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Uleksitin Hidroklorik Asit Çözeltilerindeki Çözünürlüğü

Dissolution of Ulexite in Hydrochloric Acid Solutions

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Öz: Uleksit bor bileşiklerin üretiminde kullanılan önemli bor minerallerinden bir tanesidir. Bu çalışmanın amacı uleksitin kesikli reaktörde hidroklorik asit çözeltilerindeki çözünürlüğünün incelenerek borik asit üretimi için alternatif bir prosesin belirlenmesidir. Reaksiyon sıcaklığı, parçacık boyutu, katı/sıvı oranı, asit konsantrasyonu ve karıştırma hızı parametrelerinin çözünme hızı üzerine etkisi incelenmiştir. Deneme parametleri asit konsantrasyonu için 0.75*,1,1.25,1.5,2M, reaksiyon sıcaklığı için 30*,40,50,60,70°C, parçacık boyutu için -14+18*,-18+30,-30+40,-40+50,-50+60 meş, katı/sıvı oranı için 5/100,7.5/100,10/100* g/mL, karıştırma hızı için 500*,650,800 rpm olarak belirlenmiştir. Çözünme hızının reaksiyon sıcaklığı ve karıştırma hızının artmasıyla arttığı, parçaçık boyutu, asit konsantrasyonu ve katı/sıvı oranı artışıyla ise azaldığı tespit edilmiştir.

Anahtar Kelimeler — Uleksit, Hidroklorik asit, Borik asit.

Abstract: Ulexite is an important boron mineral used for the production of boron compounds. Purpose of the study has been to investigate the dissolution of ulexite in hydrochloric acid solutions in a batch reactor and to present an alternative process to produce boric acid. Reaction temperature, particle size, solid/liquid ratio, acid concentration and stirring speed have been selected as parameters on the dissolution rate of ulexite. The parameters used in experiments are; acid concentration 0.75*,1,1.25,1.5,2M, reaction temperature 30*,40,50,60,70°C, particle size -14+18*,-18+30,-30+40,-40+50,-50+60 mesh, solid/liquid ratio 5/100,7.5/100,10/100* g/mL, stirring speed 500*,650,800 rpm. It has been found that the dissolution rate has increased the reaction temperature and the stirring speed. However, increasing the particle size, acid concentration and solid/liquid ratio have decreased the rate of dissolution.

Keywords — Ulexite, Hydrochloric acid, Boric acid.

1. Introduction

Boron is an element that commonly exists in soil, rock and water on the earth. Boron exists in high concentrations in the vicinity reposing from Mediterranean and from the west regions of the USA to Kazakhstan besides, boron content of soil is generally 10-20 ppm on average. It is between the ranges of 0.5-9.6 ppm in the sea water and 0.01 - 1.5 ppm in the fresh water. Boron is seen in high concentrations and economical dimensions that mostly exist in the arid climate, volcanic regions and the regions whose hydrothermal activity is high for Turkey and the USA such as enchained compounds of boron with oxygen (Woods, 1994). Boron is a valuable element besides its industrial importance. It is found as borates (oxides) in nature. The present boron minerals are tincal ($Na_2B_4O_7.10H_2O$), colemanite ($Ca_2B_6O_{11}.5H_2O$), ulexite ($Na_2O.2CaO.5B_2O_3.16H_2O$), kernite ($Na_2B_4O_7.4H_2O$), datolite ($Ca_2B_2O_5.Si_2O_5.H_2O$), and hydroboracid ($CaMgB_6O_{11}.6H_2O$). Turkey has a significant amount of boron minerals, and besides, the world's largest boron deposits are found in the Eskişehir-Kırka region of Turkey (Durak et. al., 2014). Boron minerals and boron products have been significant substances because of being used in many fields from agriculture to energy, from defense industry to space industry. The specific boron compounds are synthesized by using the minerals that add a premium on it in terms of both economically and usage area rather than the direct consumption of boron minerals.

Boric acid is most commonly used as a primary material in the preparation of many boron chemicals such as synthetic organic borate salts, borate esters, boron carbide, fluoroborates, and boron tri-halides. The preferred commercial method of preparing boric acid is by way of digestion of alkali and alkaline earth metal borates with concentrated mineral acids followed by crystallization of boric acid (Shiloff, 1968; Taylan et al., 2007). In Turkey, boric acid is obtained from the reaction of colemanite and sulfuric acid in accordance with heterogeneous solid–liquid reaction (Mergen et al., 2003). In this process, gypsum forms as a by-product and precipitates in the reactor while boric acid remains in the liquid phase throughout the reaction.

This process has some disadvantages such as sulphate contamination in final product, and environmental pollution. Therefore, the alternative processes have been suggested by many researchers.

Those having commercial value within 230 types of natural boron minerals in the nature are especially tincal, kernite, colemanite, inyoite, pandermite, ulexite and probertite (Kirk Othmer, 1992). Ulexite, which is a sodium–calcium borate with a chemical formula of $Na_2O\cdot2CaO\cdot5B_2O_3\cdot16H_2O$, is a structurally complex mineral. Many investigations have been performed related to the leaching of ulexite mineral in different solution. The solution kinetic of boron minerals in many solvents has been analyzed by using different parameters. Some of the studies that have been conducted about the dissolution kinetic of boron minerals are showed in Table 1.

Table 1. *Summary of dissolution kinetics and activation energy of boron minerals acid or gaseous solutions*

Boron minerals	Solutions	References
Ulexite	Borax pentahydrate solutions saturated with CO_2	(Kuşlu et al., 2010)
Ulexite	Ammonium carbonate	(Demirkıran and Künkül,2007)
Ulexite	Percloric acid	(Demirkıran and Künkül, 2007)
Ulexite	Phosphoric acid	(Doğan and Yartaşı, 2009)
Ulexite	Acetic acid	(Ekmekyapar et al., 2008)
Ulexite	CO_2 satured water	(Kocakerim et al., 1993)
Colemanite	Ammonium sulphate	(Tunç et al., 2006)
Colemanite	Potassium hydrogen sulphate solutions	(Guliyev et al., 2012)
Tincal	Phosphoric acid	(Durak and Genel, 2012)
Tincal	Oxalic acid	(Abali et al., 2006)

The process of boric acid generation, notably tincal (borax) and colemanite, some boron minerals are used like kernite, ulexite, probertite, hydroboracite, inderite, datolite and asharite. Two main raw materials are used in the process of boric acid generation in all over the world. In Europe and Turkey, colemanite is used for boric acid generation while tincal is used in the USA. Purpose of the study is to propose an alternative process to boric acid generation by analyzing the dissolution kinetic of ulexite in hydrochloric acid solutions.

2. *Methods and Materials*

Dissolution experiments have been conducted under atmospheric pressure conditions. All reagents used in the experiments have been prepared from the analytical grade chemicals (Merck) and distilled water. Ulexite ore which has been used in the study has been provided from Bigadiç, Balıkesir region of Turkey. The ore has been washed with water and dried several times after being cleaned from apparent impurities. After the process, the ore has been broken with crackers in the laboratory media then it has been separated into -14+18*,-18+30,-30+40,-40+50 and -50+60 of mesh of sieve fractions by the sieves in standards of ASTM. The result of the chemical analysis of ulexite mineral that are used in the studies is showed in Table 2. Besides XRD graphic and SEM picture of ulexite sample used in the study have also been showed in the Figure 1 and 2.

Table 2. *Chemical analysis of tincal ore used in this study*

Component	% Composition
B_2O_3	42.83
Na_2O	6.38
CaO	14.22
MgO	4.58
H_2O	29.67
Other	2.32

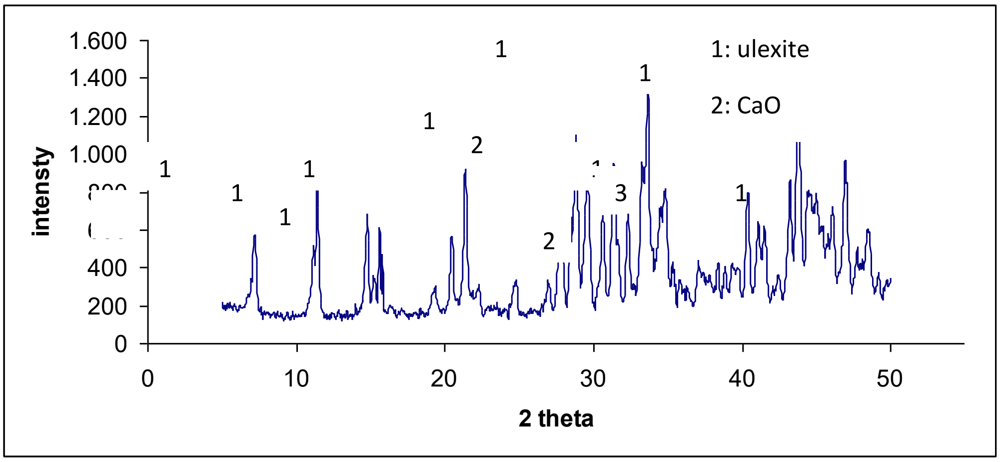

Fig. 1. *XRD diffractogram of ulexite minerals used in this study.*

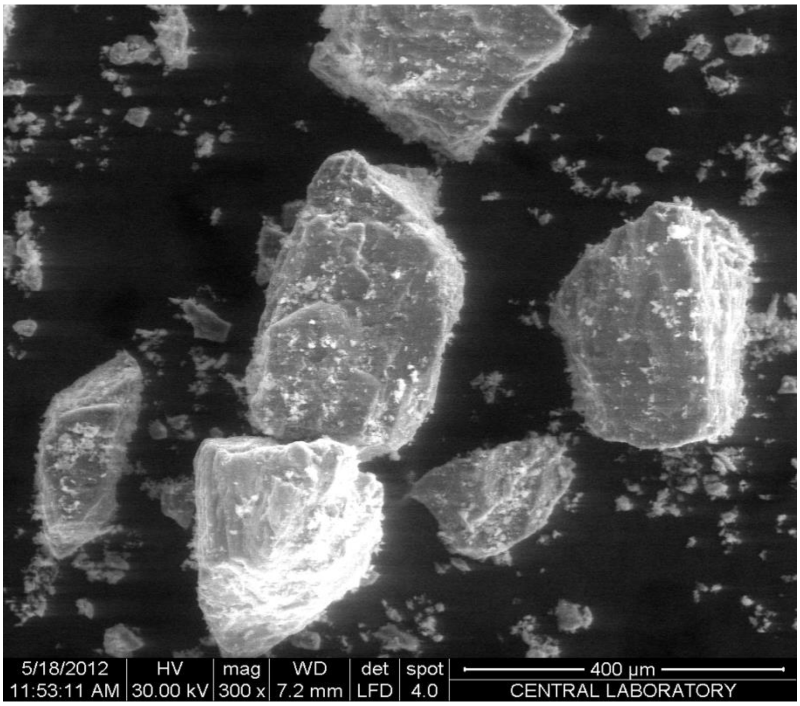

Fig. 2. *SEM photograph of ulexite minerals used in this study*

2.1. Experimental procedure

The solution treatments have been carried out in a 250mL-spherical glass reactor within atmospheric pressure. For mixing process, mechanics mixer and in order to keep temperature constant a constant heat water circulator has been used.

The parameters that have been used in solution process have been displayed in Table 3. Each experiment has been repeated twice, and the arithmetic averages of the results of the two experiments have been used in the kinetic analysis.

In solution processes, 0.1 L hydrochloric acid solution has been put in reactor for each experiment and then the mixing process has been started by closing the reactor cap. After the temperature of solution in reactor has reached to the desired value, reaction has been launched by putting a certain amount of ore.

At the end of the determined duration, mixing process has been finished and then some substances have been filtered in G-4 glass crucible by means of squinch in a short time by receiving the substance that is enough for analyzing from reactor. The B_2O_3 in the solution has been analyzed according to volumetric method with mannitol (Scott, 1963).

Table 3. *Parameters and their ranges used on the experiments*

Parameter	**Value**
Particle size (mesh)	-14+18*,-18+30,-30+40,-40+50,-50+60
Concentration of hydrochloric acid (M)	0.75*,1,1.25,1.5,2M
Solid/liquid ratio (g/mL)	5/100,7.5/100,10/100*
Stirring speed (rpm)	500*,650,800
Reaction temperature (^oC)	30*,40,50,60,70^oC

* The constant values used when the effect of other parameters was investigated

The conversion quantity has been found by transferring the dissolving H_3BO_3 quantity at the end of the reaction into B_2O_3 quantity.

The conversion fraction of the ore in terms of B_2O_3;

$$X_{B_2O_3} = \frac{\text{Amount of dissolved } B_2O_3 \text{ in the solution}}{\text{Amount of } B_2O_3 \text{ in the original sample}}$$

3. Result and analysis

3.1. Dissolution reactions

Hydrochloric acids resolve their ions in one phase. When hydrochloric acid dissolves in water, principally, it sends its 1 proton to medium.

$$HCI + H_2O \rightleftharpoons H_3O^+ + CI^- \qquad Ka= 10^7$$

It is regarded as that the dissolvability of ulexite in hydrochloric acid solutions and it originates according to the following equations.

$$Na_2O.2CaO.5B_2O_3.16H_2O + 6\ HCI_{(aq)} \rightarrow 2Na^+_{(aq)} + 2Ca^{+2}_{(aq)} + 10\ H_3BO_{3(aq)} + 6CI^-_{(aq)} + 4H_2O$$

The leach solution obtained from the dissolution of ulexite includes sodium, calcium, chlorine ions, and dissolved boric acid. Boric acid can be crystallized from the leach solution. Furthermore, NaCl and $CaCl_2$ may be obtained as by-products from the solution.

3.2. Effect of the parameters

The parameters that influences dissolution ratio of ulexite in hydrochloric acid solutions such as temperature, particle size, solid/liquid ratio, acid concentration, stirring speed have been selected and the effect of these parameters on dissolution ratio has been analyzed. Before studying on the effects of other parameters that may influence the dissolution rate, firstly effect of the stirring speed has been performed. Experiments have been carried out at stirring speeds of 500, 650 and 800 rpm to observe the effect of the stirring speed on the dissolution rate. In these experiments, hydrochloric acid concentration, particle size, solid-to-liquid ratio, and reaction temperature have been fixed at 0.75 M, -14+18 mesh, 10/100 g/mL, and 30 ^{o}C, respectively. From the obtained results, it has been observed that the dissolution rate has been practically independent of the stirring speed. Therefore, all subsequent experiments have been carried out at stirring speed of 500 rpm.

