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AN L^p HEISENBERG-PAULI-WEYL UNCERTAINTY PRINCIPLE FOR THE DUNKL TRANSFORM

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Abstract. In this paper, we give a generalization of the Heisenberg-Pauli-Weyl uncertainty inequality for the Dunkl transform on $\mathbb{R}^d$ in L^p-norm.

1. Introduction and preliminaries

In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle .,.\rangle$ and norm $|y| := \sqrt{\langle y,y\rangle}$. For $\alpha \in \mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha y := y - \frac{2\langle \alpha, y\rangle}{|\alpha|^2}\alpha.$$

A finite set $\Re \subset \mathbb{R}^d\backslash\{0\}$ is called a root system, if $\Re \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \Re = \Re$ for all $\alpha \in \Re$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in \Re$. For a root system $\Re$, the reflections σ_α, $\alpha \in \Re$, generate a finite group $G \subset O(d)$, the reflection group associated with $\Re$. All reflections in G, correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d\backslash \bigcup_{\alpha\in\Re} H_\alpha$, we fix the positive subsystem $\Re_+ := \{\alpha \in \Re : \langle \alpha, \beta\rangle > 0\}$. Then for each $\alpha \in \Re$ either $\alpha \in \Re_+$ or $-\alpha \in \Re_+$.

Let $k : \Re \to \mathbb{C}$ be a multiplicity function on $\Re$ (that is, a function which is constant on the orbits under the action of G). As an abbreviation, we introduce the index $\gamma = \gamma_k := \sum_{\alpha\in\Re_+} k(\alpha)$.

Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in \Re$. Moreover, let w_k denote the weight function $w_k(y) := \prod_{\alpha\in\Re_+} |\langle \alpha, y\rangle|^{2k(\alpha)}$, for all $y \in \mathbb{R}^d$, which is G-invariant and homogeneous of degree 2γ.

The Dunkl operators $\mathcal{D}_j$; $j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given, for a function f of class C^1 on $\mathbb{R}^d$,

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by

$$\mathcal{D}_j f(y) := \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha\in\Re_+} k(\alpha)\alpha_j \frac{f(y) - f(\sigma_\alpha y)}{\langle \alpha, y\rangle}.$$

For $y \in \mathbb{R}^d$, the initial problem $\mathcal{D}_j u(.,y)(x) = y_j u(x,y)$, $j = 1,...,d$, with $u(0,y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x,y)$ and called Dunkl kernel [4, 7]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$. In our case, $|E_k(-ix,y)| \le 1$, for all $x, y \in \mathbb{R}^d$.

Let c_k be the Mehta-type constant given by $c_k := (\int_{\mathbb{R}^d} e^{-|y|^2/2} w_k(y)\mathrm{d}y)^{-1}$. We denote by μ_k the measure on $\mathbb{R}^d$ given by $\mathrm{d}\mu_k(y) := c_k w_k(y)\mathrm{d}y$; and by $L^p(\mu_k)$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^p(\mu_k)} := \Big(\int_{\mathbb{R}^d} |f(y)|^p \mathrm{d}\mu_k(y)\Big)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|f\|_{L^\infty(\mu_k)} := \operatorname*{ess\,sup}_{y\in\mathbb{R}^d} |f(y)| < \infty.$$

If $f \in L^1(\mu_k)$ with $f(x) = F(|x|)$, then

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_k(x) = \frac{1}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})} \int_0^\infty F(t) t^{2\gamma+d-1}\mathrm{d}t. \tag{1.1}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and was introduced by Dunkl in [5], where already many basic properties were established. Dunkl's results were completed and extended later by de Jeu [7]. The Dunkl transform of a function f in $L^1(\mu_k)$, is

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix,y) f(y)\mathrm{d}\mu_k(y), \quad x \in \mathbb{R}^d.$$

Some of the properties of Dunkl transform $\mathcal{F}_k$ are collected bellow (see [5, 7]).

(a) $L^1 - L^\infty$-boundedness. For all $f \in L^1(\mu_k)$, $\mathcal{F}_k(f) \in L^\infty(\mu_k)$ and

$$\|\mathcal{F}_k(f)\|_{L^\infty(\mu_k)} \le \|f\|_{L^1(\mu_k)}. \tag{1.2}$$

(b) Inversion theorem. Let $f \in L^1(\mu_k)$, such that $\mathcal{F}_k(f) \in L^1(\mu_k)$. Then

$$f(x) = \mathcal{F}_k\Big(\mathcal{F}_k(f)\Big)(-x), \quad \text{a.e.} \quad x \in \mathbb{R}^d.$$

(c) Plancherel theorem. The Dunkl transform $\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2(\mu_k)$ onto itself. In particular,

$$\|f\|_{L^2(\mu_k)} = \|\mathcal{F}_k(f)\|_{L^2(\mu_k)}. \tag{1.3}$$

Using relations (1.2) and (1.3) with Marcinkiewicz's interpolation theorem [10, 11], we deduce that for every $1 \le p \le 2$, and for every $f \in L^p(\mu_k)$, the function $\mathcal{F}_k(f)$ belongs to the space $L^q(\mu_k)$, $q = p/(p-1)$, and

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \le \|f\|_{L^p(\mu_k)}. \tag{1.4}$$

Many uncertainty principles have already been proved for the Dunkl transform, namely by Rösler [8] and Shimeno [9] who established the Heisenberg-Pauli-Weyl inequality for the Dunkl transform, by showing that for every $f \in L^2(\mu_k)$,

$$\|f\|^2_{L^2(\mu_k)} \le \frac{2}{2\gamma+d} \|\,|x|f\|_{L^2(\mu_k)} \|\,|y|\mathcal{F}_k(f)\|_{L^2(\mu_k)}. \tag{1.5}$$

Building on the techniques of Ciatti et al. [1] we show a general form of the Heisenberg-Pauli-Weyl inequality for the Dunkl transform $\mathcal{F}_k$. More precisely, we prove that for all $f \in L^p(\mu_k)$, $1 < p \le 2$, $q = p/(p-1)$ and $0 < a < (2\gamma + d)/q$, $b > 0$,

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \le C(a,b) \| \, |x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+b}} \| \, |y|^b \mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+b}}, \tag{1.6}$$

where $C(a,b)$ is a positive constant. This inequality generalizes the Heisenberg-Pauli-Weyl inequality given by (1.5); and in the case $k = 0$ and $q = 2$, this inequality is due to Cowling-Price [2] and Hirschman [6].

We shall use the Heisenberg-Pauli-Weyl principle (1.6); and building on the techniques of Donoho and Stark [3], we show a continuous-time principle for the L^p theory, when $1 < p \le 2$.

This paper is organized as follows. In Section 2 we list some basic properties of the Dunkl transform $\mathcal{F}_k$. In Section 3 we prove a general form of the Heisenberg-Pauli-Weyl inequality for $\mathcal{F}_k$. The last section is devoted to Donoho-Stark's uncertainty principle for the Dunkl transform $\mathcal{F}_k$ in the L^p theory, when $1 < p \le 2$.

2. L^p Heisenberg-Pauli-Weyl inequality

In this section, we extend the Heisenberg-Pauli-Weyl uncertainty principle (1.5) to more general case. We need to use the method of Ciatti et al. [1], which is the counterpart in the Euclidean case. We begin by the following lemma.

Lemma 2.1. *Let $1 < p \le 2$, $q = p/(p-1)$ and $0 < a < (2\gamma + d)/q$. Then for all $f \in L^p(\mu_k)$ and $t > 0$,*

$$\|e^{-t|y|^2} \mathcal{F}_k(f)\|_{L^q(\mu_k)} \le \Big(1 + \frac{a_k}{(2q)^{(\gamma+\frac{d}{2})\frac{1}{q}}}\Big) t^{-a/2} \| \, |x|^a f\|_{L^p(\mu_k)}, \tag{2.1}$$

where

$$a_k = \left[(2\gamma + d - qa) 2^{\gamma + \frac{d}{2} - 1} \Gamma(\gamma + \frac{d}{2})\right]^{-1/q}.$$

Proof. Inequality (2.1) holds if $\| \, |x|^a f\|_{L^p(\mu_k)} = \infty$. Assume that $\| \, |x|^a f\|_{L^p(\mu_k)} < \infty$. For $r > 0$, let $B_r = \{x : \ |x| < r\}$ and $B_r^c = \mathbb{R}^d \backslash B_r$. Denote by χ_{B_r} and $\chi_{B_r^c}$ the characteristic functions. Let $f \in L^p(\mu_k)$, $1 < p \le 2$ and let $q = p/(p-1)$. Since $|(f\chi_{B_r^c})(x)| \le r^{-a}|x|^a|f(x)|$, then by (1.4),

$$\begin{aligned} \|e^{-t|y|^2} \mathcal{F}_k(f\chi_{B_r^c})\|_{L^q(\mu_k)} &\le \|e^{-t|y|^2}\|_{L^\infty(\mu_k)} \|\mathcal{F}_k(f\chi_{B_r^c})\|_{L^q(\mu_k)} \\ &\le \|f\chi_{B_r^c}\|_{L^p(\mu_k)} \le r^{-a} \| \, |x|^a f\|_{L^p(\mu_k)}. \end{aligned}$$

On the other hand, by (1.2) and Hölder's inequality,

$$\begin{aligned} \|e^{-t|y|^2} \mathcal{F}_k(f\chi_{B_r})\|_{L^q(\mu_k)} &\le \|e^{-t|y|^2}\|_{L^q(\mu_k)} \|\mathcal{F}_k(f\chi_{B_r})\|_{L^\infty(\mu_k)} \\ &\le \|e^{-t|y|^2}\|_{L^q(\mu_k)} \|f\chi_{B_r}\|_{L^1(\mu_k)} \\ &\le \|e^{-t|y|^2}\|_{L^q(\mu_k)} \| \, |x|^{-a}\chi_{B_r}\|_{L^q(\mu_k)} \| \, |x|^a f\|_{L^p(\mu_k)}. \end{aligned}$$

By (1.1), we have $\|e^{-t|y|^2}\|_{L^q(\mu_k)} = \frac{1}{(2q)^{(\gamma+\frac{d}{2})\frac{1}{q}}} t^{-(\gamma+\frac{d}{2})\frac{1}{q}}$ and $\| \, |x|^{-a}\chi_{B_r}\|_{L^q_k} = a_k r^{-a+(2\gamma+d)/q}$. Hence,

$$\|e^{-t|y|^2} \mathcal{F}_k(f\chi_{B_r})\|_{L^q_k} \le \frac{a_k}{(2q)^{(\gamma+\frac{d}{2})\frac{1}{q}}} r^{-a+(2\gamma+d)/q} t^{-(\gamma+\frac{d}{2})\frac{1}{q}} \| \, |x|^a f\|_{L^p(\mu_k)},$$

and

$$\begin{aligned}\|e^{-t|y|^2}\mathcal{F}_k(f)\|_{L^q(\mu_k)} &\le \|e^{-t|y|^2}\mathcal{F}_k(f\chi_{B_r})\|_{L^q(\mu_k)}+\|e^{-t|y|^2}\mathcal{F}_k(f\chi_{B_r^c})\|_{L^q(\mu_k)}\\ &\le r^{-a}\Big(1+\frac{a_k}{(2q)^{(\gamma+\frac{d}{2})\frac{1}{q}}}r^{(2\gamma+d)/q}t^{-(\gamma+\frac{d}{2})\frac{1}{q}}\Big)\|\,|x|^a f\|_{L^p(\mu_k)}.\end{aligned}$$

Choosing $r=t^{1/2}$, we obtain (2.1). □

Theorem 2.1. *Let $1<p\le 2$, $q=p/(p-1)$, $0<a<(2\gamma+d)/q$ and $b>0$, then for all $f\in L^p(\mu_k)$,*

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}\le C(a,b)\|\,|x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+b}}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+b}}, \tag{2.2}$$

where $C(a,b)$ is a positive constant.

Proof. Let $f\in L^p(\mu_k)$, $1<p\le 2$, such that $\|\,|x|^a f\|_{L^p(\mu_k)}+\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}<\infty$. Assume that $0<a<(2\gamma+d)/q$ and $b\le 2$. By Lemma 2.1, for all $t>0$,

$$\begin{aligned}\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} &\le \|e^{-t|y|^2}\mathcal{F}_k(f)\|_{L^q(\mu_k)}+\|(1-e^{-t|y|^2})\mathcal{F}_k(f)\|_{L^q(\mu_k)}\\ &\le \Big(1+\frac{a_k}{(2q)^{(\gamma+\frac{d}{2})\frac{1}{q}}}\Big)t^{-a/2}\|\,|x|^a f\|_{L^p(\mu_k)}+\|(1-e^{-t|y|^2})\mathcal{F}_k(f)\|_{L^q(\mu_k)}.\end{aligned}$$

On the other hand,

$$\|(1-e^{-t|y|^2})\mathcal{F}_k(f)\|_{L^q(\mu_k)}=t^{b/2}\|(t|y|^2)^{-b/2}(1-e^{-t|y|^2})|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

Since $(1-e^{-t})t^{-b/2}$ is bounded for $t\ge 0$ if $b\le 2$. Hence,

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}\le C\left(t^{-a/2}\|\,|x|^a f\|_{L^p(\mu_k)}+t^{b/2}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}\right).$$

We choose $t=\left(\frac{a}{b}\frac{\|\,|x|^a f\|_{L^p(\mu_k)}}{\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}}\right)^{\frac{2}{a+b}}$, we obtain the result

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}\le C\|\,|x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+b}}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+b}},\qquad \text{for all } b\le 2. \tag{2.3}$$

If $b>2$. For $u\ge 0$, $u\le 1+u^b$ which for $u=\frac{|y|}{\varepsilon}$ gives the inequality $\frac{|y|}{\varepsilon}\le 1+\left(\frac{|y|}{\varepsilon}\right)^b$, for all $\varepsilon>0$. It follows that

$$\|\,|y|\mathcal{F}_k(f)\|_{L^q(\mu_k)}\le \varepsilon\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}+\varepsilon^{1-b}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

We choose $\varepsilon=(b-1)^{1/b}\left(\frac{\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}}{\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}}\right)^{1/b}$, we get

$$\|\,|y|\mathcal{F}_k(f)\|_{L^q(\mu_k)}\le \frac{b}{b-1}(b-1)^{1/b}\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{b-1}{b}}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{1/b} \tag{2.4}$$

Then, by (2.3) and (2.4) we obtain

$$\begin{aligned}\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} &\le C\|\,|x|^a f\|_{L^p(\mu_k)}^{\frac{1}{a+1}}\|\,|y|\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+1}}\\ &\le C\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a(b-1)}{b(a+1)}}\||x|^a f\|_{L^p(\mu_k)}^{\frac{1}{a+1}}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{b(a+1)}}.\end{aligned}$$

Thus,

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a+b}{b(a+1)}}\le C\|\,|x|^a f\|_{L^p(\mu_k)}^{\frac{1}{a+1}}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{b(a+1)}},$$

which gives the result for $b>2$. □

Remark 2.1. When $q = 2$, by (1.3) we obtain

$$\|f\|_{L^2(\mu_k)} \leq C(a,b)\| \,|x|^a f\|_{L^2(\mu_k)}^{\frac{b}{a+b}} \| \,|y|^b \mathcal{F}_k(f)\|_{L^2(\mu_k)}^{\frac{a}{a+b}},$$

which is the general case of the inequality (1.5) proved by Rösler [8] and Shimeno [9].

Now, we give application of the L^p Heisenberg-Pauli-Weyl inequality to the Donoho-Stark uncertainty principle.

Let E be measurable subset of $\mathbb{R}^d$. We introduce the partial sum operator S_E by

$$\mathcal{F}_k(S_E f) = \mathcal{F}_k(f)\chi_E \tag{2.5}$$

Let $b > 0$. We say that a function $f \in L^p(\mu_k)$, $1 \leq p \leq 2$, is $|y|^b \mathcal{F}_k(f)$ is ε-concentrated to E in $L^q(\mu_k)$-norm, $q = p/(p-1)$, if there is a function $h(y)$ vanishing outside E with $\| \,|y|^b \mathcal{F}_k(f) - h\|_{L^q(\mu_k)} \leq \varepsilon \| \,|y|^b \mathcal{F}_k(f)\|_{L^q(\mu_k)}$.

From (2.5) it follows that $|y|^b \mathcal{F}_k(f)$ is ε_E-concentrated to E in $L^q(\mu_k)$-norm, $q = p/(p-1)$, if and only if

$$\| \,|y|^b \mathcal{F}_k(f) - |y|^b \mathcal{F}_k(S_E f)\|_{L^q(\mu_k)} \leq \varepsilon_E \| \,|y|^b \mathcal{F}_k(f)\|_{L^q(\mu_k)} \tag{2.6}$$

It is useful to have uncertainty principle for the $L^p(\mu_k)$-norm.

Theorem 2.2. *Let E be measurable subset of $\mathbb{R}^d$; and let $1 < p \leq 2$, $q = p/(p-1)$, $f \in L^p(\mu_k)$ and $b > 0$. If $|y|^b \mathcal{F}_k(f)$ is ε_E-concentrated to E in $L^q(\mu_k)$-norm, then for $0 < a < (2\gamma + d)/q$:*

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \frac{C(a,b)}{(1-\varepsilon_E)^{\frac{a}{a+b}}} \| \,|x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+b}} \| \,|y|^b \mathcal{F}_k(f)\chi_E\|_{L^q(\mu_k)}^{\frac{a}{a+b}},$$

where $C(a,b)$ is the constant given by (2.2).

Proof. Let $f \in L^p(\mu_k)$, $1 < p \leq 2$. Since $|y|^b \mathcal{F}_k(f)$ is ε_E-concentrated to E in $L^q(\mu_k)$-norm, $q = p/(p-1)$, then by (2.6),

$$\| \,|y|^b \mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \varepsilon_E \| \,|y|^b \mathcal{F}_k(f)\|_{L^q(\mu_k)} + \| \,|y|^b \mathcal{F}_k(f)\chi_E\|_{L^q(\mu_k)}.$$

Thus,

$$\| \,|y|^b \mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+b}} \leq \frac{1}{(1-\varepsilon_E)^{\frac{a}{a+b}}} \| \,|y|^b \mathcal{F}_k(f)\chi_E\|_{L^q(\mu_k)}^{\frac{a}{a+b}}.$$

Multiply this inequality by $C(a,b)\| \,|x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+b}}$ and applying Theorem 2.1 we deduce the desired inequality. $\square$

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SIMPLICIAL AND CROSSED HOM-LIE ALGEBRA

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Abstract. We introduce the simplicial Hom-Lie algebras and determine their relations among crossed modules of Hom-Lie algebras.

1. Introduction

A Hom algebra structure is a multiplication on a vector space where the structure is twisted by a homomorphism. The structure of Hom-Lie algebra was introduced in [2]. Crossed modules were introduced by Whitehead in [7] as a model for connected homotopy 2-types. After then, crossed modules were used in many branches of mathematics such as category theory, cohomology of algebraic structures, differential geometry and in physics. This makes the crossed modules one of the fundamental algebraic gadget. For some different usage, crossed modules were defined in different categories such as Lie algebras, commutative algebras etc.([5],[3]). Also the crossed modules of Hom-Lie algebras were defined in [6]. The goal of this paper is to define simplicial Hom-Lie algebras and show their relation between the crossed modules over Hom-Lie algebras and the simplicial Hom-Lie algebras.

2. Preliminaries

In the rest of this paper $\Bbbk$ will be a fixed field.

Definition 2.1. ([2]) A Hom-Lie algebra is a triple space $(L, [-,-], \alpha_L)$ consisting of a $\Bbbk$-vector space L, a skew-symmetric bilinear map $[-,-] : L \times L \longrightarrow L$ and a $\Bbbk$-linear map $\alpha_L : L \longrightarrow L$ satisfying the following hom-Jacobi identity;

$$[\alpha_L(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0,$$

for all $x, y, z \in L$.

Definition 2.2. A homomorphism of Hom-Lie algebras

$$f : (L, [-,-]_L, \alpha_L) \longrightarrow (L, [-,-]_M, \alpha_M)$$

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is a linear map $f : L \longrightarrow M$ such that

$$f([x,y]_L) = [f(x), f(y)]_M \quad , \quad f \circ \alpha_L = \alpha_M \circ f,$$

for all $x, y \in L$.

Example 2.1. If we take $\alpha = id$, then every Lie algebra L forms a Hom-Lie algebra $(L, [-,-], id)$.

We have the category **HomLie** whose objects are Hom-Lie algebras and whose morphisms are Hom-Lie algebra homomorphism.

So the category **Lie** of Lie algebras is a full subcategory of **HomLie** which gives an inclusion functor **Lie** $\hookrightarrow$ **HomLie**.

From now on we use L instead of $(L, [-,-]_L, \alpha_L)$, for shortness.

3. Crossed Modules of Hom-Lie Algebras

In this section we will recall the action in **HomLie** and the definition of crossed modules from [6]. Also we will adapt some well known examples and results from crossed modules of groups to crossed modules of Hom-Lie algebras.

Definition 3.1. Let L be a Hom-Lie algebra. A Hom-representation of L is a $\Bbbk$-vector space M together with a bilinear map $\rho : L \otimes M \longrightarrow M$, $\rho(l \otimes m) = {}^{l}m$ and a $\Bbbk$-linear map $\alpha_M : M \longrightarrow M$ such that

1. ${}^{[x,y]}\alpha_M(m) = {}^{\alpha_L(x)}({}^{y}m) - {}^{\alpha_L(y)}({}^{x}m)$,
2. $\alpha_M({}^{x}m) = {}^{\alpha_L(x)}(\alpha_M(m))$,

for all $x, y \in L$ and $m \in M$.

Definition 3.2. Let L, M be Hom-Lie algebras and L has an action on M. Then we have the Hom-Lie algebra $(M \rtimes L, \alpha)$ defined on the vector space $M \oplus L$ where $\alpha : M \rtimes L \longrightarrow M \rtimes L$ is defined by $\alpha(m, l) = (\alpha_M(m), \alpha_L(l))$ and the bracket is as follows

$$[(m,l),(m',l')] = [[m,m']_M + {}^{\alpha_L(l)}m' - {}^{\alpha_L(l')}m, [l,l']_L]$$

for all $(m,l),(m',l') \in M \oplus L$.

Definition 3.3. A crossed module of Hom-Lie algebras is Hom-Lie homomorphism $\partial : M \longrightarrow L$ where M is a Hom-representation of L such that

$$\partial({}^{x}m) = [x, \partial m], \qquad {}^{\partial(m)}m' = [m, m'],$$

for all $x \in L, m, m' \in M$.

The crossed module $\partial : M \longrightarrow L$ will be denoted by (M, L, ∂).

Definition 3.4. Let $(M, L, \partial), (M', L', \partial')$ be crossed modules. A homomorphism from (M, L, ∂) to (M', L', ∂') is a pair (μ_1, μ_0) of Hom-Lie homomorphisms such that,

$$\mu_0 \partial = \partial' \mu_1 \quad \text{and} \quad \mu_1({}^{l}m) = {}^{\mu_0(l)}(\mu_1(m)),$$

for all $l \in L, m \in M$.

Consequently, we define the category of crossed modules on Hom-Lie algebras, whose objects are crossed modules of Hom-Lie algebras and whose morphisms are homomorphisms of crossed modules. This category will be denoted by **XHomLie**.

Example 3.1. Let L be a Hom-Lie algebra and I be an ideal of L. I is a Hom-representation of L thanks to the map $\rho : L \otimes I \longrightarrow I$ defined by

$$\rho(l, i) = [l, i],$$

for all $l \in L, i \in I$. This gives rise to the crossed module $(I, L, inc.)$.

Proposition 3.1. *If (M, L, ∂) is a crossed module then $\partial(M)$ is an ideal of L (This is not the case for arbitrary homomorphisms, in general).*

Proof. Since $\partial : M \longrightarrow L$ is a crossed module, we have $[l, \partial m] = \partial({}^{l}m)$ for all $l \in L, m \in M$, as required. □

Example 3.2. Let M be a $\mathbb{k}$-vector space which is also a Hom-representation of a Hom-Lie algebra L. Then $0 : M \longrightarrow L$ is a crossed module. (Here, if M chosen as an arbitrary Hom-Lie algebra, then the Peiffer condition do not satisfied, in general.)

4. Simplicial Hom-Lie Algebras

Let $\triangle$ be the category of finite ordinals. A simplicial Hom-Lie algebra **HL** is a sequence of Hom-Lie algebras

$$\mathbf{HL} = \{HL_0, HL_1, ..., HL_n, ...\}$$

together with face and degeneracy maps

$$\begin{array}{llcll} d_i^n : & HL_n & \longrightarrow & HL_{n-1} & , \ 0 \le i \le n \quad (n \neq 0) \\ s_i^n : & HL_n & \longrightarrow & HL_{n+1} & , \ 0 \le i \le n \end{array}$$

which are Hom-Lie homomorphisms satisfying the usual simplicial identities.

4.1. **The Moore Complex.** The Moore complex **NHL** of a simplicial Hom-Lie algebra **HL** is the complex

$$\mathbf{NHL} : \cdots \longrightarrow NHL_n \xrightarrow{\partial_n} NHL_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} NHL_1 \xrightarrow{\partial_1} NHL_0$$

where $NHL_0 = HL_0$, $NHL_n = \bigcap\limits_{i=0}^{n-1} Kerd_i$ and ∂_n is the restriction of d_n to NHL_n.

We say that the Moore complex **NHL** of a simplicial Hom-Lie algebra **HL** is of length k if $NHL_n = 0$, for all $n \ge k+1$. Let $\mathbf{Simp}_{\le k}(\mathbf{HL})$ be the category whose objects are simplicial Hom-Lie algebras with Moore complex of length k.

4.2. **Truncated Simplicial Hom-Lie Algebras.** The following terminology adapted to simplicial Hom-Lie algebras from [1]. Details of the group case can be found in [1]. A k-truncated simplicial Hom-Lie algebra is a family of Hom-Lie algebras $\{HL_0, HL_1, ..., HL_k\}$ and homomorphism $d_i : HL_n \longrightarrow HL_{n-1}$, $s_i : HL_n \longrightarrow HL_{n+1}$, for each $0 \le i \le n$ which satisfy the simplicial identities. We denote the category of k-truncated simplicial Hom-Lie algebras by $\mathbf{Tr}_k\mathbf{Simp}(\mathbf{HL})$. There is a truncation functor tr_k from the category $\mathbf{Simp}(\mathbf{HL})$ to the category $\mathbf{Tr}_k\mathbf{Simp}(\mathbf{HL})$ given by restrictions. This truncation functor has a left adjoint st_k and a right adjoint $cost_k$ called as k-skeleton and k-coskeleton respectively. These adjoints can be pictured as follows;

$$\mathbf{Tr}_k\mathbf{Simp}(\mathbf{HL}) \quad \overset{tr_k}{\underset{cost_k}{\rightleftarrows}} \quad \mathbf{Simp}(\mathbf{HL}) \quad \overset{tr_k}{\underset{st_k}{\rightleftarrows}} \quad \mathbf{Tr}_k\mathbf{Simp}(\mathbf{HL}).$$

see [1] for details about the functors $cost_k$ and st_k.

Theorem 4.1. *The category* **XHomLie** *of crossed modules of Hom-Lie algebras is naturally equivalent to the category* $\mathbf{Simp}_{\leq 1}(\mathbf{HL})$ *of simplicial Hom-Lie algebras with Moore complex of length 1.*

Proof. Let **HL** be a simplicial Hom-Lie algebra with Moore complex of length 1. NHL_1 is a Hom-representation of NHL_0, thanks to the degenerate operator s_0^0. In fact, by using the map $\rho : NHL_0 \otimes NHL_1 \longrightarrow NHL_1$, $(x,a) \longmapsto {}^{x}a := [s_0(x), a]$, we have

$$\begin{aligned} {}^{[x,y]}\alpha_M(m) &= [[s_0x, s_0y], \alpha_M(m)] \\ &= -([\alpha_M(m), [[s_0x, s_0y]]]) \\ &= (\alpha_M s_0 x, [s_0y, m] + [\alpha_M s_0 y, [m, s_0 x]]) \\ &= [s_0\alpha_L x, [s_0 y, m]] - [s_0 \alpha_L y, [s_0 x, m]] \\ &= {}^{\alpha_{L(x)}}({}^{y}m) - {}^{\alpha_{L(y)}}({}^{x}m), \end{aligned}$$

and

$$\begin{aligned} \alpha_M({}^{x}m) &= \alpha_M[s_0(x), m] \\ &= [\alpha_M s_0 x, \alpha_M m] \\ &= [s_0\alpha_L(x), \alpha_M(m)], \end{aligned}$$

for all $x, y \in NHL_0, a \in NHL_1$.

Define $\partial := d_1 \mid_{Kerd_0}$. Then (NHL_1, NHL_0, ∂) is a crossed module. We have

$$\begin{aligned} \partial({}^{x}a) &= \partial[s_0(x), a] \\ &= [\partial s_0(x), \partial(a)] \\ &= [x, \partial(a)], \end{aligned}$$

since $d_1^1 s_0^0 = id$. On the other hand, we have

$$\begin{aligned} {}^{\partial(a)}b &= [s_0\partial(a), b] \\ &= [s_0 d_1(a), b] \\ &= [a - a + s_0 d_1(a), b] \\ &= [a, b] - [a + s_0 d_1(a), b] \\ &= [a, b] - [d_2^2 s_1^1 a + d_2^2 s_0^1 a, d_2^2 s_1^1 b] \\ &= [a, b] - d_2^2[s_1^1 a + s_0^1 a, s_1^1 b] \\ &= [a, b], \end{aligned}$$

for all $a, b \in NHL_1$, since $d_2^2 s_1^1 = id$, $s_0^1 d_1^1 = d_2^2 s_0^1$. Consequently (NHL_1, NHL_0, ∂) is a crossed module. So we obtain the functor

$$X : \mathbf{Simp}_{\leq 1}(\mathbf{HL}) \longrightarrow \mathbf{XHomLie}$$

Conversely, let (M, L, ∂) be a crossed module. Since M is a Hom-representation of L, we have the semi-direct product $M \rtimes L$. Define the maps $d_0 : M \rtimes L \longrightarrow L$, $d_1 : M \rtimes L \longrightarrow L$ and $s_0 : L \longrightarrow M \rtimes L$ by $(m, l) \longmapsto l$, $(m, l) \longmapsto \partial(m) + l$ and $l \longmapsto (0, l)$, respectively. It can be easily showed that these maps are Hom-Lie algebra homomorphisms. So $HL_1 = M \rtimes L$ and $HL_0 = L$. Then

$$HL_1 \quad \begin{matrix} \xrightarrow{d_1} \\ \xrightarrow{d_1} \\ \xleftarrow[s_0]{} \end{matrix} \quad HL_0$$

is a 1-trunated simplicial Hom-Lie algebra. Thus we have the functor

$$T : \mathbf{XHomLie} \longrightarrow \mathbf{Tr}_1\mathbf{Simp}(\mathbf{HL})$$

By using the functor st_k, we have

$$S := st_1T : \mathbf{XHomLie} \longrightarrow \mathbf{Simp}_{\leq 1}(\mathbf{HL})$$

which gives the natural equivalance of the categories **XHomLie** and $\mathbf{Simp}_{\leq 1}(\mathbf{HL})$ with the functor X. □

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About one non-local problem for the degenerating parabolic-hyperbolic type equation.

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Abstract.

In the present paper, the existence and uniqueness of solution of the analogue of Frankl's problem for the degenerated equation of the parabolic-hyperbolic type was investigated. Uniqueness of solution of the investigated problem are proved with principle an extremum and existence of solution with method of integral equations.

Key words.

Boundary value problem, existence and uniqueness of solution, degenerating equation, parabolic-hyperbolic type, a principle an extremum, method of integral equations.

1.Introduction.

As we know, in 1959 year in the first by I.M Gelfand [3] was offered to studying of boundary value problems for the equations parabolic-hyperbolic type.

Since A.V.Bitsadze's works, in the theory partial differential equations there was a new direction, in which the problem of the type of Frankl for the first time is formulated and investigated for the modeling equations of the mixed type. We note following works that are connected with studying Frankl problem for the mixed type equations. In the books [1],[2] the Frankl problem was discussed for the special mixed type equation of second order: $u_{xx} + signyu_{yy} = 0$. The Frankl problem for the mixed equation with parabolic degeneracy $singy|y|^m u_{xx} + u_{yy} = 0$ with is a mathematical model of problem of gas dynamic, was discussed in the book of M.M.Smirnov [8]. Existence of solution of Frankl problem for general Lavrent'ev-Bitsadze equations was proved in work of Guo-chun Wen and H.Begehr [4].

The basic review of boundary value problems for the mixed type equations with Frankl condition it is possible will receive in the work J. M. Rassias [9].

2. Initial necessary dates

Definition. Let's, the function $f(x)$ is any function from a class $L(a,b)$ $a < x < \infty$. An operator in the form

$$D_{ax}^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_a^x \dfrac{f(t)}{(x-t)^{1+c}}, & \alpha < 0, \\ \dfrac{d^{n+1}}{dx^{n+1}} D_{ax}^{\alpha-(n+1)} f(x), & \alpha > 0, \end{cases}$$

$$D_{xb}^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_x^b \dfrac{f(t)}{(t-x)^{1+\alpha}}, & \alpha < 0, \\ (-1)^{n+1} \dfrac{d^{n+1}}{dx^{n+1}} D_{xb}^{\alpha-(n+1)} f(x), & \alpha > 0, \end{cases} \tag{2.1}$$

where D_{ax}^{α} and D_{xb}^{α} is called as the integral operator of fractionally integration α, at $\alpha < 0$, and the generalized derivatives in understand of Liuvill on the order α, at $\alpha > 0$, $n = [\alpha]$; $[\alpha]$ the whole part of number α.

Some properties integral differential operators of fractionally order.

1^0. If $f(x) \in L(a,b)$, then for all $\alpha > 0$ almost for all $x \in (a,b)$

$$D_{xb}^{\alpha} D_{xb}^{-\alpha} f(x) = f(x), \tag{2.2}$$

2^0. Lets $f(x) \in L(a,b)$, then:

1) if $\beta \geq \alpha > 0$, then

$$D_{ax}^{\alpha} D_{ax}^{-\beta} f(x) = D_{ax}^{-(\beta-\alpha)} f(x), \; D_{xb}^{\alpha} D_{xb}^{-\beta} f(x) = D_{xb}^{-(\beta-\alpha)} f(x), \; x \in (a,b);$$

2) if $\alpha > \beta \geq 0$ and the function $f(x)$ have a derivative of $D_{ax}^{\alpha-\beta} f(x)$, $D_{xb}^{\alpha-\beta} f(x)$ then

$$D_{ax}^{\alpha} D_{ax}^{-\beta} f(x) = D_{ax}^{(\alpha-\beta)} f(x),$$

$$D_{xb}^{\alpha} D_{xb}^{-\beta} f(x) = D_{xb}^{(\alpha-\beta)} f(x), \; x \in (a,b);$$

3^0. Let $0 < 2\beta < 1$ $\;(b-x)^{-\beta} f(x) \in L(a,b)$. then almost everywhere on (a,b) it is fair identities:

$$D_{xb}^{\beta} (b-x)^{2\beta-1} D_{xb}^{\beta-1} (b-x)^{-\beta} f(x) = (b-x)^{\beta-1} D_{xb}^{2\beta-1} f(x). \tag{2.3}$$

4^0. *A principle of an extremum for the fractional derivative operations* D_{ax}^{α} and $D_{xb}^{\alpha} (0 < \alpha < 1)$. Let positive not decreasing function $\omega(t)$ and a function $f(t)$ continuously in $[a,b]$. Then, if the function $f(t)$ reaches the positive maximum (a negative minimum) in the segment $[a,b]$ on the point $t = x$, $a < x < b$ and in as much as small vicinity of this point derivative of function $\omega(t) f(t)$ satisfy Gelder condition with an indicator $\gamma > \alpha$, then $D_{ax}^{\alpha} \omega f > 0$, $(D_{xb}^{\alpha} \omega f < 0)$.

