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FORMULAS RELATED TO APOSTOL-GENOCCHI POLYNOMIALS AND NUMBERS

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ABSTRACT

Spline curve families have garnered significant attention in recent years due to their wide-ranging applications in fields such as mathematics, mathematical modeling, probability, statistics, and other applied sciences. Additionally, spline curves hold particular importance in computer-aided design, surface modeling, animation production, and more. In three-dimensional design, spline curve families play a crucial role. This study defines new types of spline curve families and explores their relationships with special numbers, special polynomials, and special functions. The results are expected to contribute to the theory of spline curves and extend their applications to various disciplines including mathematics, medicine, engineering, economics, and robotics.

Keywords: *Frobenius Euler numbers and polynomials, Exponential Euler Spline, Special numbers and polynomials, Generating functions, Beta-type rational functions.*

1. INTRODUCTION

The history of spline curves dates back to ancient times; however, the mathematical theory of these curves was first constructed by Isaac Schoenberg during the 1940s and 1950s. Spline curves, defined as piecewise polynomials with continuous derivatives at their knot points, are widely used in fields ranging from scientific computing and engineering to industrial design. Prominent special families of spline curves, such as cardinal splines, B-splines, exponential splines, and exponential Euler splines, are frequently employed due to their simplicity and effectiveness in curve fitting.

Among these, Bézier curves stand out as a distinguished member of the spline curve family, developed independently by Paul de Casteljau and Pierre Bézier in 1959. Bézier curves are widely used in computer-aided design (CAD), modeling applications, and the animation industry. Recent studies have further advanced the field by exploring novel spline curve types and their applications (*cf*. [1-24]).

This research contributes to the theory of spline curves by introducing new spline families, examining their mathematical properties, and demonstrating their applications in areas such as trajectory planning, velocity and acceleration analysis, and surface modeling. The implications of these findings extend beyond mathematics to practical domains, enhancing the versatility of spline curves in real-world problem-solving.

The sets of natural numbers, integers, real numbers, complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

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The Apostol- Genocchi higher-order numbers and polynomials, respectively, as follows:

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}=\sum_{n=0}^{\infty}\frac{\mathcal{G}_n^{(\alpha)}(\lambda)}{n!}t^n \tag{1}$$

and

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}e^{tx}=\sum_{n=0}^{\infty}\frac{\mathcal{G}_n^{(\alpha)}(x;\lambda)}{n!}t^n \tag{2}$$

(*cf*. [11,12,23,24]).
The Frobenius-Euler higher-order numbers and polynomials, respectively, as follows:

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha}=\sum_{n=0}^{\infty}\frac{H_n^{(\alpha)}(\lambda)}{n!}t^n \tag{3}$$

and

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha}e^{tx}=\sum_{n=0}^{\infty}\frac{H_n^{(\alpha)}(x;\lambda)}{n!}t^n \tag{4}$$

(*cf*. [2,9,10]).
The beta-type rational functions $\mathfrak{M}_{j,n}(\lambda)$ are defined by means of the following generating functions:

$$\left(\frac{\lambda}{\lambda+1}\right)^{j}e^{t(\lambda+1)}=\sum_{n=0}^{\infty}\frac{\mathfrak{M}_{j,n}(\lambda)}{n!}t^n. \tag{5}$$

The beta-type rational functions are defined by

$$\mathfrak{M}_{j,n}(\lambda)=\lambda^j(1+\lambda)^{n-j}, \tag{6}$$

where $n,j\in\mathbb{N}_0$ and $\lambda\in\mathbb{R}$, set of real numbers, (or $\mathbb{C}$, set of complex numbers) (*cf*. [17]).

The Stirling numbers of the second kind $S_2(r,d)$ are defined by

$$\frac{(e^t-1)^d}{d!}=\sum_{r=0}^{\infty}\frac{S_2(r,d)}{r!}t^r \tag{7}$$

(*cf*. [3,5,6,8,15,16]).

