Konuralp Journal of Mathematics
Volume 1 No. 1 pp. 1–7 (2013) ©KJM

CYCLIC-PARALLEL RICCI TENSOR OF ALMOST S-MANIFOLDS

YAVUZ SELIM BALKAN, NESIP AKTAN⋆ AND ERDAL ÖZÜSAĞLAM

Abstract. In this paper, we consider cyclic-parallel almost S-manifolds and we obtain some results.

1. Introduction

An extensive research about contact geometry is done in recent years. We recall the precise definitions. Let M be a $(2n+s)$-dimensional manifold. We say that M is equipped with an f-structure with a parallelizable kernel, more briefly $f.pk$-structure, if there are given on M an f-structure φ, s global vector fields $\xi_1,...,\xi_s$ and 1-forms $\eta_1,...,\eta_s$ on M satisfying the following conditions

$$\varphi\left(\xi_i\right)=0,\quad \eta_i\circ\varphi=0,\quad \varphi^2=-Id+\sum_{j=1}^{s}\eta_j\otimes\xi_j,\quad \eta_i\left(\xi_j\right)=\delta_j^i, \tag{1.1}$$

for all $i,j\in\{1,...,s\}$; we denote by D the bundle $Im\left(\varphi\right)$, and we set $\overline{\xi}:=\xi_1+...+\xi_s$, $\overline{\eta}:=\eta_1+...+\eta_s$. The structure $\left(\varphi,\xi_i,\eta_j\right)$ on M is said to be normal if and only if $N_\varphi=0$, where N_φ is the $(2,1)$-tensor on M given by $N_\varphi:=[\varphi,\varphi]+2\sum_{i=1}^{s}d\eta_i\otimes\xi_i$. On a manifold equipped with an $f.pk$-structure there always exists a compatible Riemannian metric g in the sense that for each $X,Y\in\Gamma\left(TM\right)$

$$g\left(\varphi X,\varphi Y\right)=g\left(X,Y\right)-\sum_{j=1}^{s}\eta_j\left(X\right)\eta_j\left(Y\right). \tag{1.2}$$

However such that a metric on M is not unique: we fix one of them; then the structure obtained is called a metric $f.pk$-structure. Let Φ be the Sasaki form of φ defined by $\Phi\left(X,Y\right):=g\left(X,\varphi Y\right)$ for $X,Y\in\Gamma\left(TM\right)$. It may be observed that D is the orthogonal complement of the bundle $\ker\left(\varphi\right)=\left\langle\xi_1,...,\xi_s\right\rangle$.

2000 *Mathematics Subject Classification.* 53D10, 53C15, 53C25, 53C35.
Key words and phrases. S-structures, f-structures almost S-manifolds, cyclic-parallel Ricci tensor.

⋆corresponding author.

The metric $f.pk$-manifold $(M, \varphi, \xi_i, \eta_j, g)$ is said to be an almost S-manifold if and only if $d\eta_1 = ... = d\eta_s = \Phi$. Almost S-manifold which are normal are called S-manifolds.

The study of f-manifolds was started by Blair, Goldberg, Yano, Vanzura, cf. [3], [6] [11]. Almost S-structures were studied; without being precisely named, by Cabrerizo, Fernandez, Fernandez, cf., [7]. Then Duggal, Pastore and Ianus, cf. [8], also studied such manifolds and gave them the name "almost S-manifolds". S-manifolds were introduced by Blair cf. [3], who proved that the space of a principal toroidal bundle over a Kaehler manifolds is an S-manifold. S-structures are a natural generalization of Sasakian structures, but unlike Sasakian manifolds, no S-structure can be realized on a simply connected, compact manifold cf. [5]. In [9] there is an example of an even dimensional principal toroidal bundle over a Kaehler manifold which does not carry any Sasakian structure. On the other hand, there is constructed an S-structure on the even dimensional manifold $U(2)$. It is well known that $U(2)$ does not admit a Kaehler structure. We conclude that there exist manifolds such that the best structure we can hope to obtain on them is an S-structure.

On an almost S-manifold $(M, \varphi, \xi_i, \eta_j, g)$ there are defined the $(1,1)$-tensor fields $h_i := \frac{1}{2} L_{\xi_i}\varphi$ for $i = 1, ..., s$ cf. ([7] (2.5)). We use extensively the properties of these tensor fields in the present paper. In particular these operators are self adjoint, traceless, anti-commute with φ and for each $i, j \in \{1, ..., s\}$

$$h_i\xi_j = 0, \quad \eta_i \circ h_j = 0, \tag{1.3}$$

cf. [7]. Moreover the following identities hold, cf. [8],

$$\nabla_X \xi_i = -\varphi X - \varphi h X, \quad \nabla_{\xi_i}\varphi = 0, \quad \nabla_{\xi_i}\xi_j = 0, \tag{1.4}$$

where ∇ is the Levi Civita connection of g, $X \in \Gamma(TM)$ and $i, j \in \{1, ..., s\}$. We shall sometimes use the following curvature identity related to ∇

$$R_{\xi_i X}\xi_j - \varphi\left(R_{\xi_i \varphi X}\xi_j\right) = 2\left(\left(h_i \circ h_j\right) X + \varphi^2 X\right), \tag{1.5}$$

which can be immediately obtained combining the first equation on ([7] pag. 158) and (1.4).

In 1995 Blair, Koufogiorgos and Papantonio, cf. [4], studied contact metric manifolds such that the characteristic vector field belongs to the (κ, μ)-nullity distribution. This concept is generalized for almost S-manifolds by Cappelletti Montano and Di Terlizzi in [1].

In the present paper we are concerned cyclic-parallel Ricci tensor of almost S-manifolds.

2. Preliminaries

Definition 2.1. [1] Let M be an almost S-manifold, κ, μ real constant. We say that M verifies the (κ, μ)-nullity condition if and only if for each $i \in \{1, ..., s\}$, $X, Y \in \Gamma(TM)$ the following identity holds

$$R_{XY}\xi_i = \kappa\left(\overline{\eta}(X)\varphi^2 Y - \overline{\eta}(Y)\varphi^2 X\right) + \mu\left(\overline{\eta}(Y) h_i X - \overline{\eta}(X) h_i Y\right). \tag{2.1}$$

Lemma 2.1. [1] *Let M be an almost S-manifold verifying the (κ,μ)-nullity condition. Then we have*

i) $h_i \circ h_j = h_j \circ h_i$ for each $i,j \in \{1,...,s\}$,

ii) $k \leq 1$,

iii) if $\kappa < 1$ then, for each $i \in \{1,...,s\}$, h_i has eigenvalues $0,\ \pm\sqrt{1-\kappa}$,

iv) $h_i^2 = (\kappa - 1)\varphi^2$.

Proposition 2.1. [1] *Let M be an almost S-manifold verifying the (κ,μ)-nullity condition. Then*

$$h_1 = ... = h_s. \tag{2.2}$$

Remark 2.1. [1] Throughout all this paper whenever (2.1) holds we put $h := h_1 = ... = h_s$. Then (2.1) becomes

$$R_{XY}\xi_i = \kappa\left(\overline{\eta}(X)\varphi^2 Y - \overline{\eta}(Y)\varphi^2 X\right) + \mu\left(\overline{\eta}(Y)hX - \overline{\eta}(X)hY\right). \tag{2.3}$$

Furthermore, using (2.3), the symmetries properties of the curvature tensor and the symmetry of φ^2 and h, we get

$$R_{\xi_i X}Y = \kappa\left(\overline{\eta}(Y)\varphi^2 X - g\left(X,\varphi^2 Y\right)\overline{\xi}\right) + \mu\left(g(X,hY)\overline{\xi} - \overline{\eta}(Y)hX\right). \tag{2.4}$$

Proposition 2.2. [1] *Let M be an almost S-manifold verifying the (κ,μ)-nullity condition. Then M is an S-manifold if and only if $\kappa = 1$.*

3. Properties of the Curvature

Let $\left(M^{2n+s},\varphi,\xi_i,\eta_j,g\right)$, $i,j \in \{1,...,s\}$, be an almost S-manifold. We consider the $(1,1)$-tensor fields defined by

$$l_{ij}(X) = R_{X\xi_i}\xi_j,$$

for each $i,j \in \{1,...,s\}$, $X \in \Gamma(TM)$ and put $l_i = l_{ii}$.

Lemma 3.1. [1] *Let M be an almost S-manifold. For each $i,j \in \{1,...,s\}$ the following identities hold*

$$\varphi \circ l_{ij} \circ \varphi - l_{ij} = 2\left(h_i \circ h_j + \varphi^2\right), \tag{3.1}$$

$$\eta_k \circ l_{ij} = 0, \tag{3.2}$$

$$l_{ij}(\xi_k) = 0, \tag{3.3}$$

$$\nabla_{\xi_i} h_j = \varphi - \varphi \circ l_{ij} - \varphi \circ h_i \circ h_j + \varphi \circ (h_j - h_i), \tag{3.4}$$

$$\nabla_{\xi_i} h_i = \varphi - \varphi \circ l_{ij} - \varphi \circ h_i^2. \tag{3.5}$$

Lemma 3.2. [1] *Let M be an almost S-manifold verifying the (κ,μ)-nullity condition. Then for $i,j\in\{1,...,s\}$ we have*

$$\nabla_{\xi_i}h=\mu h\circ\varphi, \tag{3.6}$$

$$l\circ\varphi-\varphi\circ l=2\mu h\circ\varphi, \tag{3.7}$$

$$l\circ\varphi+\varphi\circ l=2\kappa\varphi, \tag{3.8}$$

$$Q\xi_i=2n\kappa\overline{\xi}. \tag{3.9}$$

Lemma 3.3. [1] *Let M be an almost S-manifold verifying the (κ,μ)-nullity condition. Then the following identities hold*

$$(\nabla_X\varphi)Y=g\left(Y,hX-\varphi^2X\right)\overline{\xi}-\overline{\eta}(Y)\left(hX-\varphi^2X\right), \tag{3.10}$$

$$\begin{aligned}(\nabla_Xh)Y-(\nabla_Yh)X= &\quad (1-\kappa)\left(2g(X,\varphi Y)\overline{\xi}+\overline{\eta}(X)\varphi Y-\overline{\eta}(Y)\varphi X\right.\\ &\quad (1-\mu)\left(\overline{\eta}(X)\varphi hY-\overline{\eta}(Y)\varphi hX\right),\end{aligned} \tag{3.11}$$

$$\begin{aligned}(\nabla_Xh)Y= &\quad ((1-\kappa)g(X,\varphi Y)+g(X,h\varphi Y))\overline{\xi}\\ &\quad \overline{\eta}(Y)h(\varphi X+\varphi hX)-\mu\overline{\eta}(X)\varphi hY.\end{aligned} \tag{3.12}$$

Lemma 3.4. [1] *Let M be an almost S-manifold verifying the (κ,μ)-nullity condition with $\kappa<1$ and κ,μ are smooth function. Then the Ricci operator verifies the following identities*

$$Q=s\left[(2(1-n)+\mu n)\varphi^2+(2(n-1)+\mu)h\right]+2n\kappa\overline{\eta}\otimes\overline{\xi}, \tag{3.13}$$

$$Q\circ\varphi-\varphi\circ Q=2s(2(n-1)+\mu)h\circ\varphi. \tag{3.14}$$

Lemma 3.5. *Let M be a $(2n+s)$ dimensional almost S-manifold verifying the (κ,μ)-nullity condition, $\kappa<1$. Then*

(3.15)
$$\begin{aligned}(\nabla_XS)(Y,Z)= &\quad s(2(1-n)+\mu n)\{\overline{\eta}(Z)(g(\varphi Y,hX)-g(Y,\varphi X))\\ &\quad -\overline{\eta}(Y)(g(\varphi hX,Z)+g(\varphi X,Z))\}\\ &\quad +s(2(n-1)+\mu)\{\overline{\eta}(Z)((1-\kappa)g(\varphi Y,X)+g(h\varphi Y,X))\\ &\quad +\overline{\eta}(Y)(g(h\varphi X,Z)+(\kappa-1)g(\varphi X,Z))\\ &\quad -\mu\overline{\eta}(X)g(\varphi hY,Z)\}-2nk\overline{\eta}(Z)(g(Y,\varphi hX)+g(Y,\varphi X))\\ &\quad -2\overline{\eta}(Y)(g(\varphi hX,Z)+g(\varphi X,Z)).\end{aligned}$$

Proof. We know that the Ricci operator satisfies

$$(\nabla_X S)(Y,Z) = \nabla_X S(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{3.16}$$

Using (3.13) in (3.16) we have

(3.17)

$$\begin{aligned}(\nabla_X S)(Y,Z) = \;& s\,(2\,(1-n)+\mu n)\,(g\,(\nabla_X\varphi)\,(\varphi Y)\,,Z) + g\,(\varphi\,(\nabla_X\varphi)\,Y,Z) \\ & s\,(2\,(n-1)+\mu)\,g\,((\nabla_X h)\,Y,Z) \\ & +2n\kappa\overline{\eta}(Z)g\,(Y,\nabla_X\xi_i) + 2n\kappa\overline{\eta}\,(Y)\,g\,(Z,\nabla_X\xi_i)\,.\end{aligned}$$

By the use of (1.4), (3.10) and (3.12) in (3.17) we get (3.15). □

4. Almost S-manifolds with Cyclic-Parallel Ricci Tensor

The Ricci tensor S of Riemannian manifold M is said to be cyclic-parallel if

$$C\nabla S = 0,$$

namely

$$(\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) = 0, \tag{4.1}$$

for all vector fields X, Y, Z.

Let M be an η-Einstein manifold whose Ricci tensor S of the form

$$S(X,Y) = Ag\,(X,Y) + B\overline{\eta}(X)\overline{\eta}(Y), \tag{4.2}$$

where A, B are non-zero real numbers and X, Y are vector fields on M. So we have;

Theorem 4.1. *Let M be a $(2n+s)$ dimensional an η-Einstein almost S-manifold of the form (4.2). If the Ricci tensor S of M is cyclic parallel then M is an S-manifold.*

Proof. Let us consider M is an η-Einstein almost S-manifold of the form (4.2). If the Ricci tensor S of M is cyclic parallel then replacing Z with ξ_i in (4.1) we can write

$$(\nabla_{\xi_i} S)(X,Y) + (\nabla_X S)(Y,\xi_i) + (\nabla_Y S)(\xi_i,X) = 0. \tag{4.3}$$

Using (4.2) after some computations we get

$$(\nabla_X S)(Y,Z) = B\left[\overline{\eta}(Z)g(Y,\nabla_X\xi_i) + \overline{\eta}(Y)g(Z,\nabla_X\xi_i)\right],$$

which implies

$$(\nabla_{\xi_i} S)(X,Y) = 0, \tag{4.4}$$

$$(\nabla_X S)(Y,\xi_i) = B\left[g(Y,\nabla_X\xi_i) + \overline{\eta}(Y)g(\xi_i,\nabla_X\xi_i)\right], \tag{4.5}$$

$$(\nabla_Y S)(\xi_i,X) = B\left[g(X,\nabla_Y\xi_i) + \overline{\eta}(X)g(\xi_i,\nabla_Y\xi_i)\right]. \tag{4.6}$$

So substituting (4.4)-(4.6) in (4.3) and using (1.2) and (1.4), the equation (4.3)

$$g(\varphi X + \varphi hX, Y) + g\,(\varphi Y + \varphi hY, X) = 0, \tag{4.7}$$

which implies

$$g(\varphi Y, hX) = 0. \tag{4.8}$$

Replacing Y by φY in (4.8), we get

$$g\left(\varphi^2 Y, hX\right) = 0.$$

So by the use (1.1) and (1.3), we have

$$g(Y, hX) = 0, \tag{4.9}$$

for all vector fields X and Y and hence we have $h = 0$ which implies M is an S-manifold. □

Theorem 4.2. *Let M be a $(2n+s)$ dimensional a non-Sasakian almost S-manifold. If the Ricci tensor S of M is cyclic parallel then M is either S-manifold or $\kappa = \frac{-1}{4}\frac{s\mu^2 + s4n\mu}{n}$.*

Proof. Let M be an almost S-manifold. Then by [7], $\kappa \leq 1$. But if $\kappa = 1$ then M is Sasakian. Since we suppose M is non-Sasakian we have $\kappa < 1$. So by the use of (3.15) we have

$$(\nabla_{\xi_i} S)(X, Y) = s(2(n-1) + \mu)\mu g(hX, \varphi Y). \tag{4.10}$$

Similarly, using (3.15) we have

$$\begin{aligned}(\nabla_X S)(Y, \xi_i) = \; & (s(2(1-n) + \mu n) + s(2(n-1) + \mu)(1-\kappa) + 2n\kappa) g(X, \varphi Y) \\ & + (s(2(1-n) + \mu n) + s(2(n-1) + \mu) + 2n\kappa) g(hX, \varphi Y),\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}(\nabla_Y S)(\xi_i, X) = \; & (s(2(1-n) + \mu n) + s(2(n-1) + \mu) + 2n\kappa) g(\varphi X, hY) \\ & + (s(2(1-n) + \mu n) + s(2(n-1) + \mu)(1-\kappa) + 2n\kappa) g(\varphi X, Y).\end{aligned} \tag{4.12}$$

So substituting (4.10)- (4.12) into (4.3) we obtain

$$\left[4ns\mu + 4n\kappa + \mu^2\right] g(hX, \varphi Y) = 0. \tag{4.13}$$

Suppose $g(hX, \varphi Y) = 0$. Then replacing Y with φY, the last equation becomes $g\left(hX, \varphi^2 Y\right) = 0$. So using (1.1) we get $g(hX, Y) = 0$ for all vector fields X and Y snd hence; we have $h = 0$ which gives us M is an S-manifold (note that M is non-Sasakian since $n \neq 1$). If $4ns\mu + 4n\kappa + \mu^2 = 0$ then we get $\kappa = \frac{-1}{4}\frac{s\mu^2 + s4n\mu}{n}$. □

Corollary 4.1. *Let M be a $(2n+s)$ dimensional a non-Sasakian manifold with ξ_i, $i \in \{1, ..., s\}$, belonging to κ-nullity distribution. If M is S-manifold and the Ricci tensor S of M is cyclic parallel then M is locally isometric to the product $\mathbf{E}^{n+s} \times \mathbf{S}^n(4)$.*

Proof. Since ξ_i, $i \in \{1, ..., s\}$, belongs to κ-nullity distribution then $\mu = 0$. Hence from Theorem 2., we get $\kappa = 0$. So by [2], M is locally isometric to the product $\mathbf{E}^{n+s} \times \mathbf{S}^n(4)$. □

References

[1] B. Cappelletti Montano and L. Di Terlizzi, D-homothetic transformations for a generalization of contact metric manifolds,Bull. Belg. Math. Soc. Simon Stevin 14 (2007), 277–289.

[2] D. Blair, Contact manifolds in Riemannian Geometry, Lectures Notes in Math. Springer-Verlag, Berlin-Heidelberg-New-York, 509 (1976).

[3] D.E. Blair, Geometry of manifolds with structural group $U(n) \times O(s)$, J. Diff. Geom., 4(2) (1970), 155-167.

[4] D.E. Blair, T. Koufogiorgos, B.J. Papantoniou, Contact metric manifolds satisfying a nullity condition, Israel J. Math., 91 (1995), 189-214.

[5] G. Dileo, A. Lotta, On the structure and symmetry properties of almost S-manifolds, Geom. Dedicata 110 (2005), 191–211.

[6] J. Vanzura, Almost r-contact structures, Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. 26 (1972), 97–115.

[7] J.L. Cabrerizo, L.M. Fernandez, M. Fernandez, The curvature tensor fields on f-manifolds with complemented frames, An. Univ. 'Al.I. Cuza', Iaşi, Mat., 36 (1990), 151–161.

[8] K.L. Duggal, S. Ianus, A.M. Pastore, Maps interchanging f–structures and their harmonicity, Acta Appl. Math., 67 (1) (2001), 91-115.

[9] L. Di Terlizzi, J. Konderak, Examples of a generalization of a contact metric manifolds, Journal of Geometry, to appear.

[10] M.C. Chaki, On pseudo Ricci-symmetric manifolds, Bulgar J. Phys., 15 (1988), 526-531.

[11] S.I. Goldberg, K. Yano, On normal globally framed f-manifolds, Tohoku Math. J., 22 (1970), 362-370.