3.3. Effect of the reaction temperature

The effect of temperature on dissolution ratio of ulexite has been analyzed at 30,40,50,60 and 70^{o}C. In the experiments, particle size has been kept constant as -14+18 mesh of sieve, solid/liquid ratio as 10/100 g/mL, stirring speed as 500 rpm and 0.75M HCI acid concentration. According to the obtained results, as the temperature of reaction increases, dissolution rate increases as it is seen from the Fig. 3.

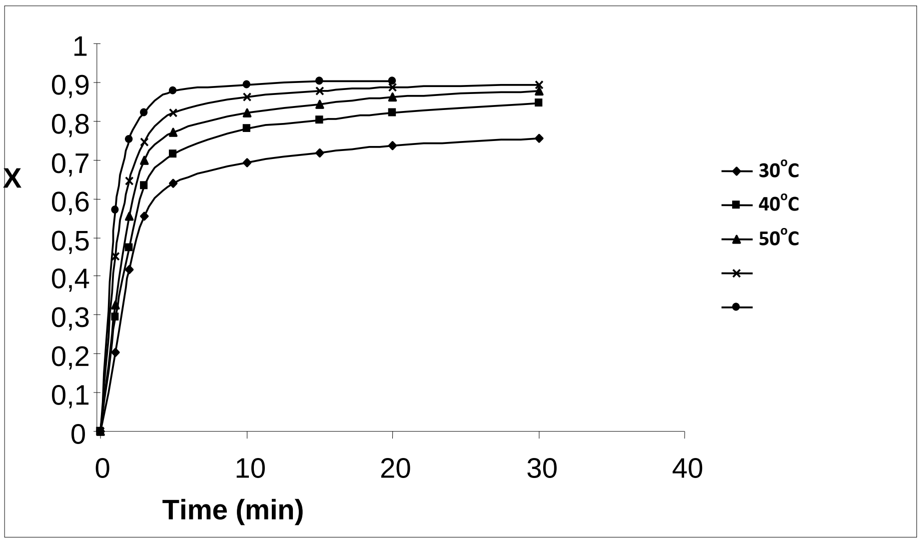

Fig. 3. *Effect of reaction temperature on dissolution rate of ulexite*

3.4. Effect of concentration of hydrochloric acid

The effect of acid concentration on dissolution rate of ulexite has been researched in the concentration of 0.75M, 1M, 1.25M, 1.5M and 2M. In the experiments, particle size has been kept constant as -14+18 mesh of sieve, temperature as 30^{o}C, solid/liquid ratio as 10/100 g/mL, stirring speed as 500 rpm.

In the experiment of acid concentration, dissolution ratio has showed decrease in inversely proportion to the increasing acid concentration. It is obviously seen in Figure 4.

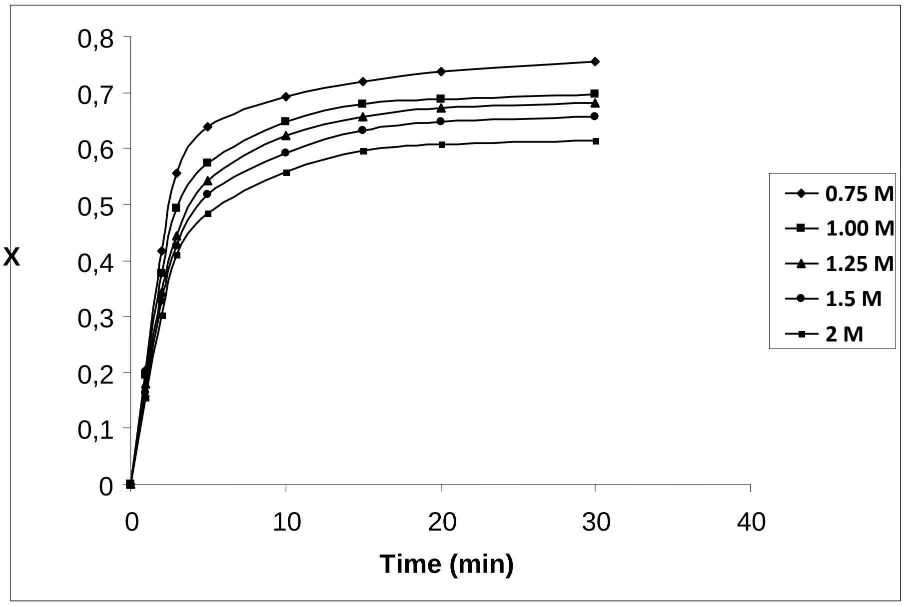

Fig. 4. Effect of concentration of hydrochloric acid on dissolution ratio of ulexite

3.5. Effect of the ulexite particle size

The effect of particle size on the dissolution ratio of ulexite ore in hydrochloric acid solutions have been studied in the fractions of -14+18, -18+30, -30+40, -40+50, -50+60 mesh of sieve. In the experiments, heat has been kept constant as 30^{o}C, acid concentration as 0.75M, solid/liquid ratio as 10/100 g/mL, stirring speed as 500 rpm.. In Figure 5, the effect of particle size on dissolution ratio is showed. Dissolution rate increases with the decrement of particle size.

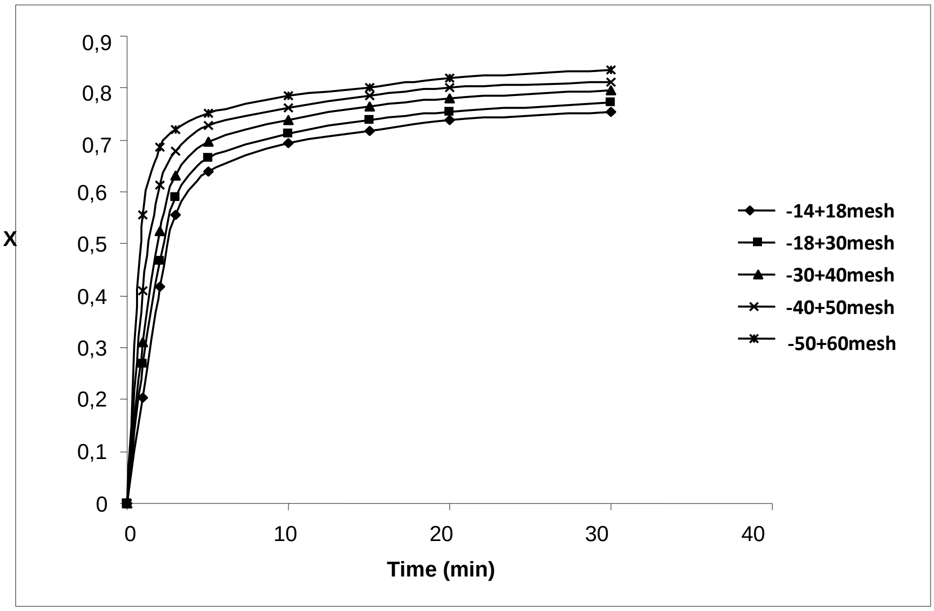

Fig. 5. *Effect of particle size on dissolution ratio of ulexite*

3.6. Effect of the solid/liquid ratio

The effect of solid/liquid ratio on the dissolution rate of ulexite ore in hydrochloric acid solutions have been studied in the values of 5/100, 7.5/100 and 10/100 g/mL. In the experiments, particle size has been kept constant as -14+18 mesh of sieve, reaction heat as 30^{o}C, acid concentration as 0.75M, stirring speed as 500 rpm. As seen in Figure 6, as solid/liquid ratio increases, dissolution rate decreases.

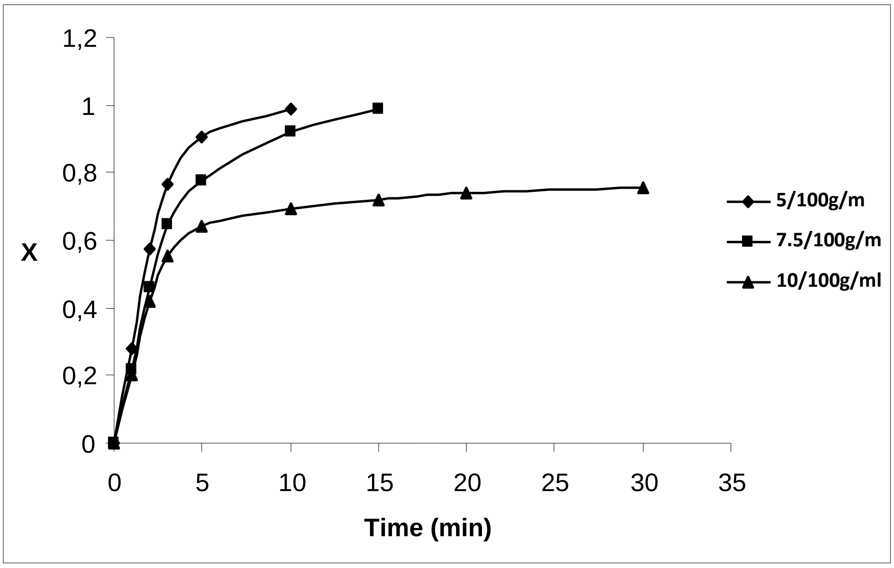

Fig. 6. *Effect of solid/liquid ratio on dissolution rate of ulexite*

3.7.Effect of the stirring speed

The effect of stirring speed on the dissolution ratio of ulexite ore in hydrochloric acid solutions have been studied at a rate of 500, 650 and 800 rpm. In the experiments, particle size has been determined as -14+18 mesh of sieve, solid/liquid ratio as 10/100 g/mL, reaction heat as 30^{o}C, acid concentration as 0.75M. According to experimental data, as seen from Figure 7, while stirring speed increases, dissolution rate increases.

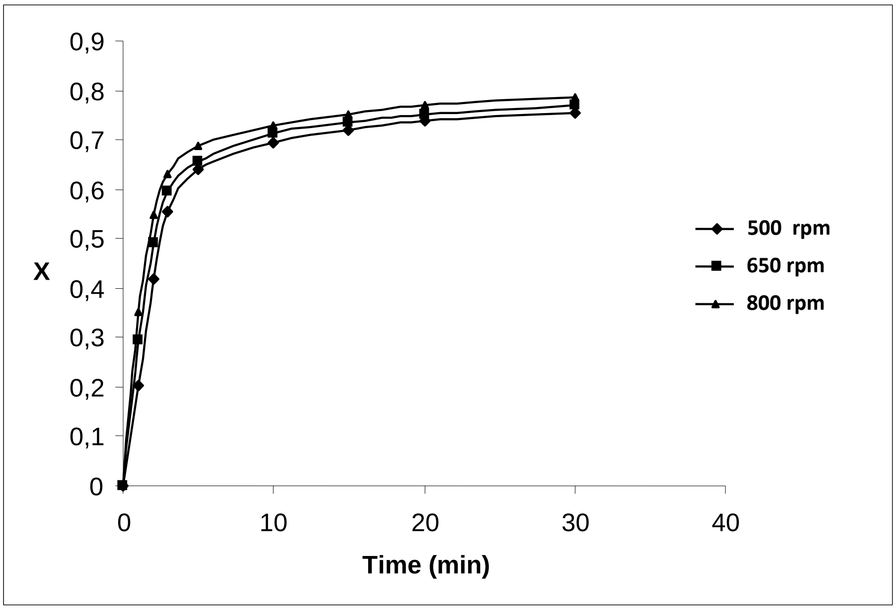

Fig. 7. *Effect of stirring speed on dissolution ratio of ulexite*

4. Discussion and conclusion

In this study, the dissolution kinetics of ulexite in hydrochloric acid solutions has been investigated in a batch reactor. Ulexite has been examined by taking into consideration to the acid concentration in hydrochloric acid solutions, stirring speed, particle size, solid/liquid rate and temperature parameters. It has been determined that the rate of solution increases by increasing temperature and stirring speed while it decreases by increasing of solid/liquid rate, acid concentration and particle size. Dissolution of ulexite in hydrochloric acid solution of different parameters has been examined in this study as a result.

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Escherichia coli' nin Ultraviyole ve Ultrases Enerjisi ile Dezenfeksiyonu

The Disinfection of *Escherichia coli* by Ultraviolet Intensity and Ultrasound Power

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Öz: Organik bileşenler ve mikroorganizmalar gibi fiziksel, kimyasal ve biyolojik bileşenler fabrikalardan, evlerden, tesislerden ve diğer bazı kaynaklardan atık su ile birlikte salıverilirler. Mikrobiyal kirlilikler arasında bakteriler, virüsler ve funguslar sayılabilir. Bu çalışmada, atık sulardan *Escherichia coli* O157:H7 suşunun ultrases enerjisi ve ultraviyole ışınları kullanılarak dezenfeksiyonu amaçlanmıştır. Parametre olarak, ultrases enerjisinin %60'lık amplitüdü ve 254 nm'de 88 W/m^2 ışık şiddeti kullanılmıştır. Ultrases kaynağı olarak ultrasonic jeneratör ve ultraviyole ışık kaynağı olarak da Pen-Ray ultraviyole lamba kullanılmıştır. Denemeler sırasında sıcaklık 37°C' de ve TiO_2 miktarı 300 mg' da sabit tutulmuştur. Ultrases enerjisinin etkisini incelemek için, ultrases enerjisi sistemde tek başına kullanılmıştır. Aynı zamanda, ultraviyole ışığın etkisini incelemek için, ultraviyole ışık sistemde tek başına kullanılmıştır. Sinerjistik etkinin gözlenmesi için ise ultrases ve ultraviyole ışığın her ikisi sistemde birlikte kullanılmıştır. Deney sonuçlarından, bakterinin tamamen yok edilmesi ultraviyole ışığın (UV) yalnız kullanıldığı sistemde 30. dakikadan sonra, ultrases enerjisinin (US) tek başına kullanıldığı denemelerde 40. dakikadan sonra ve ultrases ve ultraviyole ışığın her ikisinin birlikte eşzamanlı kullanıldığı sistemde ise 8. dakikadan sonra olduğu görülmüştür. En fazla bakteri giderimi ultrases ve ultraviyole ışığın eşzamanlı olarak birlikte kullanıldıkları sistemde olduğu tespit edilmiştir. Sesin ve ışık enerjilerinin birlikte kullanımı bakteri giderimi için daha fazla $^{\bullet}OH$ radikalinin üretilmesini sağladığından en etkili proses olmuştur. Ayrıca, 254 nm dalga boyunun daha fazla delici ve tahrip edici özellikte olmasından dolayı, bu çalışmada daha etkili bir dezenfeksiyon gözlenmiştir.

Anahtar Kelimeler — Ultrases, ultraviyole ışık, *Escherichia coli*, su kirliliği.