The similar remark takes place for the operator D_{xb}^{α}, if $\omega(t)$ positive not increasing function on the $[a,b]$.

3. The statement of problems F.

The given work is devoted research of non-local problem of the Frankl type for the equation

$$0 = \begin{cases} y^{m_0} u_{xx} - x^{n_0} u_y, & x > 0, y > 0, \\ (-y)^n u_{xx} - x^n u_{yy}, & x > 0, y < 0, \end{cases} \tag{3.1}$$

where $m_0, n_0, n = const$, $m_0 > 0, n_0 > 0, n > 0$.

Let's Ω is, domain restricted at $x<0$, $y>0$, by the segments AB, BB_0, A_0B_0, A_0A on the lines $y=0$, $x=1$, $y=1$, $x=0$ and at $x>0$, $y<0$, restricted by line $x=0$, $(-1\le y\le 0)$ and characteristics

$$BC: x^{\frac{n+2}{2}}+(-y)^{\frac{n+2}{2}}=1,$$

of equation (3.1), where $A(0,0)$, $B(1,0)$, $A_0(0,1)$, $B_0(1,1)$.

Let's to put designations:

$$J=\{(x,y):0<x<1,y=0\},\ \Omega_1=\Omega\cap\{(x,y):x>0,y>0\},$$

$$\Omega_2=\Omega\cap\{(x,y):x>0,y<0\}, \Omega_{21}=\Omega_2\cap\{(x,y):x+y>0\},$$

$$\Omega_{22}=\Omega_2\cap\{(x,y):x+y<0\}, \Omega^*=\Omega_1\cup\Omega_{21}\cup J,\ \ 2\beta=\frac{n}{n+2},\ \alpha_0=\frac{n_0+1}{n_0+2},$$

and

$$0<2\beta<1,\quad 1<2\alpha_0<2. \tag{3.2}$$

we will designate, through

$$\theta\left(x_0\right)=\left(\frac{1+x_0}{2}\right)^{\frac{2}{n+2}}-i\left(\frac{1-x_0}{2}\right)^{\frac{2}{n+2}}, \tag{3.3}$$

affix of the point of crossing characteristic BC_0 by the characteristic leaving on the point $(x_0,0)\in J$, parallel characteristic AC_0, where $C_0\left(2^{1/(n+2)},2^{-1/(n+2)}\right)$.

The Problem F. To find a function $u(x,y)$ with following conditions:
1)$u(x,y)\in C(\overline{\Omega})\cap C^{2,1}(\Omega_1)\cap C^2(\Omega_{21}\cup\Omega_{22})$;
2)$u(x,y)$ satisfies equation (3.1) in the domain $\Omega_1\cup\Omega_{21}\cup\Omega_{22}$;
3)$u_x(x,y)\in C(\Omega_1\cup AA_0)\cap C(\Omega_{22}\cup AC)$, $u_x(+0,y)\in C(\Omega_{22}\cup AC)$, $y^{-m_0}u_y\in C(\Omega_1\cup J)$, $u_y\in C(\Omega_2\cup J)$ and on AB satisfied gluing condition:

$$\lim_{y\to+0}y^{-m_0}u_y(x,y)=\lim_{y\to-0}u_y(x,y),\ \ (x,y)\in J, \tag{3.4}$$

4)$u(x,y)$ satisfies boundary conditions :

$$u(x,y)|_{AA_0}=\tau_0(y),\quad u(x,y)|_{BB_0}=\varphi_0(y),\quad 0\le y\le 1, \tag{3.5}$$

$$D^{\beta}_{x^2 1}(1-x^2)^{2\beta-1}u\left[\theta(x)\right]=a(x)u\left(x^{\frac{2}{n+2}},0\right)+$$

$$+b(x)(1-x^2)^{\beta-1}u_y\left(x^{\frac{2}{n+2}},0\right)+c(x),\quad x\in(0,1), \tag{3.6}$$

$$u_x(0,+y)=u_x(0,-y),\quad 0<y<1, \tag{3.7}$$

where $\varphi_0(y)$,$\tau_0(y)$, $a(x)$, $b(x)$, $c(x)$ are given continuous functions, at that

$$\tau_0(y),\varphi_0(y)\in C[0,1]\cap C^1(0,1), \tag{3.8}$$

$$a(x),b(x),c(x)\in C\left[0,1\right]\cap C^3(0,1). \tag{3.9}$$

3.1. Reduction of main functional relations.

A solution of the Cauchy problem satisfying the following conditions $\tau^-(x)=u(x,-0)$, $0\le x\le 1$, $\nu^-(x)=u_y(x,-0)$, $0<x<1$, for the equation(3.1) in the domain of Ω_{21}, looks like[7]:

$$u(x,y)=\gamma_1\int_0^1\tau^-\left(z_1^{\frac{1}{n+2}}\right)z^{\beta-1}(1-z)^{\beta-1}dz-$$

$$-\gamma_2 xy \int_0^1 z_1^{-\frac{1}{n+2}} \nu^- \left(z_1^{\frac{1}{n+2}} \right) z^{-\beta}(1-z)^{-\beta} dz, \tag{3.10}$$

where , $\gamma_1 = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)}$, $\gamma_2 = \frac{\Gamma(2-2\beta)}{\Gamma^2(1-\beta)}$, $z_1 = x^{n+2} + (-y)^{n+2} + 2x^{\frac{n+2}{2}}(-y)^{\frac{n+2}{2}}(2z-1)$.

By virtue (3.3), from (3.10), we have

$$u\left[\theta(x)\right] = \gamma_1 \int_0^1 \tau^- \left[\left(x^2 + (1-x^2)z\right)^{\frac{1}{2}-\beta} \right] (z(1-z))^{\beta-1} dz + \gamma_2 \left(\frac{1-x^2}{4} \right)^{1-2\beta} \times$$

$$\times \int_0^1 \left(x^2 + (1-x^2)z\right)^{\beta-\frac{1}{2}} \nu^- \left[(x^2 + (1-x^2)z)^{\frac{1}{2}-\beta} \right] (z(1-z))^{-\beta} dz.$$

From here, owing to replacement $x^2 + (1-x^2)z = s$, we will receive

$$u[\theta(x)] = \gamma_1 (1-x^2)^{1-2\beta} \int_{x^2}^1 (s-x^2)^{\beta-1}(1-s)^{\beta-1} \tau^- \left(s^{\frac{1}{n+2}} \right) ds +$$

$$+\gamma_2 4^{2\beta-1} \int_{x^2}^1 (s-x^2)^{-\beta}(1-s)^{-\beta} s^{\beta-\frac{1}{2}} \nu^- \left(s^{\frac{1}{n+2}} \right) ds.$$

Further,taking properties of integro-differential operators into account (2.1)[8],we have

$$u[\theta(x)] = \gamma_1 \Gamma(\beta)(1-x^2)^{1-2\beta} D_{x^2 1}^{-\beta} \tau^- \left(x^{\frac{2}{n+2}} \right) (1-x^2)^{\beta-1} +$$

$$+\gamma_2 \Gamma(1-\beta) 4^{2\beta-1} D_{x^2 1}^{\beta-1} (1-x^2)^{-\beta} x^{2\beta-1} \nu^- \left(x^{\frac{2}{n+2}} \right). \tag{3.11}$$

Substituting (3.11),(2.2) to the condition (3.6), and replacing x^2 to x, we have

$$\left[\overline{a}(x) - \gamma_1 \Gamma(\beta)(1-x)^{\beta-1} \right] \tau^- \left(x^{\frac{1}{n+2}} \right) + \overline{c}(x) = \gamma_2 \Gamma(1-\beta) 4^{2\beta-1} \times$$

$$\times D_{x1}^{\beta} (1-x)^{2\beta-1} D_{x1}^{\beta-1} (1-x)^{-\beta} x^{\beta-\frac{1}{2}} \nu^- \left(x^{\frac{1}{n+2}} \right) - (1-x)^{\beta-1} \overline{b}(x) \nu^- \left(x^{\frac{1}{n+2}} \right). \tag{3.12}$$

From (3.12) and (2.3), we have

$$\overline{a}_1(x) \widetilde{\tau}^-(x) = \gamma_2 \Gamma(1-\beta) 4^{2\beta-1} D_{x1}^{2\beta-1} x^{\beta-\frac{1}{2}} \widetilde{\nu}^-(x) -$$

$$-\overline{b}(x) \widetilde{\nu}^-(x) - \overline{c}(x)(1-x)^{1-\beta}, \quad 0 < x < 1, \tag{3.13}$$

where, $\widetilde{\tau}^-(x) = \tau^- \left(x^{\frac{1}{n+2}} \right)$, $\widetilde{\nu}^-(x) = \nu^- \left(x^{\frac{1}{n+2}} \right)$,

$$\overline{a}_1(x) = \overline{a}(x)(1-x)^{1-\beta} - \gamma_1 \Gamma(\beta),\ \overline{a}(x) = a(\sqrt{x}),\ \overline{b}(x) = b(\sqrt{x}),\ \overline{c}(x) = c(\sqrt{x}). \tag{3.14}$$

Let's consider three cases:

I. Let's $b(x) = 0$, $a(x) \neq 0$. Then from (3.13), receive

$$\gamma_2 \Gamma(1-\beta) 4^{2\beta-1} D_{x1}^{2\beta-1} x^{\beta-\frac{1}{2}} \widetilde{\nu}^-(x) = \overline{a}_1(x) \widetilde{\tau}^-(x) + (1-x)^{1-\beta} \overline{c}(x). \tag{3.15}$$

Applying the operator $D_{x1}^{1-2\beta}[.]$ to both parts of equality (3.15), we will obtain the basic functional relation between $\widetilde{\tau}^-(x)$ and $\widetilde{\nu}^-(x)$:

$$\gamma_2 \Gamma(1-\beta) 4^{2\beta-1} \widetilde{\nu}^-(x) = x^{\frac{1}{2}-\beta} D_{x1}^{1-2\beta} \overline{a}_1(x) \widetilde{\tau}^-(x) +$$

$$+x^{\frac{1}{2}-\beta}D_{x1}^{1-2\beta}(1-x)^{1-\beta}\overline{c}\,(x)\,. \tag{3.16}$$

Further, from the equation $u_{xx}-x^{n_0}y^{-m_0}u_y=0$ at the $y\to+0$ we have receive ordinary differential equation

$$\tau^{''+}(x)-x^{n_0}\nu^{+}(x)=0,\quad 0<x<1, \tag{3.17}$$

where $\tau^{+}(x)=u(x,+0)$ and $\nu^{+}(x)=\lim\limits_{y\to+0}y^{-m_0}u_y(x,y)$.

Solving this equation with conditions $\tau^{+}(0)=\tau_0(0)$ and $\tau^{+}(1)=\varphi_0(0)$, deduce functional relation between $\tau^{+}(x)$ and $\nu^{+}(x)$:

$$\tau^{+}(x)=\int\limits_0^1 G(x,t)t^{n_0}\nu^{+}(t)dt+f(x),\qquad 0x1,$$

here

$$G(x,t)=\begin{cases} t(x-t), & 0\le t\le x,\\ (t-1)x, & x\le t\le 1.\end{cases} \tag{3.18}$$

$$f(x)=\tau_0(0)+x(\varphi_0(0)-\tau_0(0)). \tag{3.19}$$

Further, by virtue replace $x\sim x^{\frac{1}{n+2}}$ and $t\sim t^{\frac{1}{n+2}}$ receive functional relation between $\widetilde{\tau}^{+}(x)$ $\widetilde{\nu}^{+}(x)$:

$$\widetilde{\tau}^{+}(x)=\int\limits_0^1 \widetilde{G}(x,t)\widetilde{\nu}^{+}(t)dt+\widetilde{f}(x),\qquad 0x1, \tag{3.20}$$

where, $\widetilde{f}(x)=f\left(x^{\frac{1}{n+2}}\right),\qquad \widetilde{\tau}^{+}(x)=\tau^{+}\left(x^{\frac{1}{n+2}}\right),\qquad \widetilde{\nu}^{+}(t)=\nu^{+}\left(t^{\frac{1}{n+2}}\right),$

$$G(x,t)=\frac{1}{n+2}t^{\frac{n_0+1}{n+2}-1}G\left(x^{\frac{1}{n+2}},t^{\frac{1}{n+2}}\right). \tag{3.21}$$

3.2. Uniqueness of the solution.

Theorem 1. If satisfying the conditions (3.2),$b(x)=0$, $a(x)\neq 0$ and

$$\overline{a}_1(x)>0,\quad x\in(0,1), \tag{3.22}$$

then a solution $u(x,y)$ of the problem F is unique.

The Proof. According to the extremum principle for the parabolic equations [5], [10], the solution $u(x,y)$ of the equation(3.1) cannot reach the positive maximum and negative minimum in the domain of Ω_1 and on a piece A_0B_0. We will denote, that the solution $u(x,y)$ does not reach the positive maximum and negative minimum on an interval AB.

Let's assume the return, i.e. let in some point $E(x_0,0)$ function $u(x,y)$ reaches the positive maximum (negative minimum). Then from (3.16), at $\overline{c}(x)\equiv 0$ we have:

$$\gamma_2\Gamma(1-\beta)4^{2\beta-1}\widetilde{\nu}^{-}\,(x_0)=x_0^{\frac{1}{2}-\beta}D_{x_01}^{1-2\beta}\overline{a}_1(x_0)\widetilde{\tau}^{-}\,(x_0)\,. \tag{3.23}$$

From here, owing to a principle for the differential operators fractional order [8], on the point of positive maximum (negative minimum) strictly positively (negatively) $D_{x_01}^{1-2\beta}\overline{a}_1(x_0)\widetilde{\tau}^{-}\,(x_0)>0$, $\left(D_{x_01}^{1-2\beta}\overline{a}_1(x_0)\widetilde{\tau}^{-}\,(x_0)<0\right)$. Accordingly, owing to that $x_0>0$, $\gamma_2>0$ from (3.23), receive, $\widetilde{\nu}^{-}\,(x_0)>0$, $(\widetilde{\nu}^{-}\,(x_0)<0)$. From here, by virtue (3.4) we have $\widetilde{\nu}^{+}\,(x_0)>0$, $(\widetilde{\nu}^{+}\,(x_0)<0)$. This inequality contradicts an inequality $\widetilde{\nu}^{+}\,(x_0)\,0$, $(\widetilde{\nu}^{+}\,(x_0)\geq 0)$, which is direct appears from (3.17).

Thus, the solution $u(x, y)$ of equation (3.1) can't reach the positive maximum and negative minimum on an interval AB. Hence, $u(x, y)$ can to reach the positive maximum (a negative minimum) on the piece of $\overline{AA_0}$ and $\overline{BB_0}$.

From here owing to (3.5), considering continuity of the function $u(x, y)$ in $\overline{\Omega}_1$, a solution of the first boundary value problem for the equation (3.1) in the domain of Ω_1 to identically equally zero at $\varphi_0(y) \equiv \tau_0(y) \equiv 0$.

As $u(x, y) \equiv 0$ in domain $\overline{\Omega}_1$ we have $\widetilde{\tau}(x) \equiv 0$, and by virtue (3.23), $\widetilde{\nu}(x) \equiv 0$. Hence, owing to unequivocal solvability of Cauchy problem it is had $u(x, y) \equiv 0$ in the domain $\overline{\Omega}_{21}$, from here $u(x, -x) \equiv 0$.

Further, from solution homogeneous first boundary value problem for the equation (3.1) in domain of Ω_1 taking into account a condition (3.7), we will receive $u_x(0, y) = u_x(0, -y) = 0,\ 0 < y < 1$. Hence, the solution of the Cauchy-Gaursat problem for the equation (3.1) with zero given identically equally to zero in the domain of Ω_{22}, i.e. $u(x, y) \equiv 0$ in the domain $\overline{\Omega}_{22}$.

Thus, from the above-stated we will receive, that $u(x, y) \equiv 0$ in the domain $\overline{\Omega}$. Hence, the solution of a problem F in the domain of Ω is unique. **The theorem 1 was proved.**

3.3. Existence of the solution.

Theorem 2. If satisfying the conditions (3.2), (3.8), (3.9) and $b(x) = 0$, $a(x) \neq 0$ then a solution $u(x, y)$ of the problem F is exist in the domain of Ω.

Proof. Considering a continuity of the solution of a problem F, excluding $\widetilde{\tau}(x) = \widetilde{\tau}^-(x) = \widetilde{\tau}^+(x)$ from (3.16) and (3.20), we have

$$\gamma_2 4^{2\beta-1}\Gamma(1-\beta)\widetilde{\nu}(x) = x^{\frac{1}{2}-\beta} D_{x1}^{1-2\beta}\overline{a}_1(x)\left[\int_0^1 \widetilde{G}(x,t)\widetilde{\nu}(t)dt + \widetilde{f}(x)\right] +$$

$$+x^{\frac{1}{2}-\beta} D_{x1}^{1-2\beta}\overline{c}(x)(1-x)^{1-\beta}.$$

Further, taking into account properties of integro-differential operators (2.1) [11], we find

$$\widetilde{\nu}(x) = k_1 x^{\frac{1}{2}-\beta}\frac{d}{dx}\left(\int_x^1 (t-x)^{2\beta-1}\overline{a}_1(t)dt\int_0^1 \widetilde{G}(t,s)\widetilde{\nu}(s)ds\right) +$$

$$+k_1 x^{\frac{1}{2}-\beta}\frac{d}{dx}\left(\int_x^1 (t-x)^{2\beta-1}\overline{a}_1(t)\widetilde{f}(t)dt + \int_x^1 (t-x)^{2\beta-1}(1-t)^{1-\beta}\overline{c}(t)dt\right), \tag{3.24}$$

where, $k_1 = 1/\gamma_2 4^{2\beta-1}\Gamma(1-\beta)\Gamma(2\beta)$

Having executed replacement $t = x+(1-x)\sigma$ and changing an order of integration from (3.24), we have

$$\widetilde{\nu}(x) = k_1 x^{\frac{1}{2}-\beta}\frac{d}{dx}\left((1-x^{2\beta}\int_0^1 \sigma^{2\beta-1}\overline{a}_1(x+(1-x)\sigma)d\sigma\int_0^1 \widetilde{G}\left(x+(1-x)\sigma, s)\right)\widetilde{\nu}(s)ds\right) +$$

$$+k_1 x^{\frac{1}{2}-\beta}\frac{d}{dx}\left((1-x)^{2\beta}\int_0^1 \sigma^{2\beta-1}\overline{a}_1(x+(1-x)\sigma)\widetilde{f}(x+(1-x)\sigma)d\sigma\right) +$$

$$+k_1x^{\frac{1}{2}-\beta}\frac{d}{dx}\left((1-x)^{1+\beta}\int\limits_0^1\sigma^{2\beta-1}(1-\sigma)^{1-\beta}\overline{c}(x+(1-x)\sigma)d\sigma\right).$$

From here, after some evaluations, we will obtain the integral equation

$$\widetilde{\nu}(x)=\int\limits_0^1\widetilde{K}(x,s)\widetilde{\nu}(s)ds+\widetilde{\Phi}(x), \tag{3.25}$$

where

$$\widetilde{K}(x,s)=\widetilde{K}_1(x,s)+\widetilde{K}_2(x,s), \tag{3.26}$$

$$\widetilde{K}_1(x,s)=-2\beta k_1x^{\frac{1}{2}-\beta}\left(1-x\right)^{2\beta-1}\int\limits_0^1\sigma^{2\beta-1}\overline{a}_1\left(x+(1-x)\sigma\right)\times$$

$$\times\widetilde{G}\left(x+(1-x)\sigma,s\right)d\sigma, \tag{3.27}$$

$$\widetilde{K}_2(x,s)=k_1x^{\frac{1}{2}-\beta}\left(1-x\right)^{2\beta}\frac{d}{dx}\int\limits_0^1\sigma^{2\beta-1}\overline{a}_1(x+(1-x)\sigma)\times$$

$$\times\widetilde{G}(x+(1-x)\sigma,s)d\sigma, \tag{3.28}$$

$$\widetilde{\Phi}(x)=-2\beta k_1x^{\frac{1}{2}-\beta}\int\limits_0^1\sigma^{2\beta-1}\overline{a}_1(x+(1-x)\sigma)\widetilde{f}(x+(1-x)\sigma)d\sigma+$$

$$+k_1x^{\frac{1}{2}-\beta}(1-x)^{2\beta}\int\limits_0^1\sigma^{2\beta-1}\frac{d}{dx}\left[\overline{a}_1(x+(1-x)\sigma)\widetilde{f}(x+(1-x)\sigma)\right]d\sigma-$$

$$-(1+\beta)k_1x^{\frac{1}{2}-\beta}(1-x)^{\beta}\int\limits_0^1\sigma^{2\beta-1}(1-\sigma)^{1-\beta}\overline{c}(x+(1-x)\sigma)d\sigma+$$

$$+k_1x^{\frac{1}{2}-\beta}(1-x)^{1+\beta}\int\limits_0^1\sigma^{2\beta-1}(1-\sigma)^{1-\beta}\frac{d}{dx}\left[\overline{c}(x+(1-x)\sigma)\right]d\sigma. \tag{3.29}$$

From here, owing to continuity the functions $G(x,t)\in C\left([0,1]\times[0,1]\right)$ and $a(x)$, we have

$$\left|\widetilde{K}_1(x,s)\right|c_1s^{\frac{n_0+1}{n+2}-1}\left(1-x\right)^{2\beta-1}. \tag{3.30}$$

Also, considering (3.9),(3.14), (3.18),(3.21) from (3.28) we will receive

$$\left|\widetilde{K}_2(x,s)\right|c_2s^{\frac{n_0+2}{n+2}-1}(1-x)^{2\beta-1}. \tag{3.31}$$

Thus, by virtue (3.30) and (3.31) from (3.26), we have

$$\left|\widetilde{K}(x,s)\right|c_3s^{\frac{n_0+1}{n+2}-1}(1-x)^{2\beta-1}. \tag{3.32}$$

There under (3.2), (3.9), (3.14), (3.19) appear from (3.29) that the function $\widetilde{\Phi}(x)$. Supposes an estimate

$$\left|\widetilde{\Phi}(x)\right|c_4(1-x)^{2\beta-1}. \tag{3.33}$$

where, $c_1,c_2,c_3,c_4=const.$

Thus, by virtue (3.32), (3.33) integral equation (3.25) constitute Fredholm integral equation of the second kind[6], with the weak feature which unequivocal solvability appears from the uniqueness of the solution of investigated problem, i.e. the equation (3.25) has the unique solution, and $\nu^{+}(x) \in C^2(0,1)$.

Hence, it is possible to present its solution on the form of[6]:

$$\widetilde{\nu}^{-}(x) = \widetilde{\Phi}(x) + \int_0^1 R(x,s)\widetilde{\Phi}(s)ds, \tag{3.34}$$

where $R(x,s)$- resolvent the kernel of $K(x,s)$.

From here, according to gluing condition (3.4) taking into account (3.34) and (3.20) we find function $\widetilde{\tau}^{+}(x)$,

$$\widetilde{\tau}^{+}(x) = \int_0^1 \widetilde{G}(x,t)\left[\widetilde{\Phi}(t) + \int_0^1 R(t,z)\widetilde{\Phi}(z)dz\right]dt + \widetilde{f}(x),$$

Further, designating, $\overline{\Phi}(t) = \widetilde{\Phi}(t) + \int\limits_0^1 R(t,z)\widetilde{\Phi}(z)dz$, we have

$$\widetilde{\tau}^{+}(x) = \int_0^1 \widetilde{G}(x,t)\overline{\Phi}(t)dt + \widetilde{f}(x), \qquad 0x1. \tag{3.35}$$

Hence, by virtue (3.2), (3.9) owing to (3.35) and (3.21), (3.19) conclude, that the function $\tau^{+}(x)$ in $C[0,1] \cap C^2(0,1)$.

II. Let's $b(x) \neq 0,\ \ a(x) \neq 0$.

From (3.13), we will receive

$$\widetilde{\nu}^{-}(x) = \frac{\gamma_2\Gamma(1-\beta)4^{2\beta-1}}{\overline{b}\,(x)\Gamma(1-2\beta)}\int_x^1 (t-x)^{-2\beta}t^{\beta-\frac{1}{2}}\widetilde{\nu}^{-}(t)dt -$$

$$-\frac{\overline{a}_1(x)}{\overline{b}\,(x)}\widetilde{\tau}^{-}(x) - \frac{\overline{c}(x)}{\overline{b}\,(x)}(1-x)^{1-\beta}. \tag{3.36}$$

Let's notice, that the integral equation (2.36) is integrated Equation Volterra of the second kind

$$\widetilde{\nu}^{-}(x) = \lambda\int_x^1 N(x,t)\widetilde{\nu}^{-}(t)dt + F(x), \tag{3.37}$$

where, $\lambda = \frac{\gamma_2\Gamma(1-\beta)4^{2\beta-1}}{\Gamma(1-2\beta)}$,

$$F(x) = -\frac{\overline{a}_1(x)}{\overline{b}\,(x)}\widetilde{\tau}^{-}(x) - \frac{\overline{c}(x)}{\overline{b}(x)}(1-x)^{1-\beta}, \tag{3.38}$$

$$N(x,t) = \frac{1}{\overline{b}(x)}(t-x)^{-2\beta}t^{\beta-\frac{1}{2}}. \tag{3.39}$$

By virtue (3.9), from (3.38) and (3.39) accordingly

$$|N(x,t)| \leq M, \qquad 0 \leq x \leq 1, \tag{3.40}$$

and

$$|F(x)| \leq const \tag{3.41}$$

further, owing to the theory of integrated equations Volterra of the second kind [9], taking into account (3.40) and (3.41), we have $|R(x,s;\lambda)| \leq const$, i. the solution of equation (3.37) it is possible will present on the form of

$$\widetilde{\nu}^-(x) = -\lambda \int\limits_x^1 \frac{\overline{a}_1(s)}{\overline{b}(x)} R(x,s;\lambda)\widetilde{\tau}^-(s)ds - \lambda \int\limits_x^1 \frac{R(x,s;\lambda)\overline{c}(s)}{\overline{b}(s)} ds -$$

$$-\frac{\overline{a}_1(x)}{\overline{b}(x)}\tau^-(x) - \frac{\overline{c}(x)}{\overline{b}(x)}(1-x)^{1-\beta}. \tag{3.42}$$

3.4. Uniqueness of the solution.

Theorem 3. If satisfying the conditions (3.2),(3.8), (3.9), $b(x) \neq 0$, $a(x) \neq 0$ and

$$\overline{a}_1(x) > 0, \quad \overline{b}(x) < 0, \quad 0 < x < 1, \tag{3.43}$$

then the solution $u(x,y)$ of the problem F is unique.

Proof. Let's notice, that justice of the theorem 3 the follows at once from the theorem 1, if is proved, than the solution $u(x,y)$ of the equations (3.1) cannot reach the positive maximum and negative minimum in domain of Ω_1 and on a piece A_0B_0. And this statement is similarly proved as the theorem 1, i.e. by virtue principle of an extremum for the parabolic equations [5], the solution $u(x,y)$ the equation (3.1) cannot reach the positive maximum and a negative minimum in domain Ω_1 and on a piece A_0B_0. Let's show, that the solution $u(x,y)$ does not reach the positive maximum and negative minimum on an interval AB. We will assume the return, i.e. let in some point $(x_0,0)$ function $u(x,y)$ reaches the positive maximum (negative minimum). Then from (3.42), at $\overline{c}(x) \equiv 0$ we have:

$$\widetilde{\nu}^-(x_0) = -\lambda \int\limits_{x_0}^1 \frac{\overline{a}_1(s)}{\overline{b}(x_0)} R(x_0,s;\lambda)\widetilde{\tau}^-(s)ds - \frac{\overline{a}_1(x_0)}{\overline{b}(x_0)}\widetilde{\tau}^-(x_0).$$

From here considering (3.43), owing to, that $R(x_0,s;\lambda) > 0$ in the point of positive maximum (negative minimum) $\widetilde{\tau}^-(x_0) \geq 0$ ($\widetilde{\tau}^-(x_0) \leq 0$) we will receive $\widetilde{\nu}^-(x_0) \geq 0$ ($\widetilde{\nu}^-(x_0) \leq 0$), and this inequality contradicts an inequality $\widetilde{\nu}^+(x_0)\,0$, ($\widetilde{\nu}^+(x_0) \geq 0$), which directly follows from (3.17). Hence the solution $u(x,y)$ the equation (3.1) can't reach the positive maximum and negative minimum in domain Ω_1 and on a piece A_0B_0. **The theorem 3 is proved.**

3.5. Existence of the solution.

Theorem 4. If satisfying the conditions (3.2), (3.8), (3.9) and $b(x) \neq 0$, $a(x) \neq 0$ then the solution $u(x,y)$ of the problem F is exist.

Proof. Substituting (3.42) in (3.20), we have

$$\widetilde{\tau}(x) = \int\limits_0^1 \widetilde{K}(x,s)\widetilde{\tau}(s)ds + \widetilde{f}(x), \qquad 0x1, \tag{3.44}$$

where,

$$\widetilde{K}(x,s) = -\frac{\overline{a}_1(s)}{\overline{b}(s)}\left[\widetilde{G}(x,s) + \lambda \int\limits_0^s \widetilde{G}(x,t)R(t,s;\lambda)dt\right].$$

The equation (3.44) is Fredholm integral equation the second kind[6] and it unequivocal resolubility follows from the uniqueness of the solution the problems F.

III. Let's $a(x) \equiv 0,\ b(x) \neq 0$.

3.6. Uniqueness and existence of the solution.

On the case of $a(x) \equiv 0,\ b(x) \neq 0$ takes place the following uniqueness theorem:

Theorem 5. If satisfying the conditions (3.2) and

$$\bar{b}(x) > 0, \quad 0 < x < 1, \tag{3.45}$$

then the solution $u(x,y)$ of the problem F is unique.

Proof. From the integral equation (3.36) at $a(x) \equiv 0$, taking into account (3.14) we will receive:

$$\widetilde{\nu}^{-}(x) = \frac{\gamma_2 \Gamma(1-\beta) 4^{2\beta-1}}{\bar{b}(x)\Gamma(1-2\beta)} \int\limits_{x}^{1} (t-x)^{-2\beta} t^{\beta-\frac{1}{2}} \widetilde{\nu}^{-}(s) ds \ + F_1(x)$$

where

$$F_1(x) = \frac{\gamma_1 \Gamma(\beta)}{\bar{b}(x)} \widetilde{\tau}^{-}(x) - \frac{\overline{c}(x)}{\bar{b}(x)} (1-x)^{1-\beta},$$

and $|F_1(x)|\, const$.

Hence, from (3.42), at $a(x) \equiv 0$ we obtain main functional relation between $\widetilde{\tau}^{-}(x)$ and $\widetilde{\nu}^{-}(x)$:

$$\widetilde{\nu}^{-}(x) = \lambda \gamma_1 \Gamma(\beta) \int\limits_{x}^{1} \frac{1}{\bar{b}(x)} R(x,s;\lambda) \widetilde{\tau}^{-}(s) ds - \lambda \int\limits_{x}^{1} \frac{R(x,s;\lambda)\overline{c}(s)}{\bar{b}(s)} ds +$$

$$+ \frac{\gamma_1 \Gamma(\beta)}{\bar{b}(x)} \tau^{-}(x) - \frac{\overline{c}(x)}{\bar{b}(x)} (1-x)^{1-\beta}. \tag{3.46}$$

Let's show, that the solution $u(x,y)$ does not reach the positive maximum and negative minimum on an interval AB. We will assume the return, i.e. let in some point $(x_0, 0)$ function $u(x,y)$ reach the positive maximum (negative minimum). Then from (3.46), at $\overline{c}(x) \equiv 0$ we have:

$$\widetilde{\nu}^{-}(x_0) = \lambda \gamma_1 \Gamma(\beta) \int\limits_{x_0}^{1} \frac{1}{\bar{b}(x_0)} R(x_0,s;\lambda) \widetilde{\tau}^{-}(s) ds + \frac{\gamma_1 \Gamma(\beta)}{\bar{b}(x_0)} \widetilde{\tau}^{-}(x_0).$$

From here considering (3.45), owing to, that $R(x_0,s;\lambda) > 0$ in the point of positive maximum (negative minimum) $\widetilde{\tau}^{-}(x_0) \geq 0$ ($\widetilde{\tau}^{-}(x_0) \leq 0$) we will receive $\widetilde{\nu}^{-}(x_0) \geq 0$ ($\widetilde{\nu}^{-}(x_0) \leq 0$), and this inequality contradicts an inequality $\widetilde{\nu}^{+}(x_0)\, 0$, ($\widetilde{\nu}^{+}(x_0) \geq 0$), which directly follows from (3.17). Hence the solution $u(x,y)$ the equation (3.1) can't reach the positive maximum and negative minimum in domain Ω_1 and on a piece A_0B_0. further, let's notice, that justice of the theorem 4 the follows at once from the theorem 1 and theorem 3. **The theorem 5 is proved.**

Theorem 6. If satisfying the conditions (3.2), (3.8), (3.9) and $a(x) \equiv 0$, $b(x) \neq 0$ then the solution $u(x,y)$ of the problem F is exist.

Proof. Substituting (3.46) in (3.20), we have

$$\widetilde{\tau}(x) = \int\limits_0^1 \widetilde{K}_1(x,s)\widetilde{\tau}(s)ds + \widetilde{f}(x), \qquad 0x1, \tag{3.47}$$

where,

$$\widetilde{K}_1(x,s) = \frac{\gamma_1\Gamma(\beta)}{\bar{b}(s)}\left[\widetilde{G}(x,s) + \lambda\int\limits_0^s \widetilde{G}(x,t)R(t,s;\lambda)dt\right].$$

The equation (3.47) is Fredholm integral equation the second kind[6], and it unequivocal solubility follows from the uniqueness of the solution the problems F.

Thus, the solution of the investigated problem in the domain of Ω_1 is restored as the solution of the first boundary problem which has kind of [11]:

$$u(x,y) = \int\limits_0^1 G_1(x,\xi;y,\alpha_0)\tau^+(\xi)\xi^{n_0}d\xi + y^{-m_0}\frac{\partial}{\partial y}\int\limits_0^y G_2(x,y-t,\alpha_0)\tau_0^+(t)t^{m_0}dt +$$

$$+y^{-m_0}\frac{\partial}{\partial y}\int\limits_0^y G_3(x,y-t,\alpha_0)\varphi_0(t)t^{m_0}dt,$$

where

$$G_3(x,y,\alpha_0) = (1-\alpha_0)^{2(1-\alpha_0)}x - \int\limits_0^1 G_1(x,\xi;y,\alpha_0)\left[(1-\alpha_0)^{2(1-\alpha_0)}\right]\xi^{n_0}d\xi,$$

$$G_2(x,y,\alpha_0) = 1-(1-\alpha_0)^{2(1-\alpha_0)}x - \int\limits_0^1 G_1(x,\xi;y,\alpha_0)\left[1-(1-\alpha_0)^{2(1-\alpha_0)}\xi\right]\xi^{n_0}d\xi,$$

$$G_1(x,\xi,y;\alpha_0) = \sum_{k=0}^{\infty} e^{-\frac{\lambda_k^2 y^{m_0+1}}{4(m_0+1)}}(1-\alpha_0)\sqrt{x\xi}\times$$

$$\times\frac{J_{1-\alpha_0}(\lambda_k(1-\alpha_0)(\sqrt{x})^{\frac{1}{1-\alpha_0}})J_{1-\alpha_0}(\lambda_k(1-\alpha_0)(\sqrt{\xi})^{\frac{1}{1-\alpha_0}})}{J^2_{2-\alpha_0}(\lambda_k)},$$

$J_\nu(z) = \sum\limits_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$ is the function of Bessel on the first kind , λ_k are positive solutions of equation $J_{1-\alpha_0}(\lambda_k) = 0$, $k = 0,1,2$. $G_1(x,\xi;y,\alpha_0)$ - the Grin function of the first boundary value problem.

