Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 1–8 (2014) ©KJM

MODULES WITH VALUES IN THE SPACE OF ALL DERIVATIONS OF AN ALGEBRA

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Abstract. In this paper, we construct a groupoid associated to a module with values in the space of all derivations of a unital algebra. More precisely, for a pair $(\mathcal{A}, \mathcal{G})$ consisting of an algebra $\mathcal{A}$ with a unit, a module $\mathcal{G}$ over the center $Z(\mathcal{A})$ of $\mathcal{A}$ together with a homomorphism of $Z(\mathcal{A})$-modules from $\mathcal{G}$ to the space of all derivations $Der(\mathcal{A})$ of $\mathcal{A}$, we associate a groupoid. We discuss on the equivalence relation induced from this groupoid.

1. Introduction

The concept of a groupoid is a generalization of the concept of a group, the main difference being that not any two elements of a groupoid are composable. Note that groupoids generalize not only the notion of a group but also the notion of a group action. A groupoid can be endowed with the algebraic, geometric or topological structures and in this case we can study the compatibility among these structures and groupoid.

Note that the theory of groupoids has developed in different fields of mathematics. The algebraic, topological and differentiable groupoids play an important role in algebra, measure theory, harmonic analysis, differential geometry and symplectic geometry. This can also be seen from a look at the list of references (see [3, 5, 6, 7, 8, 9]).

A set $\mathcal{H}^{(1)}$ has the structure of a *groupoid* with the set of units $\mathcal{H}^{(0)}$, if there are defined maps $\Delta : \mathcal{H}^{(0)} \to \mathcal{H}^{(1)}$, an involution $\imath : \mathcal{H}^{(1)} \to \mathcal{H}^{(1)}$ and denoted by $\imath(\alpha) = \alpha^{-1}$, a map $r : \mathcal{H}^{(1)} \to \mathcal{H}^{(0)}$, a map $s : \mathcal{H}^{(1)} \to \mathcal{H}^{(0)}$ and an associative multiplication $(\alpha, \beta) \mapsto \alpha\beta$ defined on the set

$$\mathcal{H}^{(2)} = \{(\alpha, \beta) \in \mathcal{H}^{(1)} \times \mathcal{H}^{(1)} \mid s(\alpha) = r(\beta)\},$$

satisfying the conditions

(i) $s(\alpha) = r(\alpha^{-1}), \;\; \alpha\alpha^{-1} = \Delta(r(\alpha))$,

(ii) $r(\Delta(t)) = t = s(\Delta(t)), \;\; \alpha\Delta(s(\alpha)) = \alpha, \;\; \Delta(r(\alpha))\alpha = \alpha$,

for all $\alpha \in \mathcal{H}^{(1)}$ and $t \in \mathcal{H}^{(0)}$.

2000 *Mathematics Subject Classification.* 53C12, 58B34.
Key words and phrases. Groupoid, Lie algebra, Lie algebroid.

It is known that for an arbitrary groupoid $(\mathcal{H}^{(1)}, \mathcal{H}^{(0)})$ there is an equivalence relation on the unit set $\mathcal{H}^{(0)}$. Namely, for two elements $x, y \in \mathcal{H}^{(0)}$ the relation $x \sim y$ iff $s^{-1}(x) \cap r^{-1}(y) \neq \emptyset$ is an equivalence relation on the unit set $\mathcal{H}^{(0)}$.

In [1], a method of associating a groupoid to a smooth manifold was introduced. In this paper, we use the same method to construct a groupoid associated to a module with values in the space of all derivations of a unital algebra. The focus in this paper is on the several examples.

Let $\mathcal{A}$ be an algebra with a unit. Let $Der(\mathcal{A})$ denote the space of all derivations of $\mathcal{A}$, i.e., the space of all linear mappings X of $\mathcal{A}$ into itself satisfying the Leibniz rule $X(ab) = X(a)b + aX(b)$. The space $Der(\mathcal{A})$ is in a natural way a module over the center $Z(\mathcal{A})$ of $\mathcal{A}$. Furthermore, the space $Der(\mathcal{A})$ is also a Lie algebra with Lie bracket $[X, Y] = XY - YX$.

Consider a pair $(\mathcal{A}, \mathcal{G})$ consisting of a module $\mathcal{G}$ over the center $Z(\mathcal{A})$ of $\mathcal{A}$ together with a linear map from $\mathcal{G}$ to $Der(\mathcal{A})$, which is also a homomorphism of $Z(\mathcal{A})$-modules. In this paper, such pairs are called $\mathcal{A}$-pairs.

We now give a brief summary of how the paper is organized.

In **Section** 2, we begin with our basic construction. We construct a groupoid associated to an $\mathcal{A}$-pair and we shall discuss on the equivalence relation induced from this groupoid. In the case when $\mathcal{G}$ is a Lie algebra, we will give the conditions that the equivalence classes are abelian Lie subalgebras of the Lie algebra $\mathcal{G}$.

In **Section** 3, we compute and investigate the equivalence classes for several examples. This section is devoted to the central algebras, foliation manifolds and the endomorphism algebra of a vector bundle.

Our basic reference for groupoids is [2], and for an extensive use of them one can refer to [5].

Throughout this paper, all smooth manifolds are assumed to be real, Hausdorff, and finite-dimensional. All vector fields on manifolds are assumed to be smooth. If M is a smooth manifold, let $\Im(M)$ be the Lie algebra of all vector fields on M and let $C^\infty(M)$ be the algebra of all smooth functions on M.

2. Groupoid associated to an $\mathcal{A}$-pair

In this section, we will introduce and construct a groupoid associated to an $\mathcal{A}$-pair. Let $\mathcal{A}$ be an algebra with a unit.

Definition 2.1. By an $\mathcal{A}$*-pair* we mean a pair $(\mathcal{A}, \mathcal{G})$, where $\mathcal{G}$ is a module over the center $Z(\mathcal{A})$ of $\mathcal{A}$ together with a homomorphism of $Z(\mathcal{A})$-modules $T : \mathcal{G} \to Der(\mathcal{A})$.

In this paper, $X(a)$ denotes $T(X)(a)$, for all $X \in \mathcal{G}$ and $a \in \mathcal{A}$. Using Definition 2.1, for all $X \in \mathcal{G}$ and $a \in Z(\mathcal{A})$ we have $X(a) \in Z(\mathcal{A})$.

Example 2.1. The pair $(\mathcal{A}, Der(\mathcal{A}))$ is an $\mathcal{A}$-pair.

Consider an $\mathcal{A}$-pair $(\mathcal{A}, \mathcal{G})$. The set of all invertible elements in the algebra $\mathcal{A}$ is denoted by $Inv(\mathcal{A})$, that is, for all $a \in Inv(\mathcal{A})$ there exists an element $a^{-1} \in \mathcal{A}$ such that $aa^{-1} = 1 = a^{-1}a$.

Let $\Gamma_{\mathcal{A}} = (Z(\mathcal{A}) \cap Inv(\mathcal{A})) \times Z(\mathcal{A})$. Fix an element $X \in \mathcal{G}$. Let

$$\mathcal{G}_X^{(1)}(\mathcal{A}) = \{(Y, a, b) \mid Y \in \mathcal{G},\ (a, b) \in \Gamma_{\mathcal{A}},\ Y(a) = X(b) + b\}.$$

We have to show that the pair $(\mathcal{G}_X^{(1)}(\mathcal{A}), \mathcal{G})$ has the structure of a groupoid. Define the map $\Delta : \mathcal{G} \to \mathcal{G}_X^{(1)}(\mathcal{A})$ by

$$\Delta(Y) := (Y, 1, 0).$$

Since $Y(1) = X(0) + 0$, it follows that Δ is well-defined. Define $r, s : \mathcal{G}_X^{(1)}(\mathcal{A}) \to \mathcal{G}$ by

$$r(Y, a, b) := Y, \qquad s(Y, a, b) := aY - bX.$$

Define the involution $\imath : \mathcal{G}_X^{(1)}(\mathcal{A}) \to \mathcal{G}_X^{(1)}(\mathcal{A})$ by

$$\imath(Y, a, b) = (Y, a, b)^{-1} := (aY - bX, a^{-1}, -a^{-1}b).$$

We have

$$\begin{aligned}(aY - bX)(a^{-1}) - X(-a^{-1}b) &= aY(a^{-1}) + a^{-1}Y(a) - a^{-1}b \\ &= Y(1) - a^{-1}b = -a^{-1}b.\end{aligned}$$

This shows that $\imath$ is well-defined. Define the multiplication by

$$(Y, a, b)(aY - bX, c, d) := (Y, ac, bc + d).$$

We have

$$\begin{aligned}Y(ac) - X(bc + d) &= (Y(a) - X(b))c + aY(c) - bX(c) - X(d) \\ &= bc + (aY - bX)(c) - X(d) = bc + d.\end{aligned}$$

It is straightforward to check the following axioms are true:

$$\begin{aligned}&s(\alpha) = r(\alpha^{-1}), \qquad r(\Delta(Y)) = Y = s(\Delta(Y)), \\ &\alpha\alpha^{-1} = \Delta(r(\alpha)), \\ &\alpha\Delta(s(\alpha)) = \alpha, \qquad \Delta(r(\alpha))\alpha = \alpha,\end{aligned}$$

for all $\alpha \in \mathcal{G}_X^{(1)}(\mathcal{A})$ and $Y \in \mathcal{G}$.

For any fixed $X \in \mathcal{G}$, we obtain an equivalence relation on the module $\mathcal{G}$ (see Section 1 above). We say that two elements Y and W of $\mathcal{G}$ are equivalent iff there exists a pair $(a, b) \in \Gamma_{\mathcal{A}}$ such that

$$aW = bX + Y, \qquad W(a) = X(b) + b.$$

Let $[(X, Y)]_{\mathcal{A}}$ be the equivalence class of any $Y \in \mathcal{G}$.

Let us give an example. In the following, we consider the case for $(C^\infty(\mathbb{R}^2), \Im(\mathbb{R}^2))$ (see Example 2.1 above).

Example 2.2. The vector field X on $\mathbb{R}^2$ defined in terms of the identity chart x by

$$X = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}$$

has integral curves $\gamma(t) = (z_1 \exp t, z_2 \exp t)$ starting at the point (z_1, z_2). Let $\zeta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ be the zero vector field, defined by $\zeta(f) = 0$ for each $f \in C^\infty(\mathbb{R}^2)$. We have

$$[(X, \zeta)]_{C^\infty(\mathbb{R}^2)} = \{\frac{g}{f} X \mid (f, g) \in \Gamma_{C^\infty(\mathbb{R}^2)},\ gX(f) - fX(g) = fg\}.$$

Assume that $W = \frac{g}{f} X \in [(X, \zeta)]_{C^\infty(\mathbb{R}^2)}$ such that g is a non-zero function. Since $\frac{d}{dt}(\frac{g}{f} \circ \gamma) = X(\frac{g}{f}) \circ \gamma$, it follows that

$$f(z_1, z_2)g(z_1 \exp t, z_2 \exp t) \exp t = f(z_1 \exp t, z_2 \exp t)g(z_1, z_2),$$

on $\mathbb{R}$, for all $(z_1, z_2) \in \mathbb{R}^2$. Hence, we have

$$f(z_1, z_2)g(z_1 s, z_2 s)s = f(z_1 s, z_2 s)g(z_1, z_2),$$

for all $(z_1, z_2) \in \mathbb{R}^2$ and all $s > 0$. Since $\frac{g}{f} \neq 0$ we can choose $z \in \mathbb{R}^2$ such that $\frac{g(z)}{f(z)} \neq 0$. Then $\lim_{s \to 0} |\frac{g(sz)}{f(sz)}|$ would be infinite and this would imply that $\frac{g}{f}$ is not continuous at 0 and this is a contradiction. Hence, we have $[(X, \zeta)]_{C^\infty(\mathbb{R}^2)} = \{\zeta\}$.

In addition, in the case when $\mathcal{G}$ is a Lie algebra we will give the conditions that the equivalence classes are Lie subalgebras of the Lie algebra $\mathcal{G}$.

The following theorem is the main result of this paper.

Theorem 2.1. *Consider an $\mathcal{A}$-pair $(\mathcal{A}, \mathcal{G})$, where $\mathcal{G}$ is also a Lie algebra with a Lie bracket $[,]$. Moreover, assume that for all $a \in Z(\mathcal{A})$ and $X, Y \in \mathcal{G}$ we have $[X, aY] = X(a)Y + a[X, Y]$. Let $X, Y \in \mathcal{G}$. Then the following properties are equivalent:*
(i) $[(X, Y)]_{\mathcal{A}}$ is an abelian Lie subalgebra of the Lie algebra $\mathcal{G}$,
(ii) $[(X, Y)]_{\mathcal{A}} = \{aX \mid a \in Z(\mathcal{A}),\ X(a) = -a\}$,
(iii) there is an element $t \in Z(\mathcal{A})$ such that $Y = -tX$ and $X(t) = -t$.

Proof. It is easy to check that $(i) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (iii)$. It suffices to prove that $(iii) \Longrightarrow (i)$. Let $t \in Z(\mathcal{A})$ such that $Y = -tX$ and $X(t) = -t$. Assume that $W, Z \in [(X, Y)]_{\mathcal{A}}$ and $\lambda \in \mathbb{C}$. We have to show that $W + Z \in [(X, Y)]_{\mathcal{A}}, \lambda W \in [(X, Y)]_{\mathcal{A}}$ and $[W, Z] = 0$. Choose (a, b) and (c, d) in $\Gamma_{\mathcal{A}}$ such that

$$cZ = (d - t)X, \qquad aW = (b - t)X,$$

and

$$Z(c) = X(d) + d, \qquad W(a) = X(b) + b.$$

Let $(s, h) = (ac, ad + bc + t(1 - c - a))$. It follows that

$$s(W + Z) - hX = Y.$$

Also, we have

$$\begin{aligned}(W + Z)(s) - X(h) &= X(b)c + cb + (b - t)X(c) + (d - t)X(a) \\ &\quad + aX(d) + ad - X(a)d - aX(d) - X(b)c \\ &\quad - bX(c) - X(t) + X(t)c + tX(c) + X(t)a \\ &\quad + tX(a) \\ &= ad + bc - X(t) + X(t)c + X(t)a = h.\end{aligned}$$

So, we have $W + Z \in [(X, Y)]_{\mathcal{A}}$. On the other hand, it is simple to see that $\lambda W \in [(X, Y)]_{\mathcal{A}}$. Also, we have

$$\begin{aligned}ac[W, Z] &= (b - t)X(d - t)X - (b - t)\underbrace{X(c)c^{-1}(d - t)}_{X(d)+d}X \\ &\quad - (d - t)X(b - t)X + (d - t)\underbrace{X(a)a^{-1}(b - t)}_{X(b)+b}X \\ &= (b - t)X(d - t)X - (b - t)(X(d) + d)X \\ &\quad - (d - t)X(b - t)X + (d - t)(X(b) + b)X = 0.\end{aligned}$$

Since $ac \in Inv(\mathcal{A})$, it follows that $[W, Z] = 0$. ☐

Let M be a smooth manifold and X a vector field on it. A non-zero function $h \in C^\infty(M)$ such that $X(h) = \lambda h$, for some real number λ, is said to be an *eigenfunction* of the vector field X and λ is called the corresponding *eigenvalue*. Note that a non-zero function $h \in C^\infty(M)$ is an eigenfunction of a vector field X corresponding to a zero eigenvalue if and only if it is constant on the range of every integral curve.

Example 2.3. (i) Any vector field X on a compact manifold M has all its eigenvalues zero. Let $\zeta : C^\infty(M) \to C^\infty(M)$ be the zero vector field. Using Theorem 2.1, one gets $[(X,\zeta)]_{C^\infty(M)} = \{\zeta\}$.

(ii) The vector field X on $\mathbb{R}^2$ defined in terms of the identity chart x by

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$$

has every eigenvalue zero, since its integral curves

$$\gamma(t) = (a\sin(t+b), a\cos(t+b)), \quad \gamma(0) = (a\sin(b), a\cos(b)),$$

are all periodic, that is, there exists $r > 0$ such that $\gamma(t_1) = \gamma(t_2)$ if and only if $t_1 - t_2 = kr$, for some $k \in \mathbb{Z}$. Let ζ be the zero vector field on $\mathbb{R}^2$. Hence, using Theorem 2.1, we have $[(X,\zeta)]_{C^\infty(\mathbb{R}^2)} = \{\zeta\}$.

3. Examples

Let us compute and investigate the equivalence classes for some examples. First, we consider the case where the center of the algebra $\mathcal{A}$ is the set $Z(\mathcal{A}) = \mathbb{C}1$. For example, consider the algebra of $n \times n$ complex matrices $M_n(\mathbb{C})$.

Example 3.1. Consider an $\mathcal{A}$-pair $(\mathcal{A},\mathcal{G})$, assume that $Z(\mathcal{A}) = \mathbb{C}1$. We see that $X(Z(\mathcal{A})) = 0$, for all $X \in \mathcal{G}$. Let $X, Y \in \mathcal{G}$ and $W \in [(X,Y)]_{\mathcal{A}}$. It follows that

$$W(a) = X(b) + b, \qquad aW = bX + Y,$$

for $a = \alpha 1, b = \beta 1$, where $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq 0$. Therefore, we have $b = 0$ and $W = \frac{1}{\alpha} Y$. Thus $[(X,Y)]_{\mathcal{A}} \subset (\mathbb{C} - \{0\})Y$. On the other hand, it is simple to see that $(\mathbb{C} - \{0\})Y \subset [(X,Y)]_{\mathcal{A}}$. Hence, we have

$$[(X,Y)]_{\mathcal{A}} = (\mathbb{C} - \{0\})Y,$$

for all $X, Y \in \mathcal{G}$.

Recall that a p-dimensional *foliation* $\mathcal{F}$ on a n-dimensional smooth manifold M consists of the partition of M into maximal integral submanifolds (*leaves*) of an integrable, p-dimensional subbundle $F = T\mathcal{F}$ of the tangent bundle TM. The vector fields on M which are tangent to the leaves of $\mathcal{F}$ form a Lie subalgebra of the Lie algebra $\Im(M)$, which we denote by $\Im(\mathcal{F})$. In other words, $\Im(\mathcal{F})$ consists of the sections of the tangent bundle $T\mathcal{F}$ of the foliation $\mathcal{F}$. A smooth function φ on M is called *basic* if it is constant along the leaves. Equivalently, a function φ is basic if $X(\varphi) = 0$ whenever $X \in \Im(\mathcal{F})$, briefly $\Im(\mathcal{F})(\varphi) = 0$. We refer to [4], for details on foliations.

Example 3.2. Let $(M,\mathcal{F})$ be a foliation manifold. The basic functions on $(M,\mathcal{F})$ form a subalgebra $\mathcal{A}$ of $C^\infty(M)$:

$$\mathcal{A} = \{\varphi \in C^\infty(M) \mid \Im(\mathcal{F})(\varphi) = 0\}.$$

In general, the Lie subalgebra $\Im(\mathcal{F})$ is not a Lie ideal in $\Im(M)$, but it is clearly a Lie ideal in the Lie subalgebra

$$\mathcal{G} = \{Z \in \Im(M) \bigm| [\Im(\mathcal{F}), Z] \subset \Im(\mathcal{F})\}.$$

Remark that $\mathcal{G}$ is a module over the algebra of basic functions. Also, the definition of the Lie bracket implies that the derivative of a basic function in the direction of a vector field in $\mathcal{G}$ is again basic. Therefore, we can define a linear map $T : \mathcal{G} \to Der(\mathcal{A})$ by $T(Z)(\varphi) = Z(\varphi)$. We obtain that the pair $(\mathcal{A}, \mathcal{G})$ is an $\mathcal{A}$-pair. Let $X \in \Im(\mathcal{F}) \subset \mathcal{G}$ and one gets

$$[(X, X)]_{\mathcal{A}} = \{\varphi X \bigm| \varphi \in Inv(\mathcal{A})\}.$$

In the following, we consider the case for the pair

$$(End(\mathcal{E}), Der(End(\mathcal{E}))),$$

which $End(\mathcal{E})$ is the algebra of the endomorphisms of a vector bundle $\mathcal{E}$ over a smooth manifold M. We compute the equivalence class for $X, Y \in Der(End(\mathcal{E}))$ which are also homomorphisms of $C^\infty(M)$-modules.

Example 3.3. Let $\mathcal{E}$ be a finite-dimensional complex (or real) vector bundle over a smooth manifold M. We denote by $End(\mathcal{E})$ the algebra of the endomorphisms of this bundle. Any element $\varphi \in End(\mathcal{E})$ can be considered as a section of the bundle of endomorphisms. Therefore, for any element $\varphi \in End(\mathcal{E})$ and any point $p \in M$ we have $\varphi_p \in End(\mathcal{E}_p)$. The center of the algebra $End(\mathcal{E})$ is the set

$$C^\infty(M) \cdot 1 = Z(End(\mathcal{E})).$$

Assume that derivations $X, Y : End(\mathcal{E}) \to End(\mathcal{E})$ are homomorphisms of $C^\infty(M)$-modules, i.e., $X(f \cdot \varphi) = f \cdot X(\varphi)$ and $Y(f \cdot \varphi) = f \cdot Y(\varphi)$, for all $f \in C^\infty(M)$ and $\varphi \in End(\mathcal{E})$. Hence, we obtain

$$X(Z(End(\mathcal{E}))) = 0 = Y(Z(End(\mathcal{E}))).$$

Let $W \in [(X, Y)]_{End(\mathcal{E})}$. Hence, there exists a pair (a, b) such that

$$W(a) = X(b) + b, \qquad aW = bX + Y,$$

where $a = f \cdot 1, b = g \cdot 1,\ f \in Inv(C^\infty(M))$ and $g \in C^\infty(M)$. Therefore, we have $W(Z(End(\mathcal{E}))) = 0$ and

$$W = (\frac{1}{f} \cdot 1)Y \in (Inv(C^\infty(M)) \cdot 1)Y.$$

Also, it is simple to see that $(Inv(C^\infty(M)) \cdot 1)Y \subset [(X, Y)]_{End(\mathcal{E})}$. Hence, we obtain

$$[(X, Y)]_{End(\mathcal{E})} = (Inv(C^\infty(M)) \cdot 1)Y,$$

for all derivations $X, Y : End(\mathcal{E}) \to End(\mathcal{E})$ that are homomorphisms of $C^\infty(M)$-modules.

As another example we would like to investigate the equivalence classes for derivations of the endomorphism algebra $End(\mathcal{E})$ of a vector bundle $\mathcal{E}$ over a smooth manifold M. In the following, we investigate the equivalence classes for a type of derivations of the algebra $End(\mathcal{E})$ that are not homomorphisms of $C^\infty(M)$-modules.

Recall that a Lie algebroid may be thought of as a generalization of the tangent bundle of a manifold. Just as Lie algebras are in some sense the infinitesimal versions of Lie groups, Lie algebroids are objects that play a similar role for Lie groupoids. A *Lie algebroid* over the manifold M is the triple $(\mathcal{K}, [,], \mu)$ where $\mathcal{K}$

is a vector bundle over M, whose $C^\infty(M)$-module of sections $\Gamma(\mathcal{K})$ is equipped with a Lie algebra structure $[,]$ and $\mu : \mathcal{K} \to TM$ is a bundle map which induces a Lie algebra homomorphism (also denoted μ) from $\Gamma(\mathcal{K})$ to $\Im(M)$, satisfying the Leibnitz rule

$$[X, fY] = \mu(X)(f)Y + f[X, Y],$$

for all $f \in C^\infty(M)$ and $X, Y \in \Gamma(\mathcal{K})$. Here, the map $\mu : \mathcal{K} \to TM$ is called an *anchor map* (see [10, 11]).

Example 3.4. Let M be a smooth manifold and $(\mathcal{K}, [,], \mu)$ be a Lie algebroid over M. Consider a $\mathcal{K}$-connection on a vector bundle $\mathcal{E}$ over M. So, there exists a $\mathbb{R}$-bilinear map $\nabla : \Gamma(\mathcal{K}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ such that

$$\nabla_{fX}(s) = f\nabla_X(s), \qquad \nabla_X(fs) = \mu(X)(f)s + f\nabla_X(s),$$

for all $f \in C^\infty(M)$, $X \in \Gamma(\mathcal{K})$ and $s \in \Gamma(\mathcal{E})$. For any $X \in \Gamma(\mathcal{K})$ define a derivation $D_X : End(\mathcal{E}) \to End(\mathcal{E})$ as

$$D_X(\varphi)(s) = \nabla_X(\varphi(s)) - \varphi(\nabla_X(s)),$$

for all $\varphi \in End(\mathcal{E})$ and $s \in \Gamma(\mathcal{E})$. It is simple to see that

$$D_X(f \cdot \varphi) = \mu(X)(f) \cdot \varphi + f \cdot D_X(\varphi),$$

for all $f \in C^\infty(M)$ and $\varphi \in End(\mathcal{E})$. Thus, one gets

$$D_X(f \cdot 1) = \mu(X)(f) \cdot 1,$$

for all $f \in C^\infty(M)$. Take $X, Y \in \Gamma(\mathcal{K})$ and $W \in [(X, Y)]_{C^\infty(M)}$. There exists a pair $(f, g) \in \Gamma_{C^\infty(M)}$ such that

$$fW = gX + Y, \qquad \mu(W)(f) = \mu(X)(g) + g.$$

We show that $D_W \in [(D_X, D_Y)]_{End(\mathcal{E})}$. For all $\varphi \in End(\mathcal{E})$ and $s \in \Gamma(\mathcal{E})$ we have

$$\begin{aligned}((f \cdot 1)D_W - (g \cdot 1)D_X)(\varphi)(s) &= \nabla_{fW-gX}(\varphi(s)) - \varphi(\nabla_{fW-gX}(s)) \\ &= \nabla_Y(\varphi(s)) - \varphi(\nabla_Y(s)) \\ &= D_Y(\varphi)(s),\end{aligned}$$

which implies that $(f \cdot 1)D_W - (g \cdot 1)D_X = D_Y$. Also, we have

$$\begin{aligned}D_W(f \cdot 1) - D_X(g \cdot 1) &= \mu(W)(f) \cdot 1 - \mu(X)(g) \cdot 1 \\ &= (\mu(W)(f) - \mu(X)(g)) \cdot 1 \\ &= g \cdot 1,\end{aligned}$$

hence one gets $D_W \in [(D_X, D_Y)]_{End(\mathcal{E})}$. So, we can define the surjective map $F : [(X, Y)]_{C^\infty(M)} \to [(D_X, D_Y)]_{End(\mathcal{E})}$ by $F(W) = D_W$. Now, we can check that if the anchor map μ is injective then

$$[(X, Y)]_{C^\infty(M)} \simeq [(D_X, D_Y)]_{End(\mathcal{E})},$$

for all $X, Y \in \Gamma(\mathcal{K})$.

References

[1] H. Abbasi, GH. Haghighatdoost, Groupoid associated to a smooth manifold (preprint).
[2] R. Brown, From groups to groupoids, a brief survey, *Bull. London Math. Soc.*, **19** (1987) 113-134.
[3] H. Bursztyn, O. Radko, Gauge equivalence of Dirac structures and symplectic groupoids, *Ann. Inst. Fourier*, **53** (2003) 309-337.
[4] C. Camacho, A. Neto, *Geometric theory of foliations*, Birkhauser, Boston, Massachusetts, (1985).
[5] A. Connes, *Noncommutative geometry*, Academic Press, San Diego, (1994).
[6] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, Springer Verlag, Berlin, (1980).
[7] A. Weinstein, Symplectic groupoids and Poisson manifolds, *Bull. Amer. Math. Soc.*, **16** (1987) 101-104.
[8] A. Weinstein, Coisotropic calculus and Poisson groupoids, *J. Math. Soc. Japan*, **40** (1988) 705-727.
[9] K. Mikami, A. Weinstein, Moments and reduction for symplectic groupoid actions, *Publ. Rims Kyoto University*, **24** (1988) 121-140.
[10] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society, Lecture Note Series, Cambridge, no., **124** (1987).
[11] E. Martinez, Lagrangian mechanics on Lie algebroids, *Acta Appl. Math.*, **67** (2001) 295-320.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 9–21 (2014) ©KJM

FRAMED-COMPLEX SUBMERSIONS

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Abstract. In this paper, we introduce the concept of framed-complex submersion from a framed metric manifold onto an almost Hermitian manifold. We investigate the influence of a given structure defined on the total manifold on the determination of the corresponding structure on the base manifold. Moreover, we provide an example, investigate various properties of the O'Neill's tensors for such submersions, find the integrability of the horizontal distribution. We also obtain curvature relations between the base manifold and the total manifold.

1. Introduction

The theory of Riemannian submersion was introduced by O'Neill and Gray in [15] and [11], respectively. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. Riemannian submersions were considered between almost complex manifolds by Watson in [19] under the name of almost Hermitian submersion. He showed that if the total manifold is a Kähler manifold, the base manifold is also a Kähler manifold. Riemannian submersions between almost contact manifolds were studied by Chinea in [2] under the name of almost contact submersions. Since then, Riemannian submersions have been used as an effective tool to describe the structure of a Riemannian manifold equipped with a differentiable structure. For instance, Riemannian submersions have been also considered for quaternionic Kähler manifolds [12]. This kind of submersions have been studied with different names by many authors(see [7], [8],[9], [10], [13],[14], [18], and more).

On the other hand, let (M, g) be a Riemannian manifold equipped with a framed metric structure, i.e. an endomorphism φ of the tangent bundle such that $\varphi^3+\varphi = 0$ and which is compatible with g; the compatibility means that for each $X, Y \in TM$ we have $g(\varphi X, Y) = -g(X, \varphi Y)$ [21]. Moreover we assume that the kernel of φ is of constant rank and parallelizable, i.e. there exist global vector fields $\xi_1, ..., \xi_s$ spanning kerφ. Such manifolds are necessarily of dimension $2m + s$ where $2m$ is the

2000 *Mathematics Subject Classification.* 53C15, 53C40.

Key words and phrases. Framed metric manifold, Almost Hermitian manifold, Riemannian submersion, Framed-complex submersion.

rank of φ. The study of such manifolds was started by Blair, Goldberg and Yano ([1], [5], [6]). In this paper, we define framed-complex submersions from a framed metric manifold onto an almost Hermitian manifold and study the geometry of such submersions. We observe that framed-complex submersion has also rich geometric properties.

The paper is organized as follows. In section 2, we collect basic definitions, some formulas and results for later use. In section 3, we introduce the notion of framed-complex submersions and give an example of framed-complex submersion. Moreover, we investigate properties of O'Neill's tensors and show that such tensors have nice algebraic properties for framed-complex submersions. We find the integrability of the horizontal distribution. In section 4 is focused on the transference of structures defined on the total manifold. Finally, we obtain relations between bisectional curvatures and sectional curvatures of the base manifold, the total manifold and the fibres of a framed-complex submersion.

2. Preliminaries

In this section, we are going to recall main definitions and properties of framed metric manifolds, almost Hermitian manifolds and Riemannian submersions.

2.1. Framed metric manifolds. Let M be a $(2m+s)$- dimensional framed metric manifold [20](or almost s-contact metric manifold[17]) with a framed metric structure $(\varphi, \xi_j, \eta_j, g)$, $j \in \{1, ..., s\}$, that is, φ is a $(1,1)-$tensor field defining a $\varphi-$structure of rank $2m$; $\xi_1, ..., \xi_s$ are s vector fields; $\eta_1, ..., \eta_s$ are s 1-forms and g is a Riemannian metric on M such that

$$\varphi^2 = -I + \sum_{j=1}^{s} \eta_j \otimes \xi_j, \quad \eta_j(\xi_i) = \delta_i^j, \quad \varphi(\xi_j) = 0, \quad \eta_j \circ \varphi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{j=1}^{s} \eta_j(X)\eta_j(Y), \tag{2.2}$$

$$\Phi(X, Y) = g(X, \varphi Y) = -\Phi(Y, X), \tag{2.3}$$

$$g(X, \xi_j) = \eta_j(X) \tag{2.4}$$

for all $X, Y \in \Gamma(TM)$ and $i, j \in \{1, ..., s\}$[20].

A framed metric structure is called normal[20]if

$$[\varphi, \varphi] + 2d\eta_j \otimes \xi_j = 0, \tag{2.5}$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{2.6}$$

We note that a framed metric manifold $(M^{2m+s}, g, \varphi, \xi_j, \eta_j)$ is called

(a) *almost* $\mathcal{S}-$*manifold*, if $d\eta_j = \Phi$;
(b) $\mathcal{K}-$*manifold*, if $d\Phi = 0$ and normal;
(c) $\mathcal{S}-$*manifold*, if $d\eta_j = d\Phi$ and normal;
(d) *almost* $\mathcal{C}-$*manifold*, if $d\eta_j = 0$, $d\Phi = 0$;

(e) $\mathcal{C}-$*manifold*, if $d\eta_j = 0$, $d\Phi = 0$ and normal([1], [3]).

We have the following relation between the Levi-Civita connection and fundamental 2-form of M.

Lemma 2.1. [3]. *Let $(M, \varphi, \xi_j, \eta_j, g)$ be a framed metric manifold. Then we have*

$$\begin{aligned} 2g((\nabla_X \varphi)Y, Z) &= 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) + +g(N^{(1)}(Y, Z), \varphi X) \\ &\quad + \sum_{j=1}^{s} \{N^{(2)}(Y, Z)\eta_j(X) + 2d\eta_j(\varphi Y, X)\eta_j(Z) \\ &\quad -2d\eta_j(\varphi Z, X)\eta_j(Y)\}, \end{aligned} \tag{2.7}$$

where the tensor field $N^{(2)}$ is defined defined by $N^{(2)}(X, Y) = (L_{\varphi X}\eta_j)(Y) - (L_{\varphi Y}\eta_j)(X)$, where $\mathcal{L}$ denotes the Lie derivative, for any X, Y, Z vector fields on M.

On $\mathcal{S}-$ manifolds we have[16]

$$\begin{aligned} (\nabla_X \Phi)(Y, Z) &= \frac{1}{2}\sum_{i=1}^{s}[\eta_i(Y)g(X, Z) - \eta_i(Z)g(X, Y)] \\ &\quad -\frac{1}{2}\sum_{i,j=1}^{s} \eta_j(X)[\eta_i(Y)\eta_j(Z) - \eta_i(Z)\eta_j(Y)]. \end{aligned} \tag{2.8}$$

It is easy to see that if M is a framed metric manifold, then the following identities are well known:

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) + 2\sum_{j=1}^{s} d\eta_j(X, Y)\xi_j, \tag{2.9}$$

$$(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi(\nabla_X Y), \tag{2.10}$$

$$(\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi)Z) = -g(Z, (\nabla_X \varphi)Y), \tag{2.11}$$

$$(\nabla_X \eta_j)Y = g(Y, \nabla_X \xi_j). \tag{2.12}$$

2.2. Almost Hermitian manifolds. Let M be an even-dimensional differentiable manifold. An almost Hermitian structure on M is by definition a pair (J, g) of an almost complex structure J and a Riemannian metric g satisfying

$$J^2(X) = -X, \quad g(JX, JY) = g(X, Y) \tag{2.13}$$

for any vector fields X, Y on M. A manifold with such a structure (J, g) is called an almost Hermitian manifold. The fundamental 2-from Φ of an almost Hermitian structure is defined by

$$\Phi(X, Y) = g(X, JY)$$

for any vector fields X, Y and is skew-symmetric[20].

The Nijenhuis(or the torsion) tensor of an almost complex structure J is defined by

$$N(X, Y) = -[X, Y] + [JX, JY] - J[X, JY] - J[JX, Y] \tag{2.14}$$

for any vector fields X, Y on M.
An almost Hermitian manifold (M, g, J) is called

(a) *Kähler* if $\nabla J = 0$;
(b) *almost Kähler* if $d\Phi = 0$;
(c) *nearly Kähler* if $(\nabla_X J)X = 0$;
(d) *Hermitian* if $N = 0$[20], where N is the Nijenhuis tensor of J.

2.3. Riemannian Submersions. Let (M, g) and (B, g') be two Riemannian manifolds. A surjective $C^\infty-$map $\pi : M \to B$ is a $C^\infty-$submersion if it has maximal rank at any point of M. Putting $\mathcal{V}_x = Ker\pi_{*x}$, for any $x \in M$, we obtain an integrable distribution $\mathcal{V}$, which is called vertical distribution and corresponds to the foliation of M determined by the fibres of π. The complementary distribution $\mathcal{H}$ of $\mathcal{V}$, determined by the Riemannian metric g, is called horizontal distribution. A $C^\infty-$submersion $\pi : M \to B$ between two Riemannian manifolds (M, g) and (B, g') is called a Riemannian submersion if, at each point x of M, π_{*x} preserves the length of the horizontal vectors. A horizontal vector field X on M is said to be basic if X is $\pi-$related to a vector field X' on B. It is clear that every vector field X' on B has a unique horizontal lift X to M and X is basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion $\pi : M \to B$ determines two $(1, 2)$ tensor fields T and A on M, by the formulas:

$$T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{2.15}$$

and

$$A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{2.16}$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projections (see [4]). From (2.15) and (2.16), one can obtain

$$\nabla_U X = T_U X + h(\nabla_U X); \tag{2.17}$$

$$\nabla_X U = v(\nabla_X U) + A_X U; \tag{2.18}$$

$$\nabla_X Y = A_X Y + h(\nabla_X Y), \tag{2.19}$$

for any $X, Y \in \Gamma(\mathcal{H})$, $U \in \Gamma(\mathcal{V})$. Moreover, if X is basic then

$$h(\nabla_U X) = h(\nabla_X U) = A_X U. \tag{2.20}$$

We note that for $U, V \in \Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma(\mathcal{H})$, $A_X Y = \frac{1}{2} v[X, Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X, Y \in \Gamma(\mathcal{H})$ and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U, V \in \Gamma(\mathcal{V})$.

We now recall the following result which will be useful for later.

Lemma 2.2. *(see* [4],[15]*). If $\pi : M \to B$ is a Riemannian submersion and X, Y basic vector fields on M, $\pi-$related to X' and Y' on B, then we have the following properties*

(1) *$h[X,Y]$ is a basic vector field and $\pi_* h[X,Y] = [X',Y'] \circ \pi$;*
(2) *$h(\nabla_X Y)$ is a basic vector field $\pi-$related to $(\nabla'_{X'} Y')$, where ∇ and ∇' are the Levi-Civita connection on M and B;*
(3) *$[E,U] \in \Gamma(\mathcal{V})$, for any $U \in \Gamma(\mathcal{V})$ and for any basic vector field E.*

3. Framed-Complex Submersions

In this section, we define the notion of framed-complex submersion, give an example and study the geometry of such submersions. We now define a $(\varphi, J)-$holomorphic map between framed metric manifolds and almost Hermitian manifolds.

Definition 3.1. Let $(M^{2m+s}, \varphi, (\xi_j, \eta_j)_{j=1}^s, g)$ be a framed metric manifold and (B^{2n}, J) be an almost complex manifold, respectively. The map $\pi : M \to B$ is $(\varphi, J)-$holomorphic if $\pi_* \circ \varphi = J \circ \pi_*$.

By using the above definition, we are ready to give the following notion.

Definition 3.2. Let $(M^{2m+s}, \varphi, (\xi_j, \eta_j)_{j=1}^s, g)$ be a framed metric manifold and (B, J, g') be an almost Hermitian manifold. A Riemannian submersion $\pi : M \to B$ is called a framed-complex submersion if it is (φ, J)-holomorphic, as well.

Consider R^{2m+s} with its standard coordinates $x_1, ..., x_m, y_1, ..., y_m, z_1, ..., z_s$. We introduce on R^{2m+s} a framed metric structure $(\varphi, \xi_j, \eta_j, g)$ by setting

$$\eta_j = dz_j, \ \ \xi_j = \frac{\partial}{\partial z_j}, \ \ g = 2\sum_{i=1}^{m}((dx_i)^2 + (dy_i)^2) + \sum_{j=1}^{s}(\eta_j \otimes \eta_j)$$

and φ given, with respect to the frame $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_m}, \frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_s})$ by the $(2m+s) \times (2m+s)-$matrix

$$\begin{pmatrix} 0 & -I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On the other hand, the canonical almost complex structure on R^{2n} is given by

$$J(x_1, ..., x_{2n}) = (-x_{2n}, -x_{2n-1}, ..., x_2, x_1),$$

where Riemannian metric is standard inner product defined on R^{2n}.

We now give an example for a framed-complex submersion.

Example 3.1. Consider the following submersion defined by

$$\begin{array}{rcl} \pi : R^{4+2} & \to & R^2 \\ (x_1, x_2, y_1, y_2, z_1, z_2) & \to & (x_1 + x_2, y_1 + y_2). \end{array}$$

Then, the kernel of π_* is

$$\mathcal{V} = Ker\pi_* = Span\{V_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, V_2 = -\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, \xi_1 = \frac{\partial}{\partial z_1}, \xi_2 = \frac{\partial}{\partial z_2}\}$$

and the horizontal distribution is spanned by

$$\mathcal{H} = (Ker\pi_*)^{\perp} = Span\{X = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Y = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\}.$$

Hence, we have

$$g(X,X)=g'(\pi_*X,\pi_*X)=4,\quad g(Y,Y)=g'(\pi_*Y,\pi_*Y)=4.$$

Thus, π is a Riemannnian submersion. Moreover, we can easily obtain that π satisfies

$$\pi_*\varphi X=J\pi_*X,\quad \pi_*\varphi Y=J\pi_*Y.$$

Thus, π is a framed-complex submersion.

As an obvious consequence of Definition 3.2 we obtain:

Proposition 3.1. *Let $\pi:M\to B$ be a framed-complex submersion from a framed metric manifold M onto an almost Hermitian manifold B. If X,Y are basic vector fields on M, $\pi-$related to X',Y' on B, then, we have*

(i) *$h(\nabla_X\varphi)Y$ is the basic vector field $\pi-$related to $(\nabla'_{X'}J)Y'$;*
(ii) *φX is the basic vector field $\pi-$related to JX'.*

Next proposition shows that a framed-complex submersion puts some restrictions on the distributions $\mathcal{V}$ and $\mathcal{H}$.

Proposition 3.2. *Let $\pi:M\to B$ be a framed-complex submersion from a framed metric manifold M onto an almost Hermitian manifold B. Then, the horizontal and vertical distributions are $\varphi-$ invariant.*

Proof. Consider a vertical vector field U; it is known that $\pi_*(\varphi U)=J(\pi_*U)$. Since U is vertical and π is a Riemannian submersion, we have $\pi_*U=0$ from which $\pi_*(\varphi U)=0$ follows and implies that φU is vertical, being in the kernel of π_*. As concerns the horizontal distribution, let X be a horizontal vector field. We have $g(\varphi X,U)=-g(X,\varphi U)=0$ because φU is vertical and X is horizontal. From $g(\varphi X,U)=0$ we deduce that φX is orthogonal to U and then φX is horizontal. □

Proposition 3.3. *Let $\pi:M\to B$ be a framed-complex submersion from a framed metric manifold M onto an almost Hermitian manifold B. Then, we have*

(i) *$\pi^*\Phi'=\Phi$ holds on the horizontal distribution, only;*
(ii) *Each ξ_j is vertical vector field, $j\in\{1,...,s\}$;*
(iii) *$\eta_j(X)=0$, for all horizontal vector fields X.*

Proof. We prove only statement (i), the other assertions can be obtained in a similar way. If X and Y are basic vector fields on M, $\pi-$related to X',Y' on B, then using the definition of a framed-complex submersion, we have

$$\begin{aligned}\pi^*\Phi'(X,Y) &= \Phi'(\pi_*X,\pi_*Y)=g'(\pi_*X,J\pi_*Y)=g'(\pi_*X,\pi_*\varphi Y)\\ &= \pi^*g'(X,\varphi Y)=g(X,\varphi Y)=\Phi(X,Y)\end{aligned}$$

which gives the proof of assertion(i). □

We now check the properties of the tensor fields T and A for a framed-complex submersion, we will see that such tensors have extra properties for such submersions.

Lemma 3.1. *Let $\pi:M\to B$ be a framed-complex submersion. If the total space is a $\mathcal{C}-$manifold, then we have*

(i) $T_U\varphi V=\varphi T_UV$;

(ii) $T_{\varphi U}V = \varphi T_U V$,

for $U, V \in \Gamma(\mathcal{V})$.

Proof. We only prove (i), the other assertion can be obtained in a similar way. Let U and V be vertical vector fields, and X horizontal. Since M is a $\mathcal{C}-$manifold, from (2.7) we get

$$2g((\nabla_U \varphi)V, X) = 0.$$

Then, since the vertical and the horizontal distributions are $\varphi-$invariant, from (2.15) we obtain

$$g(T_U \varphi V - \varphi T_U V, X) = 0.$$

Hence, we have

$$T_U \varphi V = \varphi T_U V.$$

□

For the tensor field A we have the following.

Lemma 3.2. *Let $\pi : M \to B$ be a framed-complex submersion. If the total space is a $\mathcal{C}-$manifold, then we have*

(i) $A_X \varphi Y = \varphi A_X Y$;

(ii) $A_{\varphi X} Y = \varphi A_X Y$;

(iii) $A_{\varphi X} X = 0$;

for $X, Y \in \Gamma(\mathcal{H})$.

Using (2.8) we have the following.

Lemma 3.3. *Let $\pi : M \to B$ be a framed-complex submersion. If the total space is a $\mathcal{S}-$manifold, then we have*

(i) $T_U \varphi V = \varphi T_U V$;

(ii) $T_{\varphi U} V = \varphi T_U V$,

(iii) $A_X \varphi Y = \varphi A_X Y$;

(iv) $A_{\varphi X} Y = \varphi A_X Y$;

(v) $A_{\varphi X} X = 0$;

for $X, Y \in \Gamma(\mathcal{H})$ *and* $U, V \in \Gamma(\mathcal{V})$.

We shall be interested with the tensor T which is a usefully tool in the study of the fibres.
If $T_U \varphi V = \varphi T_U V$, then the fibres are minimal and if $T = 0$ they are totally geodesic.

Theorem 3.1. *Let $\pi : M \to B$ be a framed-complex submersion from a $\mathcal{S}-$manifold or a $\mathcal{C}-$manifold M onto an almost Hermitian manifold B. If for all U, V, $T_{\varphi U}\varphi V + T_U V = 0$, then T=0.*

Proof. Let U and V be vertical vector fields. From Lemma 3.1, we get $T_{\varphi U}\varphi V = \varphi T_{\varphi U} V$. Using again Lemma 3.1, we have

$$T_{\varphi U}\varphi V = \varphi T_{\varphi U} V = \varphi^2 T_U V = -T_U V + \sum_{j=1}^{s} \eta_j(T_U V)\xi_j, j \in \{1, ..., s\}.$$

On the other hand, it can be shown that $T_U\xi_j = 0$ and then $\eta_j(T_U V) = 0$ which gives $T_{\varphi U}\varphi V = -T_U V$ from which $T_{\varphi U}\varphi V + T_U V = 0$ follows. Thus, we get $T = 0$. □

Corollary 3.1. *Let $\pi : M \to B$ be a framed-complex submersion. If the total space is a $\mathcal{K}-$ manifold or an almost $\mathcal{C}-$manifold, then the fibres are totally geodesic.*

We now investigate the integrability of the horizontal distribution $\mathcal{H}$.

Theorem 3.2. *Let $\pi : M \to B$ be a framed-complex submersion from a $\mathcal{K}-$manifold M onto an almost Hermitian manifold B. Then, the horizontal distribution is integrable.*

Proof. Let X and Y be basic vector fields. It suffices to prove that $v([X, Y]) = 0$, for basic vector fields on M. Since M is a $\mathcal{K}-$manifold, it implies $d\Phi(X, Y, V) = 0$, for any vertical vector V. Then, one obtains

$$\begin{gathered} X(\Phi(Y, V)) - Y(\Phi(X, V)) + V(\Phi(X, Y)) \\ -\Phi([X, Y], V) + \Phi([X, V], Y) - \Phi([Y, V], X) = 0. \end{gathered}$$

Since $[X, V], [Y, V]$ are vertical and the two distributions are $\varphi-$invariant, the last two and the first two terms vanish. Thus, one gets

$$g([X, Y], \varphi V) = V(g(X, \varphi Y)).$$

On the other hand, if X is basic then $h(\nabla_V X) = h(\nabla_X V) = A_X V$, thus we have

$$\begin{aligned} V(g(X, \varphi Y)) &= g(\nabla_V X, \varphi Y) + g(\nabla_V \varphi Y, X) \\ &= g(A_X V, \varphi Y) + g(A_{\varphi Y} V, X). \end{aligned}$$

Since A are skew-symmetric and alternating operator, we get $V(g(X, \varphi Y)) = 0$. This proves the assertion. □

Corollary 3.2. *Let $\pi : M \to B$ be a framed-complex submersion. If the total space is a $\mathcal{S}-$manifold, an almost $\mathcal{C}-$manifold or a $\mathcal{C}-$manifold, then the horizontal distribution is integrable.*

4. Transference of Structures

In this section, we investigate what kind of almost Hermitian structures are defined on the base manifold, when the total manifold has some special framed structures.

As the fibres of a framed-complex submersion is an invariant submanifold of M with respect to φ, we have the following.

Proposition 4.1. *Let $\pi : (M^{2m+s}, \varphi, \xi, \eta, g) \to (B^{2n}, J, g')$ be a framed-complex submersion from a framed metric manifold M onto an almost Hermitian manifold B. Then, the fibres are framed metric manifolds.*

Proof. Denoting by F the fibres, it is clear that $dimF = 2(m-n)+s = 2r+s$, where $r = m-n$. We define a framed metric structure $(\hat{g}, \hat{\varphi}, \hat{\eta_j}, \hat{\xi_j})$, where $j = 1, ..., s$, by setting $\varphi = \hat{\varphi}$, $\eta_j = \hat{\eta_j}$and $\xi_j = \hat{\xi_j}$. Then, we get

$$\hat{\varphi}^2 U = \varphi^2 U = -U + \sum_{j=1}^{s} \eta_j(U)\xi_j,$$

for $U \in \Gamma(\mathcal{V})$.

On the other hand, for $U, V \in \Gamma(\mathcal{V})$ we obtain

$$\begin{aligned} \hat{g}(\hat{\varphi}V, \hat{\varphi}U) &= \hat{g}(\varphi V, \varphi U) = -\hat{g}(V, \varphi^2 U) = -\hat{g}(V, -U + \sum_{j=1}^{s} \eta_j(U)\xi_j) \\ &= \hat{g}(V, U) - \sum_{j=1}^{s} \hat{\eta}_j(U)\hat{\eta}_j(V), \end{aligned}$$

which gives the proof of assertion. □

In the sequel, we show that base space is a Hermitian manifold if the total space is a normal.

