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ON A NEW CLASS OF s-TYPE OPERATORS

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Abstract. In this paper, we introduce a new class of operators by using s-numbers and the sequence space $Z(u, v; \ell_p)$ for $1 < p < \infty$. We prove that this class is a quasi-Banach operator ideal. Also, we give some properties of the quasi-Banach operator ideal. Lastly, we establish some inclusion relations among the operator ideals formed by different s-number sequences.

1. Introduction

By ω, we denote the space of all real-valued sequences. Any vector subspace of ω is called a sequence space. We write ℓ_p for the sequence space of p-absolutely convergent series.

Maddox [6] defined the linear space $\ell(p)$ as follows:

$$\ell(p) = \left\{ x \in \omega : \sum_{n=1}^{\infty} |x_n|^{p_n} < \infty \right\},$$

where (p_n) is a bounded sequence of strictly positive real numbers.

Altay and Baar [1] introduced the sequence space $\ell(u, v; p)$ which is the set of all sequences whose generalized weighted mean transforms are in the space $\ell(p)$, that is,

$$\ell(u, v; p) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left| u_n \sum_{k=1}^{n} v_k x_k \right|^{p_n} < \infty \right\},$$

where $u_n, v_k \neq 0$ for all $n, k \in \mathbb{N}$.

If $(p_n) = (p)$, $\ell(u, v; p) = Z(u, v; \ell_p)$ which is defined by Malkowsky and Sava [8] as follows:

$$Z(u, v; \ell_p) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left| u_n \sum_{k=1}^{n} v_k x_k \right|^{p} < \infty \right\},$$

where $1 < p < \infty$.

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Cesàro sequence space was defined by Shiue [13] as

$$ces_p = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}$$

for $1 < p < \infty$.

In [7], the weighted Cesàro sequence space $ces(p,q)$ is defined as

$$ces(p,q) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} |q_k x_k| \right)^p < \infty \right\},$$

where $q = (q_k)$ is a bounded sequence of positive real numbers, $Q_n = \sum_{k=1}^{n} q_k$ and $1 < p < \infty$.

In the literature, various operator ideals were defined by using sequences of different s-numbers of bounded linear operators. For example, Pietsch [9] defined the class of ℓ_p type operators for $0 < p < \infty$. A bounded linear operator T is in this class if $\sum_{n=1}^{\infty} (a_n(T))^p < \infty$. By using the Cesàro sequence space, Constantin [3] introduced the class of $ces - p$ type operators which satisfy the following condition:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k(T) \right)^p < \infty$$

for $1 < p < \infty$. s-type $ces(p,q)$ operators studied by Maji and Srivastava [7] as a general case of $ces - p$ type operators. A bounded linear operator T is of s-type $ces(p,q)$ operator if

$$\sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} q_k s_k(T) \right)^p < \infty$$

for $1 < p < \infty$.

The main purpose of this paper is to introduce a more general class of s-type operators by using the sequence space $Z(u,v;\ell_p)$. We show that the class of s-type $Z(u,v;\ell_p)$ operators is an operator ideal and a quasi-norm is defined on this class. Moreover, we give some properties and inclusion relations related to the operator ideals formed by different s-number sequences.

2. Preliminaries and Background

Firstly, we give basic notations used throughout this paper. By $\mathcal{B}$, we denote the class of all bounded linear operators between any two Banach spaces. $\mathcal{B}(X,Y)$ is the space of all bounded linear operators from X to Y, where X and Y Banach spaces. X' is composed of continuous linear functionals on X, that is, X' is the dual of X. The map $x' \otimes y : X \to Y$ is defined by $(x' \otimes y)(x) = x'(x)y$, where $x' \in X'$ and $y \in Y$. By $\mathbb{N}$ and $\mathbb{R}^+$, we denote the set of all natural numbers and all nonnegative real nubers, respectively.

Now, we give some definitions and results about s-number sequences and operator ideals.

Definition 2.1. [7] A finite rank operator is a bounded linear operator whose dimension of the range space is finite.