Duzce University, Faculty of Art and Sciences, Department of Mathematics, Duzce/TURKEY
E-mail address: `y.selimbalkan@gmail.com`

Duzce University, Faculty of Art and Sciences, Department of Mathematics, Duzce/TURKEY
E-mail address: `nesipaktan@gmail.com`

Aksaray University, Faculty of Art and Sciences, Department of Mathematics, Aksaray/TURKEY
E-mail address: `materdalo@gmail.com`

Konuralp Journal of Mathematics
Volume 1 No. 1 pp. 8–17 (2013) ©KJM

COMPLEX TORSIONS AND HOLOMORPHIC HELICES

F.NEJAT EKMEKCİ, PELİN POŞPOŞ, AND D. PINAR DENİZ

Abstract. Recently, properties of holomorphic helix of Kahler Frenet curves on n- dimensional M Kahler manifold studied by S. Maeda, H. Tanabe and T. Adachi. In this paper we give some characterizasions for complex torsions by $\tau_{i,j}$ in the Kahler manifold to be general helix, and by considering κ_1, κ_2 curvatures of order 3. Curvatures of Frenet curve on M Kahler manifold are not constant but their ratios are constant. We investigate relationship between $\tau_{1,2}$ and $\tau_{2,3}$ complex torsions which are not seperately constant but their ratios are constant.

1. Introduction

Let M be a n-dimentional Kahler manifold, with complex structure J and Riemannian metric g. For a helix γ on M of order $d(\leq 2n)$ with the associated Frenet frame $\{V_1, \ldots, V_d\}$ and we define $\tau_{i,j}$ called complex torsions by $\tau_{i,j} = g(V_i(s), JV_j(s))$ for $1 \leq i < j \leq d$, γ is a holomorphic helix if all the complex torsions are constant [4]. They are used curvatures κ_i and complex torsions $\tau_{i,j}$ which are constant. A classical result stated by M. A. Lancert in 1802 and first proved by B. De Saint Venant in 1845 is a necessary and sufficient condition that a curve be a general helix is the ratio of curvature of torsion to be constant [7, 8]. In a Kahler manifold, a Frenet curve is called a general helix if $\dfrac{\tau_{1,2}}{\tau_{2,3}}$ is constant and its first and second curvatures are not constant.

If its first and second curvatures are constant and its third curvature is zero then the Frenet curve is called a *helix*. We obtained the relations between the complex torsions and their own derivations.

2. Preliminaries

2.1. **Complex Torsions.** A smooth curve $\gamma = \gamma(s)$ parametrized by its arclenght s is called a *helix of proper order* d if there exist an orthonormal system $\{V_1 = \dot{\gamma}, V_2, \ldots, V_d\}$ of vector fields along γ and positive constants $\kappa_1(s), \kappa_2(s), \ldots, \kappa_{d-1}(s)$

1991 *Mathematics Subject Classification.* 53A04, 30D05, 32A20.
Key words and phrases. complex torsions, holomorphic helices, kahler manifolds.

Which satisfy the system of ordinary differential equations

$$D_{\dot{\gamma}} V_j(s) = -\kappa_{j-1}(s) V_{j-1}(s) + \kappa_j(s) V_{j+1}(s), \qquad j = 1, 2, \ldots, d$$

where $V_0 \equiv V_{d+1} \equiv 0$ and $\kappa_0 = \kappa_d = 0$ [1].

Let M be a complex n-dimensional Kahler manifold (K- manifold) with complex structure J. $\{V_1, \ldots, V_d, JV_1, \ldots, JV_d\}$ system is a basis of tangent space of M. A smooth curve $\gamma = \gamma(s)$ on M parametrized by its arclength s is called a *Kahler Frenet curve*, if it satisfies the following diferential equation

$$D_{\dot{\gamma}} \dot{\gamma} = \kappa(s) J \dot{\gamma} \quad \text{or} \quad D_{\dot{\gamma}} \dot{\gamma} = -\kappa(s) J \dot{\gamma}$$

for some positive C^{∞} function $\kappa = \kappa(s)$, where $D_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the Riemannian connection D of M [5].

For a Frenet curve γ in a K-manifold M of order d with associated Frenet frame $\{V_1, \ldots, V_d, JV_1, \ldots, JV_d\}$, we define functions $\tau_{i,j}$called *complex torsions* by

$$\tau_{i,j}(s) = \begin{cases} 0 & , i = j, i = 0, j > d \\ \langle V_i(s), JV_j(s) \rangle & , 1 \leq i < j \leq d \end{cases} \qquad , \|\tau_{i,j}(s)\| \leq 1$$

[5].

Definition 2.1. For a curve γ on a K-manifold M of order d we call a *holomophic helix* (H - helis) if all its complex torsions are constant functions.

Let a curve γ on a K-manifold M of order d. In this stuation for

$$D_{\dot{\gamma}} V_j(s) = -\kappa_{j-1}(s) V_{j-1}(s) + \kappa_j(s) V_{j+1}(s), \qquad j = 1, 2, \ldots, d \quad \text{and} \quad \tau_{i,j}(s) = \langle V_i(s), JV_j(s) \rangle$$

$$D_{\dot{\gamma}} \tau_{i,j}(s) = -\kappa_{i-1} \tau_{i-1,j}(s) + \kappa_i \tau_{i+1,j}(s) - \kappa_{j-1} \tau_{i,j-1}(s) + \kappa_j \tau_{i,j+1}(s) \tag{2.1}$$

[2].

For complex torsions of helix on K-manifold of order 3 from $d = 3$, $1 \leq i < j \leq 3$, $i = j = 0$, $\quad i = 1, 2 \quad j = 1, 2, 3$ and from (2.1) we obtain

$$D_{\dot{\gamma}} \tau_{1,2} = \kappa_2 \tau_{1,3} \quad , \quad D_{\dot{\gamma}} \tau_{1,3} = -\kappa_2 \tau_{1,2} + \kappa_1 \tau_{2,3} \quad , \quad D_{\dot{\gamma}} \tau_{2,3} = -\kappa_1 \tau_{1,3}$$

or

$$\begin{bmatrix} D_{\dot{\gamma}} \tau_{1,2} \\ D_{\dot{\gamma}} \tau_{1,3} \\ D_{\dot{\gamma}} \tau_{2,3} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & 0 \\ -\kappa_2 & 0 & \kappa_1 \\ 0 & -\kappa_1 & 0 \end{bmatrix} \begin{bmatrix} \tau_{1,2} \\ \tau_{1,3} \\ \tau_{2,3} \end{bmatrix}$$

When γ a Frenet curve on K-manifold M of order 2 and $\tau_{1,2}$ is constant. Really

for $\tau_{1,2} = \langle V_1, JV_2 \rangle$

$$D_{\dot{\gamma}} \langle V_1, JV_2 \rangle = \langle D_{\dot{\gamma}} V_1, JV_2 \rangle + \langle V_1, J D_{\dot{\gamma}} V_2 \rangle = \kappa \langle V_2, JV_2 \rangle - \kappa \langle V_1, JV_1 \rangle = 0$$

Then a Frenet curve of order 2 is a H-helix.

3. Holomorphic Helices

If we give theorems and results which they known related to holomorphic helices of order 3 and 4.

Theorem 3.1. *The complex torsions of a H-helix of proper order on a K-manifold satisfy*

$$\sum_{j=1}^{i-1} \tau_{j,i}^2 + \sum_{j=i+1}^{d} \tau_{i,j}^2 \leq 1$$

For every i [4].

We take H-helices of order3 we need to choose orthonormal vectors $\{V_1, V_2, V_3\}$ which satisfy

$$\begin{array}{ccccccc} \langle V_1, V_1 \rangle & = & \langle V_2, V_2 \rangle & = & \langle V_3, V_3 \rangle & = & 1 \\ \langle V_1, V_2 \rangle & = & \langle V_1, V_3 \rangle & = & \langle V_2, V_3 \rangle & = & 0 \\ \langle JV_1, JV_1 \rangle & = & \langle JV_2, JV_2 \rangle & = & \langle JV_3, JV_3 \rangle & = & 1 \\ \langle JV_1, JV_2 \rangle & = & \langle JV_1, JV_3 \rangle & = & \langle JV_2, JV_3 \rangle & = & 0 \end{array}$$

And then we set $V_1, V_2$and V_3 as

$$\begin{aligned} V_1 &= (1, 0, \ldots, 0) \\ V_2 &= (-i\tau, \sqrt{1-\tau^2}, 0, \ldots 0) \\ V_3 &= (0, \tfrac{-i\rho}{\sqrt{1-\tau^2}}, \tfrac{\sqrt{1-\tau^2-\rho^2}}{\sqrt{1-\tau^2}}, 0, \ldots, 0) \end{aligned}$$

For positive constants $\tau = \tau_{1,2}$ and $\rho = \tau_{2,3}$ with $|\tau| \leq 1$, $\tau^2 + \rho^2 \leq 1$ then we

obtain orthonormal vectors and satisfy $\langle V_1, JV_2 \rangle = \tau, \langle V_2, JV_3 \rangle = \rho$, $\langle V_1, JV_3 \rangle = 0$.

Corollary 3.1. *The complex torsions $\tau_{i,j}$ of a H-helix γ, $\tau_{i,j} = 0$ when $i + j$ is even* [6].

Theorem 3.2. *The complex torsions of a holomorphic helix of odd and even proper order d on a Kahler manifold satisfy the following relations.*

$$\begin{array}{lll} \tau_{i,j+2k} = 0 & i = 1, 2, \ldots, d-2k, & k = 1, 2, \ldots, (d-1)/2 \ (d \ odd) \\ & & k = 1, 2, \ldots, (d-2)/2 \ (d \ even) \\ \kappa_1 \tau_{2,d} = \kappa_{d-1} \tau_{1,d-1} & & \\ \kappa_1 \tau_{2,j} + \kappa_j \tau_{1,j+1} = \kappa_{j-1} \tau_{1,j-1} & j = 3, 5, \ldots, d-2 \ (d \ odd), & j = j = 3, 5, \ldots, d-1 \ (d \ even) \\ \kappa_{i-1} \tau_{i-1,d} + \kappa_{d-1} \tau_{i,d-1} = \kappa_i \tau_{i+1,d} & i = 3, 5, \ldots, d-2 \ (d \ odd), & i = 2, 4, \ldots, d-2 \ (d \ even) \\ \kappa_{i-1} \tau_{i-1,j} + \kappa_{j-1} \tau_{i,j-1} = \kappa_j \tau_{i,j+1} + \kappa_i \tau_{i+1,j} & i = 2, 3, \ldots, d-3 & j = i+2, i+4, \ldots, d-1 \end{array}$$

[4].

3.1. **Holomorphic helices of order 3.**

Theorem 3.3. *For $\{V_1, V_2, V_3\}$ orthonormal frame and κ_1, κ_2 pozitive constant on a K-manifold M . There is a H-helix γ with curvatures κ_1, κ_2 if and only if*

$$\begin{cases} \kappa_1\tau_{3,2} + \kappa_2\tau_{1,2} = 0 \\ \tau_{1,3} = 0 \end{cases}, \quad \tau_{1,2} \leq \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}} \text{ for } n \geq 3 \text{ and } \tau_{1,2} = \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}} \text{ for } n = 2$$

[4].

Theorem 3.4. *K-manifold M of order 2 and all complex torsions of H-helix of order 3 with curvatures κ_1 and κ_2 satisfy*

$$\tau_{1,2} = \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}, \quad \tau_{1,3} = 0, \quad \tau_{2,3} = \frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}}$$

or

$$\tau_{1,2} = -\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}, \quad \tau_{1,3} = 0, \quad \tau_{2,3} = -\frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}}$$

[4].

A classical result stated by M. A. Lancert in 1802 and first proved by B. De Saint Venant in 1845 is a necessary and sufficient condition that a curve be a general helix is the ratio of curvature of torsion to be constant [7, 8]. Adhering to this definition we will give the following definition.

Definition 3.1. For Frenet curve γ on a K-manifold of order 3, if the ratio of $\frac{\tau_{1,2}}{\tau_{2,3}}$ is constant, then γ is called a holomorphic helix.

Theorem 3.5. *If γ is a general helices of order 3 on K-manifold. $\frac{\kappa_1}{\kappa_2}$ is constant.*

Proof. $\tau_{i,j} = -\tau_{j,i}$, $\tau_{i,j} = 0$ $(i+j$ even$)$, $-\kappa_2\tau_{1,2} + \kappa_1\tau_{2,3} = 0$. then $\frac{\tau_{1,2}}{\tau_{2,3}} = \frac{\kappa_1}{\kappa_2}$ from hypethesis $\frac{\tau_{1,2}}{\tau_{2,3}}$ =constant then $\frac{\kappa_1}{\kappa_2}$ =constant. □

Theorem 3.6. *γ be a general helix on K-manifold of order 3. Then γ is a general helix if and only if*

$$D_{\dot\gamma}^{(3)}\tau_{1,2} + \lambda D_{\dot\gamma}^{(2)}\tau_{1,2} + \mu D_{\dot\gamma}\tau_{1,2} = 0$$

here $\lambda = -\frac{3\kappa_2'}{\kappa_2}$ *and* $\mu = (\kappa_1^2+\kappa_2^2) - \frac{\kappa_2''}{\kappa_2} - \frac{3(\kappa_2')^2}{\kappa_2^2}$.

Proof. if γ is a general helix

$$\begin{aligned} D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3} \\ D_{\dot\gamma}^{(2)}\tau_{1,2} &= \kappa_2'\tau_{1,3} - \kappa_2^2\tau_{1,2} + \kappa_1\kappa_2\tau_{2,3} \\ D_{\dot\gamma}^{(3)}\tau_{1,2} &= \kappa_2''\tau_{1,3} + \kappa_2'(-\kappa_2\tau_{1,2} + \kappa_1\tau_{2,3}) - 2\kappa_2'\kappa_2\tau_{1,2} \\ &= 3\kappa_2'\left(\frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}\tau_{1,2} + \frac{1}{\kappa_2}D_{\dot\gamma}^{(2)}\tau_{1,2}\right) + \left\{\frac{\kappa_2''}{\kappa_2} - (\kappa_1^2+\kappa_2^2)\right\} \end{aligned}$$

And we obtain

$$D_{\dot\gamma}^{(3)}\tau_{1,2}-\frac{3\kappa_2'}{\kappa_2}D_{\dot\gamma}^{(2)}\tau_{1,2}+\left\{(\kappa_1^2+\kappa_2^2)-\frac{\kappa_2''}{\kappa_2}-\frac{3(\kappa_2')^2}{\kappa_2^2}\right\}D_{\dot\gamma}\tau_{1,2}=0$$

conversely

$$\begin{aligned}D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3}\Longrightarrow\tau_{1,3}=\frac{1}{\kappa_2}D_{\dot\gamma}\tau_{1,2}\\ D_{\dot\gamma}\tau_{1,3} &= -\frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}\tau_{1,2}+\frac{1}{\kappa_2}D_{\dot\gamma}^{(2)}\tau_{1,2}\end{aligned}$$

and

$$D_{\dot\gamma}^{(2)}\tau_{1,2} = \left(-\frac{\kappa_2'}{\kappa_2^2}\right)'D_{\dot\gamma}\tau_{1,2}-\frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}^{(2)}\tau_{1,2}-\frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}^{(2)}\tau_{1,2}+\frac{1}{\kappa_2}D_{\dot\gamma}^{(3)}\tau_{1,2}\tag{3.1}$$

we know that

$$\begin{aligned}D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3}\\ D_{\dot\gamma}^{(2)}\tau_{1,2} &= \kappa_2'\tau_{1,3}-\kappa_2^2\tau_{1,2}+\kappa_1\kappa_2\tau_{2,3}\\ D_{\dot\gamma}^{(3)}\tau_{1,2} &= 3\kappa_2'D_{\dot\gamma}\tau_{1,3}+\Delta D_{\dot\gamma}\tau_{1,2}\end{aligned}$$

Where $\Delta=\frac{\kappa_2''}{\kappa_2}-(\kappa_2^2+\kappa_1^2)$, from (3.1)

$$D_{\dot\gamma}^{(2)}\tau_{1,3}=\left\{\left(-\frac{\kappa_2'}{\kappa_2^2}\right)'+\frac{\Delta}{\kappa_2}\right\}D_{\dot\gamma}\tau_{1,2}-\kappa_2'\tau_{1,2}-\frac{2(\kappa_2')^2}{\kappa_2}\tau_{1,3}+\frac{\kappa_2'\kappa_1}{\kappa_2}\tau_{2,3}\tag{3.2}$$

$D_{\dot\gamma}\tau_{1,3} = -\kappa_2\tau_{1,2}+\kappa_1\tau_{2,3}$ if we find the derivative of the given equation

$D_{\dot\gamma}^{(2)}\tau_{1,3} = -\kappa_2'\tau_{1,2}-\kappa_2D_{\dot\gamma}\tau_{1,2}+\kappa_1'\tau_{2,3}+\kappa_1D_{\dot\gamma}\tau_{2,3}$ and using $D_{\dot\gamma}\tau_{2,3}=-\kappa_1\tau_{1,3}$ we have

$$D_{\dot\gamma}^{(2)}\tau_{1,3}=-\kappa_2'\tau_{1,2}-\kappa_2D_{\dot\gamma}\tau_{1,2}+\kappa_1'\tau_{2,3}-\kappa_1^2\tau_{1,3}$$

By using the equality of (3.2) and (3.3)

$$\begin{aligned}-\kappa_2'\tau_{1,2}-\kappa_2D_{\dot\gamma}\tau_{1,2}+\kappa_1'\tau_{2,3}-\kappa_1^2\tau_{1,3} &= \left\{\left(-\frac{\kappa_2'}{\kappa_2^2}\right)'+\frac{\Delta}{\kappa_2}\right\}D_{\dot\gamma}\tau_{1,2}-\kappa_2'\tau_{1,2}\\ &\quad-\frac{2(\kappa_2')^2}{\kappa_2}\tau_{1,3}+\frac{\kappa_2'\kappa_1}{\kappa_2}\tau_{2,3}\end{aligned}$$

If we product the both sides of the equation with $\tau_{2,3}$ we have the $\kappa_1'=\frac{\kappa_2'\kappa_1}{\kappa_2}$ and then $\kappa_1'\kappa_2-\kappa_2'\kappa_1=0$ and since $\frac{\kappa_1}{\kappa_2}$ is constant then we obtain $\frac{\kappa_1}{\kappa_2}=\frac{\tau_{1,2}}{\tau_{2,3}}$ =constant. □

Theorem 3.7. *If γ is a helix of order 3 on K-manifold then*

$$D_{\dot\gamma}^{(3)}\tau_{1,2}+(\kappa_2^2+\kappa_1^2)D_{\dot\gamma}\tau_{1,2}=0$$

Proof. Since κ_1, κ_2 are constants and for $d=3$

$$D_{\dot\gamma}\tau_{1,2}=\kappa_2\tau_{1,3},\qquad D_{\dot\gamma}\tau_{1,3}=-\kappa_2\tau_{1,2}+\kappa_1\tau_{2,3},\qquad D_{\dot\gamma}\tau_{2,3}=-\kappa_1\tau_{1,3}$$

then we obtain

$$\begin{array}{rcl} D_{\dot\gamma}\tau_{1,2} &=& \kappa_2\tau_{1,3}\\ D^{(2)}_{\dot\gamma}\tau_{1,2} &=& \kappa_2 D_{\dot\gamma}\tau_{1,3}\\ &=& =-\kappa_2^2\tau_{1,2}+\kappa_1\kappa_2\tau_{2,3}\\ D^{(3)}_{\dot\gamma}\tau_{1,2} &=& -\kappa_2^2(\kappa_2\tau_{1,3})+\kappa_1\kappa_2(-\kappa_1\tau_{1,3})\\ &=& -(\kappa_2^2+\kappa_1^2)D_{\dot\gamma}\tau_{1,2} \end{array}$$

where

$$D^{(3)}_{\dot\gamma}\tau_{1,2}+(\kappa_2^2+\kappa_1^2)D_{\dot\gamma}\tau_{1,2}=0$$

□

Corollary 3.2. *If γ is a holomorphic helix κ_1, κ_2 separately constants then $\kappa_1'=0,\kappa_2'=0$. From there we find*

$$D^{(3)}_{\dot\gamma}\tau_{1,2}+(\kappa_2^2+\kappa_1^2)D_{\dot\gamma}\tau_{1,2}=0$$

3.2. **Holomorphic helices of order 4.** From $D_{\dot\gamma}V_j(s)=-\kappa_{j-1}V_{j-1}(s)+\kappa_jV_{j+1}(s)$, $j=1,2,\dots,d$ and $\tau_{i,j}=\langle V_i,JV_j\rangle$ also for curve of order 4 ($i=1,2,3\quad j=1,2,3,4$) then we have

$$\begin{array}{rcl} D_{\dot\gamma}\tau_{1,2} &=& \kappa_2\tau_{1,3}\\ D_{\dot\gamma}\tau_{1,3} &=& -\kappa_2\tau_{1,2}+\kappa_3\tau_{1,4}+\kappa_1\tau_{2,3}\\ D_{\dot\gamma}\tau_{1,4} &=& -\kappa_3\tau_{1,3}+\kappa_1\tau_{2,4}\\ D_{\dot\gamma}\tau_{2,3} &=& -\kappa_1\tau_{1,3}+\kappa_3\tau_{2,4}\\ D_{\dot\gamma}\tau_{2,4} &=& -\kappa_1\tau_{1,4}-\kappa_3\tau_{2,3}+\kappa_2\tau_{3,4}\\ D_{\dot\gamma}\tau_{3,4} &=& -\kappa_2\tau_{2,4} \end{array}$$

so, the matrix form is

$$\begin{bmatrix} D_{\dot\gamma}\tau_{1,2}\\ D_{\dot\gamma}\tau_{1,3}\\ D_{\dot\gamma}\tau_{1,4}\\ D_{\dot\gamma}\tau_{2,3}\\ D_{\dot\gamma}\tau_{2,4}\\ D_{\dot\gamma}\tau_{3,4}\end{bmatrix}=\begin{bmatrix} 0 & \kappa_2 & 0 & 0 & 0 & 0\\ -\kappa_2 & 0 & \kappa_3 & \kappa_1 & 0 & 0\\ 0 & -\kappa_3 & 0 & 0 & \kappa_1 & 0\\ 0 & -\kappa_1 & 0 & 0 & \kappa_3 & 0\\ 0 & 0 & -\kappa_1 & -\kappa_3 & 0 & \kappa_2\\ 0 & 0 & 0 & 0 & -\kappa_2 & 0\end{bmatrix}\begin{bmatrix}\tau_{1,2}\\ \tau_{1,3}\\ \tau_{1,4}\\ \tau_{2,3}\\ \tau_{2,4}\\ \tau_{3,4}\end{bmatrix}$$

and

$$\begin{array}{ccccc} \tau_{3,1} &=& \tau_{4,2} &=& 0\\ \kappa_2\tau_{2,1} &=& \kappa_3\tau_{4,1} &+& \kappa_1\tau_{3,2}\\ \kappa_2\tau_{4,3} &=& \kappa_1\tau_{4,1} &+& \kappa_3\tau_{3,2} \end{array}$$

Theorem 3.8. *Let M is a 2-dimentional K- manifold. For all H- helix of order 4 of complex torsions with curvatures κ_1,κ_2 and κ_3, satisfy the following equations*

$$\tau_{1,2}=\tau_{3,4}=\tau,\quad \tau_{2,3}=\tau_{1,4}=\frac{\kappa_2\tau}{\kappa_1+\kappa_3},\quad \tau_{1,3}=\tau_{2,4}=0$$

where $\tau=\pm\dfrac{\kappa_1+\kappa_3}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}}$

$$\tau_{1,2}=-\tau_{3,4}=\tau,\quad \tau_{2,3}=-\tau_{1,4}=\frac{\kappa_2\tau}{\kappa_1-\kappa_3},\quad \tau_{1,3}=\tau_{2,4}=0$$

when $\kappa_1 \neq \kappa_3$
$\tau = \pm\dfrac{\kappa_1-\kappa_3}{\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}}$ *or* $\tau_{1,2}=\tau_{3,4}=\tau_{1,3}=\tau_{2,4}=0, \quad \tau_{2,3}=-\tau_{1,4}=\pm 1$ *where* $\kappa_1=\kappa_3$[4].

