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# RELATIONS AND INTEGRATION OF HERMITE-BASED MILNE-THOMSON AND FUBINI TYPE POLYNOMIALS

**Neslihan KILAR**[1,*]

[1] Department of Computer Technologies, Bor Vocational School, Niğde Ömer Halisdemir University, TR-51700, Niğde, Türkiye

## ABSTRACT

The purpose of this paper is to present a lot of formulas for the $r$-parametric Hermite-based Milne-Thomson type polynomials. By applying functional equation method of generating functions, we also present a lot of relations and integral formulas incorporated the Fubini type polynomials, the $r$-parametric Hermite-based cosine-and sine-Milne-Thomson type Fubini polynomials, Gould-Hopper polynomials, and other special polynomials. Furthermore, we show that the special values of these results reduce to connections to previously known results.

***Keywords:*** *Fubini type numbers and polynomials, Gould-Hopper polynomials, Hermite-based Milne-Thomson type polynomials, Trigonometric functions, Generating functions.*

## 1. INTRODUCTION

Special numbers and polynomials, such as, Fubini numbers, Hermite polynomials, their corresponding generating functions, and trigonometric functions, play important roles in various branches of pure and applied sciences. For instance, the Fubini numbers are used to count combinatorial problems, while the Hermite polynomials are particularly used in combinatorics, probability theory, numerical analysis, computational science, and the seismic waves of earthquake. Consequently, they have numerous applications in many disciplines, such as engineering, mathematics, physics, and other sciences. Moreover, many formulas and identities, involving some special polynomials and their parametric forms, have been examined by many authors (*cf.* [1-20]).

This paper focuses on investigating the $r$-parametric Hermite-based Milne-Thomson type polynomials and the Fubini type polynomials using generating function methods. From these functions and integral equations, we derive some formulas and relations involving these polynomials and the first kind Gould-Hopper polynomials. These type of polynomials have wide applications in variety of areas, especially mathematics and engineering. As a result, formulas of derived from this paper have significant potential for use in many areas such as solving mathematical modeling problems, combinatorial problems, linear differential equations, and etc.

We now begin by the following notations, definitions, and relations in order to use in the following sections.

Let

$$\mathbb{N} = \{1,2,3,\ldots\}, \qquad \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

and the sets of integers: $\mathbb{Z}$, real numbers: $\mathbb{R}$, complex numbers: $\mathbb{C}$, and also

$$e^t = \exp(t).$$

*Corresponding Author: neslihankilar@ohu.edu.tr; Neslihan.kilar@gmail.com
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The Fubini numbers, represented by $\omega_g(w)$, are defined by

$$\frac{1}{2-\exp(t)}=\sum_{w=0}^{\infty}\omega_g(w)\frac{t^w}{w!} \tag{1}$$

(*cf.* [3]).

From (1), one has

$$\omega_g(w)=\sum_{k=0}^{w}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^w$$

(*cf.* [3,10]).

The Fubini type polynomials of order $z$, represented by $a_w^{(z)}(x)$, are defined by

$$\frac{2^z\exp(xt)}{\left(2-\exp(t)\right)^{2z}}=\sum_{w=0}^{\infty}a_w^{(z)}(x)\frac{t^w}{w!} \tag{2}$$

(*cf.* [10]).

Setting $x=0$ yields the Fubini type numbers of order $z$:

$$a_w^{(z)}(0)=a_w^{(z)}.$$

When $z=1$ and $x=0$ in (2), one has

$$a_w^{(1)}(0)=a_w. \tag{3}$$

From (1) and (3), we get

$$a_w=2\sum_{k=0}^{w}\binom{w}{k}\omega_g(k)\omega_g(w-k)$$

(*cf.* [10]).

The first kind Gould-Hopper polynomials, represented by $H_w^m(x,\varphi)$, are defined by

$$\exp(xt+\varphi t^m)=\sum_{w=0}^{\infty}H_w^m(x,\varphi)\frac{t^w}{w!}, \tag{4}$$

and their explicit formula is given as follows:

$$H_w^m(x,\varphi)=\sum_{k=0}^{\left[\frac{w}{m}\right]}\frac{w!\,\varphi^k x^{w-mk}}{k!\,(w-km)!} \tag{5}$$

in which $[d]$ represents the largest integer $d$ (*cf.* [2,4,5]).

The polynomials $H_w(\vec{u},r)$ are defined by

$$\exp\left(\sum_{k=1}^{r}u_k t^k\right)=\sum_{w=0}^{\infty}H_w(\vec{u},r)\frac{t^w}{w!}, \tag{6}$$

and their explicit formula is given as follows:

$$H_w(\vec{u},r)=\sum_{k=0}^{\left[\frac{w}{r}\right]}\frac{w!\,u_r^k H_{w-rk}(\vec{u},r-1)}{k!\,(w-kr)!},$$

where $\vec{u}=(u_1,u_2,\dots,u_r)$ (*cf.* [2,4,5,13]). Here, we note that the polynomials $H_w(\vec{u},r)$ represent generalized Hermite-Kampè de Fèriet polynomials.

