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ADVANCEMENTS IN SOLVING HIGHER-ORDER ORDINARY DIFFERENTIAL EQUATIONS VIA THE VARIATIONAL ITERATIVE METHOD

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ABSTRACT

This study presents advancements in solving higher-order ordinary differential equations (ODEs) using the Variational Iterative Method (VIM) and compares its performance with the New Iteration Method (NIM) and Adomian Decomposition Method (ADM). ODEs are critical in modeling the rate of change in various systems over time, and many do not have exact solutions, necessitating the use of numerical methods to obtain approximate results. While several iterative methods exist, a detailed comparison of VIM with other techniques, particularly for higher-order ODEs, is still lacking. This research focuses on understanding the principles and methodology of VIM and applying it to solve higher-order linear and nonlinear ODEs. The study evaluates the accuracy, convergence rate, and computational efficiency of VIM compared to NIM and ADM through the solution of third, fourth, and fifth-order differential problems. The results show that VIM outperforms NIM and ADM, with faster convergence and higher efficiency. Error analysis in Figures 1, 2, and 3 highlights the strengths and limitations of each method, providing valuable insights for researchers and practitioners in selecting the most appropriate technique for solving higher-order ODEs. These findings advance the development of iterative methods in numerical analysis and contribute to progress in the field of differential equations.

Keywords: *Variational Iterative Method, Higher-Order equations, Comparative analysis, Iterative Technique, Numerical analysis.*

1. INTRODUCTION

Higher-Order ordinary differential equations are equations that involves the derivatives of a function with respect to one variable, where the highest derivative is of order greater than one [1]. These equations are crucial in mathematical modelling because they frequently arise in various fields of science and engineering [2, 3, 4, 5].
The general form of a higher-order differential equation with initial conditions is given by

$$f^{n}(x) + a_{(n-1)f^{n-1}(x)+\cdots+a_1 f'(x)+a_0 f(x)=g(x)} \quad (1)$$

With initial conditions $f(x_0) = y_0$,$f'(x_0)= y_0$ …$f^{n-1}(x_0)$=y_{n-1}

These equations become difficult to solve using the traditional methods as the order increases so also the complexity. So, over the years, modified methods such as the Variational Iterative method (VIM), Adomian Decomposition Method (ADM), New Iterative Methods (NIM) and other methods has been proposed by numerous researchers to address both linear and nonlinear differential equations without requiring small parameters assumption [6, 7, 8, 9].

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The Variational Iterative Method (VIM) is an analytical technique developed for solving differential equations including Higher-Order ordinary differential equations. This method leverages on the principles of variational theory to construct correction functionals, incorporating Langrange multipliers to iteratively approximate solutions [10, 11, 12, 13].
However, despite the existence of various Iterative methods, a thorough comparison between VIM and other numerical methods for higher order is lacking.
Current studies shows a huge research gap in the comparative analysis of these elite numercial methods, recent works have only focused on VIM and ADM for lower order. This study aims to bridge that gap by expanding into VIM, NIM and ADM for complex higher-order ODEs, focusing on their computational power as well as precision in solving higher-order ODEs' [14, 15, 16, 17, 18].
The Novelty of this study lies in the need to evaluate the effectiveness and applicability of the VIM when applied to Higher-Oder ODEs and to provide a comparative analysis between VIM, NIM and ADM for higher-order differential equations in order to understand how these methods react when applied to higher-order ordinary differential equations and to compare the accuracy and efficiency of these methods, highlighting their strengths and limitations, hence guiding their optimal application in complex systems and phenomena.