Abstract: Physical, chemical and biological constituents such as organic compounds and microorganisms are released by wastewaters from fabrics, homes, facilities and other resources. Bacteria, viruses and fungi can be described as microbial pollutants. In this study the disinfection of Escherichia coli O157:H7 strain from the wastewater was aimed by using ultrasound power and ultraviolet light intensity. The 60% amplitude of ultrasound energy and also, 88 W/m2 light intensity at 254 nm wavelength was used. The ultrasound source was an ultrasonic generator and the ultraviolet light source was Pen-Ray ultraviolet lamp. The temperature was constant at 37°C temperature and the amount of TiO2 was 300 mg during the experiments. For investigating the effect of ultrasound energy, it was used alone to the system. Also, for the effect of ultraviolet light, it was used alone to the system. For the synergistic effect of ultrasound and ultraviolet, they were used together to the system. The results showed that the completely disappearance was seen at 30 min. when the ultraviolet light (UV) was used alone, 40 min. when the ultrasound (US) was used alone and 8 min when the ultrasound and the ultraviolet light were used simultaneously. The most disinfection was determined when the ultrasound and the ultraviolet light were used simultaneously. Together using of the sound and light energies proved to be the most effective process on bacterial disinfection by generating greater •OH radicals. Also, having more piercing and devastating properties of 254 nm wavelength, it was seen more effective disinfection in this study.

Keywords — Ultrasound, ultraviolet light, Escherichia coli, water pollution

1.Introduction

There are several types of pollutant in wastewaters. The release of treated and untreated effluent to water-bodies is a global occurrence and can be a significant source of pathogenic microorganisms (Barrett *et al.* 2016). Microbial pollutants such as bacteria, viruses, fungi are one of the significant harmful sources of wastewaters. They have damaged health of livings. It can be pronounced that the *Escherichia coli* is the most important contamination markers among the bacteria for the health. *E. coli* is normally present in the gastrointestinal systems of people and animals as a part of the natural microflora. Currently, and largely as a result of *Escherichia coli* O157:H7,producing *E. coli*, outbreaks associated with lettuce, baby spinach, other tender greens, and leafy culinary herbs, many growers and fresh processors have adopted preharvest testing and/or a finished goods 'test to release' approach before accepting a field lot for harvest or transporting product to processing operations or commercial distribution channels following a negative finding of target pathogens (D'Lima and Suslow, 2009; Velasco *et al.,* 2015). *E. coli* is a member of *Enterobacteriaceae*, is a fecal coliform. These bacteria are the most widely adopted indicator of fecal pollution in food and water (Tavakoli *et al.,* 2008). The existence of these microorganisms in fresh water arises from the release of wastewater to the natural water sources. Accordingly, various methods have been developed for eliminating the microorganisms from wastewaters before they reach the natural water sources. The physical methods are commonly used for removing the pollutants. However, these methods are not destructive enough and they forward the organic compounds to another phase. The biological processes such as slow sand filters and biologically active carbon are expensive and difficult applications. Some traditional disinfection methods such as chlorine based technologies lead to the formation of undesirable chloro-organic disinfection by-products like trihalomethanes and haloacetic acids with carcinogenic and mutagenic effects on mammals (Rook, 1974, Nissinen *et al.,* 2002). Accordingly, the use of alternative disinfection systems should be evaluated as possible alternative to these methods.

In recent years, there is great effort to find suitable technologies for treatment of wastewaters (Casani *et al.*, 2005; Artés *et al.*, 2009; Olmez and Kretzschmar, 2009; Anese *et al.*, 2015) and also, some new methods have gained great attention for disinfection of wastewaters. Power ultrasound has been suggested as an alternative technology for wastewater decontamination (Neis and Blume, 2002; Piyasena *et al.*, 2003; Anese *et al.*, 2015). Ultrasound frequencies higher than 20 kHz are actually considered safe, non-toxic and environmentally friendly (Kentish and Ashokkumar, 2011; Anese *et al.*, 2015). During ultrasound treatment cavitation phenomena occur into the liquid medium causing a rapidly alternating compression and decompression zones, which are in turn responsible for generating shock waves with associated local very high temperatures and pressures, as well as free radicals and hydrogen peroxide (Leighton, 1994; Mason *et al.*, 2003; Anese *et al.*, 2015). Improved efficiency of ultrasound technology can be obtained by its combination with ultraviolet irradiation. In UV mechanism, upon irradiation, valence band electrons are promoted to the conduction band leaving a hole behind (Eq. (I)). The hydroxyl radicals can be produced as (Daneshvar *et al.*, 2004; Konstantinou and Albanis, 2004; Behnajanady *et al.*, 2006; Saygi and Tekin, 2013)

$$TiO_2 + h\nu \longrightarrow e^- + h^+ \qquad (I)$$
$$e^- + h^+ \longrightarrow heat \qquad (II)$$
$$h^+ + H_2O \longrightarrow {}^{\bullet}OH + H^+ \qquad (III)$$
$$h^+ + OH^- \longrightarrow {}^{\bullet}OH \qquad (IV)$$

In US mechanism, the hydroxyl radicals can be produced by the following equations (Wu, 2008; Ertugay and Acar, 2014; Monteagudo *et al*., 2014). When water is irradiated with US, $^{\bullet}OH$ radicals are formed by thermolysis of H_2O in the collapsing bubble (Eq. (V)).

$$H_2O \longrightarrow {}^{\bullet}H + {}^{\bullet}OH, \qquad (V)$$
$${}^{\bullet}H + {}^{\bullet}OH \longrightarrow H_2O, \qquad (VI)$$
$${}^{\bullet}H + O_2 \longrightarrow {}^{\bullet}HO_2, \qquad (VII)$$
$$2\,{}^{\bullet}HO \longrightarrow H_2O_2, \qquad (VIII)$$
$$2\,{}^{\bullet}HO_2 \longrightarrow H_2O_2 + O_2, \qquad (IX)$$
$${}^{\bullet}H + H_2O_2 \longrightarrow {}^{\bullet}OH + H_2O. \qquad (X)$$

Free radicals generated by the hemolysis of water have important role in destroying bacteria (Crum *et al*., 1999; Ince *et al*., 2001; Tezcanli *et al*., 2004; Anese *et al*., 2015). In recent years, the simultaneous use of ultrasound and ultraviolet light, *i.e*. the so-called UV+US has been studied regarding the efficiency of microbial disinfection. Simultaneous uses of them proved to be more effective than their individual usages (Mrowetz *et al*., 2003; Sahu and Parida, 2012; Yetim and Tekin, 2012).

In this study, advanced bacterial disinfection was aimed by generating more free radicals using ultrasound and ultraviolet light methods simultaneously. On this purpose, *E.coli* was exposed to UV, US and UV+US processes in order to appoint the most effective methods.

2. *Materials & Methods*

Experiments were performed in a Pyrex glass reactor. US was carried out by an ultrasonic generator (Cole Parmer, Ultrasonic homogenizer, 750 W, 20 kHz) with a cup horn probe. Pen-Ray UV lamp (Cole Parmer, 365 nm) was used as the radiation source in UV. The water was circulated continuously within the water jacket reactor by the constant temperature water circulator to keep the temperature stabile. The required O_2 for system was provided by a vacuum pump.

The commercial TiO_2 supplied by Degussa (P25) was used as photocatalyst. According to the manufacturer's specifications, P25 has an elementary particle size of 30 nm. a BET specific surface area of 50 m^2/g and its crystalline mode was 80 % anatase and 20 % rutile.

Escherichia coli O157:H7 strain was used throughout the study. The strain was grown aerobically in 250 mL flasks containing 100 mL of nutrient broth at 37 °C on a rotary shaker overnight. The final bacterial count was adjusted to 10^8 cfu/mL in bacterial suspension for further use. The cell viability of bacteria was tested on the Plate Count Agar (PCA). During the course of reactions, samples were taken from suspension at regular intervals for 60 min in triplicates. The viable count was performed on PCA plates after serial dilutions of the sample in phosphate-buffered solutions. All plates were incubated at 37 °C for 24-48-72 h.

Amplitude of ultrasound energy of 60 % (55.03 W), 254 nm wavelength, 88 W/m^2 light intensity and 37 °C temperature were used as the constant parameters for the experiments. The reactor was isolated from the outside light. Bacterial solution and 300 mg TiO_2 were introduced to the reactor. Experiments were carried out by continuously stirring with the magnetic stirrer. 5 ml samples of suspension were withdrawn at regular intervals. Samples were diluted nine times and each dilution was spread on PCA. US, UV, UV+US experiments were studied for displaying the most effective process on disinfection.

3. *Results & Discussion*

Bacterial disinfection data of the US, UV and UV+US at 88 W/m^2 light intensity and 254 nm wavelength was graphed in Fig.1. After 6 min the disinfection values were approximately 1.4×10^4, 2.7×10^7 and 8×10^0 for UV, US, and UV+US processes, respectively. The complete disappearance was seen after 30 min for UV, 40 min for US and 8 min for UV+US.

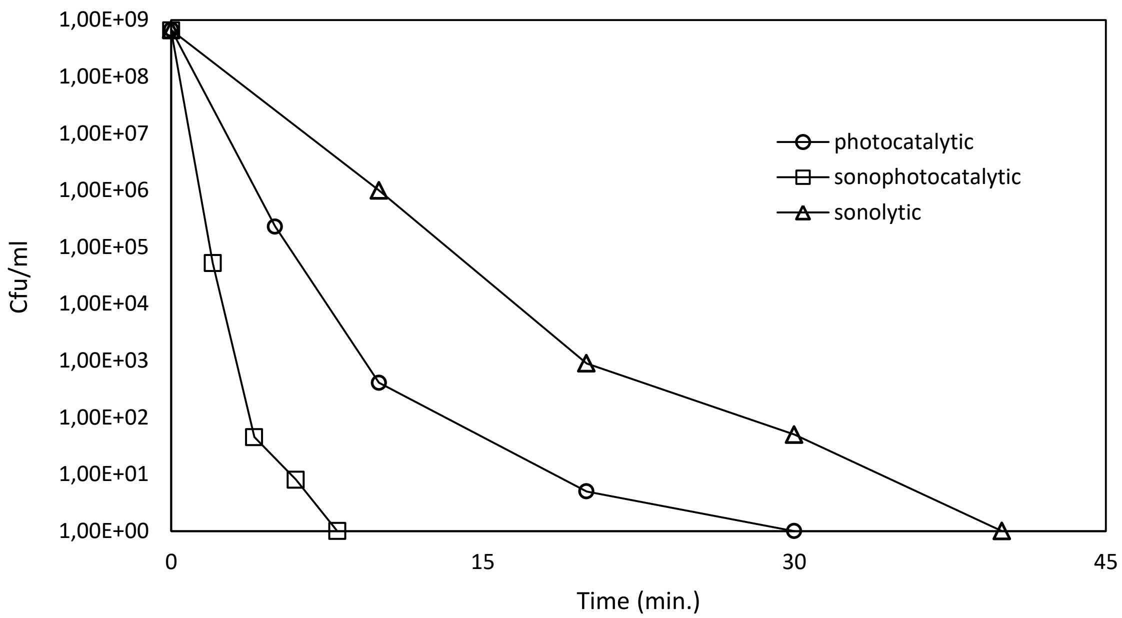

Figure 1. *The disinfection of E. coli in US, UV and UV+US processes at 254 nm wavelength and 88 W/m^2 light intensity.*

The highest disinfection rate was obtained by the UV+US process. As seen from the Fig. 1, much more disinfection can be determined at UV and UV+US processes than US process. This can be attributed to the amount of light intensity. Increasing the light intensity the excited electron amount increases, and also, the occurring $^{\bullet}OH$ radicals amount increase. Also, it can be attributed to the shortness of the wavelength too. It is a general knowledge that the frequency increases by the decreasing of the wavelength value. Therefore, the number of the wave passed in a second will increase. Thus, the number of waves effected to the bacteria increase. It is known that as the wavelength decrease, the piercing and the devastating properties increase. So that at 254 nm and at 88 W/m^2 light intensity more bacteria can be disinfected. Besides that, in US process bubbles collapse strongly on the catalytic surface. The collapsing of bubbles occurred by US at high temperatures and pressures cause the formation of $^{\bullet}OH$ radicals. Hence, more holes and pores are formed. The surface area of the catalyst can also increase by these holes and pores. Also, catalyst surfaces can be cleaner by US for forming more $^{\bullet}OH$ radicals. However, the wavelength and the light intensity are more effective than the collapse of bubbles on the disinfection. When these reasons are considered at the same time the synergistic effect of them would be strongest on the disinfection. So that the most disinfection is seen in UV+US process.

4. Conclusion

In this study, *Escherichia coli* O157:H7 strain was used for investigating the disinfection of bacteria by comparing the US, UV and UV+US processes. The obtained results from this study can be summarized as follows:

- At 60 % (55.03 W) amplitude of ultrasound energy, 254 nm wavelength and 88 W/m^2 light intensity the most effective process was found to be UV+US on the disinfection due to generating more $^{\bullet}OH$ radicals in this process.
- In US process the collapsing of bubbles high temperatures and pressures occur and cause the formation of $^{\bullet}OH$ radicals.
- At 88 W/m^2 and the short wavelength as 254 nm, UV and UV+US processes are more effective than US process on the disinfection, due to irradiating more photons to the medium for generating much more $^{\bullet}OH$ radicals.

