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Non-null Translation-Homothetical surfaces in four-dimensional Minkowski space

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Abstract. In the present work, we deal with non-null translation-homothetical surfaces in Minkowski 4−space. Initially, we describe non-null $TH-$type surface (Translation-Homothetical surface). Then, we yield the normal curvature, mean curvature vector and Gaussian curvature functions. Using these concepts, we characterize the non-null semiumbilical, minimal and flat translation-homothetical surfaces in $\mathbb{E}_1^4$.

1. Introduction

In physics literature, special relativity is a scientific theory that explains the relationship between space and time. According to the theory, all objects and physical phenomena are relative. Time, space and motion are not independent of each other. Minkowski space-time is the geometry that mathematically describes the four-dimensional structure of special relativity. Minkowski 4−space (or Minkowski space-time) is defined with the help of a Lorentzian metric as

$$g(x, y) = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3. \tag{1}$$

Any arbitrary vector is known as spacelike, lightlike or timelike, if the Lorentzian metric $g(x, x)$ is positive definite, zero or negative definite, respectively. In Minkowski space-time, all surfaces are also divided into three categories in a similar way. Any surface in $\mathbb{E}_1^4$ is known as a spacelike surface (or timelike surface), given that its all tangent vectors are spacelike (timelike).

Let $M : \psi = \psi(s, t)$ be a non-lightlike (spacelike or timelike) surface in $\mathbb{E}_1^4$. Four-dimensional Minkowski space can be decomposed into tangent space and normal space of M, at each point p as:

$$\mathbb{E}_1^4 = T_p^{\perp} M \oplus T_p M. \tag{2}$$

Levi-Civita connections are indicated by $\widetilde{\nabla}$ and ∇ on $\mathbb{E}_1^4$ and M. Assume: X and Y are tangent vector fields and N is a normal vector field of M. The vector fields $\widetilde{\nabla}_X N$ and $\widetilde{\nabla}_X Y$ are decomposed into normal and

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tangent components by Weingarten and Gauss formulas:

$$\begin{aligned} \widetilde{\nabla}_X N &= -A_N X + D_X N, \\ \widetilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \end{aligned} \tag{3}$$

where D, h and A_N are the normal connection, the second fundamental form and the shape operator, respectively. [7, 12].

Let $\psi = \psi(s, t)$ be a parametrization for a non-null surface M in $\mathbb{E}^4_1$. Then, $T_pM = span\{\psi_s, \psi_t\}$ corresponds to the tangent space at a point p of M. The standard indications $E = g(\psi_s, \psi_s)$, $F = g(\psi_s, \psi_t)$, $G = g(\psi_t, \psi_t)$ are known as first fundamental form coefficients

$$I = Eds^2 + 2Fdsdt + Gdt^2, \tag{4}$$

We can choose the tangent vector fields, for the timelike surface, as $g(\psi_s, \psi_s) < 0$, $g(\psi_t, \psi_t) > 0$. In addition, we settle a normal frame field $\{N_1, N_2\}$ for the spacelike surface as $g(N_1, N_1) = -1$, $g(N_2, N_2) = 1$, i.e. $\{\psi_s, \psi_t, N_1, N_2\}$ is oriented positively in $\mathbb{E}^4_1$. For the later use, we set

$$\xi = \begin{cases} 1, & \text{if } M \text{ is spacelike} \\ -1, & \text{if } M \text{ is timelike} \end{cases} \tag{5}$$

Thus, we present $W = \sqrt{\xi(EG - F^2)}$. It means; $EG - F^2$ is positive or negative definite with respect to being the surface spacelike or timelike.

H:the mean curvature vector field can be computed by $H = \frac{1}{2}trh$. In other words, using the tangent bundle's orthonormal frame $\{X, Y\}$, it can be written as $H = \frac{1}{2}(\xi h(X, X) + h(Y, Y))$. The second fundamental form coefficients can be calculated as

$$\begin{aligned} c^1_{11} &= g(\psi_{ss}, N_1), \quad c^1_{12} = g(\psi_{st}, N_1), \quad c^1_{22} = g(\psi_{tt}, N_1), \\ c^2_{11} &= g(\psi_{ss}, N_2), \quad c^2_{12} = g(\psi_{st}, N_2), \quad c^1_{22} = g(\psi_{tt}, N_2). \end{aligned} \tag{6}$$

One can write the second fundamental tensor as

$$\begin{aligned} h(\psi_s, \psi_s) &= -\xi c^1_{11} N_1 + c^2_{11} N_2, \\ h(\psi_s, \psi_t) &= -\xi c^1_{12} N_1 + c^2_{12} N_2, \\ h(\psi_t, \psi_t) &= -\xi c^1_{22} N_1 + c^2_{22} N_2. \end{aligned} \tag{7}$$

Another way of representing it;

$$h(X, Y) = -\xi g(A_{N_1}(X), Y) N_1 + g(A_{N_2}(X), Y) N_2. \tag{8}$$

H_k is used for $k-$th mean curvature function and calculated by $H_k = g(H, N_k) = \frac{tr(A_{N_k})}{2}$, hence we obtain

$$H_k = \frac{c^k_{11}G - 2c^k_{12}F + c^k_{22}E}{2(EG - F^2)}. \tag{9}$$

According to the basis $\{N_1, N_2\}$, the mean curvature vector field H turns into

$$H = -\xi H_1 N_1 + H_2 N_2, \tag{10}$$

(see, [7, 12])

The mean curvature of M is congruent to the norm of the mean curvature vector $(\left\|\vec{H}\right\|)$. The surface is called as minimal, if the mean curvature vector of it is identically zero [5].

Gaussian curvature of $M : \psi(s,t)$ can be stated by using the first and second fundamental forms' coefficients:

$$K = \frac{-\xi \det(A_{N_1}) + \det(A_{N_2})}{W^2}. \tag{11}$$

In case of zero Gaussian curvature, M is called as a flat surface.

Furthermore, with the help of orthonormal tangent vectors $\{\psi_1, \psi_2\}$ and unit normal vectors $\{N_1, N_2\}$, the normal curvature of a surface is

$$K_N = g\left(R^{\perp}\left(\psi_1, \psi_2\right) N_2,\ N_1\right). \tag{12}$$

This relation can be given by the entries of shape operator matrices:

$$K_N = h_{12}^1\left(h_{22}^2 - h_{11}^2\right) + h_{12}^2\left(h_{11}^1 - h_{22}^1\right).$$

Regarding the previous equation, a surface M is known as semiumbilical surface if the normal curvature is zero, for all points on M [8].

In [1], Yu A. Aminov focused on the notion of Monge Patch in $\mathbb{E}^4$ with the representation

$$f = f(s,t),\ g = g(s,t). \tag{13}$$

Also, in [3], the authors studied some surfaces given by the parametrization

$$\psi(s,t) = (s, t, f(s,t), g(s,t)). \tag{14}$$

Two special surfaces, called translation surfaces and homothetical (factorable) surfaces are interesting classes in differential geometry. These surfaces have been studied from many viewpoints, theoretically [2, 4, 9, 10, 13].

A new surface named $TH-$ type surface (or translation-homothetical surface) is first handled by Difi et. al. in $3-$dimensional Euclidean spaces [6]. The parameterization of this surface is given with the help of the sum and multiplication of differentiable functions. Some studies on $TH-$type surfaces can be found in [6, 11]. Recently, the authors have defined $TF-$ type ($TH-$ type) surface in $4-$dimensional Euclidean space[11]. They investigated the structure of this type of surface in $\mathbb{E}^4$.

In this study, we deal with the non-null translation-homothetical surfaces in 4-dimensional Minkowski space. First, we describe the non-lightlike (non-null) Translation-Homothetical surface in $\mathbb{E}_1^4$. Then, we yield the normal curvature, mean curvature vector and Gaussian curvature for spacelike and timelike surfaces. Further, we characterize some non-null semiumbilical, minimal and flat $TH-$type surfaces in Minkowski space-time.

2. Classification of Non-null Translation-Homothetical Surfaces in $\mathbb{E}_1^4$

Definition 2.1. *[2] The surface which is defined by the sum of two curves $\alpha(s) = (s, 0, z_1(s), z_2(s))$ and $\beta(t) = (0, t, w_1(t), w_2(t))$ is called as translation surface. Thus, the translation surface in $4-$dimensional space has the parametrization*

$$\psi(s,t) = (s, t, z_1(s) + w_1(t), z_2(s) + w_2(t)). \tag{15}$$

Definition 2.2. *[4] The surface which is given by an explicit form $f(s,t) = z_1(s) w_1(t)$, $g(s,t) = z_2(s) w_2(t)$ is called as homothetical (or factorable) surface where s, t, f, g are Cartesian coordinates. Thus, the homothetical surface in $4-$dimensional space has the parametrization*

$$\psi(s,t) = (s, t, z_1(s) w_1(t), z_2(s) w_2(t)). \tag{16}$$

With respect to these definitions, the translation-homothetical surface is defined as the following:

Definition 2.3. *The surface is called TH−type surface (or translation-homothetical surface) if it is given by the Monge patch*

$$\psi(s,t) = (s, t, \lambda(z_1(s)+w_1(t)) + \mu(z_1(s)w_1(t)), \sigma(z_2(s)+w_2(t)) + \rho(z_2(s)w_1(t))) \tag{17}$$

where λ, μ, σ and ρ are non-zero real constants.

$TH-$type surface in Minkowski space-time can be considered by the representation

$$\psi(s,t) = (s, t, z_1(s) + w_1(t) + z_1(s)w_1(t), z_2(s) + w_2(t) + z_2(s)w_2(t)). \tag{18}$$

Thus, in this study, we investigate some properties of non-null (spacelike and timelike) $TH-$type surfaces given by the parameterization (18). Let M be a non-null $TH-$type surface in $\mathbb{E}^4_1$, then we have the followings:

The following vector fields span the tangent space of M:

$$\begin{aligned} \psi_s &= \left(1, 0, z_1'(s) + z_1'(s)w_1(t), z_2'(s) + z_2'(s)w_2(t)\right), \\ \psi_t &= \left(0, 1, w_1'(t) + z_1(s)w_1'(t), w_2'(t) + z_2(s)w_2'(t)\right). \end{aligned} \tag{19}$$

Therefore, the first fundamental form coefficients can be yielded by the Lorentzian inner product

$$\begin{aligned} E &= -1 + \left(z_1' + z_1'w_1\right)^2 + \left(z_2' + z_2'w_2\right)^2, \\ F &= \left(z_1' + z_1'w_1\right)\left(w_1' + z_1w_1'\right) + \left(z_2' + z_2'w_2\right)\left(w_2' + z_2w_2'\right), \\ G &= 1 + \left(w_1' + z_1w_1'\right)^2 + \left(w_2' + z_2w_2'\right)^2. \end{aligned} \tag{20}$$

Choosing the surface as timelike or spacelike with respect to being $E < 0$ (or $E > 0$), one can determine $W = \sqrt{\xi(EG - F^2)}$.

Two times derivatives of $\psi(s,t)$ are

$$\begin{aligned} \psi_{ss} &= \left(0, 0, z_1''(s) + z_1''(s)w_1(t), z_2''(s) + z_2''(s)w_2(t)\right), \\ \psi_{st} &= \left(0, 0, z_1'(s)\, w_1'(t), z_2'(s)\, w_2'(t)\right), \\ \psi_{tt} &= \left(0, 0, w_1''(t) + z_1(s)w_1''(t), w_2''(t) + z_2(s)w_2''(t)\right). \end{aligned} \tag{21}$$

The orthonormal vector fields $\{N_1, N_2\}$ spans the normal space of non-null surface:

$$N_1 = \frac{1}{\sqrt{|A_1|}}\left(z_1' + z_1'w_1, -\left(w_1' + z_1w_1'\right), 1, 0\right), \tag{22}$$

$$N_2 = \frac{1}{\sqrt{A_1W^*}}\left(A_1\left(z_2'+z_2'w_2\right) - A_3\left(z_1'+z_1'w_1\right), A_3\left(w_1'+z_1w_1'\right) - A_1\left(w_2'+z_2w_2'\right), -A_3, A_1\right),$$

where

$$\begin{aligned} A_1 &= 1 - \left(z_1' + z_1'w_1\right)^2 + \left(w_1' + z_1w_1'\right)^2, \\ A_2 &= 1 - \left(z_2' + z_2'w_2\right)^2 + \left(w_2' + z_2w_2'\right)^2 \\ A_3 &= \left(w_1' + z_1w_1'\right)\left(w_2' + z_2w_2'\right) - \left(z_1' + z_1'w_1\right)\left(z_2' + z_2'w_2\right), \\ W^* &= A_1A_2 - (A_3)^2. \end{aligned}$$

and by using (29) and (30), c_{ij}^k, $(i, j, k = 1, 2)$ are given as

$$\begin{aligned}
c_{11}^1 &= \frac{z_1'' + z_1'' w_1}{\sqrt{|A_1|}}, \qquad c_{11}^2 = \frac{\left(z_2'' + z_2'' w_2\right) A_1 - \left(z_1'' + z_1'' w_1\right) A_3}{\sqrt{A_1 W^*}}, \\
c_{12}^1 &= \frac{z_1' w_1'}{\sqrt{|A_1|}}, \qquad c_{12}^2 = \frac{z_2' w_2' A_1 - z_1' w_1' A_3}{\sqrt{A_1 W^*}}, \\
c_{22}^1 &= \frac{w_1'' + z_1 w_1''}{\sqrt{|A_1|}}, \qquad c_{22}^2 = \frac{\left(w_2'' + z_2 w_2''\right) A_1 - \left(w_1'' + z_1 w_1''\right) A_3}{\sqrt{A_1 W^*}}.
\end{aligned} \tag{23}$$

we can write the orthonormal tangent vector by using Gram-Schmidt orthonormalization method for ψ_s and ψ_t,

$$\begin{aligned}
X &= \frac{\psi_s}{\sqrt{|E|}}, \\
Y &= \frac{\sqrt{|E|}}{W}\left(\psi_t - \frac{F}{E}\psi_s\right).
\end{aligned} \tag{24}$$

By the use of (6), (7), (24) and (8), the shape operator matrices can be presented as

$$\begin{bmatrix} h_{11}^1 & h_{12}^1 \\ h_{12}^1 & h_{22}^1 \end{bmatrix}, \qquad \begin{bmatrix} h_{11}^2 & h_{12}^2 \\ h_{12}^2 & h_{22}^1 \end{bmatrix}, \tag{25}$$

where the functions h_{ij}^k are given by

$$\begin{aligned}
h_{11}^1 &= \xi \frac{\left(z_1'' + z_1'' w_1\right)}{E\sqrt{|A_1|}}, \quad h_{12}^1 = \frac{E z_1' w_1' - F\left(z_1'' + z_1'' w_1\right)}{EW\sqrt{|A_1|}}, \\
h_{22}^1 &= \xi \frac{\left(w_1'' + z_1 w_1''\right) E^2 - 2 z_1' w_1' EF + \left(z_1'' + z_1'' w_1\right) F^2}{E\sqrt{|A_1|}}, \\
h_{11}^2 &= \xi \frac{A_1\left(z_2'' + z_2'' w_2\right) - A_3\left(z_1'' + z_1'' w_1\right)}{E\sqrt{A_1 W^*}}, \\
h_{12}^2 &= \frac{\left(z_2' w_2' A_1 - z_1' w_1' A_3\right) E - \left[A_1\left(z_2'' + z_2'' w_2\right) - A_3\left(z_1'' + z_1'' w_1\right)\right] F}{EW\sqrt{A_1 W^*}}, \\
h_{22}^2 &= \xi \frac{\begin{array}{c}\left[\left(w_2'' + z_2 w_2''\right) A_1 - \left(w_1'' + z_1 w_1''\right) A_3\right] E^2 - 2\left(z_2' w_2' A_1 - z_1' w_1' A_3\right) EF \\ + \left[\left(z_2'' + z_2'' w_2\right) A_1 - A_3\left(z_1'' + z_1'' w_1\right)\right] F^2\end{array}}{EW^2\sqrt{A_1 W^*}}.
\end{aligned} \tag{26}$$

2.1. *Non-null Flat Translation-Homothetical Surfaces*

Theorem 2.4. *Let M be a non-null translation-homothetical surface with the parameterization (18) in $\mathbb{E}_1^4$. Then, its Gaussian curvature is given as*

$$K = \frac{\begin{array}{c} A_1\left(\left(z_2'' + z_2'' w_2\right)\left(w_2'' + z_2 w_2''\right) - \left(z_2' w_2'\right)^2\right) \\ + A_2\left(\left(z_1'' + z_1'' w_1\right)\left(w_1'' + z_1 w_1''\right) - \left(z_1' w_1'\right)^2\right) \\ - A_3\left(\left(z_2'' + z_2'' w_2\right)\left(w_1'' + z_1 w_1''\right) + \left(z_1'' + z_1'' w_1\right)\left(w_2'' + z_2 w_2''\right) - 2 z_1' w_1' z_2' w_2'\right)\end{array}}{W^* W^2}.$$

where W and W^* are defined as $W^2 = \xi\left(EG - F^2\right)$, $W^* = A_1 A_2 - A_3^2$, respectively.

Proof. By using (11) and (26), we obtain the desired result. □

Theorem 2.5. *Let M be a non-null translation-homothetical surface with the parameterization (18) in* $\mathbb{E}_1^4$. *Then M has zero Gaussian curvature if and only if*

$$\begin{aligned} 0 &= A_1\left(\left(z_2'' + z_2'' w_2\right)\left(w_2'' + z_2 w_2''\right) - \left(z_2' w_2'\right)^2\right) \\ &\quad + A_2\left(\left(z_1'' + z_1'' w_1\right)\left(w_1'' + z_1 w_1''\right) - \left(z_1' w_1'\right)^2\right) \\ &\quad - A_3\left(\left(z_2'' + z_2'' w_2\right)\left(w_1'' + z_1 w_1''\right) + \left(z_1'' + z_1'' w_1\right)\left(w_2'' + z_2 w_2''\right) - 2 z_1' w_1' z_2' w_2'\right). \end{aligned} \tag{27}$$

Theorem 2.6. *Let M be a non-null translation-homothetical surface with the parameterization (18) in* $\mathbb{E}_1^4$. *If M is given by one of the following parametrizations, then it is flat surface:*

(1) $f(s,t) = a_1 w_1(t) + a_1 + w_1(t)$, $g(s,t) = a_2 w_2(t) + a_2 + w_2(t)$;
(2) $f(s,t) = a_1 z_1(s) + a_1 + z_1(s)$, $g(s,t) = a_2 z_2(s) + a_2 + z_2(s)$;
(3) $f(s,t) = a_1 w_1(t) + a_1 + w_1(t)$, $g(s,t) = a_2 z_2(s) + a_2 + z_2(s)$;
(4) $f(s,t) = a_1 z_1(s) + a_1 + z_1(s)$, $g(s,t) = a_2 w_2(t) + a_2 + w_2(t)$;
(5) $f(s,t) = a_1$, $g(s,t) = a_2 w_2(t) + a_2 + w_2(t)$;
(6) $f(s,t) = a_1$, $g(s,t) = a_2 z_2(s) + a_2 + z_2(s)$;
(7) $f(s,t) = a$, $g(s,t) = b e^{a_1 s} e^{a_2 t} - 1$;
(8) $f(s,t) = a$, $g(s,t) = z_2(s) + w_2(t) + z_2 w_2(t)$ satisfying

$$\begin{aligned} z_2(s) &= \left((1-c)(a_3 s + a_4)^{\frac{1}{1-c}} - 1\right), \\ w_2(t) &= \left(\frac{(c-1)(a_5 t + a_6)}{c}\right)^{\frac{c}{c-1}} - 1, \end{aligned} \tag{28}$$

(9) $f(s,t) = a_1 a_5 e^{a_2 s} e^{a_6 t} - 1$, $g(s,t) = a_3 a_7 e^{a_4 s} e^{a_8 t} - 1$; $a_4 a_6 = a_2 a_8$,
(10) $f(s,t) = a_1 a_5 e^{a_2 s} e^{a_6 t} - 1$, $g(s,t) = a_3 a_7 e^{a_4 s} e^{a_8 t} - 1$; $a_2 a_4 = a_6 a_8$,
where a, b, c, a_i are real constants, $i = 1, .., 8$, $c \neq 0, 1$.