2. MATERIALS AND METHOD

2.1. Description of VIM

The following authors explained the VIM method, its formula and algorithm [10, 19, 20, 21, 22, 23]
Consider the following general function equation for nonlinear equation

$$L_p(B_1, B_2, \ldots, B_p) + N_p(B_1, B_2, \cdots, B_p) = G_p \tag{2}$$

where L_p represent a linear operator and N_p is a nonlinear operator, G_p is a given function.
So, lets consider a differential equation of the form;

$$L_p(t) + N_p(t) = g_p \tag{3}$$

an initial approximation $B_{p,n+1}$ is assumed and the correction functional is constructed below;

$$B_{p,n+1} = B_{p,n} + \int_0^t \lambda_p [L_p(\tilde{B}_{1,n}(t)) + N_p(\tilde{B}_{1,n}(t)), \cdots, \tilde{B}_{m,n}(t)) - g_i(m))dt \tag{4}$$

where λ_i is a langrange multiplier to be determined using the variational theory which requires the fuctional to be stationary. The subscript $_m$ represent the mth-order approximation where B_m is taken as the bounded variation and $\delta B_m = 0$. The value of the Lagrange Multiplier is computed using the formula

$$\lambda = \frac{(-1)^m}{(m-1)!}(s-x)^{m-1} \tag{5}$$

$$L_p\left(B_p\right) + N_p\left(B_1, \cdots, B_m\right) = g_i\left(t\right), \quad i = 1, 2, \cdots, m \tag{6}$$

And t is the count of occurrencies of differentials, optimal value of λ is best determined through variational theorry, by fixing parameters defined by equation (2), the following conditions are established;

$$\lambda_i '(t)|_{t=B} = 0$$
$$\lambda_i(t) + 1|_{t=B} = 0, \quad i = 1, 2, \ \dots \ , \ m$$

$$B_{i,n+1} = B_{i,n} + \int_0^t \lambda_i [L_i(B_{i,n}(t)) + N_i(B_{1,n+1}(t)), \dots \ , B_{i-1,n+1}(t), B_{i,n}(t), \dots, B_{m,n}(t)) - g_i(t)]dt \tag{7}$$

for $i = 1, 2, \dots, m$ the revised value of $J_{i-1,n+1}$ is used to compute $J_{i,n+1}$, thus quickening the convergence of the equations.

3. NUMERICAL EXPERIMENTS

In this section, the Variational Iteration Method (VIM) is applied to solve third-, fourth-, and fifth-order ordinary differential equations (ODEs). The computations are performed using Python, and the resulting solutions are compared with those obtained from the Adomian Decomposition Method (ADM) and the New Iteration Method (NIM). To provide clear insights into the comparative performance of these methods, the results are presented in both tabulated and graphical formats.

Numerical Problem 1: Consider the third-order linear ODE below and apply the VIM[24]

$$\frac{d^3u}{dt^3} + \frac{du}{dt} = 0 \tag{8}$$

With initial conditions given as: $u(0) = 0, \quad u'(0) = 1, \ u''(0) = 2, \qquad 0 \le t \le 1$. And exact solution is provided as: $u(t) = 2(1 - cos(t)) + sin(t)$. The appropriate Langrange multiplier for third order is selected by $\lambda(k) = \frac{-(t-x)^2}{2}$ and correction functional is constructed by (9)

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{-(t-x)^2}{2} [u'''(t) + u'(t) - g(t)] dt \tag{9}$$

And the following iteration is obtained;

$$\begin{aligned}
u_0 &= t + t^2 \\
u_1 &= t + t^2 - \frac{t^3}{2} - \frac{t^4}{12} \\
u_2 &= t + t^2 - \frac{t^3}{6} - \frac{t^5}{60} \\
u_3 &= t + t^2 - \frac{t^3}{6} - \frac{t^5}{120} \\
u_4 &= t + t^2 - \frac{t^3}{6} - \frac{t^5}{120} + t^7
\end{aligned} \tag{10}$$

Numerical Problem 2: Compute the solutions of the fourth-order ODE given below using the Variational Iteration Method [25].

$$\frac{d^4u}{dt} + \frac{d^3u}{dt^3} - 10u(t) = 0 \tag{11}$$

with initial conditions $u(0)=0, u'(0)=0, u''(0)=1, u'''(0)=2, \quad 0 \le t \le 1$. Proceeding according to the method explained earlier, we select the langrange multiplier for fourth-order $\lambda(x)=\frac{-(t-x)^4}{4!}$ and the correction functional is constructed

$$u_{n+1}(t)=u_n(t)+\int_0^t \frac{-(t-x)^4}{24}\left[\frac{d^4u}{dt}+\frac{d^3u}{dt^3}-10u(t)-g(t)\right]dt \tag{12}$$

