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Research Article

Proximal Lusternik-Schnirelmann, Császár and Bornology Categories

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Abstract: This paper introduces new forms of proximity space categories, namely, proxLS and dproxLS, spatial and descriptive proximity refinements of the Lusternik-Schnirelmann category LS and **desProx**, which is a descriptive proximity extension of the Császár **Prox** category. A main result given here is that a proximity space (X, δ) is proximally contractible if and only if the proximal Lusternik-Schnirelmann category $\mathrm{LS}_\delta(X) = 1$. Several other results are also given, namely, **Prox** has all nonempty products and coproducts and **ProxBor** has all nonempty coproducts.

Keywords: Čech proximity, descriptive proximity, Lusternik-Schnirelmann category, bornology.

1. Introduction

This paper introduces new forms of spatial and descriptive proximities, originally introduced in [18, 19] and elaborated in [7, 22]. These proximities include spatially proximal as well as descriptively proximal Lusternik-Schnirelmann categories $\mathrm{LS}_\delta(X)$ and $\mathrm{LS}_{\delta_\Phi}(X)$ as well as the proximal Császár categories **Prox** and **desProx** and **ProxBor** (proximal bornological spaces), which is an extension of the category Born of bornological spaces.

The oldest of these categories is LS-cat, a numerical invariant of a manifold M introduced by L.A. Lusternik and L.G. Schnirelmann in their study of critical points, homotopy classes, principal of stationary points and the category of a closed set with respect to a compact manifold and category of a projective space [10, 15, 16], leading to recent results for spaces of persistence modules [29], a digital LS-cat [1] and LS-cat of a simplicial map [25]. Under suitable restrictions, the LS-cat of M gives a lower bound for the number of critical points for any smooth function on M [5]. O. Cornea, G. Lupton, J. Oprea and D. Tanré observe that a Lusternik-Schnirelmann

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category is like a Picasso painting [5], *i.e.*, completely different perspectives of the LS-cat lead to completely different impressions of the beauty and applicability of the LS-cat.

The Lusternik-Schnirelmann category admits two common normalization conventions. In the normalized convention, $\mathrm{cat}(X)$ is the least integer $n \geq 0$ such that X admits a cover by $n+1$ subsets that are categorical in X. In the unnormalized convention, it is the least positive integer k such that X admits a cover by k subsets that are categorical in X. A subset $U \subseteq X$ is categorical in X provided that the inclusion map $U \hookrightarrow X$ is homotopic to a constant map. Throughout this paper, we use the unnormalized convention; consequently, a contractible space has LS-category equal to 1. If no finite categorical cover exists, the LS-category is defined to be ∞ [5, 15]. The LS-category is an invariant of homotopy type [2].

The category **Prox** whose objects are proximity spaces is a subcategory of the semi-proximities category **Sprox** introduced by Á. Császár [6, Corollary 3.6, p. 12]. An object in **Prox** is a proximity relation δ provided δ is a Čech proximity (see Appendix A).

This paper also introduces the category **ProxBor** of proximal bornological spaces and proximally continuous proper maps [24, pp. 20-21]. **ProxBor** has objects that are proximal bornological spaces and morphisms that are proximally continuous maps between proximal bornological spaces.

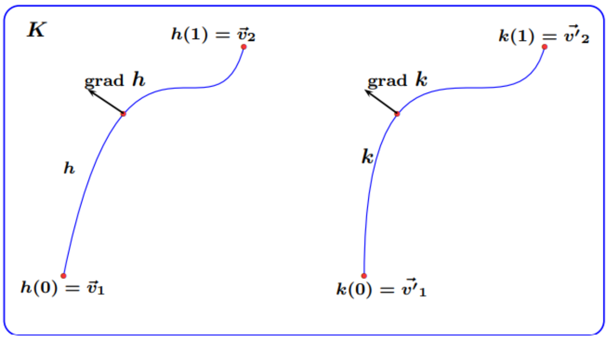

Figure 1: Descriptively proximal paths

2. Preliminaries

2.1. Importance of Descriptive Proximal Homotopy

Descriptive proximal homotopy is important, since this paves the way for homotopies between disjoint paths that are descriptively close. For an introduction to descriptive Čech proximity, see Appendix C.

2.2. Basic Definitions

Definition 2.1 *Let $I = [0,1]$. A path in a topological space X from x_0 to x_1 is a continuous map $h : I \to X$ such that*

$$h(0) = x_0, \qquad h(1) = x_1.$$

Its geometric realization is the image

$$|h| = h(I).$$

Example 2.2 *Let the description of a planar path h be defined in terms of the gradient* $\mathbf{grad}(h)$, *which corresponds to the volume derivative (representing the local rate of space occupancy) describing a small movement along path h,* i.e.,

$$\mathbf{grad}(h) = \frac{\partial h}{\partial x} i + \frac{\partial h}{\partial y} j.$$

In Figure 1, paths h, k are descriptively proximal, since $\mathbf{grad}(h) = \mathbf{grad}(k)$ *at corresponding points along paths h and k.*

Definition 2.3 [27, p. 5] *A map $f : (X, \delta_1) \to (Y, \delta_2)$ between two proximity spaces is proximally continuous (pc) provided f preserves proximity,* i.e., *$A \ \delta_1 \ B$ implies $f(A) \ \delta_2 \ f(B)$ for $A, B \in 2^X$. Further, a bijective pc map f with a pc converse is called a proximal isomorphism.*

Descriptive proximally continuous maps on descriptive proximity spaces are useful in this work. Recall that in a descriptive Čech proximity space (X, δ_Φ), every $A \in 2^X$ has a description $\Phi(A) \in \mathbb{R}^n$ and there is a relation $A \ \delta_\Phi \ B$ for every pair $A, B \in 2^X$ [18, §4.3.1]. In other words, Čech proximity space equipped with descriptive proximity relation δ_Φ is a descriptive proximity space. For a pure Čech proximity space, see Appendix A. Throughout this paper when we say descriptive proximity space, descriptive Čech proximity space will be understood.

Definition 2.4 [20, §1.2.1, p. 48] *Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be a pair of descriptive proximity spaces. A map $f : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is descriptive proximally continuous (dpc) provided $A \ \delta_{\Phi_1} \ B$ implies $f(A) \ \delta_{\Phi_2} \ f(B)$ for $A, B \subset X$. Further, a bijective dpc map f with a dpc inverse is called a descriptive proximal isomorphism.*

Definition 2.5 [23] *Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces and $f, g : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ dpc maps. Then we say f and g are descriptive proximally homotopic provided there exists a dpc map $H : X \times [0,1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Such a map H is called a descriptive proximal homotopy between f and g. We denote $f \underset{\Phi}{\sim} g$ provided there exists a dph between them.*

Definition 2.6 [23] *A map $d : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ between descriptive proximity spaces is said to be a descriptive constant provided $d(x) = y_0$ for all $x \in X$ and for some $y_0 \in Y$.*

Definition 2.7 [23] *A descriptive proximity space is descriptively contractible provided the identity map on it is descriptive proximally homotopic to a descriptive constant map.*

Definition 2.8 [23] *A map $d : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ between descriptive proximity spaces is said to be a degenerate descriptive constant provided $\Phi_2(d(x_0)) = \Phi_2(d(x_1))$ for all $x_0, x_1 \in X$.*

Note that a degenerate descriptive constant map is a dpc map [23, Theorem 3].

Definition 2.9 [23] *A descriptive proximity space is a degenerate descriptively contractible provided the identity map on it is descriptive proximally homotopic to a degenerate descriptive constant map.*

A degenerate descriptively contractible space is also a descriptively contractible [23, Corollary 2].

3. Fibrations on Proximity Spaces

This section introduces fibration on a proximity space. Before going into that, we should discuss the classical concept of fibration. Recall that a map $p : E \to B$ has the homotopy lifting property (HLP) with respect to X provided for every map $f : X \to E$ and homotopy $G : X \times I \to B$ such that $p \circ f(x) = G(x, 0)$, there exists a homotopy $H : X \times I \to E$ with $p \circ H = G$ and $H(x, 0) = f(x)$. The diagram is as follows.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "G"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

The map p is said to be a fibration provided it has the HLP for all spaces X [28, p. 52].

For a fibration $p : E \to B$, a fiber is an inverse image of elements b of B and for two distinct elements in B, their corresponding fibers are homotopy equivalent and the fiber of the fibration is denoted by F. We also denote a fibration by $F \to E \to B$, where the first map is an inclusion and the second map is the fibration.

Definition 3.1 [28, p. 56] *A continuous surjective map $p : E \to B$ is said to be a fibre bundle with fibre F provided $p^{-1}(b) = F$ for every $b \in B$ and for every $b \in B$ there exist an open neighborhood $b \in U_b \subset B$ and a homeomorphism $\varphi : p^{-1}(U_b) \to U_b \times F$ in such a way that p agrees with the projection onto the first factor. The diagram is as follows.*

$$\begin{tikzcd} p^{-1}(U_b) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U_b \times F \arrow[dl, "pr_1"] \\ & U_b & \end{tikzcd}$$

For spaces X and Y, let Y^X be a space of all continuous maps $X \to Y$. Then let $\hat{P}(X,x_0) \subset X^I$ be the subspace of paths ending at x_0 and $\hat{\Omega}(X,x_0) \subset \hat{P}(X,x_0)$ be the subspace of paths that begin and end at x_0. Then, from [9, p. 29], we obtain the fibration

$$\hat{\Omega}(X,x_0) \longrightarrow \hat{P}(X,x_0) \xrightarrow{p} X$$

$p : \gamma \mapsto \gamma(0)$, and $\hat{P}X$ is contractible.

In the context of proximity spaces, the concepts of fibre bundle and fibration are introduced in the following manner.

Definition 3.2 *A proximally continuous surjective map $p : (E,\delta_1) \to (B,\delta_2)$ is said to be a fibre bundle with fibre F provided $p^{-1}(b) = F$ for every $b \in B$ and for every $b \in B$, there exist an open neighborhood $b \in U_b \subset B$ and a proximal isomorphism $\varphi : p^{-1}(U_b) \to U_b \times F$ in such a way that p agrees with the projection onto the first factor. The diagram is as follows.*

$$\begin{tikzcd} p^{-1}(U_b) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U_b \times F \arrow[dl, "pr_1"] \\ & U_b & \end{tikzcd}$$

Definition 3.3 *A map $p : (E,\delta_1) \to (B,\delta_2)$ has the proximal homotopy lifting property (pHLP) with respect to a proximity space (X,δ_3) provided for every map $f : X \to E$ and proximal homotopy (see Definition B.3) $G : X \times I \to B$ such that $p \circ f(x) = G(x,0)$, there exists a proximal homotopy $H : X \times I \to E$ with $p \circ H = G$ and $H(x,0) = f(x)$. The diagram is as follows.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "G"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

The map p is said to be a proximal fibration provided it has the pHLP for all proximity spaces X.

3.1. NDR and DR Pairs for (Descriptive) Proximity Spaces

Recall that for a topological space X and a pair $A \subset X$, the pair (X,A) is an NDR pair (neighborhood deformation retract pair) provided there exist continuous maps $h : X \to I$ and $H : X \times I \to X$ such that

- $h^{-1}(0) = A$,
- $H(x,0) = x$ for every $x \in X$,
- $H(a,t) = a$ for every $a \in A$ and $t \in I$,
- $H(x,1) \in A$ whenever $h(x) < 1$

and (X,A) is a DR pair (deformation retract pair) provided it is an NDR pair and $h(x) < 1$ for all $x \in X$.

Proposition 3.4 [9, p. 15] *If (X,A) and (Y,B) are NDR pairs, then $(X \times Y, X \times B \cup A \times Y)$ is also an NDR pair.*

Definition 3.5 *Let (X,δ) be a proximity space and $A \subset X$. The pair (X,A) is an NpDR pair (neighborhood proximal deformation retract pair) provided there exist proximally continuous maps $h : X \to I$ and $H : X \times I \to X$ such that*

- $h^{-1}(0) = A$,
- $H(x,0) = x$ *for every* $x \in X$,
- $H(a,t) = a$ *for every* $a \in A$ *and* $t \in I$,
- $H(x,1) \in A$ *whenever* $h(x) < 1$

and (X,A) is a pDR pair (proximal deformation retract pair) if $h(x) < 1$ for all $x \in X$.

Proposition 3.6 *Suppose that (X,δ_1) and (Y,δ_2) are Čech proximity spaces, $A \subseteq X$, and $B \subseteq Y$. If (X,A) and (Y,B) are NpDR pairs, then*

$$\big(X \times Y, (A \times Y) \cup (X \times B)\big)$$

is an NpDR pair.

Proof Let (h,H) and (k,K) be NpDR data for (X,A) and (Y,B), respectively. Define

$$m : X \times Y \longrightarrow I, \qquad m(x,y) = \min\{h(x), k(y)\}.$$

Then

$$\begin{aligned} m(x,y) = 0 &\Leftrightarrow h(x) = 0 \text{ or } k(y) = 0 \\ &\Leftrightarrow x \in A \text{ or } y \in B \\ &\Leftrightarrow (x,y) \in (A \times Y) \cup (X \times B). \end{aligned}$$

Consequently,

$$m^{-1}(0) = (A \times Y) \cup (X \times B).$$

Set

$$R = \{(x,y,t) \in (X \times Y) \times I : k(y) \geq h(x)\}$$

and

$$S = \{(x,y,t) \in (X \times Y) \times I : h(x) \geq k(y)\}.$$

Since proximally continuous maps are continuous with respect to the induced topologies, the map

$$(x,y,t) \longmapsto \big(h(x), k(y)\big)$$

is continuous. Moreover,

$$R = \big\{(x,y,t) : k(y) \geq h(x)\big\}$$

and

$$S = \big\{(x,y,t) : h(x) \geq k(y)\big\}$$

are the inverse images of closed subsets of $I \times I$. Consequently, R and S are closed subsets of $(X \times Y) \times I$. Furthermore,

$$R \cup S = (X \times Y) \times I.$$

Define $M_R : R \to X \times Y$ by

$$M_R(x,y,t) = \begin{cases} \left(H(x,t), K\left(y, t\dfrac{h(x)}{k(y)}\right)\right), & k(y) > 0, \\ (x,y), & k(y) = 0, \end{cases}$$

and define $M_S : S \to X \times Y$ by

$$M_S(x,y,t) = \begin{cases} \left(H\left(x, t\dfrac{k(y)}{h(x)}\right), K(y,t)\right), & h(x) > 0, \\ (x,y), & h(x) = 0. \end{cases}$$

If $k(y) = 0$ on R, then

$$0 = k(y) \geq h(x) \geq 0,$$

and hence $h(x) = 0$. Therefore $x \in A$ and $y \in B$. Similarly, if $h(x) = 0$ on S, then $k(y) = 0$. Thus the definitions above remove the possible $0/0$ expressions.

The restrictions M_R and M_S are proximally continuous on their respective domains. On $R \cap S$, one has

$$h(x) = k(y).$$

If this common value is positive, both formulas give

$$\big(H(x,t), K(y,t)\big).$$

If the common value is zero, both formulas give (x, y). Hence $M_R = M_S$ on $R \cap S$.

Since R and S are closed, satisfy

$$R \cup S = (X \times Y) \times I,$$

and $M_R = M_S$ on $R \cap S$, the proximal gluing lemma [23, Lemma 2.4, p. 28] shows that M_R and M_S determine a proximally continuous map

$$M : (X \times Y) \times I \longrightarrow X \times Y.$$

We now verify the NpDR conditions. First,

$$M(x, y, 0) = (x, y)$$

for every $(x, y) \in X \times Y$.

Next, suppose that $(x, y) \in (A \times Y) \cup (X \times B)$. If $x \in A$, then $h(x) = 0$. Hence the first coordinate remains x, while the second coordinate is evaluated at time 0 and remains y. Similarly, if $y \in B$, then $k(y) = 0$, and both coordinates remain fixed. Therefore

$$M((x, y), t) = (x, y)$$

for every $(x, y) \in (A \times Y) \cup (X \times B)$ and $t \in I$.

Finally, suppose that

$$m(x, y) < 1.$$

If $k(y) \geq h(x)$, then $h(x) < 1$, and at $t = 1$ the first coordinate is

$$H(x, 1) \in A.$$

If $h(x) \geq k(y)$, then $k(y) < 1$, and at $t = 1$ the second coordinate is

$$K(y, 1) \in B.$$

Thus

$$M(x, y, 1) \in (A \times Y) \cup (X \times B).$$

Therefore (m, M) gives NpDR data for

$$\big(X \times Y, (A \times Y) \cup (X \times B)\big).$$

$\square$

Remark 3.7 *The expressions*

$$t\frac{h(x)}{k(y)} \qquad \text{and} \qquad t\frac{k(y)}{h(x)}$$

occur as time reparametrizations inside the full restrictions M_R and M_S. The relevant proximal continuity statement concerns these complete maps, rather than the scalar reparametrizations considered separately.

At the common zero set

$$h(x) = k(y) = 0,$$

we define

$$M(x, y, t) = (x, y).$$

This is consistent because $x \in A$, $y \in B$, and the homotopies H and K fix A and B, respectively, throughout the homotopies.

Definition 3.8 *Let (X, δ_Φ) be a descriptive proximity space and $A \subset X$. The pair (X, A) is an NdpDR pair (neighborhood descriptive proximal deformation retract pair) provided there exist descriptive proximally continuous maps $h : X \to I$ and $H : X \times I \to X$ such that*

- $h^{-1}(0) = A$,
- $H(x, 0) = x$ *for every* $x \in X$,
- $H(a, t) = a$ *for every* $a \in A$ *and* $t \in I$,
- $H(x, 1) \in A$ *whenever* $h(x) < 1$

and (X, A) is a dpDR pair (descriptive proximal deformation retract pair) if $h(x) < 1$ for all $x \in X$.

Proposition 3.9 *Suppose that (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) are descriptive proximity spaces, $A \subset X$ and $B \subset Y$. If (X, A) and (Y, B) are NdpDR pairs, then $(X \times Y, X \times B \cup A \times Y)$ is an NdpDR pair.*

Proof The result follows from the same piecewise construction used in the proof of Proposition 3.6. Once the two restrictions are verified to be descriptive proximally continuous on the corresponding descriptively closed subsets and to agree on their intersection, the descriptive gluing theorem [23, Theorem 2.9, p. 29] shows that the resulting map M is descriptive proximally continuous. The pair (m, M) therefore satisfies the NdpDR conditions. □

4. Lusternik-Schnirelmann Category of Proximity Spaces

Throughout this section, the unnormalized convention is used. Accordingly, the LS-category is the minimum number of categorical sets in a suitable open cover, and a contractible space has

LS-category equal to 1.

4.1. Spatial Proximity Case: Prox Refinement

An open cover of a proximity space (X, δ) is a family of open subsets $\{U_i\}_{i \in I}$ such that

$$X = \bigcup_{i \in I} U_i.$$

Let U be a nonempty subset of a proximity space (X, δ), equipped with the proximity induced from X. The subset U is said to be proximally categorical in X provided that the inclusion map

$$\iota_U : U \hookrightarrow X$$

is proximally homotopic to a constant map

$$c_{x_U} : U \longrightarrow X, \qquad c_{x_U}(u) = x_U$$

for some $x_U \in X$.

Definition 4.1 *Let (X, δ) be a proximity space. The proximal Lusternik–Schnirelmann category of X, denoted by $\mathrm{LS}_\delta(X)$ and briefly called the proximal LS-category or proxLS, is the least positive integer k for which there exists an open cover*

$$X = U_1 \cup \cdots \cup U_k$$

such that every U_i is proximally categorical in X. If no such finite open cover exists, we set

$$\mathrm{LS}_\delta(X) = \infty.$$

Theorem 4.2 *A proximity space (X, δ) is proximally contractible if and only if*

$$\mathrm{LS}_\delta(X) = 1.$$

Proof Suppose first that X is proximally contractible. Then the identity map

$$1_X : X \longrightarrow X$$

is proximally homotopic to a constant map. Hence X is proximally categorical in itself. The one-member open cover $\{X\}$ therefore shows that

$$\mathrm{LS}_\delta(X) \le 1.$$

Since $\mathrm{LS}_\delta(X)$ is defined as a positive integer whenever it is finite, it follows that

$$\mathrm{LS}_\delta(X) = 1.$$

Conversely, suppose that

$$\mathrm{LS}_{\delta}(X) = 1.$$

Then X admits a one-member open cover $\{U_1\}$ such that U_1 is proximally categorical in X. Since U_1 covers X, we have $U_1 = X$. Consequently, the inclusion

$$\iota_{U_1} : U_1 \hookrightarrow X$$

is the identity map 1_X, and it is proximally homotopic to a constant map. Therefore, X is proximally contractible. □

4.2. Descriptive Proximity Case: desProx

A descriptive open cover of a descriptive proximity space (X, δ_Φ) is a family of descriptively open subsets $\{U_i\}_{i \in I}$ such that

$$X = \bigcup_{i \in I} U_i.$$

Let U be a nonempty subset of a descriptive proximity space (X, δ_Φ), equipped with the descriptive proximity induced from X. The subset U is said to be descriptively categorical in X provided that the inclusion map

$$\iota_U : U \hookrightarrow X$$

is descriptive proximally homotopic to a descriptive constant map

$$c_{x_U} : U \longrightarrow X, \qquad c_{x_U}(u) = x_U$$

for some $x_U \in X$.

Definition 4.3 *Let (X, δ_Φ) be a descriptive proximity space. The descriptively proximal Lusternik–Schnirelmann category of X, denoted by $\mathrm{LS}_{\delta_\Phi}(X)$ and briefly called the descriptive LS-category or dproxLS, is the least positive integer k for which there exists a descriptive open cover*

$$X = U_1 \cup \cdots \cup U_k$$

such that every U_i is descriptively categorical in X. If no such finite descriptive open cover exists, we set

$$\mathrm{LS}_{\delta_\Phi}(X) = \infty.$$

Theorem 4.4 *A descriptive proximity space (X, δ_Φ) is descriptively contractible if and only if*

$$\mathrm{LS}_{\delta_\Phi}(X) = 1.$$

Proof Suppose first that X is descriptively contractible. Then the identity map

$$1_X : X \longrightarrow X$$

is descriptive proximally homotopic to a descriptive constant map. Hence X is descriptively categorical in itself, and the one-member descriptive open cover $\{X\}$ shows that

$$\mathrm{LS}_{\delta_\Phi}(X) \le 1.$$

Since $\mathrm{LS}_{\delta_\Phi}(X)$ is defined as a positive integer whenever it is finite, we obtain

$$\mathrm{LS}_{\delta_\Phi}(X) = 1.$$

Conversely, suppose that

$$\mathrm{LS}_{\delta_\Phi}(X) = 1.$$

Then X admits a one-member descriptive open cover $\{U_1\}$ such that U_1 is descriptively categorical in X. Since $U_1 = X$, its inclusion into X is the identity map 1_X. Therefore, 1_X is descriptive proximally homotopic to a descriptive constant map, and hence X is descriptively contractible. □

Let U be a nonempty subset of a descriptive proximity space (X, δ_Φ), equipped with the descriptive proximity induced from X. The subset U is said to be degenerately descriptively categorical in X provided that the inclusion map

$$\iota_U : U \hookrightarrow X$$

is descriptive proximally homotopic to a degenerate descriptive constant map.

Definition 4.5 *Let (X, δ_Φ) be a descriptive proximity space. The degenerate descriptively proximal Lusternik–Schnirelmann category of X, denoted by*

$$\mathrm{dLS}_{\delta_\Phi}(X),$$

is the least positive integer k for which there exists a descriptive open cover

$$X = U_1 \cup \cdots \cup U_k$$

such that every U_i is degenerately descriptively categorical in X. If no such finite descriptive open cover exists, we set

$$\mathrm{dLS}_{\delta_\Phi}(X) = \infty.$$

Theorem 4.6 *A descriptive proximity space (X, δ_Φ) is degenerate descriptively contractible if and only if*

$$\mathrm{dLS}_{\delta_\Phi}(X) = 1.$$

Proof If X is degenerate descriptively contractible, then X is degenerately descriptively categorical in itself. Hence the one-member descriptive open cover $\{X\}$ gives

$$\mathrm{dLS}_{\delta_\Phi}(X) = 1.$$

Conversely, if

$$\mathrm{dLS}_{\delta_\Phi}(X) = 1,$$

then X admits a one-member descriptive open cover $\{U_1\}$ in which U_1 is degenerately descriptively categorical in X. Since $U_1 = X$, the identity map on X is descriptive proximally homotopic to a degenerate descriptive constant map. Therefore, X is degenerate descriptively contractible. $\square$

5. desProx & Császár Prox Categories

This section introduces a refinement of the Császár **Prox** category and its extension, namely, the descriptive proximity **desProx** category, which includes descriptively proximally continuous maps between descriptive proximity spaces (see Appendix C) as morphisms.

5.1. Császár Prox Category

Let δ_1 and δ_2 be two nearness relations defined on the same set X. We say that δ_2 is finer than δ_1 provided for all subsets A, B of X, $A\ \delta_1\ B$ implies $A\ \delta_2\ B$ [11].

Let (X, δ) be a proximity space and Y be a nonempty set. For a surjective function $f : X \to Y$, the finest proximity, δ^f, on Y that makes f proximally continuous is defined as follows: Two subsets C and D in Y are near, $C\ \delta^f\ D$ provided $f^{-1}(C)\ \delta\ f^{-1}(D)$.

Definition 5.1 *A nearness relation δ^f on Y is called an identification proximity on Y and such a function f is called a proximal identification .*

Proposition 5.2 *δ^f is a proximity on Y.*

Proof We show that (Y, δ^f) satisfies the proximity axioms. Let C, D, and E be subsets of Y.

$(P_f.0)$ $C\ \not\delta^f\ \varnothing$ since X is a proximity space so that $f^{-1}(C)\ \not\delta^f\ \varnothing$.

$(P_f.1)$ It is clear that if $C\ \delta^f\ D$ implies $D\ \delta^f\ C$.

$(P_f.2)$ If $C \cap D \neq \varnothing$, then $f^{-1}(C) \cap f^{-1}D \neq \varnothing$ so that $f^{-1}(C)\ \delta\ f^{-1}D$. It follows that $C\ \delta^f\ D$.

$(P_f.3)$ Let $C\ \delta^f\ (D \cup E)$. It follows that $f^{-1}(C)\ \delta\ f^{-1}(D \cup E) = f^{-1}(D) \cup f^{-1}(E)$. Since X is a proximity space, we have $f^{-1}(C)\ \delta\ f^{-1}(D)$ or $f^{-1}(C)\ \delta\ f^{-1}(E)$. This shows that $C\ \delta^f\ D$ or $C\ \delta^f\ E$.

□

Here, the Császár **Prox** category is refined to include proximal continuous maps between proximity spaces as morphisms. In this category, a singleton set with a proximity relation is a terminal object.

Definition 5.3 *Let $\{(X_i, \delta_i)\}_{i \in I}$ be a nonempty family of Čech proximity spaces and let $Z = \prod_{i \in I} X_i$. Define a relation β_Π on subsets of Z by*

$$A \, \beta_\Pi \, B \quad \Longleftrightarrow \quad \mathrm{pr}_i(A) \, \delta_i \, \mathrm{pr}_i(B) \quad \textit{for every } i \in I.$$

The product proximity δ_Π is the Čech proximity generated by β_Π through the standard finite-cover construction [26]: for nonempty $A, B \subseteq Z$,

$$A \, \delta_\Pi \, B$$

if and only if, for every pair of finite covers

$$A = \bigcup_{r=1}^{m} A_r, \qquad B = \bigcup_{s=1}^{n} B_s,$$

there exist r and s such that $A_r \, \beta_\Pi \, B_s$. If either set is empty, they are declared far.

Proposition 5.4 *The pair $(\prod_{i \in I} X_i, \delta_\Pi)$ is a Čech proximity space, and every projection*

$$\mathrm{pr}_i : (\prod_{i \in I} X_i, \delta_\Pi) \to (X_i, \delta_i)$$

is proximally continuous.