Theorem 3.9. *Let γ be a general helix on K- manifold M of order 4, so*

$$D_{\dot\gamma}^{(3)}\tau_{1,2}+\lambda D_{\dot\gamma}^{(2)}\tau_{1,2}+\mu D_{\dot\gamma}\tau_{1,2}=0$$

here $\lambda=-\dfrac{3\kappa_2'}{\kappa_2}$ *and* $\mu=\dfrac{3(\kappa_2')^2}{\kappa_2^3}-\dfrac{\kappa_2''}{\kappa_2}+\kappa_1^2+\kappa_2^2-\kappa_3^2$.

Proof.

$$\begin{array}{lcl} D_{\dot\gamma}\tau_{1,2} & = & \kappa_2\tau_{1,3} \\ D_{\dot\gamma}^{(2)}\tau_{1,2} & = & \kappa_2'\tau_{1,3}-\kappa_2^2\tau_{1,2}+\kappa_2\kappa_3\tau_{1,4}+\kappa_2\kappa_1\tau_{2,3} \\ \textit{and} & & \\ D_{\dot\gamma}^{(3)}\tau_{1,2} & = & 3\kappa_2'D_{\dot\gamma}\tau_{1,3}+(\kappa_2''-\kappa_2^3-\kappa_2\kappa_3^2-\kappa_1^2\kappa_2)\tau_{1,3}+2\kappa_1\kappa_2\kappa_3\tau_{2,4} \end{array}$$

$\kappa_1\tau_{2,d}=\kappa_{d-1}\tau_{1,d-1}$ using this relation, $\kappa_1\tau_{2,4}=\kappa_3\tau_{1,3}$ is obtained and in the above expression

$$2\kappa_1\kappa_2\kappa_3\tau_{2,4} = 2\kappa_2\kappa_3^2\tau_{1,3}$$

is written,

$$\begin{array}{lcl} D_{\dot\gamma}^{(3)}\tau_{1,2} & = & 3\kappa_2'D_{\dot\gamma}\tau_{1,3}+(\kappa_2''-\kappa_2^3-\kappa_2\kappa_3^2-\kappa_1^2\kappa_2)\tau_{1,3}+2\kappa_2\kappa_3^2\tau_{1,3} \\ & = & 3\kappa_2'D_{\dot\gamma}\tau_{1,3}(\kappa_2''-\kappa_2^3+\kappa_2\kappa_3^2-\kappa_1^2\kappa_2)\tau_{1,3} \end{array}$$

is obtained and for

$$D_{\dot\gamma}\tau_{1,2} = \kappa_2\tau_{1,3} \Longrightarrow \tau_{1,3}=\frac{1}{\kappa_2}D_{\dot\gamma}\tau_{1,2}$$

$$\Longrightarrow D_{\dot\gamma}\tau_{1,3}=\left(\frac{1}{\kappa_2}\right)'D_{\dot\gamma}\tau_{1,2}+\frac{1}{\kappa_2}D_{\dot\gamma}^{(2)}\tau_{1,2}$$

we find

$$D_{\dot\gamma}^{(3)}\tau_{1,2} = \frac{3\kappa_2'}{\kappa_2^2}D_{\dot\gamma}^{(2)}\tau_{1,2}+\left\{-\frac{3(\kappa_2')^2}{\kappa_2^2}+\frac{\kappa_2''}{\kappa_2}-\kappa_2^2+\kappa_3^2-\kappa_1^2\right\}D_{\dot\gamma}\tau_{1,2}$$

or

$$D_{\dot\gamma}^{(3)}\tau_{1,2}+\lambda D_{\dot\gamma}^{(2)}\tau_{1,2}+\mu D_{\dot\gamma}\tau_{1,2}=0$$

Where $\lambda=-\dfrac{3\kappa_2'}{\kappa_2}$ and $\mu=\dfrac{3(\kappa_2')^2}{\kappa_2^3}-\dfrac{\kappa_2''}{\kappa_2}+\kappa_1^2+\kappa_2^2-\kappa_3^2$

□

Theorem 3.10. *If γ is a helix on K- manifold of order 4*

$$D_{\dot\gamma}^{(3)}\tau_{1,2}+\left\{\kappa_1^2+\kappa_2^2-\kappa_3^2\right\}D_{\dot\gamma}\tau_{1,2}=0$$

Proof.

$$\begin{aligned}
D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3}\\
D^{(2)}_{\dot\gamma}\tau_{1,2} &= -\kappa_2^2\tau_{1,2}+\kappa_2\kappa_3\tau_{1,4}+\kappa_2\kappa_1\tau_{2,3}\\
D^{(3)}_{\dot\gamma}\tau_{1,2} &= -\kappa_2^2 D_{\dot\gamma}\tau_{1,2}-\kappa_2\kappa_3^2\tau_{1,3}-\kappa_1^2\kappa_2\tau_{1,3}+2\kappa_1\kappa_2\kappa_3\tau_{2,4}
\end{aligned}$$

using the equation $\kappa_1\tau_{2,d}=\kappa_{d-1}\tau_{1,d-1}$, $\kappa_1\tau_{2,4}=\kappa_3\tau_{1,3}$ is obtained and from the above equation

$$2\kappa_1\kappa_2\kappa_3\tau_{2,4} = 2\kappa_2\kappa_3^2\tau_{1,3}$$

and

$$D_{\dot\gamma}\tau_{1,2} = \kappa_2\tau_{1,3}\Longrightarrow \tau_{1,3}=\frac{1}{\kappa_2}D_{\dot\gamma}\tau_{1,2}$$

using the equations,

$$\begin{aligned}
D^{(3)}_{\dot\gamma}\tau_{1,2} &= -\kappa_2^2 D_{\dot\gamma}\tau_{1,2}-\frac{\kappa_2\kappa_3^2}{\kappa_2}D_{\dot\gamma}\tau_{1,2}-\frac{\kappa_2\kappa_1^2}{\kappa_2}D_{\dot\gamma}\tau_{1,2}+2\kappa_2\kappa_3^2\frac{1}{\kappa_2}D_{\dot\gamma}\tau_{1,2}\\
&= (-\kappa_2^2+\kappa_3^2-\kappa_1^2)D_{\dot\gamma}\tau_{1,2}
\end{aligned}$$

is obtained

$$D^{(3)}_{\dot\gamma}\tau_{1,2}+\left\{\kappa_1^2+\kappa_2^2-\kappa_3^2\right\}D_{\dot\gamma}\tau_{1,2}=0.$$

□

Corollary 3.3. *If γ is a helix, because of κ_1,κ_2 will be constants sperately, $\kappa_1'=0,\kappa_2'=0$. Then we obtain*

$$D^{(3)}_{\dot\gamma}\tau_{1,2}+\left\{\kappa_1^2+\kappa_2^2-\kappa_3^2\right\}D_{\dot\gamma}\tau_{1,2}=0.$$

References

[1] Adachi, T. *Killing Helices on a symmetric space of rank one*, J. Geom. 84 (2005), 1-12.

[2] Adachi, T. and Madea, S. *Holomorphic helix of proper order 3 on a complex hyperbolic plane*, Topology Appl. 146-147 (2005), 201-207.

[3] Kobayashi, S. and Nomizu, K. Foundations of Differential Geometry, Vol. II, Interscience Publishers., 2009.

[4] Maeda, S. and Adachi, T. Holomorphic helices in a complex space forms, Proc. A.M. S. 125 (1997), 1197-1202.

[5] Maeda, S. and Tanabe, H. Totally geodesic immersions of Kahler Manifolds and Kahler Frenet curves, Math. Z. 252 (2006), 787-795.

[6] Maeda S. and Ohnita Y. Helical geodesic immersions into complex space forms, Geom. Dedicata 30 (1983), 93-114.

[7] Millman and G. D. Parker, Elements of Differential Geometry, Prentice Hall, Englewood Cliffs, New Jersey, 1987.

[8] Struik, D. J. Lectures on Classical Differential Geometry, Addison-Wesley Press, Inc. Cambridge, Mass., 1950.

Ankara University, Faculty of Sciences, Department of Mathematics, 06100 Tandoğan, Ankara, TURKEY

E-mail address: ekmekci@science.ankara.edu.tr

Aksaray University, Faculty of Sciences and Letters, Department of Mathematics, 68100 Aksaray, TURKEY

E-mail address: pelinpospos@gmail.com,

Karamanoğlu Mehmetbey University, Faculty of Sciences, Department of Mathematics, 70100 Karaman, TURKEY

E-mail address: pinarmaths@gmail.com

Konuralp Journal of Mathematics
Volume 1 No. 1 pp. 17–23 (2013) ©KJM

NEW ERROR ESTIMATIONS FOR THE MILNE'S QUADRATURE FORMULA IN TERMS OF AT MOST FIRST DERIVATIVES

MOHAMMAD W. ALOMARI[1,*] AND ZHENG LIU[2]

Abstract. Error estimations for the Milne's rule for mappings of bounded variation and for absolutely continuous mappings whose first derivatives are belong to $L_p[a,b]$ $(1 < p \leq \infty)$, are established. Some numerical applications are provided.

1. Introduction

Suppose $f : [a,b] \to \mathbb{R}$, is a four times continuously differentiable mapping on (a,b) and

$$\left\| f^{(4)} \right\|_{\infty} := \sup_{x \in (a,b)} \left| f^{(4)}(x) \right| < \infty.$$

Then the Simpson's inequality is known as:

$$\left| \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{(b-a)^4}{2880}\left\| f^{(4)} \right\|_{\infty}. \tag{1.1}$$

In the recent years, modern theory of inequalities is used at large and many efforts devoted to establish several generalizations of the Simpson's inequality and other inequalities for mappings of bounded variation and for monotonic, absolutely continuous and Lipschitzian mappings, as well as n-times differentiable via kernels to refine the error bounds of these inequalities. For recent results and generalizations concerning Simpson's inequality see [1]–[2], [4]–[18] and the references therein.

In terms of Newton–Cotes formulas, the Milne's formula which is of open type is parallel to the Simpson's formula which is of closed type, since they are hold under

2000 *Mathematics Subject Classification.* 26D15, 26D20.
Key words and phrases. Milne's rule, Simpson's rule, Newton–Cotes formulae.
*Corresponding author.

the same conditions. Let f as above. Then we consider the Milne's inequality as follows:

$$\left|\frac{b-a}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\int_a^b f(t)\,dt\right|\leq\frac{7(b-a)^5}{23040}\left\|f^{(4)}\right\|_\infty . \tag{1.2}$$

Indeed, Milne recommends to use the three point Newton–Cotes open formula (1.2) as a predictor and three point Newton–Cotes closed formula (1.1) as a corrector (see [3]).

The aim of this paper is to discuss the Milne's inequality for mappings of bounded variation and for absolutely continuous mappings whose first derivatives are belong to $L_p[a,b]$ $(1<p\leq\infty)$.

2. Inequalities for mappings of bounded variation

We begin with the following result:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then for all $x\in[a,b]$, we have the inequality*

$$\left|\frac{b-a}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\int_a^b f(t)\,dt\right|\leq\frac{2(b-a)}{3}\cdot\bigvee_a^b(f), \tag{2.1}$$

where $\bigvee_a^b(f)$, denotes to total variation of f over $[a,b]$. The constant $\frac{2}{3}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Proof. Define the mapping

$$k(t)=\begin{cases} t-\frac{a+2b}{3}, & t\in\left[a,\frac{a+b}{2}\right] \\ t-\frac{2a+b}{3}, & t\in\left(\frac{a+b}{2},b\right]\end{cases}.$$

Using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\int_a^{\frac{a+b}{2}}k(t)\,df(t)=-\left(\frac{b-a}{6}\right)f\left(\frac{a+b}{2}\right)+2\left(\frac{b-a}{3}\right)f(a)-\int_a^{\frac{a+b}{2}}f(t)\,dt,$$

and

$$\int_{\frac{a+b}{2}}^b k(t)\,df(t)=2\left(\frac{b-a}{3}\right)f(b)-\left(\frac{b-a}{6}\right)f\left(\frac{a+b}{2}\right)-\int_{\frac{a+b}{2}}^b f(t)\,dt.$$

If we add the above equalities, we get

$$\int_a^b k(t)\,df(t)=\frac{b-a}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\int_a^b f(t)\,dt$$

Now, assume that $\delta_n: a=x_0^{(n)}<x_1^{(n)}<\cdots<x_{n-1}^{(n)}<x_n^{(n)}=b$, is a sequence of divisions, with $\nu(\delta_n)\to 0$ as $n\to\infty$, where $\nu(\delta_n):=\max\limits_{i\in\{0,1\dots,n-1\}}\left(x_{i+1}^{(n)}-x_i^{(n)}\right)$ and $\xi_i^{(n)}\in\left[x_i^{(n)},x_{i+1}^{(n)}\right]$.

If $s : [a,b] \to \mathbb{R}$, is a piecewise continuous on $[a,b]$, and $\nu : [a,b] \to \mathbb{R}$, is of bounded variation on $[a,b]$, then

$$\begin{aligned}
&\left|\int_a^b s(t)\,d\nu(t)\right| \\
&\le \left|\lim_{\nu(\delta_n)\to 0}\sum_{i=0}^{n-1} s\left(\xi_i^{(n)}\right)\left[\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right]\right| \\
&\le \lim_{\nu(\delta_n)\to 0}\sum_{i=0}^{n-1}\left|s\left(\xi_i^{(n)}\right)\right|\left|\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right| \\
&\le \lim_{\nu(\delta_n)\to 0}\sum_{i=0}^{n-1}\left|s\left(\xi_i^{(n)}\right)\right|\sum_{i=0}^{n-1}\left|\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right| \\
&\le \sup_{x\in[a,b]}|s(x)|\cdot\sup_{\delta_n}\sum_{i=0}^{n-1}\left|\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right| \\
&= \sup_{x\in[a,b]}|s(x)|\cdot\bigvee_a^b(\nu)\,. \qquad (2.2)
\end{aligned}$$

Applying the inequality (2.2) for $s(t) = k(t)$ as above and $\nu(t) = f(t)$, $t \in [a,b]$, we get

$$\left|\int_a^b k(t)df(t)\right| \le \sup_{t\in[a,b]}|k(t)|\cdot\bigvee_a^b(f) \le \frac{2(b-a)}{3}\cdot\bigvee_a^b(f)\,.$$

To show that $2/3$ is the best possible (2.1). Assume (2.1) holds with constant $C > 0$, i.e.,

$$\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \le C(b-a)\cdot\bigvee_a^b(f)\,. \tag{2.3}$$

Consider the function

$$f(t) = \begin{cases} 0, & t \in (a,b) \\ \\ 1, & t = a,b \end{cases}$$

then $\int_a^b f(t)\,dt = 0$ and $\bigvee_a^b(f) = 2$. Using (2.3), we get

$$\frac{4}{3}(b-a) \le 2C(b-a)\,,$$

which gives $\frac{2}{3} \le C$, and thus $\frac{2}{3}$ is the best possible, which completes the proof. $\square$

Therefore, we may write the following result regarding monotonic mappings:

Corollary 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a monotonous mapping on $[a,b]$. Then for all $x \in [a,b]$, we have the inequality*

$$\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \le \frac{2(b-a)}{3}\cdot|f(b) - f(a)|\,. \tag{2.4}$$

The following result holds for L–lipschitz mappings:

Corollary 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a L–lipschitz mapping on $[a,b]$. Then for all $x\in[a,b]$, we have the inequality*

$$\left|\frac{b-a}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\int_a^b f(t)\,dt\right|\le\frac{2L}{3}(b-a)^2. \tag{2.5}$$

Remark 2.1. If we assume that f is continuous differentiable on (a,b) and f' is integrable on (a,b), then we have

$$\left|\frac{b-a}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\int_a^b f(t)\,dt\right|\le\frac{2(b-a)}{3}\left\|f'\right\|_1 \tag{2.6}$$

3. Inequalities involving derivatives belong to $L_p[a,b]$ $(1<p\le\infty)$

The following Milne's type inequality holds for absolutely continuous mappings whose first derivatives belong to $L_\infty[a,b]$.

Theorem 3.1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a,b\in I$ with $a<b$, such that $f'\in L_1[a,b]$. If f' is bounded on $[a,b]$, i.e., $\|f'\|:=\sup_{t\in[a,b]}|f'(t)|<\infty$, then we have the following inequality:*

$$\left|\frac{b-a}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\int_a^b f(t)\,dt\right|\le\frac{5}{12}(b-a)^2\left\|f'\right\|_\infty. \tag{3.1}$$

The constant $\frac{5}{12}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Proof. Integrating by parts

$$\frac{b-a}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\int_a^b f(t)\,dt=\int_a^b k(t)\,f'(t)\,dt \tag{3.2}$$

where,

$$k(t)=\begin{cases} t-\frac{a+2b}{3}, & t\in\left[a,\frac{a+b}{2}\right] \\ t-\frac{2a+b}{3}, & t\in\left(\frac{a+b}{2},b\right]\end{cases}.$$

We get

$$\begin{aligned}
&\left|\frac{b-a}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\int_a^b f(t)\,dt\right| \\
&\le\int_a^b|k(t)|\,|f(t)|\,dt \\
&=\int_a^{\frac{a+b}{2}}\left|t-\frac{a+2b}{3}\right|\left|f'(t)\right|dt+\int_{\frac{a+b}{2}}^b\left|t-\frac{2a+b}{3}\right||f(t)|\,dt \\
&\le\left\|f'\right\|_\infty\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+2b}{3}-t\right)dt+\int_{\frac{a+b}{2}}^b\left(t-\frac{2a+b}{3}\right)dt\right] \\
&=\frac{5}{12}(b-a)^2\left\|f'\right\|_\infty.
\end{aligned}$$

To show that $\frac{5}{12}$ is the best possible. Assume that (3.1) holds with constant $C > 0$, i.e.,

$$\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \le C(b-a)^2 \|f'\|_\infty . \tag{3.3}$$

Consider the function $f(t) = \left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$, then $\int_a^b f(t)\,dt = \frac{(b-a)^2}{4}$ and $\|f'\|_\infty = 1$. Using (3.3), we get

$$\frac{5}{12}(b-a)^2 \le C(b-a)^2 ,$$

which gives $\frac{5}{12} \le C$, and thus $\frac{5}{12}$ is the best possible, which completes the proof. □

Next result investigate Milne's formula for absolutely continuous mappings whose first derivatives are belong to $L_p[a,b]$, $p > 1$.