using $u_o=t+\frac{1}{2}t^2+\frac{1}{3}t^3$ from the initial conditions, the following successive estimations is obtained;

$$\begin{aligned}
u_o &= t+\frac{1}{2}t^2+\frac{1}{3}t^3 \\
u_1 &= t+\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{5}{6}t^4 \\
u_2 &= t+\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{5}{6}t^4+\frac{25}{12}t^5 \\
u_3 &= t+\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{5}{6}t^4+\frac{25}{9}t^5+\frac{95}{12}t^6 \\
u_4 &= t+\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{5}{6}t^4+\frac{179}{12}t^6+\frac{250}{63}t^7+\frac{1175}{73}t^8
\end{aligned} \tag{13}$$

Numerical Problem 3: Compute the solutions of the fifth-order nonlinear Ordinary Differential Equation given below using the VIM [24].

$$\frac{d^5u}{dt^5}-u\frac{d^3u}{dt^3}-\frac{du}{dt}=0 \tag{14}$$

with initial conditions given as: $u(0)=1, \quad u'(0)=2, \; u''(0)=1, \quad u'''(0)=0, \quad u^{iv}(0)=3, \quad 0 \le t \le 1$ the exact solution is provided as: $u(t)=1+t+t^2$. Proceeding according to the method explained earlier, we select the langrange multiplier for fourth-order $\lambda(x)=\frac{-(t-x)^4}{4!}$ and the correction functional is constructed by equation (15) as

$$u_{n+1}(t)=u_n(t)+\int_0^t \frac{-(t-x)^4}{24}\left[\frac{d^5u}{dt^5}-u\frac{d^3u}{dt^3}-\frac{du}{dt}-g(t)\right]dt \tag{15}$$

using $u_o=1+2t+\frac{t^3}{2}+\frac{t^4}{8}$ from the initial conditions, the following successive approximations is obtained;

$$
\begin{aligned}
u_0 &= 1+2t+\frac{t^3}{2}+\frac{t^4}{8} \\
u_1 &= 1+2t+\frac{t^3}{2}+\frac{t^4}{8}-\frac{t^5}{40} \\
u_2 &= 1+2t+\frac{t^3}{2}+\frac{t^4}{8}-\frac{t^5}{40}+\frac{t^6}{120} \\
u_3 &= 1+2t+\frac{t^3}{2}+\frac{t^4}{8}-\frac{t^5}{40}+\frac{t^6}{120}-\frac{t^7}{504} \\
u_4 &= 1+2t+\frac{t^3}{2}+\frac{t^4}{8}-\frac{t^5}{40}+\frac{t^6}{120}-\frac{t^7}{504}+\frac{t^8}{4052}
\end{aligned}
\tag{16}
$$

The computed results are presented in the tables below for clarity and ease of comparison. These tables provide a detailed breakdown of the numerical outcomes, allowing for a straightforward evaluation of the methods applied in solving the differential equations.

Table 1. Computational Comparison for the two methods (VIM and NIM) on Problem one

X	VIM	NIM	Exact Solution
0.00	0.00000000000000	0.000000000000000	0.00000000000000
0.10	0.20965850200000	0.208189860000000	0.10982508600000
0.20	0.22000000000000	0.238553475000000	0.15385361750000
0.30	0.35800000000000	0.384851298000000	0.27848472290000
0.40	0.51210000000000	0.548342654000000	0.43729635400000
0.50	0.68047000000000	0.725428915000000	0.52426041500000
0.60	0.86132500000000	0.914657244000000	0.69397124300000
0.70	1.05290440000000	1.115535236300000	0.88453329400000
0.80	1.25345529000000	1.324632552000000	1.05394264400000
0.90	1.46121919600000	1.543127073000000	1.23010693300000
1.00	1.67442191200000	1.762839373000000	1.41086631800000

The given Table above shows the Variational Iterative Method approximate solution, New Iteration Method solution [15] and exact solution for Problem 1.