Proof The Čech proximity axioms follow from the finite-cover construction in Definition 5.3. In particular, the union axiom is obtained by refining finite covers, as described above. If $A \delta_\Pi B$, the one-member covers $\{A\}$ and $\{B\}$ imply $A \beta_\Pi B$. Hence $\mathrm{pr}_i(A) \delta_i \, \mathrm{pr}_i(B)$ for every i, and each projection is proximally continuous. □

Definition 5.5 [11, Definition 3.1] *Let $\{(X_j, \delta_j)\}_{j \in J}$ be a family of proximity spaces for an indexed set J and $\sqcup X_j$ the disjoint union of X_j's. Then define a relation $\delta^\sqcup$ on the subsets A and B of $\sqcup X_j$ by declaring that $A \; \delta^\sqcup \; B$ if and only if $A_i \; \delta_i \; B_i$ for some $i \in J$, where $A_i = \sqcup X_j \cap X_i$.*

Proposition 5.6 [11, Proposition 3.2] *Space $(\sqcup X_j, \delta^\sqcup)$ is a proximity space.*

Remark 5.7 *Proposition 5.6 is still valid for Čech proximity spaces, since Efremovic proximity implies Čech proximity.*

Theorem 5.8 *The category* **Prox** *has nonempty products and coproducts.*

Proof Let $\{(X_i, \delta_i)\}_{i\in I}$ be a nonempty family of Čech proximity spaces, and let

$$\left(\prod_{i\in I} X_i, \delta_\Pi\right)$$

be the product proximity space introduced in Definition 5.3, with coordinate projections

$$\mathrm{pr}_i : \prod_{i\in I} X_i \longrightarrow X_i.$$

Let (Y, δ_Y) be a Čech proximity space and let

$$f_i : (Y, \delta_Y) \longrightarrow (X_i, \delta_i)$$

be proximally continuous for every $i \in I$. Define

$$f : Y \longrightarrow \prod_{i\in I} X_i, \qquad f(y) = \big(f_i(y)\big)_{i\in I}.$$

Then

$$\mathrm{pr}_i \circ f = f_i$$

for every $i \in I$.

We show that f is proximally continuous. Let $C, D \subseteq Y$ satisfy

$$C\, \delta_Y\, D.$$

Consider arbitrary finite covers

$$f(C) = \bigcup_{r=1}^{m} A_r, \qquad f(D) = \bigcup_{s=1}^{n} B_s.$$

Define

$$C_r = C \cap f^{-1}(A_r), \qquad D_s = D \cap f^{-1}(B_s).$$

Then $\{C_r\}_{r=1}^m$ and $\{D_s\}_{s=1}^n$ are finite covers of C and D, respectively. By repeated application of the finite-union axiom, there exist indices r and s such that

$$C_r\, \delta_Y\, D_s.$$

Since every f_i is proximally continuous,

$$f_i(C_r)\, \delta_i\, f_i(D_s)$$

for every $i \in I$. Moreover,

$$f_i(C_r) \subseteq \mathrm{pr}_i(A_r), \qquad f_i(D_s) \subseteq \mathrm{pr}_i(B_s).$$

By monotonicity of δ_i,

$$\mathrm{pr}_i(A_r)\,\delta_i\,\mathrm{pr}_i(B_s)$$

for every $i \in I$. Hence

$$A_r\,\beta_\Pi\,B_s.$$

Since the two finite covers were arbitrary, Definition 5.3 gives

$$f(C)\,\delta_\Pi\,f(D).$$

Therefore, f is proximally continuous.

The map f is unique, since any map

$$g : Y \longrightarrow \prod_{i\in I} X_i$$

satisfying $\mathrm{pr}_i \circ g = f_i$ for every $i \in I$ must satisfy

$$g(y) = \big(f_i(y)\big)_{i\in I} = f(y)$$

for every $y \in Y$. Thus

$$\left(\prod_{i\in I} X_i, \delta_\Pi\right),$$

together with the coordinate projections, satisfies the universal property of the categorical product in **Prox**. Similarly, given a family of proximity spaces $\{(X_j, \delta_j)\}_{j\in J}$ and the coproduct (disjoint union) of its members

$$\sqcup X_j = \cup\big(X_j \times \{j\}\big)$$

with the family of inclusion maps $\iota_k : X_k \to \sqcup X_j$. Then for any proximity space Y with a family of (pc) functions $f_k : X_k \to Y$, there is a universal (pc) map $f : \sqcup X_j \to Y$ defined by $f(a,k) = f_k(a)$ so that the following diagram commutes. □

$$\begin{tikzcd}
\sqcup X_j \arrow[dr, dashed, "f"] & \\
X_k \arrow[u, "\iota_j"] \arrow[r, "f_k"'] & Y
\end{tikzcd}$$

5.2. Descriptive Proximity Case

Let δ_{Φ_1} and δ_{Φ_2} be two descriptive nearness relations defined on a set X. We say that δ_{Φ_2} is finer than δ_{Φ_1} provided for all subsets A, B of X, $A\ \delta_{\Phi_1}\ B$ implies $A\ \delta_{\Phi_2}\ B$.

Let (X, δ_Φ) be a descriptive proximity space and Y be a nonempty set. For a surjective function $f : X \to Y$, the finest descriptive proximity, δ_Φ^f, on Y that makes f descriptive proximally continuous is defined as follows: Two subsets C and D in Y are said to be descriptively near, $C \; \delta_\Phi^f \; D$, provided $f^{-1}(C) \; \delta_\Phi \; f^{-1}(D)$.

Definition 5.9 *Such a descriptive nearness relation δ_Φ^f on Y is called a descriptively identification proximity on Y and f is called a descriptive identification.*

Proposition 5.10 *δ_Φ^f is a descriptive proximity on Y.*

Proof We show that (Y, δ_Φ^f) satisfies the descriptive proximity axioms. Let C, D, and E be subsets of Y.

$(dP_f.0)$ $C \; \not\delta_\Phi^f \; \varnothing$ since X is a descriptive proximity space so that $f^{-1}(C) \; \not\delta_\Phi^f \; \varnothing$.

$(dP_f.1)$ It is clear that if $C \; \delta_\Phi^f \; D$ implies $D \; \delta_\Phi^f \; C$.

$(dP_f.2)$ If $C \underset{\Phi}{\cap} D \neq \varnothing$, then $f^{-1}(C) \underset{\Phi}{\cap} f^{-1}D \neq \varnothing$ so that $f^{-1}(C) \; \delta_\Phi \; f^{-1}D$. It follows that $C \; \delta_\Phi^f \; D$.

$(dP_f.3)$ Let $C \; \delta_\Phi^f \; (D \cup E)$. It follows that $f^{-1}(C) \; \delta_\Phi \; f^{-1}(D \cup E) = f^{-1}(D) \cup f^{-1}(E)$. Since X is a descriptive proximity space, we have $f^{-1}(C) \; \delta_\Phi \; f^{-1}(D)$ or $f^{-1}(C) \; \delta_\Phi \; f^{-1}(E)$. This shows that $C \; \delta_\Phi^f \; D$ or $C \; \delta_\Phi^f \; E$.

$\square$

Recall that a continuous map $f : X \to X$ and $A, B \subset X$, if $A \; \delta_\Phi \; B$ implies $f(A) \; \delta_\Phi \; f(B)$, then f is descriptive proximally continuous map [22, p. 386].

Definition 5.11 *The* **desProx** *category is an extension of the Császár* **Prox** *category that includes descriptive proximally continuous maps between descriptive proximity spaces as morphisms. For a brief introduction to descriptive proximity, see Appendix C.*

6. Proximity Spaces with Boundedness (Bornology)

A bornology $\mathcal{B}$ on a nonempty subset X is a collection of subsets of X such that

- $X = \cup_{B \in \mathcal{B}} B$,
- if $B \in \mathcal{B}$ and $A \subset B$, then $A \in \mathcal{B}$,
- if $B_1, B_2 \in \mathcal{B}$, then $B_1 \cup B_2 \in \mathcal{B}$.

The elements of $\mathcal{B}$ are called bounded subsets of X and the pair $(X,\mathcal{B})$ is called a bounded structure [3].

Definition 6.1 [3, p. 15] *Let $f:X\to Y$ be a map between sets equipped with bornologies $\mathcal{B}$ and $\mathcal{B}'$, respectively. Then we say that*

- *f is proper provided for every $B'\in\mathcal{B}'$, we have $f^{-1}(B')\in\mathcal{B}$, and*
- *f bornological provided for every $B\in\mathcal{B}$, we have $f(B)\in\mathcal{B}'$.*

Definition 6.2 *Let (X,δ) be a Čech proximity space and $\mathcal{B}$ a bornology on X. Then δ and $\mathcal{B}$ are said to be compatible provided every bounded subset has a bounded closure and a bounded proximal open neighborhood.*

That is,

- If $B\in\mathcal{B}$, then $c\ell B\in\mathcal{B}$, and
- For $B\in\mathcal{B}$, there is an open set $C\in\mathcal{B}$ such that $B\ll C$.

Definition 6.3 *A proximal bornological space is a triple $(X,\delta,\mathcal{B})$ consisting of a set with a nearness relation δ and a bornology $\mathcal{B}$ which are compatible.*

Definition 6.4 *A morphism f between two proximal bornological spaces $(X,\delta,\mathcal{B})$ and $(X',\delta',\mathcal{B}')$ is a map $f:X\to X'$ which is proximally continuous and proper.*

Let **ProxBor** denote the category whose objects are proximal bornological spaces and whose morphisms are proximally continuous and proper maps.

Theorem 6.5 *The category* **ProxBor** *has all nonempty coproducts.*

Proof Let $\{(X_j,\delta_j,\mathcal{B}_j)\}$ be a family of proximal bornological spaces. The coproduct of this family is the proximal bornological space $(X,\delta,\mathcal{B})$, where the pair (X,δ) is the coproduct proximity space introduced in Definition 5.5 and the bornology $\mathcal{B}$ is given by

$$\mathcal{B}:=\{B\subseteq X\ :\iota_j^{-1}(B)=B\cap X_j\in\mathcal{B}_j,\forall j\in J\}.$$

Here each $\iota_j:X_j\hookrightarrow X$ is an inclusion: It is proximally continuous and for a bounded set $B\in\mathcal{B}$, $\iota_j^{-1}(B)=B\cap X_j\in\mathcal{B}_j$, so that ι_j is proper.

Notice that δ and $\mathcal{B}$ are compatible: For a bounded set $B\in\mathcal{B}$ in X, we have $X_j\cap B\in\mathcal{B}_j$. Therefore $c\ell B\cap X_j=c\ell_{\delta_j}(B\cap X_j)\in\mathcal{B}_j$.

Further, for each $j \in J$, there is an open set $C_j \in \mathcal{B}_j$ such that

$$B \cap X_j \ll C_j.$$

Hence

$$B \ll C, \qquad C = \bigsqcup_{j \in J} C_j.$$

Therefore, δ and $\mathcal{B}$ are compatible.

It remains to verify the universal property. Let $(Y, \delta_Y, \mathcal{B}_Y)$ be a proximal bornological space and let

$$f_j : X_j \longrightarrow Y$$

be proximally continuous and proper for every $j \in J$. Define

$$f : X = \bigsqcup_{j \in J} X_j \longrightarrow Y, \qquad f|_{X_j} = f_j.$$

By the universal property of the coproduct proximity, f is proximally continuous. If $D \in \mathcal{B}_Y$, then

$$f^{-1}(D) \cap X_j = f_j^{-1}(D) \in \mathcal{B}_j$$

for every $j \in J$. Hence

$$f^{-1}(D) \in \mathcal{B},$$

so f is proper. Its uniqueness follows from its restrictions to the components X_j. Therefore, $(X, \delta, \mathcal{B})$, together with the inclusion maps $\iota_j : X_j \to X$, satisfies the categorical coproduct property in **ProxBor**. □

7. Conclusion

In this paper, we have introduced and investigated spatial and descriptive proximity extensions of fundamental topological structures, specifically the Lusternik-Schnirelmann (LS) category, Császár proximity categories, and bornological spaces. The main contributions and results of this study are summarized as follows:

1. **Proximal LS-Categories (proxLS and dproxLS):** We formulated spatial and descriptive forms of the Lusternik-Schnirelmann category, denoted by $LS_\delta(X)$ and $LS_{\delta_\Phi}(X)$, respectively. We established a fundamental characterization proving that a proximity space (X, δ) is proximally contractible if and only if $LS_\delta(X) = 1$ (and similarly, $LS_{\delta_\Phi}(X) = 1$ for descriptive contractibility).

2. **Fibration and Deformation Retracts in Proximity Spaces:** We extended the classical notions of fibrations, fibre bundles, and neighborhood deformation retracts (NDR pairs) to

proximity spaces, introducing pHLP, NpDR, and NdpDR pairs, and proved that products of such pairs preserve these structural properties.

3. **Category Prox and desProx:** We defined desProx, an extension of Császár's Prox category that incorporates descriptive proximally continuous maps. We proved that Prox admits all nonempty products and coproducts, and we introduced desProx as a category of descriptive proximity spaces and descriptive proximally continuous maps.

4. **Category ProxBor:** By integrating Čech proximity with bornological structures, we introduced the category ProxBor of proximal bornological spaces and showed that ProxBor possesses all nonempty coproducts.

Overall, these results bridge classical homotopy-theoretic invariants with spatial and descriptive proximity structures, offering potential applications in digital topology, topological data analysis, and space physics.

A. Čech Proximity

A nonempty set X equipped with the relation δ is a Čech proximity space (denoted by (X,δ)) [4, §2.5, p. 439] provided the following axioms are satisfied.

ČP.0 All nonempty subsets in X are far from the empty set, *i.e.*, $A \not\delta \varnothing$ for all $A \subseteq X$.

ČP.1 $A \ \delta \ B \Rightarrow B \ \delta \ A$.

ČP.2 $A \cap B \neq \varnothing \Rightarrow A \ \delta \ B$.

ČP.3 $A\delta(B \cup C)$ if and only if $A\delta B$ or $A\delta C$.

Example A.1 (Čech proximities derived from the inverse Hilbert transform)
The inverse Hilbert transform [13] $H(t)$ *is defined by*

$$H(t) = \frac{\pi}{t}(\cos(\pi t) - 1).$$

Consider each pair of geometric A, B *shapes in Figure 2 be defined by*

$$A = 2H(t) \quad \textit{and} \quad B = -H(t).$$

Each of the Čech axioms are satisfied by $\{A, B\}$. *For example,*

$$A \ \delta \ B, \ \textit{since} \ A \cap B \neq \varnothing.$$

A.1. Closure of a Subset in Čech Proximity Space

For a Čech proximity space (X, δ), the closure of a subset A of X is defined as

$$c\ell A = \{x \in X \ : \ \{x\} \ \delta \ A\}.$$

A set is said to be closed provided it is equal to its closure and it is said to be open provided its complement is closed.

Let (X, δ) be a Čech proximity space and B, C are nonvoid subsets of X. We say that C is a proximal neighbour of B provided B and the complement of C are remote, i.e., $B \not\delta \ C^c$. We write $B \ll C$ provided C is a proximal neighbor of B [18, p. 75].

B. Spatial Proximal Homotopy

Basic results for the spatial form of proximal homotopy maps are introduced, here.

We consider the nearness relation on $\mathbb{R}$ defined as follows [18, §1.7, p. 48]. Two subsets A and B of $\mathbb{R}$ are near, denoted $A \delta_d B$, if and only if the infimum distance between the sets [12] $D(A, B) = 0$, where

$$D(A,B) = \begin{cases} \inf\{|a-b| \ : \ a \in A \text{ and } \ b \in B\}, & \text{if } \ A, B \neq \varnothing, \\ \infty, & \text{if } \ A = \varnothing \ \text{ or } \ B = \varnothing. \end{cases}$$

The nearness relation δ_d on $\mathbb{R}$ satisfies Čech proximity axioms so that $(\mathbb{R}, \delta_d)$ is a Čech proximity space.

Remark B.1 *Let $I = [0, 1]$ denote the closed interval in $\mathbb{R}$. Then two nonvoid subsets A and B of I are near provided $D(A, B) = 0$* [17].

Example B.2 *In Figure 2, the sets $A = C$ and $B = D$ are near, since $D(A, B) = 0$. However, the pair of sets $A = C$ and $B = E$ are not near, since $D(A, B) \neq 0$.*

Let I be the unit interval $[0, 1]$. Recall that a homotopy from X to Y is a continuous map $H : X \times I \to Y$ with $H_t : X \to Y$ defined by $H_t(x) = H(x, t)$ for all $t \in I$. The functions F_t are called stages of the homotopy [28, p. 2].

Definition B.3 [21, 23] *Let (X, δ_1) and (Y, δ_2) be Čech proximity spaces and $f, g : (X, \delta_1) \to (Y, \delta_2)$ proximally continuous maps. Then we say f and g are proximally homotopic, provided there exists a proximally continuous map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Such a map H is called a proximal homotopy between f and g. In keeping with Hilton's notation* [14]*, we write $f \underset{\delta}{\sim} g$ provided there is a proximal homotopy between them.*

Definition B.4 [23] *A Čech proximity space is proximally contractible provided the identity map on it is proximally homotopic to a constant map.*

Proposition B.5 *The Čech proximity space* $(\mathbb{R}, \delta_d)$ *is proximally contractible.*

Proof Consider the homotopy $H : \mathbb{R} \times I \to \mathbb{R}$ defined by $H(x,t) = tx$. Observe that $H(x,0) = 0$ and $H(x,1) = x$. Further, for two near subsets $A \times J$ and $B \times K$ of $\mathbb{R} \times I$ satisfy $D(A,B) = 0$ and $D(J,K) = 0$ so that $D(H(A \times J), H(B \times K)) = \inf\{|ja - kb| \ : \ a \in A, \ b \in B, \ j \in J, \ k \in K\} = 0$. This shows that H is proximally continuous. Hence H is a proximal homotopy between the identity map on $\mathbb{R}$ and the constant map at the point 0. □

Proposition B.6 *A proximal retract of a proximally contractible space is proximally contractible.*

Proof Let H be the proximal homotopy between the identity map 1_X on X and a constant map c_x for $x \in X$, $r : X \to A$ be a retraction, and $i : A \to X$ be the inclusion map. Then the composition

$$A \times I \xrightarrow{i \times id} X \times I \xrightarrow{H} X \xrightarrow{r} X$$

is the desired proximal homotopy between the identity map 1_A on A and a constant map at a point in A. □

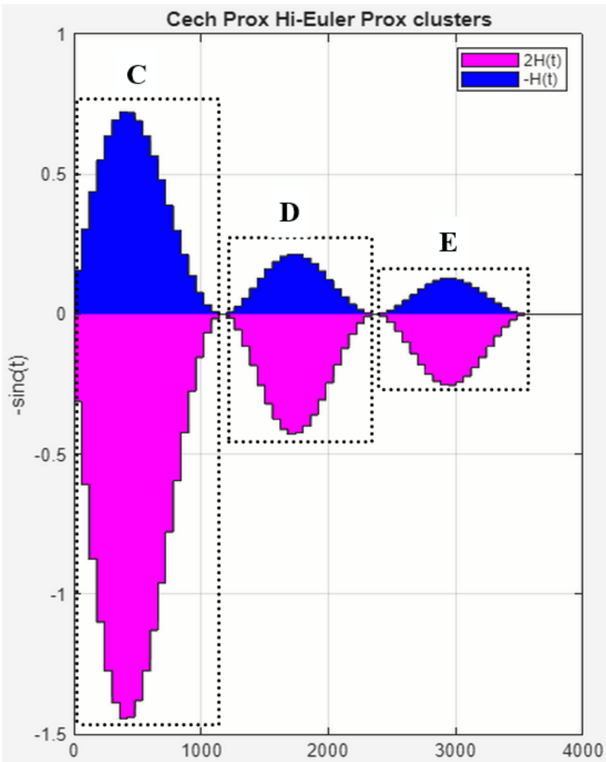

Figure 2: Čech proximal geometric shapes

C. Descriptive Čech Proximity

This section gives the axioms for a descriptive proximity space (X, δ_Φ) in which δ_Φ is a descriptive proximity relation on a nonempty set X. Nonempty sets $A, B \subset X$ with overlapping descriptions are descriptively proximal (denoted by $A\ \delta_\Phi\ B$). The descriptive intersection [7] of nonempty subsets in $A \cup B$ (denoted by $A \underset{\Phi}{\cap} B$) is defined by

$$A \underset{\Phi}{\cap} B = \overbrace{\{x \in A \cup B\ :\ \Phi(x) \in \Phi(A)\ \cap\ \Phi(B)\}}^{\textit{i.e.}, \text{ Descriptions } \Phi(A)\ \&\ \Phi(B) \text{ overlap}}.$$

Let 2^X denote the collection of all subsets in a nonempty set X. A nonempty set X equipped with the relation δ_Φ with nonempty subsets $A, B, C \in 2^X$ is a descriptive proximity space provided the following descriptive forms of the Čech axioms are satisfied.

Descriptive Proximity Axioms [7, pp. 97-98].

(dP.0) All nonempty subsets in 2^X are descriptively far from the empty set, *i.e.*, $A\ \not\delta_\Phi\ \emptyset$ for all $A \in 2^X$.

(dP.1) $A\ \delta_\Phi\ B \Rightarrow B\ \delta_\Phi\ A$.

(dP.2) $A \underset{\Phi}{\cap} B \neq \emptyset \Rightarrow A\ \delta_\Phi\ B$.

(dP.3) $A\ \delta_\Phi\ (B \cup C)$ if and only if $A\ \delta_\Phi\ B$ or $A\ \delta_\Phi\ C$.

For a descriptive Čech proximity space (X, δ_Φ), the descriptive closure of a subset A of X is defined as

$$c_\Phi \ell A = \{x \in X\ :\ \{x\}\ \delta_\Phi\ A\}.$$

A set is said to be descriptively closed provided it is equal to its closure and it is said to be descriptively open provided its complement is descriptively closed.

Example C.1 (Sample descriptively near geometric shapes)
Let C, D, E be the shapes shown in Figure 2. Observe that $C\delta_\Phi D$, $C\delta_\Phi E$, and $D\delta_\Phi E$ are descriptively near, since each pair of shapes has a matching description, if we limit the description such that each of the shapes C, D, E has a jagged edge. Then, for example, observe

$$C \underset{\Phi}{\cap} D = \overbrace{\Phi(C)\ \cap\ \Phi(D) \neq \emptyset}^{\textit{i.e.}, \text{ Descriptions } \Phi(C)\ \&\ \Phi(D) \text{ overlap}}.$$

A similar observation can be made for $C\delta_\Phi E$ and $D\delta_\Phi E$.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and legal special permission.

Authors Contributions

Author [James Francis Peters]: Thought and designed the research/problem, contributed to research method or evaluation of data, and wrote the manuscript (%35).

Author [Tane Vergili]: Thought and designed the research/problem, contributed to research method or evaluation of data, and wrote the manuscript (%35).

Author [Ebubekir İnan]: Contributed to research method or evaluation of data (%30).

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Borat A., Vergili T., *Digital Lusternik-Schnirelmann category*, Turkish Journal of Mathematics, 42(4), 1845-1852, 2018.

[2] Borsuk K., *On the Lusternik-Schnirelmann category in the theory of shape*, Fundamenta Mathematicae, 99(1), 35-42, 1978.

[3] Bunke U., Engel A., Homotopy Theory with Bornological Coarse Spaces, Springer, 2020.

[4] Čech E., Topological Spaces, John Wiley & Sons Ltd., 1966.

[5] Cornea O., Lupton G., Oprea J., Tanré D., Lusternik-Schnirelmann Category, Mathematical Surveys and Monographs, 103, American Mathematical Society, Providence, RI, 2003.

[6] Császár Á., *Proximities, screens, merotopies, uniformities. I*, Acta Mathematica Hungarica, 49(3-4), 459-479, 1987.

[7] Di Concilio A., Guadagni C., Peters J.F., Ramanna S., *Descriptive proximities. Properties and interplay between classical proximities and overlap*, Mathematics in Computer Science, 12(1), 91-106, 2018.

[8] Efremovič V.A., *The geometry of proximity I*, Matematicheskii Sbornik, 31(73)(1), 189-200, 1952.

[9] Félix Y., Halperin S., Thomas J.-C., Rational Homotopy Theory, Graduate Texts in Mathematics, 205, Springer-Verlag, 2001.

[10] Fox R.H., *On the Lusternik-Schnirelmann category*, Annals of Mathematics, 42(2), 333-370, 1941.

[11] Grzegrzolka P., *Coproducts of proximity spaces*, Afrika Matematika, 31, 751-770, 2020.

[12] Hausdorff F., Grundzüge der Mengenlehre, Veit and Company, 1914.

[13] Hilbert D., Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Teubner, 1912; reprinted by Chelsea Publishing Company, 1953.

[14] Hilton P.J., *The Hopf invariant and homotopy groups of spheres*, Proceedings of the Cambridge Philosophical Society, 48, 547-554, 1952.

[15] Lusternik L.A., Schnirelmann L.G., Méthodes Topologiques dans les Problèmes Variationnels. I. Espaces à un Nombre Fini de Dimensions, Hermann, 1934.

[16] Lusternik L.A., Schnirelmann L.G., *Topological methods in variational problems and their application to the differential geometry of surfaces*, Uspekhi Matematicheskikh Nauk, 2(1)(17), 166-217, 1947.

[17] Naimpally S.A., Warrack B.D., Proximity Spaces, Cambridge Tracts in Mathematics and Mathematical Physics, 59, Cambridge University Press, 1970.

[18] Naimpally S.A., Peters J.F., Topology with Applications: Topological Spaces via Near and Far, World Scientific, 2013.

[19] Peters J.F., *Near sets. Special theory about nearness of objects*, Fundamenta Informaticae, 75, 407-433, 2007.

[20] Peters J.F., Topology of Digital Images: Visual Pattern Discovery in Proximity Spaces, Intelligent Systems Reference Library, 63, Springer, 2014.

[21] Peters J.F., Vergili T., *Descriptive fixed set properties for ribbon complexes*, arXiv:2007.04394, 2020.

[22] Peters J.F., Vergili T., *Fixed point property of amenable planar vortexes*, Applied General Topology, 22(2), 385-397, 2021.

[23] Peters J.F., Vergili T., *Good coverings of proximal Alexandrov spaces. Path cycles in the extension of the Mitsuishi-Yamaguchi good covering and Jordan curve theorem*, Applied General Topology, 24(1), 25-45, 2023.

[24] Preuss G., Theory of Topological Structures, D. Reidel Publishing Company, 1988.

[25] Scoville N.A., *On the Lusternik-Schnirelmann category of a simplicial map*, Topology and Its Applications, 216, 116-128, 2017.

[26] Sharma P.L., *Proximity bases and subbases*, Pacific Journal of Mathematics, 37(2), 515-526, 1971.

[27] Smirnov J.M., *On proximity spaces*, Matematicheskii Sbornik, 31(73)(3), 543-574, 1952.

[28] Switzer R.M., Algebraic Topology: Homology and Homotopy, Classics in Mathematics, 212, Springer, 2002.

[29] Vergili T., *Connectedness and Lusternik-Schnirelmann categories of the spaces of persistence modules*, New York Journal of Mathematics, 26, 598-606, 2020.

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Research Article

The q-Analogues of Min and Max Matrices and their Algebraic Properties

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Abstract: This study introduces a new family of matrices, the q-analogues, by generalizing the classical Min and Max matrices with entries from the q-calculus. The fundamental algebraic properties of these matrices, such as determinant, inverse, characteristic polynomial, LU decomposition, and Frobenius and spectral norms, are systematically investigated. The results are supported by low-dimensional numerical examples demonstrating the validity of the formulas. This study makes a significant contribution to the literature on Min and Max matrices by providing a comprehensive analysis of a q-dependent parametric extension of these matrices.

Keywords: Min matrix, Max matrix, q-calculus, algebraic properties.

1. Introduction

The q-calculus, often called quantum calculus, is a generalization of classical calculus that eliminates the notation of limits and provides new ways to define derivatives and integrals. It includes a q-parameter to extend fundamental mathematical concepts. For example, the q-analogue of an integer k is defined as

$$[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$$

[15]. By convention, $[0]_q = 0$. In q-calculus, q-integers serve the role of ordinary integers in classical calculus. Many fundamental concepts, such as the q-factorial and the q-binomial coefficients, are based on q-integers. They are defined by

$$[k]_q! = [1]_q[2]_q \cdots [k]_q \quad \text{and} \quad \binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

[15]. Kızılateş et al. [20] introduced a new family of hybrid numbers defined via q-integers and higher-order generalized Fibonacci numbers and established their comprehensive algebraic

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structure, deriving a Binet-like formula, novel identities, generating functions, and summation properties. Yener and Emiroğlu [36] introduced the q-analogues of fundamental concepts in multiplicative calculus, establishing basic theorems for derivatives, integrals, and infinite products within this framework. This framework extends to q-derivatives and q-integrals, creating a rich language with deep applications in combinatorics, special functions, and quantum groups. In recent decades, this language has been productively imported into linear algebra, giving rise to a wide array of structured q-matrices. For instance, the q-analogue of the classical Pascal matrix was introduced by Zheng [37]. Tuğlu and Kuş [34] later defined q-Bernoulli and q-Bernoulli polynomial matrices, investigating their properties. Other notable examples include the q-Catalan matrix studied by Yaying et al. [35], and the q-distance and q-Laplacian matrices for trees, whose algebraic properties were analyzed by Bapat et al. [2] and later extended to weighted trees by Barik et al. [3]. Cheon et al. [5] generalized the concept of exponential Riordan matrices to a q-setting. A more systematic approach to q-deformed matrix computation was presented by Ernst [7], who subsequently developed related techniques for q-Appell, q-Bernoulli, and q-Euler polynomials [8] and established connections among q-Cauchy, q-Bernoulli, and q-Euler matrices [9]. Recently, Mersin [23] defined the q-derivative operator for matrix functions, introducing new structures such as the q-matrix exponential function, q-binomial of two matrices, q-trigonometric and q-hyperbolic matrix functions, and examined their fundamental properties in detail. Moreover, the q-analogue of the Frank matrix, which is a special type of Max matrix, was defined, and some of its properties are investigated, such as determinant, inverse, characteristic polynomials and eigenvalues in [24]. For another generalization of the Frank matrix, called the r-Frank matrix, see [30]. Further examples of q-analogues can be found in [18] on (p,q)-Chebyshev polynomials and in [19] on q-Fibonacci and q-Lucas octonions.