Proof. Let M be a non-null $TH-$type surface given by the parametrization (18) in $\mathbb{E}_1^4$. If $z_1'(s) = 0$, $z_2'(s) = 0$ or $w_1'(t) = 0$, $w_2'(t) = 0$ or $z_1'(s) = 0$, $w_2'(t) = 0$ ($z_2'(s) = 0$, $w_1'(t) = 0$) in (27), then we obtain the cases (1), (2), (3) and (4). If $z_1'(s) = 0$ and $w_1'(t) = 0$,then we have

$$\left(z_2'' + z_2'' w_2\right)\left(w_2'' + z_2 w_2''\right) - \left(z_2' w_2'\right)^2 = 0. \tag{29}$$

In this equation, if $z_2' = 0$ (or $w_2' = 0$), then we obtain the surfaces (5) and (6). If $z_1'(s) w_1'(t) \neq 0$, then we get

$$\frac{z_2''(s) z_2(s) + z_2''(s)}{\left(z_2'(s)\right)^2} = \frac{\left(w_2'(t)\right)^2}{w_2''(t) w_2(t) + w_2''(t)} = c, \tag{30}$$

where $c \in IR$. If $c = 1$, from (30) we get the differential equations $z_2''(s) z_2(s) + z_2''(s) = \left(z_2'(s)\right)^2$ and $w_2''(t) w_2(t) + w_2''(t) = \left(w_2'(t)\right)^2$ which have the solutions $z_2(s) = a_3 e^{a_4 s} - 1$ and $w_2(t) = a_5 e^{a_6 t} - 1$. Then, we obtain the surface parameterization (7).

If $c \neq 1$, we yield the solution of the differential equation (30) as $z_2(s) = (1-c)(a_3 s + a_4)^{\frac{1}{1-c}} - 1$ and $w_2(t) = \frac{(c-1)(a_5 t + a_6)^{\frac{c}{c-1}} - 1}{c}$ -1. Hence, we get the surface (8). Also, in equation (27) we suppose

$$\left(z_1'' + z_1'' w_1\right)\left(w_1'' + z_1 w_1''\right) - \left(z_1' w_1'\right)^2 = 0. \tag{31}$$

$$\left(z_2'' + z_2''w_2\right)\left(w_2'' + z_2w_2''\right) - \left(z_2'w_2'\right)^2 = 0. \tag{32}$$

and

$$\left(z_2'' + z_2''w_2\right)\left(w_1'' + z_1w_1''\right) + \left(z_1'' + z_1''w_1\right)\left(w_2'' + z_2w_2''\right) - 2z_1'w_1'z_2'w_2' = 0 \text{ or } A_3 = 0. \tag{33}$$

In a similar way, we obtain the solutions of the differential equations (31) and (32) as

$$\begin{aligned} z_1(s) &= a_1e^{a_2s} - 1, \quad w_1(t) = a_5e^{a_6t} - 1, \\ z_2(s) &= a_3e^{a_4s} - 1, \quad w_2(t) = a_7e^{a_8t} - 1. \end{aligned} \tag{34}$$

Substituting these functions into (33), we yield the surface parametrizations (9) and (10). □

2.2. *Non-null Minimal Translation-Homothetical Surfaces*

Theorem 2.7. *Let M be a non-null translation-homothetical surface with the parameterization (18) in $\mathbb{E}_1^4$. Then, its mean curvature vector is given as*

$$H = -\xi\frac{\left(z_1'' + z_1''w_1\right)G - 2z_1'w_1'F + \left(w_1'' + z_1w_1''\right)E}{2W^2\sqrt{|A_1|}}N_1 + \frac{\begin{array}{c}\left[\left(z_2'' + z_2''w_2\right)G - 2z_2'w_2'F + \left(w_2'' + z_2w_2''\right)E\right]A_1 \\ -\left[\left(z_1'' + z_1''w_1\right)G - 2z_1'w_1'F + w_1'' + z_1w_1''E\right]A_3\end{array}}{2W^2\sqrt{A_1W^*}}N_2. \tag{35}$$

Proof. By the use of (9), (10) and (23), we obtain the desired result. □

Theorem 2.8. *Let M be a non-null translation-homothetical surface with the parameterization (18) in $\mathbb{E}_1^4$. Then, M has zero mean curvature if and only if*

$$\left(z_i'' + z_i''w_i\right)G - 2z_i'w_i'F + \left(w_i'' + z_iw_i''\right)E = 0. \tag{36}$$

Theorem 2.9. *Let M be a non-null TH−type surface with the parameterization (18) in $\mathbb{E}_1^4$. Then, M is minimal if it is given by one of the following parametrizations :*

(1) $f(s,t) = a_1t + a_2, \quad g(s,t) = a_3t + a_4,$
(2) $f(s,t) = a_1s + a_2, \quad g(s,t) = a_3s + a_4,$
(3) $f(s,t) = a_1s + a_2, \quad g(s,t) = a_3t + a_4,$
(4) $f(s,t) = a_1t + a_2, \quad g(s,t) = a_3s + a_4,$
(5) $f(s,t) = a, \quad g(s,t) = (s+b)\tan(ct+d) - 1,$
(6) $f(s,t) = a, \quad g(s,t) = (t+b)\tan(cs+d) - 1,$
(7) $f(s,t) = (s+b)\tan(ct+d) - 1, \quad g(s,t) = (s+b)\tan(ct+d) - 1,$
(8) $f(s,t) = (t+b)\tan(cs+d) - 1, \quad g(s,t) = (t+b)\tan(cs+d) - 1,$
(9) $f(s,t) = z_1(s) + w_1(t) + z_1(s)w_1(t), \quad g(s,t) = z_2(s) + w_2(t) + z_2(s)w_2(t)$
where the functions satisfy

$$\begin{aligned} s &= \pm\int\frac{dz_i(s)}{\sqrt{2c\ln(z_i(s)+1) - 2ca_1}}, \\ t &= \pm\int\frac{dw_i(t)}{\sqrt{a_2(w_i(t)+1)^4 - \frac{d}{2}}}, \end{aligned} \tag{37}$$

or

$$\begin{aligned} s &= \pm\int\frac{dz_i(s)}{\sqrt{a_1(z_i(s)+1)^4 - \frac{c}{2}}}, \\ t &= \pm\int\frac{dw_i(t)}{\sqrt{2d\ln(w_i(s)+1) - 2da_2}}, \end{aligned} \tag{38}$$

or

$$\begin{aligned} s &= \pm \int \frac{dz_i(s)}{\sqrt{a_1 (z_i(s)+1)^{2(1+k)} - a_2}}, \\ t &= \pm \int \frac{dw_i(t)}{\sqrt{a_3 (w_i(t)+1)^{2(1+k)} + a_4}}. \end{aligned} \tag{39}$$

Proof. Let M is $TH-$type surface given by the parameterization (18) in $\mathbb{E}_1^4$. If M is minimal, then the equation (36) is hold. Hence, we write

$$\begin{aligned} 0 &= \left(z_1'' + z_1'' w_1\right)\left(1 + \left(w_1' + z_1 w_1'\right)^2 + \left(w_2' + z_2 w_2'\right)^2\right) \\ &\quad -2 z_1' w_1' \left(\left(z_1' + z_1' w_1\right)\left(w_1' + z_1 w_1'\right) + \left(z_2' + z_2' w_2\right)\left(w_2' + z_2 w_2'\right)\right) \\ &\quad + \left(w_1'' + z_1 w_1''\right)\left(-1 + \left(z_1' + z_1' w_1\right)^2 + \left(z_2' + z_2' w_2\right)\right), \end{aligned} \tag{40}$$

and

$$\begin{aligned} 0 &= \left(z_2'' + z_2'' w_2\right)\left(1 + \left(w_1' + z_1 w_1'\right)^2 + \left(w_2' + z_2 w_2'\right)^2\right) \\ &\quad -2 z_2' w_2' \left(\left(z_1' + z_1' w_1\right)\left(w_1' + z_1 w_1'\right) + \left(z_2' + z_2' w_2\right)\left(w_2' + z_2 w_2'\right)\right) \\ &\quad + \left(w_2'' + z_2 w_2''\right)\left(-1 + \left(z_1' + z_1' w_1\right)^2 + \left(z_2' + z_2' w_2\right)\right), \end{aligned} \tag{41}$$

The surface parametrizations (1), (2), (3) and (4) are obtained by taking $z_1'(s) = 0$, $z_2'(s) = 0$ or $w_1'(t) = 0$, $w_2'(t) = 0$ or $z_1'(s) = 0, w_2'(t) = 0$ or $z_2'(s) = 0, w_1'(t) = 0$, respectively. By taking $z_1'(s) = 0$ and $w_1'(t) = 0$, then we get

$$-\frac{w_2''}{w_2+1} + \frac{z_2''}{z_2+1} + \left(z_2'\right)^2 \left(w_2''(w_2+1) - \left(w_2'\right)^2\right) + \left(w_2'\right)^2 \left(z_2''(z_2+1) - \left(z_2'\right)^2\right) = 0. \tag{42}$$

In this equation, if we suppose $z_2''(s) = 0$ or $w_2''(t) = 0$, then we yield $w_2(t) = \frac{\tan(ct+d)}{a_1} - 1$ and $z_2(s) = \frac{\tan(ct+d)}{a_2} - 1$. Hence, the surfaces (5) and (6) are obtained. Also, in (42), if $z_2''(s)\, w_2''(t) \neq 0$, the derivatives of (42) with regards to s and t, one after another are obtained as

$$\frac{\left(z_2''(z_2+1) - \left(z_2'\right)^2\right)'}{\left(\left(z_2'\right)^2\right)'} = -\frac{\left(w_2''(w_2+1) - \left(w_2'\right)^2\right)'}{\left(\left(w_2'\right)^2\right)'} = c \tag{43}$$

where $c \in IR$. Therefore, integrating this equation regarding s or t, we get

$$\begin{aligned} z_2''(z_2+1) - (1+c)\left(z_2'\right)^2 &= k, \\ w_2''(w_2+1) - (1-c)\left(w_2'\right)^2 &= l, \end{aligned} \tag{44}$$

where $k, l \in IR$. By taking $c = 1$ and $c = -1$ in (44) respectively, we get

$$\begin{aligned} z_2''(z_2+1) &= k, \\ w_2''(w_2+1) - 2\left(w_2'\right)^2 &= l, \end{aligned} \tag{45}$$

and

$$\begin{aligned} z_2''(z_2+1) - 2\left(z_2'\right)^2 &= k, \\ w_2''(w_2+1) &= l. \end{aligned} \tag{46}$$

Thus, if we solve these differential equations, the results (37) and (38) are obtained. If $c \neq 1$ in (44), the solution of the differential equation is congruent to the last result of (9). Finally, by taking $z_1(s) = z_2(s)$ and $w_1(t) = w_2(t)$, then we have the surfaces (7) and (8). This completes the proof. □

2.3. Non-null Semiumbilical Translation-Homothetical Surfaces

Theorem 2.10. *Let M be a non-null translation-homothetical surface in $\mathbb{E}^4_1$. Then, M has the normal curvature as*

$$K_N = -\xi \frac{\begin{array}{c} E\left(z_1' w_1'\left(w_2'' + z_2 w_2''\right) - z_2' w_2'\left(w_1'' + z_1 w_1''\right)\right) \\ -F\left(\left(z_1'' + z_1'' w_1\right)\left(w_2'' + z_2 w_2''\right) - \left(z_2'' + z_2'' w_2\right)\left(w_1'' + z_1 w_1''\right)\right) \\ +G\left(z_2' w_2'\left(z_1'' + z_1'' w_1\right) - z_1' w_1'\left(z_2'' + z_2'' w_2\right)\right) \end{array}}{W^3 \sqrt{W^*}}. \tag{47}$$

Proof. Let M be a non-null $TH-$type surface with (15) in $\mathbb{E}^4_1$. Substituting the second fundamental form coefficients h^k_{ij} into (12), we get the result. □

Corollary 2.11. *Let M be a non-null translation-homothetical surface with the parameterization (15). If the functions z_i, w_i, $(i = 1, 2)$ are linear polynomial functions, then M corresponds to semiumbilical surface in $\mathbb{E}^4_1$.*

Proof. Let M be a non-null $TH-$type surface and suppose z_i, w_i, $(i = 1, 2)$ are the linear polynomial functions as

$$\begin{aligned} z_i &= a_i s + b_i, \\ w_i &= c_i t + d_i. \end{aligned} \tag{48}$$

Thus, by the use of (47) and (48), we get $z_i''(s) = 0$, $w_i''(t) = 0$, i.e , $K_N = 0$. This completes the proof. □

Example. The surface given by the parameterization

$$\psi(s,t) = (s, t, 2st - 2s + 3t - 3, -st + 2s + 4t - 8)$$

is a semiumbilical $TH-$type surface and can be plotted by projection in $3-$dimension with command *plot3d* $([s + t, f(s,t), g(s,t)] : \; s = -2..2, \; t = 0..1)$:

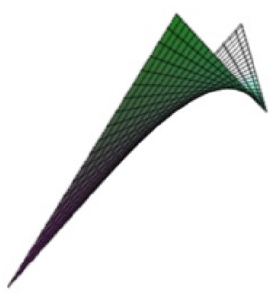

Figure 1: Semiumbilical $TH-$type surface

3. Conclusion

$TH-$type surfaces (or Translation-Homothetical surfaces) have been previously discussed by Difi et al.(2018) and Pamuk et al.(2021). They considered $3-$dimensional spaces and $4-$dimensional Euclidean space. In this article, we define non-null translation-homothetical surfaces in Minkowski space-time and classify these surfaces with respect to being flat, minimal and especially semiumbilical. The results provide valuable insights into the nature of surfaces in Minkowski space and will be of interest to researchers and scholars in the fields of mathematics, physics, and astronomy.

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New General Inequalities For Exponential Type Convex Function

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Abstract. In this paper, we introduce the concept of an exponential type convex function. We establish new integral inequalities of the Hermite-Hadamard type by using the Power-Mean and Hölder inequalities. Additionally, we give definitions of the Riemann-Liouville fractional integrals. We use these Riemann-Liouville fractional integrals to establish a new integral inequalities for exponential type convex function.

1. Introduction

The mathematical branches heavily rely on mathematical inequalities. Numerous scientists investigated the characteristics of convexity and came up with several integral inequalities (see references [3]-[8]). Hermite-Hadamard inequality is among the most well-known integral inequalities for convex functions. This double integral inequality is stated as follows:

Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

for all $a, b \in I$ with $a < b$.

Convex functions take a significant place in the Mathematical Inequalities. Many researchers have carried out studies on different definitions of convex functions. Previous studies have focused on convexity types such as $s-$convex, $m-$convex, $(\alpha, m)-$convex and quasi-convex (see references [9]-[12]). However, recent studies have found that many new types of convexity have been obtained. One of these new types of convexity is exponential type convex functions. A new definition is given as follows:

Definition 1.1. *[2] A nonnegative function $f : I \to \mathbb{R}$ is called exponential type convex function if, for every $a, b \in I$ and $k \in [0, 1]$,*

$$f(ka + (1-k)b) \leq \left(e^{k} - 1\right) f(a) + \left(e^{1-k} - 1\right) f(b)$$

The class of all exponential type convex functions on interval I is indicated by $EXPC(I)$.

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In [2], Kadakal and İşcan have defined exponential type convex functions and obtained new inequalities related to this definition as follows:

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be an exponential type convex function. If $a < b$ and $f \in L[a, b]$, then the following Hermite-Hadamard type inequalities hold:*

$$\frac{1}{2\left(\sqrt{e}-1\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq (e-2)\left[f(a)+f(b)\right]$$

Theorem 1.3. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$, and assume that $f' \in L[a, b]$. If $\left|f'\right|$ is an exponential type convex function on $[a, b]$, then the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left(4\sqrt{e}-e-\frac{7}{2}\right) A\left(\left|f'(a)\right|, \left|f'(b)\right|\right)$$

holds for $k \in [0, 1]$, where $A(u, v)$ is the arithmetic mean of u and v.

Theorem 1.4. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$, $q > 1$ and assume that $f' \in L[a, b]$. If $\left|f'\right|^q$ is an exponential type convex function on $[a, b]$, then the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{2}\left[2(e-2)\right]^{\frac{1}{q}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)$$

holds for $k \in [0, 1]$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $A(u, v)$ is the arithmetic mean of u and v.

Theorem 1.5. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$, $q \geq 1$ and assume that $f' \in L[a, b]$. If $\left|f'\right|^q$ is an exponential type convex function on $[a, b]$, then the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{2^{2-\frac{1}{q}}}\left[2\left(4\sqrt{e}-e-\frac{7}{2}\right)\right]^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)$$

holds for $k \in [0, 1]$, where $A(u, v)$ is the arithmetic mean of u and v.

In [1], Alomari *et al.* proved the following result connected with the right part of Hermite-Hadamard Inequality:

Lemma 1.6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on I°, where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\frac{f(a)+rf(b)}{r+1} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{r+1}\int_0^1 \left[(r+1)t-1\right] f'(tb+(1-t)a)\,dt.$$

for every fixed $r \in [0, 1]$.

As a result of the research exists, we investigate new inequalities that are connected to right hand side of Hermite-Hadamard integral inequalities for some exponential type convex functions utilizing the Hölder inequality, properties of modulus, power mean inequality, and elementary calculations.

The aim of this paper is to establish some Hermite-Hadamard type inequalities for exponential type convex functions. In order to obtain our results, we utilized Lemma 1.6 and Lemma 3.2.