Satisfying condition $\nu_0^+(y) = u_x(0,y)$, $(0 < y < 1)$ to solution of the first boundary value problem, we have:

$$\nu_0^+(y) = \lim_{x\to+0}\frac{\partial}{\partial x}\int\limits_0^1 G_1(x,\xi;y,\alpha_0)\tau_1^+(\xi)\xi^{n_0}d\xi +$$

$$+\lim_{x\to+0}\frac{\partial}{\partial x}\left[y^{-m_0}\frac{\partial}{\partial y}\int\limits_0^y G_2(x,y-t,\alpha_0)\tau_0^+(t)t^{m_0}dt\right]+$$

$$+\lim_{x\to+0}\frac{\partial}{\partial x}\left[y^{-m_0}\frac{\partial}{\partial y}\int\limits_0^y G_3(x,y-t,\alpha_0)\varphi_0(t)t^{m_0}dt\right].$$

From here, by virtue condition(3.7), the solution of the problem F on domain of Ω_{22}, it is restored as the solution of problem Cauchy-Gaursat, satisfying to conditions $\nu^+(y)=\nu^-(y)=u_x(0,y)$, $-1<y<0\quad u(-y,y)=h(y)$, where $h(y)$ is the trace of solution of problem Cauchy in domain of Ω_{21} on the characteristics $y=-x$. Thus, the existence of solution of the problem F is proved.

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CONJUGATE TANGENT VECTORS AND ASYMPTOTIC DIRECTIONS FOR SURFACES AT A CONSTANT DISTANCE FROM EDGE OF REGRESSION ON A SURFACE IN E_1^3

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Abstract. In this paper we give conjugate tangent vectors and asymptotic directions for surfaces at a constant distance from edge of regression on a surface in E_1^3.

1. Introduction

Conjugate tangent vectors and asymptotic directions in Euclidean space E^3 can be found in [9]. In 1984, A. Kılıç and H. H. Hacısalihoğlu found the Euler theorem and Dupin indicatrix for parallel hypersurfaces in E^n [13]. Also the Euler theorem and Dupin indicatrix are obtained for the parallel hypersurfaces in pseudo-Euclidean spaces E_1^{n+1} and E_ν^{n+1} in the papers ([5], [7], [8]).

In 2005 H. H. Hacısalihoğlu and Ö. Tarakçı introduced surfaces at a constant distance from edge of regression on a surface. These surfaces are a generalization of parallel surfaces in E^3. Because the authors took any vector instead of normal vector [17]. Euler theorem and Dupin indicatrix for these surfaces are given in [2]. Conjugate tangent vectors and asymptotic directions are given in [1]. In 2010 we obtained the surfaces at a constant distance from edge of regression on a surface in E_1^3 [15]. We obtained the Euler theorem and Dupin indicatrix for these surfaces in E_1^3 [16].

In this paper we give conjugate tangent vectors and asymptotic directions for surfaces at a constant distance from edge of regression on a surface in E_1^3.

2. Preliminaries

Let E_1^3 be the Minkowski 3-space is the real vector space R^3 endowed with the standard flat Lorentzian metric given by

$$\langle , \rangle = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2$$

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where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. An arbitrary vector $x \in E_1^3$ is called spacelike if $\langle x, x \rangle > 0$ or $x = 0$, timelike if $\langle x, x \rangle < 0$ and lightlike (null) if $\langle x, x \rangle = 0$ and $x \neq 0$.

The timelike-cone of E_1^3 is defined as the set of all timelike vectors of E_1^3, that is

$$\mathcal{T} = \{(x, y, z) \in E_1^3;\ x^2 + y^2 - z^2 < 0\}.$$

The set of lightlike vectors is defined by $\mathcal{C}$ and it is the following set:

$$\mathcal{C} = \{(x, y, z) \in E_1^3;\ x^2 + y^2 - z^2 = 0\} - \{0, 0, 0\}.$$

The cross product $x \times y$ of vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in E_1^3 is defined as

$$\langle x \times y, z \rangle = \det(x, y, z) \quad \text{for all } z = (z_1, z_2, z_3) \in E_1^3.$$

More explicitly, if x, y belong to E_1^3, then

$$\begin{aligned}
\langle x, y \rangle &= -x_1y_1 + x_2y_2 + x_3y_3 \\
x \times y &= (-(x_2y_3 - x_3y_2), x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) \\
\langle a \times b, x \times y \rangle &= -\begin{vmatrix} \langle a, x \rangle & \langle b, x \rangle \\ \langle a, y \rangle & \langle b, y \rangle \end{vmatrix}
\end{aligned}$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in E_1^3 (Lagrange identity in E_1^3).

Let $e_1, e_2 \in E_1^3$ be such that $< e_i, e_i >= \pm 1$ and $\langle e_1, e_2 \rangle = 0$ and $e_3 = e_1 \times e_2$. Then these three vectors form an orthonormal frame. If $\langle e_1, e_1 \rangle = \varepsilon_1$ and $\langle e_2, e_2 \rangle = \varepsilon_2$ where $\varepsilon_1, \varepsilon_2 = \pm 1$, it follows from the Lagrange identity that $\langle e_3, e_3 \rangle = -\varepsilon_1\varepsilon_2$. Each vector $x \in E_1^3$ can be written uniquely in terms of e_1, e_2, e_3 by

$$x = \varepsilon_1 \langle x, e_1 \rangle e_1 + \varepsilon_2 \langle x, e_2 \rangle e_2 - \varepsilon_1\varepsilon_2 \langle x, e_3 \rangle e_3.$$

The angle between two vectors in Minkowski 3-space is defined by ([3], [10], [11], [12]):

Definition 2.1. i. Hyperbolic angle: Let x and y be timelike vectors in the same timecone of Minkowski space. Then there is a unique real number $\theta \geq 0$, called the hyperbolic angle between x and y, such that

$$< x, y >= -\|x\| \|y\| \cosh \theta.$$

ii. Central angle: Let x and y be spacelike vectors in Minkowski space that span a timelike vector subspace. Then there is a unique real number $\theta \geq 0$, called the central angle between x and y, such that

$$|< x, y >| = \|x\| \|y\| \cosh \theta.$$

iii. Spacelike angle: Let x and y be spacelike vectors in Minkowski space that span a spacelike vector subspace. Then there is a unique real number θ between 0 and π called the spacelike angle between x and y, such that

$$< x, y >= \|x\| \|y\| \cos \theta.$$

iv. Lorentzian timelike angle: Let x be a spacelike vector and y be a timelike vector in Minkowski space. Then there is a unique real number $\theta \geq 0$, called the Lorentzian timelike angle between x and y, such that

$$|< x, y >| = \|x\| \|y\| \sinh \theta.$$

Definition 2.2. Let M and M^f be two surfaces in E_1^3 and N_p be a unit normal vector of M at the point $P \in M$. Let T_pM be tangent space at $P \in M$ and $\{X_p, Y_p\}$ be an orthonormal bases of T_pM. Let $Z_p = d_1X_p + d_2Y_p + d_3N_p$ be a unit vector, where $d_1, d_2, d_3 \in R$ are constant numbers and $\varepsilon_1 d_1^2 + \varepsilon_2 d_2^2 - \varepsilon_1\varepsilon_2 d_3^2 = \pm 1$. If a function f exists and satisfies the condition $f : M \to M^f$, $f(P) = P + rZ_p$, r constant, M^f is called as the surface at a constant distance from the edge of regression on M and M^f denoted by the pair (M, M^f).

If $d_1 = d_2 = 0$, then we have $Z_p = N_p$ and $f(P) = P + rN_p$. In this case M and M^f are parallel surfaces [15].

Theorem 2.1. *Let the pair (M, M^f) be given in E_1^3. For any $W \in \chi(M)$, we have $f_*(W) = \overline{W} + r\overline{D_W Z}$, where $W = \sum\limits_{i=1}^{3} w_i \dfrac{\partial}{\partial x_i}$,$\overline{W} = \sum\limits_{i=1}^{3} \overline{w_i} \dfrac{\partial}{\partial x_i}$ and $\forall P \in M$, $w_i(P) = \overline{w_i}(f(p))$, $1 \le i \le 3$ [15].*

Let (ϕ, U) be a parametrization of M, so we can write that

$$\phi : \underset{(u,v)}{U} \subset E_1^3 \to \underset{P=\phi(u,v)}{M}.$$

In this case $\{\phi_u|_p, \phi_v|_p\}$ is a basis of T_pM. Let N_p is a unit normal vector at $P \in M$ and $d_1, d_2, d_3 \in R$ be constant numbers then we can write that $Z_p = d_1\phi_u|_p + d_2\phi_v|_p + d_3N_p$. Since $M^f = \{f(P) \mid f(P) = P + rZ_p\}$, a parametric representation of M^f is $\psi(u,v) = \phi(u,v) + rZ(u,v)$. Thus we can write

$$\begin{aligned} M^f \;=\; & \{\psi(u,v) \mid \psi(u,v) = \phi(u,v) + r(d_1\phi_u(u,v) + d_2\phi_v(u,v) + d_3N(u,v)), \\ & \qquad d_1,\ d_2,\ d_3, r \text{ are constant}, \quad \varepsilon_1 d_1^2 + \varepsilon_2 d_2^2 - \varepsilon_1\varepsilon_2 d_3^2 = \pm 1\big\}. \end{aligned}$$

If we take $rd_1 = \lambda_1,\ rd_2 = \lambda_2,\ rd_3 = \lambda_3$ then we have

$$M^f = \{\psi(u,v)|\psi(u,v) = \phi(u,v)+\lambda_1\phi_u(u,v)+\lambda_2\phi_v(u,v)+\lambda_3N(u,v), \quad \lambda_1, \lambda_2, \lambda_3 \text{ are constant}\}.$$

Let $\{\phi_u, \phi_v\}$ is basis of $\chi(M^f)$. If we take $\langle\phi_u, \phi_u\rangle = \varepsilon_1,\ \langle\phi_v, \phi_v\rangle = \varepsilon_2$ and $\langle N, N\rangle = -\varepsilon_1\varepsilon_2$, then

$$\begin{aligned} \psi_u &= (1 + \lambda_3 k_1)\phi_u + \varepsilon_2\lambda_1 k_1 N, \\ \psi_v &= (1 + \lambda_3 k_2)\phi_v + \varepsilon_1\lambda_2 k_2 N \end{aligned}$$

is a basis of $\chi(M^f)$, where N is the unit normal vector field on M and k_1, k_2 are principal curvatures of M [15].

Theorem 2.2. *Let the pair (M, M^f) be given in E_1^3. Let $\{\phi_u, \phi_v\}$ (orthonormal and principal vector fields on M) be basis of $\chi(M)$ and k_1, k_2 be principal curvatures of M. The matrix of the shape operator of M^f with respect to the basis $\{\psi_u = (1 + \lambda_3 k_1)\phi_u + \varepsilon_2\lambda_1 k_1 N,\ \psi_v = (1 + \lambda_3 k_2)\phi_v + \varepsilon_1\lambda_2 k_2 N\}$ of $\chi(M^f)$ is*

$$S^f = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{bmatrix}$$

where

$$
\begin{aligned}
\mu_1 &= \frac{(1+\lambda_3 k_2)}{A^3}\left\{\varepsilon\lambda_1\frac{\partial k_1}{\partial u}(\lambda_2^2k_2^2-\varepsilon_1(1+\lambda_3k_2)^2)+k_1A^2\right\},\\
\mu_2 &= \frac{\varepsilon\lambda_1^2\lambda_2k_1k_2(1+\lambda_3k_2)}{A^3}\frac{\partial k_1}{\partial u},\\
\mu_3 &= \frac{-\varepsilon\lambda_1\lambda_2^2k_1k_2(1+\lambda_3k_1)}{A^3}\frac{\partial k_2}{\partial v},\\
\mu_4 &= \frac{(1+\lambda_3k_1)}{A^3}\left\{-\varepsilon\lambda_2\frac{\partial k_2}{\partial v}(\lambda_1^2k_1^2-\varepsilon_2(1+\lambda_3k_1)^2)+k_2A^2\right\}
\end{aligned}
$$

and $A=\sqrt{\varepsilon\left(\varepsilon_1\lambda_1^2k_1^2(1+\lambda_3k_2)^2+\varepsilon_2\lambda_2^2k_2^2(1+\lambda_3k_1)^2-\varepsilon_1\varepsilon_2(1+\lambda_3k_1)^2(1+\lambda_3k_2)^2\right)}$ [15].

Definition 2.3. Let M be an Euclidean surface in E^3 and S be shape operator of M. For any $X_p, Y_p \in T_pM$, if

$$\langle S(X_p), Y_p\rangle = 0 \tag{2.1}$$

then X_p and Y_p are called conjugate tangent vectors of M at p [9].

Definition 2.4. Let M be an Euclidean surface in E^3 and S be shape operator of M. For any $X_p \in T_pM$, if

$$\langle S(X_p), X_p\rangle = 0 \tag{2.2}$$

then X_p is called an asymptotic direction of M at p [9].

We can get the definitions of conjugate tangent vectors and asymptotic direction in Minkowski 3-space similar to Definition 2.3 and 2.4 as below:

Definition 2.5. Let M be a surface in E_1^3 and S be shape operator of M. For any $X_p, Y_p \in T_pM$, if

$$\langle S(X_p), Y_p\rangle = 0 \tag{2.3}$$

then X_p and Y_p are called conjugate tangent vectors of M at p.

Definition 2.6. M be a surface in E_1^3 and S be shape operator of M. For any $X_p \in T_pM$, if

$$\langle S(X_p), X_p\rangle = 0 \tag{2.4}$$

then X_p is called an asymptotic direction of M at p.

3. Conjugate tangent vectors for surfaces at a constant distance from edge of regression on a surface in E_1^3

Theorem 3.1. *Let M^f be a surface at a constant distance from edge of regression on a M in E_1^3. Let k_1 and k_2 denote principal curvature function of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. For $X_p, Y_p \in T_pM$, $f_*(X_p)$ and $f_*(Y_p)$ are conjugate tangent vectors if and only if*

$$\varepsilon_1\mu_1^*x_1y_1+\varepsilon_1\mu_2^*x_1y_2+\varepsilon_2\mu_3^*x_2y_1+\varepsilon_2\mu_4^*x_2y_2=0 \tag{3.1}$$

where

$$
\begin{aligned}
x_1 &= \langle X_p, \phi_u \rangle, \qquad x_2 = \langle X_p, \phi_v \rangle, \\
y_1 &= \langle Y_p, \phi_u \rangle, \qquad y_2 = \langle Y_p, \phi_v \rangle, \\
\mu_1^* &= \mu_1(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_2\mu_1\lambda_1 k_1 + \varepsilon_1\mu_2\lambda_2 k_2), \\
\mu_2^* &= \mu_2(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\varepsilon_2\mu_1\lambda_1 k_1 + \varepsilon_1\mu_2\lambda_2 k_2), \\
\mu_3^* &= \mu_3(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_2\mu_3\lambda_1 k_1 + \varepsilon_1\mu_4\lambda_2 k_2), \\
\mu_4^* &= \mu_4(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\varepsilon_2\mu_3\lambda_1 k_1 + \varepsilon_1\mu_4\lambda_2 k_2).
\end{aligned} \tag{3.2}
$$

Proof. Let $f_*(X_p) \in T_{f(p)}M^f$. Then let us calculate $f_*(X_p)$ and $S^f(f_*(X_p))$. Since ϕ_u and ϕ_v are orthonormal we have

$$
\begin{aligned}
X_p &= \varepsilon_1 \langle X_p, \phi_u \rangle \phi_u + \varepsilon_2 \langle X_p, \phi_v \rangle \phi_v \\
&= \varepsilon_1 x_1 \phi_u + \varepsilon_2 x_2 \phi_v.
\end{aligned}
$$

Further without lost of generality, we suppose that X_p is a unit vector. Then

$$
\begin{aligned}
f_*(X_p) &= \varepsilon_1 x_1 f_*(\phi_u) + \varepsilon_2 x_2 f_*(\phi_v) \\
&= \varepsilon_1 x_1 \psi_u + \varepsilon_2 x_2 \psi_v.
\end{aligned} \tag{3.3}
$$

On the other hand we find that

$$
\begin{aligned}
S^f(f_*(X_p)) &= \varepsilon_1 x_1 S^f(\psi_u) + \varepsilon_2 x_2 S^f(\psi_v) \\
&= \varepsilon_1 x_1 \left(\mu_1(1+\lambda_3 k_1)\phi_u + \mu_2(1+\lambda_3 k_2)\phi_v + (\mu_1\varepsilon_2\lambda_1 k_1 + \mu_2\varepsilon_1\lambda_2 k_2)N\right) \\
&\quad + \varepsilon_2 x_2 \left(\mu_3(1+\lambda_3 k_1)\phi_u + \mu_4(1+\lambda_3 k_2)\phi_v + (\mu_3\varepsilon_2\lambda_1 k_1 + \mu_4\varepsilon_1\lambda_2 k_2)N\right)
\end{aligned} \tag{3.4}
$$

and for $Y_p \in T_pM$ we have

$$
\begin{aligned}
Y_p &= \varepsilon_1 \langle Y_p, \phi_u \rangle \phi_u + \varepsilon_2 \langle Y_p, \phi_v \rangle \phi_v \\
&= \varepsilon_1 y_1 \phi_u + \varepsilon_2 y_2 \phi_v.
\end{aligned} \tag{3.5}
$$

Then

$$
\begin{aligned}
f_*(Y_p) &= \varepsilon_1 y_1 f_*(\phi_u) + \varepsilon_2 y_2 f_*(\phi_v) \\
&= \varepsilon_1 y_1 \psi_u + \varepsilon_2 y_2 \psi_v.
\end{aligned} \tag{3.6}
$$

Thus using equations (3.4) and (3.6) in equation (2.3) we obtain (3.1). □

Theorem 3.2. *Let M^f be a surface at a constant distance from edge of regression on M in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $X_p \in T_pM$ and ϕ_u, ϕ_v by θ_1, θ_2 respectively and the angle between $Y_p \in T_pM$ and ϕ_u, ϕ_v by θ_1', θ_2' respectively. $f_*(X_p)$ and $f_*(Y_p)$ are conjugate tangent vectors if and only if*
(a)Let N_p be a timelike vector then

$$
\mu_1^* \cos\theta_1 \cos\theta_1' + \mu_2^* \cos\theta_1 \cos\theta_2' + \mu_3^* \cos\theta_2 \cos\theta_1' + \mu_4^* \cos\theta_2 \cos\theta_2' = 0.
$$

(b) Let ϕ_u be a timelike vector.
(b.1) If X_p and Y_p are spacelike vectors then

$$
\begin{aligned}
0 = &-\delta_1\delta_1'\mu_1^* \sinh\theta_1 \sinh\theta_1' - \delta_1\delta_2'\mu_2^* \sinh\theta_1 \cosh\theta_2' \\
&+\delta_1'\delta_2\mu_3^* \cosh\theta_2 \sinh\theta_1' + \delta_2\delta_2'\mu_4^* \cosh\theta_2 \cosh\theta_2'.
\end{aligned}
$$

(b.2) If X_p, Y_p and ϕ_u are timelike vectors in the same timecone then

$$\begin{aligned}0 &= \mu_1^* \cosh\theta_1 \cosh\theta_1' + \delta_2' \mu_2^* \cosh\theta_1 \sinh\theta_2' \\ &\quad -\delta_2 \mu_3^* \sinh\theta_2 \cosh\theta_1' + \delta_2 \delta_2' \mu_4^* \sinh\theta_2 \sinh\theta_2'.\end{aligned}$$

(b.3) If X_p, ϕ_u are timelike vectors in the same timecone and Y_p is spacelike vector then

$$\begin{aligned}0 &= \delta_1' \mu_1^* \cosh\theta_1 \sinh\theta_1' + \delta_2' \mu_2^* \cosh\theta_1 \cosh\theta_2' \\ &\quad +\delta_1' \delta_2 \mu_3^* \sinh\theta_2 \sinh\theta_1' + \delta_2 \delta_2' \mu_4^* \sinh\theta_2 \cosh\theta_2'.\end{aligned}$$

(b.4) If Y_p and ϕ_u are timelike vectors in the same timecone and X_p is spacelike vector then

$$\begin{aligned}0 &= \delta_1 \mu_1^* \sinh\theta_1 \cosh\theta_1' - \delta_1 \delta_2' \mu_2^* \sinh\theta_1 \sinh\theta_2' \\ &\quad -\delta_2 \mu_3^* \cosh\theta_2 \cosh\theta_1' + \delta_2 \delta_2' \mu_4^* \cosh\theta_2 \sinh\theta_2'.\end{aligned}$$

(c) Let ϕ_v be a timelike vector.
(c.1) If X_p and Y_p are spacelike vectors then

$$\begin{aligned}0 &= \delta_1 \delta_1' \mu_1^* \cosh\theta_1 \cosh\theta_1' + \delta_1 \delta_2' \mu_2^* \cosh\theta_1 \sinh\theta_2' \\ &\quad -\delta_1' \delta_2 \mu_3^* \sinh\theta_2 \cosh\theta_1' - \delta_2 \delta_2' \mu_4^* \sinh\theta_2 \sinh\theta_2'.\end{aligned}$$

(c.2) If X_p, Y_p and ϕ_v are timelike vectors in the same timecone then

$$\begin{aligned}0 &= \delta_1 \delta_1' \mu_1^* \sinh\theta_1 \sinh\theta_1' - \delta_1 \mu_2^* \sinh\theta_1 \cosh\theta_2' \\ &\quad -\delta_1' \mu_3^* \cosh\theta_2 \sinh\theta_1' - \mu_4^* \cosh\theta_2 \cosh\theta_2'.\end{aligned}$$

(c.3) If X_p and ϕ_v are timelike vectors in the same timecone and Y_p is spacelike vector then

$$\begin{aligned}0 &= \delta_1 \delta_1' \mu_1^* \sinh\theta_1 \cosh\theta_1' + \delta_1 \delta_2' \mu_2^* \sinh\theta_1 \sinh\theta_2' \\ &\quad +\delta_1' \mu_3^* \cosh\theta_2 \cosh\theta_1' + \delta_2' \mu_4^* \cosh\theta_2 \sinh\theta_2'.\end{aligned}$$

(c.4) If Y_p and ϕ_v are timelike vectors in the same timecone and X_p is spacelike vector then

$$\begin{aligned}0 &= \delta_1 \delta_1' \mu_1^* \cosh\theta_1 \sinh\theta_1' - \delta_1 \mu_2^* \cosh\theta_1 \cosh\theta_2' \\ &\quad -\delta_2 \delta_1' \mu_3^* \sinh\theta_2 \sinh\theta_1' + \delta_2 \mu_4^* \sinh\theta_2 \cosh\theta_2'.\end{aligned}$$

Abovementioned $\mu_1^*, \mu_2^*, \mu_3^*$ and μ_4^* are given in (3.2),

$$\delta_i = \begin{cases} 1, & x_i \text{ is positive} \\ -1, & x_i \text{ is negative} \end{cases}, \quad i = (1, 2)$$

and

$$\delta_i' = \begin{cases} 1, & y_i \text{ is positive} \\ -1, & y_i \text{ is negative} \end{cases}, \quad i = (1, 2).$$

Proof. (a) Let N_p be a timelike vector. In this case θ_1, θ_2, θ_1', θ_2' are spacelike angles then

$$\begin{aligned} x_1 &= \langle X_p, \phi_u \rangle = \cos\theta_1 \\ x_2 &= \langle X_p, \phi_v \rangle = \cos\theta_2. \end{aligned}$$

and

$$\begin{aligned} y_1 &= \langle Y_p, \phi_u \rangle = \cos\theta_1' \\ y_2 &= \langle Y_p, \phi_v \rangle = \cos\theta_2'. \end{aligned}$$

Substituting these equations in (3.1) the proof is obvious.

(b) Let ϕ_u be a timelike vector.
(b.1) If X_p and Y_p are spacelike vectors and ϕ_u is timelike vector then there are Lorentzian timelike angles θ_1, θ_1' and central angles θ_2, θ_2'. Thus

$$\begin{aligned} x_1 &= \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \cosh\theta_2 \\ y_1 &= \delta_1' \sinh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \cosh\theta_2'. \end{aligned}$$

(b.2) If X_p, Y_p and ϕ_u are timelike vectors in the same timecone then there are hyperbolic angles θ_1, θ_1' and Lorentzian timelike angles θ_2, θ_2'. Thus

$$\begin{aligned} x_1 &= -\cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2 \\ y_1 &= -\cosh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \sinh\theta_2'. \end{aligned}$$

(b.3) If X_p and ϕ_u are timelike vectors in the same timecone and Y_p is spacelike vector then there is a hyperbolic angle θ_1, a central angle θ_2' and there are Lorentzian timelike angles θ_2, θ_1'. Thus

$$\begin{aligned} x_1 &= -\cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2 \\ y_1 &= \delta_1' \sinh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \cosh\theta_2'. \end{aligned}$$

(b.4) If Y_p and ϕ_u are timelike vectors in the same timecone and X_p is spacelike vector then there is a central angle θ_2, a hyperbolic angle θ_1' and there are Lorentzian timelike angles θ_1, θ_2'. Thus

$$\begin{aligned} x_1 &= \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \cosh\theta_2 \\ y_1 &= -\cosh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \sinh\theta_2'. \end{aligned}$$

(c) Let ϕ_v be a timelike vector.
(c.1) If X_p and Y_p are spacelike vectors and ϕ_v is timelike vector then there are central angles θ_1, θ_1' and Lorentzian timelike angles θ_2, θ_2'.Thus

$$\begin{aligned} x_1 &= \delta_1 \cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2 \\ y_1 &= \delta_1' \cosh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \sinh\theta_2'. \end{aligned}$$

(c.2) If X_p, Y_p and ϕ_v are timelike vectors in the same timecone then there are Lorentzian timelike angles θ_1, θ_1' and hyperbolic angles θ_2, θ_2'. Thus

$$\begin{aligned} x_1 &= \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = -\cosh\theta_2 \\ y_1 &= \delta_1' \sinh\theta_1' \quad \text{and} \quad y_2 = -\cosh\theta_2'. \end{aligned}$$

(c.3) If X_p and ϕ_v are timelike vectors in the same timecone and Y_p is spacelike vector then there is a hyperbolic angle θ_2, a central angle θ_1' and there are Lorentzian timelike vectors θ_1, θ_2'. Thus

$$\begin{aligned} x_1 &= \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = -\cosh\theta_2 \\ y_1 &= \delta_1' \cosh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \sinh\theta_2'. \end{aligned}$$

(c.4) If Y_p and ϕ_v are timelike vectors in the same timecone and X_p is spacelike vector then then there is a central angle θ_1, a hyperbolic angle θ_2' and there are Lorentzian timelike angles θ_1', θ_2. Thus

$$\begin{aligned} x_1 &= \delta_1 \cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2 \\ y_1 &= \delta_1' \sinh\theta_1' \quad \text{and} \quad y_2 = -\cosh\theta_2'. \end{aligned}$$

□

As a special case if we take $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = r = constant$, then we obtain that M and M^f are parallel surfaces. Hence we give the following corollaries.

Corollary 3.1. *Let M and M_r be parallel surfaces in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $X_p \in T_pM$ and ϕ_u, ϕ_v by θ_1, θ_2 respectively and the angle between $Y_p \in T_pM$ and ϕ_u, ϕ_v by θ_1', θ_2' respectively. $f_*(X_p)$ and $f_*(Y_p)$ are conjugate tangent vectors if and only if*

$$\varepsilon_1 k_1(1 + rk_1)x_1y_1 + \varepsilon_2 k_2(1 + rk_2)x_2y_2 = 0. \tag{3.7}$$

Proof. Since

$$\begin{aligned} \mu_1^* &= k_1(1 + rk_1), \\ \mu_2^* &= 0, \qquad \mu_3^* = 0, \\ \mu_4^* &= k_2(1 + rk_2) \end{aligned}$$

from (3.1) we find (3.7). □

Corollary 3.2. *Let M and M_r be parallel surfaces in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $X_p \in T_pM$ and ϕ_u, ϕ_v by θ_1, θ_2 respectively and the angle between $Y_p \in T_pM$ and ϕ_u, ϕ_v by θ_1', θ_2' respectively. $f_*(Y_p)$ are conjugate tangent vectors if and only if*
(a)Let N_p be a timelike vector then

$$k_1(1 + rk_1)\cos\theta_1\cos\theta_1' + k_2(1 + rk_2)\cos\theta_2\cos\theta_2' = 0.$$

(b) Let ϕ_u be a timelike vector.
(b.1) If X_p and Y_p are spacelike vectors then

$$-\delta_1\delta_1' k_1(1 + rk_1)\sinh\theta_1\sinh\theta_1' + \delta_2\delta_2' k_2(1 + rk_2)\cosh\theta_2\cosh\theta_2' = 0.$$

(b.2) If X_p, Y_p and ϕ_u are timelike vectors in the same timecone then

$$-k_1(1 + rk_1)\cosh\theta_1\cosh\theta_1' + k_2(1 + rk_2)\sinh\theta_2\sinh\theta_2' = 0.$$

(b.3) If X_p and ϕ_u are timelike vectors in the same timecone and Y_p is spacelike vector then

$$\delta_1' k_1(1 + rk_1)\cosh\theta_1\sinh\theta_1' + \delta_2\delta_2' k_2(1 + rk_2)\sinh\theta_2\cosh\theta_2' = 0.$$

(b.4) If Y_p and ϕ_u are timelike vectors in the same timecone and X_p is spacelike vector then

$$\delta_1 k_1(1 + rk_1)\sinh\theta_1\cosh\theta_1' + \delta_2\delta_2' k_2(1 + rk_2)\cosh\theta_2\sinh\theta_2' = 0.$$

(c) Let ϕ_v be a timelike vector.
(c.1) If X_p and Y_p are spacelike vectors then

$$\delta_1\delta_1' k_1(1 + rk_1)\cosh\theta_1\cosh\theta_1' - \delta_2\delta_2' k_2(1 + rk_2)\sinh\theta_2\sinh\theta_2' = 0.$$

(c.2) If X_p, Y_p and ϕ_v are timelike vectors in the same timecone then

$$\delta_1\delta_1' k_1(1 + rk_1)\sinh\theta_1\sinh\theta_1' - k_2(1 + rk_2)\cosh\theta_2\cosh\theta_2' = 0.$$

(c.3) If X_p and ϕ_v are timelike vectors in the same timecone and Y_p is spacelike vector then

$$\delta_1\delta_1' k_1(1 + rk_1)\sinh\theta_1\cosh\theta_1' + \delta_2' k_2(1 + rk_2)\cosh\theta_2\sinh\theta_2' = 0.$$

(c.4) If Y_p and ϕ_v are timelike vectors in the same timecone and X_p is spacelike vector then

$$\delta_1\delta_1' k_1(1+rk_1)\cosh\theta_1\sinh\theta_1' + \delta_2 k_2(1+rk_2)\sinh\theta_2\cosh\theta_2' = 0.$$

For the above equations

$$\delta_i = \begin{cases} 1, & x_i \text{ is positive} \\ -1, & x_i \text{ is negative} \end{cases}, \quad i = (1,2)$$

and

$$\delta_i' = \begin{cases} 1, & y_i \text{ is positive} \\ -1, & y_i \text{ is negative} \end{cases}, \quad i = (1,2).$$

4. Asymptotic directions for surfaces at a constant distance from edge of regression on a surface in E_1^3

Theorem 4.1. *Let M^f be a surface at a constant distance from edge of regression on a M in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. $f_*(X_p) \in T_{f(p)}(M^f)$ is an asymptotic direction if and only if*

$$\mu_1^* x_1^2 + \varepsilon_1\varepsilon_2\mu_2^* x_1 x_2 + \mu_3^* x_2^2 = 0 \tag{4.1}$$

where

$$\begin{aligned} x_1 &= \langle X_p, \phi_u\rangle, \qquad x_2 = \langle X_p, \phi_v\rangle, \\ \mu_1^* &= \varepsilon_1\mu_1(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_1\varepsilon_2\mu_1\lambda_1 k_1 + \mu_2\lambda_2 k_2), \\ \mu_2^* &= \varepsilon_2\mu_2(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\mu_1\lambda_1 k_1 + \varepsilon_1\varepsilon_2\mu_2\lambda_2 k_2) \\ &\quad + \varepsilon_1\mu_3(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_1\varepsilon_2\mu_3\lambda_1 k_1 + \mu_4\lambda_2 k_2), \\ \mu_3^* &= \varepsilon_2\mu_4(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\mu_3\lambda_1 k_1 + \varepsilon_1\varepsilon_2\mu_4\lambda_2 k_2). \end{aligned} \tag{4.2}$$

Proof. Let $f_*(X_p) \in T_{f(p)}(M^f)$. Then let us calculate $f_*(X_p)$ and $S^f(f_*(X_p))$. Since ϕ_u and ϕ_v are orthonormal we have

$$\begin{aligned} X_p &= \varepsilon_1\langle X_p, \phi_u\rangle\phi_u + \varepsilon_2\langle X_p, \phi_v\rangle\phi_v \\ &= \varepsilon_1 x_1\phi_u + \varepsilon_2 x_2\phi_v \end{aligned}$$

Further without lost of generality, we suppose that X_p is a unit vector. Then

$$\begin{aligned} f_*(X_p) &= \varepsilon_1 x_1 f_*(\phi_u) + \varepsilon_2 x_2 f_*(\phi_v) \\ &= \varepsilon_1 x_1\psi_u + \varepsilon_2 x_2\psi_v. \end{aligned} \tag{4.3}$$

On the other hand we find that

$$\begin{aligned} S^f(f_*(X_p)) &= \varepsilon_1 x_1 S^f(\psi_u) + \varepsilon_2 x_2 S^f(\psi_v) \\ &= \varepsilon_1 x_1\left(\mu_1(1+\lambda_3 k_1)\phi_u + \mu_2(1+\lambda_3 k_2)\phi_v + (\mu_1\varepsilon_2\lambda_1 k_1 + \mu_2\varepsilon_1\lambda_2 k_2)N\right) \\ &\quad + \varepsilon_2 x_2\left(\mu_3(1+\lambda_3 k_1)\phi_u + \mu_4(1+\lambda_3 k_2)\phi_v + (\mu_3\varepsilon_2\lambda_1 k_1 + \mu_4\varepsilon_1\lambda_2 k_2)N\right) \end{aligned} \tag{4.4}$$

Thus using equations (4.3) and (4.4) in equation (2.4) we obtain (4.1). □

Corollary 4.1. *Let M^f be a surface at a constant distance from edge of regression on M in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $X_p \in T_pM$ and ϕ_u, ϕ_v by θ_1, θ_2 respectively. $f_*(X_p) \in T_{f(p)}M^f$ is an asymptotic direction if and only if* *(a)Let N_p be a timelike vector then*

$$\mu_1^* \cos^2\theta_1 + \mu_2^* \cos\theta_1 \cos\theta_2 + \mu_3^* \cos^2\theta_2 = 0.$$

(b) Let N_p be a spacelike vector.
(b.1) If X_p and ϕ_u are timelike vectors in the same timecone then

$$\mu_1^* \cosh^2\theta_1 + \delta_2\mu_2^* \cosh\theta_1 \sinh\theta_2 + \mu_3^* \sinh^2\theta_2 = 0.$$

(b.2) If X_p and ϕ_v are timelike vectors in the same timecone then

$$\mu_1^* \sinh^2\theta_1 + \delta_1\mu_2^* \sinh\theta_1 \cosh\theta_2 + \mu_3^* \cosh^2\theta_2 = 0.$$

(b.3) If X_p is a spacelike vector and ϕ_u is timelike vector then

$$\mu_1^* \sinh^2\theta_1 - \delta_1\delta_2\mu_2^* \sinh\theta_1 \cosh\theta_2 + \mu_3^* \cosh^2\theta_2 = 0.$$

(b.4) If X_p is a spacelike vector and ϕ_v is timelike vector then

$$\mu_1^* \cosh^2\theta_1 - \delta_1\delta_2\mu_2^* \cosh\theta_1 \sinh\theta_2 + \mu_3^* \sinh^2\theta_2 = 0.$$

Abovementioned μ_1^*, μ_2^* and μ_3^* are given in (4.2) and

$$\delta_i = \begin{cases} 1, & x_i \text{ is positive} \\ -1, & x_i \text{ is negative} \end{cases}, \quad i = (1,2).$$

Proof. (a) Let N_p be a timelike vector. In this case θ_1 and θ_2 are spacelike angles then

$$\begin{aligned} x_1 &= \langle X_p, \phi_u \rangle = \cos\theta_1 \\ x_2 &= \langle X_p, \phi_v \rangle = \cos\theta_2. \end{aligned}$$

Substituting these equations in (4.1) the proof is obvious.
(b) Let N_p be a spacelike vector.
(b.1) If X_p and ϕ_u are timelike vectors in the same timecone then there is a hyperbolic angle θ_1 and a Lorentzian timelike angle θ_2. Since

$$x_1 = -\cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2$$

the proof is obvious.
(b.2) If X_p and ϕ_v are timelike vectors in the same timecone then there is a Lorentzian timelike angle θ_1 and a hyperbolic angle θ_2. Thus

$$x_1 = \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = -\cosh\theta_2.$$

(b.3) If X_p is a spacelike vector and ϕ_u is timelike vector then there is a Lorentzian timelike angle θ_1 and a central angle θ_2. Thus

$$x_1 = \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \cosh\theta_2.$$

(b.4) If X_p is a spacelike vector and ϕ_v is timelike vector then there is a central angle θ_1 and a Lorentzian timelike angle θ_2. Thus

$$x_1 = \delta_1 \cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2.$$

□

As a special case if M and M_r be parallel surfaces from (4.1) and (4.2) we obtain that $f_*(X_p) \in T_{f(p)}M_r$ is an asymptotic direction if and only if

$$\varepsilon_1 k_1 (1 + rk_1) x_1^2 + \varepsilon_2 k_2 (1 + rk_2) x_2^2 = 0.$$

Corollary 4.2. *Let M and M_r be parallel surfaces in E_1^3. Let k_1 and k_2 denote principal curvature function of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $X_p \in T_pM$ and ϕ_u, ϕ_v by θ_1, θ_2 respectively. $f_*(X_p) \in T_{f(p)}M_r$ is an asymptotic direction if and only if*
(a)Let N_p be a timelike vector then

$$k_1(1+rk_1)\cos^2\theta_1 + k_2(1+rk_2)\cos^2\theta_2 = 0.$$

(b) Let N_p be a spacelike vector.
(b.1) If X_p and ϕ_u are timelike vectors in the same timecone then

$$-k_1(1+rk_1)\cosh^2\theta_1 + k_2(1+rk_2)\sinh^2\theta_2 = 0.$$

(b.2) If X_p and ϕ_v are timelike vectors in the same timecone then

$$k_1(1+rk_1)\sinh^2\theta_1 - k_2(1+rk_2)\cosh^2\theta_2 = 0.$$

(b.3) If X_p is a spacelike vector and ϕ_u is timelike vector then

$$-k_1(1+rk_1)\sinh^2\theta_1 + k_2(1+rk_2)\cosh^2\theta_2 = 0.$$

(b.4) If X_p is a spacelike vector and ϕ_v is timelike vector then

$$k_1(1+rk_1)\cosh^2\theta_1 - k_2(1+rk_2)\sinh^2\theta_2 = 0.$$

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SOME GRÜSS TYPE INEQUALITIES FOR THE RIEMANN–STIELTJES INTEGRAL WITH LIPSCHITZIAN INTEGRATORS

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Abstract. In this paper several new inequalities of Grüss' type for the Riemann–Stieltjes integral with Lipschitzian integrators are proved.