Definition 2.2. [2] A map

$$s : T \to (s_n(T)),$$

which assigns a non-negative scalar sequence to each operator, is called an s-number sequence if for all Banach spaces X, Y, Z and W the following conditions are satisfied:

(i) $\|T\| = s_1(T) \geq s_2(T) \geq ... \geq 0$ for all $T \in \mathcal{B}(X,Y)$.
(ii) $s_{n+m-1}(T+S) \leq s_n(T) + s_m(S)$ for $T, S \in \mathcal{B}(X,Y)$.
(iii)$s_n(RST) \leq \|R\| s_n(S) \|T\|$ for all $R \in \mathcal{B}(Z,W)$, $S \in \mathcal{B}(Y,Z)$, $T \in \mathcal{B}(X,Y)$.
(iv) If rank$(T) < n$, then $s_n(T) = 0$ for all $T \in \mathcal{B}(X,Y)$.
(v) $s_n(I_n) = 1$, where I_n is the identity map of n-dimensional Hilbert space ℓ_2^n to itself.

$s_n(T)$ is called the nth s-number of T.

Let $T \in \mathcal{B}(X,Y)$ and $n \in \mathbb{N}$. $(a_n(T))$, $(c_n(T))$, $(d_n(T))$, $(x_n(T))$, $(y_n(T))$ and $(h_n(T))$ are the sequences of nth approximation number, nth Gel'fand number, nth Kolmogorov number, Weyl number, Chang number and Hilbert number, respectively. These sequences are some examples of s-number sequences of a bounded linear operator. For the definition of these sequences, see [7, 2].

Definition 2.3. [4, p. 440] A subcollection $\mathcal{M}$ of $\mathcal{B}$ is said to be an operator ideal if the following conditions are satisfied:
(OI-1) $x' \otimes y : X \to Y \in \mathcal{M}(X,Y)$ for $x' \in X'$ and $y \in Y$.
(OI-2) $T + S \in \mathcal{M}(X,Y)$ for $T, S \in \mathcal{M}(X,Y)$.
(OI-3) $RST \in \mathcal{M}(X_0, Y_0)$ for $S \in \mathcal{M}(X,Y)$, $T \in \mathcal{M}(X_0,X)$ and $R \in \mathcal{M}(Y,Y_0)$.

Definition 2.4. [10] A function $\alpha : \mathcal{M} \to \mathbb{R}^+$ is said to be a quasi-norm on the operator ideal $\mathcal{M}$ if the following conditions hold:
(QN-1) If $x' \in X'$ and $y \in Y$, then $\alpha(x' \otimes y) = \|x'\| \|y\|$.
(QN-2) If $S, T \in \mathcal{M}(X,Y)$, then there exists a constant $C \geq 1$ such that $\alpha(S+T) \leq C[\alpha(S) + \alpha(T)]$.
(QN-3) If $S \in \mathcal{M}(X,Y)$, $T \in \mathcal{M}(X_0,X)$ and $R \in \mathcal{M}(Y,Y_0)$, then $\alpha(RST) \leq \|R\| \alpha(S) \|T\|$.

In particular if $C = 1$ then α becomes a norm on the operator ideal $\mathcal{M}$.

Let $\mathcal{M}$ be an ideal and α be a quasi-norm on the ideal $\mathcal{M}$. $[\mathcal{M}, \alpha]$ is said to be a quasi-Banach operator ideal if each $\mathcal{M}(X,Y)$ is complete under the quasi-norm α.

Lemma 2.1. [5] *If Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space.*

Lemma 2.2. [11] *Let $T, S \in \mathcal{B}(X,Y)$. Then $|s_n(T) - s_n(S)| \leq \|T - S\|$ for $n = 1, 2, ...$*

Definition 2.5. [12, p. 90] An s-number sequence $s = (s_n)$ is called injective if, given any metric injection $I \in \mathcal{B}(Y, Y_0)$, $s_n(T) = s_n(IT)$ for all $T \in \mathcal{B}(X,Y)$.

A quasi-normed operator ideal $[\mathcal{M}, \alpha]$ is called injective if $T \in \mathcal{M}(X,Y)$ and $\alpha(IT) = \alpha(T)$ as $IT \in \mathcal{M}(X, Y_0)$, where $T \in \mathcal{B}(X,Y)$ and $I \in \mathcal{B}(Y, Y_0)$ is a metric injection.