2. Hermite-Hadamard Inequality For Exponential Type Convex Functions

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a differentiable function on I^o, $a, b \in I^o$ with $a < b$ and assume that $f' \in L[a,b]$. If $|f'|$ is an exponential type convex function on $[a,b]$, then the inequality*

$$\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{r+1}\left[\left((2r+2)e^{\frac{1}{r+1}}-\frac{3r^2+6r+2re+2e+5}{2r+2}\right)\left|f'(b)\right| + \left((2r+2)e^{\frac{r}{r+1}}-\frac{5r^2+6r+2er^2+2er+3}{2r+2}\right)\left|f'(a)\right|\right]$$

holds for every fixed $r \in [0,1]$.

Proof. Using Lemma 1.6 and the exponential type convexity of $|f'|$, it follows that

$$\begin{aligned}
&\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{r+1}\int_0^1 |(r+1)t-1|\left|f'(tb+(1-t)a)\right|dt \\
\leq\ & \frac{b-a}{r+1}\int_0^1 |(r+1)t-1|\left[\left(e^t-1\right)\left|f'(b)\right|+\left(e^{1-t}-1\right)\left|f'(a)\right|\right]dt.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{r+1}\left(\int_0^{\frac{1}{r+1}}(1-(r+1)t)\left[\left(e^t-1\right)\left|f'(b)\right|+\left(e^{1-t}-1\right)\left|f'(a)\right|\right]dt\right. \\
& \left.+\int_{\frac{1}{r+1}}^1((r+1)t-1)\left[\left(e^t-1\right)\left|f'(b)\right|+\left(e^{1-t}-1\right)\left|f'(a)\right|\right]dt\right) \\
=\ & \frac{b-a}{r+1}\left[\left((2r+2)e^{\frac{1}{r+1}}-\frac{3r^2+6r+2re+2e+5}{2r+2}\right)\left|f'(b)\right|\right. \\
& \left.+\left((2r+2)e^{\frac{r}{r+1}}-\frac{2r^2e+2re+5r^2+6r+3}{2r+2}\right)\left|f'(a)\right|\right]
\end{aligned}$$

which completes the proof. □

Remark 2.2. *Under the assumptions of Theorem 2.1 with $r = 1$, we get the conclusion of Theorem 1.3.*

Corollary 2.3. *Under the assumptions of Theorem 2.1 with $r = 0$, we obtain*

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left[\left|f'(a)\right|+(2e-5)\left|f'(b)\right|\right].$$

Theorem 2.4. *Let $f : I \to \mathbb{R}$ be a differentiable function on I^o, $a, b \in I^o$ with $a < b$, $q > 1$ assume that $f' \in L[a, b]$. If $\left|f'\right|^q$ is an exponential type convex function on $[a, b]$, then the inequality*

$$\left|\frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{r+1}\left[\frac{r^{p+1}+1}{(r+1)(p+1)}\right]^{\frac{1}{p}} (2(e-2))^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)$$

holds for every fixed $r \in [0, 1]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1.6 and using Hölder inequality with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{r+1}\int_0^1 |(r+1)t - 1|\left|f'(tb + (1-t)a)\right| dt \\
\leq\ & \frac{b-a}{r+1}\left(\int_0^1 |(r+1)t-1|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{r+1}\left(\int_0^{\frac{1}{r+1}} (1-(r+1)t)^p dt + \int_{\frac{1}{r+1}}^1 ((r+1)t-1)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f'\right|^q$ is exponential type convex function on $[a, b]$, we get

$$\begin{aligned}
\left|\frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq\ & \frac{b-a}{r+1}\left(\int_0^{\frac{1}{r+1}} (1-(r+1)t)^p dt + \int_{\frac{1}{r+1}}^1 ((r+1)t-1)^p dt\right)^{\frac{1}{p}} \\
& \times\left(\left|f'(b)\right|^q \int_0^1 \left(e^t - 1\right)dt + \left|f'(a)\right|^q \int_0^1 \left(e^{1-t} - 1\right)dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{r+1}\left[\frac{r^{p+1}+1}{(r+1)(p+1)}\right]^{\frac{1}{p}} (2(e-2))^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)
\end{aligned}$$

which completes the proof. □

Remark 2.5. *Under the assumptions of Theorem 2.4 with $r = 1$, we get the conclusion of Theorem 1.4.*

Corollary 2.6. *Under the assumptions of Theorem 2.4 with $r = 0$, we obtain*

$$\left|f(a) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}} (2(e-2))^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right).$$

Theorem 2.7. *Let $f : I \to \mathbb{R}$ be a differentiable function on I^o, $a, b \in I^o$ with $a < b$, $q \geq 1$ and assume that $f' \in L[a,b]$. If $|f'|^q$ is an exponential type convex function on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{r+1}\left(\frac{r^2+1}{2r+2}\right)^{1-\frac{1}{q}}\left[\left(e^{\frac{1}{r+1}}(2r+2)-\frac{3r^2+6r+2re+2e+5}{2r+2}\right)|f'(b)|^q\right. \\
&\left.+\left(e^{\frac{r}{r+1}}(2r+2)-\frac{5r^2+6r+2er^2+2er+3}{2r+2}\right)|f'(a)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 1.6 and using the well known power mean inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{r+1}\int_0^1 |(r+1)t-1|\,|f'(ta+(1-t)b)|\,dt \\
\leq\ & \frac{b-a}{r+1}\left(\int_0^1 |(r+1)t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |(r+1)t-1|\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}.
\end{aligned}$$

On the other hand, we obtain

$$\int_0^1 |(r+1)t-1|\,dt=\int_0^{\frac{1}{r+1}}[1-(r+1)t]\,dt+\int_{\frac{1}{r+1}}^1[(r+1)t-1]\,dt=\frac{r^2+1}{2r+2}.$$

Since $|f'|^q$ is exponential type convex function on $[a,b]$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{r+1}\left(\frac{r^2+1}{2r+2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |(r+1)t-1|\left[\left(e^t-1\right)|f'(a)|^q+\left(e^{1-t}-1\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{r+1}\left(\frac{r^2+1}{2r+2}\right)^{1-\frac{1}{q}} \\
&\times\left[\left(e^{\frac{1}{r+1}}(2r+2)-\frac{3r^2+6r+2re+2e+5}{2r+2}\right)|f'(b)|^q\right. \\
&\left.+\left(e^{\frac{r}{r+1}}(2r+2)-\frac{5r^2+6r+2er^2+2er+3}{2r+2}\right)|f'(a)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

which is required. □

Remark 2.8. *Under the assumptions of Theorem 2.7 with $r = 1$, we get the conclusion of Theorem 1.5.*

Corollary 2.9. *Under the assumptions of Theorem 2.7 with $q = 1$, we obtain the Theorem 2.1.*

Corollary 2.10. *Under the assumptions of Theorem 2.7 with $r = 0$, we have*

$$\left| f(a) - \frac{1}{b-a} \int_a^b f(x)dx \right| \le \frac{b-a}{2} \left[\left| f'(a) \right|^q + (2e-5) \left| f'(b) \right|^q \right]^{\frac{1}{q}}.$$

3. Hermite-Hadamard Inequalities for Fractional Integrals

Now we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 3.1. *[13] Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$. Here is $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

In [14], Özdemir *et al.* proved the following result for fractional integrals. Also, different results have been obtained for different values of r.

Lemma 3.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I with $a < r, a, r \in I$. If $f' \in L[a,r]$, then the following equality for fractional integrals holds:*

$$\begin{aligned} & \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}} \left[J_{r^-}^{\alpha} f(a) + J_{a^+}^{\alpha} f(r) \right] \\ = & \frac{r-a}{2} \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha} \right] f'(r + (a-r)t) dt. \end{aligned}$$

Theorem 3.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I with $a < r, a, r \in I$ and $f' \in L[a,r]$. If $\left| f' \right|^q$ is an exponential type convex function on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left| \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}} \left[J_{r^-}^{\alpha} f(a) + J_{a^+}^{\alpha} f(r) \right] \right| \\ \le & \frac{r-a}{2(\alpha p + 1)^{\frac{1}{p}}} \left(2(e-2) \right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(\left| f'(a) \right|^q, \left| f'(b) \right|^q \right). \end{aligned}$$

Proof. From Lemma 3.2 and using Hölder inequality with properties of modulus, we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}} \left[J_{r^-}^{\alpha} f(a) + J_{a^+}^{\alpha} f(r) \right] \right| \\ \le & \frac{r-a}{2} \int_0^1 \left| (1-t)^{\alpha} - t^{\alpha} \right| \left| f'(r + (a-r)t) \right| dt \\ \le & \frac{r-a}{2} \left(\int_0^1 \left| (1-t)^{\alpha} - t^{\alpha} \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(r + (a-r)t) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

We know that for $\alpha \in [0,1]$ and $\forall t_1, t_2 \in [0,1]$,

$$\left|t_1^\alpha - t_2^\alpha\right| \leq |t_1 - t_2|^\alpha,$$

therefore

$$\begin{aligned}
\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|^p dt &\leq \int_0^1 |1-2t|^{\alpha p} dt \\
&= \int_0^{\frac{1}{2}} [1-2t]^{\alpha p} dt + \int_{\frac{1}{2}}^1 [2t-1]^{\alpha p} dt \\
&= \frac{1}{\alpha p + 1}.
\end{aligned}$$

Also, $\left|f'\right|^q$ is exponential type convex function on $[a,b]$,we have

$$\begin{aligned}
\left|f'(r + (a-r)t)\right|^q &= \left|f'(ta + (1-t)r)\right|^q \\
&\leq (e^t - 1)\left|f'(a)\right|^q + (e^{1-t} - 1)\left|f'(r)\right|^q, \ t \in (0,1).
\end{aligned}$$

and consequently

$$\begin{aligned}
&\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^\alpha}\left[J_{r^-}^\alpha f(a) + J_{a^+}^\alpha f(r)\right]\right| \\
\leq\ & \frac{r-a}{2(\alpha p+1)^{\frac{1}{p}}} \left(2\left(e-2\right)\right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(\left|f'\left(a\right)\right|^q, \left|f'\left(b\right)\right|^q\right)
\end{aligned}$$

which completes the proof. □

Remark 3.4. *If in Theorem 3.3, we choose $\alpha = 1$ and $r = b$, then we obtain Theorem 1.4.*

4. Conclusion

In this paper, we obtained new general integral inequalities for exponential type convex functions. We proved the Hermite-Hadamard type integral inequalities and obtained new theorems with the Hölder inequality. With this definition, many new integral inequalities can be obtained. Also, by using Hölder-İşcan (see reference [15]) inequality and different Lemmas, new results can be obtained for exponential type convex functions.

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Analytical Solutions of Coupled Boiti-Leon-Pempinelli Equation with Fractional Derivative

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Abstract. In this study, the sub-equation method is used as a tool for finding the analytical solutions of Coupled Boiti-Leon-Pempinelli (CBLP) equation where the derivatives are in conformable form with the fractional term. In the introduction section, the advantages of the conformable derivative are expressed. By using the fractional wave transform and chain rule for conformable derivative, the nonlinear fractional partial differential equation turns into a nonlinear integer order differential equation. This translation gives us a great advantage in obtaining analytical solutions and interpreting the physical behavior of the acquired solutions. In the rest of article, the sub-equation method is applied to Coupled Boiti-Leon-Pempinelli equation, and the analytical results are derived successfully. This means that our method is effective and powerful for constructing exact and explicit analytic solutions to nonlinear PDEs with the fractional term. While this process, symbolic computation such as Mathematica is used. It is shown that, with the help of symbolic computation, sub-equation method ensures a powerful and straightforward mathematical tool for solving nonlinear partial differential equations.

1. Introduction

Nonlinear phenomena draws great attraction in the last decades. Understanding the physical nature of the nonlinear mathematical models allures scientists because the only way for interpreting natural events arises as a result of this curiosity [3–6, 16–18]. For this aim, many methods are developed such as homotopy analysis method [7], differential transform method [8], exp-function method [9], Jacobi elliptic function expansion method [10] and etc. As we see, both numerical methods and analytical methods are applied to get the results. But analytical solutions of very few of the differential equations that arise as a result of mathematical models of events encountered in nature can be obtained. This makes the analytical method valuable. Because numerical methods give the approximate value for the expected solution and give us a restricted chance to understand the physical nature of the solution. In spite of that the solutions which are obtained as a result of analytical methods give us extensive perspective for explaining the behavior of the solution.

In the beginning, integer order derivative and integral are used for modeling the natural event. But by the time it is understood that Newtonian type derivative and integral fall short of modeling the event that arises in the nonlinear nature. So the survey for fractional differentiation and integration is started. By the time

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it is understood that fractional calculus has a clearer physical meaning and a simpler statement compared with the integer order models while describing complicated physical mechanics problems. This motivation helped fractional calculus to be improved faster. First of all, scientists need to give an efficient and applicable definition of fractional differentiation and integration. Riemann-Liouville, Caputo and Grünwald-Letnikov definitions were the popular definition. But there were some deficiencies while describing the mathematical model. For instance, the Riemann-Liouville fractional derivative of a constant is not zero. In addition to this fractional initial/boundary conditions of problems which are described as mathematical models of different physical, chemical, or engineering problems must be expressed in fractional form. More than these, basic properties such as a derivative of the quotient of two functions, derivative of the product of two functions, chain rule and etc. are not satisfied by Riemann-Liouville, Caputo, and Grünwald-Letnikov definitions. As a result, Khalil et al. expressed a new definition that obeys the basic properties.

Definition 1.1. *α^{th} order conformable derivative(CFD) of a function f can be expressed as*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - (f)(t)}{\varepsilon},$$

for $f : [0, \infty) \to R$ and for all $t > 0$, $\alpha \in (0, 1)$.

Definition 1.2. *Let f function is defined with n variables $x_1, ..., x_n$. The fractional partial derivatives of f of order $\alpha \in (0, 1]$ in conformable sense with respect to x_i is given by [19]*

$$\frac{d^\alpha}{dx_i^\alpha} f(x_1, ..., x_n) = \lim_{\varepsilon \to 0} \frac{f(x_1, ..., x_{i-1}, x_i + \varepsilon x_i^{1-\alpha}, ..., x_n) - f(x_1, ..., x_n)}{\varepsilon}.$$

Definition 1.3. *Let $\alpha \geqslant 0$. Then conformable fractional integral of a function is defined as [13]*

$$I_\alpha^a(f)(s) = \int_a^s \frac{f(t)}{t^{1-\alpha}} dt.$$

In this article, the analytical solutions of Coupled Boiti-Leon-Pempinelli equation where the derivatives are in the conformable sense with the fractional term are obtained. To the best of our knowledge, these solutions are seen firstly in the literature. Also, some graphical representations of the solutions are given to understand the physical behavior of the solutions.

2. Brief Description of Considered Method

2.1. Sub-Equation Method

In this section sub-equation method [11] which established on the Riccati equation

$$\varphi'(\xi) = \sigma + (\varphi(\xi))^2. \tag{1}$$

is going to be described. Regarding the general form of nonlinear time fractional partial differential equation as

$$P\left(u, D_t^\alpha u, D_x u, D_x^2 u, ...\right) = 0 \tag{2}$$

where $D_t^\alpha u$ indicates fractional order differentiation in conformable sense. The fractional wave transformation [12] could be expressed

$$u(x, y, t) = U(\xi), \ \xi = kx + wy + c\frac{t^\alpha}{\alpha} \tag{3}$$

where c is a constant to be calculated later and by the help of the chain rule [14], Eq. (2) can turn into an nonlinear differential equation with integer order

$$G(U(\xi), U'(\xi), U''(\xi), ...) = 0. \tag{4}$$

Suppose that the solution of the reduced Eq. (4) can be expressed as

$$U(\xi) = \sum_{i=0}^{N} a_i \varphi^i(\xi), \ a_N \neq 0, \tag{5}$$

where a_i $(0 \leq i \leq N)$ are constant coefficients to be calculated and positive integer N is going to be obtained by using balancing principle [15] in equation (4) and $\varphi(\xi)$ is the solution of Riccati equation (1). Some solutions of equation (1) is given as follows.

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\xi\right), \ \sigma < 0 \\ -\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\xi\right), \ \sigma < 0 \\ \sqrt{\sigma}\tan\left(\sqrt{\sigma}\xi\right), \qquad \sigma > 0 \\ -\sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right), \qquad \sigma > 0 \\ -\frac{1}{\xi+\varpi}, \ \varpi \text{ is a cons.}, \ \sigma = 0 \end{cases} \tag{6}$$

After the all solutions procedure we get a polynomial due to $\varphi(\xi)$. Equating zero to all the coefficients of $\varphi^i(\xi)$ $(i = 0, 1, ..., N)$ ends with a nonlinear algebraic equation system depending on c, a_i $(i = 0, 1, ..., N)$. By solving this algebraic equation system we have the values of c, a_i $(i = 0, 1, ..., N)$. Substituting all the results in the formulas (6) we get the exact solutions for equation (2).

3. Solutions of the Equation

Consider (CBLP) equation in conformable sense as

$$\begin{aligned} &D_y D_t^{\alpha} u - D_x D_y (u^2 - D_x u) - 2D_x^3 v = 0, \\ &D_t^{\alpha} v - D_x^2 v + 2u D_x v = 0. \end{aligned} \tag{7}$$

Using the following wave transformation

$$u(x, y, t) = U(\xi), v(x, y, t) = V(\xi), \ \xi = kx + wy + c\frac{t^{\alpha}}{\alpha} \tag{8}$$

and chain rule [14] (7) turns into nonlinear differential equation system

$$\begin{aligned} &cwU'' - kw(U^2 - kU')'' - 2k^3 V''' = 0, \\ &cV' - k^2 V'' - 2kUV' = 0 \end{aligned} \tag{9}$$

where the derivatives described in integer order. Now integrating twice the first equation in (7), we have

$$V' = \frac{cw}{2k^3}U - \frac{w}{2k^2}(U^2 - kU'). \tag{10}$$

Using this result in the second equation of Eq. (7) we have the following equation.

$$c^2 U - 3kcU^2 + 2k^2 U^3 - k^4 U'' = 0. \tag{11}$$

Let the solution of Eq. (11) is given in terms of $\varphi(\xi)$ as

$$U(\xi) = \sum_{i=0}^{N} a_i \varphi^i(\xi), \ a_N \neq 0. \tag{12}$$

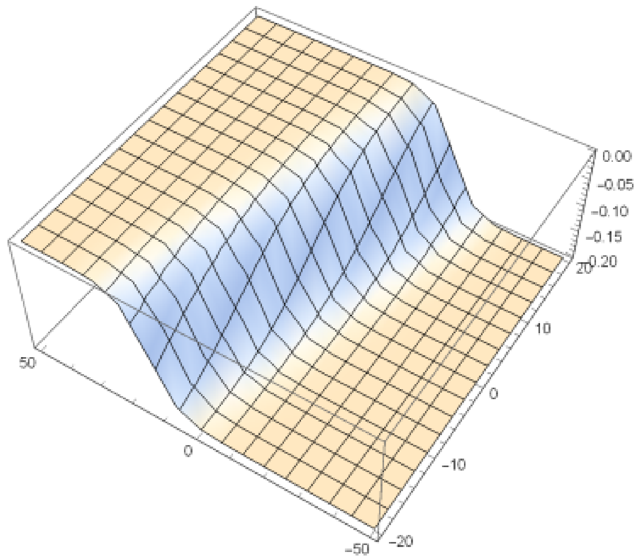

Figure 1: Surface plot of the exact solution $u_1(x, y, t)$ for $w = 0.1, a_0 = -0.1, \sigma = -2, \alpha = 0.75, t = 0.1$

Using the balancing principle [15], we calculate $N = 1$. Collecting all the obtained results in Eq. (11), an algebraic equation system come to exist with respect to w, k, c, a_0, a_1. Solving this obtained system led to following solution set

$$a_1 = \frac{a_0}{\sqrt{-\sigma}}, c = -\frac{2a_0^2}{\sqrt{-\sigma}}, k = \frac{-a_0}{\sqrt{-\sigma}} \tag{13}$$

where $\sigma < 0$ and a_0 and w are free constants. Using (6) and (3) the traveling wave solutions of Eq. (7) can be deducted

$$u_1(x, y, t) = a_0 - a_0 \tanh\left(\xi \sqrt{-\sigma}\right),$$

$$v_1(x, y, t) = -\frac{w\sigma \tanh\left(\xi \sqrt{-\sigma}\right)}{\sqrt{-\sigma}},$$

$$u_2(x, y, t) = a_0 - a_0 \coth\left(\xi \sqrt{-\sigma}\right),$$

$$v_2(x, y, t) = -\frac{w\sigma \coth\left(\xi \sqrt{-\sigma}\right)}{\sqrt{-\sigma}}$$

where $\xi = wy - \frac{a_0 x}{\sqrt{-\sigma}} - \frac{2a_0^2 t^\alpha}{\alpha \sqrt{-\sigma}}$. Some graphical representations of the obtained results are given in Figure 1 and Figure 2.