Recently, we [7] defined the polynomials $u_n^{(\alpha)}(x;\lambda)$ by means of the following generating function:

$$\left(\frac{1+\lambda}{\lambda e^t+1}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} \frac{u_n^{(\alpha)}(x;\lambda)}{n!} t^n. \tag{8}$$

A relation between the polynomials $u_n^{(\alpha)}(x;\lambda)$ and $H_n^{(\alpha)}(x;\lambda)$ is given by

$$u_n^{(\alpha)}(x;\lambda) = H_n^{(\alpha)}\left(x;-\frac{1}{\lambda}\right) \tag{9}$$

(*cf*. [7]).

The Apostol Euler numbers of higher order are defined by

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} = \sum_{n=0}^{\infty} \frac{\mathcal{E}_n^{(\alpha)}(\lambda)}{n!} t^n \tag{10}$$

(*cf*. [2,4,7,9,10,12,15,18,23]).
Substituting (10) into above equation, we have the Apostol Euler numbers of higher negative order are defined by

$$\left(\frac{\lambda e^t+1}{2}\right)^{\alpha} = \sum_{n=0}^{\infty} \frac{\mathcal{E}_n^{(-\alpha)}(\lambda)}{n!} t^n \tag{11}$$

and also

$$\mathcal{E}_n^{(\alpha)}(\lambda) = \sum_{j=1}^{n} 2^{\alpha}(-1)^j S_2(n,j)\lambda^j \frac{(\alpha)^{(j)}}{(\lambda+1)^{j+\alpha}} \tag{12}$$

(*cf*. [2,4,7,9,10,12,15,18,23]).
$(\alpha)^{(n)}$ denotes the rising factorial (the Pochhammer symbol) defined by

$$(\alpha)^{(n)} = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+n-1), & n \in \mathbb{N} \\ 1, & n=0 \end{cases} \tag{13}$$

for $(\alpha)^{(n)}$ see also [5].
Combining (8) and (10), we have

$$u_n^{(\alpha)}(x;\lambda) = \left(\frac{1+\lambda}{2}\right)^{\alpha} \mathcal{E}_n^{(\alpha)}(x;\lambda) \tag{14}$$

and

$$u_n^{(\alpha)}(\lambda) = \left(\frac{1+\lambda}{2}\right)^{\alpha} \mathcal{E}_n^{(\alpha)}(\lambda) \tag{15}$$

(*cf*. [2,4,7,9,10,12,15,18,23]).

The following integral formulas for the function $\mathfrak{M}_{j,n}(\lambda)$, as given by Simsek [17, Eqs. (18)-(19)], are presented below:

$$\int_{-1}^{0}\mathfrak{M}_{j,n}(\lambda)d\lambda=\sum_{k=0}^{n-j}(-1)^{n-k}\binom{n-j}{k}\frac{1}{n+1-k} \tag{16}$$

and

$$\int_{-1}^{0}\mathfrak{M}_{j,n}(\lambda)d\lambda=(-1)^{j}\frac{1}{(n+1)\binom{n}{j}} \tag{17}$$

(*cf.* [7,17]).
In Reference (7), the newly defined exponential splines are presented as follows:

$$Y(x;\lambda;n,\alpha)=\frac{u_n^{(\alpha)}\left(x;-\frac{1}{\lambda}\right)}{u_n^{(\alpha)}\left(-\frac{1}{\lambda}\right)} \tag{18}$$

and

$$Y(x;\lambda;n,\alpha)=\frac{H_n^{(\alpha)}(x;\lambda)}{H_n^{(\alpha)}(\lambda)} \tag{19}$$

(*cf.* [7]).

2. CALCULATION FORMULAS FOR APOSTOL-GENOCCHI NUMBERS AND POLYNOMIALS

In this section, new formulas for some combinatorial sums involving Beta-type rational functions, Apostol-Genocchi numbers of order $-\boldsymbol{m}$, and Stirling numbers of the second kind will be presented. Additionally, integral formulas using Beta-type rational integrals will be provided for certain finite combinatorial sums.

The primary focus of this study is the introduction of new spline curve families and their mathematical and practical relevance.