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Baklagil Yem Bitkilerinde Zararlı Olan Curculionidae Familyasına Ait Bazı Türlerden Entomopatojen Bakterilerin İzolasyonu ve Tanısı

Isolation and Identification of Entomopathogenic Bacteria from Some Species of Curculionidae Familia Harmful to Leguminous Forage Plants

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Öz: Bu çalışmada; Erzurum ili ve bazı ilçelerinde (Aşkale, Pasinler, Tortum ve Oltu) baklagil yem bitkilerinde zararlı olan *Curculionidae* familyasına ait bazı böceklerden entomopatojen bakteriyel mikroorganizmaların izole edilerek tanılanması amaçlanmıştır. Araştırmada kullanılan hastalıklı ve ölmüş zararlı böceklerin farklı dönemleri ilaç kullanılmayan alanlardan haziran ve temmuz aylarında toplanarak uygun şartlarda laboratuvara getirilmiştir. Laboratuvara getirilen bu örneklerden izolasyonlar yapılarak çok sayıda bakteri izolatı elde edilmiştir. Bu izolatlardan toplam 88 bakteri izolatı saflaştırılmış ve -80˚C'de muhafaza edilmiştir. Saflaştırılan her bir bakteri izolatının tanısında bazı klasik ve moleküler yöntemler Microbial Identification System (MIS) kullanılmıştır. Tanılanan izolatlar incelendiğinde yapılmış birçok çalışmada biyopestisit olarak kullanılmış 3 *Bacillus thuringiensis*, 3 *Bacillus pumilus*, 4 *Brevibacillus brevis*, 4 *Paenibacillus* spp. ve 2 *Serratia* spp., olmak üzere çok sayıda izolat elde edilmiştir. Bakteriyel izolatların tanılanması amacı ile yapılmış olan klasik ve biyokimyasal test sonuçlarına göre; bütün izolatların katalaz testi sonucunda pozitif olduğu ve KOH testleri sonucunda ise izolatlar arasında hem negatif hem de pozitif sonuçlar elde edilmiştir. Tütün bitkisinde yapılan aşırı duyarlılık testleri (HR) sonucunda ise elde edilen izolatların negatif sonuç verdiği belirlenmiştir. Sonuç olarak yapılmış olan bu çalışmadan elde edilen entomopatojenlerin belirlenen hedef doğrultusunda tanısı yapılmış olup, izolatlar içerisinde biyopestisit olarak pratikte özellikle baklagil yem bitkilerinde zararlı olan *Curculionidae* familyasına ait bazı zararlılara karşı kullanılabilme potansiyeline sahip türlerin olabileceği düşünülmektedir.

Anahtar Kelimeler — Biyopestisit, Baklagil yem bitkileri, MIS

Abstract: In this study; It is aimed to isolate and identify entomopathogenic bacterial microorganisms from some insects belonging to Curculionidae family which are harmful in leguminous forage plants in Erzurum province and some districts (Aşkale, Pasinler, Tortum and Oltu). Different periods of diseased and dead pest insects used in the research were collected in June and July from areas unused chemicals and brought to the laboratory under suitable conditions. Many isolates of bacteria have been obtained by isolating these samples brought from the laboratory. A total of 88 isolates from these isolates were isolated and stored at -80 ° C. Some classical and molecular methods Microbial Identification System (MIS) were used in the identification of each isolate of bacteria. When the identified isolates were examined, they were used as biopesticides in many studies. 3 Bacillus thuringiensis, 3 Bacillus pumilus, 4 Brevibacillus brevis, 4 Paenibacillus spp. And 2 Serratia spp., Were obtained. According to the results of classical and biochemical tests carried out with the aim of identifying bacterial isolates All isolates were positive for the catalase test and for KOH tests, both negative and positive results were obtained between the isolates. As a result of the hypersensitivity tests (HR) performed on tobacco plants, it was determined that the obtained isolates gave a negative result. As a result, entomopathogens obtained from this study were diagnosed in the determined target direction, It is thought that there are potentially usable species as biopesticides in isolates, in particular against some damages of the Curculionidae family which are harmful in leguminous forage plants.

Keywords — Biopesticides, Leguminous forage plants, MIS

1.Giriş

Baklagiller olarak adlandırılan *Leguminosae* familyası dünya üzerindeki en geniş familyalardan birisidir. 250 000 çiçekli bitki türünün 12 000'i baklagiller olup, yaklaşık 600 cins içerisine dağılmışlardır. Baklagil yem bitkileri; bitkiler alemi (*Plantae*), çiçekli bitkiler (*Embriyophyta*) bölümüne, kapalı tohumlular (*Angiospermae*) alt bölümüne, çift çenekliler (*Dicotyledonae*) sınıfına, gülgiller (*Rosales*) takımına, baklagiller (*Leguminosae*) familyasına dahildirler (Elçi 2005; Tan ve Serin 2008). *Leguminosae* familyasının üç alt familyaya ayrıldığı bildirilmekte olup; yurdumuzda yem bitkileri olarak kullanılan baklagillerin tümü *Papillionoideae* alt familyası içerisinde yer almaktadır (Elçi 2005). Tarımsal açıdan birçok önemli özelliğe sahip olan baklagil yem bitkileri uzun yıllardan beri yeryüzünde yetiştirilmiş olup; insanlar bu bitkilerden çeşitli şekillerde yararlanmıştır. Yem bitkileri içerisinde en fazla yetiştiriciliği yapılan yonca (*Medicago sativa*) tarihi belgelere göre M.Ö. 490 yıllarında Yunanistan'da tanınmış ve buradan Avrupa'ya yayılmıştır. En eski kayıtlara göre yoncanın Türkiye'de 3300 yıl önce kullanıldığı belirtilmektedir (Bolton *et al*. 1972). Gıda ve Tarım Teşkilatı (Food and Agriculture Organization (FAO)) tarafından yayınlanan kaynaklar incelendiğinde; bir çok ülkede toplam tarla arazisi içerisindeki yem bitkilerinin payının büyük boyutlara ulaştığı, örneğin Avustralya'da bu oran % 50'ye yaklaşırken, birçok Kuzey Avrupa ülkesinde % 25 'ler düzeyinde bulunduğu gözlenmektedir. İncelenen Avrupa ülkelerinde en düşük pay Türkiye'de bulunmaktadır(Anonymous 2002). Ülkemizde ise Türkiye İstatistik Kurumu (TÜİK) tarafından yayınlanan istatistiklere göre; 5 210 000 ton yonca, 2 896 000 ton fiğ ve 716 000 ton korunga üretilmektedir (Anonim 2007). Son yıllarda ülkemizde başta yonca olmak üzere yem bitkileri ekim alanları genişlemekte ancak gelişme hızı oldukça yavaş seyretmektedir. Sonuç olarak ülkemiz birçok ülkeye göre yem bitkileri yetiştiriciliği açısından çok az bir üretime sahiptir. Baklagil yem bitkilerinin ülkemizin tarımında bu günkü üretim alanından çok daha geniş alanlarda üretilmesi tarımımızın gelişmesi ve teknolojinin gösterdiği yolda başarıya ulaşması için zorunludur (Elçi 2005). Bu nedenle ülkemizde hem tarım hem de ekonomik açıdan büyük faydalar sağlayacak yem bitkileri yetiştiriciliğine gereken önem verilmeli ve bu amaçla yapılacak olan çalışmalar her yönden desteklenmelidir.

Kültür bitkilerinin verim ve kalitesini azaltan organizmalar arasında; hastalık etmeni olarak bilinen patojenler (fungus, bakteri, virüs ve mikoplazma) ve zararlılar olarak bilinen hayvansal organizmalar (böcekler, akarlar, nematodlar, salyangozlar, sümüklü böcekler, kemirgenler, memeliler ve kuşlar) sayılabilir. Bu organizmalar kültür bitkilerinde verim ve kaliteyi düşürmek suretiyle ekonomik kayıplara sebep olmaktadırlar. Hızla artan nüfusu besleyebilmek için yeni tarım alanlarının açılması gerekirken maalesef erozyon, yeni yerleşim alanlarının açılması, yeni yolların açılması gibi sebeplerle tarımsal üretime elverişli sahalar giderek azalmaktadır. Bu nedenle tarım alanları içerisinde zaten yetersiz bir alana sahip olan baklagil yem bitkileri ve diğer bitkilerin üretimi için gerekli olan arazi ihtiyacını karşılamak çok zor hatta imkansızdır. Bu durum karşısında yapılacak iş, birim alandan elde edilecek ürün miktarının artırılması, mevcut tarımsal ürünlerin verim kapasitesinin artırılmasının yanı sıra, bu ürünlerin verim ve kalitesini azaltan başta hastalık ve zararlılar olmak üzere birçok olumsuz faktöründe mümkün olabildiğince asgariye indirilmesi gerekmektedir (Yıldırım 2008). Yapılan birçok çalışmada özellikle *Curculionidae* familyasına ait *Hypera* spp., *Sitona* spp., ve *Apion* spp. cinslerinin baklagil yem bitkilerinde büyük oranda kayba neden olduğu bildirilmektedir (Lykouressıs *et al*. 1991; Pisarek 1995; Vasil'eva 2004; Özbek ve Hayat 2008). *Sitona* spp. ve *Apion* spp. cinsleri içerisinde *Sitona lineatus, Sitona humeralis, Sitona puncticollis, Apion trifolii, Apion aestimatum* başta olmak üzere bir çok tür hem dünyada hem ülkemizde çok büyük populasyonlara sahip olup; yem bitkilerinde önemli zararlar oluşturmaktadır (Barrad 1996; Strbac 2005; Toshova *et al*. 2009; Akkaya 1995; Kıvan 1995; Tamer vd. 1997; Özbek ve Hayat 2008). Tüm bunlar dikkate alındığında hem dünyada hem de ülkemizde baklagil yem bitkileri ve diğer tarımsal ürünlerde ciddi boyutta kayıplara sebep olan zararlı türleri ve zarar şekilleri tespit edilerek en uygun mücadele yöntemlerinin geliştirilmesine gereken destek ve önem verilmelidir.

Bu çalışmanın amacı, baklagil yem bitkilerindeki zararlılara karşı patojen olan bakterileri izole ederek bunların laboratuarda tanısını ve karakterizasyonunu yapmaktır. Ayrıca elde edilen potansiyel bakteriyel izolatlar ile ileriki çalışmalarda bu zararlılara karşı biyolojik mücadelede kullanılabilecek alt yapıyı oluşturmaktır.

2. *Materyal ve Yöntem*

Hastalıklı ve Ölmüş Böceklerin Larva ve Erginlerinin Toplanması ve Laboratuara Getirilmesi

Araştırma için kullanılan zararlı böceklerin farklı dönemleri Erzurum ili ve çevre ilçelerden Aşkale, Pasinler, Tortum ve Oltu ilçelerinde ilaç kullanılmayan baklagil yem bitkileri tarlalarından haziran ve temmuz aylarında toplanmıştır. Toplanan numuneler etiketlenerek ve polietilen torbalar içerisinde araç buzdolabına konularak laboratuara getirilmiştir.

Bakteri İzolatlarının İzolasyonu ve Stoklanması

Hastalıklı ve ölmüş böceklerin larva ve erginleri önce %95'lik etil alkol ile 5 dk süreyle yüzeysel sterilizasyona tabi tutulup steril fizyolojik su (SFS) içerisinde steril bir havanda ezilerek homojenize edilmiştir. Homojenattan steril pipetle alınarak Nutrient Agar (NA) ve Tripticase Soy

Agar (TSA) besi ortamlarına çizgi ekimi yöntemiyle ekimleri yapılmıştır. Kültürler 25-30°C de inkübe edilerek ve 24-72 saat sonunda oluşan kolonilerden mümkün olduğunca öncelikle yoğun gelişenlerden olmak üzere her farklı karakterdeki koloniden yeni besi yerlerine transfer edilip saflaştırılmıştır.

Her bir izolata ayrı bir kod numarası verilerek, izolasyonla ilgili bilgiler (izole edildiği lokasyon, böcek evresi, tarih vs.) kaydedilmiş; tanı ve karakterizasyon işlemleri ve yapılacak diğer çalışmalarda kullanılmak üzere %30 gliserol ve Lauryl Broth (LB) içeren stok besiyerlerinde -86 ^{0}C' de muhafazaya alınmıştır.

Bakteri İzolatlarının Tanısı

Gram Özelliği

Hem Gram boyama hem de KOH testi uygulanarak tanısı yapılan bakteriyel izolatların gram + mi yoksa gram – mi olduğu tespit edilmiştir.

Katalaz Testi

Elde edilen izolatların katalaz enzimine sahip olup olmadıkları bu testle belirlenmiştir. Test için 48 saatlik bakteri kültüründen bir öze dolusu alınıp, üzerine 1 damla %5'lik H_2O_2 ilave edilmiş kabarcık oluşumu pozitif sonuç olarak değerlendirilmiştir.

İzolatların MIS Sistemi İle Yağ Asiti Profillerinin Belirlenmesi ve Tanılanması

Saflaştırılarak muhafaza edilen bakteri izolatlarının yağ asiti metil esterleri elde edilmiş, Mikrobial Identification Sistemi=MIS (MIDI, Inc., Newark, DE) kullanılarak tanılanmıştır.

Örneklerin Mikrobiyal Tanılama Sistemi ile (MIS) Analiz Edilmesi

Yukarıdaki protokole göre hazırlanan örnekler MIS cihazı üzerindeki örnek depolama tepsisine yerleştirildikten sonra, cihaz çalıştırıldı. MIS sistem kılavuzunda belirtildiği gibi örnekler tek tek analiz edilerek, bilgisayar ortamında tanı sonuçları alınmıştır.

3. Bulgular ve Tartışma

Erzurum ili ve bazı ilçelerinde (Aşkale, Pasinler, Tortum ve Oltu) baklagil yem bitkilerinde zararlı olan *Curculionidae* familyasına ait bazı böceklerden entomopatojen bakteriyel mikroorganizmalar izole edilerek tanılanmıştır. Bu izolatlardan toplam 86 bakteri izolatı saflaştırılmış ve -80˚C'de muhafaza edilmiştir. Saflaştırılan her bir bakteri izolatının tanısında bazı klasik ve moleküler yöntemler Microbial Identification System (MIS) kullanılmıştır.

Tanılanan izolatlar incelendiğinde yapılmış birçok çalışmada biyopestisit olarak kullanılmış 3 *Bacillus thuringiensis*, 3 *Bacillus pumilus*, 4 *Brevibacillus brevis*, 4 *Paenibacillus* spp. ve 2 *Serratia* spp., olmak üzere çok sayıda izolat elde edilmiş ve sonuçlar Çizelge 1'de verilmiştir. Elde edilen izolatların tanılanması amacı ile yapılmış olan klasik test sonuçlarına göre; bütün izolatların katalaz testinin pozitif olduğu ve KOH testleri sonucunda ise izolatlar arasında hem negatif hem de pozitif sonuçlar elde edilmiş olup sonuçlar Çizelge 1'de verilmiştir. Bilindiği gibi klasik yöntemler kullanılarak bakterilerin tanılanması tek başına yeterli olmamaktadır. Ancak klasik yöntemler hem bakterilerin ön tanısı için hem de kesin tanı ve karakterizasyonda kullanacağımız moleküler metotların belirlenmesi açısından büyük önem arz etmektedir. Bu nedenle mikroorganizmaların tanı ve karakterizasyonu konusunda çalışan bir çok araştırmacı klasik yöntemlerden her zaman yararlanmaktadırlar (Saygılı 2006; Dhingani ve ark., 2013). Yapılmış olan bu çalışmada da bakterilerin tanısında farklı klasik yöntemler kullanılmıştır.