4. Conclusion

In this article, it is obtained that sub-equation method shows great performance while obtaining the exact solutions of the nonlinear partial differential equations where the derivatives are in conformable sense with fractional term. While obtaining the solution symbolic computer software called Mathematica is used. Also some graphical simulations of the obtained solutions are given. This article may give an insight to the researchers who study on obtaining the analytical solutions of nonlinear fractional partial differential equations.

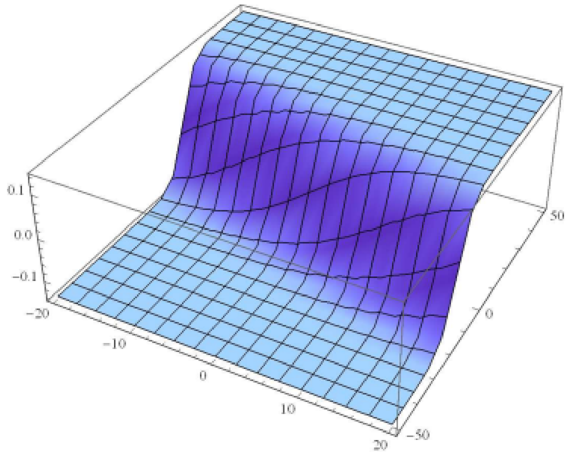

Figure 2: Surface plot of the exact solution $v_1(x, y, t)$ for $w = 0.1, a_0 = -0.1, \sigma = -2, \alpha = 0.75, t = 0.1$

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Boundary Value Problems for Differential Equations Involving the Generalized Caputo-Fabrizio Fractional Derivative in λ-Metric Space

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Abstract. In this paper, by using the fixed point results of α-φ-Geraghty type mappings, the existence and uniqueness results for solutions to differential equations involving the generalized Caputo-Fabrizio derivative are investigated in λ-metric spaces. As application, an illustrative example is given to show the applicability of our theoretical results.

1. Introduction

In recent years, fractional calculus has attracted the attention of many researchers from various disciplines (physics, biology, chemistry, applied sciences,...). Indeed, The use of fractional derivatives has been observed to be beneficial for modeling many problems in engineering sciences (see, for example, [1, 2, 17, 21, 33, 35, 36].

Various there are several notions about fractional derivatives in the literature. Caputo and Riemann-Liouville introduced the basic notions (see for example [10, 27]), which imply the singular kernel $k(t,s) = \frac{(t-s)^{-q}}{\Gamma(1-q)}$, $0 < q < 1$. These derivatives play an important role in modeling phenomena in physics. However, as introduced by Fabrizio and Caputo [8], some phenomena related to material heterogeneities cannot be well modeled using fractional Caputo derivatives or Riemann-Liouville. Therefore, Fabrizio and Caputo [8] proposed a new fractional derivative with non-singular kernel $k(t,s) = e^{\frac{-q(t-s)}{1-q}}$, $0 < q < 1$. Later fractional derivative of Caputo-Fabrizio was used by many researchers to model several problems in engineering sciences (see [3, 4, 7, 11, 18, 26, 29, 30, 37]). Additionally, other fractional order derivatives with non-singular kernels have been introduced by some researchers (more details see [9, 10, 19, 20, 25, 32]).

In 1993, Czerwik proposed the notion of λ-metric (see [14, 15]). Following these initial works, the existence of a fixed point for the different operators in the definition of λ-metric spaces has been widely studied (see [12, 16, 22–24, 28, 31]).

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In this paper, we study the existence-uniqueness of solutions for problems of generalized fractional order differential equations of the Caputo-Fabrizio in λ-metric space.

$$\begin{cases} \left(D^q_{0,d,c}z\right)(\xi) = f\left(\xi, z(\xi), (D^q_{0,d,c}z)(\xi)\right), & \xi \in J = [0,\Lambda] \quad d > 0, c \geq 0 \\ z(0) = z_0 \end{cases} \tag{1}$$

Where $\Lambda > 0$, $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is derivative function, $D^q_{0,d,c}$ is generalized Caputo-Fabrizio fractional derivative with $q \in (0,1)$.

This work is arranged as follows. In the second section, we recall the notions of fractional calculus and the λ-metric space. The third Section is concerned to prove the main result. Finally, We provide an example illustrating the main result.

2. Preliminaries

We start with definition of λ-metric spaces, which was introduced by Afshari, Aydi and Karapinar [5, 6].

Definition 2.1. *Let Υ be a nonempty set, $\lambda \in \mathbb{R}^{*+}$ and $M : \Upsilon \times \Upsilon \to [0,\infty)$ such that for all $\varsigma, \gamma, \epsilon \in \Upsilon$*

(i) $M(\varsigma,\gamma) = 0 \Leftrightarrow \varsigma = \gamma$;
(ii) $M(\varsigma,\gamma) = M(\gamma,\varsigma)$;
(iii) $M(\varsigma,\gamma) \leq \lambda[M(\varsigma,\epsilon) + M(\epsilon,\gamma)]$.

Then, the triple (Υ, M, λ) is called a λ-metric space.

Example 2.2. *[5, 6] let $M : [0,1] \times [0,1] \to [0,\infty)$ by defined by*

$$M(\gamma,\epsilon) = \mid \gamma^2 - \epsilon^2 \mid, \textit{ for all } \gamma, \epsilon \in [0,1].$$

It is clear that the triple (Υ, M, λ) is a λ-metric space with $\lambda \geq 2$, but it is easy to see that the pair $([0,1], M)$ is not a metric space.

Example 2.3. *[5, 6] let $\Upsilon = C(\mathbb{R})$ and $M : \Upsilon \times \Upsilon \to \mathbb{R}^+$ defined by*

$$M(\vartheta,\eta) = \| \left(\vartheta - \eta\right)^2 \|_{L^\infty(\mathbb{R})}, \textit{ for all } \vartheta, \eta \in C(\mathbb{R}).$$

Then, the triple $(C(\mathbb{R}), M, 2)$ is a λ-metric space.

In 2012, B. Samet and Erdal Karapinar [22] originated the concept of α-admissibility presented in [30].

Definition 2.4. *[30] Let $\mathcal{P} : \Upsilon \longrightarrow \Upsilon$ be a self-mapping and $\alpha : \Upsilon \times \Upsilon \longrightarrow [0,\infty)$ be a function. We say that $\mathcal{P}$ is a α-admissible if*

$$\alpha(\vartheta,\eta) \geq 1 \Longrightarrow \alpha(\mathcal{P}\vartheta, \mathcal{P}\eta) \geq 1 \textit{ for all } \vartheta, \eta \in \Upsilon.$$

Example 2.5. *[29] Let $\Upsilon = \mathbb{R}^*_+$. Define $\mathcal{P} : \Upsilon \longrightarrow \Upsilon$ and $\alpha : \Upsilon \times \Upsilon \longrightarrow \mathbb{R}^+$ as follows $\mathcal{P}\vartheta = \ln(\vartheta)$ for all $\vartheta \in Y$, and*

$$\alpha(\vartheta,\eta) = \begin{cases} 0 & \textit{if } \vartheta < \eta, \\ 2 & \textit{if } \vartheta \geq \eta. \end{cases}$$

Then, $\mathcal{P}$ is α-admissible.

Example 2.6. *[30] We define the mappings $\mathcal{P} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $\alpha : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, as follows $\mathcal{P}\vartheta = \sqrt{\vartheta}$ for all $\vartheta \in \mathbb{R}^+$, and*

$$\alpha(\vartheta,\eta) = \begin{cases} 0 & \textit{if } \vartheta < \eta, \\ e^{\vartheta-\eta} & \textit{if } \vartheta \geq \eta. \end{cases}$$

Then, $\mathcal{P}$ is α-admissible.

Definition 2.7. *[6] Let (Υ, M, λ) be a λ-metric space and $\alpha : \Upsilon\times\Upsilon \longrightarrow \mathbb{R}^+$ be a function. We say that Υ is α-regular if*

$$(\theta_n)_{n\in\mathbb{N}} \subset \Upsilon \text{ suth that, } \alpha(\theta_n, \theta_{n+1}) \geq 1, \quad \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} \theta_n = \theta,$$

there exists a subsequence $(\theta_{n_k})_{k\in\mathbb{N}}$ of $(\theta_n)_{n\in\mathbb{N}}$, such that

$$\alpha(\theta_{n_k}, \theta) \geq 1, \quad \forall k \in \mathbb{N}.$$

We denote by Ψ the set of all increasing functions $\mu : \mathbb{R}^+ \to [0, \frac{1}{c^2})$, $\quad c \geq 1$ and Φ the set of all continuous and nondecreasing functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\varphi(ct) \leq c\varphi(t) \leq ct \quad \text{ for } c > 1.$$

Definition 2.8. *[5] Let (Υ, M, λ) be a λ-metrice space. An operator $\mathcal{P} : \Upsilon \longrightarrow \Upsilon$ is a generalized α-φ-Geraghty contraction, if there exists $\alpha : \Upsilon \times \Upsilon \longrightarrow [0, \infty)$ such that*

$$\alpha(\vartheta, \eta)\varphi\Big(\lambda^3\, M(\mathcal{P}\vartheta, \mathcal{P}\eta)\Big) \leq \mu\Big(\varphi(M(\vartheta, \eta))\Big)\varphi(M(\vartheta, \eta)), \quad \forall \vartheta, \eta \in \Upsilon,$$

where $\mu \in \Psi$ and $\varphi \in \Phi$.

Theorem 2.9. *[5] Let (Υ, M, λ) be a λ-metrice space, and $\mathcal{P} : \Upsilon \longrightarrow \Upsilon$ be a generalized α-φ-Geraghty contraction. Assume that*

1) *$\mathcal{P}$ is α-admissible;*
2) *there exists $\theta_0 \in \Upsilon$ such that $\alpha(\theta_0, \mathcal{P}\theta_0) \geq 1$;*
3) *either $\mathcal{P}$ is continous or Υ is α-regular.*

Then $\mathcal{P}$ has a fixed point. Moreover, if

4) *for all fixed point ϑ, η of $\mathcal{P}$, either*

$$\alpha(\vartheta, \eta) \geq 1 \text{ or } \alpha(\eta, \vartheta) \geq 1,$$

then $\mathcal{P}$ has a uniques fixed point.

Now, we introduce definitions of generalized Caputo-Fabrizio fractional derivatives which are used throughout this paper.

Definition 2.10. *[8] Let $d > 0, c \geq 0, 0 < q < 1$, $m \in \mathbb{N}\bigcup\{0\}$ and $f \in C^{m+1}(\mathbb{R}^+)$. The fractional derivative of order $q + m$ of f with respect to Kernel function $K_{d,c}$, where*

$$K_{d,c}(\xi) = (\frac{d^2 + c^2}{d})e^{-d\xi}cos(c\xi), \;\; \xi \geq 0$$

is defined by

$$D^{q+m}_{0,d,c}\Big(f\Big)(\xi) = \Big(\frac{1}{1-q}\Big)\Big(\frac{d^2 + c^2}{d}\Big)\int_0^{\xi} e^{\frac{-dq(\xi-\tau)}{1-q}} cos\Big(\frac{cq(\xi - \tau)}{1-q}\Big) f^{m+1}(\tau)d\tau.$$

Definition 2.11. *[8] Let $h \in C[0, T]$. The fractional integral of h is given by*

$$\Big(I^q_{0,d,c}h\Big)(\xi) = \eta_q h(\xi) + q\int_0^{\xi} g(\tau)d\tau - \delta_q \frac{c^2}{d^2 + c^2}\int_0^{\xi} e^{\frac{-dq(\xi-\tau)}{1-q}} h(\tau)d\tau$$

where $\eta_q = \frac{d(1-q)}{d^2+c^2}$ and $\delta_q = \frac{c^2 q}{d^2+c^2}$.

3. Main Result

Let $(C^1(\Lambda), \| \cdot \|)$ be the Banach space of all continuous functions on J, where $\| z \| = \sup\limits_{\xi \in \Lambda} \mid z(\xi) \mid$ and $M : C^1(\Lambda) \times C^1(\Lambda) \to \mathbb{R}^*_+$ be defined by

$$M(y, z) = \sup_{\xi \in \Lambda} \Big(y(\xi) - z(\xi) \Big)^2.$$

Then $(C^1(\Lambda), M, 2)$ is a complete $\lambda-$metrice space with $\lambda = 2$.

In this paper, we make use of the following assumptions:

(A_1**)** There exists a function $\nu : C^1(\Lambda) \times C^1(\Lambda) \to]0, \infty)$ and $\xi_0 \in C^1(\Lambda)$ such that

$$\nu\Big(\xi_0(t), \theta_h + \eta_q h(t) + q \int_0^t h(\tau)d\tau + \delta_q \int_0^t \exp\{\frac{-aq(t-\tau)}{1-q}\} h(\tau)d\tau\Big) \geq 0,$$

$h \in C^1(\Lambda)$, with $h(t) = f(t, \xi_0(t), h(t))$ and $\theta_h = x_0 + \eta_q h(0)$.

(A_2**)** There exists $\varphi \in \Phi$ and $\sigma : C^1(\Lambda) \times C^1(\Lambda) \to \mathbb{R}^{*+}$ and $\chi : \Lambda \to]0, 1[$ such that for each $z, y, z_1, y_1 \in C^1(\Lambda)$ and $\tau \in \Lambda$

$$\mid f(\tau, z, y) - f(\tau, z_1, y_1) \mid \leq \sigma(z, y) \mid z - z_1 \mid + \chi(\tau) \mid y - y_1 \mid,$$

with

$$\| 2\eta_q \frac{\sigma(z, y)}{1 - \chi_s} + (q + \delta_q) \int_0^t \frac{\sigma(z, y)}{1 - \chi_s} d\tau \|^2_\infty \leq \frac{1}{4} \varphi\Big(\| (z - y)^2 \|_\infty \Big),$$

where $\chi_s = \sup\limits_{\tau \in J} \mid \chi(\tau) \mid$.

(A_3**)** For each $t \in \Lambda$ and $z, y \in C^1(\Lambda)$, we have

$$\nu\Big(z(t) \quad , \quad y(t)\Big) \geq 0 \quad \Rightarrow \quad \nu\Big(A_g, A_h\Big) \geq 0,$$

where ν is defined in assumption **(**A_1**)** and

$$\begin{aligned} A_h =& \theta_g + \eta_q g(t) + q \int_0^t g(\tau)d\tau + \delta_q \int_0^t \exp\{\frac{-aq(t-\tau)}{1-q}\} g(\tau)d\tau, \\ A_g =& \theta_h + \eta_q h(t) + q \int_0^t h(\tau)d\tau + \delta_q \int_0^t \exp\{\frac{-aq(t-\tau)}{1-q}\} h(\tau)d\tau, \end{aligned}$$

and $h, g \in C^1(\Lambda)$, with $h(\tau) = f(\tau, y(\tau), h(\tau),\ g(\tau) = f(\tau, z(\tau), g(\tau)))$ and $\theta_h = u_0 + \eta_q h(0),\ \theta_g = u_0 + \eta_q g(0)$.

(A_4**)** If $(p_n)_{n \in \mathbb{N}} \subset C^1(\Lambda)$ such that $\lim\limits_{n \to \infty} p_n = p$ and $\nu(p_n, p_{n+1}) \geq 0$, then $\nu(p_n, p) \geq 0$.