1. Introduction

The *Čebyšev functional*

$$\mathcal{T}(f,g)=\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt, \tag{1.1}$$

has interesting applications in the approximation of weighted integrals as one can has from the literature below.

Bounding Čebyšev functional has a long history, starting with Grüss inequality [14] in 1935, where Grüss had proved that for two integrable mappings f,g such that $\phi\le f(x)\le\Phi$ and $\gamma\le f(x)\le\Gamma$, the inequality

$$|\mathcal{T}(f,g)|\le\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma) \tag{1.2}$$

holds, and the constant $\frac{1}{4}$ is the best possible.

After that many authors have studied the functional (1.1) and several bounds under various assumptions for the functions involved have been obtained. For new results and generalizations the reader may refer to [2]–[15].

A generalization of (1.1) for Riemann–Stieltjes integral was considered by Dragomir in [10]. Namely, the author has introduced the following *Čebyšev functional for the*

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Riemann–Stieltjes integral:

$$\begin{aligned}(1.3)\quad \mathcal{T}(f,g;u) := {} & \frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) \\ & -\frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t)\end{aligned}$$

under the assumptions that, f, g are continuous on $[a, b]$ and u is of bounded variation on $[a, b]$ with $u(b) \neq u(a)$.

By simple computations with Riemann–Stieltjes integral, Dragomir [10] has introduced the identity,

$$\begin{aligned}(1.4)\quad \mathcal{T}(f,g;u) := {} & \frac{1}{u(b)-u(a)}\int_a^b \left[f(t)-\frac{f(a)+f(b)}{2}\right] \\ & \times\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t),\end{aligned}$$

to obtain several sharp bounds of the Čebyšev functional for the Riemann–Stieltjes integral (1.3).

In this work, several sharp inequalities of Grüss' type for the Riemann–Stieltjes integral with Lipschitzian integrators are proved.

2. The Results

We recall that a function $f : [a, b] \to \mathbb{C}$ is p–H_f–Holder continuous on $[a, b]$, if

$$|f(t) - f(s)| \leq H_f\,|t - s|^p$$

for all $t, s \in [a, b]$, where $p \in (0, 1]$ and $H_f > 0$ are given. If $p = 1$ we call f H_f–Lipschitzian.

We are ready to state our first result as follows:

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a p–H_f–Hölder continuous on $[a, b]$, where $p \in (0, 1]$ and $H_f > 0$; are given. Let $g, u : [a, b] \to \mathbb{R}$ be such that g is Lebesgue integrable on $[a, b]$ and there exists the real numbers m, M such that $m \leq g(x) \leq M$ for all $x \in [a, b]$, and u is L_u–Lipschitzian on $[a, b]$ then*

$$(2.1)\qquad |\mathcal{T}(f,g;u)| \leq \frac{L_u H_f}{(p+1)}\cdot\frac{(M-m)}{|u(b)-u(a)|}\cdot(b-a)^{p+1}.$$

Proof. Taking the modulus in (1.4) and utilizing the triangle inequality, we get

$$\begin{aligned}
\left|\mathcal{T}\left(f,g;u\right)\right| &= \left|\frac{1}{u\left(b\right)-u\left(a\right)}\int_a^b\left[f\left(t\right)-\frac{f\left(a\right)+f\left(b\right)}{2}\right]\right. \\
&\qquad \left.\times\left[g\left(t\right)-\frac{1}{u\left(b\right)-u\left(a\right)}\int_a^b g\left(s\right)du\left(s\right)\right]du\left(t\right)\right| \\
&\le \frac{L_u}{\left|u\left(b\right)-u\left(a\right)\right|}\cdot\int_a^b\left|f\left(t\right)-\frac{f\left(a\right)+f\left(b\right)}{2}\right| \\
&\qquad \times\left|g\left(t\right)-\frac{1}{u\left(b\right)-u\left(a\right)}\int_a^b g\left(s\right)du\left(s\right)\right|dt \\
&\le \frac{L_u}{\left|u\left(b\right)-u\left(a\right)\right|}\cdot\sup_{t\in[a,b]}\left|g\left(t\right)-\frac{1}{u\left(b\right)-u\left(a\right)}\int_a^b g\left(s\right)du\left(s\right)\right| \\
&\qquad \times\int_a^b\left|f\left(t\right)-\frac{f\left(a\right)+f\left(b\right)}{2}\right|dt
\end{aligned}$$

$$\begin{aligned}
&\le \frac{L_u}{\left|u\left(b\right)-u\left(a\right)\right|}\cdot\frac{L_u\left(M-m\right)}{\left|u\left(b\right)-u\left(a\right)\right|}\left(b-a\right)\cdot\frac{H_f}{2}\int_a^b\left[(t-a)^p+(b-t)^p\right]dt \\
&= \frac{L_u^2 H_f}{p+1}\cdot\frac{\left(M-m\right)}{\left(u\left(b\right)-u\left(a\right)\right)^2}\left(b-a\right)^{p+2},
\end{aligned}$$

since $m\le g(x)\le M$, for all $x\in[a,b]$, then

$$\begin{aligned}
\left|g\left(t\right)-\frac{1}{u\left(b\right)-u\left(a\right)}\int_a^b g\left(s\right)du\left(s\right)\right| &\le \left|\frac{\int_a^b\left[g\left(t\right)-g\left(s\right)\right]du\left(s\right)}{u\left(b\right)-u\left(a\right)}\right| \\
&\le \frac{L_u}{\left|u\left(b\right)-u\left(a\right)\right|}\int_a^b\left|g\left(t\right)-g\left(s\right)\right|ds \\
&\le \frac{L_u\left(M-m\right)}{\left|u\left(b\right)-u\left(a\right)\right|}\left(b-a\right), \qquad (2.2)
\end{aligned}$$

which completes the proof. □

Corollary 2.1. *Let g,u be as in Theorem 2.1. If $f:[a,b]\to\mathbb{R}$ is L_f–Lipschitzian on $[a,b]$, then*

$$\left|T\left(f,g;u\right)\right|\le\frac{L_u^2 L_f\left(M-m\right)}{2\left(u\left(b\right)-u\left(a\right)\right)^2}\left(b-a\right)^3 \qquad (2.3)$$

Remark 2.1. Under the assumptions of Theorem 2.1, we have

$$\left|\mathcal{T}\left(f,g\right)\right|\le\frac{H_f}{\left(p+1\right)}\left(M-m\right)\cdot\left(b-a\right)^p. \qquad (2.4)$$

In particular, if f is L_f–Lipschitzian, then

$$\left|T\left(f,g\right)\right|\le\frac{1}{2}L_f\left(b-a\right)\left(M-m\right). \qquad (2.5)$$

Theorem 2.2. *Let g, u be as in Theorem 2.1. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$, then we have*

$$|\mathcal{T}(f, g; u)| \leq \frac{1}{2} \frac{L_u (M - m)}{(u(b) - u(a))^2} (b - a) \cdot \bigvee_a^b (f). \tag{2.6}$$

Proof. Since u is L_u–Lipschitzian on $[a, b]$, as in Theorem 2.1, we have

$$\begin{aligned} |\mathcal{T}(f, g; u)| \leq \frac{L_u}{|u(b) - u(a)|} \cdot \sup_{t \in [a,b]} \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s)\, du(s) \right| \\ \times \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| dt \end{aligned}$$

Since $m \leq g \leq M$, by (2.2) we have

$$\begin{aligned} \frac{1}{|u(b) - u(a)|} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s)\, du(s) \right| du(t) \\ \leq \frac{L_u (M - m)}{(u(b) - u(a))^2} (b - a). \end{aligned} \tag{2.7}$$

Now as f is of bounded variation on $[a, b]$, we have

$$\begin{aligned} \sup_{t \in [a,b]} \left| f(t) - \frac{f(a) + f(b)}{2} \right| &= \sup_{t \in [a,b]} \left| \frac{f(t) - f(a) + f(t) - f(b)}{2} \right| \\ &\leq \frac{1}{2} \sup_{t \in [a,b]} \left[|f(t) - f(a)| + |f(t) - f(b)| \right] \leq \frac{1}{2} \bigvee_a^b (f), \end{aligned} \tag{2.8}$$

for all $t \in [a, b]$. Finally, combining the inequalities (2.7)–(2.8), we obtain the required result (2.6). □

Theorem 2.3. *Let $g, u : [a, b] \to \mathbb{R}$ be such that g is of bounded variation on $[a, b]$, and u be L_u–Lipschitzian on $[a, b]$, then we have*

$$\begin{aligned} &|\mathcal{T}(f, g; u)| \\ &\leq \begin{cases} \frac{H_f L_u^2 (b-a)^{p+2}}{(p+1)(u(b)-u(a))^2} \cdot \bigvee_a^b (g), & if\ f\ is\ H_f - p\text{–Hölder} \\ \\ \frac{L_u^2 (b-a)^2}{2(u(b)-u(a))^2} \bigvee_a^b (g) \cdot \bigvee_a^b (f), & if\ f\ is\ of\ bounded\ variation \end{cases} \end{aligned} \tag{2.9}$$

where, $L_u, H_f > 0$ and $p \in (0, 1]$ are given.

Proof. Using (1.4) we may write

$$
\begin{aligned}
&|\mathcal{T}(f,g;u)| \qquad (2.10)\\
&\leq \frac{L_u}{|u(b)-u(a)|}\int_a^b \left|f(t)-\frac{f(a)+f(b)}{2}\right|\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt\\
&= \frac{L_u}{(u(b)-u(a))^2}\int_a^b \left|f(t)-\frac{f(a)+f(b)}{2}\right|\left|\int_a^b [g(t)-g(s)]\,du(s)\right|dt\\
&\leq \frac{L_u^2}{(u(b)-u(a))^2}\int_a^b \left[\left|f(t)-\frac{f(a)+f(b)}{2}\right|\cdot\int_a^b |g(t)-g(s)|\,ds\right]dt,
\end{aligned}
$$

but since g is of bounded variation then we have,

$$
\int_a^b |g(t)-g(s)|\,ds \leq \sup_{s\in[a,b]}|g(t)-g(s)|\cdot\int_a^b ds \leq (b-a)\bigvee_a^b(g). \tag{2.11}
$$

Therefore, if f is of p–Hölder type, then we have

$$
\begin{aligned}
&|\mathcal{T}(f,g;u)|\\
&\leq \frac{1}{2}\cdot\frac{L_u^2(b-a)}{(u(b)-u(a))^2}\cdot\bigvee_a^b(g)\cdot\int_a^b [|f(t)-f(a)|+|f(t)-f(b)|]\,dt\\
&\leq \frac{H_f}{2}\cdot\frac{L_u^2(b-a)}{(u(b)-u(a))^2}\cdot\bigvee_a^b(g)\cdot\int_a^b [|t-a|^p+|t-b|^p]\,dt\\
&= \frac{H_f}{(p+1)}\cdot\frac{L_u^2(b-a)^{p+2}}{(u(b)-u(a))^2}\cdot\bigvee_a^b(g),
\end{aligned}
$$

which prove the first part of inequality (2.9).

To prove the second part of (2.9), assume that f is of bounded variation, then by (2.10) we have

$$
\begin{aligned}
&|\mathcal{T}(f,g;u)|\\
&\leq \frac{L_u^2}{(u(b)-u(a))^2}\int_a^b \left[\left|f(t)-\frac{f(a)+f(b)}{2}\right|\cdot\int_a^b |g(t)-g(s)|\,ds\right]dt\\
&\leq \frac{L_u^2(b-a)^2}{2(u(b)-u(a))^2}\bigvee_a^b(g)\cdot\bigvee_a^b(f),
\end{aligned}
$$

and thus the theorem is proved. □

Remark 2.2. Under the assumptions of Theorem 2.3, we have

$$
|\mathcal{T}(f,g)| \leq \begin{cases} \frac{H_f}{p+1}(b-a)^p\cdot\bigvee_a^b(g), & \textit{if } f \textit{ is } H_f-p\text{–Hölder}\\[2ex] \frac{1}{2}\bigvee_a^b(g)\cdot\bigvee_a^b(f), & \textit{if } f \textit{ is of bounded variation}\end{cases} \tag{2.12}
$$

where, $H_f, > 0$ and $p \in (0,1]$ are given.

An improvement for the first inequality in (2.9) may be stated as follows:

Corollary 2.2. *Let g, u be as in Theorem 2.3 and $f:[a,b]\to\mathbb{R}$ be of p-H_f–Holder type on $[a,b]$, then*

$$|\mathcal{T}(f,g;u)| \le \frac{L_u^2 H_f (b-a)^{p+2}}{2^p (u(b)-u(a))^2} \bigvee_a^b (g). \tag{2.13}$$

Proof. By Theorem 2.3 we have

$$\begin{aligned}
&|\mathcal{T}(f,g;u)| \\
&\le \frac{L_u^2}{(u(b)-u(a))^2} \int_a^b \left[\left| f(t) - \frac{f(a)+f(b)}{2} \right| \cdot \int_a^b |g(t)-g(s)|\, ds \right] dt \\
&\le \frac{L_u^2 (b-a)}{(u(b)-u(a))^2} \bigvee_a^b (g) \int_a^b \left| f(t) - \frac{f(a)+f(b)}{2} \right| dt \\
&\le \frac{L_u^2 (b-a)^2}{(u(b)-u(a))^2} \bigvee_a^b (g) \cdot \sup_{t\in[a,b]} \left| f(t) - \frac{f(a)+f(b)}{2} \right| \\
&\le \frac{L_u^2 (b-a)^2}{(u(b)-u(a))^2} \bigvee_a^b (g) \cdot H_f \left(\frac{b-a}{2} \right)^p,
\end{aligned}$$

and since f is of p–H_f–Holder type on $[a,b]$, we have

$$\begin{aligned}
\left| f(t) - \frac{f(a)+f(b)}{2} \right| &= \left| \frac{f(t)-f(a)+f(t)-f(b)}{2} \right| \\
&\le \frac{1}{2}|f(t)-f(a)| + \frac{1}{2}|f(t)-f(b)| \\
&\le \frac{H_f}{2}\left[(t-a)^p + (b-t)^p\right],
\end{aligned}$$

it follows that

$$\sup_{t\in[a,b]} \left| f(t) - \frac{f(a)+f(b)}{2} \right| \le H_f \left(\frac{b-a}{2} \right)^p. \tag{2.14}$$

which completes the proof. $\square$

Remark 2.3. Under the assumptions of Corollary 2.2, we have

$$|\mathcal{T}(f,g)| \le \frac{1}{2^p} H_f (b-a)^p \bigvee_a^b (g), \tag{2.15}$$

which improves the first inequality in (2.12), where $H_f > 0$ and $p \in (0,1]$ are given.

Theorem 2.4. *Let $g, u:[a,b]\to\mathbb{R}$ be such that g is of q-H_g–Hölder type on $[a,b]$, and and u be L_u–Lipschitzian on $[a,b]$, then we have*

$$\begin{aligned}
&|\mathcal{T}(f,g;u)| \\
&\le L_u^2 H_g \cdot \begin{cases} \frac{(b-a)^{q+2}}{(q+1)(q+2)(u(b)-u(a))^2} \cdot \bigvee_a^b (f), & \textit{if } f \textit{ is of bounded variation} \\ \\ \frac{H_f (b-a)^{p+q+2}}{2^p (q+1)(q+2)(u(b)-u(a))^2}, & \textit{if } f \textit{ is } H_f - p\textit{–Hölder} \end{cases}
\end{aligned} \tag{2.16}$$

where, $L_u, H_g, H_f > 0$ and $p, q \in (0,1]$ are given.

Proof. Assume that g is of q-H_g–Hölder type on $[a,b]$ and f is of bounded variation on $[a,b]$. Using (1.4), then we may write

$$\begin{aligned}
|\mathcal{T}(f,g;u)| &\le \frac{L_u}{|u(b)-u(a)|}\int_a^b \left|f(t)-\frac{f(a)+f(b)}{2}\right| \\
&\qquad\times\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt \\
&= \frac{L_u}{(u(b)-u(a))^2}\int_a^b \left|f(t)-\frac{f(a)+f(b)}{2}\right|\left|\int_a^b [g(t)-g(s)]\,du(s)\right|dt \\
&\le \frac{L_u^2}{(u(b)-u(a))^2}\int_a^b \left[\left|f(t)-\frac{f(a)+f(b)}{2}\right|\cdot\int_a^b |g(t)-g(s)|\,ds\right]dt \\
&\le \frac{L_u^2}{(u(b)-u(a))^2}\cdot\sup_{t\in[a,b]}\left|f(t)-\frac{f(a)+f(b)}{2}\right|\cdot\int_a^b\left[\int_a^b |g(t)-g(s)|\,ds\right]dt \qquad (2.17)\\
&\le \frac{L_u^2 H_g}{2(u(b)-u(a))^2}\cdot\bigvee_a^b(f)\cdot\int_a^b\left[\int_a^b |t-s|^q\,ds\right]dt.
\end{aligned}$$

$$\begin{aligned}
&= \frac{L_u^2 H_g}{2(u(b)-u(a))^2}\cdot\bigvee_a^b(f)\cdot\int_a^b\left[\int_a^t (s-a)^q\,ds+\int_t^b (b-s)^q\,ds\right]dt \\
&= \frac{L_u^2 H_g}{2(u(b)-u(a))^2}\cdot\bigvee_a^b(f)\cdot\int_a^b\left[\frac{(t-a)^{q+1}+(b-t)^{q+1}}{q+1}\right]dt \\
&= \frac{L_u^2 H_g}{(u(b)-u(a))^2}\cdot\bigvee_a^b(f)\cdot\frac{(b-a)^{q+2}}{(q+1)(q+2)},
\end{aligned}$$

which proves the first inequality in (2.16).

To prove the second inequality in (2.16), assume that f is of p–H_f–Hölder type on $[a,b]$, then by (2.17) we have

$$\sup_{t\in[a,b]}\left|f(t)-\frac{f(a)+f(b)}{2}\right| \le H_f\left(\frac{b-a}{2}\right)^p \qquad (2.18)$$

which together with (2.17) proves the second part of (2.16), and thus the proof is established. □

Corollary 2.3. *Let $g,u:[a,b]\to\mathbb{R}$ be respectively; L_g, L_u–Lipschitzian on $[a,b]$, then we have*

$$\begin{aligned}
&|\mathcal{T}(f,g;u)| \qquad (2.19)\\
&\le L_u^2 L_g\cdot\begin{cases}\frac{(b-a)^3}{6(u(b)-u(a))^2}\cdot\bigvee_a^b(f), & \text{if } f \text{ is of bounded variation}\\[2mm] \frac{H_f(b-a)^{p+3}}{2^{p+1}\cdot 3(u(b)-u(a))^2}, & \text{if } f \text{ is } H_f-p-Holder\end{cases}
\end{aligned}$$

where, $H_g,H_f>0$ and $p,q\in(0,1]$ are given.

Remark 2.4. Under the assumptions of Theorem 2.4, we have

$$|\mathcal{T}(f,g)| \le H_g \cdot \begin{cases} \frac{(b-a)^q}{(q+1)(q+2)} \cdot \bigvee_a^b (f), & if\ f\ is\ of\ bounded\ variation \\ \frac{H_f (b-a)^{p+q}}{2^p (q+1)(q+2)}, & if\ f\ is\ H_f - p - Holder \end{cases} \tag{2.20}$$

where, $H_f > 0$ and $p, q \in (0,1]$ are given. In particular, if g is L_g–Lipschitzian, then

$$|\mathcal{T}(f,g)| \le L_g \cdot \begin{cases} \frac{1}{6}(b-a) \cdot \bigvee_a^b (f), & if\ f\ is\ of\ bounded\ variation \\ \frac{1}{12} L_f (b-a)^2, & if\ f\ is\ L_f - Lipschitzian \end{cases} \tag{2.21}$$

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INTEGRAL TRANSFORM METHOD FOR SOLVING DIFFERENT F.S.I.ES AND P.F.D.ES

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Abstract. In this work, the authors used Laplace transform to obtain formal solution to some systems of singular integral equations of fractional type. In the last section, the authors considered certain non homogeneous fractional system of heat equations with different orders which is a generalization to the problem of heat transferring from metallic bar through the surrounding media. Illustrative examples are also provided.

1. Introduction and Definitions

Fractional differential equations have been the focus of many studies due to their frequent appearance in various fields such as chemistry and engineering, physics. The main reason for success of applications fractional calculus is that these new fractional order models are more accurate than integer order models, i.e. there are more degrees of freedom in the fractional order models. The Laplace transform technique is one of most useful tools of applied mathematics. Typical applications include heat transfer, diffusion, waves, vibrations and fluid motion problems. However, contrary to expectations, it is surprising to find that the popularity of Laplace transforms, in comparison to numerical or other methods, is gradually diminishing and Laplace transform is less fashionable today than they were a few decades ago. Nevertheless, the applications of Laplace transforms continue to be an important part of the mathematical education received by students in various fields of natural sciences and engineering. The fractional diffusion equation, the fractional wave equation, the fractional advection-dispersion equation, the fractional kinetic equation and other fractional PDEs have been studied and explicit solutions have been achieved by Mainardi, Pagnini and Saxena [18], Langlands [13], Mainardi, Pagnini and Gorenflo [17], Mainardi and Pagnini [15,16], Yu and Zhang [25], Liu, Anh, Turner and Zhang [14], Saichev and Zaslavsky [21], Saxena, Mathai and Haubold [22], Wyss [24] and several other research works can be found in other literatures. In these works, the techniques of using integral transforms were used to obtain the

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formal solutions of fractional PDEs. Integral transforms are extensively used in solving boundary value problems and integral equations. The problem related to partial differential equations can be solved by using a special integral transform thus many authors solved the boundary value problems by using single Laplace transform. Laplace transform is very useful in applied mathematics, for instance for solving some differential equations and partial differential equations, and in automatic control, where it defines a transfer function.

The Caputo fractional derivatives of order $\alpha > 0$ $(n-1 < \alpha \leq n, n \in N)$ is defined by

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx. $$

The Laplace transform of a function $f(t)$ denoted by $F(s)$, is defined by the integral equation

$$ L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt := F(s). $$

Definition 1.1. The inverse Laplace transform is given by the contour integral

$$ f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, $$

where $F(s)$ is analytic in the region $Re(s) > c$.

Theorem 1.1. *For $n-1 < \alpha \leq n$, we can get*

$$ L\{{}_0^C D_t^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0). $$

Two-parameter Mittag-Leffler function and Wright function is given by

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, $$

$$ W(\alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\alpha n + \beta)}. $$

when $\alpha, \beta, z \in C$.

Theorem 1.2. ***Schouten-Van der Pol Theorem****: Consider a function $f(t)$ which has the Laplace transform $F(s)$ which is analytic in the half-plane $Re(s) > s_0$. We can use this knowledge to find $g(t)$ whose Laplace transform $G(s)$ equals $F(\phi(s))$, where $\phi(s)$ is also analytic for $Re(s) > s_0$. This means that if*

$$ G(s) = F(\phi(s)) = \int_0^{\infty} f(\tau) \exp(-\phi(s)\tau) d\tau, $$

and

$$ g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\phi(s)) \exp(ts) ds, $$

then

$$g(t) = \int_0^\infty f(\tau) \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(-\phi(s)\tau) \exp(ts) ds \right) d\tau.$$

Proof. See [10]

2. Fractional Order Singular Integral Equations

The mathematical formulation of physical phenomena often involves Cauchy type, or more severe, singular integral equations. There are many applications in many important fields, like fracture mechanics, elastic contact problems, the theory of porous filtering contain integral and integro- differential equation with singular kernel. In following section, Laplace transform has been used to solve certain types of singular integral equations of fractional order. We solve a fractional order singular integral equation system. Special examples are mentioned.

Lemma 2.1. *The fractional Fredholm singular integro-differential equation of the form*

$$ {}_0^C D_x^\alpha \varphi(x) = f(x) + \lambda \int_0^\infty (\frac{x}{t})^{\frac{\nu}{2}} J_\nu(2\sqrt{xt}) \varphi(t) dt, \tag{2.1}$$

where $\varphi(0) = 0$, $0 \leq \alpha \leq 1$ *and* $\nu > -1$ *has the formal solution as*

$$\varphi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s^{-\alpha} F(s) + \frac{\lambda}{s^{\nu+1}} F(\frac{1}{s})}{1 - \lambda^2} e^{sx} ds. \tag{2.2}$$

Proof. Let $L(\varphi(x)) = \Phi(s)$ and $L(f(x)) = F(s)$, then by using the Laplace transform of (2-1) we have the following relation

$$s^\alpha \Phi(s) = F(s) + \lambda \frac{1}{s^{\nu+1}} \Phi(\frac{1}{s}). \tag{2.3}$$

In relation (2-3) we replace s by $\frac{1}{s}$, to obtain

$$s^{-\alpha} \Phi(\frac{1}{s}) = F(\frac{1}{s}) + \lambda s^{\nu+1}\ \Phi(s). \tag{2.4}$$

Combination of (2-3) and (2-4), $\Phi(s)$ can be obtained as

$$\Phi(s) = \frac{s^{-\alpha} F(s) + \frac{\lambda}{s^{\nu+1}} F(\frac{1}{s})}{1 - \lambda^2}. \tag{2.5}$$

By using the complex inversion formula, relation (2-5) leads to the following,

$$\varphi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s^{-\alpha} F(s) + \frac{\lambda}{s^{\nu+1}} F(\frac{1}{s})}{1 - \lambda^2} e^{sx} ds.$$

Example 2.1. Solve the following fractional singular integral equation

$$ {}_0^C D_x^{\frac{2}{3}} \varphi(x) = \frac{1}{\sqrt{\pi x}} + \lambda \int_0^\infty (\frac{x}{t})^{\frac{1}{4}} J_{\frac{1}{2}}(2\sqrt{xt}) \varphi(t) dt,$$

Solution. By using the formula (2-2), we get

$$\begin{aligned}\varphi(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{s^{-\frac{2}{3}}F(s)+\frac{\lambda}{s^{\frac{3}{2}}}F(\frac{1}{s})}{1-\lambda^2}e^{sx}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\frac{1}{s^{\frac{7}{6}}}+\frac{\lambda}{s}}{1-\lambda^2}e^{sx}ds\\ &= \frac{1}{1-\lambda^2}(\frac{x^{\frac{1}{6}}}{\Gamma(\frac{7}{6})}+\lambda).\end{aligned}$$

Lemma 2.2. *The system of fractional Fredholm singular integro-differential equation of the form*

$${}_0^C D_x^{\alpha}\varphi_1(x) = f(x) + \lambda\int_0^{\infty}(\frac{x}{t})^{\frac{\nu}{2}}J_{\nu}(2\sqrt{xt})\varphi_2(t)dt,$$

$${}_0^C D_x^{\alpha}\varphi_2(x) = g(x) + \lambda\int_0^{\infty}(\frac{x}{t})^{\frac{\mu}{2}}J_{\mu}(2\sqrt{xt})\varphi_2(t)dt,$$

where $\varphi_1(0)=\varphi_2(0)=0$ *and* $0<\alpha,\beta\leq 1$ *has the formal solutions*

(2.6)
$$\varphi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(s^{-\alpha}\left(F(s)+\frac{\lambda^2}{1-\lambda^2}\frac{G(s)}{s^{\nu-\mu}}\right)+\frac{\lambda}{1-\lambda^2}\frac{1}{s^{\nu+1}}G(\frac{1}{s})\right)e^{xs}ds,$$

$$\varphi_2(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{s^{-\alpha}G(s)+\frac{\lambda}{s^{\mu+1}}G(\frac{1}{s})}{1-\lambda^2}e^{xs}ds. \tag{2.7}$$

Proof. Applying the Laplace transform term wise to both equations and using the initial conditions yields

$$s^{\alpha}\Phi_1(s) = F(s) + \frac{\lambda}{s^{\nu+1}}\Phi_2(\frac{1}{s}), \tag{2.8}$$

$$s^{\alpha}\Phi_2(s) = G(s) + \frac{\lambda}{s^{\mu+1}}\Phi_2(\frac{1}{s}). \tag{2.9}$$

Following the same procedure as in lemma 2.1, we get $\Phi_2(s)$ as

$$\Phi_2(s) = \frac{s^{-\alpha}G(s)+\frac{\lambda}{s^{\mu+1}}G(\frac{1}{s})}{1-\lambda^2},$$

then, changing s to $\frac{1}{s}$ leads to

$$\Phi_2(\frac{1}{s}) = \frac{s^{\alpha}G(\frac{1}{s})+\lambda s^{\mu+1}G(s)}{1-\lambda^2}.$$

By replacing $\Phi_2(\frac{1}{s})$ in (2-8), we will have

$$\Phi_1(s) = s^{-\alpha}\left(F(s)+\frac{\lambda^2}{1-\lambda^2}\frac{G(s)}{s^{\nu-\mu}}\right)+\frac{\lambda}{1-\lambda^2}\frac{1}{s^{\nu+1}}G(\frac{1}{s}).$$

At this point, using the complex inversion formula, the final solutions are as follows

$$\varphi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(s^{-\alpha}\left(F(s)+\frac{\lambda^2}{1-\lambda^2}\frac{G(s)}{s^{\nu-\mu}}\right)+\frac{\lambda}{1-\lambda^2}\frac{1}{s^{\nu+1}}G(\frac{1}{s})\right)e^{x\,s}ds,$$

$$\varphi_2(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{s^{-\alpha}G(s)+\frac{\lambda}{s^{\mu+1}}G(\frac{1}{s})}{1-\lambda^2}e^{xs}ds.$$

Example 2.2. Let us solve the system

$${}_0^CD_x^{\frac{1}{2}}\varphi_1(x)=\frac{e^{-\frac{1}{4x}}}{2\sqrt{\pi x^3}}+\lambda\int_0^\infty(\frac{x}{t})^{\frac{3}{4}}J_{\frac{3}{2}}(2\sqrt{xt})\varphi_2(t)dt,$$

$${}_0^CD_x^{\frac{1}{2}}\varphi_2(x)=1+\lambda\int_0^\infty(\frac{x}{t})^{\frac{1}{4}}J_{\frac{1}{2}}(2\sqrt{xt})\varphi_2(t)dt,$$

where $\varphi_1(0)=\varphi_2(0)=0$ and $0<\alpha,\beta\leq 1$. Direct use of relations (2-6) and (2-7), leads to

$$\begin{aligned}\varphi_1(x) &= L^{-1}\left\{\frac{e^{-\sqrt{s}}}{\sqrt{s}}+\frac{\lambda^2}{1-\lambda^2}\frac{1}{s^{\frac{5}{2}}}+\frac{\lambda}{1-\lambda^2}\frac{1}{s^{\frac{3}{2}}}\right\}\\ &= \frac{e^{-\frac{1}{4x}}}{\sqrt{\pi x}}+\frac{4\lambda^2x^{\frac{3}{2}}}{3\sqrt{\pi}(1-\lambda^2)}+\frac{2\lambda x^{\frac{1}{2}}}{\sqrt{\pi}(1-\lambda^2)},\end{aligned}$$

$$\varphi_2(x)=L^{-1}\left\{\frac{s^{-\frac{3}{2}}+\lambda s^{-\frac{1}{2}}}{1-\lambda^2}\right\}=\frac{\frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}+\frac{\lambda}{\sqrt{\pi x}}}{1-\lambda^2}.$$

2.1. **Evaluation of the Integrals.** In applied mathematics, the **Kelvin functions** $Ber_\nu(x)$ and $Bei_\nu(x)$ are the real and imaginary parts, respectively, of $J_\nu(xe^{3\pi i/4})$,where x is real, and $J_\nu(z)$ is the ν-th order Bessel function of the first kind. Similarly, the functions $Ker_\nu(x)$ and $Kei_\nu(x)$ are the real and imaginary parts, respectively, of $K_\nu(xe^{\pi i/4})$, where $K_\nu(z)$is the ν-th order modified Bessel function of the second kind. These functions are named after William Thomson, 1st Baron Kelvin. The Kelvin functions were investigated because they are involved in solutions of various engineering problems occurring in the theory of electrical currents, elasticity and in fluid mechanics. One of the main applications of Laplace transform is evaluating the integrals as discussed in the following.