Theorem 4.1. *Let $\pi : M \to B$ be a framed-complex submersion. If the framed metric structure of M is normal, then the base space B is a Hermitian manifold.*

Proof. Let X and Y be basic vector fields on M, $\pi-$related to X' and Y' on B. From (2.5), we have

$$\pi_*(N^{(1)}(X, Y)) = \pi_*([\varphi, \varphi](X, Y) + \sum_{j=1}^{s} 2d\eta_j(X, Y)\xi_j).$$

On the other hand, $\pi_*\varphi = J\pi_*$ implies that

$$\begin{aligned} \pi_*([\varphi, \varphi](X, Y)) &= \pi_*(\varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]) \\ &= -[\pi_* X, \pi_* Y] + \sum_{j=1}^{s} \eta_j([X, Y])\pi_*\xi_j + [\pi_*\varphi X, \pi_*\varphi Y] - J\pi_*[\varphi X, Y] \\ &- J\pi_*[X, \varphi Y] \\ &= -[X', Y'] + [JX', JY'] - J[JX', Y'] \\ &- J[X', JY']. \end{aligned}$$

Then, we have

$$\pi_*([\varphi, \varphi](X, Y)) = N'(X', Y') = 0,$$

which shows that B is a Hermitian manifold. □

Proposition 4.2. *Let $\pi : M \to B$ be a framed-complex submersion. If the total space M is an almost $\mathcal{C}-$ manifold, then the base space B is an almost Kähler manifold.*

Proof. Let X, Y and Z be basic vector fields on M $\pi-$related to X', Y' and Z' on B. Since M is an almost $\mathcal{C}-$manifold, we have $d\Phi(X, Y, Z) = 0$. Then, we obtain

$$\begin{gathered} X(\Phi(Y, Z)) - Y(\Phi(X, Z)) + Z(\Phi(X, Y)) \\ -\Phi([X, Y], Z) + \Phi([X, Z], Y) - \Phi([Y, Z], X) = 0. \end{gathered}$$

On the other hand, by direct calculations, we get

$$\begin{aligned} 0 &= g(\nabla_X Y, \varphi Z) + g(Y, \nabla_X \varphi Z) - g(\nabla_Y X, \varphi Z) - g(X, \nabla_Y \varphi Z) \\ &+ g(\nabla_Z X, \varphi Y) + g(X, \nabla_Z \varphi Y) - g([X,Y], \varphi Z) \\ &+ g([X,Z], \varphi Y) - g([Y,Z], \varphi X). \end{aligned}$$

Then, by using $\pi_*\varphi = J\pi_*$, we get

$$\begin{aligned} &0 = g'(\nabla'_{X'}Y', JZ') + g'(Y', \nabla'_{X'}JZ') - g'(\nabla'_{Y'}X', JZ') - g'(X', \nabla'_{Y'}JZ') \\ &+g'(\nabla'_{Z'}X', JY') + g'(X', \nabla'_{Z'}JY') - g'([X',Y'], JZ') \\ &+g'([X',Z'], JY') - g'([Y',Z'], JX') \\ &0 = X'(\Phi'(Y',Z')) - Y'(\Phi'(X',Z')) + Z'(\Phi'(X',Y')) \\ &-\Phi'([X',Y'],Z') + \Phi'([X',Z'],Y') - \Phi'([Y',Z'],X') \\ &0 = d\Phi'(X',Y',Z'). \end{aligned}$$

Thus, if the total space M is an almost $\mathcal{C}-$manifold, then the base space B is an almost Kähler manifold. □

Corollary 4.1. *Let $\pi : M \to B$ be a framed-complex submersion. If the total space M is a $\mathcal{K}-$ manifold or a $\mathcal{S}-$ manifold , then the base space B is an almost Kähler manifold.*

We also have the following result which shows that the other structures can be mapped onto the base manifold.

Proposition 4.3. *Let $\pi : M \to B$ be a framed-complex submersion. If the total space M is a $\mathcal{C}-$ manifold, then the base space B is a Kähler manifold.*

Proof. Let X, Y and Z be basic vector fields on M $\pi-$related to X', Y' and Z' on B. Since M is a $\mathcal{C}-$ manifold, from (2.7) we get

$$g((\nabla_X\varphi)Y, Z) = g(\nabla_X\varphi Y - \varphi\nabla_X Y, Z) = 0.$$

Since π is a Riemannian submersion, we obtain

$$g'(\pi_*(\nabla_X\varphi Y - \varphi\nabla_X Y), \pi_*(Z)) = 0.$$

Then, by using $\pi_*\varphi = J\pi_*$, we get

$$g'(\pi_*(\nabla_X\varphi Y - J\pi_*(\nabla_X Y)), \pi_*(Z)) = 0.$$

On the other hand, from Proposition 3.1, we know that if X is $\pi-$related to X', then φX is $\pi-$related to JX'. Also, from Lemma 2.2, it follows $h(\nabla_X\varphi Y)$ and $h(\nabla_X Y)$ are $\pi-$related to $\nabla'_{X'}JY'$ and $\nabla'_{X'}Y'$. Thus, we have

$$g'(\nabla'_{X'}JY' - J\nabla'_{X'}Y', Z') = 0.$$

Then, from (2.10) we get $g'(\nabla'_{X'}J)Y', Z') = 0$. Hence, we have $(\nabla'_{X'}J)Y' = 0$ which proves the assertion. □

Corollary 4.2. *Let $\pi : M \to B$ be a framed-complex submersion. If the total space M is an almost $\mathcal{C}-$manifold, a $\mathcal{K}-$ manifold, an almost $\mathcal{S}-$ manifold, a $\mathcal{C}-$manifold or a $\mathcal{S}-$ manifold , then the fibres inherit from the total space a structure of the same type.*

5. Curvature Relations for Framed-Complex Submersions

We begin this section relating the $\varphi-$holomorphic bisectional and sectional curvatures of the total space, the base space and the fibres of a framed-complex submersion.

Let π be a framed-complex submersion between a framed metric manifold M and an almost Hermitian manifold N. We denote the Riemannian curvatures of M, N and any fibre $\pi^{-1}(x)$ by R, R' and $\hat{R}$, respectively. For $X, Y, Z, W \in \Gamma(\mathcal{H})$, we have

$$R^*(X,Y,Z,W) = R'(\pi_*X, \pi_*Y, \pi_*Z, \pi_*W) \circ \pi.$$

Let $\pi : M \to N$ be a framed-complex submersion from a framed metric manifold $(M, \varphi, \xi_j, \eta_j, g)$ onto an almost Hermitian manifold (N, J, g'). We denote by B the $\varphi-$holomorphic bisectional curvature, defined for any pair of vectors X and Y on M orthogonal to ξ_j by the formula:

$$B(X,Y) = \frac{R(X, \varphi X, Y, \varphi Y)}{\|X\|^2 \|Y\|^2}.$$

The $\varphi-$holomorphic sectional curvature is $H(X) = B(X,X)$ for any vector X orthogonal to ξ_j, $j \in \{1, ..., s\}$. We denote by B' and H' the $\varphi-$holomorphic bisectional and $\varphi-$holomorphic sectional curvatures of N. Similarly, $\hat{B}$ and $\hat{H}$ denote the bisectional and the sectional holomorphic curvatures of a fibre.

The following is a translation of the results of Gray[11] and O'Neill[15] to the present situation:

Proposition 5.1. *Let $\pi : M \to N$ a framed-complex submersion from a framed metric manifold M onto an almost Hermitian manifold N. Let U and V be unit vertical vectors, and X and Y unit horizontal vectors orthogonal to ξ_j. Then, we have*

$$\begin{aligned}
(a) B(U,V) &= \hat{B}(U,V) - g(T_U V, T_{\varphi U} \varphi V) + g(T_{\varphi U} V, T_U \varphi V);\\
(b) B(X,U) &= g((\nabla_U A)_X \varphi X, \varphi U) - g((\nabla_{\varphi U} A)_X \varphi X, U)\\
&+ g(A_X U, A_{\varphi X} \varphi U) - g(A_X \varphi U, A_{\varphi X} U)\\
&- g(T_U X, T_{\varphi U} \varphi X) + g(T_{\varphi U} X, T_U \varphi X);\\
(c) B(X,Y) &= B'(X', Y') \circ \pi - 2g(A_X \varphi X, A_Y \varphi Y)\\
&+ g(A_{\varphi X} Y, A_X \varphi Y) - g(A_X Y, A_{\varphi X} \varphi Y).
\end{aligned}$$

Using Proposition 5.1, we have the following result.

Proposition 5.2. *Let $\pi : M \to N$ a framed-complex submersion from a framed metric manifold M onto an almost Hermitian manifold N. Let U be unit vertical vector, and X unit horizontal vector orthogonal to ξ_j. Then, one has:*

(a) $H(U) = \hat{H}(U) + \|T_U \varphi U\|^2 - g(T_{\varphi U} \varphi U, T_U U)$;
(b) $H(X) = H'(X') \circ \pi - 3\|A_X \varphi X\|^2$.

Theorem 5.1. *Let $\pi : M \to N$ a framed-complex submersion. If the total space is a $\mathcal{K}-$manifold, a$\mathcal{S}-$manifold, an almost $\mathcal{C}-$manifold or a $\mathcal{C}-$manifold, then we have*

(a) *$B(U,V) = \hat{B}(U,V)$;*

(b) $H(U) = \hat{H}(U)$,

where U and V are unit vertical vectors orthogonal to ξ_j.

Proof. (a) Since the fibres are totally geodesic, we have $T = 0$. Then using Proposition 5.1(a) we get $B(U,V) = \hat{B}(U,V)$.
In a similar way, we obtain (b). $\square$

Since the horizontal distribution $\mathcal{H}$ is integrable, we get $A = 0$. Then, we have the following result.

Theorem 5.2. *Let $\pi : M \to N$ a framed-complex submersion. If the total space is a $\mathcal{K}-$manifold, a $\mathcal{S}-$manifold, an almost $\mathcal{C}-$manifold or a $\mathcal{C}-$manifold, then we have*

(a) $B(X,Y) = B'(X',Y') \circ \pi$;
(b) $H(X) = H'(X') \circ \pi$,

where X and Y are unit horizontal vectors orthogonal to ξ_j.

References

[1] D.E. Blair, Geometry of manifolds with structural group $\mathcal{U}(n) \times \mathcal{O}(s)$, J. Differential Geom. 4(2)(1970), 155-167.
[2] D. Chinea, Almost contact metric submersions, Rend. Circ. Mat. Palermo, II Ser. 34 (1985), 89-104.
[3] K.L. Duggal, S. Ianus, A.M. Pastore, Maps interchanging f-structures and their harmonicity, Acta Applicandae Mathematicae 67 (2001), 91-115.
[4] M. Falcitelli, S. Ianus, A.M. Pastore, Riemannian submersions and related topics, World Scientific, 2004.
[5] S.I. Goldberg, K. Yano, On normal globally framed $f-$manifolds, Tôhoku Math. Journal 22 (1970), 362-370.
[6] S.I. Goldberg, K. Yano, Globally framed $f-$manifolds, Illinois Math. Journal 22 (1971), 456-474.
[7] Y. Gündüzalp, B. Şahin, Paracontact semi-Riemannian submersions, Turkish J.Math. 37(1) (2013), 114-128.
[8] Y. Gündüzalp, B. Şahin, Para-contact para-complex semi-Riemannian submersions, Bull. Malays. Math. Sci. Soc. 37(1)(2014), 139-152.
[9] Y. Gündüzalp, Slant submersions from almost product Riemannian manifolds, Turkish J.Math. 37(5) (2013), 863-873.
[10] Y. Gündüzalp, Anti-Invariant Semi-Riemannian Submersions from Almost Para-Hermitian Manifolds, Journal of Function Spaces and Applications, ID 720623, 2013.
[11] A. Gray, Pseudo-Riemannian almost product manifolds and submersions, J. Math. Mech. 16 (1967), 715-737.
[12] S. Ianus, R. Mazzocco, G.E. Vilcu, Riemannian submersions from quaternionic manifolds, Acta Appl. Math. 104 (2008), 83-89.
[13] S. Ianus, S. Marchiafava, G.E. Vilcu, Para-quaternionic CR-submanifolds of para-quaternionic Kähler Manifols and semi-Riemannian submersions, Central European Journal of Mathematics 8, 4 (2010),735-753.
[14] S. Ianus, A.M. Ionescu, R. Mocanu, G.E. Vilcu, Riemannian submersions from almost contact metric Manifols, Abh. Math. Semin. Univ. Hamburg 81 (2011), 101-114.
[15] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J. 13 (1966), 459-469.
[16] L.D. Terlizzi, On invariant submanifolds of C-and S-manifolds, Acta Math. Hungar. 85(1999), 229-239.
[17] J. Vanzura, Almost s-contact structures, Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. 26(1972), 97-115.
[18] G.E. Vilcu, 3-submersions from QR-hypersurfaces of quaternionic Kähler manifolds, Ann. Polon. Math. 98 (2010), 301-309.

[19] B. Watson, Almost Hermitian submersions, J. Differential Geom. 11(1976), 147-165.
[20] K. Yano, M. Kon, Structures on manifolds, World Scientific, 1984.
[21] K. Yano, On a structure defined by a tensor field f satisfying $f^3 + f = 0$, Tensor 14 (1963), 99-109.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 22–34 (2014) ©KJM

APPROXIMATING THE RIEMANN–STIELTJES INTEGRAL BY A THREE–POINT QUADRATURE RULE AND APPLICATIONS

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Abstract. In this paper, a three–point quadrature rule for the Riemann–Stieltjes integral is introduced. As application; an error estimate for the obtained quadrature rule is provided as well.

1. Introduction

The Riemann–Stieltjes integral $\int_a^b f(t)\,dg(t)$ is an important concept in Mathematics with multiple applications in several subfields including Probability Theory & Statistics, Complex Analysis, Functional Analysis, Operator Theory and others.

In 2008, Mercer [27] has introduced new midpoint and trapezoid type rules for the Riemann–Stieltjes integral which engender a natural generalization of Hadamard's integral inequality, as follows:

Theorem 1.1. *Let g be continuous and increasing on $[a,b]$, let $c \in [a,b]$ which satisfies*

$$\int_a^b g(t)\,dt = (c-a)\,g(a) + (b-c)\,g(b).$$

If $f'' \geq 0$, then we have

$$f(c)\,[g(b)-g(a)] \leq \int_a^b f dg \leq [G-g(a)]\,f(a) + [g(b)-G]\,f(b) \tag{1.1}$$

where, $G := \frac{1}{b-a}\int_a^b g(t)\,dt$.

In fact, Mercer established the following quadrature rule for the Riemann–Stieltjes integral.

$$\int_a^b f dg \cong [G-g(a)]\,f(a) + [g(b)-G]\,f(b), \tag{1.2}$$

and so that, he obtained the error as follows:

Date: January 1, 2013 and, in revised form, February 2, 2013.
2000 *Mathematics Subject Classification.* 26D10, 26D15.
Key words and phrases. Riemann–Stieltjes integral, Quadrature rule.

Theorem 1.2. *Suppose that f'' and g' are continuous on $[a,b]$ and that g is monotonic there. Let $G := \frac{1}{b-a}\int_a^b g(t)\,dt$. Then there exist $\eta, \sigma \in (a,b)$ such that*

$$\int_a^b f dg - [G - g(a)] f(a) - [g(b) - G] f(b) = -f''(\eta) g'(\sigma) \frac{(b-a)^3}{12}. \tag{1.3}$$

Recently, Alomari and Dragomir [7], proved several new error bounds for the Mercer–Trapezoid quadrature rule (1.2) for the Riemann-Stieltjes integral under various assumptions for the integrand (and integrator) involved.

After that, and motivated by the method used in [27], Alomari and Dragomir [8] introduced the following quadrature formula:

Theorem 1.3. *Suppose that f'' and g' are continuous on $[a,b]$ and that g is monotonic on $[a,x]$ and $[x,b]$. Then there exist $\xi_1, \eta_1 \in (a,x)$ and $\xi_2, \eta_2 \in (x,b)$ such that*

$$\begin{aligned}\int_a^b f(t) g'(t)\,dt &= [G(a,x) - g(a)] f(a) + [G(x,b) - G(a,x)] f(x) \\ &\quad + [g(b) - G(x,b)] f(b) \\ &\quad - \frac{1}{12}\left[f''(\xi_1) g'(\eta_1)(x-a)^3 + f''(\xi_2) g'(\eta_2)(b-x)^3\right],\end{aligned} \tag{1.4}$$

for all $a < x < b$, where $G(\alpha,\beta) := \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)\,dt$.

For other quadrature rules for Riemann–Stieltjes integral under various assumptions to the function involved the reader may refer to [1]–[6], [9]–[26] and [28].

In this work, we study the quadrature rule

$$\begin{aligned}\int_a^b f(t)\,dg(t) \cong [G(a,x) - g(a)] f(a) + [G(x,b) - G(a,x)] f(x) \\ + [g(b) - G(x,b)] f(b)\end{aligned}$$

for all $x \in (a,b)$, by relaxing the conditions in Theorem 1.3. Various error estimates for the above quadrature rule are proved. As application an error estimate for the new three–point quadrature rule for Riemann–Stieltjes integral is given.

2. The case when f is of bounded variation

Theorem 2.1. *Fix $x \in (a,b)$. Let $f, g : [a,b] \to \mathbb{R}$ be such that f is of bounded variation on $[a,b]$ and g is continuous. If g is increasing on the both intervals $[a,x]$ and $[x,b]$, then*

$$|\mathcal{R}(f,g;x)| \le \left[\frac{g(b) - g(a)}{2} + \left|g(x) - \frac{g(a) + g(b)}{2}\right|\right] \cdot \bigvee_a^b (f) \tag{2.1}$$

for all $a < x < b$, where $G(\alpha,\beta) := \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)\,dt$.

Proof. It is easy to observe that

$$\mathcal{R}(f,g;x) = \int_a^x [g(t) - G(a,x)]\,df(t) + \int_x^b [g(t) - G(x,b)]\,df(t). \tag{2.2}$$

Using the fact that for a continuous function $p:[c,d]\to\mathbb{R}$ and a function $\nu:[c,d]\to\mathbb{R}$ of bounded variation, then the Riemann–Stieltjes integral $\int_c^d p(t)\,d\nu(t)$ exists and one has the inequality

$$\left|\int_c^d p(t)\,d\nu(t)\right|\le\sup_{t\in[c,d]}|p(t)|\bigvee_c^d(\nu)\,. \tag{2.3}$$

As f is of bounded variation on $[a,b]$, by (2.3) we have

$$\begin{aligned}|\mathcal{R}(f,g;x)|&\le\left|\int_a^x[g(t)-G(a,x)]\,df(t)\right|+\left|\int_x^b[g(t)-G(x,b)]\,df(t)\right|\\&\le\sup_{t\in[a,x]}|g(t)-G(a,x)|\cdot\bigvee_a^x(f)+\sup_{t\in(x,b]}|g(t)-G(x,b)|\cdot\bigvee_x^b(f)\,,\end{aligned}\tag{2.4}$$

but since g is increasing on $[a,x]$ and $[x,b]$, then

$$\begin{aligned}\sup_{t\in[a,x]}|g(t)-G(a,x)|&=\max\{g(x)-G(a,x)\,,G(a,x)-g(a)\}\\&=\frac{1}{2}[g(x)-g(a)+|g(x)-2G(a,x)+g(a)|]\,,\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}\sup_{t\in(x,b]}|g(t)-G(x,b)|&=\max\{g(b)-G(x,b)\,,G(x,b)-g(x)\}\\&=\frac{1}{2}[g(b)-g(x)+|g(b)-2G(x,b)+g(x)|]\,.\end{aligned}\tag{2.6}$$

Also, since

$$g(a)\le G(a,x)\le g(x)\,,$$

and

$$g(x)\le G(x,b)\le g(b)\,,$$

so that from (2.5) and (2.6), we have

$$\sup_{t\in[a,x]}|g(t)-G(a,x)|\le g(x)-g(a)$$

and

$$\sup_{t\in(x,b]}|g(t)-G(x,b)|\le g(b)-g(x)$$

which gives by (2.4) that

$$\begin{aligned}|\mathcal{R}(f,g;x)|&\le\sup_{t\in[a,x]}|g(t)-G(a,x)|\cdot\bigvee_a^x(f)+\sup_{t\in(x,b]}|g(t)-G(x,b)|\cdot\bigvee_x^b(f)\\&\le[g(x)-g(a)]\cdot\bigvee_a^x(f)+[g(b)-g(x)]\cdot\bigvee_x^b(f)\\&\le\left[\frac{g(b)-g(a)}{2}+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]\cdot\bigvee_a^b(f)\,,\end{aligned}$$

and thus the theorem is proved. □

Corollary 2.1. *In Theorem 2.1, choose $g(t) = t$, $t \in [a,b]$, then we have the inequality:*

$$\left| \frac{1}{2} \left[f(x) + \frac{(x-a) f(a) + (b-x) f(b)}{b-a} \right] - \int_a^b f(t)\, dt \right| \leq \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \cdot \bigvee_a^b (f), \tag{2.7}$$

for all $a < x < b$. Moreover, if we choose $x = \frac{a+b}{2}$, then we get

$$\left| \frac{1}{2} \left[f\left(\frac{a+b}{2} \right) + \frac{f(a) + f(b)}{2} \right] - \int_a^b f(t)\, dt \right| \leq \frac{1}{2} (b-a) \bigvee_a^b (f). \tag{2.8}$$

Theorem 2.2. *Fix $x \in (a,b)$. Let $f, g : [a,b] \to \mathbb{R}$ be such that f is of bounded variation on $[a,b]$ and g is continuous.*

(1) *If g is of bounded variation on $[a,b]$, then*

$$|\mathcal{R}(f,g;x)| \leq \left[\bigvee_a^b (g) + \left| \bigvee_a^x (g) - \bigvee_x^b (g) \right| \right] \cdot \bigvee_a^b (f). \tag{2.9}$$

(2) *If g is of L_g–Lipschitzian on $[a,b]$, then*

$$|\mathcal{R}(f,g;x)| \leq \frac{L_g}{2} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \cdot \bigvee_a^b (f). \tag{2.10}$$

for all $a < x < b$.

Proof. (1) Since f is of bounded variation on $[a,b]$, then by (2.4) we have

$$|\mathcal{R}(f,g;x)| \leq \sup_{t \in [a,x]} |g(t) - G(a,x)| \cdot \bigvee_a^x (f) + \sup_{t \in (x,b]} |g(t) - G(x,b)| \cdot \bigvee_x^b (f). \tag{2.11}$$

In [17], the author proved the following Ostrowski type inequality for functions of bounded variation

$$|g(t) - G(a,x)| = \left| g(t) - \frac{1}{x-a} \int_a^x g(s)\, ds \right| \leq \left[\frac{1}{2} + \left| \frac{t - \frac{a+x}{2}}{x-a} \right| \right] \bigvee_a^x (g),$$

it follows that,

$$\sup_{t \in [a,x]} |g(t) - G(a,x)| \leq \sup_{t \in [a,x]} \left[\frac{1}{2} + \left| \frac{t - \frac{a+x}{2}}{x-a} \right| \right] \bigvee_a^x (g) = \bigvee_a^x (g).$$

Similarly, one may observe that

$$\sup_{t \in [x,b]} |g(t) - G(x,b)| \leq \sup_{t \in [x,b]} \left[\frac{1}{2} + \left| \frac{t - \frac{x+b}{2}}{b-x} \right| \right] \bigvee_x^b (g) = \bigvee_x^b (g).$$

Combining the above two inequalities with (2.11), we get

$$\begin{aligned}\left|\mathcal{R}\left(f,g;x\right)\right| &\le \bigvee_a^x\left(g\right)\cdot\bigvee_a^x\left(f\right)+\bigvee_x^b\left(g\right)\cdot\bigvee_x^b\left(f\right)\\ &\le \left[\bigvee_a^b\left(g\right)+\left|\bigvee_a^x\left(g\right)-\bigvee_x^b\left(g\right)\right|\right]\cdot\bigvee_a^b\left(f\right)\end{aligned}$$

which proves (2.9).

(2) In [25], the author proved the following Ostrowski type inequality for Lipschitzian functions

$$\begin{aligned}\left|g\left(t\right)-G\left(a,x\right)\right| &= \left|g\left(t\right)-\frac{1}{x-a}\int_a^x g\left(s\right)ds\right|\\ &\le L_g\left[\frac{1}{4}+\left(\frac{t-\frac{a+x}{2}}{x-a}\right)^2\right]\left(x-a\right),\end{aligned}$$

it follows that,

$$\begin{aligned}\sup_{t\in[a,x]}\left|g\left(t\right)-G\left(a,x\right)\right| &\le L_g\sup_{t\in[a,x]}\left|g\left(t\right)-\frac{1}{x-a}\int_a^x g\left(s\right)ds\right|\\ &\le L_g\sup_{t\in[a,x]}\left[\frac{1}{4}+\left(\frac{t-\frac{a+x}{2}}{x-a}\right)^2\right]\left(x-a\right)=\frac{1}{2}L_g\left(x-a\right).\end{aligned}$$

Similarly, one may observe that

$$\begin{aligned}\sup_{t\in[x,b]}\left|g\left(t\right)-G\left(x,b\right)\right| &\le L_g\sup_{t\in[x,b]}\left[\frac{1}{4}+\left(\frac{t-\frac{x+b}{2}}{b-x}\right)^2\right]\left(b-x\right)\\ &=\frac{1}{2}L_g\left(b-x\right).\end{aligned}$$

Combining the above two inequalities with (2.11), we get

$$\begin{aligned}\left|\mathcal{R}\left(f,g;x\right)\right| &\le \frac{1}{2}L_g\left(x-a\right)\cdot\bigvee_a^x\left(f\right)+\frac{1}{2}L_g\left(b-x\right)\cdot\bigvee_x^b\left(f\right)\\ &\le \frac{L_g}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\bigvee_a^b\left(f\right).\end{aligned}$$

which proves (2.10).

Thus the theorem is completely proved. □

3. The case when f is of Lipschitz type

Theorem 3.1. *Fix $x\in(a,b)$. Let $f,g:[a,b]\to\mathbb{R}$ be such that f is L_f–Lipschitzian on $[a,b]$and g is a Riemann integrable on $[a,b]$. If there exists positive constants γ,Γ,ϕ,Φ such that*

$$\gamma\le g(t)\le\Gamma,\quad\forall t\in[a,x],$$

and

$$\phi\le g(t)\le\Phi,\quad\forall t\in(x,b],$$

for some $x \in (a,b)$. *Then,*

$$|\mathcal{R}(f,g;x)| \leq \frac{1}{2} L \left[(x-a)(\Gamma - \gamma) + (b-x)(\Phi - \phi) \right] \tag{3.1}$$

for all $a < x < b$.

Proof. Since

$$\mathcal{R}(f,g;x) = \int_a^x \left[g(t) - G(a,x) \right] df(t) + \int_x^b \left[g(t) - G(x,b) \right] df(t),$$

using the fact that for a Riemann integrable function $p : [c,d] \to \mathbb{R}$ and L-Lipschitzian function $\nu : [c,d] \to \mathbb{R}$, the inequality one has the inequality

$$\left| \int_c^d p(t)\, d\nu(t) \right| \leq L \int_c^d |p(t)|\, dt. \tag{3.2}$$

As f is L_f–Lipschitzian on $[a,b]$, by (3.2) we have

$$\begin{aligned} |\mathcal{R}(f,g;x)| &\leq \left| \int_a^x \left[g(t) - G(a,x) \right] df(t) \right| + \left| \int_x^b \left[g(t) - G(x,b) \right] df(t) \right| \\ &\leq L_f \left[\int_a^x |g(t) - G(a,x)|\, dt + \int_x^b |g(t) - G(x,b)|\, dt \right]. \end{aligned} \tag{3.3}$$

Now, using the same techniques applied in [26], we define

$$I_1(g) := \frac{1}{x-a} \int_a^x \left(g(t) - \frac{1}{x-a} \int_a^x g(s)\, ds \right)^2 dt.$$

Then, we have

$$\begin{aligned} I_1(g) : &= \frac{1}{x-a} \int_a^x \left[g^2(t) - 2g(t) \frac{1}{x-a} \int_a^x g(s)\, ds + \left(\frac{1}{x-a} \int_a^x g(s)\, ds \right)^2 \right] dt \\ &= \frac{1}{x-a} \int_a^x g^2(t)\, dt - \left(\frac{1}{x-a} \int_a^x g(s)\, ds \right)^2 \end{aligned}$$

and

$$\begin{aligned} I_1(g) := &\left(\Gamma - \frac{1}{x-a} \int_a^x g(s)\, ds \right) \left(\frac{1}{x-a} \int_a^x g(s)\, ds - \gamma \right) \\ &- \frac{1}{x-a} \int_a^x (\Gamma - g(t))(g(t) - \gamma)\, dt. \end{aligned}$$

As $\gamma \leq g(t) \leq \Gamma$, for all $t \in [a,b]$, then

$$\int_a^x (\Gamma - g(t))(g(t) - \gamma)\, dt \geq 0,$$

which implies

$$\begin{aligned} I_1(g) &\leq \left(\Gamma - \frac{1}{x-a} \int_a^x g(s)\, ds \right) \left(\frac{1}{x-a} \int_a^x g(s)\, ds - \gamma \right) \\ &\leq \frac{1}{4} \left[\left(\Gamma - \frac{1}{x-a} \int_a^x g(s)\, ds \right) + \left(\frac{1}{x-a} \int_a^x g(s)\, ds - \gamma \right) \right]^2 \\ &= \frac{1}{4} (\Gamma - \gamma)^2 \end{aligned} \tag{3.4}$$

Using Cauchy–Buniakowski–Schwarz's integral inequality we have

$$I_1(g) \geq \left[\frac{1}{x-a}\int_a^x \left|g(t) - \frac{1}{x-a}\int_a^x g(s)\,ds\right| dt\right]^2$$

and thus by (3.4) we get

$$\int_a^x \left|g(t) - \frac{1}{x-a}\int_a^x g(s)\,ds\right| dt \leq \frac{1}{2}(\Gamma-\gamma)(x-a). \tag{3.5}$$

Similarly, define

$$I_2(g) := \frac{1}{b-x}\int_x^b \left(g(t) - \frac{1}{b-x}\int_x^b g(s)\,ds\right)^2 dt,$$

then one can observe that

$$\int_x^b \left|g(t) - \frac{1}{b-x}\int_x^b g(s)\,ds\right| dt \leq \frac{1}{2}(\Phi-\phi)(b-x). \tag{3.6}$$

Therefore, from (3.3) we have

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\leq L_f\left[\int_a^x |g(t) - G(a,x)|\,dt + \int_x^b |g(t) - G(x,b)|\,dt\right]\\ &\leq \frac{1}{2}L_f\left[(x-a)(\Gamma-\gamma) + (b-x)(\Phi-\phi)\right],\end{aligned}$$

which gives the inequality (3.1). □

Remark 3.1. In Theorem 3.1, if $\gamma \leq g(t) \leq \Gamma$ for all $t \in [a,b]$, then we have

$$|\mathcal{R}(f,g;x)| \leq \frac{1}{2}L_f(b-a)(\Gamma-\gamma). \tag{3.7}$$

for all $x \in (a,b)$.

Corollary 3.1. *In Theorem 3.1, choose $g(t) = t$, $t \in [a,b]$, then we have the inequality:*

$$\left|\frac{1}{2}\left[f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a}\right] - \int_a^b f(t)\,dt\right| \leq \frac{1}{2}L_f(b-a)^2 \tag{3.8}$$

for all $x \in (a,b)$. Moreover, if we choose $x = \frac{a+b}{2}$, then we get

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \int_a^b f(t)\,dt\right| \leq \frac{1}{2}L_f(b-a)^2. \tag{3.9}$$

Theorem 3.2. *Let $f,g : [a,b] \to \mathbb{R}$ be such that f is L_f–Lipschitzian on $[a,b]$ and g is of r-H_g–Hölder type on $[a,b]$, where $r \in (0,1]$ and $H_g > 0$ are given. Then,*

$$|\mathcal{R}(f,g;x)| \leq \frac{2L_f H_g}{(r+1)(r+2)}\left[(x-a)^{r+1} + (b-x)^{r+1}\right], \tag{3.10}$$

for all $x \in (a,b)$.

Proof. Since f is L_f–Lipschitzian on $[a,b]$, then (3.3) holds; that is,

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\leq \left|\int_a^x [g(t)-G(a,x)]\,df(t)\right| + \left|\int_x^b [g(t)-G(x,b)]\,df(t)\right| \\ &\leq L_f \left[\int_a^x |g(t)-G(a,x)|\,dt + \int_x^b |g(t)-G(x,b)|\,dt\right].\end{aligned}$$

Also, since g is of r-H_g–Hölder type on $[a,b]$, then we have

$$\begin{aligned}|g(t)-G(a,x)| &= \left|g(t) - \frac{1}{x-a}\int_a^x g(s)\,ds\right| \\ &\leq \frac{1}{x-a}\int_a^x |g(t)-g(s)|\,ds \\ &\leq \frac{H_g}{x-a}\int_a^x |t-s|^r\,ds = \frac{H_g}{x-a}\cdot\frac{(t-a)^{r+1}+(x-t)^{r+1}}{r+1},\end{aligned}$$

and

$$\begin{aligned}|g(t)-G(x,b)| &= \left|g(t) - \frac{1}{b-x}\int_x^b g(s)\,ds\right| \\ &\leq \frac{1}{b-x}\int_x^b |g(t)-g(s)|\,ds \\ &\leq \frac{H_g}{b-x}\int_x^b |t-s|^r\,ds = \frac{H_g}{b-x}\cdot\frac{(t-x)^{r+1}+(b-t)^{r+1}}{r+1},\end{aligned}$$

which gives by (3.3), we have

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\leq \frac{L_f H_g}{x-a}\cdot\int_a^x \frac{(t-a)^{r+1}+(x-t)^{r+1}}{r+1}dt \\ &\quad + \frac{L_f H_g}{b-x}\cdot\int_x^b \frac{(t-x)^{r+1}+(b-t)^{r+1}}{r+1}dt \\ &= \frac{2L_f H_g}{(r+1)(r+2)}\left[(x-a)^{r+1}+(b-x)^{r+1}\right],\end{aligned}$$

and thus the proof is completed. □

Corollary 3.2. *In Theorem 3.2, if g is L_g–Lipschitzian on $[a,b]$, then we have*

$$|\mathcal{R}(f,g;x)| \leq \frac{1}{3}L_f L_g\left[(x-a)^2+(b-x)^2\right], \tag{3.11}$$

for all $x \in (a,b)$. Moreover, if we choose $x = \frac{a+b}{2}$, then

$$\left|\mathcal{R}\left(f,g;\frac{a+b}{2}\right)\right| \leq \frac{1}{6}L_f L_g (b-a)^2. \tag{3.12}$$

Corollary 3.3. *In Theorem 3.2, choose $g(t) = t$, $t \in [a,b]$, then we have the inequality:*

$$\left|\frac{1}{2}\left[f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a}\right] - \int_a^b f(t)\,dt\right| \le \frac{2L_f}{(r+1)(r+2)}\left[(x-a)^{r+1} + (b-x)^{r+1}\right] \tag{3.13}$$

for all $a < x < b$. Moreover, if we choose $x = \frac{a+b}{2}$, then we get

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \int_a^b f(t)\,dt\right| \le \frac{1}{6}L_f(b-a)^2. \tag{3.14}$$

Theorem 3.3. *Let $f, g : [a,b] \to \mathbb{R}$ be such that f is L_f–Lipschitzian on $[a,b]$ and g is of bounded variation on $[a,b]$. Then,*

$$|\mathcal{R}(f,g;x)| \le \frac{3}{4}L_f\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]\bigvee_a^b(g), \tag{3.15}$$

for all $x \in (a,b)$.

Proof. Since f is L_f–Lipschitzian on $[a,b]$, then (3.3) holds; that is,

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\le \left|\int_a^x [g(t) - G(a,x)]\,df(t)\right| + \left|\int_x^b [g(t) - G(x,b)]\,df(t)\right| \\ &\le L_f\left[\int_a^x |g(t) - G(a,x)|\,dt + \int_x^b |g(t) - G(x,b)|\,dt\right].\end{aligned}$$

Using the Ostrowski integral inequality for the bounded variation function g we have

$$\begin{aligned}\int_a^x |g(t) - G(a,x)|\,dt &= \int_a^x \left|g(t) - \frac{1}{x-a}\int_a^x g(s)\,ds\right| dt \\ &\le \int_a^x \left[\frac{1}{2} + \left|\frac{t - \frac{a+x}{2}}{x-a}\right|\right] dt \bigvee_a^x(g) \\ &\le \frac{3}{4}(x-a)\bigvee_a^x(g),\end{aligned}$$

similarly, we observe

$$\int_x^b |g(t) - G(x,b)|\,dt \le \frac{3}{4}(b-x)\bigvee_x^b(g),$$

which gives by (3.3), we have

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\le \frac{3}{4}L_f\left[(x-a)\bigvee_a^x(g) + (b-x)\bigvee_x^b(g)\right] \\ &\le \frac{3}{4}L_f\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]\bigvee_a^b(g),\end{aligned}$$

for all $x \in (a,b)$, and thus the proof is completed. $\square$

Remark 3.2. Let f be a monotonic nondecreasing in the theorems above. By applying the same techniques used in the corresponding proofs of each theorem, we may obtain several inequalities for monotonic non-decreasing integrator f using the fact that for a monotonic non-decreasing function $\nu : [a,b] \to \mathbb{R}$ and continuous function $p : [a,b] \to \mathbb{R}$, one has the inequality

$$\left| \int_a^b p(t)\, d\nu(t) \right| \leq \int_a^b |p(t)|\, d\nu(t).$$

We leave the details to the interested reader.

4. Applications to A Three-point Quadrature rule

Consider $I_n : a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$, be a division of the interval $[a,b]$, $L_i := g(x_{i+1}) - g(x_i)$, $(i = 0, 1, \ldots n-1)$ and $\nu(L) := \max\{L_i | i = 0, 1, \ldots n-1\}$. Consider the following Three–point quadrature rule as

(4.1)
$$S(f, g, I_n, \xi) = \sum_{i=0}^{n} [G(x_i, \xi_i) - g(x_i)] f(x_i) + [G(\xi_i, x_{i+1}) - G(x_i, \xi_i)] f(\xi_i) + [g(x_{i+1}) - G(\xi_i, x_{i+1})] f(x_{i+1})$$

for all $\xi_i \in (x_i, x_{i+1})$, where $G(\alpha, \beta) := \frac{1}{\beta - \alpha} \int_\alpha^\beta g(t)\, dt$.

In the following, we establish an upper bound for the error approximation of the Riemann–Stieltjes integral $\int_a^b f(t)\, dg(t)$ by its Riemann sum $S(f, g, I_n, \xi)$. As a sample we consider (2.1).

Theorem 4.1. *Under the assumptions of Theorem 2.1, we have*

$$\int_a^b f(t)\, dg(t) = S(f, g, I_n, \xi) + R(f, g, I_n, \xi)$$

where, $S(f, g, I_n, \xi)$ is given in (4.1) and the remainder $R(f, g, I_n, \xi)$ satisfies the bound

$$|R(f, g, I_n, \xi)| \leq \left[\frac{1}{2} \nu(L) + \max_{0, n-1} \left| g(\xi_i) - \frac{g(x_i) + g(x_{i+1})}{2} \right| \right] \cdot \bigvee_a^b (f)$$

$$\leq \nu(L) \cdot \bigvee_a^b (f). \tag{4.2}$$

Proof. Fix $\xi_i \in (x_i, x_{i+1})$. Applying Theorem 2.1 on the intervals $[x_i, x_{i+1}]$, we may state that

$$\Bigg| [G(x_i, \xi_i) - g(x_i)] f(x_i) + [G(\xi_i, x_{i+1}) - G(x_i, \xi_i)] f(\xi_i) + [g(x_{i+1}) - G(\xi_i, x_{i+1})] f(x_{i+1}) - \int_{x_i}^{x_{i+1}} f(t)\, dg(t) \Bigg|$$
$$\leq \left[\frac{g(x_{i+1}) - g(x_i)}{2} + \left| g(\xi_i) - \frac{g(x_i) + g(x_{i+1})}{2} \right| \right] \cdot \bigvee_{x_i}^{x_{i+1}} (f),$$

for all $i \in \{0, 1, 2, \cdots, n-1\}$. Summing the above inequality over i from 0 to $n-1$, we deduce

$$\begin{aligned}
&|R(f, g, I_n, \xi)| \\
&= \sum_{i=0}^{n-1} \Big\{ \Big| [G(x_i, \xi_i) - g(x_i)] f(x_i) + [G(\xi_i, x_{i+1}) - G(x_i, \xi_i)] f(\xi_i) \\
&\qquad + [g(x_{i+1}) - G(\xi_i, x_{i+1})] f(x_{i+1}) - \int_{x_i}^{x_{i+1}} f(t)\, dg(t) \Big| \Big\} \\
&\le \sum_{i=0}^{n-1} \left[\frac{g(x_{i+1}) - g(x_i)}{2} + \left| g(\xi_i) - \frac{g(x_i) + g(x_{i+1})}{2} \right| \right] \cdot \bigvee_{x_i}^{x_{i+1}} (f) \\
&\le \max_{0,n-1} \left[\frac{g(x_{i+1}) - g(x_i)}{2} + \left| g(\xi_i) - \frac{g(x_i) + g(x_{i+1})}{2} \right| \right] \cdot \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) \\
&\le [g(b) - g(a)] \cdot \bigvee_a^b (f),
\end{aligned}$$

since

$$\begin{aligned}
\max_{0,n-1} &\left[\frac{g(x_{i+1}) - g(x_i)}{2} + \left| g(\xi_i) - \frac{g(x_i) + g(x_{i+1})}{2} \right| \right] \\
&\le \frac{1}{2}\nu(L) + \max_{0,n-1} \left| g(\xi_i) - \frac{g(x_i) + g(x_{i+1})}{2} \right|
\end{aligned}$$

and

$$\sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) = \bigvee_a^b (f).$$

For the second inequality, we observe that

$$\max_{0,n-1} \left| g(\xi_i) - \frac{g(x_i) + g(x_{i+1})}{2} \right| \le \frac{1}{2} \max_{0,n-1} L_i = \frac{1}{2}\nu(L)$$

which completes the proof. □

Remark 4.1. Several error estimations for the quadrature $S(f, g, I_n, \xi)$ (4.1) by using the results in section 2, we shall omit the details.

References

[1] M.W. Alomari, Some Grüss type inequalities for Riemann-Stieltjes integral and applications, *Acta Mathematica Universitatis Comenianae*, 81 (2) (2012), 211–220.

[2] M.W. Alomari, A companion of Dragomir's generalization of Ostrowski's inequality and applications in numerical integration, *Ukrainian Math. J.*, 64(4) 2012, 491–510.

[3] M.W. Alomari, A companion of Ostrowski's inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$, where f is of bounded variation and u is of r-H-Hölder type and applications, *Appl. Math. Comput.*, 219 (2013), 4792–4799.

[4] M.W. Alomari, A companion of Ostrowski's inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is of r-H-Hölder type and u is of bounded variation and applications, submitted. Avalibale at: http://ajmaa.org/RGMIA/papers/v14/v14a59.pdf.

[5] M.W. Alomari, A new generalization of Grüss type inequalities for the Stieltjes integral and Applications, submitted. Avaliable at: http://ajmaa.org/RGMIA/papers/v15/v15a36.pdf

[6] M.W. Alomari, New sharp inequalities of Ostrowski and generalized trapezoid type for the Riemann–Stieltjes integrals and applications, *Ukrainian Mathematical Journal*, 65 (7) 2013, 995–1018.

[7] M.W. Alomari and S.S. Dragomir, Mercer–Trapezoid rule for the Riemann–Stieltjes integral with applications, *Journal of Advances in Mathematics*, 2 (2) (2013), 67–85.

[8] M.W. Alomari and S.S. Dragomir, A three–point quadrature rule for the Riemann–Stieltjes integral, *Southeast Bulletin Journal of Mathematics*, accepted.

[9] N.S. Barnett, S.S. Dragomir and I. Gomma, A companion for the Ostrowski and the generalised trapezoid inequalities, *Math. and Comp. Mode.*, 50 (2009), 179–187.

[10] N.S. Barnett, W.-S. Cheung, S.S. Dragomir, A. Sofo, Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators, *Comp. Math. Appl.* , 57 (2009), 195–201.

[11] P. Cerone and S.S. Dragomir, Midpoint-type rules from an inequalities point of view, Handbook of Analytic-Computational Methods in Applied Mathematics, Editor: G.A. Anastassiou, CRC Press, N.Y. (2000), 135-200.

[12] P. Cerone, W.S. Cheung, S.S. Dragomir, On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation, *Comp. Math. Appl.*, 54 (2007), 183–191.

[13] P. Cerone, S.S. Dragomir, New bounds for the three-point rule involving the Riemann–Stieltjes integrals, in: C. Gulati, et al. (Eds.), Advances in Statistics Combinatorics and Related Areas, World Science Publishing, 2002, pp. 53–62.

[14] P. Cerone, S.S. Dragomir, Approximating the Riemann–Stieltjes integral via some moments of the integrand, *Mathematical and Computer Modelling*, 49 (2009), 242–248.

[15] P. Cerone, S.S. Dragomir and C.E.M. Pearce, A generalized trapezoid inequality for functions of bounded variation, *Turk. J. Math.*, 24 (2000), 147–163.

[16] W.-S. Cheung and S.S. Dragomir, Two Ostrowski type inequalities for the Stieltjes integral of monotonic functions, *Bull. Austral. Math. Soc.*, 75 (2007), 299–311.

[17] S.S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. & Appl.*, 4(1) (2001), 59–66.

[18] S.S. Dragomir, On the Ostrowski inequality for Riemann–Stieltjes integral $\int_a^b f(t)du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM*, 5 (2001), 35–45.

[19] S.S. Dragomir, On the Ostrowski's inequality for Riemann–Stieltes integral and applications, *Korean J. Comput. & Appl. Math.*, 7 (2000), 611–627.

[20] S.S. Dragomir, C. Buşe, M.V. Boldea and L. Braescu, A generalisation of the trapezoid rule for the Riemann–Stieltjes integral and applications, *Nonlinear Anal. Forum*, 6 (2) (2001) 337–351.

[21] S.S. Dragomir, Some inequalities of midpoint and trapezoid type for the Riemann-Stieltjes integral, *Nonlinear Anal.*, 47 (4) (2001), 2333–2340.

[22] S.S. Dragomir, Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation, *Arch. Math.*, 91 (2008), 450–460

[23] S.S. Dragomir, Approximating the Riemann–Stieltjes integral in terms of generalised trapezoidal rules, *Nonlinear Anal.* TMA 71 (2009), e62–e72.

[24] S.S. Dragomir, Approximating the Riemann–Stieltjes integral by a trapezoidal quadrature rule with applications, *Mathematical and Computer Modelling* 54 (2011), 243–260.

[25] S.S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.*, 38 (1999), 33–37.

[26] S.S. Dragomir and I. Fedotov, An inequality of Gruss type for Riemann—Stieltjes integral and applications for special means, *Tamkang J. Math.*, 29(4) (1998), 287–292.

[27] R.P. Mercer, Hadamard's inequality and trapezoid rules for the Riemann–Stieltjes integral, *J. Math. Anal. Appl.*, (344) (2008), 921-927.

[28] M. Tortorella, Closed Newton–Cotes quadrature rules for Stieltjes integrals and numerical convolution of life distributions. *SIAM J. Sci. Statist. Comput.* 11 (1990), no. 4, 732–748.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 35–41 (2014) ©KJM

SEMIRADICAL EQUALITY

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Abstract. Semiprime radical of a module is defined and the relation between the intersection of prime submodules and the intersection of semiprime submodules is investigated. Semiradical formula is defined and it is shown that cartesian product of $M_1 \times M_2$ satisfies the semiradical formula if and only if M_1 and M_2 satisfy the semiradical formula.

1. Introduction

Throughout all rings are commutative and all modules are unitary. Let R be a ring and M be an R-module. A proper submodule N of M is prime if whenever $rm \in N$, for some $r \in R, m \in M$ then $m \in N$ or $rM \subseteq N$. A proper submodule N of an R-module M is semiprime, if whenever $r^k m \in N$ for some $r \in R, m \in M$ and $k \in \mathbb{Z}^+$, then $rm \in N$. Also, for any submodule N of M the envelope of N in M is defined as the set

$$E_M(N) = \{rm : r \in R, m \in M \quad and \quad r^k m \in N \quad for \quad some \quad k \in \mathbb{Z}^+\}.$$

It is easy to show that a proper submodule N is semiprime if and only if $\langle E_M(N)\rangle = N$. Also, it is clear that every prime submodule is semiprime but the converse is not true in general; to show this with an example let's give the following Theorem of Ylmaz and Klarslan Cansu.

Theorem 1.1. *([4], Theorem 2.5) Let $N = Q_1 \cap Q_2 \cap \cdots \cap Q_k$ be minimal primary decomposition of N where $\sqrt{Q_i : M} = p_i$ for all $i = 1, 2, \ldots, k$ and $S = \{1, 2, \ldots, k\}$. Then*

$$\langle E_M(N)\rangle = N + (\bigcap_{i=1}^{k} p_i)M + \sum_{T\subset S}(\bigcap_{i\in T} p_i)(\bigcap_{i\in S\setminus T} Q_i)$$

where the summation runs over each non-empty subset T of S.

Date: March 13, 2014 and, in revised form, July 3, 2014.
2000 *Mathematics Subject Classification.* 13C99, 13A99.
Key words and phrases. Semiprime submodules, semiprime radical.

Now, we can give the example. The computer algebra system **SINGULAR** was used during the computations.

If $R = \mathbb{Q}[x,y,z]$, $M = R^3$ and $N = \langle z\mathbf{e}_1, y\mathbf{e}_1, xy\mathbf{e}_2, xy\mathbf{e}_3, xz\mathbf{e}_2 + x^2z\mathbf{e}_3\rangle$. Then primary decompostion of N is $N = Q_1 \cap Q_2 \cap Q_3$ where

$$\begin{aligned} &Q_1 = \langle \mathbf{e}_1, y\mathbf{e}_2, y\mathbf{e}_3, x\mathbf{e}_3 + \mathbf{e}_2\rangle \text{ is } \langle y\rangle - \text{primary},\\ &Q_2 = \langle z\mathbf{e}_1, z\mathbf{e}_2, z\mathbf{e}_3, y\mathbf{e}_1, y\mathbf{e}_2, y\mathbf{e}_3\rangle \text{ is } \langle z,y\rangle - \text{primary and}\\ &Q_3 = \langle \mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3\rangle \text{ is } \langle x\rangle - \text{primary.}\end{aligned}$$

By Theorem 1.1,

$$\begin{aligned}\langle E_M(N)\rangle &= N + (p_1 \cap p_2 \cap p_3)M + p_1(Q_2 \cap Q_3) + p_2(Q_1 \cap Q_3) + p_3(Q_1 \cap Q_2)\\ &\quad +(p_1 \cap p_2)Q_3 + (p_1 \cap p_3)Q_2 + (p_2 \cap p_3)Q_1\\ &= \langle z\mathbf{e}_1, y\mathbf{e}_1, xy\mathbf{e}_2, xy\mathbf{e}_3, xz\mathbf{e}_2 + x^2z\mathbf{e}_3\rangle = N.\end{aligned}$$

Hence, N is a semiprime submodule of M with $N : M = \langle xy\rangle$. On the other hand N is not a prime submodule; since $r = z$ and $m = (0, x, x^2)$ gives $rm = z(0, x, x^2) = (0, xz, x^2z) \in N$ but $r = z \notin N : M$ and $m = (0, x, x^2) \notin N$.