(A_5**)** If u, v two fiexd solutions of problem **(1)**, either

$$\nu(u, v) \geq 0 \quad \text{or} \quad \nu(v, u) \geq 0.$$

Lemma 3.1. *Let $g \in C^1[0,T]$. A function $x \in C^1[0,T]$ is solution of problem*

$$\begin{cases} \left(D^q_{0,d,c} z\right)(t) = g(t), \qquad \forall t \in \Lambda = [0,T] \quad 0 < q < 1, \quad d > 0, c \geq 0 \\ z(0) = z_0 \end{cases} \tag{2}$$

if and only if z satisfies the following equation

$$z(t) = z_0 - \frac{d(1-q)}{d^2+c^2} g(0) + \left(I^q_{0,d;c} g(.)\right)(t) \quad t \in [0,T] \tag{3}$$

Proof. Let $z \in C^1[0,T]$ be a solution of **(2)**. One has

$$\left(D^q_{0,a,b} x\right)'(t) \quad = \quad g'(t) \quad , \quad \forall t \in [0,T].$$

By Definition **2.10**, we obtain

$$\begin{aligned} \left(D^q_{0,d,c} z\right)'(t) &= \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right) \\ & \quad \left\{ z'(t) + \int_0^t \frac{d}{dt}\left(e^{\frac{-dq(t-s)}{1-q}} \cos\left(\frac{cq(t-s)}{1-q}\right)\right) z'(t) ds \right\} \\ &= \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right) z'(t) \\ &- \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right)\left(\frac{qd}{1-q}\right) \int_0^t e^{\frac{-dq(t-s)}{1-q}} cos\left(\frac{cq(t-s)}{1-q}\right) z'(t) ds \\ &- \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right)\left(\frac{qc}{1-q}\right) \int_0^t e^{\frac{-dq(t-s)}{1-q}} sin\left(\frac{cq(t-s)}{1-q}\right) z'(t) ds \\ &= \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right) z'(t) \\ &- \left(\frac{qd}{1-q}\right) g(t, z(t)) - \left(\frac{qc}{1-q}\right)\left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right) \gamma(t), \end{aligned} \tag{4}$$

where

$$\gamma(t) = \int_0^t e^{\frac{-dq(t-s)}{1-q}} sin\left(\frac{cq(t-s)}{1-q}\right) z'(t) ds.$$

On the other hand

$$\gamma'(t) = \int_0^t \frac{d}{dt}\left(e^{\frac{-dq(t-s)}{1-q}} sin\left(\frac{cq(t-s)}{1-q}\right)\right) z'(t) ds.$$

Then,

$$\begin{aligned} \gamma(t)' &= \frac{-dq}{1-q} \int_0^t e^{\frac{-dq(t-s)}{1-q}} \sin\left(\frac{cq(t-s)}{1-q}\right) z'(t) ds \\ &+ \frac{cq}{1-q} \int_0^t e^{\frac{-dq(t-s)}{1-q}} \cos\left(\frac{cq(t-s)}{1-q}\right) z'(t) ds, \\ &= \frac{-dq}{1-q} \gamma(t) + \frac{dcq}{d^2+c^2} g(t). \end{aligned} \tag{5}$$

Using that $\gamma(0) = 0$ and integrating the equality **(5)**, we get

$$\gamma(t) = \frac{dcq}{d^2+c^2} \int_0^t e^{\frac{-dq(t-s)}{1-q}} g(s) ds.$$

Hence by **(4)**, we deduce that

$$\begin{aligned}\left(D^{q}_{0,d,c}z\right)'(t) &= \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right)\\ &\quad \left\{z'(t)-\left(\frac{qd}{1-q}\right)g(t)-\left(\frac{qc}{1-q}\right)^2\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\right\}\end{aligned}$$

By using

$$\left(D^{q}_{0,d,c}x\right)'(t)=g'(t)\ ,\qquad t\in[0,T].$$

We obtain that

$$\begin{aligned}g'(t) &= \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right)\\ &\quad \left\{z'(t)-\left(\frac{qd}{1-q}\right)g(t)-\left(\frac{qc}{1-q}\right)^2\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\right\}.\end{aligned}$$

Then,

$$\begin{aligned}z'(t) &= \frac{d(1-q)}{d^2+c^2}g'(t)\\ &+ \frac{qd^2}{d^2+c^2}g(t)+\frac{dc^2q^2}{(d^2+c^2)(1-q)}\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds.\end{aligned} \tag{6}$$

Using that $z(0)=z_0$ and integrating the **(6)**, we have

$$\begin{aligned}z(t)-z_0 &= \frac{qc^2}{d^2+c^2}\int_0^t g(\tau)d\tau+\frac{d(1-q)}{d^2+c^2}g(t)-\frac{d(1-q)}{d^2+c^2}g(0)\\ &+ \frac{dc^2q^2}{(d^2+c^2)(1-q)}\int_0^t\int_0^\tau e^{\frac{-dq(\tau-s)}{1-q}}g(s)dsd\tau.\end{aligned}$$

From Fubini's theorem, we have

$$\begin{aligned}\int_0^t\int_0^\tau e^{\frac{-dq(\tau-s)}{1-q}}g(s)dsd\tau &= \int_0^t e^{\frac{dqs}{1-q}}g(s)\left(\int_s^t e^{\frac{-dq\tau}{1-q}}d\tau\right)ds\\ &= \left(\frac{1-q}{dq}\right)\int_0^t g(s)ds-\left(\frac{1-q}{dq}\right)\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\end{aligned}$$

Then,

$$\begin{aligned}z(t)-z_0 &= \frac{qd^2}{d^2+c^2}\int_0^t g(\tau)d\tau+\frac{d(1-q)}{d^2+c^2}g(t)-\frac{d(1-q)}{d^2+c^2}g(0)\\ &+ \frac{dc^2q^2}{(d^2+c^2)(1-q)}\left(\left(\frac{1-q}{dq}\right)\int_0^t g(s)ds-\left(\frac{1-q}{dq}\right)\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\right)\\ &= \frac{qd^2}{d^2+c^2}\int_0^t g(\tau)d\tau+\frac{d(1-q)}{d^2+c^2}g(t)-\frac{d(1-q)}{d^2+c^2}g(0)+\frac{c^2q}{d^2+c^2}\int_0^t g(s)ds\\ &- \frac{c^2q}{d^2+c^2}\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\\ &= \frac{d(1-q)}{d^2+c^2}g(t)-\frac{d(1-q)}{d^2+c^2}g(0)+q\left(\int_0^t g(s)ds-\frac{c^2}{d^2+c^2}\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\right)\end{aligned}$$

So, we get **(3)**.

Conversely, if z satisfies **(3)**, then $\left(D^q_{0,d,c}z\right)(t) = g(t), \qquad \forall t \in \Lambda = [0,T]$ and $z(0) = z_0$. $\square$

We can deduce the following result

Lemma 3.2. *A function x is a solution of problem 1, if and only if x satisfies the following integral equation*

$$z(t) = \theta_g + \left(I^q_{0,a;b}g(.)\right)(t) \quad t \in \Lambda = [0,T]$$

with $g(t) = f(t, z(t), g(t)$ *and* $\theta_g = z_0 + \eta_q g(0) = z_0 - \frac{a(1-q)}{a^2+b^2}g(0)$.

Theorem 3.3. *Under assumptions* (A_1)*-*(A_5)*, the problem*(**1**) *has a unique solution.*

Proof. consider the maping $Q : C^1(\Lambda) \to C^1(\Lambda)$ with

$$\begin{aligned} Q : C^1(\Lambda) &\to C^1(\Lambda) \\ x &\mapsto Qz(t) = \theta_h + \eta_q h(t) + q\int_0^t h(s)ds + \delta_q\int_0^t \exp\{\frac{-dq(t-s)}{1-q}\}h(s)ds, \end{aligned}$$

where $h \in C^1(\Lambda)$, such that $h(t) = f(t, z(t), h(t))$ and $\theta_h = z_0 + \eta_q h(0)$.

Using Lemma **3.2**, the problem reduces to finding a fixed point of the map Q.

Let $\alpha : C^1(\Lambda) \times C^1(\Lambda) \to [0,\infty)$ be the function defined by

$$\alpha(x,y) = \begin{cases} 1 \quad \text{if } \ v(x(t), y(t)) \geq 0 \quad t \in J, \\ \\ 0 \qquad\qquad \textit{otherwise}. \end{cases}$$

We have to prove that Q is a generalized α-φ-Geraghty operator:

Lets $x, y \in C^1(\Lambda)$ and $t \in \Lambda$, we have

$$\begin{aligned} Qz(t) - Qy(t) = \theta_g - \theta_h + \eta_q[g(t) - h(t)] + q\int_0^t g(s) - h(s)ds \\ + \delta_q\int_0^t \exp\{\frac{-dq(t-s)}{1-q}\}g(s) - h(s)ds, \end{aligned}$$

where $h, g \in C^1(\Lambda)$, such that $h(t) = f(t, y(t), h(t))$, $g(t) = f(t, z(t), g(t))$ and

$$\begin{aligned} \theta_g &= u_0 + \eta_q g(0), \\ \theta_h &= u_0 + \eta_q h(0). \end{aligned}$$

Then

$$\begin{aligned} \mid Qz(t) - Qy(t) \mid &\leq \mid \theta_g - \theta_h \mid + \eta_q \mid g(t) - h(t) \mid + q\int_0^t \mid g(s) - h(s) \mid ds \\ &\quad + \delta_q\int_0^t \exp\{\frac{-dq(t-s)}{1-q}\} \mid g(s) - h(s) \mid ds \\ &\leq +\eta_q \mid g(0) - h(0) \mid + \eta_q \mid g(t) - h(t) \mid \\ &\quad + \int_0^t \left(q + \delta_q \exp\{\frac{-dq(t-s)}{1-q}\}\right) \mid g(s) - h(s) \mid ds. \end{aligned}$$

By **(**A_2**)**, we get

$$\begin{aligned} \mid g(t)-h(t) \mid &= \mid f(t,z(t),g(t))-f(t,y(t),h(t)) \mid \\ &\leq \sigma(x,y) \mid z(t)-y(t) \mid +\chi(t) \mid g(t)-h(t) \mid \\ &\leq \sigma(z,y) \mid \Big(z(t)-y(t)\Big)^2 \mid^{1/2} +\chi(t) \mid g(t)-h(t) \mid . \end{aligned}$$

Thus,

$$\| g-h \|_\infty \leq \frac{\sigma(z,y)}{1-\chi_s} \| \Big(z-y\Big)^2 \|_\infty^{1/2} .$$

Next, we have

$$\begin{aligned} \mid Qz(t)-Qy(t) \mid &\leq 2\eta_q \frac{\sigma(x,y)}{1-\chi_s} \| \Big(z-y\Big)^2 \|_\infty^{1/2} \\ &+ \int_0^t \Big(q+\delta_q \exp\{\frac{-dq(t-s)}{1-q}\}\Big)\frac{\sigma(z,y)}{1-\chi_s} \| \Big(z-y\Big)^2 \|_\infty^{1/2} ds \\ &\leq 2\eta_q \frac{\sigma(z,y)}{1-\chi_s} \| \Big(z-y\Big)^2 \|_\infty^{1/2} \\ &+ \int_0^t \Big(q+\delta_q\Big)\frac{\sigma(z,y)}{1-\chi_s} \| \Big(z-y\Big)^2 \|_\infty^{1/2} ds \\ &\leq \| \Big(z-y\Big)^2 \|_\infty^{1/2} \Big[2\eta_q \frac{\sigma(z,y)}{1-\chi_s} + \Big(q+\delta_q\Big)\int_0^t \frac{\sigma(z,y)}{1-\chi_s} ds\Big] \\ &\leq \| \Big(z-y\Big)^2 \|_\infty^{1/2} \| \Big[2\eta_q \frac{\sigma(z,y)}{1-\chi_s} + \Big(q+\delta_q\Big)\int_0^t \frac{\sigma(z,y)}{1-\chi_s} ds\Big] \|_\infty . \end{aligned}$$

So,

$$\begin{aligned} \mid Qz(t)-Qy(t) \mid^2 &\leq \| \Big(z-y\Big)^2 \|_\infty \| \Big[2\eta_q \frac{\sigma(z,y)}{1-\chi_s} + \Big(q+\delta_q\Big)\int_0^t \frac{\sigma(z,y)}{1-\chi_s} ds\Big] \|_\infty^2 \\ &\leq \| \Big(z-y\Big)^2 \|_\infty \| \Big[2\eta_q \frac{\sigma(z,y)}{1-\chi_s} + \Big(q+\delta_q\Big)\int_0^t \frac{\sigma(z,y)}{1-\chi_s} ds\Big] \|_\infty^2 . \end{aligned}$$

This implies

$$\begin{aligned} \mid Qz(t)-Qy(t) \mid^2 &\leq \| \Big(z-y\Big)^2 \|_\infty \frac{1}{4}\varphi(\| \Big(z-y\Big)^2 \|_\infty) \\ &\leq \frac{1}{4}M(z,y)\varphi(M(z,y)). \end{aligned}$$

Then,

$$M(Qz,Qy) \leq \frac{1}{4}M(z,y)\varphi(M(z,y)).$$

And thus,

$$2^3M(Qz(t),Qy(t)) \leq \frac{1}{32}M(z,y)\varphi(M(z,y)).$$

Since $\varphi \in \Phi$, we have

$$\begin{aligned} \alpha(z,y)\varphi\Big(2^3M(Qz(t),Qy(t)\Big) &\leq \alpha(z,y)\varphi\Big(\frac{1}{32}M(z,y)\varphi(M(z,y))\Big) \\ &\leq \varphi\Big(\frac{1}{32}M(z,y)\Big)\varphi(M(z,y)) \\ &\leq \frac{1}{32}\varphi\Big(M(z,y)\Big)\varphi(M(z,y)). \end{aligned}$$

Hence,

$$\alpha(z,y)\varphi\Big(c^3 M(Qz(t),Qy(t)\Big) \le \mu\Big(\varphi\Big(M(z,y)\Big)\Big)\varphi(M(z,y)) + L\psi(N(z,y)),$$

where $\mu(t) = \frac{t}{32}$, $\varphi \in \Phi$, $L = 0$ and $c = 2$.

So, Q is generalized α-φ-Geraghty operator.

Lets $z, y \in C^1(\Lambda)$ such that $\alpha(z,y) \ge 1$.
Thus, for each $t \in \Lambda$, we have

$$\nu(z(t), y(t)) \ge 0.$$

By **(**A_3**)**, then

$$\nu(Qz(t), Qy(t)) \ge 0,$$

this implies that

$$\alpha(Qz, Qy) \ge 1.$$

Hence, Q is a α-admissible.

From **(**A_1**)**, there exist $\xi_0 \in C^1(\Lambda)$ such that such that

$$\nu\Big(\xi_0(t), \theta_h + \eta_q h(t) + q\int_0^t h(s)ds + \delta_q \int_0^t \exp\{\frac{-dq(t-s)}{1-q}\}h(s)ds\Big) \ge 0,$$

this implies that

$$\nu(\xi_0, Q\xi_0) \ge 0.$$

Thus,

$$\alpha(\xi_0, Q\xi_0) \ge 1.$$

So, there exist $\xi_0 \in C^1(\Lambda)$ such that

$$\alpha(\xi_0, Q\xi_0) \ge 1.$$

Finally, if $(p_n)_{n\in\mathbb{N}} \subset C^1(\Lambda)$ such that $\lim_{n\to\infty} p_n = p$ and $\alpha(p_n, p_{n+1}) \ge 1$, which gives

$$\nu(p_n, p_{n+1}) \ge 0.$$

Then, from **(**A_4**)** we have $\nu(p_n, p) \ge 0$.
And thus,

$$\nu(p_n, p) \ge 0.$$

This implies that

$$\alpha(p_n, p) \ge 1.$$

Therefore, by applying Theorem **2.9**, we conclude that if Q has a fixed point in $C^1(\Lambda)$, then it is a solution of the fractional problem **(1)**.
Moreover, **(**A_5**)**, if u and v are two fixed points of Q, then either

$$\nu(u,v) \ge 0 \text{ or } \nu(v,u) \ge 0.$$

This implies that either

$$\alpha(u,v) \ge 1 \text{ or } \alpha(v,u) \ge 1.$$

From an application of Theorem **2.9**, then the problem **(1)** has the uniqueness solution. □

4. Example

We consider the following Caputo-Fabrizio fractional problem.

$$\begin{cases} ({}^{CF}\mathcal{D}^{q}_{0,1,0}\, z)(t) &= g\Big(t, z(t), ({}^{CF}\mathcal{D}^{q}_{0,1,0}\, z)(t)\Big); \quad t \in \Lambda := [0,1], \\ z(0) = 0, \end{cases} \tag{7}$$

Where ${}^{CF}\mathcal{D}^{q}_{0,1,0}$ is Generalized of Caputo-Fabrizio fractional derivative of order $q \in (0,1)$ and $g : \Lambda \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function definied by the following expression

$$g(t,z,y) = \frac{1}{5}\Big[\frac{1+z+\sin(z)}{2+z} - \frac{e^{-t}}{1+y}\Big].$$

Let $(C^1(\Lambda), M, 2)$ is a complete $\lambda-$metrice space with $c = 2$, such that

$$\begin{aligned} d : C^1(\Lambda) \times C^1(\Lambda) &\to \mathbb{R}_+ \\ (z,y) &\mapsto M(z,y) = \sup_{t\in\Lambda} \Big(z(t) - y(t)\Big)^2 \\ &\qquad\qquad\quad = \| \Big(z - y\Big)^2 \|_\infty . \end{aligned}$$

Lets $z, y, v, u \in C^1(\Lambda)$ and $t \in \Lambda$, we have

$$\begin{aligned} g\Big(t, z(t), u(t)\Big) - g\Big(t, y(t), v(t)\Big) &= \frac{1}{5}\Big[\frac{1+z(t)+\sin(x(t))}{2+z(t)} - \frac{e^{-t}}{1+u(t)}\Big] \\ &- \frac{1}{5}\Big[\frac{1+y(t)+\sin(y(t))}{2+y(t)} - \frac{e^{-t}}{1+v(t)}\Big] \\ &= \frac{1}{5}\Big[\frac{z(t)-y(t)}{(1+z(t))(1+y(t))} \\ &+ \frac{(2+y(t))\sin(z(t)) - (2+z(t))\sin(y(t))}{(1+z(t))(1+y(t))}\Big] \\ &+ \frac{e^{-t}}{5}\frac{u(t)-v(t)}{(1+u(t))(1+v(t))}. \end{aligned}$$

And thus,

$$\begin{aligned} \mid g\Big(t, z(t), u(t)\Big) - g\Big(t, y(t), v(t)\Big) \mid &\leq \frac{1}{5} \mid z(t) - y(t) \mid \\ &+ \mid (2+y(t))\sin(z(t)) - (2+z(t))\sin(y(t)) \mid \\ &+ \frac{e^{-t}}{5} \mid u(t) - v(t) \mid . \end{aligned}$$

Case-1: if $y(t) \leq z(t)$, we get

$$\begin{aligned} \mid g\Big(t, z(t), u(t)\Big) - g\Big(t, y(t), v(t)\Big) \mid &\leq \mid z(t) - y(t) \mid \\ &+ \mid (2+z(t))\Big(\sin(z(t)) - \sin(y(t))\Big) \mid \\ &+ \frac{e^{-t}}{5} \mid u(t) - v(t) \mid \\ &\leq \mid z(t) - y(t) \mid + 2(2+ \mid x(t) \mid) \\ &\quad \mid \cos\Big(\frac{z(t)+y(t)}{2}\Big) \| \sin\Big(\frac{z(t)-y(t)}{2}\Big) \mid \\ &+ \frac{e^{-t}}{5} \mid u(t) - v(t) \mid . \end{aligned}$$

Since $\sin z \leq x$ for all $z \geq 0$, then

$$\begin{aligned} \mid g\Big(t, z(t), u(t)\Big) - g\Big(t, y(t), v(t)\Big) \mid &\leq \mid z(t) - y(t) \mid + (2 + \mid z(t) \mid) \mid z(t) - y(t) \mid \\ &+ \frac{e^{-t}}{5} \mid u(t) - v(t) \mid \\ &\leq \Big(3 + \parallel z \parallel_\infty \Big) \parallel z - y \parallel_\infty + \frac{e^{-t}}{5} \parallel u - v \parallel_\infty . \end{aligned}$$

Case-2: if $y(t) > z(t)$, we obtain

$$\mid g\Big(t, z(t), u(t)\Big) - g\Big(t, y(t), v(t)\Big) \mid \leq \Big(3 + \parallel y \parallel_\infty \Big) \parallel z - y \parallel_\infty + \frac{e^{-t}}{5} \parallel u - v \parallel_\infty .$$

So,

$$\begin{aligned} \mid g\Big(t, z(t), u(t)\Big) - g\Big(t, y(t), v(t)\Big) \mid &\leq \min\{3 + \parallel y \parallel_\infty, 3 + \parallel z \parallel_\infty\} \parallel z - y \parallel_\infty \\ &+ \frac{e^{-t}}{5} \parallel u - v \parallel_\infty . \end{aligned}$$

Then hypothesis **(A_2)** is satisfied

$$\mid g\Big(t, z(t), u(t)\Big) - g\Big(t, y(t), v(t)\Big) \mid \leq \sigma(z, y) \mid z(t) - y(t) \mid + \chi(t) \mid u(t) - v(t) \mid,$$

where

$$\begin{aligned} \sigma(z, y) &= \min\{3 + \parallel y \parallel_\infty, 3 + \parallel z \parallel_\infty\}, \\ \chi(t) &= \frac{1}{5} e^{-t}. \end{aligned}$$

We define the function $\alpha : C(\Lambda) \times C(\Lambda) \to \mathbb{R}_+^*$ by

$$\alpha(z, y) = \begin{cases} 1 & \text{if } \varrho(z(t), y(t)) \geq 0 \quad t \in \Lambda, \\ 0 & \qquad \textit{otherwise}, \end{cases}$$

and

$$\begin{aligned} \varrho : C(\Lambda) \times C(\Lambda) &\to \mathbb{R} \\ (z, y) &\mapsto \varrho(z, y) = \parallel z - y \parallel_\infty . \end{aligned}$$

Thus, hypothesis **(A_3)** is satisfied with

$$\xi_0(t) = z(0).$$

Moreover **(A_4)** holds from the definitions of the ϱ.