Lemma 2.3. *The following integral relationship holds true*

$$\int_1^\infty\frac{bei(\sqrt{2\lambda})d\lambda}{\sqrt{\lambda^2-1}}=\frac{\pi}{2}J_0(1)I_0(1).$$

Proof. Let us define the following function

$$I(x)=\int_1^\infty\frac{bei(\sqrt{2x\lambda})d\lambda}{\sqrt{\lambda^2-1}}.$$

Laplace transform of $I(x)$ leads to

$$L\{I(x)\}=\int_0^\infty e^{-sx}\left(\int_1^\infty\frac{bei(\sqrt{2x\lambda})d\lambda}{\sqrt{\lambda^2-1}}\right)dx.$$

By changing the order of integration, which is permissible, we obtain

$$L\{I(x)\} = \int_1^\infty \frac{1}{\sqrt{\lambda^2-1}} \left(\int_0^\infty e^{-sx} bei(\sqrt{2x\lambda})dx \right) d\lambda,$$

or

$$L\{I(x)\} = \int_1^\infty \frac{1}{\sqrt{\lambda^2-1}} (\frac{1}{s} \sin \frac{\lambda}{2s})d\lambda.$$

At this point, let us introduce the new variable $\lambda = \cosh \xi$, we get the following

$$L\{I(x)\} = \frac{1}{s}\int_0^\infty \sin((2s)^{-1}\cosh\xi)d\xi,$$

using the following well-known integral representation for $J_0(\varphi)$

$$J_0(\varphi) = \frac{2}{\pi}\int_0^\infty \sin(\varphi\cosh\vartheta)d\vartheta.$$

One gets finally

$$L\{I(x)\} = \frac{\pi}{2s}J_0(\frac{1}{2s}),$$

now, taking inverse Laplace transform of the above relationship leads to

$$I(x) = L^{-1}\{\frac{\pi}{2s}J_0(\frac{1}{2s})\} = \frac{\pi}{2}J_0(\sqrt{x})I_0(\sqrt{x}).$$

Letting $x = 1$ we get

$$\int_1^\infty \frac{bei(\sqrt{2\lambda})d\lambda}{\sqrt{\lambda^2-1}} = \frac{\pi}{2}J_0(1)I_0(1).$$

Lemma 2.4. *The following integral relations hold true*

$$\int_0^1 x^{\mu-1} ber(2\sqrt{\ln x})dx = \frac{1}{\mu}\cos\frac{1}{\mu},$$

$$\int_0^1 \frac{ber(2\sqrt{\ln x})}{\sqrt{x}}dx = 2\cos 2.$$

Proof. Let us define the following function

$$I(\xi) = \int_0^1 x^{\mu-1} ber(2\sqrt{(\ln x)\xi})dx.$$

Laplace transform of $I(\xi)$ leads to

$$L\{I(\xi)\} = \int_0^\infty e^{-s\xi}\left(\int_0^1 x^{\mu-1} ber(2\sqrt{(\ln x)\xi})dx\right)d\xi.$$

By changing the order of integration, which is permissible, we will have

$$L\{I(\xi)\} = \int_0^1 x^{\mu-1}\int_0^\infty e^{-s\xi} ber(2\sqrt{(\ln x)\xi})d\xi dx.$$

But the value of inner integral is as following

$$\int_0^\infty e^{-s\xi} ber(2\sqrt{(\ln x)\xi})d\xi = \frac{1}{s}\cos\frac{(\ln x)}{s}.$$

To prove the second relationship, by setting this value in the integral, one gets

$$L\{I(\xi)\} = \int_0^1 x^{\mu-1}\frac{1}{s}\cos\frac{(\ln x)}{s}dx = \frac{1}{s}\int_0^1 x^{\mu-1}\cos\frac{(\ln x)}{s}dx.$$

At this point, we introduce the new variable $\ln x = -w$. One gets after easy calculation

$$L\{I(\xi)\} = \frac{1}{s}\int_0^\infty e^{-\mu w}\cos(\frac{w}{s})dw = \frac{1}{\mu}\{\frac{s}{s^2+(\mu^{-1})^2}\}.$$

Taking inverse Laplace transform to obtain

$$I(\xi) = \int_0^1 x^{\mu-1}ber(2\sqrt{(\ln x)\xi})dx = \frac{1}{\mu}\cos\frac{\xi}{\mu},$$

from the above relationship, we get

$$I(1) = I_0(\mu) = \int_0^1 x^{\mu-1}ber(2\sqrt{\ln x})dx = \frac{1}{\mu}\cos\frac{1}{\mu}.$$

In the above integral, by setting 0.5 for the parameter,we obtain the second assertion

$$I_0(0.5) = \int_0^1 \frac{ber(2\sqrt{\ln x})dx}{\sqrt{x}} = 2\cos 2.$$

3. Bobylev-Cercignani Theorem and Their Applications

Bobylev and Cercignani developed a theorem [8] concerning the inversion of multivalued transforms that are analytic everywhere in the $s-$ plane except along the negative real axis. The theorem is as follows:

Theorem 3.1. ***Bobylev-Cercignani Theorem****: Let $f(t)$ denote a real-valued function, where its Laplace transform $F(s)$ exists. Let $F(s)$ satisfy the following hypothesis:*

1) $F(s)$ is a multi-valued function which has no singularities in the cut $s-$ plane. The branch cut lies along the negative real axis $(-\infty, 0]$.

2) $F^*(s) = F(s^*)$, where the star denotes the complex conjugate.

3) $F^{\pm}(\eta) = \lim\limits_{\phi\to\pi^-} F(\eta e^{\pm\phi i})$ and $F^+(\eta) = (F_-(\eta))^*$.

4) $F(s\ ) = o(1)$ as $|s| \to \infty$ and $F(s) = o(\frac{1}{|s|})$ as $|s| \to 0$, uniformly in any sector $|\arg(s)| < \pi - \eta$, $0 < \eta < \pi$.

5) There exists $\varepsilon > 0$, such that for every $\pi - \varepsilon < \phi \leq \pi$, $\frac{F(re^{\pm\phi i})}{1+r} \in L_1(R^+)$ and $|F(re^{\pm\phi i})| < a(r)$, where $a(r)$ does not depend on ϕ and $a(r)e^{-\delta r} \in L_1(R^+)$ for any $\delta > 0$.Then

$$f(t) = \frac{1}{\pi}\int_0^\infty Im(\ F^-(\eta))e^{-t\eta}d\eta.$$

In following lemma, we apply this theorem.

Lemma 3.1. *The following relationship holds true*

$$L^{-1}\left\{\frac{1}{s+1}\exp\left(-x\sqrt{\frac{\mu+s^{\alpha}}{\lambda+s^{\alpha}}}\right)\right\}=\frac{1}{\pi}\int_{0}^{\infty}Im(F^{-}(\eta))e^{-t\eta}d\eta.$$

where $0<\alpha<1,\ \lambda,\mu>0$ *and*

$$Im(F^{-}(\eta))=\frac{e^{-x\sqrt{\frac{\rho_1}{\rho_2}}\cos(\frac{\theta_1-\theta_2}{2})}}{\eta-1}\sin\left(x\sqrt{\frac{\rho_1}{\rho_2}}\sin(\frac{\theta_1-\theta_2}{2})\right).$$

Proof. $F(s)$ satisfies all of the conditions listed in the theorem 3.1. Then

$$\begin{aligned}F^{-}(\eta) &= \lim_{\phi\to\pi}F(\eta e^{-\phi i})=\frac{1}{\eta e^{-\pi i}+1}\exp\left(-x\sqrt{\frac{\eta^{\alpha}e^{-\pi\alpha i}+\mu}{\eta^{\alpha}e^{-\pi\alpha i}+\lambda}}\right)\\ &= \frac{1}{1-\eta}\exp\left(-x\sqrt{\frac{\rho_1}{\rho_2}}e^{\frac{i(\theta_1-\theta_2)}{2}}\right)\\ &= \frac{1}{1-\eta}\exp\left(-x\sqrt{\frac{\rho_1}{\rho_2}}(\cos(\frac{\theta_1-\theta_2}{2})+i\sin(\frac{\theta_1-\theta_2}{2}))\right),\end{aligned}$$

where

$$\rho_1=\sqrt{\eta^{2\alpha}+2\mu\eta^{\alpha}\cos\pi\alpha+\mu^2},\rho_2=\sqrt{\eta^{2\alpha}+2\lambda\eta^{\alpha}\cos\pi\alpha+\lambda^2},$$

$$\theta_1=-\tan^{-1}\left(\frac{\eta^{\alpha}\sin\alpha\pi}{\eta^{\alpha}\cos\alpha\pi+\mu}\right),\theta_2=-\tan^{-1}\left(\frac{\eta^{\alpha}\sin\alpha\pi}{\eta^{\alpha}\cos\alpha\pi+\lambda}\right)\ \ (0<\theta<\pi).$$

Image part of $F^{-}(\eta)$ is founded as

$$Im(F^{-}(\eta))=\frac{e^{-x\sqrt{\frac{\rho_1}{\rho_2}}\cos(\frac{\theta_1-\theta_2}{2})}}{\eta-1}\sin\left(x\sqrt{\frac{\rho_1}{\rho_2}}\sin(\frac{\theta_1-\theta_2}{2})\right).$$

Finally, the inverse Laplace transform is as

$$f(t)=\frac{1}{\pi}\int_{0}^{\infty}Im(F^{-}(\eta))e^{-t\eta}d\eta.$$

Problem 1. *Let us consider the following* ***four terms*** *partial fractional differential equation*

$$\frac{\partial}{\partial x}\{\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}\}+a\frac{\partial^{\beta}u(x,t)}{\partial t^{\beta}}=\lambda u(x,t)-b\frac{\partial u(x,t)}{\partial x},$$

where $0<\alpha<1,\ 0<\beta\leq 1,\ 0<x<\infty,\ t,a,b>0$ *with the boundary conditions*

$$u(0,t)=\frac{t^{\gamma-1}}{\Gamma(\gamma)}\ \ (\gamma>0),\lim_{x\to\infty}|u(x,t)|<\infty,$$

and the initial conditions $u(x,0)=u_x(x,0)=0.$

Solution. Applying the Laplace transform of the equation and using the boundary and initial conditions leads to differential equation with respect to x as

$$U_x(x,t) + \frac{as^\beta - \lambda}{s^\alpha + b} U(x,t) = 0,$$

when $L\{u(x,t)\} = U(x,s)$. Solution of the above equation yields

$$U(x,s) = \frac{1}{s^\gamma} \exp\left(-x \frac{as^\beta - \lambda}{s^\alpha + b}\right).$$

$U(x,s)$ satisfies all of the conditions explained in the theorem 3.1. Hence

$$\begin{aligned} U^-(x,\eta) &= \lim_{\phi \to \pi} U(x, \eta e^{-\phi i}) = \frac{1}{\eta^\gamma e^{-\pi\gamma i}} \exp\left(-x \frac{a\eta^\beta e^{-\pi\beta i} - \lambda}{\eta^\alpha e^{-\pi\alpha i} + b}\right) \\ &= \frac{e^{\pi\gamma i}}{\eta^\gamma} \exp\left(-x \frac{(a\eta^\beta e^{-\pi\beta i} - \lambda)(\eta^\alpha e^{\pi\alpha i} + b)}{\rho}\right), \end{aligned}$$

where $\rho = \eta^{2\alpha} + 2b\eta^\alpha \cos\pi\alpha + b^2$. Therefore

$$Im(U^-(x,\eta)) =$$

$$\frac{1}{\eta^\gamma} \exp\left\{-x \frac{ab\eta^\beta \cos\beta\pi + a\eta^{\alpha+\beta}\cos(\alpha-\beta)\pi - \lambda\eta^\alpha \cos\alpha\pi - \lambda b}{\rho}\right\}$$
$$\times \sin\left\{\pi\gamma - x \frac{a\eta^{\alpha+\beta}\sin(\alpha-\beta)\pi - ab\eta^\beta \sin\beta\pi - \lambda\eta^\alpha \sin\alpha\pi}{\rho}\right\}.$$

Finally, $u(x,\, t)$ is found to be

$$u(x,t) = \frac{1}{\pi} \int_0^\infty Im(U^-(x,\eta)) e^{-t\eta} d\eta.$$

4. Partial Fractional Differential Equation (PFDE) with Moving Boundary

In PFDE problems, Laplace transforms are particularly useful when the boundary conditions are time dependent. We consider now the case when one of the boundaries is moving. This type of problem arises in combustion problems where the boundary moves due to the burning of the fuel [10]. Such fractional partial differential equations have not been studied in the literature.

Problem 2. *Let us solve the following three terms time-fractional heat equation with moving boundaries*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \lambda \frac{\partial u(x,t)}{\partial x} \; (0 < \alpha \le 1), \tag{4.1}$$

where $\lambda \in R$, $\beta t < x < \infty$, $t > 0$ *and subject to the boundary conditions*

$$u(x,t)\Big|_{x=\beta t} = \frac{1}{\sqrt{\pi t}} \exp(-\frac{1}{4t}), \; \lim_{x\to\infty} |u(x,t)| < \infty,$$

and the initial condition $u(x,0) = 0$ $(0 < x < \infty)$.

Solution: We introduce the change of variable $\eta = x - \beta t$. The above equation can be reformulated as

$$\frac{\partial^\alpha w(\eta,t)}{\partial t^\alpha} - \beta\frac{\partial}{\partial\eta}\left({}_0I_t^{1-\alpha}w(\eta,t)\right) = a^2\frac{\partial^2 w(\eta,t)}{\partial\eta^2} + \lambda\frac{\partial w(\eta,t)}{\partial\eta}, \tag{4.2}$$

where $0 < \eta < \infty$, $t > 0$ subject to the boundary conditions

$$w(0,t) = \frac{1}{\sqrt{\pi t}}\exp(-\frac{1}{4t}),\ \lim_{\eta\to\infty}|w(\eta,t)| < \infty,$$

and the initial condition $w(\eta, 0) = 0$ $(0 < \eta < \infty)$. By applying the Laplace transform of the equation (4-2), we obtain

$$\frac{\partial^2 W(\eta,s)}{\partial\eta^2} + \frac{1}{a^2}(\frac{\beta}{s^{1-\alpha}} + \lambda)\frac{\partial W(\eta,s)}{\partial\eta} - \frac{s^\alpha}{a^2}W(\eta,s) = 0, \tag{4.3}$$

with conditions

$$W(0,s) = \frac{e^{-\sqrt{s}}}{\sqrt{s}},\ \lim_{\eta\to\infty}|W(\eta,t)| < \infty.$$

Differential equation (4-3) has the solution as

$$W(\eta,s) = \frac{e^{-\sqrt{s}}}{\sqrt{s}}\exp\left(-\frac{\lambda\eta}{2a^2} - \frac{\beta\eta}{2a^2s^{1-\alpha}} - \frac{\eta}{2}\sqrt{\frac{1}{a^4}(\frac{\beta}{s^{1-\alpha}} + \lambda)^2 + \frac{4s^\alpha}{a^2}}\right).$$

Case 1: If $\alpha = 1$, then

$$W(\eta,s) = e^{-(\lambda+\beta)\frac{\eta}{2a^2}}\frac{e^{-\sqrt{s}}}{\sqrt{s}}\exp\left(-\frac{\eta}{a}\sqrt{\frac{1}{4\,a^2}(\beta+\lambda)^2 + s}\right).$$

Using the fact that

$$L^{-1}\left\{\exp\left(-\frac{\eta}{a}\sqrt{\frac{1}{4a^2}(\beta+\lambda)^2 + s}\right)\right\} = e^{-\frac{1}{4a^2}(\beta+\lambda)^2t}\frac{\eta}{2a\sqrt{\pi t^3}}e^{-\frac{\eta^2}{4\,a^2t}},$$

and using the Laplace transform inversion and then applying the convolution theorem in this transform, we get $w(\eta,t)$ as

$$\begin{aligned} w(\eta,t) &= L^{-1}\{W(\eta,s)\} \\ &= \frac{\eta}{2a\pi}e^{-(\lambda+\beta)\frac{\eta}{2a^2}}\int_0^t\frac{e^{-\frac{1}{4(t-\tau)}}}{\sqrt{\tau^3(t-\tau)}}e^{-\frac{1}{4a^2}\beta+\lambda)^2\tau}e^{-\frac{\eta^2}{4\,a^2\tau}}d\tau. \end{aligned}$$

Therefore we obtain $u(x,t)$ as following

$$u(x,t) = \frac{x-\beta t}{2a\pi}e^{-(\lambda+\beta)\frac{x-\beta t}{2a^2}}\int_0^t\frac{e^{-\frac{1}{4(t-\tau)}}}{\sqrt{\tau^3(t-\tau)}}e^{-\frac{1}{4a^2}(\beta+\lambda)^2\tau}e^{-\frac{(x-\beta t)^2}{4a^2\tau}}d\tau.$$

Case 2: If $\alpha \neq 1$, then

$$W(\eta,s)=e^{-\frac{\lambda\eta}{2a^2}}\frac{1}{\sqrt{s}}\exp\left(-\sqrt{s}-\frac{\beta\eta}{2a^2s^{1-\alpha}}-\frac{\eta}{2}\sqrt{\frac{\beta^2}{a^4s^{2-2\alpha}}+\frac{2\beta\lambda}{a^4s^{1-\alpha}}+\frac{4s^\alpha}{a^2}+\lambda^2}\right),$$

and we can use the theorem 3.1, hence

$$W^-(\eta,\xi)=\lim_{\phi\to\pi}W(\eta,\xi e^{-\phi i})=\frac{e^{-\frac{\lambda\eta}{2a^2}}e^{\sqrt{\xi}e^{-\frac{\pi i}{2}}}}{\sqrt{\xi}e^{-\frac{\pi i}{2}}}\times$$

$$\exp\left(-\frac{\beta\eta}{2a^2\xi^{1-\alpha}e^{(\alpha-1)\pi i}}-\frac{\eta}{2}\sqrt{\frac{\beta^2}{a^4\xi^{2-2\alpha}e^{(2\alpha-2)\pi i}}+\frac{2\beta\lambda}{a^4\xi^{1-\alpha}e^{(\alpha-1)\pi i}}+\frac{4\xi^\alpha e^{-\alpha\pi i}}{a^2}+\lambda^2}\right)$$

$$=\frac{e^{-\frac{\lambda\eta}{2a^2}}e^{i(\frac{\pi}{2}-\sqrt{\xi})}}{\sqrt{\xi}}\exp\left(\frac{\beta\eta e^{-\alpha\pi i}}{2a^2\xi^{1-\alpha}}-\frac{\eta}{2}\sqrt{\frac{\beta^2e^{-2\alpha\pi i}}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})e^{-\alpha\pi i}+\lambda^2}\right)$$

$$=\frac{e^{-\frac{\lambda\eta}{2a^2}}e^{i(\frac{\pi}{2}-\sqrt{\xi})}}{\sqrt{\xi}}\times$$

$$e^{\frac{\beta\eta(\cos\alpha\pi-i\sin\alpha\pi)}{2a^2\xi^{1-\alpha}}-\frac{\eta}{2}\sqrt{\left[\frac{\beta^2\cos2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\cos\alpha\pi+\lambda^2\right]-i\left[\frac{\beta^2\sin2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\sin\alpha\pi\right]}}$$

$$=\frac{e^{-\frac{\lambda\eta}{2a^2}}e^{i(\frac{\pi}{2}-\sqrt{\xi})}}{\sqrt{\xi}}e^{\frac{\beta\eta(\cos\alpha\pi-i\sin\alpha\pi)}{2a^2\xi^{1-\alpha}}-\frac{\eta}{2}\sqrt{\rho}e^{\frac{\theta i}{2}}},$$

where

$$\rho=\sqrt{\left\{\frac{\beta^2\cos2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\cos\alpha\pi+\lambda^2\right\}^2+\left\{\frac{\beta^2\sin2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\sin\alpha\pi\right\}^2}$$

$$\theta=-\tan^{-1}\left(\frac{\frac{\beta^2\sin2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\sin\alpha\pi}{\frac{\beta^2\cos2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\cos\alpha\pi+\lambda^2}\right)\quad(0<\theta<\pi).$$

Then imaginary part of $W^-(\eta,\xi)$ is

$$Im(W^-(\eta,\xi))=\frac{e^{-\frac{\lambda\eta}{2a^2}}}{\sqrt{\xi}}e^{\frac{\beta\eta\cos\alpha\pi}{2a^2\xi^{1-\alpha}}-\frac{\eta}{2}\sqrt{\rho}\cos\frac{\theta}{2}}\cos(\sqrt{\xi}+\frac{\beta\eta}{2a^2\xi^{1-\alpha}}\sin\alpha\pi+\frac{\eta}{2}\sqrt{\rho}\sin\frac{\theta}{2}).$$

The formal solution will be as follows,

$$u(x,t)=\frac{1}{\pi}\int_0^\infty Im(W^-(x-\beta\xi,\xi))e^{-t\xi}d\xi.$$

5. A Non-Homogenous System of Fractional Heat equations with different orders

In this section, we consider certain non-homogeneous fractional system of heat equations (different orders) which is a generalization to the problem of heat transferring from metallic bar through the surrounding media studied by V.A. Ditkin, P.A. Prudnikov [9]. The basic goal of this work has been to implement the Laplace transform method for studying the above mentioned problem. The goal has been achieved by formally deriving exact analytical solution.

Problem 3. *We consider the following system of fractional PDE with different orders in Caputo sense*

$$ {}^{c}D_t^{\alpha} u + \gamma u = 1 + \frac{\partial^2 u}{\partial x^2} + \lambda a \frac{\partial v}{\partial r}\bigg|_{r=a}, \tag{5.1}$$

$$ {}^{c}D_t^{\delta} v - \beta v = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}, \tag{5.2}$$

where $0 < \alpha, \delta < 1$, $t > 0$, $-l \le x \le l$, $r \ge a$ *and* $\beta, \gamma \in R$ *with the boundary conditions*

$$u(x,0) = v(x,r,0) = 0, u(-l,t) = u(l,t) = 0,$$

and

$$v(x,a,t) = u(x,t), \lim_{r\to\infty} v(x,r,t) = 0.$$

Solution: By taking the Laplace transform of relation (5-2), we get

$$r^2 V_{rr} + rV_r + (i\sqrt{s^\delta - \beta})^2 r^2 V = 0.$$

Let us assume that $L\{v(x,r,t)\} = V(x,r,s)$, then one has

$$V(x,r,s) = c_1 J_0(i\sqrt{s^\delta - \beta} r) + c_2 Y_0(i\sqrt{s^\delta - \beta} r),$$

where J_0 and Y_0 are Bessel functions of the first and second kind of order zero, respectively. Using this fact that $\lim_{r\to\infty} v(x,r,t) = 0$, we get

$$V(x,r,s) = c_1 J_0(i\sqrt{s^\delta - \beta} r).$$

But $v(x,a,t) = u(x,t)$, therefore

$$V(x,r,s) = \frac{J_0(i\sqrt{s^\delta - \beta} r)}{J_0(i\sqrt{s^\delta - \beta} a)} U(x,s),$$

where $L\{u(x,t)\} = U(x,s)$. On the other hand, we have

$$\frac{\partial V}{\partial r}\bigg|_{r=a} = U(x,s)\left(-i\sqrt{s^\delta - \beta}\frac{J_1(i\sqrt{s^\delta - \beta} a)}{J_0(i\sqrt{s^\delta - \beta} a)}\right).$$

Applying the Laplace transform term wise to relation (5-1), we obtain

$$s^\alpha U = U_{xx} - i\lambda a\sqrt{s^\delta - \beta}\frac{J_1(i\sqrt{s^\delta - \beta} a)}{J_0(i\sqrt{s^\delta - \beta} a)} U + \frac{1}{s} - \gamma U,$$

or

$$U_{xx} - h(s)U = -\frac{1}{s}, \tag{5.3}$$

where

$$h(s) = s^{\alpha} + \gamma + i\lambda a\sqrt{s^{\delta} - \beta}\frac{J_1(i\sqrt{s^{\delta} - \beta}a)}{J_0(i\sqrt{s^{\delta} - \beta}a)}.$$

Differential equation (5-3) has the following solution

$$U(x,s) = c_1 \cosh(\sqrt{h(s)}x) + c_2 \sinh(\sqrt{h(s)}x) + \frac{1}{sh(s)}.$$

Using the boundary conditions $u(-l,t) = u(l,t) = 0$ leads to

$$U(x,s) = \frac{1}{sh(s)}\left(1 - \frac{\cosh(\sqrt{h(s)}x)}{\cosh(\sqrt{h(s)}l)}\right).$$

Let us assume that

$$F(x,h(s)) = 1 - \frac{\cosh(\sqrt{h(s)}x)}{\cosh(\sqrt{h(s)}l)},$$

then we get

$$U(x,s) = \frac{F(x,h(s))}{sh(s)}.$$

Now, if

$$L_t\{\phi(x,t)\} = \frac{F(x,s)}{s}, L_t\{\psi(\xi,t)\} = \frac{e^{-\xi h(s)}}{s},$$

then

$$u(x,t) = L_t^{-1}\{U(x,s)\} = L^{-1}\{\frac{F(x,h(s))}{sh(s)}\} = \int_0^{\infty} \psi(\xi,t)\phi(x,\xi)d\xi.$$

Finally, we will have

$$\begin{aligned}
\phi(x,t) &= L_t^{-1}\{\frac{F(x,s)}{s}\} = L_t^{-1}\left\{\frac{1}{s}(1 - \frac{\cosh(\sqrt{s}x)}{\cosh(\sqrt{s}l)})\right\} \\
&= 1 - L_t^{-1}\left\{\frac{\cosh(\sqrt{s}x)}{s\cosh(\sqrt{s}l)}\right\} = 1 - L_t^{-1}\left\{e^{\sqrt{s}(x-l)}\frac{1+e^{-2\sqrt{s}x}}{s(1+e^{-2\sqrt{s}l})}\right\} \\
&= 1 - \sum_{n=0}^{\infty} L_t^{-1}\left\{\frac{\exp(-((2n+1)l-x)\sqrt{s})}{s} - \frac{\exp(-((2n+1)l+x)\sqrt{s})}{s}\right\} \\
&= 1 - \sum_{n=0}^{\infty}\left(erfc(\frac{(2n+1)l-x}{2\sqrt{t}}) - erfc(\frac{(2n+1)l+x}{2\sqrt{t}})\right).
\end{aligned}$$

Also,

Figure 1

$$h(s) = s^{\alpha} + \gamma + i\lambda a\sqrt{s^{\delta}-\beta}\frac{J_1(i\sqrt{s^{\delta}-\beta}a)}{J_0(i\sqrt{s^{\delta}-\beta}a)},$$

hence

$$\begin{aligned}
\psi(\xi,t) &= L_t^{-1}\{\frac{e^{-\xi h(s)}}{s}\} \\
&= L_t^{-1}\left\{e^{-\xi\gamma}\frac{e^{-\xi s^{\alpha}}}{s}\exp\left(-i\xi\lambda a\sqrt{s^{\delta}-\beta}\frac{J_1(i\sqrt{s^{\delta}-\beta}a)}{J_0(i\sqrt{s^{\delta}-\beta}a)}\right)\right\} \\
&= e^{-\xi\gamma}L_t^{-1}\left\{\frac{\sqrt{s^{\delta}-\beta}}{s}\frac{e^{-\xi s^{\alpha}}}{\sqrt{s^{\delta}-\beta}}\exp\left(-i\xi\lambda a\sqrt{s^{\delta}-\beta}\frac{J_1(i\sqrt{s^{\delta}-\beta}a)}{J_0(i\sqrt{s^{\delta}-\beta}a)}\right)\right\}.
\end{aligned}$$

Case 1: Assume that $\delta = 1$, therefore

$$\begin{aligned}
f_1(\xi,t) &= L_t^{-1}\left\{\frac{1}{\sqrt{s-\beta}}\exp\left(-i\xi\lambda a\sqrt{s-\beta}\frac{J_1(i\sqrt{s-\beta}a)}{J_0(i\sqrt{s-\beta}a)}\right)\right\} \\
&= e^{\beta t}L_t^{-1}\left\{\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)\right\}.
\end{aligned}$$

The inverse Laplace transform is given by

$$L_t^{-1}\left\{\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds.$$

The integrand has a branch point at the origin and it is thus necessary to choose a contour which does not contain the origin. We deform the Bromwich contour so that the circular arc BDE is terminated just short of the horizontal axis and the arc LNA starts just below the horizontal axis. In between the contour follows an inclined path EH followed by a circular arcHJK enclosing the origin and a return section KL meeting the arc LNA (see figure). As there are no singularities inside this contour C, we have

$$\int_C \frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds = 0.$$

Now on BDE and LNA, we get

$$\left|\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)\right| \le \frac{1}{\sqrt{s}},$$

so that the integrals over these arcs tend to zero as $R \to \infty$. Over the circular arc HJK as its radius $\varepsilon \to 0$, we have $s = \varepsilon e^{i\theta}$, $\phi \le \theta \le -\phi$. Thus

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{HJK}\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds = 0.$$

Along EH, $s = ue^{i\phi}$, $\sqrt{s} = \sqrt{u}e^{\frac{i\phi}{2}}$, hence

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0,\phi\to\pi}} \int_{EH}\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds =$$
$$\int_0^\infty \frac{1}{i\sqrt{u}}\exp\left(-\xi\lambda a\sqrt{u}\frac{J_1(\sqrt{u}a)}{J_0(\sqrt{u}a)}\right)e^{-tu}du.$$

Similarly, along KL, $s = ue^{-i\phi}$, $\sqrt{s} = \sqrt{u}e^{-\frac{i\phi}{2}}$, then

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0,\phi\to\pi}} \int_{KL}\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds =$$
$$\int_0^\infty \frac{1}{i\sqrt{u}}\exp\left(-\xi\lambda a\sqrt{u}\frac{J_1(\sqrt{u}a)}{J_0(\sqrt{u}a)}\right)e^{-tu}du.$$

Consequently, we have

$$\frac{1}{2\pi i}\int_C \frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds = \frac{1}{2\pi i}\int_{AB}ds + \frac{1}{2\pi i}\int_{BDE}ds$$
$$+\frac{1}{2\pi i}\int_{EH}ds + \frac{1}{2\pi i}\int_{HJK}ds + \frac{1}{2\pi i}\int_{KL}ds + \frac{1}{2\pi i}\int_{LNA}ds = 0.$$

The final result is as

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds =$$
$$\frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{u}}\exp\left(-\xi\lambda a\sqrt{u}\frac{J_1(\sqrt{u}a)}{J_0(\sqrt{u}a)}\right)e^{-tu}du.$$

Thus we obtain

$$\begin{aligned} f_1(\xi,t) &= L_t^{-1}\left\{\frac{1}{\sqrt{s-\beta}}\exp\left(-i\xi\lambda a\sqrt{s-\beta}\frac{J_1(i\sqrt{s-\beta}a)}{J_0(i\sqrt{s-\beta}a)}\right)\right\} \\ &= \frac{1}{\pi}e^{\beta t}\int_0^\infty \frac{1}{\sqrt{u}}\exp\left(-\xi\lambda a\sqrt{u}\frac{J_1(\sqrt{u}a)}{J_0(\sqrt{u}a)}\right)e^{-tu}du. \end{aligned}$$

In case of $0 < \delta < 1$, we get

$$\begin{aligned} f_2(\xi,t) &= L_t^{-1}\left\{\frac{1}{\sqrt{s^\delta-\beta}}\exp\left(-i\xi\lambda a\sqrt{s^\delta-\beta}\frac{J_1(i\sqrt{s^\delta-\beta}a)}{J_0(i\sqrt{s^\delta-\beta}a)}\right)\right\} \\ &= \frac{1}{t}\int_0^\infty f_1(\xi,\tau)W(-\delta,0;-\tau t^{-\delta})d\tau. \end{aligned}$$

Also, for $0<\delta<1$,

$$\begin{aligned} f_3(t) &= L_t^{-1}\left\{\frac{\sqrt{s^\delta-\beta}e^{-\xi s^\alpha}}{s}\right\} = L_t^{-1}\left\{s^{\frac{\delta}{2}-1}(1-\beta s^{-\delta})^{\frac{1}{2}}e^{-\xi s^\alpha}\right\} \\ &= \sum_{n=0}^\infty(-\beta)^n\begin{pmatrix}\frac{1}{2}\\ n\end{pmatrix}L_t^{-1}\{s^{-\delta n+\frac{\delta}{2}-1}e^{-\xi s^\alpha}\} \\ &= \sum_{n=0}^\infty(-\beta)^n\begin{pmatrix}\frac{1}{2}\\ n\end{pmatrix}L_t^{-1}\left\{s^{-\delta n+\frac{\delta}{2}-1}\sum_{k=0}^\infty\frac{(-\xi)^k s^{\alpha k}}{k!}\right\} \\ &= \sum_{n=0}^\infty(-\beta)^n\begin{pmatrix}\frac{1}{2}\\ n\end{pmatrix}\left\{\sum_{k=0}^\infty\frac{(-\xi)^k}{k!}\frac{t^{\delta n-\alpha k-\frac{\delta}{2}}}{\Gamma(\delta n-\alpha k-\frac{\delta}{2}+1)}\right\}. \end{aligned}$$

Consequently

$$\psi(\xi,t)=L_t^{-1}\{\frac{1}{s}\exp(-\xi h(s))\}=e^{-\xi\gamma}\int_0^t f_2(\xi,\eta)f_3(t-\eta)d\eta \quad : 0<\alpha,\delta<1$$

Finally, we obtain $u(x,t)$ as follows

$$\begin{aligned} u(x,t) &= \int_0^\infty \psi(\xi,t)\phi(x,\xi)d\xi \\ &= \int_0^\infty e^{-\xi\gamma}\left(\int_0^t f_2(\xi,\eta)f_3(t-\eta)d\eta\right) \end{aligned}$$

$$\times\left(1-\sum_{n=0}^\infty\left(erfc(\frac{(2n+1)l-x}{2\sqrt{\xi}})-erfc(\frac{(2n+1)l+x}{2\sqrt{\xi}})\right)\right)d\xi.$$

Now, we should determine the inverse Laplace transform of the following term

$$V(x,r,s)=U(x,s)\frac{J_0(i\sqrt{s^\delta-\beta}r)}{J_0(i\sqrt{s^\delta-\beta}a)}.$$

If $\delta=1$, we obtain

$$g_1(r,t)=L_t^{-1}\left\{\frac{J_0(i\sqrt{s-\beta}r)}{J_0(i\sqrt{s-\beta}a)}\right\}=\frac{2}{a^2}\sum_{k=0}^\infty\frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}\exp(-(\frac{\lambda_k^2}{a^2}-\beta)t),$$

where $\lambda_1,\lambda_2,\lambda_3,\ldots$ are roots of $J_0(i\sqrt{s-\beta}a)$. For $0<\delta<1$, we conclude that

$$
\begin{aligned}
g_2(r,t) &= L_t^{-1}\left\{\frac{J_0(i\sqrt{s^\delta-\beta}r)}{J_0(i\sqrt{s^\delta-\beta}a)}\right\} \\
&= \frac{2}{a^2 t}\sum_{k=0}^{\infty}\frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}\int_0^\infty \exp(-(\frac{\lambda_k^2}{a^2}-\beta)\tau)W(-\delta,0;-\tau t^{-\delta})d\tau \\
&= \frac{2}{a^2 t}\sum_{k=0}^{\infty}\frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}L\left\{W(-\delta,0;-\tau t^{-\delta});\tau\to s\right\}\Bigg|\; s=\frac{\lambda_k^2}{a^2}-\beta \\
&= \frac{2}{t}\sum_{k=0}^{\infty}\frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}\left(\sum_{n=0}^{\infty}\frac{(-a^2)^n t^{-\delta n}}{\Gamma(-\delta n)(\lambda_k^2-a^2\beta)^{n+1}}\right) \\
&= \frac{2}{t}\sum_{k=0}^{\infty}\frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)(\lambda_k^2-a^2\beta)}E_{-\delta,0}(-\frac{a^2t^{-\delta}}{\lambda_k^2-a^2\beta}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
v(x,r,t) &= L_t^{-1}\{V(x,r,s)\} = L_t^{-1}\left(U(x,s)\frac{J_0(i\sqrt{s^\delta-\beta}r)}{J_0(i\sqrt{s^\delta-\beta}a)}\right) \\
&= \int_0^t u(x,\eta)g_2(r,t-\eta)d\eta \quad : 0<\alpha,\delta<1.
\end{aligned}
$$

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7. Conclusion

The paper is devoted to study and applications of Laplace transform. The main purpose of this work is to develop a method for finding formal solution of certain systems of Fredholm fractional singular integral equations of second kind, analytic solution of the time fractional heat equation and system of partial fractional differential equations with different orders, which is a generalization to certain types of problems in the literature. Numerous non trivial examples and exercises provided throughout the paper. We hope that it will also benefit many researchers in the disciplines of applied mathematics, mathematical physics and engineering.

