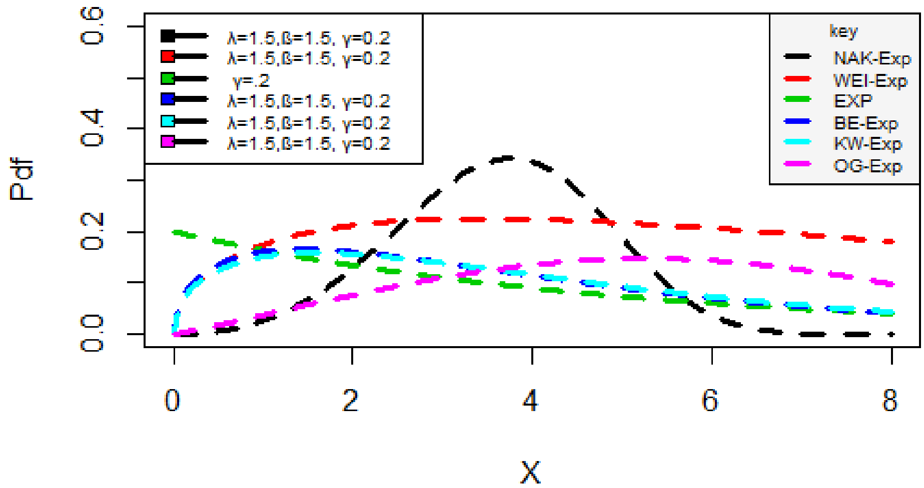

Figure 2: Graph of the Six Distributions Nak-Exp Wei-Exp, KW-Exp, BE-Exp, OG-Exp and Exp (λ= 1.5 (shape parameter) and β, γ= 1.5, 0.2 (scale parameters))

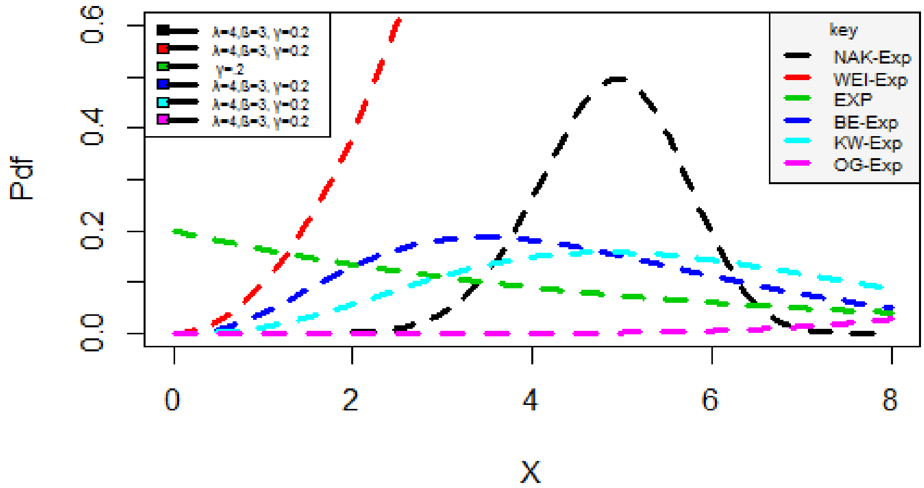

Figure 3: Graph of the Six Distributions Nak-Exp Wei-Exp, KW-Exp, BE-Exp, OG-Exp andExp (λ= 4 (shape parameter) and β, γ= 3, 0.2 (scale parameters))

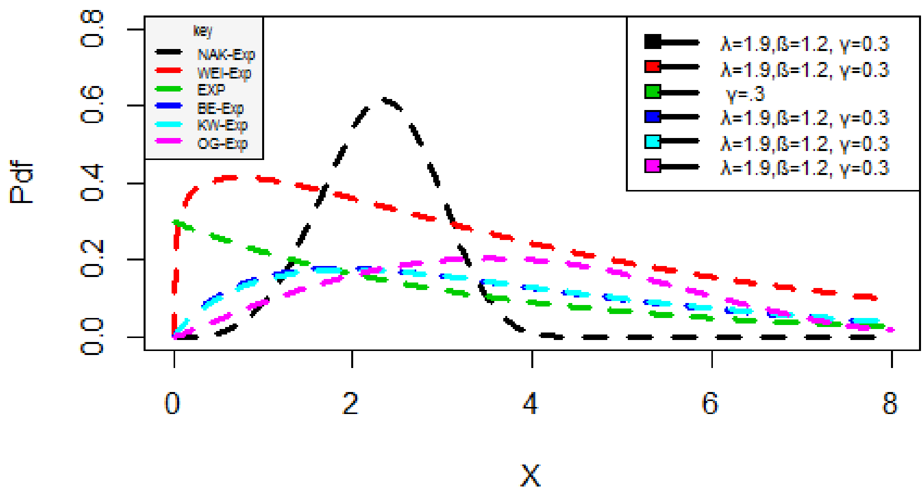

Figure 4: Graph of the Six Distributions Nak-Exp Wei-Exp, KW-Exp, BE-Exp, OG-Exp and Exp (λ= 1.9 (shape parameter) and β, γ =1.2, 0.3 (scale parameters))

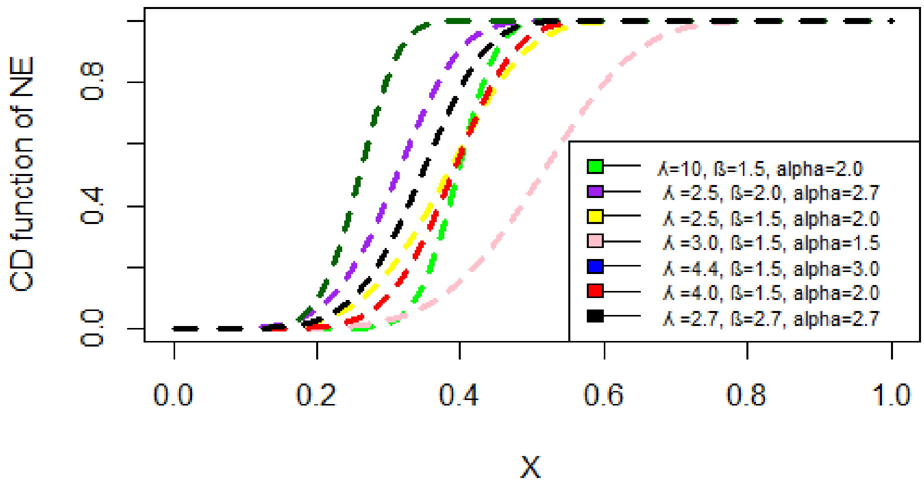

Figure 5: Graph of the Cummulative Distribution of Nak-Exp (λ= shape parameter and β, γ = scale parameters)

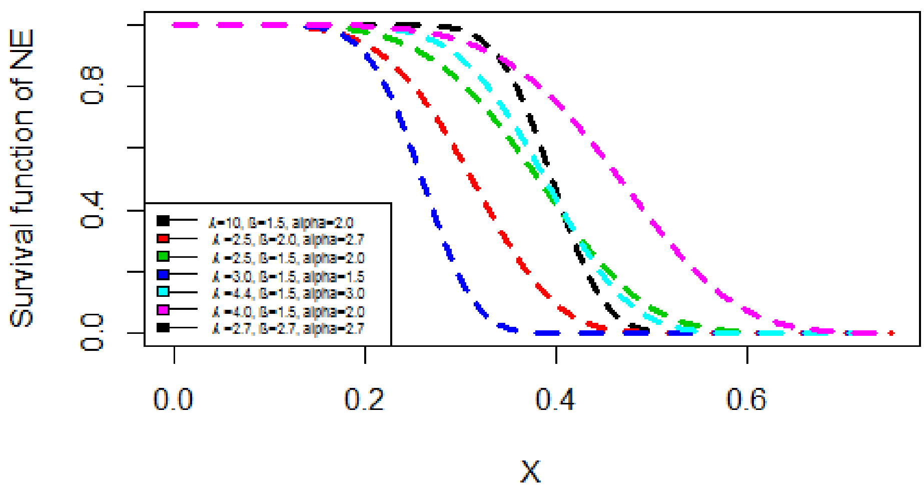

Figure 6: Graph of the Survival Function of Nak-Exp (λ= shape parameter and β, γ = scale parameters)

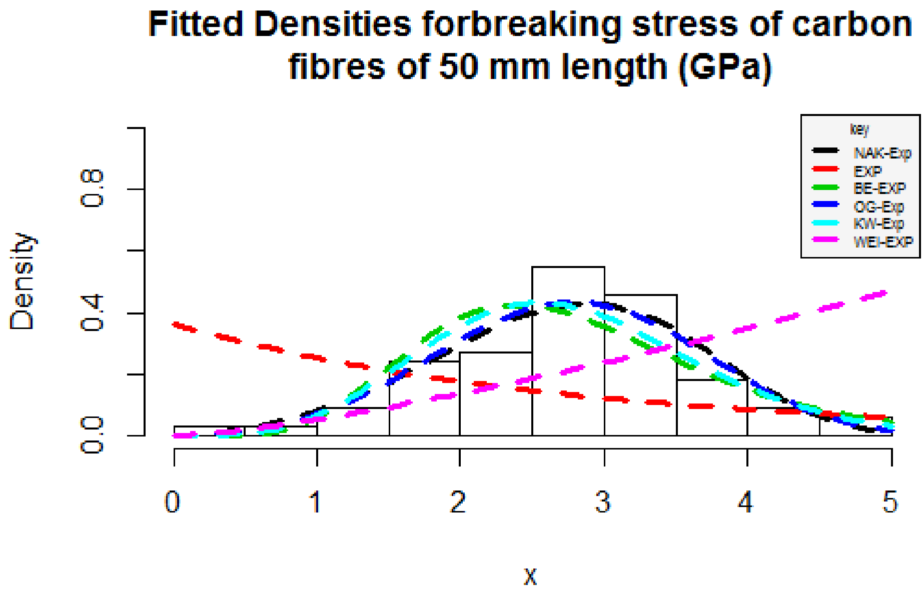

Figure 7: fitted Models on histogram of the first data set

4.1. CONCLUSION

For the first time, we propose a new family of Nakagami-G distributions by add two parameter to Exponential distribution called Nakagami Exponential distribution and some of its statistical properties of the new family were studied. The model parameters were estimated by using the maximum likelihood estimation technique. We finally fit the proposed model among others to real life data show that Nakagami Exponential distribution was found to provide a better fit than its competitors

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The Hadamard-type Padovan-p Sequences

Yeşim Akü zümᵃ

[a]*Department of Mathematics, Faculty of Science and Letters, Kafkas University, 36100 Kars, TURKEY*

Abstract. In this paper, we define the Hadamard-type Padovan-p sequence by using the Hadamard-type product of characteristic polynomials of the Padovan sequence and the Padovan-p sequence. Also, we derive the generating matrices for these sequences. Then using the roots of characteristic polynomial of the Hadamard-type Padovan-p sequence, we produce the Binet formula for the Hadamard-type Padovan-p numbers. Also, we give the permanental, determinantal, combinatorial, exponential representations and the sums of the Hadamard-type Padovan-p numbers.

1. Introduction

It is well-known that Padovan sequence is defined by the following equation:

$$P(n) = P(n-2) + P(n-3)$$

for $n \geq 3$, where $P(0) = P(1) = P(2) = 1$.

Deveci and Karaduman defined [8] the Padovan p-numbers as shown:

$$Pap(n+p+2) = Pap(n+p) + Pap(n)$$

for any given $p\,(p = 2, 3, 4, \ldots)$ and $n \geq 1$ with initial conditions $Pap(1) = Pap(2) = \cdots = Pap(p) = 0$, $Pap(p+1) = 1$ and $Pap(p+2) = 0$.

It is clear that the characteristic polynomials of Padovan sequence and the Padovan-p sequence are $P(x) = x^3 - x - 1$ and $P_p(x) = x^{p+2} - x^p - 1$, respectively.

Akuzum and Deveci [1] defined the Hadamard-type product of polynomials f and g as follows:

$$f(x) * g(x) = \sum_{i=0}^{\infty} (a_i * b_i)\, x^i, \text{ where } a_i * b_i = \begin{cases} a_i b_i & \text{if} \quad a_i b_i \neq 0 \\ a_i + b_i & \text{if} \quad a_i b_i = 0 \end{cases},$$

such that $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ and $g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$.

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

Corresponding author: YA mail address: yesim_036@hotmail.com ORCID: https://orcid.org/0000-0001-7168-8429
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where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [13], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

$$A = \left[a_{i,j}\right]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument, he obtained that

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geq 0$.

Recently, many authors studied number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant [2, 5–12, 14–20]. In [1], Akuzum and Deveci defined the Hadamard-type product of two polynomials and they obtained the Hadamard-type k-step Fibonacci sequence by the aid of this the Hadamard-type product. Then they studied properties of this sequence in detail. In this paper, we define the Hadamard-type Padovan-p sequence by using the definition of Hadamard-type product in [1]. Also, we produce the generating matrix of this sequence. Then we give relationships between the Hadamard-type Padovan-p numbers and the permanents and the determinants of certain matrices which are produced by using the generating matrix of the Hadamard-type Padovan-p sequence. Also, we obtain the combinatorial representations, the generating function, the exponential representation and the sums of the Hadamard-type Padovan-p numbers.

2. The Hadamard-type Padovan-p Sequences

We define a new sequence which is defined by using Hadamard-type product of characteristic polynomials of Padovan sequence and the Padovan-p sequence and is called the Hadamard-type Padovan-p sequence. This sequence is defined by integer constants $P_0^h = P_1^h = \cdots = P_p^h = 0$ and $P_{p+1}^h = 1$ and the recurrence relation

$$P_{n+p+2}^h = P_{n+p}^h - P_{n+3}^h + P_{n+1}^h - P_n^h \tag{1}$$

for the integers $n \geq 0$ and $p \geq 4$.

By relation (1), we can write the following companion matrix:

$$M_p = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{(p+2)\times(p+2)}.$$

The matrix M_p is said to be a Hadamard-type Padovan-p matrix.

It can be readily established by an inductive argument that

$$\left(M_p\right)^n = \begin{bmatrix} P^h_{n+p+1} & P^h_{n+p+2} & & P^h_{n+p-1} - P^h_{n+p-2} & P^h_{n+p} - P^h_{n+p-1} & -P^h_{n+p} \\ P^h_{n+p} & P^h_{n+p+1} & & P^h_{n+p-2} - P^h_{n+p-3} & P^h_{n+p-1} - P^h_{n+p-2} & -P^h_{n+p-1} \\ P^h_{n+p-1} & P^h_{n+p} & & P^h_{n+p-3} - P^h_{n+p-4} & P^h_{n+p-2} - P^h_{n+p-3} & -P^h_{n+p-2} \\ \vdots & \vdots & M_p^* & \vdots & \vdots & \vdots \\ P^h_{n+1} & P^h_{n+2} & & P^h_{n-1} - P^h_{n-2} & P^h_{n} - P^h_{n-1} & -P^h_{n} \\ P^h_{n} & P^h_{n+1} & & P^h_{n-2} - P^h_{n-3} & P^h_{n-1} - P^h_{n-2} & -P^h_{n-1} \end{bmatrix} \tag{2}$$

where M_p^* is a $(p-3)\times(p-3)$ matrix as follows:

$$\begin{bmatrix} P^h_{n+p+3} - P^h_{n+p+1} & P^h_{n+p+4} - P^h_{n+p+2} & \cdots & P^h_{n+2p-1} - P^h_{n+2p-3} \\ P^h_{n+p+2} - P^h_{n+p} & P^h_{n+p+3} - P^h_{n+p+1} & \cdots & P^h_{n+2p-2} - P^h_{n+2p-4} \\ P^h_{n+p+1} - P^h_{n+p-1} & P^h_{n+p+2} - P^h_{n+p} & \cdots & P^h_{n+2p-3} - P^h_{n+2p-5} \\ \vdots & \vdots & & \vdots \\ P^h_{n+3} - P^h_{n+1} & P^h_{n+4} - P^h_{n+2} & \cdots & P^h_{n+p-1} - P^h_{n+p-3} \\ P^h_{n+2} - P^h_{n} & P^h_{n+3} - P^h_{n+1} & \cdots & P^h_{n+p-2} - P^h_{n+p-4} \end{bmatrix}$$

for $n \geq 3$. Also, It is easy to see that $\det M_p = (-1)^p$.

Now we concentrate on finding a Binet formula for the Hadamard-type Padovan-p numbers.

Lemma 2.1. *The characteristic equation of the Hadamard-type Padovan-p sequence $x^{p+2} - x^p + x^3 - x + 1 = 0$ does not have multiple roots.*

Proof. Let $f(x) = x^{p+2} - x^p + x^3 - x + 1$. It is clear that $f(0) \neq 0$ and $f(1) \neq 0$ for all $p \geq 4$. Let λ be a multiple root of $f(x)$, then $\lambda \notin \{0,1\}$. If it is possible that λ is a multiple root of $f(x)$ then it follows that $f(\lambda) = 0$ and $f'(\lambda) = 0$. Now, we consider $f(\lambda) = \lambda^{p+2} - \lambda^p + \lambda^3 - \lambda + 1$. So, we obtain

$$\lambda^p = \frac{-\lambda^3 + \lambda - 1}{\lambda^2 - 1}. \tag{3}$$

Moreover, we may write $f'(\lambda) = (p+2)\lambda^{p+1} - p\lambda^{p-1} + 3\lambda^2 - 1$ and hence we get

$$\lambda^p = \frac{-3\lambda^3 + \lambda}{(p+2)\lambda^2 - p}. \tag{4}$$

From (3) and (4), the following equation can be obtained:

$$p = 1 + \frac{3\lambda^2 - 1}{-\lambda^5 + 2\lambda^3 - \lambda^2 - \lambda + 1}.$$

Using appropriate softwares such as Mathematica Wolfram 10.0 [21], we obtain that there is no solution for $p \geq 4$. Since all p's are integers with $p \geq 4$, it is a contradiction. So, the equation $f(x) = 0$ does not have multiple roots. □

If $x_1, x_2, \ldots, x_{p+2}$ are roots of the equation $x^{p+2} - x^p + x^3 - x + 1$, then by Lemma 2.1, it is known that $x_1, x_2, \ldots, x_{p+2}$ are distinct. Define the $(p+2)\times(p+2)$ Vandermonde matrix V^{p+2} as shown:

$$V^{p+2} = \begin{bmatrix} (x_1)^{p+1} & (x_2)^{p+1} & \cdots & \left(x_{p+2}\right)^{p+1} \\ (x_1)^{p} & (x_2)^{p} & \cdots & \left(x_{p+2}\right)^{p} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & & x_{p+2} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Assume that

$$W^{p+2}(i,j) = \begin{bmatrix} x_1^{n+p+2-i} \\ x_2^{n+p+2-i} \\ \vdots \\ x_{p+2}^{n+p+2-i} \end{bmatrix}$$

and $V^{p+2}(i,j)$ is a $(p+2)\times(p+2)$ matrix obtained from V^{p+2} by replacing the jth column of V^{p+2} by $W^{p+2}(i,j)$.

Theorem 2.2. *Let $(M_P)^n = \left[m_{i,j}^{p,n}\right]$, then*

$$m_{i,j}^{p,n} = \frac{\det V^{p+2}(i,j)}{\det V^{p+2}},$$

for $n \geq 3$ and $p \geq 4$.

Proof. Since the eigenvalues of the matrix M_P, $x_1, x_2, \ldots, x_{p+2}$ are distinct, the matrix M_P is diagonalizable. Let $D^{p+2} = \left(x_1, x_2, \ldots, x_{p+2}\right)$, then we easily see that $M_P V^{p+2} = V^{p+2} D^{p+2}$. Since V^{p+2} is invertible, we can write $\left(V^{p+2}\right)^{-1} M_P V^k = D^{p+2}$. Then, the matrix M_P is similar to D^{p+2} and so $(M_P)^n V^{p+2} = V^{p+2}\left(D^{p+2}\right)^n$. Hence we have the following linear system of equations:

$$\begin{cases} m_{i,1}^{p,n} x_1^{p+1} + m_{i,2}^{p,n} x_1^{p} + \cdots + m_{i,p+2}^{p,n} = x_1^{n+p+2-i} \\ m_{i,1}^{p,n} x_2^{p+1} + m_{i,2}^{p,n} x_2^{p} + \cdots + m_{i,p+2}^{p,n} = x_2^{n+p+2-i} \\ \vdots \\ m_{i,1}^{p,n} x_{p+2}^{p+1} + m_{i,2}^{p,n} x_{p+2}^{p} + \cdots + m_{i,p+2}^{p,n} = x_{p+2}^{n+p+2-i} \end{cases}$$

Therefore, for each $i, j = 1, 2, \ldots, k$, we obtain

$$m_{i,j}^{p,n} = \frac{\det V^{p+2}(i,j)}{\det V^{p+2}}.$$

□

From this result we immediately deduce:

Corollary 2.3. *Let P_n^h be the nth the Hadamard-type Padovan-p number, then*

$$P_n^h = \frac{\det V^{p+2}(p+2,1)}{\det V^{p+2}} = -\frac{\det V^{p+2}(p+1,p+2)}{\det V^{p+2}}$$

for $n \geq 3$ and $p \geq 4$.

Now we concentrate on finding the permanental representations of the Hadamard-type Padovan-p numbers.

Definition 2.4. *A $u \times v$ real matrix $M = \left[m_{i,j}\right]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.*

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [3], Brualdi and Gibson obtained that $per(M) = per(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Let $\alpha \geq p+2$ be a integer and let $A^{p,\alpha} = \left[a_{i,j}^{p,\alpha}\right]$ be the $\alpha \times \alpha$ super-diagonal matrix, defined by

$$a_{i,j}^{p,\alpha} = \begin{cases} 1 & \begin{array}{c} \text{if } i = r \text{ and } j = r+1 \text{ for } 1 \leq r \leq \alpha - 1, \\ i = r \text{ and } j = r-1 \text{ for } 2 \leq r \leq \alpha \\ \text{and} \\ i = r \text{ and } j = r+p \text{ for } 1 \leq r \leq \alpha - p, \end{array} \\ -1 & \begin{array}{c} \text{if } i = r \text{ and } j = r+p-2 \text{ for } 1 \leq r \leq \alpha - p + 2 \\ \text{and} \\ i = r \text{ and } j = r+p+1 \text{ for } 1 \leq r \leq \alpha - p - 1, \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following Theorem.