Finally, by Theorem **3.3**, we get the existence of solutions and the uniqueness of problem **(7)**.

Conclusion

This paper presents contributions to the study of differential equations involving the generalized Caputo-Fabrizio fractional derivative in the λ-Metric Space, using fixed point theory of α-φ-Geraghty type. Furthermore, we have concluded this study with an illustrative example of our theoretical results

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Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest

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Some Attributes of the Matrix Operators about the Weighted Generalized Difference Sequence Space

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Abstract. We can describe the norm for an operator given as $T : X \to Y$ as follows: It is the most appropriate value of U that satisfies the following inequality

$$||Tx||_Y \leq U||x||_X$$

and also for the lower bound of T we can say that the value of L agrees with the following inequality

$$||Tx||_Y \geq L||x||_X,$$

where $||.||_X$ and $||.||_Y$ stand for the norms corresponding to the spaces X and Y. The main feature of this article is that it converts the norms and lower bounds of those matrix operators used as weighted sequence space $\ell_p(w)$ into a new space. This new sequence space is the generalized weighted sequence space. For this purpose, the double sequential band matrix $\tilde{B}(\tilde{r}, \tilde{s})$ and also the space consisting of those sequences whose $\tilde{B}(\tilde{r}, \tilde{s})$ transforms lie inside $\ell_p(\tilde{w})$, where $\tilde{r} = (r_n)$, $\tilde{s} = (s_n)$ are convergent sequences of positive real numbers. When comparing with the corresponding results in the literature, it can be seen that the results of the present study are more general and comprehensive.

1. Introduction

Let us outline some fundamental definitions and results, which we will largely be used in the following sections. Primarily, we will offer the concept of the sequence, the details of which are well known in elementary analysis. Although there are many different ways to describe the sequence, all of which mean the same thing, we have chosen to give the following definition here. The sentence "x is a sequence" means $x := \{x_n\} := \{x_0, x_1, \ldots, x_n, \ldots\}$, where each x_n is a complex number. In other words, a sequence is easily introduced as an ordered list of complex numbers. Thus if x is a sequence, then it can be viewed as a mapping of $x : \mathbb{N} := \{1, 2, \ldots\} \to \mathbb{C}$. More generally terms, every sequence x in X is a transformation $x : \mathbb{N} \to X$, where X is a non-empty set. The collection of all real or complex number sequences forms a vector space which we denote by w, under the operations of coordinate-wise addition and the familiar scalar multiplication. The subspaces of ω are significant in such applications because each of them is called a sequence space.

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Given an infinite matrix $A = (a_{nk})$ having complex numbers a_{nk} as entries in which $n, k \in \mathbb{N}$, it can be written for a sequence x, as follows

$$(Ax)_n := \sum a_{nk} x_k; \quad (n \in \mathbb{N}, x \in D_{00}(A)),$$

in which $D_{00}(A)$ describes the defined subspace of ω consisting of $x \in \omega$ for which the summation exists as a finite sum. For a simple notation, the summation ranges without limits from 0 to ∞.

The X_A is known to be the matrix domain of an infinite matrix A for any subspace X of the all real-valued sequence space w is described as

$$X_A := \{x = (x_k) \in \omega : Ax \in X\}$$

which is a sequence space. There are several techniques to create new sequence spaces from old ones like X. One of them is to use an arbitrary matrix domain generated by an infinite matrix A such as X_A. To briefly explain the topic, these sequence spaces, namely X and X_A, may overlap but in any case either of them may contain the other one. The reader can find detailed information in the book "Summability Theory and Its Applications" by Başar [1] and therein.

Recently, we have seen a significant increase in the construction of new sequence spaces using matrix domain in summability areas such as sequence spaces.

Many of the works [2–12] we have studied so far have something in common, they use the matrix domain.

Attempts have been made to find the best upper bound for some well-known matrix operators denoted by T from $\ell_p(w)$ to $F_{w,p}$. In the context of this statement, note that an upper bound for a matrix operator denoted by T defined from one sequence space X into another denoted by Y can be given by the following value of U

$$\|Tx\|_Y \leq U\|x\|_X,$$

in which $\|.\|_X$ and $\|.\|_Y$ denote the commonly known norms prescribed for spaces X and Y, respectively. Here, U does not dependent on x. Among them, the best value of U can be called the operator norm for T.

In addition, several researchers have tried to figure out the lower bounds for these matrix operators. This concept was first discussed in Ref [13] on the Cesàro matrix. But after that, others such as in Refs [14, 15] and [16, 17] have studied the lower bounds for some matrix operators defined on the sequence space denoted by ℓ_p and simultaneously on the weighted sequence space denoted by $\ell_p(w)$ with the Lorentz sequence space. Similarly, a lower bound of a matrix operator defined as $T : X \to Y$ is defined as the value of L satisfying the following inequality

$$\|Tx\|_Y \geq L\|x\|_X.$$

This inequality can also be used for some applications of functional analysis. For example, for finding the necessary and sufficient conditions under which an operator has its inverse, and for simultaneously finding the operator kernel containing only the zero vector for this case. For these reasons, knowing the lower bound for an operator is significant. In recent years, Dehghan and Talebi [18] have worked on the largest possible lower bound for some matrices on the Fibonacci sequence spaces. Furthermore, Foroutannia and Roopaei [19] have considered the problem of computing both the norm and lower and upper bounds for some operators defined on weighted difference sequence spaces. One can refer to these works [20–26] and those contained therein for related problems over some classical sequence spaces.

In this article, it is assumed that $w = (w_n)$ and also $\tilde{w} = (\tilde{w}_n)$ are sequences consisting of positive real terms. In this paper, a new space the generalized weighted difference sequence space, is introduced via the generalized difference matrix. Moreover, some properties of this sequence space are investigated. Among other things, it was found that although this space is semi-normed, it is not necessarily a normed space. Recall that a semi-normed satisfies every axiom of a norm, but the semi-norm of a vector must be zero without including the zero vector. Again, this is a semi-inner product space for the value of $p = 2$. Moreover, one obtains an isomorphism when using this space. Next, the norm for some matrix operators on the generalized weighted difference sequence space is defined. In the next step, we address the lower bound problem for the described operators of $\ell_p(w)$ in the generalized weighted difference sequence space.

2. The Sequence Space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$

We examined in the former chapter that many topic lead to building new sequence spaces. Moreover, the concepts we offered were inherently large. Let us start by presenting the following matrix $\tilde{B} = (\tilde{b}_{nk}(\tilde{r}, \tilde{s}))$;

$$\tilde{b}_{nk}(\tilde{r}, \tilde{s}) = \begin{cases} s_n, & k = n+1 \\ r_n, & k = n \\ 0, & 0 \leq k < n \text{ or } k > n+1 \end{cases}$$

where $\tilde{r} = (r_n)$, $\tilde{s} = (s_n)$ are convergent sequences of positive real numbers. It should be noted at this point that many authors have described various sequence spaces and studied many different aspects of these spaces, using a different matrix similar to this matrix but actually different. Some of them are available in references [2–5].

We will see later that this matrix allows us to construct an efficient structure for solving algebraic and topological properties. Applying the definition of matrix domain to this matrix, we define the new sequence space whose result lies in the $\ell_p(\tilde{w})$ space, as follows:

$$\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s})) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \tilde{w}_n \left| r_n x_n + s_n x_{n+1} \right|^p < \infty \right\},$$

in which $1 \leq p < \infty$. For detailed information, the reader is advised to look at the references and therein [27, 28]. We note here that, the space is a semi-normed space with the semi-norm defined by

$$\|x\|_{p,\tilde{w},\tilde{B}} = \left(\sum_{n=1}^{\infty} \tilde{w}_n \left| r_n x_n + s_n x_{n+1} \right|^p \right)^{1/p}.$$

To calculate the truth of this assertion, we now give an example. If we consider the sequence $x_n = \frac{1}{r_n} \prod_{i=1}^{n-1} \left(\frac{-r_{i+1}}{s_i} \right)$, so due to $r_n x_n + s_n x_{n+1} = 0$ we obtain $\|x\|_{p,\tilde{w},\tilde{B}} = 0$, then it follows, from the definition of the norm, that $\|.\|_{p,\tilde{w},\tilde{B}}$ defined on $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ is not a norm.

Before we begin with the general theory, we will first state the following basic theorem, which indicate that the set just described plays a significant role in its algebraic structure.

Theorem 2.1. *The set $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ is linear space, that is, sequence space.*

Proof. We omit the proof which can be found in standard procedure. □

Let us proceed with the following theorem about an algebraic property of this newly defined sequence space.

Theorem 2.2. *It is true that the inclusion relation $\ell_p(\tilde{w}) \subset \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ is strictly valid.*

Proof. If we take any $x \in \ell_p(\tilde{w})$, then the following calculation shows that the inclusion is valid

$$\begin{aligned} \tilde{w}_n \left| r_n x_n + s_n x_{n+1} \right|^p &\leq \tilde{w}_n 2^{p-1} \left(|r_n x_n|^p + |s_n x_{n+1}|^p \right) \\ &\leq 2^{p-1} max \left[|sup_{n\in\mathbb{N}} r_n|^p, |sup_{n\in\mathbb{N}} s_n|^p \right] \tilde{w}_n \left(|x_n|^p + |x_{n+1}|^p \right) \end{aligned}$$

by summing of n from 1 to ∞, in which $1 \leq p < \infty$.

To show that the inclusion relation is strictly valid. If the sequence $\tilde{w}$ with $(1, 1, 1, ...)$, we consider again the sequence $(x_n) = \left(\frac{1}{r_n} \prod_{i=1}^{n-1} \left(\frac{-r_{i+1}}{s_i} \right) \right) \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$. From this it is easy to deduce that $(x_n) \notin \ell_p(\tilde{w})$. □

Theorem 2.3. *If $H = \left\{ x = (x_n) \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s})) : r_n x_n + s_n x_{n+1} = 0 \text{ for all } n \in \mathbb{N} \right\}$, the quotient space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))/H$ is linearly isomorphic to the space $\ell_p(\tilde{w})$.*

Proof. The basic approach to proving this theorem is to define a new T transformation from the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r},\tilde{s}))$ to $\ell_p(\tilde{w})$ that exploits the definition of the fundamental matrix transformation, for all $x \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r},\tilde{s}))$ uniquely $Tx = ((Tx)_n) = (r_n x_n + s_n x_{n+1})$. Since it is fairly obvious that T is linear, the first issue here is to show that T is surjective. One of the ways to accomplish this for any $y = (y_k) \in \ell_p(\tilde{w})$ is to say $x_n = \frac{1}{r_n}\sum_{k=n}^{\infty}\prod_{i=n}^{k-1}\left(\frac{-s_i}{r_{i+1}}\right) y_k$ for all $n \in \mathbb{N}$ in the norm of $\ell_p(\tilde{w}, \tilde{B}(\tilde{r},\tilde{s}))$. In this case, by simple calculations, we obtain the following equations

$$\begin{aligned}
\|x\|^p_{p,\tilde{w},\tilde{B}} &= \sum_{n=1}^{\infty}\tilde{w}_n\left|\frac{r_n}{r_n}\sum_{k=n}^{\infty}\prod_{i=n}^{k-1}\left(\frac{-s_i}{r_{i+1}}\right)y_k + \frac{s_n}{r_{n+1}}\sum_{k=n+1}^{\infty}\prod_{i=n+1}^{k-1}\left(\frac{-s_i}{r_{i+1}}\right)y_k\right|^p \\
&= \sum_{n=1}^{\infty}\tilde{w}_n\left|y_n + \left[\sum_{k=n+1}^{\infty}\prod_{i=n}^{k-1}\left(\frac{-s_i}{r_{i+1}}\right)y_k - \sum_{k=n+1}^{\infty}\prod_{i=n}^{k-1}\left(\frac{-s_i}{r_{i+1}}\right)y_k\right]\right|^p \\
&= \sum_{n=1}^{\infty}\tilde{w}_n\left|y_n\right|^p \\
&= \|y\|^p_{p,\tilde{w}} \\
&< \infty
\end{aligned}$$

which implies that $x = (x_n) \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r},\tilde{s}))$. Returning back to the T transformation described above, it is very simple to say that $Tx = y$. Due to the fact that the image of the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r},\tilde{s}))$ under the transformation T is $\ell_p(\tilde{w})$ and also $\ker T = H$, we have that $\ell_p(\tilde{w}, \tilde{B}(\tilde{r},\tilde{s}))/H$ is linearly isomorphic to the space $\ell_p(\tilde{w})$ under the first isomorphism theorem. □

We will use an example to show that the transformation T defined above is not injective. Namely, for $x = (x_n) = \left(\frac{1}{r_n}\prod_{i=1}^{n-1}\left(\frac{-r_{i+1}}{s_i}\right)\right)$ we get $Tx = 0$; in other words, $\ker T \neq \{0\}$.

Theorem 2.4. *If p is not equal to 2 and at the same time the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r},\tilde{s}))$ is not given as a semi-inner product space, then it is concluded that the space $\ell_2(\tilde{w}, \tilde{B}(\tilde{r},\tilde{s}))$ is defined as a semi-inner product space.*

Proof. First, we will answer the question whether the semi-norm $\|.\|_{2,\tilde{w},\tilde{B}}$ can be induced with a semi-inner product. It is convenient at this point to use the notation $z_k = \tilde{w}_k^{1/2}(r_k x_k + s_k x_{k+1})$ for all $k \in \mathbb{N}$ and $\langle z, z\rangle_2 = \sum_{k=1}^{\infty}|z_k|^2$. Indeed taken arbitrary, $x \in \ell_2(\tilde{w}, \tilde{B}(\tilde{r},\tilde{s}))$, we get

$$\|x\|_{2,\tilde{w},\tilde{B}} = \sqrt{\langle z, z\rangle_2}.$$

Moreover, it is easy to verify from the following equations that the semi-norm $\|.\|_{p,\tilde{w},\tilde{B}}$ cannot be obtained when considering a semi-inner product just defined as

$$\begin{aligned}
\|x+y\|^2_{p,\tilde{w},\tilde{B}} + \|x-y\|^2_{p,\tilde{w},\tilde{B}} &= 4\tilde{w}_1^{2/p} + \tilde{w}_2^{2/p}\left(\frac{r_2}{r_1}\right)^2 \\
&\neq 4\left(\tilde{w}_1 + \frac{\tilde{w}_2}{2^p}\left|\frac{r_2}{r_1}\right|^p\right)^{2/p} \\
&= 2(\|x\|^2_{p,\tilde{w},\tilde{B}} + \|y\|^2_{p,\tilde{w},\tilde{B}}),
\end{aligned}$$

in which $x = \left(\frac{2r_1+s_1}{2r_1^2}, -\frac{1}{2r_1}, 0, 0, ...\right)$, $y = \left(\frac{2r_1-s_1}{2r_1^2}, \frac{1}{2r_1}, 0, 0, ...\right)$ and $p \neq 2$. □

We examined in the former chapter that many topic lead to building new sequence spaces. Moreover, the concepts we offered were inherently large. Let us start by presenting the following matrix $\tilde{B} = (\tilde{b}_{nk}(\tilde{r},\tilde{s}))$;

$$\tilde{b}_{nk}(\tilde{r},\tilde{s}) = \begin{cases} s_n, & k = n+1 \\ r_n, & k = n \\ 0, & 0 \leq k < n \text{ or } k > n+1 \end{cases}$$

where $\tilde{r} = (r_n)$, $\tilde{s} = (s_n)$ are convergent sequences of positive real numbers. It should be noted at this point that many authors have described various sequence spaces and studied many different aspects of these spaces, using a different matrix similar to this matrix but actually different. Some of them are available in references [2–5].

We will see later that this matrix allows us to construct an efficient structure for solving algebraic and topological properties. Applying the definition of matrix domain to this matrix, we define the new sequence space whose result lies in the $\ell_p(\tilde{w})$ space, as follows:

$$\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s})) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \tilde{w}_n \left| r_n x_n + s_n x_{n+1} \right|^p < \infty \right\},$$

in which $1 \le p < \infty$. For detailed information, the reader is advised to look at the references and therein [27, 28]. We note here that, the space is a semi-normed space with the semi-norm defined by

$$\|x\|_{p,\tilde{w},\tilde{B}} = \left(\sum_{n=1}^{\infty} \tilde{w}_n \left| r_n x_n + s_n x_{n+1} \right|^p \right)^{1/p}.$$

To calculate the truth of this assertion, we now give an example. If we consider the sequence $x_n = \frac{1}{r_n} \prod_{i=1}^{n-1} \left(\frac{-r_{i+1}}{s_i} \right)$, so due to $r_n x_n + s_n x_{n+1} = 0$ we obtain $\|x\|_{p,\tilde{w},\tilde{B}} = 0$, then it follows, from the definition of the norm, that $\|.\|_{p,\tilde{w},\tilde{B}}$ defined on $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ is not a norm.

Before we begin with the general theory, we will first state the following basic theorem, which indicate that the set just described plays a significant role in its algebraic structure.