Theorem 3.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a, b \in I$ with $a < b$. If f' is belong to $L_p[a,b]$, $p > 1$, then we have the following inequality:*

$$\begin{aligned}\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \\ \le 2 \cdot \frac{\left(2^{q+1} - 2^{-q-1}\right)^{1/q}}{(q+1)^{1/q}} \cdot \left(\frac{b-a}{3}\right)^{1+\frac{1}{q}} \|f'\|_p .\end{aligned} \tag{3.4}$$

Proof. By (3.2) and using the well known Hölder inequality, we have

$$\begin{aligned}&\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \\ &\le \int_a^b |k(t)|\,|f(t)|\,dt \\ &\le \left(\int_a^b |k(t)|^q\,dt\right)^{1/q} \left(\int_a^b |f(t)|^p\,dt\right)^{1/p} \\ &= \|f'\|_p \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+2b}{3} - t\right)^q dt + \int_{\frac{a+b}{2}}^b \left(t - \frac{2a+b}{3}\right)^q dt\right]^{1/q} \\ &= 2 \cdot \frac{\left(2^{q+1} - 2^{-q-1}\right)^{1/q}}{(q+1)^{1/q}} \cdot \left(\frac{b-a}{3}\right)^{1+\frac{1}{q}} \|f'\|_p ,\end{aligned}$$

which is required. □

Remark 3.1. One may generalizes Theorem 3.1 and gives different approaches for Theorem 3.2, by applying the Hölder inequality in a different way and we shall left the details to the interested reader.

Remark 3.2. One may write new inequalities for mappings whose $|f'|$ is convex on $[a,b]$, using the inequality

$$|f'(t)| \le \frac{t-a}{b-a}|f'(b)| + \frac{b-t}{b-a}|f'(a)| ,$$

for any $t \in [a,b]$. Also, the corresponding version for powers $|f'|^q$ $(q > 1)$ may be considered by applying the well–known Hölder inequality in two different ways. We left the details to the interested reader.

4. Estimations for the error bound in the Milne's formula

Consider $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of $[a,b]$ and let $h_i = x_{i+1} - x_i$. In what follows, we point out some upper bounds for the error approximation of the Milne's formula.

$$S(f, I_n) := \sum_{i=0}^{n-1} \frac{h_i}{3}\left[2f(x_i) - f\left(\frac{x_i + x_{i+1}}{2}\right) + 2f(x_{i+1})\right] \tag{4.1}$$

Theorem 4.1. *Assume that the assumptions of Theorem 2.1 hold. Then, we have*

$$\int_a^b f(t)\,dt = S(f, I_n) + R(f, I_n)$$

where, $S(f, I_n)$ is given in (4.1) and the remainder $R(f, I_n)$ satisfies the bound

$$|R(f, I_n)| \le \frac{2}{3}\sum_{i=0}^{n-1}\left(h_i \cdot \bigvee_{x_i}^{x_{i+1}}(f)\right). \tag{4.2}$$

Proof. Applying Theorem 2.1 on the subintervals $[x_i, x_{i+1}]$, we have

$$\left|\frac{h_i}{3}\left[2f(x_i) - f\left(\frac{x_i + x_{i+1}}{2}\right) + 2f(x_{i+1})\right] - \int_{x_i}^{x_{i+1}} f(t)dt\right| \le \frac{2}{3}h_i \cdot \bigvee_{x_i}^{x_{i+1}}(f).$$

Summing the obtained inequalities over $i = 0, \cdots, n-1$, we get,

$$\left|S(f, I_n) - \int_a^b f(t)\,dt\right| \le \frac{2}{3}\sum_{i=0}^{n-1}\left(h_i \cdot \bigvee_{x_i}^{x_{i+1}}(f)\right),$$

which is required. □

Theorem 4.2. *Assume that the assumptions of Theorem 3.1 hold. Then, we have*

$$\int_a^b f(t)\,dt = S(f, I_n) + R(f, I_n)$$

where, $S(f, I_n)$ is given in (4.1) and the remainder $R(f, I_n)$ satisfies the bound

$$|R(f, I_n)| \le \frac{5}{12}(b-a)\left\|f'\right\|_\infty. \tag{4.3}$$

Proof. Applying Theorem 3.1 on the subintervals $[x_i, x_{i+1}]$ and then summing the obtained inequalities over $i = 0, \cdots, n-1$, we get the required result. We shall omit the details. □

Theorem 4.3. *Assume that the assumptions of Theorem 3.2 hold. Then, we have*

$$\int_a^b f(t)\,dt = S(f, I_n) + R(f, I_n)$$

where, $S(f,I_n)$ is given in (4.1) and the remainder $R(f,I_n)$ satisfies the bound

$$|R(f,I_n)| \leq \frac{2}{3^{\frac{q+1}{q}}} \cdot \frac{\left(2^{q+1}-2^{-q-1}\right)^{1/q}}{(q+1)^{1/q}} \cdot \|f'\|_p \cdot \sum_{i=0}^{n-1} h_i^{\frac{q+1}{q}}. \tag{4.4}$$

Proof. Applying Theorem 3.2 on the subintervals $[x_i, x_{i+1}]$ and then summing the obtained inequalities over $i = 0, \cdots, n-1$, we get the required result. We shall omit the details. □

References

[1] Alomari, M. and Hussain, S., Two inequalities of Simpson type for quasi-convex functions and applications, *Appl. Math. E-Notes*, 11 (2011), 110–117.

[2] Alomari, M., and Darus, M., On some inequalities of Simpson-type via quasi-convex functions with applications, *Tran. J. Math. Mech.*, 2(2010), 15–24.

[3] Booth, A.D., Numerical methods, 3rd Ed., Butterworths, California, 1966.

[4] Dragomir, S.S., On Simpson's quadrature formula for mappings of bounded variation and applications, *Tamkang J. Mathematics*, 30 (1999), 53–58.

[5] Dragomir, S.S. On Simpson's quadrature formula for Lipschitzian mappings and applications, *Soochow J. Mathematics*, 25 (1999), 175–180.

[6] Dragomir, S.S., On Simpson's quadrature formula for differentiable mappings whose derivatives belong to L_p spaces and applications, *J. KSIAM*, 2 (1998), 57–65.

[7] Dragomir, S.S., Agarwal R.P., and Cerone, P., On Simpson's inequality and applications, *J. of Inequal. Appl.*, 5 (2000), 533–579.

[8] Dragomir, S.S., Pečarić, J.E., and Wang, S., The unified treatment of trapezoid, Simpson and Ostrowski type inequalities for monotonic mappings and applications, *J. of Inequal. Appl.*, 31 (2000), 61–70.

[9] Dragomir, S.S. and Rassias, Th. M., (Eds) *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht, 2002.

[10] Fedotov, I., and Dragomir, S.S., An inequality of Ostrowski type and its applications for Simpson's rule and special means, *Preprint, RGMIA Res. Rep. Coll.*, 2 (1999), 13–20. http://ajmaa.org/RGMIA/v2n1.php

[11] Liu, Z., An inequality of Simpson type, *Proc R. Soc. London Ser. A*, 461(2005), 2155–2158.

[12] Liu, Z., More on inequalities of Simpson type, *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis* 23 (2007), 15–22.

[13] Shi Y., and Liu, Z., Some sharp Simpson type inequalities and applications, *Appl. Math. E-Notes,* 9(2009), 205–215.

[14] Pečarić, J., and Varošanec, S., Simpson's formula for functions whose derivatives belong to L_p spaces, *Appl. Math. Lett.*, 14 (2001), 131-135.

[15] Ujević, N., Two sharp inequalities of Simpson type and applications, *Georgian Math. J.*, 1 (11) (2004), 187–194.

[16] Ujević, N., Sharp inequalities of Simpson type and Ostrowski type, *Comp. Math. Appl.*, 48(2004), 145–151.

[17] Ujević, N., A generalization of the modi.ed Simpson.s rule and error bounds, *ANZIAM J.*, 47(2005), E1–E13.

[18] Ujević, N., New error bounds for the Simpson's quadrature rule and applications, *Comp. Math. Appl.*, 53(2007), 64–72. $\mathbb{E}_1^{n+1}$, Journ. Inst. Math. and Comp. Sci. (Math. Series) Vol:6, No.2 (1993), 161–165.

[1]Department of Mathematics, Faculty of Science, Jerash University, 26150 Jerash, Jordan

E-mail address: `mwomath@gmail.com`

[2]Institute of Applied Mathematics, School of Science, University of Science and Technology, Liaoning Anshan 114051, Liaoning, China.

E-mail address: `lewzheng@163.net`

Konuralp Journal of Mathematics
Volume 1 No. 1 pp. 24–33 (2013) ©KJM

ON TIMELIKE PARALLEL RULED SURFACES WITH SPACELIKE RULING

YASIN ÜNLÜTÜRK⋆ AND CUMALI EKICI

Abstract. In this paper, first, timelike parallel surfaces and their some basic properties are presented in Minkowski 3-space. Then the main theorem for timelike parallel ruled surface is given to understand how parallel surfaces of a timelike ruled surface with spacelike ruling become again a timelike ruled surface with spacelike ruling. Additionally, some basic properties of that kind ruled surface are given in Minkowski 3-space.

1. Introduction

Parallel surfaces as a subject of differential geometry have been intriguing for mathematicians throughout history and so it has been a research field. In theory of surfaces, there are some special surfaces such as ruled surfaces, minimal surfaces and surfaces of constant curvature in which differential geometers are interested. Among these surfaces, parallel surfaces have been also studied in many papers [2, 3, 6, 8, 12, 14]. Craig had studied to find parallel of ellipsoid in [3]. Eisenhart gave a chapter for parallel surfaces in his famous *A treatise of differential geometry* [5]. Nizamoğlu stated parallel ruled surface as a curve depending on one-parameter and gave some geometric properties of such a surface [12].

A surface M^r whose points are at a constant distance along the normal from another surface M is said to be parallel to M. So, there are infinite number of surfaces because we choose the constant distance along the normal arbitrarily. From the definition it follows that a parallel surface can be regarded as the locus of point which are on the normals to M at a non-zero constant distance r from M [18].

In this paper, it has been shown that parallel surfaces of a non-developable ruled surface are not ruled surfaces by using fundamental forms, however that parallel surfaces of a timelike developable ruled surface are timelike developable ruled surface. After construction of timelike parallel ruled surface, some properties of that kind surface such as drall, striction curve and orthogonal trajectory have

2000 *Mathematics Subject Classification.* 53A05, 53B25, 53B30.

Key words and phrases. Timelike ruled surface with spacelike ruling, timelike parallel surface, timelike parallel ruled surface, developable surface.

⋆corresponding author.

been given for timelike parallel ruled surfaces of timelike ruled surface with spacelike ruling.

2. Preliminaries

Let $\mathbb{E}_1^3$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric

$$< d\mathbf{x}, d\mathbf{x} >= dx_1^2 + dx_2^2 - dx_3^2$$

where (x_1, x_2, x_3) denotes the canonical coordinates in $\mathbb{R}^3$. An arbitrary vector $\mathbf{x}$ of $\mathbb{E}_1^3$ is said to be spacelike if $< \mathbf{x}, \mathbf{x} > > 0$ or $\mathbf{x} = \mathbf{0}$, timelike if $< \mathbf{x}, \mathbf{x} > < 0$ and lightlike or null if $< \mathbf{x}, \mathbf{x} >= 0$ and $\mathbf{x} \neq \mathbf{0}$. A timelike or lightlike vector in $\mathbb{E}_1^3$ is said to be causal. For $\mathbf{x} \in \mathbb{E}_1^3$, the norm is defined by $\|\mathbf{x}\| = \sqrt{|< \mathbf{x}, \mathbf{x} >|}$, then the vector $\mathbf{x}$ is called a spacelike unit vector if $< \mathbf{x}, \mathbf{x} >= 1$ and a timelike unit vector if $< \mathbf{x}, \mathbf{x} >= -1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [13]. For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ of $\mathbb{E}_1^3$, the inner product is the real number $< \mathbf{x}, \mathbf{y} >= x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $\mathbf{x} \times \mathbf{y} = ((x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), -(x_1y_2 - x_2y_1))$ [11].

A (differentiable) one-parameter family of (straight) lines $\{\alpha(u), X(u)\}$ is a correspondence that assigns each $u \in I$ to a point $\alpha(u) \in \mathbb{R}_1^3$ and a vector $X(u) \in \mathbb{R}_1^3$, $X(u) \neq 0$, so that both $\alpha(u)$ and $X(u)$ can be differentiated in terms of the variable u. For each $u \in I$, the line L_u which passes through $\alpha(u)$ and is parallel to $X(u)$ is called the *line of the family* at u.

Given a one-parameter family of lines $\{\alpha(u), X(u)\}$, the parametrized surface

$$\varphi(u, v) = \alpha(u) + vX(u), \qquad u \in I, \qquad v \in \mathbb{R} \tag{2.1}$$

is called the ruled surface generated by the family $\{\alpha(u), X(u)\}$. The lines L_u are called the *rulings* and the curve $\alpha(u)$ is called a *directrix* of the surface φ. The normal vector of surface is denoted by $\overrightarrow{N}$. Let us take timelike ruled surface φ with timelike directrix and spacelike ruling. So the system $\{T, X, N\}$ establishes an orthonormal frame such that $T = \alpha'(u)$. Therefore

$$< T, T >= -1, \;\; < X, X >= 1, \;\; < N, N >= 1. \tag{2.2}$$

Derivative equations of the frame $\{T, X, N\}$ are

$$D_T T = aX + bN, \quad D_T X = aT + cN, \quad D_T N = bT - cX. \tag{2.3}$$

Also the reciprocal cross products of the vectors T, X, N are

$$T \wedge X = N, \; T \wedge N = -X, \; X \wedge N = -T. \tag{2.4}$$

The parameter of distribution is expressed as follows

$$\lambda = \frac{\det(\alpha', X, X')}{|X'|^2} \tag{2.5}$$

where, as usual, (α', X, X') is a short for $\langle \alpha' \wedge X, X' \rangle$ [16].

Theorem 2.1. *A surface in Minkowski 3-space is called a timelike surface if the induced metric on the surface is a Lorentzian metric, i.e., the normal on the surface is a spacelike vector* [1].

Theorem 2.2. *Using standard parameters, a ruled surface is up to Lorentzian motions, uniquely determined by the following quantities:*

$$Q = \langle \alpha', X \wedge X' \rangle, \quad J = \langle X, X'' \wedge X' \rangle, \quad F = \langle \alpha', X \rangle \tag{2.6}$$

each of which is a function of u. Conversely, every choice of these three quantities uniquely determines a ruled surface [10].

Theorem 2.3. *The Gaussian K and mean H curvatures of the timelike ruled surface φ in terms of the parameters Q, J, F, D in $\mathbb{E}_1^3$ are obtained as follows:*

$$K = -\frac{Q^2}{D^4} \quad \text{and} \quad H = \frac{1}{2D^3}(-QF + Q^2 J + vQ' + v^2 J), \tag{2.7}$$

where $D = \sqrt{-Q^2 - v^2}$, respectively [4].

Theorem 2.4. *Parameter curves are lines of curvature if and only if $F = f = 0$ in $\mathbb{E}_1^3$* [11].

Theorem 2.5. *Let $\varphi(u, v)$ be a surface in $\mathbb{E}_1^3$ with the normal vector field of N. Then the shape operator S of φ is given in terms of the base $\{\varphi_u, \varphi_v\}$ by*

$$\begin{aligned} -S(\varphi_u) = N_u &= \frac{mF - lG}{EG - F^2}\varphi_u + \frac{lF - mE}{EG - F^2}\varphi_v \\ -S(\varphi_v) = N_v &= \frac{nF - mG}{EG - F^2}\varphi_u + \frac{mF - nE}{EG - F^2}\varphi_v \end{aligned} \tag{2.8}$$

[15].

The parallel surface of the timelike surface $\varphi(u, v)$, is denoted by $\varphi^r(u, v)$, is defined in $\mathbb{E}_1^3$ as follows:

$$\varphi^r(u, v) = \varphi(u, v) + rN(u, v), \tag{2.9}$$

where N is the unit normal vector of $\varphi(u, v)$ such that $\langle N, N \rangle = 1$ and $r \in R$. The coefficients of the first and second fundamental forms I^r and II^r of timelike parallel surfaces can be given in terms of the coefficients of the timelike surface's fundamental forms:

$$\begin{aligned} E^r &= E - 2rl + r^2 \langle N_u, N_u \rangle, & l^r &= l - r \langle N_u, N_u \rangle \\ F^r &= F - 2rm + r^2 \langle N_u, N_v \rangle, & m^r &= m - r \langle N_u, N_v \rangle \\ G^r &= G - 2rn + r^2 \langle N_v, N_v \rangle, & n^r &= n - r \langle N_v, N_v \rangle, \end{aligned} \tag{2.10}$$

where E, F, G are the coefficients of the first fundamental form for the surface φ and l, m, n are the coefficients of the second fundamental form for the surface φ and E^r, F^r, G^r are the coefficients of the first fundamental form for the parallel surface φ^r and l^r, m^r, n^r are the coefficients of the second fundamental form for the parallel surface φ^r [17].

Definition 2.1. Let M and M^r be two surfaces in Minkowski 3-space. The function

$$\begin{aligned} f: \quad M &\longrightarrow \quad M^r \\ p &\longrightarrow \quad f(p) = p + r\mathbf{N}_p \end{aligned}$$

is called the parallellization function between M and M^r and furthermore M^r is called parallel surface to M in $\mathbb{E}_1^3$ where r is a given positive real number and $\mathbf{N}$ is the unit normal vector field on M [6].

Theorem 2.6. *Let M be a surface and M^r be a parallel surface of M in Minkowski 3-space. Let $f : M \to M^r$ be the parallelization function. Then for $X \in \chi(M)$,*

1. $f_*(X) = X + rS(X)$,
2. $S^r(f_*(X)) = S(X)$,
3. f preserves principal directions of curvature, that is

$$S^r(f_*(X)) = \frac{k}{1+rk} f_*(X)$$

where S^r is the shape operator on M^r, and k is a principal curvature of M at p in direction X [6].

Definition 2.2. Let M be a hypersurface of $\overline{M}$- manifold and M^r be parallel hypersurface of M in $\mathbb{E}^3_1$. If σ is a curve passing through p on M and T is the tangent vector field of σ on M, then $\sigma^r = f \circ \sigma$ is a curve passing through a point $f(p)$ on M^r and $f_*(T) \in T_{f(p)}M^r$ is a tangent of σ^r at $f(p)$. The connection D^r belongs to the parallel surface M^r of M and the vector N^r is the unit normal vector of M^r, where $\langle N^r, N^r \rangle = \varepsilon = \pm 1$, therefore the Gauss equation is as follows:

$$\overline{D}_{f_*(T)} f_*(T) = D^r_{f_*(T)} f_*(T) - \varepsilon \langle S^r(f_*(T)), f_*(T) \rangle N^r \tag{2.11}$$

[8, 13].

Definition 2.3. Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}^3_1$. Let N^r and S^r be, respectively, the unit normal vector field and the shape operator of M^r. The Gaussian and mean curvature functions are defined as follows:

$$\begin{array}{cclcl} K^r & : & M^r & \to & \mathbb{R} \\ & & f(P) & \to & K^r(f(P)) = \det S^r_{f(P)} \\ \\ H^r & : & M^r & \to & \mathbb{R} \\ & & f(P) & \to & H^r(f(P)) = \frac{1}{2}\ iz S^r_{f(P)}, \end{array} \tag{2.12}$$

where $P \in M$, $f(P) \in M^r$ and $\langle N, N \rangle = 1$, respectively [17].

Theorem 2.7. *Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}^3_1$. Let N^r and S^r be the unit normal vector field and the shape operator of M^r, respectively. The Gaussian and mean curvatures are given in terms of coefficients of fundamental forms I^r and II^r as follows:*

$$K^r = \frac{e^r g^r - f^{r2}}{E^r G^r - F^{r2}} \quad \textit{and} \quad H^r = \frac{e^r G^r - 2f^r F^r + g^r E^r}{2(E^r G^r - F^{r2})}, \tag{2.13}$$

respectively [17].

Lemma 2.1. *Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}^3_1$. The surface M is timelike one if and only if the surface M^r is timelike parallel surface* [17].

Theorem 2.8. *Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}^3_1$. Then we have*

$$K^r = \frac{K}{1+2rH+r^2K} \quad \textit{and} \quad H^r = \frac{H+rK}{1+2rH+r^2K} \tag{2.14}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r, respectively [17].