Theorem 2.5. *For $\alpha \geq p+2$ and $p \geq 4$,*

$$perA^{p,\alpha} = P_{\alpha+p+1}^{h}.$$

Proof. The assertion may be proved by induction on α. Let the equation be hold for $\alpha \geq p+2$, then we show that the equation holds for $\alpha+1$. If we expand the $perA^{p,\alpha}$ by the Laplace expansion of permanent according to the first row, then we obtain

$$perA^{p,\alpha+1} = perA^{p,\alpha-1} - perA^{p,\alpha-p+2} + perA^{p,\alpha-p} - perA^{p,\alpha-p-1}.$$

Since $perA^{p,\alpha-1} = P_{\alpha+p}^{h}$, $perA^{p,\alpha-p+2} = P_{\alpha+3}^{h}$, $perA^{p,\alpha-p} = P_{\alpha+1}^{h}$ and $perA^{p,\alpha-p-1} = P_{\alpha}^{h}$, it is easy to see that $perA^{p,\alpha+1} = P_{\alpha+p+2}^{h}$. Thus, the proof is complete. □

Let $\alpha \geq p+2$ and let $B^{p,\alpha} = \left[b_{i,j}^{p,\alpha}\right]$ be the $\alpha \times \alpha$ matrix, defined by

$$b_{i,j}^{p,\alpha} = \begin{cases} 1 & \begin{array}{c} \text{if } i = r \text{ and } j = r+1 \text{ for } 1 \leq r \leq \alpha - p - 1, \\ i = r \text{ and } j = r-1 \text{ for } 2 \leq r \leq \alpha \\ \text{and} \\ i = r \text{ and } j = r+p \text{ for } 1 \leq r \leq \alpha - p - 1, \end{array} \\ -1 & \begin{array}{c} \text{if } i = r \text{ and } j = r+p-2 \text{ for } 1 \leq r \leq \alpha - p - 1 \\ \text{and} \\ i = r \text{ and } j = r+p+1 \text{ for } 1 \leq r \leq \alpha - p - 1, \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Now we define the $\alpha \times \alpha$ matrix $C^{p,\alpha} = \left[c_{i,j}^{p,\alpha}\right]$ as follows:

$$C^{p,\alpha} = \begin{array}{c} \begin{array}{cccccc} & & \overset{(\alpha-p-2)\text{th}}{\downarrow} & & & \end{array} \\ \left[\begin{array}{cccccc} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & B^{p,\alpha-1} & & \\ \vdots & & & & & \\ 0 & & & & & \end{array}\right] \end{array}.$$

Then we can give the following Theorem by using the permanental representations.

Theorem 2.6. (i). *For $\alpha \geq p+2$,*

$$perB^{p,\alpha} = -P_{\alpha-1}^{h}.$$

(ii). *For $\alpha > p+2$,*

$$perC^{p,\alpha} = -\sum_{i=0}^{\alpha-2} P_{i}^{h}.$$

Proof. (*i*) .Let the equation be hold for $\alpha \geq p+2$, then we show equation hold for $\alpha+1$. If we expand the $perB^{p,\alpha}$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\begin{aligned} perB^{p,\alpha+1} &= perB^{p,\alpha-1} - perB^{p,\alpha-p+2} + perB^{p,\alpha-p} - perB^{p,\alpha-p-1} \\ &= -P^h_{\alpha-2} + P^h_{\alpha-p+1} - P^h_{\alpha-p-1} + P^h_{\alpha-p-2}. \end{aligned}$$

So, we have the conclusion.

(*ii*) . If we expand the $perC^{p,\alpha}$ with respect to the first row, we write

$$perC^{p,\alpha} = perC^{p,\alpha-1} + perB^{p,\alpha-1}.$$

From Theorem 2.5 and Theorem 2.6. (i) and induction on α, the proof follows directly. □

Let the notation $M \circ K$ denotes the Hadamard product of M and K. A matrix M is called convertible if there is an $u \times u$ (1, -1)-matrix K such that per $M = \det(M \circ K)$.

Let G be the $\alpha \times \alpha$ matrix, defined by

$$G = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}$$

for $\alpha > p+2$.

Corollary 2.7. *For $\alpha > p+2$ and $p \geq 4$*

$$\det(A^{p,\alpha} \circ G) = P^h_{\alpha+p+1},$$

$$\det(B^{p,\alpha} \circ G) = -P^h_{\alpha-1}$$

and

$$\det(C^{p,\alpha} \circ G) = -\sum_{i=0}^{\alpha-2} P^h_i.$$

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2.8. *(Chen and Louck [4]).The (i, j) entry $k^{(u)}_{i,j}(k_1, k_2, \ldots, k_v)$ in the matrix $K^u(k_1, k_2, \ldots, k_v)$ is given by the following formula:*

$$k^{(u)}_{i,j}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v} \tag{5}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = u - i + j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v} = \frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ is a multinomial coefficient, and the coefficients in (5) are defined to be 1 if $u = i - j$.

Then we have the following Corollary for the Hadamard-type Padovan-p numbers.

Corollary 2.9. *For $p \geq 4$, let P_n^h be the nth the Hadamard-type Padovan-p number. Then*
i.

$$P_n^h = \sum_{(t_1, t_2 \ldots, t_{p+2})} \binom{t_1 + \cdots + t_{p+2}}{t_1, \ldots, t_{p+2}} (-1)^{t_{p-1} + t_{p+2}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+2)\, t_{p+2} = n - p - 1$.
ii.

$$P_n^h = - \sum_{(t_1, t_2 \ldots, t_k)} \frac{t_{p+2}}{t_1 + t_2 + \cdots + t_{p+2}} \times \binom{t_1 + \cdots + t_{p+2}}{t_1, \ldots, t_{p+2}} (-1)^{t_{p-1} + t_{p+2}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+2)\, t_{p+2} = n + 1$.

Proof. In Theorem 2.8, If we take $i = p + 2$ and $j = 1$, for case *i.* and $i = p + 1$, $j = p + 2$, for case *ii.*, then the proof is immediately seen from $\left(M_p\right)^n$. □

The generating function of the Hadamard-type Padovan-p sequence is given by:

$$f_p(x) = \frac{x^{p+1}}{1 - x^2 + x^{p-1} - x^{p+1} + x^{p+2}}.$$

It can be readily established that the Hadamard-type Padovan-p sequences have the following exponential representation.

Theorem 2.10. *The Hadamard-type Padovan-p numbers have the following exponential representation:*

$$f_p(x) = x^{p+1} \exp\left(\sum_{i=1}^{\infty} \frac{\left(x^2\right)^i}{i} \left(1 - x^{p-3} + x^{p-1} - x^p\right)^i \right)$$

where $p \geq 4$.

Proof. It is clear that

$$\ln \frac{f_p(x)}{x^{p+1}} = -\ln\left(1 - x^2 + x^{p-1} - x^{p+1} + x^{p+2}\right)$$

and

$$\begin{aligned} -\ln\left(1 - x^2 + x^{p-1} - x^{p+1} + x^{p+2}\right) = & -[-x^2\left(1 - x^{p-3} + x^{p-1} - x^p\right) - \\ & \frac{1}{2}x^4\left(1 - x^{p-3} + x^{p-1} - x^p\right)^2 - \cdots - \\ & \frac{1}{n}x^{2n}\left(1 - x^{p-3} + x^{p-1} - x^p\right)^n - \cdots]. \end{aligned}$$

A simple calculation shows that

$$\ln \frac{f_p(x)}{x^{p+1}} = \sum_{i=1}^{\infty} \frac{\left(x^2\right)^i}{i} \left(1 - x^{p-3} + x^{p-1} - x^p\right)^i.$$

Thus the conclusion is obtained. □

Now we consider the sums of the Hadamard-type Padovan-p numbers.
Let

$$T_n = \sum_{i=0}^{n} P_n^h$$

for $n \geq 3$ and $p \geq 4$, and let Q_p be the $(p+3) \times (p+3)$ matrix, such that

$$Q_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & M_p & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

Then it can be shown by induction that

$$\left(Q_p\right)^n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ T_{n+p} & & & \\ T_{n+p-1} & & \left(M_p\right)^n & \\ \vdots & & & \\ T_{n-1} & & & \end{bmatrix}.$$

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A New View on Topological Polygroups

Gulay Oguz[a]

[a]*Department of Mathematics, Faculty of Arts and Sciences, Harran University, Şanlıurfa, TURKEY*

Abstract. Soft set theory, defined by Molodtsov as a novel mathematical tool modeling uncertainty, has been combined with many different discipline fields. In this article, the concept of soft topological polygroups is proposed by examining polygroups, a special class of hypergroups, with a soft topological approach. Also, several results have been obtained by establishing important characterizations related to this concept. In last, by presenting the definition of soft topological subpolygroups, some of their properties are examined.

1. Introduction

Hyperstructure theory, as a generalization of classical algebraic theory, was initiated by F. Marty at the eighth congress of Scandinavian Mathematicians in 1934 [2]. Although it does not have a long history, this theory has been used successfully in both applied and theoretical branches of mathematics. A special subclass of hypergroups, one of the most important hyperstructures, is polygroups. Polygroups studied by many researchers were defined by Ioulidis in 1981 [17]. Some algebraic and topological properties were investigated in detail. Davvaz and Poursalavati in [16] described matrix representations of polygroups over hyperrings. Subsequently, Davvaz introduced permutation polygroups and notions related to it [15]. Also, by examining the topological properties of this concept, the concept of topological polygroups was presented by Heidari et al. as a generalization of topological groups [19].

Another important theory in the basis of this study is soft set theory. In 1999, soft set theory was proposed by Molodtsov to resolve some complex problems involving uncertain data in engineering, medical science, economics, environment science [1]. This theory, which is a powerful mathematical approach for modeling uncertainties, has been studied algebraically and topologically by many mathematicians. Aktas and cagman presented the definition of soft groups [3]. Later on, Jun defined the notion of soft ideals on BCK/ BCI-algebras [8]. By defining the actions of soft groups, Oguz *et al*. examined the relation between the soft action and soft symmetric group [9]. Also, topological studies on soft sets were introduced by Shabir and Naz [6]. By proposing the definition of a soft topological space, they studied the separation axioms in a soft topological space. Aygunoglu and Aygun described soft product topologies and soft compactness [11]. Oguz *et al*. defined soft topological categories and obtained some important properties [7]. After that, Oguz proposed the concept of soft topological transformation groups [10]. On the other hand, soft hyperstructures are introduced by applying soft set theory to hyperstructures. Leoreanu-Fotea and Corsini [13] defined the concept of soft hypergroups. Yamak *et. al.* [12] introduced the notion of soft hypergroupoids. Morever, soft polygroups were studied by Wanga *et. al*. [14].

Corresponding author: GO, mail address: gulay.oguz@harran.edu.tr ORCID: https://orcid.org/0000-0003-4302-8401
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The main purpose of this study is to introduce the notion of soft topological polygroups by applying soft set theory to topological polygroups. In addition, some important properties of soft topological polygroups are examined and soft topological subpolygroups are studied.

2. Preliminaries

In this section, we review some fundamental notions and properties of soft sets and topological polygroups for the sake of completeness. See [1-4, 18].

Assume that X is an initial universe set and E is a set of parameters. Also, $P(X)$ denotes the power set of X and $A \subset E$. Then, Molodtsov defined the soft set follow as:

Definition 2.1. [1] *A pair $(\mathcal{F}, A)$ is said to be a soft set over X, where $\mathcal{F}$ is a mapping defined by*

$$\mathcal{F} : A \longrightarrow P(X)$$

Clearly, a soft set over X can be regarded as a parametrized family of subsets of the universe X.

Definition 2.2. [4] *Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two soft sets over the common universe X. Then, $(\mathcal{F}, A)$ is said to be a soft subset of $(\mathcal{G}, B)$ if*
i) *$A \subseteq B$,*
ii) *$\mathcal{F}(a)$ and $\mathcal{G}(a)$ are identical approximations for all $\alpha \in A$.*
We denote it as $(\mathcal{F}, A)\widetilde{\subset}(\mathcal{G}, B)$.

Definition 2.3. [4] *A soft set $(\mathcal{F}, A)$ over X is is said to be a null soft set denoted by Φ, if $\mathcal{F}(\alpha) = \emptyset$ for all $\alpha \in A$.*

Definition 2.4. [4] *A soft set $(\mathcal{F}, A)$ over X is is said to be an absolute soft set denoted by $\tilde{A}$, if $\mathcal{F}(\alpha) = X$ for all $\alpha \in A$.*

From an general perspective, the following notions are presented for the nonempty family $\{(\mathcal{F}_i, A_i)|i \in I\}$ of soft sets over the common universe X

Definition 2.5. [5] *The restricted intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = \widetilde{\bigcap}_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \bigcap_{i\in I} A_i \neq \emptyset$ and $\mathcal{F}(a) = \bigcap_{i\in I} \mathcal{F}_i(a)$ for all $a \in A_i$.*

Definition 2.6. [5] *The restricted union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = (\bigcup_{\mathcal{R}})_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \bigcap_{i\in I} A_i \neq \emptyset$ and $\mathcal{F}(a) = \bigcup_{i\in I} \mathcal{F}_i(a)$ for all $a \in A_i$.*

Definition 2.7. [5] *The extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = \widetilde{\bigcup}_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \bigcup_{i\in I} A_i$ and $\mathcal{F}(a) = \bigcup_{i\in I(a)} \mathcal{F}_i(a)$, $I(a) = \{i \in I : a \in A_i\}$ for all $a \in A_i$.*

Definition 2.8. [5] *The extended intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = (\bigcap_{\mathcal{E}})_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \bigcup_{i\in I} A_i$ and $\mathcal{F}(a) = \bigcap_{i\in I(a)} \mathcal{F}_i(a)$, $I(a) = \{i \in I : a \in A_i\}$ for all $a \in A_i$*

Definition 2.9. [5] *The $\wedge-$intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = \widetilde{\bigwedge}_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \Pi_{i\in I}A_i$ and $\mathcal{F}((a_i)_{i\in I}) = \bigcap_{i\in I} \mathcal{F}_i(a_i)$ for all $(a_i)_{i\in I} \in A_i$.*

Definition 2.10. [5] *The $\vee-$intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = \widetilde{\bigvee}_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \Pi_{i\in I}A_i$ and $\mathcal{F}((a_i)_{i\in I}) = \bigcup_{i\in I} \mathcal{F}_i(a_i)$ for all $(a_i)_{i\in I} \in A_i$.*

Now, we recall the definitions of polygroup and topological polygroup. Assume $P^*(P)$ be the set of all non-empty subsets of P.

Definition 2.11. [18] *A polygroup is a multi-valued system* $\mathcal{P} =< P, \circ, e,^{-1} >$, *where* $\circ : P \times P \longrightarrow P^*(P)$, $e \in P$, $^{-1}$ *is a unitary operation on* P *and the following conditions hold for all* $x, y, z \in P$: **i**. $(x \circ y) \circ z = x \circ (y \circ z)$, **ii**. $e \circ x = x \circ e = x$, **iii**. $x \in y \circ z$ *implies* $y \in x \circ z^{-1}$ *and* $z \in y^{-1} \circ x$.

Definition 2.12. [14] *Let* $\mathcal{P} =< P, \circ, e,^{-1} >$ *be a polygroup and* K *be a non-empty subset of* P. *Then* K *is said to be a subpolygroup if* $< K, \circ, e,^{-1} >$ *is itself a polygroup.*

The concept of polygroup is examined with the soft set theory and the concept of soft polygroup is defined as follows:

Definition 2.13. [14] *For a non-null soft set* $(\mathcal{F}, A)$ *over the polygroup* $\mathcal{P} =< P, \circ, e,^{-1} >$, $(\mathcal{F}, A)$ *is said to be a soft polygroup over* $\mathcal{P}$ *if and only if* $\mathcal{F}(a)$ *is a subpolygroup of* $\mathcal{P}$ *for all* $a \in Supp(F, A)$.

Definition 2.14. [19] *Let* $\mathcal{P} =< P, \circ, e,^{-1} >$ *be a polygroup and* (P, τ) *be a topological space. Then multi-valued system* $\mathcal{P} =< P, \circ, e,^{-1}, \tau >$ *is said to be a topological polygroup if the mappings* $^{-1} : P \longrightarrow P$ *and* $\circ : P \times P \longrightarrow P^*(P)$ *are continuous with respect to the the product topology on* $\tau \times \tau$ *and the topology* τ^* *on* $P^*(P)$ *which is generated by* $\mathfrak{B} = \{S_V | V \in \tau\}$, *where* $S_V = \{U \in P^*(P) | U \subseteq V, U \in \tau\}$.

Definition 2.15. [19] *Let* $\mathcal{P} =< P, \circ, e,^{-1}, \tau >$ *and* $\mathcal{P}' =< P', \circ', e',^{-1}, \tau' >$ *be two topological polygroups. A mapping* $\theta : P \longrightarrow P'$ *is called a* ***good topological homomorphism*** *if the following conditions are satisfied for all* $x, y \in P$:
i. $\theta(e) = e'$
ii. $\theta(x \circ y) = \theta(x) \circ' \theta(y)$
iii. θ *is continuous and open.*

Note that a good topological homomorphism is a topological isomorphism if the mapping θ is one to one and onto.

3. Soft Topological Polygroups

In this section, we define soft topological polygroups and present some of their features. From now on, $\mathcal{P}^*$ denotes the set of all subpolygroups of a polygroup $\mathcal{P} =< P, \circ, e,^{-1} >$ and $P^*(P)$ denotes the set of all non-empty subsets of P.

Definition 3.1. *Let* τ *be a topology on the polygroup* $\mathcal{P} =< P, \circ, e,^{-1} >$ *such that and* τ^* *be a topology on* P^*, *which is generated by* $\mathfrak{B} = \{S_V | V \in \tau\}$, *where* $S_V = \{U \in \mathcal{P}^* | U \subseteq V, U \in \tau\}$. *Let* $(\mathcal{F}, A)$ *be a non-null soft set over* $\mathcal{P}$. *The pair* $(\mathcal{F}, A)$ *is said to be a soft topological polygroup over* $\mathcal{P}$ *with the topology* τ *if the following axioms hold:*
i. $F(a)$ *is a subhpolygroup of* $\mathcal{P}$ *for all* $a \in Supp(F, A)$.
ii. *The mappings* $\circ : \mathcal{F}(a) \times \mathcal{F}(a) \longrightarrow P^*(\mathcal{F}(a))$ *and* $^{-1} : \mathcal{F}(a) \longrightarrow \mathcal{F}(a)$ *are continuous with respect to the topologies induced by* $\tau \times \tau$ *and* τ^* *for all* $a \in Supp(F, A)$.
It is to be noted that if $\mathcal{P}$ *is a topological polygroup, it is sufficient that only the first condition of the above definition is satisfied in order to the pair* $(\mathcal{F}, A)$ *to be defined as a soft topological polygroup. Namely, the soft topological polygroup* $(\mathcal{F}, A)$ *can be considered as a parameterized family of subpolygroups of the topological polygroup* $\mathcal{P}$.

Theorem 3.2. *Every soft polygroup on a topological polygroup is a soft topological polygroup.*

Proof. Let $\mathcal{P}$ be a topological polygroup and let $(\mathcal{F}, A)$ be a soft polygroup over $\mathcal{P}$ with the topology τ. Then $\mathcal{F}(a)$ is a subpolygroup of $\mathcal{P}$ for all $a \in A$. Hence, $\mathcal{F}(a)$ is a topological subpolygroup of $\mathcal{P}$ with recpect to the topologies induced by τ and τ^* for all $a \in A$. Therefore, $(\mathcal{F}, A)$ is also a soft topological polygroup over $\mathcal{P}$. □

Remark 3.3. *Each soft polygroup* $\mathcal{P}$ *can be transformed into a soft topological polygroup by equipping both* $\mathcal{P}$ *and* $P^*(P)$ *with discrete or indiscrete topology. However, every soft polygroup over a polygroup is not a soft topological polygroup.*

Theorem 3.4. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ.*
i. *The restricted intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ with $\bigcap_{i\in I} A_i \neq \emptyset$ is a soft topological polygroup over $\mathcal{P}$ if $\widetilde{\bigcap}_{i\in I}(\mathcal{F}_i, A_i) \neq \emptyset$*
ii. *The extended intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological polygroup over $\mathcal{P}$ if $(\bigcap_{\mathcal{E}})_{i\in I}(\mathcal{F}_i, A_i) \neq \emptyset$*

Proof. **i**. The restricted intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ with $\bigcap_{i\in I} A_i \neq \emptyset$ defined as the soft set $\widetilde{\bigcap}_{i\in I}(\mathcal{F}_i, A_i) = (\mathcal{F}, A)$ such that $\bigcap_{i\in I} \mathcal{F}_i(a)$ for all $a \in A$. Choose $a \in Supp(F, A)$. Suppose $\bigcap_{i\in I} F_i(a) \neq \emptyset$ so that $\mathcal{F}_i(a) \neq \emptyset$ for all $i \in I$. Since $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a non-empty family of soft topological polygroup over $\mathcal{P}$ with the topology τ, $\mathcal{F}_i(a)$ is a topological polygroup of $\mathcal{P}$ for all $i \in I$. Then, $\bigcap_{i\in I} \mathcal{F}_i(a)$ is a topological subpolygroup of $\mathcal{P}$. Thus, $(\mathcal{F}, A)$ is a soft topological polygroup over $\mathcal{P}$ with the topology τ.
ii. The proof is similar to **i**. □

Theorem 3.5. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ.*
i. *The extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological polygroup over $\mathcal{P}$ if $\mathcal{F}_i(x) \subseteq \mathcal{F}_j(x)$ or $\mathcal{F}_j(x) \subseteq \mathcal{F}_i(x)$ for all $i, j \in I$, $x \in \bigcup_{i\in I} A_i$*
ii. *The restricted union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological polygroup over $\mathcal{P}$ if $\mathcal{F}_i(x) \subseteq \mathcal{F}_j(x)$ or $\mathcal{F}_j(x) \subseteq \mathcal{F}_i(x)$ for all $i, j \in I$, $x \in \cap_{i\in I} A_i$ with $\bigcap_{i\in I} A_i \neq \emptyset$.*

Proof. **i**. Assume $(\mathcal{F}, A) = \widetilde{\bigcup}_{i\in I}(\mathcal{F}_i, A_i)$ as the extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ with $\bigcap_{i\in I} A_i \neq \emptyset$. Let $\mathcal{F}_i(x) \subseteq \mathcal{F}_j(x)$ or $\mathcal{F}_j(x) \subseteq \mathcal{F}_i(x)$ for all $i, j \in I$, $x \in \bigcup_{i\in I} A_i$. Choose $a \in Supp(\mathcal{F}, A)$. Since each $(\mathcal{F}_i, A_i)$ is non-null soft sets over $\mathcal{P}$, then $\bigcup_{i\in I}(\mathcal{F}_i, A_i)$ is also a non-null soft set over $\mathcal{P}$ for all $i \in I$. By the hypothesis, $\mathcal{F}_i(x) \subseteq \mathcal{F}_j(x)$ or $\mathcal{F}_j(x) \subseteq \mathcal{F}_i(x)$ for all $i, j \in I$, $x \in \bigcap_{i\in I} A_i$ with $\bigcap_{i\in I} A_i \neq \emptyset$ such that $\mathcal{F}_i(x)$ and $\mathcal{F}_j(x)$ are the topological subpolygroups of $\mathcal{P}$ and thus their union must be non-null too. Therefore, $\mathcal{F}(x)$ is a topological subpolygroup of $\mathcal{P}$. Hence, $(\mathcal{F}, A)$ is a soft topological polygroup over $\mathcal{P}$ with the topology τ.
ii. The proof is similar to that of **i**. □