The success of q-calculus in generating structured matrices naturally leads to a question: can this powerful framework be applied to other fundamental matrix families, such as Min and Max matrices?

Min and Max matrices, whose entries are given by the minimum and maximum of their row and column indices, respectively, were introduced by Pólya and Szegő [29] as follows

$$A_{\min} = [\min(i,j)]_{i,j=1}^{n} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix}$$

and

$$A_{\max} = [\max(i,j)]_{i,j=1}^{n} = \begin{bmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 2 & 3 & \cdots & n-1 & n \\ 3 & 3 & 3 & \cdots & n-1 & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-1 & n-1 & \cdots & n-1 & n \\ n & n & n & \cdots & n & n \end{bmatrix}.$$

Based on a real sequence $\{a_k\}$ with $a_1 < a_2 < \ldots < a_n$, the well-known generalizations $\mathcal{A}_{\min} = \left[a_{\min(i,j)}\right]_{i,j=1}^{n}$ and $\mathcal{A}_{\max} = \left[a_{\max(i,j)}\right]_{i,j=1}^{n}$ have been introduced, and their properties such as determinants, inverses, LU decompositions, eigenvalues, and norms have been studied [6, 16, 21, 25, 26]. Petroudi and Pirouz [12, 13] studied Min and Max matrices whose entries are taken from the Fibonacci and Lucas sequences, and derived their properties in terms of these numbers. Similarly, the characteristic polynomials, determinants, and inverses of Min and Max matrices with entries from the hyper-Fibonacci and hyper-Lucas sequences were investigated in [27] and [32]. In addition, certain norms of Min matrices with harmonic number entries can be found in [14]. Some factorizations of generalized Min and Max matrices have been obtained by Kılıç and Arıkan [16], while the works of [33] and [21] also contain significant results related to Min and Max matrices. Another generalization, the r-Min and r-Max matrices, was introduced by Kızılateş and Terzioğlu [17], who examined their algebraic properties in detail. Further studies on the generalizations of r-Min and r-Max matrices, and hence Min and Max matrices, can be found in [1, 4, 10, 22, 28, 31].

Motivated by the diverse generalizations of Min and Max matrices and the established framework of q-calculus in matrix theory, we introduce their q-analogues. We derive explicit formulas for their determinants, inverses, characteristic polynomials, and LU decompositions. Furthermore, we investigate their Frobenius and spectral norms, establishing upper bounds via Hadamard factorization. To ground our theoretical results, we include a numerical example for low dimensions. Our work systematically extends the classical theory of Min and Max matrices into the q-domain, offering a new family of structured matrices with explicitly computable properties. Although the general framework of q-analogues has been applied to other matrices such as the q-Frank matrix, the dense and symmetric structure of Min and Max matrices leads to significantly simpler algebraic properties, including tridiagonal inverses and elementary LU decompositions. These features distinguish the q-Min and q-Max matrices from previously studied q-matrices such as the q-Frank matrix.

2. Preliminaries

The following definitions and results related to matrix norms and the Hadamard product can be found in [11].

Let $A \in \mathbb{C}^{m\times n}$ and let A^H denote its conjugate transpose.

The spectral norm of A is defined as

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i(A^H A)},$$

where $\lambda_i(A^H A)$ are the eigenvalues of $A^H A$.

The Frobenius (Euclidean) norm of A is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}.$$

For any matrix A, the following inequality holds

$$\|A\|_2 \le \|A\|_F.$$

The maximum row length norm and maximum column length norm are given by

$$r_1(A) = \max_{1\le i\le m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2}, \qquad c_1(A) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{m} |a_{ij}|^2}.$$

The following inequality is known

$$\|X \circ Y\|_2 \le r_1(X)\, c_1(Y),$$

where $X \circ Y$ denotes the Hadamard product of X and Y, where $X, Y \in \mathbb{C}^{m\times n}$.

3. Main Results

Definition 3.1 *The matrix* $A_{\min}^{(q)} = \left[[\min(i,j)]_q\right]_{i,j=1}^{n}$ *is called* q*-Min matrix whose explicit form is*

$$A_{\min}^{(q)} = \begin{bmatrix} [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [2]_q & [2]_q & \cdots & [2]_q & [2]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [3]_q & [3]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n-1]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1+q & 1+q & \cdots & 1+q & 1+q \\ 1 & 1+q & 1+q+q^2 & \cdots & 1+q+q^2 & 1+q+q^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1+q & 1+q+q^2 & \cdots & 1+q+q^2+\cdots+q^{n-2} & 1+q+q^2+\cdots+q^{n-2} \\ 1 & 1+q & 1+q+q^2 & \cdots & 1+q+q^2+\cdots+q^{n-2} & 1+q+q^2+\cdots+q^{n-1} \end{bmatrix}.$$

For $q \to 1$ the q-Min matrix $A_{\min}^{(q)}$ corresponds to the well-known Min matrix $A_{\min}$, whose entries are given by $[\min(i,j)]_{i,j=1}^{n}$.

Theorem 3.2 *The determinant of the matrix $A_{\min}^{(q)}$ is calculated by the rule*

$$\det\left(A_{\min}^{(q)}\right) = \prod_{k=1}^{n-1} q^k.$$

Proof We perform the column operations $C_{i+1} \leftarrow C_{i+1} - C_i$ for $i = 1, 2, \ldots, n-1$, where C_i denotes the i-th column of the matrix. Then, we have

$$\begin{aligned}
\det\left(A_{\min}^{(q)}\right) &= \begin{vmatrix}
[1]_q & 0 & 0 & \cdots & 0 & 0 \\
[1]_q & [2]_q - [1]_q & 0 & \cdots & 0 & 0 \\
[1]_q & [2]_q - [1]_q & [3]_q - [2]_q & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
[1]_q & [2]_q - [1]_q & [3]_q - [2]_q & \cdots & [n-1]_q - [n-2]_q & 0 \\
[1]_q & [2]_q - [1]_q & [3]_q - [2]_q & \cdots & [n-1]_q - [n-2]_q & [n]_q - [n-1]_q
\end{vmatrix} \\
&= [1]_q \left([2]_q - [1]_q\right)\left([3]_q - [2]_q\right) \cdots \left([n]_q - [n-1]_q\right) \\
&= \prod_{k=1}^{n-1} \left([k+1]_q - [k]_q\right) \\
&= \prod_{k=1}^{n-1} q^k.
\end{aligned}$$

$\square$

Theorem 3.3 *The inverse of the matrix $A_{\min}^{(q)}$, denoted by $R^{(q)} = [r_{ij}]_{i,j=1}^{n}$, has entries given by*

$$r_{ij} = \begin{cases}
1 + \dfrac{1}{q}, & i = j = 1 \\
-\dfrac{1}{q^i}, & j = i + 1 \\
-\dfrac{1}{q^j}, & i = j + 1 \\
\dfrac{1 + \dfrac{1}{q}}{q^{i-1}}, & i = j \neq 1, n \\
\dfrac{1}{q^{i-1}}, & i = j = n \\
[0]_q, & \textit{otherwise}.
\end{cases}$$

Proof The proof is done by induction on n. For $n = 2$, we have

$$A_{\min}^{(q)} = \begin{bmatrix} [1]_q & [1]_q \\ [1]_q & [2]_q \end{bmatrix},$$

whose inverse is

$$R_2^{(q)} = \begin{bmatrix} 1 + \dfrac{1}{q} & -\dfrac{1}{q} \\ -\dfrac{1}{q} & \dfrac{1}{q} \end{bmatrix}.$$

The hypothesis holds for $n = 2$. Assuming the hypothesis holds for $n - 1$, we show that it also holds for n. For this purpose, we partition the matrices $A_{\min}^{(q)}$ and $R^{(q)}$ into block matrices as follows

$$A^{(q)}_{\min} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad \text{and} \quad R^{(q)} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix},$$

where

$$Q_{11} = \begin{bmatrix} [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [2]_q & [2]_q & \cdots & [2]_q & [2]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [3]_q & [3]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-2]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \end{bmatrix},$$

$$Q_{12} = \begin{bmatrix} [1]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \end{bmatrix}^T,$$

$$Q_{21} = \begin{bmatrix} [1]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \end{bmatrix},$$

and

$$Q_{22} = \begin{bmatrix} [n]_q \end{bmatrix}.$$

Note that, according to the assumption, Q_{22}^{-1} equals the matrix $R^{(q)}$ of order $n-1$. The equation

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

where I and 0 represent the appropriately sized identity and zero matrices, yields the following results

$$R_{22} = \left(Q_{22} - Q_{21}Q_{11}^{-1}Q_{12}\right)^{-1} = \begin{bmatrix} \frac{1}{q^{n-1}} \end{bmatrix},$$

$$R_{21} = -R_{22}Q_{21}Q_{11}^{-1} = \begin{bmatrix} [0]_q & [0]_q & [0]_q & \cdots & -\frac{1}{q^{n-1}} \end{bmatrix},$$

$$R_{12} = -Q_{11}^{-1}Q_{12}R_{22} = \begin{bmatrix} [0]_q & [0]_q & [0]_q & \cdots & -\frac{1}{q^{n-1}} \end{bmatrix}^T,$$

and

$$R_{11} = Q_{11}^{-1} - Q_{11}^{-1}Q_{12}R_{22}Q_{21}Q_{11}^{-1} = \begin{bmatrix} 1+\frac{1}{q} & -\frac{1}{q} & [0]_q & \cdots & [0]_q \\ -\frac{1}{q} & \frac{1+\frac{1}{q}}{q} & -\frac{1}{q^2} & \cdots & [0]_q \\ [0]_q & -\frac{1}{q^2} & \frac{1+\frac{1}{q}}{q^2} & \cdots & [0]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [0]_q & [0]_q & [0]_q & \cdots & -\frac{1}{q^{n-2}} \\ [0]_q & [0]_q & [0]_q & \cdots & \frac{1+\frac{1}{q}}{q^{n-2}} \end{bmatrix}.$$

Hence, we obtain

$$R^{(q)} = \begin{bmatrix} 1+\frac{1}{q} & -\frac{1}{q} & [0]_q & \cdots & [0]_q & [0]_q \\ -\frac{1}{q} & \frac{1+\frac{1}{q}}{q} & -\frac{1}{q^2} & \cdots & [0]_q & [0]_q \\ [0]_q & -\frac{1}{q^2} & \frac{1+\frac{1}{q}}{q^2} & \cdots & [0]_q & [0]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [0]_q & [0]_q & [0]_q & \cdots & \frac{1+\frac{1}{q}}{q^{n-2}} & -\frac{1}{q^{n-1}} \\ [0]_q & [0]_q & [0]_q & \cdots & -\frac{1}{q^{n-1}} & \frac{1}{q^{n-1}} \end{bmatrix},$$

which completes the proof. □

Theorem 3.4 *The characteristic polynomial of the matrix $A^{(q)}_{\min}$ satisfies the recurrence relation for $n \geq 2$,*

$$P_n(\lambda) = \left(q^{n-1} - 2\lambda\right)P_{n-1}(\lambda) - \lambda^2 P_{n-2}(\lambda)$$

with initials $P_0(\lambda) = 1$ and $P_1(\lambda) = 1 - \lambda$.

Proof To obtain the characteristic polynomial, we use the equality $P_n(\lambda) = |A^{(q)}_{\min} - \lambda I|$, which gives

$$P_n(\lambda) = \begin{vmatrix} [1]_q - \lambda & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [2]_q - \lambda & [2]_q & \cdots & [2]_q & [2]_q \\ [1]_q & [2]_q & [3]_q - \lambda & \cdots & [3]_q & [3]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q - \lambda & [n-1]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q - \lambda \end{vmatrix}.$$

To compute this determinant, for $i = 1, 2, \dots, n-1$, we subtract the ith column from the $(i+1)$th column, and then subtract the ith row from the $(i+1)$th row. This produces the following determinant

$$P_n(\lambda) = \begin{vmatrix} [1]_q - \lambda & \lambda & 0 & \cdots & 0 & 0 \\ \lambda & [2]_q - [1]_q - 2\lambda & \lambda & \cdots & 0 & 0 \\ 0 & \lambda & [3]_q - [2]_q - 2\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-1]_q - [n-2]_q - 2\lambda & \lambda \\ 0 & 0 & 0 & \cdots & \lambda & [n]_q - [n-1]_q - 2\lambda \end{vmatrix}.$$

By expanding along the last row, we obtain

$$P_n(\lambda) = \left([n]_q - [n-1]_q - 2\lambda\right)P_{n-1}(\lambda) - \lambda^2 P_{n-2}(\lambda),$$

which completes the proof. □

Theorem 3.5 *The components of the LU decomposition of the matrix* $A_{\min}^{(q)}$ *are* $L = [l_{ij}]_{i,j=1}^{n}$ *and* $U = [u_{ij}]_{i,j=1}^{n}$*, whose entries are given as follows*

$$l_{ij} = \begin{cases} [0]_q, & i < j \\ [1]_q, & i \geq j \end{cases}$$

and

$$u_{ij} = \begin{cases} [0]_q, & i > j \\ q^{i-1}, & i \leq j. \end{cases}$$

Proof The proof is obtained directly from the matrix multiplication $A_{\min}^{(q)} = LU$. □

Theorem 3.6 *For the matrix* $A_{\min}^{(q)}$*, we have the following results.*

(i) The Frobenius norm satisfies the formula

$$\|A_{\min}^{(q)}\|_F = \sqrt{\sum_{i=1}^{n} (2i-1)\,[n+1-i]_q^2},$$

(ii) The spectral norm has the upper bound

$$\|A_{\min}^{(q)}\|_2 \leq \sqrt{\left(\sum_{i=1}^{n} [i]_q^2\right)\left(\sum_{i=1}^{n} [i]_q^2 + 1\right)}.$$

Proof

(i) For the Frobenius norm of $A_{\min}^{(q)}$, we have

$$\begin{aligned} \|A_{\min}^{(q)}\|_F^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} [\min(i,j)]_q^2 \\ &= 1[n]_q^2 + 3[n-1]_q^2 + 5[n-2]_q^2 + \cdots + (2n-1)[1]_q^2 \\ &= \sum_{i=1}^{n} (2i-1)\,[n+1-i]_q^2. \end{aligned}$$

(ii) For the spectral norm of $A_{\min}^{(q)}$, we use the Hadamard product as

$$A_{\min}^{(q)} = A_1 \circ A_2,$$

where

$$A_1 = \begin{bmatrix} [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [2]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [1]_q & [1]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [1]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [1]_q & [2]_q & \cdots & [2]_q & [2]_q \\ [1]_q & [1]_q & [1]_q & \cdots & [3]_q & [3]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [n-1]_q \\ [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \end{bmatrix}.$$

Then, we have

$$r_1(A_1) = \sqrt{\sum_{i=1}^{n} [i]_q^2} \quad \text{and} \quad c_1(A_2) = \sqrt{\sum_{i=1}^{n-1} [i]_q^2 + [1]_q^2}.$$

Hence, the inequality

$$\|A_{\min}^{(q)}\|_2 \leq r_1(A_1)\, c_1(A_2)$$

completes the proof.

□

Definition 3.7 *The matrix*

$$A_{\max}^{(q)} = \left[[\max(i,j)]_q\right]_{i,j=1}^{n}$$

is called the q-Max matrix*. Its explicit form can be expressed as*

$$A_{\max}^{(q)} = \begin{bmatrix} [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ [2]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ [3]_q & [3]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [n-1]_q & [n-1]_q & [n-1]_q & \cdots & [n-1]_q & [n]_q \\ [n]_q & [n]_q & [n]_q & \cdots & [n]_q & [n]_q \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+q & 1+q+q^2 & \cdots & 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-1} \\ 1+q & 1+q & 1+q+q^2 & \cdots & 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-1} \\ 1+q+q^2 & 1+q+q^2 & 1+q+q^2 & \cdots & 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-2} & \cdots & 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-1} \\ 1+q+\cdots+q^{n-1} & 1+q+\cdots+q^{n-1} & 1+q+\cdots+q^{n-1} & \cdots & 1+q+\cdots+q^{n-1} & 1+q+\cdots+q^{n-1} \end{bmatrix}.$$

As $q \to 1$, the q-Max matrix $A_{\max}^{(q)}$ reduces to the well-known Max matrix $A_{\max}$, whose entries are given by $[\max(i,j)]_{i,j=1}^{n}$.

Theorem 3.8 *The determinant of* $A_{\max}^{(q)}$ *is given by*

$$\det\left(A_{\max}^{(q)}\right) = [n]_q \prod_{k=1}^{n-1} -q^k.$$

Proof We perform the row operations $R_i \leftarrow R_i - R_{i+1}$ for $i = 1, 2, \ldots, n-1$, where R_i denotes the i-th row of the matrix. Then, we obtain

$$\begin{aligned}\det\left(A^{(q)}_{\max}\right) &= \begin{bmatrix} [1]_q - [2]_q & 0 & 0 & \cdots & 0 & 0 \\ [2]_q - [3]_q & [2]_q - [3]_q & 0 & \cdots & 0 & 0 \\ [3]_q - [4]_q & [3]_q - [4]_q & [3]_q - [4]_q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [n-1]_q - [n]_q & [n-1]_q - [n]_q & [n-1]_q - [n]_q & \cdots & [n-1]_q - [n]_q & 0 \\ [n]_q & [n]_q & [n]_q & \cdots & [n]_q & [n]_q \end{bmatrix} \\ &= [n]_q \prod_{k=1}^{n-1} \left([k]_q - [k+1]_q\right) \\ &= [n]_q \prod_{k=1}^{n-1} -q^k. \end{aligned}$$

□

Theorem 3.9 *The inverse of the matrix $A^{(q)}_{\max}$, denoted by $S^{(q)} = [s_{ij}]$, has entries characterized as follows*

$$s_{ij} = \begin{cases} -\dfrac{1}{q}, & i = j = 1 \\ \dfrac{1}{q^i}, & j = i+1 \\ \dfrac{1}{q^j}, & i = j+1 \\ -\dfrac{1 + \dfrac{1}{q}}{q^{i-1}}, & i = j \neq 1, n \\ -\dfrac{[i-1]_q}{q^{i-1}[i]_q}, & i = j = n \\ [0]_q & \textit{otherwise.} \end{cases}$$

Proof The proof proceeds by induction on n. In the base case $n = 2$, the theorem is satisfied, as

$$A^{(q)}_{\max} = \begin{bmatrix} [1]_q & [2]_q \\ [2]_q & [2]_q \end{bmatrix} \quad \text{and} \quad S^{(q)}_2 = \begin{bmatrix} -\dfrac{1}{q} & \dfrac{1}{q} \\ \dfrac{1}{q} & -\dfrac{1}{q(q+1)} \end{bmatrix}.$$

Assume the statement holds for $n-1$. To prove it for n, we apply the induction hypothesis to the leading principal submatrix of order $n-1$. For this purpose, we partition the matrices as follows

$$A^{(q)}_{\max} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \quad \text{and} \quad S^{(q)} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where

$$U_{11}=\begin{bmatrix} [1]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \\ [2]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \\ [3]_q & [3]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [n-2]_q & [n-2]_q & [n-2]_q & \cdots & [n-2]_q & [n-1]_q \\ [n-1]_q & [n-1]_q & [n-1]_q & \cdots & [n-1]_q & [n-1]_q \end{bmatrix},$$

$$U_{12}=\begin{bmatrix} [n]_q & [n]_q & [n]_q & \cdots & [n]_q & [n]_q \end{bmatrix}^T,$$

$$U_{21}=\begin{bmatrix} [n]_q & [n]_q & [n]_q & \cdots & [n]_q & [n]_q \end{bmatrix},$$

and

$$U_{22}=\begin{bmatrix} [n]_q \end{bmatrix}.$$

According to assumption, U_{11}^{-1} corresponds to the matrix $S^{(q)}$ of order $n-1$. For appropriately sized identity matrix I and zero matrix 0, the equation

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

yields the block matrices of $S^{(q)}$ as follows

$$S_{22}=\left(U_{22}-U_{21}U_{11}^{-1}U_{12}\right)^{-1}=\begin{bmatrix} -\dfrac{[n-1]_q}{q^{n-1}[n]_q} \end{bmatrix},$$

$$S_{21}=-S_{22}U_{21}U_{11}^{-1}=\begin{bmatrix} [0]_q & [0]_q & [0]_q & \cdots & \dfrac{1}{q^{n-1}} \end{bmatrix},$$

$$S_{12}=-U_{11}^{-1}U_{12}S_{22}=\begin{bmatrix} [0]_q & [0]_q & [0]_q & \cdots & \dfrac{1}{q^{n-1}} \end{bmatrix}^T,$$

and

$$S_{11}=U_{11}^{-1}-U_{11}^{-1}U_{12}S_{22}U_{21}U_{11}^{-1}=\begin{bmatrix} -\dfrac{1}{q} & \dfrac{1}{q} & [0]_q & \cdots & [0]_q \\ \dfrac{1}{q} & -\dfrac{1+\dfrac{1}{q}}{q} & \dfrac{1}{q^2} & \cdots & [0]_q \\ [0]_q & \dfrac{1}{q^2} & -\dfrac{1+\dfrac{1}{q}}{q^2} & \cdots & [0]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [0]_q & [0]_q & [0]_q & \cdots & \dfrac{1}{q^{n-2}} \\ [0]_q & [0]_q & [0]_q & \cdots & -\dfrac{1+\dfrac{1}{q}}{q^{n-2}} \end{bmatrix}.$$

Hence, we obtain

$$S^{(q)} = \begin{bmatrix} -\frac{1}{q} & \frac{1}{q} & [0]_q & \cdots & [0]_q & [0]_q \\ \frac{1}{q} & -\frac{1+\frac{1}{q}}{q} & \frac{1}{q^2} & \cdots & [0]_q & [0]_q \\ [0]_q & \frac{1}{q^2} & -\frac{1+\frac{1}{q}}{q^2} & \cdots & [0]_q & [0]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [0]_q & [0]_q & [0]_q & \cdots & -\frac{1+\frac{1}{q}}{q^{n-2}} & \frac{1}{q^{n-1}} \\ [0]_q & [0]_q & [0]_q & \cdots & \frac{1}{q^{n-1}} & -\frac{[n-1]_q}{q^{n-1}[n]_q} \end{bmatrix}.$$

$\square$

Theorem 3.10 *For the characteristic polynomial of the matrix $A_{\max}^{(q)}$, the following recurrence relation holds*

$$\psi_n(\mu) = ([n]_q - \mu)\,\phi_{n-1}(\mu) - \mu^2\phi_{n-2}(\mu),$$

where

$$\phi_{n-1}(\mu) = (-q^{n-1} - 2\mu)\phi_{n-2}(\mu) - \mu^2\phi_{n-3}(\mu)$$

with initials $\phi_0(\mu) = 1$ and $\phi_1(\mu) = -q - 2\mu$.

Proof Using the equality $\psi_n(\mu) = |A_{\max}^{(q)} - \mu I|$, we have

$$\psi_n(\mu) = \begin{vmatrix} [1]_q - \mu & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ [2]_q & [2]_q - \mu & [3]_q & \cdots & [n-1]_q & [n]_q \\ [3]_q & [3]_q & [3]_q - \mu & \cdots & [n-1]_q & [n]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [n-1]_q & [n-1]_q & [n-1]_q & \cdots & [n-1]_q - \mu & [n]_q \\ [n]_q & [n]_q & [n]_q & \cdots & [n-1]_q & [n]_q - \mu \end{vmatrix}.$$

For $i = 1, 2, \ldots, n-1$, performing $R_i \leftarrow R_i - R_{i+1}$ and $C_i \leftarrow C_i - C_{i+1}$ yields

$$\psi_n(\mu) = \begin{vmatrix} [1]_q - [2]_q - 2\mu & \mu & 0 & \cdots & 0 & 0 \\ \mu & [2]_q - [3]_q - 2\mu & \mu & \cdots & 0 & 0 \\ 0 & \mu & [3]_q - [4]_q - 2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-1]_q - [n]_q - 2\mu & \mu \\ 0 & 0 & 0 & \cdots & \mu & [n]_q - \mu \end{vmatrix}.$$

We now expand the determinant along the last row, and then we obtain

$$\psi_n(\mu) = ([n]_q - \mu)\begin{vmatrix} [1]_q - [2]_q - 2\mu & \mu & 0 & \cdots & 0 & 0 \\ \mu & [2]_q - [3]_q - 2\mu & \mu & \cdots & 0 & 0 \\ 0 & \mu & [3]_q - [4]_q - 2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-2]_q - [n-1]_q - 2\mu & \mu \\ 0 & 0 & 0 & \cdots & \mu & [n-1]_q - [n]_q - 2\mu \end{vmatrix}$$
$$-\mu\begin{vmatrix} [1]_q - [2]_q - 2\mu & \mu & 0 & \cdots & 0 & 0 \\ \mu & [2]_q - [3]_q - 2\mu & \mu & \cdots & 0 & 0 \\ 0 & \mu & [3]_q - [4]_q - 2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-2]_q - [n-1]_q - 2\mu & 0 \\ 0 & 0 & 0 & \cdots & \mu & \mu \end{vmatrix}.$$

In the resulting expression, let the first determinant on the right-hand side be denoted by $\phi_{n-1}(\mu)$ and the second one by $X(\mu)$. When $\phi_{n-1}(\mu)$ is expanded along the last row, we obtain

$$\phi_{n-1}(\mu) = ([n-1]_q - [n]_q - 2\mu)\begin{vmatrix} [1]_q - [2]_q - 2\mu & \mu & 0 & \cdots & 0 & 0 \\ \mu & [2]_q - [3]_q - 2\mu & \mu & \cdots & 0 & 0 \\ 0 & \mu & [3]_q - [4]_q - 2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-3]_q - [n-2]_q - 2\mu & \mu \\ 0 & 0 & 0 & \cdots & \mu & [n-2]_q - [n-1]_q - 2\mu \end{vmatrix}$$
$$-\mu\begin{vmatrix} [1]_q - [2]_q - 2\mu & \mu & 0 & \cdots & 0 & 0 \\ \mu & [2]_q - [3]_q - 2\mu & \mu & \cdots & 0 & 0 \\ 0 & \mu & [3]_q - [4]_q - 2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-3]_q - [n-2]_q - 2\mu & 0 \\ 0 & 0 & 0 & \cdots & \mu & \mu \end{vmatrix}.$$

In this case, the first determinant on the right-hand side is evidently $\phi_{n-2}(\mu)$. Expanding the second determinant along the last column yields

$$\begin{aligned} \phi_{n-1}(\mu) &= ([n-1]_q - [n]_q - 2\mu)\,\phi_{n-2}(\mu) - \mu^2\phi_{n-3}(\mu) \\ &= (-q^{n-1} - 2\mu)\phi_{n-2}(\mu) - \mu^2\phi_{n-3}(\mu) \end{aligned}$$

with initial conditions $\phi_0(\mu) = 1$ and $\phi_1(\mu) = -q - 2\mu$. Moreover, expanding $X(\mu)$ along the last column shows that it equals $\phi_{n-2}(\mu)$. Hence,

$$\psi_n(\mu) = ([n]_q - \mu)\,\phi_{n-1}(\mu) - \mu^2\phi_{n-2}(\mu).$$

□

Theorem 3.11 *For the matrix $A^{(q)}_{\max}$, let its LU decomposition be given by $L = [l^*_{ij}]^n_{i,j=1}$ and $U = [u^*_{ij}]^n_{i,j=1}$. The entries of these matrices are characterized as follows*

$$l^*_{ij} = \begin{cases} [1]_q, & i = j = 1 \\ [0]_q, & i < j \\ \dfrac{[i]_q}{[j]_q}, & i > j \end{cases}$$

and

$$u^*_{ij} = \begin{cases} [j]_q, & i = 1 \\ -\dfrac{[i]_q q^{i-1}}{[i-1]_q}, & i = j \neq 1 \\ -\dfrac{[j]_q q^{i-1}}{[i-1]_q}, & i < j \\ [0]_q, & i > j. \end{cases}$$

Proof This result is obtained by verifying the equality $A^{(q)}_{\max} = LU$ through matrix multiplication. □

Theorem 3.12 *For the matrix* $A^{(q)}_{\max}$, *the following hold*

(i) The Frobenius norm satisfies the formula

$$\|A^{(q)}_{\max}\|_F = \sqrt{\sum_{i=1}^{n}(2i-1)\,[i]_q^2},$$

(ii) The spectral norm admits the upper bound

$$\|A^{(q)}_{\max}\|_2 \le \sqrt{n[n]_q^2\left((n-1)[n]_q^2+1\right)}.$$

Proof

(i) For the Frobenius norm of the matrix $A^{(q)}_{\max}$, we obtain

$$\begin{aligned}\|A^{(q)}_{\max}\|_F^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}[\max(i,j)]_q^2 \\ &= 1[1]_q^2 + 3[2]_q^2 + 5[3]_q^2 + \cdots + (2n-1)[n]_q^2 \\ &= \sum_{i=1}^{n}(2i-1)\,[i]_q^2.\end{aligned}$$

(ii) To obtain the upper bound for the spectral norm of the matrix $A^{(q)}_{\max}$, we consider the Hadamard product

$$A^{(q)}_{\max} = A_1^* \circ A_2^*,$$

where

$$A_1^* = \begin{bmatrix} [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [2]_q & [2]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [3]_q & [3]_q & [3]_q & \cdots & [1]_q & [1]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [n-1]_q & [n-1]_q & [n-1]_q & \cdots & [n-1]_q & [1]_q \\ [n]_q & [n]_q & [n]_q & \cdots & [n]_q & [n]_q \end{bmatrix}$$

and

$$A_2^* = \begin{bmatrix} [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ [1]_q & [1]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ [1]_q & [1]_q & [1]_q & \cdots & [n-1]_q & [n]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [n]_q \\ [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \end{bmatrix}.$$

Then, we obtain

$$r_1(A_1^*) = \sqrt{n[n]_q^2}, \qquad c_1(A_2^*) = \sqrt{(n-1)[n]_q^2+1}.$$

It follows that

$$\|A^{(q)}_{\max}\|_2 \leq r_1(A_1^*)\, c_1(A_2^*),$$

which completes the proof.