Corollary 2.1. *Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Then we have*

$$K = \frac{K^r}{1 - 2rH^r + r^2K^r} \quad \text{and} \quad H = \frac{H^r - rK^r}{1 - 2rH^r + r^2K^r} \tag{2.15}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r, respectively [17].

Theorem 2.9. *Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Curves on the timelike parallel surface M^r which correspond to lines of curvature on the timelike surface M are also the lines of curvature* [17].

3. Timelike parallel ruled surfaces with spacelike ruling

The timelike ruled surface M with spacelike ruling, is parameterized as

$$\varphi(u,v) = \alpha(u) + vX(u), \;\; \langle \alpha', \alpha' \rangle = -1, \;\; \langle X, X \rangle = 1, \;\; \langle X', X' \rangle = -1. \tag{3.1}$$

The normal vector of the surface M is as follows:

$$N = \alpha' \wedge X + vX' \wedge X. \tag{3.2}$$

For the normal vector of a developable ruled surface which is constant along its ruling and is independent from the parameter v, the expressions $\alpha' \wedge X$ and $X' \wedge X$ in (3.2) are linearly dependent, that is, the following equation is obtained

$$\alpha' \wedge X = \lambda X' \wedge X,$$

where $\lambda \in \mathbb{R}$. Also, from the equation (3.2), the normal vector of the surface M can be obtained as

$$N = (\lambda + v)X' \wedge X. \tag{3.3}$$

The unit normal vector of the surface becomes as follows

$$\mathbf{N} = X' \wedge X. \tag{3.4}$$

We get parallel surface of the ruled surface parameterized as $\varphi(u,v) = \alpha(u) + vX(u)$ as

$$\varphi^r(u,v) = \alpha(u) + rX'(u) \wedge X(u) + vX(u). \tag{3.5}$$

We call the surface obtained in (3.5) as the parallel ruled surface. The ruling of parallel ruled surface is

$$f_*(X) = f_*(T) \wedge N^r = (T + rbT) \wedge N = -(1 + rb)X. \tag{3.6}$$

And we also get

$$f \circ \alpha(u) = \alpha(u) + rN(u) = \alpha(u) + rX'(u) \wedge X(u). \tag{3.7}$$

The coefficient g^r of the second fundamental form II^r of the parallel surface M^r is

$$g^r = -\langle \varphi_v^r, \mathbf{N}_v \rangle = -\langle X, 0 \rangle = 0.$$

The drall of parallel ruled surface is obtained from the following formula

$$P^r = < \frac{df \circ \alpha}{du}, f_*'(X) \wedge f_*(X) > . \tag{3.8}$$

From (3.8), the value of drall is found as follows:

$$P^r = \left\langle \alpha' + rX'' \wedge X, (1 + rb)^2 X' \wedge X \right\rangle = 0 \tag{3.9}$$

Finally, the parallel ruled surface given in (3.5) is a developable ruled surface. Therefore, we can give the following theorem:

Theorem 3.1. *Let M be a timelike ruled surface with spacelike ruling and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Parallel surface of a timelike developable ruled surface is again a timelike ruled surface.*

The coefficients of the first and second fundamental forms I^r and II^r for the parallel surfaces of timelike ruled surface parameterized in (3.5) are as such:

$$\begin{aligned} E^r = \langle \alpha', \alpha' \rangle + 2r \langle \alpha', X'' \wedge X \rangle + 2v \langle \alpha', X' \rangle + r^2 \langle X'' \wedge X, X'' \wedge X \rangle \\ +2rv \langle X', X'' \wedge X \rangle + v^2 \langle X', X' \rangle . \end{aligned} \tag{3.10}$$

Since $\langle X', X' \rangle = -1$ and $\langle X'', X' \rangle = 0$, X'' lies in the plane spanned by the vectors X and $X' \wedge X$. Therefore

$$X'' = mX + nX' \wedge X, \tag{3.11}$$

where $m, n \in \mathbb{R}$. By using (3.11), we have

$$X'' \wedge X = (mX + nX' \wedge X) \wedge X = (nX' \wedge X) \wedge X = nX'. \tag{3.12}$$

Substituting (3.12) into (3.10), the coefficients E^r, F^r and G^r of the first fundamental form I^r for the parallel surface M^r are found as

$$E^r = -1 - (rn - v)^2$$

$$F^r = \langle \varphi_u^r, \varphi_v^r \rangle = \langle \alpha' + rX'' \wedge X + vX', X \rangle = \langle \alpha', X \rangle$$

$$G^r = \langle \varphi_v^r, \varphi_v^r \rangle = \langle X, X \rangle = 1.$$

Also, the normal vector of the surface is

$$N^r = N = \varphi_u \wedge \varphi_v = \alpha' \wedge X + vX' \wedge X.$$

Let us find the coefficients of the second fundamental form II^r. The coefficients l^r, m^r and n^r of the second fundamental form II^r for the parallel surface $M^r$are computed as follows:

$$l^r = -\langle \varphi_u^r, N_u^r \rangle = -\langle \alpha', \alpha'' \wedge X \rangle - \langle X', \alpha'' \wedge X \rangle (rn + v) + v^2 n + rvn^2$$

$$m^r = \langle -\varphi_u^r, N_v^r \rangle = -\langle \alpha', X' \wedge X \rangle$$

$$n^r = -\langle \varphi_v^r, N_v^r \rangle = -\langle X, X' \wedge X \rangle = 0.$$

Corollary 3.1. *Let M^r be a timelike parallel ruled surface of a timelike ruled surface with spacelike ruling in $\mathbb{E}_1^3$. Then the directrix and the ruling of the timelike parallel ruled surface are a timelike curve and its ruling is a timelike vector, respectively.*

Proof. The ruling of timelike parallel ruled surface M^r given in (3.5) is a spacelike vector since $\langle X, X \rangle = 1$. The causal character of the directrix is seen by the following computations:

$$\begin{aligned} \left\langle \frac{df \circ \alpha(u)}{du}, \frac{df \circ \alpha(u)}{du} \right\rangle &= \langle \alpha' + rX'' \wedge X, \alpha' + rX'' \wedge X \rangle \\ &= 1\text{-}2rn \langle \alpha', X' \rangle + r^2 n^2 \langle X', X' \rangle \langle X, X \rangle . \end{aligned} \tag{3.13}$$

By using $\langle X', X' \rangle = -1$ and $\langle X', X'' \rangle = 0$ in (3.13), it becomes

$$\left\langle \frac{df \circ \alpha(u)}{du}, \frac{df \circ \alpha(u)}{du} \right\rangle = -1 - r^2 n^2 < 0. \tag{3.14}$$

That the causal character of the directrix is timelike is seen from □

Theorem 3.2. *Let M be a developable timelike ruled surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Let $f_*(T)$, $f_*(X)$ and N^r be, the directrix, the ruling and the normal vector of the parallel surface M^r, respectively. Hence the reciprocal cross products of these three vectors are as follows:*

$$f_*(T) \wedge N^r = f_*(X)$$

$$f_*(T) \wedge f_*(X) = -(1 + rb)^2 N^r$$

$$f_*(X) \wedge N^r = -f_*(T).$$

Proof. Frenet equations for the timelike developable ruled surface M are obtained in (2.3) by taking $c = 0$. And also the unit vectors T, X, N for the timelike developable ruled surface M are as in (2.4). By means of these information, we have the following results:

$$f_*(T) \wedge N^r = (T + rS(T)) \wedge N = (T + rbT) \wedge N = \text{-}(1+rb)X = f_*(X),$$

$$f_*(T) \wedge f_*(X) = (T + rS(T)) \wedge f_*(X) = (T + rbT) \wedge (\text{-}1\text{-}rb)X = \text{-}(1+rb)^2 N = \text{-}(1+rb)^2 N^r,$$

$$f_*(X) \wedge N^r = -(1+rb)X \wedge N = -(1 + rb)T = -f_*(T).$$

□

Theorem 3.3. *The vectors $f_*(T)$, $f_*(X), N^r$ for the timelike parallel ruled surface M^r are timelike, spacelike and spacelike vectors, respectively, while the unit vectors T, X, N for the timelike developable ruled surface with spacelike ruling M are timelike, spacelike and spacelike vectors, respectively.*

Proof. The normal vector of the timelike parallel ruled surface M^r is a spacelike vector because

$$\langle N^r, N^r \rangle = \langle N, N \rangle = 1.$$

The tangent vector field of the directrix is a timelike vector because

$$\langle f_*(T), f_*(T) \rangle = -(1 - rb)^2 < 0.$$

From (3.6), the vector $f_*(X)$ is a spacelike vector because

$$\langle f_*(X), f_*(X) \rangle = \langle -(1 + rb)X, -(1 + rb)X \rangle = (1 + rb)^2 > 0.$$

□

Position vector of striction curve on the timelike parallel ruled surface M^r is written as

$$\overrightarrow{O\gamma} = \overrightarrow{Of \circ \alpha} + \overrightarrow{\theta f_*(X)}. \tag{3.15}$$

By using $f_*(X) = X^r$ in (3.15), we get

$$\gamma(u) = f \circ \alpha(u) + \theta X^r(u) \text{ and } \theta = \theta(u). \tag{3.16}$$

After making the required computations in (3.16), we get the function $\theta = \theta(u)$ as follows:

$$\begin{aligned}\theta &= -\frac{\left\langle \dfrac{df\circ\alpha}{du}, \dfrac{dX^r}{du}\right\rangle}{\left\langle \dfrac{dX^r}{du}, \dfrac{dX^r}{du}\right\rangle} \\ &= \frac{\langle \alpha', X'\rangle + r\langle X''\wedge X, X'\rangle}{(1+rb)\langle X', X'\rangle}.\end{aligned} \tag{3.17}$$

Hence using (3.16) and (3.17), the striction curve is given as follows:

$$\gamma(u) = \alpha(u) + rX'(u)\wedge X(u) + \frac{\langle \alpha', X'\rangle + r\langle X''\wedge X, X'\rangle}{\langle X', X'\rangle}X. \tag{3.18}$$

After some calculations, the equation (3.18) becomes

$$\gamma(u) = \alpha(u) + rX'(u)\wedge X(u) + \frac{1+rna}{a}X. \tag{3.19}$$

Corollary 3.2. *Striction curve of the timelike parallel ruled surface is also a directrix provided that* $\langle \alpha', X'\rangle = 0$ *and* $\langle X''\wedge X, X'\rangle = 0$.

Proof. Straightforward calculation by using (3.18). □

Corollary 3.3. *Striction curve of the timelike parallel ruled surface is also a directrix provided that* $1 + rna = 0$.

Proof. Straightforward calculation by using (3.19). □

Theorem 3.4. *The striction curve γ of the timelike parallel ruled surface M^r is a timelike curve.*

Proof. The normal vector field of the timelike parallel ruled surface M^r is

$$N^r = N = \varphi_u \wedge \varphi_v = \alpha'\wedge X + vX'\wedge X.$$

For $v = 0$, we get

$$N^r(u,0) = \alpha'(u)\wedge X(u). \tag{3.20}$$

From (3.20), we have

$$\begin{aligned}\langle N^r(u,0), N^r(u,0)\rangle &= \langle \alpha'(u)\wedge X(u), \alpha'(u)\wedge X(u)\rangle \\ &= F^2 + 1 > 0.\end{aligned} \tag{3.21}$$

The result obtained in (3.21) means that the striction curve is timelike because the vector which is normal to it is spacelike vector. □

Theorem 3.5. *Striction curve of the timelike parallel ruled surface M^r does not depend on the choice of the base curve $f\circ\alpha$.*

Proof. Let $f\circ\alpha$ and ρ be two different directrices of the timelike parallel ruled surface. Then the timelike parallel ruled surface is

$$\varphi^r(u,v) = f\circ\alpha(u) + vX^r(u) = \rho(u) + sX^r(u) \tag{3.22}$$

for some function $s = s(v)$. Assume that the curves $\gamma(u)$ and $\overline{\gamma}(u)$ are the striction curves of the surfaces given in (3.22). Then as analogous to (3.18) by (3.22) we get

$$\gamma(u) - \overline{\gamma}(u) = (v-s)X^r - \frac{\langle (v-s)X^{r\prime}, X'\rangle}{\langle X', X'\rangle}X(u) = 0. \tag{3.23}$$

The proof is completed by the result obtained in (3.23). □

Theorem 3.6. *Given a timelike parallel ruled surface M^r which is parallel to developable timelike ruled surface M with spacelike ruling. There exists a unique orthogonal trajectory of M^r through each point of M. This orthogonal trajectory in terms of magnitudes of the timelike ruled surface M is as follows:*

$$\beta(s) = \alpha(s) + rX'(s) \wedge X(s) + g(s)X(s).$$

Here, the function $g(s)$ has been taken instead of $-v(1 + rb)$.

Proof. Let

$$\begin{aligned} \varphi^r : \quad & I \times J \longrightarrow \mathbb{E}^3_1 \\ & (u, v) \longrightarrow \varphi^r(u, v) = f \circ \alpha(u) + vX^r(u) \\ & \qquad\qquad\qquad = \alpha(u) + rX'(u) \wedge X(u) - v(1 + rb)X. \end{aligned} \tag{3.24}$$

An orthogonal trajectory of M^r is given by

$$\begin{aligned} \beta : \quad & \widetilde{I} \longrightarrow M^r \\ & s \longrightarrow \beta(s) = f \circ \alpha(s) + g(s)X^r(s). \end{aligned} \tag{3.25}$$

We may assume $\widetilde{I} \subset I$. Since

$$\langle \beta'(s), X^r(s)\rangle = \langle \alpha'(s), X(s)\rangle + g'(s) = 0, \tag{3.26}$$

we obtain

$$g(s) = -\int \langle \alpha'(s), X(s)\rangle \, ds + h,$$

where h is a real constant. Hence $h = g(s_0) - F(s_0)$, where

$$-\int \langle \alpha'(s), X(s)\rangle \, ds = F(s).$$

Therefore the orthogonal trajectory of M^r through the point P_0 is unique. Thus, we have $\widetilde{I} = I$ since the orthogonal trajectory of the surface M^r meets each one of the rulings of M^r. □

Corollary 3.4. *Let M be a timelike ruled surface with spacelike ruling and M^r be a timelike parallel surface of M in $\mathbb{E}^3_1$. The Gaussian and mean curvatures K^r and H^r in terms of the parameters Q, J, F, D are as follows:*

$$\begin{aligned} K^r &= \frac{-Q^2}{D^4 - rQFD + rQ^2JD + rvQ'D + rv^2JD - r^2Q^2} \\ H^r &= \frac{-QFD + Q^2JD + vQ'D + v^2JD - 2rQ^2}{2D^4 - 2rQFD + 2rQ^2JD + 2rvQ'D + 2rv^2JD - 2r^2Q^2}, \end{aligned} \tag{3.27}$$

respectively.

Proof. Using (2.7) in (2.14), the values of Gaussian K^r and mean curvatures H^r are obtained in (3.27). □

4. Conclusion

In this study, timelike parallel ruled surfaces have been introduced in Minkowski 3-space. Additionally timelike parallel ruled surface have been constructed by using basic features of timelike ruled surfaces with spacelike ruling. Also some properties of timelike parallel ruled surface have been given in Minkowski 3-space.

References

[1] J. K. Beem, P. E. Ehrlich and K. E. Easley, *Global Lorentzian Geometry*, Marcel Dekker, New York, 1996.

[2] A. C. Coken, U. Ciftci and C. Ekici, On Parallel Timelike Ruled Surfaces withTimelike Rulings, *Kuwait Journal of Science & Engineering*, 35(2008), 21–31.

[3] T. Craig, Note on Parallel Surfaces, *Journal für die Reine und Angewandte Mathematik (Crelle's journal)*, 94(1883) 162–170.

[4] F. Dillen and W. Kuhnel, W., Ruled Weingarten Surfaces in Minkowski 3-Space, *Manuscripta Math.*, 98(1999) 307–320.

[5] L. P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Ginn and Company, Boston: New York, 1909.

[6] A. Gorgulu and A. C. Coken, The Dupin Indicatrix for Parallel Pseudo-Euclidean Hypersurfaces in Pseudo-Euclidean Space in Semi-Euclidean Space R_1^n, *Journ. Inst. Math. and Comp. Sci. (Math Series)*, 7(3)(1994), 221-225.

[7] A. Gray, *Modern Differential Geometry of Curves and Surfaces*, CRC Press, Inc., Florida, 1993.

[8] R. Gunes, On the Properties of Curves Under the Parallel Map Preserving the Connection, *Erciyes Üniv. Fen Bil. Derg.,* 12, 1-2(1996), 50-59.

[9] Z. H. Hou and F. Ji, Helicoidal Surfaces with $H^2 = K$ in Minkowski 3-Space, *J. Math. Anal. Appl.,* 325(2007), 101–113.

[10] Kühnel, W., *Differential Geometry, Curves-Surfaces-Manifolds*, American Mathematical Society, 2002.

[11] Lopez, R., *Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space, Mini-Course taught at the Instituto de Matematica e Estatistica (IME-USP)*, University of Sao Paulo, Brasil, 2008.

[12] S. Nizamoglu, 1986, Surfaces Réglées Parallèles, *Ege Üniv. Fen Fak. Derg.*, 9 (Ser. A)(1986), 37-48.

[13] B. O'Neill, *Semi Riemannian Geometry with Applications to Relativity*, Academic Press, Inc. New York, 1983.

[14] K. R. Park and G. I. Kim, Offsets of Ruled Surfaces, *J. Korean Computer Graphics Society*, 4(1998), 69-75.

[15] W. Sodsiri, *Ruled Surfaces of Weingarten Type in Minkowski 3-Space, PhD. thesis*, Katholieke Universiteit Leuven, Belgium, 2005.

[16] A. Turgut and H. H. Hacısalihoğlu,, Timelike Ruled Surfaces in the Minkowski 3-Space, *Commun. Fac. Sci. Univ. Ank. Series A1*, v. 46(1997), 83-91.

[17] Y. Unluturk, *On parallel ruled Weingarten surfaces in 3-dimensional Minkowski space, (in Turkish), PhD thesis*, Eskisehir Osmangazi University, Graduate School of Natural Sciences, Eskisehir, 2011.

[18] D. W. Yoon, Some Properties of Parallel Surfaces in Euclidean 3-Spaces, *Honam Mathematical J.,* 30, No. 4(2008),637-644.

Kirklareli University, Science and Art Faculty, Department of Mathematics, 39100, Kirklareli-TURKEY

E-mail address: yasinunluturk@kirklareli.edu.tr

Eskişehir Osmangazi University, Science and Art Faculty, Department of Mathematics and Computer Sciences, 26480, Eskişehir - TURKEY

E-mail address: cekici@ogu.edu.tr

Konuralp Journal of Mathematics
Volume 1 No. 1 pp. 34–39 (2013) ©KJM

SOME PROPERTIES OF FINITE {0,1}-GRAPHS

İ. GÜNALTILI*, A. ULUKAN AND Ş. OLGUN

Abstract. Let G=(V,E) be a connected graph , X be a subset of V, A be a finite subset of non-negative integers and $n\,(x,y)$ be the total number of neighbours of any two vertices x,y of X. The set X is called A-semiset if $n\,(x,y) \in A$ for any two vertices x ande y of X. If X is a A-semiset, but not B-semiset for any subset B of A ,the set X is called A-set. The graph G=(V,E) is a A-semigraph and A-graph if V is the A-semiset and A-set, respectively. Mulder [2] observed that $\{0,\lambda\}$-semigraphs(these graphs are called $(0,\lambda)$-graphs by Mulder [2]), $(\lambda \geq 2)$, are regular. Furthermore a lower bound for the degree of $\{0,\lambda\}$-semigraphs with diameter at least four was derived by Mulder [2].

In this paper, we determined basic properties of finite bigraphs with at least one {0,1}-part.

1. Introduction

Let us first recall some definitions and results. For more details, (see [1]). To facilitate the general definition of a graph, we first introduce the concept of the unordered product of a set V with itself. Recall that the ordered product or cartesian product of a set V with itself,denoted by $V \times V$, is defined to be the set of all ordered pairs (s,t), where $s \in V$ and $t \in V$. The symbol $\{s,t\}$ will denote an unordered pair.

A graph G=(V,E) consists of a finite nonempty set V of v vertices together with a prescribed set E of e unordered pairs of distinct vertices of V. If a pair u=$\{x,y\}$ is an edge of G, u is said to joins x and y. We write $u = xy$ and say that vertices x and y are adjacent vertices; the vertex x and the edge u are incident with each other, If two distinct edges u and v are incident with a common vertex, then they are adjacent edges. A vertex z which adjacents to two distinct vertices x and y is called neighbour of x and y. The neighborhood of a vertex x is the set $N(x)$ consists of all vertices which are adjacent to x. The set $N_i(x)$ is the set of vertices at distance i from $x.G\,[W]$ is the subgraph of G induced by the vertex or edge set W. The degree of a vertex p is the number $d\,(p)$ of edges which are incident with

2000 *Mathematics Subject Classification.* 53D10, 53C15, 53C25, 53C35.
Key words and phrases. graph, bipartite graph, A-semiset, convex graph.
*corresponding author.

it. Let X be a subset of V. The integer n , where $n+1 = \max\{d(p) : p \in X\}$, is called the order of the set X. The minimum degree among the vertices of G=(V,E) is denoted by $\delta(G)$. If G=(V,E) contains a cycle, the girth of G=(V,E) denoted $g(G)$ is the lenght of its shortest cycle.