If N is a proper submodule of an R-module M, then the prime radical of N, $rad_M(N)$, is the intersection of all prime submodules containing N. If it is necessary to indicate the underlying ring, the prime radical of N is denoted by $rad_{_RM}(N)$. The semiprime radical of N, denoted by $srad_M(N)(srad_{_RM}(N))$, is defined as the intersection of all semiprime submodules of M containing N. If there is no semiprime submodule containing N, then $srad_M(N) = M$.

A module M satisfies the radical formula (s.t.r.f.) if for any submodule N of M, $rad_M(N) = \langle E_M(N)\rangle$. In the same manner we define, an R-module M satisfies the semiradical formula (s.t.s.r.f.) if for any submodule N of M, $srad_M(N) = \langle E_M(N)\rangle$. Since intersection of semiprime submodules is semiprime, $srad_M(N)$ is the unique smallest semiprime submodule of M containing N.

We know that for an ideal I of R, $\sqrt{\sqrt{I}} = \sqrt{I}$; but the envelope of a submodule does not satisfy an equation similiar to this one. If $R = \mathbb{Q}[x,y,z]$, M is an R-module $R \oplus R$ and $N = \langle z^2\mathbf{e}_1, z^2\mathbf{e}_2, yz\mathbf{e}_2, y^2\mathbf{e}_1 + z\mathbf{e}_2, y^2\mathbf{e}_2, y\mathbf{e}_1 + x^3\mathbf{e}_2\rangle$ is an R-submodule of M. Since N is $\langle z,y\rangle$-primary,

$$\langle E_M(N)\rangle = N + \langle z,y\rangle M = \langle z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2, x^3\mathbf{e}_2\rangle.$$

Since $\langle E_M(N)\rangle = Q_1 \cap Q_2$, where

$$\begin{aligned}&Q_1 = \langle \mathbf{e}_2, z\mathbf{e}_1, y\mathbf{e}_1\rangle \text{ is } \langle z,y\rangle - \text{primary},\\ &Q_2 = \langle z\mathbf{e}_1, z\mathbf{e}_2, z\mathbf{e}_3, y\mathbf{e}_1, y\mathbf{e}_2, x^3\mathbf{e}_1, x^3\mathbf{e}_2\rangle \text{ is } \langle x,y,z\rangle - \text{primary,}\end{aligned}$$

Theorem 1.1 implies that $\langle E_M(\langle E_M(N)\rangle)\rangle = \langle z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2, x\mathbf{e}_2\rangle \neq \langle E_M(N)\rangle$.

In [2], Azizi and Nikseresht defined the kth envelope of N recursively by $E_0(N) = N, E_1(N) = E_M(N), E_2(N) = E_M(\langle E_M(N)\rangle)$ and $E_k(N) = E_M(\langle E_{k-1}(N)\rangle)$ for every submodule N of M. It is easy to show that

$$N = \langle E_0(N)\rangle \subseteq \langle E_1(N)\rangle \subseteq \langle E_2(N)\rangle \subseteq \cdots\cdots \subseteq \langle E_\infty(N)\rangle \subseteq srad_M(N) \subseteq rad_M(N)$$

where $\langle E_\infty(N)\rangle = \bigcup_{k=0}^{\infty} \langle E_k(N)\rangle$. It is clear that $\langle E_\infty(N)\rangle$ is semiprime and thus $\langle E_\infty(N)\rangle = srad_M(N)$.

When we consider the chain

$$N = \langle E_0(N)\rangle \subseteq \langle E_1(N)\rangle \subseteq \langle E_2(N)\rangle \subseteq \cdots\cdots \subseteq \langle E_\infty(N)\rangle = srad_M(N) \subseteq rad_M(N),$$

it seems meaningfull to focus on the submodules $srad_M(N)$ and $rad_M(N)$ and investigate the conditions where the equality $srad_M(N) = rad_M(N)$ occurs.

Here, semiradical equality is defined and some equivalent conditions for a ring to satisfy the semiradical equality are stated.

2. Semiradical Equality

It is clear that intersection of prime submodules is semiprime, but the converse is not true in general, see [1].

Lemma 2.1. *Let M be an R-module. Then every semiprime submodule is an intersection of prime submodules if and only if $srad_M(N) = rad_M(N)$ for any submodule N of M.*

Proof. ($\Rightarrow$) Since intersection of semiprime submodules is semiprime , $srad_M(N)$ is a semiprime submodule. Hence it is obvious.

($\Leftarrow$) Let K be a semiprime submodule of M. Then $K = srad_M(K) = rad_M(K)$. Hence K is an intersection of prime submodules. □

Lemma 2.2. *Let N be a submodule of an R-module M such that M/N is projective. Then $srad_M(N) = rad_M(N)$.*

Proof. Since M/N is projective, $rad_{M/N}(0) = \langle E_{M/N}(0)\rangle$ by [1] Lemma 8. Then we have,$rad_M(N) = \langle E_M(N)\rangle$ which implies that $srad_M(N) = rad_M(N)$. □

Corollary 2.1. *Let N be a submodule of an R-module M such that M/N is projective. Then $srad_M(N) = radRM + N$.*

Proof. Clear by [5] Theorem 2.7 and the above lemma. □

We say that a module M satisfies the semiradical equality if for every submodule N of M, $srad_M(N) = rad_M(N)$. It is said that a ring R satisfies the semiradical equality if every R-module satisfies the semiradical equality. Since arithmetical rings satisfy the radical formula [3], an arithmetical ring satisfies the semiradical equality.

Proposition 2.1. *The followings are equivalent.*

(i) *The ring R satisfies the semiradical equality.*
(ii) *for any ideal I of R, the ring R/I satisfies the semiradical equality.*
(iii) *for any non-maximal semiprime ideal P of R, the ring R/P satisfies the semiradical equality.*

Proof. ($i \Rightarrow ii$) Let M be an R/I-module. By Lemma 2.1, it is enough to show that every semiprime R/I-module is an intersection of prime submodules. Let K be a semiprime submodule of an R/I-module M. Then K is a semiprime submodule of M as an R-module. So, $K = srad_{_R M}(K) = rad_{_R M}(K)$.

It is easy to see that every submodule of M is a prime R-submodule if and only if it is a prime R/I-submodule. Hence $rad_{_R M}(K) = rad_{_{R/I} M}(K)$ and thus $K = rad_{_{R/I} M}(K)$.

($iii \Rightarrow i$) Let N be a semiprime submodule of an R-module M with $N : M = P$ where P is a non-maximal semiprime ideal. Consider M/N as an R/P-module. Then by our assumption, $srad_{M/N}(0) = rad_{M/N}(0)$. Hence $srad_M(N) = rad_M(N)$.

□

Corollary 2.2. *If for any non-maximal semiprime ideal P of R; R/P is a Prüfer domain, then R satisfies the semiradical equality.*

Lemma 2.3. *A ring R satisfies the semiradical equality if and only if every free R-module satisfies the semiradical equality.*

Proof. Let M be an R-module. Then there exists a free R-module F such that $M \cong F/K$. By our assumption, for any submodule N of M

$$\begin{aligned} srad_{F/K}(N/K) &= srad_F(N)/K \\ &= rad_F(N)/K \\ &= rad_{F/K}(N/K). \end{aligned}$$

Hence M satisfies the semiradical equality. □

3. Semiprime Submodules of Cartesian Product of Modules

Let $R = R_1 \times R_2$ where each R_i is a commutative ring with nonzero identity. Let M_i be an R_i-module for $i = 1, 2$ and $M = M_1 \times M_2$ be the R-module with action $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$ where $r_i \in R_i, m_i \in M_i$. These notations are fixed for this section.

Note that since our action is $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$ where $r_i \in R_i, m_i \in M_i$, every submodule of $M_1 \times M_2$ is of the form $N_1 \times N_2$ with N_1 is a submodule of M_1 and N_2 is a submodule of M_2.

Proposition 3.1. *Let R and M be as above. Then*

(i) *If N_1 is semiprime submodule of M_1, then $N_1 \times M_2$ is semiprime submodule of $M_1 \times M_2$.*
(ii) *If N_2 is semiprime submodule of M_2, then $M_1 \times N_2$ is semiprime submodule of $M_1 \times M_2$.*

Proof. (i) Let $r = (r_1, r_2) \in R$, $m = (m_1, m_2) \in M$ and $r^k m \in N_1 \times M_2$ for some $k \in \mathbb{Z}^+$. Since N_1 is semiprime submodule of M_1, $r_1 m_1 \in N_1$. Then $(r_1 m_1, r_2 m_2) = rm \in N_1 \times M_2$ which implies that $N_1 \times M_2$ is semiprime submodule of M.

(ii) Similiar to case (i). □

Lemma 3.1. *Let R and M be as above. Then $Q_1 \times Q_2$ is a semiprime submodule of M if and only if Q_i is semiprime submodule of M_i for all $i = 1, 2$.*

Proof. Let $(r_1, r_2)^k(m_1, m_2) \in Q_1 \times Q_2$ where $m_i \in M_i$, $r_i \in R_i$ and $k \in \mathbb{Z}^+$ and Q_1 and Q_2 be semiprime submodules of M_1 and M_2 respectively. Since Q_1 and Q_2 are semiprime, $r_i m_i \in Q_i$ for $i = 1, 2$ which implies that $Q_1 \times Q_2$ is semiprime.
Now assume that $Q_1 \times Q_2$ is semiprime submodule of $M_1 \times M_2$. Let $r_1 \in R_1$, $m_1 \in M_1$ with $r_1^k m_1 \in Q_1$. Then $(r_1, 1)^k(m_1, 0) \in Q_1 \times Q_2$. Since $Q_1 \times Q_2$ is semiprime, $(r_1, 1)(m_1, 0) = (r_1 m_1, 0) \in Q_1 \times Q_2$ implies that Q_1 is semiprime submodule of M_1. Similarly it can be shown that Q_2 is semiprime submodule of M_2. □

Lemma 3.2. *Let $N = N_1 \times N_2$ be a submodule of M where N_i is a submodule of M_i for $i = 1, 2$. Then $N : M = (N_1 : M_1) \times (N_2 : M_2)$*

Proof. Let $x = (x_1, x_2) \in (N : M)$. Then $xM \subseteq N$ which means that

$$(x_1, x_2)(m_1, m_2) = (x_1 m_1, x_2 m_2) \in N_1 \times N_2$$

for all $m_1 \in M_1$ and $m_2 \in M_2$. So, $x_1 m_1 \in N_1$ and $x_2 m_2 \in N_2$. Hence

$$x_1 \in (N_1 : M_1), \quad x_2 \in (N_2 : M_2)$$

and thus $x = (x_1, x_2) \in (N_1 : M_1) \times (N_2 : M_2)$.

Conversely, let $y = (y_1, y_2) \in (N_1 : M_1) \times (N_2 : M_2)$. Then $y_1 M_1 \subseteq N_1$ and $y_2 M_2 \subseteq N_2$. Hence for all $m_1 \in M_1, m_2 \in M_2$,

$$(y_1, y_2)(m_1, m_2) = (y_1 m_1, y_2 m_2) \in N_1 \times N_2.$$

This implies that $y \in (N_1 \times N_2) : (M_1 \times M_2) = (N : M)$. □

Let N be a semiprime submodule of an R-module M. If $p = \sqrt{N : M}$ is a prime ideal, then N is called p -semiprime submodule.

Lemma 3.3. *Let $N = N_1 \times N_2$ be a submodule of M. Then*

(i) *N is $p \times R_2$ semiprime submodule of M iff N_1 is p-semiprime submodule of M_1 and $N_2 = M_2$.*
(ii) *N is $R_1 \times p$ semiprime submodule of M iff N_2 is p-semiprime submodule of M_2 and $N_1 = M_1$.*

Proof. (i) Suppose $N = N_1 \times N_2$ is semiprime submodule of $M_1 \times M_2$. By Lemma 3.1, N_1 is semiprime submodule of M_1.

Since $N : M = p \times R_2$, N_1 is p-semiprime and $N_2 : M_2 = R_2$ implies that $N_2 = M_2$.

Other side is clear by Proposition 3.1 and Lemma 3.2.

(ii) Similiar to case (i). □

If p_1 and p_2 are prime ideals of R_1 and R_2 respectively, it is not true in general that $p_1 \times p_2$ is prime ideal of $R_1 \times R_2$, for example if we take $R_1 = R_2 = \mathbb{Z}$, $p_1 = 2\mathbb{Z}$ and $p_2 = 3\mathbb{Z}$, then $2\mathbb{Z} \times 3\mathbb{Z}$ is not a prime ideal of $\mathbb{Z} \times \mathbb{Z}$ since $\mathbb{Z}_2 \times \mathbb{Z}_3$ is not an integral domain. So, if we try to generalize Lemma 3.3, we only get the following lemma.

Lemma 3.4. *Let $N = N_1 \times N_2$ be a submodule of M. If $N_1 \times N_2$ is $p_1 \times p_2$-semiprime submodule, then N_i is p_i-semiprime submodule of M_i for $i = 1, 2$.*

Proof. Assume that $N_1 \times N_2$ is $p_1 \times p_2$-semiprime submodule of M. By Lemma 3.1, N_1 and N_2 are semiprime submodules of M_1 and M_2 respectively. Since $p_1 \times p_2$ is prime ideal, by Lemma 3.2

$$\begin{aligned}(N_1 \times N_2) : (M_1 \times M_2) &= \sqrt{N : M} = p_1 \times p_2 \\ (N_1 : M_1) \times (N_2 : M_2) &= p_1 \times p_2.\end{aligned}$$

Hence, $N_1 : M_1 = p_1$ and $N_2 : M_2 = p_2$. Since p_1 and p_2 are prime ideals,

$$N_1 : M_1 = \sqrt{N_1 : M_1} = p_1 \quad \text{and}$$
$$N_2 : M_2 = \sqrt{N_2 : M_2} = p_2.$$

Thus, N_i is p_i-semiprime submodule of M_i for $i = 1, 2$. □

Proposition 3.2. *Let $N = N_1 \times N_2$ be a submodule of M. Then*

$$srad_M(N) = srad_{M_1}(N_1) \times srad_{M_2}(N_2)$$

Proof. Let $Q_1 \times Q_2$ be a semiprime submodule of M containing $N_1 \times N_2$. By Lemma 3.1, Q_i is semiprime submodule of M_i containing N_i for $i = 1, 2$. Then

$$srad_{M_1}(N_1) \times srad_{M_2}(N_2) \subseteq srad_M(N_1 \times N_2)$$

since $srad_{M_1}(N_1) \times srad_{M_2}(N_2) \subseteq Q_1 \times Q_2$.

Since $srad_{M_i}(N_i)$ is the minimal semiprime submodule of M_i containing N_i, Lemma 3.1 implies that $srad_{M_1}(N_1) \times srad_{M_2}(N_2)$ is a semiprime submodule of $M_1 \times M_2$ which contains $N_1 \times N_2$. Hence

$$srad_M(N) \subseteq srad_{M_1}(N_1) \times srad_{M_2}(N_2)$$

□

Corollary 3.1. *Let $N = N_1 \times N_2$ be a submodule of M. Then*

(i) $srad_M(N_1 \times M_2) = srad_{M_1}(N_1) \times M_2$
(ii) $srad_M(M_1 \times N_2) = M_1 \times srad_{M_2}(N_2)$

Proof. Clear by Proposition 3.2.

□

Proposition 3.3. *([6], Proposition 2.12) Let $N = N_1 \times N_2$ be a submodule of M. Then $\langle E_M(N)\rangle = \langle E_{M_1}(N_1)\rangle \times \langle E_{M_2}(N_2)\rangle$.*

Theorem 3.1. *M s.t.s.r.f. if and only if M_i s.t.s.r.f. for all $i = 1, 2$.*

Proof. Assume M s.t.s.r.f.. Take a submodule N_1 of M_1. Then $N_1 \times M_2$ s.t.s.r.f., so that $srad_{M_1}(N_1) \times M_2 = \langle E_{M_1}(N_1)\rangle \times \langle E_{M_2}(M_2)\rangle$. Now, let $x \in srad_{M_1}(N_1)$. Then $(x, m) \in srad_{M_1}(N_1) \times M_2$ and hence $x \in \langle E_{M_1}(N_1)\rangle$. Similiarly it can be shown that $srad_{M_2}(N_2) = \langle E_{M_2}(N_2)\rangle$.

Conversely assume that M_1 and M_2 s.t.s.r.f.. Take any submodule $N_1 \times N_2$ of $M_1 \times M_2$. Then

$$\begin{aligned}srad_M(N_1 \times N_2) &= srad_{M_1}(N_1) \times srad_{M_2}(N_2) \\ &= \langle E_{M_1}(N_1)\rangle \times \langle E_{M_2}(N_2)\rangle \\ &= \langle E_M(N_1 \times N_2)\rangle\end{aligned}$$

Thus, $M = M_1 \times M_2$ s.t.s.r.f.. □

References

[1] J. Jenkins and P. F. Smith, On the prime radical of a module over a commutative ring, Comm. in Algebra. Vol:20, No.12 (1992), 3593 – 3602.
[2] A. Azizi and A. Nikseresht, On radical formula in modules, Glasgow. Math. J. Vol:53, No.3 (2011), 657 – 668.
[3] A. Parkash, Arithmetical rings satisfy the radical formula, Journal of Commutative Algebra. Vol:4, No.2 (2012), 293 – 296.
[4] E. Ylmaz and S. Klarslan Cansu, Baer's lower nilradical and classical prime submodules, Bul. Iran Math. Soc., to appear.
[5] M. Alkan and Y. Tra, On prime submodules, Rocky Mountain Journal of Mathematics, Vol:37, No.3 (2007), 709 – 722.
[6] S. Atani and F. K. Saraei, Modules which satisfy the radical formula, Int. J. Contemp. Math. Sci. Vol:2, No.1 (2007), 13 – 18.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 42–52 (2014) ©KJM

UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING TWO FINITE SETS IN $\mathbb{C}$ WITH FINITE WEIGHT

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Abstract. With the aid of the notion of weighted sharing of sets of meromorphic functions we improve some previous results concerning a particular range set.

1. Introduction, Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \quad (r \longrightarrow \infty, r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z)-a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a\in S}\{z : f(z)-a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values.

In connection with the famous "Gross Question" {see [8]} in the uniqueness literature Gross and Yang [9] (see also [16]) made a vital contribution by introducing the new idea of unique range set for meromophic function (URSM in brief). We recall that "Gross's Question" was the first one which deal with the uniqueness of

2000 *Mathematics Subject Classification.* 30D35.
Key words and phrases. eromorphic function, Uniqueness, Shared Set, Weighted sharing.
The first author is thankful to DST-PURSE programme for financial assistance.

two functions that share sets of distinct elements instead of values. Initially Gross and Yang proved that if f and g are two non-constant entire functions and S_1, S_2 and S_3 are three distinct finite sets such that $f^{-1}(S_i) = g^{-1}(S_i)$ for $i = 1, 2, 3$, then $f \equiv g$. In [8] Gross posed the following question:

Question A. *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

In 2003, the following question was asked by Lin and Yi [18] which is also pertinent with that of Gross.

Question B. *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2$ must be identical ?*

During the last two decades, the main investigations on two set sharing problems of entire and meromorphic functions have been oriented on the basis of this two questions. Gradually the research in this direction has somehow been shifted to give explicitly a set S with n elements and make n as small as possible such that any two meromorphic functions f and g that share the value ∞ and the set S must be equal {cf.[1]-[7], [11], [15], [17]-[18], [20]-[23]}.

In this connection, we recall the following theorem of Yi [20].

Theorem A. [20] *Let $S = \{z : z^n + az^{n-m} + b = 0\}$ where n and m are two positive integers such that $m \geq 2$, $n \geq 2m + 7$ with n and m having no common factor, a and b be two nonzero constants such that $z^n + az^{n-m} + b = 0$ has no multiple root. If f and g are two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

In the same paper Yi [19] also asked the following question:
What can be said if $m = 1$ in *Theorem A*?
To answer this question Yi [20] proved the following theorem.

Theorem B. [20] *Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n (\geq 9)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g be two non-constant meromorphic functions such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then either $f \equiv g$ or $f \equiv \frac{-ah(h^{n-1}-1)}{h^n-1}$ and $g \equiv \frac{-a(h^{n-1}-1)}{h^n-1}$, where h is a non-constant meromorphic function.*

In 2001 the idea of gradation of sharing of values and sets known as weighted sharing has been introduced in [13, 14] which measures how close a shared value is to being shared IM or to being shared CM. We now give the definition.

Definition 1.1. [13, 14] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [13] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\cup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

The notion of weighted sharing of set has immense applications to deal with the *Questions A* and *B*. In particular there are many refinements and improvements of *Theorem B* {[2]-[4], [15]} using this notion. But in all the papers, to serve the purpose, the variations over different deficiency conditions have been taken under considerations.

In 1996, Yi [21] proved that the set S as defined in *Theorem A* is an URSM when $m \geq 2$ and $n \geq 2m+9$. Clearly in that case $S = \{z : z^{13} + z^{11} + 1 = 0\}$ is a URSM. So it would be natural to explore the analogous situation in the direction of *Question B*, corresponding to the set S as defined in *Theorem B* such that the uniqueness of meromorphic functions only depends on the sharing of the range sets in $\mathbb{C}$. The purpose of the paper is to find a suitable range set, together with S as defined in *Theorem B*, such that for the uniqueness of two non-constant meromorphic functions sharing two sets with finite weight, the conditions over deficiencies will no longer required. Following two theorems are the main results of the paper which will improve all the subsequent improvements of *Theorem B* in some sense.

Theorem 1.1. *Let* $S_1 = \{0, -a\frac{n-1}{n}\}$, $S_2 = \{z : z^n + az^{n-1} + b = 0\}$ *where* $n(\geq 7)$ *be an integer and* a *and* b *be two nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no multiple root. If* $E_f(S_1, 2) = E_g(S_1, 2)$, *and* $E_f(S_2, 3) = E_g(S_2, 3)$, *then* $f \equiv g$.

Theorem 1.2. *Let* S_i, $i = 1, 2$ *be given as in* Theorem 1.1 *where* $n(\geq 8)$ *be an integer. If* $E_f(S_1, m) = E_g(S_1, m)$, $E_f(S_2, p) = E_g(S_2, p)$, *then* $f \equiv g$, *where* $\frac{7}{2m} + \frac{m+1}{m(2p+1)} < 2$, *with* $\frac{7}{2m} + \frac{1}{m(2p+1)} > 1$.

Corollary 1.1. Theorem 1.2 *holds for the following pairs of least values of* p *and* m*: (i) p=1, m=3; (ii) p=3, m=2.*

Though for the standard definitions and notations of the value distribution theory we refer to [10], we now explain some notations which are used in the paper.

Definition 1.3. [12] For $a \in \mathbb{C} \cup \{\infty\}$we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Definition 1.4. [14] We denote by $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$.

Definition 1.5. [13, 14] Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and in particular if f and g share (a, p) then $\overline{N}_*(r, a; f, g) \leq \overline{N}(r, a; f \mid\geq p+1) = \overline{N}(r, a; g \mid\geq p+1)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$ as follows

$$F = \frac{f^{n-1}(f+a)}{-b}, \quad G = \frac{g^{n-1}(g+a)}{-b}, \tag{2.1}$$

where a, b two nonzero constants defined as in *Theorem B*. Henceforth we shall denote by H and Φ the following two functions

$$H = \left(\frac{F^{''}}{F^{'}} - \frac{2F^{'}}{F-1}\right) - \left(\frac{G^{''}}{G^{'}} - \frac{2G^{'}}{G-1}\right) \tag{2.2}$$

and

$$\Phi = \frac{F^{'}}{F-1} - \frac{G^{'}}{G-1}. \tag{2.3}$$

Lemma 2.1. *(*[14]*, Lemma 1) Let F, G be two non-constant meromorphic functions sharing $(1,1)$ and $H \not\equiv 0$. Then*

$$N(r,1;F \mid= 1) = N(r,1;G \mid= 1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. *Let S_1 and S_2 be defined as in* Theorem 1.1 *and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_f(S_1,p) = E_g(S_1,p)$, $E_f(S_2,0) = E_g(S_2,0)$, where $0 \leq p < \infty$ and $H \not\equiv 0$ then*

$$\begin{aligned} N(r,H) \quad \leq \quad & \overline{N}(r,0;f \mid\geq p+1) + \overline{N}\left(r, -a\frac{n-1}{n}; f \mid\geq p+1\right) + \overline{N}_*(r,1;F,G) \\ & + \overline{N}(r,\infty;f) + +\overline{N}(r,\infty;g) + \overline{N}_0(r,0;f^{'}) + \overline{N}_0(r,0;g^{'}), \end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ is the reduced counting function of those zeros of $f^{'}$ which are not the zeros of $f\left(f - a\frac{n-1}{n}\right)(F-1)$ and $\overline{N}_0(r,0;g^{'})$ is similarly defined.

Proof. We note that $F^{'} = \frac{f^{n-2}(nf+a(n-1))f^{'}}{-b}$, $G^{'} = \frac{g^{n-2}(ng+a(n-1))g^{'}}{-b}$ and

$$F^{''} = \frac{f^{n-2}(nf+a(n-1))f^{''} + f^{n-3}(n(n-1)f + a(n-1)(n-2))f^{'2}}{-b},$$

$$G^{''} = \frac{g^{n-2}(ng+a(n-1))g^{''} + g^{n-3}(n(n-1)g + a(n-1)(n-2))g^{'2}}{-b}.$$

So

$$\begin{aligned} H \quad = \quad & \frac{(n-1)(nf+a(n-2))f^{'}}{f(nf+a(n-1))} - \frac{(n-1)(ng+a(n-2))g^{'}}{g(ng+a(n-1))} \\ & + \frac{f^{''}}{f^{'}} - \frac{g^{''}}{g^{'}} - \left(\frac{2F^{'}}{F-1} - \frac{2G^{'}}{G-1}\right). \end{aligned}$$

Since $E_f(S_1,0) = E_g(S_1,0)$ it follows that if z_0 is a 0-point of f (g) then either $g(z_0) = 0$ $(f(z_0) = 0)$ or $g(z_0) = -a\frac{n-1}{n}$ $(f(z_0) = -a\frac{n-1}{n})$. Clearly F and G share $(1,0)$. Since H has only simple poles, the lemma can easily be proved by simple calculations. $\square$

Lemma 2.3. [5] *Let f and g be two meromorphic functions sharing $(1,m)$, where $1 \leq m < \infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\\ \leq\ &\frac{1}{2}\left[N(r,1;f)+N(r,1;g)\right]\end{aligned}$$

Lemma 2.4. [19] *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{k=0}^{n}a_kf^k}{\sum\limits_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$ Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.5. *Let S_1 and S_2 be defined as in* Theorem 1.1 *with $n \geq 3$ and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_f(S_1,p)=E_g(S_1,p)$, $E_f(S_2,m)=E_g(S_2,m)$, $0 \leq p < \infty$ and $\Phi \not\equiv 0$ then*

$$\begin{aligned}&(2p+1)\left\{\overline{N}\left(r,0;f\mid\geq p+1\right)+\overline{N}\left(r,-a\frac{n-1}{n};f\mid\geq p+1\right)\right\}\\ \leq\ &\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Proof. By the given condition clearly F and G share $(1,m)$. Also we see that

$$\Phi=\frac{f^{n-2}\left(nf+a(n-1)\right)f'}{-b(F-1)}-\frac{f^{n-2}\left(nf+a(n-1)\right)f'}{-b(G-1)}.$$

Let z_0 be a zero or a $-a\frac{n-1}{n}$- point of f with multiplicity r. Since $E_f(S_1,p)=E_g(S_1,p)$ then that would be a zero of Φ of multiplicity $min\ \{(n-2)r+r-1,\ r+r-1\}$ i.e., of multiplicity $min\ \{(n-1)r-1,\ 2r-1\}$ if $r \leq p$ and a zero of multiplicity at least $min\{(n-2)(p+1)+p,\ p+1+p\}$ i.e., a zero of multiplicity at least $min\{(n-1)p+(n-2),\ 2p+1\}$ if $r > p$. So using *Lemma 2.4* by a simple calculation we can write

$$\begin{aligned}&\min\{(n-1)p+(n-2),(2p+1)\}\left\{\overline{N}(r,0;f\mid\geq p+1)\right.\\ &\left.+\overline{N}(r,-a\frac{n-1}{n};f\mid\geq p+1)\right\}\\ \leq\ &N(r,0;\Phi)\\ \leq\ &T(r,\Phi)\\ \leq\ &N(r,\infty;\Phi)+S(r,F)+S(r,G)\\ \leq\ &\overline{N}_*(r,1;F,G)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f)+S(r,g).\end{aligned}$$

□

Lemma 2.6. *Let S_1, S_2 be defined as in* Theorem 1.1 *and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_f(S_1,p) = E_g(S_1,p)$, $E_f(S_2,m) = E_g(S_2,m)$, where $0 \le p < \infty$, $2 \le m < \infty$ and $H \not\equiv 0$, then*

$$\begin{aligned}
& (n+1)\left\{T(r,f)+T(r,g)\right\} \\
\le\ & 2\left\{\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)\right\}+\overline{N}(r,0;f\mid\ge p+1) \\
& +\overline{N}\left(r,-a\frac{n-1}{n};f\mid\ge p+1\right)+2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\} \\
& +\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned}
(2.4)\qquad & (n+1)\{T(r,f)+T(r,g)\} \\
\le\ & \overline{N}(r,1;F)+\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G) \\
& +\overline{N}(r,0;g)+\overline{N}\left(r,-a\frac{n}{n-1};g\right)+\overline{N}(r,\infty;g)-N_0(r,0;f^{'}) \\
& -N_0(r,0;g^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Using *Lemmas 2.1*, *2.2*, *2.3* and *2.4* we note that

$$\begin{aligned}
(2.5)\qquad & \overline{N}(r,1;F)+\overline{N}(r,1;G) \\
\le\ & \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+N(r,1;F\mid=1)-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G) \\
\le\ & \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+\overline{N}(r,0;f\mid\ge p+1) \\
& +\overline{N}\left(r,-a\frac{n-1}{n};f\mid\ge p+1\right)+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G) \\
& +\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Using (2.5) in (2.4) and noting that $\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right) = \overline{N}(r,0;g)+\overline{N}\left(r,-a\frac{n-1}{n};g\right)$ the lemma follows. $\square$

Lemma 2.7. *Let f, g be two non-constant meromorphic functions such that $E_f(\{0,-a\frac{n-1}{n}\},0) = E_g(\{0,-a\frac{n-1}{n}\},0)$ then $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ implies $f \equiv g$, where $n\ (\ge 2)$ is an integer and a is a nonzero finite constant.*

Proof. Let

$$f^{n-1}(f+a) \equiv g^{n-1}(g+a)$$

and suppose $f \not\equiv g$. We consider two cases:

Case I Let $y = \frac{g}{f}$ be a constant. Then from (2.6) it follows that $y \ne 1$, $y^{n-1} \ne 1$, $y^n \ne 1$ and $f \equiv -a\frac{y^{n-1}-1}{y^n-1}$, a constant, which is impossible.

Case II Let $y = \frac{g}{f}$ be non-constant. Suppose none of 0 and $-a\frac{n-1}{n}$ is an exceptional value of Picard (e.v.P.) of f and g. Then from (2.6) we see that if z_0 is a 0 $(-a\frac{n-1}{n})$-point of f then that must be a 0 $(-a\frac{n-1}{n})$-point of g. That is f, g share $(0,\infty)$ (∞,∞). So y has no zero and pole. Again from (2.6) we observe that

$$f(y^n-1) \equiv -a(\ y^{n-1}-1).$$

Clearly $y \not\equiv 1$. So eliminating this common factor we are left with

$$f(y-\alpha_1)(y-\alpha_2)\ldots(y-\alpha_{n-1}) \equiv -a(y-\beta_1)(y-\beta_2)\ldots(y-\beta_{n-2}),$$

where $\alpha_j = exp\ (\frac{2j\pi i}{n})$ for $j = 1, 2, \ldots, n-1$ and $\beta_k = exp\ (\frac{2k\pi i}{n-1})$ for $k = 1, 2, \ldots, n-2$. Clearly none of the α_j's coincides with β_k's. First we observe that y can not omit any of the $2n-3$ distinct values α_j or β_k for $j = 1, 2, \ldots, n-1$ and $k = 1, 2, \ldots, n-2$, since otherwise y will have more than two Picard exceptional value, a contradiction.

So if z_0 is a point such that $y(z_0) = \alpha_j$, then we have $(y(z_0)-\beta_1)(y(z_0)-\beta_2)\ldots(y(z_0)-\beta_{n-2}) \equiv 0$, a contradiction. On the otherhand if z_1 is a point such that $y(z_1) = \frac{g(z_1)}{f(z_1)} = \beta_k \neq 0$, $k = 1, 2, \ldots, n-2$, then we must have $f(z_1) = 0$ which is impossible as f and g share $(0, \infty)$.

If 0 is an e.v.P. or 0 and $-a\frac{n-1}{n}$ both are e.v.P. of f and g then by the same argument as above we can obtain a contradiction.

If $-a\frac{n-1}{n}$ is an e.v.P. of f and g, then we have from (2.7) that

$$\left(f + a\frac{(n-1)}{n}\right)\{n(y^n - 1)\} \equiv a\{(n-1)y^n - ny^{n-1} + 1\}.$$

If we assume $p(z) = (n-1)z^n - nz^{n-1} + 1$, then $p(0) \neq 0$ and $p(1) = p^{'}(1) = 0$. From above we see that $p(y)$ has $n-1$ distinct zeros none of which coincides with α_j, $j = 1, 2, \ldots, n-1$. Then again by the same argument as above we have at last left with a point say z_2 such that $f(z_2) = -a\frac{n-1}{n}$, a contradiction.

Hence $f \equiv g$ and this proves the lemma. □

Lemma 2.8. *Let f, g be two non-constant meromorphic functions such that $E_f(\{0, -a\frac{n-1}{n}\}, 0) = E_g(\{0, -a\frac{n-1}{n}\}, 0)$ and suppose $n\ (\geq 3)$ be an integer. Then*

$$f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2,$$

where a, b are finite nonzero constants.

Proof. If possible, let us suppose

$$f^{n-1}(f+a)g^{n-1}(g+a) \equiv b^2. \tag{2.6}$$

Let z_0 be a zero of f (g). Then z_0 must be either a 0-point or a $-a\frac{n-1}{n}$ point of g (f), which is impossible from (2.6). It follows that f (g) has no zero.

Next let z_0 be a zero of $f+a$ with multiplicity p. Then z_0 is a pole of g with multiplicity q such that $p = (n-1)q + q = nq \geq n$.

Since the poles of f are the zeros of $g+a$ only, we get

$$\overline{N}(r, \infty; f) \leq \overline{N}(r, -a; g) \leq \frac{1}{n}T(r, g).$$

By the second fundamental theorem we get

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \overline{N}(r, -a; f) + S(r, f) \\ &\leq \frac{1}{n}\,N(r, -a; f) + \frac{1}{n}T(r, g) + S(r, f) \\ &\leq \frac{1}{n}T(r, f) + \frac{1}{n}T(r, g) + S(r, f). \end{aligned}$$

i.e.,

$$(1 - \frac{1}{n})\ T(r, f) \leq \frac{1}{n}\ T(r, g) + S(r, f).$$

Similarly

$$(1-\frac{1}{n})\,T(r,g)\leq\frac{1}{n}\,T(r,f)+S(r,g)$$

Adding (2.9) and (2.10) we get

$$(1-\frac{2}{n})\,\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

a contradiction for $n\geq 3$. This proves the lemma. □

Lemma 2.9. *Let F, G be given by (2.1) and they share $(1,m)$. Also let $\omega_1,\omega_2\ldots\omega_n$ are the members of the set $S_2=\{z: z^n+az^{n-1}+b=0\}$, where a, b are nonzero constants such that $z^n+az^{n-1}+b=0$ has no repeated root and n (≥ 3) is an integer. Then*

$$\overline{N}_*(r,1;F,G)\leq\frac{1}{m}\left[\overline{N}(r,0;f)+\overline{N}(r,-a\frac{n-1}{n};f)\right]+S(r,f).$$

Proof. First we note that since S_2 has distinct elements, $-a\frac{n-1}{n}$ can not be a member of S_2. So

$$\begin{aligned}
&\overline{N}_*(r,1;F,G)\\
\leq\;&\overline{N}(r,1;F\mid\geq m+1)\\
\leq\;&\frac{1}{m}\,\big(N(r,1;F)-\overline{N}(r,1;F)\big)\\
\leq\;&\frac{1}{m}\left[\sum_{j=1}^{n}\big(N(r,\omega_j;f)-\overline{N}(r,\omega_j;f)\big)\right]\\
\leq\;&\frac{1}{m}\left[N\left(r,0;f^{'}\mid f\neq 0,\;-a\frac{n-1}{n}\right)\right]]\\
\leq\;&\frac{1}{m}\left[\overline{N}\left(r,\infty;\frac{f(f+a\frac{n-1}{n})}{f'}\right)\right]\\
\leq\;&\frac{1}{m}\left[N\left(r,\infty;\frac{f^{'}}{f(f+a\frac{n-1}{n})}\right)\right]+S(r,f)\\
\leq\;&\frac{1}{m}\left[\overline{N}(r,0;f)+\overline{N}(r,-a\frac{n-1}{n};f)\right]+S(r,f)
\end{aligned}$$

□

Lemma 2.10. [2] *Let F, G be given by (2.1) where $n\geq 7$ is an integer. If $H\equiv 0$ then either $f^{n-1}(f+a)g^{n-1}(g+a)\equiv b^2$ or $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$*

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1). Then F and G share $(1,3)$. We consider the following cases.

Case 1. Suppose that $\Phi\not\equiv 0$.

Subcase 1.1 Let $H\not\equiv 0$. Then using *Lemma 2.6* for $m=3$ and $p=2$, *Lemma 2.5* for $p=0$, *Lemma 2.4* and *Lemma 2.9* for $m=3$ we obtain

$$
\begin{aligned}
(3.1)\quad & (n+1)\left\{T(r,f)+T(r,g)\right\}\\
\leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)\right\}+2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}\\
& +\overline{N}(r,0;f\mid\geq 3)+\overline{N}\left(r,-a\frac{n-1}{n};f\mid\geq 3\right)+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]\\
& -\frac{3}{2}\overline{N}_{*}(r,1;F,G)+S(r,f)+S(r,g)\\
\leq\ & (4+\frac{1}{5})\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]\\
& +\frac{7}{60}\left\{\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)+\overline{N}(r,0;g)+\overline{N}\left(r,-a\frac{n-1}{n};g\right)\right\}\\
& +S(r,f)+S(r,g)\\
\leq\ & (\frac{n}{2}+4+\frac{13}{30})\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).
\end{aligned}
$$

(3.1) gives a contradiction for $n\geq 7$.
Subcase 1.2 Let $H\equiv 0$. Now the conclusion of the theorem can be obtain from *Lemmas 2.10*, *2.8* and *2.7*.
Case 2. Suppose that $\Phi\equiv 0$. On integration we get $(F-1)\equiv A(G-1)$ for some non zero constant A. So in view of *Lemma 2.4* we have

$$T(r,f)=T(r,g)+O(1). \tag{3.2}$$

Since by the given condition of the theorem $E_f(S_1,0)=E_g(S_1,0)$ we consider the following cases.
Subcase 2.1. Let us first assume f and g share $(0,0)$ and $(-a\frac{n-1}{n},0)$. If one of 0 or $-a\frac{n-1}{n}$ is an e.v.P. of both f and g, then we get $A=1$ and we have $F\equiv G$, which in view of *Lemma 2.7* implies $f\equiv g$. Let both 0 and $-a\frac{n-1}{n}$ are e.v.P. of f as well as g then noting that here $F\equiv AG+(1-A)$, suppose $A\neq 1$. Using *Lemma 2.4*, (3.2) and the second fundamental theorem we get

$$
\begin{aligned}
& nT(r,f)\\
\leq\ & \overline{N}(r,0;F)+\overline{N}(r,1-A;F)+\overline{N}(r,\infty;F)+S(r,F)\\
\leq\ & \overline{N}(r,0;f)+\overline{N}(r,-a;f)+\overline{N}(r,0;G)+\overline{N}(r,\infty;f)+S(r,f)\\
\leq\ & 2T(r,f)+T(r,g)+S(r,f)\\
\leq\ & 3T(r,f)+S(r,f),
\end{aligned}
$$

which implies a contradiction since $n\geq 7$.
Subcase 2.2. Next suppose that there is at least one point z_0 such that $f(z_0)=0$ and $g(z_0)=-a\frac{n-1}{n}$. At the point z_0, we have $F(z_0)=0$ and $G(z_0)=\beta$ (say). So $A=\frac{1}{1-\beta}$. Putting this values we obtain from above

$$F\equiv\frac{1}{1-\beta}G+\frac{\beta}{\beta-1}.$$

If $\beta\neq 0$ then again noting that $\overline{N}(r,\frac{\beta}{\beta-1};F)=\overline{N}(r,0;G)$, we can again get a contradiction as above when $n\geq 7$. $\square$

Proof of Theorem 1.2. Let F, G be given by (2.1). Then F and G share $(1,3)$. We consider the following cases.
Case 1. Suppose that $\Phi \not\equiv 0$.
Subcase 1.1. Let $H \not\equiv 0$. Then using *Lemma 2.6*, *Lemma 2.5* for $p = 0$, *Lemma 2.4* and *Lemma 2.9* we obtain

$$\begin{aligned}
(3.3)\quad & (n+1)\left\{T(r,f)+T(r,g)\right\} \\
\leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)\right\}+2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\} \\
& +\overline{N}(r,0;f\mid\geq p+1)+\overline{N}\left(r,-a\frac{n-1}{n};f\mid\geq p+1\right)+\frac{1}{2}\left[N(r,1;F)\right. \\
& \left.+N(r,1;G)\right]+\left(\frac{3}{2}-m\right)\overline{N}_{*}(r,1;F,G)+S(r,f)+S(r,g) \\
\leq\ & \left(4+\frac{1}{2p+1}\right)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right] \\
& +\left(\frac{7}{4m}+\frac{1}{2m(2p+1)}-\frac{1}{2}\right)\left\{\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)+\overline{N}(r,0;g)\right. \\
& \left.+\overline{N}\left(r,-a\frac{n-1}{n};g\right)\right\}+S(r,f)+S(r,g) \\
\leq\ & \left(\frac{n}{2}+3+\frac{7}{2m}+\frac{m+1}{m(2p+1)}\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).
\end{aligned}$$

Since $\frac{7}{2m}+\frac{m+1}{m(2p+1)}<2$, with $\frac{7}{2m}+\frac{1}{m(2p+1)}>1$ and $n\geq 8$, (3.2) gives a contradiction.
We now omit the rest of the proof since the same is similar to that of *Theorem 1.1*. □

References

[1] A. Banerjee, *On the uniqueness of meromorphic functions that share two sets*, Georgian Math., Vol:15, No.1 (2008), 21-38.

[2] A. Banerjee, *Uniqueness of meromorphic functions sharing two sets with finite weight*, Portugal. Math. (N.S.), Vol:65, No.1 (2008), 81-93.

[3] A. Banerjee, *Some further results ona question of Yi*, Publ. De Inst. Math., Vol:92, No.6 (2012), 177-187.

[4] A. Banerjee, and S. Mukherjee, *Uniqueness of meromorphic functions sharing two or three sets*, Hokkaido Math. J., Vol:37, No.3 (2008), 507-530.

[5] A. Banerjee and P. Bhattacharajee, *Uniqueness and set sharing of derivatives of meromorphic functions*, Math. Slovaca, Vol:61, No.2 (2011), pp. 197-214.

[6] M. Fang and H.Guo, *On meromorphic functions sharing two values*, Analysis, Vol:17 (1997), 355-366.

[7] M. Fang and I. Lahiri, *Unique range set for certain meromorphic functions*, Indian J. Math., Vol:45, No.2 (2003), 141-150.

[8] F. Gross, *Factorization of meromorphic functions and some open problems*, Proc. Conf. Univ. Kentucky, Leixngton, Ky(1976); Lecture Notes in Math., 599 (1977), 51-69, Springer(Berlin).

[9] F. Gross and C. C. Yang, *On preimage and range sets of meromorphic functions*, Proc. Japan Acad., Vol:58 (1982), 17-20.

[10] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[11] I. Lahiri, *The range set of meromorphic derivatives*, Northeast. Math. J., Vol:14, No.3 (1998), 353-360.

[12] I. Lahiri, *Value distribution of certain differential polynomials*, Int. J. Math. Math. Sci., Vol:28, No.2 (2001), 83-91.
[13] I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., Vol:161 (2001), 193-206.
[14] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., Vol:46 (2001), 241-253.
[15] I. Lahiri, *On a question of Hong Xun Yi*, Arch. Math. (Brno), Vol:38 (2002), 119-128.
[16] P. Li and C. C. Yang, *Some further results on the unique range sets for meromorphic functions*, Kodai Math. J.,Vol:18 (1995), 437-450.
[17] P. Li and C. C. Yang, *On the unique range sets for meromorphic functions*, Proc. Amer. Math. Soc., Vol:124 (1996) 177-185.
[18] W. C. Lin and H. X. Yi, *Some further results on meromorphic functions that share two sets*, Kyungpook Math. J., Vol:43 (2003), 73-85.
[19] A. Z. Mohon'ko, *On the Nevanlinna characteristics of some meromorphic functions*, Theory of Functions. Funct. Anal. Appl., Vol:14 (1971), 83-87.
[20] H. X. Yi, *Unicity theorems for meromorphic or entire functions II*, Bull. Austral. Math. Soc., Vol:52 (1995), 215-224.
[21] H. X. Yi, *Unicity theorems for meromorphic or entire functions III*, Bull. Austral. Math. Soc.,Vol:53 (1996), 71-82.
[22] H. X. Yi and W. C. Lin, *Uniqueness of meromorphic functions and a question of Gross*, Kyungpook Math. J, Vol:46 (2006), 437-444.
[23] H. X. Yi and W. R. Lü, *Meromorphic functions that share two sets II*, Acta Math. Sci. Ser.B Engl. Ed., Vol:24 No. 1 (2004), 83-90.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 53–000 (2014) ©KJM

ON PURE $\mathcal{LA}$-SEMIHYPERGROUPS

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Abstract. We generalize the existing theory of an associative structure [6] by studying it in a non-associative hyper-structure called an LA-semihypergroup. The results obtained will also generalize the results on LA-semigroup without hyper theory. As an application of our results we characterize $(0,2)$-hyperideals of an LA-semihypergroup H and prove that A is a $(0,2)$-hyperideal of H if and only if A is a left hyperideal of some left hyperideal of H. We also show that an LA-semihypergroup H is 0-$(0,2)$-bisimple if and only if H is right 0-simple. Finally we give the connection of ordered and hyper theories of an LA-semigroup.

1. Introduction

Hyperstructure theory was introduced in 1934, when F. Marty [9] defined hypergroups, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science and they are studied in many countries of the world. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. A lot of papers and several books have been written on hyperstructure theory, see [2]. Many authors studied different aspects of semihypergroups, for instance, Corsini et al. [1], Davvaz et al. [3], Hila et al. [5] and Leoreanu [8].

A left almost semigroup ($\mathcal{LA}$-semigroup) is a groupoid $\mathcal{S}$ whose elements satisfy the following left invertive law $(ab)c = (cb)a$ for all $a, b, c \in \mathcal{S}$.This concept was first given by Kazim and Naseeruddin in 1972 [7]. In an $\mathcal{LA}$-semigroup, the medial law [7] $(ab)(cd) = (ac)(bd)$ holds, $\forall\ a, b, c, d \in \mathcal{S}$. An $\mathcal{LA}$-semigroup may or may not contain a left identity. The left identity of an $\mathcal{LA}$-semigroup allow us to introduce the inverses of elements in an $\mathcal{LA}$-semigroup. If an $\mathcal{LA}$-semigroup contains a left

Date: January 1, 2013 and, in revised form, February 2, 2013.
2000 *Mathematics Subject Classification.* 20N20.
Key words and phrases. $\mathcal{LA}$-semihypergroup, left invertive law, pure left identity and 0-minimal hyperideals.

identity, then it is unique [10]. In an $\mathcal{LA}$-semigroup $\mathcal{S}$ with left identity, the paramedial law $(ab)(cd) = (dc)(ba)$ holds, $\forall\ a, b, c, d \in \mathcal{S}$. By using medial law with left identity, we get $a(bc) = b(ac)$ for all $a, b, c \in \mathcal{S}$.

An $\mathcal{LA}$-semigroup is a non-associative and non-commutative algebraic structure mid way between a groupoid and a commutative semigroup. This structure is closely related to a commutative semigroup; indeed if an $\mathcal{LA}$-semigroup contains a right identity, then it becomes a commutative semigroup [10]. The connection between a commutative inverse semigroup and an $\mathcal{LA}$-semigroup was established by Yousafzai et al. in [18] as follows: a commutative inverse semigroup $(\mathcal{S}, .)$ becomes an $\mathcal{LA}$-semigroup $(\mathcal{S}, *)$ under $a * b = ba^{-1}r^{-1}$ for all $a, b, r \in \mathcal{S}$. An $\mathcal{LA}$-semigroup $\mathcal{S}$ with a left identity becomes a semigroup under the binary operation "$\circ_e$" defined as follows: $x \circ_e y = (xe)y$ for all $x, y \in \mathcal{S}$ [18]. There are lot of results which have been added to the theory of an $\mathcal{LA}$-semigroup by Mushtaq, Kamran, Holgate, Jezek, Protic, Madad, Yousafzai and many other researchers. An $\mathcal{LA}$-semigroup is a generalization of a semigroup [10] and it has vast applications in semigroups, as well as in other branches of mathematics. Yaqoob et al. [11, 12] and Yousafzai et al. [15, 16, 17] studied some aspects of fuzzy $\mathcal{LA}$-semigroups and fuzzy $\mathcal{AG}$-groupoids.