□

Next, we present an illustrative example.

Example 3.13 *For $n = 5$ and $q = 2$, the matrices $A^{(q)}_{\min}$ and $A^{(q)}_{\max}$ are as follows*

$$A^{(2)}_{\min} = \begin{bmatrix} [1]_2 & [1]_2 & [1]_2 & [1]_2 & [1]_2 \\ [1]_2 & [2]_2 & [2]_2 & [2]_2 & [2]_2 \\ [1]_2 & [2]_2 & [3]_2 & [3]_2 & [3]_2 \\ [1]_2 & [2]_2 & [3]_2 & [4]_2 & [4]_2 \\ [1]_2 & [2]_2 & [3]_2 & [4]_2 & [5]_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 3 & 7 & 7 & 7 \\ 1 & 3 & 7 & 15 & 15 \\ 1 & 3 & 7 & 15 & 31 \end{bmatrix}$$

and

$$A^{(2)}_{\max} = \begin{bmatrix} [1]_2 & [2]_2 & [3]_2 & [4]_2 & [5]_2 \\ [2]_2 & [2]_2 & [3]_2 & [4]_2 & [5]_2 \\ [3]_2 & [3]_2 & [3]_2 & [4]_2 & [5]_2 \\ [4]_2 & [4]_2 & [4]_2 & [4]_2 & [5]_2 \\ [5]_2 & [5]_2 & [5]_2 & [5]_2 & [5]_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 7 & 15 & 31 \\ 3 & 3 & 7 & 15 & 31 \\ 7 & 7 & 7 & 15 & 31 \\ 15 & 15 & 15 & 15 & 31 \\ 31 & 31 & 31 & 31 & 31 \end{bmatrix}.$$

Their determinants are

$$\det\left(A^{(2)}_{\min}\right) = \prod_{k=1}^{4} 2^k = 1024$$

and

$$\det\left(A^{(2)}_{\max}\right) = [5]_q \prod_{k=1}^{4} -2^k = 31744.$$

Their inverse matrices are

$$\left(A^{(2)}_{\min}\right)^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{3}{8} & -\frac{1}{8} & 0 \\ 0 & 0 & -\frac{1}{8} & \frac{3}{16} & -\frac{1}{16} \\ 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{16} \end{bmatrix}$$

and

$$\left(A^{(2)}_{\max}\right)^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{3}{8} & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{8} & -\frac{3}{16} & \frac{1}{16} \\ 0 & 0 & 0 & \frac{1}{16} & -\frac{15}{496} \end{bmatrix}.$$

The characteristic polynomials of the matrices $A_{\min}^{(2)}$ *for* $2 \le n \le 5$ *are*

$$\begin{aligned}
P_2(\lambda) &= \lambda^2 - 4\lambda + 2,\\
P_3(\lambda) &= -\lambda^3 + 11\lambda^2 - 20\lambda + 8,\\
P_4(\lambda) &= \lambda^4 - 26\lambda^3 + 126\lambda^2 - 176\lambda + 64,\\
P_5(\lambda) &= -\lambda^5 + 57\lambda^4 - 648\lambda^3 + 2360\lambda^2 - 2944\lambda + 1024.
\end{aligned}$$

The characteristic polynomials of the matrices $A_{\max}^{(2)}$ *for* $2 \le n \le 5$ *are*

$$\begin{aligned}
\psi_2(\mu) &= \mu^2 - 4\mu - 6,\\
\psi_3(\mu) &= -\mu^3 + 11\mu^2 + 76\mu + 56,\\
\psi_4(\mu) &= \mu^4 - 26\mu^3 - 586\mu^2 - 1616\mu - 960,\\
\psi_5(\mu) &= -\mu^5 + 57\mu^4 + 3624\mu^3 + 27656,\mu^2 + 58496\mu + 31744,
\end{aligned}$$

where

$$\begin{aligned}
\phi_1(\mu) &= -2 - 2\mu,\\
\phi_2(\mu) &= 3\mu^2 + 12\mu + 8,\\
\phi_3(\mu) &= -4\mu^3 - 46\mu^2 - 112\mu - 64,\\
\phi_4(\mu) &= 5\mu^4 + 144\mu^3 + 952\mu^2 + 1920\mu + 1024.
\end{aligned}$$

The LU decompositions of the matrices $A_{\min}^{(2)}$ *and* $A_{\max}^{(2)}$ *are*

$$A_{\min}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 0 & 2 & 2 & 2 & 2\\ 0 & 0 & 4 & 4 & 4\\ 0 & 0 & 0 & 8 & 8\\ 0 & 0 & 0 & 0 & 16 \end{bmatrix}$$

and

$$A_{\max}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 3 & 1 & 0 & 0 & 0\\ 7 & \frac{7}{3} & 1 & 0 & 0\\ 15 & 5 & \frac{15}{7} & 1 & 0\\ 31 & \frac{31}{3} & \frac{31}{7} & \frac{31}{15} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 & 15 & 31\\ 0 & -6 & -14 & -30 & -62\\ 0 & 0 & \frac{-28}{3} & -20 & \frac{-124}{3}\\ 0 & 0 & 0 & \frac{-120}{7} & \frac{-248}{7}\\ 0 & 0 & 0 & 0 & \frac{-496}{15} \end{bmatrix}.$$

The Frobenius norms of these matrices are

$$\|A_{\min}^{(2)}\|_F = \sqrt{\sum_{i=1}^{5} (2i-1)\,[6-i]_2^2} = 44.19$$

and

$$\|A_{\max}^{(2)}\|_F = \sqrt{\sum_{i=1}^{5} (2i-1)\,[i]_2^2} = 102.45.$$

The spectral norms of the matrices $A_{\min}^{(2)}$ *and* $A_{\max}^{(2)}$ *are*

$$\|A_{\min}^{(2)}\|_2 = 43.24$$

and

$$\|A^{(2)}_{\max}\|_2 = 97.25,$$

computed numerically using Maple. Now, we present the upper bounds for these spectral norms.

The matrices A_1 and A_2 according to Theorem 3.6 (ii) are

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 3 & 7 & 1 & 1 \\ 1 & 3 & 7 & 15 & 1 \\ 1 & 3 & 7 & 15 & 31 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 7 & 7 \\ 1 & 1 & 1 & 1 & 15 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then

$$r_1(A_1) = \sqrt{\sum_{i=1}^{5} [i]_2^2} = \sqrt{1245}$$

and

$$c_1(A_2) = \sqrt{\sum_{i=1}^{4} [i]_2^2 + 1} = \sqrt{285}.$$

Hence,

$$\|A^{(2)}_{\min}\|_2 \le r_1(A_1)\, c_1(A_2) \approx 595.67.$$

Also,

$$\|A^{(2)}_{\min}\|_2 \le \|A^{(2)}_{\min}\|_F = 44.19.$$

The matrices A_1^* and A_2^* in Theorem 3.12 (ii) are

$$A_1^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 1 & 1 & 1 \\ 7 & 7 & 7 & 1 & 1 \\ 15 & 15 & 15 & 15 & 1 \\ 31 & 31 & 31 & 31 & 31 \end{bmatrix}, \qquad A_2^* = \begin{bmatrix} 1 & 3 & 7 & 15 & 31 \\ 1 & 1 & 7 & 15 & 31 \\ 1 & 1 & 1 & 15 & 31 \\ 1 & 1 & 1 & 1 & 31 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then

$$r_1(A_1^*) = \sqrt{5[5]_2^2} = \sqrt{4805}$$

and

$$c_1(A_2^*) = \sqrt{4[5]_2^2 + 1} = \sqrt{3845}.$$

Hence,

$$\|A^{(2)}_{\max}\|_2 \le r_1(A_1^*)\, c_1(A_2^*) \approx 4298.28.$$

In addition,

$$\|A^{(2)}_{\max}\|_2 \le \|A^{(2)}_{\max}\|_F = 102.45.$$

The numerical results for the spectral norms of $A^{(2)}_{\min}$ and $A^{(2)}_{\max}$ for $n = 2, 3, 4, 5$ are summarised in Tables 1 and 2, respectively. The actual spectral norms and the Frobenius norms were computed numerically using Maple.

Table 1: Actual spectral norms and upper bounds for $\|A^{(2)}_{\min}\|_2$

n	$\|A^{(2)}_{\min}\|_2$ (actual)	Upper bound (Theorem 3.6)	$\|A^{(2)}_{\min}\|_F$
2	3.41	4.47	3.46
3	8.84	25.48	9.00
4	20.18	130.54	20.59
5	43.24	595.67	44.19

Table 2: Actual spectral norms and upper bounds for $\|A^{(2)}_{\max}\|_2$

n	$\|A^{(2)}_{\max}\|_2$ (actual)	Upper bound (Theorem 3.12)	$\|A^{(2)}_{\max}\|_F$
2	5.16	13.42	5.29
3	15.98	120.64	16.52
4	41.19	780.00	42.99
5	97.25	4298.28	102.45

The upper bounds in Tables 1 and 2, obtained from Theorems 3.6 and 3.12 respectively, are very loose for the spectral norm, especially for larger n. In contrast, the Frobenius norm provides a much sharper estimate, almost equal to the actual spectral norm. Thus, the Frobenius norm is a much better practical upper bound. The numerical values also show that the upper bounds grow rapidly with n, becoming increasingly loose.

Now, we compare the q-Max matrix introduced in this paper with the q-Frank matrix, which is a special type of Max matrix studied in [24].

3.1. Comparison with the q-Frank Matrix

The q-Frank matrix defined in [24] has a lower Hessenberg structure, while the q-Max matrix introduced in this paper is dense and symmetric. Their determinants are $\prod_{k=1}^{n} q^{k-1}$ and $[n]_q \prod_{k=1}^{n-1}(-q^k)$, respectively. The inverse of the q-Frank matrix is upper Hessenberg, whereas the inverse of the q-Max matrix is tridiagonal. Both matrices satisfy second-order recurrence relations for their characteristic polynomials, but their explicit forms are different. These differences show that the present work is not a mere repetition but a natural extension to a different family of q-matrices.

4. Conclusion

In this paper, we have introduced the q-analogues of Min and Max matrices by defining their entries as $[\min(i,j)]_q$ and $[\max(i,j)]_q$, the q-integers. Thereby, we have embedded these well-studied matrix families into the richer framework of q-calculus. We have obtained explicit formulas for the determinants, inverses, characteristic polynomials, LU decompositions, Frobenius norms, and spectral norm bounds of both matrices. A numerical example confirms the validity of the formulas.

The results show that the q-analogues preserve the structural simplicity of the classical Min and Max matrices, such as tridiagonal inverses and elementary LU factors, while introducing a q-dependence that allows further flexibility. This demonstrates that Min and Max structure leads to simpler algebraic properties compared to other q-matrices such as the q-Frank matrix, making the q-Min and q-Max matrices potentially useful in numerical linear algebra.

Future work includes a detailed spectral analysis and the investigation of sharper norm bounds for these matrices.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

References

[1] Anđelić M., da Fonseca C.M., Kızılateş C., Terzioğlu N., *r-min and r-max matrices with harmonic higher order Gauss Fibonacci numbers entries*, Journal of Applied Mathematics and Computing, 71(5), 7437-7461, 2025.

[2] Bapat R.B., Lal A.K., Pati S., *A q-analogue of the distance matrix of a tree*, Linear Algebra and Its Applications, 416(2-3), 799-814, 2006.

[3] Barik S., Mondal M., Pan S., *A q-analogue of the distance matrix of a tree with matrix weights*, Linear and Multilinear Algebra, 73(7), 1423-1447, 2025.

[4] Carrasco-Olivera D., Kızılateş C., Herrera C.M., Miranda H., *On the linear algebraic properties of (r,n)-min and (r,n)-max matrices*, Filomat, 39(33), 11837-11852, 2025.

[5] Cheon G.S., Jung J.H., Lim Y., *A q-analogue of the Riordan group*, Linear Algebra and Its Applications, 439(12), 4119-4129, 2013.

[6] Chu K., Puntanen S., Styan G., *Problem section*, Statistical Papers, 52, 257-262, 2011.

[7] Ernst T., *An umbral approach to find q-analogues of matrix formulas*, Linear Algebra and Its Applications, 439(4), 1167-1182, 2013.

[8] Ernst T., *q-Pascal and q-Wronskian matrices with implications to q-Appell polynomials*, Journal of Discrete Mathematics, 2013(1), 450481, 2013.

[9] Ernst T., *On several q-special matrices, including the q-Bernoulli and q-Euler matrices*, Linear Algebra and Its Applications, 542, 422-440, 2018.

[10] da Fonseca C.M., Kızılateş C., Terzioğlu N., *A new generalization of min and max matrices and their reciprocals counterparts*, Filomat, 38(2), 421-435, 2024.

[11] Horn R.A., Johnson C.R., Topics in Matrix Analysis, Cambridge University Press, 1991.

[12] Jafari-Petroudi S., Pirouz B., *A note on Hadamard inverse and Hadamard exponential of a matrix with Fibonacci numbers*, The 7th National Conference on Mathematics of Payame Noor University, Tabriz, Iran, 135-138, 28-29 October 2015.

[13] Jafari-Petroudi S., Pirouz M., *On the bounds for the spectral norm of particular matrices with Fibonacci and Lucas numbers*, The International Journal of Advances in Applied Mathematics and Mechanics, 3, 82-90, 2016.

[14] Jafari-Petroudi S., Pirouz M., *Toward special symmetric matrices with harmonic numbers*, The 8th National Conference on Mathematics of Payame Noor University, Lorestan, Iran, 109-113, 11-12 April 2016.

[15] Kac V.G., Cheung P., Quantum Calculus, Springer, 2002.

[16] Kılıç E., Arıkan T., *Studying new generalizations of max-min matrices with a novel approach*, Turkish Journal of Mathematics, 43(4), 2010-2024, 2019.

[17] Kızılateş C., Terzioğlu N., *On r-min and r-max matrices*, Journal of Applied Mathematics and Computing, 68(6), 4559-4588, 2022.

[18] Kızılateş C., Tuğlu N., Çekim B., *On the (p,q)-Chebyshev polynomials and related polynomials*, Mathematics, 7(2), 136, 2019.

[19] Kızılateş C., Polatlı E., *New families of Fibonacci and Lucas octonions with q-integer components*, Indian Journal of Pure and Applied Mathematics, 52(1), 231-240, 2021.

[20] Kızılateş C., Polatlı E., Terzioğlu N., Du W.S., *On higher-order generalized Fibonacci hybrid numbers with q-integer components: New properties, recurrence relations, and matrix representations*, Symmetry, 17(4), 584, 2025.

[21] Mattila M., Haukkanen P., *Studying the various properties of MIN and MAX matrices-elementary vs. more advanced methods*, Special Matrices, 4(1), 101-109, 2016.

[22] Mersin E.Ö., *Geometric min-max matrices*, Kyungpook Mathematical Journal, 65(3), 369-393, 2025.

[23] Mersin E.Ö., *Quantum matrix calculus*, Kyungpook Mathematical Journal, 65(4), 587-604, 2025.

[24] Mersin E.Ö., *The q-analogue of the Frank matrix*, Computational and Applied Mathematics, 45(4), 163, 2026.

[25] Mersin E.Ö., *Sturm's theorem for min matrices*, AIMS Mathematics, 8(7), 17229-17245, 2023.

[26] Neudecker H., Trenkler G., Liu S., *Problem section*, Statistical Papers, 50, 221-223, 2009.

[27] Özgül D., Bahşi M., *Min matrices with hyper-Lucas numbers*, Journal of Science and Arts, 4, 855-864, 2020.

[28] Polatlı E., *On some properties of a generalized min matrix*, AIMS Mathematics, 8(11), 26199-26212, 2023.

[29] Pólya G., Szegő G., Problems and Theorems in Analysis II, Springer, 1998.

[30] Shi B., Kızılateş C., *A new generalization of the Frank matrix and Its some properties*, Computational and Applied Mathematics, 43(1), 19, 2024.

[31] Shi B., Kızılateş C., *Geometric r-Frank matrix: Some properties and applications*, Computational and Applied Mathematics, 44(8), 388, 2025.

[32] Solmaz T., Bahşi M., *Max and min matrices with hyper-Fibonacci numbers*, Asian-European Journal of Mathematics, 15(5), 2250084, 2022.

[33] Stuart J.L., *Nested matrices and inverse M-matrices*, Czechoslovak Mathematical Journal, 65(2), 537-544, 2015.

[34] Tuğlu N., Kuş S., *q-Bernoulli matrices and their some properties*, Gazi University Journal of Science, 28(2), 269-273, 2015.

[35] Yaying T., Kara M.I., Hazarika B., Kara E.E., *A study on q-analogue of Catalan sequence spaces*, Filomat, 37(3), 839-850, 2023.

[36] Yener G., Emiroğlu I., *A q-analogue of the multiplicative calculus: q-multiplicative calculus*, Discrete and Continuous Dynamical Systems - Series S, 8, 1435-1450, 2015.

[37] Zheng D., *q-analogue of the Pascal matrix*, Ars Combinatoria, 87, 321-336, 2008.

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Research Article

A KKT-Based Approach to Kantorovich-Type Inequalities

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Abstract: We develop a unified optimization-based framework for Kantorovich-type inequalities using Karush–Kuhn–Tucker (KKT) conditions. For two commuting positive definite matrices A and B, we derive a sharp eigenvalue-based upper bound for the product $(Ax,x)(Bx,x)$, which recovers the classical Kantorovich inequality as a special case when $B = A^{-1}$.

A central feature of our approach is a structural characterization of extremal configurations: We show that every global maximizer is supported on at most two indices. This reveals that the reduction to pairs of eigenvalues arises naturally from first-order optimality conditions.

In addition, we establish a weighted inequality involving explicit optimal constants, extending known results related to the generalized Kantorovich constant $K(h,p)$.

Our results provide a systematic variational framework for deriving and understanding Kantorovich-type inequalities, offering both alternative proofs and a basis for further generalizations in matrix analysis.

Keywords: Kantorovich inequality, KKT conditions, quadratic forms, positive definite matrices.

1. Introduction

With several applications in statistics, numerical analysis, and matrix theory, the Kantorovich inequality is a fundamental result in mathematical analysis and optimization. The inequality was first established by Leonid Vital'evich Kantorovich [8] in 1948 and has subsequently been generalized and expanded in many ways. This inequality provides an upper bound for the product of two quadratic forms involving A and its inverse A^{-1}. The classical Kantorovich inequality states that for a real symmetric positive definite matrix A with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$ and any unit vector x, the following holds

$$(Ax,x)\left(A^{-1}x,x\right) \leq \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}. \tag{1}$$

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By replacing unit vector x by $\frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}$ in inequality (1), Kantorovich inequality can also be expressed in terms of A^2 as:

$$(A^2x,x) \leq \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}(Ax,x)^2. \tag{2}$$

Another way to express the Kantorovich inequalities (1) and (2) is in the "normalized reduced" form:

$$\left(\sum_{i=1}^{n}\lambda_i x_i^2\right)\cdot\left(\sum_{i=1}^{n}\lambda_i^{-1}x_i^2\right) \leq \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}, \tag{3}$$

and

$$\left(\sum_{i=1}^{n}\lambda_i^2 x_i^2\right) \leq \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}\left(\sum_{i=1}^{n}\lambda_i x_i^2\right)^2 \tag{4}$$

respectively. For a detailed analysis of the Kantorovich inequality, its equivalent formulations, and related inequalities, we refer the reader to the comprehensive studies [1, 12] and also [3, 5, 9, 10, 14].

An extension of the Kantorovich inequality in a different direction was given in [4] as: Let A be an $n\times n$ positive definite matrix satisfying $MI_n \geq A \geq mI_n > 0$, where I_n is $n\times n$ identity matrix. Then the following inequalities hold for every unit vector x:

$$\langle Ax,x\rangle^p \leq \langle A^p x,x\rangle \leq K(m,M,p)\langle Ax,x\rangle^p \tag{5}$$

for $p \notin [0,1]$, where

$$K(m,M,p) = \frac{(p-1)^{p-1}}{p^p}\frac{(M^p-m^p)^p}{(M-m)(mM^p-Mm^p)^{p-1}}.$$

By using spectral decomposition theorem, normalized reduced form corresponding to the right-hand side of inequality (5) can be written as

$$\sum_{i=1}^{n}\lambda_i^p z_i^2 \leq K(m,M,p)\left(\sum_{i=1}^{n}\lambda_i z_i^2\right)^p \tag{6}$$

with $\sum_{i=1}^{n} z_i^2 = 1$. Also, note that when p is in the interval $[0,1]$, the reverse inequality holds.

While Kantorovich-type inequalities are traditionally established via convexity arguments, spectral techniques, or direct algebraic manipulations, the Karush–Kuhn–Tucker (KKT) framework provides a natural optimization-based perspective. By reformulating the problem as a constrained optimization problem, the KKT conditions make the underlying variational structure explicit and characterize extremal configurations through first-order optimality conditions. Moreover, this approach connects matrix inequalities with optimization theory, particularly Lagrangian duality and multiplier methods, and has been effectively used in deriving eigenvalue bounds and related inequalities (see, e.g., [6, 7, 13]). This viewpoint not only recovers known inequalities but also offers a systematic way to derive extensions.

Our KKT-based framework provides a precise structural characterization of extremal configurations in Kantorovich-type inequalities. In particular, we show that all global maximizers are supported on at most two indices, revealing that the commonly observed reduction to eigenvalue pairs is not merely technical, but an intrinsic consequence of first-order optimality conditions. Moreover, our results recover known inequalities, such as inequality (6), as special cases while providing a unified variational derivation and explicit expressions for the optimal constants. This perspective not only clarifies the role of eigenvalue interactions in determining sharp bounds, but also offers a systematic methodology that can be extended to broader classes of matrix inequalities.

We now state our main results. We use the notation $(\cdot,\cdot)$ to denote the standard inner product in $\mathbb{R}^n$, i.e., $(Ax, x) = x^T Ax$ for any symmetric matrix A.

2. Main Results

To start our analysis, we need the following lemmas.

Lemma 2.1 *[11, Theorem 12.1] Suppose that x^* is a local minimizer of*

$$\min_{x\in\mathbb{R}^n} f(x) \tag{7}$$

$$s.t.\ \ c_i(x) = 0, \quad i \in E$$

$$c_i(x) \le 0, \quad i \in I$$

and linear inequality constraint qualification (LICQ) holds at x^*. Then there is a Lagrange multiplier vector λ^* with components $\lambda_i^* \in E \cup I$ such that the following conditions are satisfied at (x^*, λ^*)

$$\nabla_x L(x^*, \lambda^*) = 0,$$

$$c_i(x^*) = 0, \quad i \in E$$

$$c_i(x^*) \le 0, \quad i \in I$$

$$\lambda_i^* \ge 0, \quad i \in I$$

$$\lambda_i^* c_i(x^*) = 0, \quad i \in E \cup I.$$

These conditions are often known as the Karush–Kuhn–Tucker (KKT) conditions. Since $c_i(x^*) \le 0$ and $\lambda_i^* \ge 0$ for all $i \in I$, $\lambda_i^* c_i(x^*) = 0$ imply that either $\lambda_i^* = 0$ or $c_i(x^*) = 0$. These conditions are specifically called complementary conditions.

Lemma 2.2 *[6, Corollary 9.10] Let x^* be a local solution of optimization problem in (7). The KKT conditions hold at every local solution if all constraint functions are affine functions in $\mathbb{R}^n$.*

In the following theorem, we derive an extension of inequality (1) using the KKT conditions.

Theorem 2.3 *Let A and B be two $n \times n$ positive definite matrices such that $AB = BA$. Assume that their eigenvalues are ordered as*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \qquad \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n.$$

Then for any unit vector $x \in \mathbb{R}^n$,

$$(Ax, x)\,(Bx, x) \leq \max_{\substack{i<j \\ \lambda_i \neq \lambda_j,\ \mu_i \neq \mu_j \\ \frac{\lambda_i}{\mu_i} > \frac{\lambda_j}{\mu_j}}} \frac{(\lambda_i \mu_j - \mu_i \lambda_j)^2}{4(\lambda_i - \lambda_j)(\mu_j - \mu_i)}. \tag{8}$$

Proof We start by examining the constrained optimization problem:

$$\max \left\{ (Ax, x)\,(Bx, x) \;\middle|\; \|x\|^2 = 1 \right\}.$$

Since A and B are positive definite and commute $(AB = BA)$, they are simultaneously orthogonally diagonalizable (see [2]). Thus, there exists an orthogonal matrix U such that $A = U^T D_1 U$, $B = U^T D_2 U$, where $D_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, $D_2 = \operatorname{diag}(\mu_1, \ldots, \mu_n)$.

Let $y = Ux$. Then $\|y\| = 1$ and the problem reduces to

$$\max \left\{ \left(\sum_{i=1}^n \lambda_i y_i^2 \right) \left(\sum_{i=1}^n \mu_i y_i^2 \right) \;\middle|\; \sum_{i=1}^n y_i^2 = 1 \right\}.$$

Setting $z_i = y_i^2$, we obtain

$$\max \left\{ \left(\sum_{i=1}^n \lambda_i z_i \right) \left(\sum_{i=1}^n \mu_i z_i \right) \;\middle|\; \sum_{i=1}^n z_i = 1,\ z \geq 0 \right\}.$$

The feasible set is the standard simplex in $\mathbb{R}^n$, which is compact. Since the objective function is continuous, a global maximizer exists. Any global maximizer is also a local maximizer. Because the constraints are affine, Lemma 2.2 ensures that the Karush–Kuhn–Tucker conditions are satisfied at such a point. The Lagrangian is

$$\mathcal{L}(z, h, \theta) = -\left(\sum_{i=1}^n \lambda_i z_i \right) \left(\sum_{i=1}^n \mu_i z_i \right) + h \left(\sum_{i=1}^n z_i - 1 \right) - \langle \theta, z \rangle.$$

From the KKT conditions we obtain

$$-\lambda_i \sum_{k=1}^{n} \mu_k z_k - \mu_i \sum_{k=1}^{n} \lambda_k z_k + h - \theta_i = 0. \tag{9}$$

If we multiply both sides of (9) by z_i and sum over i, then the terms involving θ_i vanish due to the complementarity condition $\theta_i z_i = 0$. Using also the feasibility condition $\sum_{i=1}^{n} z_i = 1$, we obtain

$$h = 2\left(\sum_{k=1}^{n} \lambda_k z_k\right)\left(\sum_{k=1}^{n} \mu_k z_k\right). \tag{10}$$

Now define

$$N := \sum_{k=1}^{n} \lambda_k z_k, \qquad M := \sum_{k=1}^{n} \mu_k z_k.$$

Let $I = \{i : z_i > 0\}$. For every $i \in I$, the complementarity condition implies $\theta_i = 0$. Substituting this into (9) and dividing by the product NM in view of (10), we obtain

$$\frac{\lambda_i}{N} + \frac{\mu_i}{M} = 2. \tag{11}$$

Now let's take two indices $i, j \in I$. If we subtract the corresponding equations in (11), we obtain

$$\frac{\lambda_i - \lambda_j}{N} = \frac{\mu_j - \mu_i}{M}. \tag{12}$$

This relation allows us to express the ratio $\frac{N}{M}$ as

$$\frac{N}{M} = \frac{\lambda_i - \lambda_j}{\mu_j - \mu_i}.$$

If we substitute this expression into (11) and solve the resulting system, we obtain

$$M = \frac{\lambda_i \mu_j - \mu_i \lambda_j}{2(\lambda_i - \lambda_j)}, \qquad N = \frac{\lambda_i \mu_j - \mu_i \lambda_j}{2(\mu_j - \mu_i)}.$$

We now justify that it is sufficient to consider solutions supported on at most two indices. Let z^* be a global maximizer and let $I = \{i : z_i^* > 0\}$ be its active set. If $|I| \le 2$, there is nothing to prove. Suppose that $|I| \ge 3$.