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ON SPECTRAL PROPERTIES FOR A REGULAR STURM-LIOUVILLE PROBLEM

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Abstract. In this work we study a discontinuous boundary-value problem with retarded argument which contains a spectral parameter in the transmission conditions. We firstly prove the existence theorem and then obtain asymptotic representation of eigenvalues and eigenfunctions.

1. Introduction

The theory of differential equations with retarded arguments is one of the actual branch of the theory of ordinary differential equations. Particularly, there has been increasing interest in spectral analysis of boundary value problems. There is quite substantial literature concerning such problems. Here we mention the results of [1–19].

In this paper we study the eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument and spectral parameters in the transmission conditions. Namely we consider the boundary value problem for the differential equation

$$y''(x) + q(x)y(x - \Delta(x)) + \lambda y(x) = 0 \tag{1.1}$$

on $[0, h_1) \cup (h_1, h_2) \cup (h_2, \pi]$, with boundary conditions

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \tag{1.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \tag{1.3}$$

and transmission conditions

$$y(h_1 - 0) - \sqrt[3]{\lambda}\delta y(h_1 + 0) = 0, \tag{1.4}$$

$$y'(h_1 - 0) - \sqrt[3]{\lambda}\delta y'(h_1 + 0) = 0, \tag{1.5}$$

$$y(h_2 - 0) - \gamma y(h_2 + 0) = 0, \tag{1.6}$$

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$$y'(h_2 - 0) - \gamma y'(h_2 + 0) = 0, \tag{1.7}$$

where the real-valued function $q(x)$ is continuous in $[0, h_1) \cup (h_1, h_2) \cup (h_2, \pi]$ and has finite limits $q(h_1 \pm 0) = \lim_{x \to h_1 \pm 0} q(x)$, $q(h_2 \pm 0) = \lim_{x \to h_2 \pm 0} q(x)$ the real valued function $\Delta(x) \geq 0$ continuous in $[0, h_1) \cup (h_1, h_2) \cup (h_2, \pi]$ and has finite limits $\Delta(h_1 \pm 0) = \lim_{x \to h_1 \pm 0} \Delta(x)$, $\Delta(h_2 \pm 0) = \lim_{x \to h_2 \pm 0} \Delta(x)$, $x - \Delta(x) \geq 0$, if $x \in [0, h_1)$; $x - \Delta(x) \geq h_1$, if $x \in (h_1, h_2)$, $x - \Delta(x) \geq h_2$, if $x \in (h_2, \pi]$; λ is a real spectral parameter; δ, γ are arbitrary real numbers and $\sin \alpha \sin \beta \neq 0$.

Let $w_1(x, \lambda)$ be a solution of Equation (1.1) on $[0, h_1]$, satisfying the initial conditions

$$w_1 (0, \lambda) = \sin \alpha, w_1' (0, \lambda) = -\cos \alpha. \tag{1.8}$$

The conditions (1.8) define a unique solution of Equation (1.1) on $[0, h_1]$ ([2], p. 12).

After defining the above solution we shall define the solution $w_2 (x, \lambda)$ of Equation (1.1) on $[h_1, h_2]$ by means of the solution $w_1 (x, \lambda)$ using the initial conditions

$$w_2 (h_1, \lambda) = \lambda^{-1/3} \delta^{-1} w_1 (h_1, \lambda) , \quad w_2'(h_1, \lambda) = \lambda^{-1/3} \delta^{-1} w_1'(h_1, \lambda). \tag{1.9}$$

The conditions (1.9) are defined as a unique solution of Equation (1.1) on $[h_1, h_2]$.

After defining the above solution we shall define the solution $w_3 (x, \lambda)$ of Equation (1.1) on $[h_2, \pi]$ by means of the solution $w_2 (x, \lambda)$ using the initial conditions

$$w_3 (h_2, \lambda) = \gamma^{-1} w_2 (h_2, \lambda) , \quad w_3'(h_2, \lambda) = \gamma^{-1} w_2'(h_2, \lambda). \tag{1.10}$$

The conditions (1.10) are defined as a unique solution of Equation (1.1) on $[h_2, \pi]$.

Consequently, the function $w (x, \lambda)$ is defined on $[0, h_1) \cup (h_1, h_2) \cup (h_2, \pi]$ by the equality

$$w(x, \lambda) = \begin{cases} w_1(x, \lambda), & x \in [0, h_1) , \\ w_2(x, \lambda), & x \in (h_1, h_2) , \\ w_3(x, \lambda), & x \in (h_2, \pi] \end{cases}$$

is a solution of the Equation (1.1) on $[0, h_1) \cup (h_1, h_2) \cup (h_2, \pi]$;which satisfies one of the.boundary conditions and transmission conditions (1.4)-(1.5).

Lemma 1.1. *Let $w (x, \lambda)$ be a solution of Equation* (1.1) *and $\lambda > 0$. Then the following integral equations hold:*

$$\begin{aligned} w_1(x, \lambda) &= \sin \alpha \cos sx - \frac{\cos \alpha}{s} \sin sx \\ &- \frac{1}{s} \int_0^x q (\tau) \sin s (x - \tau) w_1 (\tau - \Delta (\tau) , \lambda) d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0 \right), \end{aligned} \tag{1.11}$$

$$\begin{aligned} w_2(x, \lambda) &= \frac{1}{s^{2/3} \delta} w_1 (h_1, \lambda) \cos s (x - h_1) + \frac{w_1' (h_1, \lambda)}{s^{5/3} \delta} \sin s (x - h_1) \\ &- \frac{1}{s} \int_{h_1}^x q (\tau) \sin s (x - \tau) w_2 (\tau - \Delta (\tau) , \lambda) d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0 \right), \end{aligned} \tag{1.12}$$

$$w_3(x,\lambda) = \frac{1}{\gamma} w_2(h_2,\lambda)\cos s(x-h_2) + \frac{w_2'(h_2,\lambda)}{s\gamma}\sin s(x-h_2)$$

$$- \frac{1}{s}\int_{h_2}^{x} q(\tau)\sin s(x-\tau)\, w_3(\tau-\Delta(\tau),\lambda)\, d\tau \quad \left(s=\sqrt{\lambda}, \lambda>0\right). \tag{1.13}$$

Proof. To prove this, it is enough to substitute $-s^2 w_1(\tau,\lambda) - w_1''(\tau,\lambda)$, $-s^2 w_2(\tau,\lambda) - w_2''(\tau,\lambda)$ and $-s^2 w_3(\tau,\lambda) - w_3''(\tau,\lambda)$ instead of $-q(\tau)w_1(\tau-\Delta(\tau),\lambda)$, $-q(\tau)w_2(\tau-\Delta(\tau),\lambda)$ and $-q(\tau)w_3(\tau-\Delta(\tau),\lambda)$ in the integrals in (1.11), (1.12) and (1.13) respectively and integrate by parts twice. □

Theorem 1.1. *The problem* (1.1) – (1.7) *can have only simple eigenvalues.*

Proof. Let $\widetilde{\lambda}$ be an eigenvalue of the problem (1.1) – (1.7) and

$$\widetilde{y}(x,\widetilde{\lambda}) = \begin{cases} \widetilde{y}_1(x,\widetilde{\lambda}), & x\in[0,h_1), \\ \widetilde{y}_2(x,\widetilde{\lambda}), & x\in(h_1,h_2), \\ \widetilde{y}_3(x,\widetilde{\lambda}), & x\in(h_2,\pi] \end{cases}$$

be a corresponding eigenfunction. Then from (1.2) and (1.8) it follows that the determinant

$$W\left[\widetilde{y}_1(0,\widetilde{\lambda}), w_1(0,\widetilde{\lambda})\right] = \begin{vmatrix} \widetilde{y}_1(0,\widetilde{\lambda}) & \sin\alpha \\ \widetilde{y}_1'(0,\widetilde{\lambda}) & -\cos\alpha \end{vmatrix} = 0,$$

and by Theorem 2.2 in [2] the functions $\widetilde{y}_1(x,\widetilde{\lambda})$ and $w_1(x,\widetilde{\lambda})$ are linearly dependent on $[0,h_1]$. We can also prove that the functions $\widetilde{y}_2(x,\widetilde{\lambda})$, $w_2(x,\widetilde{\lambda})$ are linearly dependent on $[h_1,h_2]$ and $\widetilde{y}_3(x,\widetilde{\lambda})$, $w_3(x,\widetilde{\lambda})$ are linearly dependent on $[h_2,\pi]$. Hence

$$\widetilde{y}_i(x,\widetilde{\lambda}) = K_i w_i(x,\widetilde{\lambda}) \quad \left(i=\overline{1,3}\right) \tag{1.14}$$

for some $K_1\neq 0$, $K_2\neq 0$ and $K_3\neq 0$. We must show that $K_1=K_2=K_3$. Suppose that $K_1\neq K_2$. From the equalities (1.4) and (1.14), we have

$$\begin{aligned} \widetilde{y}(h_1-0,\widetilde{\lambda}) - \sqrt[3]{\widetilde{\lambda}}\delta\widetilde{y}(h_1+0,\widetilde{\lambda}) &= \widetilde{y}_1(h_1,\widetilde{\lambda}) - \sqrt[3]{\widetilde{\lambda}}\delta\widetilde{y}_2(h_1,\widetilde{\lambda}) \\ &= K_1 w_1(h_1,\widetilde{\lambda}) - \sqrt[3]{\widetilde{\lambda}}\delta K_2 w_2(h_1,\widetilde{\lambda}) \\ &= \sqrt[3]{\widetilde{\lambda}}\delta K_1 w_2(h_1,\widetilde{\lambda}) - \sqrt[3]{\widetilde{\lambda}}\delta K_2 w_2(h_1,\widetilde{\lambda}) \\ &= \sqrt[3]{\widetilde{\lambda}}\delta\left(K_1-K_2\right) w_2(h_1,\widetilde{\lambda}) = 0. \end{aligned}$$

Since $\delta\left(K_1-K_2\right)\neq 0$ it follows that

$$w_2\left(h_1,\widetilde{\lambda}\right) = 0. \tag{1.15}$$

By the same procedure from equality (1.5), we can derive that

$$w_2'\left(h_1,\widetilde{\lambda}\right) = 0. \tag{1.16}$$

From the fact that $w_2(x,\widetilde{\lambda})$ is a solution of the differential Equation (1.1) on $[h_1,h_2]$ and satisfies the initial conditions (1.15) and (1.16) it follows that $w_2(x,\widetilde{\lambda})=0$ identically on $[h_1,h_2]$ (cf. [2, p. 12, Theorem 1.2.1]).

By using this, we may also find

$$w_1\left(h_1,\widetilde{\lambda}\right) = w_1'\left(h_1,\widetilde{\lambda}\right) = 0$$

From the latter discussions of $w_2(x,\widetilde{\lambda})$ it follows that $w_1(x,\widetilde{\lambda}) = 0$ identically on $[0,h_1]$. But this contradicts (1.8), thus completing the proof. Analogically we can show that $K_2 = K_3$. □

2. An existence theorem

The function $w(x,\lambda)$ defined in section 1 is a nontrivial solution of Equation (1.1) satisfying conditions (1.2) and (1.4)-(1.7). Putting $w(x,\lambda)$ into (1.3), we get the characteristic equation

$$F(\lambda) \equiv w(\pi,\lambda)\cos\beta + w'(\pi,\lambda)\sin\beta = 0. \tag{2.1}$$

By Theorem 1.1, the set of eigenvalues of boundary-value problem (1.1)-(1.7) coincides with the set of real roots of Eq. (2.1). Let $q_1 = \int\limits_0^{h_1} |q(\tau)|d\tau$, $q_2 = \int\limits_{h_1}^{h_2} |q(\tau)|\, d\tau$ and $q_3 = \int\limits_{h_2}^{\pi} |q(\tau)|\, d\tau$.

Lemma 2.1. (1) *Let $\lambda \geq 4q_1^2$. Then for the solution $w_1(x,\lambda)$ of Equation* (1.11)*, the following inequality holds*:

$$|w_1(x,\lambda)| \leq \frac{1}{|q_1|}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}, \quad x \in [0,h_1]. \tag{2.2}$$

(2) *Let $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$. Then for the solution $w_2(x,\lambda)$ of Equation* (1.12)*, the following inequality holds*:

$$|w_2(x,\lambda)| \leq \frac{2.5198421}{\sqrt[3]{q_1^5}\delta}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}, \quad x \in [h_1,h_2]. \tag{2.3}$$

(3) *Let $\lambda \geq \max\left\{4q_1^2, 4q_2^2, 4q_3^2\right\}$. Then for the solution $w_2(x,\lambda)$ of Equation* (1.13)*, the following inequality holds*:

$$|w_3(x,\lambda)| \leq \frac{10.0793684}{\sqrt[3]{q_1^5}\delta\gamma}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}, \quad x \in [h_2,\pi]. \tag{2.4}$$

Proof. Let $B_{1\lambda} = \max_{[0,h_1]}|w_1(x,\lambda)|$. Then from (1.11), it follows that, for every $\lambda > 0$, the following inequality holds:

$$B_{1\lambda} \leq \sqrt{\sin^2\alpha + \frac{\cos^2\alpha}{s^2}} + \frac{1}{s}B_{1\lambda}q_1.$$

If $s \geq 2q_1$ we get (2.2). Differentiating (1.11) with respect to x, we have

$$w_1'(x,\lambda) = -s\sin\alpha\sin sx - \cos\alpha\cos sx - \int\limits_0^x q(\tau)\cos s(x-\tau)\, w_1(\tau - \Delta(\tau),\lambda)d\tau. \tag{2.5}$$

From (2.5) and (2.2), it follows that, for $s \geq 2q_1$, the following inequality holds:.

$$\frac{|w_1'(x,\lambda)|}{s^{5/3}} \leq \frac{1}{\sqrt[3]{4q_1^5}}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}. \tag{2.6}$$

Let $B_{2\lambda} = \max_{[h_1,h_2]} |w_2(x,\lambda)|$. Then from (1.12), (2.2) and (2.6) it follows that, for $s \geq 2q_1$ and $s \geq 2q_2$, the following inequalities hold:

$$B_{2\lambda} \leq \frac{2}{\sqrt[3]{4q_1^5}\delta}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha} + \frac{1}{2q_2}B_{2\lambda}q_2,$$

$$B_{2\lambda} \leq \frac{2\sqrt[3]{2}}{\sqrt[3]{q_1^5}\delta}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}.$$

Hence if $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$ we get (2.3).

Differentiating (1.12) with respect to x, we have

$$w_2'(x,\lambda) = -\frac{\sqrt[3]{s}}{\delta}w_1(h_1,\lambda)\sin s(x-h_1) + \frac{w_1'(h_1,\lambda)}{\sqrt[3]{s^2}\delta}\cos s(x-h_1)$$

$$-\int_{h_1}^{x} q(\tau)\cos s(x-\tau)\,w_2(\tau-\Delta(\tau),\lambda)d\tau. \tag{2.7}$$

From (2.7) and (2.3), it follows that, for $s \geq 2q_1$, the following inequality holds:.

$$\frac{|w_2'(x,\lambda)|}{s} \leq \frac{\sqrt[3]{16}}{\delta\sqrt[3]{q_1^5}}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}. \tag{2.8}$$

Let $B_{3\lambda} = \max_{[h_2,\pi]} |w_3(x,\lambda)|$. Then from (1.13), (2.2), (2.3) and (2.8) it follows that, for $s \geq 2q_1$, $s \geq 2q_2$ and $s \geq 2q_3$, the following inequalities hold:

$$B_{3\lambda} \leq \frac{\sqrt[3]{2^4}}{\sqrt[3]{q_1^5}\delta\gamma}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha} + \frac{\sqrt[3]{2^4}}{\sqrt[3]{q_1^5}\delta\gamma}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha} + \frac{1}{2q_3}B_{3\lambda}q_3,$$

$$B_{3\lambda} \leq \frac{\sqrt[3]{2^{10}}}{\sqrt[3]{q_1^5}\delta\gamma}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}.$$

Hence if $\lambda \geq \max\left\{4q_1^2, 4q_2^2, 4q_3^2\right\}$ we get (2.4). □

Theorem 2.1. *The problem* (1.1)-(1.7) *has an infinite set of positive eigenvalues.*

Proof. Differentiating (1.9) with respect to$x$, we get

$$w_3'(x,\lambda) = -\frac{s}{\gamma}w_2(h_2,\lambda)\sin s(x-h_2) + \frac{w_2'(h_2,\lambda)}{\gamma}\cos s(x-h_2)$$

$$-\int_{h_2}^{x} q(\tau)\cos s(x-\tau)\,w_3(\tau-\Delta(\tau),\lambda)d\tau. \tag{2.9}$$

From (1.11), (1.12), (1.13), (2.1), (2.5), (2.7) and (2.9), we get

$$\left[\frac{1}{\gamma}\left\{\frac{1}{s^{2/3}\delta}\left(\sin\alpha\cos sh_1 - \frac{\cos\alpha}{s}\sin sh_1 - \frac{1}{s}\int_0^{h_1} q(\tau)\sin s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\right.\right.$$

$$\times\cos s(h_2-h_1)$$

$$-\frac{1}{s^{5/3}\delta}\left(s\sin\alpha\sin sh_1 + \cos\alpha\cos sh_1 + \int_0^{h_1} q(\tau)\cos s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$

$$\left.\times \sin s\left(h_2-h_1\right)-\frac{1}{s}\int_{h_1}^{h_2} q(\tau)\sin s(h_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\right\}\cos s\left(\pi-h_2\right)+$$

$$\frac{1}{s\gamma}\left\{-\frac{s^{1/3}}{\delta}\left(\sin\alpha\cos sh_1-\frac{\cos\alpha}{s}\sin sh_1-\frac{1}{s}\int_0^{h_1} q(\tau)\sin s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\right.$$

$$\times \sin s\left(h_2-h_1\right)$$

$$-\frac{1}{s^{2/3}\delta}\left(s\sin\alpha\sin sh_1+\cos\alpha\cos sh_1+\int_0^{h_1} q(\tau)\cos s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$

$$\left.\times \cos s\left(h_2-h_1\right)-\int_{h_1}^{h_2} q(\tau)\cos s(h_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\right\}\sin s\left(\pi-h_2\right)$$

$$\left.-\frac{1}{s}\int_{h_2}^{\pi} q(\tau)\sin s\left(\pi-\tau\right)w_3(\tau-\Delta(\tau),\lambda)d\tau\right]\cos\beta+$$

$$\left[-\frac{s}{\gamma}\left\{\frac{1}{s^{2/3}\delta}\left(\sin\alpha\cos sh_1-\frac{\cos\alpha}{s}\sin sh_1-\frac{1}{s}\int_0^{h_1} q(\tau)\sin s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\right.\right.$$

$$\times \cos s\left(h_2-h_1\right)$$

$$-\frac{1}{s^{5/3}\delta}\left(s\sin\alpha\sin sh_1+\cos\alpha\cos sh_1+\int_0^{h_1} q(\tau)\cos s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$

$$\left.\times \sin s\left(h_2-h_1\right)-\frac{1}{s}\int_{h_1}^{h_2} q(\tau)\sin s(h_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\right\}\sin s\left(\pi-h_2\right)+$$

$$\frac{1}{\gamma}\left\{-\frac{s^{1/3}}{\delta}\left(\sin\alpha\cos sh_1-\frac{\cos\alpha}{s}\sin sh_1-\frac{1}{s}\int_0^{h_1} q(\tau)\sin s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\right.$$

$$\times \sin s\left(h_2-h_1\right)$$

$$-\frac{1}{s^{2/3}\delta}\left(s\sin\alpha\sin sh_1+\cos\alpha\cos sh_1+\int_0^{h_1} q(\tau)\cos s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$

$$\left.\times \cos s\left(h_2-h_1\right)-\int_{h_1}^{h_2} q(\tau)\cos s(h_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\right\}\cos s\left(\pi-h_2\right)$$

$$\left.-\int_{h_2}^{\pi} q(\tau)\cos s\left(\pi-\tau\right)w_3(\tau-\Delta(\tau),\lambda)d\tau\right]\sin\beta=0. \tag{2.10}$$

Let λ be sufficiently large. Then, by (2.2)-(2.4), Equation (2.10) may be rewritten in the form

$$\sqrt[3]{s}\sin s\pi+O(1)=0. \tag{2.11}$$

Obviously, for large s Equation (2.11) has an infinite set of roots. Thus the theorem is proved. □

3. Asymptotic Formulas for Eigenvalues and Eigenfunctions

Now we begin to study asymptotic properties of eigenvalues and eigenfunctions. In the following we shall assume that$s$is sufficiently large. From (1.11) and (2.2), we get

$$w_1(x,\lambda) = O(1) \quad \text{on} \quad [0, h_1]. \tag{3.1}$$

From (1.12) and (2.3), we get

$$w_2(x,\lambda) = O(1) \quad \text{on} \quad [h_1, h_2]. \tag{3.2}$$

From (1.13) and (2.4), we get

$$w_3(x,\lambda) = O(1) \quad \text{on} \quad [h_2, \pi]. \tag{3.3}$$

The existence and continuity of the derivatives $w'_{1s}(x,\lambda)$ for $0 \le x \le h_1, |\lambda| < \infty$, $w'_{2s}(x,\lambda)$ for $h_1 \le x \le h_2, |\lambda| < \infty$ and $w'_{3s}(x,\lambda)$ for $h_2 \le x \le \pi, |\lambda| < \infty$ follows from Theorem 1.4.1 in [2].

$$w'_{1s}(x,\lambda) = O(1), \quad x \in [0, h_1], \tag{3.4}$$

$$w'_{2s}(x,\lambda) = O(1), \quad x \in [h_1, h_2], \tag{3.5}$$

$$w'_{3s}(x,\lambda) = O(1), \quad x \in [h_2, \pi] \tag{3.6}$$

hold.

Proof. By differentiating (1.13) with respect to s, we get, by (3.3)

(3.7)

$$w'_{3s}(x,\lambda) = -\frac{1}{s}\int_0^x q(\tau)\cos s(x-\tau) w'_{3s}\left(\tau - \Delta\left(\tau\right), \lambda\right) + \theta(x,\lambda), \quad (|\theta(x,\lambda)| \le \theta_0).$$

Let $D_\lambda = \max_{[h_2,\pi]} \left| w'_{3s}(x,\lambda) \right|$. Then the existance of D_λ follows from continuity of derivation for $x \in [h_2, \pi]$. From (3.7)

$$D_\lambda \le \frac{1}{s} q_3 D_\lambda + \theta_0.$$

Now let $s \ge 2q_3$. Then $D_\lambda \le 2\theta_0$ and the validity of the asymptotic formula (3.6) follows. Formulas (3.4) and (3.5) may be proved analogically. □

Theorem 3.1. *Let n be a natural number. For each sufficiently large n there is exactly one eigenvalue of the problem* (1.1)-(1.7) *near n^2.*

Proof. We consider the expression which is denoted by $O(1)$ in Equation (2.11):

$$\begin{aligned}
&\frac{\delta\gamma}{\sin\alpha\sin\beta}\left\{ -\frac{\sin(\alpha-\beta)}{s^{2/3}\delta\gamma}\cos s\pi + \frac{\cos\alpha\cos\beta}{s^{5/3}\delta\gamma}\sin s\pi \right. \\
&- \frac{1}{\delta\gamma}\int_0^{h_1}\left[\frac{\cos\beta}{s^{5/3}}\sin s(\pi-\tau) + \frac{\sin\beta}{s^{2/3}}\cos s\left(\pi-\tau\right)\right] q\left(\tau\right) w_1\left(\tau - \Delta\left(\tau\right), \lambda\right) d\tau \\
&+ \frac{1}{\gamma}\int_{h_1}^{h_2}\left[\frac{\cos\beta}{s^{5/3}}\sin s(\pi-\tau) + \frac{\sin\beta}{s^{2/3}}\cos s\left(\pi-\tau\right)\right] q\left(\tau\right) w_2\left(\tau - \Delta\left(\tau\right), \lambda\right) d\tau \\
&\left. + \int_{h_2}^{\pi}\left[\frac{\cos\beta}{s}\sin s(\pi-\tau) + \sin\beta\cos s\left(\pi-\tau\right)\right] q\left(\tau\right) w_3\left(\tau - \Delta\left(\tau\right), \lambda\right) d\tau \right\}.
\end{aligned}$$

If formulas (3.1)-(3.6) are taken into consideration, it can be shown by differentiation with respect to s that for large s this expression has bounded derivative. It is obvious that for large s the roots of Equation (2.11) are situated close to entire numbers. We shall show that, for large n, only one root (2.11) lies near to each n. We consider the function $\phi(s) = \sqrt[3]{s}\sin s\pi + O(1)$. Its derivative, which has the form $\phi'(s) = \frac{1}{3\sqrt[3]{s^2}}\sin s\pi + \sqrt[3]{s}\pi\cos\pi + O(1)$, does not vanish for s close to n for sufficiently large n. Thus our assertion follows by Rolle's Theorem. □

Let n be sufficiently large. In what follows we shall denote by$\lambda_n = s_n^2$ the eigenvalue of the problem (1.1)-(1.7) situated near n^2. We set $s_n = n + \delta_n$. From (2.11) it follows that $\delta_n = O\left(\frac{1}{n^{1/3}}\right)$.

Consequently

$$s_n = n + O\left(\frac{1}{n^{1/3}}\right). \tag{3.8}$$

Formula (3.8) make it possible to obtain asymptotic expressions for eigenfunction of the problem (1.1)-(1.7). From (1.11), (2.5) and (3.1), we get

$$w_1(x,\lambda) = \sin\alpha\cos sx + O\left(\frac{1}{s}\right), \tag{3.9}$$

$$w_1'(x,\lambda) = -s\sin\alpha\sin sx + O\left(1\right). \tag{3.10}$$

From (1.12), 2.6), (3.2), (3.9) and (3.10), we get

$$w_2(x,\lambda) = \frac{\sin\alpha}{s^{2/3}\delta}\cos sx + O\left(\frac{1}{s}\right), \tag{3.11}$$

$$w_2'(x,\lambda) = -\frac{s^{1/3}\sin\alpha}{\delta}\sin sx + O\left(1\right). \tag{3.12}$$

From (1.13), (2.7), (3.3), (3.11) and (3.12), we get

$$w_3(x,\lambda) = \frac{\sin\alpha}{s^{2/3}\delta\gamma}\cos sh_2\cos s\left(x-h_2\right) - \frac{\sin\alpha}{s^{2/3}\delta\gamma}\sin sh_2\sin s\left(x-h_2\right) + O\left(\frac{1}{s}\right),$$

(3.13)

$$w_3(x,\lambda) = \frac{\sin\alpha}{s^{2/3}\delta\gamma}\cos sx + O\left(\frac{1}{s}\right).$$

By substituting (3.8) into (3.9), (3.11) and (3.12), we find that

$$\begin{aligned} u_{1n} &= w_1\left(x,\lambda_n\right) = \sin\alpha\cos nx + O\left(\frac{1}{n^{1/3}}\right), \\ u_{2n} &= w_2\left(x,\lambda_n\right) = \frac{\sin\alpha}{\delta n^{2/3}}\cos nx + O\left(\frac{1}{n}\right), \\ u_{3n} &= w_3\left(x,\lambda_n\right) = \frac{\sin\alpha}{\delta\gamma n^{2/3}}\cos nx + O\left(\frac{1}{n}\right). \end{aligned}$$

Hence the eigenfunctions $u_n(x)$ have the following asymptotic representation:

$$u_n(x) = \begin{cases} \sin\alpha\cos nx + O\left(\frac{1}{n^{1/3}}\right), & x\in[0,h_1), \\ \frac{\sin\alpha}{\delta n^{2/3}}\cos nx + O\left(\frac{1}{n}\right), & x\in(h_1,h_2), \\ \frac{\sin\alpha}{\delta\gamma n^{2/3}}\cos nx + O\left(\frac{1}{n}\right), & x\in(h_2,\pi]. \end{cases}$$

Under some additional conditions the more exact asymptotic formulas which depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled:

a) The derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded in $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$ and have finite limits $q'(h_i\pm 0)=\lim\limits_{x\to h_i\pm 0} q'(x)$ and $\Delta''(h_i\pm 0)=\lim\limits_{x\to h_i\pm 0}\Delta''(x)$ $(i=1,2)$, respectively.

b) $\Delta'(x)\le 1$ in $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$, $\Delta(0)=0$ and $\lim\limits_{x\to h_i+0}\Delta(x)=0$ $(i=1,2)$.

By using b), we have

$$x-\Delta(x)\ge 0, x\in[0,h_1)\,, \tag{3.14}$$

$$x-\Delta(x)\ge h_1, x\in(h_1,h_2)\,, \tag{3.15}$$

$$x-\Delta(x)\ge h_1, x\in(h_2,\pi]\,. \tag{3.16}$$

From Equations (3.9), (3.11) and (3.13)-(3.16) we have

$$w_1\left(\tau-\Delta\left(\tau\right),\lambda\right)=\sin\alpha\cos s\left(\tau-\Delta\left(\tau\right)\right)+O\left(\frac{1}{s}\right), \tag{3.17}$$

$$w_2\left(\tau-\Delta\left(\tau\right),\lambda\right)=\frac{\sin\alpha}{s^{2/3}\delta}\cos s\left(\tau-\Delta\left(\tau\right)\right)+O\left(\frac{1}{s}\right), \tag{3.18}$$

$$w_3\left(\tau-\Delta\left(\tau\right),\lambda\right)=\frac{\sin\alpha}{s^{2/3}\delta\gamma}\cos s\left(\tau-\Delta\left(\tau\right)\right)+O\left(\frac{1}{s}\right). \tag{3.19}$$

Putting these expressions into Equation (2.10), we have

$$\begin{aligned}&-\frac{s^{1/3}}{\delta\gamma}\sin\alpha\sin\beta\sin s\pi+\frac{\sin\left(\alpha-\beta\right)}{s^{2/3}\delta\gamma}\cos s\pi-\frac{\sin\alpha\sin\beta}{s^{2/3}\delta\gamma}\\&\times\left\{\cos s\pi_0^\pi\frac{q\left(\tau\right)}{2}\left[\cos s\Delta(\tau)+\cos s\left(2\tau-\Delta(\tau)\right)\right]d\tau\right.\\&\left.+\sin s\pi_0^\pi\frac{q\left(\tau\right)}{2}\left[\sin s\Delta(\tau)+\sin s\left(2\tau-\Delta(\tau)\right)\right]d\tau\right\}+O\left(\frac{1}{s^{5/3}}\right)=0.\end{aligned} \tag{3.20}$$

Let

$$\left\{\begin{array}{l}A\left(x,s,\Delta(\tau)\right)=\frac{1}{2}\int\limits_0^x q(\tau)\sin\left(s\Delta(\tau)\right)d\tau,\\ B(x,s,\Delta(\tau))=\frac{1}{2}\int\limits_0^x q(\tau)\cos\left(s\Delta(\tau)\right)d\tau.\end{array}\right. \tag{3.21}$$

It is obvious that these functions are bounded for $0\le x\le\pi, 0<s<+\infty$.

Under the conditions a) and b) the following formulas

$$\int\limits_0^x q(\tau)\cos s(2\tau-\Delta(\tau))d\tau=O\left(\frac{1}{s}\right),\quad\int\limits_0^x q(\tau)\sin s(2\tau-\Delta(\tau))d\tau=O\left(\frac{1}{s}\right) \tag{3.22}$$

can be proved by the same technique in Lemma 3.3 in [2]. From Equations (3.20), (3.21) and(3.22),we have

$$\sin s\pi\left[s\sin\alpha\sin\beta+A\left(\pi,s,\Delta\left(\tau\right)\right)\sin\alpha\sin\beta\right]-$$

$$\cos s\pi\left[\sin\alpha\cos\beta-\cos\alpha\sin\beta-B\left(\pi,s,\Delta\left(\tau\right)\right)\sin\alpha\sin\beta\right]+O\left(\frac{1}{s}\right)=0.$$

Hence

$$\tan s\pi = \frac{1}{s}\left[\cot\beta - \cot\alpha - B\left(\pi, s, \Delta\left(\tau\right)\right)\right] + O\left(\frac{1}{s^2}\right).$$

Again if we take $s_n = n + \delta_n$, then

$$\tan\left(n + \delta_n\right)\pi = \tan\delta_n\pi = \frac{1}{n}\left[\cot\beta - \cot\alpha - B\left(\pi, n, \Delta\left(\tau\right)\right)\right] + O\left(\frac{1}{n^2}\right).$$

Hence for large n,

$$\delta_n = \frac{1}{n\pi}\left[\cot\beta - \cot\alpha - B\left(\pi, n, \Delta\left(\tau\right)\right)\right] + O\left(\frac{1}{n^2}\right)$$

and finally

$$s_n = n + \frac{1}{n\pi}\left[\cot\beta - \cot\alpha - B\left(\pi, n, \Delta\left(\tau\right)\right)\right] + O\left(\frac{1}{n^2}\right). \tag{3.23}$$

Thus, we have proven the following theorem.