Recently, Hila et al. introduced the notion of $\mathcal{LA}$-semihypergroups [4]. They investigated several properties of hyperideals of $\mathcal{LA}$-semihypergroup and defined the topological space and study the topological structure of $\mathcal{LA}$-semihypergroups using hyperideal theory. In [13], Yaqoob et al. have characterized intra-regular $\mathcal{LA}$-semihypergroups by using the properties of their left and right hyperideals, and investigated some useful conditions for an $\mathcal{LA}$-semihypergroup to become an intra-regular $\mathcal{LA}$-semihypergroup. This non-associative hyper structure has been further explored by Yousafzai et al. in [19] and [21]. Yaqoob and Gulistan [14] defined partial order relations on $\mathcal{LA}$-semihypergroups.

In this paper, we discuss 0-minimal hyperideals and $(0, 2)$-hyperideals. We characterize an $\mathcal{LA}$-semihypergroup in terms of $(1, 2)$-hyperideals and show that a $(1, 2)$-hyperideal of a pure $\mathcal{LA}$-semihypergroup is a left hyperideal of some bi-hyperideal. We give the necessary and sufficient condition for an $\mathcal{LA}$-semihypergroup to become right 0-simple.

2. Preliminaries and examples

A map $\circ : \mathcal{H} \times \mathcal{H} \to \mathcal{P}^*(\mathcal{H})$ is called *hyperoperation* or *join operation* on the set $\mathcal{H}$, where $\mathcal{H}$ is a non-empty set and $\mathcal{P}^*(\mathcal{H}) = \mathcal{P}(\mathcal{H})\backslash\{\emptyset\}$ denotes the set of all non-empty subsets of $\mathcal{H}$. A hypergroupoid is a set $\mathcal{H}$ together with a (binary) hyperoperation.

Let A and B be two non-empty subsets of $\mathcal{H}$, then we denote

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, \quad a \circ A = \{a\} \circ A \ \text{ and } a \circ B = \{a\} \circ B.$$

A hypergroupoid $(\mathcal{H}, \circ)$ is called an $\mathcal{LA}$-semihypergroup [4] if $(x \circ y) \circ z = (z \circ y) \circ x$ holds for all $x, y, z \in \mathcal{H}$. The law is called a left invertive law. A hypergroupoid $(\mathcal{H}, \circ)$ is called a right almost semihypergroup ($\mathcal{RA}$-semihypergroup) if $x \circ (y \circ z) = z \circ (y \circ x)$ hold for all $x, y, z \in \mathcal{H}$. The law is called a right invertive law. A hypergroupoid $(\mathcal{H}, \circ)$ is called an almost semihypergroup ($\mathcal{A}$-semihypergroup) if it is both an $\mathcal{LA}$-semihypergroup and an $\mathcal{RA}$-semihypergroup.

Every $\mathcal{LA}$-semihypergroup satisfies the law $(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w)$ for all $w, x, y, z \in \mathcal{H}$. This law is known as medial law (cf. [4]).

Let $\mathcal{H}$ be an $\mathcal{LA}$-semihypergroup [13], then an element $e \in \mathcal{H}$ is called

(i) a left identity (resp. pure left identity) if for all $a \in \mathcal{H}$, $a \in e \circ a$ (resp. $a = e \circ a$),

(ii) a right identity (resp. pure right identity) if for all $a \in \mathcal{H}$, $a \in a \circ e$ (resp. $a = a \circ e$),

(iii) an identity (resp. pure identity) if for all $a \in \mathcal{H}$, $a \in e \circ a \cap a \circ e$ (resp. $a = e \circ a \cap a \circ e$).

We have shown in [19] that unlike an $\mathcal{LA}$-semigroup, an $\mathcal{LA}$-semihypergroup may have a right identity or an identity. This fact can lead us to the following major remark.

Remark 2.1. The right identity of an $\mathcal{LA}$-semihypergroup need not to be a left identity in general. An $\mathcal{LA}$-semihypergroup may have a left identity or a right identity or an identity. Moreover, an $\mathcal{LA}$-semihypergroup with a right identity need not to be associative.

However an $\mathcal{LA}$-semihypergroup with pure right identity becomes a commutative semigroup with identity [19].

An $\mathcal{LA}$-semihypergroup with pure left identity e is called a *pure $\mathcal{LA}$-semihypergroup.* A pure $\mathcal{LA}$-semihypergroup $(\mathcal{H}, \circ)$ satisfy the following laws for all $w, x, y, z \in \mathcal{H}$:

$$(x \circ y) \circ (z \circ w) = (w \circ z) \circ (y \circ x),$$

called a paramedial law, and

$$x \circ (y \circ z) = y \circ (x \circ z).$$

Example 2.1. Let $(\mathcal{H}, \circ)$ be an $\mathcal{LA}$-semihypergroup with pure left identity e. Define a binary hyperoperation $\hat{o}$ (e-sandwich hyperoperation) as follows:

$$a \ \hat{o} \ b = (a \circ e) \circ b \text{ for all } a, b \in \mathcal{H}.$$

Then $(\mathcal{H}, \hat{o})$ *becomes a commutative semihypergroup with pure identity.*

Example 2.2. An $\mathcal{A}$-semihypergroup $\mathcal{H}$ with pure left identity becomes an abelian hypergroup.

If there is an element 0 of an $\mathcal{LA}$-semihypergroup $(H, \circ)$ such that $x \circ 0 = 0 \circ x = x$ $\forall \ x \in H$, we call 0 a *zero element* of H.

Example 2.3. Let us consider the following table for $H = \{a, b, c, d, e\}$ with a pure left identity d. It is easy to see that $(H, \circ)$ is a pure unitary $\mathcal{LA}$-semigroup with a zero element a.

$\circ$	a	b	c	d	e
a	a	a	a	a	a
b	a	$\{a, e\}$	$\{a, e\}$	$\{a, c\}$	$\{a, e\}$
c	a	$\{a, e\}$	$\{a, e\}$	$\{a, b\}$	$\{a, e\}$
d	a	b	c	d	e
e	a	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$

A subset A of an $\mathcal{LA}$-semihypergroup H is called a *left* (*right*) hyperideal of H if $H \circ A \subseteq A$ ($A \circ H \subseteq A$), and is called a *hyperideal* of H if it is both left and right hyperideal of H. A subset A of an $\mathcal{LA}$-semihypergroup H is called an $\mathcal{LA}$-*subsemihypergroup* of H if $A^2 \subseteq A$. A hyperideal A of an $\mathcal{LA}$-semihypergroup H with zero is said to be 0-*minimal* if $A \neq \{0\}$ and $\{0\}$ is the only hyperideal of H properly contained in A. An $\mathcal{LA}$-semihypergroup H with zero is said to be 0-$(0,2)$-*bisimple* if $H^2 \neq \{0\}$ and $\{0\}$ is the only proper $(0,2)$-bi-hyperideal of H.

3. 0-minimal $(0,2)$-bi-hyperideals in pure $\mathcal{LA}$-semihypergroups

If H is a pure $\mathcal{LA}$-semihypergroup, then it is easy to see that $H \circ H = H$, $H \circ A^2 = A^2 \circ H$ and $A \subseteq H \circ A \; \forall \; A \subseteq H$. Note that every right hyperideal of a pure $\mathcal{LA}$-semihypergroup H is a left hyperideal of H but the converse is not true in general. Example 2.3 shows that there exists a subset $\{a, b, e\}$ of H which is a left hyperideal of H but not a right hyperideal of H. It is easy to see that $H \circ A$ and $H \circ A^2$ are the left and right hyperideals of a pure $\mathcal{LA}$-semihypergroup H. Thus $H \circ A^2$ is a hyperideal of a pure $\mathcal{LA}$-semihypergroup H.

Lemma 3.1. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then A is a $(0,2)$-hyperideal of H if and only if A is a hyperideal of some left hyperideal of H.*

Proof. Let A be a $(0,2)$-hyperideal of H, then

$$(H \circ A) \circ A = (A \circ A) \circ H = H \circ A^2 \subseteq A,$$

and

$$A \circ (H \circ A) = H \circ (A \circ A) = (H \circ H) \circ (A \circ A) = H \circ A^2 \subseteq A.$$

Hence A is a hyperideal of a left hyperideal $H \circ A$ of H.

Conversely, assume that A is a left hyperideal of a left hyperideal L of H, then

$$H \circ A^2 = (A \circ A) \circ H = (H \circ A) \circ A \subseteq (H \circ L) \circ A \subseteq L \circ A \subseteq A,$$

and clearly A is an $\mathcal{LA}$-subsemihypergroup of H, therefore A is a $(0,2)$-hyperideal of H. $\square$

Corollary 3.1. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then A is a $(0,2)$-hyperideal of H if and only if A is a left hyperideal of some left hyperideal of H.*

Lemma 3.2. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then A is a $(0,2)$-bi-hyperideal of H if and only if A is a hyperideal of some right hyperideal of H.*

Proof. Let A be a $(0,2)$-bi-hyperideal of H, then

$$(H \circ A^2) \circ A = (A^2 \circ H) \circ A = (A \circ H) \circ A^2 \subseteq H \circ A^2 \subseteq A,$$

and

$$\begin{aligned} A \circ (H \circ A^2) &= (H \circ H) \circ (A \circ A^2) = (A^2 \circ A) \circ (H \circ H) \\ &= (H \circ A) \circ A^2 \subseteq H \circ A^2 \subseteq A. \end{aligned}$$

Hence A is a hyperideal of some right hyperideal $H \circ A^2$ of H.

Conversely, assume that A is a hyperideal of a right hyperideal R of H, then

$$\begin{aligned} H \circ A^2 &= A \circ (H \circ A) = A \circ ((H \circ H) \circ A) = A \circ ((A \circ H) \circ H) \\ &\subseteq A \circ ((R \circ H) \circ R) \subseteq A \circ R \subseteq A, \end{aligned}$$

and

$$(A \circ H) \circ A \subseteq (R \circ H) \circ A \subseteq R \circ A \subseteq A,$$

which shows that A is a $(0,2)$-hyperideal of H. $\square$

Theorem 3.1. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then the following statements are equivalent.*

(i) *A is a $(1,2)$-hyperideal of H;*
(ii) *A is a left hyperideal of some bi-hyperideal of H;*
(iii) *A is a bi-hyperideal of some hyperideal of H;*
(iv) *A is a $(0,2)$-hyperideal of some right hyperideal of H;*
(v) *A is a left hyperideal of some $(0,2)$-hyperideal of H.*

Proof. $(i) \Longrightarrow (ii)$. It is easy to see that $(H \circ A^2) \circ H$ is a bi-hyperideal of H. Let A be a $(1,2)$-hyperideal of H, then

$$\begin{aligned}\{(H \circ A^2) \circ H\} \circ A &= \{(H \circ A^2) \circ (H \circ H)\} \circ A = \{(H \circ H) \circ (A^2 \circ H)\} \circ A \\ &= \{H \circ (A^2 \circ H)\} \circ A = (A^2 \circ H) \circ A \\ &= (A \circ H) \circ A^2 \subseteq A,\end{aligned}$$

which shows that A is a left hyperideal of a bi-hyperideal $(H \circ A^2) \circ H$ of H.

$(ii) \Longrightarrow (iii)$. Let A be a left hyperideal of a bi-hyperideal B of H, then

$$\begin{aligned}\{A \circ (H \circ A^2)\} \circ A &= \{H \circ (A \circ A^2)\} \circ A \\ &\subseteq [H \circ \{(H \circ A) \circ (A \circ A)\}] \circ A \\ &= [H \circ \{(A \circ A) \circ (A \circ H)\}] \circ A \\ &= [(A \circ A) \circ \{H \circ (A \circ H)\}] \circ A \\ &= [\{(H \circ (A \circ H)) \circ A\} \circ A] \circ A \\ &= [\{(A \circ H) \circ A\} \circ A] \circ A \\ &\subseteq [\{(B \circ H) \circ B\} \circ A] \circ A \\ &\subseteq (B \circ A) \circ A \subseteq A,\end{aligned}$$

which shows that A is a bi-hyperideal of a hyperideal $H \circ A^2$ of H.

$(iii) \Longrightarrow (iv)$. Let A be a bi-hyperideal of a hyperideal I of H, then

$$\begin{aligned}(H \circ A^2) \circ A^2 &= \{A^2 \circ (A \circ A)\} \circ H = \{A \circ (A^2 \circ A)\} \circ H \\ &\subseteq [A \circ \{(A \circ I) \circ A\}] \circ H = (A \circ A) \circ H \\ &= (H \circ A) \circ A \subseteq (H \circ I) \circ H \subseteq I,\end{aligned}$$

which shows that A is a $(0,2)$-hyperideal of a right hyperideal $H \circ A^2$ of H.

$(iv) \Longrightarrow (v)$. It is easy to see that $H \circ A^3$ is a $(0,2)$-hyperideal of H. Let A be a $(0,2)$-hyperideal of a right hyperideal R of H, then

$$\begin{aligned}A \circ (H \circ A^3) &= A \circ \{(H \circ H) \circ (A^2 \circ A)\} \\ &= A \circ \{(A \circ A^2) \circ H\} \\ &\subseteq A \circ [\{(H \circ A) \circ (A \circ A)\} \circ H] \\ &= A \circ [\{(A \circ A) \circ (A \circ H)\} \circ H] \\ &= (A \circ A) \circ [\{A \circ (A \circ H)\} \circ H] \\ &= [H \circ \{A \circ (A \circ H)\}] \circ A^2 \\ &= [A \circ \{H \circ (A \circ H)\}] \circ A^2 \\ &\subseteq (R \circ H) \circ A^2 \subseteq R \circ A^2 \subseteq A,\end{aligned}$$

which shows that A is a left hyperideal of a $(0,2)$-hyperideal $H \circ A^3$ of H.

$(v) \Longrightarrow (i)$. Let A be a left hyperideal of a $(0,2)$-hyperideal O of H, then

$$\begin{aligned} (A \circ H) \circ A^2 &= \{(A \circ A) \circ (H \circ H)\} \circ A = (H \circ A^2) \circ A \\ &\subseteq (H \circ O^2) \circ A \subseteq O \circ A \subseteq A, \end{aligned}$$

which shows that A is a $(1,2)$-hyperideal of H. □

Lemma 3.3. *Let H be a pure $\mathcal{LA}$-semihypergroup and A be an idempotent subset of H. Then A is a $(1,2)$-hyperideal of H if and only if there exist a left hyperideal L and a right hyperideal R of H such that $R \circ L \subseteq A \subseteq R \cap L$.*

Proof. Assume that A is a $(1,2)$-hyperideal of H such that A is idempotent. Setting $L = H \circ A$ and $R = H \circ A^2$, then

$$\begin{aligned} R \circ L &= (H \circ A^2) \circ (H \circ A) \\ &= (A^2 \circ H) \circ (H \circ A) \\ &= \{(H \circ A) \circ (H \circ H)\} \circ A^2 \\ &= \{(H \circ H) \circ (A \circ H)\} \circ A^2 \\ &= [H \circ \{(A \circ A) \circ (H \circ H)\}] \circ A^2 \\ &= [H \circ \{(H \circ H) \circ (A \circ A)\}] \circ A^2 \\ &= [H \circ [A \circ \{(H \circ H) \circ A\}]] \circ A^2 \\ &= [A \circ \{H \circ (H \circ A)\}] \circ A^2 \\ &\subseteq (A \circ H) \circ A^2 \subseteq A. \end{aligned}$$

It is clear that $A \subseteq R \cap L$.

Conversely, let R be a right hyperideal and L be a left hyperideal of H such that $R \circ L \subseteq A \subseteq R \cap L$, then

$$(A \circ H) \circ A^2 = (A \circ H) \circ (A \circ A) \subseteq (R \circ H) \circ (H \circ L) \subseteq R \circ L \subseteq A.$$

□

Assume that H is a pure unitary $\mathcal{LA}$-semihypergroup with zero. Then it is easy to see that every left (right) hyperideal of H is a $(0,2)$-hyperideal of H. Hence if O is a 0-minimal $(0,2)$-hyperideal of H and A is a left (right) hyperideal of H contained in O, then either $A = \{0\}$ or $A = O$.

Lemma 3.4. *Let H be a pure $\mathcal{LA}$-semihypergroup with zero. Assume that A is a 0-minimal hyperideal of H and O is an $\mathcal{LA}$-subsemihypergroup of A. Then O is a $(0,2)$-hyperideal of H contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

Proof. Let O be a $(0,2)$-hyperideal of H contained in a 0-minimal hyperideal A of H. Then $H \circ O^2 \subseteq O \subseteq A$. Since $H \circ O^2$ is a hyperideal of H, therefore by minimality of A, $H \circ O^2 = \{0\}$ or $H \circ O^2 = A$. If $H \circ O^2 = A$, then $A = H \circ O^2 \subseteq O$ and therefore $O = A$. Let $H \circ O^2 = \{0\}$, then

$$O^2 \circ H = H \circ O^2 = \{0\} \subseteq O^2,$$

which shows that O^2 is a right hyperideal of H, and hence a hyperideal of H contained in A, therefore by minimality of A, we have $O^2 = \{0\}$ or $O^2 = A$. Now if $O^2 = A$, then $O = A$.

Conversely, let $O^2 = \{0\}$, then

$$H \circ O^2 = O^2 \circ H = \{0\} \circ H = \{0\} = O^2.$$

Now if $O = A$, then

$$H \circ O^2 = (H \circ H) \circ (O \circ O) = (H \circ A) \circ (H \circ A) \subseteq A = O,$$

which shows that O is a $(0,2)$-hyperideal of H contained in A. $\square$

Corollary 3.2. *Let H be a pure $\mathcal{LA}$-semihypergroup with zero. Assume that A is a 0-minimal left hyperideal of H and O is an $\mathcal{LA}$-subsemihypergroup of A. Then O is a $(0,2)$-hyperideal of H contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

Lemma 3.5. *Let H be a pure $\mathcal{LA}$-semihypergroup with zero and O be a 0-minimal $(0,2)$-hyperideal of H. Then $O^2 = \{0\}$ or O is a 0-minimal right (left) hyperideal of H.*

Proof. Let O be a 0-minimal $(0,2)$-hyperideal of H, then

$$\begin{aligned} H \circ (O^2)^2 &= (H \circ H) \circ (O^2 \circ O^2) = (O^2 \circ O^2) \circ H \\ &= (H \circ O^2) \circ O^2 \subseteq O \circ O^2 \subseteq O^2, \end{aligned}$$

which shows that O^2 is a $(0,2)$-hyperideal of H contained in O, therefore by minimality of O, $O^2 = \{0\}$ or $O^2 = O$. Suppose that $O^2 = O$, then $O \circ H = (O \circ O) \circ (H \circ H) = H \circ O^2 \subseteq O$, which shows that O is a right hyperideal of H. Let R be a right hyperideal of H contained in O, then $R^2 \circ H = (R \circ R) \circ H \subseteq (R \circ H) \circ H \subseteq R$. Thus R is a $(0,2)$-hyperideal of H contained in O, and again by minimality of O, $R = \{0\}$ or $R = O$. $\square$

The following Corollary follows from Lemma 3.4 and Corollary 3.2.

Corollary 3.3. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then O is a minimal $(0,2)$-hyperideal of H if and only if O is a minimal left hyperideal of H.*

Theorem 3.2. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then A is a minimal $(2,1)$-hyperideal of H if and only if A is a minimal bi-hyperideal of H.*

Proof. Let A be a minimal $(2,1)$-hyperideal of H. Then

$$\begin{aligned} [\{(A^2 \circ H) \circ A\}^2 \circ H] \circ \{(A^2 \circ H) \circ A\} &= [[\{(A^2 \circ H) \circ A\} \circ \{(A^2 \circ H) \circ A\}] \circ H] \\ &\quad \circ \{(A^2 \circ H) \circ A\} \\ &\subseteq [[\{(A \circ H) \circ A\} \circ \{(A \circ H) \circ A\}] \circ H] \\ &\quad \circ \{(A \circ H) \circ A\} \\ &= [[\{(A \circ H) \circ (A \circ H)\} \circ (A \circ A)] \circ H] \\ &\quad \circ \{(A \circ H) \circ A\} \\ &= [\{(A^2 \circ H) \circ (A \circ A)\} \circ H] \circ \{(A \circ H) \circ A\} \\ &\subseteq [\{(A \circ H) \circ (A \circ H)\} \circ H] \circ \{(A \circ H) \circ A\} \\ &= \{(A^2 \circ H) \circ H\} \circ \{(A \circ H) \circ A\} \\ &\subseteq \{(A \circ H) \circ H\} \circ \{(A \circ H) \circ A\} \\ &= \{(A \circ H) \circ (A \circ H)\}(H \circ A) \\ &= (A^2 \circ H) \circ (H \circ A) = (A \circ H) \circ (H \circ A^2) \\ &= \{(H \circ A^2) \circ H\} \circ A = \{(A^2 \circ H) \circ H\} \circ A \\ &= \{(H \circ H) \circ (A \circ A)\} \circ A = (A^2 \circ H) \circ A, \end{aligned}$$

and similarly we can show that $\{(A^2 \circ H) \circ A\}^2 \subseteq (A^2 \circ H) \circ A$. Thus $(A^2 \circ H) \circ A$ is a $(2,1)$-hyperideal of H contained in A, therefore by minimality of A, $(A^2 \circ H) \circ A = A$. Now

$$\begin{aligned}
(A \circ H) \circ A &= (A \circ H) \circ \{(A^2 \circ H) \circ A\} = [\{(A^2 \circ H) \circ A\} \circ H] \circ A \\
&= \{(H \circ A) \circ (A^2 \circ H)\} \circ A = [A^2 \circ \{(H \circ A) \circ H\}] \circ A \\
&\subseteq (A^2 \circ H) \circ A = A,
\end{aligned}$$

It follows that A is a bi-hyperideal of H. Suppose that there exists a bi-hyperideal B of H contained in A, then $(B^2 \circ H) \circ B \subseteq (B \circ H) \circ B \subseteq B$, so B is a $(2,1)$-hyperideal of H contained in A, therefore $B = A$.

Conversely, assume that A is a minimal bi-hyperideal of H, then it is easy to see that A is a $(2,1)$-hyperideal of H. Let C be a $(2,1)$-hyperideal of H contained in A, then

$$\begin{aligned}
[\{(C^2 \circ H) \circ C\} \circ H] \circ \{(C^2 \circ H) \circ C\} &= \{(H \circ C) \circ (C^2 \circ H)\} \circ \{(C^2 \circ H) \circ C\} \\
&= \{(H \circ C^2) \circ (C \circ H)\} \circ \{(C^2 \circ H) \circ C\} \\
&= [C \circ \{(H \circ C^2) \circ H\}] \circ \{(C^2 \circ H) \circ C\} \\
&= [\{(C^2 \circ H) \circ C\} \circ \{(H \circ C^2) \\
&\quad \circ (H \circ H)\}] \circ C \\
&= [\{(C^2 \circ H) \circ C\} \circ \{H \circ (C^2 \circ H)\}] \circ C \\
&= [\{(C^2 \circ H) \circ C\} \circ (C^2 \circ H)] \circ C \\
&= [C^2 \circ [\{(C^2 \circ H) \circ C\} \circ H]] \circ C \\
&\subseteq (C^2 \circ H) \circ C.
\end{aligned}$$

This shows that $(C^2 \circ H) \circ C$ is a bi-hyperideal of H, and by minimality of A, $(C^2 \circ H) \circ C = A$. Thus $A = (C^2 \circ H) \circ C \subseteq C$, and therefore A is a minimal $(2,1)$-hyperideal of H. □

Theorem 3.3. *Let A be a 0-minimal $(0,2)$-bi-hyperideal of a pure $\mathcal{LA}$-semihypergroup H with zero. Then exactly one of the following cases occurs*:

(*i*) $A = \{0, a\}$, $a^2 = 0$;
(*ii*) $\forall\ a \in A \backslash \{0\}$, $H \circ \circ a^2 = A$.

Proof. Assume that A is a 0-minimal $(0,2)$-bi-hyperideal of H. Let $a \in A\backslash\{0\}$, then $H \circ a^2 \subseteq A$. Also $H \circ a^2$ is a $(0,2)$-bi-hyperideal of H, therefore $H \circ a^2 = \{0\}$ or $H \circ a^2 = A$.

Let $H \circ a^2 = \{0\}$. Since $a^2 \subseteq A$, we have either $a^2 = a$ or $a^2 = 0$ or $a^2 \subseteq A\backslash\{0, a\}$. If $a^2 = a$, then $a^3 = a^2 \circ a = a$, which is impossible because $a^3 \subseteq a^2 \circ H = H \circ a^2 = \{0\}$. Let $a^2 \subseteq A\backslash\{0, a\}$, we have

$$H \circ [\{0, a^2\}\{0, a^2\}] = (H \circ H) \circ (a^2 \circ a^2) = (H \circ a^2) \circ (H \circ a^2) = \{0\} \subseteq \{0, a^2\},$$

and

$$[\{0, a^2\}H]\{0, a^2\} = \{0, a^2H\}\{0, a^2\} = a^2H \cdot a^2 \subseteq Ha^2 = \{0\} \subseteq \{0, a^2\}.$$

Therefore $\{0, a^2\}$ is a $(0,2)$-bi-hyperideal of H contained in A. We observe that $\{0, a^2\} \neq \{0\}$ and $\{0, a^2\} \neq A$. This is a contradiction to the fact that A is a 0-minimal $(0,2)$-bi-hyperideal of H. Therefore $a^2 = 0$ and $A = \{0, a\}$.

If $H \circ a^2 \neq \{0\}$, then $H \circ a^2 = A$. □

Corollary 3.4. *Let A be a 0-minimal $(0,2)$-bi-hyperideal of a pure $\mathcal{LA}$-semihypergroup H with zero such that $A^2 \neq 0$. Then $A = H \circ a^2$ for every $a \in A \backslash \{0\}$.*

Lemma 3.6. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then every right hyperideal of H is a $(0,2)$-bi-hyperideal of H.*

Proof. Assume that A is a right hyperideal of H, then

$$H \circ A^2 = (A \circ A) \circ (H \circ H) = (A \circ H) \circ (A \circ H) \subseteq A \circ A \subseteq A \circ H \subseteq A, \ (A \circ H) \circ A \subseteq A,$$

and clearly $A^2 \subseteq A$, therefore A is a $(0,2)$-bi-hyperideal of H. □

The converse of Lemma 3.6 is not true in general. Example 2.3 shows that there exists a $(0,2)$-bi-hyperideal $A = \{a, c, e\}$ of H which is not a right hyperideal of H.

Theorem 3.4. *Let H be a pure $\mathcal{LA}$-semihypergroup with zero. Then $H \circ a^2 = H$ $\forall\ a \in H \backslash \{0\}$ if and only if H is 0-$(0,2)$-bisimple if and only if H is right 0-simple.*

Proof. Assume that $H \circ a^2 = H$ for every $a \in H \backslash \{0\}$. Let A be a $(0,2)$-bi-hyperideal of H such that $A \neq \{0\}$. Let $a \in A \backslash \{0\}$, then $H = H \circ a^2 \subseteq H \circ A^2 \subseteq A$. Therefore $H = A$. Since $H = H \circ a^2 \subseteq H \circ H = H^2$, we have $H^2 = H \neq \{0\}$. Thus H is 0-$(0,2)$-bisimple. The converse statement follows from Corollary 3.4.

Let R be a right hyperideal of 0-$(0,2)$-bisimple H. Then by Lemma 3.6, R is a $(0,2)$-bi-hyperideal of H and so $R = \{0\}$ or $R = H$.

Conversely, assume that H is right 0-simple. Let $a \in H \backslash \{0\}$, then $H \circ a^2 = H$. Hence H is 0-$(0,2)$-bisimple. □

Theorem 3.5. *Let A be a 0-minimal $(0,2)$-bi-hyperideal of a pure $\mathcal{LA}$-semihypergroup H with zero. Then either $A^2 = \{0\}$ or A is right 0-simple.*

Proof. Assume that A is 0-minimal $(0,2)$-bi-hyperideal of H such that $A^2 \neq \{0\}$. Then by using Corollary 3.4, $H \circ a^2 = A$ for every $a \in A \backslash \{0\}$. Since $a^2 \subseteq A \backslash \{0\}$ for every $a \in A \backslash \{0\}$, we have $a^4 = (a^2)^2 \subseteq A \backslash \{0\}$ for every $a \in A \backslash \{0\}$. Let $a \in A \backslash \{0\}$, then

$$\begin{aligned} \{(A \circ a^2) \circ H\} \circ (A \circ a^2) &= (a^2 \circ A) \circ \{H \circ (A \circ a^2)\} = [\{H \circ (A \circ a^2)\} \circ A] \circ a^2 \\ &\subseteq \{(H \circ A) \circ A\} \circ a^2 = \{(A \circ A) \circ (H \circ H)\} \circ a^2 \\ &= (H \circ A^2) \circ a^2 \subseteq A \circ a^2, \end{aligned}$$

and

$$\begin{aligned} H \circ (A \circ a^2)^2 &= H \circ \{(A \circ a^2) \circ (A \circ a^2)\} = H \circ \{(a^2 \circ A) \circ (a^2 \circ A)\} \\ &= H \circ [a^2 \circ \{(a^2 \circ A) \circ A\}] = (a \circ a) \circ [H \circ \{(a^2 \circ A) \circ A\}] \\ &= [\{(a^2 \circ A) \circ A\} \circ H] \circ a^2 \subseteq \{(A \circ A) \circ (H \circ H)\} \circ a^2 \\ &= (H \circ A^2) \circ a^2 \subseteq A \circ a^2, \end{aligned}$$

which shows that $A \circ a^2$ is a $(0,2)$-bi-hyperideal of H contained in A. Hence $A \circ a^2 = \{0\}$ or $A \circ a^2 = A$. Since $a^4 \subseteq A \circ a^2$ and $a^4 \subseteq A \backslash \{0\}$, we get $A \circ a^2 = A$. Thus by using Theorem 3.4, A is right 0-simple. □

4. Construction of $\mathcal{LA}$-semihypergroups

Let $(H, \cdot, \leq)$ be any ordered $\mathcal{LA}$-semigroup [20]. Define a hyperoperation $\circ$ on H by:

$$x \circ y = \{z \in H : z \leq xy\} = (xy] \text{ for all } x, y \in H.$$

Then for all $a, b, c \in H$, we claim that $(a \circ b) \circ c = ((ab)c]$. Let $x \in (a \circ b) \circ c$, then $x \in y \circ c$ for some $y \in a \circ b$, which shows that $x \leq yc$ and $y \leq ab$. Hence $x \leq (ab)c$, so $(a \circ b) \circ c \subseteq ((ab)c]$. Let $x \in ((ab)c]$, then $x \leq (ab)c$. Thus $x \in ab \circ c \subseteq \bigcup\limits_{y \in a \circ b} y \circ c = (a \circ b) \circ c$. Consequently $(a \circ b) \circ c = ((ab)c]$. Similarly we can show that $(c \circ b) \circ a = ((cb)a]$, which shows that $(a \circ b) \circ c = (c \circ b) \circ a$ for all $a, b, c \in H$. Thus $(H, \circ)$ becomes an $\mathcal{LA}$-semihypergroup.

Let us consider an ordered $\mathcal{LA}$-semigroup $H = \{a, b, c\}$ in the following multiplication table and ordered below:

$\cdot$	a	b	c
a	b	c	a
b	a	b	c
c	c	a	b

$$\leq := \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}.$$

The hyperoperation $\circ$ is defined in the following table.

$\circ$	a	b	c
a	$\{a, b\}$	$\{a, b, c\}$	a
b	a	$\{a, b\}$	$\{a, b, c\}$
c	$\{a, b, c\}$	a	$\{a, b\}$

Then $(H, \circ)$ is an $\mathcal{LA}$-semihypergroup because $(a \circ b) \circ c = (c \circ b) \circ a$ for all $a, b, c \in H$ and $(c \circ b) \circ c \neq c \circ (b \circ c)$ for $b, c \in H$.

Acknowledgements: The authors are highly thankful to CAS-TWAS President's Fellowship.

References

[1] P. Bonansinga and P. Corsini, On semihypergroup and hypergroup homomorphisms, Boll. Un. Mat. Ital. Vol:6 (1982), 717-727.
[2] P. Corsini, Prolegomena of hypergroup theory, Aviani editor, (1993).
[3] B. Davvaz, Some results on congruences on semihypergroups, Bull. Malays. Math. Sci. Soc. Vol:23 (2000), 53-58.
[4] K. Hila and J. Dine, On hyperideals in left almost semihypergroups, ISRN Algebra, Vol:2011, Article ID953124 (2011), 8 pages.
[5] K. Hila, B. Davvaz and K. Naka, On quasi-hyperideals in semihypergroups, Commun. Algebra. Vol:39, No.11 (2011), 4183-4194.
[6] D. N. Kargovic, On 0-minimal $(0, 2)$-bi-ideals of semigroups, Publ. Inst. Math. (Beograd) Vol:31, No.45 (1982), 103-107.
[7] M. A. Kazim and M. Naseeruddin, On almost semigroups, The Alig. Bull. Math. Vol:2 (1972), 1-7.
[8] V. Leoreanu, About the simplifiable cyclic semihypergroups, Ital. J. Pure Appl. Math. Vol:7 (2000), 69-76.
[9] F. Marty, Sur une generalization de la notion de groupe, 8^{iem} congres Math. Scandinaves. Stockholm., (1934), 45-49.
[10] Q. Mushtaq and S. M. Yusuf, On LA-semigroups, The Alig. Bull. Math. Vol:8 (1978), 65-70.

[11] N. Yaqoob, Interval-valued intuitionistic fuzzy ideals of regular LA-semigroups, Thai J. Math. Vol:11, No.3 (2013), 683-695.
[12] N. Yaqoob, R. Chinram, A. Ghareeb and M. Aslam, Left almost semigroups characterized by their interval valued fuzzy ideals, Afr. Mat. Vol:24, No.2 (2013), 231-245.
[13] N. Yaqoob, P. Corsini and F. Yousafzai, On intra-regular left almost semihypergroups with pure left identity, Journal of Mathematics. Vol:2013 (2013), Article ID 510790, 10 pages.
[14] N. Yaqoob and M. Gulistan, Partially ordered left almost semihypergroups, J. Egyptian Math. Soc. in press, doi: 10.1016/j.joems.2014.05.012.
[15] F. Yousafzai, N. Yaqoob and A. Ghareeb, Left regular $\mathcal{AG}$-groupoids in terms of fuzzy interior ideals, Afr. Mat. Vil:24, No.4 (2013), 577-587.
[16] F. Yousafzai, N. Yaqoob and A.B. Saeid, Some results in bipolar-valued fuzzy ordered $\mathcal{AG}$-groupoids, Discus. Math. Gen. Algebra Appl. Vol:32 (2012), 55-76.
[17] F. Yousafzai, N. Yaqoob and K. Hila, On fuzzy $(2,2)$-regular ordered Γ-$\mathcal{AG}^{**}$-groupoids, UPB Sci. Bull. Series A. Vol:74, No.2 (2012), 87-104.
[18] F. Yousafzai, A. Khan and B. Davvaz, On fully regular $\mathcal{AG}$-groupoids, Afr. Mat. Vol:25 (2014), 449–459.
[19] F. Yousafzai and P. Corsini, Some characterization problems in LA-semihypergroups, J. Algebra, Number Theory: Adv. Appl. Vol:10 (2013), 41-55.
[20] F. Yousafzai, A. Khan, V. Amjad and A. Zeb, On fuzzy fully regular ordered AG-groupoids, J. Intell. Fuzzy Syst. Vol:26 (2014), 2973–2982.
[21] F. Yousafzai, K. Hila, P. Corsini and A. Zeb, Existence of non-associative algebraic hyperstructures and related problems, Afr. Mat. in press, doi: 10.1007/s13370-014-0259-6.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 64–69 (2014) ©KJM

ON THE DIOPHANTINE EQUATION $3^x + 5^y + 7^z = w^2$

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Abstract. We exhaust all solutions of the Diophantine equation

$$3^x + 5^y + 7^z = w^2$$

in non-negative integers using elementary methods.

1. Introduction

One of the many interesting equations that are being studied by number theorists is the class of Diophantine equations. Diophantine equations are usually polynomial equations in two or more variables and mathematicians are searching only for integer solutions. The simplest type of such is the so-called linear Diophantine equation, which is of the form

$$ax + by = 1, \tag{1.1}$$

where x and y are unknowns while a and b are constants. An equation that can not be transformed into (1.1) is usually referred to as nonlinear diophantine equation. Pell's equation is an example, which is of the form $x^2 - by^2 = 1$, where b is not a perfect square integer, and we are searching for the integer values of x and y that satisfy the equation. Another example is the equation

$$x^n + y^n = z^n, \tag{1.2}$$

which has infinitely many integer solutions (x, y, z) if $n = 2$ but no integer solutions whenever $n > 2$. Then there is the class of the so-called exponential Diophantine equation which is formed when one or more exponents serve as unknowns as well. An example is the equation

$$a^x - b^y = 1, \tag{1.3}$$

where a, b, x, and y are all positive integers greater than 1. In 1844, Charles Catalan conjectured that the only solution to (1.3) is the 4-tuple $(a, b, x, y) = (3, 2, 2, 3)$ and

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Date: Received on August 13, 2014 and, in revised form, September 24, 2014. Accepted on October 2, 2014.

2000 *Mathematics Subject Classification.* 11D61.

Key words and phrases. Exponential Diophantine equation, integer solutions.

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this was finally proven in 2002 by Preda Mihǎilescu (cf. [6]). In 2007, D. Acu [1] showed that the Diophantine equation $2^x+5^y = z^2$ has exactly two solutions in non-negative integers. In 2011, A. Suvarnamani, A. Singta, and S. Chotchaisthit [2] used P. Mihǎilescu's theorem (Catalan's conjecture) to show that the two Diophantine equations $4^x+7^y = z^2$ and $4^x+11^y = z^2$ have no solution in non-negative integers. In [3], S. Chotchaisthit studied the Diophantine equation $4^x + p^y = z^2$ in non-negative integers, and in [4], he obtained a complete solution to the Diophantine equation $2^x + 11^y = z^2$ in non-negative integers. In fact, the latter equation has already been studied by A. D. Nicoară and C. E. Pumnea [7] without the use of an auxilliary result (Mihǎilescu's Theorem). Banyat Sroysang also studied several exponential Diophantine equations of $a^x + b^y = z^2$ type (cf. [11, 12, 13, 14, 15, 16]). In [16], Sroysang was asking for the set of all solutions (x, y, z, w) for the Diophantine equation

$$3^x + 5^y + 7^z = w^2 \tag{1.4}$$

in non-negative integers. As a response, we present in this paper that the only non-negative integer solutions (x, y, z, w) to (1.4) are $(0, 0, 1, 3), (1, 1, 0, 3)$ and $(3, 1, 2, 9)$. Related exponential Diophantine equations in the form $p^x \pm q^y \pm r^z = c$ where p, q, r are primes, x, y, z are non-negative integers, and c an integer have been studied in [5] and [10]. Particularly, J. Leitner [5] solved the equation $3^a + 5^b - 7^c = 1$ for non-negative integers a, b, c and the equation $y^2 = 3^a + 2^b + 1$ for non-negative integers a, b and integer y. R. Scott and R. Styer [10] studied, among other things, the Diophantine equation $p^x \pm q^y \pm 2^z = 0$ for primes p and q and integer c in positive integers x, y, and z. They used elementary methods to show that, with a few explicitly listed exceptions, there are at most two solutions (x, y) to $|p^x \pm q^y| = c$ and at most two solutions (x, y, z) to $p^x \pm q^y \pm 2^z = 0$ in positive integers. In the following section we shall use elementary methods to prove our main result.

2. Main Result

Theorem $(0, 0, 1, 3), (1, 1, 0, 3)$ *and* $(3, 1, 2, 9)$ *are the only solutions* (x, y, z, w) *to the exponential Diophantine equation* (1.4) *in non-negative integers.*

The proof considers separate cases where at least one of the three exponents is zero or where all of them are strictly positive. Throughout the discussion, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive and non-negative integers, respectively.

2.1. The case $\min\{x, y, z\} = 0$.

We consider the following cases:

$$3^x + 5^y = w^2 - 1, \tag{2.1}$$

$$3^x + 7^z = w^2 - 1, \tag{2.2}$$

$$5^y + 7^z = w^2 - 1. \tag{2.3}$$

We prove the following lemmas.

LEMMA 1: $(1, 1, 3)$ is the unique solution to equation (2.1) in $\mathbb{N}_0$.

Note that by considering only the case when $w > 1$ is sufficient to study the problem since the left hand side (LHS) of (1.4) is greater than or equal to three. So we proceed as follows.

Proof of Lemma 1. Let $x, y, w \in \mathbb{N}_0$ with $w > 1$ and suppose that $3^x + 5^y = w^2 - 1$ has a solution in $\mathbb{N}_0$. First, we let $x = 0$. So, we have $2+5^y = w^2$ which is impossible because $2 + 5^y \equiv 3 \pmod 4$ whereas $w^2 \equiv 0, 1 \pmod 4$. If $y = 0$, then we obtain $3^x + 2 = w^2$ which is also impossible since $3^x + 2 \equiv 2 \pmod 3$. This leaves to consider $\min\{x, y\} > 0$. Note that $3^x \equiv 1 \pmod 4$ when x is even and $3^x \equiv 3$ (mod 4) when x is odd. Also, note that $5^y \equiv 1 \pmod 4$ for any $y \in \mathbb{N}_0$ so, x is odd, i.e. $x = 2r + 1$ for some $r \in \mathbb{N}_0$. Furthermore, since $w^2 \equiv 0, 1, 4 \pmod 8$, $3^{2r+1} + 1 \equiv 4$ (mod 8) and, $5^y \equiv 1 \pmod 8$ when y is odd and $5^y \equiv 5 \pmod 8$ when y is even, we conclude that y is odd, i.e. $y = 2s + 1$ for some $s \in \mathbb{N}_0$. We also conclude that w is odd.

Since x and y are odd, we can now express (2.1) as $3^{2r+1} + 5^{2s+1} = w^2 - 1$. Writing w as $w = 2m + 1$ for some $m \in \mathbb{N}$, we further express (2.1) as

$$3^{2r+1} + 5^{2s+1} = 4m^2 + 4m = 4m(m + 1).$$

Dividing this equation by 8, we obtain

$$\frac{3^{2r+1} + 5^{2s+1}}{8} = \frac{m(m + 1)}{2}. \tag{2.4}$$

Notice that the LHS of (2.4) is a triangular number. It is easy to see that the equation is true for $r = s = 0$, *i.e.* $(3^1 + 5^1)/8 = (1)(2)/2$ giving us $(x, y, w) = (1, 1, 3)$ as a solution to (2.1). Now suppose that there is another solution (r', s') such that $\min\{r', s'\} > 0$. Hence, $w^2 \equiv 0, 1, 4, 7 \pmod 9$, $3^{2r'+1} + 1 \equiv 1 \pmod 9$, and $5^{2s'+1} \equiv 1, 4, 7 \pmod 9$. It follows that $3^{2r'+1} + 5^{2s'+1} + 1 \equiv 2, 5, 8 \not\equiv w^2$ (mod 9). Thus, $(r, s, w) = (0, 0, 3)$ is the only solution to $3^{2r+1} + 5^{2s+1} = w^2 - 1$ and this completes the proof of the lemma.

□

Note: In equation (2.4), the triangular number is 8 times a perfect square plus 1. So (2.4) has the form $1 + p^{\alpha_1} + \cdots + p^{\alpha_k} = A^2$ where p is prime and $A, a_1, \ldots, a_k$ are positive integers. This equation has already been studied by Rotkiewicz and Złotkowski in 1987 (cf. [9]). Also, Scott and Styer already proved that the equation $3^{2r+1} + 5^{2s+1} = 8 \times c$, where c is a triangular number, has at most 2 solutions (cf. Theorem 7 of [10]).

LEMMA 2: $(0, 1, 3)$ is the unique solution to equation (2.2) in $\mathbb{N}_0$.

Proof of Lemma 2. Consider equation (2.2) modulo 4. Observe that $w^2 - 1 \equiv 0, 3$ (mod 4) and $7^z \equiv 1 \pmod 4$ if z is even and $7^z \equiv 3 \pmod 4$ if z is odd. Hence, x and z in (2.2) are of different parity. Suppose x is odd and z is even. Then, $3^x \equiv 0$ (mod 3) and $7^z \equiv 1 \pmod 3$. It follows that $3^x + 7^z + 1 \equiv 2 \not\equiv w^2 \pmod 3$ because $w^2 \equiv 0, 1 \pmod 3$. Thus, the equation (2.2) has no solution. Suppose now that $x = 2r$ and $z = 2t + 1$ for some $r, t \in \mathbb{N}_0$. We can express (2.2) as

$$8(7^{2t} - 7^{2t-1} + \cdots + 1) = (w + 3^r)(w - 3^r). \tag{2.5}$$

For $t = 0$ we can distribute the factor 8 as $8 = (w + 3^r)(w - 3^r)$ with either $w + 3^r = 8, w - 3^r = 1$ or $w + 3^r = 4, w - 3^r = 2$ (but not $w + 3^r = 2$ and $w - 3^r = 4$, etc because $w + 3^r > w - 3^r$). If $w + 3^r = 8$ and $w - 3^r = 1$, then by addition, $2w = 9$ which is clearly impossible. If $w + 3^r = 4$ and $w - 3^r = 2$, then $w = 3, r = 0$ and here we obtain $(x, y, z, w) = (0, 0, 1, 3)$ which is a solution to (1.4). Now to deal with the two conditions; namely,

i) $t > 0, 2|(w + 3^r)$, and $4|(w - 3^r)$, and
ii) $t > 0, 4|(w + 3^r)$, and $2|(w - 3^r)$,

it suffices to assume that $r > 0$. We treat these two cases at once. Note that for $r > 0, 3^{2r} \equiv 0 \pmod 3$ and $7^z \equiv 1 \pmod 3$. Hence, we see that $3^x + 7^z + 1 \equiv 2 \not\equiv w^2$ since $w^2 \equiv 0, 1 \pmod 3$. The conclusion follows. □

LEMMA 3: $(0, 1, 3)$ is the unique solution to equation (2.3) in $\mathbb{N}_0$.

Proof of Lemma 3. Consider now equation (2.3) modulo 4. Noting that w is odd we have $w^2 - 1 \equiv 0, 3 \pmod 4$ which implies that z is odd. Since $w^2 \equiv 1 \pmod 8$, $7^{2t+1} + 1 \equiv 7 + 1 \equiv 0 \pmod 8$, and $5^y \equiv 1 \pmod 8$ when y is even and $5^y \equiv 5 \pmod 8$ when y is odd, we conclude that y is even.

If $y = 2s$ and $z = 2t+1$ for some $s, t \in \mathbb{N}_0$, then we obtain $8(7^{2t} - 7^{2t-1} + \cdots + 1) = (w + 5^s)(w - 5^s)$. Note that $w + 5^s > w - 5^s$. So for $t = 0$, we can only distribute 8 as factors of $(w + 5^s)(w - 5^s)$ as follows: $w + 5^s = 8$ and $w - 5^s = 1$ or $w + 5^s = 4$ and $w - 5^s = 2$. The first pair of equations is clearly impossible since, by addition, $2w = 9$. However, the second pair of equations gives us $2w = (w + 5^s) + (w - 5^s) = 6$, or equivalently, $w = 3$. Here we obtain $(x, y, z, w) = (0, 0, 1, 3)$ as a solution to (1.4) and so it follows that $(0, 1, 3)$ is a solution to (2.3).

For $t > 0$, we consider the following cases:

i) $4|(w + 5^s)$ and $2|(w - 5^s)$; and
ii) $2|(w + 5^s)$ and $4|(w - 5^s)$.

We treat these cases at once and we may assume (WLOG) that $s > 0$.

For $\min\{s, t\} > 0$ we have,

$$(5^{2s} - 1) + (7^{2t} + 1) = w^2 - 1 \quad \Leftrightarrow \quad 8\left[\frac{(5^{2s} - 1) + (7^{2t} + 1)}{8}\right] = (w + 1)(w - 1).$$

Since w is odd then $w + 1$ and $w - 1$ were both even then the LHS of the latter equation is 8 times an odd integer. Hence, if $4|(w \pm 1)$ and $2|(w \mp 1)$ then $w \pm 1 = 4(2k + 1)$ and $w \mp 1 = 2(2l + 1)$ for some $k, l \in \mathbb{N}$. Subtracting these two equations yields $2 = 2[2(2k + 1) - (2l + 1)]$ or equivalently, $1 = 2(2k + 1) - (2l + 1)$ which implies that $k = l = 0$. This contradicts our assumption that $k, l \in \mathbb{N}$. Thus, $5^{2s} + 7^{2s+1} = w^2 - 1$ has no solution for $\min\{s, t\} > 0$ which completes the proof. □

2.2. **The case $\min\{x, y, z\} > 0$.** Let $\min\{x, y, z\} > 0$. We first determine a possible parity of x, y, z so that equation (1.4) has a solution in positive integers. Taking modulo 4 of both sides of (1.4) we see that x and z must be of different parity and w is odd. If we take modulo 3 of (1.4) both sides, then y must be odd. Lastly, taking modulo 8 of (1.4) both sides we conclude that x is odd and z is even. Hence, (1.4) is only possible in positive integers provided x is odd, y is odd, z is even, and w is odd.