Moleküler yöntemler; karbonhidratları, lipitleri, proteinleri ve genetik materyalleri (DNA ve RNA) çalışma materyali olarak kabul etmekte ve bunlardan birinin veya kombinasyonlarının kullanımı ile mikroorganizmaların tanı ve karakterizasyonunun yapılmasını sağlamaktadır. Yağ asit analizleri (Mikrobiyal Identifikasyon Sistemi (MIS)), metabolik enzim profillerinin belirlenmesi (BIOLOG), protein profillerinin belirlenmesi (SDS-PAGE), serolojik reaksiyonlar (Immunofloresans, Radioimmunoassay, Immuno Blot, Dot İmmunobinding Assay ve Enzim Linked Immunosorbent Assay (ELISA) ve genetik profillere (r DNA-PCR, Rep-PCR, Eric-PCR, Box-PCR ve Spesifik PCR) göre mikroorganizmaları tanılayan sistemlerin tamamı moleküler sistemler olarak kabul edilir (Jackman 1985, Kersters 1985, Miller ve Berger 1985, Miller ve Martin 1988, Guillorit-Rondeau ve ark., 1996, Scortichini ve ark.,1996, Zhang ve Geider 1997). Moleküler yöntemler; içerisinde MIS, BIOLOG, ELISA ve PCR yöntemleri mikroorganizmaların tanısında en fazla kullanılan yöntemler olmaktadır. Bu nedenle yapılmış olan bu çalışmada da yukarıda belirtilen yöntemlerden MIS sistemi kullanılmıştır. İzole edilerek saflaştırılan toplam 88 izolatdan 86 izolat MIS sisitemi kullanılarak tanılanmış 2 izolat ise tanılanamamıştır. Tanılanan izolatlar içerisinde biyopestisit olarak yaygın bir şekilde kullanılan *Bacillus*, *Serratia* ve *Paenibacillus* cinslerine ait olmak üzere birçok bakteri türü elde edilmiştir.

Sonuç olarak, yapılmış olan bu çalışmada bölgemizde yaygın olarak yetiştiriciliği yapılan baklagil yem bitkilerinde zararlı olan *Curculionidae* familyasına ait türlerden biyopestisit olma özelliğine sahip çok sayıda bakteri izolatı izole edilerek tanılanmıştır.

Tablo 1. *Böceklerden izole edilen bakteriyel izolatların konukçu, yıl ve lokasyon verileri ile MIS ve klasik yöntemlerden bazıları kullanılarak elde edilen tanı sonuçları*

İzolat no	**MIS tanı sonuçları**	**MIS %**	**Konukçu**	**Lokasyon**	**Yıl**	**KOH**	**K**
FDP-1	*Paenibacillus alvei*	58	*Hypera postica*	Erzurum	2009	+	K^+
FDP-2	*Brevibacillus brevis*	31	*Hypera postica*	Erzurum	2009	+	K^+
FDP-3	*Bacillus-cereus*-GC subgroup B	34	*Hypera postica*	Erzurum	2009	+	K^+
FDP-4	*Bacillus-cereus*-GC subgroup B	45	*Hypera postica*	Erzurum	2009	+	+
FDP-5	*Bacillus lentimorbus*	50	*Hypera postica*	Erzurum	2009	+	K^+
FDP-6	*Bacillus marinus*	44	*Hypera postica*	Erzurum	2009	+	K^+
FDP-7	*Bacillus-cereus*-GC subgroup B	45	*Hypera postica*	Erzurum	2009	+	+
FDP-8	*Bacillus-thuringiensis kenyae*	32	*Hypera postica*	Erzurum	2009	+	K^+
FDP-9	*Vibrio-alginolyticus*-GC	59	*Hypera postica*	Erzurum	2009	-	+
FDP-10	*Photobacterium-leiognathi*	70	*Hypera postica*	Erzurum	2009	-	+
FDP-11	*Kocuria-rosea*	44	*Hypera postica*	Erzurum	2009	+	K^+
FDP-12	*Bacillus-cereus*-GC subgroup B	50	*Hypera postica*	Erzurum	2009	+	+
FDP-13	*Pseudomonas-putida*-biotype A	87	*Hypera postica*	Erzurum	2009	-	+
FDP-14	*Staphylococcus-kloosii*	37	*Hypera postica*	Erzurum	2009	-	+
FDP-15	*Bacillus-cereus*-GC subgroup B	37	*Hypera postica*	Erzurum	2009	+	+
FDP-16	*Photobacterium-leiognathi*	66	*Hypera postica*	Erzurum	2009	-	+
FDP-17	*Bacillus-cereus*-GC subgroup B	40	*Hypera postica*	Erzurum	2009	+	K^+
FDP-18	*Bacillus-cereus*-GC subgroup B	34	*Hypera postica*	Erzurum	2009	+	+
FDP-19	*Pantoea agglomerans*	40	*Hypera postica*	Erzurum	2009	-	+
FDP-20	*Brevibacillus brevis*	47	*Hypera postica*	Erzurum	2009	+	K^+
FDP-21	*Bacillus-megaterium*-GC	63	*Hypera postica*	Erzurum	2009	+	+
FDP-22	*Bacillus-megaterium*-GC	68	*Hypera postica*	Erzurum	2009	+	K^+
FDP-23	*Kocuria varians*	55	*Hypera postica*	Erzurum	2009	+	K^+
FDP-24	*Bacillus-cereus*-GC subgroup B	38	*Hypera postica*	Erzurum	2009	+	K^+
FDP-25	*Photobacterium-leiognathi*	69	*Hypera postica*	Erzurum	2009	-	+
FDP-26	*Lactococcus plantarum*	61	*Hypera postica*	Erzurum	2009	+	-
FDP-27	*Erwinia-chrysanthemi*-biotype	63	*Hypera postica*	Erzurum	2009	-	+
FDP-28	*Bacillus-cereus*-GC subgroup B	56	*Hypera postica*	Erzurum	2009	+	K^+
FDP-29	*Brevibacillus brevis*	39	*Hypera postica*	Erzurum	2009	+	+
FDP-30	*Brevibacillus brevis*	21	*Hypera postica*	Erzurum	2009	+	+
FDP-31	*Bacillus-cereus*-GC subgroup B	35	*Hypera postica*	Erzurum	2009	+	K^+
FDP-32	*Bacillus-pumilus*-GC subgroup	45	*Hypera postica*	Erzurum	2009	+	K^+
FDP-33	*Bacillus-cereus*-GC subgroup B	20	*Hypera postica*	Erzurum	2009	+	K^+
FDP-34	*Bacillus-megaterium*-GC	60	*Hypera postica*	Erzurum	2009	+	K^+
FDP-35	*Nesterenkonia halobia*	25	*Hypera postica*	Erzurum	2009	+	K^+
FDP-36	*Staphylococcus-kloosii*	36	*Hypera postica*	Erzurum	2009	+	+
FDP-37	*Bacillus-megaterium*-GC	40	*Hypera postica*	Erzurum	2009	+	K^+
FDP-38			*Apion* sp. (ergin)	Erzurum	2009		
FDP-39	*Staphylococcus-arlettae*	14	*Apion* sp. (ergin)	Erzurum	2009	+	K^+
FDP-40	*Bacillus-megaterium*-GC	58	*Apion* sp. (ergin)	Erzurum	2009	+	K^+
FDP-41	*Bacillus-thuringiensis kurstakii*	55	*Apion* sp. (ergin)	Erzurum	2009	+	+
FDP-42	*Bacillus-thuringiensis kenyae*	35	*Apion* sp. (ergin)	Erzurum	2009	+	+

FDP-43	*Pantoea agglomerans*	66	*Hypera postica*	Oltu	2009	-	K^+
FDP-44	*Enterobacter-intermedius*	84	*Hypera postica*	Tortum	2009	-	+
FDP-45	*Providencia-alcalifaciens*	88	*Sitona* sp. (ergin)	Oltu	2009	-	K^+
FDP-46	*Serratia-fonticola*	39	*Sitona* sp. (ergin)	Oltu	2009	-	K^+
FDP-47	*Enterobacter-cloacae*	72	*Hypera postica*	Pasinler	2009	-	+
FDP-48	*Serratia-liquefaciens*	84	*Hypera postica*	Oltu	2009	-	+
FDP-49	*Pantoea agglomerans*	70	*Hypera postica*	Oltu	2009	-	K^+
FDP-50	*Providencia-alcalifaciens*	97	*Hypera postica*	Oltu	2009	-	+
FDP-51	*Pantoea agglomerans*	66	*Hypera postica*	Oltu	2009	-	+
FDP-52	*Proteus myxofaciens*	77	*Hypera postica*	Oltu	2009	-	K^+
FDP-53	*Acinetobacter radioresistens*	58	*Hypera postica*	Pasinler	2009	-	K^+
FDP-54	*Enterobacter agglomerans*-GC	84	*Sitona* sp. (ergin)	Oltu	2009	-	+
FDP-55	*Pantoea agglomerans*	58	*Hypera postica*	Tortum	2009	-	+
FDP-56	*Acinetobacter radioresistens*	59	*Apion* sp. (ergin)	Tortum	2009	-	K^+
FDP-57	*Bacillus lentimorbus*	61	*Apion* sp. (ergin)	Tortum	2009	+	K^+
FDP-58	*Stenotrophomonas maltophilia*	73	*Hypera postica*	Pasinler	2009	-	+
FDP-59	*Paenibacillus apiarius*	50	*Hypera postica*	Oltu	2009	+	K^+
FDP-60	*Bacillus-megaterium*-GC	41	*Hypera postica*	Oltu	2009	+	K^+
FDP-61	*Paenibacillus apiarius*	49	*Hypera postica*	Oltu	2009	+	K^+
FDP-62	*Brevibacterium epidermidis*	51	*Sitona* sp. (ergin)	Oltu	2009	+	K^+
FDP-63	*Bacillus-pumilus*-GC subgroup	49	*Hypera postica*	Oltu	2009	+	K^+
FDP-64	*Bacillus-megaterium*-GC	83	*Sitona sp*.(ergin)	Oltu	2009	+	+
FDP-65	*Stenotrophomonas maltophilia*	65	*Hypera postica*	Oltu	2009	-	+
FDP-66	*Brevibacterium linens*	79	*Hypera postica*	Oltu	2009	+	+
FDP-67	*Bacillus-pumilus*-GC subgroup	63	*Hypera postica*	Oltu	2009	+	+
FDP-68	*Bacillus marinus*	51	*Hypera postica*	Oltu	2009	+	+
FDP-69	*Bacillus-megaterium*-GC	45	*Hypera postica*	Pasinler	2009	+	+
FDP-70	*Stenotrophomonas maltophilia*	73	*Apion* sp. (ergin)	Tortum	2009	-	-
FDP-71	*Bacillus halodenitrificans*	42	*Hypera postica*	Oltu	2009	+	K^+
FDP-72	*Bacillus subtilis*	68	*Hypera postica*	Oltu	2009	+	+
FDP-73	*Paenibacillus alvei*	35	*Sitona sp*.(ergin)	Erzurum	2009	+	K^+
FDP-74	*Enterebacter intermedius*	80	*Sitona sp*.(ergin)	Erzurum	2009	-	K^+
FDP-75	*Enterobacter agglomerans*-GC	78	*Sitona sp*.(ergin)	Erzurum	2009	-	+
FDP-76	*Corynebacterium bovis*	34	*Sitona sp*.(ergin)	Erzurum	2009	-	K^+
FDP-77	*Pseudomonas-putida*-biotype A	46	*Sitona sp*.(ergin)	Erzurum	2009	-	K^+
FDP-78	*Bacillus lentimorbus*	63	*Sitona sp*.(ergin)	Aşkale	2009	+	K^+
FDP-79	*Bacillus lentimorbus*	57	*Sitona sp*.(ergin)	Aşkale	2009	+	K^+
FDP-80	*Bacillus-cereus*-GC subgroup A	31	*Sitona sp*.(ergin)	Aşkale	2009	+	K^+
FDP-81	*Micrococcus-luteus*-GC	63	*Sitona sp*.(ergin)	Aşkale	2009	+	+
FDP-82	*Enterebacter intermedius*	82	*Sitona sp*.(ergin)	Aşkale	2009	-	K^+
FDP-83	*Cedecea davisae*	40	*Sitona sp*.(ergin)	Aşkale	2009	-	K^+
FDP-84	*Bacillus-cereus*-GC subgroup B	36	*Sitona sp*.(ergin)	Aşkale	2009	+	K^+
FDP-85	*Enterebacter intermedius*	85	*Sitona sp*.(ergin)	Aşkale	2009	-	K^+
FDP-86	*Alcaligenes faecalis*	75	*Sitona sp*.(ergin)	Aşkale	2009	-	+
FDP-87	*Kocuria varians*	50	*Sitona sp*.(ergin)	Aşkale	2009	+	K^+
FDP-88			*Sitona sp*.(ergin)	Aşkale	2009		

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s −Konveks ve s −Konkav Fonksiyonlar İçin Kesirli İntegraller Yardımıyla Hermite-Hadamard Tipli Eşitsizlikler

Hermite-Hadamard type inequalities for s −convex and s −concave functions via fractional integrals

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Öz: Kesirli integraller için yeni bir integral özdeşliği tanımlandı. Bu özdeşlik yardımıyla Riemann-Liouville kesirli integralleri için bazı yeni Hermite-Hadamard tipli eşitsizlikler geliştirildi. Elde edilen sonuçların Avcı vd. [4, Appl. Math. Comput., 217 (2011) 5171-5176] adlı makalede ispat edilen sonuçlarla ilişkili olduğu belirlendi.

Anahtar Kelimeler — s-konveks fonksiyon, Hölder eşitsizliği, Power-Mean eşitsizliği, Riemann-Liouville kesirli integral, Euler Gama fonksiyonu, Euler Beta fonksiyonu.

Abstract: New identity for fractional integrals have been defined. By using this identity, some new Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral have been developed. It has been determined that the results are related to the results of Avcı et al., proved in [4, published in Appl. Math. Comput., 217 (2011) 5171-5176].

Keywords — s −convex function, Hölder inequality, Power-mean inequality, Riemann Liouville fractional integral, Euler Gamma function, Euler Beta function.

1.Introduction

The following double inequality, named Hermite-Hadamard inequality, is one of the best known results in the literature.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I of real numbers and $a,b \in I$ with $a < b$. Then the following double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

The above double inequality is reversed if f is concave.

In [6], Hudzik and Maligranda considered among others the class of functions which are s-convex in the second sense.

Definition 1. *A function* $f:\mathbb{R}^+\to\mathbb{R}$, *where* $\mathbb{R}^+=[0,\infty)$, *is said to be* $s-$ *convex in the second sense if*

$$f(\alpha x+\beta y)\le\alpha^s f(x)+\beta^s f(y)$$

for all $x,y\in 0,\infty)$, $\alpha,\beta\ge 0$ with $\alpha+\beta=1$ and for some fixed $s\in(0,1]$. This class of s-convex functions in the second sense is usually denoted by K_s^2.