Theorem 3.2. *If conditions a) and b) are satisfied then, the positive eigenvalues $\lambda_n = s_n^2$ of the problem (1.1)-(1.7) have the asymptotic representation of (3.23) forn $\to \infty$.*

We now may obtain a more exact asymptotic formula for the eigenfunctions. From Equations (1.11), (3.17), (3.21) and (3.22)

$$w_1(x,\lambda) = \sin\alpha\cos sx\left[1 + A\left(x, \rangle\, s, \rangle\, \Delta(\tau)\right.\right.$$

$\bar{s}$

$$-\frac{\sin sx}{s}\left[\cos\alpha + \sin\alpha B\left(x, \rangle\, s, \rangle\, \Delta(\tau) + O\left(\frac{1}{s^2}\right)\right.\right. . \tag{3.24}$$

Replacing s by s_n and using Equation (3.23), we have

$$u_{1n}(x) = w_1(x,\lambda_n) = \sin\alpha\left\{\cos nx\left[1 + \frac{A\left(x, n, \Delta\left(\tau\right)\right)}{n}\right]\right.$$

(3.25)

$$\left.-\frac{\sin nx}{n\pi}\left[\left(\cot\beta - \cot\alpha - B\left(\pi, n, \Delta\left(\tau\right)\right)\right)x + \left(\cot\alpha + B\left(x, n, \Delta\left(\tau\right)\right)\right)\pi\right]\right\} + O\left(\frac{1}{n^2}\right).$$

From (2.5), (3.18), (3.21), (3.22) and (3.24), we have

$$w_2\left(x,\lambda\right) = \frac{\sin\alpha}{s^{2/3}\delta}\cos sx\left[1 + \frac{A\left(x, s, \Delta(\tau)\right)}{s}\right]$$

$$-\frac{\sin sx}{s^{5/3}\delta}\left(\cos\alpha + \sin\alpha B\left(x, s, \Delta(\tau)\right)\right) + O\left(\frac{1}{s^2}\right), \tag{3.26}$$

Now, replacing s by s_n and using Equation (3.23), we have

$$u_{2n}(x) = \frac{\sin\alpha}{n^{2/3}\delta}\left\{\cos nx\left[1 + \frac{A\left(x, n, \Delta\left(\tau\right)\right)}{n}\right] - \frac{\sin nx}{n^{5/3}\pi}\right.$$

$$\left.\times\left[\left(\cot\beta - \cot\alpha - B\left(\pi, n, \Delta\left(\tau\right)\right)\right)x + \left(\cot\alpha + B\left(x, n, \Delta\left(\tau\right)\right)\right)\pi\right]\right\} + O\left(\frac{1}{n^2}\right).$$

From (2.9), (3.19), (3.21), (3.22) and (3.26), we have

$$\begin{aligned} w_3(x,\lambda) &= \frac{\sin\alpha}{s^{2/3}\delta\gamma}\cos sx\left[1+\frac{A(x,s,\Delta(\tau))}{s}\right] \\ &- \frac{\sin sx}{s^{5/3}\delta\gamma}\left(\cos\alpha+\sin\alpha B(x,s,\Delta(\tau))\right)+O\left(\frac{1}{s^2}\right), \end{aligned}$$

Now, replacing s by s_n and using Equation (3.23)

$$u_{3n}(x) = \frac{\sin\alpha}{n^{2/3}\delta\gamma}\left\{\cos nx\left[1+\frac{A(x,n,\Delta(\tau))}{n}\right]-\frac{\sin nx}{n^{5/3}\pi}\right.$$

$$\left.\times\left[\left(\cot\beta-\cot\alpha-B(\pi,n,\Delta(\tau))\right)x+\left(\cot\alpha+B(x,n,\Delta(\tau))\right)\pi\right]\right\}+O\left(\frac{1}{n^2}\right).$$

Thus, we have proven the following theorem.

Theorem 3.3. *If conditions a) and b) are satisfied then, the eigenfunctions $u_n(x)$ of the problem (1.1)-(1.7) have the following asymptotic representation for $n\to\infty$:*

$$u_n(x) = \begin{cases} u_{1n}(x), & x\in[0,h_1), \\ u_{2n}(x), & x\in(h_1,h_2), \\ u_{3n}(x), & x\in(h_2,\pi], \end{cases}$$

where $u_{1n}(x)$, $u_{2n}(x)$ and $u_{3n}(x)$ are defined as in (3.12) and (3.14) respectively.

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SOME INEQUALITIES OF OSTROWSKI TYPE IN THREE INDEPENDENT VARIABLES

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Abstract. Some new inequalities of Ostrowski type involving functions of three independent variables are established.

1. INTRODUCTION

In 2001, Cheng in [3] proved the following integral inequality:

Theorem 1.1. *Let $f : [a, b] \to \mathbf{R}$ be an absolutely continuous function such that there exist constants $\gamma, \Gamma \in \mathbf{R}$ with $\gamma \leq f'(t) \leq \Gamma$, $t \in [a, b]$. Then for all $x \in [a, b]$, we have*

$$\begin{aligned} &|\tfrac{1}{2} f(x) - \tfrac{1}{b-a}\textstyle\int_a^b f(t)\,dt - \tfrac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}| \\ &\leq \tfrac{1}{8(b-a)}((x-a)^2 + (x-b)^2)(\Gamma - \gamma). \end{aligned} \tag{1.1}$$

The constant $\frac{1}{8}$ is sharp (see [4]).

Remark 1.1. If we take $x = a$ or $x = b$ in (1), then we get a sharp trapezoid inequality

$$|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt| \leq \frac{1}{8}(b-a)(\Gamma - \gamma).$$

In 2010, Sarikaya in [5] established the following inequality of Ostrowski type involving functions of two independent variables.

Theorem 1.2. *Let $f : [a, b] \times [c, d] \to \mathbf{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and suppose that there exist constants $\gamma, \Gamma \in \mathbf{R}$ with $\gamma \leq \frac{\partial^2 f(t,s)}{\partial t \partial s} \leq \Gamma$ for all $(t, s) \in [a, b] \times [c, d]$. Then we have*

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$$
(1.2)\quad
\begin{aligned}
&|\tfrac{1}{4}f(x,y)+\tfrac{1}{4}H(x,y)-\tfrac{1}{2(b-a)}\int_a^b f(t,y)\,dt-\tfrac{1}{2(d-c)}\int_c^d f(x,s)\,ds\\
&-\tfrac{1}{2(b-a)(d-c)}\int_a^b[(y-c)f(t,c)+(d-y)f(t,d)]\,dt\\
&-\tfrac{1}{2(b-a)(d-c)}\int_c^d[(x-a)f(a,s)+(b-x)f(b,s)]\,ds\\
&+\tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,ds\,dt|\\
&\le \tfrac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{32(b-a)(d-c)}(\Gamma-\gamma)
\end{aligned}
$$

for all $(x,y)\in[a,b]\times[c,d]$, *where*

$$
\begin{aligned}
H(x,y)=&\tfrac{(x-a)[(y-c)f(a,c)+(d-y)f(a,d)]+(b-x)[(y-c)f(b,c)+(d-y)f(b,d)]}{(b-a)(d-c)}\\
&+\tfrac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a}+\tfrac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}.
\end{aligned}
$$

Here we have given a revised version for (2) since the expression in [5] and [6] contained a misprint.

Remark 1.2. If we take any one of the four cases $x=a, y=c$; $x=a, y=d$; $x=b, y=c$ and $x=b, y=d$ in (2), then we get a trapezoid type inequality for double integrals.

$$
(1.3)\quad
\begin{aligned}
&|\tfrac{f(a,c)+f(b,d)+f(b,c)+f(a,d)}{4}-\tfrac{1}{2(b-a)}\int_a^b[f(t,c)+f(t,d)]\,dt\\
&-\tfrac{1}{2(d-c)}\int_c^d[f(a,s)+f(b,s)]\,ds+\tfrac{1}{2(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,ds\,dt|\\
&\le \tfrac{(b-a)(d-c)}{32}(\Gamma-\gamma).
\end{aligned}
$$

It is interesting to compare this inequality (3) with the result in [2].

In the literature, we find that Pachpatte was the first author who has established an inequality of Ostrowski type in three independent variables as follows:

Theorem 1.3. *Let* $f:[a,k]\times[b,l]\times[c,m]\to\mathbf{R}$ *be an absolutely continuous function such that the partial derivative of order* 3 *exists and continuous for all* $(t,s,u)\in[a,k]\times[b,l]\times[c,m]$. *Then we have*

$$
(1.4)\quad
\begin{aligned}
&|\tfrac{(k-a)(l-b)(m-c)}{8}[f(a,b,c)+f(a,b,m)+f(a,l,c)+f(a,l,m)\\
&+f(k,b,c)+f(k,b,m)+f(k,l,c)+f(k,l,m)]\\
&-\tfrac{(l-b)(m-c)}{4}\int_a^k[f(t,b,c)+f(t,l,c)+f(t,b,m)+f(t,l,m)]\,dt\\
&-\tfrac{(k-a)(m-c)}{4}\int_b^l[f(a,s,c)+f(k,s,c)+f(a,s,m)+f(k,s,m)]\,ds\\
&-\tfrac{(k-a)(l-b)}{4}\int_c^m[f(a,b,u)+f(a,l,u)+f(k,b,u)+f(k,l,u)]\,du\\
&+\tfrac{(m-c)}{2}\int_a^k\int_b^l[f(t,s,m)+f(t,s,c)]\,ds\,dt\\
&+\tfrac{(k-a)}{2}\int_b^l\int_c^m[f(k,s,u)+f(a,s,u)]\,du\,ds\\
&+\tfrac{(l-b)}{2}\int_a^k\int_c^m[f(t,l,u)+f(t,b,u)]\,du\,dt\\
&-\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt|\\
&\le \tfrac{(k-a)(l-b)(m-c)}{8}\int_a^k\int_b^l\int_c^m|\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}|\,du\,ds\,dt.
\end{aligned}
$$

Here we also have given a revised version for (4) since the expression in [1] contained misprints.

In this paper, we will extend the above result to establish some new Ostrowski type inequalities involving functions of three independent variables.

2. MAIN RESULTS

Theorem 2.1. *Let $f : [a,k] \times [b,l] \times [c,m] \to \mathbf{R}$ be an absolutely continuous function such that the partial derivative of order 3 exists and suppose that there exist constants $\gamma, \Gamma \in \mathbf{R}$ with $\gamma \leq \frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u} \leq \Gamma$ for all $(t,s,u) \in [a,k] \times [b,l] \times [c,m]$. Then we have*

(2.1)
$$\begin{aligned}
&|\tfrac{1}{8} f(x,y,z) + \tfrac{1}{8} H(x,y,z) - \tfrac{1}{4}\int_a^k G_1(t,y,z)\,dt - \tfrac{1}{4}\int_b^l G_2(x,s,z)\,ds - \tfrac{1}{4}\int_c^m G_3(x,y,u)\,du \\
&+ \tfrac{1}{2(k-a)(l-b)(m-c)}\{\int_a^k \int_b^l [(z-c)f(t,s,c) + (m-z)f(t,s,m) + (m-c)f(t,s,z)]\,ds\,dt \\
&+ \int_b^l \int_c^m [(x-a)f(a,s,u) + (k-x)f(k,s,u) + (k-a)f(x,s,u)]\,du\,ds \\
&+ \int_a^k \int_c^m [(y-b)f(t,b,u) + (l-y)f(t,l,u) + (l-b)f(t,y,u)]\,du\,dt\} \\
&- \tfrac{1}{(k-a)(l-b)(m-c)} \int_a^k \int_b^l \int_c^m f(t,s,u)\,du\,ds\,dt| \\
&\leq \tfrac{[(x-a)^2+(k-x)^2][(y-b)^2+(l-y)^2][(z-c)^2+(m-z)^2]}{128(k-a)(l-b)(m-c)}(\Gamma - \gamma)
\end{aligned}$$

for all $(x,y,z) \in [a,k] \times [b,l] \times [c,m]$, where

$$\begin{aligned}
&H(x,y,z) = \tfrac{(z-c)f(x,y,c)+(m-z)f(x,y,m)}{m-c} + \tfrac{(x-a)f(a,y,z)+(k-x)f(k,y,z)}{k-a} \\
&+ \tfrac{(y-b)f(x,b,z)+(l-y)f(x,l,z)}{l-b} \\
&+ \tfrac{(x-a)(y-b)f(a,b,z)+(k-x)(y-b)f(k,b,z)+(x-a)(l-y)f(a,l,z)+(k-x)(l-y)f(k,l,z)}{(k-a)(l-b)} \\
&+ \tfrac{(y-b)(z-c)f(x,b,c)+(l-y)(z-c)f(x,l,c)+(y-b)(m-z)f(x,b,m)+(l-y)(m-z)f(x,l,m)}{(l-b)(m-c)} \\
&+ \tfrac{(x-a)(z-c)f(a,y,c)+(k-x)(z-c)f(k,y,c)+(x-a)(m-z)f(a,y,m)+(k-x)(m-z)f(k,y,m)}{(k-a)(m-c)} \\
&+ \tfrac{1}{(k-a)(l-b)(m-c)}\{(y-b)(z-c)[(x-a)f(a,b,c) + (k-x)f(k,b,c)] \\
&+ (l-y)(z-c)[(x-a)f(a,l,c) + (k-x)f(k,l,c)] \\
&+ (y-b)(m-z)[(x-a)f(a,b,m) + (k-x)f(k,b,m)] \\
&+ (l-y)(m-z)[(x-a)f(a,l,m) + (k-x)f(k,l,m)]\},
\end{aligned}$$

$$\begin{aligned}
G_1(t,y,z) &= \tfrac{f(t,y,z)}{k-a} + \tfrac{(y-b)f(t,b,z)+(l-y)f(t,l,z)}{(k-a)(l-b)} + \tfrac{(z-c)f(t,y,c)+(m-z)f(t,y,m)}{(k-a)(m-c)} \\
&+ \tfrac{(z-c)[(y-b)f(t,b,c)+(l-y)f(t,l,c)]+(m-z)[(y-b)f(t,b,m)+(l-y)f(t,l,m)]}{(k-a)(l-b)(m-c)},
\end{aligned}$$

$$\begin{aligned}
G_2(x,s,z) &= \tfrac{f(x,s,z)}{l-b} + \tfrac{(x-a)f(a,s,z)+(k-x)f(k,s,z)}{(k-a)(l-b)} + \tfrac{(z-c)f(x,s,c)+(m-z)f(x,s,m)}{(l-b)(m-c)} \\
&+ \tfrac{(z-c)[(x-a)f(a,s,c)+(k-x)f(k,s,c)]+(m-z)[(x-a)f(a,s,m)+(k-x)f(k,s,m)]}{(k-a)(l-b)(m-c)},
\end{aligned}$$

$$\begin{aligned}
G_3(x,y,u) &= \tfrac{f(x,y,u)}{m-c} + \tfrac{(y-b)f(x,b,u)+(l-y)f(x,l,u)}{(l-b)(m-c)} + \tfrac{(x-a)f(a,y,u)+(k-x)f(k,y,u)}{(k-a)(m-c)} \\
&+ \tfrac{(x-a)[(y-b)f(a,b,u)+(l-y)f(a,l,u)]+(k-x)[(y-b)f(k,b,u)+(l-y)f(k,l,u)]}{(k-a)(l-b)(m-c)}.
\end{aligned}$$

Proof. Put

$$p(x,t) := \begin{cases} t - \frac{a+x}{2}, & t \in [a,x], \\ t - \frac{k+x}{2}, & t \in (x,k], \end{cases}$$

$$q(y,s) := \begin{cases} s - \frac{b+y}{2}, & s \in [b,y], \\ s - \frac{l+y}{2}, & s \in (y,l], \end{cases}$$

and

$$r(z,u) := \begin{cases} u - \frac{c+z}{2}, & u \in [c,z], \\ u - \frac{m+z}{2}, & u \in (z,m]. \end{cases}$$

We have

$$\begin{aligned}
&\int_a^k \int_b^l \int_c^m p(x,t)q(y,s)r(z,u)\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&= \int_a^x \int_b^y \int_c^z (t-\tfrac{a+x}{2})(s-\tfrac{b+y}{2})(u-\tfrac{c+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&+ \int_a^x \int_b^y \int_z^m (t-\tfrac{a+x}{2})(s-\tfrac{b+y}{2})(u-\tfrac{m+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&+ \int_a^x \int_y^l \int_c^z (t-\tfrac{a+x}{2})(s-\tfrac{l+y}{2})(u-\tfrac{c+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&+ \int_a^x \int_y^l \int_z^m (t-\tfrac{a+x}{2})(s-\tfrac{l+y}{2})(u-\tfrac{m+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&+ \int_x^k \int_b^y \int_c^z (t-\tfrac{k+x}{2})(s-\tfrac{b+y}{2})(u-\tfrac{c+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&+ \int_x^k \int_b^y \int_z^m (t-\tfrac{k+x}{2})(s-\tfrac{b+y}{2})(u-\tfrac{m+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&+ \int_x^k \int_y^l \int_c^z (t-\tfrac{k+x}{2})(s-\tfrac{l+y}{2})(u-\tfrac{c+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&+ \int_x^k \int_y^l \int_z^m (t-\tfrac{k+x}{2})(s-\tfrac{l+y}{2})(u-\tfrac{m+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt.
\end{aligned} \tag{2.2}$$

Integrating by parts three times, we can state:

$$\begin{aligned}
&\int_a^x \int_b^y \int_c^z (t-\tfrac{a+x}{2})(s-\tfrac{b+y}{2})(u-\tfrac{c+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&= \tfrac{(x-a)(y-b)(z-c)}{8}[f(x,y,z)+f(x,y,c)+f(x,b,z)+f(x,b,c) \\
&+f(a,y,z)+f(a,y,c)+f(a,b,z)+f(a,b,c)] \\
&-\tfrac{(y-b)(z-c)}{4}\int_a^x [f(t,y,z)+f(t,y,c)+f(t,b,z)+f(t,b,c)]\,dt \\
&-\tfrac{(x-a)(z-c)}{4}\int_b^y [f(x,s,z)+f(x,s,c)+f(a,s,z)+f(a,s,c)]\,ds \\
&-\tfrac{(x-a)(y-b)}{4}\int_c^z [f(x,y,u)+f(x,b,u)+f(a,y,u)+f(a,b,u)]\,du \\
&+\tfrac{x-a}{2}\int_b^y \int_c^z [f(x,s,u)+f(a,s,u)]\,du\,ds \\
&+\tfrac{y-b}{2}\int_a^x \int_c^z [f(t,y,u)+f(t,b,u)]\,du\,dt \\
&+\tfrac{z-c}{2}\int_a^x \int_b^y [f(t,s,z)+f(t,s,c)]\,ds\,dt \\
&-\int_a^x \int_b^y \int_c^z f(t,s,u)\,du\,ds\,dt.
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\int_a^x \int_b^y \int_z^m (t-\tfrac{a+x}{2})(s-\tfrac{b+y}{2})(u-\tfrac{m+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&= \tfrac{(x-a)(y-b)(m-z)}{8}[f(x,y,m)+f(x,y,z)+f(x,b,m)+f(x,b,z) \\
&+f(a,y,m)+f(a,y,z)+f(a,b,m)+f(a,b,z)] \\
&-\tfrac{(y-b)(m-z)}{4}\int_a^x [f(t,y,m)+f(t,y,z)+f(t,b,m)+f(t,b,z)]\,dt \\
&-\tfrac{(x-a)(m-z)}{4}\int_b^y [f(x,s,m)+f(x,s,z)+f(a,s,m)+f(a,s,z)]\,ds \\
&-\tfrac{(x-a)(y-b)}{4}\int_z^m [f(x,y,u)+f(x,b,u)+f(a,y,u)+f(a,b,u)]\,du \\
&+\tfrac{x-a}{2}\int_b^y \int_z^m [f(x,s,u)+f(a,s,u)]\,du\,ds \\
&+\tfrac{y-b}{2}\int_a^x \int_z^m [f(t,y,u)+f(t,b,u)]\,du\,dt \\
&+\tfrac{m-z}{2}\int_a^x \int_b^y [f(t,s,m)+f(t,s,z)]\,ds\,dt \\
&-\int_a^x \int_b^y \int_z^m f(t,s,u)\,du\,ds\,dt.
\end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&\int_a^x \int_y^l \int_c^z (t-\tfrac{a+x}{2})(s-\tfrac{l+y}{2})(u-\tfrac{c+z}{2})\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt\\
&=\tfrac{(x-a)(l-y)(z-c)}{8}[f(x,l,z)+f(x,l,c)+f(x,y,z)+f(x,y,c)\\
&+f(a,l,z)+f(a,l,c)+f(a,y,z)+f(a,y,c)]\\
&-\tfrac{(l-y)(z-c)}{4}\int_a^x[f(t,l,z)+f(t,l,c)+f(t,y,z)+f(t,y,c)]\,dt\\
&-\tfrac{(x-a)(z-c)}{4}\int_y^l[f(x,s,z)+f(x,s,c)+f(a,s,z)+f(a,s,c)]\,ds\\
&-\tfrac{(x-a)(l-y)}{4}\int_c^z[f(x,l,u)+f(x,y,u)+f(a,l,u)+f(a,y,u)]\,du\\
&+\tfrac{x-a}{2}\int_y^l\int_c^z[f(x,s,u)+f(a,s,u)]\,du\,ds\\
&+\tfrac{l-y}{2}\int_a^x\int_c^z[f(t,l,u)+f(t,y,u)]\,du\,dt\\
&+\tfrac{z-c}{2}\int_a^x\int_y^l[f(t,s,z)+f(t,s,c)]\,ds\,dt\\
&-\int_a^x\int_y^l\int_c^z f(t,s,u)\,du\,ds\,dt.
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
&\int_a^x \int_y^l \int_z^m (t-\tfrac{a+x}{2})(s-\tfrac{l+y}{2})(u-\tfrac{m+z}{2})\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt\\
&=\tfrac{(x-a)(l-y)(m-z)}{8}[f(x,l,m)+f(x,l,z)+f(x,y,m)+f(x,y,z)\\
&+f(a,l,m)+f(a,l,z)+f(a,y,m)+f(a,y,z)]\\
&-\tfrac{(l-y)(m-z)}{4}\int_a^x[f(t,l,m)+f(t,l,z)+f(t,y,m)+f(t,y,z)]\,dt\\
&-\tfrac{(x-a)(m-z)}{4}\int_y^l[f(x,s,m)+f(x,s,z)+f(a,s,m)+f(a,s,z)]\,ds\\
&-\tfrac{(x-a)(l-y)}{4}\int_z^m[f(x,l,u)+f(x,y,u)+f(a,l,u)+f(a,y,u)]\,du\\
&+\tfrac{x-a}{2}\int_y^l\int_z^m[f(x,s,u)+f(a,s,u)]\,du\,ds\\
&+\tfrac{l-y}{2}\int_a^x\int_z^m[f(t,l,u)+f(t,y,u)]\,du\,dt\\
&+\tfrac{m-z}{2}\int_a^x\int_y^l[f(t,s,m)+f(t,s,z)]\,ds\,dt\\
&-\int_a^x\int_y^l\int_z^m f(t,s,u)\,du\,ds\,dt.
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
&\int_x^k \int_b^y \int_c^z (t-\tfrac{k+x}{2})(s-\tfrac{b+y}{2})(u-\tfrac{c+z}{2})\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt\\
&=\tfrac{(k-x)(y-b)(z-c)}{8}[f(k,y,z)+f(k,y,c)+f(k,b,z)+f(k,b,c)\\
&+f(x,y,z)+f(x,y,c)+f(x,b,z)+f(x,b,c)]\\
&-\tfrac{(y-b)(z-c)}{4}\int_x^k[f(t,y,z)+f(t,y,c)+f(t,b,z)+f(t,b,c)]\,dt\\
&-\tfrac{(k-x)(z-c)}{4}\int_b^y[f(k,s,z)+f(k,s,c)+f(x,s,z)+f(x,s,c)]\,ds\\
&-\tfrac{(k-x)(y-b)}{4}\int_c^z[f(k,y,u)+f(k,b,u)+f(x,y,u)+f(x,b,u)]\,du\\
&+\tfrac{k-x}{2}\int_b^y\int_c^z[f(k,s,u)+f(x,s,u)]\,du\,ds\\
&+\tfrac{y-b}{2}\int_x^k\int_c^z[f(t,y,u)+f(t,b,u)]\,du\,dt\\
&+\tfrac{z-c}{2}\int_x^k\int_b^y[f(t,s,z)+f(t,s,c)]\,ds\,dt\\
&-\int_x^k\int_b^y\int_c^z f(t,s,u)\,du\,ds\,dt.
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&\int_x^k \int_b^y \int_z^m (t-\tfrac{k+x}{2})(s-\tfrac{b+y}{2})(u-\tfrac{m+z}{2})\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt\\
&= \tfrac{(k-x)(y-b)(m-z)}{8}[f(k,y,m)+f(k,y,z)+f(k,b,m)+f(k,b,z)\\
&+f(x,y,m)+f(x,y,z)+f(x,b,m)+f(x,b,z)]\\
&-\tfrac{(y-b)(m-z)}{4}\int_x^k[f(t,y,m)+f(t,y,z)+f(t,b,m)+f(t,b,z)]\,dt\\
&-\tfrac{(k-x)(m-z)}{4}\int_b^y[f(k,s,m)+f(k,s,z)+f(x,s,m)+f(x,s,z)]\,ds\\
&-\tfrac{(k-x)(y-b)}{4}\int_z^m[f(k,y,u)+f(k,b,u)+f(x,y,u)+f(x,b,u)]\,du\\
&+\tfrac{k-x}{2}\int_b^y\int_z^m[f(k,s,u)+f(x,s,u)]\,du\,ds\\
&+\tfrac{y-b}{2}\int_x^k\int_z^m[f(t,y,u)+f(t,b,u)]\,du\,dt\\
&+\tfrac{m-z}{2}\int_x^k\int_b^y[f(t,s,m)+f(t,s,z)]\,ds\,dt\\
&-\int_x^k\int_b^y\int_z^m f(t,s,u)\,du\,ds\,dt.
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&\int_x^k \int_y^l \int_c^z (t-\tfrac{k+x}{2})(s-\tfrac{l+y}{2})(u-\tfrac{c+z}{2})\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt\\
&= \tfrac{(k-x)(l-y)(z-c)}{8}[f(k,l,z)+f(k,l,c)+f(k,y,z)+f(k,y,c)\\
&+f(x,l,z)+f(x,l,c)+f(x,y,z)+f(x,y,c)]\\
&-\tfrac{(l-y)(z-c)}{4}\int_x^k[f(t,l,z)+f(t,l,c)+f(t,y,z)+f(t,y,c)]\,dt\\
&-\tfrac{(k-x)(z-c)}{4}\int_b^y[f(k,s,z)+f(k,s,c)+f(x,s,z)+f(x,s,c)]\,ds\\
&-\tfrac{(k-x)(l-y)}{4}\int_c^z[f(k,l,u)+f(k,y,u)+f(x,l,u)+f(x,y,u)]\,du\\
&+\tfrac{k-x}{2}\int_y^l\int_c^z[f(k,s,u)+f(x,s,u)]\,du\,ds\\
&+\tfrac{l-y}{2}\int_x^k\int_c^z[f(t,l,u)+f(t,y,u)]\,du\,dt\\
&+\tfrac{z-c}{2}\int_x^k\int_y^l[f(t,s,z)+f(t,s,c)]\,ds\,dt\\
&-\int_x^k\int_y^l\int_c^z f(t,s,u)\,du\,ds\,dt.
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&\int_x^k \int_y^l \int_z^m (t-\tfrac{k+x}{2})(s-\tfrac{l+y}{2})(u-\tfrac{m+z}{2})\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt\\
&= \tfrac{(k-x)(l-y)(m-z)}{8}[f(k,l,m)+f(k,l,z)+f(k,y,m)+f(k,y,z)\\
&+f(x,l,m)+f(x,l,z)+f(x,y,m)+f(x,y,z)]\\
&-\tfrac{(l-y)(m-z)}{4}\int_x^k[f(t,l,m)+f(t,l,z)+f(t,y,m)+f(t,y,z)]\,dt\\
&-\tfrac{(k-x)(m-z)}{4}\int_y^l[f(k,s,m)+f(k,s,z)+f(x,s,m)+f(x,s,z)]\,ds\\
&-\tfrac{(k-x)(l-y)}{4}\int_z^m[f(k,l,u)+f(k,y,u)+f(x,l,u)+f(x,y,u)]\,du\\
&+\tfrac{k-x}{2}\int_y^l\int_z^m[f(k,s,u)+f(x,s,u)]\,du\,ds\\
&+\tfrac{l-y}{2}\int_x^k\int_z^m[f(t,l,u)+f(t,y,u)]\,du\,dt\\
&+\tfrac{m-z}{2}\int_x^k\int_y^l[f(t,s,m)+f(t,s,z)]\,ds\,dt\\
&-\int_x^k\int_y^l\int_z^m f(t,s,u)\,du\,ds\,dt.
\end{aligned}
\tag{2.10}
$$

From (6)-(14), we can easily deduce that

$$
\begin{aligned}
&\int_a^k \int_b^l \int_c^m p(x,t)q(y,s)r(z,u)\tfrac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&= \tfrac{1}{8}\{(k-a)(l-b)(m-c)f(x,y,z) + (k-a)(l-b)[(z-c)f(x,y,c)+(m-z)f(x,y,m)] \\
&+(l-b)(m-c)[(x-a)f(a,y,z)+(k-x)f(k,y,z)] \\
&+(k-a)(m-c)[(y-b)f(x,b,z)+(l-y)f(x,l,z)] \\
&+(m-c)[(x-a)(y-b)f(a,b,z)+(k-x)(y-b)f(k,b,z) \\
&+(x-a)(l-y)f(a,l,z)+(k-x)(l-y)f(k,l,z)] \\
&+(k-a)[(y-b)(z-c)f(x,b,c)+(l-y)(z-c)f(x,l,c) \\
&+(y-b)(m-z)f(x,b,m)+(l-y)(m-z)f(x,l,m)] \\
&+(l-b)[(x-a)(z-c)f(a,y,c)+(k-x)(z-c)f(k,y,c) \\
&+(x-a)(m-z)f(a,y,m)+(k-x)(m-z)f(k,y,m)] \\
&+(y-b)(z-c)[(x-a)f(a,b,c)+(k-x)f(k,b,c)] \\
&+(l-y)(z-c)[(x-a)f(a,l,c)+(k-x)f(k,l,c)] \\
&+(y-b)(m-z)[(x-a)f(a,b,m)+(k-x)f(k,b,m)] \\
&+(l-y)(m-z)[(x-a)f(a,l,m)+(k-x)f(k,l,m)]\} \\
&-\tfrac{1}{4}\int_a^k\{(l-b)(m-c)f(t,y,z)+(m-c)[(y-b)f(t,b,z)+(l-y)f(t,l,z)] \\
&+(l-b)[(z-c)f(t,y,c)+(m-z)f(t,y,m)]+(z-c)[(y-b)f(t,y,c)+(l-y)f(t,l,c)] \\
&+(m-z)[(y-b)f(t,b,m)+(l-y)f(t,l,m)]\}\,dt \\
&-\tfrac{1}{4}\int_b^l\{(k-a)(m-c)f(x,s,z)+(m-c)[(x-a)f(a,s,z)+(k-x)f(k,s,z)] \\
&+(k-a)[(z-c)f(x,s,c)+(m-z)f(x,s,m)]+(z-c)[(x-a)f(a,s,c)+(k-x)f(k,s,c)] \\
&+(m-z)[(x-a)f(a,s,m)+(k-x)f(k,s,m)]\}\,ds \\
&-\tfrac{1}{4}\int_c^m\{(k-a)(l-b)f(x,y,u)+(k-a)[(y-b)f(x,b,u)+(l-y)f(x,l,u)] \\
&+(l-b)[(x-a)f(a,y,u)+(k-x)f(k,y,u)]+(y-b)[(x-a)f(a,b,u)+(k-x)f(k,b,u)] \\
&+(l-y)[(x-a)f(a,l,u)+(k-x)f(k,l,u)]\}\,du \\
&+\tfrac{1}{2}\int_a^k\int_b^l[(z-c)f(t,s,c)+(m-z)f(t,s,m)+(m-c)f(t,s,z)]\,ds\,dt \\
&+\tfrac{1}{2}\int_b^l\int_c^m[(x-a)f(a,s,u)+(k-x)f(k,s,u)+(k-a)f(x,s,u)]\,du\,ds \\
&+\tfrac{1}{2}\int_a^k\int_c^m[(y-b)f(t,b,u)+(l-y)f(t,l,u)+(l-b)f(t,y,u)]\,du\,dt\} \\
&-\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt,
\end{aligned}
$$

and it follows that

(2.11)
$$
\begin{aligned}
&\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k \int_b^l \int_c^m p(x,t)q(y,s)r(z,u)\tfrac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&= \tfrac{1}{8}f(x,y,z)+\tfrac{1}{8}H(x,y,z)-\tfrac{1}{4}\int_a^k G_1(t,y,z)\,dt-\tfrac{1}{4}\int_b^l G_2(x,s,z)\,ds-\tfrac{1}{4}\int_c^m G_3(x,y,u)\,du \\
&+\tfrac{1}{2(k-a)(l-b)(m-c)}\{\int_a^k\int_b^l[(z-c)f(t,s,c)+(m-z)f(t,s,m)+(m-c)f(t,s,z)]\,ds\,dt \\
&+\int_b^l\int_c^m[(x-a)f(a,s,u)+(k-x)f(k,s,u)+(k-a)f(x,s,u)]\,du\,ds \\
&+\int_a^k\int_c^m[(y-b)f(t,b,u)+(l-y)f(t,l,u)+(l-b)f(t,y,u)]\,du\,dt\} \\
&-\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt.
\end{aligned}
$$

We also have

$$
\int_a^k \int_b^l \int_c^m p(x,t)q(y,s)r(z,u)\,du\,ds\,dt = 0. \tag{2.12}
$$

Let $M=\frac{\Gamma+\gamma}{2}$. From (16), it follows that

$$\begin{aligned}(2.13)\qquad &\int_a^k\int_b^l\int_c^m p(x,t)q(y,s)r(z,u)[\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}-M]\,du\,ds\,dt\\ &=\int_a^k\int_b^l\int_c^m p(x,t)q(y,s)r(z,u)\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt.\end{aligned}$$

On the other hand, we have

(2.14)

$$\begin{aligned}&|\int_a^k\int_b^l\int_c^m p(x,t)q(y,s)r(z,u)[\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}-M]\,du\,ds\,dt|\\ \le{}& \max_{(t,s,u)\in[a,k]\times\in[b,l]\times[c,m]}|\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}-M|\int_a^k\int_b^l\int_c^m |p(x,t)q(y,s)r(z,u)|\,du\,ds\,dt.\end{aligned}$$

Moreover,

$$(2.15)\qquad \max_{(t,s,u)\in[a,k]\times\in[b,l]\times[c,m]}|\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}-M|\le\frac{\Gamma-\gamma}{2}$$

and

(2.16)

$$\int_a^k\int_b^l\int_c^m |p(x,t)q(y,s)r(z,u)|\,du\,ds\,dt=\frac{[(x-a)^2+(k-x)^2][(y-b)^2+(l-y)^2][(z-c)^2+(m-z)^2]}{64}.$$

From (18)-(20), we get

$$\begin{aligned}(2.17)\qquad &|\int_a^k\int_b^l\int_c^m p(x,t)q(y,s)r(z,u)[\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}-M]\,du\,ds\,dt|\\ &\le\tfrac{[(x-a)^2+(k-x)^2][(y-b)^2+(l-y)^2][(z-c)^2+(m-z)^2]}{128}(\Gamma-\gamma).\end{aligned}$$

Finally, from (15), (17) and (21), we see that the inequality (5) holds.