Let $w = 2m + 1$ where $m \in \mathbb{N}$. Suppose that

$$3^{2r+1} + 5^{2s+1} + 7^{2t} = w^2, \tag{2.6}$$

for some $r, s, t \in \mathbb{N}_0$. Since $w^2 \equiv 0 \pmod 3$ and $w^2 \equiv 1 \pmod 8$ then $w = 24n - 15$ for some $n \in \mathbb{N}$. The least possible value of w would be 9. Letting $w = 9$ we see that $3^{2r+1} + 5^{2s+1} = 9^2 - 7^{2t} = (9 + 7^t)(9 - 7^t)$ in which follows that $t = 0$ or $z = 2$. Now we have $3^{2r+1} + 5^{2s+1} = 32$. Since $\min\{r, s\} > 0$ and, $2r + 1 \leq 3$ and $2s + 1 \leq 2$ it follows that $r = 1$ and $s = 0$ giving us $(1, 0, 1, 9)$ as a solution to (2.6) or equivalently, $(3,1,2,9)$ as a solution to (1.4). Now suppose that there is another solution to (2.6) with $s>0$ then we can express (2.6) as

$$4[(3^{2r}-3^{2r-1}+\cdots+1)+(5^{2s}+5^{2s-1}+\cdots+1)]=(w+7^t)(w-7^t). \tag{2.7}$$

Note that the term inside $[\ \cdot\ ]$ is even and $w+7^t$ and $w-7^t$ were both even. In addition, $w=8B+1$ and $7^t=(8-1)^t=8A\pm 1$ for some $A,B\in\mathbb{N}$. So $w^2-7^{2t}=(8B+1+8A\pm 1)(8B+1-(8A\pm 1))=16(A+B)(4(B-A)+1)$. Since $A+B$ is even then the LHS of (2.7) is 32 times an odd integer. Hence, we can rewrite equation (2.7) as

$$32\left[\frac{(3^{2r}-3^{2r-1}+\cdots+1)+(5^{2s}+5^{2s-1}+\cdots+1)}{8}\right]=(w+7^t)(w-7^t). \tag{2.8}$$

Now we consider the following possibilities: (i) $(3^{2r}-3^{2r-1}+\cdots+1)+(5^{2s}+5^{2s-1}+\cdots+1)=8$ where $16|(w+7^s)$ or $16|(w-7^s)$; and (ii) $8|(w+7^t)$ and $4|(w-7^t)$ or $8|(w-7^t)$ and $4|(w+7^t)$, which are the ways of distributing the factor 16 of the LHS across the two factors of the RHS. If $16|(w+7^t)$ or $16|(w-7^t)$, then we have $(3^{2r}-3^{2r-1}+\cdots+1)+(5^{2s}+5^{2s-1}+\cdots+1)=8$ which is impossible since $s>0$. If $8|(w\pm 7^t)$ and $4|(w\mp 7^t)$, then the requirements of $w\pm 7^t$ being both odd mean $2m+1+7^t=8(2k+1)$ and $2m+1-7^t=4(2l+1)$ for some $k,l\in\mathbb{N}$. Subtracting these two equations yields $2\cdot 7^t=4(2(2k+1)-(2l+1))$ or equivalently $7^t=2(2(2k+1)-(2l+1))$ which is clearly impossible. Therefore the only solutions in $\mathbb{N}_0$ to the exponential Diophantine equation (1.4) are $(0,0,1,3),(1,1,0,3)$ and $(3,1,2,9)$.

ACKNOWLEDGEMENT We would like to express our deepest gratitude to the anonymous referee for carefully examining this paper and providing it a number of important remarks and for pointing out papers [2], [5], [6], and [10]. The proof of our results were substaintially improved due to the referee's comments and suggestions.

References

[1] D. Acu, On a Diophantine equation $2^x+5^y=z^2$, *Gen. Math.,* **15**, 2007, 145-148.

[2] S. Chotchaisthit, A. Singta, and A. Suvarnamani , On two Diophantine Equations $4^x+7^y=z^2$ and $4^x+11^y=z^2$, *Sci. Technol. RMUTT J.,* **1**, 2011, 25-28.

[3] S. Chotchaisthit, On a Diophantine equation $4^x+p^y=z^2$ where p is a prime number, *Amer. J. Math. Sci.,* **1** (2012), 191-193.

[4] S. Chotchaisthit, On a Diophantine equation $2^x+11^y=z^2$, *Maejo Int. J. Sci. Technol.,* **7** (2013) 291-293.

[5] D. J. Leitner, Two exponential Diophantine equation, *Journal de Théorie des Nombres de Bordeaux,* **23** (2011), 479-487.

[6] P. Mihăilescu, Primary cycolotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.,* **27** (2004), 167-195.

[7] A. D. Nicoară and C. E. Pumnea, On a Diophantine equation of $a^x+b^y=z^2$ type, *Educaţia Matematică,* **4** (2008), no. 1, 65-75.

[8] K. H. Rosen, Elementary Number Theory and its applications, fifth edition, Pearson Addison-Wesley, 2005.

[9] A. Rotkiewicz and W. Złotkowski, On the Diophantine equation $1+p^{\alpha_1}+\cdots+p^{\alpha_k}=y^2$, *Colloq. Math. Soc. János Bolyai,* **51** (1990), 917937.

[10] R. Scott and R. Styer, On $p^x-q^y=c$ and related three term exponential Diophantine equations with prime bases, *J. Number Theory,* **105** (2004), 212-234.

[11] B. Sroysang, On the Diophantine equation $3^x+5^y=z^2$, *Int. J. Pure Appl. Math.,* **81**, (2012), 605-608.

[12] B. Sroysang, On the Diophantine equation $8^x + 19^y = z^2$, *Int. J. Pure Appl. Math.,* **81**, (2012), 601-604.

[13] B. Sroysang, On the Diophantine equation $31^x + 32^y = z^2$, *Int. J. Pure Appl. Math.,* **81**, (2012), 609-612.

[14] B. Sroysang, On the Diophantine equation $7^x + 8^y = z^2$, *Int. J. Pure Appl. Math.,* **84**, (2013), 111-114.

[15] B. Sroysang, On the Diophantine equation $2^x + 3^y = z^2$, *Int. J. Pure Appl. Math.,* **84**, (2013), No. 1, 133-137.

[16] B. Sroysang, On the Diophantine equation $5^x + 7^y = z^2$, *Int. J. Pure Appl. Math.,* **89**, (2013), No. 1, 115-118.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 70–75 (2014) ©KJM

GROWTH PROPERTIES OF GENERALIZED ITERATED ENTIRE AND COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. In this paper we consider generalized iteration of entire functions and prove some growth properties of generalized iterated entire functions and composition of entire and meromorphic functions under certain restrictions on (p,q) orders of functions.

1. INTRODUCTION AND DEFINITIONS

It is well known that for any two transcendental entire functions $f(z)$ and $g(z)$, $\lim\limits_{r\to\infty} \frac{M(r,f\circ g)}{M(r,f)} = \infty$. In a paper [10] Clunie proved that the same is also true when maximum modulus functions are replaced by their characteristic functions. Singh [15] proved some results dealing with the ratios of $\log T(r, f \circ g)$ and $T(r, f)$ under some restrictions on the order of f and g. After this several authors {see [9], [13] } made close investigation on comparative growth of $\log T(r, f \circ g)$ and $T(r, g)$ by imposing certain restrictions on order of f and g.

If $f(z)$ and $g(z)$ be entire functions then following the iteration process of Lahiri and Banerjee [12], we write

$f_1(z) = f(z)$
$f_2(z) = f(g(z)) = f(g_1(z))$
$f_3(z) = f(g(f(z))) = f(g_2(z))$
$f_4(z) = f(g(f(g(z)))) = f(g_3(z))$
$\ldots \quad \ldots \quad \ldots$
$f_n(z) = f(g(f(g(....(f(z)$ or $g(z))$ according as n is odd or even)))
and so are
$g_1(z) = g(z)$
$g_2(z) = g(f(z)) = g(f_1(z))$
$g_3(z) = g(f(g(z))) = g(f_2(z))$
$g_4(z) = g(f(g(f(z)))) = g(f_3(z))$

2010 *Mathematics Subject Classification.* 30D35.
Key words and phrases. Entire function, Meromorphic function, Generalized iteration, Composition, Growth.

$\ldots \qquad \ldots \qquad \ldots$
$g_n(z) = g(f(g(f(....(g(z)$ or $f(z))$ according as n is odd or even$)))$.
Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

Following this iteration process several papers {see [1], [2], [3], [4] } on growth properties of entire functions have appeared in the literature where growing interest of researchers on this topic has been noticed.

Recently Banerjee and Mondal [5] introduced another type of iteration called generalised iteration to study {see [5], [6] } some growth properties of entire functions.

Let f and g be two non-constant entire functions and α be any real number satisfying $0 < \alpha \leq 1$. Then the generalized iteration of f with respect to g is defined as follows:

$$f_{1,g}(z) = (1-\alpha)z + \alpha f(z)$$
$$f_{2,g}(z) = (1-\alpha)g_{1,f}(z) + \alpha f(g_{1,f}(z))$$
$$f_{3,g}(z) = (1-\alpha)g_{2,f}(z) + \alpha f(g_{2,f}(z))$$

$$\ldots. \qquad \ldots. \qquad \ldots.$$
$$f_{n,g}(z) = (1-\alpha)g_{n-1,f}(z) + \alpha f(g_{n-1,f}(z))$$

and so are

$$g_{1,f}(z) = (1-\alpha)z + \alpha g(z)$$
$$g_{2,f}(z) = (1-\alpha)f_{1,g}(z) + \alpha g(f_{1,g}(z))$$
$$g_{3,f}(z) = (1-\alpha)f_{2,g}(z) + \alpha g(f_{2,g}(z))$$

$$\ldots. \qquad \ldots. \qquad \ldots.$$
$$g_{n,f}(z) = (1-\alpha)f_{n-1,g}(z) + \alpha g(f_{n-1,g}(z)).$$

Following Sato [14], we write $\log^{[0]} x = x$, $\exp^{[0]} x = x$ and for positive integer m, $\log^{[m]} x = \log(\log^{[m-1]} x)$, $\exp^{[m]} x = \exp(\exp^{[m-1]} x)$.

Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ be an entire function.Then the (p,q)-order and lower (p,q)-order of $f(z)$ are denoted by $\rho_{(p,q)}(f)$ and $\lambda_{(p,q)}(f)$ respectively and defined by[7]

$$\rho_{(p,q)}(f) = \lim_{r\to\infty} \sup \frac{\log^{[p]} T(r,f)}{\log^{[q]} r} \text{ and } \lambda_{(p,q)}(f) = \lim_{r\to\infty} \inf \frac{\log^{[p]} T(r,f)}{\log^{[q]} r},\ p \geq q \geq 1.$$

Definition 1.1. A real valued function $\varphi(r)$ is said to have the property P_1 if
i) $\varphi(r)$ is non negative ;
ii) $\varphi(r)$ is strictly increasing and $\varphi(r) \to \infty$ as $r \to \infty$;
iii) $\log \varphi(r) \leq \delta\varphi(\frac{r}{4})$ holds for every $\delta > 0$ and for all sufficiently large values of r.

Remark 1.1. If $\varphi(r)$ satisfies the property P_1 then it is clear that $log^{[p]}\varphi(r) \leq \delta\varphi(\frac{r}{4})$ holds for every $p \geq 1$.

The purpose of this paper is to compare the growth of generalized iterated entire functions with composition of a meromorphic function and an entire function imposing certain restrictions on (p,q)-order and lower (p,q)-order. Throughout the paper we assume that f is a meromorphic function and g, h and k are non-constant entire functions such that the maximum modulus functions of h, k and all of their generalized functions satisfy property P_1. We do not explain the standard notations and definitions of the theory of meromorphic functions as those are available in [11].

2. LEMMAS

In this section we state some known results in the form of lemma which will be needed in the sequel.

Lemma 2.1. *([11]) If $f(z)$ be regular in $|z| \leq R$, then for $0 \leq r < R$*
$T(r,f) \leq \log^{+} M(r,f) \leq \frac{R+r}{R-r} T(R,f)$.
In particular if f be entire, then for all large values of r
$T(r,f) \leq \log^{+} M(r,f) \leq 3T(2r,f)$.

Lemma 2.2. *([8]) If f is meromorphic and g is entire then for all large values of r*
$T(r, f \circ g) \leq (1+o(1)) \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f)$.
Since g is entire so using Lemma 2.1, we have
$T(r, f \circ g) \leq (1+o(1)) T(M(r,g), f)$.

Lemma 2.3. *([10]) Let $f(z)$ and $g(z)$ be entire functions with $g(0) = 0$. Let β satisfy $0 < \beta < 1$ and let $C(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for $r > 0$*
$M(r, f \circ g) \geq M(C(\beta) M(\beta r, g), f)$.
Further if $g(z)$ is any entire function, then with $\beta = 1/2$, for sufficiently large values of r
$M(r, f \circ g) \geq M(\frac{1}{8} M(\frac{r}{2}, g) - |g(0)|, f)$.
Clearly $M(r, f \circ g) \geq M(\frac{1}{16} M(\frac{r}{2}, g), f)$.

3. MAIN THEOREMS

In this section, we present the main results of this paper.

Theorem 3.1. *Let g, h and k be three entire functions and f be a meromorphic function with $\rho_{(p,q)}(f) < \infty, \rho_{(p,q)}(g) < \infty$ and $\lambda_{(p,q)}(g) < \min\{\lambda_{(p,q)}(h), \lambda_{(p,q)}(k)\}$. Then*
$\limsup_{r \to \infty} \frac{\log^{[p+(n-2)(p+1-q)]} T(r, h_{n,k})}{\log^{[p]} T(r, f_{2,g})} = \infty$*, where $f_{2,g}(z)$ is the generalized composition of f with respect to g.*

Proof. Since $\lambda_{(p,q)}(g) < \min\{\lambda_{(p,q)}(h), \lambda_{(p,q)}(k)\}$ we can choose $\epsilon > 0$ in such a way that $\lambda_{(p,q)}(g) + \epsilon < \min\{\lambda_{(p,q)}(h) - \epsilon, \lambda_{(p,q)}(k) - \epsilon\}$. Using Lemma 2.1 and 2.3, we have for all large values of r

$$\begin{aligned} T(r, h_{n,k}) &\geq \frac{1}{3} \log M(\tfrac{r}{2}, h_{n,k}) \\ &\geq \frac{1}{3} \log\{M(\tfrac{r}{2}, \alpha h(k_{n-1,h})) - M(\tfrac{r}{2}, (1-\alpha) k_{n-1,h})\} \\ &\geq \frac{1}{3} \log\{\alpha M(\tfrac{1}{16} M(\tfrac{r}{4}, k_{n-1,h}), h) - (1-\alpha) M(\tfrac{r}{2}, k_{n-1,h})\} + O(1) \text{ [for } \alpha \neq 1] \\ &= \frac{1}{3} [\log M(\tfrac{1}{16} M(\tfrac{r}{4}, k_{n-1,h}), h) - \log M(\tfrac{r}{2}, k_{n-1,h})] + O(1). \end{aligned}$$

So for all sufficiently large values of r, we get

$$\begin{aligned} \log^{[p]} T(r, h_{n,k}) &\geq \log^{[p+1]} M(\tfrac{1}{16} M(\tfrac{r}{4}, k_{n-1,h}), h) - \log^{[p+1]} M(\tfrac{r}{2}, k_{n-1,h}) + O(1) \\ &> (\lambda_{(p,q)}(h) - \epsilon) \log^{[q]}(\tfrac{1}{16} M(\tfrac{r}{4}, k_{n-1,h})) - \log^{[p+1]} M(\tfrac{r}{2}, k_{n-1,h}) + O(1) \\ &> (\lambda_{(p,q)}(h) - \epsilon) \log^{[q]} M(\tfrac{r}{4}, k_{n-1,h}) - \\ &\quad \tfrac{1}{2}(\lambda_{(p,q)}(h) - \epsilon) \log^{[q]} M(\tfrac{r}{4}, k_{n-1,h}) + O(1), \text{using property } P_1 \\ &= \tfrac{1}{2}(\lambda_{(p,q)}(h) - \epsilon) \log^{[q]} M(\tfrac{r}{4}, k_{n-1,h}) + O(1) \qquad (3.1) \end{aligned}$$

$$\begin{aligned}
\text{or, } \log^{[p+(p+1-q)]} T(r,h_{n,k}) &> \log^{[p]}\{\log M(\tfrac{r}{4},k_{n-1,h})\} + O(1)\\
&\geq \log^{[p]} T(\tfrac{r}{4},k_{n-1,h}) + O(1)\\
&> \tfrac{1}{2}(\lambda_{(p,q)}(k)-\epsilon)\log^{[q]} M(\tfrac{r}{4^2},h_{n-2,k}) + O(1), \text{using}(3.1)\\
\text{or, } \log^{[p+2(p+1-q)]} T(r,h_{n,k}) &> \log^{[p]} T(\tfrac{r}{4^2},h_{n-2,k}) + O(1)\\
&> \tfrac{1}{2}(\lambda_{(p,q)}(h)-\epsilon)\log^{[q]} M(\tfrac{r}{4^3},k_{n-3,h}) + O(1).
\end{aligned}$$

Proceeding similarly after some steps we get for even n

$$\begin{aligned}
\log^{[p+(n-2)(p+1-q)]} T(r,h_{n,k}) &> \tfrac{1}{2}(\lambda_{(p,q)}(h)-\epsilon)\log^{[q]} M(\tfrac{r}{4^{n-1}},k_{1,h}) + O(1)\\
&= \tfrac{1}{2}(\lambda_{(p,q)}(h)-\epsilon)\log^{[q]} M\{\tfrac{r}{4^{n-1}},(1-\alpha)z+\alpha k\} + O(1)\\
&\geq \tfrac{1}{2}(\lambda_{(p,q)}(h)-\epsilon)\{\log^{[q]} M(\tfrac{r}{4^{n-1}},k) - \log^{[q]} M(\tfrac{r}{4^{n-1}},z)\} + O(1)\\
&\geq \tfrac{1}{2}(\lambda_{(p,q)}(h)-\epsilon)[\exp^{[p-q]}\{\log^{[q-1]}(\tfrac{r}{4^{n-1}})\}^{\lambda_{(p,q)}(k)-\epsilon} - \log^{[q]}(\tfrac{r}{4^{n-1}})] + O(1). \qquad (3.2)
\end{aligned}$$

On the other hand using Lemma 2.2 for a sequence of values of r tending to infinity

$$\begin{aligned}
T(r,f_{2,g}) &= T\{r,(1-\alpha)g_{1,f}+\alpha f(g_{1,f})\}\\
&\leq T(r,g_{1,f}) + T(r,f(g_{1,f})) + O(1)\\
&\leq T(r,g_{1,f}) + (1+o(1))T(M(r,g_{1,f}),f) + O(1)\\
\text{or, } \log^{[p]} T(r,f_{2,g}) &\leq \log^{[p]} T(r,g_{1,f}) + \log^{[p]} T(M(r,g_{1,f}),f) + O(1)\\
&< \log^{[p]} T(r,g_{1,f}) + (\rho_{(p,q)}(f)+\epsilon)\log^{[q]} M(r,g_{1,f}) + O(1)\\
&\leq \log^{[p]} T(r,z) + \log^{[p]} T(r,g) + (\rho_{(p,q)}(f)+\epsilon)[\log^{[q]} M(r,z)\\
&\qquad + \log^{[q]} M(r,g)] + O(1) \qquad (3.3)\\
&< \log^{[p+1]} r + (\rho_{(p,q)}(g)+\rho_{(p,q)}(f)+2\epsilon)\log^{[q]} r +\\
&\qquad \exp^{[p-q]}(\log^{[q-1]} r)^{\lambda_{(p,q)}(g)+\epsilon} + O(1). \qquad (3.4)
\end{aligned}$$

From (3.2) and (3.4) we get for a sequence of values of r tending to infinity

$$\frac{\log^{[p+(n-2)(p+1-q)]} T(r,h_{n,k})}{\log^{[p]} T(r,f_{2,g})} > \frac{\frac{1}{2}(\lambda_{(p,q)}(h)-\epsilon)[\exp^{[p-q]}\{\log^{[q-1]}(\frac{r}{4^{n-1}})\}^{\lambda_{(p,q)}(k)-\epsilon} - \log^{[q]}(\frac{r}{4^{n-1}})] + O(1)}{\log^{[p+1]} r + (\rho_{(p,q)}(g)+\rho_{(p,q)}(f)+2\epsilon)\log^{[q]} r + \exp^{[p-q]}(\log^{[q-1]} r)^{\lambda_{(p,q)}(g)+\epsilon} + O(1)}. \qquad (3.5)$$

When n is odd as in (3.2) we get

$$\begin{aligned}
\log^{[p+(n-2)(p+1-q)]} T(r,h_{n,k}) &> \tfrac{1}{2}(\lambda_{(p,q)}(k)-\epsilon)[\exp^{[p-q]}\{\log^{[q-1]}(\tfrac{r}{4^{n-1}})\}^{\lambda_{(p,q)}(h)-\epsilon}\\
&\qquad - \log^{[q]}(\tfrac{r}{4^{n-1}})] + O(1). \qquad (3.6)
\end{aligned}$$

From (3.4) and (3.6) we get for a sequence of values of r tending to infinity

$$\frac{\log^{[p+(n-2)(p+1-q)]} T(r,h_{n,k})}{\log^{[p]} T(r,f_{2,g})} > \frac{\frac{1}{2}(\lambda_{(p,q)}(k)-\epsilon)[\exp^{[p-q]}\{\log^{[q-1]}(\frac{r}{4^{n-1}})\}^{\lambda_{(p,q)}(h)-\epsilon} - \log^{[q]}(\frac{r}{4^{n-1}})] + O(1)}{\log^{[p+1]} r + (\rho_{(p,q)}(g)+\rho_{(p,q)}(f)+2\epsilon)\log^{[q]} r + \exp^{[p-q]}(\log^{[q-1]} r)^{\lambda_{(p,q)}(g)+\epsilon} + O(1)} \qquad (3.7)$$

From (3.5) and (3.7) we get

$$\limsup_{r\to\infty} \frac{\log^{[p+(n-2)(p+1-q)]} T(r,h_{n,k})}{\log^{[p]} T(r,f_{2,g})} = \infty.$$

Remark 3.1. In the above theorem if we take $\rho_{(p,q)}(g) < \min\{\lambda_{(p,q)}(h), \lambda_{(p,q)}(k)\}$ instead of $\lambda_{(p,q)}(g) < \min\{\lambda_{(p,q)}(h), \lambda_{(p,q)}(k)\}$ then limit superior is replaced by limit inferior.

Theorem 3.2. *Let g, h and k be three entire functions and f be a meromorphic function such that $\rho_{(p,q)}(f) < \infty$, $\rho_{(p,q)}(g) < \infty, \lambda_{(p,q)}(h) > 0$ and $\lambda_{(p,q)}(k) > 0$. Then*

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\log^{[2p+1-q]} T(r,f_{2,g})}{\log^{[p+(n-1)(p+1-q)]} T(r,h_{n,k})} &\leq \frac{\rho_{(p,q)}(g)}{\lambda_{(p,q)}(k)}, \qquad \textit{if } n \textit{ is even.}\\
&\leq \frac{\rho_{(p,q)}(g)}{\lambda_{(p,q)}(h)}, \qquad \textit{if } n \textit{ is odd.}
\end{aligned}$$

Proof**.** We may suppose that $\rho_{(p,q)}(g)$ is finite. Otherwise the result is obvious. First suppose that n is even. Then by (3.3) we get for large values of r

$$\log^{[p]} T(r, f_{2,g}) < \log^{[p]} T(r,z) + \log^{[p]} T(r,g) + (\rho_{(p,q)}(f)+\epsilon)[\log^{[q]} M(r,z) + \log^{[q]} M(r,g)] + O(1)$$
$$< \log^{[p+1]} r + (\rho_{(p,q)}(g) + \rho_{(p,q)}(f) + 2\epsilon)\log^{[q]} r + \exp^{[p-q]}(\log^{[q-1]} r)^{\rho_{(p,q)}(g)+\epsilon} + O(1).$$

So,

(3.8) $$\log^{[2p+1-q]} T(r, f_{2,g}) < \log^{[2p+2-q]} r + \log^{[p+1]} r + (\rho_{(p,q)}(g)+\epsilon)\log^{[q]} r + O(1).$$

From (3.2) we get for large values of r

$$\log^{[p+(n-2)(p+1-q)]} T(r, h_{n,k}) > \tfrac{1}{2}(\lambda_{(p,q)}(h)-\epsilon)[\exp^{[p-q]}\{\log^{[q-1]}(\tfrac{r}{4^{n-1}})\}^{\lambda_{(p,q)}(k)-\epsilon} - \log^{[q]}(\tfrac{r}{4^{n-1}})] + O(1).$$

So,

(3.9) $$\log^{[p+(n-1)(p+1-q)]} T(r, h_{n,k}) > (\lambda_{(p,q)}(k)-\epsilon)\log^{[q]} r - \log^{[p+1]}(\tfrac{r}{4^{n-1}}) + O(1).$$

Now from (3.8) and (3.9) we get for all large values of r

$$\frac{\log^{[2p+1-q]} T(r,f_{2,g})}{\log^{[p+(n-1)(p+1-q)]} T(r,h_{n,k})} < \frac{\log^{[2p+2-q]} r + \log^{[p+1]} r + (\rho_{(p,q)}(g)+\epsilon)\log^{[q]} r + O(1)}{(\lambda_{(p,q)}(k)-\epsilon)\log^{[q]} r - \log^{p+1}(\frac{r}{4^{n-1}}) + O(1)}$$
$$= \frac{(\rho_{(p,q)}(g)+\epsilon)\log^{[q]} r[1 + \frac{\log^{[2p+2-q]} r + \log^{[p+1]} r + O(1)}{(\rho_{(p,q)}(g)+\epsilon)\log^{[q]} r}]}{(\lambda_{(p,q)}(k)-\epsilon)\log^{[q]} r[1 - \frac{\log^{[p+1]}(\frac{r}{4^{n-1}}) + O(1)}{(\lambda_{(p,q)}(k)-\epsilon)\log^{[q]} r}]}.$$

Therefore, $\limsup_{r\to\infty} \frac{\log^{[2p+1-q]} T(r,f_{2,g})}{\log^{[p+(n-1)(p+1-q)]} T(r,h_{n,k})} \le \frac{\rho_{(p,q)}(g)}{\lambda_{(p,q)}(k)}$.

When n is odd we get as in (3.9)

(3.10) $$\log^{[p+(n-1)(p+1-q)]} T(r, h_{n,k}) > (\lambda_{(p,q)}(h)-\epsilon)\log^{[q]} r - \log^{[p+1]}(\tfrac{r}{4^{n-1}}) + O(1).$$

Now from (3.8) and (3.10) the remaining part of the theorem easily follows.

Theorem 3.3. *Let f be a meromorphic function and g, h and k be three entire functions such that $\rho_{(p,q)}(f) < \infty$, $\rho_{(p,q)}(g) < \infty, \lambda_{(p,q)}(h) > 0$ and $\lambda_{(p,q)}(k) > 0$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[2p+2-q]} T(r,f_{2,g})}{\log^{[p+1+(n-1)(p+1-q)]} T(r,h_{n,k})} \le 1.$$

Proof**.** From (3.8) we get for all large values of r

$$\log^{[2p+1-q]} T(r, f_{2,g}) < \log^{[2p+2-q]} r + \log^{[p+1]} r + (\rho_{(p,q)}(g)+\epsilon)\log^{[q]} r + O(1)$$
$$= (\rho_{(p,q)}(g)+\epsilon)\log^{[q]} r[1 + \frac{\log^{[2p+2-q]} r + \log^{[p+1]} r + O(1)}{(\rho_{(p,q)}(g)+\epsilon)\log^{[q]} r}].$$

Therefore,

(3.11) $$\log^{[2p+2-q]} T(r, f_{2,g}) < \log^{[q+1]} r + O(1).$$

Again from (3.9) we get for all large values of r and for even n

$$\log^{[p+(n-1)(p+1-q)]} T(r, h_{n,k}) > (\lambda_{(p,q)}(k)-\epsilon)\log^{[q]} r - \log^{[p+1]}(\tfrac{r}{4^{n-1}}) + O(1)$$
$$= (\lambda_{(p,q)}(k)-\epsilon)\log^{[q]} r[1 - \frac{\log^{[p+1]}(\frac{r}{4^{n-1}}) + O(1)}{(\lambda_{(p,q)}(k)-\epsilon)\log^{[q]} r}].$$

Therefore,

(3.12) $$\log^{[p+1+(n-1)(p+1-q)]} T(r, h_{n,k}) > \log^{[q+1]} r + O(1).$$

When n is odd we get from (3.10)

$$\log^{[p+(n-1)(p+1-q)]} T(r, h_{n,k}) > (\lambda_{(p,q)}(h)-\epsilon)\log^{[q]} r - \log^{[p+1]}(\tfrac{r}{4^{n-1}}) + O(1)$$

$$= (\lambda_{(p,q)}(h) - \epsilon) \log^{[q]} r[1 - \frac{\log^{[p+1]}(\frac{r}{4^{n-1}}) + O(1)}{(\lambda_{(p,q)}(h) - \epsilon) \log^{[q]} r}].$$

So,

$$\log^{[p+1+(n-1)(p+1-q)]} T(r, h_{n,k}) > \log^{[q+1]} r + O(1). \tag{3.13}$$

Therefore from (3.11), (3.12) and (3.13) we get

$$\limsup_{r\to\infty} \frac{\log^{[2p+2-q]} T(r, f_{2,g})}{\log^{[p+1+(n-1)(p+1-q)]} T(r, h_{n,k})} \leq 1.$$

Remark 3.2. In the above theorems if we take relative iteration instead of generalized iteration and take $q = 1$ then the results coincide with the results of Banerjee and Jana [3].

References

[1] Banerjee, D. and Dutta, R. K., *The growth of iterated entire functions*, Journal of Indian Acad. Maths.Vol.30, No.2 (2008), 489-495.

[2] Banerjee, D. and Jana, S., *Comparative growth of maximum modulus and maximum term of iterated entire functions*, Journal of Indian Acad. of Math., Vol.31, No.2 (2009), 351-360.

[3] Banerjee, D. and Jana, S., *On growth properties of iterated entire and composite entire and meromorphic functions*, Journal of Indian Acad Math., Vol.32, No.1 (2010), 295-309.

[4] Banerjee, D. and Dutta, R. K., *On growth of iterated entire functions*,Bulletin of Allahabad Mathematical Society, Vol.25, No.2 (2010), 333-340.

[5] Banerjee, D. and Mondal, N., *Maximum modulus and maximum term of generalized iterated entire functions*, Bulletin of the Allahabad Mathematical Society, Vol.27,No.1 (2012), 117-131.

[6] Banerjee, D. and Mondal, N., *Growth of generalized iterated entire functions*, Bulletin of the Allahabad Mathematical Society, Vol.27,No.2 (2012), 239-254.

[7] Bergweiler, W., Jank Gerhard and Volkmann Lutz, *Wachstumsverhalten zusammengesetzter Funktionen*, Results Math. 7 (1984), 35-53.

[8] Bergweiler, W., *On the Nevanlinna characteristic of a composite function*, Complex variables, Vol.10 (1988), 225-236.

[9] Bhoosnurmath, S. S. and Prabhaiah, V. S., *On the generalized growth properties of composite entire and meromorphic functions*, Journal of Indian Acad Math., Vol.29, No.2 (2007), 343-369.

[10] Clunie, J., *The composition of entire and meromorphic functions*, Mathematical Essays dedicated to Macintyre, Ohio Univ. Press (1970), 75-92.

[11] Hayman,W. K., *Meromorphic functions*, Oxford University Press,1964.

[12] Lahiri, B. K. and Banerjee, D. , *Relative fix points of entire functions*, J. Indian Acad. Math., Vol.19, No.1 (1997), 87-97.

[13] Lahiri, I., *Growth of composite integral functions*, Indian J. pure appl. Math., 35(4) (2004), 525-543.

[14] Sato,D., *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc., 69 (1963), 411-414.

[15] Singh, A. P., *Growth of composite entire functions*, Kodai Math. J., Vol 8 (1985), 99-102.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 76–84 (2014) ©KJM

HERMITE-HADAMARD'S INEQUALITIES FOR PREQUASIINVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

IMDAT ISCAN

Abstract. In this paper, we extend some estimates of the right hand side of Hermite-Hadamard type inequality for prequasiinvex functions via fractional integrals.

1. Introduction and Preliminaries

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This doubly inequality is known in the literature as Hermite-Hadamard integral inequality for convex mapping. For several recent results concerning the inequality (1.1) we refer the interested reader to [2, 3, 4, 5, 7, 17, 18].

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if inequality

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\},$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Clearly, any convex function is quasi-convex function. Futhemore there exist quasi-convex functions which are not convex (see [7]).

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.1. Let $f \in L[a, b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of oder $\alpha > 0$ with $a \geq 0$ are defined by

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2000 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.

Key words and phrases. Hermite-Hadamard type inequalities, prequasiinvex function, fractional integral.

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent result connected with fractional integral inequalities see ([13, 16, 18, 19]).

In [13], Ozdemir and Yıldız proved the Hadamard inequality for quasi-convex functions via Riemann-Liouville fractional integrals as follows:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$, be positive function with $0 \le a < b$ and $f \in L[a,b]$. If f is a quasi-convex function on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \le \max\left\{f(a), f(b)\right\} \tag{1.2}$$

with $\alpha > 0$.

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is quasi-convex on $[a,b]$, $\alpha > 0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \right| \\ \le\ & \frac{b-a}{\alpha+1} \left(1 - \frac{1}{2^{\alpha}}\right) \max\left\{|f'(a)|, |f'(b)|\right\}. \end{aligned} \tag{1.3}$$

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$, and $q > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \right| \\ \le\ & \frac{b-a}{2(\alpha p+1)^{\frac{1}{p}}} \left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} \end{aligned} \tag{1.4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0,1]$.

Theorem 1.4. *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$, and $q \ge 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \right| \\ \le\ & \frac{b-a}{(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}}\right) \left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} \end{aligned} \tag{1.5}$$

with $\alpha > 0$.

In recent years several extentions and generalizations have been considered for classical convexity. A significant generalization of convex functions is the invex functions introduced by Hanson in [6]. Weir and Mond [20] introduced the concept of preinvex functions and applied it to the establisment of the sufficient optimality conditions and duality in nonlinear programming. Pini [15] introduced the concept of prequasiinvex function as a generalization of invex functions. Later, Mohan and Neogy [9] obtained some properties of generalized preinvex functions. Noor ([11]-[12])has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions. In recent papers Yang et al. in [22] studied prequasiinvex functions and semistrictly prequasiinvex functions, Barani et al. in [4] presented some generalizations of the right hand side of a Hermite-Hadamard type inequality for prequasiinvex functions and Park in [14] established generalized Simpson-like and Hermite-Hadamard-like type integral inequalities for functions whose second derivatives in absolutely value at certain powers are preinvex and prequasiinvex.

In this paper we generalized the results in [13] for prequasiinvex functions. Now we recall some notions in invexity analysis which will be used throught the paper (see [1, 21] and references therein)

Let $f : A\rightarrow \mathbb{R}$ and $\eta : A\times A \rightarrow \mathbb{R}$,where A is a nonempty set in $\mathbb{R}^n$, be continuous functions.

Definition 1.2. The set $A \subseteq \mathbb{R}^n$ is said to be invex with respect to $\eta(.,.)$, if for every $x, y \in A$ and $t \in [0,1]$,

$$x + t\eta(y,x) \in A.$$

The invex set A is also called a $\eta-$connected set.

It is obvious that every convex set is invex with respect to $\eta(y,x) = y - x$, but there exist invex sets which are not convex [1].

Definition 1.3. The function f on the invex set A is said to be preinvex with respect to η if

$$f\left(x + t\eta(y,x)\right) \leq (1-t)\, f(x) + tf(y),\ \forall x, y \in A,\ t \in [0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

Definition 1.4. The function f on the invex set A is said to be prequasiinvex with respect to η if

$$f\left(x + t\eta(y,x)\right) \leq \max\left\{f(x), f(y)\right\},\ \forall x, y \in A,\ t \in [0,1].$$

Every quasi-convex function is a prequasinvex with respect to $\eta(y,x) = y - x$, but the converse does not holds (see example 1.1 in [22])

We also need the following assumption regarding the function η which is due to Mohan and Neogy [9]:

Condition C: Let $A \subseteq \mathbb{R}^n$ be an invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$. For any $x, y \in A$ and any $t \in [0,1]$,

$$\begin{aligned} \eta\left(y, y + t\eta(x,y)\right) &= -t\eta(x,y) \\ \eta\left(x, y + t\eta(x,y)\right) &= (1-t)\eta(x,y). \end{aligned}$$

Note that for every $x, y \in A$ and every $t \in [0,1]$ from condition C, we have

$$\eta\left(y+t_2\eta(x,y),y+t_1\eta(x,y)\right)=(t_2-t_1)\eta(x,y). \tag{1.6}$$

In [4] Barani et al. proved the Hermite-Hadamard type inequality for prequasiinvex as follows:

Theorem 1.5. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $|f'|$ is prequasiinvex on A then, for every $a,b\in A$ the following inequalities holds*

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|\\ \leq\ &\frac{|\eta(b,a)|}{4}\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}.\end{aligned} \tag{1.7}$$

Theorem 1.6. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. Assume that $p\in\mathbb{R}$ with $p>1$. If $|f'|^{\frac{p}{p-1}}$ is preinvex on A then, for every $a,b\in A$ the following inequalities holds*

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|\\ \leq\ &\frac{|\eta(b,a)|}{2(p+1)^{\frac{1}{p}}}\left(\max\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}.\end{aligned} \tag{1.8}$$

In order to prove our main results we need the following lemma:

Lemma 1.1. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If f' is preinvex function on A and $f'\in L\left[a,a+\eta(b,a)\right]$ then, the following equality holds:*

$$\begin{aligned}&\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\\ =\ &\frac{\eta(b,a)}{2}\int_0^1\left[t^{\alpha}-(1-t)^{\alpha}\right]f'\left(a+t\eta(b,a)\right)dt\end{aligned} \tag{1.9}$$

By using partial integration in right hand of (1.9) equality, the proof is obvious (see [8]).

In this paper, using Lemma 1.1 we obtained new inequalities related to the right side of Hermite-Hadamard inequalities for prequasiinvex functions via fractional integrals.

2. Main Results

Theorem 2.1. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$. If $f:\left[a,a+\eta(b,a)\right]\to(0,\infty)$ is a prequasiinvex function, $f\in L\left[a,a+\eta(b,a)\right]$ and η satisfies condition C then, the following*

inequalities for fractional integrals holds:

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right] \\ \leq \quad & \max\{f(a),f(a+\eta(b,a)\}\leq \max\{f(a),f(b)\} \end{aligned} \tag{2.1}$$

Proof. Since $a,b\in A$ and A is an invex set with respect to η, for every $t\in[0,1]$, we have $a+t\eta(b,a)\in A$. By prequasiinvexity of f and inequality (1.6) for every $t\in[0,1]$ we get

$$\begin{aligned} f(a+t\eta(b,a)) &= f(a+\eta(b,a)+(1-t)\eta(a,a+\eta(b,a))) \\ &\leq \max\{f(a),f(a+\eta(b,a)\} \end{aligned} \tag{2.2}$$

and similarly

$$\begin{aligned} f(a+(1-t)\eta(b,a)) &= f(a+\eta(b,a)+t\eta(a,a+\eta(b,a))) \\ &\leq \max\{f(a),f(a+\eta(b,a)\}. \end{aligned}$$

By adding these inequalities we have

$$f(a+t\eta(b,a))+f(a+(1-t)\eta(b,a))\leq 2\max\{f(a),f(a+\eta(b,a)\} \tag{2.3}$$

Then multiplying both (2.3) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1}f(a+t\eta(b,a))\,dt+\int_0^1 t^{\alpha-1}f(a+(1-t)\eta(b,a))\,dt\leq 2\max\{f(a),f(a+\eta(b,a)\}\int_0^1 t^{\alpha-1}dt.$$

i.e.

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\leq\frac{2\max\{f(a),f(a+\eta(b,a)\}}{\alpha}.$$

Using the mapping η satisfies condition C the proof is completed. $\square$

Remark 2.1. In Theorem 2.1, if we take $\eta(b,a)=b-a$, then inequality (2.1) become inequality (1.2) of Theorem 1.1.

Theorem 2.2. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L[a,a+\eta(b,a)]$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $|f'|$ is prequasiinvex function on $[a,a+\eta(b,a)]$ then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$\begin{aligned} & \left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right| \\ \leq \quad & \frac{\eta(b,a)}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\{|f'(a)|,|f'(b)|\}. \end{aligned} \tag{2.4}$$

Proof. Using Lemma 1.1 and the prequasiinvexity of $|f'|$ we get

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{\eta(b,a)}{2}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|dt \\
\leq\ & \frac{\eta(b,a)}{2}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt \\
\leq\ & \frac{\eta(b,a)}{2}\left\{\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt\right\} \\
=\ & \eta(b,a)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\left(\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt\right) \\
=\ & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\},
\end{aligned}$$

which completes the proof. □

Remark 2.2. a) In Theorem 2.2, if we take $\eta(b,a)=b-a$, then inequality (2.4) become inequality (1.3) of Theorem 1.2

b) In Theorem2.2, if we take $\alpha=1$, then inequality (2.4) become inequality (1.7) of Theorem 1.5.

c) In Theorem2.2, assume that η satisfies condition C.Using inequality (2.2) we get

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|f'(a)\right|,\left|f'(a+\eta(b,a))\right|\right\}
\end{aligned}$$

Theorem 2.3. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L\left[a,a+\eta(b,a)\right]$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $\left|f'\right|^{q}$ is prequasiinvex function on $\left[a,a+\eta(b,a)\right]$ for some fixed $q\geq1$ then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}
\end{aligned}\tag{2.5}$$

where $\alpha>0$.

Proof. From Lemma1.1 and using Power-mean inequality with properties of modulus, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{\eta(b,a)}{2}\int_0^1 \left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|dt \\
\leq\ & \frac{\eta(b,a)}{2}\left(\int_0^1 \left|t^{\alpha}-(1-t)^{\alpha}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\int_0^1 \left|t^{\alpha}-(1-t)^{\alpha}\right|dt &= \int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]dt \\
&= \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right).
\end{aligned}
$$

Since $|f'|^q$ is prequasiinvex function on $[a,a+\eta(b,a)]$, we obtain

$$
\left|f'\left(a+t\eta(b,a)\right)\right|^q \leq \max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\},\quad t\in[0,1]
$$

and

$$
\begin{aligned}
\int_0^1 \left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|^q dt &\leq \int_0^1 \left|t^{\alpha}-(1-t)^{\alpha}\right|\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}dt \\
&= \max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}.\int_0^1 \left|t^{\alpha}-(1-t)^{\alpha}\right|dt \\
&= \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}
\end{aligned}
$$

from here we obtain inequality (2.5). This completes the proof. □

Remark 2.3. a) In Theorem2.3, if we take $\eta(b,a)=b-a$ then inequality (2.5)become inequality (1.5) Theorem1.4.

b) In Theorem2.3, assume that η satisfies condition C. Using inequality (2.2) we get

$$
\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\max\left\{\left|f'(a)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 2.4. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L\left[a,a+\eta(b,a)\right]$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $|f'|^q$ is prequasiinvex function on $[a,a+\eta(b,a)]$ for*

some fixed $q>1$ then the following inequality holds:

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \tag{2.6}$$
$$\leq \frac{\eta(b,a)}{2\left(\alpha p+1\right)^{\frac{1}{p}}}\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(a+\eta(b,a))\right|^{q}\right\}\right)^{\frac{1}{q}}.$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha\in[0,1]$.

Proof. From Lemma1.1 and using Hölder inequality with properties of modulus, we have

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{\eta(b,a)}{2}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|dt$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|f'\left(a+t\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}.$$

We know that for $\alpha\in[0,1]$ and $\forall t_1,t_2\in[0,1]$,

$$\left|t_1^{\alpha}-t_2^{\alpha}\right|\leq\left|t_1-t_2\right|^{\alpha},$$

therefore

$$\begin{aligned}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|^{p}dt &\leq \int_{0}^{1}\left|1-2t\right|^{\alpha p}dt\\ &= \int_{0}^{\frac{1}{2}}\left[1-2t\right]^{\alpha p}dt+\int_{\frac{1}{2}}^{1}\left[2t-1\right]^{\alpha p}dt\\ &= \frac{1}{\alpha p+1}.\end{aligned}$$

Since $|f'|^{q}$ is prequasiinvex on $[a,a+\eta(b,a)]$, we have inequality (2.6). This completes the proof. □

Remark 2.4. a) In Theorem 2.4, if we take $\eta(b,a)=b-a$ then inequality (2.6) become inequality (1.4) of Theorem 1.3.

b) In Theorem 2.4, if we take $\alpha=1$then inequality (2.6) become inequality (1.8) of Theorem 1.6.

c) In Theorem 2.4, assume that η satisfies condition C. Using inequality (2.2) we get

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{\eta(b,a)}{2\left(\alpha p+1\right)^{\frac{1}{p}}}\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(a+\eta(b,a))\right|^{q}\right\}\right)^{\frac{1}{q}}.$$

References

[1] T. Antczak, Mean value in invexity analysis, *Nonlinear Analysis,* 60 (2005) 1471-1484.
[2] M. Alomari, M. Darus and U.S. Kırmacı, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, *Comp.and Math. with Applications*, 59 (2010), 225-232.
[3] M.K. Bakula, M.E. Ozdemir and J. Pečarić, Hadamard type inequalities for m-convex and (α, m)-convex functions, *J. Inequal. Pure Appl. Math.* 9 (2008) Article 96. [Online: http://jipam.vu.edu.au].
[4] A. Barani, A.G. Ghazanfari and S.S. Dragomir, Hermite-Hadamard inequality through prequasiinvex functions, *RGMIA Res. Rep. Coll.*, 14 (2011), Article 48.
[5] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs,Victoria University, 2000.
[6] M.A. Hanson, On sufficiency of the Kuhn-Tucker conditions, *J. Math. Anal. Appl.,* 80 (1981) 545-550.
[7] D.A. Ion, Some estimates on the Hermite-Hadamard inequalities through quasi-convex functions, *Annals of University of Craiova, Math. Comp. Sci. Ser.*, 34 (2007), 82-87.
[8] I. Iscan, Hermite-Hadamard's inequalities for preinvex functions via fractional integrals and related fractional inequalities, arXiv:1204.0272, submitted.
[9] S.R.Mohan and S.K. Neogy, On invex sets and preinvex functions, *J. Math. Anal. Appl.,* 189 (1995), 901-908.
[10] M. Aslam Noor, Some new classes of nonconvex functionss, *Nonl. Funct. Anal. Appl.,* 11 (2006), 165-171.
[11] M. Aslam Noor, On Hadamard integral inequalities invoving two log-preinvex functions, *J. Inequal. Pure Appl. Math.*, 8 (2007), No. 3, 1-6, Article 75.
[12] M. Aslam Noor, Hadamard integral inequalities for product of two preinvex function, *Nonl. anal. Forum*, 14 (2009), 167-173.
[13] M.E. Özdemir and Ç. Yıldız, The Hadamard's inequality for quasi-convex functions via fractional integrals, *RGMIA Res. Rep. Coll.*, 14 (2011), Article 101.
[14] J. Park, Simpson-like and Hermite-Hadamard-like type integral inequalities for twice differentiable preinvex functions, *Int. Journal of Pure and Appl. Math.,*79 (4) (2012), 623-640.
[15] R. Pini, Invexity and generalized Convexity, *Optimization,* 22 (1991) 513-525.
[16] M.Z. Sarıkaya and H. Ogunmez, On new inequalities via Riemann-Liouville fractional integration, *Abstract and Applied Analysis,* 2012 (2012), Article ID 428983, 10 pages, doi:10.1155/2012/428983.
[17] M.Z. Sarıkaya, E. Set and M.E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions, *Acta Nath. Univ. Comenianae* vol. LXXIX, 2 (2010), pp. 265-272.
[18] M.Z. Sarıkaya, E. Set, H. Yaldız and N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Mathematical and Computer Modelling*, DOI:10.1016/j.mcm.2011.12.048.
[19] E. Set, New inequalities of Ostrowski type for mapping whose derivatives are s-convex in the second sense via fractional integrals, *Computers and Math. with Appl.,* 63 (2012) 1147-1154.
[20] T. Weir, and B. Mond, Preinvex functions in multiple objective optimization, *Journal of Mathematical Analysis and Applications*, 136, (1998) 29-38.
[21] X.M. Yang and D. Li, On properties of preinvex functions, *J. Math. Anal. Appl.* 256 (2001) 229-241.
[22] X.M. Yang, X.Q. Yang and K.L. Teo, Characterizations and applications of prequasiinvex functions, properties of preinvex functions, *J. Optim. Theo. Appl.*, 110 (2001) 645-668.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 85–95 (2014) ©KJM

AREA FORMULAS FOR A TRIANGLE IN THE $m-$PLANE

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Abstract. In this paper, we give three area formulas for a triangle in the m-plane in terms of the $m-$distance. The two of them are $m-$version of the standart area formula for a triangle in the Euclidean plane, and the third one is a $m-$version of the well-known Heron′s formula.

1. Introduction

If one want to measure the distance between two points on a plane, then one can use frequently Euclidean distance which is defined as the length of segment between these points. Although it is the most popular distance function, it is not practical when we measure the distance which we actually move in the real world. So taxicab distance and Chinese checkers distance were introduced. Taxicab and Chinese checkers distance functions are similar to moving with a car or Chinese chess in the real world. Later, Tian [16] introduced α-distance function which includes the taxicab and Chinese checkers metrics as special cases. Then, some authors developed and studied on these topics (see [7], [8], [10]). In [5] Colakoğlu and Kaya gave a new distance function in the real plane which includes alpha, Chinese checkers, taxicab distances as special cases. The distance function is called m-distance. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points in $\mathbb{R}^2$, then for each real numbers u, v and m such that $u \geq v \geq 0 \neq u$, the distance function

$$d_m : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$$

defined by

$$d_m(P, Q) = (u\Delta_{PQ} + v\delta_{PQ}) \,/\, \left(\sqrt{1 + m^2}\right)$$

where $\Delta_{PQ} = \max\{|(x_1 - x_2) + m(y_1 - y_2)|, |m(x_1 - x_2) - (y_1 - y_2)|\}$ and $\delta_{PQ} = \min\{|(x_1 - x_2) + m(y_1 - y_2)|, |m(x_1 - x_2) - (y_1 - y_2)|\}$. Obviously, there are infinitely many different distance function depending on values u, v and m. But we suppose that values u and v are initially determined and fixed unless otherwise stated.

Date: January 1, 2013 and, in revised form, February 2, 2013.
2000 *Mathematics Subject Classification.* 51K05, 51K99.
Key words and phrases. $m-$distance, $m-$plane, Heron′s formula, area of a triangle.

According to m-distance function, the $m-$distance between points P and Q is constant u multiple of the Euclidean length of one of the shortest paths from P to Q composed of line segments each parallel to one of lines with slope m, $-1/m$, $\left[m(u^2-v^2)+2uv\right] / \left[(u^2-v^2)-2uvm\right]$ or $\left[m(u^2-v^2)-2uv\right] / \left[(u^2-v^2)+2uvm\right]$. See Figure 1.