Definition 2.6. [12, p. 95] An s-number sequence $s = (s_n)$ is called surjective if, given any metric surjection $S \in \mathcal{B}(X_0, X)$, $s_n(T) = s_n(TS)$ for all $T \in \mathcal{B}(X,Y)$.

A quasi-normed operator ideal $[\mathcal{M}, \alpha]$ is called surjective if $T \in \mathcal{M}(X,Y)$ and $\alpha(TS) = \alpha(T)$ as $TS \in \mathcal{M}(X_0, Y)$, where $T \in \mathcal{B}(X,Y)$ and $S \in \mathcal{B}(X_0, X)$ is a metric surjection.

Definition 2.7. [10, p. 152] Let T' be the dual of T. An s-number sequence is called symmetric if $s(T) \geq s_n(T')$ for all $T \in \mathcal{B}$. If $s(T) = s_n(T')$ then the s-number sequence is said to be completely symmetric.

Definition 2.8. [10] For every operator ideal $\mathcal{M}$, the dual operator ideal denoted by $\mathcal{M}'$ is defined as

$$\mathcal{M}'(X,Y) = T \in \mathcal{B}(X,Y) : T' \in \mathcal{M}'(Y', X'),$$

where T' is the dual of T, X' and Y' are duals of X and Y, respectively.

Definition 2.9. [10, p. 68] An operator ideal $\mathcal{M}$ is called symmetric if $\mathcal{M} \subset \mathcal{M}'$. If $\mathcal{M} = \mathcal{M}'$, the operator ideal $\mathcal{M}$ is called completely symmetric.

3. s-type $Z(u, v; \ell_p)$ operators

Let $u = (u_n)$ and $v = (v_n)$ be sequences of positive real numbers. An operator $T \in \mathcal{B}(X,Y)$ is in the class of s-type $Z(u, v; \ell_p)$ if

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) \right)^p < \infty, \qquad 1 < p < \infty.$$

The class of all s-type $Z(u, v; \ell_p)$ operators is denoted by $\mathcal{G}_p^{(s)}$.

If $u_n = \frac{1}{Q_n}$ and $v_k = q_k$ are taken for all $n, k \in \mathbb{N}$, then the class of s-type $Z(u, v; \ell_p)$ operators reduces to the class of s-type $ces(p, q)$ operators.

Theorem 3.1. *Let $v = (v_k)$ be a sequence of positive numbers such that*

$$v_{2k-1} + v_{2k} \leq M v_k \quad \textit{for all } k = 1, 2, ..., \tag{3.1}$$

where $M > 0$. If $\sum_{n=1}^{\infty} (u_n)^p < \infty$, then the class $\mathcal{G}_p^{(s)}$ is an operator ideal for $1 < p < \infty$.

Proof. Let X and Y be any two Banach spaces and $1 < p < \infty$. For $x' \in X'$ and $y \in Y$, the rank of the operator $x' \otimes y$ is one which means $s_n(x' \otimes y) = 0$ for all $n \geq 2$. We obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(x' \otimes y) \right)^p &= \sum_{n=1}^{\infty} (u_n v_1 s_1(x' \otimes y))^p \\ &= (v_1 s_1(x' \otimes y))^p \sum_{n=1}^{\infty} (u_n)^p < \infty. \end{aligned}$$

Hence $x' \otimes y \in \mathcal{G}_p^{(s)}(X,Y)$.