It can be easily seen that for $s=1$, $s-$ convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [7], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$ convex functions in the second sense as following.

Theorem 2. *Suppose that f:* $[0,\infty)\to[0,\infty)$ *is an* $s-$ *convex functions in the second sense, where* $s\in(0,1)$, *and let a,b∈*$[0,\infty)$ $a<b$. *If* $f\in L^1[a,b]$, *then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{s+1}. \tag{1.1}$$

The constant $k=\dfrac{1}{s+1}$ is the best possible in the second inequality in (1.1).

In [5], Kavurmacı *et al*. proved the following identity.

Lemma 1. *Let* $f:I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable function on* I°, *where* $a,b\in I$ *with* $a<b$. *If* $f'\in L[a,b]$, *then the following equality holds:*

$$\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du$$

$$=\frac{(x-a)^2}{b-a}\int_0^1(t-1)f'(tx+(1-t)a)dt+\frac{(b-x)^2}{b-a}\int_0^1(1-t)f'(tx+(1-t)b)dt.$$

In [4], Avcı *et al.* obtained the following results by using the above Lemma.

Theorem 3. *Let f:I*$\subset[0,\infty)\to$R *be a differentiable function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b<\infty$. *If* $|f'|$ *is* $s-$ *convex on* $[a,b]$ *for some fixed* $s\in(0,1]$, *then*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \tag{1.2}$$

$$\leq \frac{1}{(s+1)(s+2)}\left|f'(x)\right|\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]+\frac{1}{(s+2)}\left[\frac{(x-a)^2}{b-a}\left|f'(a)\right|+\frac{(b-x)^2}{b-a}\left|f'(b)\right|\right].$$

Theorem 4. *Let* $f: I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a,b \in I$ *with* $a < b < \infty$. *If* $\left|f'\right|^q$ *is* $s-$*convex on* $[a,b]$ *for some fixed* $s \in (0,1]$, $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \tag{1.3}$$

$$\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{s+1}\right]^{\frac{1}{q}}+\frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}.$$

Theorem 5. *Suppose that all the assumptions of Theorem 4 are satisfied. Then*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \tag{1.4}$$

$$\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'(x)\right|^q\frac{1}{(s+1)(s+2)}+\left|f'(a)\right|^q\frac{1}{s+2}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'(x)\right|^q\frac{1}{(s+1)(s+2)}+\left|f'(b)\right|^q\frac{1}{s+2}\right)^{\frac{1}{q}}.$$

Theorem 6. *Let* $f: I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a,b \in I$ *with* $a < b$. *If* $\left|f'\right|^q$ *is* $s-$*concave on* $[a,b]$ *for* $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \tag{1.5}$$

$$\leq \frac{2^{\frac{s-1}{q}}}{(1+p)^{\frac{1}{p}}(b-a)}\left\{(x-a)^2\left|f'\left(\frac{x+a}{2}\right)\right|+(b-x)^2\left|f'\left(\frac{x+b}{2}\right)\right|\right\}.$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 2. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha}(f)$ and $J_{b^-}^{\alpha}(f)$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ \ x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ \ b > x$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha-1} du$. Here $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties of this operator can be found in [1]-[3].

The main aim of this paper is to establish Hermite-Hadamard type inequalities for $s-$convex and $s-$concave functions in the second sense via Riemann-Liouville fractional integral.

2. Hermite-Hadamard type inequalities for fractional integrals

In order to prove our main results we need the following Lemma.

Lemma 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, the interior of I, where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then for all $x \in [a,b]$ and $\alpha > 0$ we have:*

$$\frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)\right]$$

$$= \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 (t^{\alpha}-1) f'(tx+(1-t)a)dt + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 (1-t^{\alpha}) f'(tx+(1-t)b)dt$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha-1} du$.

Proof. By integration by parts, we can state

$$\int_0^1 (t^{\alpha}-1) f'(tx+(1-t)a)dt \qquad (2.1)$$

$$= \left(t^{\alpha}-1\right)\frac{f\left(tx+(1-t)a\right)}{x-a}\Bigg|_0^1 - \int_0^1 \alpha t^{\alpha-1}\frac{f\left(tx+(1-t)a\right)}{x-a}dt$$

$$= \frac{f(a)}{x-a} - \frac{\alpha}{x-a}\int_a^x \left(\frac{u-a}{x-a}\right)^{\alpha-1}\frac{f(u)}{x-a}du$$

$$= \frac{f(a)}{x-a} - \frac{\alpha\Gamma(\alpha)}{(x-a)^{\alpha+1}} J^{\alpha}_{x^-} f(a)$$

and

$$\int_0^1 \left(1-t^{\alpha}\right) f'\left(tx+(1-t)b\right)dt \tag{2.2}$$

$$= \left(1-t^{\alpha}\right)\frac{f\left(tx+(1-t)b\right)}{x-b}\Bigg|_0^1 - \int_0^1 \alpha t^{\alpha-1}\frac{f\left(tx+(1-t)b\right)}{x-b}dt$$

$$= \frac{f(b)}{b-x} - \frac{\alpha}{b-x}\int_x^b \left(\frac{u-b}{x-b}\right)^{\alpha-1}\frac{f(u)}{x-b}du$$

$$= \frac{f(b)}{b-x} - \frac{\alpha\Gamma(\alpha)}{(b-x)^{\alpha+1}} J^{\alpha}_{x^+} f(b).$$

Multiplying the both sides of (2.1) and (2.2) by $\frac{(x-a)^{\alpha+1}}{b-a}$ and $\frac{(b-x)^{\alpha+1}}{b-a}$, respectively, and then adding the resulting identities we obtain the desired result.

Theorem 7. *Let $f: I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $\left|f'\right|$ is $s-$convex on $[a,b]$ for some fixed $s \in (0,1]$ and $x \in [a,b]$ then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\left|\frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x^-} f(a) + J^{\alpha}_{x^+} f(b)\right]\right|$$

$$\leq \frac{\alpha}{(s+1)(\alpha+s+1)}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]\left|f'(x)\right|$$

$$+\left[\frac{1}{s+1} - \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]\left[\frac{(x-a)^{\alpha+1}\left|f'(a)\right| + (b-x)^{\alpha+1}\left|f'(b)\right|}{b-a}\right]$$

where Γ is Euler Gamma function.

Proof. From Lemma 2, property of the modulus and using the $s-$ convexity of $\left|f'\right|$, we have

$$\left|\frac{(x-a)^{\alpha} f(a)+(b-x)^{\alpha} f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha} f(a)+J_{x^+}^{\alpha} f(b)\right]\right|$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\left|t^{\alpha}-1\right|\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1\left|1-t^{\alpha}\right|\left|f'(tx+(1-t)b)\right|dt$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\left(1-t^{\alpha}\right)\left[t^s\left|f'(x)\right|+(1-t)^s\left|f'(a)\right|\right]dt$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1\left(1-t^{\alpha}\right)\left[t^s\left|f'(x)\right|+(1-t)^s\left|f'(b)\right|\right]dt$$

$$=\frac{(x-a)^{\alpha+1}}{b-a}\left\{\int_0^1\left(1-t^{\alpha}\right)t^s\left|f'(x)\right|dt+\int_0^1\left(1-t^{\alpha}\right)(1-t)^s\left|f'(a)\right|dt\right\}$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\left\{\int_0^1\left(1-t^{\alpha}\right)t^s\left|f'(x)\right|dt+\int_0^1\left(1-t^{\alpha}\right)(1-t)^s\left|f'(b)\right|dt\right\}$$

$$=\frac{\alpha}{(s+1)(\alpha+s+1)}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]\left|f'(x)\right|$$

$$+\left[\frac{1}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]\left[\frac{(x-a)^{\alpha+1}\left|f'(a)\right|+(b-x)^{\alpha+1}\left|f'(b)\right|}{b-a}\right].$$

We have used the facts that

$$\int_0^1\left(1-t^{\alpha}\right)t^s dt=\frac{\alpha}{(s+1)(\alpha+s+1)}$$

and

$$\int_0^1\left(1-t^{\alpha}\right)(1-t)^s dt=\left[\frac{1}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]$$

where β is Euler Beta function defined by

$$\beta(x,y)=\int_0^1 t^x(1-t)^y dt,\quad x,y>0$$

and

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The proof is completed.

Remark 1. *In Theorem 7, if we choose* $\alpha=1$, *we get the inequality in (1.2).*

Theorem 8. *Let* $f:I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $|f'|^{q}$ *is* $s-$*convex on* $[a,b]$ *for some fixed* $s\in(0,1]$, $p,q>1$, $x\in[a,b]$, *then the following inequality for fractional integrals holds:*

$$\left|\frac{(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right]\right|$$

$$\leq\left(\frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^{\alpha+1}}{b-a}\left(\frac{|f'(x)|^{q}+|f'(a)|^{q}}{s+1}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{(b-x)^{\alpha+1}}{b-a}\left(\frac{|f'(x)|^{q}+|f'(b)|^{q}}{s+1}\right)^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$ and Γ is Euler Gamma function.

Proof. From Lemma 2, property of the modulus and using the Hölder inequality we have

$$\left|\frac{(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right]\right|$$

$$\leq\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\left|t^{\alpha}-1\right|\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1\left|1-t^{\alpha}\right|\left|f'(tx+(1-t)b)\right|dt$$

$$\leq\frac{(x-a)^{\alpha+1}}{b-a}\left\{\left(\int_0^1(1-t^{\alpha})^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}\right\}$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\left\{\left(\int_0^1(1-t^{\alpha})^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f'\right|^q$ is $s-$convex on $[a,b]$, we get

$$\int_0^1 \left|f'(tx+(1-t)a)\right|^q dt \le \frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{s+1},$$

$$\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt \le \frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1}$$

and by simple computation

$$\int_0^1 (1-t^\alpha)^p dt = \frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)}.$$

Hence we have

$$\left|\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(a)+J_{x^+}^\alpha f(b)\right]\right|$$

$$\le \frac{(x-a)^{\alpha+1}}{b-a}\left(\frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)}\right)^{\frac{1}{p}}\left(\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\left(\frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)}\right)^{\frac{1}{p}}\left(\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

which completes the proof.

Remark 2. *In Theorem 8, if we choose* $\alpha=1$, *we get the inequality in (1.3).*

Corollary 1. *In Theorem 8, if we choose* $x=\frac{a+b}{2}$, *we obtain the following inequality:*

$$\left|(b-a)^{\alpha-1}\frac{f(a)+f(b)}{2^\alpha}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{\frac{a+b}{2}^-}^\alpha f(a)+J_{\frac{a+b}{2}^+}^\alpha f(b)\right]\right|$$

$$\leq \left(\frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)} \right)^{\frac{1}{p}} \frac{(b-a)^{\alpha}}{2^{\alpha+1}}$$

$$\times \left\{ \left(\frac{\left| f'\left(\frac{a+b}{2}\right)\right|^{q} + \left| f'(a)\right|^{q}}{s+1} \right)^{\frac{1}{q}} + \left(\frac{\left| f'\left(\frac{a+b}{2}\right)\right|^{q} + \left| f'(b)\right|^{q}}{s+1} \right)^{\frac{1}{q}} \right\}.$$

Theorem 9. *Let $f{:}I\subset[0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b\in I$ with $a<b$. If $\left|f'\right|^{q}$ is $s-$convex on $[a,b]$ for some fixed $s\in(0,1]$, $q\geq 1$, $x\in[a,b]$, then the following inequality for fractional integrals holds:*

$$\left| \frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x^-} f(a) + J^{\alpha}_{x^+} f(b) \right] \right|$$

$$\leq \left(\frac{\alpha}{\alpha+1} \right)^{1-\frac{1}{q}}$$

$$\times \left\{ \frac{(x-a)^{\alpha+1}}{b-a} \left(\frac{\alpha}{(s+1)(\alpha+s+1)} \left| f'(x)\right|^{q} + \left[\frac{1}{s+1} - \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)} \right] \left| f'(a)\right|^{q} \right)^{\frac{1}{q}} \right.$$

$$\left. + \frac{(b-x)^{\alpha+1}}{b-a} \left(\frac{\alpha}{(s+1)(\alpha+s+1)} \left| f'(x)\right|^{q} + \left[\frac{1}{s+1} - \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)} \right] \left| f'(b)\right|^{q} \right)^{\frac{1}{q}} \right\}$$

where $\alpha>0$ and Γ is Euler Gamma function.

Proof. From Lemma 2, property of the modulus and using the power-mean inequality we have

$$\left| \frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x^-} f(a) + J^{\alpha}_{x^+} f(b) \right] \right| \tag{2.3}$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \left| t^{\alpha}-1 \right| \left| f'(tx+(1-t)a) \right| dt + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 \left| 1-t^{\alpha} \right| \left| f'(tx+(1-t)b) \right| dt$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\left\{\left(\int_0^1 (1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t^{\alpha})\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\left\{\left(\int_0^1 (1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t^{\alpha})\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f'\right|^q$ is $s-$ convex on $[a,b]$, we get

$$\int_0^1 (1-t^{\alpha})\left|f'(tx+(1-t)a)\right|^q dt \tag{2.4}$$

$$\leq \int_0^1 (1-t^{\alpha})\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q\right]$$

$$=\frac{\alpha}{(s+1)(\alpha+s+1)}\left|f'(x)\right|^q+\left[\frac{1}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]\left|f'(a)\right|^q$$

and

$$\int_0^1 (1-t^{\alpha})\left|f'(tx+(1-t)b)\right|^q dt \tag{2.5}$$

$$\leq \int_0^1 (1-t^{\alpha})\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]$$

$$=\frac{\alpha}{(s+1)(\alpha+s+1)}\left|f'(x)\right|^q+\left[\frac{1}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]\left|f'(a)\right|^q.$$

If we use (2.4) and (2.5) in (2.3), we obtain the desired result.

Remark 3. *In Theorem 9, if we choose* $\alpha=1$, *we get the inequality in (1.4).*

Theorem 10. *Let* $f:I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $\left|f'\right|^q$ *is* $s-$*concave on* $[a,b]$ *for some fixed* $s\in(0,1]$, $q>1$, $x\in[a,b]$, *then the following inequality for fractional integrals holds:*

$$\left|\frac{(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right]\right|$$

$$\leq \left(\frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)} \right)^{\frac{1}{p}} \frac{2^{\frac{s-1}{q}}}{b-a} \times \left\{ (x-a)^{\alpha+1} \left| f'\left(\frac{x+a}{2}\right) \right| + (b-x)^{\alpha+1} \left| f'\left(\frac{x+b}{2}\right) \right| \right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$ and Γ is Euler Gamma function.