The $r$-parametric Hermite-based Milne-Thomson type polynomials, represented by $\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)$ and $\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)$, are defined by, respectively,

$$2\big(b+f(t,p)\big)^z\exp(xt)\exp\left(\sum_{k=1}^{r}u_k t^k\right)\cos(\varphi t)=\sum_{w=0}^{\infty}\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!}\qquad(7)$$

and

$$2\big(b+f(t,p)\big)^z\exp(xt)\exp\left(\sum_{k=1}^{r}u_k t^k\right)\sin(\varphi t)=\sum_{w=0}^{\infty}\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!},\qquad(8)$$

where $f(t,p)$ represents an analytic function or a meromorphic function and $p,b\in\mathbb{R}$ (*cf.* [6,13]).

When $b=0$ and $f(t,1)=\frac{2}{(2-\exp(t))^2}$ in (7) and (8), we have

$$\frac{2^{z+1}}{\big(2-\exp(t)\big)^{2z}}\exp(xt)\exp\left(\sum_{k=1}^{r}u_k t^k\right)\cos(\varphi t)=\sum_{w=0}^{\infty}{}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r)\frac{t^w}{w!}\qquad(9)$$

and

$$\frac{2^{z+1}}{\big(2-\exp(t)\big)^{2z}}\exp(xt)\exp\left(\sum_{k=1}^{r}u_k t^k\right)\sin(\varphi t)=\sum_{w=0}^{\infty}{}_F\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r)\frac{t^w}{w!},\qquad(10)$$

where ${}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r)$ and ${}_F\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r)$ are called $r$-parametric Hermite-based cosine-Milne-Thomson type Fubini polynomials of order $z$ and $r$-parametric Hermite-based sine-Milne-Thomson type Fubini polynomials of order $z$, respectively (*cf.* [6,13]).

For $\vec{u}=(0,0,\dots,0)=\vec{0}$, and combining (9), (10) with (2), we have

$${}_F\mathfrak{h}_1\big(w,x,\varphi,z;\vec{0},r\big)=2\sum_{k=0}^{\left[\frac{w}{2}\right]}(-1)^k\binom{w}{2k}\varphi^{2k}a_{w-2k}^{(z)}(x)$$

and

$${}_F\mathfrak{h}_2\big(w,x,\varphi,z;\vec{0},r\big)=2\sum_{k=0}^{\left[\frac{w-1}{2}\right]}(-1)^k\binom{w}{2k+1}\varphi^{2k+1}a_{w-1-2k}^{(z)}(x)$$

(*cf.* [6, Theorems 3.29 and 3.38]).

When $\varphi=0$ and $\vec{u}=\vec{0}$ in (9), one has [6]:

$${}_F\mathfrak{h}_1\big(w,x,0,z;\vec{0},r\big)=2a_w^{(z)}(x).$$

## 2. RELATIONS FOR $r$-PARAMETRIC HERMITE-BASED MILNE-THOMSON TYPE POLYNOMIALS

The aim of this section is to utilize generating functions for the polynomials $\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,a,b)$ and $\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,a,b)$ in order to obtain some relations including these polynomials with the first kind Gould-Hopper polynomials, and also the polynomials ${}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r)$ and ${}_F\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r)$.

**Theorem 2.1.** For $w \in \mathbb{N}_0$, we have

$$\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)=\sum_{k=0}^{w}\binom{w}{k}\mathfrak{h}_1(k,0,\varphi,z;\vec{u},r-1,p,b)H_{w-k}^{r}(x,u_r). \tag{11}$$

*Proof.* By combining (7) with (4), we obtain

$$\sum_{w=0}^{\infty}\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!}=\sum_{w=0}^{\infty}\mathfrak{h}_1(w,0,\varphi,z;\vec{u},r-1,p,b)\frac{t^w}{w!}\sum_{w=0}^{\infty}H_w^{r}(x,u_r)\frac{t^w}{w!}.$$

Therefore

$$\sum_{w=0}^{\infty}\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!}=\sum_{w=0}^{\infty}\sum_{k=0}^{w}\binom{w}{k}\mathfrak{h}_1(k,0,\varphi,z;\vec{u},r-1,p,b)H_{w-k}^{r}(x,u_r)\frac{t^w}{w!}.$$

Matching the terms of $\frac{t^w}{w!}$ in both expressions brings us to the intended result.

**Theorem 2.2.** For $w \in \mathbb{N}_0$, we have

$$\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)=\sum_{k=0}^{w}\binom{w}{k}\mathfrak{h}_2(k,0,\varphi,z;\vec{u},r-1,p,b)H_{w-k}^{r}(x,u_r). \tag{12}$$

*Proof.* Combining (8) with (4), we get

$$\sum_{w=0}^{\infty}\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!}=\sum_{w=0}^{\infty}\mathfrak{h}_2(w,0,\varphi,z;\vec{u},r-1,p,b)\frac{t^w}{w!}\sum_{w=0}^{\infty}H_w^{r}(x,u_r)\frac{t^w}{w!}$$

and consequently

$$\sum_{w=0}^{\infty}\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!}=\sum_{w=0}^{\infty}\sum_{k=0}^{w}\binom{w}{k}\mathfrak{h}_2(k,0,\varphi,z;\vec{u},r-1,p,b)H_{w-k}^{r}(x,u_r)\frac{t^w}{w!}.$$

Matching the terms of $\frac{t^w}{w!}$ in both expressions brings us to the intended result.