For each $i \in I$, the complementarity condition implies $\theta_i = 0$, and hence the KKT condition (11) holds:

$$\frac{\lambda_i}{N} + \frac{\mu_i}{M} = 2.$$

Thus, all active indices satisfy the same first-order optimality relation.

Hence, on the face

$$\mathrm{conv}\{e_i : i \in I\},$$

the partial derivatives of the objective function with respect to the active variables are equal. Consequently, the restriction of the objective function to this face is affine. Since an affine function on a compact convex set attains its maximum at an extreme point, there exists a maximizer supported on at most two indices.

Since the boundary of a face of the simplex consists of lower-dimensional faces, this argument can be iterated to conclude that there exists a global maximizer supported on at most two indices. Therefore, it suffices to consider pairs of indices (i, j).

Since $N > 0$ and $M > 0$, the above expressions must be positive. This requires

$$\frac{\lambda_i \mu_j - \mu_i \lambda_j}{\lambda_i - \lambda_j} > 0, \quad \frac{\lambda_i \mu_j - \mu_i \lambda_j}{\mu_j - \mu_i} > 0,$$

which is equivalent to

$$\frac{\lambda_i}{\mu_i} > \frac{\lambda_j}{\mu_j}.$$

Thus, the objective function value is

$$\left(\sum_{k=1}^{n} \lambda_k z_k\right)\left(\sum_{k=1}^{n} \mu_k z_k\right) = \frac{(\lambda_i \mu_j - \mu_i \lambda_j)^2}{4(\lambda_i - \lambda_j)(\mu_j - \mu_i)}.$$

Taking the maximum over all admissible pairs (i, j) completes the proof. □

Remark 2.4 *When we take $B = A^{-1}$ in the given theorem (so that $\mu_i = \lambda_i^{-1}$), the inequality simplifies to:*

$$(Ax, x)(A^{-1}x, x) \leq \max \frac{\left(\lambda_i \lambda_j^{-1} - \lambda_i^{-1} \lambda_j\right)^2}{4(\lambda_i - \lambda_j)\left(\lambda_j^{-1} - \lambda_i^{-1}\right)}.$$

By making the substitution $t = \lambda_i/\lambda_j$, the right-hand side expression simplifies to the single-variable function $\frac{(t+1)^2}{4t}$, where t ranges over the interval $[\lambda_n/\lambda_1, \lambda_1/\lambda_n]$. Remarkably, the maximum value of this function yields the well-known Kantorovich constant:

$$\frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n},$$

which corresponds precisely to the classical Kantorovich inequality. In this context, the inequality we obtained in Theorem 2.3 can be considered as a generalization of the Kantorovich inequality.

The following example illustrates both the applicability and sharpness of Theorem 2.3 and shows how the bound can be computed explicitly for a concrete pair of commuting positive definite matrices.

Example 2.5 *Consider the matrices*

$$A = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{5}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{7}{4} & -\frac{5}{4} \\ -\frac{5}{4} & \frac{7}{4} \end{pmatrix}.$$

Both A and B are symmetric positive definite matrices and satisfy $AB = BA$. Their eigenvalues are

$$\lambda_1 = 4,\ \lambda_2 = 1, \qquad \mu_1 = \tfrac{1}{2},\ \mu_2 = 3.$$

Moreover,

$$\frac{\lambda_1}{\mu_1} = 8 > \frac{\lambda_2}{\mu_2} = \frac{1}{3},$$

so the ordering condition in Theorem 2.3 is satisfied.

For this pair, the upper bound in inequality (8) is given by

$$\max_{i \neq j} \frac{(\lambda_i \mu_j - \mu_i \lambda_j)^2}{4(\lambda_i - \lambda_j)(\mu_j - \mu_i)} = \frac{\left(4 \cdot 3 - \frac{1}{2} \cdot 1\right)^2}{4(4-1)\left(3 - \frac{1}{2}\right)} = \frac{529}{120}.$$

Choosing

$$z_1 = \frac{13}{30}, \qquad z_2 = \frac{17}{30}$$

(which correspond to a unit vector x through the relations $y = Ux$ and $z_i = y_i^2$), one obtains

$$(Ax, x) = \frac{23}{10}, \qquad (Bx, x) = \frac{23}{12},$$

and therefore

$$(Ax, x)(Bx, x) = \frac{23}{10} \cdot \frac{23}{12} = \frac{529}{120}.$$

Thus, the bound in Theorem 2.3 is attained for this example.

Now consider the special case $B = A^{-1}$. In this case, the eigenvalues of B are $\mu_i = \lambda_i^{-1}$, so that

$$\mu_1 = \frac{1}{4}, \qquad \mu_2 = 1.$$

Substituting into Theorem 2.3, we obtain

$$(Ax, x)(A^{-1}x, x) \leq \frac{(\lambda_1 \mu_2 - \mu_1 \lambda_2)^2}{4(\lambda_1 - \lambda_2)(\mu_2 - \mu_1)} = \frac{\left(4 \cdot 1 - \frac{1}{4} \cdot 1\right)^2}{4(4-1)\left(1 - \frac{1}{4}\right)} = \frac{25}{16}.$$

This coincides with the classical Kantorovich inequality

$$(Ax,x)(A^{-1}x,x) \leq \frac{(\lambda_1+\lambda_2)^2}{4\lambda_1\lambda_2},$$

with $\lambda_1 = 4$ and $\lambda_2 = 1$. Hence, the classical Kantorovich inequality is recovered as a special case of Theorem 2.3.

The following Definition 2.6 and Theorem 2.7 can be found in [12, p. 74].

Definition 2.6 *Let $h > 0$. A generalized Kantorovich constant $K(h,p)$ is defined by*

$$K(h,p) = \frac{h^p - h}{(p-1)(h-1)} \left(\frac{p-1}{p}\right)^p \left(\frac{h^p-1}{h^p-h}\right)^p$$

for any real number $p \in \mathbb{R}$ and $K(h,p)$ is sometimes denoted by $K(p)$ briefly.

We remark that $K(m,M,p)$ just coincides with $K(h,p)$ by putting $h = \frac{M}{m} > 1$ and $K(m,M,p)$ is an extension of the Kantorovich constant $\frac{(M+m)^2}{4Mm}$, in fact, $K(m,M,2) = K(m,M,-1) = \frac{(M+m)^2}{4Mm}$.

We now recall some important properties of $K(h,p)$.

Theorem 2.7 (Properties of $K(h,p)$) *Let $h > 0$ be given. Then a generalized Kantorovich constant $K(h,p)$ has the following properties:*

(i) $K(h,p) = K\left(\frac{1}{h},p\right)$ *for all* $p \in \mathbb{R}$.

(ii) $K(h,p) = K(h,1-p)$ *for all* $p \in \mathbb{R}$.

(iii) $K(h,0) = K(h,1) = 1$ *and* $K(1,p) = 1$ *for all* $p \in \mathbb{R}$.

(iv) $K(h,p)$ *is increasing for* $p > \frac{1}{2}$ *and decreasing for* $p < \frac{1}{2}$.

(v) $K\left(h^r,\frac{p}{r}\right)^{\frac{1}{p}} = K\left(h^p,\frac{r}{p}\right)^{-\frac{1}{r}}$ *for* $pr \neq 0$.

(vi) $K(h,p) \leq h^{p-1}$ *for all* $p \geq 1$ *and* $h > 1$.

We now use the KKT conditions to derive a broader form of inequality (6) in the theorem below.

Theorem 2.8 *Let $\{\mu_i\}_{i=1}^n$ and $\{\lambda_i\}_{i=1}^n$ be positive real numbers, and let $p > 1$. Assume that*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \qquad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n.$$

Then for every $z \in \mathbb{R}^n$ *with* $\sum_{i=1}^n z_i^2 = 1$*, the following inequality holds:*

$$\sum_{i=1}^n \mu_i^p z_i^2 \le \max_{\substack{i<j \\ \lambda_i \neq \lambda_j \\ \frac{\mu_i^p}{\lambda_i} > \frac{\mu_j^p}{\lambda_j}}} \frac{(p-1)^{p-1}(\mu_i^p - \mu_j^p)^p}{p^p(\lambda_i - \lambda_j)(\mu_i^p \lambda_j - \mu_j^p \lambda_i)^{p-1}} \left(\sum_{i=1}^n \lambda_i z_i^2 \right)^p . \tag{13}$$

Proof We consider the constrained optimization problem

$$\max \left\{ \frac{\sum_{j=1}^n \mu_j^p z_j^2}{\left(\sum_{j=1}^n \lambda_j z_j^2 \right)^p} \;\middle|\; \sum_{j=1}^n z_j^2 = 1 \right\}.$$

If we set $y_j = z_j^2$, then the problem becomes

$$\max \left\{ \frac{\sum_{j=1}^n \mu_j^p y_j}{\left(\sum_{j=1}^n \lambda_j y_j \right)^p} \;\middle|\; \sum_{j=1}^n y_j = 1, \quad y_j \ge 0 \right\}.$$

The feasible set is the standard simplex in $\mathbb{R}^n$, which is compact. Since the objective function is continuous, a global maximizer exists. Any global maximizer is also a local maximizer. Because the constraints are affine, Lemma 2.2 ensures that the Karush–Kuhn–Tucker conditions are satisfied at such a point.

We now write the Lagrangian function:

$$\mathcal{L}(y, \delta, \theta) = -\frac{\sum_{j=1}^n \mu_j^p y_j}{\left(\sum_{j=1}^n \lambda_j y_j \right)^p} + \delta \left(\sum_{j=1}^n y_j - 1 \right) - \sum_{j=1}^n \theta_j y_j,$$

where $\delta \in \mathbb{R}$ and $\theta_j \ge 0$.

We now compute the partial derivatives. If we differentiate with respect to y_i, we obtain

$$\frac{\partial \mathcal{L}}{\partial y_i} = \frac{-\mu_i^p \left(\sum_{j=1}^n \lambda_j y_j \right)^p + p\lambda_i \left(\sum_{j=1}^n \lambda_j y_j \right)^{p-1} \sum_{j=1}^n \mu_j^p y_j}{\left(\sum_{j=1}^n \lambda_j y_j \right)^{2p}} + \delta - \theta_i = 0. \tag{14}$$

For convenience, introducing the notation

$$S = \sum_{j=1}^n \lambda_j y_j, \qquad T = \sum_{j=1}^n \mu_j^p y_j,$$

(14) can be rewritten equivalently as

$$-\frac{\mu_i^p}{S^p} + \frac{p\lambda_i T}{S^{p+1}} + \delta - \theta_i = 0.$$

The KKT conditions are therefore:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial y_i} = 0, \quad i = 1, \dots, n, \\ \displaystyle\sum_{j=1}^{n} y_j = 1, \\ y_j \geq 0, \quad \theta_j \geq 0, \\ \theta_j y_j = 0, \quad j = 1, \dots, n. \end{cases}$$

If we multiply (14) by y_i and sum over all i, then the terms involving θ_i vanish due to complementarity. Carrying out this computation, we obtain

$$\delta = \frac{(1-p)\sum_{j=1}^{n} \mu_j^p y_j}{\left(\sum_{j=1}^{n} \lambda_j y_j\right)^p}.$$

Let us define the active index set

$$I = \{\, i \mid y_i > 0 \,\}.$$

For every $i \in I$, the complementarity condition implies $\theta_i = 0$.

We note that $\delta \neq 0$. Indeed, if $\delta = 0$, then (14) would lead to a contradiction, since $p > 1$ and $y \neq 0$. Therefore, we may safely divide both sides of (14) by δ. This yields

$$\frac{\mu_i^p}{(1-p)\sum_{k=1}^{n} \mu_k^p y_k} - \frac{p\lambda_i}{(1-p)\sum_{k=1}^{n} \lambda_k y_k} = 1$$

for all $i \in I$.

Now, taking two indices $i, j \in I$ and subtracting the corresponding equations eliminates the constant term on the right-hand side. As a result, we obtain

$$\frac{\sum_{k=1}^{n} \lambda_k y_k}{\sum_{k=1}^{n} \mu_k^p y_k} = \frac{p(\lambda_i - \lambda_j)}{\mu_i^p - \mu_j^p}.$$

Using this relation, we can solve for S and T in terms of μ_i, μ_j, λ_i, and λ_j. This gives

$$T = \frac{\mu_j^p \lambda_i - \mu_i^p \lambda_j}{(1-p)(\lambda_i - \lambda_j)}, \qquad S = \frac{p(\mu_i^p \lambda_j - \mu_j^p \lambda_i)}{(1-p)(\mu_i^p - \mu_j^p)}.$$

We now justify that it is sufficient to consider solutions supported on at most two indices. Let y^* be a global maximizer and define the active set

$$I = \{\, i \mid y_i^* > 0 \,\}.$$

If $|I| \leq 2$, there is nothing to prove. Suppose that $|I| \geq 3$. Then, for any $i \in I$, the complementarity condition implies $\theta_i = 0$. Hence, all active indices satisfy the same relations arising from the KKT conditions, which in turn imply that the partial derivatives of the objective function with respect to y_i, $i \in I$, are equal.

Thus, all active indices satisfy the same first-order optimality relations arising from the KKT conditions. In particular, the partial derivatives of the objective function with respect to the active variables are equal.

Hence, on the face

$$\mathrm{conv}\{e_i :\ i \in I\},$$

where e_i denotes the i-th standard unit vector in $\mathbb{R}^n$, the restriction of the objective function is affine. Since an affine function on a compact convex set attains its maximum at an extreme point, there exists a maximizer supported on at most two indices. Therefore, it suffices to consider pairs of indices (i, j).

Since both S and T are positive, the numerators and denominators in the above expressions must have consistent signs. In particular, this leads to the condition

$$\mu_i^p \lambda_j - \mu_j^p \lambda_i > 0,$$

which is equivalent to

$$\frac{\mu_i^p}{\lambda_i} > \frac{\mu_j^p}{\lambda_j}.$$

Thus, for any admissible pair (i, j), the objective value is given by

$$\frac{(p-1)^{p-1}(\mu_i^p - \mu_j^p)^p}{p^p(\lambda_i - \lambda_j)(\mu_i^p \lambda_j - \mu_j^p \lambda_i)^{p-1}}.$$

Taking the maximum over all admissible pairs (i, j) completes the proof. □

Remark 2.9 *If we replace μ with λ in inequality (13), we obtain:*

$$\sum_{i=1}^{n} \lambda_i^p z_i^2 \leq \max_{\substack{i<j \\ \lambda_i \neq \lambda_j}} \frac{(p-1)^{p-1}(\lambda_i^p - \lambda_j^p)^p}{p^p(\lambda_i - \lambda_j)(\lambda_i^p \lambda_j - \lambda_j^p \lambda_i)^{p-1}} \left(\sum_{i=1}^{n} \lambda_i z_i^2 \right)^p.$$

By substituting $\lambda_i = h\lambda_j$ (with $h > 0$), the coefficient reduces to:

$$\frac{(p-1)^{p-1}(h^p - 1)^p}{p^p(h-1)(h^p - h)^{p-1}}.$$

This expression matches the generalized Kantorovich constant $K(h,p)$ defined in Definition 2.6:

$$K(h,p) = \frac{(h^p - h)}{(p-1)(h-1)}\left(\frac{p-1}{p}\frac{h^p-1}{h^p-h}\right)^p.$$

Given the ordered sequence of eigenvalues $M = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = m > 0$, the ratio $\frac{\lambda_i}{\lambda_j}$ is bounded below by $\frac{m}{M}$, and above by $\frac{M}{m}$ for any pair (i,j). Then, based on the symmetry property (i) in Theorem 2.7 and the monotonicity result (iv) in Theorem 2.7, it follows that the generalized Kantorovich constant $K(h,p)$ attains its maximum over this interval at either endpoint. That is, for $p>1$ or $p<0$, we have

$$\max_{h\in\left[\frac{m}{M},\frac{M}{m}\right]} K(h,p) = \frac{(p-1)^{p-1}}{p^p}\frac{(M^p-m^p)^p}{(M-m)(mM^p-Mm^p)^{p-1}},$$

which gives us the constant in inequality (6). In this context, inequality (13) can be regarded as a generalization of inequality (6).

3. Conclusion

In this work, we presented a unified optimization-based framework for deriving Kantorovich-type inequalities. By reformulating the problem in a spectral setting and analyzing the associated optimization problem, we showed that the extremal structure can be reduced to configurations supported on at most two indices. This reduction provides a transparent and systematic way to recover and extend several known inequalities.

A central feature of our approach is that the reduction to two-point supports follows directly from the Karush–Kuhn–Tucker conditions. In this sense, the appearance of eigenvalue pairs is not merely a technical device, but an intrinsic consequence of first-order optimality conditions.

Several directions remain open for further investigation. One is the case $p \in (0,1)$, where the geometry of the optimization problem changes substantially. Another is the extension of the present method to non-commuting matrices, where simultaneous diagonalization is no longer available. It would also be interesting to study whether similar optimization-based techniques can be developed for broader classes of functions beyond power functions.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and legal special permission.

Authors Contributions

Author [İbrahim Halil Gümüş]: Collected the data, contributed to completing the research and solving the problem, wrote the manuscript (%70).

Author [Asiye Gezen]: Contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%30).

Conflicts of Interest
The authors declare no conflict of interest.

References

[1] Alpargu G., The Kantorovich Inequality, with Some Extensions and with Some Statistical Applications, M.Sc. Thesis, McGill University, Montreal, Canada, 1996.

[2] Bhatia R., Positive Definite Matrices, Princeton University Press, 2007.

[3] Dannan F.M., *Generalized Kantorovich constant, a new formulation and properties*, Problemy Analiza - Issues of Analysis, 9(27)(1), 38-51, 2020.

[4] Furuta T., *Operator inequalities associated with Hölder–McCarthy and Kantorovich inequalities*, Journal of Inequalities and Applications, 2(2), 137-148, 1998.

[5] Furuichi S., Moradi H.R., Advances in Mathematical Inequalities, De Gruyter, 2020.

[6] Güler O., Foundations of Optimization, Graduate Texts in Mathematics, 258, Springer, 2010.

[7] Huang J., Zhou J., *A direct proof and a generalization for a Kantorovich type inequality*, Linear Algebra and Its Applications, 397, 185-192, 2005.

[8] Kantorovich L.V., *Functional analysis and applied mathematics*, Uspekhi Matematicheskikh Nauk, 3(6), 89-185, 1948.

[9] Moradi H.R., Gümüş İ.H., Heydarbeygi Z., *A glimpse at the operator Kantorovich inequality*, Linear and Multilinear Algebra, 67(5), 1031-1036, 2019.

[10] Moslehian M.S., *Recent developments of the operator Kantorovich inequality*, Expositiones Mathematicae, 30(4), 376-388, 2012.

[11] Nocedal J., Wright S.J., Numerical Optimization, Springer, 1999.

[12] Pečarić J., Furuta T., Mićić Hot J., Seo Y., Mond-Pečarić Method in Operator Inequalities, Element, 2005.

[13] Wolkowicz H., *Generating eigenvalue bounds using optimization*, Nonlinear Analysis and Variational Problems: In Honor of George Isac, Springer, 465-490, 2010.

[14] Xie Z., Liu J., Zhang M., *A Kantorovich type inequality for positive matrices*, Linear and Multilinear Algebra, 73(8), 1777-1791, 2025.

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Research Article

Approximate Solution of Time-Fractional Kawahara Equation Using the RBF Collocation Technique

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Abstract: The paper presents an approximate solution to the time-fractional Kawahara equation (TFKE), marking a significant advancement in the field. We decisively utilize the Liouville-Caputo fractional derivative (FD) as the foundation of our analytical framework. To discretize the equations in time, we expertly implement the Crank-Nicolson difference scheme, which guarantees precision and effectiveness. Additionally, we conduct a comprehensive stability analysis of the present scheme, applying the von-Neumann approach designed explicitly for fractional equations. The findings from the numerical experiments are clearly illustrated in the figures, effectively showcasing the robustness and reliability of our approach.

Keywords: Fractional Kawahara equation, Crank-Nicolson method, RBF collocation technique, Liouville-Caputo fractional derivative operator, von-Neumann stability.

1. Introduction

Fractional calculus is a compelling and advanced extension of traditional calculus. In this domain, we generalize familiar integer-order derivatives and integral operators into their fractional counterparts, enabling us to compute derivatives and integrals of arbitrary order. This powerful concept not only expands our understanding but also delivers profound insights across numerous fields of science and engineering. This intriguing field has been evolving since the 17th century, capturing the imagination of researchers and gaining remarkable traction in academic discourse [27, 30, 32]. Its applications are vast and varied, illuminating its potential across a spectrum of disciplines, from physics and biology to chemistry, viscoelasticity, fluid mechanics, acoustic waves, fractal dynamics, and control theory. Each application reveals a facet of its powerful versatility, making it a topic of fervent interest in scholarly literature. A wealth of studies has delved into

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the existence and validity of problems modeled within this compelling framework. In recent years, the integration of fractional-order derivatives into mathematical models has emerged as a powerful approach, thanks to their capability to accurately represent memory and hereditary effects found in complex systems. Consequently, fractional partial differential equations (PDEs) such as the Kawahara equation, the Fokker-Planck equation, the Rosenau equation, the KdV equation, and diffusion equations have made significant inroads into the literature, enriching the landscape of mathematical research [4, 7, 12, 20, 28]. While finding exact solutions to these equations can present formidable challenges, the quest has spurred the development of several innovative numerical methods for their resolution. Among the most widely employed approaches are finite element methods, the Crank-Nicolson technique, spectral methods, meshless Galerkin methods, moving least squares techniques, and radial basis function (RBF), and RBF collocation methods. There are many papers about the solution of fractional partial differential equations, such as [15, 16, 18, 31].

Notably, the time-fractional Kawahara equation provides an advanced framework for effectively modeling anomalous diffusion and nonlocal temporal dynamics evident in viscoelastic materials, fluid flows, and biological systems. However, the combination of significant nonlinearity and high-order spatial derivatives, along with the unique nonlocal characteristics of fractional derivatives, presents substantial challenges, often rendering analytical solutions either extremely difficult or impossible to achieve [23–25]. Embracing these complexities is essential for advancing our understanding and application of these sophisticated models [14].

As a result, developing efficient and accurate numerical methods for solving these equations has become a significant research topic. Traditional grid-based methods, such as finite difference and finite element approaches, often have limitations including mesh dependency, reduced accuracy for higher-order derivatives, and increased computational cost, particularly in higher dimensions [36]. To address these issues, we will employ a powerful alternative: a mesh-free method based on RBFs [5, 8, 17]. The RBF collocation method features high accuracy, geometric flexibility, and easy implementation without needing mesh generation [9]. Motivated by these benefits, this study aims to solve the time-fractional Kawahara equation using the RBF collocation method combined with an appropriate time discretization scheme to approximate its solutions.

The organization of this paper is structured as follows: In Section 1, we briefly present the governing equation and some recent references related to it. Section 2 discusses the application of the RBF method for obtaining numerical solutions to the aforementioned equations. In Section 4, we examine the stability of the proposed method by employing techniques similar to classical von-Neumann analysis. Section 5 includes a numerical example to demonstrate the validity of the suggested scheme, alongside several numerical experiments presented in tabular format. Finally, we conclude with our findings and provide a list of references.

2. Statement of the Governing Equation

This part gives us the information of the mentioned time-fractional Kawahara equation (TFKE). The Kawahara equation (KE) was depicted and introduced by Kawahara. The KE is one of the effective equations in physics and ocean technology [19]. Signal dispersion across lines of transmission is indicated by this equation. Furthermore, it enhances the process by incorporating memory effects, leading to more precise formulations of waves propagating in complex temporal environments [29]. There are many numerical studies about the mentioned equation. For example, a spectral collocation algorithm is used to obtain approximate solutions of the time-fractional generalized Kawahara equation by [2]. Another study, which mentions that the numerical solution of the TFKE by using the collocation method based on a certain Lucas polynomial sequence is given by [1]. Ahmed et al. gives a novel numerical technique to solve the TFKE numerically in [3].

General form of the KE is given as follows:

$$z_t(x,t) + \alpha z(x,t) z_x(x,t) + \beta z_{xxx}(x,t) - \gamma z_{xxxxx}(x,t) = 0, \tag{1}$$

with the initilal condition

$$z(x,0) = h(x), \tag{2}$$

in here $z(x,t)$ descripts the dependent variable, t is the time variable, and x is the spatial variable. α, β and γ are real constants. The constants β and γ are dispersion parameters related to third and fifth-order spatial derivatives. The zz_x term depicts nonlinear convection, while z_{xxx} and z_{xxxxx} account for dispersive effects. The interaction between nonlinearity and dispersion plays a crucial role in shaping the behavior of solitary and periodic waves. The dispersive term represents wave dispersion; the higher-order dispersion term captures additional dispersion effects that arise in specific media, while the convective term describes the advection of the wave packet due to its own velocity [2].

Let's now consider the TFKE

$$D_t^{\delta} z(x,t) + \alpha z(x,t) z_x(x,t) + \beta z_{xxx}(x,t) - \gamma z_{xxxxx}(x,t) = f(x,t), \quad x \in [a,b], \quad t > 0, \quad 0 \le \delta < 1, \tag{3}$$

with the initial and homogeneous boundary conditions respectively,

$$z(x,0) = h(x), \quad x \in [a,b],$$

$$z(a,t) = z(b,t) = 0, \quad t > 0,$$

$$z_x(a,t) = z_x(b,t) = 0, \quad t > 0,$$

$$z_{xx}(a,t) = z_{xx}(b,t) = 0, \quad t > 0,$$

where $f(x,t)$ is a source term, $h(x)$ is a sufficiently smooth function, the operator D_t^{δ} represents the Liouville-Caputo FD and $z(x,t)$ is the unknown function. The fifth-order derivative z_{xxxxx} represents higher-order dispersion effects and the nonlinear term zz_x denotes wave steepening. The fractional derivative can be said to introduce memory effects into the system. The selection of the Liouville-Caputo fractional derivative in this study is driven by its physical relevance and computational convenience, especially regarding time-dependent nonlinear partial differential equations, such as the time-fractional Kawahara equation. The Liouville-Caputo FD presents compelling advantages for various applications.

First, it facilitates the formulation of initial conditions through integer-order derivatives, which offer clear and intuitive physical interpretations. This clarity is crucial for ensuring accurate modeling in practical scenarios. Second, it excels at capturing memory and hereditary effects within complex systems, making it an indispensable tool for researchers and practitioners alike. Finally, from a numerical perspective, the Liouville-Caputo FD supports stable and widely adopted discretization schemes, such as the L1 approximation and its variants, ensuring reliable results in computational analyses. These schemes are relatively straightforward to implement and integrate efficiently with spatial discretization methods like the RBF collocation method [11, 21, 30]. Ultimately, the Liouville-Caputo FD is one of the most commonly used definitions in the literature on fractional calculus. This definition facilitates comparisons with existing studies and ensures consistency with previously developed numerical methods [23, 32].

The definition of Liouville-Caputo FD is provided below:

Definition 2.1. *The Liouville-Caputo FD $D_t^{\delta} z(x,t)$ of order $\delta > 0$ is denoted as [32]*

$$D_t^{\delta} z(x,t) = \begin{cases} \frac{1}{\Gamma(m-\delta)} \int_0^t \frac{\partial^m z(x,\xi)}{\partial \xi^m} (t-\xi)^{m-\delta-1} d\xi, & m-1 < \delta < m, \\ \frac{\partial^m z(x,t)}{\partial t^m}, & \delta = m, \quad m \in \mathbb{N}. \end{cases} \tag{4}$$

This FD has the following property for $m-1 \leq \delta < m$, $m \in \mathbb{N}$,

$$D_t^{\delta} z(x,t) = \begin{cases} 0, & m \in \mathbb{N}_0 \quad \text{and} \quad m < \lceil \delta \rceil, \\ \frac{\Gamma(m+1)}{\Gamma(m-\delta+1)} t^{m-\delta}, & m \in \mathbb{N}_0 \quad \text{and} \quad m \geq \lceil \delta \rceil, \end{cases} \tag{5}$$

where $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$, the function $\Gamma(.)$ is called a Gamma function and the notation $\lceil \delta \rceil$ denotes the ceiling function.