Proof. From Lemma 2, property of the modulus and using the Hölder inequality we have

$$\left| \frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a} \left[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b) \right] \right| \tag{2.6}$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \left| t^{\alpha} - 1 \right| \left| f'(tx+(1-t)a) \right| dt + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 \left| 1-t^{\alpha} \right| \left| f'(tx+(1-t)b) \right| dt$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a} \left\{ \left(\int_0^1 (1-t^{\alpha})^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(tx+(1-t)a) \right|^q dt \right)^{\frac{1}{q}} \right\}$$

$$+ \frac{(b-x)^{\alpha+1}}{b-a} \left\{ \left(\int_0^1 (1-t^{\alpha})^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(tx+(1-t)b) \right|^q dt \right)^{\frac{1}{q}} \right\}.$$

Since $\left| f' \right|^q$ is $s-$concave on $[a,b]$, using the inequality (1.1), we have

$$\int_0^1 \left| f'(tx+(1-t)a) \right|^q dt \leq 2^{s-1} \left| f'\left(\frac{x+a}{2}\right) \right|^q \tag{2.7}$$

and

$$\int_0^1 \left| f'(tx+(1-t)b) \right|^q dt \leq 2^{s-1} \left| f'\left(\frac{x+b}{2}\right) \right|^q . \tag{2.8}$$

From (2.6)-(2.8), we complete the proof.

Remark 4. *In Theorem 10, if we choose* $\alpha=1$, *we get the inequality in (1.5).*

3. *Applications for P.D.F.'s*

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ with the cumulative distribution function $F(x)=Pr(X\leq x)=\int_a^b f(t)dt$.

Proposition 1. *With the assumptions of Theorem 7 with* $\alpha=1$, *we have the inequality*

$$\left|\frac{(b-x)F(b)+(x-a)F(a)}{b-a}-\frac{b-E(X)}{b-a}\right|$$

$$\leq\frac{1}{(s+1)(s+2)}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left|F^{'}(x)\right|+\frac{1}{s+2}\left[\frac{(x-a)^2\left|F^{'}(a)\right|+(b-x)^2\left|F^{'}(b)\right|}{b-a}\right]$$

for all $x\in[a,b]$ and $E(X)$ is the expectation of X where

$$E(X)=\int_a^b tdF(t)=b-\int_a^b F(t)dt.$$

Proof. If we write the inequality in Theorem 7 with $\alpha=1$ for F, we get the desired result.

Proposition 2. *With the assumptions of Theorem 8 with* $\alpha=1$, *we have the inequality*

$$\left|\frac{(b-x)F(b)+(x-a)F(a)}{b-a}-\frac{b-E(X)}{b-a}\right|$$

$$\leq\frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{\left|F^{'}(x)\right|^q+\left|F^{'}(a)\right|^q}{s+1}\right]^{\frac{1}{q}}+\frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{\left|F^{'}(x)\right|^q+\left|F^{'}(b)\right|^q}{s+1}\right]^{\frac{1}{q}}$$

for all $x\in[a,b]$ and $E(X)$ is the expectation of X.

Proof. If we write the inequality in Theorem 8 with $\alpha=1$ for F, we get the desired result.

Proposition 3. *With the assumptions of Theorem 9 with* $\alpha=1$, *we have the inequality*

$$\left|\frac{(b-x)F(b)+(x-a)F(a)}{b-a}-\frac{b-E(X)}{b-a}\right|$$

$$\leq\frac{(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left|F^{'}(x)\right|^q\frac{1}{(s+1)(s+2)}+\left|F^{'}(a)\right|^q\frac{1}{s+2}\right]^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left|F^{'}(x)\right|^q\frac{1}{(s+1)(s+2)}+\left|F^{'}(b)\right|^q\frac{1}{s+2}\right]^{\frac{1}{q}}$$

for all $x\in[a,b]$ and $E(X)$ is the expectation of X.

Proof. If we write the inequality in Theorem 9 with $\alpha = 1$ for F, we get the desired result.

Proposition 4. *With the assumptions of Theorem 10 with* $\alpha = 1$, *we have the inequality*

$$\left|\frac{(b-x)F(b)+(x-a)F(a)}{b-a}-\frac{b-E(X)}{b-a}\right|$$

$$\leq \frac{2^{\frac{s-1}{q}}}{(1+p)^{\frac{1}{p}}(b-a)}\left\{(x-a)^2\left|F'\left(\frac{x+a}{2}\right)\right|+(b-x)^2\left|F'\left(\frac{x+b}{2}\right)\right|\right\}$$

for all $x\in[a,b]$ and $E(X)$ is the expectation of X.

Proof. If we write the inequality in Theorem 10 with $\alpha = 1$ for F, we get the desired result.

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Koordinatlarda h −Konveks İki Fonksiyonun Çarpımı İçin Bazı Hermite-Hadamard Tipli Eşitsizlikler Üzerine

On Some Hadamard-Type Inequalities for Product of Two h −Convex Functions On the Co-ordinates

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Öz: Bu çalışmada, koordinatlarda h-konveks fonksiyonların çarpımı için Hadamard tipli eşitsizlikler oluşturulmuştur. Elde edilen sonuçlar literatürde bazı iyi bilinen sonuçları genelleştirmiştir.

Anahtar Kelimeler — Koordinatlar, Hadamard eşitsizliği, h-konveks fonksiyonlar.

Abstract: In this paper, Hadamard-type inequalities for product of h-convex functions on the co-ordinates on the rectangle from the plane are established. Obtained results generalize the corresponding to some well-known results given before now.

Keywords — co-ordinates, Hadamard's inequality, h-convex functions

1.Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a,b \in I$ with $a < b$. Then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hadamard's inequality for convex mapping. For particular choice of the function f in (1.1) yields some classical inequalities of means.

Definition 1. *(See [11]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to Godunova-Levin function or f is said to belong to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and for $\alpha \in (0,1)$ we have the inequality:*

$$f(\alpha x + (1-\alpha)y) \le \frac{f(x)}{\alpha} + \frac{f(y)}{1-\alpha}.$$

The class $Q(I)$ was firstly described in [11] by Godunova-Levin. Some further properties of it can be found in [10], [15] and [16]. Among others, it is noted that non-negative monotone and non-negative convex functions belongs to this class of functions. In [6], Breckner introduced $s-$convex functions as a generalization of convex functions. In [7], he proved the important fact that the set-valued map is $s-$convex only if associated support function is $s-$convex. A number of properties and connections with $s-$convexity in the first sense are discussed in paper [12]. It is clear that $s-$convexity is merely convexity for $s=1$.

Definition 2. *(See [6]) Let* $s\in(0,1]$ *be fixed real number. A function* $f:[0,\infty)\to[0,\infty)$ *is said to be* $s-$*convex in the second sense, or that* f *belongs to the class* K_s^2, *if*

$$f(\alpha x+(1-\alpha)y)\leq\alpha^s f(x)+(1-\alpha)^s f(y)$$

for all $x,y\in[0,\infty)$ and $\alpha\in[0,1]$.

Definition 3. *(See [10]) A function* $f:I\subseteq\mathbb{R}\to\mathbb{R}$ *is said to be* $P-$*function or that* f *is said to belong to the class* $P(I)$ *if* f *is non-negative and for all* $x,y\in I$ *and* $\alpha\in[0,1]$, *if*

$$f(\alpha x+(1-\alpha)y)\leq f(x)+f(y).$$

In [9], Dragomir and Fitzpatrick proved the following variant of Hadamard's inequality which holds for $s-$convex function in the second sense:

Theorem 1. *Suppose that* $f:[0,\infty)\to[0,\infty)$ *is an* $s-$*convex function in the second sense, where* $s\in(0,1)$ *and let* $a,b\in[0,\infty)$, $a<b$. *If* $f\in L_1([a,b])$ *then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{s+1} \tag{1.2}$$

The constant $k=\dfrac{1}{s+1}$ is the best possible in the second inequality in (1.2).

In [9], Dragomir and Fitzpatrick also proved the following Hadamard-type inequality which holds for $s-$convex functions in the first sense:

Theorem 2. *Suppose that* $f:[0,\infty)\to[0,\infty)$ *is an* $s-$*convex function in the first sense, where* $s\in(0,1)$ *and let* $a,b\in[0,\infty)$. *If* $f\in L_1([a,b])$ *then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+sf(b)}{s+1} \tag{1.3}$$

The above inequalities are sharp.

A modification for convex functions which is also known as co-ordinated convex(concave) functions was introduced by Dragomir in [8] as following:

Let us now consider a bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. A mapping $f:\Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(\alpha x+(1-\alpha)z, \alpha y+(1-\alpha)w) \le \alpha f(x,y)+(1-\alpha)f(z,w)$$

holds, for all $(x,y),(z,w) \in \Delta$ and $\alpha \in [0,1]$. If the inequality reversed then f is said to be concave on Δ. A function $f:\Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b] \to \mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d] \to \mathbb{R}$, $f_x(v)=f(x,v)$ are convex where defined for all $x \in a,b]$, $y \in c,d]$.

A formal definition for co-ordinated convex functions may be stated as follow [see [23]]:

Definition 4. *A function* $f:\Delta \to \mathbb{R}$ *is said to be convex on the co-ordinates on* Δ *if the following inequality:*

$$f(tx+(1-t)y, su+(1-s)w) \le tsf(x,u)+t(1-s)f(x,w)+s(1-t)f(y,u)+(1-t)(1-s)f(y,w)$$

holds for all $t,s \in 0,1]$ and $(x,u),(x,w),(y,u),(y,w) \in \Delta$.

Clearly, every convex mapping $f:\Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex. In [8], Dragomir established the following inequalities of Hadamard's type for convex functions on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 3. *Suppose* $f:\Delta \to \mathbb{R}$ *is convex function on the co-ordinates on* Δ. *Then one has the inequalities:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \quad (1.4)$$

$$\le \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}$$

In [1], Alomari and Darus proved the following inequalities of Hadamard-type as above for $s-$ convex functions in the second sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 4. *Suppose* $f:\Delta\to\mathbb{R}$ *is* $s-$*convex function (in the second sense) on the co-ordinates on* Δ. *Then one has the inequalities:*

$$4^{s-1}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \tag{1.5}$$

$$\le\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{(s+1)^2}$$

Also in [4] (see also [5]), Alomari and Darus established the following inequalities of Hadamard-type similar to (1.5) for $s-$convex functions in the first sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 5. *Suppose* $f:\Delta\to\mathbb{R}$ *is* $s-$*convex function on the co-ordinates on* Δ *in the first sense. Then one has the inequalities:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \tag{1.6}$$

$$\le\frac{f(a,c)+sf(b,c)+sf(a,d)+s^2f(b,d)}{(s+1)^2}$$

The above inequalities are sharp.

For refinements, counterparts, generalizations and new Hadamard-type inequalities see the papers [1, 2, 3, 4, 5, 8, 9, 10, 12, 21, 22, 23, 24].

In [17], Pachpatte established two Hadamard-type inequalities for product of convex functions. An analogous results for $s-$convex functions is due to Kırmacı *et al.* [13].

Theorem 6. *Let* $f,g:[a,b]\subset R\to[0,\infty)$ *be convex functions on* $[a,b]$, $a<b$. *Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\le\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b) \tag{1.7}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b) \tag{1.8}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Theorem 7. *Let* $f,g:[a,b]\subseteq \mathbb{R}\to\mathbb{R}$ $a,b\in[a,b]$, $a<b$, *be functions such that* g *and* fg *are in* $L_1([a,b])$. *If* f *is convex and non-negative on* $[a,b]$ *and if* g *is* s*-convex on* $[a,b]$ *for some* $s\in(0,1)$. *Then*

$$2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \tag{1.9}$$

$$\le \frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{(s+1)(s+2)}M(a,b)+\frac{1}{s+2}N(a,b)$$

and

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\le \frac{1}{s+2}M(a,b)+\frac{1}{(s+1)(s+2)}N(a,b) \tag{1.10}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

The class of $h-$convex functions was introduced by S. Varosanec in [19] (see [19] for further properties of $h-$convex functions).

Definition 5. *Let* $h:J\subseteq\mathbb{R}\to\mathbb{R}$, *where* $(0,1)\subseteq J$, *be a positive function. A function* $f:I\subseteq\mathbb{R}\to\mathbb{R}$ *is said to be* $h-$*convex or that* f *is said to belong to the class* $SX(h,I)$, *if* f *is non-negative and for all* $x,y\in I$ *and* $\alpha\in(0,1)$, *we have*

$$f(\alpha x+(1-\alpha)y)\le h(\alpha)f(x)+h(1-\alpha)f(y)$$

if the inequality is reversed then f is said to be $h-$concave and we say that f belongs to the class $SV(h,I)$.

Remark 1. *Obviously, if* $h(\alpha)=\alpha$, *then all the non-negative convex functions belong to the class* $SX(h,I)$ *and all non-negative concave functions belong to the class* $SV(h,I)$. *Also note that if* $h(\alpha)=\frac{1}{\alpha}$, *then* $SX(h,I)=Q(I)$; *if* $h(\alpha)=1$, *then* $SX(h,I)\supseteq P(I)$; *and if* $h(\alpha)=\alpha^s$, *where* $s\in(0,1)$, *then* $SX(h,I)\supseteq K_s^2$.

In [18], Sarıkaya *et al.* established the following inequalities of Hadamard's type for product of $h-$convex functions.

Theorem 8. *Let* $f\in SX(h_1,I)$, $g\in SX(h_2,I)$, $a,b\in I$, $a<b$, *be functions such that* $fg\in L_1([a,b])$ *and* $h_1h_2\in L_1([0,1])$, *then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le M(a,b)\int_0^1 h_1(t)h_2(t)dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)dt \tag{1.11}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 9. *Let $f \in SX(h_1, I)$, $g \in SX(h_2, I)$, $a,b \in I$, $a < b$, be functions such that $fg \in L_1([a,b])$ and $h_1h_2 \in L_1([0,1])$, then*

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)dx \tag{1.12}$$

$$\le M(a,b)\int_0^1 h_1(t)h_2(1-t)dt + N(a,b)\int_0^1 h_1(t)h_2(t)dt$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

In [20], Sarıkaya *et al.* established the following inequality of Hadamard's type which involving $h-$ convex functions:

Theorem 10. *Let $f \in SX(h, I)$, $a,b \in I$ with $a < b$, $f \in L_1([a,b])$ and $g:[a,b] \to \mathbb{R}$ is non-negative, integrable and symmetric about $\frac{a+b}{2}$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b \left(h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right)\right)g(x)dx. \tag{1.13}$$

In [14], authors proved the following results for product of two convex functions on the co-ordinates on rectagle from the plane $\mathbb{R}^2$.