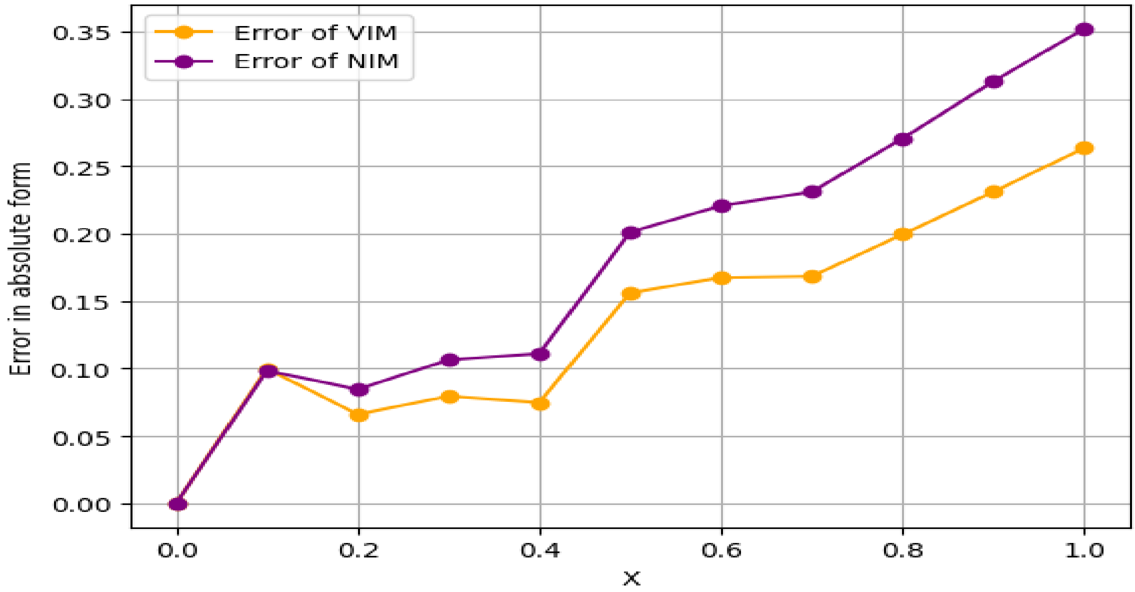

Figure 1: Plot illustrating Errors of the Compared Methods for Problem 1

The given plot shows the computed errors of the Variational Iterative and New Iteration Methods for problem one. It is observed that the error of VIM solution is lesser than that of NIM.

Table 2. Computational Comparison for the two methods (VIM and ADM) on Problem two

X	VIM	ADM	Exact Solution
0.10	0.009621383691244	0.005333333334933	0.012345678901235
0.20	0.035849143276779	0.022666667106314	0.045678901234568
0.30	0.067342991683491	0.054000012063215	0.089012345678901
0.40	0.125472524156772	0.101333462064786	0.154321098765432
0.50	0.197532855221887	0.166667484775190	0.234567890123456
0.60	0.273848076021479	0.231001507485594	0.345678901234567
0.70	0.356978715924633	0.295335530196998	0.478901234567890
0.80	0.445597930124879	0.359669552908402	0.634567890123456
0.90	0.531609842927041	0.424003575619806	0.812345678901234
1.00	0.612282933119536	0.488337598331210	1.012345678901230

The given Table above shows the Variational Iterative Method approximate solution, Adomian Decomposition Method solution and exact solution for Problem 2.