A useful approximation was presented by Sun and Wu [33] for computing the fractional derivative concerning t. Now, let's state the following lemma about this approximation.

Lemma 2.2. *[33] Let* $z(x,t) \in C^2[0,t_l]$*. Then*

$$\begin{aligned} |R(z(x,t_l))| &= \Bigg| \frac{1}{\gamma(1-\delta)} \int_0^{t_l} \frac{z'(x,s)}{(t_l-s)^\delta} ds \\ &- \frac{\Delta t^{-\delta}}{\gamma(2-\delta)} \Big[b_0 z(x,t_l) - \sum_{i=1}^{l-1} (b_{l-i-1} - b_{l-i}) z(x,t_i) - b_{l-1} z(x,t_0) \Big] \Bigg| \\ &\leq \frac{1}{\gamma(2-\delta)} \left[\frac{1-\delta}{12} + \frac{2^{2-\delta}}{2-\delta} - (1+2^{-\delta}) \right] \max_{0\leq t\leq t_l} |z''(x,t)| \Delta t^{2-\delta}, \end{aligned}$$

where $\delta \in (0,1)$ *ve* $b_i = (i+1)^{1-\delta} - i^{1-\delta}$*.*

Let's use the L1 approximation for this derivative.

$$D_t^\delta z(x_j, t_{n+1}) = \mu \left[b_0 z_j^{n+1} - \sum_{i=1}^{n} (b_{n-i} - b_{n-i+1}) z_j^i - b_n z_j^0 \right] + \mathcal{O}((\Delta t)^{2-\delta}), \tag{6}$$

with

$$b_i = (i+1)^{1-\delta} - i^{1-\delta}, \tag{7}$$

$$\mu = \frac{(\Delta t)^{-\delta}}{\Gamma(2-\delta)}. \tag{8}$$

3. Implementation of the RBF Collocation Technique to the TFKE

This part introduces the RBF collocation technique. First, let's delve into the captivating concept of RBFs and uncover their significance.

Definition 3.1. *Let* s *be a positive integer. A RBF* $\Phi : \mathbb{R}^s \to \mathbb{R}$ *is introduced as*

$$\Phi(\mathbf{x}) = \phi(||x - x_j||), \tag{9}$$

where $\phi : [0,\infty) \to \mathbb{R}$ *is a function with one variable and* $||\cdot||$ *is a norm in* $\mathbb{R}^s$*.*

It depends on the distance from a specific center point.

This collocation technique relies on point collocation of a straightforward series expression, which simplifies the mathematical algorithm and makes it easy to implement. In comparison to classical finite difference methods, the RBF collocation approach offers several advantages: it is mesh-free, it efficiently handles high-order derivatives, it provides spectral-like spatial accuracy for smooth solutions, and it is flexible for nonlinear fractional models. Kansa's RBF collocation technique was developed to solve PDEs using RBF [17]. Among the available approximation schemes, the multiquadric approach has been favored due to its remarkable accuracy. Hardy was initially introduced to this scheme [13]. Researchers established the existence, uniqueness, and convergence of this approximation [10, 22, 26]. Recently, there has been increasing interest in applying this function to approximate solutions for fractional PDEs. The mentioned method has been applied to solve fractional PDEs by authors [6, 35]. We shall now proceed to discuss the implementation of the RBF collocation technique in detail.

Utilizing the method, our objective is to approximate the value of $z(x_j, t^n)$:

$$z(x_j, t^n) = \sum_{k=1}^{M} \lambda_k^n \phi_k(x_j), \tag{10}$$

where λ_k^n are the unkown coefficients and $\phi_k(x_j)$ is a RBF. The most widely utilized RBFs are Gaussian (G), Multiquadric (MQ) which are presented below:

- $\phi(r) = e^{-cr^2}$,
- $\phi(r) = \sqrt{r^2 + c^2}$.

This paper presents an approach that utilizes both Gaussian basis function to derive numerical solutions.

First, we apply Liouville-Caputo FD which is given Equation (6), and then Crank-Nicolson technique to the Equation (3), we get

$$\begin{aligned}
\mu\Bigg[b_0 z_j^{n+1} \quad &- \quad \sum_{i=1}^{n}(b_{n-i} - b_{n-i+1}) z_j^i - b_n z_j^0\Bigg] \\
&+ \quad \alpha \frac{(z)_j^{n+1}(z_x)_j^n + (z)_j^n (z_x)_j^{n+1}}{2} + \beta \frac{(z_{3x})_j^{n+1} + (z_{3x})_j^n}{2} \\
&- \quad \gamma \frac{(z_{5x})_j^{n+1} + (z_{5x})_j^n}{2} = 0,
\end{aligned} \tag{11}$$

where $t^{n+1} = (n+1)\Delta t, \quad n = 0, 1, 2, \cdots, M$ and the step size of the time variable t is depicted by Δt. The approximate solution of $z(x,t)$ at the point (x_j, t^n) are denoted by z_j^n. The approximation of $z(x_j, t^n)$ and its spatial derivative at each time step can be stated as:

$$z(x_j, t^n) = \sum_{k=1}^{M} \lambda_k^n \phi_k(x_j), \tag{12}$$

$$\frac{\partial^3 z(x_j, t^n)}{\partial x^3} = \sum_{k=1}^{M} \lambda_k^n \phi_k'''(x_j), \tag{13}$$

$$\frac{\partial^5 z(x_j, t^n)}{\partial x^5} = \sum_{k=1}^{M} \lambda_k^n \phi_k^{(v)}(x_j), \tag{14}$$

where the coefficients, denoted as λ_k^n, need to be computed at every time step. By substituting Equations (12)-(14) into the Equation (11) and rearranging, we obtain the following algebraic equation:

$$\begin{aligned}
& \mu \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k(x_j) \\
+ \quad & \frac{\alpha}{2}\left[\sum_{k=1}^{M} \lambda_k^{n+1} \phi_k(x_j) \sum_{k=1}^{M} \lambda_k^{n} \phi_k'(x_j) + \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k'(x_j) \sum_{k=1}^{M} \lambda_k^{n} \phi_k(x_j) \right] \\
+ \quad & \frac{\beta}{2} \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k'''(x_j) - \frac{\gamma}{2} \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k^{(v)}(x_j) \\
= \quad & \mu \sum_{i=1}^{n} (b_{n-i} - b_{n-i+1}) \sum_{k=1}^{M} \lambda_k^{i} \phi_k(x_j) + \mu b_n \sum_{k=1}^{M} \lambda_k^{0} \phi_k(x_j) \\
- \quad & \frac{\beta}{2} \sum_{k=1}^{M} \lambda_k^{n} \phi_k'''(x_j) + \frac{\gamma}{2} \sum_{k=1}^{M} \lambda_k^{n} \phi_k^{(v)}(x_j), \quad j = 1, 2, \cdots, M.
\end{aligned} \tag{15}$$

By systematically solving the system at each time step, we can elegantly construct the solution through Equation (12). This process is efficiently executed using MATLAB, allowing us to uncover insightful results with precision and ease. Let's now give the algorithm for the system (15).

Algorithm

- Choose the fractional order δ, final time T, spatial nodes N_x, and time levels N_t.
- Generate the spatial collocation points

  $$x_j, \qquad j = 1, 2, \cdots, N_x,$$

  and define the time step size

  $$\Delta t = \frac{T}{N_t}.$$

- Construct the Gaussian RBFs

  $$\phi_k(x) = \exp\left(-c(x - x_k)^2\right).$$

- Assemble the interpolation matrix

  $$A_{kj} = \phi_k(x_j).$$

- Compute the RBF derivative matrices such as

  $$D_1 = A_x A^{-1}, \qquad D_3 = A_{xxx} A^{-1}, \qquad D_5 = A_{xxxxx} A^{-1}.$$

- Initialize the numerical solution using the initial condition

  $$z^0 = z(x, 0).$$

- Calculate fractional terms and then solve nonlinear system for each time step by using Newton-Raphson method.
- Compute the exact solution, absolute errors.
- Plot the numerical solutions, exact solutions, and error surfaces.

4. Stability Analysis

At first, Equation (3) can be linearized by assuming the quantity z in the nonlinear term zz_x as locally constant q. Then, we get

$$D_t^{\delta} z(x,t) + \alpha q z_x(x,t) + \beta z_{xxx}(x,t) - \gamma z_{xxxxx}(x,t) = 0, \tag{16}$$

$$\begin{aligned} D_t^{\delta} z(x,t) &= \mu[z_j^{n+1} - z_j^n] \\ &+ \mu b_1 z_j^n + \mu \sum_{i=1}^{n-1} b_i^* z_j^{n-i} - \mu b_n z_j^0, \end{aligned} \tag{17}$$

where $b_i^* = b_{i+1} - b_i, \quad i = 1, 2, \cdots, n-1$. It can be expressed into the difference equation same with Equation (15), as

$$\begin{aligned} \mu \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k(x_j) &+ \frac{q\alpha}{2} \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k'(x_j) \\ &+ \frac{\beta}{2} \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k'''(x_j) - \frac{\gamma}{2} \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k^{v}(x_j) \\ &= -\mu \sum_{i=1}^{n} b_i^* \sum_{k=1}^{M} \lambda_k^{n-i} \phi_i(x_j) + \mu b_n \sum_{k=1}^{M} \lambda_k^{0} \phi_k(x_j) - \mu b_1 \sum_{k=1}^{M} \lambda_k^{n} \phi_k(x_j) + \mu \sum_{k=1}^{M} \lambda_k^{n} \phi_k(x_j) \\ &- \frac{q\alpha}{2} \sum_{k=1}^{M} \lambda_k^{n} \phi_k'(x_j) - \frac{\beta}{2} \sum_{k=1}^{M} \lambda_k^{n} \phi_k'''(x_j) + \frac{\gamma}{2} \sum_{k=1}^{M} \lambda_k^{n} \phi_k^{v}(x_j), \quad j = 1, 2, \cdots, M. \end{aligned} \tag{18}$$

As in the von-Neumann stability analysis method, we assume that

$$\sum_{k=1}^{M} \lambda_k^n \phi_k(x_j) = \xi^n e^{i\theta x_j}, \tag{19}$$

in here $i^2 = -1$, the amplification factor is shown by ξ and θ is the wave number. If this equation is plugged into the linear difference equation, then equating the coefficients of $e^{i\theta x_j}$ gives this equality

$$\xi = \frac{-\mu \sum_{j=1}^{n} b_j^* \xi^{-j} + \mu b_n \xi^{-n} - \mu b_1 + \mu + \frac{i}{2}(-q\alpha\theta + \beta\theta^3 + \gamma\theta^5)}{\mu - \frac{i}{2}(-q\alpha\theta + \beta\theta^3 + \gamma\theta^5)}. \tag{20}$$

Let's consider the time independent limit value $\xi = -1$ as in [34], then

$$\mu \sum_{j=1}^{n} b_j^* \xi^{-j} + \mu b_n \xi^{-n} = -\mu(2^{1-\delta} - 1),$$

and

$$\mu b_1 = \mu(2^{1-\delta} - 1).$$

Equation (20) can be rewritten as

$$\xi = \frac{\mu + \frac{i}{2}(-q\alpha\theta + \beta\theta^3 + \gamma\theta^5)}{\mu - \frac{i}{2}(-q\alpha\theta + \beta\theta^3 + \gamma\theta^5)}. \tag{21}$$

In conclusion, we find that the method is unconditionally stable since $|\xi| = 1$.

5. Numerical Outcomes and Discussion

This part gives numerical results of the TFKE on the two test problem.

Test Problem 1:

For $\delta = 1$ and $f(x,t) = 0$, the exact solution of Equation (3) is

$$z(x,t) = \frac{105}{169}\mathrm{sech}^4\left(\frac{1}{2\sqrt{13}}(x - \frac{36t}{169})\right),$$

with the initial condition is given as

$$z(x,0) = \frac{105}{169}\mathrm{sech}^4\left(\frac{x}{2\sqrt{13}}\right). \tag{22}$$

Numerical calculation is done over the solution domain $-20 \le x \le 20$ in the time period $0 \le t \le 10$ with parameters $\Delta x = 0.05$ and $\Delta t = 0.1$. Numerical and exact solutions are presented in Figures 1, 3, 4, and 5 for the choice of $\delta = 1$. Additionaly, absolute errors are illustrated in Figures 2, and 6. All numerical results show that the method is effective and reliable.

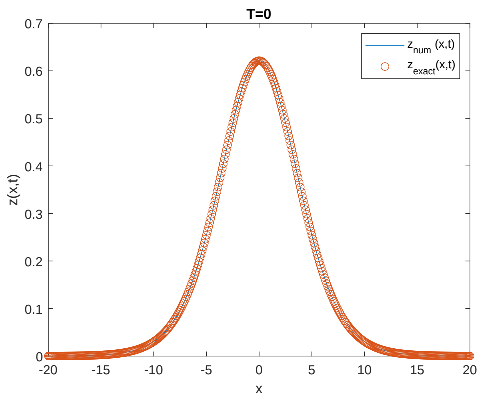

Figure 1: Numerical and exact solutions for $\delta = 1$

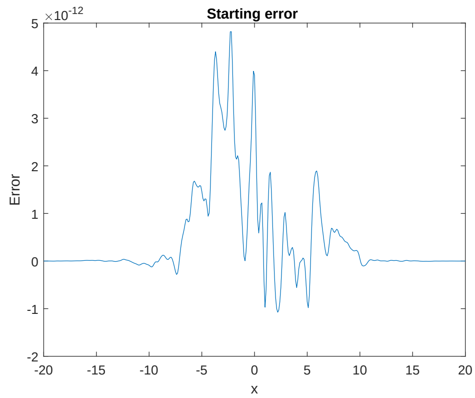

Figure 2: Starting error at $T = 0$ for Test Problem 1

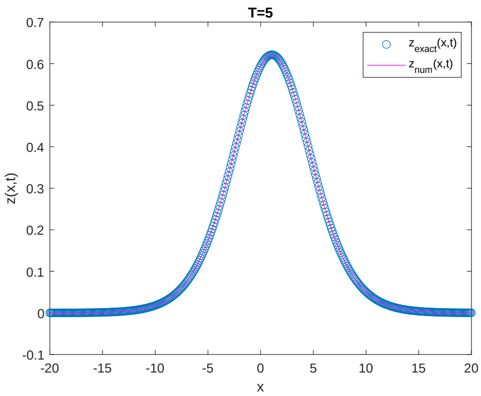

Figure 3: Numerical and exact solutions at $T = 5$ for Test Problem 1

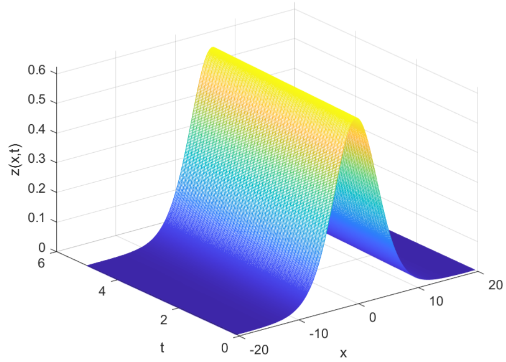

Figure 4: Numerical solution for $\delta = 1$, $\Delta x = 0.05$, and $\Delta t = 0.1$

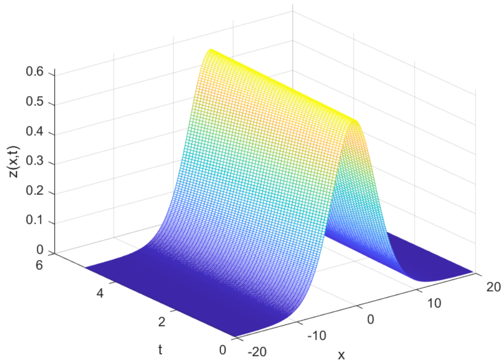

Figure 5: Exact solution for $\delta = 1$, $\Delta x = 0.05$, and $\Delta t = 0.1$

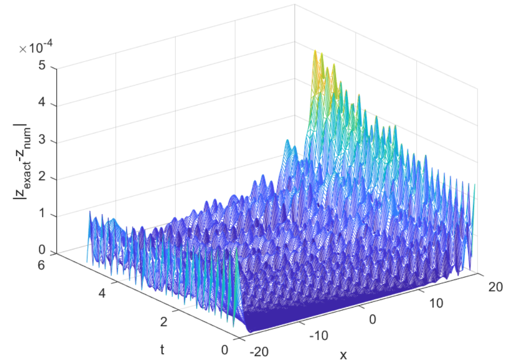

Figure 6: Absolute error for $\delta = 1$

Test Problem 2: To effectively validate the proposed numerical method, we employ a manufactured solution approach. In this process, we strategically select the exact solution as

$$z(x,t) = t^{2+\delta}\sin(\pi x),$$

and carefully construct the corresponding source term to guarantee that the equation is satisfied precisely. This recognized scheme serves as a strong means of verifying numerical techniques for fractional differential equations, ensuring reliability and accuracy. The forcing term $f(x,t)$ is also computed by substituting the exact solution into the equation.

The numerical simulations conducted using the RBF collocation method provide valuable insights into the effectiveness of this approach for approximating solutions to the TFKE across a range of fractional-order parameters δ. The strong agreement between the numerical and exact solutions throughout the computational domain reflects the method's high accuracy and stability.

The three-dimensional surface plots 7, 8, 9, 10 offer a clear illustration of how the solution evolves smoothly in both space and time. As the fractional order transitions from $\delta = 0.4$ to $\delta = 1.0$, we observe an increase in the amplitude of the solution and more pronounced wave propagation.

This behavior aligns with expectations, as a higher δ reduces the memory effect associated with the fractional derivative, guiding the model toward the classical Kawahara equation when $\delta = 1$.

The comparison plots at the final time level 11 reinforce the RBF collocation technique's ability to effectively capture the complex nonlinear and dispersive dynamics of the equation. The near overlap of the numerical curves with the exact solutions across all tested fractional orders highlights the algorithm's reliability and effectiveness.

As shown in Figure 12, the numerical error consistently decreases steadily as the numbers of collocation points and time levels increases. In summary, the results suggest that the RBF collocation method serves as a promising and accurate numerical framework for solving the time-fractional Kawahara equation across various fractional orders. The established error bounds further support its reliability, making it an appealing choice for practical applications in the field.

Table 1 shows the maximum absolute errors achieved by the RBF collocation method for different spatial and temporal discretizations. The results also show that the fully implicit scheme remains stable and efficient, even when dealing with nonlinear and fifth-order dispersive terms.

Ultimately, the Table 1 confirms that the proposed method demonstrates reliable numerical performance with minimal errors.

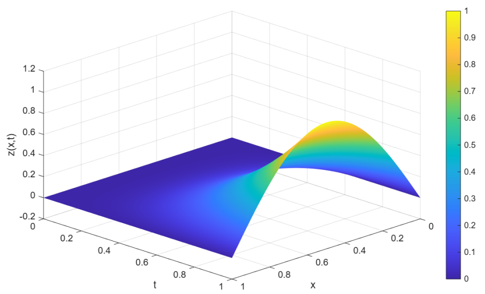

Figure 7: Numerical solution for $\delta = 0.4$

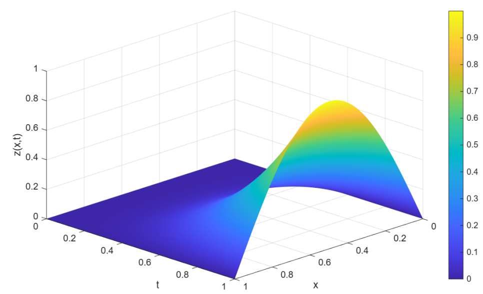

Figure 8: Numerical solution for $\delta = 0.6$

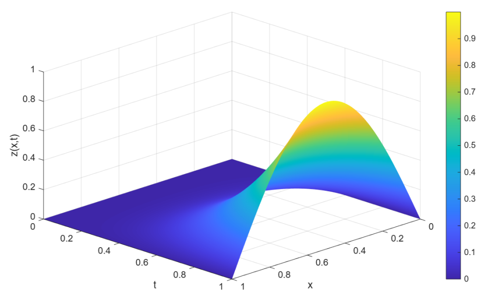

Figure 9: Numerical solution for $\delta = 0.8$

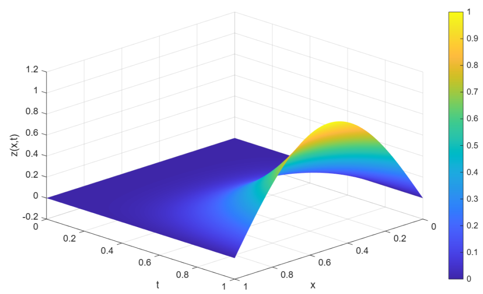

Figure 10: Numerical solution for $\delta = 1$

In here, N_x represents the number of spatial collocation points employed in the RBF approximation over the spatial domain, while N_t denotes the number of temporal discretization

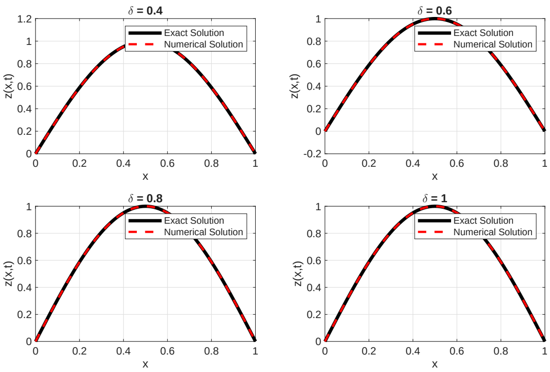

Figure 11: Comparison of numerical and exact solutions for $\delta = 0.4,\ 0.6,\ 0.8$, and 1

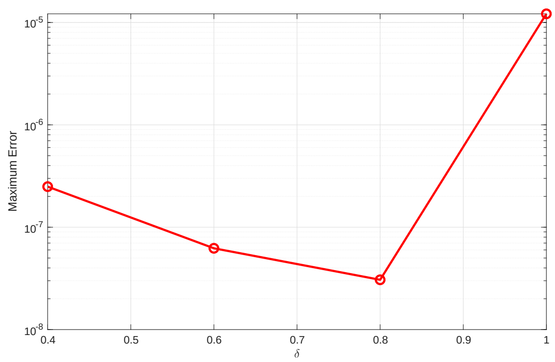

Figure 12: Maximum error for different values of δ

Table 1: Maximum absolute errors for the RBF collocation method applied to Test Problem 2

N_x	N_t	L_∞
21	300	$1.144899e-08$
41	600	$3.991269e-08$
61	900	$3.810005e-08$
81	1200	$2.840317e-08$

points utilized in the L1 finite difference approximation of the Liouville-Caputo FD.

Recent research utilizing spectral collocation methods for the nonlinear time-fractional generalized Kawahara equation has revealed impressive accuracy and robust convergence properties. Notably, eighth-kind Chebyshev collocation schemes [2] and operational matrix techniques have demonstrated the ability to provide highly precise approximations with minimal computational errors.

Our current findings are consistent with these advanced spectral methods, yet they also offer the advantage of simpler implementation through RBF interpolation. Moreover, unlike many conventional polynomial-based spectral methods, the RBF collocation approach eliminates the need for structured basis functions or complex orthogonal polynomial constructions, making it a more accessible and efficient choice for practitioners. The numerical error behavior observed in our simulations aligns closely with established theoretical analyses for fractional differential equations. Studies on spectral and finite difference methods for time-fractional models have shown that the L1 approximation typically achieves a temporal accuracy of $O(\Delta t^{2-\delta})$. This finding supports the convergence trend we've identified in our computations.

Additionally, prior research on local RBF-FD methods for KE has demonstrated that RBF-based discretizations maintain stability and accuracy, even when faced with equations containing fifth-order dispersive terms. Our current results reinforce the idea that RBF collocation is highly effective at approximating nonlinear fractional KE across various fractional orders.

In summary, the evidence clearly indicates that our proposed method stands out as competitive compared to existing spectral, finite difference, and operational matrix techniques, while delivering:

- Exceptional spatial accuracy,
- Effortless implementation,
- Flexibility for nonlinear fractional PDEs,
- Consistent convergence behavior across diverse fractional orders.

Choosing our approach ensures you harness the full potential of modern numerical methods for complex problems.

6. Conclusion

This paper begins by demonstrating a mesh-free approximation using the RBF collocation method to numerically solve the TFKE. The time-fractional derivative is expressed by the Liouville-Caputo FD. We use two test problems to obtain the numerical solutions for the present equation.

In the first test problem, we compute numerical solution for the fractional derivative with a value of $\delta = 1$. Graphical results are given in figures.

In the second test problem, a well-designed manufactured test problem with an exact analytical solution is used to validate the effectiveness of the method. Numerical simulations are conducted for various fractional-order values, and the resulting numerical solutions display an excellent match with the corresponding exact solutions. The graphical outcomes indicate that the proposed method effectively captures the nonlinear and dispersive behavior of the equation while maintaining numerical stability throughout the computational domain.

The error analysis approves the method's convergence and reliability. Maximum absolute errors significantly decrease as the number of spatial collocation points and temporal discretization levels increases. Furthermore, the stability analysis of the approach is performed using a method akin to the traditional von-Neumann analysis. The consequence indicates that it is unconditionally stable.

Our main results display the idea that RBF collocation is highly effective at approximating nonlinear fractional Kawahara equations across various fractional orders. Thus, it can be said that the mentioned method provides an efficient alternative for solving these fractional PDEs due to its straightforward implementation.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and legal special permission.

Authors Contributions

Author [Bahar Karaman]: Collected the data, contributed to completing the research and solving the problem, wrote the manuscript (%45).

Author [Emrah Karaman]: Contributed to completing the research and solving the problem (%25).

Author [Yılmaz Dereli]: Contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%30).

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Abd-Elhameed W.M., Al-Harbi A.K., Alqubori O.M., Alharbi M.H., Atta A.G., *Collocation method for the time-fractional generalized Kawahara equation using a certain Lucas polynomial sequence*, Axioms, 14(2), 114, 2025.

[2] Abd-Elhameed W.M., Youssri Y.H., Amin A.K., Atta A.G., *Eighth-kind Chebyshev polynomials collocation algorithm for the nonlinear time-fractional generalized Kawahara equation*, Fractal and Fractional, 7(9), 652, 2023.

[3] Ahmed H.M., Izadi M., *An accurate approximation technique based on Schröder operational matrices for numerical treatments of time-fractional nonlinear generalized Kawahara equation*, Boundary Value Problems, 2025, 90, 2025.

[4] Barkai E., Metzler R., Klafter J., *From continuous time random walks to the fractional Fokker-Planck equation*, Physical Review E, 61, 132, 2000.

[5] Buhmann M.D., Radial Basis Functions: Theory and Implementations, Cambridge University Press, 2003.

[6] Chen W., Ye L., Sun H., *Fractional diffusion equations by the Kansa method*, Computers & Mathematics with Applications, 59, 1614-1620, 2010.

[7] Diego M.A., *Implicit finite difference approximation for time fractional diffusion equations*, Computers & Mathematics with Applications, 56, 1138-1145, 2008.

[8] Fasshauer G.E., Meshfree Approximation Methods with MATLAB, World Scientific, 2007.

[9] Flyer N., Fornberg B., *Radial basis functions: Developments and applications*, Acta Numerica, 24, 215-258, 2015.

[10] Franke C., Schaback R., *Convergence order estimates of meshless collocation methods using radial basis functions*, Advances in Computational Mathematics, 8, 381-399, 1998.

[11] Gao G.-H., Sun Z.-Z., Zhang H.-W., *A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications*, Journal of Computational Physics, 259, 33-50, 2014.

[12] Gorenflo R., Mainardi F., Essentials of Fractional Calculus, Preprint Submitted to MaPhySto Center, January 28, 2000.

[13] Hardy R.L., *Multiquadric equations of topography and other irregular surfaces*, Journal of Geophysical Research, 76, 1905-1915, 1971.

[14] He J.H., *Approximate analytical solution for nonlinear fractional differential equations*, Communications in Nonlinear Science and Numerical Simulation, 4, 230-245, 1999.

[15] Izadi M., Patel Y.F., *Analytical study of models in quantum field theory: Fractional Kundu-Eckhaus and massive Thirring equations*, Nonlinear Dynamics, 113, 26583-26602, 2025.

[16] Izadi M., Samei M.E., *Time accurate solution to Benjamin-Bona-Mahony-Burgers equation via Taylor-Boubaker series scheme*, Boundary Value Problems, 2022, 17, 2022.