From the above proposition, the following resultf is easily obtained:

Corollary 3.6. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ. Then the extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological polygroup over $\mathcal{P}$ with the topology τ if $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$, $i \neq j$.*

Theorem 3.7. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ.*
i. *The $\wedge-$intersection $\widetilde{\bigwedge}_{i\in I}(\mathcal{F}_i, A_i)$ is a soft topological polygroup over $\mathcal{P}$ if it is non-null.*
ii. *The $\vee-$union $\widetilde{\bigvee}_{i\in I}(\mathcal{F}_i, A_i)$ is a soft topological polygroup over $\mathcal{P}$ if $\mathcal{F}_i(x_i) \subseteq \mathcal{F}_j(x_j)$ or $\mathcal{F}_j(x_j) \subseteq \mathcal{F}_i(x_i)$ for all $i, j \in I$, $x_i \in A_i$.*

Proof. **i**. Write $(\mathcal{F}, A) = \widetilde{\bigwedge}_{i\in I}(\mathcal{F}_i, A_i)$ for a non-empty family $\{(\mathcal{F}_i, A_i)|i \in I\}$ of soft topological polygroups over $\mathcal{P}$ with the topology τ. Let $a \in Supp(F, A)$. By the assumption, $\bigcap_{i\in I} \mathcal{F}_i(a_i) \neq \emptyset$ so that $\mathcal{F}_i(a_i) \neq \emptyset$ for all $i \in I$ and $(a_i)_{i\in I} \in A_i$. Hence, $\mathcal{F}_i(a_i)$ is a topological subpolygroup of $\mathcal{P}$ for all $i \in I$ so that their intersection must be a topological subpolygroup of $\mathcal{P}$ too. Thus, $(\mathcal{F}, A)$ is a soft topological polygroup over H with the topology τ.
□

Definition 3.8. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}_i$ with the topologies τ_i. Then the cartesian product of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ over $\Pi_{i\in I} H_i$ with the product topology $\prod_{i\in I} \tau_i$ is denoted by $\Pi_{i\in I}(\mathcal{F}_i, A_i)$, is defined as $\Pi_{i\in I}(\mathcal{F}_i, A_i) = (\mathcal{F}, A)$ where $A = \Pi_{i\in I} A_i$ and $\mathcal{F}(x_i) = \Pi_{i\in I}\mathcal{F}_i(x_i)$ for all $(x_i)_{i\in I} \in A$.*

Theorem 3.9. *The cartesian product of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological polygroup over $\Pi_{i\in I} H_i$ with the product topology $\Pi_{i\in I}\tau_i$.*

Proof. Assume that $(\mathcal{F}_i, A_i)$ is a soft topological polygroup over $\mathcal{P}_i$ with the topology τ_i for all $i \in I$. Then, $\mathcal{F}_i(a) \neq \emptyset$ and $\mathcal{F}_i(a_i)$ a topological subpolygroup of $\mathcal{P}_i$ for all $(a_i)_{i\in I} \in Supp(\mathcal{F}_i, A_i)$. Thus, $\Pi_{i\in I}\mathcal{F}_i(a_i) \neq \emptyset$ and $\Pi_{i\in I}\mathcal{F}_i(a_i)$ a topological subpolygroup of $\Pi_{i\in I}\mathcal{P}_i$ with the product topology $\Pi_{i\in I}\tau_i$. Therefore, $\Pi_{i\in I}(Fi, Ai)$ is a soft topological polygroup over $\Pi_{i\in I}\mathcal{P}_i$. □

3.1. Soft Topological Polygroup Homomorphisms

Definition 3.10. *Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be soft topological polygroups over $\mathcal{P}$ and $\mathcal{P}'$ with the topologies τ and τ', respectively. Let $\varphi : A \longrightarrow B$ and $\psi : \mathcal{P} \longrightarrow \mathcal{P}'$ be two mappings. Then, the pair (ψ, φ) is said to be a soft topological homomorphism if the following axioms hold:*
i. *ψ is a good homomorphism.*
ii. *$\psi(\mathcal{F}(a)) = \mathcal{K}(\varphi(a))$ for all $a \in Supp(\mathcal{F}, A)$.*
ii. *$\psi_a : (\mathcal{F}(a), \tau_{\mathcal{F}(a)}) \longrightarrow (\mathcal{K}(\varphi(a)), \tau'_{\mathcal{K}(\varphi(a))})$ is continuous and open for all $a \in Supp(\mathcal{F}, A)$.*

In this perspective,, it follows that a soft topological homomorphism (ψ, φ) is a mapping of soft topological polygroups. Therefore, we define a new category whose objects are soft topological polygroups and whose arrows are soft topological homomorphisms.

In addition, it can be said that $(\mathcal{F}, A)$ is soft topologically isomorphic to $(\mathcal{K}, B)$ if the mappings ψ and φ are one to one and onto.

Example 3.11. *Let $(\mathcal{K}, B)$ be a soft topological subpolygroup of $(\mathcal{F}, A)$ over $\mathcal{P}$. Together with the inclusion map $i : B \longrightarrow A$ and the identity map $\mathcal{I} : P \longrightarrow P$, the pair $(\mathcal{I}, i)$ is a soft topological homomorphism from $(\mathcal{K}, B)$ to $(\mathcal{F}, A)$.*

Example 3.12. *Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be the two soft good homomorphic polygroups defined over $\mathcal{P}$ and $\mathcal{P}'$, respectively. Then $(\mathcal{F}, A)$ is soft topological homomorphic to $(\mathcal{K}, B)$ with discrete or anti-discrete topology. Thus, any soft good homomorphic polygroups can be regarded as soft topological homomorphic polygroups in the discrete or anti-discrete topology.*

Theorem 3.13. *Let the pair (ψ, φ) be a soft topological homomorphism from $(\mathcal{F}, A)$ to $(\mathcal{K}, B)$, where $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ are two soft topological polygroups over $\mathcal{P}$ and $\mathcal{P}'$, respectively. Then, $(\psi(\mathcal{F}), B)$ is a soft topological polygroup over $\mathcal{P}'$ if $\varphi : A \longrightarrow B$ be an injective mapping.*

Proof. Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be two soft topological polygroups over $\mathcal{P}$ and $\mathcal{P}'$ with the topologies τ and τ', respectively. Then, $\mathcal{F}(a)$ is a topological subpolygroup of $\mathcal{P}$ for all $a \in Supp(F, A)$. Since $(\psi, \varphi) : (\mathcal{F}, A) \longrightarrow (\mathcal{K}, B)$ is a soft topological homomorphism, we have $\varphi(Supp(\mathcal{F}, A)) = Supp(\psi(\mathcal{F}), B)$. Choose $b \in Supp(\psi(\mathcal{F}), B)$. So there exist $a \in Supp(\mathcal{F}, A)$ such that $\varphi(a) = b$, thus we have $\mathcal{F}(a) \neq \emptyset$. Further, $\mathcal{F}(a)$ is a topological subpolygroup of $\mathcal{P}$ with respect to the topology induced by τ. Since ψ is a good topological homomorphism, then $\psi(\mathcal{F}(x))$ is a topological subpolygroup of $\mathcal{P}'$ with respect to the topology induced by τ'. Therefore, $(\psi(\mathcal{F}), B)$ is a soft topological polygroup over $\mathcal{P}'$ with the topology τ'. □

Theorem 3.14. *Let the pair (ψ, φ) be a soft topological homomorphism from $(\mathcal{F}, A)$ to $(\mathcal{K}, B)$, where $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ are two soft topological polygroups over $\mathcal{P}$ and $\mathcal{P}'$, respectively. Then, $(\psi^{-1}(\mathcal{K}), A)$ is a soft topological polygroup over $\mathcal{P}$ if it is non-null.*

Proof. Assume that $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ are two soft topological polygroups over $\mathcal{P}$ and $\mathcal{P}'$ with the topologies τ and τ', respectively. So for all $b \in Supp(\mathcal{K}, B)$, it is easy to show that $\varphi(Supp(\psi^{-1}(\mathcal{K}), A)) = \varphi^{-1}(Supp(\mathcal{K}, B))$. Let $a \in Supp(\psi^{-1}(\mathcal{K}), A)$, thus $\varphi(a) \in Supp(\mathcal{K}, B)$. Hence, the nonempty set $\mathcal{K}(\varphi(a))$ is a topological subpolygroup of $\mathcal{P}'$ with respect to the topology induced by τ'. Since ψ is a good topological homomorphism, then $\psi^{-1}(\mathcal{K}(\varphi(b))) = \psi^{-1}(\mathcal{K}(a))$ is a topological subpolygroup of $\mathcal{P}$ with respect to the topology induced by τ. Thus, it has been proven that the pair $(\psi^{-1}(\mathcal{K}), A)$ is a soft topological polygroup over $\mathcal{P}$ with the topology τ. □

Theorem 3.15. *Let $(\mathcal{F}, A)$, $(\mathcal{K}, B)$ and $(\mathcal{N}, C)$ be soft topological polygroups over $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}''$ with the topologies τ, τ' and τ'', respectively. Then, $(\psi' \circ \psi, \varphi' \circ \varphi) : (\mathcal{F}, A) \longrightarrow (\mathcal{N}, C)$ is a soft topological homomorphism if $(\psi, \varphi) : (\mathcal{F}, A) \longrightarrow (\mathcal{K}, B)$ and $(\psi', \varphi') : (\mathcal{K}, B) \longrightarrow (\mathcal{N}, C)$ are two soft topological homomorphisms.*

Proof. Suppose that $(\psi, \varphi) : (\mathcal{F}, A) \longrightarrow (\mathcal{K}, B)$ and $(\psi', \varphi') : (\mathcal{K}, B) \longrightarrow (\mathcal{N}, C)$ are two soft topological homomorphisms. Then, $\psi : P \longrightarrow P'$ and $\psi' : P' \longrightarrow P''$ are two good topological homomorphisms, and $\varphi : A \longrightarrow B$ and $\varphi' : B \longrightarrow C$ are two mappings such that the equalities $\psi(\mathcal{F}(a)) = \mathcal{K}(\varphi(a))$ and $\psi'(\mathcal{K}(b)) = \mathcal{N}(\varphi'(b))$ hold for all $a \in Supp(\mathcal{F}, A)$, $b \in Supp(\mathcal{K}, B)$. Obviously, $\psi' \circ \psi : \mathcal{P} \longrightarrow \mathcal{P}''$ is also good topological homomorphism and $\varphi' \circ \varphi : A \longrightarrow C$ is a mapping so that the equality

$$(\psi' \circ \psi)(\mathcal{F}(a)) = \psi'(\psi(\mathcal{F}(a))) = \psi'(\mathcal{K}(\varphi(a))) = \mathcal{N}(\varphi'(\varphi(a))) = \mathcal{N}((\varphi' \circ \varphi)(a))$$

holds for all $a \in Supp(\mathcal{F}, A)$. Thus, the pair $(\psi' \circ \psi, \varphi' \circ \varphi)$ is a soft topological homomorphism from $(\mathcal{F}, A)$ to $(\mathcal{N}, C)$. □

3.2. *Soft Topological Subpolygroups*

Definition 3.16. *Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be soft topological polygroups over $\mathcal{P}$ with the topology τ. Then the pair $(\mathcal{K}, B)$ is said to be a soft topological subpolygroup of $(\mathcal{F}, A)$ if the following axioms hold :*
i. $B \subseteq A$.
ii. *$\mathcal{K}(b)$ is a subpolygroup of $\mathcal{F}(b)$ for all $b \in Supp(K, B)$.*
iii. *The mappings $\cdot : \mathcal{K}(b) \times \mathcal{K}(b) \longrightarrow P^*(\mathcal{K}(b))$ and*

$$^{-1} : \mathcal{K}(b) \longrightarrow \mathcal{K}(b)$$

are continuous for all $b \in Supp(\mathcal{K}, B)$.

Example 3.17. *Take a soft topological polygroup $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ. Then, $(\mathcal{F}|_B, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $B \subseteq A$.*

Theorem 3.18. *If $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ and $(\mathcal{N}, C)$ is a soft topological subpolygroup of $(\mathcal{K}, B)$, then $(\mathcal{N}, C)$ is the soft topological subpolygroup of $(\mathcal{F}, A)$.*

Proof. The proof follows from Definition 3.16. □

Theorem 3.19. *Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be two soft topological polygroups over $\mathcal{P}$ with the topology τ. Then, $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $(\mathcal{K}, B)$ is a soft subset of $(\mathcal{F}, A)$.*

Proof. Suppose that $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ are two soft topological polygroups over $\mathcal{P}$ with the topology τ. Then, the nonempty sets $\mathcal{F}(x)$ and $\mathcal{K}(x)$ are the topological subpolygroup of $\mathcal{P}$. By the assumption, if $(\mathcal{K}, B)$ is a soft subset of $(\mathcal{F}, A)$, then $B \subseteq A$ and $\mathcal{K}(b) \subseteq \mathcal{F}(b)$ for all $b \in Supp(\mathcal{K}, B)$. So, $\mathcal{K}(b)$ is a topological subpolygroup of $\mathcal{F}(b)$ with respect to the topology induced by τ. From this fact, we conclude that $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ. □

Theorem 3.20. *Let $(\mathcal{F}, A)$ be a soft topological polygroup over $\mathcal{P}$ with the topology τ and $\{(\mathcal{F}_i, A_i) | i \in I\}$ be a non-empty family of soft topological subpolygroups of $(\mathcal{F}, A)$.*
i. *The restricted intersection of the family $\{(\mathcal{F}_i, A_i) | i \in I\}$ with $\bigcap_{i \in I} A_i \neq \emptyset$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $\tilde{\bigcap}_{i \in I}(\mathcal{F}_i, A_i) \neq \emptyset$*
ii. *The extended intersection of the family $\{(\mathcal{F}_i, A_i) | i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $(\bigcap_{\mathcal{E}})_{i \in I} \neq \emptyset$*

Proof. **i**. The restricted intersection of the family $\{(\mathcal{F}_i, A_i) | i \in I\}$ with $\bigcap_{i \in I} A_i \neq \emptyset$ defined as the soft set $\tilde{\bigcap}_{i \in I}(\mathcal{F}_i, A_i) = (\mathcal{F}, A)$ such that $\mathcal{F}(a) = \bigcap_{i \in I} \mathcal{F}_i(a)$ for all $a \in A$. Let $a \in Supp(\mathcal{F}, A)$. Suppose $\bigcap_{i \in I} \mathcal{F}_i(a) \neq \emptyset$, which implies $\mathcal{F}_i(a) \neq \emptyset$ for all $i \in I$. Since $\{(\mathcal{F}_i, A_i) | i \in I\}$ is a non-empty family of soft topological subpolygroups of $(\mathcal{F}, A)$, we get $A_i \subseteq A$ and $\mathcal{F}_i(a)$ is a topological subpolygroup of $\mathcal{F}(a)$ with respect to the topology induced by τ for all $i \in I$. Hence, $\bigcap_{i \in I} A_i \subseteq A$ and $\bigcap_{i \in I} \mathcal{F}_i(a)$ is a topological subpolygroup of $\mathcal{F}(a)$. Consequently, the family $\{(\mathcal{F}_i, A_i) | i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$
ii. The proof is similar to **i**. □

Theorem 3.21. *Let $\{(\mathcal{F}_i, A_i) | i \in I\}$ be a non-empty family of soft topological subpolygroups of a soft topological polygroup $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ.*
i. *The extended union of the family $\{(\mathcal{F}_i, A_i) | i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $f_i(x) \subseteq f_j(x)$ or $f_j(x) \subseteq f_i(x)$ for all $i, j \in I$, $x \in \bigcup_{i \in I} A_i$*
ii. *The restricted union of the family $\{(\mathcal{F}_i, A_i) | i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $f_i(x) \subseteq f_j(x)$ or $f_j(x) \subseteq f_i(x)$ for all $i, j \in I$, $x \in \bigcap_{i \in I} A_i$ with $\bigcap_{i \in I} A_i \neq \emptyset$.*

Proof. **i**. Suppose that $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a non-empty family of soft topological subpolygroups of a soft topological polygroup $(\mathcal{F}, A)$ with $\bigcap_{i\in I} A_i \neq \emptyset$. Let $\mathcal{F}_i(x) \subseteq \mathcal{F}_j(x)$ or $\mathcal{F}_j(x) \subseteq \mathcal{F}_i(x)$ for all $i, j \in I$, $x \in \bigcup_{i\in I} A_i$. Take $a \in Supp(F, A)$. Since each $(\mathcal{F}_i, A_i)$ is non-null soft sets over $\mathcal{P}$, then $\widetilde{\bigcup}_{i\in I}(\mathcal{F}_i, A_i)$ is also a non-null soft set over $\mathcal{P}$ for all $i \in I$. By assumption, $\mathcal{F}_i(a) \subseteq \mathcal{F}_j(a)$ or $\mathcal{F}_j(a) \subseteq \mathcal{F}_i(a)$ for all $i, j \in I$, $a \in \bigcap_{i\in I} A_i$ with $\bigcap_{i\in I} A_i \neq \emptyset$ such that $\mathcal{F}_i(a)$ and $\mathcal{F}_j(a)$ are the topological subpolygroups of $\mathcal{F}(a)$ with respect to the topology induced by τ and so their union must be non-null too. This show that the extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ.
ii. The proof is similar to **i**.
□

Corollary 3.22. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological subpolygroups of a soft topological polygroup $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ. Then the extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ if $A_i \cap A_i \neq \emptyset$ for all $i, j \in I$, $i \neq j$.*

Theorem 3.23. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ and let $(\mathcal{K}_i, B_i)$ be a soft topological subpolygroup of $(\mathcal{F}_i, A_i)$ for all $i \in I$.*
i. *The $\wedge-$intersection $\widetilde{\bigwedge}_{i\in I}(\mathcal{K}_i, B_i)$ is a soft topological subpolygroup of $\widetilde{\bigwedge}_{i\in I}(\mathcal{F}_i, A_i)$ if it is non-null.*
ii. *The $\vee-$union $\widetilde{\bigvee}_{i\in I}(\mathcal{K}_i, B_i)$ is a soft topological subpolygroup of $\widetilde{\bigvee}_{i\in I}(\mathcal{F}_i, A_i)$ if $\mathcal{K}_i(b_i) \subseteq \mathcal{K}_j(b_j)$ or $\mathcal{K}_j(b_j) \subseteq \mathcal{K}_i(b_i)$ for all $i, j \in I$, $b_i \in B_i$.*

Proof. **i**. Consider $\{(\mathcal{F}_i, A_i)|i \in I\}$ as a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ. By 3.5 Theorem (ii), $\widetilde{\bigvee}_{i\in I}(\mathcal{F}_i, A_i)$ is also a soft topological polygroup over $\mathcal{P}$ with the topology τ. Choose $b_i \in Supp(\mathcal{K}_i, B_i)$. By the assumption, $\bigcap_{i\in I} \mathcal{K}_i(b_i) \neq \emptyset$ such that $\mathcal{K}_i(b_i) \neq \emptyset$ for all $i \in I$ and $(b_i)_{i\in I} \in B_i$. Also, $B_i \subseteq A_i$ and $\mathcal{K}_i(b_i)$ is a topological subpolygroup of $\mathcal{F}_i(b_i)$ with respect to the topology induced by τ for all $i \in I$ so that $\bigcap_{i\in I} B_i \subseteq \bigcap_{i\in I} A_i$ and $\bigvee_{i\in I}(\mathcal{K}_i(b_i))$ must be a topological subpolygroup of $\bigvee_{i\in I}(\mathcal{F}_i(b_i))$ too. So, $\widetilde{\bigwedge}_{i\in I}(\mathcal{K}_i, B_i)$ is a soft topological subpolygroup of $\widetilde{\bigwedge}_{i\in I}(\mathcal{F}_i, A_i)$ with the topology τ
ii. The proof is similar to **i**. □

Theorem 3.24. *Let $(\mathcal{F}, A)$ be a soft topological polygroup over $\mathcal{P}$ with the topology τ and $(\mathcal{K}, B)$ be a soft topological subpolygroup of $(\mathcal{F}, A)$.*
i. *The restricted intersection of $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if it is non-null.*
ii. *The restricted union of $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if it is non-null.*

Proof. **i**. Assume that $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ. If it is non-null, it follows that $B \subseteq A$ and $\mathcal{K}(b)$ is a topological subpolygroup of $\mathcal{F}(b)$ with respect to the topology induced by τ for all $b \in Supp(\mathcal{K}, B)$. Thus, it is easy to see that $A \cap B \subseteq A$ and $\mathcal{K}(b) \cap \mathcal{F}(b)$ is also a topological subhypergroupoid of $\mathcal{F}(b)$ with respect to the topology induced by τ for all $b \in Supp(\mathcal{K}, B)$. Therefore, the restricted intersection $(\mathcal{F}, A)\tilde{\cap}(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ.
ii. The proof is similar to **i**. □

Theorem 3.25. *Let $f : P \longrightarrow P'$ be a good homomorphism of topological polygroups with the topologies τ and τ', respectively, and let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be two soft topological polygroups over $\mathcal{P}'$. Then, $(f^{-1}(\mathcal{K}), B)$ is a soft topological subpolygroup of $(f^{-1}(\mathcal{F}), A)$ if $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ.*

Proof. Assume $(\mathcal{K}, B)$ be a soft topological subpolygroup of $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ'. Take $b \in Supp(f^{-1}(\mathcal{K}), B)$. Since $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$, it follows that $B \subseteq A$ and $(\mathcal{K}(b))$ is a topological subpolygroup of $(\mathcal{F}(b)$ with respect to the topology induced by τ' for all $b \in Supp(f^{-1}(\mathcal{K}), B)$. Morever, since $f : \mathcal{P} \longrightarrow \mathcal{P}'$ be a good topological homomorphism, then $f^{-1}(\mathcal{F})(b) = f^{-1}(\mathcal{F}(b))$ is a topological subpolygroup of $f^{-1}(\mathcal{K})(b) = f^{-1}(\mathcal{K}(b))$ with respect to the topology induced by τ for all $b \in Supp(f(\mathcal{K}), B)$. This proves that $(f^{-1}(\mathcal{K}), B)$ is a soft topological subpolygroup of $(f^{-1}(\mathcal{F}), A)$ with the topology τ.
□

Theorem 3.26. *Let $f : P \longrightarrow P'$ be a good homomorphism of topological polygroups with the topologies τ and τ', respectively, and let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be two soft topological polygroups over $\mathcal{P}$. Then, $(f(\mathcal{K}), B)$ is a soft topological subpolygroup of $(f(\mathcal{F}), A)$ over P' with the topology τ' if $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ.*

Proof. Suppose that $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ. If $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$, it follows that $B \subseteq A$ and $(\mathcal{K}(b))$ is a topological subpolygroup of $(\mathcal{F}(b)$ with respect to the topology induced by τ for all $b \in Supp(\mathcal{K}, B)$. Furthermore, since $f : P \longrightarrow P'$ be a good topological homomorphism, so $f(\mathcal{F})(b) = f(\mathcal{F}(b))$ is a topological subpolygroup of $f(\mathcal{K})(b) = f(\mathcal{K}(b))$ with respect to the topology induced by τ' for all $b \in Supp(f(\mathcal{K}), B)$. Therefore, $(f(\mathcal{K}), B)$ is a soft topological subpolygroup of $(f(\mathcal{F}), A)$. □

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An Examination on the Striction Curves in terms of Special Ruled Surfaces

Şeyda Kılıçoğlu[a]

[a]*Faculty of Education, Department of Mathematics, Başkent University, Ankara/Turkey.*

Abstract. In this paper, we firstly express ruled surfaces drawn by Frenet and Darboux vectors of Bertrand mate depending on Bertrand curve. Then, the tangent vectors of the striciton curves on these surfaces are calculated. Finally, we give some results with these vectors.