Theorem 2.5. *The set $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ is linear space, that is, sequence space.*

Proof. We omit the proof which can be found in standard procedure. □

Let us proceed with the following theorem about an algebraic property of this newly defined sequence space.

Theorem 2.6. *It is true that the inclusion relation $\ell_p(\tilde{w}) \subset \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ is strictly valid.*

Proof. If we take any $x \in \ell_p(\tilde{w})$, then the following calculation shows that the inclusion is valid

$$\begin{aligned} \tilde{w}_n \left| r_n x_n + s_n x_{n+1} \right|^p &\le \tilde{w}_n 2^{p-1} \left(|r_n x_n|^p + |s_n x_{n+1}|^p \right) \\ &\le 2^{p-1} max \left[|sup_{n\in\mathbb{N}} r_n|^p, |sup_{n\in\mathbb{N}} s_n|^p \right] \tilde{w}_n \left(|x_n|^p + |x_{n+1}|^p \right) \end{aligned}$$

by summing of n from 1 to ∞, in which $1 \le p < \infty$.

To show that the inclusion relation is strictly valid. If the sequence $\tilde{w}$ with $(1, 1, 1, ...)$, we consider again the sequence $(x_n) = \left(\frac{1}{r_n} \prod_{i=1}^{n-1} \left(\frac{-r_{i+1}}{s_i} \right) \right) \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$. From this it is easy to deduce that $(x_n) \notin \ell_p(\tilde{w})$. □

Theorem 2.7. *If $H = \left\{ x = (x_n) \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s})) : r_n x_n + s_n x_{n+1} = 0 \text{ for all } n \in \mathbb{N} \right\}$, the quotient space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))/H$ is linearly isomorphic to the space $\ell_p(\tilde{w})$.*

Proof. The basic approach to proving this theorem is to define a new T transformation from the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ to $\ell_p(\tilde{w})$ that exploits the definition of the fundamental matrix transformation, for all $x \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ uniquely $Tx = ((Tx)_n) = (r_n x_n + s_n x_{n+1})$. Since it is fairly obvious that T is linear, the first issue here is to show that T is surjective. One of the ways to accomplish this for any $y = (y_k) \in \ell_p(\tilde{w})$ is to say

$x_n = \frac{1}{r_n}\sum_{k=n}^{\infty}\prod_{i=n}^{k-1}\left(\frac{-s_i}{r_{i+1}}\right)y_k$ for all $n \in \mathbb{N}$ in the norm of $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$. In this case, by simple calculations, we obtain the following equations

$$\begin{aligned}
\|x\|^p_{p,\tilde{w},\tilde{B}} &= \sum_{n=1}^{\infty}\tilde{w}_n\left|\frac{r_n}{r_n}\sum_{k=n}^{\infty}\prod_{i=n}^{k-1}\left(\frac{-s_i}{r_{i+1}}\right)y_k + \frac{s_n}{r_{n+1}}\sum_{k=n+1}^{\infty}\prod_{i=n+1}^{k-1}\left(\frac{-s_i}{r_{i+1}}\right)y_k\right|^p \\
&= \sum_{n=1}^{\infty}\tilde{w}_n\left|y_n + \left[\sum_{k=n+1}^{\infty}\prod_{i=n}^{k-1}\left(\frac{-s_i}{r_{i+1}}\right)y_k - \sum_{k=n+1}^{\infty}\prod_{i=n}^{k-1}\left(\frac{-s_i}{r_{i+1}}\right)y_k\right]\right|^p \\
&= \sum_{n=1}^{\infty}\tilde{w}_n\left|y_n\right|^p \\
&= \|y\|^p_{p,\tilde{w}} \\
&< \infty
\end{aligned}$$

which implies that $x = (x_n) \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$. Returning back to the T transformation described above, it is very simple to say that $Tx = y$. Due to the fact that the image of the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ under the transformation T is $\ell_p(\tilde{w})$ and also $\ker T = H$, we have that $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))/H$ is linearly isomorphic to the space $\ell_p(\tilde{w})$ under the first isomorphism theorem. □

We will use an example to show that the transformation T defined above is not injective. Namely, for $x = (x_n) = \left(\frac{1}{r_n}\prod_{i=1}^{n-1}\left(\frac{-r_{i+1}}{s_i}\right)\right)$ we get $Tx = 0$; in other words, $\ker T \neq \{0\}$.

3. The Norm of Matrix Operators from $\ell_1(w)$ to $\ell_1(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$

Having defined a function from the space $\ell_1(w)$ to the space $\ell_1(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$, we will compute in this chapter that it is a norm. Before proceeding with the development of the general theory, let us start by presenting a very simple definition.

The matrix $A = (a_{nk})$ is said to be quasi-summable if A is an upper triangular matrix, namely, $a_{nk} = 0$ for $n > k$. As it can be clearly seen, the matrix satisfies $\sum_{n=1}^{k} a_{nk} = 1$ for all $k \in \mathbb{N}$.

Theorem 3.1. *The matrix $T = (t_{nk})$ is a bounded matrix operator from the space $\ell_1(w)$ to the space $\ell_1(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ if $M = \sup_{k\in\mathbb{N}}\frac{\lambda_k}{w_k} < \infty$, in which $\lambda_k = \sum_{n=1}^{\infty}\tilde{w}_n\left|r_n t_{nk} + s_n t_{n+1,k}\right|$. In that case, the norm of operator is obtained as $\|T\|_{1,w,\tilde{w},\tilde{B}} = M$.*

For all $n \in \mathbb{N}$, taking both $w_n = 1$ and $\tilde{w}_n = 1$ specially, the transformation T is a bounded operator from the space ℓ_1 to the space $\ell_1(\tilde{B}(\tilde{r}, \tilde{s}))$ and also $\|T\|_{1,\tilde{B}} = \sup_{k\in\mathbb{N}}\lambda_k$.

Proof. We take into consideration a sequence $x = (x_n)$ in $\ell_1(w)$, thus

$$\begin{aligned}
\|Tx\|_{1,\tilde{w},\tilde{B}} &= \sum_{n=1}^{\infty} \tilde{w}_n \left| \sum_{k=1}^{\infty} (r_n t_{nk} + s_n t_{n+1,k})\, x_k \right| \\
&\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \tilde{w}_n \left| r_n t_{nk} + s_n t_{n+1,k} \right| |x_k| \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \tilde{w}_n \left| r_n t_{nk} + s_n t_{n+1,k} \right| |x_k| \\
&= \sum_{k=1}^{\infty} \lambda_k |x_k| \\
&\leq M \sum_{k=1}^{\infty} w_k |x_k| \\
&= M \|x\|_{1,w}.
\end{aligned}$$

From these equations it follows that $\|T\|_{1,w,\tilde{w},\tilde{B}} \leq M$ since $\frac{\|Tx\|_{1,\tilde{w},\tilde{B}}}{\|x\|_{1,w}} \leq M$. We introduce the sequence $e^i = (0, 0, ..., 0, \overset{i.}{1}, 0, ...)$ for each $i \in \mathbb{N}$ to compute the inverse inequality, and then obtain $\|e^i\|_{1,w} = w_i$ and also $\|Te^i\|_{1,\tilde{w},\tilde{B}} = \lambda_i$. Therefore, it is easy to see that $\|T\|_{1,w,\tilde{w},\tilde{B}} \geq M$, and then $\|T\|_{1,w,\tilde{w},\tilde{B}} = M$.

Since special choices are made in the proof of the remaining part, no proof will be given here. □

Theorem 3.2. *Let us assume that $T = (t_{nk})$ is the upper triangular matrix having the non-negative entries and also assume that (w_n) is an increasing given sequence. When the inequality $t_{nk} \geq t_{n+1,k}$ is valid for each values of $n \in \mathbb{N}$, constant $k \in \mathbb{N}$ and $M' = \sup_{k\in\mathbb{N}} \sum_{k=1}^{n} t_{nk} < \infty$, then T is defined as a bounded operator described from $\ell_1(w)$ to $\ell_1(w, \tilde{B}(\tilde{r}, \tilde{s}))$. At the same time, the norm of this given operator satisfies the inequality given in the form $\|T\|_{1,w,\tilde{B}} \leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|)M'$. When the specific condition of T is being quasi summable matrix, also $r_k \geq -s_k > 0$ and $s_{k-1} + r_k = 1$ is taken into consideration, thus the condition $\|T\|_{1,w,\tilde{B}} = 1$ is satisfied.*

Proof. Given the hypothesis, we must say that the matrix $T = (t_{nk})$ satisfying the condition $t_{nk} \geq t_{n+1,k}$ (for all $n, k = 1, 2, ...$) is an upper triangular and also the sequence (w_n) is increasing. With simple calculations, we can derive the following

$$\begin{aligned}
\lambda_k &= \sum_{n=1}^{\infty} w_n \left| r_n t_{nk} + s_n t_{n+1,k} \right| \\
&= \sum_{n=1}^{k-1} w_n \left| r_n t_{nk} + s_n t_{n+1,k} \right| + w_k |r_k| t_{kk} \\
&\leq w_k \left[\sum_{n=1}^{k-1} (|r_n| t_{nk} + |s_n| t_{n+1,k}) + |r_k| t_{kk} \right] \\
&= w_k \left[(|r_1| t_{1k} + |s_1| t_{2k}) + ... + (|r_{k-1}| t_{k-1,k} + |s_{k-1}| t_{kk}) + |r_k| t_{kk} \right] \\
&= w_k \left[|r_1| t_{1k} + (|s_1| + |r_2|)\, t_{2k} + ... + (|s_{k-1}| + |r_k|)\, t_{kk} \right] \\
&\leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|) w_k \sum_{n=1}^{k} t_{nk}.
\end{aligned}$$

Obviously, $\|T\|_{1,w,\tilde{B}} = \sup_{k\in\mathbb{N}} \frac{\lambda_k}{w_k} \leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|) \sup_{k\in\mathbb{N}} \sum_{n=1}^{k} t_{nk} = (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|)M'$ from Theorem 3.1.

Let us suppose that T is a quasi summable matrix, so $M' = 1$. If $r_k \geq -s_k > 0$ holds, then of course $r_n t_{nk} + s_n t_{n+1,k} > 0$ holds for every $k, n \in \mathbb{N}$ and also if the equality $s_{k-1} + r_k = 1$ is satisfied, then we can easily write $\lambda_k \leq w_k \sum_{n=1}^{k} t_{nk}$ thus $\|T\|_{1,w,\tilde{B}} \leq 1$. To obtain the inverse inequality, let us consider the sequence $e^1 = (1, 0, 0, ...)$. It follows that $\|e^1\|_{1,w} = w_1$ and $\|Te^1\|_{1,w,\tilde{B}} = w_1$, namely $\|T\|_{1,w,\tilde{B}} \geq 1$. As a result, we obtain $\|T\|_{1,w,\tilde{B}} = 1$. $\square$

In the light of the above theorems, we are concerned here with the computation of the norm of some specific quasi summable matrices. First, we consider the transpose of the well-known Riesz matrix $\tilde{R} = (\tilde{r}_{nk})$ which is described as follows:

$$\tilde{r}_{nk} = \begin{cases} \frac{q_n}{Q_k}, & n \leq k \\ 0, & n > k, \end{cases} \tag{1}$$

where (q_n) is a non-negative sequence with $q_1 > 0$ and $Q_k = q_1 + ... + q_k$ for all $k \in \mathbb{N}$.

Taking $q_n = 1$ for all $n \in \mathbb{N}$, we derive the transpose of the Cesáro matrix of order one, also known as the Copson matrix (see [17]). We denote this particular matrix by $\tilde{C} = (\tilde{c}_{nk})$, where

$$\tilde{c}_{nk} = \begin{cases} \frac{1}{k}, & n \leq k \\ 0, & n > k. \end{cases}$$

Corollary 3.3. *When (q_n) is a decreasing sequence and (w_n) is an increasing sequence, in that case $\tilde{R}$ is a bounded operator from the space $\ell_1(w)$ into the space $\ell_1(w, \tilde{B}(\tilde{r}, \tilde{s}))$ and, also $\|\tilde{R}\|_{1,w,\tilde{B}} = 1$ for $r_n \geq -s_n > 0$ and $s_{n-1} + r_n = 1$ for every $n \in \mathbb{N}$.*

Proof. First of all, since (q_n) is a decreasing sequence from the hypothesis the following inequality $\tilde{r}_{nk} = \frac{q_n}{Q_k} \geq \frac{q_{n+1}}{Q_k} = \tilde{r}_{n+1,k}$ holds for all $n \in \mathbb{N}$, each fixed $k \in \mathbb{N}$. For $\tilde{R}$ is a non-negative upper triangular matrix and (w_n) is an increasing sequence, it follows from Theorem 3.2 that $\tilde{R}$ is a bounded operator from $\ell_1(w)$ into $\ell_1(w, \tilde{B}(\tilde{r}, \tilde{s}))$. Also due to the fact that $\sum_{n=1}^{k} \tilde{r}_{nk} = 1$ for every $k \in \mathbb{N}$, $\tilde{R}$ is a quasi summable matrix. If $r_n \geq -s_n > 0$ and $s_{n-1} + r_n = 1$ for every $n \in \mathbb{N}$, then it is clear that $\|\tilde{R}\|_{1,w,\tilde{B}} = 1$ from Theorem 3.2. $\square$

Corollary 3.4. *If $\sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k} < \infty$, then the matrix $\tilde{C}$ defined just above is a bounded operator from the space $\ell_1(w)$ into $\ell_1(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ and $\|\tilde{C}\|_{1,w,\tilde{w},\tilde{B}} \leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|) \sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k}$.*

Proof. We get the following inequality

$$\begin{aligned} \lambda_k &= \sum_{n=1}^{\infty} \tilde{w}_n \left| r_n \tilde{c}_{nk} + s_n \tilde{c}_{n+1,k} \right| \\ &\leq \frac{1}{k} \left[\sum_{n=1}^{k-1} \tilde{w}_n \left(|r_n| + |s_n| \right) + \tilde{w}_k |r_k| \right] \\ &= \frac{\sup_{k\in\mathbb{N}} |r_k|}{k} \sum_{n=1}^{k} \tilde{w}_n + \frac{\sup_{k\in\mathbb{N}} |s_k|}{k} \sum_{n=1}^{k-1} \tilde{w}_n \\ &\leq \frac{\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|}{k} \sum_{n=1}^{k} \tilde{w}_n. \end{aligned}$$

Therefore, we obtain that $\|\tilde{C}\|_{1,w,\tilde{w},B} \leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|) \sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k}$ from Theorem 3.1. $\square$

Theorem 3.5. *Let us suppose that $T = (t_{nk})$ is a matrix having the non-negative entries and the inequalities $t_{nk} \geq t_{n+1,k}$ hold for all $n \in \mathbb{N}$ and each fixed $k \in \mathbb{N}$. If $\sum_{n=1}^{\infty} t_{nk} < \infty$ for each $k \in \mathbb{N}$ and also $M'' = \sup_{k\in\mathbb{N}} \sum_{n=1}^{\infty} t_{nk} < \infty$, then the matrix T is a bounded operator from the space ℓ_1 to $\ell_1(\tilde{B}(\tilde{r}, \tilde{s}))$ and the norm of operator is $\|T\|_{1,\tilde{B}} \leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|) M''$. When the fact that the specific condition of T is being quasi summable matrix is taken into consideration for $r_k \geq -s_k > 0$ and $s_{k-1} + r_k = 1$ (for all $k \in \mathbb{N}$), then the condition $\|T\|_{1,\tilde{B}} = 1$ is derived.*

Proof. For any $k \in \mathbb{N}$, we get

$$\lambda_k = \sum_{n=1}^{\infty} \left| r_n t_{nk} + s_n t_{n+1,k} \right| = (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|) \sum_{n=1}^{\infty} t_{nk}.$$

Using Theorem 3.1 here, we find that the norm $\|T\|_{1,\tilde{B}} \leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|)M''$. The rest of the proof can be done similarly to the proof of Theorem 3.2. □

The matrix $H = (h_{nk})$ defined as $h_{nk} = \frac{1}{n+k}$ for all $n, k \in \mathbb{N}$ is known to be the Hilbert matrix operator. Here, we will discover the norm of the operator just mentioned.

Now, let us give the following integral to be used in the proofs:

$$\int_0^{\infty} \frac{1}{t^{\alpha}(t+c)} dt = \frac{\pi}{c^{\alpha} \sin \alpha\pi},$$

in which $0 < \alpha < 1$.

Theorem 3.6. *Let $w_n = \frac{1}{n^{\alpha}}$ for all $n \in \mathbb{N}$, in which $0 < \alpha < 1$. In this case, the Hilbert matrix operator H just described is bounded from the space $\ell_1(w)$ to the space $\ell_1(w, \tilde{B}(\tilde{r}, \tilde{s}))$ and also the norm $\|H\|_{1,w,\tilde{B}} \leq \frac{\pi}{\sin \alpha\pi}(\sup_{i\in\mathbb{N}} |r_i| + \sup_{i\in\mathbb{N}} |s_i|)$.*

Proof. For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\lambda_n &= \sum_{i=1}^{\infty} w_i \left| r_i h_{in} + s_i h_{i+1,n} \right| \\
&\leq \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} \left(\frac{|r_i|}{i+n} + \frac{|s_i|}{i+n+1} \right) \\
&\leq \int_0^{\infty} \frac{1}{t^{\alpha}} \left(\frac{\sup_{i\in\mathbb{N}} |r_i|}{t+n} + \frac{\sup_{i\in\mathbb{N}} |s_i|}{t+n+1} \right) dt \\
&= \frac{\pi}{\sin \alpha\pi} \left(\frac{\sup_{i\in\mathbb{N}} |r_i|}{n^{\alpha}} + \frac{\sup_{i\in\mathbb{N}} |s_i|}{(n+1)^{\alpha}} \right).
\end{aligned}$$

It follows that

$$n^{\alpha} \lambda_n \leq \frac{\pi}{\sin \alpha\pi} \left[\sup_{i\in\mathbb{N}} |r_i| + \sup_{i\in\mathbb{N}} |s_i| \left(\frac{n}{n+1} \right)^{\alpha} \right] \leq \frac{\pi}{\sin \alpha\pi} \left(\sup_{i\in\mathbb{N}} |r_i| + \sup_{i\in\mathbb{N}} |s_i| \right).$$

Considering Theorem 3.1, this means that $\|H\|_{1,w,\tilde{B}} \leq \frac{\pi}{\sin \alpha\pi} \left(\sup_{i\in\mathbb{N}} |r_i| + \sup_{i\in\mathbb{N}} |s_i| \right)$. □

4. The Norm of Matrix Operators from $\ell_p(w)$ to $\ell_p(w, \tilde{B}(\tilde{r}, \tilde{s}))$

In this section, we are going to discuss calculating the norm of some matrix operators from the space $\ell_p(w)$ to the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$. We now present an essential lemma which is obtained by Jameson and Lashkaripour, since this important result is used in the proofs.