[17] Kansa E.J., *Multiquadrics - a scattered data approximation scheme with applications to computational fluid dynamics, II: Solutions to parabolic, hyperbolic and elliptic partial differential equations*, Computers & Mathematics with Applications, 19(8-9), 147-161, 1990.

[18] Karaman B., Dereli Y., *Numerical simulation for a time-fractional coupled nonlinear Schrödinger equations*, International Journal of Computer Mathematics, 98(6), 1233-1253, 2021.

[19] Kawahara T., *Oscillatory solitary waves in dispersive media*, Journal of the Physical Society of Japan, 33, 260-264, 1972.

[20] Laskin N., *Time fractional quantum mechanics*, Chaos, 102, 16-28, 2017.

[21] Lin Y., Xu C., *Finite difference/spectral approximations for the time-fractional diffusion equation*, Journal of Computational Physics, 225(2), 1533-1552, 2007.

[22] Madcy W.R., Nelson S.A., *Multivariate interpolation and conditionally positive definite functions II*, Mathematics of Computation, 54, 211-230, 1990.

[23] Magin R.L., Fractional Calculus in Bioengineering, Begell House, 2006.

[24] Mainardi F., Fractional Calculus and Waves in Linear Viscoelasticity, Imperial College Press, 2010.

[25] Metzler R., Klafter J., *The random walk's guide to anomalous diffusion*, Physics Reports, 339, 1-77, 2000.

[26] Michelli C.A., *Interpolation of scattered data: Distance matrices and conditionally positive definite functions*, Constructive Approximation, 2, 11-22, 1986.

[27] Miller K.S., Ross B., An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, 1993.

[28] Momani S., *An explicit and numerical solutions of the fractional KdV equation*, Mathematics and Computers in Simulation, 70, 110-118, 2005.

[29] Nadeem M., Khan A., Javeed M.A., Yubin Z., *Analytical scheme for time fractional Kawahara and modified Kawahara problems in shallow water waves*, Fractal and Fractional, 8(7), 395, 2024.

[30] Oldham K.B., Spanier J., The Fractional Calculus, Academic Press, 1974.

[31] Patel Y.F., Izadi M., *Exact solutions of the generalized time-fractional order Burgers-Huxley equation with a robust semi-analytical technique*, Journal of Umm Al-Qura University for Applied Sciences, 2025.

[32] Podlubny I., Fractional Differential Equations, Academic Press, 1999.

[33] Sun Z., Wu X., *A fully discrete difference scheme for a diffusion-wave system*, Applied Numerical Mathematics, 56, 193-209, 2006.

[34] Sweilam N.H., Khader M.M., Ade M., *On the stability analysis of weighted average finite difference methods for fractional wave equation*, Fractional Differential Calculus, 2(1), 17-29, 2012.

[35] Uddin M., Haq S., *RBFs approximation method for time fractional partial differential equations*, Communications in Nonlinear Science and Numerical Simulation, 16, 4208-4214, 2011.

[36] Zayernouri M., Karniadakis G.E., *Fractional spectral collocation methods*, SIAM Journal on Scientific Computing, 36, A40-A62, 2014.

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Research Article

Soft $\mathcal{N}$-Clogroups over Near-Rings and their Extended Operations

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Abstract: In this paper, we introduce the notion of a soft $\mathcal{N}$-clogroup over a near-ring by combining the classical concept of $\mathcal{N}$-clogroups with soft set theory. This framework provides a parameter-dependent extension of the subgroup-generation mechanism induced by multiplicative action in near-rings. We define soft $\mathcal{N}$-clogroups and soft sub-$\mathcal{N}$-clogroups and present illustrative examples clarifying their structural behavior. We further investigate the behavior of these structures under extended union and extended intersection operations. The associativity of these extended operations is established within the soft $\mathcal{N}$-clogroup framework. The obtained results provide a systematic structural analysis of soft $\mathcal{N}$-clogroups and clarify the behavior of extended operations in this setting. Moreover, we examine the behavior of soft $\mathcal{N}$-clogroups under near-ring homomorphisms and show that preimages preserve the soft $\mathcal{N}$-clogroup structure, whereas homomorphic images do not necessarily preserve it.

Keywords: Near-rings, $\mathcal{N}$-clogroups, soft sets.

1. Introduction

Near-ring theory provides a flexible algebraic framework in which additive and multiplicative structures interact without the full symmetry of rings. A structural phenomenon in near-rings arises from the interaction between multiplication and the additive group. In this context, the concept of an $\mathcal{N}$-clogroup was introduced to describe subsets whose multiplicative action generates additive subgroups [5]. This notion captures a subgroup-generation mechanism determined by multiplication.

On the other hand, soft set theory, introduced and further developed by Molodtsov [13–15], offers a parameter-based formalism for describing mathematical objects that vary with respect to external attributes. Soft set theory has also been applied to decision-making problems, information systems, and soft information recognition [11, 18, 22]. Instead of studying a single subset of a

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universe, a soft set assigns to each parameter a corresponding subset, thereby allowing a structured parameterization of algebraic data.

Since its introduction, soft set theory has been adapted to various algebraic systems, including groups, rings, semirings, near-rings, BCK/BCI-algebras, WS-algebras, and soft modules [1, 2, 4, 8–10, 16, 19, 21].

In recent years, there has been increasing interest in extending classical algebraic structures into parameterized settings using soft set theory and related uncertainty-based frameworks, such as rough set theory [17]. Recent studies have focused on the development of soft algebraic structures and their generalizations, including soft near-ring frameworks and their applications [6, 7]. These developments indicate that soft algebraic systems remain an active area of research and continue to provide new perspectives for algebraic generalizations.

Although $\mathcal{N}$-clogroups have been studied within classical near-ring theory and soft algebraic structures have been extensively developed, the integration of these two frameworks has not been addressed. There is currently no systematic treatment of $\mathcal{N}$-clogroups in a parameterized (soft) setting.

The aim of this paper is to establish such a framework. We introduce the concept of a soft $\mathcal{N}$-clogroup over a near-ring and investigate its basic structural properties. We define soft sub-$\mathcal{N}$-clogroups and provide explicit examples illustrating how parameterization interacts with the classical subgroup-generation condition.

Furthermore, we examine the behavior of these structures under extended union and extended intersection operations in the sense of soft set theory [3, 12]. We analyze how these operations interact with the defining subgroup-generation condition of soft $\mathcal{N}$-clogroups and investigate their structural properties within this framework.

The paper is organized as follows. Section 2 recalls the necessary preliminaries on near-rings, $\mathcal{N}$-clogroups, and soft sets. In Section 3 we introduce soft $\mathcal{N}$-clogroups and study their fundamental properties. Section 4 is devoted to the analysis of extended union and extended intersection operations.

2. Preliminaries

2.1. Near-rings and $\mathcal{N}$-clogroups

We first recall the notion of a near-ring and the related concept of an $\mathcal{N}$-clogroup (see [5]).

Definition 2.1 ([5]) *Let $\mathcal{N}$ be a nonempty set endowed with two binary operations, denoted by "$+$" and "$\cdot$". The structure $(\mathcal{N}, +, \cdot)$ is called a* near-ring *provided that:*

- *The algebraic system $(\mathcal{N}, +)$ forms a group (commutativity is not assumed).*

- *The pair* $(\mathcal{N},\cdot)$ *constitutes a semigroup.*
- *The multiplication is right distributive with respect to addition; that is,*

$$(a+b)\cdot c = a\cdot c + b\cdot c \tag{1}$$

holds for every $a,b,c\in\mathcal{N}$.

A nonempty subset $\mathcal{M}\subseteq\mathcal{N}$ *is said to be a* subnear-ring *of* $\mathcal{N}$ *if* $(\mathcal{M},+)$ *is a subgroup of* $(\mathcal{N},+)$ *and* $\mathcal{M}$ *is stable under the multiplication of* $\mathcal{N}$.

Moreover, for arbitrary subsets $\mathcal{M},\mathcal{S}\subseteq\mathcal{N}$, *we introduce the product set*

$$\mathcal{M}\mathcal{S} = \{\, ms \mid m\in\mathcal{M},\ s\in\mathcal{S}\,\}. \tag{2}$$

Definition 2.2 *[5] Assume that* $\mathcal{N}$ *is a near-ring and that* $\mathcal{M}$ *is a subnear-ring of* $\mathcal{N}$. *A nonempty subset* $\mathcal{S}\subseteq\mathcal{N}$ *is said to be an* $\mathcal{M}$*-clogroup whenever the additive structure* $(\mathcal{M}\mathcal{S},+)$ *forms a subgroup of* $(\mathcal{N},+)$.

As a special case, if $\mathcal{M}=\mathcal{N}$, *then* $\mathcal{S}$ *is referred to as an* $\mathcal{N}$-clogroup.

Remark 2.3 *Unless otherwise specified, all near-rings considered in this paper are assumed to satisfy the right distributive law.*

2.2. Soft Sets

We provide a concise overview of the main notions from soft set theory, as initiated by Molodtsov [13] that form the conceptual framework for the subsequent results.

Definition 2.4 *[13] Consider a universe* $\mathcal{U}$ *together with a parameter set* $\mathcal{E}$. *Given* $\mathcal{S}\subseteq\mathcal{E}$, *a soft set on* $\mathcal{U}$ *determined by* $\mathcal{S}$ *is a pair* $(\mathcal{J},\mathcal{S})$ *such that*

$$\mathcal{J}:\mathcal{S}\to\mathcal{P}(\mathcal{U}) \tag{3}$$

is a function assigning to every element of $\mathcal{S}$ *a subset of* $\mathcal{U}$.

Definition 2.5 *Let* $(\mathcal{J},\mathcal{S})$ *and* $(\mathcal{K},\mathcal{T})$ *be soft sets defined on the same universe* $\mathcal{U}$. *We say that* $(\mathcal{J},\mathcal{S})$ *is a* soft subset *of* $(\mathcal{K},\mathcal{T})$, *and write*

$$(\mathcal{J},\mathcal{S})\,\widetilde{\subseteq}\,(\mathcal{K},\mathcal{T}), \tag{4}$$

whenever the parameter inclusion $\mathcal{S}\subseteq\mathcal{T}$ *holds and, for each* $s\in\mathcal{S}$, *the corresponding images satisfy*

$$\mathcal{J}(s)\subseteq\mathcal{K}(s). \tag{5}$$

In addition, we briefly summarize the standard extended operations on soft sets introduced in the literature [3, 12, 20].

Definition 2.6 *[3, 12] Let $(\mathcal{J},\mathcal{S})$ and $(\mathcal{K},\mathcal{T})$ be soft sets over $\mathcal{U}$. The* extended union *of $(\mathcal{J},\mathcal{S})$ and $(\mathcal{K},\mathcal{T})$ is defined as*

$$(\mathcal{J},\mathcal{S}) \cup_c (\mathcal{K},\mathcal{T}) = (\mathcal{L},\mathcal{S}\cup\mathcal{T}), \tag{6}$$

where

$$\mathcal{L}(s) = \begin{cases} \mathcal{J}(s), & s \in \mathcal{S}\setminus\mathcal{T}, \\ \mathcal{K}(s), & s \in \mathcal{T}\setminus\mathcal{S}, \\ \mathcal{J}(s)\cup\mathcal{K}(s), & s \in \mathcal{S}\cap\mathcal{T}. \end{cases} \tag{7}$$

The extended intersection *of $(\mathcal{J},\mathcal{S})$ and $(\mathcal{K},\mathcal{T})$ is defined as*

$$(\mathcal{J},\mathcal{S}) \cap_c (\mathcal{K},\mathcal{T}) = (\mathcal{L},\mathcal{S}\cup\mathcal{T}), \tag{8}$$

where

$$\mathcal{L}(s) = \begin{cases} \mathcal{J}(s), & s \in \mathcal{S}\setminus\mathcal{T}, \\ \mathcal{K}(s), & s \in \mathcal{T}\setminus\mathcal{S}, \\ \mathcal{J}(s)\cap\mathcal{K}(s), & s \in \mathcal{S}\cap\mathcal{T}. \end{cases} \tag{9}$$

3. Soft $\mathcal{N}$-Clogroups

3.1. Definition and Basic Properties

We now introduce the central notion of this paper.

Definition 3.1 *Let $\mathcal{N}$ be a near-ring and let $\mathcal{E}$ be a set of parameters. Let $\mathcal{S}\subseteq\mathcal{E}$. A soft set $(\mathcal{J},\mathcal{S})$ over $\mathcal{N}$ is called a* soft $\mathcal{N}$-clogroup *if, for every $s\in\mathcal{S}$,*

$$(\mathcal{N}\mathcal{J}(s),+) \leq (\mathcal{N},+), \tag{10}$$

where

$$\mathcal{N}\mathcal{J}(s) = \{na \mid n\in\mathcal{N},\ a\in\mathcal{J}(s)\}. \tag{11}$$

Definition 3.2 *Let $(\mathcal{J},\mathcal{S})$ and $(\mathcal{K},\mathcal{T})$ be soft $\mathcal{N}$-clogroups over $\mathcal{N}$. We say that $(\mathcal{J},\mathcal{S})$ is a* soft sub-$\mathcal{N}$-clogroup *of $(\mathcal{K},\mathcal{T})$, denoted by*

$$(\mathcal{J},\mathcal{S})\widetilde{\leqq}_c(\mathcal{K},\mathcal{T}) \tag{12}$$

if

$$(\mathcal{J},\mathcal{S})\widetilde{\subseteq}(\mathcal{K},\mathcal{T}). \tag{13}$$

Remark 3.3 *If $(\mathcal{J}, \mathcal{S})$ is a soft $\mathcal{N}$-clogroup, then for each $s \in \mathcal{S}$, the subset $\mathcal{J}(s)$ satisfies the subgroup generation condition under the multiplicative action of $\mathcal{N}$. Thus, for each fixed parameter, the corresponding value behaves as a classical $\mathcal{N}$-clogroup under the multiplicative action of $\mathcal{N}$.*

3.2. Examples

We present illustrative examples to clarify the distinction between $\mathcal{M}$-clogroups and $\mathcal{N}$-clogroups and to demonstrate the structure of soft $\mathcal{N}$-clogroups.

Example 3.4 *Let $\mathcal{N} = \mathbb{Z}_{12}$ be the near-ring under usual addition and multiplication modulo 12. Let $\mathcal{M} = \{0, 6\}$, which is a subnear-ring of $\mathcal{N}$, and let $\mathcal{S} = \{0, 4, 8\}$.*

We compute

$$\mathcal{MS} = \{ms \mid m \in \mathcal{M},\ s \in \mathcal{S}\}. \tag{14}$$

Since

$$6 \cdot 0 = 0, \quad 6 \cdot 4 = 0, \quad 6 \cdot 8 = 0 \quad (mod\ 12), \tag{15}$$

we obtain

$$\mathcal{MS} = \{0\}. \tag{16}$$

Hence $(\mathcal{MS}, +)$ is a subgroup of $(\mathbb{Z}_{12}, +)$, and therefore $\mathcal{S}$ is an $\mathcal{M}$-clogroup.

On the other hand,

$$\mathcal{NS} = \{ns \mid n \in \mathbb{Z}_{12},\ s \in \mathcal{S}\} = \{0, 4, 8\}. \tag{17}$$

Since $\{0, 4, 8\}$ is a subgroup of $(\mathbb{Z}_{12}, +)$, $\mathcal{S}$ is also an $\mathcal{N}$-clogroup.

Example 3.5 *Let $\mathcal{N} = \mathbb{Z}_6$ under usual addition and multiplication modulo 6, and let $\mathcal{S} = \{1, 3\}$.*

Then

$$\mathcal{NS} = \{ns \mid n \in \mathbb{Z}_6,\ s \in \mathcal{S}\} = \mathbb{Z}_6. \tag{18}$$

Since $(\mathbb{Z}_6, +)$ is a group, $\mathcal{S}$ is an $\mathcal{N}$-clogroup.

We now provide a soft version.

Example 3.6 *Let $\mathcal{N} = \mathbb{Z}_6$ and let $\mathcal{S} = \{e_1, e_2\}$. Define a soft set $(\mathcal{J}, \mathcal{S})$ over $\mathcal{N}$ by*

$$\mathcal{J}(e_1) = \{1, 3\}, \qquad \mathcal{J}(e_2) = \{2, 4\}. \tag{19}$$

Since

$$\mathcal{N}\mathcal{J}(e_1) = \mathbb{Z}_6 \quad \textit{and} \quad \mathcal{N}\mathcal{J}(e_2) = \{0, 2, 4\}, \tag{20}$$

and both sets form subgroups of $(\mathbb{Z}_6, +)$, it follows that $(\mathcal{J}, \mathcal{S})$ is a soft $\mathcal{N}$-clogroup.

3.3. Extended Union

We analyze the extended union of two soft $\mathcal{N}$-clogroups and examine its structural behavior within the soft framework.

Definition 3.7 *Let $(\mathcal{J},\mathcal{S})$ and $(\mathcal{K},\mathcal{T})$ be soft $\mathcal{N}$-clogroups over a near-ring $\mathcal{N}$. The extended union of $(\mathcal{J},\mathcal{S})$ and $(\mathcal{K},\mathcal{T})$ is defined as*

$$(\mathcal{J},\mathcal{S}) \cup_c (\mathcal{K},\mathcal{T}) = (\mathcal{L},\mathcal{S} \cup \mathcal{T}), \tag{21}$$

where $\mathcal{L} : \mathcal{S} \cup \mathcal{T} \to \mathcal{P}(\mathcal{N})$ is given by

$$\mathcal{L}(s) = \begin{cases} \mathcal{J}(s), & s \in \mathcal{S} \setminus \mathcal{T}, \\ \mathcal{K}(s), & s \in \mathcal{T} \setminus \mathcal{S}, \\ \mathcal{J}(s) \cup \mathcal{K}(s), & s \in \mathcal{S} \cap \mathcal{T}. \end{cases} \tag{22}$$

Example 3.8 *Let $\mathcal{N} = (\mathbb{Z}_6, +, \cdot)$ be the near-ring under usual addition and multiplication modulo 6. Let $\mathcal{S} = \{1,2,3\}$ and $\mathcal{T} = \{0,3,4\}$. Define the soft sets $(\mathcal{J},\mathcal{S})$ and $(\mathcal{K},\mathcal{T})$ over $\mathcal{N}$ by*

$$\mathcal{J}(1) = \mathbb{Z}_6, \quad \mathcal{J}(2) = \{0,2,4\}, \quad \mathcal{J}(3) = \{0,3\},$$

and

$$\mathcal{K}(0) = \{0\}, \quad \mathcal{K}(3) = \{0,3\}, \quad \mathcal{K}(4) = \{0,2,4\}.$$

We first verify that $(\mathcal{J},\mathcal{S})$ and $(\mathcal{K},\mathcal{T})$ are soft $\mathcal{N}$-clogroups. Indeed,

$$\mathcal{N}\mathcal{J}(1) = \mathbb{Z}_6, \quad \mathcal{N}\mathcal{J}(2) = \{0,2,4\}, \quad \mathcal{N}\mathcal{J}(3) = \{0,3\},$$

and

$$\mathcal{N}\mathcal{K}(0) = \{0\}, \quad \mathcal{N}\mathcal{K}(3) = \{0,3\}, \quad \mathcal{N}\mathcal{K}(4) = \{0,2,4\}.$$

Each of these sets is a subgroup of $(\mathbb{Z}_6,+)$. Hence both $(\mathcal{J},\mathcal{S})$ and $(\mathcal{K},\mathcal{T})$ are soft $\mathcal{N}$-clogroups.

Now consider their extended union. The parameter set of the extended union is

$$\mathcal{V} = \mathcal{S} \cup \mathcal{T} = \{0,1,2,3,4\}.$$

Let

$$(\mathcal{L},\mathcal{V}) = (\mathcal{J},\mathcal{S}) \cup_c (\mathcal{K},\mathcal{T}).$$

Then the mapping $\mathcal{L} : \mathcal{V} \to \mathcal{P}(\mathcal{N})$ is defined by

$$\mathcal{L}(s) = \begin{cases} \mathcal{J}(s), & s \in \mathcal{S} \setminus \mathcal{T}, \\ \mathcal{K}(s), & s \in \mathcal{T} \setminus \mathcal{S}, \\ \mathcal{J}(s) \cup \mathcal{K}(s), & s \in \mathcal{S} \cap \mathcal{T}. \end{cases}$$

Accordingly,

$$\mathcal{L}(0) = \{0\}, \quad \mathcal{L}(1) = \mathbb{Z}_6, \quad \mathcal{L}(2) = \{0,2,4\},$$

and

$$\mathcal{L}(3) = \{0,3\} \cup \{0,3\} = \{0,3\}, \quad \mathcal{L}(4) = \{0,2,4\}.$$

Moreover,

$$\mathcal{N}\mathcal{L}(0) = \{0\}, \quad \mathcal{N}\mathcal{L}(1) = \mathbb{Z}_6, \quad \mathcal{N}\mathcal{L}(2) = \{0,2,4\},$$

and

$$\mathcal{N}\mathcal{L}(3) = \{0,3\}, \quad \mathcal{N}\mathcal{L}(4) = \{0,2,4\}.$$

Since all these sets are subgroups of $(\mathbb{Z}_6,+)$, *it follows that* $(\mathcal{L},\mathcal{V})$ *is a soft* $\mathcal{N}$*-clogroup.*

3.4. Extended Intersection

Definition 3.9 *Let* $(\mathcal{J},\mathcal{S})$ *and* $(\mathcal{K},\mathcal{T})$ *be soft* $\mathcal{N}$*-clogroups over a near-ring* $\mathcal{N}$. *The extended intersection of* $(\mathcal{J},\mathcal{S})$ *and* $(\mathcal{K},\mathcal{T})$ *is defined by*

$$(\mathcal{J},\mathcal{S}) \cap_c (\mathcal{K},\mathcal{T}) = (\mathcal{L},\mathcal{V}), \tag{23}$$

where $\mathcal{V} = \mathcal{S} \cup \mathcal{T}$ *and the mapping* $\mathcal{L}: \mathcal{V} \to \mathcal{P}(\mathcal{N})$ *is given by*

$$\mathcal{L}(s) = \begin{cases} \mathcal{J}(s), & s \in \mathcal{S} \setminus \mathcal{T}, \\ \mathcal{K}(s), & s \in \mathcal{T} \setminus \mathcal{S}, \\ \mathcal{J}(s) \cap \mathcal{K}(s), & s \in \mathcal{S} \cap \mathcal{T}. \end{cases} \tag{24}$$

Example 3.10 *Let* $\mathcal{N} = (\mathbb{Z}_6,+,\cdot)$ *be the near-ring under usual addition and multiplication modulo* 6. *Let* $\mathcal{S} = \{1,2,3\}$ *and* $\mathcal{T} = \{0,3,4\}$. *Define the soft sets* $(\mathcal{J},\mathcal{S})$ *and* $(\mathcal{K},\mathcal{T})$ *over* $\mathcal{N}$ *by*

$$\mathcal{J}(1) = \mathbb{Z}_6, \quad \mathcal{J}(2) = \{0,2,4\}, \quad \mathcal{J}(3) = \{0,3\},$$

and

$$\mathcal{K}(0) = \{0\}, \quad \mathcal{K}(3) = \{0,3\}, \quad \mathcal{K}(4) = \{0,2,4\}.$$

As in the previous example, we have

$$\mathcal{N}\mathcal{J}(1) = \mathbb{Z}_6, \quad \mathcal{N}\mathcal{J}(2) = \{0,2,4\}, \quad \mathcal{N}\mathcal{J}(3) = \{0,3\},$$

and

$$\mathcal{N}\mathcal{K}(0) = \{0\}, \quad \mathcal{N}\mathcal{K}(3) = \{0,3\}, \quad \mathcal{N}\mathcal{K}(4) = \{0,2,4\}.$$

Thus, both $(\mathcal{J},\mathcal{S})$ *and* $(\mathcal{K},\mathcal{T})$ *are soft* $\mathcal{N}$*-clogroups.*

Now consider their extended intersection. Let

$$(\mathcal{L},\mathcal{V}) = (\mathcal{J},\mathcal{S}) \cap_c (\mathcal{K},\mathcal{T}),$$

where

$$\mathcal{V} = \mathcal{S} \cup \mathcal{T} = \{0,1,2,3,4\}.$$

For each $s \in \mathcal{V}$*, the mapping* $\mathcal{L} : \mathcal{V} \to \mathcal{P}(\mathcal{N})$ *is defined by*

$$\mathcal{L}(s) = \begin{cases} \mathcal{J}(s), & s \in \mathcal{S} \setminus \mathcal{T}, \\ \mathcal{K}(s), & s \in \mathcal{T} \setminus \mathcal{S}, \\ \mathcal{J}(s) \cap \mathcal{K}(s), & s \in \mathcal{S} \cap \mathcal{T}. \end{cases}$$

Accordingly, we obtain

$$\mathcal{L}(0) = \{0\}, \quad \mathcal{L}(1) = \mathbb{Z}_6, \quad \mathcal{L}(2) = \{0,2,4\},$$

and

$$\mathcal{L}(3) = \{0,3\} \cap \{0,3\} = \{0,3\}, \quad \mathcal{L}(4) = \{0,2,4\}.$$

Moreover,

$$\mathcal{N}\mathcal{L}(0) = \{0\}, \quad \mathcal{N}\mathcal{L}(1) = \mathbb{Z}_6, \quad \mathcal{N}\mathcal{L}(2) = \{0,2,4\},$$

and

$$\mathcal{N}\mathcal{L}(3) = \{0,3\}, \quad \mathcal{N}\mathcal{L}(4) = \{0,2,4\}.$$

Since all these sets are subgroups of $(\mathbb{Z}_6,+)$*, it follows that* $(\mathcal{L},\mathcal{V})$ *is a soft* $\mathcal{N}$*-clogroup.*

Theorem 3.11 *(Associativity of the soft* $\mathcal{N}$*-clogroup extended union) Let* $(\mathcal{J},\mathcal{S})$*,* $(\mathcal{K},\mathcal{T})$ *and* $(\mathcal{L},\mathcal{V})$ *be soft* $\mathcal{N}$*-clogroups over the same near-ring* $\mathcal{N}$*. Then*

$$(\mathcal{J},\mathcal{S}) \cup_c \big((\mathcal{K},\mathcal{T}) \cup_c (\mathcal{L},\mathcal{V})\big) = \big((\mathcal{J},\mathcal{S}) \cup_c (\mathcal{K},\mathcal{T})\big) \cup_c (\mathcal{L},\mathcal{V}). \tag{25}$$

Proof First, consider the left-hand side of (25). Assume that

$$(\mathcal{K},\mathcal{T}) \cup_c (\mathcal{L},\mathcal{V}) = (\mathcal{O},\mathcal{T} \cup \mathcal{V}), \tag{26}$$

where $\mathcal{O}(s)$ is defined, for all $s \in \mathcal{T} \cup \mathcal{V}$, by

$$\mathcal{O}(s) = \begin{cases} \mathcal{K}(s), & s \in \mathcal{T} \setminus \mathcal{V}, \\ \mathcal{L}(s), & s \in \mathcal{V} \setminus \mathcal{T}, \\ \mathcal{K}(s) \cup \mathcal{L}(s), & s \in \mathcal{T} \cap \mathcal{V}. \end{cases} \tag{27}$$

Next, assume that

$$(\mathcal{J},\mathcal{S}) \cup_c (\mathcal{O},\mathcal{T} \cup \mathcal{V}) = (\mathcal{M},\mathcal{S} \cup (\mathcal{T} \cup \mathcal{V})), \tag{28}$$

where, for every $s \in \mathcal{S} \cup (\mathcal{T} \cup \mathcal{V})$,

$$\mathcal{M}(s) = \begin{cases} \mathcal{J}(s), & s \in \mathcal{S} \setminus (\mathcal{T} \cup \mathcal{V}), \\ \mathcal{O}(s), & s \in (\mathcal{T} \cup \mathcal{V}) \setminus \mathcal{S}, \\ \mathcal{J}(s) \cup \mathcal{O}(s), & s \in \mathcal{S} \cap (\mathcal{T} \cup \mathcal{V}). \end{cases} \tag{29}$$

By using (27) and (29) and taking into account the piecewise nature of $\mathcal{O}$ and $\mathcal{M}$, we may rewrite $\mathcal{M}(s)$ (for all $s \in \mathcal{S} \cup (\mathcal{T} \cup \mathcal{V})$) in the compact form

$$\mathcal{M}(s) = \begin{cases} \mathcal{J}(s), & s \in (\mathcal{S} \setminus \mathcal{T}) \cap (\mathcal{S} \setminus \mathcal{V}), \\ \mathcal{K}(s), & s \in (\mathcal{T} \setminus \mathcal{V}) \cap (\mathcal{T} \setminus \mathcal{S}), \\ \mathcal{L}(s), & s \in (\mathcal{V} \setminus \mathcal{T}) \cap (\mathcal{V} \setminus \mathcal{S}), \\ \mathcal{J}(s) \cup \mathcal{K}(s), & s \in \mathcal{S} \cap (\mathcal{T} \setminus \mathcal{V}), \\ \mathcal{J}(s) \cup \mathcal{L}(s), & s \in \mathcal{S} \cap (\mathcal{V} \setminus \mathcal{T}), \\ \mathcal{K}(s) \cup \mathcal{L}(s), & s \in (\mathcal{T} \cap \mathcal{V}) \setminus \mathcal{S}, \\ \mathcal{J}(s) \cup (\mathcal{K}(s) \cup \mathcal{L}(s)), & s \in \mathcal{S} \cap \mathcal{T} \cap \mathcal{V}. \end{cases} \tag{30}$$

Now consider the right-hand side of (25). Assume that

$$(\mathcal{J}, \mathcal{S}) \cup_c (\mathcal{K}, \mathcal{T}) = (\mathcal{R}, \mathcal{S} \cup \mathcal{T}), \tag{31}$$

where, for all $s \in \mathcal{S} \cup \mathcal{T}$,

$$\mathcal{R}(s) = \begin{cases} \mathcal{J}(s), & s \in \mathcal{S} \setminus \mathcal{T}, \\ \mathcal{K}(s), & s \in \mathcal{T} \setminus \mathcal{S}, \\ \mathcal{J}(s) \cup \mathcal{K}(s), & s \in \mathcal{S} \cap \mathcal{T}. \end{cases} \tag{32}$$

Further, assume that

$$(\mathcal{R}, \mathcal{S} \cup \mathcal{T}) \cup_c (\mathcal{L}, \mathcal{V}) = (\mathcal{Q}, (\mathcal{S} \cup \mathcal{T}) \cup \mathcal{V}), \tag{33}$$

where, for every $s \in (\mathcal{S} \cup \mathcal{T}) \cup \mathcal{V}$,

$$\mathcal{Q}(s) = \begin{cases} \mathcal{R}(s), & s \in (\mathcal{S} \cup \mathcal{T}) \setminus \mathcal{V}, \\ \mathcal{L}(s), & s \in \mathcal{V} \setminus (\mathcal{S} \cup \mathcal{T}), \\ \mathcal{R}(s) \cup \mathcal{L}(s), & s \in (\mathcal{S} \cup \mathcal{T}) \cap \mathcal{V}. \end{cases} \tag{34}$$

Using (32) and (34), and again considering the piecewise definition of $\mathcal{R}$, we obtain that $\mathcal{Q}(s)$ coincides with $\mathcal{M}(s)$ for every $s \in (\mathcal{S} \cup \mathcal{T}) \cup \mathcal{V}$. Indeed, in the common part $\mathcal{S} \cap \mathcal{T} \cap \mathcal{V}$ we have

$$(\mathcal{J}(s) \cup_c \mathcal{K}(s)) \cup_c \mathcal{L}(s) = \mathcal{J}(s) \cup_c (\mathcal{K}(s) \cup_c \mathcal{L}(s))$$

by associativity of the ordinary set union, and in all remaining disjoint regions the two constructions select the same images.