1. Introduction and Preliminaries

Many results on ruled surfaces have been obtained by mathematicians (see [1, 5, 9, 11, 12]). In [11], authors examine spatial quaternionic ruled surfaces. Another study, authors express some results about Bertrand offsets in Minkowski space [5]. A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation

$$\varphi(s, v) = \alpha(s) + ve(s) \tag{1}$$

where α base curve and e generator vector [3]. The striction curve is given by [3]

$$c(s) = \alpha(s) - \frac{\langle \alpha_s, e_s \rangle}{\langle e_s, e_s \rangle} e(s). \tag{2}$$

The notion of Bertrand curves was discovered by J. Bertrand in 1850. There are many studies on the Bertrand curve Bertrand curves in different areas. In [6], authors examine the Bertrand curves in the Euclidean 4-space as quaternionic. J. Monterde characterize Bertrand curves defined from Salkowski curves [10].
Let α be a unit speed curve in E^3, and $\{V_1(s), V_2(s), V_3(s)\}$ denote the Frenet frame of α. The Frenet formulas are given by

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where k_1 and k_2 denote the curvature and the torsion of α, respectively. On the other hand, the Darboux vector is [2]

$$D(s) = k_2(s)V_1(s) + k_1(s)V_3(s), \tag{3}$$

Corresponding author: ŞK: seyda@baskent.edu.tr ORCID: https://orcid.org/0000-0001-8535-944X
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The modified Darboux vector [4]

$$\tilde{D}(s) = \frac{k_2(s)}{k_1(s)}(s)V_1(s) + V_3(s). \tag{4}$$

Let α and α^* be the unit speed two curves and let $V_1(s), V_2(s), V_3(s)$ and $V_1^*(s), V_2^*(s), V_3^*(s)$ be the Frenet frames of the curves α and α^*, respectively. If the principal normal vector of the curve α is linearly dependent on the principal normal vector of the curve α^*, then the pair $\{\alpha, \alpha^*\}$ are called Bertrand pair and α^* is called Bertrand mate. [3]. The parametrization of Bertrand mate is [3]

$$\alpha^*(s) = \alpha(s) + \lambda V_2(s) \tag{5}$$

Theorem 1.1. *[3] The distance between corresponding points of the Bertrand pair in $\mathbb{E}^3$ is constant.*

Theorem 1.2. *[3]. If $k_2(s) \neq 0$ along $\alpha(s)$, then $\alpha(s)$ is a Bertrand curve if and only if there exist nonzero real numbers λ and β such that constant*

$$\lambda k_1 + \beta k_2 = 1. \tag{6}$$

Theorem 1.3. *[3] Let α and α^* be the unit speed two curves. $\left\{V_1, V_2, V_3, \tilde{D}, k_1, k_2\right\}$ and $\left\{V_1^*, V_2^*, V_3^*, \tilde{D}^*, k_1^*, k_2^*\right\}$ are Frenet-Serret apparatus of the Bertrand curve and the Bertrand mate, respectively. Then, the formulas are given by*

$$V_1^* = \frac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2 + \beta^2}}, \quad V_2^* = V_2, \quad V_3^* = \frac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2 + \beta^2}}, \quad \tilde{D}^* = \frac{k_1 \sqrt{\lambda^2 + \beta^2}}{(\beta k_1 - \lambda k_2)} \tilde{D}.$$

The first and second curvatures of Bertrand mate are given by

$$k_1^* = \frac{\beta k_1 - \lambda k_2}{(\lambda^2 + \beta^2) k_2}, \quad k_2^* = \frac{1}{(\lambda^2 + \beta^2) k_2}.$$

Let $\alpha : I \to \mathbb{E}^3$ be differentiable unit speed curve and let $\{V_1(s), V_2(s), V_3(s), \tilde{D}\}$ be the Frenet-Serret apparatus of this curve. The equations

$$\begin{aligned} \varphi_1(s, u_1) &= \alpha(s) + u_1 V_1(s) \\ \varphi_2(s, u_2) &= \alpha(s) + u_2 V_2(s) \\ \varphi_3(s, u_3) &= \alpha(s) + u_3 V_3(s) \\ \varphi_4(s, u_4) &= \alpha(s) + u_4 \tilde{D}(s) \end{aligned} \tag{7}$$

are the parametrization of the ruled surface which are called tangent ruled surface, normal ruled surface, binormal ruled surface, modified Darboux ruled surface, respectively. For the sake of shortness, we write Frenet ruled surfaces instead of the above all ruled surfaces.

Theorem 1.4. *[8] The tangent vectors of the striction curves on Frenet ruled surfaces are given by the following matrix*

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{k_2^2}{\eta \left\|c_2'(s)\right\|} & \dfrac{\left(\frac{k_1}{\eta}\right)'}{\left\|c_2'(s)\right\|} & \dfrac{k_1 k_2}{\eta \left\|c_2'(s)\right\|} \\ 1 & 0 & 0 \\ \dfrac{\mu - \mu' - \frac{k_2}{k_1}}{\mu \left\|c_4'(s)\right\|} & 0 & \dfrac{\mu'}{\mu^2 \left\|c_4'(s)\right\|} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

where $\eta = k_1^2 + k_2^2$, $\mu = \left(\frac{k_2}{k_1}\right)'$.

Definition 1.5. *[9] Let $\alpha^* : I \to \mathbb{E}^3$ be differentiable unit speed curve and let $\{V_1^*(s), V_2^*(s), V_3^*(s), \tilde{D}^*\}$ be the Frenet-Serret apparatus of this curve. The equations*

$$\begin{aligned}
\varphi_1^*(s,w_1) &= \alpha^*(s) + w_1 V_1^*(s) = \alpha + \lambda V_2 + w_1 \frac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2+\beta^2}} \\
\varphi_2^*(s,w_2) &= \alpha^*(s) + w_2 V_2^*(s) = \alpha + (\lambda + w_2) V_2 \\
\varphi_3^*(s,w_3) &= \alpha^*(s) + w_3 V_3^*(s) = \alpha + \lambda V_2 + w_3 \left(\frac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2+\beta^2}} \right) \\
\varphi_4^*(s,w_4) &= \alpha^*(s) + w_4 \tilde{D}^*(s) = \alpha + \lambda V_2 + w_4 \frac{k_1 \sqrt{\lambda^2+\beta^2}}{(\beta k_1 - \lambda k_2)} \tilde{D}
\end{aligned} \tag{8}$$

*are the parametrization of the ruled surface which are called **Bertrandian tangent ruled surface**, **Bertrandian normal ruled surface**, **Bertrandian binormal ruled surface** and **Bertrandian modified Darboux ruled surface**, respectively.*

For the sake of shortness, we write Bertrand ruled surfaces instead of the above all ruled surfaces.

Theorem 1.6. *[7] The tangent vectors of striction curves on Bertrand ruled surfaces are given by the following matrix*

$$\begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \\ T_4^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d^* & 0 & e^* \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}$$

where

$$a^* = \frac{{k_2^*}^2}{\eta^* \left\| {c_2^*}'(s) \right\|}, \quad b^* = \frac{\left(\frac{k_1^*}{\eta^*}\right)'}{\left\| {c_2^*}'(s) \right\|}, \quad c^* = \frac{k_1^* k_2^*}{\eta^* \left\| {c_2^*}'(s) \right\|}, \quad d^* = \frac{\mu^* - {\mu^*}' - \frac{k_2^*}{k_1^*}}{\mu^* \| {c_4^*}'(s) \|} = \frac{-m' - \left(\frac{-m'}{m^2 k_2 \sqrt{\lambda^2+\beta^2}}\right)' m^2 - m k_2 \sqrt{\lambda^2+\beta^2}}{-m' \| {c_4^*}'(s) \|},$$

$$e^* = \frac{{\mu^*}'}{{\mu^*}^2 \left\| {c_4^*}'(s) \right\|} = \frac{\left(\frac{-m'}{m^2 k_2 \sqrt{\lambda^2+\beta^2}}\right)' \frac{1}{k_2 \sqrt{\lambda^2+\beta^2}}}{\left(\frac{-m'}{m^2 k_2 \sqrt{\lambda^2+\beta^2}}\right)^2 \| {c_4^*}'(s) \|}, \quad \eta^* = {k_1^*}^2 + {k_2^*}^2, \quad \mu^* = \left(\frac{k_2^*}{k_1^*}\right)'.$$

2. An Examination on the Striction Curves in terms of Special Ruled Surfaces

In this section Then, the tangent vectors of the striciton curves on Frenet and Bertrandian ruled surfaces are calculated. We give some results with these vectors.

Theorem 2.1. *The relationship between the tangent vectors of the striciton curves on the Frenet and Bertrandian ruled surfaces is*

$$[T]\,[T^*]^{\mathrm{T}} = \frac{1}{\sqrt{\lambda^2+\beta^2}} \begin{bmatrix} \beta & a^*\beta - c^*\lambda & \beta & d^*\beta - e^*\lambda \\ x & a^* x + b^* \sqrt{\lambda^2+\beta^2} + a^* y & x & d^* x + e^* y \\ \beta & a^*\beta - c^*\lambda & \beta & d^*\beta - e^*\lambda \\ z & a^* z + c^* t & z & d^* z + e^* t \end{bmatrix}$$

where

$$x = \frac{k_2\big(\beta k_2 + \lambda k_1\big)}{\eta \left\| c_2'(s) \right\|}, \quad y = \frac{k_2\big(-\lambda k_2 + \beta k_1\big)}{\eta \left\| c_2'(s) \right\|}, \quad z = \frac{\big(\mu - \mu' - \frac{k_2}{k_1}\big)\beta + \mu'\lambda}{\mu \left\| c_4'(s) \right\|}, \quad t = \frac{\big(-\mu + \mu' + \frac{k_2}{k_1}\big)\lambda + \mu'\beta}{\mu \left\| c_4'(s) \right\|}.$$

Proof. Let $[T] = [A][V]$ and $[T^*] = [A^*][V^*]$ hence, by using the properties of the matrix, we can write

$$\begin{aligned}
[T][T^*]^{\mathbf{T}} &= [A][V]([A^*][V^*])^{\mathbf{T}} \\
&= [A]\left([V][V^*]^{\mathbf{T}}\right)[A^*]^{\mathbf{T}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{k_2^2}{\eta\left\|c_2'(s)\right\|} & \dfrac{\left(\frac{k_1}{\eta}\right)'}{\left\|c_2'(s)\right\|} & \frac{k_1k_2}{\eta\left\|c_2'(s)\right\|} \\ 1 & 0 & 0 \\ \frac{\mu-\mu'-\frac{k_2}{k_1}}{\mu\left\|c_4'(s)\right\|} & 0 & \frac{\mu'}{\mu^2\left\|c_4'(s)\right\|} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d^* & 0 & e^* \end{bmatrix} \begin{bmatrix} V_1{}^* \\ V_2{}^* \\ V_3{}^* \end{bmatrix} \right)^{\mathbf{T}} \\
&= \frac{1}{\sqrt{\lambda^2+\beta^2}} \begin{bmatrix} \beta & 0 & -\lambda \\ x & b\sqrt{\lambda^2+\beta^2} & y \\ \beta & 0 & -\lambda \\ z & 0 & t \end{bmatrix} \begin{bmatrix} 1 & a^* & 1 & d^* \\ 0 & b^* & 0 & 0 \\ 0 & c^* & 0 & e^* \end{bmatrix} \\
&= \frac{1}{\sqrt{\lambda^2+\beta^2}} \begin{bmatrix} \beta & a^*\beta - c^*\lambda & \beta & d^*\beta - e^*\lambda \\ x & a^*x + b^*\sqrt{\lambda^2+\beta^2} + a^*y & x & d^*x + e^*y \\ \beta & a^*\beta - c^*\lambda & \beta & d^*\beta - e^*\lambda \\ z & a^*z + c^*t & z & d^*z + e^*t \end{bmatrix} \\
&= \begin{bmatrix} \left\langle T_1, T_1^* \right\rangle & \left\langle T_1, T_2^* \right\rangle & \left\langle T_1, T_3^* \right\rangle & \left\langle T_1, T_4^* \right\rangle \\ \left\langle T_2, T_1^* \right\rangle & \left\langle T_2, T_2^* \right\rangle & \left\langle T_2, T_3^* \right\rangle & \left\langle T_2, T_4^* \right\rangle \\ \left\langle T_3, T_1^* \right\rangle & \left\langle T_3, T_2^* \right\rangle & \left\langle T_3, T_3^* \right\rangle & \left\langle T_3, T_4^* \right\rangle \\ \left\langle T_4, T_1^* \right\rangle & \left\langle T_4, T_2^* \right\rangle & \left\langle T_4, T_3^* \right\rangle & \left\langle T_4, T_4^* \right\rangle \end{bmatrix}.
\end{aligned}$$

□

Corollary 2.2. *There are four pairs of tangent vector fields equal to each other of the striction curves on Frenet and Bertrandian ruled surfaces.*

Proof. Since $\left\langle T_1, T_1^* \right\rangle = \left\langle T_1, T_3^* \right\rangle = \left\langle T_3, T_1^* \right\rangle = \left\langle T_3, T_3^* \right\rangle = \dfrac{\beta}{\sqrt{\lambda^2+\beta^2}}$, it is trivial. □

Corollary 2.3. ***i)****Tangent vectors of striction curves on tangent ruled surface and Bertrandian normal ruled surface are perpendicular if $\beta = \lambda m$ where $m = \beta k_1 - \lambda k_2$.*
ii)*Tangent vectors of striction curves on binormal ruled surface and Bertrandian normal ruled surface are perpendicular if $\beta = \lambda m$.*

Proof. **i)** Since $\langle T_1, T_2^* \rangle = \dfrac{a^*\beta - c^*\lambda}{\sqrt{\lambda^2+\beta^2}}$ and $\langle T_1, T_2^* \rangle = 0$

$$\begin{aligned}
a^*\beta - c^*\lambda &= 0, \\
\beta - \lambda(\beta k_1 - \lambda k_2) &= 0, \\
\beta &= \lambda m,
\end{aligned}$$

this completes the proof.
ii) Since $\langle T_1, T_2^* \rangle = \langle T_3, T_2^* \rangle$, it is trivial. □

Corollary 2.4. ***i)****Tangent vectors of striction curves on tangent ruled surface and Bertrandian modified Darboux ruled surface are perpendicular if*

$$\left(\frac{1}{m}\right)'\left[\left(\frac{1}{m}\right)' - \left(\frac{1}{m}\right)'' - \frac{1}{m}\right]\beta = \left(\frac{1}{m}\right)''\lambda.$$

ii)*Tangent vectors of striction curves on binormal ruled surface and Bertrandian modified Darboux ruled surface are perpendicular if*

$$\left(\frac{1}{m}\right)'\left[\left(\frac{1}{m}\right)' - \left(\frac{1}{m}\right)'' - \frac{1}{m}\right]\beta = \left(\frac{1}{m}\right)''\lambda.$$

Proof. **i)** Since $\langle T_1, T_4^* \rangle = \dfrac{d^*\beta - e^*\lambda}{\sqrt{\lambda^2 + \beta^2}}$ and $\langle T_1, T_4^* \rangle = 0$

$$\begin{aligned}
d^*\beta - e^*\lambda &= 0 \\
\left(\frac{1}{m}\right)'\left[\left(\frac{1}{m}\right)' - \left(\frac{1}{m}\right)'' - \frac{1}{m}\right]\beta - \left(\frac{1}{m}\right)''\lambda &= 0 \\
\left(\frac{1}{m}\right)'\left[\left(\frac{1}{m}\right)' - \left(\frac{1}{m}\right)'' - \frac{1}{m}\right]\beta &= \left(\frac{1}{m}\right)''\lambda
\end{aligned}$$

this completes the proof.
ii) Since $\langle T_1, T_4^* \rangle = \langle T_3, T_4^* \rangle$, it is trivial. □

The following corollaries are obtained similar to Corollary 2.5.

Corollary 2.5. ***i)****Tangent vectors of striction curves on normal ruled surface and Bertrandian tangent ruled surface have orthogonal under the condition $k_2 = 0$.*
ii)*Tangent vectors of striction curves on normal ruled surface and Bertrandian binormal ruled surface are perpendicular if $k_2 = 0$.*

Corollary 2.6. ***i)****Tangent vectors of striction curves on modified Darboux ruled surface and Bertrandian tangent ruled surface are perpendicular if*

$$k_1 = \frac{\beta(\frac{k_2}{k_1})'\left[k_1(\frac{k_2}{k_1})' - k_2\right]}{(\frac{k_2}{k_1})''\left[\beta(\frac{k_2}{k_1})' + \lambda\right]}.$$

ii)*Tangent vectors of striction curves on modified Darboux ruled surface and Bertrandian binormal ruled surface are perpendicular if*

$$k_1 = \frac{\beta(\frac{k_2}{k_1})'\left[k_1(\frac{k_2}{k_1})' - k_2\right]}{(\frac{k_2}{k_1})''\left[\beta(\frac{k_2}{k_1})' + \lambda\right]}.$$

Corollary 2.7. *Tangent vectors of striction curves on normal ruled surface and Bertrandian normal ruled surface are perpendicular if*

$$k_2 = -\frac{m\big(x + y\big)}{(\lambda^2 + \beta^2)^{\frac{3}{2}}}.$$

Corollary 2.8. *Tangent vectors of striction curves on normal ruled surface and Bertrandian modified Darboux ruled surface are perpendicular if*

$$\left(\frac{1}{m}\right)'\left[\left(\frac{1}{m}\right)' - \left(\frac{1}{m}\right)'' - \frac{1}{m}\right]x = -\left(\frac{1}{m}\right)''y.$$

Corollary 2.9. *Tangent vectors of striction curves on modified Darboux ruled surface and Bertrandian normal ruled surface are perpendicular if*

$$k_2 = \beta k_1 - \frac{z}{\lambda\left(\lambda\frac{(\mu-\mu'-\frac{k_2}{k_1})}{\mu\|c_4'(s)\|} - \beta\frac{k_1k_2}{\eta\|c_2'(s)\|}\right)}.$$

Corollary 2.10. *Tangent vectors of striction curves on modified Darboux ruled surface and Bertrandian modified Darboux ruled surface are perpendicular if*

$$\left(\frac{1}{m}\right)'\left[\left(\frac{1}{m}\right)' - \left(\frac{1}{m}\right)'' - \frac{1}{m}\right]z = -\left(\frac{1}{m}\right)''t.$$

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Numerical approximation for the spread of SIQR model with Caputo fractional order derivative

Ilknur Koca[a], Eyüp Akçetin[b], Pelin Yaprakdal[c]

[a]*Department of Accounting and Financial Management, Seydikemer Applied Science School, Muğla Sitki Koçman University, Muğla, Turkey*
[b]*Department of Accounting and Financial Management, Seydikemer Applied Science School, Muğla Sitki Koçman University, Muğla, Turkey*
[c]*Mehmet Akif Ersoy University, Department of Mathematics, Burdur, Turkey.*

Abstract. In our paper, the spread of SIQR model with fractional order differential equation is considered. We have evaluated the system with fractional way and investigated stability of the non-virus equilibrium point and virus equilibrium points. Also, the existence of the solutions are proved. Finally, the efficient numerical method for finding solutions of system is given.

1. Introduction

Fractional calculus is a very efficient way for researchers while studying real world phenomena problems like astronomy, biology, physics also in the social sciences e.g. education, history, sociology, life sciences . In recent years, fractional order differential equations have become an important tool in mathematical modelling. The most useful way to work on modelling is considering models again with their fractional order version. The most commonly used definitions are Riemann and Caputo fractional order derivatives. The Riemann-Liouville derivative is historically the first but there are some difficulties while applying it to real life problems. In order to overcome these difficulties, the latter concept, fractional order Caputo type derivative is defined [3, 5, 6, 8, 16].

Some disease models which are an important area in mathematical modelling are discussed [1, 9, 10, 13]. In our paper, we have investigated the system of equations involving fractional derivatives. But especially we are interested in investigating the spread of fractional order SIQR model using the concept of fractional operator of Caputo differentiations. After considered SIQR model with Caputo type, disease free equilibrium and endemic equilibrium points are computed. Also we have applied the next generation matrix method to calculated the basic reproduction number R_0 [19]. The stability analysis of SIQR model and the existence and uniqueness of its solutions have been obtained. Finally a suitable iteration for the solutions of the SIQR model is obtained by Atangana-Toufik method [18].

2. Preliminaries

In this section, let us give important definitions of fractional derivatives and their useful properties [7 – 17].

Corresponding author: IK mail address: ilknurkoca@mu.edu.tr ORCID: https://orcid.org/0000-0003-4393-1588, EA ORCID: https://orcid.org/0000-0001-7232-2154, PY ORCID: https://orcid.org/0000-0002-3261-2509

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Definition 2.1. *The Gamma function* $\Gamma(x)$ *is defined by the integral as below:*

$$\Gamma(x) = e^{-t}t^{x-1}dt. \tag{1}$$

One the basic properties of the gamma function is that it satifies the following equation :

$$\Gamma(x+1) = x.\Gamma(x) = z.(z-1)! = z!. \tag{2}$$

Definition 2.2. *The Grünwald-Letnikov definition is given as*

$$ {}_aD_t^\alpha f(t) = \lim_{h\longrightarrow 0}\frac{1}{h^\alpha}\sum_{k=0}^{\frac{(t-a)}{h}}(-1)^k\binom{\alpha}{k}f(t-kh). \tag{3}$$

Fractional derivative operator is non-local in nature and fractional equations provides an useful tool to describe phenomenas comprising memory and hereditary features. Such a phenomena can also appear in biological processes, population dinamics.