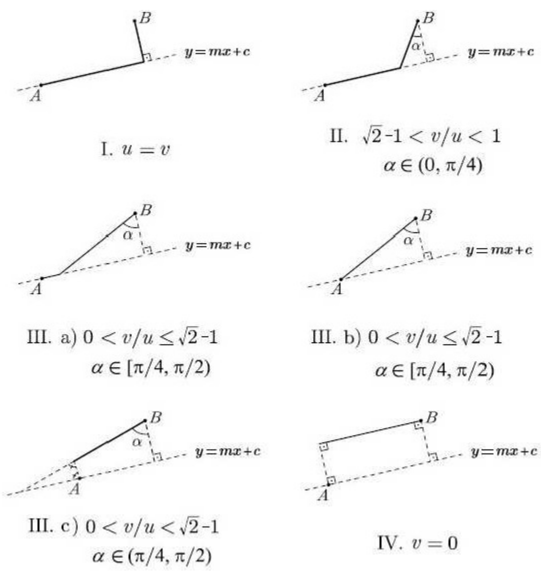

Figure 1

In this paper, we give area formulas for a triangle in the m-plane in terms of the $m-$distance. In this study, we use the usual Euclidean area notion. One can easily see that in the m-plane, there are triangles whose -lengths of corresponding sides are the same, while areas of these triangles are different (see Figure 2).This fact arises a natural question: How can one compute the area of a triangle in the m-plane? It is obvious that every formula to compute the area of a triangle depends on some parameters, and using different parameters gives different formulas. Here we give three formulas to compute the area of a triangle in the m-plane, using different parameters.

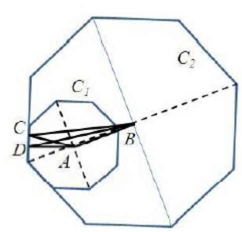

Figure 2

Let the line AB be parallel to the line $y=mx$, let C_1 be a $m-$circle with center A and radius b, C_2 a $m-$circle with center B and radius $b+c$, and C and D two points in $C_1 \cap C_2$. For different C and D such that C and D are not symmetric to the line AB, $Area(ABC) \neq Area(ABD)$, while $d_m(A,C) = d_m(A,D)$ and $d_m(B,C) = d_m(B,D)$.

2. Area of a Triangle in the m-plane

It is well-known that if ABC is a triangle with the area $\mathcal{A}$ in the Euclidean plane, and H is the point of orthogonal projection of the point A on the line BC, then standard area formula for the triangle ABC is $\mathcal{A} = \mathbf{ah}/2$, where $\mathbf{a} = d_E(B,C)$ and $\mathbf{h} = d_E(A,H)$ or $\mathbf{h} = d_E(A,BC)$ (see Figure 3). In this section, we give two m-versions of standard area formula in terms of m-distance. Clearly, a m-version of standard area formula for triangle ABC would be an equation that relates the two m-distances a and h, where $a = d_m(B,C)$, $h = d_m(A,H)$ or $h = d_m(A,BC)$ and area $\mathcal{A}$ of triangle ABC. Here, we give two m-versions of the area formula that depend on one parameter, namely, the slope of the base segment, in addition to the other parameters. Note that the real numbers u, v and m are fixed.

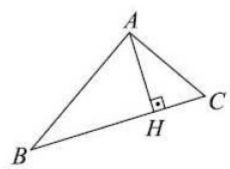

Figure 3

The following equation, which relates the Euclidean distance to the m-distance between two points in the Cartesian coordinate plane, plays an important role in the first m-version of the area formula. Following two proposition are given without proofs. One can see [2] for proofs.

Proposition 2.1. *For any two points P and Q in the Cartesian plane that do not lie on a vertical line, if n is the slope of the line through P and Q, then*

$$d_E(P,Q) = \rho(n) d_m(P,Q)$$

where $\rho(n) = \dfrac{\sqrt{(1+n^2)(1+m^2)}}{u\max\{|1+mn|, |m-n|\} + v\min\{|1+mn|, |m-n|\}}$.
If P and Q lie on a vertical line, then by definition,

$$d_E(P,Q) = \frac{\sqrt{1+m^2}}{u\max\{1,|m|\} + v\min\{1,|m|\}}\ d_m(P,Q).$$

If P and Q lie on the lines $y = mx$ or $y = \frac{-1}{m}x$, then

$$d_E(P,Q) = \frac{1}{u} d_m(P,Q).$$

Another useful fact that can be verified by direct calculation is:

Proposition 2.2. *For any real number $n \neq 0$*

$$\rho(n) = \rho(-1/n).$$

We first note by Proposition 1 and Proposition 2 that the m-distance between two points is invariant under all translations. If $b/a \neq \sqrt{2}-1$ in m-plane, the rotations of $\pi/2$, π and $3\pi/2$ radians around a point, and the reflections about the lines parallel to $y = nx+c$ such that $n \in \left\{m, \dfrac{-1}{m}, \dfrac{m-1}{1+m}, \dfrac{1+m}{1-m}\right\}$ preserve the $m-$distance. If $b/a = \sqrt{2}-1$ in the $m-$plane, the rotations of $\pi/4$, $\pi/2$, $3\pi/4$, π, $5\pi/4$, $3\pi/2$ and $7\pi/4$ radians around a point, and the reflections about the lines parallel to $y = nx+c$

such that $n \in \left\{ m, \dfrac{-1}{m}, \dfrac{m-1}{1+m}, \dfrac{1+m}{1-m}, \dfrac{(1-\sqrt{2})m-1}{(1-\sqrt{2})+m}, \dfrac{(1+\sqrt{2})m-1}{(1+\sqrt{2})+m}, \dfrac{(1-\sqrt{2})m+1}{(1-\sqrt{2})-m}, \dfrac{(1+\sqrt{2})m+1}{(1+\sqrt{2})-m} \right\}$ preserve the $m-$distance (see [5]).

The following theorem gives a m-version of the well-known Euclidean area formula of a triangle:

Theorem 2.1. *Let ABC be a triangle with the area $\mathcal{A}$ in the m-plane, H be orthogonal projection (in the Euclidean sense) of the point A on the line BC, n be the slope of the line BC, and let $a = d_m(B, C)$ and $h = d_m(A, H)$.*
(i) If BC is parallel to one of the lines $y = mx$ or $y = \frac{-1}{m}x$, then

$$\mathcal{A} = \frac{1}{u^2}\frac{ah}{2}.$$

(ii) If BC is parallel to a coordinate axis, then

$$\mathcal{A} = [\rho(n)]^2\,\frac{ah}{2},$$

where $\rho(n) = \dfrac{\sqrt{1+n^2}}{u\max\{|n|, 1\} + v\min\{|n|, 1\}}$.
(iii) f BC is not parallel to any one of coordinate axes or the lines $y = mx$ or $y = \frac{-1}{m}x$, then

$$\mathcal{A} = [\rho(n)]^2\,\frac{ah}{2}$$

where $\rho(n) = \dfrac{\sqrt{(1+n^2)(1+m^2)}}{u\max\{|1+mn|, |m-n|\} + v\min\{|1+mn|, |m-n|\}}$.

Proof. Let $\mathbf{a} = d_E(B, C)$ and $\mathbf{h} = d_E(A, H)$. Then $\mathcal{A} = \dfrac{\mathbf{ah}}{2}$.
(i) If BC is parallel to one of the lines $y = mx$ or $y = \frac{-1}{m}x$, then obviously $\mathbf{a} = \frac{1}{u}a$ and $\mathbf{h} = \frac{1}{u}h$. Hence $\mathcal{A} = \frac{1}{u^2}\frac{ah}{2}$.
(ii) If BC not be parallel to any one of the lines $y = mx$ or $y = \frac{-1}{m}x$, and let the slpe of the line BC be n. Then the slope of the line AH is $\frac{-1}{n}$. By proposition 1 and Proposition 2, $\mathbf{a} = \rho(n)a$, $\mathbf{h} = \rho(n)h$, hence $\mathcal{A} = [\rho(n)]^2\,\frac{ah}{2}$. □

In the m-plane, m-distance from a point P to a line l is defined by

$$d_m(P, l) = \min_{Q\in l}\{d_m(P, Q)\}$$

as in the Euclidean plane. It is well-known that in the Euclidean plane, Euclidean distance from a point $P = (x_0, y_0)$ to a line $l : ax + by + c = 0$ can be calculated by the following formula:

$$d_E(P, l) = |ax_0 + by_0 + c| \,/\, (a^2 + b^2)^{1/2}. \tag{2.1}$$

In Proposition 4 we give a similar formula for $d_m(P, l)$, using m-circles (see [2]). One can see by calculation that if $0 < v/u < 1$, then the unit $m-$circle is an octagon with vertices $A_1 = (\frac{1}{uk}, \frac{m}{uk})$, $A_2 = (\frac{1-m}{(u+v)k}, \frac{1+m}{(u+v)k})$, $A_3 = (\frac{-m}{uk}, \frac{1}{uk})$, $A_4 = (\frac{-1-m}{(u+v)k}, \frac{1-m}{(u+v)k})$, $A_5 = (\frac{-1}{uk}, \frac{-m}{uk})$, $A_6 = (\frac{m-1}{(u+v)k}, \frac{-1-m}{(u+v)k})$, $A_7 = (\frac{m}{uk}, \frac{-1}{uk})$, $A_8 = (\frac{1+m}{(u+v)k}, \frac{m-1}{(u+v)k})$, where $k = \sqrt{1+m^2}$. If $u = v$ or $v = 0$, then unit $m-$circle is a square with vertices A_1, A_3, A_5, A_7 or A_2, A_4, A_6, A_8, respectively (See figure 4).

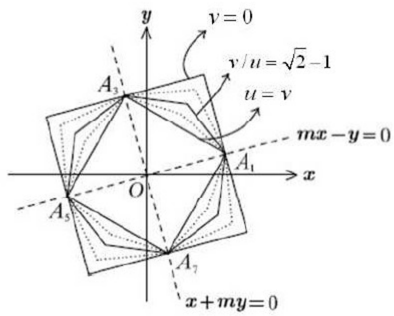

Figure 4

The next proposition introduced $m-$distance from a point P to a line l. For the proof of proposition, one can see in [2].

Proposition 2.3. *Given a point $P = (x_0, y_0)$, and a line $l : ax + by + c = 0$ in the m-plane. Then the m-distance from the point P to the line l can be calculated by the following formula:*

(2.2)
$$d_m(P,l) = \begin{cases} \dfrac{|ax_0+by_0+c|\sqrt{1+m^2}}{\max\left\{\frac{|a+bm|}{u}, \frac{|am-b|}{u}\right\}} & ,u=v \\ \dfrac{|ax_0+by_0+c|\sqrt{1+m^2}}{\max\left\{\frac{|a(1-m)+b(1+m)|}{u}, \frac{|a(1+m)-b(1-m)|}{u}\right\}} & ,v=0 \\ \dfrac{|ax_0+by_0+c|\sqrt{1+m^2}}{\max\left\{\frac{|a+bm|}{u}, \frac{|am-b|}{u}, \frac{|a(1-m)+b(1+m)|}{u+v}, \frac{|a(1+m)-b(1-m)|}{u+v}\right\}} & ,0<v/u<1. \end{cases}$$

The following equation, which relates the Euclidean distance to the m-distance from a point to a line in the Cartesian coordinate plane, plays an important role in the second m-version of the area formula.

Proposition 2.4. *Given a point P, and a line l in the Cartesian plane that is not a vertical line, if n is the slope of the line l, then*

$$d_E(P,l) = \tau(n) d_\alpha(P,l) \tag{2.3}$$

$$
\textit{where } \tau(n)=\begin{cases}
\dfrac{\max\left\{\dfrac{|1+nm|}{u},\dfrac{|m-n|}{u}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ u=v\\
\dfrac{\max\left\{\dfrac{|n(1-m)-(1+m)|}{u},\dfrac{|n(1+m)+(1-m)|}{u}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ v=0\\
\dfrac{\max\left\{\dfrac{|1+nm|}{u},\dfrac{|m-n|}{u},\dfrac{|n(1-m)-(1+m)|}{u+v},\dfrac{|n(1+m)+(1-m)|}{u+v}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ 0<v/u<1
\end{cases}
$$

If l is a vertical line, then

$$
\tau(n)=\begin{cases}
\dfrac{\max\left\{\dfrac{|m|}{u},\dfrac{1}{u}\right\}}{\sqrt{1+m^2}} & ,\ u=v\\
\dfrac{\max\left\{\dfrac{|1-m|}{u},\dfrac{|1+m|}{u}\right\}}{\sqrt{1+m^2}} & ,\ v=0\\
\dfrac{\max\left\{\dfrac{|m|}{u},\dfrac{1}{u},\dfrac{|1-m|}{u+v},\dfrac{|1+m|}{u+v}\right\}}{\sqrt{1+m^2}} & ,\ 0<v/u<1
\end{cases}.
$$

Proof. Let $P=(x_0,y_0)$ be a point, and $l: ax+by+c=0$ be a line in the Cartesian plane. If l is not a vertical line, then $b\neq 0$ and $n=-\frac{a}{b}$. Using n in equation 2.1 and equation 2.2, one gets $d_E(P,l)=|ax_0+by_0+c| \diagup |b|\,(1+n^2)^{1/2}$ and

$$
d_m(P,l)=\begin{cases}
\dfrac{|ax_0+by_0+c|\sqrt{1+m^2}}{|b|\max\left\{\frac{|1+nm|}{u},\frac{|m-n|}{u}\right\}} & ,u=v\\
\dfrac{|ax_0+by_0+c|\sqrt{1+m^2}}{|b|\max\left\{\frac{|n(1-m)-(1+m)|}{u},\frac{|n(1+m)+(1-m)|}{u}\right\}} & ,v=0\\
\dfrac{|ax_0+by_0+c|\sqrt{1+m^2}}{|b|\max\left\{\frac{|1+nm|}{u},\frac{|m-n|}{u},\frac{|n(1-m)-(1+m)|}{u+v},\frac{|n(1+m)+(1-m)|}{u+v}\right\}} & ,0<v/u<1.
\end{cases}
$$

Hence, $d_E(P,l)$=$\tau(n)d_m(P,l)$ where

$$
\tau(n)=\begin{cases}
\dfrac{\max\left\{\dfrac{|1+nm|}{u},\dfrac{|m-n|}{u}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ u=v\\
\dfrac{\max\left\{\dfrac{|n(1-m)-(1+m)|}{u},\dfrac{|n(1+m)+(1-m)|}{u}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ v=0\\
\dfrac{\max\left\{\dfrac{|1+nm|}{u},\dfrac{|m-n|}{u},\dfrac{|n(1-m)-(1+m)|}{u+v},\dfrac{|n(1+m)+(1-m)|}{u+v}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ 0<v/u<1.
\end{cases}
$$

If l is a vertical line, then $b = 0$ and $a \neq 0$. Therefore, $d_E(P,l) = |ax_0 + c| \diagup |a|$ and

$$d_m(P,l) = \begin{cases} \dfrac{|ax_0+c|\sqrt{1+m^2}}{|a|\max\left\{\frac{1}{u}, \frac{|m|}{u}\right\}} & , u = v \\ \dfrac{|ax_0+c|\sqrt{1+m^2}}{|a|\max\left\{\frac{|1-m|}{u}, \frac{|1+m|}{u}\right\}} & , v = 0 \\ \dfrac{|ax_0+c|\sqrt{1+m^2}}{|a|\max\left\{\frac{1}{u}, \frac{|m|}{u}, \frac{|1-m|}{u}, \frac{|1+m|}{u}\right\}} & , 0 < v/u < 1 \end{cases},$$

hence

$$\tau(n) = \begin{cases} \dfrac{\max\left\{\dfrac{|m|}{u}, \dfrac{1}{u}\right\}}{\sqrt{1+m^2}} & , \ u = v \\ \dfrac{\max\left\{\dfrac{|1-m|}{u}, \dfrac{|1+m|}{u}\right\}}{\sqrt{1+m^2}} & , \ v = 0 \\ \dfrac{\max\left\{\dfrac{|m|}{u}, \dfrac{1}{u}, \dfrac{|1-m|}{u+v}, \dfrac{|1+m|}{u+v}\right\}}{\sqrt{1+m^2}} & , \ 0 < v/u < 1. \end{cases}$$

□

The following theorem gives another α-version of the well-known Euclidean area formula of a triangle:

Theorem 2.2. *Let ABC be a triangle with area $\mathcal{A}$ in the m-plane, n be the slope of the line BC, and let $a = d_\alpha(B,C)$ and $h = d_\alpha(A, BC)$. Then the area of ABC is*

$$\mathcal{A} = \sigma(n)ah/2$$

(i) If BC is parallel to lines $y = mx$ or $y = \frac{-1}{m}x$, then

$$\sigma(n) = \begin{cases} 1/u^2 & , \ u = v \text{ or } v = 0 \\ 1/u(u+v) & , \ 0 < v/u < 1 \end{cases}$$

(ii) If BC is not parallel to any one of the lines $y = mx$ or $y = \frac{-1}{m}x$, then

$$\sigma(n) = \begin{cases} \dfrac{\max\left\{\dfrac{|1+nm|}{u}, \dfrac{|m-n|}{u}\right\}}{u\max\{|1+nm|, \ |m-n|\} + u\min\{|1+nm|, \ |m-n|\}} & , \ u = v \\ \dfrac{\max\left\{\dfrac{|n(1-m)-(1+m)|}{u}, \dfrac{|n(1+m)+(1-m)|}{u}\right\}}{u\max\{|1+nm|, \ |m-n|\}} & , \ v = 0 \\ \dfrac{\max\left\{\dfrac{|1+nm|}{u}, \dfrac{|m-n|}{u}, \dfrac{|n(1-m)-(1+m)|}{u+v}, \dfrac{|n(1+m)+(1-m)|}{u+v}\right\}}{u\max\{|1+nm|, \ |m-n|\} + v\min\{|1+nm|, \ |m-n|\}} & , \ 0 < v/u < 1. \end{cases}$$

Proof. Let $\mathbf{a} = d_E(B,C)$ and $\mathbf{h} = d_E(A, BC)$. Then, $\mathcal{A} = \mathbf{ah}/2$.
(i) If BC is parallel to lines $y = mx$ or $y = \frac{-1}{m}x$, then clearly $a = \frac{1}{u}\mathbf{a}$ and $h = \tau(n)\mathbf{h}$,

where

$$\tau(n)=\begin{cases}\dfrac{1}{u} & ,\ u=v \text{ or } v=0\\ \dfrac{1}{\max\{u,u+v\}} & ,\ 0<v/u<1.\end{cases}$$

Hence, $\mathcal{A}=\sigma(n)ah/2$.

(ii) Let BC not be parallel to any one of the coordinate axes, and let the slope of the line BC be n. Then, by Proposition 1 and Proposition 5, $\mathbf{a}=\rho(n)a$, $\mathbf{h}=\tau(n)h$, hence $\mathcal{A}=\rho(n)\tau(n)ah/2$. Since $\rho(n)\tau(n)=\sigma(n)$, we get $\mathcal{A}=\sigma(n)ah/2$. □

3. m Version of Heron's Formula

It is well-known that if ABC is a triangle with the area $\mathcal{A}$ in the Euclidean plane, and $\mathbf{a}=d_E(B,C)$, $\mathbf{b}=d_E(A,C)$, $\mathbf{c}=d_E(A,B)$, and $\mathbf{p}=(\mathbf{a}+\mathbf{b}+\mathbf{c})/2$, then $\mathcal{A}=[\mathbf{p}(\mathbf{p}-\mathbf{a})(\mathbf{p}-\mathbf{b})(\mathbf{p}-\mathbf{c})]^{1/2}$, which is known as *Heron's formula*. In this section, we give an m-version of this formula in terms of m-distance. Clearly, an m-version of Heron's formula for triangle ABC would be an equation that relates the three m-distances a, b and c, where $a=d_\alpha(B,C)$, $b=d_\alpha(A,C)$, $c=d_\alpha(A,B)$, and the area $\mathcal{A}$ of triangle ABC. Here, we give an m-version of Heron's formula that depend on three new parameters in addition to a, b, c and $\mathcal{A}$.

We need following two definitions which is revised according to given in [15] and [13] respectively, to give an m-version of Heron's formula:

Definition 3.1. Let ABC be any triangle in the m-plane. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to lines $y=mx$ or $y=\frac{-1}{m}x$. A line l is called a **base line** of ABC if and only if
(1) l passes through a vertex,
(2) l is parallel to lines $y=mx$ or $y=\frac{-1}{m}x$,
(3) l intersects the opposite side (as a line segment) of the vertex in (1).

Clearly, at least one of vertices of the triangle always has one or two base lines. Such a vertex of the triangle is called a **basic vertex**. A **base segment** is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

Definition 3.2. A line with slope n is called a **steep line**, a **gradual line** and a **separator** if $n>\frac{1+m}{1-m}$ or $n<\frac{-1+m}{1+m}$ or $n\to\infty$, $\frac{-1+m}{1+m}<n<\frac{1+m}{1-m}$ and $n=m$ or $n=\frac{-1}{m}$ for $0\le m\le 1$, respectively.

The following theorem gives an α-version of Heron's formula:

Theorem 3.1. *Let ABC be a triangle with area $\mathcal{A}$ in the m-plane, such that C is a basic vertex, $a=d_m(B,C)$, $b=d_m(A,C)$ and $c=d_m(A,B)$. Let D be the intersection point of a base line and AB, the opposite side of the basic vertex C. Let H_1 and H_2 be orthogonal projections (in the Euclidean sense) of A and B on the base line CD, respectively. Then,*

$$\mathcal{A}=\begin{cases}\frac{l}{2u}\left[\sqrt{1+m^2}(2p-c)-v(l_1+l_2)\right] & ;\ \textit{if } C_1 \textit{ is valid}\\ \frac{l}{2v}\left[\sqrt{1+m^2}(2p-c)-u(l_1+l_2)\right] & ;\ \textit{if } C_2 \textit{ is valid}\\ \frac{l}{2uv}\left[\sqrt{1+m^2}\left(2p-c+(v-1)\,b+(u-1)\,a\right)-\left(v^2l_1+u^2l_2\right)\right] & ;\ \textit{if } C_3 \textit{ is valid}\\ \frac{l}{2uv}\left[\sqrt{1+m^2}\left(2p-c+(u-1)\,b+(v-1)\,a\right)-\left(u^2l_1+v^2l_2\right)\right] & ;\ \textit{if } C_4 \textit{ is valid}\end{cases}$$

where $p=(a+b+c)/2$, $l=d_m(C,D)$, $l_1=d_m(C,H_1)$, $l_2=d_m(C,H_2)$,

C_1 : *lines* AC *and* BC *are not gradual and base line* CD *is horizontal, or lines* AC *and* BC *are not steep and base line* CD *is vertical,*

C_2 : *lines* AC *and* BC *are not steep and base line* CD *is horizontal, or lines* AC *and* BC *are not gradual and base line* CD *is vertical,*

C_3 : *line* AC *is not gradual, line* BC *is not steep and base line* CD *is horizontal, or line* AC *is not steep, line* BC *is not gradual and base line* CD *is vertical,*

C_4 : *line* AC *is not steep, line* BC *is not gradual and base line* CD *is horizontal, or line* AC *is not gradual, line* BC *is not steep and base line* CD *is vertical.*

Proof. Let ABC be a triangle with area $\mathcal{A}$ in the m-plane, such that C is a basic vertex, $a = d_m(B, C)$, $b = d_m(A, C)$ and $c = d_m(A, B)$. Let D be the intersection point of a base line and AB, the opposite side of the basic vertex C. Let H_1 and H_2 be orthogonal projections of A and B on the base line CD, respectively. And let $p = (a + b + c)/2$, $l = d_m(C, D)$, $l_1 = d_m(C, H_1)$, $l_2 = d_m(C, H_2)$, $h_1 = d_m(A, H_1)$, $h_2 = d_m(B, H_2)$. The m-distance between two points is invariant under all translations. If $b/a \neq \sqrt{2} - 1$ in m-plane, the rotations of $\pi/2$, π and $3\pi/2$ radians around a point, and the reflections about the lines parallel to $y = nx + c$ such that $n \in \left\{ m, \frac{-1}{m}, \frac{m-1}{1+m}, \frac{1+m}{1-m} \right\}$ preserve the $m-$distance. If $b/a = \sqrt{2} - 1$ in the $m-$plane, the rotations of $\pi/4$, $\pi/2$, $3\pi/4$, π, $5\pi/4$, $3\pi/2$ and $7\pi/4$ radians around a point, and the reflections about the lines parallel to $y = nx + c$ such that

$$n \in \left\{ m, \frac{-1}{m}, \frac{m-1}{1+m}, \frac{1+m}{1-m}, \frac{(1-\sqrt{2})m-1}{(1-\sqrt{2})+m}, \frac{(1+\sqrt{2})m-1}{(1+\sqrt{2})+m}, \frac{(1-\sqrt{2})m+1}{(1-\sqrt{2})-m}, \frac{(1+\sqrt{2})m+1}{(1+\sqrt{2})-m} \right\}$$

preserve the $m-$distance. Therefore Figure 5 represent all triangles for which C_1 holds, Figure 6 represent all triangles for which C_2 holds, Figure 7 represent all triangles for which C_3 holds, and finally Figure 8 represent all triangles for which C_4 holds.

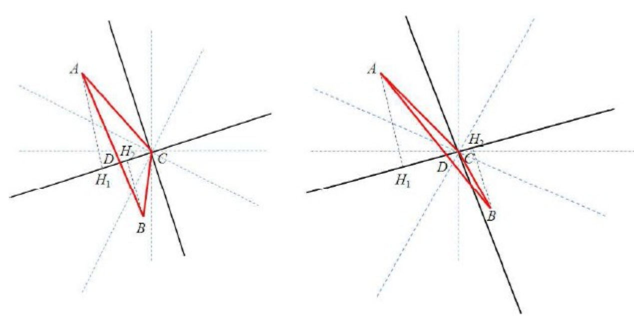

Figure 5

In Figure 5, $a = (uh_2 + vl_2)/\sqrt{1+m^2}$ and $b = (uh_1 + vl_1)/\sqrt{1+m^2}$ by m-distance definition. Since $A(ABC) = A(ADC) + A(BDC) = \frac{l}{2}(h_1 + h_2)$, using h_1 and h_2

values, one gets $\mathcal{A} = \dfrac{l}{2u}\left[\sqrt{1+m^2}\,(2p-c) - v(l_1+l_2)\right]$.

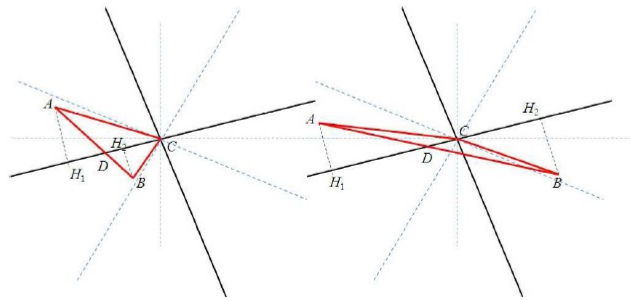

Figure 6

In Figure 6, $a = (ul_2 + vh_2)/\sqrt{1+m^2}$ and $b = (ul_1 + vh_1)/\sqrt{1+m^2}$ by m-distance definition. Since $A(ABC) = A(ADC) + A(BDC) = \frac{l}{2}(h_1 + h_2)$, using h_1 and h_2 values, one gets $\mathcal{A} = \dfrac{l}{2v}\left[\sqrt{1+m^2}\,(2p-c) - u(l_1+l_2)\right]$.

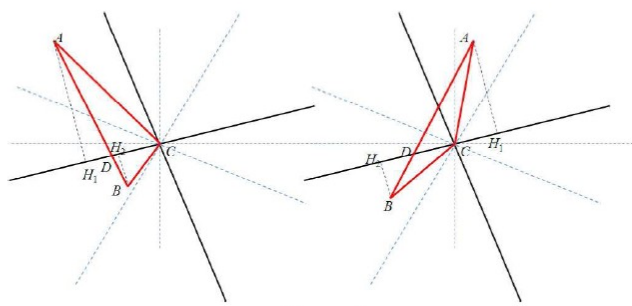

Figure 7

In Figure 7, $a = (ul_2 + vh_2)/\sqrt{1+m^2}$ and $b = (uh_1 + vl_1)/\sqrt{1+m^2}$ by m-distance definition. Since $A(ABC) = A(ADC) + A(BDC) = \frac{l}{2}(h_1 + h_2)$, using h_1 and h_2 values, one gets $\mathcal{A} = \dfrac{l}{2uv}\left[\sqrt{1+m^2}\,(2p - c + (v-1)b + (u-1)\,a) - v^2 l_1 - u^2 l_2\right]$.

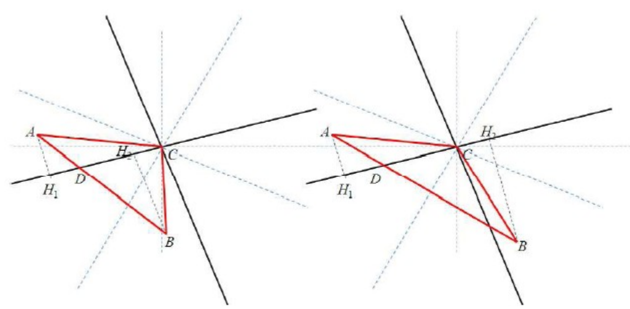

Figure 8

In Figure 8, $a = (uh_2 + vl_2)/\sqrt{1+m^2}$ and $b = (ul_1 + vh_1)/\sqrt{1+m^2}$ by m-distance definition. Since $A(ABC) = A(ADC) + A(BDC) = \frac{l}{2}(h_1 + h_2)$, using h_1 and h_2 values, one gets $\mathcal{A} = \dfrac{l}{2uv}\left[\sqrt{1+m^2}\,(2p - c + (u-1)b + (v-1)\,a) - u^2 l_1 - v^2 l_2\right]$. $\square$

Since well-known taxicab, Chinese Checker and $\alpha-$distances are special cases of m-distance for $m = 0$ and $u = v$, $v/u = \sqrt{2} - 1$ and $0 < v/u < 1$, respectively, Theorem 3, Theorem 6 and Theorem 7 give also taxicab, Chinese Checker and $\alpha-$versions of area formulas for a triangle, when $m = 0$ and $u = v$, $v/u = \sqrt{2} - 1$ and $0 < v/u < 1$, respectively, (see [12], [15], [11] and [6]).

References

[1] Chen G., Lines and Circles in Taxicab Geometry, Master Thesis, Department of Mathematics and Computer Science, University of Central Missouri, 1992.

[2] Çolakoğlu H. B., Taksi, Maksimum, Çin dama ve Alfa Düzlemlerinin Bazı Özellikleri ve Bir Genelleştirilmesi, PhD thesis, Eskişehir Osmangazi Üniversitesi, 2009..

[3] Çolakoğlu H. B. and Kaya R., Taxicab Versions of the Pythagorean Theorem, Pi Mu Epsilon J. (PMEJ), (in press).

[4] Çolakoğlu H. B. and Kaya R., Chinese Checker Versions of the Pythagorean Theorem, Int. J. Contemp. Math. Sciences. (IJCMS), 4 (2) (2009), 61-69.

[5] Çolakoğlu H. B. and Kaya R., A Generalization of Some Well-Known Distances and Related Isometries, Mathematical Communications, 16 (1), (2011), 21-35.

[6] Çolakoğlu H. B.,Gelişgen Ö. and Kaya R., Area formulas for a triangle in the alpha plane, Mathematical Communications, 18 (1), (2013), 123-132.

[7] Gelişgen Ö. and Kaya R., On α-distance in Three Dimensional Space, Applied Sciences (APPS), 8 (2006), 65-69.

[8] Gelişgen Ö. and Kaya R., Generalization of α-distance to n-dimensional Space, Scientific-Professional Information Journal of Croatian Society for Constructive Geometry and Computer Graphics (KoG), 10 (2006), 33-35.

[9] Gelişgen Ö. and Kaya R., CC Analog of the Theorem of Pythagoras, Algebras, Groups and Geometries (AGG), 23 (2) (2006), 179-188.

[10] Gelişgen Ö. and Kaya R., Alpha(i) Distance in n-dimensional Space, Applied Sciences, 10, (2008), 88-93.

[11] Gelişgen Ö. and Kaya R., CC-Version of The Heron's Formula, Missouri Journal of Mathematical Sciences, 21 (2), (2009), 94-110.

[12] Kaya R. and Colakoglu H. B., Taxicab Versions of Some Euclidean Theorems, Int. Jour. of Pure and Appl. Math. (IJPAM), 26 (1), (2006), 69-81.

[13] Krause E. F., Taxicab Geometry, Addison-Wesley, Menlo Park, California, 1975; Dover Publications, New York, 1987.

[14] Minkowski H., Gesammelte Abhandlungen, Chelsa Publishing Co. New York, 1967.

[15] Özcan M. and Kaya R., Area of a Triangle in terms of the Taxicab Distance, Missouri J. of Math. Sci., 15 (3) (2003), 178-185.

[16] Tian S., Alpha Distance-A Generalization of Chinese Checker Distance and Taxicab Distance, Missouri J. of Math. Sci. (MJMS), 17 (1), (2005), 35-40.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 96–101 (2014) ©KJM

FIXED POINT THEOREMS IN CONVEX PARTIAL METRIC SPACES

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Abstract. Partial metric spaces were introduced by S. G. Matthews [1] as a part of the study of denotational semantics of dataflow networks, the author introduced and studied the concept of partial metric space, and obtained a Banach type fixed point theorem on complete partial metric spaces. In this paper, we study some fixed point theorems for self-mappings satisfying certain contraction principles on a convex complete partial metric space, these theorem generalize previously obtained results in convex metric space.

1. Introduction

In 1970, Takahashi [2] introduced the notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. A convex metric space is a generalized space. For example, every normed space and cone Banach space is a convex metric space and convex complete metric space, Subsequently, Beg [3], Beg and Abbas [4, 5], Chang, Kim and Jin [6], Ciric [7], Shimizu and Takahashi [8], Tian [9], Ding [10], and many others studied fixed point theorems in convex metric spaces.

There exist many generalizations of the concept of metric spaces in the literature. In particular, Matthews [1] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, showing that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification.

After that, fixed point results in partial metric spaces were studied by many other authors. Refs. [11,12] are some works in this line of research. The existence of several connections between partial metrics and topological aspects of domain theory have been pointed out in, e.g., [13–18].

The purpose of this paper is to study the existence of a fixed point for self-mappings defined on a nonempty closed convex subset of a convex complete partial metric space that satisfies certain conditions, and knowing that "the partial metric space is a generalization of a metric space" from [13], our result improves and

Key words and phrases. Partial metric spaces; convex metric spaces; fixed point theorems.

extends M. Moosaei result in [19] from a convex complete metric space to a convex complete partial metric space.

2. Preliminaries

Definition 2.1. Let X be a nonempty set and let p: $X \times X \to \mathbb{R}^+$ satisfy
$(A1)$ $0 \leq p(x,x) \leq p(x,y)$ (nonnegativity and small self-distances),
$(A2)$ $x = y \Longleftrightarrow p(x,x) = p(y,y) = p(x,y)$ (indistancy implies equality),
$(A3)$ $p(x,y) = p(y,x)$ (symmetry), and
$(A4)$ $p(x,y) \leq p(x,z) + p(y,z) - p(z,z)$ (triangularity).
for all x,y and z $\in X$. then tha pair (X,p) is called a partial matric space and p is called a partial metric on X.

Remark 2.1. It is clear that, if $p(x,y) = 0$, then from $(A1)$ and $(A2)$, $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls

$\{B_p(x,\varepsilon), x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + B_p(x,\varepsilon)\}$ for all $x \in X$ and $\varepsilon > 0$.

For a partial metric p on X, the function $d_p : X \times X \to \mathbb{R}^+$ defined as

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

satisfies the conditions of a metric on X; therefore it is a (usual) metric on X.

Example 2.1. Let $max(a,b)$ be the maximum of any two nonnegative real numbers a and b; then max is a partial metric over $\mathbb{R}^+ = [0,\infty)$.

Example 2.2. If $X := \{[a,b] \,/\, a,b \in \mathbb{R},\ a \leq b\}$ then

$$p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$$

defines a partial metric p on X.

Example 2.3. If (X,d) is a metric space and $c \geq 0$ is arbitrary, then

$$p(x,y) = d(x,y) + c$$

defines a partial metric on X and the corresponding metric is $d_p(x,y) = 2d(x,y)$.

Definition 2.2. Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then
(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = lim_{n\to+\infty}p(x,x_n)$,
(ii) $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $lim_{n,m\to+\infty}p(x_n,x_m)$.

Definition 2.3. A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, such that $p(x,x) = lim_{n,m\to+\infty}p(x_n,x_m)$.

Remark 2.2. It is easy to see that every closed subset of a complete partial metric space is complete.

Theorem 2.1. [1]. *Let f be a mapping of a complete partial metric space (X, p) into itself such that there is a real number c with $0 \leq c < 1$, satisfying for all $x, y \in X$:*

$$p(fx, fy) < cp(x, y).$$

Then f has a unique fixed point.

Definition 2.4. Let (X, p) be a partial metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$p(u, W(x, y, \lambda)) \leq \lambda p(u, x) + (1 - \lambda)p(u, y)$$

A metric space (X, p) together with a convex structure W is called a convex partial metric space, which is denoted by (X, p, W).

Example 2.4. Let $(X, |.|)$ is a metric space, then

$$p(x, y) = \frac{|x - y| + x + y}{2}$$

defines a partial metric p on X and it can shown that (X, p) is a convex partial metric space.

Definition 2.5. Let (X, p, W) be a convex partial metric space. A nonempty subset C of X is said to be convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times I$.

Definition 2.6. Let $f : X \to X$. A point $x \in X$ is called a fixed point of f if $f(x) = x$.

F(f), C(f, g), and F(f, g) denote the set of all fixed points of f, coincidence points of the pair (f, g), and common fixed points of the pair (f, g), respectively.

Theorem 2.2. [19] . *Let C be a nonempty closed convex subset of a convex complete metric space (X, d, W) and f be a self-mapping of C. If there exist a, b, c, k such that*

$$2b - |c| \leq k < 2(a + b + c) - |c|,$$

$$ad(x, f(x)) + bd(y, f(y)) + cd(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in C$, then f has at least one fixed point.

3. Main result

The following theorem improves and extends Theorem 3.2 in **[19].**

Theorem 3.1. *Let C be a nonempty closed convex subset of a* ***convex complete partial metric space*** *(X, p, W) and f be a self-mapping of C. If there exist k such that*

$$0 \leq k < \frac{1}{4},$$

$$p(x, f(y)) + p(f(x), f(y)) \leq kp(y, f(x)) \tag{3.1}$$

for all $x, y \in C$, then f has at least one fixed point.

Proof. From definition 4 and by using $(A1)$ and $(A3)$,we have

$$p(x,W(x,y,\frac{1}{2})) \leq \frac{1}{2}p(x,x)+\frac{1}{2}p(x,y), \tag{3.2}$$

$$p(y,W(x,y,\frac{1}{2})) \leq \frac{1}{2}p(y,y)+\frac{1}{2}p(x,y). \tag{3.3}$$

so, we find

$$p(x,W(x,y,\frac{1}{2}))+p(y,W(x,y,\frac{1}{2}))\leq 2p(x,y). \tag{3.4}$$

Suppose $x_0 \in C$ is arbitrary. We define a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$x_n = W(x_{n-1}, f(x_{n-1}, \frac{1}{2}),\ n=1,... \tag{3.5}$$

As C is convex, $x_n \in C$ for all $n \in \mathbb{N}$. From (3.5) and (3.6) , we have

$$p(x_n,x_{n+1})+p(x_{n+1},f\left(x_n\right))\leq 2p(x_n,f\left(x_n\right)). \tag{3.6}$$

By using $(A4)$ and $(A1)$,we have

$$\begin{aligned} p\left(x_n,f\left(x_n\right)\right) &\leq p\left(x_n,f\left(x_{n-1}\right)\right)+p\left(f\left(x_n\right),f\left(x_{n-1}\right)\right)-p\left(f\left(x_{n-1}\right),f\left(x_{n-1}\right)\right) \\ &\leq p\left(x_n,f\left(x_{n-1}\right)\right)+p\left(f\left(x_n\right),f\left(x_{n-1}\right)\right) \end{aligned}$$

then, we get

$$2p\left(x_n,f\left(x_n\right)\right)-2p\left(x_n,f\left(x_{n-1}\right)\right)\leq 2p\left(f\left(x_n\right),f\left(x_{n-1}\right)\right), \tag{3.7}$$

from (3.7) and (3.8) , we obtain

$$p(x_n,x_{n+1})+p(x_{n+1},f\left(x_n\right))-2p\left(x_n,f\left(x_{n-1}\right)\right)\leq 2p\left(f\left(x_n\right),f\left(x_{n-1}\right)\right). \tag{3.8}$$

For all $n \in \mathbb{N}$. Now by substituting x with x_n and y with x_{n-1} in (3.2) , we get

$$p(x_n,f(x_{n-1}))+p(f(x_n),f(x_{n-1})\leq kp(x_{n-1},f(x_n)),$$

for all $n \in \mathbb{N}$. Therefore, from (3.9) , it follows that

$$p(x_n,x_{n+1})+p(x_{n+1},f\left(x_n\right))\leq 2kp(x_{n-1},f(x_n)), \tag{3.9}$$

from $(A4)$ and $(A1)$, we have

$$p(x_{n-1},f(x_n)\leq p(x_n,f(x_n))+p(x_{n-1},x_n),$$

by this inequality and (3.10) , we get

$$p(x_n,f\left(x_n\right))\leq p(x_n,x_{n+1})+p(x_{n+1},f\left(x_n\right))\leq 2kp(x_n,f(x_n))+2kp(x_{n-1},x_n),$$

so, we obtain

$$\left(1-2k\right)p(x_n,f\left(x_n\right))\leq 2kp(x_{n-1},x_n). \tag{3.10}$$

Now, from (3.3) and (3.6) , we have

$$p(x_n,f\left(x_n\right))\geq 2p(x_n,x_{n+1})-p(x_n,x_n)$$

for all $n \in \mathbb{N}$. from (3.11) , it follows that

$$\left(2-4k\right)p(x_n,x_{n+1})\leq 2kp(x_{n-1},x_n)+\left(1-2k\right)p(x_n,x_n)$$

by using $(A1)$, we obtain

$$p(x_n,x_{n+1})\leq \frac{1}{2-4k}p(x_{n-1},x_n). \tag{3.11}$$

for all $n \in \mathbb{N}$. from (3.11) , $\frac{1}{2-4k} \in [0,1)$, and hence, $\{x_n\}_{n=1}^{\infty}$ is a contraction sequence in C. Therefore, it is a cauchy sequence. Since C is a closed subset of a complete space, there exists $v \in C$ such that $lim_{n\to+\infty} x_n = v$.

Therefore, by using (3.4) and (3.6) , we have

$$p\left(x_n, f\left(x_{n-1}\right)\right) \leq \frac{1}{2} p\left(f\left(x_{n-1}\right), f\left(x_{n-1}\right)\right) + \frac{1}{2} p\left(x_{n-1}, f\left(x_{n-1}\right)\right),$$

we put $lim_{n\to+\infty} f\left(x_{n-1}\right) = \alpha$, letting $n \to +\infty$ in the above inequality, it follows that

$$p\left(v, \alpha\right) \leq \frac{1}{2} p\left(\alpha, \alpha\right) + \frac{1}{2} p\left(v, \alpha\right) \Longrightarrow p\left(v, \alpha\right) \leq p\left(\alpha, \alpha\right) \tag{3.12}$$

and from $(A1)$, we have

$$p\left(\alpha, \alpha\right) \leq p\left(v, \alpha\right). \tag{3.13}$$

Then from (3.13) and (3.14) , we obtain $(v, \alpha) = p\left(\alpha, \alpha\right)$, so $lim_{n\to+\infty} f\left(x_n\right) = v$. Now by substituting x with v and y with x_n in (3.2) , we obtain

$$p(v, f(x_n)) + p(f(v), f(x_n) \leq kp(x_n, f(v))$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in the above inequality, it follows that

$$p(v, v) + (1-k)\, p(v, f\left(v\right)) \leq 0.$$

Since $(1-k)$ is positive from (3.1) and from $(A2)$, we get

$$p(v, v) = p(v, f\left(v\right)) = 0,$$

then from $(A2)$, this implies $f\left(v\right) = v$, and the proof of the theorem is complete. □

Corollary 3.1. *From theorem 3, we deduce that for all x, $y \in C$, then F(f) is a nonempty set.*

Corollary 3.2. *Let $(X,\ d, W)$ be a convex complete partial metric space and C be a nonempty subset of X. Suppose that f, g are self-mappings of C, and there exist a, b, c, k such that*

$$0 \leq k < \frac{1}{4}, \tag{3.14}$$

$$p(g\left(x\right), f(y)) + p(f(x), f(y)) \leq kp(g\left(y\right), f(x)) \tag{3.15}$$

for all x, $y \in C$. If g has the property

$$g(W(x, y, \lambda)) = W(g(x), g(y), \lambda) \text{ for each } x, y \in C \text{ and } \lambda \in I = [0, 1].$$

and F(g) is a nonempty closed subset of C, then F(f,g) is nonempty.

References

[1] Matthews, SG: Partial metric topology. In: Proc. 8th Summer Conference on General Topology and Applications. Ann. New York Acad. Sci., vol. 728, pp. 183-197 (1994).

[2] Takahashi, T: A convexity in metric spaces and nonexpansive mapping I. Kodai Math. Sem. Rep. 22, 142-149(1970).

[3] Beg, I: An iteration scheme for asymptotically nonexpansive mappings on uniformly convex metric spaces. Nonlinear Analysis Forum. 6:1, 27-34(2001).

[4] Beg, I, Abbas, M: Common fixed points and best approximation in convex metric spaces. Soochow Journal of Mathematics. 33:4, 729-738(2007).

[5] Beg, I, Abbas, M: Fixed-point theorem for weakly inward multivalued maps on a convex metric space. Demonstratio Mathematica. 39:1, 149-160(2006).

[6] Chang, SS, Kim, JK, Jin, DS: Iterative sequences with errors for asymptotically quasi nonexpansive mappings in convex metric spaces. Arch. Inequal. Appl. 2, 365-374(2004).
[7] Ciric, L: On some discontinuous fixed point theorems in convex metric spaces. Czech. Math. J. 43:188, 319-326(1993).
[8] Shimizu, T, Takahashi, W: Fixed point theorems in certain convex metric spaces. Math. Japon. 37, 855-859(1992).
[9] Tian, YX: Convergence of an Ishikawa type iterative scheme for asymptotically quasi nonexpansive mappings. Computers and Maths. with Applications. 49, 1905-1912(2005).
[10] Ding, XP: Iteration processes for nonlinear mappings in convex metric spaces. J. Math. Anal. Appl. 132, 114-122(1998).
[11] I. Altun, F. Sola, H. Simsek, Generalized contractions on partial metric spaces, Topology Appl. 157 (18) (2010) 2778–2785.
[12] S. Romaguera, A kirk type characterization of completeness for partial metric spaces, Fixed Point Theory Appl. (2010) doi:10.1155/2010/493298. 6
[13] M. Bukatin, R. Kopperman, S. Matthews, H. Pajoohesh, Partial metric spaces, Amer. Math. Monthly 116 (2009) 708–718.
[14] A.F. Rabarison, in: Hans-Peter A. Künzi (Ed.), Partial Metrics, African Institute for Mathematical Sciences, 2007, Supervised.
[15] M.A. Bukatin, S.Yu. Shorina, Partial metrics and co-continuous valuations, in: M. Nivat, et al. (Eds.), Foundations of Software Science and Computation Structure, in: Lecture Notes in Computer Science, vol. 1378, Springer, 1998, pp. 125–139.
[16] S.G. Matthews, An extensional treatment of lazy data flow deadlock, Theoret. Comput. Sci. 151 (1995) 195–205.
[17] S. Romaguera, M. Schellekens, Duality and quasi-normability for complexity spaces, Appl. Gen. Topol. 3 (2002) 91–112.
[18] M.P. Schellekens, The correspondence between partial metrics and semivaluations, Theoret. Comput. Sci. 315 (2004) 135–149.
[19] M. Moosaei, Fixed Point Theorems in Convex Metric Spaces, Fixed Point Theory and Applications 2012, 2012:164 doi:10.1186/1687-1812-2012-164 Published: 25 September 2012.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 102–108 (2014) ©KJM

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NEIGHBOURHOODS OF A SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS

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Abstract. In this paper, we investigate the properties of neighbourhoods of functions for the classes $UCV(\alpha)$ and $Sp(\alpha)$. First we established an inclusion relationship between them and proved a necessary and sufficient condition interms of convolutions for a function f to be in $Sp(\alpha)$. Next we show that the class $Sp(\alpha)$ is closed under convolution with functions f(z) which are convex univalent. The results obtained in this which generalizes the results of Padmanabhan [8] and Ronning [9].

1. Introduction:

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $E = \{z\colon |z| < 1\}$. Further, let S be the subclass of A consisting of those functions that are univalent in E. Let CV and ST denote the subclasses of S consisting of convex and starlike functions respectively.

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ then the convolution or Hadamard product of $f(z)$ and $g(z)$ denoted by $f * g$ is defined by
$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$. Clearly
$f(z) * \frac{z}{(1-z)^2} = zf'(z)$ and $f(z) * \frac{z}{(1-z)} = f(z)$

Goodman[3,4] defined the following subclasses of CV and ST.

Definition A: A function f is uniformly convex (Starlike) in E if f is in CV (ST) and has the property that for every circular arc γ contained in E with centre ξ also in E, the arc $f(\gamma)$ is convex (Starlike w.r.t $f(\xi)$.

Goodman [3,4] then gave the following two variable analytic characterizations of these classes, denoted by UCV and UST.

2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Neighbourhood, Subordination, Hadamard Product.

Theorem A: A function f of the form (1.1) is in UCV if and only if

$$Re\left\{1+(z-\xi)\,\frac{f''(z)}{f'}\right\}\geq 0,\ (z,\xi)\in EXE \tag{1.2}$$

and is in UST if any only if

$$Re\left\{\frac{f(z)-f(\xi)}{(z-\xi)\,f'(z)}\right\}\geq 0,\ (z,\xi)\in EXE \tag{1.3}$$

The classical Alexander result that $f\in CV$ if and only if $zf\prime\in ST$ does not hold between the classes UCV and UST. Ronning [7] defined a subclass of starlike functions Sp with the property that a function $f\in UCV$ if and only if $zf\prime\in Sp$.
Definition B: Let $Sp=\{F\in ST/F(z)=zf'(z),f\in UCV\}$

Ma and Minda [6] and Ronning [10] independently found a more applicable one variable characterization for UCV.
Theorem B: A function f is in UCV if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geq\left|\frac{zf''(z)}{f'(z)}\right|,\quad z\in E. \tag{1.4}$$

Ronning [10] proved a one variable characterization for Sp as follows:
Theorem C: A function f is in Sp if and only if

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq Re\left\{\frac{zf'(z)}{f(z)}\right\},\ z\in E. \tag{1.5}$$

A function $f\in A$ is uniformly convex of order α for -1$\leq\alpha<1$ if and only if $1+\frac{zf''(z)}{f'(z)}$ lies in the parabolic region
(**??**) Re $\{\omega - \alpha\} > |\omega - 1|$

In otherwords, the function f is uniformly convex of order α if

$$1+\frac{zf''(z)}{f'(z)}\prec 1+\frac{2(1-\alpha)}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2,\ z\in E \tag{1.6}$$

where the symbol $\prec$ denotes subordination. This class was introduced by Ronning [9] and it is denoted by $UCV(\alpha)$. The class of all analytic functions $f(z)\in A$ for which $\frac{zf'(z)}{f(z)}$ lies in the parabolic region is denoted by $Sp(\alpha)$ and defined as follows.
Definition C: A function $f(z)$ is said to be in the class $Sp(\alpha)$ if for all $z\in E$,

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq Re\left\{\frac{zf'(z)}{f(z)}\right\}-\alpha,\ \text{for}\ -1<\alpha<1. \tag{1.7}$$

This implies $f\in Sp(\alpha)$ for $z\in E$ if and only if $\frac{zf'(z)}{f(z)}$ lies in the region $\Omega\alpha$ bounded by a parabola with vertex at $\left(\frac{1+\alpha}{2},\,0\right)$ and parameterized by $\frac{t^2+1-\alpha^2+2\,it\,(1-\alpha)}{2(1-\alpha)}$ for any real t.