Let $T, S \in \mathcal{G}_p^{(s)}(X,Y)$. Then

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) \right)^p < \infty, \quad \sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(S) \right)^p < \infty.$$

By using the inequality (3.1) with monotonicity and additivity of s-number sequence,

$$\begin{aligned}\sum_{k=1}^{n} v_k s_k(T+S) &= \sum_{k=1}^{n} v_{2k-1} s_{2k-1}(T+S) + \sum_{k=1}^{n} v_{2k} s_{2k}(T+S)\\ &\leq \sum_{k=1}^{n} (v_{2k-1}+v_{2k}) s_{2k-1}(T+S)\\ &\leq M \sum_{k=1}^{n} v_k s_{2k-1}(T+S)\\ &\leq M \left(\sum_{k=1}^{n} v_k s_k(T) + \sum_{k=1}^{n} v_k s_k(S)\right).\end{aligned}$$

From Minkowsky inequality, we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(T+S)\right)^p\right)^{1/p} &\leq M\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(T) + u_n\sum_{k=1}^{n} v_k s_k(S)\right)^p\right)^{1/p}\\ &\leq M\left[\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(T)\right)^p\right)^{1/p} + \left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(S)\right)^p\right)^{1/p}\right].\end{aligned}$$

Thus $T+S \in \mathcal{G}_p^{(s)}(X,Y)$.

Let $S \in \mathcal{G}_p^{(s)}(X,Y)$, $T \in \mathcal{G}_p^{(s)}(X_0,X)$ and $R \in \mathcal{G}_p^{(s)}(Y,Y_0)$. Since s-number sequence has ideal property, we obtain that

$$\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(RST)\right)^p\right)^{1/p} \leq \|R\|.\|T\|.\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(S)\right)^p\right)^{1/p} < \infty.$$

Hence $RST \in \mathcal{G}_p^{(s)}(X_0,Y_0)$.

We have proved that the class $\mathcal{G}_p^{(s)}$ satisfied the conditions (OI-1) to (OI-3) and so $\mathcal{G}_p^{(s)}$ is an operator ideal. □

Proposition 3.1. *The inclusion $\mathcal{G}_p^{(s)} \subseteq \mathcal{G}_q^{(s)}$ holds for $1 < p \leq q < \infty$.*

Proof. Since $\ell_p \subseteq \ell_q$ for $1 < p \leq q < \infty$, we have $\mathcal{G}_p^{(s)} \subseteq \mathcal{G}_q^{(s)}$. □

Now, let $\mathcal{G}_p^{(s)}$ be an operator ideal. Define the maps $\Gamma_p^{(s)} : \mathcal{G}_p^{(s)} \to \mathbb{R}^+$ and $\widehat{\Gamma}_p^{(s)} : \mathcal{G}_p^{(s)} \to \mathbb{R}^+$ for $1 < p < \infty$ by

$$\Gamma_p^{(s)}(T) = \left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(T)\right)^p\right)^{1/p} \text{ and } \widehat{\Gamma}_p^{(s)}(T) = \frac{\Gamma_p^{(s)}(T)}{\left(\sum_{n=1}^{\infty}(v_1 u_n)^p\right)^{1/p}}.$$

Theorem 3.2. *Let $v = (v_k)$ be a sequence of positive numbers satisfying inequality (3.1). If $\sum_{n=1}^{\infty}(u_n)^p < \infty$, then the function $\widehat{\Gamma}_p^{(s)}$ is a quasi-norm on $\mathcal{G}_p^{(s)}$.*

Proof. Let X and Y be two Banach spaces. Then $x' \otimes y : X \to Y$ is a rank one operator, that is, $s_n(x' \otimes y) = 0$ for all $n \geq 2$. Hence, we have

$$\begin{aligned}
\Gamma_p^{(s)}(x' \otimes y) &= \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(x' \otimes y) \right)^p \right)^{1/p} \\
&= \left(\sum_{n=1}^{\infty} (u_n v_1 s_1(x' \otimes y))^p \right)^{1/p} \\
&= \|x' \otimes y\| \left(\sum_{n=1}^{\infty} (v_1 u_n)^p \right)^{1/p}.
\end{aligned}$$