Definition 2.3. *Riemann-Liouville definition of fractional order differ-integral:*

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt}\int_a^t (t-\tau)^{n-\alpha-1}f(\tau)d\tau, \tag{4}$$

where

$$n-1 < \alpha \leqslant n, n \in \mathbb{N}. \tag{5}$$

The Laplace transform of the Riemann-Liouville fractional order differ-integral is given as below:

$$L[{}_0D_t^\alpha f(t)] = \{ \begin{array}{lll} s^\alpha F(s) & for & \alpha < 0, \\ s^\alpha F(s) - F'(s) & for & \alpha > 0, \end{array} . \tag{6}$$

where $n-1 < \alpha \leqslant n, n \in \mathbb{N}$.

Definition 2.4. *Caputo's definition of fractional order differ-integral:*

$$ {}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)}\int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, \tag{7}$$

where $n-1 < \alpha \leqslant n, n \in \mathbb{N}$*,* $\alpha \in \mathbb{R}$ *is a fractional order of the differ-integral of the function* $f(t)$*.*

The Laplace transform of the Caputo fractional order differ-integral is given as follows:

$$L[{}_0^CD_t^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}f^{(k)}(0) \tag{8}$$

where $n-1 < \alpha \leqslant n, n \in \mathbb{N}$.

Now, we give some important lemmas for Riemann–Liouville derivative and Caputo derivative as following:

Lemma 2.5. *Let us take a function* $f(x)$ *and* $m, n \geqslant 0$*, then the following equations hold.*

For $R-L$ *derivative given as:*

i. Linearity rule:

$$ {}_aD_t^\alpha(cf_1+f_2) = {}_aD_t^\alpha(cf_1) + {}_aD_t^\alpha(f_2) = c\,{}_aD_t^\alpha(f_1) + {}_aD_t^\alpha(f_2). \tag{9}$$

ii. The semi-group property does not hold. Indeed, the following equation is not always true.

$$ D_t^\alpha D_t^\beta f = D_t^{\alpha+\beta} f. \tag{10}$$

For Caputo derivative given as:

i. Linearity rule:

$$ {}_a^CD_t^\alpha(cf_1+f_2) = {}_a^CD_t^\alpha(cf_1) + {}_a^CD_t^\alpha(f_2) = c\,{}_a^CD_t^\alpha(f_1) + {}_a^CD_t^\alpha(f_2). \tag{11}$$

ii. The semi-group property:

$$ {}^CD_t^\alpha\,{}^CD_t^\beta f = {}^CD_t^{\alpha+\beta} f. \tag{12}$$

Fractional derivative operator is non-local in nature and fractional equations provides an useful tool to describe phenomenas comprising memory and hereditary features. Such a phenomena can also appear in biological processes, population dinamics.

Theorem 2.6. *Consider the n-dimensional system*

$$\begin{aligned} D_a^\alpha y(t) &= f(t, y(t)),\\ y(t_0) &= y_0, \end{aligned} \tag{13}$$

where $\alpha \in (0,1)$ and D_a^α represents Caputo sense fractional derivative of order α. Let y^* be the equilibrium point of the system and $J(y^*)$ be the Jacobian matrix about the equilibrium point y^*. Then, the equilibrium point y^* is locally asymptotically stable if and only if all the eigenvalues r_i, $i = 1, 2, ..., n$ of $J(y^*)$ satify $|\arg(r_i)| > \frac{\alpha\pi}{2}$.

Theorem 2.7. *Considering the delayed fractional differential system with the Caputo fractional derivative as*

$$\begin{aligned} D^\alpha y(t) &= My(t) + Ny(t-\tau),\\ y(t) &= \psi(t), t \in [-\tau, 0], \end{aligned} \tag{14}$$

where $\alpha \in (0,1]$, $y \in R^n$, $M, N \in R^{nxn}$, and $\psi(t) \in R_+^{nxn}$. The characteristic equation of the system (14) *is given as*

$$ \det|r^\alpha I - M - Ne^{-r\tau}| = 0. \tag{15}$$

If all the roots of (15) *have negative real parts, then the zero solution of system* (14) *is locally asymptotically stable* [12, 15]

3. Model Derivation

In this paper, we proposed a SIQR epidemic model with given first version with following form [11]:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \mu S - \frac{\beta SI}{N}, \\ \frac{dI}{dt} &= \frac{\beta SI}{N} - (\mu+\gamma+\delta+\alpha) I, \\ \frac{dQ}{dt} &= \delta I - (\mu+\epsilon+\alpha) Q, \\ \frac{dR}{dt} &= \gamma I + \epsilon Q - \mu R, \end{aligned} \tag{16}$$

where S, I, R detone the numbers of susceptible, infective and removed, recpectively, Q detones the number of quarantined and $N = S + I + Q + R$ is the number of total population individuals. The parameter Λ is the recruitment rate of S correspoinding to births and immigration; β detones tha average number of adequate contacts; μ is the natural death rate; γ and ϵ detone the recover rates from grup I, Q to R, recpectively; δ detones the removal rate from I; α is the disease-caused death rate of I and Q. The parameters involved in the system (3) are all positive constans [11].

Fractional calculus which means fractional derivatives and fractional integrals is of increasing interest among the researchers. It is known that fractional operators describe the system behavior more accurate and efficiently than integer order derivatives. Because of great advantege of memory properties let us consider model given above, again with fractional order. Fractional order SIQR epidemic model given as below:

$$\begin{aligned} {}_a^C D_t^{\alpha} S(t) &= \Lambda - \mu S - \frac{\beta S I}{N}, \\ {}_a^C D_t^{\alpha} I(t) &= \frac{\beta S I}{N} - (\mu + \gamma + \delta + \alpha) I, \\ {}_a^C D_t^{\alpha} Q(t) &= \delta I - (\mu + \epsilon + \alpha) Q, \\ {}_a^C D_t^{\alpha} R(t) &= \gamma I + \epsilon Q - \mu R, \end{aligned} \tag{17}$$

with initial conditions

$$S(t_0) = S_0, I(t_0) = I_0, Q(t_0) = Q_0 \text{ and } R(t_0) = R_0.$$

A working on equilibrium points and their asymptotic stability:

In this part, we study stabilities of non-virus equilibrium, virus equilibrium, and basic reproduction number of our fractional model (18).

Let $\alpha \in (0, 1]$ and consider the Caputo differential equation system as below:

$$\begin{aligned} {}_a^C D_t^{\alpha} S(t) &= F_1(t, S(t)), \\ {}_a^C D_t^{\alpha} I(t) &= F_2(t, I(t)), \\ {}_a^C D_t^{\alpha} Q(t) &= F_3(t, Q(t)), \\ {}_a^C D_t^{\alpha} R(t) &= F_4(t, R(t)). \end{aligned} \tag{18}$$

with initial conditions

$$S(t_0) = S_0, I(t_0) = I_0, Q(t_0) = Q_0 \text{ and } R(t_0) = R_0. \tag{19}$$

Here,

$$\begin{aligned} F_1(t, S(t)) &= \Lambda - \mu S(t) - \frac{\beta S(t) I(t)}{N}, \\ F_2(t, I(t)) &= \frac{\beta S(t) I(t)}{N} - (\mu + \gamma + \delta + \alpha) I(t), \\ F_3(t, Q(t)) &= \delta I(t) - (\mu + \epsilon + \alpha) Q(t), \\ F_4(t, R(t)) &= \gamma I(t) + \epsilon Q(t) - \mu R(t). \end{aligned} \tag{20}$$

3.1. Analysis of the non-virus equilibrium point

A non-virus equilibrium point is the point with no virus infection. Clearly, the point $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ to the non-virus equilibrium point of model (18).

Here, we examine the basic reproduction number in more detail utilizing the method given in [19]. According to the next generation matrix method, the matrices $\underset{\sim}{F}$ and $\tilde{W}$ are defined as:

$$\tilde{F} = \begin{bmatrix} \frac{\beta S}{N} & 0 \\ \delta & 0 \end{bmatrix} \text{ and } \tilde{W} = \begin{bmatrix} \mu + \gamma + \delta + \alpha & 0 \\ -\delta & \mu + \epsilon + \alpha \end{bmatrix}. \tag{21}$$

For obtaining the eigenvalues of the matrix $\tilde{F}\ \tilde{W}^{-1}$ at the point $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$, we have to solve the following equation

$$\left| \tilde{F} \tilde{W}^{-1} - \lambda I \right| = 0, \tag{22}$$

where λ are the eigenvalues and I is the identity matrix. So, the reproduction number is

$$R_0 = \frac{\beta \Lambda}{N\mu (\mu + \gamma + \delta + \alpha)}. \tag{23}$$

Therefore, the disease free (non-virus) equilibrium point $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ is locally asymptotically stable if $R_0 < 1$.

3.2. *Analysis of the virus equilibrium point*

The Jacobian matrix $J(S^*, I^*, Q^*, R^*)$ for the system given in (18) is.

$$J(S^*, I^*, Q^*, R^*) = \begin{bmatrix} -\mu - \frac{\beta I^*}{N} & \frac{\beta I^*}{N} & 0 & 0 \\ -\frac{\beta S^*}{N} & \frac{\beta S^*}{N} - (\mu + \gamma + \delta + \alpha) & \delta & \gamma \\ 0 & 0 & -(\mu + \epsilon + \alpha) & \epsilon \\ 0 & 0 & 0 & -\mu \end{bmatrix}. \tag{24}$$

We now discuss the asymptoticstability of the $E = (S^*, I^*, Q^*, R^*)$ equilibrium the system given by (18),

$$\begin{aligned} S^* &= \frac{N((\mu + \gamma + \delta + \alpha))}{\beta}, \\ I^* &= \frac{\beta - \mu N(\mu + \gamma + \delta + \alpha)}{\beta(\mu + \gamma + \delta + \alpha)}, \\ Q^* &= \frac{\delta(\beta - \mu N(\mu + \gamma + \delta + \alpha))}{\beta(\mu + \epsilon + \alpha)(\mu + \gamma + \delta + \alpha)}, \\ R^* &= \frac{(\gamma + \epsilon)(\beta - \mu N(\mu + \gamma + \delta + \alpha))}{\beta\mu(\mu + \gamma + \delta + \alpha)}. \end{aligned} \tag{25}$$

The characteristic equation of system is obtained via determination of (26)

$$K(\lambda) = \det(J - \lambda I) = 0. \tag{26}$$

The characteristic roots are obtained by solving the following equation

$$K(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0. \tag{27}$$

Here

$$\begin{aligned}
a_1 &= (2\mu+\epsilon+\alpha)+\mu+\frac{\beta I^*}{N}-\frac{\beta S^*}{N}+(\mu+\gamma+\delta+\alpha), \\
a_2 &= \mu(\mu+\epsilon+\alpha)+(2\mu+\epsilon+\alpha)\left[\mu+\frac{\beta I^*}{N}-\frac{\beta S^*}{N}+(\mu+\gamma+\delta+\alpha)\right] \\
&\quad -\frac{\mu\beta I^*}{N}+\mu(\mu+\gamma+\delta+\alpha)+\frac{\beta(\mu+\gamma+\delta+\alpha)I^*}{N}, \\
a_3 &= \mu(\mu+\epsilon+\alpha)\left[\mu+\frac{\beta I^*}{N}-\frac{\beta S^*}{N}+(\mu+\gamma+\delta+\alpha)\right] \\
&\quad +(2\mu+\epsilon+\alpha)\left[-\frac{\mu\beta I^*}{N}+\mu(\mu+\gamma+\delta+\alpha)+\frac{\beta(\mu+\gamma+\delta+\alpha)I^*}{N}\right], \\
a_4 &= \mu(\mu+\epsilon+\alpha)\left[-\frac{\mu\beta I^*}{N}+\mu(\mu+\gamma+\delta+\alpha)+\frac{\beta(\mu+\gamma+\delta+\alpha)I^*}{N}\right].
\end{aligned} \tag{28}$$

For $a_1, a_2, a_3, a_4 > 0$, $a_1a_2 - a_3 > 0$ and $a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$, so by Routh-Hurwitz Criterion, all characteristics roots have negative real parts. Therefore equilibrium point is asymptotic stable.

4. Working on the existence of solutions

Let $B = \Phi(q) \times \Phi(q)$ and $\Phi(q)$ be the Banach space of continuous function defined on the interval q with the norm

$$\|S, I, Q, R\| = \|S\| + \|I\| + \|Q\| + \|R\| \tag{29}$$

Here, $\|S\| = \sup\{|S(t)| : t \in q\}$, $\|I\| = \sup\{|I(t)| : t \in q\}$, $\|Q\| = \sup\{|Q(t)| : t \in q\}$ and $\|R\| = \sup\{|R(t)| : t \in q\}$.

Let us consider the classical SIQR model again by replacing the time derivative with Caputo fractional derivative:

$$\begin{aligned}
{}_a^C D_t^\alpha S(t) &= F_1(t, S(t)), \\
{}_a^C D_t^\alpha I(t) &= F_2(t, I(t)), \\
{}_a^C D_t^\alpha Q(t) &= F_3(t, Q(t)), \\
{}_a^C D_t^\alpha R(t) &= F_4(t, R(t)).
\end{aligned} \tag{30}$$

with initial conditions

$$S(t_0) = S_0, I(t_0) = I_0, Q(t_0) = Q_0 \text{ and } R(t_0) = R_0. \tag{31}$$

Here,

$$\begin{aligned}
F_1(t, S(t)) &= \Lambda - \mu S(t) - \frac{\beta S(t) I(t)}{N}, \\
F_2(t, I(t)) &= \frac{\beta S(t) I(t)}{N} - (\mu+\gamma+\delta+\alpha) I(t), \\
F_3(t, Q(t)) &= \delta I(t) - (\mu+\epsilon+\alpha) Q(t), \\
F_4(t, R(t)) &= \gamma I(t) + \epsilon Q(t) - \mu R(t).
\end{aligned} \tag{32}$$

The above system (30)is written as below:

$$
\begin{aligned}
S(t)-S_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_1(\tau, S(\tau))\, d\tau, \\
I(t)-I_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_2(\tau, I(\tau))\, d\tau, \\
Q(t)-Q_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_3(\tau, Q(\tau))\, d\tau, \\
R(t)-R_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_4(\tau, R(\tau))\, d\tau.
\end{aligned} \tag{33}
$$

Theorem 4.1. *The kernels F_1, F_2, F_3 and F_4 satisfy the Lipschitz condition and contraction if the inequality holds as below:*

$$0 \leq L_i < 1 \textit{ for } i = 1, 2, 3, 4. \tag{34}$$

Proof. Taking S and S_1 be two functions then we have following:

$$
\begin{aligned}
\|F_1(t,S) - F_1(t,S_1(t))\| &= \left\| \Lambda - \mu S(t) - \frac{\beta S(t) I(t)}{N} - \Lambda + \mu S_1(t) + \frac{\beta S_1(t) I(t)}{N} \right\|, \\
&= \left\| \mu (S_1(t) - S(t)) + \frac{\beta I(t)}{N}(S_1(t) - S(t)) \right\|, \\
&\leq \left(\mu + \frac{\beta b}{N}\right) \|S_1(t) - S(t)\|, \\
&\leq L_1 \|S_1(t) - S(t)\|.
\end{aligned} \tag{35}
$$

Taking $L_1 = \mu + \frac{\beta b}{N}$, where $a = \max_{t\in I} \|S(t)\|, b = \max_{t\in I} \|I(t)\|, c = \max_{t\in I} \|Q(t)\|, d = \max_{t\in I} \|R(t)\|$ are bounded function, then we get

$$\|F_1(t,S) - F_1(t,S_1(t))\| \leq L_1 \|S_1(t) - S(t)\|. \tag{36}$$

So, the Lipschitz condition and contraction are satisfied for F_1 if $0 \leq L_1 < 1$ is satified. With doing same way, the other kernels also satisfy the Lipschitz condition as follows:

$$
\begin{aligned}
\|F_2(t,I) - F_2(t,I_1(t))\| &\leq L_2 \|I_1(t) - I(t)\|, \\
\|F_3(t,Q) - F_3(t,Q_1(t))\| &\leq L_3 \|Q_1(t) - Q(t)\|, \\
\|F_4(t,R) - F_4(t,R_1(t))\| &\leq L_4 \|R_1(t) - R(t)\|.
\end{aligned} \tag{37}
$$

Now we consider the kernels for the model, eq. (33) and ıt is rewritten as follows:

$$
\begin{aligned}
S(t) &= S_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_1(\tau, S(\tau))\, d\tau, \qquad (38)\\
I(t) &= I_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_2(\tau, I(\tau))\, d\tau,\\
Q(t) &= Q_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_3(\tau, Q(\tau))\, d\tau,\\
R(t) &= R_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_4(\tau, R(\tau))\, d\tau.
\end{aligned}
$$

Then we have the following recursive formula:

$$
\begin{aligned}
S_n(t) &= S_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_1(\tau, S_{n-1}(\tau))\, d\tau, \qquad (39)\\
I_n(t) &= I_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_2(\tau, I_{n-1}(\tau))\, d\tau,\\
Q_n(t) &= Q_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_3(\tau, Q_{n-1}(\tau))\, d\tau,\\
R_n(t) &= R_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_4(\tau, R_{n-1}(\tau))\, d\tau.
\end{aligned}
$$

Here initial conditions are given with $S(t_0) = S_0, I(t_0) = I_0, Q(t_0) = Q_0$ and $R(t_0) = R_0$.
The difference between the successive terms in the expression are given below:

$$
\begin{aligned}
A_n(t) &= S_n(t) - S_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_1(\tau, S_{n-1}(\tau)) - F_1(\tau, S_{n-2}(\tau)))d\tau, \qquad (40)\\
B_n(t) &= I_n(t) - I_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_2(\tau, I_{n-1}(\tau)) - F_2(\tau, I_{n-2}(\tau)))d\tau,\\
C_n(t) &= Q_n(t) - Q_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_3(\tau, Q_{n-1}(\tau)) - F_3(\tau, Q_{n-2}(\tau)))d\tau,\\
D_n(t) &= R_n(t) - R_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_4(\tau, R_{n-1}(\tau)) - F_4(\tau, R_{n-2}(\tau)))d\tau.
\end{aligned}
$$

It is worth noticing that

$$
\begin{aligned}
S_n(t) &= \sum_{i=1}^{n} A_i(t), \qquad (41)\\
I_n(t) &= \sum_{i=1}^{n} B_i(t),\\
Q_n(t) &= \sum_{i=1}^{n} C_i(t),\\
R_n(t) &= \sum_{i=1}^{n} D_i(t).
\end{aligned}
$$

It is easy to see that the equation (40) reduces to (42),

$$
\begin{aligned}
\|A_n(t)\| &= \|S_n(t) - S_{n-1}(t)\|, \qquad (42)\\
&\leq \frac{1}{\Gamma(\alpha)} \left\| \int_0^t (t-\tau)^{\alpha-1} \left(F_1(\tau, S_{n-1}(\tau)) - F_1(\tau, S_{n-2}(\tau))\right) d\tau \right\|.
\end{aligned}
$$

So we have,

$$
\|S_n(t) - S_{n-1}(t)\| \leq \frac{L_1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|S_{n-1}(\tau) - S_{n-2}(\tau)\| d\tau, \qquad (43)
$$

then we get

$$
\|A_n(t)\| \leq \frac{L_1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|A_{n-1}(\tau)\| d\tau. \qquad (44)
$$

Similarly, we get the following results:

$$
\begin{aligned}
\|B_n(t)\| &\leq \frac{L_2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|B_{n-1}(\tau)\| d\tau, \qquad (45)\\
\|C_n(t)\| &\leq \frac{L_3}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|C_{n-1}(\tau)\| d\tau,\\
\|D_n(t)\| &\leq \frac{L_4}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|D_{n-1}(\tau)\| d\tau.
\end{aligned}
$$

After the above results, let us give a now theorem. □

Theorem 4.2. *The SIQR system* (30) *has a unique solution ıf we can find* $t_{\max}$ *satisfying following condition*

$$
\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_i < 1, \text{ for } i = 1, 2, 3, 4. \qquad (46)
$$

Proof. *$S(t), I(t), Q(t)$ and $R(t)$ are bounded functions so from the equality (44), we have the succeeding relation as follows:*

$$\begin{aligned} \|A_n(t)\| &\leq \|S_0\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_1 \right]^n, \\ \|B_n(t)\| &\leq \|I_0\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_2 \right]^n, \\ \|C_n(t)\| &\leq \|Q_0\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_3 \right]^n, \\ \|D_n(t)\| &\leq \|R_0\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_4 \right]^n. \end{aligned} \tag{47}$$

Now let us assume that followings are satisfied

$$\begin{aligned} S(t) - S_0 &= S_n(t) - b_n(t), \\ I(t) - I_0 &= I_n(t) - c_n(t), \\ Q(t) - Q_0 &= Q_n(t) - d_n(t), \\ R(t) - R_0 &= R_n(t) - e_n(t). \end{aligned} \tag{48}$$

Now we have to show that the infinity term $\|b_\infty(t)\| \longrightarrow 0$, therefore we have

$$\begin{aligned} \|b_n(t)\| &\leq \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_1(\tau, S) - F_1(\tau, S_{n-1})) d\tau \right\|, \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|F_1(\tau, S) - F_1(\tau, S_{n-1})\| d\tau, \\ &\leq \frac{t^{\alpha}}{\Gamma(\alpha)} L_1 \|S - S_{n-1}\|. \end{aligned} \tag{49}$$

Repeating this process recursively, we obtain following equality

$$\|b_n(t)\| \leq \left[\frac{t^{\alpha}}{\Gamma(\alpha)} \right]^{n+1} L_1^n M. \tag{50}$$

Then at $t_{\max}$ we have

$$\|b_n(t)\| \leq \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} \right]^{n+1} L_1^n M. \tag{51}$$

If we apply the limit to both sides as n tends to infinity, we have $\|b_\infty(t)\| \longrightarrow 0$. So this completes the proof. □

4.1. Uniqueness of the special solution

To prove the uniqueness of the system of solutions We assume that by contraction there exists another system of solutions of (6), $S_1(t), I_1(t), Q_1(t)$ and $R_1(t)$. Then we have

$$\|S(t) - S_1(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_1(\tau, S) - F_1(\tau, S_1)) d\tau, \tag{52}$$

Wit applying the norm to eq. (52)*, we get*

$$\|S(t) - S_1(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|F_1(\tau, S) - F_1(\tau, S_1)\| \, d\tau, \tag{53}$$

$$\|S(t) - S_1(t)\| \leq \frac{1}{\Gamma(\alpha)} L_1 t^{\alpha} \|S(t) - S_1(t)\|. \tag{54}$$

Finally, this gives

$$\begin{aligned} \|S(t) - S_1(t)\| \left(1 - \frac{1}{\Gamma(\alpha)} L_1 t^{\alpha}\right) &\leq 0, \\ \|S(t) - S_1(t)\| &= 0 \longrightarrow S(t) = S_1(t). \end{aligned} \tag{55}$$

It is easily showed that the equation $S(t)$ and other solutions have a unique solution.