It is known [9] that the function

$$P_\alpha(z)=1+\frac{2(1-\alpha)}{\pi^2}\left[\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right]^2 \tag{1.8}$$

maps the unit disk E on to the parabolic region $\Omega\alpha$ (The branch $\sqrt{z}$ is choosen in such a way that Im $\sqrt{z} \geq 0$). Then from the above definition $f \in A$ is in the class $Sp(\alpha)$ if and only if $\frac{zf'(z)}{f(z)} \prec P_\alpha(z)$.
The notion of δ - neghbourhood was first introduced by St. Ruscheweyh [11].
Definition D: For $\delta \geq 0$, the δ - neighbourhood of f(z) $= z + \sum_{n=2}^{\infty} a_n z^n \in A$ is defined by

$$(1.9) \qquad N_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n : \sum_{n=2}^{\infty} n\,|a_n - b_n| \leq \delta \right\}.$$

Recently Padmanabhan [8] has introduced the neighbourhoods of functions in the calss Sp and studied various properties.

In this paper we studied some related work on the neighbourhood problems for k-uniformly convex functions of Kanas[5].The work of Ma and Minda [7] generalize many studies on subclasses of starlike and convex functions. we introduce a new class of functions and study the properties of neighbourhoods, of functions in this class which generalizes the recent results of Padmanabhan [8] and Ronning [9].

First let us state lammas which are needed to establish our results in the sequel.
Lemma A [2]: Let $\beta, \gamma \in C$, let $h(z)$ be analytic, univalent and convex in E with $h(0) = 1$ and Re $(\beta\ h\ (z) + \gamma) > 0$, $z \in E$ and let $p(z) = 1 + p1\ z + \ldots .\ z \in E$, then

$$(1.10) \qquad p(z) + \frac{z\,p'(z)}{\beta\,p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Lemma B [12]: Let f(z) $= \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=2}^{\infty} a_n z^n$ be in $ST\left(\frac{1+\alpha}{2}\right)$ denote by $f * g$ the Hadamard product $(f * g)(z) = \sum_{n=2}^{\infty} a_n\ b_n\ z^n$. Then for any function $F(z)$ analytic in E, we have for $z \in E$ that

$$\frac{f(z) * g(z)\ F(z)}{f(z) * g(z)} \subset \overline{Co}\ (F(E))$$

$\overline{Co}$ denotes the closed convex hull.

2. Main Results

First let us establish an inclusion relation.
Theorem 2.1: Let $f \in UCV\ (\alpha)$. Then $f \in Sp(\alpha)$.
Proof: Let $p(z) = \frac{zf'(z)}{f(z)}$. Then since $f \in UCV(\alpha)$

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} \subset \Omega_\alpha$$

Since $\Omega\alpha$ is a convex damain, an application of Lemma A gives $\frac{zp'(z)}{p(z)} = p(z) \subset \Omega_\alpha,\ z \in\ E$ which implies that $f \in Sp(\alpha)$.

Now we give a characterization of the class $Sp(\alpha)$ in terms of convolution.
Definition 2.1: Let $S_p'(\alpha)$ be the class of all functions $h\alpha(z)$ in A of the form

(1.11)
$$h_\alpha(z) = \frac{2(1-\alpha)}{(1-\alpha)^2 - t^2 - 2it(1-\alpha)} \left[\frac{2}{(1-z)^2} - \frac{t^2 + 1 - \alpha^2 + 2it(1-\alpha)}{2(1-\alpha)} \frac{z}{(1-z)} \right]$$

for -1 $\leq \alpha < 1$ and for all real t.
Theorem 2.2: A function $f(z)$ in A is in $Sp(\alpha)$ if and only if for all z in E $(z \neq 0)$ there exists a function $h\alpha(z)$ in $S_p'(\alpha)$ such that $\frac{(f * h_\alpha)(z)}{z} \neq 0$.
Proof: Let us assume that $\frac{(f * h_\alpha)(z)}{z} \neq 0$, then for all $h\alpha(z) \in S_p'(\alpha)$ and for $z \in E$ $(z \neq 0)$. From the definition of $h\alpha(z)$ it follows that

$$\frac{f(z) * h_\alpha(z)}{z} = \frac{2(1-\alpha)}{z\left[(1-\alpha)^2 - t^2 - 2it(1-\alpha)\right]}\left[f(z) * \frac{z}{(1-z)^2} - \frac{t^2+1-\alpha^2+2it(1-\alpha)}{2(1-\alpha)} f * \frac{z}{1-z}\right]$$

$$= \frac{2(1-\alpha)}{z\left[(1-\alpha)^2 - t^2 - 2it(1-\alpha)\right]}\left[zf'(z) - \frac{t^2+1-\alpha^2+2it(1-\alpha)}{2(1-\alpha)} f(z)\right]$$

$$\neq 0.$$

Equivalently $\frac{zf'(z)}{f(z)} \neq \frac{t^2+1-\alpha^2+2it(1-\alpha)}{2(1-\alpha)}, t \in R$. This means that $\frac{zf'(z)}{f(z)}$ lies completely either inside $\Omega\alpha$ or complement of $\Omega\alpha$ for all z in E. At $z = 0$, $\frac{zf'(z)}{f(z)} = 1 \in \Omega\alpha$, so $\frac{zf'(z)}{f(z)} \subset \Omega\alpha$ which means $f \in Sp(\alpha)$.

Conversely let $f \in Sp(\alpha)$. Hence $\frac{zf'(z)}{f(z)}$ lies with in the parabola with vertex at the point $\left(\frac{1+\alpha}{2}, 0\right)$ and the boundary of this is given by $\frac{t^2+1-\alpha^2+2it(1-\alpha)}{2(1-\alpha)}$ for $t \in R$. So $f \in Sp(\alpha)$ only when

$$\frac{zf'(z)}{f(z)} \neq \frac{t^2+1-\alpha^2+2it(1-\alpha)}{2(1-\alpha)}$$

Equivalently
$f(z) * \left[\frac{z}{(1-z)^2} - \frac{t^2+1-\alpha^2+2it(1-\alpha)}{2(1-\alpha)}\frac{z}{(1-z)}\right] \neq 0$ for $z \neq 0$.

Normalizing the function within the brackets we get $\frac{(f * h_\alpha)(z)}{z} \neq 0$ in E where $h\alpha(z)$ is the function defined in (1.11).

To investigate the δ neighbourhoods of functions belonging to the class $Sp(\alpha)$, we need the following lemmas.
Lemma 2.1: Let $h_\alpha(z) = z + \sum_{k=2}^{\infty} c_k z^k \in S_p'(\alpha)$. Then

$$|c_k| \leq \frac{2k-(1+\alpha)}{(1-\alpha)}, \quad k = 2, 3 \ldots.$$

Proof: Let $h_\alpha(z) \in S_p'(\alpha)$. Then for $t \in R$

$$h_\alpha(z) = \frac{2(1-\alpha)}{(1-\alpha)^2 - t^2 - 2it(1-\alpha)}\left[\frac{z}{(1-z)^2} - \frac{t^2+1-\alpha^2+2it(1-\alpha)}{2(1-\alpha)}\frac{z}{(1-z)}\right]$$

$$= \frac{2(1-\alpha)}{(1-\alpha)^2 - t^2 - 2it(1-\alpha)}\left[\left(z+2z^2+\ldots\ldots\right) - \frac{t^2+1-\alpha^2+2it(1-\alpha)}{2(1-\alpha)}\left(z + z^2 + \ldots..\right)\right]$$

$= z + \sum_{k=2}^{\infty} c_k z^k$
Now comparing the coefficients on either side we get

$$c_k = \frac{2k(1-\alpha) - t^2 - 1 + \alpha^2 - 2it\,(1-\alpha)}{(1-\alpha)^2 - t^2 - 2it\,(1-\alpha)}$$

After simplication we get

$$|c_k| \le T_k = \frac{2k-(1+\alpha)}{(1-\alpha)}, \text{ for } k = 2,\ 3 \ldots$$

Lemma 2.2: For $f \in A$ and or every $\epsilon \in C$ such that $|\epsilon| < \delta$ if $F\epsilon(z) = \frac{f(z) + \varepsilon z}{1+\varepsilon} \in Sp(\alpha)$ then for every $h\alpha(z) \in S'_p(\alpha)$.

$$\left|\frac{(f * h_\alpha)(z)}{z}\right| \ge \delta, \quad z \in E.$$

Proof: Let $F\epsilon(z) \in Sp(\alpha)$. Then by Theorem 2.2, $\frac{F_\varepsilon(z) * h_\alpha(\alpha)}{z} \ne 0$, for all $h\alpha(z) \in S'_p(\alpha)$ and $z \in E$.
Equivalently

$$\frac{(f * h_\alpha)(z) + \varepsilon z}{(1+\varepsilon) z} \ne 0 \ or \ \frac{(f * h_\alpha)(z)}{z} \ne -\varepsilon,$$

that is

$$\left|\frac{(f * h_\alpha)(z)}{z}\right| \ge \delta.$$

Theorem 2.3: Let $f \in A$, $\epsilon \in C$ and for $|\epsilon| < \delta < 1$, if $F\epsilon(z) \in Sp(\alpha)$. Then $N\delta(f) \subset Sp(\alpha)$
for the sequence

$$T = T_k = \frac{2k - (1+\alpha)}{(1+\alpha)}$$

Proof: Let $h\alpha(z) \in S'_p(\alpha)$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is in $N\delta(f)$
Then

$$\left|\frac{(g * h_\alpha)(z)}{z}\right| = \left|\frac{(f * h_\alpha)(z)}{z} + \frac{((g - f) * h_\alpha)(z)}{z}\right|$$
$$\ge \left|\frac{(f * h_\alpha)(z)}{z} - \frac{(g - f)(z) * h_\alpha(z)}{z}\right|$$

$\ge \delta - \left|\sum_{k=2}^{\infty} \frac{(b_k - a_k)\, c_k\, z^k}{z}\right|$, by lemma 2.2.
We have

$$\left|\frac{(g * h_\alpha)(z)}{z}\right| \ge \delta - |z| \sum_{k=2}^{\infty} |c_k|\ |b_k - a_k|$$

$> \delta - \sum_{k=2}^{\infty} T_k\ |b_k - a_k|$, by lemma 2.1
$> \delta - \delta = 0$.
Thus $\left|\frac{(g * h_\alpha)(z)}{z}\right| \ne 0$ in E for all $h\alpha \in S'_p(\alpha)$ and then by Theorem 2.2, we have $g \in Sp(\alpha)$. Hence we have $N\delta\ (f) \subset Sp(\alpha)$.

Next we show that the class $Sp(\alpha)$ is closed under convolution with functions f which are convex univalent in E.

Theorem 2.4: Let $f \in CV$ the class of convex functions and $g(z) \in Sp(\alpha)$. Then $(f * g)\ (z) \in Sp(\alpha)$.
Proof: The proof of Theorem is similer result of T.N.Shanmugan [13], hence we omitted.
Theorem 2.5: Let $f \in ST\ \left(\frac{\alpha+1}{2}\right)$, $g \in Sp(\alpha)$. Then $(f * g)\ (z) \in Sp(\alpha)$.
Proof: Let $g \in Sp(\alpha)$. Assume $f \in ST\ \left(\frac{\alpha+1}{2}\right)$ and $\frac{zg'(z)}{g(z)}$ play in the role of F in Lemma B, and let $\Omega\alpha = \{|\omega\text{-}1|\text{Re }(\omega\text{-}\alpha)\}$. Using the Lemma B, we get for $z \in E$ that
$\frac{z(f\ * g)'(z)}{(f\ * g)(z)} = \frac{f(z) * zg'(z)}{(f * g)(z)} = \frac{f(z)\ * g(z)\ \frac{zg'(z)}{g(z)}}{(f\ * g)(z)} \subset \overline{Co}\ \frac{zg'(z)}{g(z)} \subset \Omega_\alpha$. Since $\Omega\alpha$ is convex and $g \in Sp(\alpha)$. This proves that $(f * g)\ (z) \in Sp(\alpha)$.

Setting $\alpha = 0$, the following result of Ronning [9] follows.
Corollary 2.1: Let $f \in ST\ (1/2)$, $g \in Sp(0) = Sp$, then $(f * g)\ (z) \in Sp$.
Theorem 2.6: Let $g\in\ UCV\ (\alpha)$ and $h(z) \in ST\ \left(\frac{\alpha+1}{2}\right)$. Then $(g * h)\ (z) \in UCV(\alpha)$.
Proof: If $g \in UCV(\alpha)$, then $z\ g\prime(z) \in Sp(\alpha)$. By Theorem 2.4 it follows that $h * zg\prime \in Sp(\alpha)$. So

$z(h * g)\prime\ (z) = h(z) * zg\prime\ (z) \in Sp(\alpha)$.

This proves that $(h * g)\ (z) \in UCV\ (\alpha)$.

Setting $\alpha = 0$, the following result of Padmanabhan [8] follows.
Corollary 2.2: Let $g \in UCV$ and $h(z) \in ST\ (1/2)$. Then $(g * h)\ (z) \in UCV(\alpha)$.
Theorem 2.7 : Let $f \in UCV(\alpha)$. Then $\frac{f(z)\ +\ \varepsilon\ z}{1\ +\ \varepsilon} \in S_p\,(\alpha)$ for $|\varepsilon|<$.
Proof: Let f(z) $= z\ +\ \sum_{n=2}^{\infty} a_n z^n$ then

$$\frac{f\,(z)\ +\ \varepsilon\ z}{1+\ \varepsilon} = \frac{z\,(1+\varepsilon)\ +\sum_{n=2}^{\infty} a_n z^n}{1+\varepsilon} = \frac{f\,(z)\ *\ [z\,(1+\varepsilon)\ +\sum_{n=2}^{\infty} z^n]}{1+\varepsilon}$$

$$= f\,(z)\ *\ \frac{\left(z - \frac{\varepsilon}{1+\varepsilon}\ z^2\right)}{(1-z)} = f\,(z)\ *\ h\,(z)$$

where $h\,(z) = \frac{\left[z\ -\ \frac{\varepsilon}{1+\varepsilon}\ z^2\right]}{(1-z)}$
Now

$$\frac{zh'\,(z)}{h\,(z)} = \frac{\left[z - \frac{2\varepsilon}{1+\varepsilon} z^2\right]}{\left[z - \frac{\varepsilon}{1+\varepsilon} z^2\right]} + \frac{z}{1-z} = \frac{-\rho\ z}{1-\rho\ z} + \frac{1}{1-z}$$

where $\rho = \frac{\varepsilon}{1+\varepsilon}$. Hence $|\rho| < \frac{\varepsilon}{1-|\varepsilon|} < 1/3$ gives $|\varepsilon|< 1/4$
Thus

$$Re\left\{\frac{zh'\,(z)}{h\,(z)}\right\} \geq \frac{1 - 2\,|\rho|\ |z|\ -\ |\rho|\ |z|^2}{(1-|\rho|\ |z|)\ (1\ +\ |z|)} >\ 0$$

if $|\rho|(|z|2\ +\ 2\ |z|)$ -1 < 0. This inequality holds for all $\rho < 1/3$ and $|z|< 1$, which is true for $|\varepsilon|< 1/4$. Therefore $h(z)$ is starlike in the unit disk and so $\int_0^z \frac{h(t)}{t}\ dt$ is convex.

But $h(z) * \log\left(\frac{1}{1-z}\right) = \int_0^z \frac{h(t)}{t}\ dt$ and so $h(z) * \log\left(\frac{1}{1-z}\right)$ is convex in E and
$(f * h)\ (z) = (h * f)\ (z) = h(z) * \left[zf'\,(z)\ *\ \log\ \left(\frac{1}{1-z}\right)\right]$

$= zf\prime(z) * \left[h\,(z)\ *\ \log\ \left(\frac{1}{1-z}\right)\right]$

$f(z)\in UCV(\alpha)$ implies $zf\prime\ (z)\in Sp(\alpha)$ and $h(z)\ *\ \log\ \left(\frac{1}{1-z}\right)\ \in CV$. Now by Theorem 2.4 $h(z)\ *\ \left[zf'(z)\ *\ \log\ \left(\frac{1}{1-z}\right)\right]$ is in $Sp(\alpha)$. Thus $(f*h)\ \ (z) = \frac{f(z)\,+\,\varepsilon\, z}{1+\varepsilon}\ \in\ S_p\,(\alpha)$ for $|\varepsilon|< 1/4$.

Corollary 2. 3: If $f\in UCV(\alpha)$, then $f\in Sp(\alpha)$.

Proof: Choosing $\varepsilon = 0$ in the Theorem 2.7 we get the result.

Corollary 2. 4: If $f\in UCV(\alpha)$ then $\int_0^z \frac{f(t)}{t}\,dt\ \in\ UCV\,(\alpha)$.

Proof: $f\in UCV\ (\alpha)$ implies $f\in Sp(\alpha)$ by corollary2.3,so we can write $f(z) = zg\prime(z)$ for some $g\in UCV(\alpha)$ and $g\prime\ (z) = \frac{f(z)}{z}$ gives $g\,(z)\ =\ \int_0^z \frac{f(t)}{t}dt\in UCV(\alpha)$.

Acknowledgements The authors would like to thank the referee of the paper for his helpful comments.

References:

1. R.M.Ali and V.Ravichandran, Uniformly Convex and Uniformly Starlike functions Mathematics Newspaper, Vol. No 1, (2011),16-30.

2. P. Eenigenburg, S.S. Miller, P.T. Mocanu and M.O. Reade, On a Briot – Boquet Differential Subardination, General Inequalities 3, Birkhauser Verlag-Basel, (1983) 339-348.

3. A.W. Goodman. On uniformly convex functions. Ann.Polon. Math 56(1991), 87-92.

4. A.W. Goodman, On uniformly starlike functions, J. Math. Anal. Appl.155(1991), 364-370.

5. S.Kanas, Staility of convolution and dual sets for the class k- uniformly convex and k- starlike functions functions, Zeszyty Nauk.Politech. Rzeszowskiej Mat., (**??**) (1998) 51-64.

6. W.C. Ma and D. Minda, Uniformly convex functions, Ann. Polon, Math.

$$57(1992), 165-175.$$

7. W.C.Ma and D.Minda, A unified treatment of some special classes of univalent functions , in Proceedings of the Conference on Complex Analysis (Tianjin, 1992) , 157-169, Conf.Proc.Lecture Notes Anal.,I Int.Press ,Combridgema.MA.

8. K.S. Padmanabhan, On uniformly convex functions in the unit disk. J. Analysis, 2 (1994), 87-96.

9. F. Ronning, on starlike functions associated with parabolic regions, Ann. Univ. Mariae Curie-Sklodowska. Sect. A, 45 (1991), 117-122.

10. F. Ronning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. 118 No.1 (1993), 189-196.

11. St. Ruscheweyh, Neighbourhoods of univalent functions, Proc. Amer Math. Soc. 81 (1981), 521-527.

12. St. Ruscheweyh, Convolutions in Geometric functions theory, Sem. Math. Sup. Vol.83, Presses Univ. de Montreal, placeCityMontreal 1982.

13. T.N.Shanmugam, Convolution and Differential subordination, Internat.J.Math & Math. Sci, 12 (**??**) (1989), 333-340.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 109 –119 (2014) ©KJM

KOÇAK'S ACCELERATION METHOD SMOOTHLY GEARS UP ITERATIVE SOLVERS

MEHMET ÇETIN KOÇAK

Abstract. Consider a scalar repetitive scheme symbolically represented by $x_{k+1} = g(x_k)$ where k is the iteration count. Let z and n respectively denote the target fixed-point and convergence order of g. Koçak's method g_K accelerates g by actually solving a superior secondary solver obtained from a fixed-point preserving transformation

$g_K = x + G(g - x) = (g - mx)/(1 - m), m = 1 - 1/G, G = 1/(1 - m)$

where G is a gain and m is the slope of a straight line joining g and $g = x$ line. The method uses derivatives of g in local adjustment of m so as to push g_K towards the ideal solver $g = z$ by annihilating derivatives of g_K. If n is 1, then g_K is of third order. If n exceeds 1, then g_K is of $(n+1)$th order. Variable m improves remote behaviour also. The benefits of g_K amply compensate the cost of extra derivatives. A first-order solver with highly oscillatory divergence shows that the resultant third-order g_K renders a fast and smooth flight from a remote point to z. Newton's second-order, Chebyshev's third-order, and Ostrowski's fourth-order solvers all spin off in contrast

1. INTRODUCTION

A nonlinear equation

$$x = g(x) \tag{1.1}$$

can be solved by a repetitive scheme $x_{k+1} = g(x_k)$ where k is the iteration count. If z satisfies (1.1), then z is called a fixed-point of g. (*In this text, functions are usually written without an argument list when it contains x only.*) To find a z is to locate an intersection of the curve $g = g(x)$ with the straight line $g = x$. Each scheme starts form one or more points supposedly in the vicinity of z and usually ends when the absolute difference between successive iterates falls below a pre-specified tolerance.

Date: January 1, 2013 and, in revised form, February 2, 2013.
2010 *Mathematics Subject Classification.* 65B99, 65H05.
Key words and phrases. Non-linear equations; Iterative methods; Newton's method; Ostrowski's method; Convergence acceleration.

On a plot of x versus g, the iteration process follows a sequence of joined lines which are vertical and horizontal in turn; the first is a vertical line originating from the x-axis and ending on g, the second is a horizontal line extending to $g = x$, then comes a vertical line to g again, and so on. The "ideal" solver is the horizontal line g_{id} which needs just one trial from any starting point. Albeit, z is unavailable until the end! g_{id} can be harnessed however in post priori analysis, research, comparative studies, and troubleshooting.

Let $\varepsilon_k = x_k - z$. If there exist a real number n and nonzero constant c such that

$$\lim_{k\to+\infty} \frac{|\varepsilon_{k+1}|}{|\varepsilon_k|^n} = c,$$

then n and c are respectively called the convergence order and the asymptotic error constant. According to Traub [9], if n is integral, then

$$c = \lim_{k\to+\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^n} = \frac{g^{(n)}(z)}{n!}.$$

Linear or first order convergence ($n = 1$) means that $g\prime(z) \neq 0$. Quadratic convergence ($n = 2$) means that $g\prime(z) = 0$ but $g\prime\prime(z) \neq 0$. Generalizing, an integral convergence order $n > 1$ means that $g\prime(z) = g\prime\prime(z) = \ldots = g^{(n-1)}(z) = 0$ but $g^{(n)}(z) \neq 0$ and that ε_{k+1} is proportional to ε_k^n in the vicinity of z. (*Thus, nth order solvers are a subset of $(n-1)$th order solvers.*)

Many techniques reuse old information and/or harness new information at more than one point. In Traub's methodology [9], *methods with memory* are those which utilize past values and *multipoint solvers* are those which harness new information at a number of points. Furthermore, $g \in I_n$ indicates that g belongs to the class of solvers of order n. The condition for g to converge is that $|g\prime| < 1$ in the vicinity of z [8].

1.1. **A selection of iterative techniques.** In numerous cases, the iterative solver g actually comprises a long chain of equations involving many intermediate variables. If $x = g(x)$ is a rearrangement of another equation $f(x) = 0$, then the fixed-points of g are identical with the zeroes (roots) of f. Sometimes g comes from $g = x - fu$ where u is finite. Newton's popular second-order method [2,4]

$$g_N = f - \frac{f}{f\prime}$$

is an example of this type where u equals $1/f\prime$. (*A subscript starting with a capital letter is assigned in this paper to a solver g, convergence order n, and asymptotic error c so as to indicate the person to whom the pertinent method is attributed.*).

g_N is a piecewise linearization of f since it extends the current tangent to intersect the x-axis and suggests this value as the next approximation to z. (g_N is also called *the variable tangent method.*) As shown by Traub[9], $n_N = 2$ and $c_N = f\prime\prime(z)/(2f\prime(z))$ for simple roots. Repetition of z demotes convergence of g_N from *quadratic* to *superlinear* or *geometrical* and slows down the iteration process. If r is the multiplicity of z, then $g\prime(z) = (r-1)/r \neq 0$ and $g_{Nr} = x - rf/f\prime$ restores second order [4]. Direct differentiation of g_N gives $g'_N = ff\prime\prime/f'^2 = L$ where L is called the logarithmic degree of convexity. So, the convergence condition of g_N is that $|g'_N| = |L| < 1$ in the vicinity of z.

1.1.1. *Secondary solvers generated by partial substitution.* Partial substitution (g_{ps}) employs a variable gain G to amplify the correction to x, that is r

$$g_{ps} = x + G(g - x)$$

Applied to g_N, partial substitution gives $g_{Nps} = x - Gf/f\prime$. Note that g_{Nr} defined above is a g_{Nps} with a fixed gain $G = r$. Besides g_N, this article uses three g_{Nps} methods for comparison with newly geared-up g_K. They are Chebyshev's ($g_C \in I^3$), Halley's ($g_H \in I^3$) and Ostrowski's ($g_H \in I^4$) solvers whose respective formulas are as follows:

$$g_C = x - G\frac{f}{f\prime}, \quad G = 1 + \frac{L}{2},$$

$$g_H = x - G\frac{f}{f\prime} = x - f\frac{f\prime}{f\prime^2 - 0.5ff\prime\prime}, \quad G = \left(1 - \frac{L}{2}\right)^{-1},$$

$$g_O = x - G\frac{f}{f\prime}, \quad G = 1 + \frac{f(g_N)}{f - 2f(g_N)} = \frac{f - f(g_N)}{f - 2f(g_N)}.$$

1.1.2. *Secondary solvers generated by piecewise linearization.* Let K(x,g) be a point on g and let M(p,p) be an arbitrary point on $g = x$. The slope m of the straight line KM is given by $m = (g-p)/(x-p)$. Conversely, given the slope $m = (g-p)/(x-p) \neq 1$, a straight line through K intersects the line $g = x$ at M(p,p). Rearrangement renders $p = (g - mx)/(1 - m)$. A specified solver $g_{pl} = p$ may be regarded as a piecewise linearization of g if p is used to approximate z. Hence,

$$g_{pl} = \frac{g - mx}{1 - m}. \tag{1.2}$$

It is easy to show [5-7] that g_{pl} and g_{ps} are uniquely linked. Indeed,

$$g_{pl} = (g - mx)/(1 - m) \equiv x + G(g - x) \equiv g_{ps}$$

if $m = 1 - 1/G$ or $G = 1/(1 - m)$.

Regardless of the multiplicity of z, the ideal slope for linearization of g is $m_{id} = (g-z)/(x-z)$. If the chosen slope coincides with m_{id}, then $g_{pl} = g_{id} = z$ and M falls upon Z(z,z). Two well-known *one-point accelerators with memory*, namely Aitken's [1] and Wegstein's [3] methods, are examples of g_{pl}. Wegstein's approximating lines are secants of g going through a previous iterate (x_i,g_i) and the current iterate (x_k,g_k), that is

$$m = \frac{g_k - g_i}{x_k - x_i}, \quad g_W = \frac{g_k x_i - g_i x_k}{x_i - x_k - (g_i - g_k))}.$$

g_W is calculable first time at the end of the second iteration and is updated at each iteration afterwards. Aitken's technique similarly uses secants but its updates are every other iteration. Since $i = k - 1$ and $x_k = g_{k-1}$ here,

$$g_A = \frac{g_k x_{k-1} - g_{k-1}^2}{x_{k-1} - x_k - (g_{k-1} - g_k)} = g_k - \frac{(g_k - g_{k-1})^2}{x_{k-1} - x_k - (g_{k-1} - g_k)}.$$

2. KOÇAK'S ACCELERATOR

Koçak's method accelerates a given g by actually solving a superior secondary solver g_K generated through the transformation

$$g_K = x + G(g-x) = \frac{g-mx}{1-m}, \quad m = 1 - \frac{1}{G}, \quad G = \frac{1}{1-m}, \quad m \neq 1 \tag{2.1}$$

where G is a gain and m is a slope. Obviously, g_K is piecewise linearization and partial substitution, that is $g_K \equiv g_{pl} \equiv g_{ps}$. The transformation is the ultimate result of three successive operations on the equation $x = g(x)$, namely subtraction of the product mx from both sides, collecting terms, and rearrangement. Symbolically:

$$x - mx = g - mx \implies (1-m)x = g - mx \implies x = g_K = (g-mx)/(1-m)$$

The transformation obviously preserves fixed-points, that is $g_K(z) = g(z) = z$. Direct comparison of (2.1) and (1.2) shows that the varying slope is given by $m = (g-g_K)/(x-g_K)$.

Solver comparisons customarily focus on performance in the vicinity of z, paying attention to the number of function evaluations, the number of derivative calculations, convergence order, and asymptotic error constants. As previously published [5-7], if m is adjusted such that

$$m^{(i-1)}(z) = g^{(i)}(z)/i, \quad i = 1,2,\ldots,n_K-1, \quad n_K \geq 2, \quad m^{(0)}(z) = m(z) = g\prime(z),$$

then

$$g_K^{(i)}(z) = 0, \quad i = 1,2,\ldots,n_K-1$$

and g_K achieves an integral convergence order $n_K \geq 2$.

It is well known that, irrespective of its convergence order, an iterative method is liable to unsatisfactory performance (because of oscillation or slowness) and even to total failure if the starting point is not near enough the target z. The aim of the present phase was to improve remote behavior of g_K by successively zeroing as many of its derivatives ($g_K\prime$, $g_K\prime\prime$,...) as possible. The action forces g_K towards the ideal curve $g_{id} = z$. The case $g_K\prime = g_K\prime\prime = 0$ amply illustrates the approach.

Let $h = g_K - x$. Taylor's expansion of m around x is

$$m(g_K) = m(x+h) = m(x) + m\prime h + \frac{m\prime\prime h^2}{2!} + \frac{m\prime\prime\prime h^3}{3!} + \ldots$$

Truncating after the third term results in

$$m_h = m + m\prime h + \frac{m\prime\prime h^2}{2!} \backsimeq m(g_K).$$

In order to attain $g_K\prime = g_K\prime\prime = 0$, the following conditions must be satisfied:

$$m\prime = \frac{-(m-1)(m-g\prime)}{g-x}, \tag{2.2}$$

$$m\prime\prime = \frac{2(1-m)m\prime(m-1-g\prime) + m\prime(g-x)}{(1-m)^2} - \frac{(g\prime\prime - 2m\prime)(1-m)}{g-x}. \tag{2.3}$$

Suppose the set $\{g, g\prime, g\prime\prime\}$ is available at x. Then $\{g_K, h, m\prime, m\prime\prime\}$ and hence m_h depend on m alone. Recall that the minimal condition for $n_K \geq 2$ is that $\lim_{x\to z} m = g\prime(z)$. The problem now is to tune m so as to annihilate a discrepancy function $f_m(m) = m_h(m) - g\prime(z)$. The target is $m \backsimeq m_{id}$ which makes $g_K \backsimeq z$, $m_h \backsimeq m(g_K) \backsimeq m(z) = g\prime(z)$, and hence $f_m(m) \backsimeq 0$. It seems convenient to set up an inner loop to receive $\{g, g\prime, g\prime\prime\}$ at x, iteratively solve f_m for m (*subject to a tolerance*), and deliver the pertinent g_K to the outer loop as the new x value to test.

It is easy to see that $m = g\prime$, $m = g\prime(z)$, or a weighted average

$$m = wg\prime + (1-w)g\prime(z) = g\prime(z) + w(g\prime - g\prime(z)) \tag{2.4}$$

fulfills the minimal requirement for $n_K \geq 2$ since $\lim_{x\to z} m = g\prime(z)$. The scheme can be modified in this case such that having received $\{g, g\prime, g\prime\prime, g\prime\prime\prime\}$ at x the inner loop solves a corresponding discrepancy function $f_w(w) = w_h(w) - w_{lim}$ for w by locally adjusting w subject to a specific end-point limit w_{lim} and return the pertinent g_K. The value of w_{lim} depends on n as explained later. The extra derivative $g\prime\prime\prime$ enters the scene because differentiating (2.4) with respect to x renders

$$w\prime = \frac{m\prime - wg\prime\prime}{g\prime - g\prime(z)}$$

$$w\prime\prime = \frac{m\prime\prime - (2w\prime g\prime\prime + wg\prime\prime\prime)}{g\prime - g\prime(z)}.$$

Equations (2.2) and (2.3) supply $m\prime$ and $m\prime\prime$ as before. From truncated Taylor's expansion again,

$$w_h = w + w\prime h + \frac{w\prime\prime h^2}{2!}.$$

Variable w is the distinguishing feature of the third version of g_K. The forerunner [5] employs constant $w = 1/2$ irrespective of n whereas the second [6] selects a constant w appropriate to n, that is it couples w to n. These constant w values which are coupled to n in the second version are now w_{lim} in the third version and $\lim_{x\to z} w = w_{lim}$. This means that both n_K and c_K remain unchanged in going from the second version to the third. Three different situations exist:

- **a):** If $n = 1$, then $w_{lim} = 1/2$ is used with the result that $n_K = 3$, $c_K = g_K'''(z)/3!$, and $g_K'''(z) = -0.5g\prime\prime\prime(z)/(1 - g\prime(z))$.
- **b):** If $n = 2$, then $w_{lim} = 1/2$ is employed with the result that $n_K = 3$, $c_K = g_K'''(z)/3!$, and $g_K'''(z) = -0.5g\prime\prime\prime(z)$.
- **c):** If $n > 2$, then $w_{lim} = 1/n$ is harnessed which renders $n_K = n + 1$, $c_K = g_K^{(n+1)}(z)/(n+1)!$, and $g_K^{(n+1)}(z) = -g^{(n+1)}(z)/n$.

(Interesting results accrue [7] from the solution of $g_K' = 0$ assuming that $w\prime(g\prime - g\prime(z))(g - x) \backsimeq 0$. The new approach described above is free from this restricting assumption. There is more information in the appendix.)

2.1. **The algorithm.** The formulation hinged to w has been implemented. After many revisions and runs, the final version of the accelerator is now housed in a function that supervises the whole process once triggered by a call giving necessary initial information, namely, n, $g\prime(z)$, number of g derivatives, starting point, convergence tolerance, iteration limit, and name of the function to supply $\{g, g\prime, g\prime\prime, g\prime\prime\prime\}$at x. The supervisor function has two loops one inside the other. The outer loop is started after setting w_{lim} according to n. It sets $w = w_{lim}$, gets $\{g, g\prime, g\prime\prime, g\prime\prime\prime\}$from the named user function and begins the inner loop which employs g_N to solve $f_w(w) = 0$ for w thereby fixing the pertinent m and g_K. This g_K is then used as x for the next outer iteration.

The inner loop embeds an auxiliary function that takes $\{w, x, g, g\prime, g\prime\prime, g\prime\prime\prime\}$ and calculates $\{w_h, g_K\}$as follows. First, it obtains m from (2.4) and g_K from (2.1). It then accrues, $m\prime$,$w\prime$,$m\prime\prime$,$w\prime\prime$, and w_h. If the returned w_h exceeds 1.5 times w_{lim}, then the loop halves w and tries again. Otherwise, it checks $|f_w|$. If this is sufficiently small, then the current w and the resultant g_K are accepted and the inner loop is terminated. If not, then w is updated for the next iteration using $w = w - f_w(w)/f_w\prime(w)$ in accordance with g_N formulation. The derivative $f_w\prime(w)$ is calculated numerically which means an extra $f_w(w)$ per iteration here.

2.2. **Links to other solvers.** The accelerator naturally links to other solvers for it is both g_{ps} and g_{pl}. If $w = 1$, then $m = g\prime$ and the application [5] is equivalent to utilizing g_N to solve a secondary function $g - x = 0$. Indeed,

$$g_N = x - \frac{g - x}{g\prime - 1} = \frac{g\prime x - x + g - x}{g\prime - 1} = \frac{g - g\prime x}{1 - g\prime} = \frac{g - mx}{1 - m} = g_K, \quad m = g\prime$$

In this case, $n_K = n_N = 2$. Piecewise linearization techniques g_A and g_W are in fact a subclass of this case where the slope of a secant approximates $g\prime$. Alas, their popular implementations nullify possible beneficial contribution of $g\prime(z)$. Since secants virtually tend to the tangent as x goes to z, it can be asserted that $n_A = n_W = n_N = 2$ (provided that z is not repeated). The application of g_K converts [5] g_N to g_H if $w = 1/2$. With variable w tending to $w_{lim} = 1/2$, the result should be a *smoother* g_H.

2.3. **A highly oscillatory and divergent first-order test case.** This benchmark is in fact a member of a difficult class of problems keyed to N. Let $N = 7$ and $s = 10^N$. Suppose that $g = s/x^{N-1}$ is to be harnessed for the iterative solution of $f = x^N - s$. For this class, $g\prime = -(N-1)s/x^N$, $g\prime(z) = -(N-1)$, $g\prime\prime = -Ng\prime/x$, and $g\prime\prime\prime = N(N+1)g\prime/x^2$. The new g_K with variable w will be applied to accelerate the process. The performance of g_K will be compared with those of $g_N \in I^2$, $g_C \in I^3$, $g_H \in I^3$, and $g_O \in I4$. (Remember that Chebyshev's, Halley's and Ostrowski's solvers are partial substitution variants of g_N.) The target fixed-point is $z = 10$, of course.

3. Results and discussion

Table 1 depicts the test results using $x_1 = 2$ as starting point. Note that $w_{h1} = w + w\prime h$ and $w_{h2} = w + w\prime h + w\prime\prime h^2/2!$. The use of $w_h = w_{h2}$ limits the extra information needed to $\{g\prime\prime, g\prime\prime\prime\}$. (As can be expected, w_{h2} is better to use than w_{h1}.) It is obvious that g_K with variable w superbly pilots the iteration process; the flight from a remote point to z is so fast and smooth despite the fact that g is a first-order solver with highly oscillatory divergence! Consider the first iteration now. New g_K accrues its largest correction here taking x from 2 to 7.950162903588. Notice that there are 4 internal iterations where w respectively takes the values

0.5, 0.25, 0.125, 0.056 compared with the ideal $w_{id} = 0.041652$. (w is halved twice before applying g_N.) In contrast, the solvers $\{g, g_N, g_C, g_O\}$ all spin off at $x = 2$. Only g_H takes a small step in the right direction. (*Note the previous two versions use $w = 1/2$ when $n = 1$ and with this class of problems this is equivalent to harnessing g_H.*) This proves the immense improvement of the third version over its predecessors. g_K continues to lead the other contestants in the second iteration, taking x to 10.003627135093 which is very close to the target. Not surprisingly, close to the finish $g_O \in I^4$ overtakes $g_K \in I^3$!

Without doubt, g_K with variable w amply compensates the extra cost of $\{g\prime\prime, g\prime\prime\prime\}$. In fact, its contribution is invaluable since it converts a divergent solver to a flyer. Needless to say, Koçak's acceleration method is also an important tool to analyze scalar iterative processes.

In summary, w reaches w_{lim} as solely determined by n. Presently, g_K needs $\{g,g\prime,g\prime\prime,g\prime\prime\prime\}$ and n. If preferred, $g_K\prime$ may be estimated numerically at the expense of an extra g per iteration [5]. Alternatively [5], g'_K may be replaced by the slope of a secant when $k \geq 2$. Note that this option envelops previously introduced piecewise linearization techniques with memory, namely g_A and g_W. The original formulations of these forego the beneficial contribution of $g\prime(z)$ since they harness $w = 1$. However, hinging w to n as described above should improve both of them. The requirement of $g\prime(z)$ is a handicap only when $n = 1$ for $g\prime(z) = 0$ when $n \geq 2$.

Table 1.g_K provides a fast and smooth flight to z from a point where others fail.

k	1	2	3	4
x	2.000000000000	7.950162903588	10.003627135093	10.000000001908
g	156250	39.604436076333	9.978264790570	9.999999988554
g_N	22323	12.472201928266	10.000003943018	10.000000000000
g_C	-747327804	4.755817803051	10.000000006190	10.000000000000
g_H	2.666646755895	9.621034770843	10.000000001908	10.000000000000
g_O	11162	10.431874480623	10.000000000002	10.000000000000
w_{id}	0.041652	0.353391	0.500242	0.000000
		Inner loop iterations		
1^{st} w	0.500	0.500	0.500	0.500
w_{h1}	1.167	0.920	0.499	0.500
w_{h2}	1.833	1.341	0.499	0.500
g_K	2.666646755895	9.621034770843	**10.000000001908**	**10.000000000000**
2^{nd} w	0.250	0.250		
w_{h1}	0.667	0.172		
w_{h2}	1.083	-0.269		
g_K	3.333253692267	10.390290953486		
3^{rd} w	0.125	0.339		
w_{h1}	0.417	0.460		
w_{h2}	0.708	0.408		
g_K	4.666348122867	10.047938453298		
4^{th} w	0.056	0.352		
w_{h1}	0.278	0.501		
w_{h2}	0.498	0.499		
g_K	**7.950162903588**	**10.003627135093**		

4. Conclusions

Old g_K needs g, $g\prime$, $g\prime(z)$ and n. The previous two versions use constant w throughout. The forerunner g_K sets w to $1/2$ for any g. The second version couples w to the convergence order n. If $n = 1$, then $w = 1/2$ is used and g_K is of third order. If $n > 1$, then $w = 1/n$ is harnessed and g_K is of $(n+1)$th order. The notion in the third version is to get additional higher derivatives $\{g\prime\prime, g\prime\prime\prime, ...\}$ at each step and deploy them within an embedded loop to fix $\{w, w\prime, w\prime\prime, ...\}$ at x such that $\{g_K\prime, g_K\prime\prime, ...\}$ are zero and the projected w at $x = g_K$ tends to its *(constant)* value in the second version. This action forces g_K towards the ideal solver $g = z$. The resultant g_K is of the same order as in the second version but the move to z is now a fast and smooth flight even from a remote starting point where other solvers fail.

A highly oscillatory and divergent first-order case demonstrated the super performance of $g_K$with variable w. The rewards of utilizing a variable w amply compensate the cost of the extra derivative information. It seems sufficient to get $\{g\prime\prime$, $g\prime\prime\prime\}$ only, iteratively determine the appropriate set $\{w, w\prime, w\prime\prime\}$ at x which zeroes $g_K\prime$ and $g_K\prime\prime$ simultaneously and projects w to its expected limit at g_K.

Koçak's accelerator g_K has been upgraded with great success; it is now faster, smoother, and more robust. As implemented, g_K is a powerful one-point solver without memory. Ramifications are possible. For instance, just two iterations with g_K may be sufficient to reach a safe point from which other solvers may take over. If higher derivatives are difficult or expensive to calculate, then they may be replaced by finite difference formulae leading to one-point solvers with memory. Extension to multivariable solvers is another opening to investigate in the future. It is clear that Koçak's method provides a super tool for numerical analysis.

References

[1] Atkinson, K.E., *An introduction to numerical analysis*, John Wiley and Sons, New York, 1978.

[2] Fausette, L.V. Numerical methods: algorithms and applications, Prentice-Hall, New Jersey, 2003.

[3] Franks, R.G.E., *Modeling and simulation in chemical engineering*, John Wiley Interscience, New York, 1972.

[4] Fröberg, C-E., *Introduction to numerical analysis,* (Second ed.) Addison-Wesley Publishing Co., Reading, 1972.

[5] Koçak, M.Ç., *Simple geometry facilitates iterative solution of a nonlinear equation via a special transformation to accelerate convergence to third order,* The Proceedings of the Twelfth International Congress on Computational and Applied Mathematics (ICCAM2006), 10-14 July 2006, Leuven, Belgium. Goovaerts, M.J., Vandewalle, S., Van Daele, M., Wuytack, L. (eds) J. Comput. Appl. Math. Vol:218, (2008),350-363.

[6] Koçak, M.Ç., *Acceleration of iterative methods*, in Reports of the Third Congress of the World Mathematical Society of Turkic Countries, Almaty, June 30-July 4, 2009, Kazakhstan (B.T. Zhumagulov, Ed.), ISBN 978-601-240-063-2 (2009)

[7] Koçak, M.Ç., *Second derivative of an iterative solver boosts its acceleration by Koçak's method*, AMC, Vol: 218, No.3 (2011), 893-898.

[8] Quarteroni, A.F. , Sacco, R., Saleri, F., *Numerical mathematics,* Springer-Verlag, New York, 2000.

[9] Traub, J.F., *Iterative methods for solution of equations*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

5. Appendix

Koçak's acceleration method relies on the fixed-point preserving transformation $g_K = g_{ps} = x + G(g - x) = (g - mx)/(1 - m) = g_{pl},\ m = 1 - 1/G,\ G = 1/(1 - m)$ where G is a gain and m is the slope of a straight line joining g and $g = x$ line. Consider the link between $g_{id} = z$ and g_K . Regardless of the multiplicity of z [5], the ideal slope for linearization at the kth iteration is $m_{id} = (g_k - z)/(x_k - z) = \varepsilon_{k+1}/\varepsilon_k$ where $\varepsilon_k = x_k - z$. On the other hand, according to the mean value theorem for derivatives [5],

$$m_{id} = \frac{g - z}{x - z} = \frac{g - g\left(z\right)}{x - z} = g'\left(\xi\right), \quad \xi \in (x, z)$$

Note that this does not necessarily mean that $g'(\xi) \in (g'(x), g'(z))$.

As previously published [5-7], if m is adjusted such that

$$m^{(i-1)}\left(z\right) = g^{(i)}\left(z\right)/i,\ \ i = 1, 2, ...n_K{-}1\,, \quad n_K \geq 2, \qquad m^{(0)}\left(z\right)\ = m\left(z\right) = g^{'}\ \left(z\right),$$

then $g_K^{(i)}\left(z\right) = 0,\ \ i = 1, 2, ...n_K - 1$ and g_K achieves an integral convergence order $n_K \geq 2$. It is easy to see that a weighted average $m = wg' + (1 - w)g'(z) = g'(z) + w(g' - g'(z))$ fulfills the minimal requirement for $n_K \geq 2$, that is $\lim_{x \to z} m = g'(z)$. In this case, symbolic computations show that

$$m_{id} - m = \sum_{i} \left(\frac{1}{(i+1)!} - \frac{w}{i!} \right) g^{(i+1)}\left(z\right) \varepsilon_k^i. \tag{5.1}$$

Suppose that $n = 1$. The use of $w = 1/2$ makes $m = (g' + g'(z))$, annihilates the coefficient in the first summand here, and leads to the results

$$n_K = 3, \quad c_K = \frac{g_K'''\left(z\right)}{3!}, \quad g_K'''\left(z\right) = -0.5\frac{g'''\left(z\right)}{1 - g'\left(z\right)}$$

The improvement is remarkable since the convergence order is raised from $n = 1$ to $n_K = 3$. Amelioration is even better if $g'(z) < 0$ or $g'(z) > 2$.

If $n = 2$, then $g'(z) = 0$ and so $w = 1/2$ renders

$$n_K = 3, \quad c_K = \frac{g_K'''\left(z\right)}{3!}, \quad g_K'''\left(z\right) = -0.5g'''\left(z\right).$$

If $n = 3$, then $g''(z) = 0$ and the first summand above is already zero and the second summand can be annihilated by choosing to employ $w = 1/3$ which makes $n_K = 4$. Similar reasoning leads to the deduction that if $n \geq 2$, then $w = 1/n$ makes $n_K = n + 1$. In fact, symbolic computations reveal [6] that if $n \geq 2$, then

$$g_K^{(i)}\left(z\right) = g^{(i)}\left(z\right)\left(1 - wi\right), \quad i = 3, 4, ...$$

This equation not only corroborates that $w = 1/2$ gives a $g_K \in I_3$ when n was 1 or 2, but also shows if $w = 1/2$ is used when $n > 2$, then

$$n_K = n, \quad g_K^{(n)}\left(z\right) = (1 - \frac{n}{2})\ g^{(n)}\left(z\right), \quad c_K = \frac{g_K^{(n)}\left(z\right)}{n!} = (1 - \frac{n}{2})\ c. \tag{5.2}$$

So, it is unwise to harness $w = 1/2$ when $n > 3$. If $w = 1/n$ instead of $w = 1/2$, then

$$n_K = n + 1, \quad c_K = \frac{g_K^{(n+1}(z)}{(n+1)!}, \quad g_K^{(n+1)}(z) = -\frac{g^{(n+1)}(z)}{n!}. \tag{5.3}$$

Clearly, (5.3) is better than (5.2) when $n > 3$ since it increases n_K by 1 and greatly diminishes c_K.

The forerunner of g_K [5] employs constant $w = 1/2$ irrespective of n. The second version is identical with the first when n was 1 or 2. They differ when $n > 2$ since the second version begins to employ $w = 1/n$ (instead of $w = 1/2$) thereby improving both n_K and c_K. *Variable* w is the distinguishing feature of the third version of g_K. It locally tunes w so as to annihilate derivatives of g_K subject to the condition that w tend to its value in the second version as x approaches z. This means that the remote behavior is greatly improved but both n_K and c_K remain unchanged.