Hence $\mathcal{Q} = \mathcal{M}$ as set-valued mappings on the same parameter set, which proves (25). □

Theorem 3.12 *(Associativity of the soft $\mathcal{N}$-clogroup extended intersection) Let $(\mathcal{J}, \mathcal{S})$, $(\mathcal{K}, \mathcal{T})$ and $(\mathcal{L}, \mathcal{V})$ be soft $\mathcal{N}$-clogroups over the same near-ring $\mathcal{N}$. Then*

$$(\mathcal{J}, \mathcal{S}) \cap_c \big((\mathcal{K}, \mathcal{T}) \cap_c (\mathcal{L}, \mathcal{V})\big) = \big((\mathcal{J}, \mathcal{S}) \cap_c (\mathcal{K}, \mathcal{T})\big) \cap_c (\mathcal{L}, \mathcal{V}). \tag{35}$$

Proof First, consider the left-hand side of (35). Assume that

$$(\mathcal{K},\mathcal{T}) \cap_c (\mathcal{L},\mathcal{V}) = (\mathcal{O},\mathcal{T}\cup\mathcal{V}), \tag{36}$$

where, $\mathcal{O}(s)$ is defined, for all $s\in\mathcal{T}\cup\mathcal{V}$, by

$$\mathcal{O}(s) = \begin{cases} \mathcal{K}(s), & s\in\mathcal{T}\setminus\mathcal{V},\\ \mathcal{L}(s), & s\in\mathcal{V}\setminus\mathcal{T},\\ \mathcal{K}(s)\cap\mathcal{L}(s), & s\in\mathcal{T}\cap\mathcal{V}. \end{cases} \tag{37}$$

Next, assume that

$$(\mathcal{J},\mathcal{S}) \cap_c (\mathcal{O},\mathcal{T}\cup\mathcal{V}) = (\mathcal{M},\mathcal{S}\cup(\mathcal{T}\cup\mathcal{V})), \tag{38}$$

where, for every $s\in\mathcal{S}\cup(\mathcal{T}\cup\mathcal{V})$,

$$\mathcal{M}(s) = \begin{cases} \mathcal{J}(s), & s\in\mathcal{S}\setminus(\mathcal{T}\cup\mathcal{V}),\\ \mathcal{O}(s), & s\in(\mathcal{T}\cup\mathcal{V})\setminus\mathcal{S},\\ \mathcal{J}(s)\cap\mathcal{O}(s), & s\in\mathcal{S}\cap(\mathcal{T}\cup\mathcal{V}). \end{cases} \tag{39}$$

Using (37) and (39) and taking into account the piecewise nature of $\mathcal{O}$ and $\mathcal{M}$, we may rewrite $\mathcal{M}(s)$ (for all $s\in\mathcal{S}\cup(\mathcal{T}\cup\mathcal{V})$) in the compact form

$$\mathcal{M}(s) = \begin{cases} \mathcal{J}(s), & s\in(\mathcal{S}\setminus\mathcal{T})\cap(\mathcal{S}\setminus\mathcal{V}),\\ \mathcal{K}(s), & s\in(\mathcal{T}\setminus\mathcal{V})\cap(\mathcal{T}\setminus\mathcal{S}),\\ \mathcal{L}(s), & s\in(\mathcal{V}\setminus\mathcal{T})\cap(\mathcal{V}\setminus\mathcal{S}),\\ \mathcal{J}(s)\cap\mathcal{K}(s), & s\in\mathcal{S}\cap(\mathcal{T}\setminus\mathcal{V}),\\ \mathcal{J}(s)\cap\mathcal{L}(s), & s\in\mathcal{S}\cap(\mathcal{V}\setminus\mathcal{T}),\\ \mathcal{K}(s)\cap\mathcal{L}(s), & s\in(\mathcal{T}\cap\mathcal{V})\setminus\mathcal{S},\\ \mathcal{J}(s)\cap(\mathcal{K}(s)\cap\mathcal{L}(s)), & s\in\mathcal{S}\cap\mathcal{T}\cap\mathcal{V}. \end{cases} \tag{40}$$

Now consider the right-hand side of (35). Assume that

$$(\mathcal{J},\mathcal{S}) \cap_c (\mathcal{K},\mathcal{T}) = (\mathcal{R},\mathcal{S}\cup\mathcal{T}), \tag{41}$$

where, for all $s\in\mathcal{S}\cup\mathcal{T}$,

$$\mathcal{R}(s) = \begin{cases} \mathcal{J}(s), & s\in\mathcal{S}\setminus\mathcal{T},\\ \mathcal{K}(s), & s\in\mathcal{T}\setminus\mathcal{S},\\ \mathcal{J}(s)\cap\mathcal{K}(s), & s\in\mathcal{S}\cap\mathcal{T}. \end{cases} \tag{42}$$

Further, assume that

$$(\mathcal{R},\mathcal{S}\cup\mathcal{T}) \cap_c (\mathcal{L},\mathcal{V}) = (\mathcal{Q},(\mathcal{S}\cup\mathcal{T})\cup\mathcal{V}), \tag{43}$$

where, for every $s\in(\mathcal{S}\cup\mathcal{T})\cup\mathcal{V}$,

$$\mathcal{Q}(s) = \begin{cases} \mathcal{R}(s), & s\in(\mathcal{S}\cup\mathcal{T})\setminus\mathcal{V},\\ \mathcal{L}(s), & s\in\mathcal{V}\setminus(\mathcal{S}\cup\mathcal{T}),\\ \mathcal{R}(s)\cap\mathcal{L}(s), & s\in(\mathcal{S}\cup\mathcal{T})\cap\mathcal{V}. \end{cases} \tag{44}$$

Using (42) and (44), and again considering the piecewise definition of $\mathcal{R}$, we obtain that $\mathcal{Q}(s)$ coincides with $\mathcal{M}(s)$ for every $s \in (\mathcal{S} \cup \mathcal{T}) \cup \mathcal{V}$. Indeed, on the common part $\mathcal{S} \cap \mathcal{T} \cap \mathcal{V}$ we have

$$(\mathcal{J}(s) \cap_c \mathcal{K}(s)) \cap_c \mathcal{L}(s) = \mathcal{J}(s) \cap_c (\mathcal{K}(s) \cap_c \mathcal{L}(s)) \tag{45}$$

by associativity of the ordinary set intersection, and in all remaining disjoint regions the two constructions select the same images.

Hence $\mathcal{Q} = \mathcal{M}$ as set-valued mappings on the same parameter set, which proves (35). □

3.5. Closure Properties of Extended Operations

In this subsection, we investigate whether the class of soft $\mathcal{N}$-clogroups is closed under extended union operations. We first show that this structure is not preserved in general, and then provide a complete characterization of the conditions under which it is preserved.

Theorem 3.13 *The extended union of two soft $\mathcal{N}$-clogroups is not necessarily a soft $\mathcal{N}$-clogroup.*

Proof We provide a counterexample.

Let $\mathcal{N} = \mathbb{Z}_6$ under addition and multiplication modulo 6.

Let $\mathcal{S} = \{e\}$ and define two soft sets $(\mathcal{J}, \mathcal{S})$ and $(\mathcal{K}, \mathcal{S})$ over $\mathcal{N}$ by

$$\mathcal{J}(e) = \{0, 2, 4\}, \quad \mathcal{K}(e) = \{0, 3\}. \tag{46}$$

First, we check that both $(\mathcal{J}, \mathcal{S})$ and $(\mathcal{K}, \mathcal{S})$ are soft $\mathcal{N}$-clogroups.

We have

$$\mathcal{N}\mathcal{J}(e) = \{0, 2, 4\}, \tag{47}$$

which is a subgroup of $(\mathbb{Z}_6, +)$.

Similarly,

$$\mathcal{N}\mathcal{K}(e) = \{0, 3\}, \tag{48}$$

which is also a subgroup of $(\mathbb{Z}_6, +)$.

Thus, both $(\mathcal{J}, \mathcal{S})$ and $(\mathcal{K}, \mathcal{S})$ are soft $\mathcal{N}$-clogroups.

Now consider their extended union:

$$(\mathcal{J}, \mathcal{S}) \cup_c (\mathcal{K}, \mathcal{S}) = (\mathcal{L}, \mathcal{S}), \tag{49}$$

where

$$\mathcal{L}(e) = \mathcal{J}(e) \cup \mathcal{K}(e) = \{0, 2, 3, 4\}. \tag{50}$$

Now compute

$$\mathcal{N}\mathcal{L}(e) = \{na \mid n \in \mathbb{Z}_6, a \in \{0, 2, 3, 4\}\}. \tag{51}$$

This set is not closed under addition and does not form a subgroup of $(\mathbb{Z}_6,+)$.

Therefore, $(\mathcal{L},\mathcal{S})$ is not a soft $\mathcal{N}$-clogroup.

Hence, the extended union does not preserve the soft $\mathcal{N}$-clogroup structure in general. □

Theorem 3.14 *Let $(\mathcal{J},\mathcal{S})$ and $(\mathcal{K},\mathcal{T})$ be soft $\mathcal{N}$-clogroups over a near-ring $\mathcal{N}$, and let*

$$(\mathcal{L},\mathcal{S}\cup\mathcal{T}) = (\mathcal{J},\mathcal{S})\cup_c(\mathcal{K},\mathcal{T}) \tag{52}$$

be their extended union.

Then $(\mathcal{L},\mathcal{S}\cup\mathcal{T})$ is a soft $\mathcal{N}$-clogroup if and only if for every $s\in\mathcal{S}\cap\mathcal{T}$,

$$\mathcal{N}\mathcal{J}(s)\subseteq\mathcal{N}\mathcal{K}(s) \quad \textit{or} \quad \mathcal{N}\mathcal{K}(s)\subseteq\mathcal{N}\mathcal{J}(s). \tag{53}$$

Proof Assume first that $(\mathcal{L},\mathcal{S}\cup\mathcal{T})$ is a soft $\mathcal{N}$-clogroup.

Let $s\in\mathcal{S}\cap\mathcal{T}$. Since

$$\mathcal{L}(s)=\mathcal{J}(s)\cup\mathcal{K}(s), \tag{54}$$

we have

$$\mathcal{N}\mathcal{L}(s)=\mathcal{N}(\mathcal{J}(s)\cup\mathcal{K}(s))=\mathcal{N}\mathcal{J}(s)\cup\mathcal{N}\mathcal{K}(s). \tag{55}$$

Because $(\mathcal{L},\mathcal{S}\cup\mathcal{T})$ is a soft $\mathcal{N}$-clogroup, the set $\mathcal{N}\mathcal{L}(s)$ is a subgroup of $(\mathcal{N},+)$.

Since $(\mathcal{J},\mathcal{S})$ and $(\mathcal{K},\mathcal{T})$ are soft $\mathcal{N}$-clogroups, both $\mathcal{N}\mathcal{J}(s)$ and $\mathcal{N}\mathcal{K}(s)$ are subgroups of $(\mathcal{N},+)$. By the union of two subgroups is a subgroup if and only if one is contained in the other,

$$\mathcal{N}\mathcal{J}(s)\subseteq\mathcal{N}\mathcal{K}(s) \quad \text{or} \quad \mathcal{N}\mathcal{K}(s)\subseteq\mathcal{N}\mathcal{J}(s). \tag{56}$$

Conversely, assume that for every $s\in\mathcal{S}\cap\mathcal{T}$,

$$\mathcal{N}\mathcal{J}(s)\subseteq\mathcal{N}\mathcal{K}(s) \quad \text{or} \quad \mathcal{N}\mathcal{K}(s)\subseteq\mathcal{N}\mathcal{J}(s). \tag{57}$$

Let $s\in\mathcal{S}\cup\mathcal{T}$.

If $s\in\mathcal{S}\setminus\mathcal{T}$, then $\mathcal{L}(s)=\mathcal{J}(s)$ and

$$\mathcal{N}\mathcal{L}(s)=\mathcal{N}\mathcal{J}(s), \tag{58}$$

which is a subgroup of $(\mathcal{N},+)$.

If $s\in\mathcal{T}\setminus\mathcal{S}$, then $\mathcal{L}(s)=\mathcal{K}(s)$ and

$$\mathcal{N}\mathcal{L}(s)=\mathcal{N}\mathcal{K}(s), \tag{59}$$

which is a subgroup of $(\mathcal{N},+)$.

If $s\in\mathcal{S}\cap\mathcal{T}$, then

$$\mathcal{N}\mathcal{L}(s)=\mathcal{N}\mathcal{J}(s)\cup\mathcal{N}\mathcal{K}(s). \tag{60}$$

By assumption, one of these is contained in the other, hence

$$\mathcal{N}\mathcal{L}(s) = \mathcal{N}\mathcal{J}(s) \quad \text{or} \quad \mathcal{N}\mathcal{L}(s) = \mathcal{N}\mathcal{K}(s), \tag{61}$$

which is a subgroup of $(N, +)$.

Thus $(\mathcal{L}, \mathcal{S} \cup \mathcal{T})$ is a soft $\mathcal{N}$-clogroup. □

Remark 3.15 *The previous theorem gives a complete characterization of when the extended union preserves the soft $\mathcal{N}$-clogroup structure. The obstruction occurs precisely at parameters in $\mathcal{S} \cap \mathcal{T}$, where the generated subgroups must be comparable under inclusion.*

Corollary 3.16 *Let $(\mathcal{J}, \mathcal{S})$ and $(\mathcal{K}, \mathcal{T})$ be soft $\mathcal{N}$-clogroups over a near-ring $\mathcal{N}$. If for every $s \in \mathcal{S} \cap \mathcal{T}$ one has*

$$\mathcal{N}\mathcal{J}(s) = \mathcal{N}\mathcal{K}(s), \tag{62}$$

then $(\mathcal{J}, \mathcal{S}) \cup_c (\mathcal{K}, \mathcal{T})$ is a soft $\mathcal{N}$-clogroup.

Proof The result follows immediately from the previous theorem. □

3.6. Homomorphic Images and Preimages of Soft $\mathcal{N}$-Clogroups

In this subsection, we analyze the behavior of soft $\mathcal{N}$-clogroups under near-ring homomorphisms, focusing on images and preimages.

Definition 3.17 *Let $\mathcal{N}$ and $\mathcal{N}'$ be near-rings, and let $(\mathcal{J}, \mathcal{S})$ be a soft set over $\mathcal{N}$. Let $f : \mathcal{N} \to \mathcal{N}'$ be a near-ring homomorphism.*

The image *of $(\mathcal{J}, \mathcal{S})$ under f is defined as the soft set $(f(\mathcal{J}), \mathcal{S})$ over $\mathcal{N}'$ given by*

$$f(\mathcal{J})(s) = f(\mathcal{J}(s)) = \{f(a) \mid a \in \mathcal{J}(s)\}, \quad \forall s \in \mathcal{S}. \tag{63}$$

Similarly, let $(\mathcal{K}, \mathcal{T})$ be a soft set over $\mathcal{N}'$. The preimage *of $(\mathcal{K}, \mathcal{T})$ under f is defined as the soft set $(f^{-1}(\mathcal{K}), \mathcal{T})$ over $\mathcal{N}$ given by*

$$f^{-1}(\mathcal{K})(t) = \{a \in \mathcal{N} \mid f(a) \in \mathcal{K}(t)\}, \quad \forall t \in \mathcal{T}. \tag{64}$$

Theorem 3.18 *The homomorphic image of a soft $\mathcal{N}$-clogroup is not necessarily a soft $\mathcal{N}'$-clogroup.*

Proof Let $\mathcal{N}' = (\mathbb{Z}_{12}, +, \cdot)$ be the near-ring under usual addition and multiplication modulo 12, and let

$$\mathcal{N} = \{0, 2, 4, 6, 8, 10\}.$$

Then $\mathcal{N}$ is a subnear-ring of $\mathcal{N}'$. Consider the inclusion homomorphism

$$f:\mathcal{N}\to\mathcal{N}', \qquad f(x)=x.$$

Let $\mathcal{S}=\{e\}$ and define a soft set $(\mathcal{J},\mathcal{S})$ over $\mathcal{N}$ by

$$\mathcal{J}(e)=\{4,6\}.$$

We compute

$$\mathcal{N}\mathcal{J}(e)=\{na \mid n\in\mathcal{N},\ a\in\mathcal{J}(e)\}=\{0,4,8\}.$$

Since $\{0,4,8\}$ is a subgroup of $(\mathcal{N},+)$, it follows that $(\mathcal{J},\mathcal{S})$ is a soft $\mathcal{N}$-clogroup.

Now consider its image under f. Since f is the inclusion map, we have

$$f(\mathcal{J})(e)=f(\mathcal{J}(e))=\{4,6\}.$$

However,

$$\mathcal{N}'f(\mathcal{J})(e)=\{ma \mid m\in\mathcal{N}',\ a\in f(\mathcal{J})(e)\}=\mathbb{Z}_{12}\{4,6\}.$$

More explicitly,

$$\mathbb{Z}_{12}\{4\}=\{0,4,8\} \quad \text{and} \quad \mathbb{Z}_{12}\{6\}=\{0,6\}.$$

Hence

$$\mathcal{N}'f(\mathcal{J})(e)=\{0,4,6,8\}.$$

This set is not a subgroup of $(\mathbb{Z}_{12},+)$, since

$$4+6=10\notin\{0,4,6,8\}.$$

Therefore, $(f(\mathcal{J}),\mathcal{S})$ is not a soft $\mathcal{N}'$-clogroup.

Consequently, the homomorphic image of a soft $\mathcal{N}$-clogroup need not be a soft $\mathcal{N}'$-clogroup. □

Theorem 3.19 *Let $(\mathcal{J},\mathcal{S})$ be a soft $\mathcal{N}$-clogroup and let $f:\mathcal{N}\to\mathcal{N}'$ be a surjective near-ring homomorphism. If*

$$f(\mathcal{N}\mathcal{J}(s))=\mathcal{N}'f(\mathcal{J}(s)) \tag{65}$$

for all $s\in\mathcal{S}$, then the image $(f(\mathcal{J}),\mathcal{S})$ is a soft $\mathcal{N}'$-clogroup.

Proof Since $\mathcal{N}\mathcal{J}(s)$ is a subgroup, its image is a subgroup, and by assumption this equals $\mathcal{N}'f(\mathcal{J}(s))$. Hence the result follows. □

Theorem 3.20 *Let $(\mathcal{K},\mathcal{T})$ be a soft $\mathcal{N}'$-clogroup and let $f:\mathcal{N}\to\mathcal{N}'$ be a near-ring homomorphism. Then the preimage $(f^{-1}(\mathcal{K}),\mathcal{T})$ is a soft $\mathcal{N}$-clogroup.*

Proof Since $\mathcal{N}'\mathcal{K}(t)$ is a subgroup, its preimage is also a subgroup, hence the result follows. □

Remark 3.21 *There is an asymmetry between images and preimages: preimages always preserve the soft* $\mathcal{N}$*-clogroup structure, whereas images require additional conditions.*

Remark 3.22 *Unlike the extended union, the analysis of extended intersection is more subtle. The main difficulty arises from the fact that, in general,*

$$\mathcal{N}(\mathcal{J}(s) \cap \mathcal{K}(s)) \neq \mathcal{N}\mathcal{J}(s) \cap \mathcal{N}\mathcal{K}(s). \tag{66}$$

One always has

$$\mathcal{N}(\mathcal{J}(s) \cap \mathcal{K}(s)) \subseteq \mathcal{N}\mathcal{J}(s) \cap \mathcal{N}\mathcal{K}(s). \tag{67}$$

Indeed, if $x \in \mathcal{N}(\mathcal{J}(s) \cap \mathcal{K}(s))$*, then there exist* $n \in \mathcal{N}$ *and* $a \in \mathcal{J}(s) \cap \mathcal{K}(s)$ *such that* $x = na$*. Since* $a \in \mathcal{J}(s)$ *and* $a \in \mathcal{K}(s)$*, we have* $x \in \mathcal{N}\mathcal{J}(s)$ *and* $x \in \mathcal{N}\mathcal{K}(s)$*. Hence* $x \in \mathcal{N}\mathcal{J}(s) \cap \mathcal{N}\mathcal{K}(s)$*. The reverse inclusion does not hold in general.*

This lack of equality prevents a direct characterization similar to the extended union case. Therefore, additional structural conditions would be required to fully describe when extended intersection preserves the soft $\mathcal{N}$*-clogroup structure, and such an analysis is beyond the scope of the present work.*

4. Conclusion

In this study, we introduced the concept of soft $\mathcal{N}$-clogroups as a parameter-dependent extension of classical $\mathcal{N}$-clogroups within the framework of soft set theory. By combining the multiplicative subgroup-generation mechanism of near-rings with soft parameterization, we obtained a flexible algebraic structure and analyzed its fundamental properties in detail.

We showed that, for each parameter, the associated image satisfies the classical subgroup-generation condition, ensuring consistency with the traditional $\mathcal{N}$-clogroup structure. This confirms that the proposed framework naturally extends the classical theory into a parameterized setting.

The behavior of soft $\mathcal{N}$-clogroups under extended union was investigated in detail. It was demonstrated that extended union does not preserve the soft $\mathcal{N}$-clogroup structure in general. A complete characterization was established by providing necessary and sufficient conditions under which the structure is preserved. This result shows that the obstruction arises precisely from the interaction of parameters in the overlap region.

Furthermore, we analyzed the behavior of soft $\mathcal{N}$-clogroups under near-ring homomorphisms. It was shown that homomorphic images do not preserve the structure in general, whereas preimages always preserve it. Additional conditions ensuring the preservation of the structure under homomorphisms were also identified.

These results provide a deeper and more systematic structural understanding of soft $\mathcal{N}$-clogroups and clarify how parameterization interacts with multiplicative and additive mechanisms in near-rings. The developed framework opens new directions for further research, including the study of lattice-type structures, normality conditions, and extensions to other algebraic systems.

In particular, future work may focus on defining normal soft $\mathcal{N}$-clogroups by imposing suitable normality conditions on the generated subgroups $\mathcal{N}\mathcal{J}(s)$ for each parameter s.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and legal special permission.

Authors Contributions

Author [İncinur Yılmaz]: Thought and designed the research/problem, contributed to the research method or evaluation of data, contributed to completing the research and solving the problem, and wrote the manuscript (60%).

Author [Emin Aygün]: Thought and designed the research/problem, contributed to the research method or evaluation of data, contributed to completing the research and solving the problem, and wrote the manuscript (40%).

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Acar U., Koyuncu F., Tanay B., *Soft sets and soft rings*, Computers & Mathematics with Applications, 59(11), 3458-3463, 2010.

[2] Aktaş H., Çağman N., *Soft sets and soft groups*, Information Sciences, 177(13), 2726-2735, 2007.

[3] Ali M.I., Feng F., Liu X., Min W.K., Shabir M., *On some new operations in soft set theory*, Computers & Mathematics with Applications, 57(9), 1547-1553, 2009.

[4] Atagün A.O., Sezgin A., *Soft subnear-rings, soft ideals and soft N-subgroups of near-rings*, Mathematical Sciences Letters, 7(1), 37-42, 2018.

[5] Aygün E., *On the N-clogroups and v-prime Ideals in Near-rings*, Ph.D. Thesis, Erciyes University, Kayseri, Türkiye, 2006.

[6] Aygün E., Kamacı H., *Some new algebraic structures of soft sets*, Soft Computing, 25(13), 8609-8626, 2021.

[7] Aygün E., Kamacı H., *Two different types of soft near-fields*, Punjab University Journal of Mathematics, 54(7), 429-440, 2022.

[8] Feng F., Jun Y.B., Zhao X., *Soft semirings*, Computers & Mathematics with Applications, 56(10), 2621-2628, 2008.

[9] Jun Y.B., *Soft BCK/BCI-algebras*, Computers & Mathematics with Applications, 56(5), 1408-1413, 2008.

[10] Jun Y.B., Park C.H., *Applications of soft sets in ideal theory of BCK/BCI-algebras*, Information Sciences, 178(11), 2466-2475, 2008.

[11] Maji P.K., Roy A.R., Biswas R., *An application of soft sets in a decision making problem*, Computers & Mathematics with Applications, 44(8-9), 1077-1083, 2002.

[12] Maji P.K., Biswas R., Roy A.R., *Soft set theory*, Computers & Mathematics with Applications, 45(4-5), 555-562, 2003.

[13] Molodtsov D.A., *Soft set theory-first results*, Computers & Mathematics with Applications, 37(4-5), 19-31, 1999.

[14] Molodtsov D.A., The Theory of Soft Sets (in Russian), URSS Publishers, Moscow, Russia, 2004.

[15] Molodtsov D.A., Leonov V.Y., Kovkov D.V., *Soft sets technique and its application*, Nechetkie Sistemy I Myagkie Vychisleniya, 1(1), 8-39, 2006.

[16] Park C.H., Jun Y.B., Öztürk M.A., *Soft WS-algebras*, Communications of the Korean Mathematical Society, 23(3), 313-324, 2008.

[17] Pawlak Z., *Rough sets*, International Journal of Computer and Information Sciences, 11(5), 341-356, 1982.

[18] Pei D., Miao D., *From soft sets to information systems*, Proceedings of the 2005 IEEE International Conference on Granular Computing, 2, 617-621, 2005.

[19] Sezgin A., Atagün A.O., Aygün E., *A note on soft near-rings and idealistic soft near-rings*, Filomat, 25(1), 53-68, 2011.

[20] Sezgin A., Atagün A.O., *On operations of soft sets*, Computers & Mathematics with Applications, 61(5), 1457-1467, 2011.

[21] Sun Q.-M., Zhang Z.-L., Liu J., *Soft sets and soft modules*, Rough Sets and Knowledge Technology, Lecture Notes in Computer Science, 5009, 403-409, 2008.

[22] Xiao Z., Chen L., Zhong B., Ye S., *Recognition for soft information based on the theory of soft sets*, Proceedings of the 2005 International Conference on Services Systems and Services Management (ICSSSM 2005), 2, 1104-1106, 2005.