Definition 1.1. Let G=(V,E) be a connected graph , X be a subset of V, A be a finite subset of non-negative integers and $n\,(x, y)$ be the total number of neighbours of any two different x, y of X. The set X is called A-semiset if $n\,(x, y) \in A$ for any two vertices x ande y of X. If X is a A-semiset, but not B-semiset for any subset B of A ,the set X is called A-set. The graph G=(V,E) is a A-semigraph and A-graph if V is the A-semiset and A-set, respectively. If the set X does not contain edge will be called edge-free.

Mulder [2] observed that $\{0, \lambda\}$-semigraphs(these graphs are called $(0, \lambda)$-graphs by Mulder [2]), $(\lambda \geq 2)$, are regular. Furthermore a lower bound for the degree of $\{0, \lambda\}$-semigraphs with diameter at least four was derived by Mulder [2].

Definition 1.2. A bigraph (or bipartite graph) G=($P \cup L$,E) is a graph whose vertex set $P \cup L$ can be partitioned into subsets P and L in which a way that each edge of E joins a vertex in P to a vertex in L. Its clear that the parts P and L are edge-free.

In this paper, we determined basic properties of finite (0,1)-graphs.

2. Main Results

Corollary 2.1. *Every subgraph of a $\{0,1\}-$ semigraph is a $\{0,1\}-$ semigraph.*
Proof.*Let $G = (V, E)$ be a $\{0,1\}-$ semigraph and let $G' = (V', E')$ be subgraph of G. Since G is a $(0,1)-$ graph and $V' \subset V$ for all $x, y \in V'$, $|N(x) \cap N(y)| \leq 1$. Therefore G' is a $(0,1)-$ graph.*

Corollary 2.2. *G is a $\{0,1\}-$ semigraph if and only if G is $C_4 - free$.*
Proof.*Let G be a $(0,1)-$ graph. Suppose that, G isn't $C_4 - free$. Then, G contains at least one C_4 in which $u_1 - u_2 - u_3 - u_4 - u_1$ for $u_1, u_2, u_3, u_4 \in V$. Therefore, $N(u_1) \cap N(u_3) = \{u_2\}$ and $N(u_1) \cap N(u_3) = \{u_4\}$ .In this case, $|N(u_1) \cap N(u_3| = 2$. This case contradicts with being a $(0,1)-$ graph of G. So, G is $C_4 - free$.*
Conversely, let G be $C_4 - free$. If G isn't a $(0,1)-$ graph, then there are at least two different $u_1, u_2 \in V$ such that $|N(u_1) \cap N(u_2| \geq 2$. Let v_1 and v_2 be two different vertices in the set $N(u_1) \cap N(u_2)$. Then $u_1 - v_2 - u_1 - v_1 - u_1$ is cycle of four lenght in G. This contradicts with being $C_4 - free$ of G. So, G is a $(0,1)-$ graph.

Theorem 2.1. *Let $G = (V, E)$ be a $\{0,1\}-$ semigraph, gird $(G) \geq 8$ and $v \in V$. If $P = N_0(v) \cup N_2(v)$ and $L = N_1(v) \cup N_3(v)$, then $G' = [(P \cup L)]$ is bipartite subgraph of G.*
Proof. *Let G be a $\{0,1\}-$ semigraph. Since G' is subgraph of G, G' is a is a $(0,1)-$ graph by proposition 2.1. By the definition of P and L, $P \cap L = nothing$. In this case, we show that $G' = [(P \cup L)]$ is a bipartite $\{0,1\}-$ semigraph with parts P and L.*
(i)Suppose, there are at least two vertices x and y in P such that $\{x, y\} \in E$. If $x = v$, then $y \in N_2(v)$. Thus, there is a u vertex in L which $N(x) \cap N(y) = N(v) \cap N(y) = \{u\}$. Then , $x - u - y - x$ is cycle of three-lenght which contradicts $g(G) \geq 8$. The same contradict occurs for $y = v$, thus $x \neq v \neq y$ and $x, y \in N_2(v)$.

Let $P_1 : x - u - v$ and $P_2 : v - t - y$ be two paths. In this case, if $u \neq t$, $P_1 \cup P_2$ is a cycle of lenght 5. This contradicts $g(G) \geq 8$. Therefore $u = t$. In this case, P contains a cycle of lenght 3, contradict $g(G) \geq 8$. So, P is edge-free.

(ii)Suppose, there are at least two different x and y in L for which $\{x, y\} \in E$. There are three cases for x and y.

***Case 1:** If $x, y \in N_1(v)$ then, $x - y - v - x$ is a cycle of lenght 3. This contradicts with $g(G) \geq 8$.*

***Case 2:** If $x, y \in N_3(v)$, let us consider minumum paths*

$$P_1 : v - x_1 - x_2 - x \text{ and } P_2 : v - y_1 - y_2 - y$$

For $\forall\ i, j : 1, 2$, if $x_i \neq y_j$, then seven lenght $v - x_1 - x_2 - x - y - y_2 - y_1 - v$ cycle is obtained. This contradicts $g(G) \geq 8$. Then, there is at least one pair (i, j) for which $x_i = y_j$. Thus G contains cycle of lenght 5 or 3.

If $x_i = y_j$, then G consist at least one $x - x_2 - x_1 - y_2 - y - x$ five-lenght cycle, contradicts $g(G) \geq 8$. Similar contradiction is obtained an all other cases.

***Case 3:** Let $x \in N_1(v)$ and $y \in N_3(v)$. Suppose $P_1 : v - y_1 - y_2 - y$ is three lenght path. If $x \in P$, then $x - y_2 - y - x$ is a cycle of three-lenght in G, contradiction. Therefore, $x \notin P_1$, $x \neq y_1$ and $x \neq y_2$. Thus $v - x - y - y_2 - y_1 - v$ is a cycle of lenght 5 in G. This contradicts with $g(G) \geq 8$. So, L is edge-free.*

Theorem 2.2. *Let G=(P ∪ L,E) be a connected bigraph with parts P and L. If the part P is a {0,1}-semiset, the part L is {0,1}-semiset and G=(P ∪ L,E) is {0,1}-semibigraph.*

***Proof.** Let G=(P ∪ L,E) be a bigraph with parts P and L and let the part P be a {0,1}-semiset. Assume that the part L does not {0,1}-semiset. Then the part L has at least two distinct vertices u and w having at least two distinct common neighbours x and y in the part P. This contradict to choosen of the part P. Thus the part L is {0,1}-semiset and G=(P ∪ L,E) is {0,1}-semibigraph.*

Let G=($P \cup L$,E) be a {0,1}-semibigraph with parts P and L. and $|P| = v$, $|L| = b$, the vertices of P will be labelled $p_1, p_2, ..., p_v$ Similary, the vertices of L will be labelled $l_1, l_2, ..., l_b$. To make our notation even more concise we define,

$$r_{ij} = \begin{cases} 0, & If \quad p_i l_j \notin E \\ 1, & If \quad p_i l_j \in E \end{cases}$$

we see that the $(i, j)th$ entry of the matrix $A = [r_{ij}]_{n \times m}$ is just the number r_{ij}. The matrix A is called incidence matrix of G. Where, $n = |V(G)|$, $m = |E|$. The matrix.$A' = [r_{ij}]_{vxb}$is called blok matrix of G

Theorem 2.3. *Let $A = [r_{ij}]_{vxb}$ be the blok matrix of a {0,1}-semibigraph G, then the following equations valid.*

(i) ${}_{i=1}^{v} r_{ij} = v_j$, ${}_{j=1}^{v} r_{ij} = b_i$ and

(ii) ${}_{j=1}^{b} v_j = {}_{j=1}^{b} \left({}_{i=1}^{v} r_{ij}\right) = {}_{i=1}^{v} \left({}_{j=1}^{b} r_{ij}\right) = {}_{i=1}^{v} b_i$.

Proof. If we add the 1's in each column, column by column, we get ${}_{i=1}^{b} v_i$. If we add the 1's in each row, row by row, we get ${}_{i=1}^{v} b_i$. But obviously we are just counting the same number of 1's in two different ways so we have the equations

$(i){}_{i=1}^{v} r_{ij} = v_j, {}_{j=1}^{v} r_{ij} = b_i$ and

$(ii){}_{j=1}^{b} v_j = {}_{j=1}^{b} \left({}_{i=1}^{v} r_{ij}\right) = {}_{i=1}^{v} \left({}_{j=1}^{b} r_{ij}\right) = {}_{i=1}^{v} b_i$

Theorem 2.4. *If $r_{ij} = 0$ then the number adjacent vertices to p_i and dont have common neihbour to l_j is $d(p_i) - p_{ij}$.*

Proof. *Since $d(p_i)$ is the total number of vertices which are adjacent with p_i and by definition $p_{ij}$the result is immediate.*

Theorem 2.5. *If G=($P \cup L$,E) be a {0,1}-semibigraph with parts P and L and the part P be a {0,1}-semiset and $p_{ij} = d(l_j)$ for every vertex p_i of P and vertex l_j of L such that $r_{ij} = 0$ then P is a{1}-set.*

Proof. *Since $|L| = b \geq 1$, there is a vertex l_k of L, say. We must show that the set P is {1}-set, that is, for any distinct two vertices p_i and p_j of P, $n(p_i, p_j) = 1$. Let p_i, p_j be two distinct vertices of P. If $r_{ik} = r_{jk} = 1, n(p_i, p_j) = 1$. If $r_{ik} = 0$ and $r_{jk} = 1$ then by assumption $p_{ik} = d(l_k)$ so that p_i has a common neighbour with vertex which is adjacent to l_k. In particular, p_i and p_j have common neighbour. Thus $n(p_i, p_j) = 1$. Finally, if $r_{ik} = r_{jk} = 0$, using the hypothesis once again, for a vertex q which is adjacent with l_k $n(p_i, q) = 1$. If the vertex p_j is adjacent with common neighbour of vertices p_i and q, $n(p_i, p_j) = 1$ and otherwise, by the hypothesis one last time to get a common neighbour of vertices p_i and p_j. Therefore $n(p_i, p_j) = 1$, that is, P is {1}-set.*

Theorem 2.6. *Let G=($P \cup L$,E) be a {0,1}-semibigraph and let $|P| = v$, $|L| = b$, $\delta(L) \geq 2$. P is a$\{1\}-set$ if and only if*

$$ {}_{j=1}^{b} d(l_j)(d(l_j) - 1) \geq v(v - 1) $$

Proof. *Suppose that P is $\{1\} - set$.Counting the number of pairs of vertices of P in two different ways. First of all, there are $\begin{pmatrix} v \\ 2 \end{pmatrix}$ pairs of vertices of P (counting $\{p_i, p_j\}$ to be same pair as $\{p_j, p_i\}$). Second way, since P is $\{1\} - set$, there is a uniqe l vertex of L which $l \in N(p_i) \cap N(p_j)$. Thus, the total number of pairs of vertices of P is the total number of pairs of vertices of $N(l)$, for each $l \in L$. Summed over all vertices of L, that is ${}_s umj = 1^b d(l_j)(d(l_j) - 1)/2$.*
So,

$$ {}_{j=1}^{b} d(l_j)(d(l_j) - 1) = v(v - 1) $$

Suppose, convercely, that

$$ {}_{j=1}^{b} d(l_j)(d(l_j) - 1) \geq v(v - 1).....* $$

We prove that P is $\{1\} - set$ by induction on v.Since $\delta(L) \geq 2$, v is at least two and $b = 1$. In the case, P is $\{1\} - set$. If $v = 3$ there are exactly three possibilities , for $b = 1, 2$ or 3. Of these, only the case $b = 1$, $p_1 = v = 3$ and $b = 3$, $p_1 = p_2 = p_3 = 2$ satisfy inequality. In both of these case, P is $\{1\} - set$.
Suppose then that if the inequality holds for a partial adjacent bigraph G' with part P' and L' which P' is a part with fewer than v vertices then $P'$is $\{1\} - set$.We may assume ${}_{j=1}^{b} d(l_j)(d(l_j) - 1) \geq v(v - 1)$ in G, where $v \geq 4$. Let $p \in P$ be and consider the partial adjacent bigraph G' with part P' and L' which is the restiriction of G to $P \setminus \{p\}$. So $P' = P - \{p\}$ and $L' = \{l \in L \mid \{p, l\} \notin E \text{ and } d(l) \geq 3\}$. As $|P'| = v - 1$, we attempt to prove that P is $\{1\} - set$ by showing that approprite inequality above holds. Its right hand side becomes $(v - 1)(v - 2)$.

In G' ,

$$\begin{aligned}
\sum_{l'_j} d(l'_j)(d(l'_j)-1) &= \sum_{l_j \notin N(p)} d(l_j)(d(l_j)-1) + \sum_{l_j \in N(p), d(l_j)\geq 3} d(l'_j)(d(l'_j)-1) \\
&= \sum_{l_j \notin N(p)} d(l_j)(d(l_j)-1) + \sum_{l_j \in N(p), d(l_j)\geq 3} (d(l_j)-1)(d(l_j)-2) \\
&= \sum_{l_j \notin N(p)} d(l_j)(d(l_j)-1) + \sum_{l_j \in N(p), d(l_j)\geq 3} d(l_j)(d(l_j)-1) \\
&\quad -2\left(\sum_{l_j \in N(p), d(l_j)\geq 3} (d(l_j)-1)\right)
\end{aligned}$$

In G,

$$\begin{aligned}
\sum_{l_j} d(l_j)(d(l_j)-1) &= \sum_{l_j \notin N(p)} d(l_j)(d(l_j)-1) + \sum_{l_j \in N(p), d(l_j)\geq 3} d(l_j)(d(l_j)-1) \\
&\quad + \sum_{l_j \in N(p), d(l_j)=2} d(l_j)(d(l_j)-1) \\
\sum_{l_j \notin N(p)} d(l_j)(d(l_j)-1) &= \sum_{l_j} d(l_j)(d(l_j)-1) - \sum_{l_j \in N(p), d(l_j)\geq 3} d(l_j)(d(l_j)-1) \\
&\quad - \sum_{l_j \in N(p), d(l_j)=2} d(l_j)(d(l_j)-1)
\end{aligned}$$

Substituting in the above, we get

$$\begin{aligned}
\sum_{l'_j} d(l'_j)(d(l'_j)-1) &= \sum_{l_j} d(l_j)(d(l_j)-1) - 2\sum_{l_j \in N(p), d(l_j)\geq 3} (d(l_j)-1) \\
&\quad - \sum_{l_j \in N(p), d(l_j)=2} d(l_j)(d(l_j)-1) \\
&= \sum_{l_j} d(l_j)(d(l_j)-1) - 2\left(\sum_{l_j \in N(p)} (d(l_j)-1)\right)
\end{aligned}$$

By hypohesis, $\sum_{l_j} d(l_j)(d(l_j)-1) \geq v(v-1)$ *. By counting total degree of the vertices of* L *which adjacent to* p*, it becomes evident that* $\sum_{l_j \in N(p)}(d(l_j)-1) \leq v-1$ *and so* $-2\left(\sum_{l_j \in N(p)}(d(l_j)-1)\right) \geq -2(v-1)$*. Therefore,*

$$\sum_{l'_j} d(l'_j)(d(l'_j)-1) \geq v(v-1) - 2(v-1) = (v-1)(v-2)$$

as desired. By our induction hypothesis, $P' = P - \{p\}$ *is* $\{1\}-$ *set in* G'*. In the case, show that , for each* p' *of* P'*,there is a exactly one common vertex of* p *and* p'*. Let* p'' *be arbitrarily vertex of* P *which* $p'' \neq p$ *and* $p'' \neq p'$*.* $P'' = P - \{p''\}$ *is a restriction of* P *with* $v-1$ *vertices and the and the argument used above shows that* $P'' = P - \{p''\}$ *is* $\{1\}-$ *set. Hence there is exactly one common vertex of* p *and* p'*.*

3. References

[1] A.S. Asratian,T.M.J. Denley and R. Höggkvist, Bipartite graphs and their applications, Cambridge Uni. Press, United Kingdom, (1998).

[2] M. Mulder, $(0, \lambda)$-graph and $n-$cubes, Discrete mathematics 28 (1979) 179-188.

[3] P. Hall, On Reprensentation of Subset, J. Londan Mth. Soc. 10 (1935) 26-30

[4] J. Plesnik, A note on the complexcity of finding reguler subgraphs, Discrete Math., 49 (1984),16167

[5] F. Harary, D. Hsu and Z. Miller, The biparticity of a graph, J. Graph Theory 1 (1977) 131-133.

[6] A. Ulukan, On the Finite Adjacent Bipartite Graphs, PHD thesis.

[7] İ. Günaltılı, Finite regular bigraphs with at least one {1}-part , (preprint).

Eskişehir Osmangazi University Department of Mathematics and Computer Sciences, Eskişehir-TURKEY

E-mail address: igunalti@ogu.edu.tr

E-mail address: aulukan@anadolu.edu.tr

E-mail address: solgun@ogu.edu.tr

Konuralp Journal of Mathematics
Volume 1 No. 1 pp. 40–47 (2013) ©KJM

THE HADAMARD TYPE INEQUALITIES FOR $m-$CONVEX FUNCTIONS

ÇETIN YILDIZ*, MUSTAFA GÜRBÜZ AND AHMET OCAK AKDEMIR

Abstract. In this paper, we obtained some new Hadamard-Type inequalities for functions whose derivatives absolute values are $m-$convex. Some applications to special means of real numbers are given.

1. INTRODUCTION

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality is well known as the Hermite-Hadamard inequality for convex functions.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In recent years many authors have established several inequalities connected to Hermite-Hadamard inequality. For recent results, refinements, counterparts, generalizations and new Hadamard-type inequalities see [3], [4] and [5].

A function $f : I \to \mathbb{R}$ is said to be convex function on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In [2], G. Toader defined $m-$convexity as the following:

Definition 1.1. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m-$convex where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m-concave if -f is m-convex.

2000 *Mathematics Subject Classification.* 26D07, 26D15, 26A51.

Key words and phrases. Hermite-Hadamard's inequality, $m-$Convex, Hölder inequality, Power mean inequality.

*corresponding author.

For recent results related to above definitions we refer interest of readers to [6],[7],[8].

The following theorems which were obtained by Kavurmacı *et al.* contains the Hadamard-type integral inequalities in [1].

Theorem 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is convex function on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{b-a}{12}\left[\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(a)| + |f'(b)|\right]. \tag{1.1}$$

Theorem 1.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is convex function on $[a,b]$, for some fixed $q \ge 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \\ \le\ & \frac{b-a}{8}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + 2|f'(a)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + 2|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{1.2}$$

The main purpose of this paper is to establish refinements inequalities of right-hand side of Hadamard's type for $m-$convex functions.

2. MAIN RESULTS

In [1], in order to prove some inequalities related to Hermite-Hadamard inequality Kavurmacı *et al.* used the following lemma.

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o where $a,b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\begin{aligned} &\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du \\ =\ & \frac{(x-a)^2}{b-a}\int_a^b (1-t)f'(tx+(1-t)a)dt + \frac{(b-x)^2}{b-a}\int_a^b (t-1)f'(tx+(1-t)b)dt. \end{aligned}$$

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o where $a,b \in I$ with $a < b$. If $|f'|$ is $m-$convex function on $[a,b]$ for some fixed $m \in (0,1]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \\ \le\ & \frac{(x-a)^2}{b-a}\left[\frac{|f'(x)| + 2m\left|f'\left(\frac{a}{m}\right)\right|}{6}\right] + \frac{(b-x)^2}{b-a}\left[\frac{|f'(x)| + 2m\left|f'\left(\frac{b}{m}\right)\right|}{6}\right]. \end{aligned}$$

Proof. From Lemma 1 and using the property of modulus we get;

$$
\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
\leq\ & \frac{(x-a)^2}{b-a}\int_0^1 (1-t)\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)\left|f'(tx+(1-t)b)\right|dt.
\end{aligned}
$$

Since $|f'|$ is $m-$convex, we can write;

$$
\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
\leq\ & \frac{(x-a)^2}{b-a}\int_0^1 (1-t)\left[t\left|f'(x)\right|+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|\right]dt \\
&+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)\left[t\left|f'(x)\right|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt \\
=\ & \frac{(x-a)^2}{b-a}\left[\left|f'(x)\right|\int_0^1 (t-t^2)dt+m\left|f'\left(\frac{a}{m}\right)\right|\int_0^1 (1-t)^2dt\right] \\
&+\frac{(b-x)^2}{b-a}\left[\left|f'(x)\right|\int_0^1 (t-t^2)dt+m\left|f'\left(\frac{b}{m}\right)\right|\int_0^1 (1-t)^2dt\right] \\
=\ & \frac{(x-a)^2}{b-a}\left[\frac{\left|f'(x)\right|+2m\left|f'\left(\frac{a}{m}\right)\right|}{6}\right]+\frac{(b-x)^2}{b-a}\left[\frac{\left|f'(x)\right|+2m\left|f'\left(\frac{b}{m}\right)\right|}{6}\right].
\end{aligned}
$$

This completes the proof. □

Corollary 2.1. *In Theorem 3, if we choose $x=\frac{a+b}{2}$, we have*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{b-a}{12}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+m\left|f'\left(\frac{a}{m}\right)\right|+m\left|f'\left(\frac{b}{m}\right)\right|\right].
$$

Remark 2.1. In Corollary 1, if we choose $m=1$, the inequality in (1.1) is obtained.