Since $\sup_{\|x\|=1} \|x' \otimes y\| = \sup_{\|x\|=1} \|x'(x)y\| = \|y\| \sup_{\|x\|=1} |x'(x)| = \|x'\|\|y\|$, we have

$$\widehat{\Gamma}_p^{(s)}(x' \otimes y) = \|x'\|\|y\|.$$

Since the following inequality holds

$$\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T+S) \right)^p \right)^{1/p} \leq M \left[\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) \right)^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(S) \right)^p \right)^{1/p} \right],$$

that is, $\Gamma_p^{(s)}(T+S) \leq M \left[\Gamma_p^{(s)}(T) + \Gamma_p^{(s)}(S) \right]$ for $T, S \in \mathcal{G}_p^{(s)}(X,Y)$, we have

$$\begin{aligned}
\widehat{\Gamma}_p^{(s)}(T+S) &= \frac{\Gamma_p^{(s)}(T+S)}{\left(\sum_{n=1}^{\infty} (v_1 u_n)^p\right)^{1/p}} \\
&\leq M \frac{\left[\Gamma_p^{(s)}(T) + \Gamma_p^{(s)}(S) \right]}{\left(\sum_{n=1}^{\infty} (v_1 u_n)^p\right)^{1/p}} \\
&= M \left[\widehat{\Gamma}_p^{(s)}(T) + \widehat{\Gamma}_p^{(s)}(S) \right].
\end{aligned}$$

Let $S \in \mathcal{G}_p^{(s)}(X,Y)$, $T \in \mathcal{G}_p^{(s)}(X_0,X)$ and $R \in \mathcal{G}_p^{(s)}(Y,Y_0)$. Then, we have

$$\begin{aligned}
\Gamma_p^{(s)}(RST) &= \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(RST) \right)^p \right)^{1/p} \\
&\leq \|R\|\|T\| \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(S) \right)^p \right)^{1/p} \\
&= \|R\|\|T\|\Gamma_p^{(s)}(S).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\widehat{\Gamma}_p^{(s)}(RST) &= \frac{\Gamma_p^{(s)}(RST)}{\left(\sum_{n=1}^{\infty} (v_1 u_n)^p\right)^{1/p}} \\
&\leq \frac{\|R\|\|T\|\Gamma_p^{(s)}(S)}{\left(\sum_{n=1}^{\infty} (v_1 u_n)^p\right)^{1/p}} = \|R\|\|T\|\widehat{\Gamma}_p^{(s)}(S).
\end{aligned}$$

Consequently, $\widehat{\Gamma}_p^{(s)}$ is a quasi-norm on $\mathcal{G}_p^{(s)}$. $\square$

Theorem 3.3. *Let $1 < p < \infty$. $\left[\mathcal{G}_p^{(s)}, \widehat{\Gamma}_p^{(s)}\right]$ is a quasi-Banach operator ideal.*

Proof. Let X and Y be any two Banach spaces and $1 < p < \infty$. The following inequality holds

$$\begin{aligned}\Gamma_p^{(s)}(T) &= \left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k s_k(T)\right)^p\right)^{1/p}\\ &\geq \left(\sum_{n=1}^{\infty}(u_n v_1 s_1(T))^p\right)^{1/p} = \|T\|\left(\sum_{n=1}^{\infty}(v_1 u_n)^p\right)^{1/p}\end{aligned}$$

for $T \in \mathcal{G}_p^{(s)}(X,Y)$. Hence, we have

$$\|T\| \leq \widehat{\Gamma}_p^{(s)}(T). \tag{3.2}$$

Let (T_m) be a Cauchy sequence in $\mathcal{G}_p^{(s)}(X,Y)$. Then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\widehat{\Gamma}_p^{(s)}(T_m - T_l) < \varepsilon \tag{3.3}$$

for $\forall m, l \geq n_0$. It follows that

$$\|T_m - T_l\| \leq \widehat{\Gamma}_p^{(s)}(T_m - T_l) < \varepsilon$$

from the inequality (3.2). Then (T_m) is a Cauchy sequence in $\mathcal{B}(X,Y)$. According to Lemma 2.1, $\mathcal{B}(X,Y)$ is a Banach space since Y is a Banach space. Therefore $\| T_m - T \| \to 0$ as $m \to \infty$ for $T \in \mathcal{B}(X,Y)$. Now, we show that $\widehat{\Gamma}_p^{(s)}(T_m - T) \to 0$ as $m \to \infty$ for $T \in \mathcal{G}_p^{(s)}(X,Y)$.