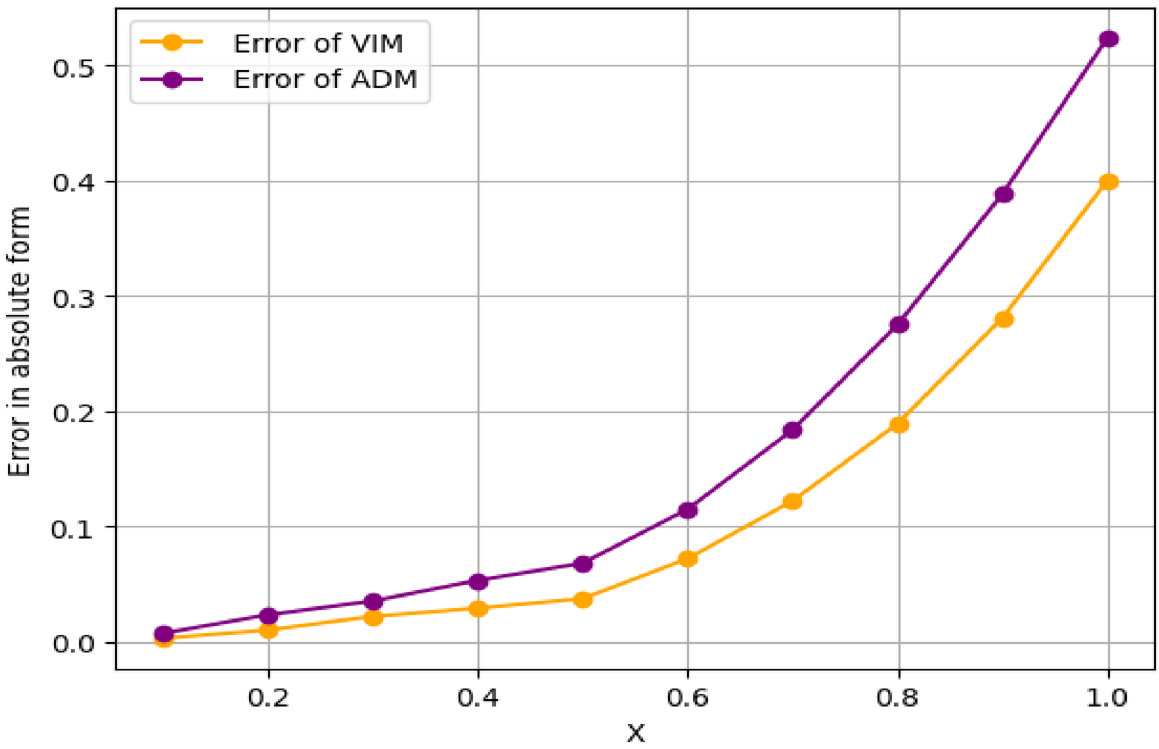

Figure 2: Plot illustrating Errors of the Compared Methods for Problem 2

The given plot shows the computed errors of the methods; Variational Iterative Method and Adomian Decomposition Method for problem four. It is observed that the error of VIM solution is lesser than the ADM.

Table 3. Computational Comparison for the two methods (VIM and NIM) on Problem Three

X	VIM	NIM	Exact Solution
0.10	1.0000000000	1.0000000000	1.0000000000
0.20	1.2232630000	1.2442610000	1.1234570000
0.30	1.4595960000	1.4883260000	1.2716050000
0.40	1.7094650000	1.7471440000	1.4444440000
0.50	1.9735460000	2.0417490000	1.6419750000
0.60	2.2526520000	2.4821350000	1.8641980000
0.70	2.5477310000	2.8923670000	2.1111110000
0.80	2.8598660000	3.3534780000	2.3827160000
0.90	3.1902640000	3.6453610000	2.6790120000
1.00	3.5401410000	3.9525350000	3.0000000000

The given Table above shows the Variational Iterative Method approximate solutions, New Iteration Method solutions and exact solutions for Problem three.