5. *Atangana-Toufik numerical scheme with Caputo derivative*

First of all, it should be emphasised that the "numerical approach" is not directly equivalent to the "approach with use of computer", although we usually use numerical approach to find the solution with use of computers. Generally, analytical solutions are possible using simplifying assumptions that may not realistically reflect reality. In many applications, analytical solutions are impossible to achieve. Numerical methods makes it possible to obtain realistic solutions without the need for simplifying assumptions. There are lots of numerical methods have been used for finding the solutions of equations [2, 4, 14].

In this section, we reconsider Atangana-Toufik method for fractional differential equations with Caputo derivative as below:

$$\begin{aligned} {}_a^C D_t^{\alpha} x(t) &= f(t, x(t)), \\ x(0) &= x_0. \end{aligned} \tag{56}$$

Caputo fractional integral of this equation is given by

$$x(t) - x(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau. \tag{57}$$

If we take $t = t_{n+1}$ for $n = 0, 1, 2, ...,$ the equation (57) is rewritten as

$$x(t_{n+1}) - x(0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau. \tag{58}$$

Here, If we use the two-step Lagrange polynomial interpolation in integral then we have following

$$P_k(\tau) = f(\tau, x(\tau)) \simeq \frac{f(t_k, x_k)(\tau - t_{k-1})}{h} - \frac{f(t_{k-1}, x_{k-1})(\tau - t_k)}{h}, \tag{59}$$

where $h = t_n - t_{n-1}$. So we have

$$\begin{aligned} & x(t_{n+1}) - x(0) \\ = \; & \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \left(\begin{array}{c} P_k(\tau) \\ + \frac{(\tau - t_k)(\tau - t_{k-1})}{2!} \frac{\partial^2}{\partial \tau^2} \left[f(\tau, x(\tau)) \right]_{\tau = \epsilon_k} \end{array} \right) (t_{n+1} - \tau)^{\alpha-1} d\tau, \end{aligned} \tag{60}$$

or

$$
\begin{aligned}
& x(t_{n+1}) - x(0) \\
= \quad & \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\begin{array}{c} \frac{f(t_k,x_k)}{h} \underbrace{\int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1} - \tau)^{\alpha-1} d\tau}_{I_1} \\ -\frac{f(t_{k-1},x_{k-1})}{h} \underbrace{\int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1} - \tau)^{\alpha-1} d\tau}_{I_2} \\ + \int_{t_k}^{t_{k+1}} \frac{(\tau-t_k)(\tau-t_{k-1})}{2!} \frac{\partial^2}{\partial \tau^2} \left[f(\tau, x(\tau)) \right]_{\tau=\epsilon_k} (t_{n+1} - \tau)^{\alpha-1} d\tau \end{array} \right].
\end{aligned} \tag{61}
$$

Finally, calculating integrals in equation above, we obtain

$$
\begin{aligned}
& x(t_{n+1}) - x(0) \\
= \quad & \frac{f(t_k, x_k)}{h} \sum_{k=0}^{n} \frac{\left(\begin{array}{c} \alpha \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \\ -(\alpha+1)(n-k+2) \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \end{array} \right)}{\Gamma(\alpha+2)} \\
& -\frac{f(t_{k-1}, x_{k-1})}{h} \sum_{k=0}^{n} \frac{\left(\begin{array}{c} (\alpha+1)(k-n-1) \left[\begin{array}{c} (n+2-k)^{\alpha} \\ -(n-k+1)^{\alpha+1} \end{array} \right] \\ -\alpha \left[\begin{array}{c} (n-k+2)^{\alpha-1} \\ -(n+1-k)^{\alpha-1} \end{array} \right] \end{array} \right)}{\Gamma(\alpha+2)} \\
& +E_n^{\alpha}
\end{aligned} \tag{62}
$$

Above ${}_1E_n^{\alpha}$ is error term and given by

$$
\begin{aligned}
& E_n^{\alpha} \\
= \quad & \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \left(\frac{(\tau - t_k)(\tau - t_{k-1})}{2!} \frac{\partial^2}{\partial \tau^2} \left[f(\tau, x(\tau)) \right]_{\tau=\epsilon_k} \right) (t_{n+1} - \tau)^{\alpha-1} d\tau.
\end{aligned} \tag{63}
$$

then we have

$$
\begin{aligned}
& \left| E_n^{\alpha} \right| \\
\leq \quad & \frac{h}{2\Gamma(2+\alpha))} \max_{[0,t_{n+1}]} \left| \frac{\partial^2 f(\tau, x(\tau))}{\partial \tau^2} \right| \times \\
& \sum_{k=0}^{n} \left(\begin{array}{c} \alpha \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] - \\ (\alpha+1)(k-n-2) \left[\begin{array}{c} (n-k)^{\alpha} \\ -(n+1-k)^{\alpha} \end{array} \right] \end{array} \right).
\end{aligned} \tag{64}
$$

The right-hand side converges as follows:

$$
\begin{aligned}
&\sum_{k=0}^{n}\left(\begin{array}{c}\alpha\left[\begin{array}{c}(n-k)^{\alpha+1}\\-(n+1-k)^{\alpha+1}\end{array}\right]-\\(\alpha+1)(k-n-2)\left[\begin{array}{c}(n-k)^{\alpha}\\-(n+1-k)^{\alpha}\end{array}\right]\end{array}\right) \\
&= \left(n^{\alpha}-(n+1)^{\alpha}\right)\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha.
\end{aligned} \tag{65}
$$

So we have error term as

$$
\begin{aligned}
\left|E_n^{\alpha}\right| \leq\ & \frac{h}{2\Gamma(2+\alpha))}\max_{[0,t_{n+1}]}\left|\frac{\partial^2 f(\tau,x(\tau))}{\partial\tau^2}\right|\left(n^{\alpha}-(n+1)^{\alpha}\right) \\
& \times\left(\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha\right).
\end{aligned} \tag{66}
$$

5.1. Application of method to system

In this part, we apply the method for fractional order Caputo system. Let us consider system with Caputo derivative.

$$
\begin{aligned}
{}_a^C D_t^{\alpha} S(t) &= F_1(t,S(t)), \\
{}_a^C D_t^{\alpha} I(t) &= F_2(t,I(t)), \\
{}_a^C D_t^{\alpha} Q(t) &= F_3(t,Q(t)), \\
{}_a^C D_t^{\alpha} R(t) &= F_4(t,R(t)).
\end{aligned} \tag{67}
$$

Then we have

$$
\begin{aligned}
S(t)-S_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_1(\tau,S(\tau))\,d\tau, \\
I(t)-I_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_2(\tau,I(\tau))\,d\tau, \\
Q(t)-Q_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_3(\tau,Q(\tau))\,d\tau, \\
R(t)-R_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_4(\tau,R(\tau))\,d\tau.
\end{aligned} \tag{68}
$$

At a given point $t = t_{n+1}$, following formula is written

$$
\begin{aligned}
& S_{n+1} - S_0 \\
= \; & \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\begin{array}{c}
\frac{F_1(t_k,S_k)}{h} \frac{\left(\begin{array}{c} \alpha \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \\ -(\alpha+1)(n-k+2) \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)} \\
- \frac{F_1(t_{k-1},S_{k-1})}{h} \frac{\left(\begin{array}{c} (\alpha+1)(k-n-1) \left[\begin{array}{c} (n+2-k)^{\alpha} \\ -(n-k+1)^{\alpha+1} \end{array} \right] \\ -\alpha \left[\begin{array}{c} (n-k+2)^{\alpha-1} - \\ (n+1-k)^{\alpha-1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)}
\end{array} \right] +_1 R_n^{\alpha},
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
& I_{n+1} - I_0 \\
= \; & \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\begin{array}{c}
\frac{F_2(t_k,I_k)}{h} \frac{\left(\begin{array}{c} \alpha \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \\ -(\alpha+1)(n-k+2) \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)} \\
- \frac{F_2(t_{k-1},I_{k-1})}{h} \frac{\left(\begin{array}{c} (\alpha+1)(k-n-1) \left[\begin{array}{c} (n+2-k)^{\alpha} - \\ (n-k+1)^{\alpha+1} \end{array} \right] \\ -\alpha \left[\begin{array}{c} (n-k+2)^{\alpha-1} - \\ (n+1-k)^{\alpha-1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)}
\end{array} \right] +_2 R_n^{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
& Q_{n+1} - Q_0 \\
= \; & \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\begin{array}{c}
\frac{F_3(t_k,Q_k)}{h} \frac{\left(\begin{array}{c} \alpha \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \\ -(\alpha+1)(n-k+2) \left[\begin{array}{c} (n-k)^{\alpha+1} - \\ (n+1-k)^{\alpha+1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)} \\
- \frac{F_3(t_{k-1},Q_{k-1})}{h} \frac{\left(\begin{array}{c} (\alpha+1)(k-n-1) \left[\begin{array}{c} (n+2-k)^{\alpha} - \\ (n-k+1)^{\alpha+1} \end{array} \right] \\ -\alpha \left[\begin{array}{c} (n-k+2)^{\alpha-1} - \\ (n+1-k)^{\alpha-1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)}
\end{array} \right] +_3 R_n^{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
&R_{n+1}-R_0\\
&= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}\left[\begin{array}{c}
\frac{F_4(t_k,R_k)}{h}\frac{\left(\begin{array}{c}\alpha\left[\begin{array}{c}(n-k)^{\alpha+1}\\-(n+1-k)^{\alpha+1}\end{array}\right]\\-(\alpha+1)(n-k+2)\left[\begin{array}{c}(n-k)^{\alpha+1}-\\(n+1-k)^{\alpha+1}\end{array}\right]\end{array}\right)}{\alpha(\alpha+1)}\\
-\frac{F_4(t_{k-1},R_{k-1})}{h}\frac{\left(\begin{array}{c}(\alpha+1)(k-n-1)\left[\begin{array}{c}(n+2-k)^{\alpha}-\\(n-k+1)^{\alpha+1}\end{array}\right]\\-\alpha\left[\begin{array}{c}(n-k+2)^{\alpha-1}-\\(n+1-k)^{\alpha-1}\end{array}\right]\end{array}\right)}{\alpha(\alpha+1)}
\end{array}\right]+{}_4R_n^{\alpha}.
\end{aligned}
$$

Where

$$
\begin{aligned}
\left|{}_1R_n^{\alpha}\right| &\leq \frac{h}{2\Gamma(2+\alpha))}\max_{[0,t_{n+1}]}\left|\frac{\partial^2F_1(\tau,S(\tau))}{\partial\tau^2}\right|(n^{\alpha}-(n+1)^{\alpha})\\
&\quad\times\left(\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha\right),
\end{aligned}
\tag{70}
$$

$$
\begin{aligned}
\left|{}_2R_n^{\alpha}\right| &\leq \frac{h}{2\Gamma(2+\alpha))}\max_{[0,t_{n+1}]}\left|\frac{\partial^2F_2(\tau,I(\tau))}{\partial\tau^2}\right|(n^{\alpha}-(n+1)^{\alpha})\\
&\quad\times\left(\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha\right),
\end{aligned}
$$

$$
\begin{aligned}
\left|{}_3R_n^{\alpha}\right| &\leq \frac{h}{2\Gamma(2+\alpha))}\max_{[0,t_{n+1}]}\left|\frac{\partial^2F_3(\tau,Q(\tau))}{\partial\tau^2}\right|(n^{\alpha}-(n+1)^{\alpha})\\
&\quad\times\left(\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha\right),
\end{aligned}
$$

$$
\begin{aligned}
\left|{}_4R_n^{\alpha}\right| &\leq \frac{h}{2\Gamma(2+\alpha))}\max_{[0,t_{n+1}]}\left|\frac{\partial^2F_4(\tau,R(\tau))}{\partial\tau^2}\right|(n^{\alpha}-(n+1)^{\alpha})\\
&\quad\times\left(\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha\right).
\end{aligned}
$$

6. Conclusion

In this paper fractional order SIQR model is considered. Here, we generalize the previous model by considering the order as fractional order. As we saw that, the fractional order model is much more efficient in modeling than its integer order version. We have applied the next generation matrix method to calculated the basic reproduction number R_0. Also, the detailed analysis such as existence ande uniqueness results of the solution and efficient numerical scheme for model are presented.

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On Caputo Fractional Derivatives via Exponential $s-$Convex Functions

Saad Ihsan Butt[a], Mehroz Nadeem[b], Ghulam Farid[c]

[a]*COMSATS University Islamabad, Lahore Campus, Pakistan*
[b]*COMSATS University Islamabad, Lahore Campus, Pakistan*
[c]*COMSATS University Islamabad, Attock Campus, Pakistan*

Abstract. In this paper, we establish several new integral inequalities including Caputo fractional derivatives for exponential s-convex functions. By using convexity for exponential $s-$convex functions of any positive integer order differentiable function some novel results are given.

1. Introduction

Convexity plays an important role in many features of mathematical programming including, for example, suficient optimality conditions and duality theorems. The topic of convex functions has been treated extensively in the classical book by Hardy, Littlewood, and Polya [5]. The study of fractional order derivatives and integrals is called fractional calculus. Fractional calculus have important applications in all fields of applied sciences. Fractional integration and fractional differentiation appear as basic part in the subject of partial differential equations [1, 12]. Many types of fractional integral as well as differential operators have been defined in literature. Classical Caputo fractional derivatives were introduced by Michele Caputo in [8] which is written in 1967.

Definition 1.1. *The function $\Psi : [u, v] \to \mathbb{R}$ is said to be convex, if we have*

$$\Psi(\tau z_1 + (1-\tau)z_2) \leq \tau\Psi(z_1) + (1-\tau)\Psi(z_2)$$

for all $z_1, z_2 \in [u, v]$ and $\tau \in [0, 1]$.

Definition 1.2. *(see[9])*
Let $\Psi : I \subseteq \Re$ is of exponential-convex, if

$$\Psi(\tau z_1 + (1-\tau)z_2) \leq \tau e^{-\alpha z_1}\Psi(z_1) + (1-\tau)e^{-\alpha z_2}\Psi(z_2)$$

for all $\tau \in [0, 1]$ and $z_1, z_2 \in I$ and $\alpha \in \Re$.

Corresponding author: SIB mail address: saadihsanbutt@gmail.com ORCID: https://orcid.org/0000-0002-4761-235X, MN ORCID: https://orcid.org/0000-0003-4462-773X, GF ORCID: https://orcid.org/0000-0002-4103-7745
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Definition 1.3. *(see[6])*
Let $\Psi : I \subset [0,\infty) \longrightarrow \Re$ *is of s-convex in second sense, with* $s \in (0,1]$, *if*

$$\Psi(\tau z_1 + (1-\tau)z_2) \leq \tau^s \Psi(z_1) + (1-\tau)^s \Psi(z_2)$$

for all $\tau \in [0,1)$ *and* $z_1, z_2 \in I$.

Definition 1.4. *(see[10])*
Let $\Psi : I \subset [0,\infty) \longrightarrow \Re$ *is of exponential s-convex in second sense, with* $s \in [0,1]$, *if*

$$\Psi(\tau z_1 + (1-\tau)z_2) \leq \tau^s e^{-\beta z_1} \Psi(z_1) + (1-\tau)^s e^{-\beta z_2} \Psi(z_2)$$

for all $\tau \in [0,1]$ *and* $z_1, z_2 \in I$ *and* $\beta \in \Re$.

The previous era of fractional calculus is as old as the history of differential calculus. They generalize the differential operators and ordinary integral. However, the fractional derivatives have some basic properties than the corresponding classical ones. On the other hand, besides the smooth requirement, Caputo derivative does not coincide with the classical derivative [2].We give the following definition of Caputo fractional derivatives, see ([1, 3, 7, 11]).

Definition 1.5. *let* $AC^n[u,v]$ *be a space of functions having nth derivatives absolutely continuous,* $\Psi \in AC^n[u,v]$, $\lambda \notin \{1,2,3,...\}$ *and* $n = [\lambda]+1$. *The right sided Caputo fractional derivative is as follows,*

$$({}^{C}D^{\lambda}_{u+}\Psi)(z) = \frac{1}{\Gamma(n-\lambda)} \int_u^z \frac{\Psi^{(n)}(\tau)}{(z-\tau)^{\lambda-n+1}} d\tau, \qquad z > u \tag{1}$$

The left sided caputo fractional derivative is as follows,

$$({}^{C}D^{\lambda}_{v-}\Psi)(z) = \frac{(-1)^n}{\Gamma(n-\lambda)} \int_z^v \frac{\Psi^{(n)}(\tau)}{(\tau-z)^{\lambda-n+1}} d\tau, \qquad z < v. \tag{2}$$

The Caputo fractional derivative $({}^{C}D^{n}_{u+}\Psi)(z)$ *coincides with* $\Psi^{(n)}(z)$ *whereas* $({}^{C}D^{n}_{v-}\Psi)(z)$ *coincides with* $\Psi^{(n)}(z)$ *with exactness to a constant multiplier* $(-1)^n$ *,if* $\lambda = n \in \{1,2,3,...\}$ *and usual derivative* $\Psi^{(n)}(z)$ *of order n exists. In particular we have*

$$({}^{C}D^{0}_{u+}\Psi)(z) = ({}^{C}D^{0}_{v-}\Psi)(z) = \Psi(z) \tag{3}$$

where $n = 1$ *and* $\lambda = 0$.

In this paper, we establish several new integral inequalities including Caputo fractional integrals for exponential s-convex functions.By using convexity for exponential s-convex functions of any integer order differentiable function some novel results are given.The purpose of this paper is to introduce some fractional inequalities for the Caputo-fractional derivatives via *s*-convex functions in second sense which have derivatives of any integer order.

2. Main Results

First we give the following estimate of the sum of left and right handed Caputo fractional derivatives for exponential s-convex function in second sense.

Theorem 2.1. *Let* $\Psi : I \subset [0,\infty) \longrightarrow \mathbb{R}$ *be a real valued n-time differentiable function where n is a positive integer. If* $\Psi^{(n)}$ *is a positive exponential s-convex function in second sense, then for* $a, b \in I; a < b$ *and* $\gamma \in \mathbb{R}$, $\alpha, \beta > 1$ *with* $n > \max\{\alpha, \beta\}$, *the following inequality for Caputo fractional derivatives holds*

$$\begin{aligned} &\Gamma(n-\alpha+1)({}^{C}D^{\alpha-1}_{a+}\Psi)(x) + \Gamma(n-\beta+1)({}^{C}D^{\beta-1}_{b-}\Psi)(x) \\ &\leq \frac{(x-a)^{n-\alpha+1}e^{-\gamma a}\Psi^{(n)}(a) + (b-x)^{n-\beta+1}e^{-\gamma b}\Psi^{(n)}(b)}{s+1} \\ &+ e^{-\gamma x}\Psi^{(n)}(x)\Big[\frac{(x-a)^{n-\alpha+1} + (b-x)^{n-\beta+1}}{s+1}\Big]. \end{aligned} \tag{4}$$

Proof. Let us consider the function Ψ on the interval $[a,x]$, $x\in[a,b]$ and n is a positive integer. For $t\in[a,x]$ and $n>\alpha$ the following inequality holds

$$(x-t)^{n-\alpha}\le(x-a)^{n-\alpha}. \tag{5}$$

Since $\Psi^{(n)}$ is exponential s-convex function in second sense therefore for $t\in[a,x]$ we have

$$\Psi^{(n)}(t)\le\left(\frac{x-t}{x-a}\right)^{s}e^{-\gamma a}\Psi^{(n)}(a)+\left(\frac{t-a}{x-a}\right)^{s}e^{-\gamma x}\Psi^{(n)}(x). \tag{6}$$

Multiplying inequalities (6) and (5), then integrating with respect to t over $[a,x]$ we have

$$\int_a^x(x-t)^{n-\alpha}\Psi^{(n)}(t)dt\le\frac{(x-a)^{n-\alpha}}{(x-a)^s}\left[e^{-\gamma a}\Psi^{(n)}(a)\int_a^x(x-t)^s dt+e^{-\gamma x}\Psi^{(n)}(x)\int_a^x(t-a)^s dt\right]$$

$$\Gamma(n-\alpha+1)({}^{C}D_{a+}^{\alpha-1}\Psi)(x)\le\frac{(x-a)^{n-\alpha+1}}{s+1}[e^{-\gamma a}\Psi^{(n)}(a)+e^{-\gamma x}\Psi^{(n)}(x)]. \tag{7}$$

Now we consider function Ψ on the interval $[x,b], x\in[a,b]$. For $t\in[x,b]$ the following inequality holds

$$(t-x)^{n-\beta}\le(b-x)^{n-\beta}. \tag{8}$$

Since $\Psi^{(n)}$ is exponential s-convex function in second sense on $[a,b]$, therefore for $t\in[x,b]$ we have

$$\Psi^{(n)}(t)\le\left(\frac{t-x}{b-x}\right)^{s}e^{-\gamma b}\Psi^{(n)}(b)+\left(\frac{b-t}{b-x}\right)^{s}e^{-\gamma x}\Psi^{(n)}(x). \tag{9}$$

Multiplying inequalities (8) and (9), then integrating with respect to t over $[x,b]$ we have

$$\int_x^b(t-x)^{n-\beta}\Psi^{(n)}(t)dt\le\frac{(b-x)^{n-\beta}}{(b-x)^s}\left[e^{-\gamma b}\Psi^{(n)}(b)\int_x^b(t-x)^s dt+e^{-\gamma x}\Psi^{(n)}(x)\int_x^b(b-t)^s dt\right]$$

$$\Gamma(n-\beta+1)({}^{C}D_{b-}^{\beta-1}\Psi)(x)\le\frac{(b-x)^{n-\beta+1}}{s+1}[e^{-\gamma b}\Psi^{(n)}(b)+e^{-\gamma x}\Psi^{(n)}(x)]. \tag{10}$$

Adding (7) and (10) we get the required inequality in (4). □

Corollary 2.2. *By setting $\alpha=\beta$ in (4) we get the following fractional integral inequality*

$$\begin{aligned}&\Gamma(n-\alpha+1)\left(({}^{C}D_{a+}^{\alpha-1}\Psi)(x)+({}^{C}D_{b-}^{\alpha-1}\Psi)(x)\right)\\&\le\frac{(x-a)^{n-\alpha+1}e^{-\gamma a}\Psi^{(n)}(a)+(b-x)^{n-\alpha+1}e^{-\gamma b}\Psi^{(n)}(b)}{s+1}\\&+e^{-\gamma x}\Psi^{(n)}(x)\left[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{s+1}\right].\end{aligned} \tag{11}$$

Remark 2.3. *By setting $\alpha=\beta$, $\gamma=0$,and $s=1$ we will get Corollary 2.1 in [4].*

Now we give the next result stated in the following theorem.