The operators $T_l - T_m$, $T - T_m$ are in the class $\mathcal{B}(X,Y)$ for $T_m, T_l, T \in \mathcal{B}(X,Y)$. From Lemma 2.2, we have

$$\begin{aligned}|s_n(T_l - T_m) - s_n(T - T_m)| &\leq \|T_l - T_m - (T - T_m)\|\\ &= \|T_l - T\|.\end{aligned}$$

Since $T_l \to T$ as $l \to \infty$, that is $\|T_l - T\| < \varepsilon$, we obtain

$$s_n(T_l - T_m) \to s_n(T - T_m) \text{ as } l \to \infty. \tag{3.4}$$

It follows from (3.3) that the statement

$$\begin{aligned}\widehat{\Gamma}_p^{(s)}(T_m - T_l) &= \frac{\Gamma_p^{(s)}(T_m - T_l)}{\left(\sum_{n=1}^{\infty}(v_1 u_n)^p\right)^{1/p}}\\ &= \frac{\left(\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n} v_k s_k(T_m - T_l)\right)^p\right)^{1/p}}{\left(\sum_{n=1}^{\infty}(v_1 u_n)^p\right)^{1/p}} < \varepsilon\end{aligned}$$

holds for $\forall m, l \geq n_0$. Then,

$$\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k s_k(T_m - T_l)\right)^p\right)^{1/p} < \varepsilon\left(\sum_{n=1}^{\infty}(v_1 u_n)^p\right)^{1/p}$$

for $\forall m, l \geq n_0$. We obtain from (3.4) that

$$\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k s_k(T_m - T)\right)^p\right)^{1/p} < \varepsilon\left(\sum_{n=1}^{\infty}(v_1 u_n)^p\right)^{1/p}$$

as $l \to \infty$ for $\forall m \geq n_0$. Hence, we have $\widehat{\Gamma}_p^{(s)}(T_m - T) < \varepsilon$ for $\forall m \geq n_0$.

Finally, we show that $T \in \mathcal{G}_p^{(s)}(X,Y)$. From the inequality (3.4) and conditions (i), (ii) of Definition 2.2, we obtain

$$\begin{aligned}\sum_{k=1}^{n} v_k s_k(T) &= \sum_{k=1}^{n} v_{2k-1} s_{2k-1}(T) + \sum_{k=1}^{n} v_{2k} s_{2k}(T)\\ &\leq \sum_{k=1}^{n} (v_{2k-1} + v_{2k}) s_{2k-1}(T)\\ &\leq M \sum_{k=1}^{n} v_k s_{2k-1}(T)\\ &= M \sum_{k=1}^{n} v_k s_{k+k-1}(T - T_m + T_m)\\ &\leq M \left[\sum_{k=1}^{n} v_k s_k(T - T_m) + \sum_{k=1}^{n} v_k s_k(T_m) \right].\end{aligned}$$

By using Minkowsky inequality, since $T_m \in \mathcal{G}_p^{(s)}(X,Y)$ for all m and $\widehat{\Gamma}_p^{(s)}(T_m - T) \to \infty$ as $m \to \infty$, we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) \right)^p \right)^{1/p} &\leq M \left[\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T - T_m) + u_n \sum_{k=1}^{n} v_k s_k(T_m) \right)^p \right]^{1/p}\\ &\leq M \left[\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T - T_m) \right)^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T_m) \right)^p \right)^{1/p} \right]\\ &< \infty\end{aligned}$$

which means $T \in \mathcal{G}_p^{(s)}(X,Y)$. $\square$

Let $\left[\mathcal{G}_p^{(a)}, \widehat{\Gamma}_p^{(a)}\right]$, $\left[\mathcal{G}_p^{(c)}, \widehat{\Gamma}_p^{(c)}\right]$, $\left[\mathcal{G}_p^{(d)}, \widehat{\Gamma}_p^{(d)}\right]$, $\left[\mathcal{G}_p^{(x)}, \widehat{\Gamma}_p^{(x)}\right]$, $\left[\mathcal{G}_p^{(y)}, \widehat{\Gamma}_p^{(y)}\right]$ and $\left[\mathcal{G}_p^{(h)}, \widehat{\Gamma}_p^{(h)}\right]$ be the quasi-Banach operator ideals corresponding to the approximation numbers $a = (a_n)$, Gel'fand numbers $c = (c_n)$, Kolmogorov numbers $d = (d_n)$, Weyl numbers $x = (x_n)$, Chang numbers $y = (y_n)$ and Hilbert numbers $h = (h_n)$, respectively.