Theorem 2.2. *Let $f:I\subset\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable mapping on I^o where $a,b\in I$ with $a<b$. If $\left|f'\right|^q$ is $m-$convex function on $[a,b]$ and $p>1$, then the following* *inequality holds:*

$$
\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int\limits_a^b f(u)du\right| \\
\leq\ & \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(x-a)^2}{b-a}\left(\frac{|f'(x)|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right. \\
&\left.+\frac{(b-x)^2}{b-a}\left(\frac{|f'(x)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

where $m\in(0,1]$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the property of modulus we can write;

$$
\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int\limits_a^b f(u)du\right| \\
\leq\ & \frac{(x-a)^2}{b-a}\int\limits_0^1 (1-t)\left|f'(tx+(1-t)a)\right|dt \\
&+\frac{(b-x)^2}{b-a}\int\limits_0^1 (1-t)\left|f'(tx+(1-t)b)\right|dt.
\end{aligned}
$$

By using the Hölder inequality we have:

$$
\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int\limits_a^b f(u)du\right| \\
\leq\ & \frac{(x-a)^2}{b-a}\left(\int\limits_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int\limits_0^1 \left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2}{b-a}\left(\int\limits_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int\limits_0^1 \left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $|f'|^q$ is $m-$convex function:

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
\leq\ & \frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f'(x)\right|^q+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f'(x)\right|^q+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(x-a)^2}{b-a}\left(\frac{\left|f'(x)\right|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right. \\
&\left.+\frac{(b-x)^2}{b-a}\left(\frac{\left|f'(x)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2.2. *In Theorem 4, if we choose $x=\frac{a+b}{2}$, we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
\leq\ & \left(\frac{1}{p+1}\right)^{\frac{1}{p}}.\frac{b-a}{4}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

In Corollary 2, if we choose $m=1$ and $\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\leq 1$, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{b-a}{4}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

Theorem 2.3. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^o where $a,b\in I$ with $a<b$. If $|f'|^q$ is $m-$convex function on $[a,b]$ for some fixed $m\in(0,1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
\leq\ & \frac{(x-a)^2}{2(b-a)}\left(\frac{\left|f'(x)\right|^q+2m\left|f'\left(\frac{a}{m}\right)\right|^q}{3}\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2}{2(b-a)}\left(\frac{\left|f'(x)\right|^q+2m\left|f'\left(\frac{b}{m}\right)\right|^q}{3}\right)^{\frac{1}{q}}
\end{aligned}$$

where $q\geq 1$.

Proof. From Lemma 1 and using the property of modulus we get;

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\int_0^1 (1-t)\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)\left|f'(tx+(1-t)b)\right|dt.$$

By using the Power-mean inequality, we have

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'(x)|^q$ is $m-$convex, we have

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left[t\left|f'(x)\right|^q+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left[t\left|f'(x)\right|^q+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$= \frac{(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'(x)\right|^q\int_0^1 (t-t^2)dt+m\left|f'\left(\frac{a}{m}\right)\right|^q\int_0^1 (1-t)^2dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'(x)\right|^q\int_0^1 (t-t^2)dt+m\left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1 (1-t)^2dt\right)^{\frac{1}{q}}$$
$$= \frac{(x-a)^2}{2(b-a)}\left(\frac{\left|f'(x)\right|^q+2m\left|f'\left(\frac{a}{m}\right)\right|^q}{3}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{2(b-a)}\left(\frac{\left|f'(x)\right|^q+2m\left|f'\left(\frac{b}{m}\right)\right|^q}{3}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 2.3. *In Theorem 5, if we choose* $x = \frac{a+b}{2}$, *we get*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{b-a}{8} \left[\left(\frac{\left| f'\left(\frac{a+b}{2}\right)\right|^q + 2m\left| f'\left(\frac{a}{m}\right)\right|^q}{3} \right)^{\frac{1}{q}} + \left(\frac{\left| f'\left(\frac{a+b}{2}\right)\right|^q + 2m\left| f'\left(\frac{b}{m}\right)\right|^q}{3} \right)^{\frac{1}{q}} \right].$$

Remark 2.2. In Corollary 3, if we choose $m = 1$, the inequality in (1.2) is obtained.

3. APPLICATIONS TO SPECIAL MEANS

We now consider the means for arbitrary real numbers α, β $(\alpha \neq \beta)$. We take

(1) *Arithmetic mean* :

$$A(\alpha, \beta) = \frac{\alpha+\beta}{2}, \quad \alpha, \beta \in \mathbb{R}^+.$$

(2) *Logarithmic mean*:

$$L(\alpha, \beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|, \ \alpha, \beta \neq 0, \ \alpha, \beta \in \mathbb{R}^+.$$

(3) *Generalized* $log - mean$:

$$L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta-\alpha)} \right]^{\frac{1}{n}}, \quad n \in \mathbb{Z} \backslash \{-1, 0\}, \ \alpha, \beta \in \mathbb{R}^+.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

Proposition 3.1. *Let* $a, b \in \mathbb{R}^+$, $a < b$, $m \in (0,1]$ *and* $n \in \mathbb{Z}, n > 1$. *Then, we have*

$$|A(a^n, b^n) - L_n^n(a,b)| \leq n.\frac{b-a}{12} \left[\left| \frac{a+b}{2} \right|^{n-1} + m\left| \frac{a}{m} \right|^{n-1} + m\left| \frac{b}{m} \right|^{n-1} \right].$$

If we choose $m = 1$, *we obtain*

$$|A(a^n, b^n) - L_n^n(a,b)| \leq n.\frac{b-a}{12} \left[\left| \frac{a+b}{2} \right|^{n-1} + |a|^{n-1} + |b|^{n-1} \right].$$

Proof. The assertion follows from Corollary 1 applied to the m-convex mapping $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. □

Proposition 3.2. *Let* $a, b \in \mathbb{R}^+$, $a < b$, $m \in (0,1]$ *and* $n \in \mathbb{Z} \backslash \{-1, 0\}$. *Then, for all* $q \geq 1$, *we have*

$$|A(a^n, b^n) - L_n^n(a,b)| \leq n.\frac{b-a}{8} \left(\left[\frac{\left|\frac{a+b}{2}\right|^{q(n-1)} + 2m\left|\frac{a}{m}\right|^{q(n-1)}}{3} \right]^{\frac{1}{q}} + \left[\frac{\left|\frac{a+b}{2}\right|^{q(n-1)} + 2m\left|\frac{b}{m}\right|^{q(n-1)}}{3} \right]^{\frac{1}{q}} \right).$$

If we choose $m = 1$, *we obtain*

$$|A(a^n, b^n) - L_n^n(a,b)| \leq n.\frac{b-a}{8} \left(\left[\frac{\left|\frac{a+b}{2}\right|^{q(n-1)} + 2\,|a|^{q(n-1)}}{3} \right]^{\frac{1}{q}} + \left[\frac{\left|\frac{a+b}{2}\right|^{q(n-1)} + 2\,|b|^{q(n-1)}}{3} \right]^{\frac{1}{q}} \right).$$

Proof. The assertion follows from Corollary 3 applied to the m-convex mapping $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. $\square$

References

[1] H. Kavurmacı, M. Avcı, M.E. Özdemir, New inequalities of Hermite-Hadamard's type for convex functions with applications, Journal of Inequalities and Applications, 2011, 2011:86.

[2] G. Toader, Some generalizations of the convexity, Proceedings of The Colloquium On Approximation and Optimization, Univ. Cluj-Napoca, Cluj-Napoca, 1984, 329-338.

[3] S.S. Dragomir, Two mappings in connection to Hadamard's inequalities, J.Math. Anal. Appl., 167(1992), 49-56.

[4] U.S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula, Appl. Math. Comp. 147(2004), 137-146.

[5] G.S. Yong, D.Y. Hwang and K.L. Tseng, Some inequalities for differentiable convex and concave mappings, Comp. Math. Appl., 47(2004), 207-216.

[6] M.K. Bakula, M.E. Özdemir and J. Pećarić, Hadamard type inequalities for $m-$convex and (α, m)-convex functions, J. Inequal. Pure and Appl. Math., 9(4) (2008), Article 96.

[7] S.S. Dragomir and G. Toader, Some inequalities for $m-$convex functions, Studia Univ. Babes-bolyai. Mathematica, 38(1993), 1, 21-28.

[8] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Type Inequalities and Applications, RGMIA Monographs, Victoria University, 2000. [ONLINE : http://ajmaa.org/RGMIA/monographs/hermite_hadamard.html].

ATATÜRK UNIVERSITY, K. K. EDUCATION FACULTY, DEPARTMENT OF MATHEMATICS,25240, CAMPUS, ERZURUM, TURKEY

E-mail address: `yildizc@atauni.edu.tr`

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES, AĞRI İBRAHİM ÇEÇEN UNIVERSITY, AĞRI, TURKEY

E-mail address: `mgurbuz@agri.edu.tr`

AĞRI İBRAHİM ÇEÇEN UNİVERSITY, FACULTY OF SCIENCE AND ARTS, DEPARTMENT OF MATHEMATICS, 04100, CAMPUS, AĞRI, TURKEY

E-mail address: `ahmetakdemir@agri.edu.tr`

Konuralp Journal of Mathematics
Volume 1 No. 1 pp. 48–53 (2013) ©KJM

ON WEIGHTED MONTOGOMERY IDENTITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

MEHMET ZEKI SARIKAYA⋆ AND HATICE YALDIZ

Abstract. In this paper, we extend the weighted Montogomery identities for the Riemann-Liouville fractional integrals. We also use this Montogomery identities to establish some new Ostrowski type integral inequalities.

1. Introduction

The inequality of Ostrowski [18] gives us an estimate for the deviation of the values of a smooth function from its mean value. More precisely, if $f : [a, b] \to \mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for every $x \in [a, b]$. Moreover the constant $1/4$ is the best possible.

For some generalizations of this classic fact see the book [8, p.468-484] by Mitrinovic, Pecaric and Fink. A simple proof of this fact can be done by using the following identity [8]:

If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ with the first derivative f' integrable on $[a, b]$, then Montgomery identity holds:

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt + \int_a^b P_1(x,t) f'(t)dt, \tag{1.1}$$

where $P_1(x, t)$ is the Peano kernel defined by

$$P_1(x,t) := \begin{cases} \dfrac{t-a}{b-a}, & a \leq t < x \\ \\ \dfrac{t-b}{b-a}, & x \leq t \leq b. \end{cases}$$

2000 *Mathematics Subject Classification.* 26D15, 41A55, 26D10 .

Key words and phrases. Riemann-Liouville fractional integral, Ostrowski inequality.

⋆corresponding author.

Recently, several generalizations of the Ostrowski integral inequality are considered by many authors; for instance covering the following concepts: functions of bounded variation, Lipschitzian, monotonic, absolutely continuous and n-times differentiable mappings with error estimates with some special means together with some numerical quadrature rules. For recent results and generalizations concerning Ostrowski's inequality, we refer the reader to the recent papers [3], [6], [9]-[11], [13]-[15].

In [1] and [16], the authors established some inequalities for differentiable mappings which are connected with Ostrowski type inequality by used the Riemann-Liouville fractional integrals, and they used the following lemma to prove their results:

Lemma 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I° with $a, b \in I$ $(a < b)$ and $f' \in L_1[a,b]$, then*
(1.2)

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^\alpha f(b) - J_a^{\alpha-1}(P_2(x,b)f(b)) + J_a^\alpha(P_2(x,b)f^{'}(b)), \quad \alpha \geq 1,$$

where $P_2(x,t)$ is the fractional Peano kernel defined by

$$P_2(x,t) = \begin{cases} \dfrac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & a \leq t < x \\ \\ \dfrac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & x \leq t \leq b. \end{cases}$$

In this article, we use the Riemann-Liouville fractional integrals to establish some new weighted integral inequalities of Ostrowski's type. From our results, the weighted and the classical Ostrowski's inequalities can be deduced as some special cases.

2. Fractional Calculus

Firstly, we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [7], [12].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ with $a \geq 0$ is defined as

$$\begin{aligned} J_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \\ J_a^0 f(x) &= f(x). \end{aligned}$$

Recently, many authors have studied a number of inequalities by used the Riemann-Liouville fractional integrals, see ([1], [2], [4], [5], [16], [17]) and the references cited therein.

3. Main Results

Throughout this work, we assume that the weight function $w : [a,b] \to [0,\infty)$, is integrable, nonnegative and

$$m(a,b) = \int_a^b w(t)dt < \infty.$$

In order to prove our main results, we need the following identities:

Lemma 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° with $a,b \in I$ $(a < b)$, $\alpha \geq 1$ and $f' \in L_1[a,b]$, then the generalization of the weighted Montgomery identity for fractional integrals holds:*

$$\begin{aligned} m(a,b) f(x) &= (b-x)^{1-\alpha} \Gamma(\alpha) J_a^\alpha (w(b) f(b)) \\ &\quad - J_a^{\alpha-1} (\Omega_w(x,b) f(b)) + J_a^\alpha \left(\Omega_w(x,b) f'(b) \right) \end{aligned} \tag{3.1}$$

where $\Omega_w(x,t)$ is the weighted fractional Peano kernel defined by

$$\Omega_w(x,t) := \begin{cases} (b-x)^{1-\alpha} \Gamma(\alpha) \int_a^t w(u)\,du, & t \in [a,x) \\ (b-x)^{1-\alpha} \Gamma(\alpha) \int_b^t w(u)\,du, & t \in [x,b]. \end{cases} \tag{3.2}$$

Proof. By definition of $\Omega_w(x,t)$, we have

$$\begin{aligned} & J_a^\alpha \left(\Omega_w(x,b) f'(b) \right) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} \Omega_w(x,t) f'(t)\,dt \\ &= (b-x)^{1-\alpha} \left[\int_a^x (b-t)^{\alpha-1} \left(\int_a^t w(u)\,du \right) f'(t)\,dt \right. \\ &= \left. + \int_x^b (b-t)^{\alpha-1} \left(\int_b^t w(u)\,du \right) f'(t)\,dt \right] \\ &= (b-x)^{1-\alpha} (J_1 + J_2). \end{aligned} \tag{3.3}$$

Integrating by parts, we can state:

$$
\begin{aligned}
J_1 &= (b-x)^{\alpha-1}\left(\int_a^x w(u)\,du\right) f(x) \\
&\quad + (\alpha-1)\int_a^x (b-t)^{\alpha-2}\left(\int_a^t w(u)\,du\right) f(t)\,dt - \int_a^x (b-t)^{\alpha-1} w(t) f(t)\,dt
\end{aligned} \tag{3.4}
$$

and similary,

$$
\begin{aligned}
J_2 &= (b-x)^{\alpha-1}\left(\int_x^b w(u)\,du\right) f(x) \\
&\quad + (\alpha-1)\int_x^b (b-t)^{\alpha-2}\left(\int_b^t w(u)\,du\right) f(t)\,dt - \int_x^b (b-t)^{\alpha-1} w(t) f(t)\,dt.
\end{aligned} \tag{3.5}
$$

Adding (3.4) and (3.5), we obtain (3.1) which this completes the proof. □

Remark 3.1. If we choose $\alpha = 1$ and $w(u) = 1$, the formula (3.1) reduces to the classical Montgomery Identity given by (1.1).

Remark 3.2. If we choose $w(u) = 1$, the formula (3.1) reduces to the fractional Montgomery Identity given by (1.2).

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable on (a,b) such that $f' \in L_1[a,b]$, where $a < b$. If $\left|f'(x)\right| \leq M$ for every $x \in [a,b]$ and $\alpha \geq 1$, then the following Ostrowski fractional inequality holds:*

$$
\begin{aligned}
&\left| m(a,b) f(x) - (b-x)^{1-\alpha}\,\Gamma(\alpha)\, J_a^{\alpha}(w(b) f(b)) - J_a^{\alpha-1}(\Omega_w(x,b) f(b)) \right| \\
&\leq \frac{M(b-x)^{1-\alpha}}{\alpha}\left[A(x) - (b-x)^{\alpha} B(x)\right]
\end{aligned} \tag{3.6}
$$

where

$$
A(x) = \int_a^x (b-u)^{\alpha-1} w(u)\,du - \int_x^b (b-u)^{\alpha} w(u)\,du
$$

and

$$
B(x) = \int_a^x w(u)\,du - \int_x^b w(u)\,du.
$$

Proof. From Lemma 3.1, we get

$$\begin{aligned}
& \left| m(a,b) f(x) - \Gamma(\alpha)(b-x)^{1-\alpha} J_a^{\alpha}(w(b) f(b)) - J_a^{\alpha-1}(\Omega_w(x,b) f(b)) \right| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \left| \int_a^b (b-t)^{\alpha-1} \Omega_w(x,t) f'(t)\, dt \right| \\
\leq\ & \frac{M}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} |\Omega_w(x,t)|\, dt \qquad (3.7)\\
=\ & M(b-x)^{1-\alpha} \left(\int_a^x (b-t)^{\alpha-1} \left(\int_a^t w(u)\, du \right) dt + \int_x^b (b-t)^{\alpha-1} \left(\int_t^b w(u)\, du \right) dt \right) \\
=\ & M(b-x)^{1-\alpha} \{ J_3 + J_4 \}.
\end{aligned}$$

Now, using the change of order of integration we get

$$\begin{aligned}
J_3 &= \int_a^x (b-t)^{\alpha-1} \left(\int_a^t w(u)\, du \right) dt \\
&= \int_a^x w(u) \int_u^x (b-t)^{\alpha-1}\, dt du \\
&= \frac{1}{\alpha} \left[\int_a^x (b-u)^{\alpha-1} w(u)\, du - (b-x)^{\alpha} \int_a^x w(u)\, du \right]
\end{aligned}$$

and similarly,

$$\begin{aligned}
J_4 &= \int_x^b (b-t)^{\alpha-1} \left(\int_t^b w(u)\, du \right) dt \\
&= \int_x^b w(u) \int_x^u (b-t)^{\alpha-1}\, dt du \\
&= \frac{1}{\alpha} \left[(b-x)^{\alpha} \int_x^b w(u)\, du - \int_x^b (b-u)^{\alpha} w(u)\, du \right].
\end{aligned}$$

Using J_3 and J_4 in (3.7), we obtain (3.6). □

Remark 3.3. We note that in the special cases, if we take $w(u) = 1$ in Theorem 3.1, then it reduces Theorem 4.1 proved by Anastassiou et. al. [1]. So, our results are generalizations of the corresponding results of Anastassiou et. al. [1].

References

[1] G. Anastassiou, M.R. Hooshmandasl, A. Ghasemi and F. Moftakharzadeh, *Montgomery identities for fractional integrals and related fractional inequalities*, J. Inequal. in Pure and Appl. Math, 10(4), 2009, Art. 97, 6 pp.

[2] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities*, J. Inequal. in Pure and Appl. Math, 10(3), 2009, Art. 97, 6 pp.

[3] P. Cerone and S.S. Dragomir, *Trapezoidal type rules from an inequalities point of view*, Handbook of Analytic-Computational Methods in Applied Mathematics, CRC Press N.Y. (2000).

[4] Z. Dahmani, L. Tabharit and S. Taf, *Some fractional integral inequalities*, Nonlinear Science Letters A, 2(1), 2010, p.155-160.

[5] Z. Dahmani, L. Tabharit and S. Taf, *New inequalities via Riemann-Liouville fractional integration*, J. Advance Research Sci. Comput., 2(1), 2010, p.40-45.

[6] J. Duoandikoetxea, *A unified approach to several inequalities involving functions and derivatives*, Czechoslovak Mathematical Journal, 51 (126) (2001), 363–376.

[7] R. Gorenflo, F. Mainardi, *Fractionalcalculus: integral and differentiable equations of fractional order*, Springer Verlag, Wien, 1997, p.223-276.

[8] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, *Inequalities involving functions and their integrals and derivatives,* Kluwer Academic Publishers, Dordrecht, 1991.

[9] S.S. Dragomir and N. S. Barnett, *An Ostrowski type inequality for mappings whose second derivatives are bounded and applications*, RGMIA Research Report Collection, V.U.T., 1(1999), 67-76.