Theorem 2.4. *Let $\Psi:I\longrightarrow\mathbb{R}$ be a real valued n-time differentiable function where n is a positive integer. If $|\Psi^{(n+1)}|$ is exponential s-convex function, then for $a,b\in I;a<b$ and $\alpha,\beta>0$, the following inequality for Caputo fractional derivatives holds*

$$\begin{aligned}&\Big|\Gamma(n-\alpha+1)({}^{C}D_{a+}^{\alpha}\Psi)(x)+\Gamma(n-\beta+1)({}^{C}D_{b-}^{\beta}\Psi)(x)\\&-\left((x-a)^{n-\alpha}\Psi^{(n)}(a)+(b-x)^{n-\beta}\Psi^{(n)}(b)\right)\Big|\\&\le\frac{(x-a)^{\alpha+1}e^{-\gamma a}|\Psi^{(n+1)}(a)|+(b-x)^{\beta+1}e^{-\gamma b}|\Psi^{(n+1)}(b)|}{s+1}\\&+\frac{e^{-\gamma x}|\Psi^{(n+1)}(x)|\left((x-a)^{\alpha+1}+(b-x)^{\beta+1}\right)}{s+1}.\end{aligned} \tag{12}$$

Proof. Since $|\Psi^{(n+1)}|$ is exponential s-convex function in second sense and n is a positive integer, therefore for $t \in [a, x]$ and $n > \alpha$ we have

$$|\Psi^{(n+1)}(t)| \leq \left(\frac{x-t}{x-a}\right)^s e^{-\gamma a}|\Psi^{(n+1)}(a)| + \left(\frac{t-a}{x-a}\right)^s e^{-\gamma x}|\Psi^{(n+1)}(x)|$$

from which we can write

$$\begin{aligned} &-\left(\left(\frac{x-t}{x-a}\right)^s e^{-\gamma a}|\Psi^{(n+1)}(a)| + \left(\frac{t-a}{x-a}\right)^s e^{-\gamma x}|\Psi^{(n+1)}(x)|\right) \\ &\leq \Psi^{(n+1)}(t) \\ &\leq \left(\frac{x-t}{x-a}\right)^s e^{-\gamma a}|\Psi^{(n+1)}(a)| + \left(\frac{t-a}{x-a}\right)^s e^{-\gamma x}|\Psi^{(n+1)}(x)|. \end{aligned} \tag{13}$$

We consider the second inequality of inequality (13)

$$\Psi^{(n+1)}(t) \leq \left(\frac{x-t}{x-a}\right)^s e^{-\gamma a}|\Psi^{(n+1)}(a)| + \left(\frac{t-a}{x-a}\right)^s e^{-\gamma x}|\Psi^{(n+1)}(x)|. \tag{14}$$

Now for $\alpha > 0$ we have

$$(x-t)^{n-\alpha} \leq (x-a)^{n-\alpha}, t \in [a, x]. \tag{15}$$

The product of last two inequalities give

$$(x-t)^{n-\alpha}\Psi^{(n+1)}(t) \leq (x-a)^{n-\alpha-s}\left((x-t)^s e^{-\gamma a}|\Psi^{(n+1)}(a)| + (t-a)^s e^{-\gamma x}|\Psi^{(n+1)}(x)|\right).$$

Integrating with respect to t over $[a, x]$ we have

$$\begin{aligned} &\int_a^x (x-t)^{n-\alpha}\Psi^{(n+1)}(t)dt \\ &\leq (x-a)^{n-\alpha-s}\left[e^{-\gamma a}|\Psi^{(n+1)}(a)| \int_a^x (x-t)^s dt + e^{-\gamma x}|\Psi^{(n+1)}(x)| \int_a^x (t-a)^s dt\right] \\ &= (x-a)^{n-\alpha+1}\left[\frac{e^{-\gamma a}|\Psi^{(n+1)}(a)| + e^{-\gamma x}|\Psi^{(n+1)}(x)|}{s+1}\right], \end{aligned} \tag{16}$$

and

$$\begin{aligned} \int_a^x (x-t)^{n-\alpha}\Psi^{(n+1)}(t)dt &= \Psi^{(n)}(t)(x-t)^{n-\alpha}|_a^x + (n-\alpha)\int_a^x (x-t)^{n-\alpha-1}\Psi^{(n)}(t)dt \\ &= -\Psi^{(n)}(a)(x-a)^{n-\alpha} + \Gamma(n-\alpha+1)({}^C D_{a+}^{\alpha}\Psi)(x). \end{aligned}$$

Therefore (16) takes the form

$$\begin{aligned} &\Gamma(n-\alpha+1)({}^C D_{a+}^{\alpha}\Psi)(x) - \Psi^{(n)}(a)(x-a)^{n-\alpha} \\ &\leq (x-a)^{n-\alpha+1}\left[\frac{e^{-\gamma a}|\Psi^{(n+1)}(a)| + e^{-\gamma x}|\Psi^{(n+1)}(x)|}{s+1}\right]. \end{aligned} \tag{17}$$

If one consider from (13) the first inequality and proceed as we did for the second inequality, then following inequality can be obtained

$$\begin{aligned} &\Psi^{(n)}(a)(x-a)^{n-\alpha} - \Gamma(n-\alpha+1)({}^C D_{a+}^{\alpha}\Psi)(x) \\ &\leq (x-a)^{n-\alpha+1}\left[\frac{e^{-\gamma a}|\Psi^{(n+1)}(a)| + e^{-\gamma x}|\Psi^{(n+1)}(x)|}{s+1}\right]. \end{aligned} \tag{18}$$

From (17) and (18) we get

$$\begin{aligned}&\left|\Gamma(n-\alpha+1)({}^{C}D^{\alpha}_{a+}\Psi)(x)-\Psi^{(n)}(a)(x-a)^{n-\alpha}\right| \\ &\leq (x-a)^{n-\alpha+1}\left[\frac{e^{-\gamma a}|\Psi^{(n+1)}(a)|+e^{-\gamma x}|\Psi^{(n+1)}(x)|}{s+1}\right].\end{aligned} \tag{19}$$

On the other hand for $t \in [x,b]$ using convexity of $|\Psi^{(n+1)}|$ as a exponential convex function we have

$$|\Psi^{(n+1)}(t)| \leq \left(\frac{t-x}{b-x}\right)^{s} e^{-\gamma b}|\Psi^{(n+1)}(b)| + \left(\frac{b-t}{b-x}\right)^{s} e^{-\gamma x}|\Psi^{(n+1)}(x)|. \tag{20}$$

Also for $t \in [x,b]$ and $\beta > 0$ we have

$$(t-x)^{n-\beta} \leq (b-x)^{n-\beta}. \tag{21}$$

By adopting the same treatment as we have done for (13) and (15) one can obtain from (20) and (21) the following inequality

$$\begin{aligned}&\left|\Gamma(n-\beta+1)({}^{C}D^{\beta}_{b-}\Psi)(x)-\Psi^{(n)}(b)(b-x)^{n-\beta}\right| \\ &\leq (b-x)^{n-\beta+1}\left[\frac{e^{-\gamma b}|\Psi^{(n+1)}(b)|+e^{-\gamma x}|\Psi^{(n+1)}(x)|}{s+1}\right].\end{aligned} \tag{22}$$

By combining the inequalities (19) and (22) via triangular inequality we get the required inequality. □

It is interesting to see the following inequalities as a special case.

Corollary 2.5. *By setting $\alpha = \beta$ in (12) we get the following fractional integral inequality*

$$\begin{aligned}&\Big|\Gamma(n-\alpha+1)[({}^{C}D^{\alpha}_{a+}\Psi)(x)+({}^{C}D^{\alpha}_{b-}\Psi)(x)] \\ &-\left((x-a)^{n-\alpha}\Psi^{(n)}(a)+(b-x)^{n-\alpha}\Psi^{(n)}(b)\right)\Big| \\ &\leq \frac{(x-a)^{n-\alpha+1}e^{-\gamma a}|\Psi^{(n+1)}(a)|+(b-x)^{n-\alpha+1}e^{-\gamma b}|\Psi^{(n+1)}(b)|}{s+1} \\ &+\frac{e^{-\gamma x}|\Psi^{(n+1)}(x)|\left[(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}\right]}{s+1}.\end{aligned}$$

Remark 2.6. *By setting $\alpha = \beta$, $\gamma = 0$,and $s = 1$ we will get Corollary 2.2 in [4].*

Before going to the next theorem we observe the following result.

Lemma 2.7. *Let $\Psi : [a,b] \longrightarrow \mathbb{R}$, be a exponential s-convex function in second sense. If Ψ is exponentially symmetric about $\frac{a+b}{2}$, then the following inequality holds*

$$\Psi\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{s-1}}\left(e^{-\gamma x}\Psi(x)\right) \qquad x \in [a,b]. \tag{23}$$

Proof. As Ψ is exponential s-convex function in second sense we have

$$\Psi\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{s}}\left[e^{-\gamma(at+(1-t)b)}\Psi(at+(1-t)b)+e^{-\gamma(a(1-t)+bt)}\Psi(a(1-t)+bt)\right] \tag{24}$$

Since Ψ is symmetric about $\frac{a+b}{2}$, therefore we get $\Psi(a+b-x) = \Psi(bt+(1-t)a)$

$$\Psi\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{s}}\left(e^{-\gamma(at+(1-t)b)}\Psi((at+(1-t)b))+e^{-\gamma(a+b-x)}\Psi(a+b-x)\right).$$

By substituting $x = at + (1-t)b$ we get

$$\Psi\left(\frac{a+b}{2}\right) \le \frac{1}{2^s}\left(e^{-\gamma x}\Psi(x) + e^{-\gamma(a+b-x)}\Psi(a+b-x)\right).$$

Also Ψ is exponentially symmetric about $\frac{a+b}{2}$, therefore we have $\Psi(a+b-x) = \Psi(x)$ and inequality in (23) holds. $\square$

Theorem 2.8. *Let $\Psi : I \longrightarrow \mathbb{R}$ be a real valued n-time differentiable function where n is a positive integer. If $\Psi^{(n)}$ is a positive exponential s- convex function in second sense and symmetric about $\frac{a+b}{2}$, then for $a, b \in I; a < b$ and $\alpha, \beta \ge 1$, the following inequality for Caputo fractional derivatives holds*

$$\begin{aligned} &\frac{h(\gamma)2^{s-1}}{2}\left(\frac{1}{n-\alpha+1} + \frac{1}{n-\beta+1}\right)\Psi^{(n)}\left(\frac{a+b}{2}\right) \\ &\le \frac{\Gamma(n-\beta+1)({}^CD_{b-}^{\beta-1}\Psi)(a)}{2(b-a)^{n-\beta+1}} + \frac{\Gamma(n-\alpha+1)({}^CD_{a+}^{\alpha-1}\Psi)(b)}{2(b-a)^{n-\alpha+1}} \\ &\le \frac{\Psi^{(n)}(a) + \Psi^{(n)}(b)}{(s+1)}. \end{aligned} \tag{25}$$

where $h(\gamma) = e^{\gamma a}$ for $\gamma < 0$ and $h(\gamma) = e^{\gamma b}$ for $\gamma \ge 0$.

Proof. For $x \in [a, b]$ we have

$$(x-a)^{n-\beta} \le (b-a)^{n-\beta}. \tag{26}$$

Also Ψ is exponential s-convex function in second sense we have

$$\Psi^{(n)}(x) \le \left(\frac{x-a}{b-a}\right)^s e^{-\gamma b}\Psi^{(n)}(b) + \left(\frac{b-x}{b-a}\right)^s e^{-\gamma a}\Psi^{(n)}(a). \tag{27}$$

Multiplying (26) and (27) and then integrating with respect to x over $[a, b]$ we have

$$\int_a^b (x-a)^{n-\beta}\Psi^{(n)}(x)dx \le \frac{(b-a)^{n-\beta}}{(b-a)^s}\left(\int_a^b e^{-\gamma b}(\Psi^{(n)}(b)(x-a)^s + e^{-\gamma a}\Psi^{(n)}(a)(b-x)^s)dx\right).$$

From which we have

$$\frac{\Gamma(n-\beta+1)({}^CD_{b-}^{\beta-1}\Psi)(a)}{(b-a)^{n-\beta+1}} \le \frac{e^{-\gamma a}\Psi^{(n)}(a) + e^{-\gamma b}\Psi^{(n)}(b)}{s+1}. \tag{28}$$

On the other hand for $x \in [a, b]$ we have

$$(b-x)^{n-\alpha} \le (b-a)^{n-\alpha}. \tag{29}$$

Multiplying (27) and (29) and then integrating with respect to x over $[a, b]$ we get

$$\int_a^b (b-x)^{n-\alpha}\Psi^{(n)}(x)dx \le (b-a)^{n-\alpha+1}\frac{e^{-\gamma a}\Psi^{(n)}(a) + e^{-\gamma b}\Psi^{(n)}(b)}{s+1}.$$

From which we have

$$\frac{\Gamma(n-\alpha+1)({}^CD_{a+}^{\alpha-1}\Psi)(b)}{(b-a)^{n-\alpha+1}} \le \frac{e^{-\gamma a}\Psi^{(n)}(a) + e^{-\gamma b}\Psi^{(n)}(b)}{s+1}. \tag{30}$$

Adding (28) and (30) we get the second inequality.

$$\frac{\Gamma(n-\beta+1)({}^CD_{b-}^{\beta-1}\Psi)(a)}{2(b-a)^{n-\beta+1}} + \frac{\Gamma(n-\alpha+1)({}^CD_{a+}^{\alpha-1}\Psi)(b)}{2(b-a)^{n-\alpha+1}} \le \frac{e^{-\gamma a}\Psi^{(n)}(a) + e^{-\gamma b}\Psi^{(n)}(b)}{s+1}.$$

Since $\Psi^{(n)}$ is exponential s-convex function in second sense and symmetric about $\frac{a+b}{2}$ using Lemma 2.7 we have

$$\Psi^{(n)}\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{s-1}}\left(e^{-\gamma x}\Psi^{n}(x)\right), \quad x \in [a,b]. \tag{31}$$

Multiplying with $(x-a)^{n-\beta}$ on both sides and then integrating over $[a,b]$ we have

$$\Psi^{(n)}\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{n-\beta}dx \leq \frac{1}{h(\gamma)2^{s-1}}\int_a^b (x-a)^{n-\beta}\Psi^{(n)}(x)dx. \tag{32}$$

By definition of Caputo fractional derivatives for exponential s-convex function one can has

$$\Psi^{(n)}\left(\frac{a+b}{2}\right)\frac{1}{2(n-\beta+1)} \leq \frac{1}{h(\gamma)2^{s-1}}\frac{\Gamma(n-\beta+1)({}^{C}D_{b-}^{\beta-1}\Psi)(a)}{2(b-a)^{n-\beta+1}}. \tag{33}$$

Multiplying (31) with $(b-x)^{n-\alpha}$, then integrating over $[a,b]$ one can get

$$\Psi^{(n)}\left(\frac{a+b}{2}\right)\frac{1}{2(n-\alpha+1)} \leq \frac{1}{h(\gamma)2^{s-1}}\frac{\Gamma(n-\alpha+1)({}^{C}D_{a+}^{\alpha-1}\Psi)(b)}{2(b-a)^{n-\alpha+1}}. \tag{34}$$

Adding (33) and (34) we get the first inequality. □

Corollary 2.9. *If we put $\alpha = \beta$ in (25), then we get*

$$\begin{aligned} &h(\gamma)2^{s-1}\Psi^{(n)}\left(\frac{a+b}{2}\right)\frac{1}{(n-\alpha+1)} \\ &\leq \frac{\Gamma(n-\alpha+1)}{2(b-a)^{\alpha+1}}\left[({}^{C}D_{b-}^{\alpha+1}\Psi)(a) + ({}^{C}D_{a+}^{\alpha+1}\Psi)(b)\right] \\ &\leq \frac{e^{-\gamma a}\Psi^{(n)}(a) + e^{-\gamma b}\Psi^{(n)}(b)}{s+1}. \end{aligned}$$

where $h(\gamma) = e^{\gamma a}$ for $\gamma < 0$ and $h(\gamma) = e^{\gamma b}$ for $\gamma \geq 0$.

Remark 2.10. *By setting $\gamma = 0$ and $m = 1$ in Theorem 2.8 we will get theorem 2.3 in [4].*

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On The Connections Between Jacobsthal Numbers and Fibonacci p-Numbers

Özgür Erdağ[a], Ömür Deveci[b]

[a]*Department of Mathematics, Faculty of Science and Letters, Kafkas University 36100, Turkey*
[b]*Department of Mathematics, Faculty of Science and Letters, Kafkas University 36100, Turkey*

Abstract. In this paper, we define the Fibonacci-Jacobsthal p-sequence and then we discuss the connection between of the Fibonacci-Jacobsthal p-sequence with the Jacobsthal and Fibonacci p-sequences. Also, we provide a new Binet formula and a new combinatorial representation of the Fibonacci-Jacobsthal p-numbers by the aid of the nth power of the generating matrix of the Fibonacci-Jacobsthal p-sequence. Furthermore, we derive some properties of the Fibonacci-Jacobsthal p-sequences such as the exponential, permanental, determinantal representations and the sums by using its generating matrix.

1. Introduction

The well-known Jacobsthal sequence $\{J_n\}$ is defined by the following recurrence relation:

$$J_n = J_{n-1} + 2J_{n-2}$$

for $n \geq 2$ in which $J_0 = 0$ and $J_1 = 1$.

There are many important generalizations of the Fibonacci sequence. The Fibonacci p-sequence $\left\{F_p(n)\right\}$ (see detailed information in [21, 22]) is one of them:

$$F_p(n) = F_p(n-1) + F_p(n-p-1)$$

for $n > p$ and $p = 1, 2, 3, \ldots$, in which $F_p(0) = 0$, $F_p(1) = \cdots F_p(p) = 1$. When $p = 1$, the Fibonacci p-sequence $\left\{F_p(n)\right\}$ is reduced to the usual Fibonacci sequence $\{F_n\}$.

It is easy to see that the characteristic polynomials of Jacobsthal sequence and Fibonacci p-sequence are $g_1(x) = x^2 - x - 2$ and $g_2(x) = x^{p+1} - x^p - 1$, respectively. We will use these in the next section.

Let the $(n+k)$th term of a sequence be defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

in which $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [12], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Corresponding author: ÖE mail address: ozgur_erdag@hotmail.com ORCID:https://orcid.org/0000-0001-8071-6794, ÖD ORCID: https://orcid.org/0000-0001-5870-5298

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Let the matrix A be defined by

$$A = \left[a_{i,j}\right]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geq 0$.

Several authors have used homogeneous linear recurrence relations to deduce miscellaneous properties for a plethora of sequences: see for example, [1, 4, 8–11, 19, 20]. In [5–7, 14–16, 21–23], the authors defined some linear recurrence sequences and gave their various properties by matrix methods. In this paper, we discuss connections between the Jacobsthal numbers and Fibonacci p-numbers. Firstly, we define the Fibonacci-Jacobsthal p-sequence and then we study recurrence relation among this sequence, Jacobsthal sequence and Fibonacci p-sequence. Also, we give the relations between the generating matrix of the Fibonacci-Jacobsthal p-numbers and the elements of Jacobsthal sequence and Fibonacci p-sequence. Furthermore, using the generating matrix the Fibonacci-Jacobsthal p-sequence, we obtain some new structural properties of the Fibonacci p-numbers such as the Binet formula and combinatorial representations. Finally, we derive the exponential, permanental, and determinantal representations and the sums of Fibonacci-Jacobsthal p-sequences.

2. On The Connections Between Jacobsthal Numbers and Fibonacci *p*-Numbers

Now we define the Fibonacci-Jacobsthal p-sequence $\left\{F_n^{J,p}\right\}$ by the following homogeneous linear recurrence relation for any given $p\,(3, 4, 5, \ldots)$ and $n \geq 0$

$$F_{n+p+3}^{J,p} = 2F_{n+p+2}^{J,p} + F_{n+p+1}^{J,p} - 2F_{n+p}^{J,p} + F_{n+2}^{J,p} - F_{n+1}^{J,p} - 2F_n^{J,p} \tag{1}$$

in which $F_0^{J,p} = \cdots = F_{p+1}^{J,p} = 0$ and $F_{p+2}^{J,p} = 1$.

First, we consider the relationship between the Fibonacci-Jacobsthal p-sequence which is defined above, Jacobsthal sequence, and Fibonacci p-sequences.

Theorem 2.1. *Let J_n, $F_p(n)$ and $F_n^{J,p}$ be the nth Jacobsthal number, Fibonacci p-number, and Fibonacci-Jacobsthal p-numbers, respectively. Then,*

$$J_n + F_p(n+1) = F_{n+p+2}^{J,p} - 3F_{n+p}^{J,p} - F_n^{J,p}$$

for $n \geq 0$ and $p \geq 3$.

Proof. The assertion may be proved by induction on n. It is clear that $J_0 + F_p(1) = F_{p+2}^{J,p} - 3F_p^{J,p} - F_0^{J,p} = 0$. Suppose that the equation holds for $n \geq 1$. Then we must show that the equation holds for $n + 1$. Since the characteristic polynomial of Fibonacci-Jacobsthal p-sequence $\left\{F_n^{J,p}\right\}$, is

$$h(x) = x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2$$

and

$$h(x) = g_1(x)\, g_2(x),$$

where $g_1(x)$ and $g_2(x)$ are the characteristic polynomials of Jacobsthal sequence and Fibonacci p-sequence, respectively, we obtain the following relations:

$$J_{n+p+3} = 2J_{n+p+2} + J_{n+p+1} - 2J_{n+p} + J_{n+2} - J_{n+1} - 2J_n$$

and

$$F_p(n+p+3) = 2F_p(n+p+2) + F_p(n+p+1) - 2F_p(n+p) + F_p(n+2) - F_p(n+1) - 2F_p(n)$$

for $n \geq 1$. Thus, by a simple calculation, we have the conclusion. □

Theorem 2.2. *Let J_n and $F_n^{J,p}$ be the nth Jacobsthal number and Fibonacci-Jacobsthal p-numbers. Then,*

i.

$$J_n = F_{n+p+1}^{J,p} - F_{n+p}^{J,p} - F_n^{J,p},$$

ii.

$$J_n + J_{n+1} = F_{n+p+2}^{J,p} - F_{n+p}^{J,p} - F_{n+1}^{J,p} - F_n^{J,p}$$

for $n \geq 0$ and $p \geq 3$.