An ideal w, symbolized by w_{id}, annihilates the sum on the right-hand side of (5.1) and renders $m = m_{id}$. The definition of m_{id} directly leads to $w_{id} = (m_{id} - g'(z)/(g' - g'(z))$. Therefore, with the help of Hospital's rule [6],

$$w_{\lim} = \lim_{x \to z} w_{id} = \begin{cases} 1/2, & n = 1 \\ 1/n, & n > 1 \end{cases}$$

This fact can be coupled to a simple problem with a known z to estimate integer convergence orders if $n \geq 2$. In the neighborhood of z:

$$n = \frac{1}{w_{\lim}}, \quad w_{\lim} \approx \frac{m_{id} - g'(z)}{g' - g'(z)}, \quad m_{id} = \frac{g - z}{x - z}.$$

6. Abbreviations

c	Asymptotic error constant
c_K	Asymptotic error constant of Koçak's solver
f	Nonlinear function to be solved
f_w	Discrepancy function, $f_w = w_z - w_{\lim}$
g	Nonlinear solver
g_A	Aitken's solver (accelerator)
g_{id}	Ideal solver, $g_{id} = z$
g_C	Chebyshev's third-order solver
g_H	Halley's third-order solver
g_K	Koçak's solver
g_N	Newton's solver
g_{Nps}	Newton's solver with partial substitution
g_{Nr}	Newton's solver for repeated roots
g_O	Ostrowski's fourth-order solver
g_{pl}	Piecewise linearization
g_{ps}	Partial substitution
g_W	Wegstein's solver (accelerator)
k	iteration counter
L	Logarithmic degree of convexity, $L = ff''/f'^2$
m	Linearization slope
m_{id}	Ideal linearization slope
n	Convergence order
n_K	Convergence order
r	Multiplicity of z
x	(Default) independent variable
w	Weight for derivative
w_h	Weight projected to g_K
$w_{\lim}$	Limit for w at z
z	Fixed-point
ε_k	Eror at the kth iteration, $\varepsilon_k = x_k - z$

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 22–34 (2014) ©KJM

SOME RESULTS ON THE SENSITIVITY OF SCHUR STABILITY OF LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract. In this work, new results on the sensitivity problem of the Schur stability of linear difference equation systems with constant coefficients and scalar-linear difference equations with order k are obtained and some examples illustrating the efficiency of the theorems are given.

1. Introduction

In this article we consider the following linear system of difference equations with constant coefficients:

$$x(n+1) = Ax(n), \qquad n \in \mathbb{Z}. \tag{1.1}$$

where A is a matrix of dimensions $N \times N$. The asymptotic stability of the system (1.1) is equivalent to the asymptotic stability of the coefficient matrix A. It is well-known that with respect to Lyapunov, a matrix A is discrete-asymptotically stable if and only if the discrete-Lyapunov matrix equations $A^*XA-X+C=0$, $C=C^*>0$ has a solution matrix X which is positive definite matrix, i.e. $X=X^*>0$. Moreover, this solution given by $X=\sum_{k=0}^{\infty}(A^*)^k CA^k$ and also according to the spectral criteria, a matrix A is discrete-asymptotically stable if and only if the eigenvalues of the coefficient matrix A lay in the unit disc, i.e. $|\lambda_i(A)|<1$ for all $i=1,2,\ldots,N$, where λ_i $(i=1,2,\ldots,N)$ stands for the eigenvalues of the coefficient matrix A [1, 2, 3, 4]. Such systems are also called as Schur stable [5, 6, 7]. Throughout the study, we focus our attention to the concept of Schur stability.

In the literature, some restrictions on the perturbation matrix B are assumed to study the Schur stability of the following system

$$y(n+1) = (A+B)y(n), \qquad n \in \mathbb{Z}, \tag{1.2}$$

where A is the coefficient matrix of the Schur stable system (1.1). So called continuation are used to study the sensitivity of the ω^*- Schur stability and the Schur stabilitiy of the system (1.1) [1, 2, 4, 8].

2000 *Mathematics Subject Classification.* 39A11.
Key words and phrases. Schur stability, difference equations, sensitivity, perturbation systems.

In this work, some results on the sensitivity of the Schur stability and the ω^*- Schur stability of the difference equation system (1.1) were presented. We have also applied the results to the delay difference equations.

2. Sensitivity of Systems

In this section, we give some results in the literature on the sensitivity of the Schur stability of the systems with constant coefficients.

Let's start with the parameter $\omega(A)$ that shows the quality of Schur stability of the system (1.1) and holds and an important place in the theory of stability.

Schur stability parameter $\omega(A)$ is defined as follows:

$$\omega(A) = ||H||; \quad H = \sum_{k=0}^{\infty} (A^*)^k A^k, \quad H = H^* > 0, \quad A^*HA - H + I = 0$$

where I is unit matrix, A^* is adjoint of the matrix A, $||A|| = \max_{\|x\|=1} \|Ax\|$ is the spectral norm of the matrix A, furthermore the norm $||x||$ is Euclidean norm for the vector $x = (x_1, x_2, \ldots, x_N)^T$. Linear difference system (1.1) is Schur stable if and only if $\omega(A) < \infty$ holds and so it is clear that the perturbed linear difference system (1.2) is Schur stable if and only if $\omega(A+B) = ||\widetilde{H}|| < \infty$ holds, where the matrix $\widetilde{H} = \sum_{k=0}^{\infty} (A^* + B^*)^k (A+B)^k$ is positive definite solution of the discrete-Lyapunov matrix equation $(A^* + B^*)\widetilde{H}(A+B) - \widetilde{H} + I = 0$. Moreover, let ω^* be the practical Schur stability parameter of the system (1.1), then the matrix A is called as practically Schur stable (ω^*- Schur stable) provided that $\omega^* > 1$ and $\omega(A) \leq \omega^*$ hold. If $\omega(A) > \omega^*$ holds, then the matrix A is called as ω^*- Schur unstable matrix [1, 3, 9].

Theorem 2.1 ([3])**.** *Let A be a Schur stable matrix ($\omega(A) < \infty$). If $\|B\| \leq \frac{1}{6\pi\omega(A)}$ then the matrix $A+B$ is Schur stable. Moreover, if $(2\|B\|\|A\| + \|B\|^2)\omega(A) < 0.5$ then the inequality*

$$|\omega(A+B) - \omega(A)| \leq 2\omega^2(A)(2\|A\| + \|B\|)\|B\|,$$

holds.

Corollary 2.1 ([10])**.** *Suppose that A is a Schur stable matrix, that is $\omega(A) < \infty$. If the matrix B satisfies $\|B\| < \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|$, then $A+B$ is Schur stable. Moreover, the inequality*

$$|\omega(A+B) - \omega(A)| \leq \frac{(2\|A\| + \|B\|)\|B\|\omega^2(A)}{1 - (2\|A\| + \|B\|)\|B\|\omega(A)},$$

holds.

Now, considering Theorem 2.1 and Corollary 2.1 we give the continuity theorem which allows the greater perturbe than others without disturbing the Schur stability for the linear difference equation systems with constant coefficients.

Theorem 2.2 ([10])**.** *Suppose that A is a Schur stable matrix, that is $\omega(A) < \infty$. If the matrix B satisfies $\|B\| < \gamma$, then $A+B$ is Schur stable. Moreover, if*

$\|B\| < \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|$, *then the following inequalities*

$$\omega(A+B) \leq \frac{\omega(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)}, \quad |\omega(A+B) - \omega(A)| \leq \frac{(2\|A\|+\|B\|)\|B\|\omega^2(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)}$$

holds, where $\gamma = \max\left\{\frac{1}{6\pi\omega(A)}, \ \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|\right\}$.

Corollary 2.2 ([10])**.** *Let* $\|A\| < 1$. *If the matrix* B *satisfies* $\|A\| + \|B\| < 1$ *then the matrix* $A+B$ *is Schur stable. Moreover, the following inequalities*

$$\omega(A+B) \leq \frac{1}{1-(\|A\|+\|B\|)^2}, \quad |\omega(A+B) - \omega(A)| \leq \frac{\|B\|}{1-\|A\|}\frac{1}{1-(\|A\|+\|B\|)^2}$$

holds.

Theorem 2.3 ([10])**.** *Let* A *be a* ω^*- *Schur stable matrix* $(\omega(A) \leq \omega^*)$. *If the matrix* B *satisfies* $\|B\| \leq \sqrt{\|A\|^2 + \frac{\omega^*-\omega(A)}{\omega^*\omega(A)}} - \|A\|$ *then* $A+B$ *is* ω^*- *Schur stable.*

Corollary 2.3. *Let* $\|A\| < 1$ *and* A *be a* ω^*- *Schur stable matrix* $(\omega(A) \leq \omega^*)$. *If the matrix* B *satisfies* $\|A\| + \|B\| < 1$ *and* $\|B\| \leq \sqrt{\frac{\omega^*-1}{\omega^*}} - \|A\|$ *then the matrix* $A+B$ *is* ω^*- *Schur stable.*

Proof. Let $\|A\| < 1$ and A be a ω^*- Schur stable. $\|A\| + \|B\| < 1$ and $\|B\| \leq \sqrt{\frac{\omega^*-1}{\omega^*}} - \|A\|$ are satisfied. From the second inequality

$$\begin{aligned} &\Longrightarrow \|A\| + \|B\| \leq \sqrt{\tfrac{\omega^*-1}{\omega^*}} \\ &\Longrightarrow (\|A\| + \|B\|)^2 \leq \tfrac{\omega^*-1}{\omega^*} \\ &\Longrightarrow 1 \leq \omega^*\left(1 - (\|A\| + \|B\|)^2\right) \end{aligned}$$

and therefore the inequality

$$\frac{1}{1 - (\|A\| + \|B\|)^2} \leq \omega^*$$

is obtained. Since $\omega(A+B) \leq \frac{1}{1-(\|A\|+\|B\|)^2}$ is valid from Corollary 2.2, the inequality $\omega(A+B) \leq \omega^*$ is found. This completes the proof. ☐

3. Some Results on the Sensitivity of Scalar-Linear Difference Equations with Order k

Consider the following scalar-linear difference equations with order k

$$x(n+1) - a_0x(n) - a_1x(n-1) - ... - a_{k-1}x(n-k+1) = 0, \qquad n \geq 0. \tag{3.1}$$

The equation (3.1) can be written as

$$y(n+1) = Cy(n), \qquad n \geq 0 \tag{3.2}$$

in matrix-vector form where the matrix C is companion matrix as follows

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_0 \end{pmatrix}, \qquad c = (a_{k-1}, a_{k-2}, \ldots, a_0).$$

Consider the perturbation of the equation (3.1) and so, of the system (3.2)

$$z(n+1) = (C+D)\, z(n), \qquad n \geq 0, \tag{3.3}$$

and the set B_γ called as the nD$-$ball, i.e. the $n-$dimensional ball [11], where

$$D = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ d_{k-1} & d_{k-2} & d_{k-3} & \cdots & d_0 \end{pmatrix}, \quad B_\gamma = \{x = (x_1, x_2, \ldots, x_n) | \ \|x\| < \gamma\}.$$

Let

- $d = (d_{k-1}, d_{k-2}, \ldots, d_0)$,
- $\gamma(C) = \max\left\{\frac{1}{6\pi\omega(C)}, \ \sqrt{\|C\|^2 + \frac{1}{\omega(C)}} - \|C\|\right\}$,
- $\delta_3^*(C) = \sqrt{\|C\|^2 + \frac{\omega^* - \omega(C)}{\omega^*\omega(C)}} - \|C\|$.

Theorem 3.1 ([10])**.** *Let the system (3.2) be a Schur stable (the companian matrix C is Schur stable). If the $k-$tuple $d \in B_{\gamma(C)}$, then the perturbed system (3.3) is a Schur stable.*

Theorem 3.2 ([10])**.** *Let the system (3.2) be ω^*-Schur stable (the companian matrix C is ω^*-Schur stable). If the $k-$tuple $d \in B_{\delta_3^*(C)}$, then the perturbed system (3.3) is also ω^*-Schur stable.*

Theorem 3.3. $\lim_{\omega\to\infty} B_{\gamma(C)} = \varnothing$.

Proof. The equality $\lim_{\omega\to\infty} \gamma(C) = 0$ holds, where $\lim_{\omega\to\infty} \frac{1}{6\pi\omega(C)} = 0$ and $\lim_{\omega\to\infty} \sqrt{\|C\|^2 + \frac{1}{\omega(C)}} - \|C\| = 0$.
Thus $\lim_{\omega\to\infty} \{x = (x_1, x_2, \ldots, x_n) | \, \|x\| < \gamma(C)\} = \varnothing$. □

Theorem 3.4. *The set sequence $\left\{B_{\gamma(C)}\right\}$ is increasing according to $\gamma(C)$.*

Proof. Let $x(n+1) = C_1 x(n)$ and $y(n+1) = C_2 y(n)$. If $\gamma_2(C) < \gamma_1(C)$ then the inequality $B_{\gamma_2(C)} \subset B_{\gamma_1(C)}$ holds. □

Theorem 3.5. *The set sequence $\left\{B_{\delta_3^*}\right\}$*
a) *is an increasing sequence the according to ω^*,*
b) *is a bounded sequence.*

Proof. **a)** $\omega_1^*, \ \omega_2^* \ (\omega_1^* < \omega_2^*)$ are practical Schur stability parameters of the system $y(n+1) = Cy(n)$. $\frac{1}{\omega(C)} - \frac{1}{\omega_1^*} = \frac{\omega_1^* - \omega(C)}{\omega_1^*\omega(C)} < \frac{1}{\omega(C)} - \frac{1}{\omega_2^*} = \frac{\omega_2^* - \omega(C)}{\omega_2^*\omega(C)}$ for $\omega_1^* < \omega_2^*$. Therefore $B_{\delta_{31}^*} \subset B_{\delta_{32}^*}$ for $\delta_{31}^* = \sqrt{\|C\|^2 + \frac{\omega_1^* - \omega(C)}{\omega_1^*\omega(C)}} - \|C\| < \delta_{32}^* = \sqrt{\|C\|^2 + \frac{\omega_2^* - \omega(C)}{\omega_2^*\omega(C)}} - \|C\|$.
b) $\varnothing \subset \left\{B_{\delta_3^*}\right\} \subset \{B_\alpha\}$ for $0 < \delta_3^* = \sqrt{\|C\|^2 + \frac{\omega^* - \omega(C)}{\omega^*\omega(C)}} - \|C\| < \alpha = \sqrt{\|C\|^2 + \frac{1}{\omega(C)}} - \|C\|$. □

Theorem 3.6. $\lim_{\omega\to\omega^*} B_{\delta_3^*} = \varnothing$.

Proof. Since $\lim_{\omega\to\omega^*} B_{\delta_3^*} = \varnothing$ for $\lim_{\omega\to\omega^*} \delta_3^* = \lim_{\omega\to\omega^*} \sqrt{\|C\|^2 + \frac{\omega^* - \omega(C)}{\omega^*\omega(C)}} - \|C\| = 0$ the proof is obtained. □

Theorem 3.7. $\lim_{\omega^*\to\infty} B_{\delta_3^*} = B_\alpha$ *where* $\alpha = \sqrt{\|C\|^2 + \frac{1}{\omega(C)}} - \|C\|$.

Proof. $\lim_{\omega^*\to\infty} B_{\delta_3^*} = B_\alpha$ for $\lim_{\omega^*\to\infty} \frac{\omega^*-\omega(C)}{\omega^*\omega(C)} = \frac{1}{\omega(C)}$ so the proof is completed. □

4. Numerical Examples

Example 4.1. $A_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0.5 & 9 \\ 0 & 0.1 \end{pmatrix}$.

- $\|A_1\| = 0.5$, $\omega(A_1) = \frac{4}{3}$

We have calculated $\delta^* = \sqrt{\frac{\omega^*-1}{\omega^*}} - \|A\| = 0.494987$ and $\alpha^* = \sqrt{\|A\|^2 + \frac{\omega^*-\omega(A)}{\omega^*\omega(A)}} - \|A\| = 0.494987$ for $\omega^* = 100$. Let us the perturbation matrix $B = \begin{pmatrix} 0.494987 & 0 \\ 0 & 0.494987 \end{pmatrix}$ for $\delta^* = \alpha^* = 0.494987$. We have $A_1 + B$, thus we see that $\omega(A_1 + B) = 99.9913 < 100$ holds, therefore the matrix $A_1 + B$ is 100−Schur stable matrix.

- $\|A_2\| = 9.01443$, $\omega(A_2) = 121.915$

For $\omega^* = 125$, $\alpha^* = 1.12284e - 05$ and Corollary 2.3 fail to apply, therefore the values δ^* cannot be calculated. Suitable perturbation matrices for these values may be selected as $B = \begin{pmatrix} 1.12284e-05 & 0 \\ 0 & 1.12284e-05 \end{pmatrix}$. Hence we obtain $\omega(A_2 + B) = 121.919 < 125$.

Example 4.2. Consider the delay difference equations $x_{n+1} - x_n = -\frac{21}{100}x_{n-1} - \frac{1}{100}x_{n-2}$, $x_{n+1} + x_n = -\frac{1}{10}x_{n-1} - \frac{1}{100}x_{n-2}$ and $x_{n+1} - x_n = -\frac{1}{2}x_{n-1} - \frac{3}{10}x_{n-2}$. The companion matrices $C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{100} & -\frac{21}{100} & 1 \end{pmatrix}$, $C_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{100} & -\frac{1}{10} & -1 \end{pmatrix}$ and $C_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{3}{10} & -\frac{1}{2} & 1 \end{pmatrix}$.
It is easy to check that $\omega(C_1) = 10.0889$, $\omega(C_2) = 16.4554$, $\omega(C_3) = 33.3264$ and $B_{\gamma_1} = \{(d_2, d_1, 0) |\, \|d\| < 0.0342529\}$, $B_{\gamma_2} = \{(d_2, d_1, 0) |\ \|d\| < 0.0212722\}$, $B_{\gamma_3} = \{(d_2, d_1, 0) |\ \|d\| < 0.0098588\}$. As is clearly seen from Figure 1, $B_{\gamma_3} \subset B_{\gamma_2} \subset B_{\gamma_1}$. Therefore Theorem 3.4 is satisfied.

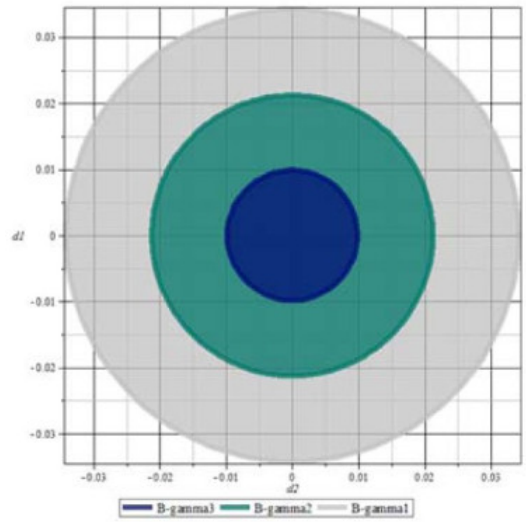

Figure 1. The regions of B_{γ_1}, B_{γ_2} and B_{γ_3}

Since $\omega(C_1) < \infty$ the equation (4) is Schur stable. Let $\omega_1^* = 15$, $\omega_2^* = 60$. It is easy to check that $B_{\delta_{31}^*} = \{(d_2, d_1, 0) | \, \|d\| < 0.0113043\}$, $B_{\delta_{32}^*} = \{(d_2, d_1, 0) | \ \|d\| < 0.0285496\}$ and it is clear that $\omega_1^* < \omega_2^* \Rightarrow B_{\delta_{31}^*} \subset B_{\delta_{32}^*}$. Therefore a option of Theorem 3.5 is satisfied.

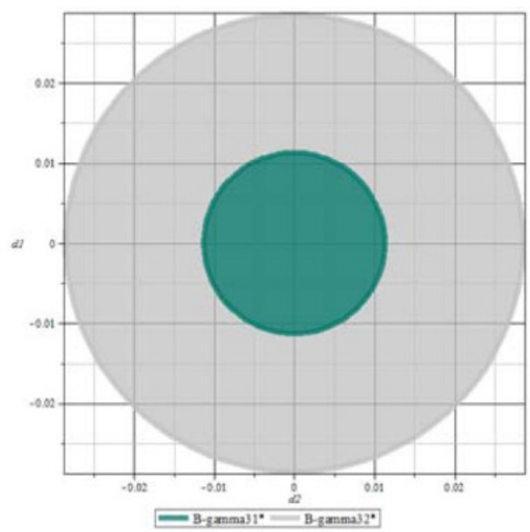

Figure 2. The regions of $B_{\delta_{31}^*}$, and $B_{\delta_{32}^*}$

Remark 4.1. The numerical examples have been computed by using matrix vector calculator MVC [12].

References

[1] Akın Ö. and Bulgak H., *Linear difference equations and stability theory.* Konya, Selçuk University, Research Center of Applied Mathematics, 1998. (in Turkish)

[2] Godunov S.K., *Modern aspects of linear algebra.* RI, American Mathematical Society, Translation of Mathematical Monographs 175. Providence, 1998.

[3] Bulgak H., Pseudoeigenvalues, spectral portrait of a matrix and their connections with different criteria of stability, *Error Control and Adaptivity in Scientific Computing, NATO Science Series, Series C: Mathematical and Physical Sciences*, in: H. Bulgak and C. Zenger(Eds), Kluwer Academic Publishers, 536, (1999), 95-124.

[4] Elaydi S.N., *An introduction to difference equations.* New York, Springer-Verlag, 1999.

[5] Wang KN., Michel AN., On sufficient conditions for the stability of interval matrices, Syst. Control Lett, 20 (5), (1993), 345-351.

[6] Rohn J., Positive definiteness and stability of interval matrices, *Siam Journal on Matrix Analysis and Applications*, 15 (1), (1994), 175-184.

[7] Voicu M. and Pastravanu O., Generalized matrix diagonal stability and linear dynamical systems, *Linear Algebra and its Applications*, 419, (2006), 299-310.

[8] Van Loan C., How near is a stable matrix to unstable matrix?, *Linear Algebra and Its Role in Systems Theory*, (1984), 465-478.

[9] A.Ya. Bulgakov (H. Bulgak), An effectively calculable parameter for the stability quality of systems of linear differential equations with constant coefficients, *Siberian Math. J.* 21 (1980), 339–347.

[10] Duman A. and Aydın K., Sensitivitiy of linear difference equation systems with constant coefficients, *Scientific Research and Essays,*6(28), (2011), 5846-5854.

[11] Roger AH., Charles RJ., *Matrix analysis.* Cambridge University Press, Cambridge, 1999.

[12] Bulgak H. and Eminov D., Computer dialogue system MVC, *Selçuk Journal Applied Mathematics*, 2, (2001), 17-38.

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Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 127–136 (2014) ©KJM

SIMILAR RULED SURFACES WITH VARIABLE TRANSFORMATIONS IN THE EUCLIDEAN 3-SPACE E^3

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Abstract. In this study, we define a family of ruled surfaces in the Euclidean 3-space E^3 and called similar ruled surfaces. We obtain some properties of these special surfaces and we show that developable ruled surfaces form a family of similar ruled surfaces if and only if the striction curves of the surfaces are similar curves with variable transformation.

1. Introduction

In local differential geometry, associated curves, the curves for which at the corresponding points of the curves one of the Frenet vectors of a curve coincides with one of the Frenet vectors of other curve, are very interesting and an important problem of the fundamental theory and the characterizations of space curves. The well-known pairs of such curves are Bertrand curves, Mannheim curves and involute-evolute curves [4,9,10]. Recently, a new definition of the associated curves was given by El-Sabbagh and Ali [1]. They have called these new curves as similar curves with variable transformation and defined as follows: Let $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ be two regular curves in E^3 parametrized by arc lengths s_α and s_β with curvatures κ_α, κ_β and torsions τ_α, τ_β and Frenet frames $\left\{\vec{T}_\alpha, \vec{N}_\alpha, \vec{B}_\alpha\right\}$ and $\left\{\vec{T}_\beta, \vec{N}_\beta, \vec{B}_\beta\right\}$, respectively. $\psi_\alpha(s_\alpha)$ and $\psi_\beta(s_\beta)$ are called similar curves with variable transformation λ_β^α if there exists a variable transformation

$$s_\alpha = \int \lambda_\beta^\alpha(s_\beta)ds_\beta,$$

of the arc lengths such that the tangent vectors are the same for two curves i.e., $\vec{T}_\alpha = \vec{T}_\beta$ for all corresponding values of parameters under the transformation λ_β^α. All curves satisfying this condition is called a family of similar curves. Moreover, they have obtained some properties of the family of similar curves.

Analogue to the special curve pairs, the surface pairs, especially ruled surface pairs (called offset surfaces), have an important positions and applications in the

2000 *Mathematics Subject Classification.* 53A25, 14J26.
Key words and phrases. Ruled surface; Frenet frame; similar curves.

study of design problems in spatial mechanisms and physics, kinematics and computer aided design (CAD) [6,7]. So, these surfaces are one of the most important topics of surface theory. In fact, ruled surface offsets are the generalization of the notion of Bertrand curves and Mannheim curves to line geometry and these surface pairs are called Bertrand offsets and Mannheim offsets [3,5,8]. In this work, we consider the notion of similar curves for ruled surfaces. We introduce a family of ruled surfaces in the Euclidean 3-space E^3 and called similar ruled surfaces with variable transformation. We give some theorems characterizing these special surfaces and we show that developable ruled surfaces form a family of similar ruled surfaces if and only if the striction curves of the surfaces are similar curves with variable transformation.

2. Ruled Surfaces in E^3

In this section, we give a brief summary of the theory of ruled surface in E^3. A more detailed information can be obtained in ref. [2].

Let I be an open interval in the real line IR, $\vec{f}=\vec{f}(u)$ be a curve in E^3 defined on I and $\vec{q}=\vec{q}(u)$ be a unit direction vector of an oriented line in E^3. Then we have the following parametrization for a ruled surface N,

$$\vec{r}(u,v)=\vec{f}(u)+v\,\vec{q}(u). \tag{2.1}$$

The curve $\vec{f}=\vec{f}(u)$ is called base curve or generating curve of the surface and various positions of the generating lines $\vec{q}=\vec{q}(u)$ are called rulings. In particular, if the direction of $\vec{q}$ is constant, then the ruled surface is said to be cylindrical, and non-cylindrical otherwise.

The distribution parameter of N is given by

$$d=\frac{\left|\dot{\vec{f}},\vec{q},\dot{\vec{q}}\right|}{\left\langle \dot{\vec{q}},\dot{\vec{q}}\right\rangle}, \tag{2.2}$$

where $\dot{\vec{f}}=\frac{d\vec{f}}{du}$, $\dot{\vec{q}}=\frac{d\vec{q}}{du}$. If $\left|\dot{\vec{f}},\vec{q},\dot{\vec{q}}\right|=0$, then normal vectors are collinear at all points of same ruling and at nonsingular points of the surface N, the tangent planes are identical. We then say that tangent plane contacts the surface along a ruling. Such a ruling is called a torsal ruling. If $\left|\dot{\vec{f}},\vec{q},\dot{\vec{q}}\right|\neq 0$, then the tangent planes of the surface N are distinct at all points of same ruling which is called nontorsal.

Definition 2.1. ([2]) A ruled surface whose all rulings are torsal is called a developable ruled surface. The remaining ruled surfaces are called skew ruled surfaces. From (2.2) it is clear that a ruled surface is developable if and only if at all its points the distribution parameter is zero.

For the unit normal vector $\vec{m}$ of a ruled surface N we have

$$\vec{m}=\frac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}=\frac{(\dot{\vec{f}}+v\dot{\vec{q}})\times\vec{q}}{\sqrt{\left\langle \dot{\vec{f}}+v\dot{\vec{q}},\ \dot{\vec{f}}+v\dot{\vec{q}}\right\rangle-\left\langle \dot{\vec{f}},\vec{q}\right\rangle^2}}. \tag{2.3}$$

The unit normal of the surface along a ruling $u = u_1$ approaches a limiting direction as v infinitely decreases. This direction is called the asymptotic normal (central tangent) direction and from (2.3) defined by

$$\vec{a} = \lim_{v \to \pm\infty} \vec{m}(u_1, v) = \frac{\vec{q} \times \dot{\vec{q}}}{\left\|\dot{\vec{q}}\right\|}.$$

The point at which the unit normal of N is perpendicular to $\vec{a}$ is called the striction point (or central point) C and the set of striction points on all rulings is called striction curve of the surface. The parametrization of the striction curve $\vec{c} = \vec{c}(u)$ on a ruled surface is given by

$$\vec{c}(u) = \vec{f}(u) + v_0\vec{q}(u) = \vec{f} - \frac{\left\langle \dot{\vec{q}}, \dot{\vec{f}} \right\rangle}{\left\langle \dot{\vec{q}}, \dot{\vec{q}} \right\rangle}\vec{q}, \tag{2.4}$$

where $v_0 = -\frac{\left\langle \dot{\vec{q}}, \dot{\vec{f}} \right\rangle}{\left\langle \dot{\vec{q}}, \dot{\vec{q}} \right\rangle}$ is called strictional distance.

The vector $\vec{h}$ defined by $\vec{h} = \vec{a} \times \vec{q}$ is called central normal which is the surface normal along the striction curve. Then the orthonormal system $\left\{C;\, \vec{q}, \vec{h}, \vec{a}\right\}$ is called Frenet frame of the ruled surface N where C is the central point of ruling of ruled surface N and $\vec{q}$, $\vec{h} = \vec{a} \times \vec{q}$, $\vec{a}$ are unit vectors of ruling, central normal and central tangent, respectively.

For the derivatives of the vectors of Frenet frame $\left\{C;\, \vec{q}, \vec{h}, \vec{a}\right\}$ of ruled surface N with respect to the arc length s of striction curve we have

$$\begin{bmatrix} d\vec{q}/ds \\ d\vec{h}/ds \\ d\vec{a}/ds \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix} \tag{2.5}$$

where $k_1 = \frac{ds_1}{ds}$, $k_2 = \frac{ds_3}{ds}$ and s_1, s_3 are the arc lengths of the spherical curves circumscribed by the bound vectors $\vec{q}$ and $\vec{a}$, respectively. The ruled surfaces satisfying $k_1 \neq 0$, $k_2 = 0$ are called conoids (For details see [2]).

Now, we can represent and prove the following theorems which are necessary for the following section.

Theorem 2.1. *Let the striction curve $\vec{c} = \vec{c}(s)$ of ruled surface N be unit speed i.e., s is arc length parameter of $\vec{c}(s)$ and let $\vec{c}(s)$ be the base curve of the surface. Then N is developable if and only if the unit tangent of the striction curve is the same with the ruling along the curve.*

Proof. Let s be arc length parameter of the striction curve. Then the unit tangent of the striction curve is given by

$$\vec{T}(s) = \frac{d\vec{c}}{ds} = (\cos\theta)\vec{q}(s) + (\sin\theta)\vec{a}(s),$$

where $\theta = \theta(s)$ is the angle between unit vectors $\vec{T}(s)$ and $\vec{q}(s)$. Since the striction curve is base curve, then from (2.2) and (2.5) the distribution parameter of the surface N is obtained as

$$d = \frac{\sin\theta}{k_1}.$$

Thus we have that N is developable if and only if $\sin\theta = 0$, i.e., $\vec{T}(s) = \vec{q}(s)$ satisfies. $\square$

Theorem 2.2. *Let the striction curve $\vec{c} = \vec{c}(s)$ of ruled surface N be unit speed i.e., s is arc length parameter of $\vec{c}(s)$. Suppose that $\vec{c} = \vec{c}(\varphi)$ is another parametrization of striction curve by the parameter $\varphi(s) = \int k_1(s)ds$. Then the ruling $\vec{q}$ satisfies a vector differential equation of third order given by*

$$\frac{d}{d\varphi}\left(\frac{1}{f(\varphi)}\frac{d^2\vec{q}}{d\varphi^2}\right) + \left(\frac{1+f^2(\varphi)}{f(\varphi)}\right)\frac{d\vec{q}}{d\varphi} - \left(\frac{1}{f^2(\varphi)}\frac{df(\varphi)}{d\varphi}\right)\vec{q} = 0, \tag{2.6}$$

where $f(\varphi) = \frac{k_2(\varphi)}{k_1(\varphi)}$.

Proof. If we write derivatives given in (2.5) according to φ, we have

$$\frac{d\vec{q}}{d\varphi} = \frac{d\vec{q}}{ds}\frac{ds}{d\varphi} = (k_1\vec{h})\frac{1}{k_1} = \vec{h},$$

$$\frac{d\vec{h}}{d\varphi} = \frac{d\vec{h}}{ds}\frac{ds}{d\varphi} = (-k_1\vec{q} + k_2\vec{a})\frac{1}{k_1} = -\vec{q} + f(\varphi)\vec{a},$$

$$\frac{d\vec{a}}{d\varphi} = \frac{d\vec{a}}{ds}\frac{ds}{d\varphi} = (-k_2\vec{h})\frac{1}{k_1} = -f(\varphi)\vec{h},$$

respectively, where $f(\varphi) = \frac{k_2(\varphi)}{k_1(\varphi)}$. Then corresponding matrix form of (2.5) can be given

$$\begin{bmatrix} d\vec{q}/d\varphi \\ d\vec{h}/d\varphi \\ d\vec{a}/d\varphi \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & f(\varphi) \\ 0 & -f(\varphi) & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix}. \tag{2.7}$$

From the first and second equations of new Frenet derivatives (2.7) we have

$$\vec{a} = \frac{1}{f(\varphi)}\left(\frac{d^2\vec{q}}{d\varphi^2} + \vec{q}\right). \tag{2.8}$$

Substituting the above equation in the last equation of (2.7) we have desired equation (2.6). $\square$

3. Similar Ruled Surfaces with Variable Transformations

In this section we introduce the definition and characterizations of similar ruled surfaces with variable transformation in E^3. First, we give the following definition.

Definition 3.1. Let N_α and N_β be two ruled surfaces in E^3 given by the parametrizations

$$\begin{cases} \vec{r}_\alpha(s_\alpha, v) = \vec{\alpha}(s_\alpha) + v\,\vec{q}_\alpha(s_\alpha), & \|\vec{q}_\alpha(s_\alpha)\| = 1, \\ \vec{r}_\beta(s_\beta, v) = \vec{\beta}(s_\beta) + v\,\vec{q}_\beta(s_\beta), & \|\vec{q}_\beta(s_\beta)\| = 1, \end{cases} \tag{3.1}$$

respectively, where $\vec{\alpha}(s_\alpha)$ and $\vec{\beta}(s_\beta)$ are striction curves of N_α and N_β and s_α, s_β are arc length parameters of $\vec{\alpha}(s_\alpha)$ and $\vec{\beta}(s_\beta)$, respectively. Let the Frenet frames and invariants of N_α and N_β be $\left\{\vec{q}_\alpha, \vec{h}_\alpha, \vec{a}_\alpha\right\}$, k_1^α, k_2^α and $\left\{\vec{q}_\beta, \vec{h}_\beta, \vec{a}_\beta\right\}$, k_1^β, k_2^β, respectively. Then, N_α and N_β are called similar ruled surfaces with variable transformation λ_β^α if there exists a variable transformation

$$s_\alpha = \int \lambda_\beta^\alpha(s_\beta) ds_\beta, \tag{3.2}$$

of the arc lengths of striction curves such that the rulings are the same for two ruled surfaces i.e.,

$$\vec{q}_\alpha(s_\alpha) = \vec{q}_\beta(s_\beta), \tag{3.3}$$

for all corresponding values of parameters under the transformation λ_β^α. All ruled surfaces satisfying equation (3.3) are called a family of similar ruled surfaces with variable transformation.

Then we can give the following theorems characterizing similar ruled surfaces. Whenever we talk about N_α and N_β we mean that these surfaces have the parametrizations as given in (3.1).

Theorem 3.1. *Let N_α and N_β be two ruled surfaces in E^3. Then N_α and N_β are similar ruled surfaces with variable transformation if and only if the central normal vectors of the surfaces are the same, i.e.,*

$$\vec{h}_\alpha(s_\alpha) = \vec{h}_\beta(s_\beta), \tag{3.4}$$

under the particular variable transformation

$$\lambda_\beta^\alpha = \frac{ds_\alpha}{ds_\beta} = \frac{k_1^\beta}{k_1^\alpha}, \tag{3.5}$$

of the arc lengths.

Proof. Let N_α and N_β be two similar ruled surfaces in E^3 with variable transformation. Then differentiating (3.3) with respect to s_β it follows

$$k_1^\alpha \lambda_\beta^\alpha \vec{h}_\alpha = k_1^\beta \vec{h}_\beta. \tag{3.6}$$

From (3.6) we obtain (3.4) and (3.5) immediately.

Conversely, let N_α and N_β be two ruled surfaces in E^3 satisfying (3.4) and (3.5). By multiplying (3.4) with k_1^β and differentiating the result equality with respect to s_β we have

$$\int k_1^\beta(s_\beta)\vec{h}_\beta(s_\beta) ds_\beta = \int k_1^\beta(s_\beta)\vec{h}_\beta(s_\beta)\frac{ds_\beta}{ds_\alpha} ds_\alpha. \tag{3.7}$$

From (3.4) and (3.5) we obtain

$$\vec{q}_\beta(s_\beta) = \int k_1^\beta(s_\beta)\vec{h}_\beta(s_\beta) ds_\beta = \int k_1^\alpha(s_\alpha)\vec{h}_\alpha(s_\alpha) ds_\alpha = \vec{q}_\alpha(s_\alpha), \tag{3.8}$$

which means that N_α and N_β are similar ruled surfaces with variable transformation. □

Theorem 3.2. *Let N_α and N_β be two ruled surfaces in E^3. Then N_α and N_β are similar ruled surfaces with variable transformation if and only if the asymptotic normal vectors of the surfaces are the same, i.e.,*

$$\vec{a}_\alpha(s_\alpha) = \vec{a}_\beta(s_\beta), \tag{3.9}$$

under the particular variable transformation

$$\lambda_\beta^\alpha = \frac{ds_\alpha}{ds_\beta} = \frac{k_2^\beta}{k_2^\alpha}, \tag{3.10}$$

of the arc lengths.

Proof. Let N_α and N_β be two similar ruled surfaces in E^3 with variable transformation. Then from Definition 3.1 and Theorem 3.1, there exists a variable transformation of the arc lengths such that the rulings and central normal vectors are the same. Then from (3.3) and (3.4) we have

$$\vec{a}_\alpha(s_\alpha) = \vec{q}_\alpha(s_\alpha) \times \vec{h}_\alpha(s_\alpha) = \vec{q}_\beta(s_\beta) \times \vec{h}_\beta(s_\beta) = \vec{a}_\beta(s_\beta). \tag{3.11}$$

Conversely, let N_α and N_β be two ruled surfaces in E^3 satisfying (3.9) and (3.10). By differentiating (3.9) with respect to s_β we obtain

$$k_2^\alpha(s_\alpha)\vec{h}_\alpha(s_\alpha)\frac{ds_\alpha}{ds_\beta} = k_2^\beta(s_\beta)\vec{h}_\beta(s_\beta), \tag{3.12}$$

which gives us

$$\lambda_\beta^\alpha = \frac{k_2^\beta}{k_2^\alpha}, \quad \vec{h}_\alpha(s_\alpha) = \vec{h}_\beta(s_\beta). \tag{3.13}$$

Then from (3.9) and (3.13) we have

$$\vec{q}_\alpha(s_\alpha) = \vec{h}_\alpha(s_\alpha) \times \vec{a}_\alpha(s_\alpha) = \vec{h}_\beta(s_\beta) \times \vec{a}_\beta(s_\beta) = \vec{q}_\beta(s_\beta), \tag{3.14}$$

which completes the proof. □

Theorem 3.3. *Let N_α and N_β be two ruled surfaces in E^3. Then N_α and N_β are similar ruled surfaces with variable transformation if and only if the ratio of curvatures are the same i.e.,*

$$\frac{k_2^\beta(s_\beta)}{k_1^\beta(s_\beta)} = \frac{k_2^\alpha(s_\alpha)}{k_1^\alpha(s_\alpha)}, \tag{3.15}$$

under the particular variable transformation keeping equal total curvatures, i.e.,

$$\varphi_\beta(s_\beta) = \int k_1^\beta(s_\beta)ds_\beta = \int k_1^\alpha(s_\alpha)ds_\alpha = \varphi_\alpha(s_\alpha) \tag{3.16}$$

of the arc lengths.

Proof. Let N_α and N_β be two similar ruled surfaces in E^3 with variable transformation. Then from (3.10) and (3.13), we have (3.15) under the variable transformation (3.16), and this transformation is also leads from (3.10) by integration.

Conversely, let N_α and N_β be two ruled surfaces in E^3 satisfying (3.15) and (3.16). From Theorem 2.2, the rulings $\vec{q}_\alpha$ and $\vec{q}_\beta$ of the surfaces N_α and N_β satisfy the following vector differential equations of third order

$$\frac{d}{d\varphi_\alpha}\left(\frac{1}{f_\alpha(\varphi_\alpha)}\frac{d^2\vec{q}_\alpha}{d\varphi_\alpha^2}\right)+\left(\frac{1+f_\alpha^2(\varphi_\alpha)}{f_\alpha(\varphi_\alpha)}\right)\frac{d\vec{q}_\alpha}{d\varphi_\alpha}-\left(\frac{1}{f_\alpha^2(\varphi_\alpha)}\frac{df_\alpha(\varphi_\alpha)}{d\varphi_\alpha}\right)\vec{q}_\alpha=0, \tag{3.17}$$

$$\frac{d}{d\varphi_\beta}\left(\frac{1}{f_\beta(\varphi_\beta)}\frac{d^2\vec{q}_\beta}{d\varphi_\beta^2}\right)+\left(\frac{1+f_\beta^2(\varphi_\beta)}{f_\beta(\varphi_\beta)}\right)\frac{d\vec{q}_\beta}{d\varphi_\beta}-\left(\frac{1}{f_\beta^2(\varphi_\beta)}\frac{df_\beta(\varphi_\beta)}{d\varphi_\beta}\right)\vec{q}_\beta=0, \tag{3.18}$$

where $f_\alpha(\varphi_\alpha) = \frac{k_2^\alpha(\varphi_\alpha)}{k_1^\alpha(\varphi_\alpha)}$, $f_\beta(\varphi_\beta) = \frac{k_2^\beta(\varphi_\beta)}{k_1^\beta(\varphi_\beta)}$, $\varphi_\alpha(s_\alpha) = \int k_1^\alpha(s_\alpha)ds_\alpha$, $\varphi_\beta(s_\beta) = \int k_1^\beta(s_\beta)ds_\beta$. From (3.15) we have $f_\alpha(\varphi_\alpha) = f_\beta(\varphi_\beta)$ under the variable transformation $\varphi_\alpha = \varphi_\beta$. Thus under the equation (3.15) and transformation (3.16), the equations (3.17) and (3.18) are the same, i.e., they have the same solutions. It means that the rulings $\vec{q}_\alpha$ and $\vec{q}_\beta$ are the same. Then N_α and N_β are two similar ruled surfaces in E^3 with variable transformation. □

Theorem 3.4. *Let the ruled surfaces N_α and N_β be developable. Then N_α and N_β are similar ruled surfaces with variable transformation if and only if the striction curves of the surfaces are similar curves with variable transformation.*

Proof. Let developable ruled surfaces N_α and N_β be two similar ruled surfaces in E^3 with variable transformation. Since the surfaces are developable, from Theorem 2.1 we have

$$\frac{d\vec{\alpha}}{ds_\alpha} = \vec{T}_\alpha(s_\alpha) = \vec{q}_\alpha(s_\alpha), \quad \frac{d\vec{\beta}}{ds_\beta} = \vec{T}_\beta(s_\beta) = \vec{q}_\beta(s_\beta). \tag{3.19}$$

where $\vec{T}_\alpha(s_\alpha)$ and $\vec{T}_\beta(s_\beta)$ are unit tangents of the curves $\vec{\alpha}(s_\alpha)$ and $\vec{\beta}(s_\beta)$, respectively. From (3.3) and (3.19) we have

$$\frac{d\vec{\alpha}}{ds_\alpha} = \vec{q}_\alpha(s_\alpha) = \vec{q}_\beta(s_\beta) = \frac{d\vec{\beta}}{ds_\beta} \tag{3.20}$$

which shows that striction curves $\vec{\alpha}(s_\alpha)$ and $\vec{\beta}(s_\beta)$ are similar curves.

Conversely, if the striction curves $\vec{\alpha}(s_\alpha)$ and $\vec{\beta}(s_\beta)$ are similar curves, then there exists a variable transformation between arc lengths such that

$$\frac{d\vec{\alpha}}{ds_\alpha} = \vec{T}_\alpha(s_\alpha) = \vec{T}_\beta(s_\beta) = \frac{d\vec{\beta}}{ds_\beta}. \tag{3.21}$$

Since the ruled surfaces are developable, from Theorem 2.1 we have $\vec{T}_\alpha(s_\alpha) = \vec{q}_\alpha(s_\alpha)$ and $\vec{T}_\beta(s_\beta) = \vec{q}_\beta(s_\beta)$. Then from (3.21) we have that $\vec{q}_\alpha(s_\alpha) = \vec{q}_\beta(s_\beta)$, i.e., N_α and N_β are similar ruled surfaces with variable transformation.

□

Let now consider some special cases. From (3.5) and (3.10) we have

$$k_1^\beta = \lambda_\beta^\alpha k_1^\alpha, \quad k_2^\beta = \lambda_\beta^\alpha k_2^\alpha, \tag{3.22}$$

respectively. From (3.22) it is clear that if N_α is a cylindrical surface i.e., $k_1^\alpha = 0$, then under the variable transformation the curvature does not change. So we have the following corollary.

Corollary 3.1. *The family of cylindrical surfaces forms a family of similar ruled surfaces with variable transformation.*

If N_α is a conoid surface i.e., $k_2^\alpha = 0$, then under the variable transformation the curvature does not change. So we have the following corollary.

Corollary 3.2. *The family of conoid surfaces forms a family of similar ruled surfaces with variable transformation.*

Example 3.1. Let consider the ruled surface N_β given by the parametrization

$$\begin{aligned}\vec{r}_\beta(s_\beta, v) &= \vec{\beta}(s_\beta) + v\,\vec{q}_\beta(s_\beta)\\ &= (0,0,s_\beta) + v\,(\cos s_\beta,\ \sin s_\beta,\ 0)\end{aligned}$$

which is plotted in Fig. 1. The Frenet vectors of N_β are

$$\begin{aligned}\vec{q}_\beta &= (\cos s_\beta,\ \sin s_\beta,\ 0),\\ \vec{h}_\beta &= (-\sin s_\beta,\ \cos s_\beta,\ 0),\\ \vec{a}_\beta &= (0,0,1),\end{aligned}$$

and curvatures are obtained as $k_1^\beta = 1,\ k_2^\beta = 0$. A similar ruled surfaces of N_β is the surface N_α given by the parametrization

$$\begin{aligned}\vec{r}_\alpha(s_\alpha, v) &= \vec{\alpha}(s_\alpha) + v\,\vec{q}_\alpha(s_\alpha)\\ &= \left(-\sin\tfrac{s_\alpha}{\sqrt{2}},\ \cos\tfrac{s_\alpha}{\sqrt{2}},\ \tfrac{s_\alpha}{\sqrt{2}}\right) + v\left(\cos\tfrac{s_\alpha}{\sqrt{2}},\ \sin\tfrac{s_\alpha}{\sqrt{2}},\ 0\right)\end{aligned}$$

which is plotted in Fig. 2. The Frenet vectors of N_α are

$$\begin{aligned}\vec{q}_\alpha &= \left(\cos\tfrac{s_\alpha}{\sqrt{2}},\ \sin\tfrac{s_\alpha}{\sqrt{2}},\ 0\right),\\ \vec{h}_\alpha &= (-\sin\tfrac{s_\alpha}{\sqrt{2}},\ \cos\tfrac{s_\alpha}{\sqrt{2}},\ 0),\\ \vec{a}_\alpha &= (0,0,1),\end{aligned}$$

and curvatures are obtained as $k_1^\alpha = \frac{1}{\sqrt{2}},\ k_2^\alpha = 0$. From Theorem 3.1 we have the particular variable transformation

$$\lambda_\beta^\alpha = \frac{ds_\alpha}{ds_\beta} = \frac{k_1^\beta}{k_1^\alpha} = \sqrt{2},$$

which means that $s_\alpha = \sqrt{2}s_\beta$.

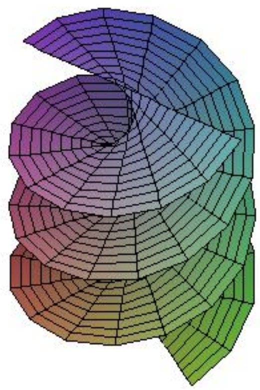

Figure 1. Helicoid surface N_β

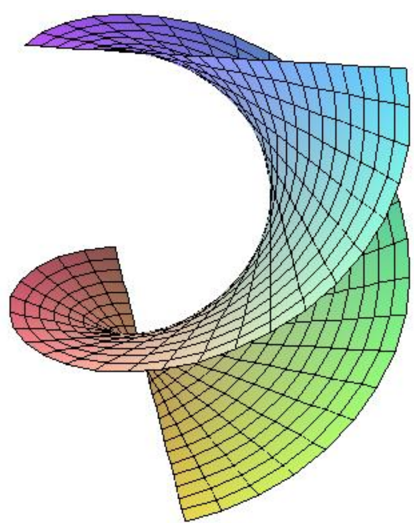

Figure 2. Similar surface N_α of N_β

4. Conclusions

A family of ruled surfaces in the Euclidean 3-space E^3 are defined and called similar ruled surfaces. Some properties of these special surfaces are obtained and it is showed that developable ruled surfaces form a family of similar ruled surfaces if and only if the striction curves of the surfaces are similar curves with variable transformation. By considering the importance of the offset surfaces we hope this paper leads new characterizations of ruled surfaces in different spaces.

References

[1] El-Sabbagh, M.F., Ali, A.T., Similar Curves with Variable Transformations, Konuralp J. of Math., Vol. 1, No. 2 (2013), 80-90.

[2] Karger, A., Novak, J., *Space Kinematics and Lie Groups*, STNL Publishers of Technical Lit., Prague, Czechoslovakia, 1978.

[3] Küçük, A., Gürsoy O., On the invariants of Bertrand trajectory surface offsets, App. Math. and Comp., 151 (2004), 763-773.
[4] Liu, H., Wang, F., Mannheim partner curves in 3-space, Journal of Geometry, Vol. 88, No. 1-2 (2008), 120-126.
[5] Orbay, K., Kasap, E., Aydemir, Î., Mannheim Offsets of Ruled Surfaces, Mathematical Problems in Engineering, Volume 2009, Article ID 160917.
[6] Papaioannou, S.G., Kiritsis, D., An application of Bertrand curves and surfaces to CAD/CAM, Computer Aided Design, Vol. 17, No. 8 (1985), 348-352.
[7] Pottmann, H., Lü, W., Ravani, B., Rational ruled surfaces and their offsets, Graphical Models and Image Processing, Vol. 58, No.6 (1996), 544-552.
[8] Ravani, B., Ku, T.S., Bertrand Offsets of ruled and developable surfaces, Comp. Aided Geom. Design, Vol. 23, No. 2 (1991), 145-152.
[9] Struik, D.J., *Lectures on Classical Differential Geometry*, 2^{nd} ed. Addison Wesley, Dover, 1988.
[10] Wang, F., Liu, H., Mannheim partner curves in 3-Euclidean space, Mathematics in Practice and Theory, Vol. 37, No. 1 (2007), 141-143.

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