Theorem 3.4. *If s-number sequence is injective, then the quasi-Banach operator ideal* $\left[\mathcal{G}_p^{(s)}, \widehat{\Gamma}_p^{(s)}\right]$ *is injective for* $1 < p < \infty$.

Proof. Let $T \in \mathcal{B}(X,Y)$ and $I \in \mathcal{B}(Y,Y_0)$ be any metric injections. If $IT \in \mathcal{G}_p^{(s)}(X,Y_0)$, then

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(IT) \right)^p < \infty.$$

Since $s = (s_n)$ is injective, we have $s_n(T) = s_n(IT)$ for all $T \in \mathcal{B}(X,Y)$. Thus, we obtain

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) \right)^p = \sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(IT) \right)^p < \infty,$$

that is, $T \in \mathcal{G}_p^{(s)}(X,Y)$. Clearly, we have $\widehat{\Gamma}_p^{(s)}(IT) = \widehat{\Gamma}_p^{(s)}(T)$. $\square$

Conclusion 3.1. *The quasi-Banach operator ideals* $\left[\mathcal{G}_p^{(c)}, \widehat{\Gamma}_p^{(c)}\right]$ *and* $\left[\mathcal{G}_p^{(x)}, \widehat{\Gamma}_p^{(x)}\right]$ *are injective since the Gel'fand numbers and the Weyl numbers are injective (See* [12, p. 90-94]*).*

Theorem 3.5. *If s-number sequence is surjective, then the quasi-Banach operator ideal* $\left[\mathcal{G}_p^{(s)}, \widehat{\Gamma}_p^{(s)}\right]$ *is surjective for* $1 < p < \infty$*.*

Proof. Let $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(X_0, X)$ be any metric surjections. If $TS \in \mathcal{G}_p^{(s)}(X_0, Y)$, then

$$\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n} v_k s_k(TS)\right)^p < \infty.$$

Since $s = (s_n)$ is surjective, we have $s_n(T) = s_n(TS)$ for all $T \in \mathcal{B}(X, Y)$. Thus, we obtain

$$\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n} v_k s_k(T)\right)^p = \sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n} v_k s_k(TS)\right)^p < \infty,$$

that is, $T \in \mathcal{G}_p^{(s)}(X, Y)$. Clearly, we have $\widehat{\Gamma}_p^{(s)}(TS) = \widehat{\Gamma}_p^{(s)}(T)$. □

Conclusion 3.2. *The quasi-Banach operator ideals* $\left[\mathcal{G}_p^{(d)}, \widehat{\Gamma}_p^{(d)}\right]$ *and* $\left[\mathcal{G}_p^{(y)}, \widehat{\Gamma}_p^{(y)}\right]$ *are surjective since the Kolmogorov numbers and the Chang numbers are surjective (See* [12, p. 95]*).*

Now, we give some inclusion relations among the operator ideals $\mathcal{G}_p^{(a)}$, $\mathcal{G}_p^{(c)}$, $\mathcal{G}_p^{(d)}$, $\mathcal{G}_p^{(x)}$, $\mathcal{G}_p^{(y)}$ and $\mathcal{G}_p^{(h)}$.