For $b=0$, putting $f(t,1)=\frac{2}{(2-\exp(t))^2}$ in (11) and (12), yields the Corollary 2.3:

**Corollary 2.3.** For $w \in \mathbb{N}_0$, we have

$${}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r)=\sum_{k=0}^{w}\binom{w}{k}\ {}_F\mathfrak{h}_1(k,0,\varphi,z;\vec{u},r-1)H_{w-k}^{r}(x,u_r) \tag{13}$$

and

$${}_F\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r)=\sum_{k=0}^{w}\binom{w}{k}\ {}_F\mathfrak{h}_2(k,0,\varphi,z;\vec{u},r-1)H_{w-k}^{r}(x,u_r). \tag{14}$$

## 3. INTEGRAL FORMULAS FOR $r$-PARAMETRIC HERMITE-BASED MILNE-THOMSON AND FUBINI TYPE POLYNOMIALS

The aim of this section is to apply integral operator to the generating functions of the polynomials $\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)$ and $\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)$ in order to give several formulas that include these polynomials, the polynomials ${}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r)$, and the Fubini type polynomials.

**Theorem 3.1 (*cf.* [6, Eq. (4.8)]).** For $w \in \mathbb{N}_0$, we have

$$\int_c^d \mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)dx = \frac{\mathfrak{h}_1(w+1,d,\varphi,z;\vec{u},r,p,b) - \mathfrak{h}_1(w+1,c,\varphi,z;\vec{u},r,p,b)}{w+1}. \quad (15)$$

*Proof.* Integrating both sides of the Eq. (7), we get

$$2\big(b+f(t,p)\big)^z \exp\left(\sum_{k=1}^{r} u_k t^k\right)\cos(\varphi t)\int_c^d \exp(xt)dx = \sum_{w=0}^{\infty}\frac{t^w}{w!}\int_c^d \mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)dx.$$

After some calculations, we obtain

$$\sum_{w=0}^{\infty}\frac{t^w}{w!}\int_c^d \mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)dx$$
$$= \sum_{w=0}^{\infty}\frac{\mathfrak{h}_1(w+1,d,\varphi,z;\vec{u},r,p,b)}{(w+1)!}t^w - \sum_{w=0}^{\infty}\frac{\mathfrak{h}_1(w+1,c,\varphi,z;\vec{u},r,p,b)}{(w+1)!}t^w.$$

Matching the terms of $\frac{t^w}{w!}$ in both expressions brings us to the intended result.

When $b=0$ and $f(t,1) = \frac{2}{\left(2-\exp(t)\right)^2}$ in (15) allows us to obtain the Corollary 3.2:

**Corollary 3.2.** For $w \in \mathbb{N}_0$, we have

$$\int_c^d {}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r)dx = \frac{{}_F\mathfrak{h}_1(w+1,d,\varphi,z;\vec{u},r) - {}_F\mathfrak{h}_1(w+1,c,\varphi,z;\vec{u},r)}{w+1}. \quad (16)$$

**Remark 3.3.** Substituting $\varphi = 0$ and $\vec{u} = \vec{0}$ into (16), and performing some calculations gives the known result:

$$\int_c^d a_w^{(z)}(x)dx = \frac{1}{w+1}\left(a_{w+1}^{(z)}(d) - a_{w+1}^{(z)}(c)\right).$$

(*cf.* [7, Eq. 20]).

**Theorem 3.4 (*cf.* [6, Eq. (4.9)]).** For $w \in \mathbb{N}_0$, we have

$$\int_c^d \mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)dx = \frac{\mathfrak{h}_2(w+1,d,\varphi,z;\vec{u},r,p,b) - \mathfrak{h}_2(w+1,c,\varphi,z;\vec{u},r,p,b)}{w+1}. \quad (17)$$

*Proof.* Integrating both sides of the Eq. (8), we get

$$2\big(b+f(t,p)\big)^z \exp\left(\sum_{k=1}^{r} u_k t^k\right)\sin(\varphi t)\int_c^d \exp(xt)dx = \sum_{w=0}^{\infty}\frac{t^w}{w!}\int_c^d \mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)dx.$$

After some calculations, we obtain

$$\sum_{w=0}^{\infty}\frac{t^w}{w!}\int_c^d \mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)dx$$
$$=\sum_{w=0}^{\infty}\frac{\mathfrak{h}_2(w+1,d,\varphi,z;\vec{u},r,p,b)}{(w+1)!}t^w-\sum_{w=0}^{\infty}\frac{\mathfrak{h}_2(w+1,c,\varphi,z;\vec{u},r,p,b)}{(w+1)!}t^w.$$

Matching the terms of $\frac{t^w}{w!}$ in both expressions brings us to the intended result.

**Remark 3.5.** Using integral methods with generating functions, we also presented some integral representations for these polynomials, see for detail, [8].

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