The relationship between Equation (12) and the Apostol-Genocchi numbers is written, and then $\alpha=-n$ is applied, the following result is obtained:

Theorem 2.1. For $d,n\in\mathbb{N}$, we have

$$\mathcal{G}_{d-n}^{(-n)}(\lambda)=\frac{2^{-n}d!}{(\mathrm{d}-\mathrm{n})!}\sum_{j=1}^{d}(-1)^{j}S_2(d,j)\lambda^{j}\frac{(-n)^{(j)}}{(\lambda+1)^{j-n}}. \tag{20}$$

The result obtained by combining the Beta-type rational function given in (6) with Theorem (2.1) is as follows:

Corollary 2.2. For $d, n \in \mathbb{N}$, we have

$$\mathcal{G}_{d-n}^{(-n)}(\lambda) = \frac{2^{-n}d!}{(\mathrm{d}-\mathrm{n})!}\sum_{j=1}^{d}\binom{n}{j}S_2(d,j)\mathrm{j}!\,\mathfrak{M}_{j,n}(\lambda). \tag{21}$$

Integrating both sides of the relation (21) for $\mathcal{G}_{d-n}^{(-n)}(\lambda)$ is taken with respect to λ from -1 to 0, the following integrals are obtained:

$$\int_{-1}^{0}\mathcal{G}_{d-n}^{(-n)}(\lambda)d\lambda = \frac{2^{-n}d!}{(\mathrm{d}-\mathrm{n})!}\sum_{j=1}^{d}\binom{n}{j}S_2(d,j)\mathrm{j}!\int_{-1}^{0}\mathfrak{M}_{j,n}(\lambda)d\lambda. \tag{22}$$

The above equation is combined with the integral formulas given by Simsek [17] for the $\mathfrak{M}_{j,n}(\lambda)$ function, the integral formulas for higher-order negative powers of Apostol-Genocchi numbers are given below:

Theorem 2.3. For $d, n \in \mathbb{N}$, we have

$$\int_{-1}^{0}\mathcal{G}_{d-n}^{(-n)}(\lambda)d\lambda = \frac{2^{-n}d!}{(\mathrm{d}-\mathrm{n})!}\sum_{j=1}^{d}\binom{n}{j}S_2(d,j)\mathrm{j}!\sum_{k=0}^{n-j}(-1)^{n-k}\binom{n-j}{k}\frac{1}{n+1-k}. \tag{23}$$

Theorem 2.4. For $d, n \in \mathbb{N}$, we have

$$\int_{-1}^{0}\mathcal{G}_{d-n}^{(-n)}(\lambda)d\lambda = \frac{2^{-n}d!}{(n+1)(\mathrm{d}-\mathrm{n})!}\sum_{j=1}^{d}(-1)^{j}j!\,S_2(d,j). \tag{24}$$

Combining (23) and (24) equation, we get

Corollary 2.5. For $d, n \in \mathbb{N}$, we have

$$(-1)^{d} = \sum_{j=1}^{d}(-1)^{j}j!\,S_2(d,j). \tag{25}$$

3. CONCLUSION

In this section, a new class of exponential Euler-type spline curves with degree n and order α will be constructed. Using the equations above, the following identities are obtained:

$$u_n^{(\alpha)}(x;\lambda) = \left(\frac{1+\lambda}{2}\right)^{\alpha} \frac{n!}{(n+\lambda)!} \mathcal{G}_{n+\alpha}^{(\alpha)}(x;\lambda) \tag{26}$$

and

$$u_n^{(\alpha)}(\lambda) = \left(\frac{1+\lambda}{2}\right)^{\alpha} \frac{n!}{(n+\lambda)!} \mathcal{G}_{n+\alpha}^{(\alpha)}(\lambda) \tag{27}$$

Using the above equations, the Apostol-Genocchi spline family will be formulated in the future. This approach will provide a deeper understanding of the relationships between Apostol-Genocchi numbers, polynomials, and spline functions, offering a new avenue for research in both number theory and combinatorics. The development of such spline families may lead to further advancements in various mathematical applications, particularly in approximation theory and the study of special numbers.

This study expands the theoretical framework of spline curves and highlights their practical utility in diverse applications. The proposed spline families not only enhance mathematical modeling capabilities but also open new avenues for research in applied sciences and engineering. Future work could focus on extending these findings to dynamic systems and real-time applications in robotics and animation.

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