Proof. Consider the case ii. The assertion may be proved by induction on n. It is clear that $J_0 + J_1 = F_5^{J,p} - F_3^{J,p} - F_1^{J,p} - F_0^{J,p} = 1$. Now we assume that the equation holds for $n > 0$. Then we show that the equation holds for $n + 1$. Since the characteristic polynomial of Jacobsthal sequence $\{J_n\}$, is

$$g_1(x) = x^2 - x - 2$$

we obtain the following relations:

$$J_{n+p+3} = 2J_{n+p+2} + J_{n+p+1} - 2J_{n+p} + J_{n+2} - J_{n+1} - 2J_n$$

for $n \geq 1$. Thus, by a simple calculation, we have the conclusion.

There is a similar proof for i. □

By the recurrence relation (1), we have

$$\begin{bmatrix} F_{n+p+2}^{J,p} \\ F_{n+p+1}^{J,p} \\ F_{n+p}^{J,p} \\ \vdots \\ F_n^{J,p} \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 0 & \cdots & 0 & 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} F_{n+p+3}^{J,p} \\ F_{n+p+2}^{J,p} \\ F_{n+p+1}^{J,p} \\ \vdots \\ F_{n+1}^{J,p} \end{bmatrix}$$

for the Fibonacci-Jacobsthal p-sequence $\left\{F_n^{J,p}\right\}$. Letting

$$M_p = \begin{bmatrix} 2 & 1 & -2 & 0 & \cdots & 0 & 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{(p+3)\times(p+3)}.$$

The companion matrix $M_p = \left[m_{i,j}\right]_{(p+3)\times(p+3)}$ is said to be the Fibonacci-Jacobsthal p-matrix. For detailed information about the companion matrices, see [17, 18]. It can be readily established by mathematical induction that for $p \geq 3$ and $\alpha \geq 2p$

$$\left(M_p\right)^\alpha = \begin{bmatrix} F_{\alpha+p+2}^{J,p} & F_{\alpha+p+3}^{J,p} - 2F_{\alpha+p+2}^{J,p} & F_p(\alpha-p+2) - 2F_{\alpha+p+1}^{J,p} & F_p(\alpha-p+3) & \cdots & \\ F_{\alpha+p+1}^{J,p} & F_{\alpha+p+2}^{J,p} - 2F_{\alpha+p+1}^{J,p} & F_p(\alpha-p+1) - 2F_{\alpha+p}^{J,p} & F_p(\alpha-p+2) & \cdots & \\ F_{\alpha+p}^{J,p} & F_{\alpha+p+1}^{J,p} - 2F_{\alpha+p}^{J,p} & F_p(\alpha-p) - 2F_{\alpha+p-1}^{J,p} & F_p(\alpha-p+1) & \cdots & M_p^* \\ \vdots & \vdots & \vdots & \vdots & & \\ F_{\alpha+1}^{J,p} & F_{\alpha+2}^{J,p} - 2F_{\alpha+1}^{J,p} & F_p(\alpha-2p+1) - 2F_{\alpha}^{J,p} & F_p(\alpha-2p+2) & \cdots & \\ F_{\alpha}^{J,p} & F_{\alpha+1}^{J,p} - 2F_{\alpha}^{J,p} & F_p(\alpha-2p) - 2F_{\alpha-1}^{J,p} & F_p(\alpha-2p+1) & \cdots & \end{bmatrix},$$

where

$$M_p^* = \begin{bmatrix} F_p(\alpha) & F_p(\alpha+1) - F_{\alpha+p+2}^{J,p} & -2F_{\alpha+p+1}^{J,p} \\ F_p(\alpha-1) & F_p(\alpha) - F_{\alpha+p+1}^{J,p} & -2F_{\alpha+p}^{J,p} \\ F_p(\alpha-2) & F_p(\alpha-1) - F_{\alpha+p}^{J,p} & -2F_{\alpha+p-1}^{J,p} \\ \vdots & \vdots & \vdots \\ F_p(\alpha-p-1) & F_p(\alpha-p) - F_{\alpha+1}^{J,p} & -2F_{\alpha}^{J,p} \\ F_p(\alpha-p-2) & F_p(\alpha-p-1) - F_{\alpha}^{J,p} & -2F_{\alpha-1}^{J,p} \end{bmatrix}.$$

We easily derive that $\det M_p = (-1)^{p+1} \cdot 2$. In [21], Stakhov defined the generalized Fibonacci p-matrix Q_p and derived the *nth* power of the matrix Q_p. In [13], Kılıc gave a Binet formula for the Fibonacci p-numbers by matrix method. Now we concentrate on finding another Binet formula for the Fibonacci-Jacobsthal p-numbers by the aid of the matrix $\left(M_p\right)^\alpha$.

Lemma 2.3. *The characteristic equation of all the Fibonacci-Jacobsthal p-numbers* $x^{p+3}-2x^{p+2}-x^{p+1}+x^p-x^2+x+2 = 0$ *does not have multiple roots for* $p \geq 3$.

Proof. It is clear that $x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2 = \left(x^{p+1} - x^p - 1\right)\left(x^2 - x - 2\right)$. In [13], it was shown that the equation $x^{p+1} - x^p - 1 = 0$ does not have multiple roots for $p > 1$. It is easy to see that the roots of the equation $x^2 - x - 2 = 0$ are 2 and −1. Since $(2)^{p+1} - (2)^p - 1 \neq 0$ and $(-1)^{p+1} - (-1)^p - 1 \neq 0$ for $p > 1$, the equation $x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2 = 0$ does not have multiple roots for $p \geq 3$. □

Let $h(x)$ be the characteristic polynomial of matrix M_p. Then we have $h(x) = x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2$, which is a well-known fact from the companion matrices. If $\lambda_1, \lambda_2, \ldots, \lambda_{p+3}$ are roots of the equation

$x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2 = 0$, then by Lemma 2.3, it is known that $\lambda_1, \lambda_2, \ldots, \lambda_{p+3}$ are distinct. Define the $(p+3) \times (p+3)$ Vandermonde matrix V_p as follows:

$$V_p = \begin{bmatrix} (\lambda_1)^{p+2} & (\lambda_2)^{p+2} & \ldots & \left(\lambda_{p+3}\right)^{p+2} \\ (\lambda_1)^{p+1} & (\lambda_2)^{p+1} & \ldots & \left(\lambda_{p+3}\right)^{p+1} \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \ldots & \lambda_{p+3} \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Assume that $V_p(i, j)$ is a $(p+3) \times (p+3)$ matrix derived from the Vandermonde matrix V_p by replacing the j^{th} column of V_p by $W_p(i)$, where, $W_p(i)$ is a $(p+3) \times 1$ matrix as follows:

$$W_p(i) = \begin{bmatrix} (\lambda_1)^{\alpha+p+3-i} \\ (\lambda_2)^{\alpha+p+3-i} \\ \vdots \\ \left(\lambda_{p+3}\right)^{\alpha+p+3-i} \end{bmatrix}.$$

Theorem 2.4. *Let p be a positive integer such that $p \geq 3$ and let $\left(M_p\right)^\alpha = m_{i,j}^{(p,\alpha)}$ for $\alpha \geq 1$, then*

$$m_{i,j}^{(p,\alpha)} = \frac{\det V_p(i, j)}{\det V_p}.$$

Proof. Since the equation $x^{p+3} - 2x^{p+2} - x^{p+1} + x^p - x^2 + x + 2 = 0$ does not have multiple roots for $p \geq 3$, the eigenvalues of the Fibonacci-Jacobsthal p-matrix M_p are distinct. Then, it is clear that M_p is diagonalizable. Let $D_p = diag\left(\lambda_1, \lambda_2, \ldots, \lambda_{p+3}\right)$, then we may write $M_p V_p = V_p D_p$. Since the matrix V_p is invertible, we obtain the equation $\left(V_p\right)^{-1} M_p V_p = D_p$. Therefore, M_p is similar to D_p; hence, $\left(M_p\right)^\alpha V_p = V_p \left(D_p\right)^\alpha$ for $\alpha \geq 1$. So we have the following linear system of equations:

$$\begin{cases} m_{i,1}^{(p,\alpha)} (\lambda_1)^{p+2} + m_{i,2}^{(p,\alpha)} (\lambda_1)^{p+1} + \cdots + m_{i,p+3}^{(p,\alpha)} = (\lambda_1)^{\alpha+p+3-i} \\ m_{i,1}^{(p,\alpha)} (\lambda_2)^{p+2} + m_{i,2}^{(p,\alpha)} (\lambda_2)^{p+1} + \cdots + m_{i,p+3}^{(p,\alpha)} = (\lambda_2)^{\alpha+p+3-i} \\ \vdots \\ m_{i,1}^{(p,\alpha)} \left(\lambda_{p+3}\right)^{p+2} + m_{i,2}^{(p,\alpha)} \left(\lambda_{p+3}\right)^{p+1} + \cdots + m_{i,p+3}^{(p,\alpha)} = \left(\lambda_{p+3}\right)^{\alpha+p+3-i} \end{cases}.$$

Then we conclude that

$$m_{i,j}^{(p,\alpha)} = \frac{\det V_p(i, j)}{\det V_p}$$

for each $i, j = 1, 2, \ldots, p+3$. □

Thus by Theorem 2.4 and the matrix $\left(M_p\right)^\alpha$, we have the following useful result for the Fibonacci-Jacobsthal p-numbers.

Corollary 2.5. *Let p be a positive integer such that $p \geq 3$ and let $F_n^{J,p}$ be the nth element of Fibonacci-Jacobsthal p-sequence, then*

$$F_n^{J,p} = \frac{\det V_p(p+3, 1)}{\det V_p}$$

and

$$F_n^{J,p} = -\frac{\det V_p\,(p+2, p+3)}{2 \cdot \det V_p}$$

for $n \geq 1$.

It is easy to see that the generating function of Fibonacci-Jacobsthal p-sequence $\left\{F_n^{J,p}\right\}$ is as follows:

$$g(x) = \frac{x^{p+2}}{1 - 2x - x^2 + 2x^3 - x^{p+1} + x^{p+2} + 2x^{p+3}},$$

where $p \geq 3$.

Then we can give an exponential representation for the Fibonacci-Jacobsthal p-numbers by the aid of the generating function with the following Theorem.

Theorem 2.6. *The Fibonacci-Jacobsthal p-sequence* $\left\{F_n^{J,p}\right\}$ *have the following exponential representation:*

$$g(x) = x^{p+2} \exp\left(\sum_{i=1}^{\infty} \frac{(x)^i}{i}\left(2 + x - 2x^2 + x^p - x^{p+1} - 2x^{p+2}\right)^i\right),$$

where $p \geq 3$.

Proof. Since

$$\ln g(x) = \ln x^{p+2} - \ln\left(1 - 2x - x^2 + 2x^3 - x^{p+1} + x^{p+2} + 2x^{p+3}\right)$$

and

$$\begin{aligned} -\ln\left(1 - 2x - x^2 + 2x^3 - x^{p+1} + x^{p+2} + 2x^{p+3}\right) &= -[-x\left(2 + x - 2x^2 + x^p - x^{p+1} - 2x^{p+2}\right) - \\ &\quad \frac{1}{2}x^2\left(2 + x - 2x^2 + x^p - x^{p+1} - 2x^{p+2}\right)^2 - \cdots \\ &\quad -\frac{1}{i}x^i\left(2 + x - 2x^2 + x^p - x^{p+1} - 2x^{p+2}\right)^i - \cdots] \end{aligned}$$

it is clear that

$$g(x) = x^{p+2} \exp\left(\sum_{i=1}^{\infty} \frac{(x)^i}{i}\left(2 + x - 2x^2 + x^p - x^{p+1} - 2x^{p+2}\right)^i\right)$$

by a simple calculation, we obtain the conclusion. □

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2.7. *(Chen and Louck [3]) The* (i, j) *entry* $k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v)$ *in the matrix* $K^n(k_1, k_2, \ldots, k_v)$ *is given by the following formula:*

$$k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v} \tag{2}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v} = \frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ *is a multinomial coefficient, and the coefficients in (2) are defined to be* 1 *if* $n = i - j$.

Then we can give other combinatorial representations than for the Fibonacci-Jacobsthal p-numbers by the following Corollary.

Corollary 2.8. *Let $F_n^{J,p}$ be the nth Fibonacci-Jacobsthal p-number for $n \geq 1$. Then*
i.
$$F_n^{J,p} = \sum_{(t_1,t_2,\ldots,t_{p+3})} \binom{t_1+t_2+\cdots+t_{p+3}}{t_1,t_2,\cdots,t_{p+3}} 2^{t_1} (-1)^{t_{p+2}} (-2)^{t_3+t_{p+3}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+3)\,t_{p+3} = n - p - 2$.
ii.
$$F_n^{J,p} = -\sum_{(t_1,t_2,\ldots,t_{p+3})} \frac{t_{p+3}}{t_1+t_2+\cdots+t_{p+3}} \times \binom{t_1+t_2+\cdots+t_{p+3}}{t_1,t_2,\cdots,t_{p+3}} 2^{t_1} (-1)^{t_{p+2}} (-2)^{t_3+t_{p+3}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+3)\,t_{p+3} = n + 1$.

Proof. If we take $i = p+3$, $j = 1$ for the case i. and $i = p+2$, $j = p+3$ for the case ii. in Theorem 2.7, then we can directly see the conclusions from $\left(M_p\right)^{\alpha}$. □

Now we consider the relationship between the Fibonacci-Jacobsthal p-numbers and the permanent of a certain matrix which is obtained using the Fibonacci-Jacobsthal p-matrix $\left(M_p\right)^{\alpha}$.

Definition 2.9. *A $u \times v$ real matrix $M = \left[m_{i,j}\right]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.*

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [2], Brualdi and Gibson obtained that $per(M) = per(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Now we concentrate on finding relationships among the Fibonacci-Jacobsthal p-numbers and the permanents of certain matrices which are obtained by using the generating matrix of Fibonacci-Jacobsthal p-numbers. Let $K_{m,p}^{F,J} = \left[k_{i,j}^{(p)}\right]$ be the $m \times m$ super-diagonal matrix, defined by

$$k_{i,j}^{(p)} = \begin{cases} 2 & \text{if } i=\tau \text{ and } j=\tau \text{ for } 1\le\tau\le m, \\ 1 & \begin{array}{c}\text{if } i=\tau \text{ and } j=\tau+1 \text{ for } 1\le\tau\le m-1,\\ i=\tau \text{ and } j=\tau+p \text{ for } 1\le\tau\le m-p\\ \text{and}\\ i=\tau+1 \text{ and } j=\tau \text{ for } 1\le\tau\le m-1,\end{array} \\ -1 & \text{if } i=\tau \text{ and } j=\tau+p+1 \text{ for } 1\le\tau\le m-p-1, \\ -2 & \begin{array}{c}\text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1\le\tau\le m-2\\ \text{and}\\ i=\tau \text{ and } j=\tau+p+2 \text{ for } 1\le\tau\le m-p-2,\end{array} \\ 0 & \text{otherwise.} \end{cases}, \text{ for } m \geq p+3.$$

Then we have the following Theorem.

Theorem 2.10. *For $m \geq p+3$,*
$$perK_{m,p}^{F,J} = F_{m+p+2}^{J,p}.$$

Proof. Let us consider matrix $K^{F,J}_{m,p}$ and let the equation be hold for $m \geq p+3$. Then we show that the equation holds for $m+1$. If we expand the $perK^{F,J}_{m,p}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perK^{F,J}_{m+1,p} = 2perK^{F,J}_{m,p} + perK^{F,J}_{m-1,p} - 2perK^{F,J}_{m-2,p} + perK^{F,J}_{m-p,p} - perK^{F,J}_{m-p-1,p} - 2perK^{F,J}_{m-p-2,p}.$$

Since

$$perK^{F,J}_{m,p} = F^{J,p}_{m+p+2},$$
$$perK^{F,J}_{m-1,p} = F^{J,p}_{m+p+1},$$
$$perK^{F,J}_{m-2,p} = F^{J,p}_{m+p},$$
$$perK^{F,J}_{m-p,p} = F^{J,p}_{m+2},$$
$$perK^{F,J}_{m-p-1,p} = F^{J,p}_{m+1}$$

and

$$perK^{F,J}_{m-p-2,p} = F^{J,p}_{m},$$

we easily obtain that $perK^{F,J}_{m+1,p} = F^{J,p}_{m+p+3}$. So the proof is complete. □

Let $L^{F,J}_{m,p} = \left[l^{(p)}_{i,j}\right]$ be the $m \times m$ matrix, defined by

$$l^{(p)}_{i,j} = \begin{cases} 2 & \text{if } i=\tau \text{ and } j=\tau \text{ for } 1 \leq \tau \leq m-3, \\ & \text{if } i=\tau \text{ and } j=\tau \text{ for } m-2 \leq \tau \leq m, \\ & i=\tau \text{ and } j=\tau+1 \text{ for } 1 \leq \tau \leq m-1, \\ 1 & i=\tau \text{ and } j=\tau+p \text{ for } 1 \leq \tau \leq m-p-2 \\ & \text{and} \\ & i=\tau+1 \text{ and } j=\tau \text{ for } 1 \leq \tau \leq m-4, \\ -1 & \text{if } i=\tau \text{ and } j=\tau+p+1 \text{ for } 1 \leq \tau \leq m-p-1, \\ & \text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1 \leq \tau \leq m-3 \\ -2 & \text{and} \\ & i=\tau \text{ and } j=\tau+p+2 \text{ for } 1 \leq \tau \leq m-p-2, \\ 0 & \text{otherwise.} \end{cases} \quad \text{, for } m \geq p+3.$$

Then we have the following Theorem.

Theorem 2.11. *For $m \geq p+3$,*

$$perL^{F,J}_{m,p} = F^{J,p}_{m+p-1}.$$

Proof. Let us consider matrix $L^{F,J}_{m,p}$ and let the equation be hold for $m \geq p+3$. Then we show that the equation holds for $m+1$. If we expand the $perL^{F,J}_{m,p}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perL^{F,J}_{m+1,p} = 2perL^{F,J}_{m,p} + perL^{F,J}_{m-1,p} - 2perL^{F,J}_{m-2,p} + perL^{F,J}_{m-p,p} - perL^{F,J}_{m-p-1,p} - 2perL^{F,J}_{m-p-2,p}.$$

Since

$$perL^{F,J}_{m,p} = F^{J,p}_{m+p-1},$$
$$perL^{F,J}_{m-1,p} = F^{J,p}_{m+p-2},$$

$$perL^{F,J}_{m-2,p} = F^{J,p}_{m+p-3},$$
$$perL^{F,J}_{m-p,p} = F^{J,p}_{m-1},$$
$$perL^{F,J}_{m-p-1,p} = F^{J,p}_{m-2}$$

and

$$perL^{F,J}_{m-p-2,p} = F^{J,p}_{m-3},$$

we easily obtain that $perL^{F,J}_{m+1,p} = F^{J,p}_{m+p}$. So the proof is complete. □

Assume that $N^{F,J}_{m,p} = \left[n^{(p)}_{i,j}\right]$ be the $m \times m$ matrix, defined by

$$N^{F,J}_{m,p} = \begin{array}{c} \quad\quad\quad (m-3)\,\text{th} \\ \quad\quad\quad \downarrow \\ \left[\begin{array}{cccccc} 1 & \cdots & 1 & 0 & 0 & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & L^{F,J}_{m-1,p} & & & \\ 0 & & & & & \\ 0 & & & & & \end{array}\right] \end{array}, \text{ for } m > p+3,$$

then we have the following results:

Theorem 2.12. *For $m > p+3$,*

$$perN^{F,J}_{m,p} = \sum_{i=0}^{m+p-2} F^{J,p}_{i}.$$

Proof. If we extend $per\ N^{F,J}_{m,p}$ with respect to the first row, we write

$$perN^{F,J}_{m,p} = perN^{F,J}_{m-1,p} + perL^{F,J}_{m-1,p}.$$

Thus, by the results and an inductive argument, the proof is easily seen. □

A matrix M is called convertible if there is an $n \times n$ $(1,-1)$-matrix K such that $perM = \det(M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K.

Now we give relationships among the Fibonacci-Jacobsthal p-numbers and the determinants of certain matrices which are obtained by using the matrix $K^{F,J}_{m,p}$, $L^{F,J}_{m,p}$ and $N^{F,J}_{m,p}$. Let $m > p+3$ and let H be the $m \times m$ matrix, defined by

$$H = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 2.13. *For $m > p+3$,*

$$\det\left(K^{F,J}_{m,p} \circ H\right) = F^{J,p}_{m+p+2},$$
$$\det\left(L^{F,J}_{m,p} \circ H\right) = F^{J,p}_{m+p-1},$$

and

$$\det\left(N^{F,J}_{m,p} \circ H\right) = \sum_{i=0}^{m+p-2} F^{J,p}_{i}.$$

Proof. *Since* $perK^{F,J}_{m,p} = \det\left(K^{F,J}_{m,p} \circ H\right)$, $perL^{F,J}_{m,p} = \det\left(L^{F,J}_{m,p} \circ H\right)$ *and* $perN^{F,J}_{m,p} = \det\left(N^{F,J}_{m,p} \circ H\right)$ *for* $m > p+3$, *by Theorem 2.10, Theorem 2.11 and Theorem 2.12, we have the conclusion.* □

Now we consider the sums of the Fibonacci-Jacobsthal p-numbers. Let

$$S_\alpha = \sum_{u=0}^{\alpha} F_u^{J,p}$$

for $\alpha > 1$ and $p \geq 3$, and let $T_p^{F,J}$ and $\left(T_p^{F,J}\right)^\alpha$ be the $(p+4)\times(p+4)$ matrix such that

$$T_p^{F,J} = \left[\begin{array}{c|c} 1 & 0 \; 0 \; \cdots \; 0 \; 0 \\ \hline 1 & \\ 0 & \\ \vdots & M_p \\ 0 & \\ 0 & \end{array}\right].$$

If we use induction on α, then we obtain

$$\left(T_p^{F,J}\right)^\alpha = \left[\begin{array}{c|c} 1 & 0 \; 0 \; \cdots \; 0 \; 0 \\ \hline S_{\alpha+p+1} & \\ S_{\alpha+p} & \\ \vdots & \left(M_p\right)^\alpha \\ S_\alpha & \\ S_{\alpha-1} & \end{array}\right].$$

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