Theorem 11. *Let $f, g:[a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be convex functions on the co-ordinates on Δ with $a < b, c < d$. Then*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \le \frac{1}{9}L(a,b,c,d) + \frac{1}{18}M(a,b,c,d) + \frac{1}{36}N(a,b,c,d) \tag{1.14}$$

where

$$L(a,b,c,d) = f(a,c)g(a,c) + f(b,c)g(b,c) + f(a,d)g(a,d) + f(b,d)g(b,d)$$

$$M(a,b,c,d) = f(a,c)g(a,d) + f(a,d)g(a,c) + f(b,c)g(b,d) + f(b,d)g(b,c)$$

$$+ f(b,c)g(a,c) + f(b,d)g(a,d) + f(a,c)g(b,c) + f(a,d)g(b,d)$$

$$N(a,b,c,d) = f(b,c)g(a,d) + f(b,d)g(a,c) + f(a,c)g(b,d) + f(a,d)g(b,c)$$

Theorem 12. *Let* $f, g : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ *be convex functions on the co-ordinates on* Δ *with* $a < b, c < d$ *. Then*

$$4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \tag{1.15}$$

$$+\frac{5}{36}L(a,b,c,d)+\frac{7}{36}M(a,b,c,d)+\frac{2}{9}N(a,b,c,d)$$

where $L(a,b,c,d), M(a,b,c,d)$, and $N(a,b,c,d)$ as in Theorem 10.

Similar to definition of co-ordinated convex functions Latif and Alomari gave the notion of $h-$ convexity of a function f on a rectangle from the plane $\mathbb{R}^2$ and $h-$convexity on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$ in [23], as follows:

Definition 6. *(See [23]) Let us consider a bidimensional interval* $\Delta = [a,b] \times [c,d]$ *in* $\mathbb{R}^2$ *with* $a < b$ *and* $c < d$. *Let* $h : J \subseteq \mathbb{R} \to \mathbb{R}$, *where* $(0,1) \subseteq J$ *, be a positive function. A mapping* $f : \Delta \to R$ *is said to be* $h-$*convex on* Δ, *if* f *is non-negative and if the following inequality:*

$$f(\alpha x+(1-\alpha)z, \alpha y+(1-\alpha)w) \le h(\alpha)f(x,y)+h(1-\alpha)f(z,w)$$

holds, for all $(x,y),(z,w) \in \Delta$ and $\alpha \in (0,1)$. Let us denote this class of functions by $SX(h,\Delta)$. The function f is said to be $h-$concave if the inequality reversed. We denote this class of functions by $SV(h,\Delta)$.

A function $f : \Delta \to \mathbb{R}$ is said to be $h-$convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are $h-$convex where defined for all $x \in a,b]$, $y \in c,d]$. A formal definition of $h-$convex functions may also be stated as follows:

Definition 7. *(See [23]) A function* $f : \Delta \to \mathbb{R}$ *is said to be* $h-$*convex on the co-ordinates on* Δ, *if the following inequality:*

$$f(tx+(1-t)y, su+(1-s)w) \le h(t)h(s)f(x,u)+h(t)h(1-s)f(x,w)$$

$$+h(s)h(1-t)f(y,u)+h(1-t)h(1-s)f(y,w)$$

holds for all $t, s \in [0,1]$ and $(x,u),(x,w),(y,u),(y,w) \in \Delta$.

Lemma 1. *(See [23]) Every $h-$convex mapping $f:\Delta\to\mathbb{R}$ is $h-$convex on the co-ordinates, but the converse is not generally true.*

The main purpose of the present paper is to establish new Hadamard-type inequalities like those given above in the Theorem 11-12, but now for product of two $h-$convex functions on the co-ordinates on rectangle from the plane $\mathbb{R}^2$.

2. *Main Results*

In this section we establish some Hadamard's type inequalities for product of two $h-$convex functions on the co-ordinates on rectangle from the plane. In the sequel of the paper h_1 and h_2 are positive functions defined on J, where $(0,1)\subseteq J\subseteq\mathbb{R}$ and f and g are non-negative functions defined on $\Delta=[a,b]\times[c,d]$.

Theorem 13. *Let $f,g:\Delta=[a,b]\times[c,d]\to R$ where $a<b$ and $c<d$, be functions such that $fg\in L^2(\Delta)$, $h_1h_2\in L_1[0,1])$. If f is h_1-convex on the co-ordinates on Δ and if g is h_2-convex on the co-ordinates on Δ, then*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \tag{2.1}$$

$$\leq p^2L(a,b,c,d)+pqM(a,b,c,d)+q^2N(a,b,c,d)$$

where $L(a,b,c,d)$, $M(a,b,c,d)$, $N(a,b,c,d)$ as in Theorem 10 and $p=\int_0^1 h_1(t)h_2(t)dt$ and $q=\int_0^1 h_1(t)h_2(1-t)dt$.

Proof. Since $f,g:\Delta=[a,b]\times[c,d]\to R$ be functions such that $fg\in L^2(\Delta)$ and f is h_1-convex on the co-ordinates on Δ and g is h_2-convex on the co-ordinates on Δ, therefore the partial mappings

$$f_y:[a,b]\to\mathbb{R},\ f_y(x)=f(x,y)$$

$$g_y:[a,b]\to\mathbb{R},\ g_y(x)=g(x,y)$$

and

$$f_x:[c,d]\to\mathbb{R},\ f_x(y)=f(x,y)$$

$$g_x:[c,d]\to\mathbb{R},\ g_x(y)=g(x,y)$$

are h_1-, h_2-convex on $[a,b]$ and $[c,d]$, respectively, for all $x \in [a,b]$ and $y \in [c,d]$,. Now by applying (1.11) to $f_x(y)g_x(y)$ on $[c,d]$ we get

$$\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy \le p[f_x(c)g_x(c)+f_x(d)g_x(d)]+q[f_x(c)g_x(d)+f_x(d)g_x(c)].$$

That is

$$\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy \le p[f(x,c)g(x,c)+f(x,d)g(x,d)]+q[f(x,c)g(x,d)+f(x,d)g(x,c)].$$

Integrating over $[a,b]$ and dividing both sides by $b-a$, we have

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \tag{2.2}$$

$$\le p\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx\right]$$

$$+q\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx\right].$$

Now by applying (1.11) to each integral on R.H.S of (2.2) again, we get

$$\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx \le p[f(a,c)g(a,c)+f(b,c)g(b,c)]+q[f(a,c)g(b,c)+f(b,c)g(a,c)]$$

$$\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx \le p[f(a,d)g(a,d)+f(b,d)g(b,d)]+q[f(a,d)g(b,d)+f(b,d)g(a,d)]$$

$$\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx \le p[f(a,c)g(a,d)+f(b,c)g(b,d)]+q[f(a,c)g(b,d)+f(b,c)g(a,d)]$$

$$\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx \le p[f(a,d)g(a,c)+f(b,d)g(b,c)]+q[f(a,d)g(b,c)+f(b,d)g(a,c)].$$

On substitution of these inequalities in (2.2) yields

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$\le p^2[f(a,c)g(a,c)+f(b,c)g(b,c)]+pq[f(a,c)g(b,c)+f(b,c)g(a,c)]$$

$$+p^2[f(a,d)g(a,d)+f(b,d)g(b,d)]+pq[f(a,d)g(b,d)+f(b,d)g(a,d)]$$

$$+pq\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right]+q^2\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right]$$

$$+pq\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right]+q^2\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right]$$

$$=p^2L(a,b,c,d)+pqM(a,b,c,d)+q^2N(a,b,c,d).$$

This completes the proof.

Remark 2. *If we take* $h_1(t)=h_2(t)=t$, *then inequality (2.1) reduces to the inequality (1.14).*

Theorem 14. *Let* $f,g:\Delta=[a,b]\times[c,d]\to R$ *where* $a<b$ *and* $c<d$, *be functions such that* $fg\in L^2(\Delta)$, $h_1h_2\in L_1[0,1])$. *If* f *is* h_1-*convex on the co-ordinates on* Δ *and if* g *is* h_2-*convex on the co-ordinates on* Δ, *then*

$$\frac{1}{4h_1^2(\frac{1}{2})h_2^2(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{2.3}$$

$$\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$

$$+(q^2+2pq)L(a,b,c,d)+(p^2+pq+q^2)M(a,b,c,d)+(p^2+2pq)N(a,b,c,d)$$

where $L(a,b,c,d),M(a,b,c,d),$ and $N(a,b,c,d)$ as in Theorem 10 and $p=\int_0^1 h_1(t)h_2(t)dt$ and $q=\int_0^1 h_1(t)h_2(1-t)dt$.

Proof. Now applying (1.12) to $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$, we get

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{2.4}$$

$$\leq\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx$$

$$+q\left[f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\right]$$

$$+p\left[f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\right]$$

and

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{2.5}$$

$$\leq \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy$$

$$+q\left[f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\right]$$

$$+p\left[f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)+f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)\right].$$

Adding (2.4) and (2.5) and multiplying both sides by $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}$, we get

$$\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{2.6}$$

$$\leq \frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx$$

$$+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy$$

$$+q\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\right]$$

$$+p\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\right]$$

$$+q\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\right]$$

$$+p\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)\right].$$

Applying (1.12) to each term within the brackets, we have

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy$$

$$+q[f(a,c)g(a,c)+f(a,d)g(a,d)]+p[f(a,c)g(a,d)+f(a,d)g(a,c)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy$$

$$+q[f(b,c)g(b,c)+f(b,d)g(b,d)]+p[f(b,c)g(b,d)+f(b,d)g(b,c)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy$$

$$+q[f(a,c)g(b,c)+f(a,d)g(b,d)]+p[f(a,c)g(b,d)+f(a,d)g(b,c)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy$$

$$+q[f(b,c)g(a,c)+f(b,d)g(a,d)]+p[f(b,c)g(a,d)+f(b,d)g(a,c)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)\leq\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx$$

$$+q[f(a,c)g(a,c)+f(b,c)g(b,c)]+p[f(a,c)g(b,c)+f(b,c)g(a,c)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\leq\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx$$

$$+q[f(a,d)g(a,d)+f(b,d)g(b,d)]+p[f(a,d)g(b,d)+f(b,d)g(a,d)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)\leq\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx$$

$$+q[f(a,c)g(a,d)+f(b,c)g(b,d)]+p[f(a,c)g(b,d)+f(b,c)g(a,d)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)\leq\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx$$

$$+q[f(a,d)g(a,c)+f(b,d)g(b,c)]+p[f(a,d)g(b,c)+f(b,d)g(a,c)].$$

Substituting these inequalities in (2.6) and simplifying we have;

$$\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{2.7}$$

$$\leq\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx$$

$$+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy$$

$$+q\frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy+q\frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy$$

$$+p\frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy+p\frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy$$

$$+q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx$$

$$+p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx$$

$$+2q^2L(a,b,c,d)+2pqM(a,b,c,d)+p^2N(a,b,c,d)$$

Now by applying (1.12) to $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)$, integrating over $[c,d]$ and dividing both sides by $d-c$, we get

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy \tag{2.8}$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy$$

$$+q\frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy+q\frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy$$

$$+p\frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy+p\frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy$$

Now again by applying (1.12) to $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)$, integrating over $[a,b]$ and dividing both sides by $b-a$, we get

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx \tag{2.9}$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$+q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx$$

$$+p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx.$$

Adding (2.8) and (2.9), we have

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_c^d f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx \tag{2.10}$$

$$+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy$$

$$\leq \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$+q\frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy+q\frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy$$

$$+p\frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy+p\frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy$$

$$+q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx$$

$$+p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx$$

Therefore from (2.7) and (2.10), we get

$$\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\leq\frac{2}{(b-a)(d-c)}\int_a^b f(x,y)g(x,y)dxdy$$

$$+2q\frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy+2q\frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy$$

$$+2p\frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy+2p\frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy$$

$$+2q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+2q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx$$

$$+2p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+2p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx$$

$$+2q^2L(a,b,c,d)+2pqM(a,b,c,d)+2p^2N(a,b,c,d)$$

By using (1.11) to each of the above integral and simplifying, we get

$$\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\leq\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$+(2q^2+4pq)L(a,b,c,d)+(2p^2+2pq+2q^2)M(a,b,c,d)+(2p^2+4pq)N(a,b,c,d)$$

Dividing both sides by 2;

$$\frac{1}{\left[2h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b f(x,y)g(x,y)dxdy$$

$$+(q^2+2pq)L(a,b,c,d)+(p^2+pq+q^2)M(a,b,c,d)+(p^2+2pq)N(a,b,c,d)$$

This completes the proof of the theorem.

Remark 3. *If we take* $h_1(t)=h_2(t)=t$, *then inequality (2.3) reduces to the inequality (1.15).*

Theorem 15. *Suppose that all the assumptions of Theorem 12 are satisfied, if* g_x *and* g_y *are symmetric about* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$, *respectively, with* $h_1=h_2=h$, *then one has the inequality;*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$\leq \frac{1}{4(b-a)}\int_a^b\int_c^d\left[f(x,c)+f(x,d)\right]\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g(x,y)dydx$$

$$+\frac{1}{4(d-c)}\int_c^d\int_a^b\left[f(a,y)+f(b,y)\right]\left(h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right)g(x,y)dxdy$$

Proof. Since the partial mappings f_x and g_x are $h-$convex, by applying to the inequality (1.13), we can write

$$\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy \leq \frac{f_x(c)+f_x(d)}{2}\int_c^d\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g_x(y)dy.$$

That is;

$$\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy \leq \frac{f(x,c)+f(x,d)}{2}\int_c^d\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g(x,y)dy.$$

Integrating the result with respect to x on $[a,b]$ and dividing both sides of inequality, we get;

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \tag{2.11}$$

$$\leq \frac{1}{2(b-a)}\int_a^b\int_c^d\left[f(x,c)+f(x,d)\right]\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g(x,y)dydx.$$

By a similar argument f_y and g_y are $h-$convex, by applying to the inequality (1.13), we get;

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \tag{2.12}$$

$$\leq \frac{1}{2(d-c)}\int_c^d\int_a^b\left[f(a,y)+f(b,y)\right]\left(h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right)g(x,y)dxdy.$$

Summing (2.11) and (2.12), we obtain the required result.

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