Lemma 4.1. *[17] Let us suppose that $A = (a_{nk})$ is a matrix operator having the nonnegative entries $a_{nk} \geq 0$, also suppose that (u_n) and (v_k) are positive sequences given such that*

$$u_n^{1/p} \sum_{k=1}^{\infty} \frac{a_{nk}}{v_k^{1/p}} \leq K_1 \qquad (\textit{for } n \in \mathbb{N},\ K_1 \in \mathbb{R})$$

and

$$\frac{1}{v_k^{(1-p)/p}} \sum_{n=1}^{\infty} u_n^{(1-p)/p} a_{nk} \leq K_2 \qquad (\textit{for } k \in \mathbb{N},\ K_2 \in \mathbb{R})$$

in that case, that inequality $\|A\|_p \leq \frac{K_2^{1/p}}{K_1^{(1-p)/p}}$ is valid, in which $p > 1$.

Now, let us state and prove another necessary lemma.

Lemma 4.2. *Let us assume that the equality* $a_{nk} = \left(\frac{\tilde{w}_n}{w_k}\right)^{1/p}(r_n t_{nk} + s_n t_{n+1,k})$ *is valid for the matrix operators* $T = (t_{nk})$ *and* $A = (a_{nk})$*. At the same time, we have* $\|T\|_{p,w,\tilde{w},\tilde{B}} = \|A\|_p$*, for* $p \geq 1$*. Under the conditions of this hypothesis, T is bounded operator from the space* $\ell_p(w)$ *to the space* $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ *iff A is bounded operator onto the space* ℓ_p*.*

Proof. If the x lying in the space $\ell_p(w)$ is taken as arbitrarily, and the sequence $y = (y_k)$ is defined as $y_k = w_k^{1/p} x_k$ for all $k \in \mathbb{N}$ by making use of it, then we derive that equality $\|x\|_{p,w} = \|y\|_p$. Therefore, the proof should be clear with the following basic calculations

$$\begin{aligned}
\|T\|^p_{p,w,\tilde{w},\tilde{B}} &= \sup_{x\in\ell_p(w),x\neq 0} \frac{\|Tx\|^p_{p,\tilde{w},\tilde{B}}}{\|x\|^p_{p,w}} \\
&= \sup_{x\in\ell_p(w),x\neq 0} \frac{\sum_{n=1}^{\infty} \tilde{w}_n \left|\sum_{k=1}^{\infty} (r_n t_{nk} + s_n t_{n+1,k}) x_k\right|^p}{\sum_{k=1}^{\infty} w_k |x_k|^p} \\
&= \sup_{y\in\ell_p} \frac{\sum_{n=1}^{\infty} \left|\sum_{k=1}^{\infty} \left(\frac{\tilde{w}_n}{w_k}\right)^{1/p} (r_n t_{nk} + s_n t_{n+1,k}) y_k\right|^p}{\sum_{k=1}^{\infty} |y_k|^p} \\
&= \sup_{y\in\ell_p} \frac{\sum_{n=1}^{\infty} \left|\sum_{k=1}^{\infty} a_{nk} y_k\right|^p}{\sum_{k=1}^{\infty} |y_k|^p} = \sup_{y\in\ell_p} \frac{\|Ay\|^p_p}{\|y\|^p_p} = \|A\|^p_p.
\end{aligned}$$

□

Theorem 4.3. *Let us assume that the matrix operator* $\tilde{R}$ *is as defined in* (1)*, and also assume that* (q_n) *is a decreasing sequence having* $q_1 = q_2 = 2$ *and* $\lim_{n\to\infty} Q_n = \infty$*. For all* $n \in \mathbb{N}$*, if the sequence* (w_n) *is taken as* $\left(\frac{2Q_{n-1}}{q_n}\right)^p$ *with* $Q_0 = 1$*, in that case,* $\tilde{R}$ *is bounded operator from the space* $\ell_p(w)$ *to the space* $\ell_p(\tilde{B}(\tilde{r}, \tilde{s}))$ *and* $\|\tilde{R}\|_{p,w,\tilde{B}} \leq \frac{\sup_{n\in\mathbb{N}} |r_n| + \sup_{n\in\mathbb{N}} |s_n|}{2}$ *for* $p > 1$*.*

Proof. In Lemma 4.2, utilize the matrix $\tilde{R}$ in place of T. So, the matrix $A = (a_{nk})$ is described by

$$a_{nk} = \begin{cases} \frac{q_k}{2Q_{k-1}Q_k}(r_n q_n + s_n q_{n+1}), & n < k \\ \frac{1}{2} r_k \frac{q_k^2}{Q_{k-1}Q_k}, & n = k \\ 0, & n > k \end{cases}$$

and besides that, $\|\tilde{R}\|_{p,w,\tilde{B}} = \|A\|_p$ is obtained.

We derive

$$\begin{aligned}
\sum_{k=1}^{\infty} a_{nk} &= \frac{r_n}{2} q_n \frac{q_n}{Q_{n-1}Q_n} + \frac{1}{2}(r_n q_n + s_n q_{n+1}) \sum_{k=n+1}^{\infty} \frac{q_k}{Q_{k-1}Q_k} \\
&= \frac{r_n}{2} q_n \left(\frac{1}{Q_{n-1}} - \frac{1}{Q_n}\right) + \frac{1}{2}(r_n q_n + s_n q_{n+1}) \frac{1}{Q_n} \\
&= \frac{r_n}{2} \frac{q_n}{Q_{n-1}} + \frac{s_n}{2} \frac{q_{n+1}}{Q_n} \\
&\leq \frac{\sup_{n\in\mathbb{N}} |r_n| + \sup_{n\in\mathbb{N}} |s_n|}{2}
\end{aligned}$$

for all $n \in \mathbb{N}$. Also, we derive

$$\begin{aligned}\sum_{n=1}^{\infty} a_{nk} &= \frac{1}{2}\frac{q_k}{Q_{k-1}Q_k}\left[\sum_{n=1}^{k-1}(r_nq_n + s_nq_{n+1})\right] + \frac{r_k}{2}\frac{q_k}{Q_{k-1}Q_k}q_k \\ &= \frac{1}{2}\frac{q_k}{Q_{k-1}Q_k}\left[r_1q_1 + \sum_{n=1}^{k-1}(r_{n+1}+s_n)\,q_{n+1}\right] \\ &\leq \frac{\sup_{k\in\mathbb{N}}|r_k| + \sup_{k\in\mathbb{N}}|s_k|}{2}\frac{q_k}{Q_{k-1}Q_k}\sum_{n=1}^{k} q_n \\ &\leq \frac{\sup_{k\in\mathbb{N}}|r_k| + \sup_{k\in\mathbb{N}}|s_k|}{2}\end{aligned}$$

for all $k \in \mathbb{N}$. Now, In Lemma 4.1, if we take $u_n = v_n = 1$ for all $n \in \mathbb{N}$, we get $K_1 \leq \frac{\sup_{n\in\mathbb{N}}|r_n| + \sup_{n\in\mathbb{N}}|s_n|}{2}$ and $K_2 \leq \frac{\sup_{k\in\mathbb{N}}|r_k| + \sup_{k\in\mathbb{N}}|s_k|}{2}$ which require that $\|\tilde{R}\|_{p,w,\tilde{B}} \leq \frac{\sup_{n\in\mathbb{N}}|r_n| + \sup_{n\in\mathbb{N}}|s_n|}{2}$ for $p > 1$. □

Theorem 4.4. *Let $w_n = \frac{1}{n^\alpha}$ for all $n \in \mathbb{N}$, in which $1-p < \alpha < 1$ and $p > 1$. In that case, the Hilbert matrix operator H is a bounded operator from the space $\ell_p(w)$ to the space $\ell_p(w, \tilde{B}(\tilde{r}, \tilde{s}))$ also following inequality*

$$\|H\|_{p,w,\tilde{B}} \leq \left(\sup_{n\in\mathbb{N}}|r_n| + \sup_{n\in\mathbb{N}}|s_n|\right)\max\left\{\frac{\pi}{\sin\beta\pi}, \frac{\pi}{\sin\gamma\pi}\right\},$$

is valid, in which $\beta = \frac{1-\alpha}{p}$ and $\gamma = \frac{p-1+\alpha}{p}$.

Proof. Let us define the matrix $A = (a_{nk})$ as follows

$$a_{nk} = \left(\frac{k}{n}\right)^{\alpha/p}\left(\frac{r_n}{n+k} + \frac{s_n}{n+k+1}\right)$$

for all $n, k \in \mathbb{N}$. In this case, $\|H\|_{p,w,\tilde{B}} = \|A\|_p$ which obtained by using Lemma 4.2. Specifically, when we choose $u_n = v_n = n$ in Lemma 4.1 for all $n \in \mathbb{N}$, we find that

$$\begin{aligned}u_n^{\frac{1}{p}}\sum_{k=1}^{\infty}\frac{a_{nk}}{v_k^{\frac{1}{p}}} &= n^{1/p}\sum_{k=1}^{\infty}\frac{1}{k^{1/p}}\left(\frac{k}{n}\right)^{\alpha/p}\left(\frac{r_n}{n+k} + \frac{s_n}{n+k+1}\right) \\ &\leq n^{\beta}\sum_{k=1}^{\infty}\frac{1}{k^{\beta}}\left(\frac{|r_n|}{n+k} + \frac{|s_n|}{n+k+1}\right) \\ &\leq n^{\beta}\int_{t=0}^{\infty}\frac{1}{t^{\beta}}\left(\frac{\sup_{n\in\mathbb{N}}|r_n|}{t+n} + \frac{\sup_{n\in\mathbb{N}}|s_n|}{t+(n+1)}\right)dt \\ &= n^{\beta}\left(\frac{\sup_{n\in\mathbb{N}}|r_n|\pi}{n^{\beta}\sin\beta\pi} + \frac{\sup_{n\in\mathbb{N}}|s_n|\pi}{(n+1)^{\beta}\sin\beta\pi}\right) \\ &\leq \frac{\pi}{\sin\beta\pi}\left(\sup_{n\in\mathbb{N}}|r_n| + \sup_{n\in\mathbb{N}}|s_n|\right)\end{aligned}$$

for all $n \in \mathbb{N}$ also

$$
\begin{aligned}
\frac{1}{v_k^{\frac{1-p}{p}}} \sum_{n=1}^{\infty} u_n^{\frac{1-p}{p}} a_{nk} &= \frac{1}{k^{(1-p)/p}} \sum_{n=1}^{\infty} n^{(1-p)/p} \left(\frac{k}{n}\right)^{\alpha/p} \left(\frac{r_n}{n+k} + \frac{s_n}{n+k+1}\right) \\
&\leq k^{\gamma} \sum_{n=1}^{\infty} \frac{1}{n^{\gamma}} \left(\frac{|r_n|}{n+k} + \frac{|s_n|}{n+k+1}\right) \\
&\leq k^{\gamma} \int_{t=0}^{\infty} \frac{1}{t^{\gamma}} \left(\frac{\sup_{n\in\mathbb{N}} |r_n|}{t+k} + \frac{\sup_{n\in\mathbb{N}} |s_n|}{t+(k+1)}\right) dt \\
&= k^{\gamma} \left(\frac{\sup_{n\in\mathbb{N}} |r_n| \pi}{k^{\gamma} \sin \gamma\pi} + \frac{\sup_{n\in\mathbb{N}} |s_n| \pi}{(k+1)^{\gamma} \sin \gamma\pi}\right) \\
&\leq \frac{\pi}{\sin \gamma\pi} \left(\sup_{n\in\mathbb{N}} |r_n| + \sup_{n\in\mathbb{N}} |s_n|\right)
\end{aligned}
$$

for all $k \in \mathbb{N}$, where $\beta = \frac{1-\alpha}{p}$ and $\gamma = \frac{p-1+\alpha}{p}$. We therefore obtain that

$$
\|H\|_{p,w,\tilde{B}} \leq \left(\sup_{n\in\mathbb{N}} |r_n| + \sup_{n\in\mathbb{N}} |s_n|\right) \max\left\{\frac{\pi}{\sin \beta\pi}, \frac{\pi}{\sin \gamma\pi}\right\}
$$

from Lemma 4.1. □

5. Lower Bounds of Matrix Operators from $\ell_p(w)$ to $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$

An important problem posed in this paper is to calculate the lower bound of an operator T from the space $\ell_p(w)$ to space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$. Thus, the goal is to obtain the lower bound of the operator T for the largest value L satisfying the following inequality

$$
\|Tx\|_{p,\tilde{w},\tilde{B}} \geq L \|x\|_{p,w}
$$

for every decreasing sequence $x = (x_k)$ with $x_k \geq 0$.

We need the following lemma to perform the calculations in the proofs in this section.

Lemma 5.1. *[17] Let us assume that both (q_n) and (x_n) are non-negative sequences, and that (x_n) is also a decreasing sequence satisfying condition $\lim_{n\to\infty} x_n = 0$. For $Q_n = \sum_{i=1}^{n} q_i$ with $Q_0 = 1$ also $R_n = \sum_{i=1}^{n} q_i x_i$, the following statements holds, in which $p \geq 1$ and $n \in \mathbb{N}$.*

1. $R_n^p - R_{n-1}^p \geq (Q_n^p - Q_{n-1}^p) x_n^p$.
2. *When the series $\sum_{i=1}^{\infty} q_i x_i$ converges, the following inequality is satisfied.*

$$
\left(\sum_{i=1}^{\infty} q_i x_i\right)^p \geq \sum_{n=1}^{\infty} Q_n^p (x_n^p - x_{n+1}^p).
$$

Theorem 5.2. *When $T = (t_{nk})$ is a matrix operator with $t_{nk} \geq 0$ from the space $\ell_p(w)$ into the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$, in which $p \geq 1$, the following inequality $t_{nk} \geq t_{n+1,k}$ is valid for all $n \in \mathbb{N}$, each fixed $k \in \mathbb{N}$ also the series $\sum_{n=1}^{\infty} w_n$ diverges to infinity, in that case, for every decreasing sequence $x = (x_k)$ having $x_k \geq 0$, we have*

$$
\|Tx\|_{p,\tilde{w},\tilde{B}} \geq L \|x\|_{p,w}
$$

in which $L^p = \inf_{n\in\mathbb{N}} \frac{S_n}{W_n}$, $W_n = \sum_{k=1}^{n} w_k$ and $S_n = \sum_{i=1}^{\infty} \tilde{w}_i \left(\sum_{k=1}^{n} (r_i t_{ik} + s_i t_{i+1,k})\right)^p$ where $r_n \geq -s_n > 0$ for all $n \in \mathbb{N}$.

Proof. Under the conditions of the hypothesis formulated in the theorem, we can give the proof as follows. Since $\sum_{n=1}^{\infty} w_n = \infty$, we obtain $\lim_{k\to\infty} x_k = 0$, and also, we can be establish that the series $\sum_{k=1}^{\infty} (r_n t_{nk} + s_n t_{n+1,k}) x_k$ is convergent for all $n \in \mathbb{N}$. On the other hands, using Lemma 5.1 and Abel summation, we have

$$\begin{aligned}
\|Tx\|_{p,\tilde{w},B}^p &= \sum_{n=1}^{\infty} \tilde{w}_n \left(\sum_{k=1}^{\infty} (r_n t_{nk} + s_n t_{n+1,k}) x_k \right)^p \\
&\geq \sum_{n=1}^{\infty} \tilde{w}_n \sum_{i=1}^{\infty} \left(\sum_{k=1}^{i} (r_n t_{nk} + s_n t_{n+1,k}) \right)^p (x_i^p - x_{i+1}^p) \\
&= \sum_{i=1}^{\infty} \left[\sum_{n=1}^{\infty} \tilde{w}_n \left(\sum_{k=1}^{i} (r_n t_{nk} + s_n t_{n+1,k}) \right)^p \right] (x_i^p - x_{i+1}^p) \\
&= \sum_{i=1}^{\infty} S_i (x_i^p - x_{i+1}^p) \geq L^p \sum_{i=1}^{\infty} W_i (x_i^p - x_{i+1}^p) = L^p \|x\|_{p,w}^p
\end{aligned}$$

which completes the proof. □

The following lemma can be verified using a technique similar to the proof of Proposition 1 in [17].

Lemma 5.3. *Let us assume that $T = (t_{nk})$ be a non-negative matrix operator defined from the space $\ell_p(w)$ to the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$, in which $p \geq 1$. If the following inequality*

$$r_n t_{nk} + s_n t_{n+1,k} \geq r_n t_{n,k+1} + s_n t_{n+1,k+1}$$

is valid also $t_{nk} \geq t_{n+1,k}$ for all $k \in \mathbb{N}$, each fixed $n \in \mathbb{N}$ and $r_n \geq -s_n > 0$, if the series $\sum_{n=1}^{\infty} w_n$ is divergent the infinity, then we have

$$L^p \geq \inf_{n\in\mathbb{N}} [n^p - (n-1)^p] \frac{t_n}{w_n},$$

in which $t_n = \sum_{i=1}^{\infty} \tilde{w}_i (r_i t_{in} + s_i t_{i+1,n})^p$.

Theorem 5.4. *Let $H = (h_{nk})$ is the Hilbert matrix operator, $w_n = \frac{1}{n^{p+\alpha}}$ and $\tilde{w}_n = \frac{1}{n^{\alpha}}$ for every $n \in \mathbb{N}$, in which $p \geq 1$, $0 \leq p + \alpha \leq 1$ and $r_n \geq -s_n > 0$. For every decreasing sequences $x = (x_k)$ that are not negative terms, we have*

$$\|Hx\|_{p,\tilde{w},\tilde{B}} \geq L \|x\|_{p,w}$$

in which $L^p \geq \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}(i+1)^p(i+2)^p}$.

Proof. It is clear that both the Hilbert matrix $H = (h_{nk})$ and the sequence (w_n) satisfy the conditions listed in Lemma 5.3, therefore, we obtain

$$\begin{aligned}
L^p &\geq \inf_{n\in\mathbb{N}} [n^p - (n-1)^p] \frac{t_n}{w_n} \\
&\geq \inf_{n\in\mathbb{N}} n^{p-1} n^{p+\alpha} \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} \left(\frac{r_i}{i+n} + \frac{s_i}{i+n+1} \right)^p \\
&\geq \inf_{n\in\mathbb{N}} n^{2p+\alpha-1} \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} \left(\frac{r_i}{i+n} + \frac{s_i}{i+n+1} \right)^p .
\end{aligned}$$

The rest of the proof can be obtained in the same way as in the proof of Theorem 4.3 in [19]. □

Conclusion

In this manuscript, we have presented the norms for matrix operators which are defined between the weighted sequence space denoted by $\ell_p(w)$ and the weighted difference sequence space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ which is valid for $1 \leq p < \infty$. To make the presentation more understandable, we have used some specific matrices like quasi summable ones (that is the transposes of Riesz and Cesàro matrices of the first order) and Hilbert matrix. Firstly, $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ space has been presented and its properties have been scrutinized. Next, we have tried to compute the lower bound for the matrix given from $\ell_p(w)$ into $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$.

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