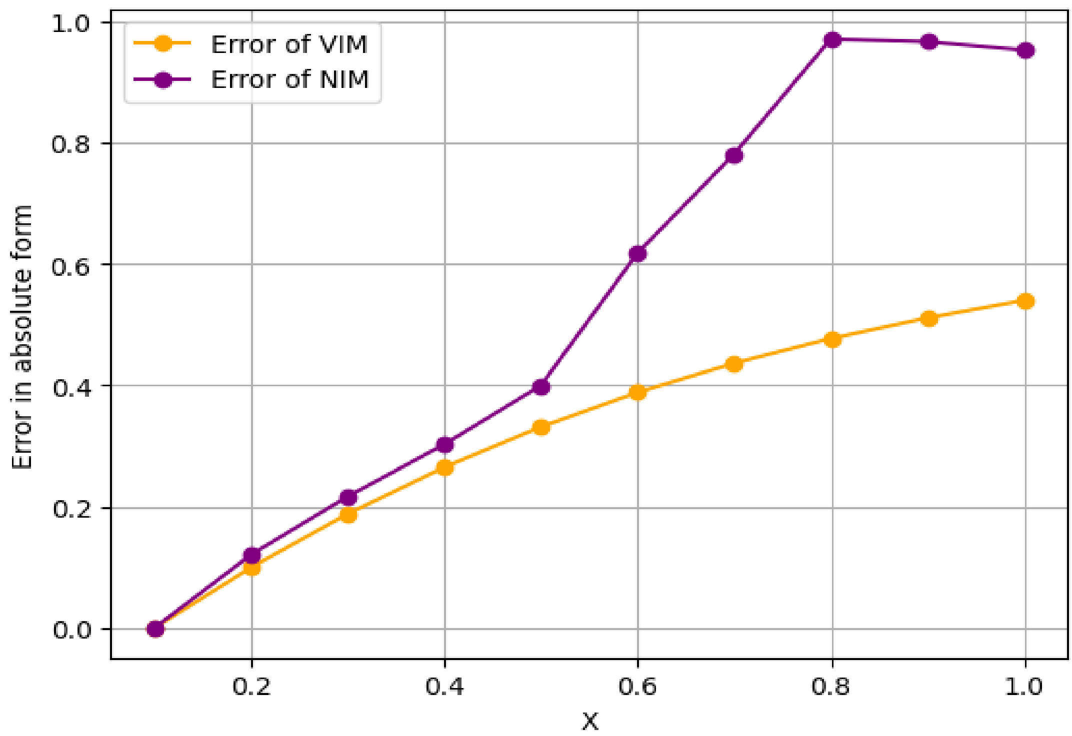

Figure 3: Plot illustrating Errors of the Compared Methods for Problem 3

The given plot shows the computed errors of the methods; Variational Iterative Method and New Iteration Method for problem three. It is observed that the error of VIM solution is lesser than that of NIM.

4. DISCUSSION OF RESULTS

Based on the criteria outlined by [26], a numerical solution is said to converge or is close to convergence if the approximate solution is closer or equal to the exact solution, if the error between the exact solution and approximate solution tends towards zero. It is based on these criteria that we can select the best method for higher-order ODE.

Comparison of Numerical Solutions (Table 1-3):

I. Concerning the comparison of numerical solutions on table 1 to 3, these tables offer great insights into the incredible solutions of VIM, NIM and ADM for problem 1 to 3.
II. The solutions gotten highlights the method's incredible performance when applied to Higher-Order ODEs.
III. Each question exposed the variations of the solution highlighting their complexities, the higher the iteration the more complex it becomes. This in turn gives us valuable information regarding how models react to different conditions.

Numerical illustration of the Error Plots (Figure 1-3)

I. The error figures show the difference between the methods and the exact solution

II. The higher the complexity of the problem, the bigger the errors become, but, on all cases, VIM outperforms the NIM and ADM as it consistently has lower errors and converges faster towards the exact solution.
III. These figures (Figure 1 to 3) aids in understanding the disparities between these methods at first glance.

Findings of the Experiments:

I. Figures (1, 2, 3) of the error analysis shows the VIM has lower errors and converges faster to the exact solution.
II. As the number of iterations increases, the error also increases highlighting the complexity of these equations towards capturing accurate solutions.
III. These observations provide valuable insights that shows areas that needs improvement

5. CONCLUSION

The Variational Iteration Method (VIM) has been successfully applied to solve three higher-order ordinary differential equation (ODE) problems, specifically of third, fourth, and fifth orders. The results obtained using VIM were compared with those from the New Iterative Method and the Adomian Decomposition Method. While all three methods produced accurate results for the ODEs, the comparison revealed that VIM outperforms the other two in terms of computational efficiency and precision. As shown in Figures (1, 2, 3), VIM proves to be superior, especially when solving higher-order ODEs. Looking ahead, we plan to extend our analysis to include additional numerical methods for higher-order ODEs. This will allow us to gain a more comprehensive understanding of the performance of various methods across different types of differential equation problems, thereby contributing to the advancement of computational mathematics and numerical analysis.

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