[10] S.S. Dragomir, *An Ostrowski type inequality for convex functions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 16 (2005), 12–25.

[11] Z. Liu, *Some companions of an Ostrowski type inequality and application*, J. Inequal. in Pure and Appl. Math, 10(2), 2009, Art. 52, 12 pp.

[12] S. G. Samko, A. A Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives Theory and Application*, Gordan and Breach Science, New York, 1993.

[13] M. Z. Sarikaya, *On the Ostrowski type integral inequality*, Acta Math. Univ. Comenianae, Vol. LXXIX, 1(2010), pp. 129-134.

[14] M. Z. Sarikaya, *On the Ostrowski type integral inequality for double integrals*, Demonstratio Mathematica, accepted.

[15] M. Z. Sarikaya and H. Ogunmez, *On the weighted Ostrowski type integral inequality for double integrals*, The Arabian Journal for Science and Engineering (AJSE)-Mathematics, (2011) 36: 1153-1160

[16] M.Z. Sarikaya and H. Ogunmez, *On new inequalities via Riemann-Liouville fractional integration*, arXiv:1005.1167v1, submitted.

[17] M.Z. Sarikaya, E. Set, H. Yaldiz and N., Basak, *Hermite -Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, DOI:10.1016/j.mcm.2011.12.048.

[18] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

E-mail address: sarikayamz@gmail.com

E-mail address: yaldizhatice@gmail.com

Konuralp Journal of Mathematics
Volume 1 No. 1 pp. 54–60 (2013) ©KJM

$\phi-$CONFORMALLY FLAT $C-$MANIFOLDS

ERDAL ÖZÜSAĞLAM

Abstract. In this paper, we have studied $\phi-$conformally flat, $\phi-$conharmonically flat and $\phi-$projectively flat $C-$manifolds.

1. Preliminaries

Let (M^n,g), $n = dimM$, $n \geq 3$, be a connected Riemannian manifold of class C^∞ and ∇ be its Riemannian connection. The Riemannian-Christoffel curvature tensor R, the Weyl conformal curvature tensor C (see [8]), the conharmonic curvature tensor K (see [13]) and the projective curvature tensor P (see [8]) of (M^n,g) are defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1}$$

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)\mathcal{S}X - g(X,Z)\mathcal{S}Y] + \\ &\quad \frac{\tau}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{1.2}$$

$$\begin{aligned} K(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)\mathcal{S}X - g(X,Z)\mathcal{S}Y] \end{aligned} \tag{1.3}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)\mathcal{S}X - g(X,Z)\mathcal{S}Y] \tag{1.4}$$

respectively, where $\mathcal{S}$ is the Ricci operator, defined by $S(X,Y) = g(\mathcal{S}X,Y)$, S is the Ricci tensor, $\tau = tr(S)$ is the scalar curvature and $X,Y,Z \in \chi(M)$, $\chi(M)$ being the vector fields of M.

2000 *Mathematics Subject Classification.* 53C21, 53C25.
Key words and phrases. $C-$manifold, $\phi-$conformally flat, $\phi-$conharmonically flat, $\phi-$projectively flat.

In this paper, we have studied $\phi-$conformally flat $C-$ manifolds. We show that there are no exist $\phi-$conformally flat and $\phi-$projectively flat $C-$manifolds unless the dimension of structure vector field s is 1. Similarly, we obtain that there is no exist $\phi-$conharmonically flat $C-$manifolds unless $s=4$.

2. $C-$MANIFOLDS

We need the following definition which is given in [4].

Let (M,g) be a Riemannian manifold with $dim(M)=2m+s$. Then M is said to be an $C-$manifold if there exist on M an $\phi-$structure ϕ [16] of rank $2m$ and s global vector fields $\xi_1,...,\ \xi_s$ (structure vector fields) such that [4]

(i) If $\eta_1,...,\eta_s$ are dual $1-$forms of $\xi_1,...,\ \xi_s$, then:

$$\phi\xi_i=0,\quad \eta_i\circ\phi=0,\quad \eta_i(\xi_i)=1,\quad \phi^2=-I+\sum_{i=1}^{s}\xi_i\otimes\eta_i \tag{2.1}$$

$$g(X,Y)=g(\phi X,\phi Y)+\sum_{i=1}^{s}\eta_i\,(X)\,\eta_i\,(Y) \tag{2.2}$$

$$g(\xi_i,X)=\eta_i(X) \tag{2.3}$$

for any $X,Y\in\chi(M)$ and $i=1,...,s$.

(ii) The $\phi-$structure ϕ is normal, that is

$$[\phi,\phi]+2\sum_{i=1}^{s}\xi_i\otimes d\eta_i=0$$

where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ.

(iii) $\eta_1\wedge...\wedge\eta_s\wedge(d\eta_i)^n\neq 0$ and $d\eta_i=0$, for any i. Examples of $C-$manifolds are given in [4].

In a $C-$manifold M, besides the relations (1.1) and (1.2) the following also hold [9] :

$$(\nabla_X\phi)\,Y=0$$

$$\nabla_X\xi_i=0$$

$$\begin{aligned} R(\xi_i,X)Y&=0\\ R(\xi_i,X)\xi_\beta&=0\end{aligned} \tag{2.4}$$

$$S(\xi_i,X)=2m\sum_{\beta=1}^{s}\eta_\beta(X) \tag{2.5}$$

$$S(\phi X,\phi Y)=S(X,Y) \tag{2.6}$$

An $C-$manifold M is said to be $\eta-$Einstein if its Ricci tensor S is of the form

$$S(X,Y)=ag(X,Y)+b\sum_{i=1}^{s}\eta_i(X)\eta_i(Y) \tag{2.7}$$

for any vector fields X and Y, where, $a,\ b$ are functions on M^n

3. Main Results

In this section we consider $\phi-$conformally flat, $\phi-$conharmonically flat and $\phi-$projectively flat $C-$ manifolds.

Let C be the Weyl conformal curvature tensor of M^n. Since at each point $p \in M^n$ the tangent space $T_P(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \phi(T_p(M^n)) \oplus L(\xi_p)$, where $L(\xi_p)$ is a $1-$dimensional linear subspace of $T_p(M^n)$ generated by ξ_p, we have a map:

$$C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \rightarrow \phi(T_p(M^n)) \oplus L(\xi_p).$$

It may be natural to consider the following particular cases:

(1) $C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \rightarrow L(\xi_p)$, that is, the projection of the image of C in $\phi(T_p(M^n))$ is zero.

(2) $C : T_p(M^n) \times T_p(M^n) \times \ T_p(M^n) \rightarrow \phi(T_p(M^n))$, that is, the projection of the image of C in $L(\xi_p)$ is zero.

(3) $C : \phi(T_p(M^n)) \times \phi(T_p(M^n)) \times \phi(T_p(M^n)) \rightarrow L(\xi_p)$, that is, when C is restricted to $(T_p(M^n)) \times \phi(T_p(M^n)) \times \phi(T_p(M^n))$, the projection of the image of in $\phi(T_p(M^n))$ is zero. This condition is equivalent to

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0 \tag{3.1}$$

(see [8]).

Definition 3.1. A differentiable manifold (M^n, g), $n > 3$, satisfying the condition (3.1) is called $\phi-$conformally flat.

The cases (1) and (2) were considered in ([18]) and ([19]) respectively. The case (3) was considered in ([8]) for the case M^n is a $K-$contact manifold.

Furthermore in [1], the authors studied $(k, \mu)-$contact metric manifolds satisfying (3.1). Now our aim is to find the characterization of $C-$manifolds satisfying the condition (3.1).

Theorem 3.1. *Let M be an $2m+s-$dimensional, $(s > 1)$, $C-$manifold. Then There is no exist $\phi-$conformally flat $C-$manifolds.*

Proof. Suppose that (M, g), $(s > 1)$, is a $\phi-$conformally flat $C-$manifold. It is easy to see that $\phi^2 C(\phi X, \phi Y)\phi Z = 0$ holds if and only if

$$g(C(\phi X, \phi Y)\phi Z, \phi W) = 0,$$

for any $X, Y, Z, W \in \chi(M)$. So by the use of (1.2) $\phi-$conformally flat means

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) \quad = \quad & \frac{1}{2m+s-2}[g(\phi Y, \phi Z)S(\phi X, \phi W) \\ & -g(\phi X, \phi Z)S(\phi Y, \phi W) + g(\phi X, \phi W)S(\phi Y, \phi Z) \\ & -g(\phi Y, \phi W)S(\phi X, \phi Z)] \\ & -\frac{\tau}{(2m+s-1)(2m+s-2)}[g(\phi Y, \phi Z)g(\phi X, \phi W) \\ & -g(\phi X, \phi Z)g(\phi Y, \phi W)] \end{aligned} \tag{3.2}$$

Let $\{w_1, ..., w_{2m}, \xi_1, ..., \xi_s\}$ be a local orthonormal basis of vector fields in M. Using that $\{\phi w_1, ..., \phi w_{2m}, \xi_1, ..., \xi_s\}$ is also a local orthonormal basis, if we put $X = W = w_i$ in (3.2) and sum up with respect to i, then

$$\begin{aligned}\sum_{i=1}^{2m} g(R(\phi w_i,\phi Y)\phi Z,\phi w_i) &= \frac{1}{2m+s-2}\sum_{i=1}^{2m}[g(\phi Y,\phi Z)S(\phi w_i,\phi w_i)\\ &\quad -g(\phi w_i,\phi Z)S(\phi Y,\phi w_i)+g(\phi w_i,\phi w_i)S(\phi Y,\phi Z)\\ &\quad -g(\phi Y,\phi w_i)S(\phi w_i,\phi Z)]\\ &\quad -\frac{\tau}{(2m+s-1)(2m+s-2)}\sum_{i=1}^{2m}[g(\phi Y,\phi Z)g(\phi w_i,\phi w_i)\\ &\quad -g(\phi w_i,\phi Z)g(\phi Y,\phi w_i)]\end{aligned} \tag{3.3}$$

It can be easily verify that

$$\sum_{i=1}^{2m} g(R(\phi w_i,\phi Y)\phi Z,\phi w_i) = S(\phi Y,\phi Z), \tag{3.4}$$

$$\sum_{i=1}^{2m} S(\phi w_i,\phi w_i) = \tau \tag{3.5}$$

$$\sum_{i=1}^{2m} g(\phi w_i,\phi Z)S(\phi Y,\phi w_i) = S(\phi Y,\phi Z) \tag{3.6}$$

$$\sum_{i=1}^{2m} g(\phi w_i,\phi w_i) = 2m \tag{3.7}$$

and

$$\sum_{i=1}^{2m} g(\phi w_i,\phi Z)g(\phi Y,\phi w_i) = g(\phi Y,\phi Z) \tag{3.8}$$

So by virtue of (3.4)–(3.8) the equation (3.3) can be written as

$$\begin{aligned}S(\phi Y,\phi Z) &= \frac{1}{2m+s-2}[\tau g(\phi Y,\phi Z)-2S(\phi Y,\phi Z)+2mS(\phi Y,\phi Z)]\\ &\quad -\frac{\tau}{(2m+s-1)(2m+s-2)}[2mg(\phi Y,\phi Z)-g(\phi Y,\phi Z)]\end{aligned}$$

$$S(\phi Y,\phi Z) = \frac{\tau}{2m+s-1}g(\phi Y,\phi Z). \tag{3.9}$$

Then by making use of (2.2) and (2.6), the equation (3.9) takes the form

$$S(Y,Z) = \frac{\tau}{2m+s-1}g(Y,Z) - \frac{\tau}{2m+s-1}\sum_{i=1}^{s}\eta_i(Y)\,\eta_i(Z)\,. \tag{3.10}$$

Therefore from (3.10), by contraction, we obtain $s=1$ which is a contradiction. This completes the proof of the theorem. □

From Theorem 3.1, we have following corollary.

Corollary 3.1. *Let M be an $(2m+1)-$dimensional $\phi-$conformally flat $C-$manifold. Then M is an $\eta-$Einstein manifold.*

Definition 3.2. A differentiable manifold (M^n, g), $n > 3$, satisfying the condition

$$\phi^2 K(\phi X, \phi Y)\phi Z = 0 \tag{3.11}$$

is called $\phi-$conharmonically flat.

In [2], the authors considered $(k, \mu)-$contact manifolds satisfying (3.11). Now we will study the condition (3.11) on $C-$ manifolds.

Theorem 3.2. *Let M be an $(2m+s)-$dimensional, $(s > 4)$, $C-$manifold There is no exist $\phi-$conharmonically flat $C-$manifold.*

Proof. Assume that (M, g), $(s > 4)$, is a $\phi-$conharmonically flat $C-$manifold. It can be easily seen that $\phi^2 K(\phi X, \phi Y)\phi Z = 0$ holds if and only if

$$g(K(\phi X, \phi Y)\phi Z, \phi W) = 0,$$

for any $X, Y, Z, W \in \chi(M)$. Using (1.3) $\phi-$conformally flat means

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) &= \frac{1}{2m+s-2}[g(\phi Y, \phi Z)S(\phi X, \phi W) - g(\phi X, \phi Z)S(\phi Y, \phi W) \\ &\quad + g(\phi X, \phi W)S(\phi Y, \phi Z) - g(\phi Y, \phi W)S(\phi X, \phi Z)]. \end{aligned} \tag{3.12}$$

Similar to the proof of Theorem 3.1, we can suppose that $\{w_1, ..., w_{2m}, \xi_1, ..., \xi_s\}$ is a local orthonormal basis of vector fields in M. By using the fact that $\{\phi w_1, ..., \phi w_{2m}, \xi_1, ..., \xi_s\}$ is also a local orthonormal basis, if we put $X = W = w_i$ in (3.12) and sum up with respect to i, then

$$\begin{aligned} \sum_{i=1}^{2m} g(R(\phi w_i, \phi Y)\phi Z, \phi w_i) &= \frac{1}{2m+s-2}\sum_{i=1}^{2m}[g(\phi Y, \phi Z)S(\phi w_i, \phi w_i) - g(\phi w_i, \phi Z)S(\phi Y, \phi w_i) \\ &\quad + g(\phi w_i, \phi w_i)S(\phi Y, \phi Z) - g(\phi Y, \phi w_i)S(\phi w_i, \phi Z)]. \end{aligned} \tag{3.13}$$

So by the use of (3.4)-(3.7) the equation (3.13) turns into

$$S(\phi Y, \phi Z) = \frac{\tau}{2m+s-2}g(\phi Y, \phi Z) - \frac{(2m-2)}{2m+s-2}S(\phi Y, \phi Z) \tag{3.14}$$

Thus applying (2.2) and (2.6) into (3.14) we get

$$S(Y, Z) = \frac{\tau}{4m+s-4}g(Y, Z) - \frac{\tau}{4m+s-4}\sum_{i=1}^{s}\eta_i(Y)\eta_i(Z) \tag{3.15}$$

from (3.10), by contraction, we obtain $n = 4$ which is a contradiction. □

From Theorem 3.2, we have following corollary.

Corollary 3.2. *Let M be an $(2m+4)-$dimensional $\phi-$conharmonically flat $C-$manifold. Then M is an $\eta-$Einstein manifold.*

Similar to Definition 3.1 and Definition 3.2 we can state the following:

Definition 3.3. A differentiable manifold (M^n, g), $n > 3$, satisfying the condition

$$\phi^2 P(\phi X, \phi Y)\phi Z = 0 \tag{3.16}$$

is called $\phi-$projectively flat.

Theorem 3.3. *Let M be an $2m+s$ dimensional, $(s \geq 2)$, $C-$manifold. There not exist $\phi-$projectively flat $C-$manifold.*

Proof. We assume that M be an $2m+s$ −dimensional, $(s \geq 2)$, $\phi-$projectively flat $C-$manifold. It can be easily seen that $\phi^2 P(\phi X, \phi Y)\phi Z = 0$ holds if and only if

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = 0,$$

for any $X, Y, Z, W \in \chi(M)$. Using ((1.1) and (1.4) $\phi-$projectively flat means

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2m+s-2}[g(\phi Y, \phi Z)S(\phi X, \phi W) - g(\phi X, \phi Z)S(\phi Y, \phi W). \tag{3.17}$$

In a manner similar to the method in the proof of Theorem 2, choosing $\{w_1, ..., w_{2m}, \xi_1, ..., \xi_s\}$ as a local orthonormal basis of vector fields in M and using the fact that $\{\phi w_1, ..., \phi w_{2m}, \xi_1, ..., \xi_s\}$ is also a local orthonormal basis, putting $X = W = w_i$ in (3.17) and summing up with respect to i, then we have

$$\sum_{i=1}^{2m} g(R(\phi w_i, \phi Y)\phi Z, \phi w_i) = \frac{1}{2m+s-2}\sum_{i=1}^{2m}[g(\phi Y, \phi Z)S(\phi w_i, \phi w_i) - g(\phi w_i, \phi Z)S(\phi Y, \phi w_i). \tag{3.18}$$

So applying (3.4)–(3.6) into (3.18) we get

$$S(\phi Y, \phi Z) = \frac{\tau}{2m+s-1} g(\phi Y, \phi Z)$$

Hence by virtue of (2.2) and (2.6) we obtain

$$S(Y, Z) = \frac{\tau}{2m+s-1} g(Y, Z) - \frac{\tau}{2m+s-1} \sum_{i=1}^{s} \eta_i(Y)\eta_i(Z) \tag{3.19}$$

Therefore from (3.19), by contraction, we obtain $s = 1$ which is a contraction. Hence, the proof is completed. □

From Theorem 3.3, we have following corollary.

Corollary 3.3. *Let M be an $(2m+1)$ −dimensional $\phi-$projectively flat $C-$manifold. Then M is an $\eta-$Einstein manifold.*

References

[1] K. Arslan, C. Murathan and C. Ozgür, *On $\phi-$Conformally flat contact metric manifolds*, Balkan J. Geom. Appl. (BJGA), 5 (2) (2000), 1–7.

[2] K. Arslan, C. Murathan and C. Ozgür, *On contact manifolds satisfying certain curvature conditions*, Proceedings of the Centennial "G. Vranceanu" and the Annual Meeting of the Faculty of Mathematics (Bucharest, 2000). An. Univ. Bucureşti Mat. Inform., 49 (2) (2000), 17–26.

[3] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2002.

[4] D. E. Blair., *Geometry of manifolds with structural group $U(n)xO(s)$*, J. *Diff. Geom.*, 4(1970), 155-167.

[5] D. E. Blair., *On a generalization of the Hopf fibration*, *An. St. Univ. "Al. I. Cuza"Iasi*, 17(1971), 171-177.

[6] D. E. Blair., G. D. Ludden and K. Yano, *Differential geometric structures on principal torodial bundles*, *Trans. Am. Math. Soc.*, 181(1973), 175-184.

[7] D. E. Blair, *Contact manifolds in Riemannian Geometry*, Lecture Notes in Math. Springer–Verlag, Berlin–Heidelberg–New–York, 509 (1976).

[8] J.L. Cabrerizo, L.M. Fernandez, M. Fernandez and G. Zhen, *The structure of a class of K–contact manifolds*, Acta Math. Hungar, 82 (4) (1999), 331–340.

[9] J.L. Cabrerizo, L.M. Fernandez, M. Fernandez, *The curvature tensor fields on $f-$manifolds with complemented frames*, An. şt. Univ. "Al. I. Cuza" Iaşi Matematica, 36(1990), 151-162.

[10] I. Mihai and R. Rosca, *On Lorentzian P–Sasakian manifolds*, Classical Analysis, World Scientific Publ, Singapore (1992), 155–169.

[11] I. Sato, *On a structure similar to almost contact structure*, Tensor N.S, 30 (1976), 219–224.

[12] I. Sato, *On a structure similar to almost contact structure II*, Tensor N.S, 31 (1977), 199–205.

[13] H. Singh, Q. Khan, *On special weakly symmetric Riemannian manifolds*, Publ. Math. Debrecen, Hungary 58(2001), 523–536.

[14] C. Özgür, *$\phi-$conformally flat Lorentzian para-Sasakian manifolds*, Radovi Mathematicki, 12(2003), 99-106.

[15] Y. Ishii, *On conharmonic transformations*, Tensor N.S, 7 (1957), 73–80.

[16] K. Yano, *On a structure defined by a tensor field f of type $(1,1)$ satisfing $f^3+f=0$ Tensor*, 14(1963), 99-109.

[17] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Math, Vol 3, World Sci, 1984.

[18] G. Zhen, *On conformal symmetric K–contact manifolds*, Chinese Quart. J. of Math, 7 (1992), 5–10.

[19] G. Zhen, J.L. Cabrerizo, L.M. Fernandez and M. Fernandez, *On $\xi-$conformally flat contact metric manifolds*, Indian J. Pure Appl. Math, 28 (1997), 725–734.

Aksaray University, Faculty of Art and Sciences, Department of Mathematics, Aksaray/TURKEY

E-mail address: materdalo@gmail.com