The proof of Theorem 4 is complete.

Remark 2.1. If we take any one of the eight cases $x=a,y=b,z=c$; $x=a,y=b,z=m$; $x=a,y=l,z=c$; $x=a,y=l,z=m$; $x=k,y=b,z=c$; $x=k,y=b,z=m$; $x=k,y=l,z=c$ and $x=k,y=l,z=m$ in (5), then we get the following inequality for triple integrals.

$$\begin{aligned}&|\tfrac{f(a,b,c)+f(a,b,m)+f(a,l,c)+f(a,l,m)+f(k,b,c)+f(k,b,m)+f(k,l,c)+f(k,l,m)}{8}\\ &-\tfrac{1}{4(k-a)}\int_a^k[f(t,b,c)+f(t,l,c)+f(t,b,m)+f(t,l,m)]\,dt\\ &-\tfrac{1}{4(l-b)}\int_b^l[f(a,s,c)+f(k,s,c)+f(a,s,m)+f(k,s,m)]\,ds\\ &-\tfrac{1}{4(m-c)}\int_c^m[f(a,b,u)+f(a,l,u)+f(k,b,u)+f(k,l,u)]\,du\\ &+\tfrac{1}{2(k-a)(l-b)}\int_a^k\int_b^l[f(t,s,m)+f(t,s,c)]\,ds\,dt\\ &+\tfrac{1}{2(l-b)(m-c)}\int_b^l\int_c^m[f(k,s,u)+f(a,s,u)]\,du\,ds\\ &+\tfrac{1}{2(k-a)(m-c)}\int_a^k\int_c^m[f(t,l,u)+f(t,b,u)]\,du\,dt\\ &-\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt|\\ &\le\tfrac{(k-a)(l-b)(m-c)}{128}(\Gamma-\gamma).\end{aligned}$$

Theorem 2.2. *Let $f:[a,k]\times[b,l]\times[c,m]\to\mathbf{R}$ be an absolutely continuous function such that the partial derivative of order 3 exists and continuous for all $(t,s,u)\in[a,k]\times[b,l]\times[c,m]$. Then we have*

(2.18)

$$
\begin{aligned}
&|\tfrac{1}{8}f(x,y,z)+\tfrac{1}{8}H(x,y,z)-\tfrac{1}{4}\int_a^k G_1(t,y,z)\,dt-\tfrac{1}{4}\int_b^l G_2(x,s,z)\,ds-\tfrac{1}{4}\int_c^m G_3(x,y,u)\,du\\
&+\tfrac{1}{2(k-a)(l-b)(m-c)}\{\int_a^k\int_b^l[(z-c)f(t,s,c)+(m-z)f(t,s,m)+(m-c)f(t,s,z)]\,ds\,dt\\
&+\int_b^l\int_c^m[(x-a)f(a,s,u)+(k-x)f(k,s,u)+(k-a)f(x,s,u)]\,du\,ds\\
&+\int_a^k\int_c^m[(y-b)f(t,b,u)+(l-y)f(t,l,u)+(l-b)f(t,y,u)]\,du\,dt\}\\
&-\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt|\\
&\le \tfrac{[\frac{k-a}{2}+|x-\frac{a+k}{2}|][\frac{l-b}{2}+|y-\frac{b+l}{2}|][\frac{m-c}{2}+|x-\frac{c+m}{2}|]}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m|\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}|\,du\,ds\,dt
\end{aligned}
$$

for all $(x,y,z)\in[a,b]\times[b,l]\times[c,m]$, *where* $H(x,y,z)$, $G_1(t,y,z)$, $G_2(x,s,z)$ *and* $G_3(x,y,u)$ *are as defined in Theorem 4.*

Proof. From (15) we get

$$
\begin{aligned}
&|\tfrac{1}{8}f(x,y,z)+\tfrac{1}{8}H(x,y,z)-\tfrac{1}{4}\int_a^k G_1(t,y,z)\,dt-\tfrac{1}{4}\int_b^l G_2(x,s,z)\,ds-\tfrac{1}{4}\int_c^m G_3(x,y,u)\,du\\
&+\tfrac{1}{2(k-a)(l-b)(m-c)}\{\int_a^k\int_b^l[(z-c)f(t,s,c)+(m-z)f(t,s,m)+(m-c)f(t,s,z)]\,ds\,dt\\
&+\int_b^l\int_c^m[(x-a)f(a,s,u)+(k-x)f(k,s,u)+(k-a)f(x,s,u)]\,du\,ds\\
&+\int_a^k\int_c^m[(y-b)f(t,b,u)+(l-y)f(t,l,u)+(l-b)f(t,y,u)]\,du\,dt\}\\
&-\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt|\\
&\le \tfrac{\max_{(t,s,u)\in[a,k]\times\in[b,l]\times[c,m]}|p(x,t)q(y,s)r(z,u)|}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m|\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}|\,du\,ds\,dt
\end{aligned}
$$

and observe that

(2.19)

$$
\max_{(t,s,u)\in[a,k]\times\in[b,l]\times[c,m]}|p(x,t)q(y,s)r(z,u)|=[\frac{k-a}{2}+|x-\frac{a+k}{2}|][\frac{l-b}{2}+|y-\frac{b+l}{2}|][\frac{m-c}{2}+|x-\frac{c+m}{2}|],
$$

we can easily obtain the inequality (22).

Remark 2.2. If we take any one of the eight cases $x=a,y=b,z=c$; $x=a,y=b,z=m$; $x=a,y=l,z=c$; $x=a,y=l,z=m$; $x=k,y=b,z=c$; $x=k,y=b,z=m$; $x=k,y=l,z=c$ and $x=k,y=l,z=m$ in (22), then we get the following inequality for triple integrals.

$$
\begin{aligned}
&|\tfrac{f(a,b,c)+f(a,b,m)+f(a,l,c)+f(a,l,m)+f(k,b,c)+f(k,b,m)+f(k,l,c)+f(k,l,m)}{8}\\
&-\tfrac{1}{4(k-a)}\int_a^k[f(t,b,c)+f(t,l,c)+f(t,b,m)+f(t,l,m)]\,dt\\
&-\tfrac{1}{4(l-b)}\int_b^l[f(a,s,c)+f(k,s,c)+f(a,s,m)+f(k,s,m)]\,ds\\
&-\tfrac{1}{4(m-c)}\int_c^m[f(a,b,u)+f(a,l,u)+f(k,b,u)+f(k,l,u)]\,du\\
&+\tfrac{1}{2(k-a)(l-b)}\int_a^k\int_b^l[f(t,s,m)+f(t,s,c)]\,ds\,dt\\
&+\tfrac{1}{2(l-b)(m-c)}\int_b^l\int_c^m[f(k,s,u)+f(a,s,u)]\,du\,ds\\
&+\tfrac{1}{2(k-a)(m-c)}\int_a^k\int_c^m[f(t,l,u)+f(t,b,u)]\,du\,dt\\
&-\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt|\\
&\le\tfrac{1}{8}\int_a^k\int_b^l\int_c^m|\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}|\,du\,ds\,dt.
\end{aligned}
\tag{2.20}
$$

It is clear that inequality (24) is just the same as inequality (4), and thus we may regard that Theorem 5 is a generalization of Theorem 3.

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ON SOME SINGULAR VALUE INEQUALITIES FOR MATRICES

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Abstract. Some singular value inequalities for matrices are given. Among other inequalities it is proved that if f and g be nonnegative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$, for all $t \in [0,\infty)$, then

$$s_j(A_1^*XB_1 + A_2^*XB_2)$$
$$\leq s_j((A_1^*f^2(\mid X^* \mid)A_1 + A_2^*f^2(\mid X^* \mid)A_2) \oplus (B_1^*g^2(\mid X \mid)B_1 + B_2^*g^2(\mid X \mid)B_2)),$$

for $j = 1,2,...,n$, where A_1, A_2, B_1, B_2, X are square matrices. Our results in this article generalize some existing singular value inequalities of matrices.

1. Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. Let $\|\cdot\|$ stand for any unitarily invariant norm on M_n, i.e., a norm with the property that $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. Any matrix $A \in M_n$ is called positive semidefinite, denoted as $A \geq 0$ if for all $x \in C^n$, $x^*Ax \geq 0$ and it is called positive definite if for all nonzero $x \in C^n$, $x^*Ax > 0$ and it is denoted as $A > 0$. The singular values of matrix A are the eigenvalues of positive semidefinite matrix $\mid A \mid = (AA^*)^{\frac{1}{2}}$, enumerated as $s_1(A) \geq s_2(A) \geq ... \geq s_n(A)$ and repeated according to multiplicity. The direct sum $A \oplus B$ represent the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

The well-known classical arithmetic-geometric mean inequality for $a, b \geq 0$ defined as

$$a^{\frac{1}{2}}b^{\frac{1}{2}} \leq \frac{a+b}{2}. \tag{1.1}$$

Arithmetic-geometric mean inequality is important in matrix theory, functional analysis, electrical networks, etc. For $A, B, X \in M_n$, such that $A, B \geq 0$, R. Bhatia and F. Kittaneh formulated some matrix versions of this inequality in [3,4] one of

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which is the following

$$\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leq \frac{1}{2}\|AX+XB\|. \tag{1.2}$$

From (1.2), for $X=I$ we have the following inequality for positive semidefinite matrices.

$$\|A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{1}{2}\|A+B\|, \tag{1.3}$$

R. Bhatia and F. Kittaneh also have proved in [5] that if $A,B\in M_n$ such that $A,B\geq 0$, then

$$\|A^{\frac{3}{2}}B^{\frac{1}{2}}+A^{\frac{1}{2}}B^{\frac{3}{2}}\| \leq \frac{1}{2}\|(A+B)^2\|. \tag{1.4}$$

From (1.3), (1.4) and also by triangle inequality, we obtain the following inequality

$$\|A^{\frac{3}{2}}B^{\frac{1}{2}}+A^{\frac{1}{2}}B^{\frac{3}{2}}+A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{1}{2}\|(A+B)^2\|+\frac{1}{2}\|A+B\|. \tag{1.5}$$

In [2] L. Zou and Y. Jiang proved that for positive semidefinite matrices $A,B\in M_n$ and $1\leq j\leq n$, the following inequality also holds

$$2s_j(A^{\frac{3}{2}}B^{\frac{1}{2}}+A^{\frac{1}{2}}B^{\frac{3}{2}}+A^{\frac{1}{2}}B^{\frac{1}{2}}) \leq s_j((A+B)^2+(A+B)), \tag{1.6}$$

and consequently,

$$\|A^{\frac{3}{2}}B^{\frac{1}{2}}+A^{\frac{1}{2}}B^{\frac{3}{2}}+A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{1}{2}\|(A+B)^2+(A+B)\|. \tag{1.7}$$

The inequality (1.7) is an improvement of the inequality (1.5).

One another interesting inequality for sum and direct sum of matrices proved by R. Bhatia and F. Kittaneh [6] is

$$s_j(A^*B+B^*A) \leq s_j((A^*A+B^*B)\oplus(A^*A+B^*B)), \tag{1.8}$$

where $A,B\in M_n$ and $1\leq j\leq n$.

In Section 2, we give generalized form of the inequality (1.6) and also, we obtain the X-version of the inequality (1.8).

2. Singular values inequalities for matrices

In this section, we generalize the inequalities (1.6) and also, we obtain X-version of the inequality (1.8). Our results based on Several lemmas. First two lemmas have been given by F. Kittaneh in [1] and Lemma 2.3 can be found in [8, Theorem 1].

Lemma 2.1. Let $T\in M_n$, then the block matrix $\begin{pmatrix} |T| & T^* \\ T & |T^*| \end{pmatrix}\geq 0$.

Lemma 2.2. Let $A,B,C\in M_n$, such that A and B are positive semidefinite, $BC=CA$ and let f and g be nonnegative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$, for all $t\in[0,\infty)$. If the block matrix $\begin{pmatrix} A & C^* \\ C & B \end{pmatrix}\geq 0$, then so $\begin{pmatrix} f^2(A) & C^* \\ C & g^2(B) \end{pmatrix}\geq 0$.

Lemma 2.3. Let $A,B,C\in M_n$ such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\geq 0$, then

$$2s_j(B)\leq s_j\begin{pmatrix} A & B \\ B^* & C \end{pmatrix},\ j=1,2,...,n. \tag{2.1}$$

The following Lemma was proved in [7].

Lemma 2.4. Let $A, B, C \in M_n$, such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$, then

$$s_j(B) \leq s_j(A \oplus C),\ j = 1, 2, ..., n. \tag{2.2}$$

To give the general form of (1.6), first we prove the following result.

Theorem 2.5. Let $A, B \in M_n$ be any two matrices and r be a positive integer, then

$2s_j(A(\mid A\mid^2 + \mid B\mid^2)^{r-1}B^* + AB^*) \leq s_j((\mid A\mid^2 + \mid B\mid^2)^r + (\mid A\mid^2 + \mid B\mid^2))$,

for $j = 1, 2, ..., n$.

Proof. Let $X = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$. Then,

$$X^*X = \begin{pmatrix} A^*A + B^*B & 0 \\ 0 & 0 \end{pmatrix},\ XX^* = \begin{pmatrix} AA^* & AB^* \\ BA^* & BB^* \end{pmatrix}.$$

So, we have

$$(X^*X)^r = \begin{pmatrix} (A^*A + B^*B)^r & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned} (XX^*)^r &= X(X^*X)^{(r-1)}X^* \\ &= \begin{pmatrix} A(A^*A + B^*B)^{(r-1)}A^* & A(A^*A + B^*B)^{(r-1)}B^* \\ B(A^*A + B^*B)^{(r-1)}A^* & B(A^*A + B^*B)^{(r-1)}B^* \end{pmatrix}. \end{aligned}$$

Therefore, we obtain

$$(X^*X)^r + X^*X = \begin{pmatrix} (A^*A + B^*B)^r + A^*A + B^*B & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned} & (XX^*)^r + XX^* \\ = & \begin{pmatrix} A(A^*A + B^*B)^{(r-1)}A^* + AA^* & A(A^*A + B^*B)^{(r-1)}B^* + AB^* \\ B(A^*A + B^*B)^{(r-1)}A^* + BA^* & B(A^*A + B^*B)^{(r-1)}B^* + BB^* \end{pmatrix}. \end{aligned}$$

So, by Lemma 2.3, from the positive semidefinite block matrix $(XX^*)^r + XX^*$, we have

$$\begin{aligned} 2s_j(A(A^*A + B^*B)^{(r-1)}B^* + AB^*) &\leq s_j((XX^*)^r + XX^*) \\ &= s_j((X^*X)^r + X^*X) \\ &= s_j((A^*A + B^*B)^r + (A^*A + B^*B)), \end{aligned}$$

for $j = 1, 2, ..., n$.

The proof is completed. $\square$

When $A, B \in M_n$ be positive semidefinite in Theorem 2.5 and A is replaced by $A^{\frac{1}{2}}$ and B is replaced by $B^{\frac{1}{2}}$, then we obtain the following promised generalization of the inequality (1.6).

Corollary 2.6. Let $A, B \in M_n$ be positive semidefinite and r be a positive integer. Then,

$$2s_j(A^{\frac{1}{2}}(A+B)^{(r-1)}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}}) \leq s_j((A+B)^r + (A+B)),$$

for $j = 1, 2, ..., n$.

Remark 2.7. When we take $r = 2$ in Corollary 2.6 , then we obtain the inequality (1.6).

To give the X-version of the inequality (1.8), first we obtain the following result.

Theorem 2.8. Let $A_1, A_2, B_1, B_2, X \in M_n$. If f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$, for all $t \in [0, \infty)$, then

$$\begin{aligned} & s_j(A_1^*XB_1 + A_2^*XB_2) \\ \leq \; & s_j((A_1^*f^2(| X^* |)A_1 + A_2^*f^2(| X^* |)A_2) \oplus (B_1^*g^2(| X |)B_1 + B_2^*g^2(| X |)B_2)), \end{aligned}$$

for $j = 1, 2, ..., n$.

Proof. Let $T_1 = \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}$, $T_2 = \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}$.

Since the block matrix $\begin{pmatrix} | X^* | & X \\ X^* & | X | \end{pmatrix}$ is positive semidefinite (by Lemma 2.1) and the block matrix $Y = \begin{pmatrix} f^2(| X^* |) & X \\ X^* & g^2(| X |) \end{pmatrix}$ is positive semidefinite (by Lemma 2.2), so, $T_1^*YT_1 = \begin{pmatrix} A_1^*f^2(| X^* |)A_1 & A_1^*XB_1 \\ B_1^*X^*A_1 & B_1^*g^2(| X |)B_1 \end{pmatrix} \geq 0$ and also, $T_2^*YT_2 = \begin{pmatrix} A_2^*f^2(| X^* |)A_2 & A_2^*XB_2 \\ B_2^*X^*A_2 & B_2^*g^2(| X |)B_2 \end{pmatrix} \geq 0$. That is, we have

$$\begin{aligned} & T_1^*YT_1 + T_2^*YT_2 \\ = \; & \begin{pmatrix} A_1^*f^2(| X^* |)A_1 + A_2^*f^2(| X^* |)A_2 & A_1^*XB_1 + A_2^*XB_2 \\ B_1^*X^*A_1 + B_2^*X^*A_2 & B_1^*g^2(| X |)B_1 + B_2^*g^2(| X |)B_2 \end{pmatrix} \geq 0 \end{aligned}$$

So, our desired result now follows by invoking inequality (2.2).
The proof is completed. □

Following is our desired X-version of the inequality (1.8).

Corollary 2.9. Let $A, B, X \in M_n$, then

$$\begin{aligned} & s_j(A^*XB + B^*XA) \\ \leq \; & s_j((A^* | X^* | A + B^* | X^* | B) \oplus (A^* | X | A + B^* | X | B)), \end{aligned}$$

for $j = 1, 2, ..., n$.

Proof. By taking $f(t) = g(t) = t^{\frac{1}{2}}$, $A_1 = B_2 = A$ and $A_2 = B_1 = B$ in Theorem 2.8, we get the desired result.
The proof is completed. □

One another important case follows from Corollary 2.9 for normal matrices.

Corollary 2.10. Let $A, B, X \in M_n$ such that X is normal matrix,then

$$\begin{aligned} & s_j(A^*XB + B^*XA) \\ \leq \quad & s_j((A^* \mid X \mid A + B^* \mid X \mid B) \oplus (A^* \mid X \mid A + B^* \mid X \mid B)), \end{aligned}$$

for $j = 1, 2, ..., n$.

In particular, when X is positive semidefinite matrix , then

$$\begin{aligned} & s_j(A^*XB + B^*XA) \\ \leq \quad & s_j((A^*XA + B^*XB) \oplus (A^*XA + B^*XB)), \end{aligned}$$

for $j = 1, 2, ..., n$.

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SIMPSON'S TYPE INEQUALITIES FOR $m-$ AND $(\alpha, m)-$GEOMETRICALLY CONVEX FUNCTIONS

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Abstract. In this paper, we establish Simpson's type inequalities for $m-$ and $(\alpha, m)-$geometrically convex functions using the lemmas.

1. Introduction

The following inequality is well-known in the literature as Simpson's inequality:

Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $[a, b]$ and $\left\|f^{(4)}\right\|_{\infty} = \sup_{x\in[a,b]} \left|f^{(4)}(x)\right| < \infty$. Then the folllowing inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

For the recent results based on the above definition see the papers [1], [4] , [7], [14], [18] and [20].

In [6], G.Toader defined the concept of m-convexity as the following;

Definition 1.1. The function $f : [0, b] \to \mathbb{R}$ is said to be $m-$convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y).$$

Denote by $K_m(b)$ the set of the $m-$convex functions on $[0, b]$ for which $f(0) \leq 0$.

In [19], Miheşan gave definition of $(\alpha, m)-$convexity as following;

Definition 1.2. The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $(\alpha, m)-$convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1 - t)y) \leq t^{\alpha}f(x) + m(1 - t^{\alpha})f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

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Denote by $K_m^\alpha(b)$ the class of all $(\alpha, m)-$convex functions on $[0, b]$ for which $f(0) \leq 0$. If we choose $(\alpha, m) = (1, m)$, it can be easily seen that $(\alpha, m)-$convexity reduces to $m-$convexity and for $(\alpha, m) = (1, 1)$, we have ordinary convex functions on $[0, b]$. For the recent results based on the $m-$ and $(\alpha, m)-$ convexity see the papers [2], [3], [5], [8]-[13] and [15]-[17].

In [2], Xi *et al.* introduced $m-$ and $(\alpha, m)-$geometrically convex functions and give a lemma as following, respectively;

Definition 1.3. Let $f(x)$ be a positive function on $[0, b]$ and $m \in (0, 1]$. If

$$f\left(x^t y^{m(1-t)}\right) \leq [f(x)]^t [f(y)]^{m(1-t)}$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that the function $f(x)$ is $m-$geometrically convex on $[0, b]$.

It is clear that when $m = 1$, $m-$geometrically convex functions become geometrically convex functions.

Definition 1.4. Let $f(x)$ be a positive function on $[0, b]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$. If

$$f\left(x^t y^{m(1-t)}\right) \leq [f(x)]^{t^\alpha} [f(y)]^{m(1-t^\alpha)}$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that the function $f(x)$ is $(\alpha, m)-$geometrically convex on $[0, b]$.

Lemma 1.1. *For $x, y \in [0, \infty)$ and $m, t \in (0, 1]$, if $x < y$ and $y \geq 1$, then*

$$x^t y^{m(1-t)} \leq tx + (1-t) y.$$

In this paper, we recite two lemmas in the literature, then we obtaine Simpson's type inequalities using the lemmas for $m-$ and $(\alpha, m)-$geometrically convex functions.

2. Results

Lemma 2.1. [[1], *pp.3*] *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I° where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\begin{aligned} &\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx \\ = \quad &(b-a)\int_0^1 p(t) f'(tb + (1-t)a)\,dt, \end{aligned}$$

where

$$p(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right), \\ t - \frac{5}{6}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Theorem 2.1. *Let $f : I \subset [0, \infty) \to (0, \infty)$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)|$ is decreasing and $(\alpha, m)-$geometrically convex on $[\min\{1, a\}, b]$, for $b \geq 1$, and for $(\alpha, m) \in (0, 1]^2$, then the following inequality holds;*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq (b-a)|f'(b)|^m M_1(\alpha, m)$$

where

(2.1)
$$M_1(\alpha,m)=\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha}dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha}dt.$$

Proof. From Lemma 2, Lemma 1 and since f is decreasing, then

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\ &(b-a)\int_0^1|p(t)|\,|f'(tb+(1-t)a)|\,dt\\ \leq\ &(b-a)\int_0^1|p(t)|\,\left|f'\left(a^{1-t}b^{mt}\right)\right|dt.\end{aligned}$$

Using the $(\alpha,m)-$geometrically convexity of $|f'(x)|$, we have,

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\ &(b-a)\int_0^1|p(t)|\,|f'(a)|^{(1-t)^\alpha}\,|f'(b)|^{m(1-(1-t)^\alpha)}\,dt\\ =\ &(b-a)\,|f'(b)|^m\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha}dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha}dt\right\}.\end{aligned}$$

So, the proof is completed. □

Corollary 2.1. *Let $f:I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|$ is decreasing and $m-$geometrically convex on $[\min\{1,a\},b]$ for $b\geq 1$, and for $m\in(0,1]$, then the following inequality holds;*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq(b-a)\,|f'(b)|^m\,M_1(1,m)$$

where $M_1(1,m)$ is the term in (2.1).

Theorem 2.2. *Let $f:I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|^{\frac{p}{p-1}}$ is decreasing and $(\alpha,m)-$geometrically convex on $[\min\{1,a\},b]$, for $b\geq 1$, and for $(\alpha,m)\in(0,1]^2$, $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds;*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\ &(b-a)\,|f'(b)|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}M_2(\alpha,m,p)\end{aligned}$$

where
(2.2)

$$M_2(\alpha,m,p)=\left(\int_0^{\frac{1}{2}}\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}+\left(\int_{\frac{1}{2}}^{1}\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}.$$

Proof. By using Lemma 2 and Hölder integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
\leq\ & (b-a)\int_0^1|p(t)|\,|f'(tb+(1-t)a)|\,dt\\
=\ & (b-a)\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right||f'(tb+(1-t)a)|\,dt+(b-a)\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right||f'(tb+(1-t)a)|\,dt\\
\leq\ & (b-a)\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f'(tb+(1-t)a)|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}\\
&+(b-a)\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|f'(tb+(1-t)a)|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Since $|f'(x)|$ is decreasing by using Lemma 1 and $(\alpha,m)-$geometrically convex, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
\leq\ & (b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{2}}\left|f'\left(a^{1-t}b^{mt}\right)\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}+\left(\int_{\frac{1}{2}}^1\left|f'\left(a^{1-t}b^{mt}\right)\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}\right\}\\
\leq\ & (b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}|f'(b)|^m\\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}+\left(\int_{\frac{1}{2}}^1\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}\right\}.
\end{aligned}$$

So, the desired result is obtained. □

Corollary 2.2. *Let $f:I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|^{\frac{p}{p-1}}$ is decreasing and $m-$geometrically convex on $[\min\{1,a\},b]$, for $b\geq1$, and for $m\in(0,1]$, $p>1$ with*

$\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds;*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ &(b-a)\left|f'(b)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} M_2(1,m,p)
\end{aligned}$$

where $M_2(1,m,p)$ is the term in (2.2).

Theorem 2.3. *Let $f: I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|^q$ is decreasing and $(\alpha,m)-$geometrically convex on $[\min\{1,a\},b]$, for $b\geq 1$, and for $(\alpha,m)\in(0,1]^2$, $q\geq 1$, then the following inequality holds;*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ &(b-a)\left|f'(b)\right|^m\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left[M_3(\alpha,m,q)\right]^{\frac{1}{q}}
\end{aligned}$$

where

(2.3)

$$M_3(\alpha,m,q)=\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha q}dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha q}dt.$$

Proof. From Lemma 2 and using the well-known power mean integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ &(b-a)\int_0^1|p(t)|\left|f'(tb+(1-t)a)\right|dt \\
\leq\ &(b-a)\left(\int_0^1|p(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|p(t)|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ &(b-a)\left(\int_0^1|p(t)|\,dt\right)^{1-\frac{1}{q}}\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt\right\}^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'(x)|^q$ is decreasing and $(\alpha, m)-$geometrically convex on $[\min\{1,a\}, b]$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & (b-a)\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'\left(a^{1-t}b^{mt}\right)\right|^q dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'\left(a^{1-t}b^{mt}\right)\right|^q dt\right\}^{\frac{1}{q}} \\
\leq\ & (b-a)\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(a)\right|^{(1-t)^\alpha q}\left|f'(b)\right|^{m(1-(1-t)^\alpha)q} dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(a)\right|^{(1-t)^\alpha q}\left|f'(b)\right|^{m(1-(1-t)^\alpha)q} dt\right\}^{\frac{1}{q}} \\
=\ & (b-a)\left|f'(b)\right|^m\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha q} dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha q} dt\right\}.
\end{aligned}$$

So, the proof is completed. □

Corollary 2.3. *Let $f : I \subset [0,\infty) \to (0,\infty)$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'(x)|^q$ is decreasing and $m-$geometrically convex on $[\min\{1,a\}, b]$, for $b \geq 1$, and for $m \in (0,1]$, $q \geq 1$, then the following inequality holds;*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & (b-a)\left|f'(b)\right|^m\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left[M_3(1,m,q)\right]^{\frac{1}{q}}
\end{aligned}$$

where $M_3(1,m,q)$ is the term in (2.3).

Now, we obtain Simpson's type inequalities for twice differentiable functions using the following lemma.

Lemma 2.2. [[14], *pp.2*]*Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx \\
=\ & (b-a)^2\int_0^1 k(t)\,f''(tb+(1-t)a)\,dt
\end{aligned}$$

where

$$k(t)=\begin{cases}\frac{t}{2}\left(\frac{1}{3}-t\right), & t\in\left[0,\frac{1}{2}\right),\\ (1-t)\left(\frac{t}{2}-\frac{1}{3}\right), & t\in\left(\frac{1}{2},1\right].\end{cases}$$

Theorem 2.4. *Let $f: I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f''\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f''(x)|$ is decreasing and $(\alpha,m)-$geometrically convex on $[\min\{1,a\},b]$, for $b\geq 1$, and for $(\alpha,m)\in(0,1]^2$, then the following inequality holds;*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\;& (b-a)^2\left|f''(b)\right|^m\left[M_4(\alpha,m)\right]\end{aligned}$$

where

$$M_4(\alpha,m)=\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{(1-t)^{\alpha}}dt+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{(1-t)^{\alpha}}dt. \tag{2.4}$$

Proof. From Lemma 3, Lemma 1 and since $|f''(x)|$ is decreasing, then

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\;&(b-a)^2\int_0^1|k(t)|\left|f''(tb+(1-t)a)\right|dt\\ \leq\;&(b-a)^2\int_0^1|k(t)|\left|f''\left(a^{1-t}b^{mt}\right)\right|dt.\end{aligned}$$

Using the $(\alpha,m)-$geometrically convexity of $|f''(x)|$, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\;&(b-a)^2\int_0^1|k(t)|\left|f''(a)\right|^{(1-t)^{\alpha}}\left|f''(b)\right|^{m(1-(1-t)^{\alpha})}dt\\ =\;&(b-a)^2\left|f''(b)\right|^m\left\{\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{(1-t)^{\alpha}}dt\right.\\ &\left.+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{(1-t)^{\alpha}}dt\right\}.\end{aligned}$$

So, the proof is completed. □

Corollary 2.4. *Let $f: I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f''\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f''(x)|$ is decreasing and*

$m-$geometrically convex on $[\min\{1,a\},b]$ for $b\geq 1$, and for $m\in(0,1]$, then the following inequality holds;

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq (b-a)^2\left|f''(b)\right|^m M_4(1,m)$$

where $M_4(1,m)$ is the term in (2.4).

Theorem 2.5. *Let $f:I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f''\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f''(x)|^q$ is decreasing and $(\alpha,m)-$geometrically convex on $[\min\{1,a\},b]$, for $b\geq 1$, and for $(\alpha,m)\in(0,1]^2$, $q\geq 1$, then the following inequality holds;*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\ &(b-a)^2\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left(M_6(\alpha,m,q)^{\frac{1}{q}}+M_7(\alpha,m,q)^{\frac{1}{q}}\right)\end{aligned}$$

where

$$M_6(\alpha,m,q)=\left|f''(b)\right|^{mq}\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{q(1-t)^\alpha}dt$$

and

$$M_7(\alpha,m,q)=\left|f''(b)\right|^{mq}\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{q(1-t)^\alpha}dt.$$

Proof. Suppose that $q \geq 1$. From Lemma 3 and using the well-known power mean integral inequality, we have

$$\begin{aligned}
& \left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
\leq\ & (b-a)^2 \int_0^1 |k(t)|\,|f''(tb+(1-t)a)|\,dt \\
=\ & (b-a)^2 \left\{ \int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3}-t\right)\right| |f''(tb+(1-t)a)|\,dt \right. \\
& \left. + \int_{\frac{1}{2}}^1 \left| (1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| |f''(tb+(1-t)a)|\,dt \right\} \\
\leq\ & (b-a)^2 \left\{ \left(\int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3}-t\right)\right| dt \right)^{1-\frac{1}{q}} \right. \\
& \times \left(\int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3}-t\right)\right| |f''(tb+(1-t)a)|^q\,dt \right)^{\frac{1}{q}} \\
& + \left(\int_{\frac{1}{2}}^1 \left| (1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| dt \right)^{1-\frac{1}{q}} \\
& \left. \times \left(\int_{\frac{1}{2}}^1 \left| (1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| |f''(tb+(1-t)a)|^q\,dt \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Since $|f''(x)|^q$ is decreasing using Lemma 1 and $(\alpha,m)-$geometrically convex on $[\min\{1,a\},b]$, we have

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3}-t\right)\right| |f''(tb+(1-t)a)|^q\,dt \\
\leq\ & \int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3}-t\right)\right| \left|f''\left(a^{1-t}b^{mt}\right)\right|^q dt \\
\leq\ & \int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3}-t\right)\right| |f''(a)|^{q(1-t)^\alpha} |f''(b)|^{mq(1-(1-t)^\alpha)}\,dt \\
=\ & |f''(b)|^{mq} \int_0^{\frac{1}{2}} \left| \frac{t}{2}\left(\frac{1}{3}-t\right)\right| \left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{q(1-t)^\alpha} dt \\
=\ & M_6(\alpha,m,q)
\end{aligned} \tag{2.5}$$

and

$$
\begin{aligned}
(2.6)\qquad & \int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|f''\left(tb+(1-t)a\right)\right|^{q}dt \\
\leq\ & \int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|f''\left(a^{1-t}b^{mt}\right)\right|^{q}dt \\
\leq\ & \int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|f''(a)\right|^{q(1-t)^{\alpha}}\left|f''(b)\right|^{mq\left(1-(1-t)^{\alpha}\right)}dt \\
=\ & \left|f''(b)\right|^{mq}\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left(\frac{\left|f''(a)\right|}{\left|f''(b)\right|^{m}}\right)^{q(1-t)^{\alpha}}dt \\
=\ & M_{7}(\alpha,m,q)
\end{aligned}
$$

From (2.5) and (2.6), we have

$$
\begin{aligned}
& \left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right| \\
\leq\ & (b-a)^{2}\left\{\left(\int_{0}^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|dt\right)^{1-\frac{1}{q}}M_{6}(\alpha,m,q)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|dt\right)^{1-\frac{1}{q}}M_{7}(\alpha,m,q)^{\frac{1}{q}}\right\} \\
=\ & (b-a)^{2}\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left(M_{6}(\alpha,m,q)^{\frac{1}{q}}+M_{7}(\alpha,m,q)^{\frac{1}{q}}\right)
\end{aligned}
$$

where we use the fact that

$$
\int_{0}^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|dt=\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|dt=\frac{1}{162}.
$$

So, the proof is completed. □

Corollary 2.5. *Let $f : I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f''\in L[a,b]$, where $a,b\in I$ with $a<b$. If $\left|f''(x)\right|^{q}$ is decreasing and $m-$geometrically convex on $[\min\{1,a\},b]$, for $b\geq 1$, and for $m\in(0,1]$, $q\geq 1$, then the following inequality holds;*

$$
\begin{aligned}
& \left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right| \\
\leq\ & (b-a)^{2}\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left(M_{6}(1,m,q)^{\frac{1}{q}}+M_{7}(1,m,q)^{\frac{1}{q}}\right)
\end{aligned}
$$

where $M_{6}(1,m,q)$ and $M_{7}(1,m,q)$ are in the Theorem 5.

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