Theorem 3.6. *The following inclusion relations*

(i) $\mathcal{G}_p^{(a)} \subseteq \mathcal{G}_p^{(c)} \subseteq \mathcal{G}_p^{(x)} \subseteq \mathcal{G}_p^{(h)}$,
(ii) $\mathcal{G}_p^{(a)} \subseteq \mathcal{G}_p^{(d)} \subseteq \mathcal{G}_p^{(y)} \subseteq \mathcal{G}_p^{(h)}$

hold for $1 < p < \infty$*.*

Proof. Let $T \in \mathcal{G}_p^{(a)}$. Then

$$\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n} v_k a_k(T)\right)^p < \infty,$$

where $1 < p < \infty$. It follows from [12, p. 115] that $h_n(T) \leq x_n(T) \leq c_n(T) \leq a_n(T)$ and $h_n(T) \leq y_n(T) \leq d_n(T) \leq a_n(T)$. Hence, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n} v_k h_k(T)\right)^p &\leq \sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n} v_k x_k(T)\right)^p \\
&\leq \sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n} v_k c_k(T)\right)^p \\
&\leq \sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n} v_k a_k(T)\right)^p < \infty
\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k h_k(T)\right)^p &\le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k y_k(T)\right)^p\\ &\le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k v_k(T)\right)^p\\ &\le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k a_k(T)\right)^p < \infty.\end{aligned}$$

Thus, the inclusions are clear. □

Theorem 3.7. *Let $1 < p < \infty$. The operator ideal $\mathcal{G}_p^{(a)}$ is symmetric and the operator ideal $\mathcal{G}_p^{(h)}$ is completely symmetric.*

Proof. Let $1 < p < \infty$.

Firstly, we prove that the inclusion $\mathcal{G}_p^{(a)} \subseteq \left(\mathcal{G}_p^{(a)}\right)'$ holds. Let $T \in \mathcal{G}_p^{(a)}$. Then

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k a_k(T)\right)^p < \infty.$$

It follows from [10, p. 152] that $a_n(T') \le a_n(T)$ for $T \in \mathcal{B}$. Hence, we have

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k a_k(T')\right)^p \le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k a_k(T)\right)^p < \infty,$$

that is, $T \in \left(\mathcal{G}_p^{(a)}\right)'$. Thus, $\mathcal{G}_p^{(a)}$ is symmetric.

Now, we prove that the equation $\mathcal{G}_p^{(h)} = \left(\mathcal{G}_p^{(h)}\right)'$ holds. It follows from [12, p. 97] that $h_n(T') = h_n(T)$ for $T \in \mathcal{B}$. Hence, we have

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k h_k(T')\right)^p = \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}h_k a_k(T)\right)^p.$$

Thus, $\mathcal{G}_p^{(h)}$ is completely symmetric. □

Theorem 3.8. *Let $1 < p < \infty$. The equation $\mathcal{G}_p^{(c)} = \left(\mathcal{G}_p^{(d)}\right)'$ and the inclusion $\mathcal{G}_p^{(d)} \subseteq \left(\mathcal{G}_p^{(c)}\right)'$ hold. Also, the equation $\mathcal{G}_p^{(d)} = \left(\mathcal{G}_p^{(c)}\right)'$ holds for any compact operators.*

Proof. Let $1 < p < \infty$. We have from [12, p. 95] that $c_n(T) = d_n(T')$ and $c_n(T') \le d_n(T)$ for $T \in \mathcal{B}$. Also, the equality $c_n(T') = d_n(T)$ holds, where T is a compact operator. Thus the proof is clear. □

Theorem 3.9. *Let $1 < p < \infty$. The equations $\mathcal{G}_p^{(x)} = \left(\mathcal{G}_p^{(y)}\right)'$ and $\mathcal{G}_p^{(y)} = \left(\mathcal{G}_p^{(x)}\right)'$ hold.*

Proof. Let $1 < p < \infty$. We have from [12, p. 96] that $x_n(T) = y_n(T')$ and $y_n(T) = x_n(T')$ for $T \in \mathcal{B}$. Thus the proof is clear. □

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ON SOME CLASSICAL THEOREMS IN INTUITIONISTIC FUZZY PROJECTIVE PLANE

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Abstract. In this work, we introduce that intuitionistic fuzzy versions of some classical configurations in projective plane are valid in intuitionistic fuzzy projective plane with base Desarguesian or Pappian plane.

1. Introduction

After the introduction of Fuzzy set theory by Zadeh [12] several researches were conducted on generalizations of this theory.

A model of fuzzy projective geometries was introduced by Kuijken, Van Maldeghem and Kerre [10]. This provided a link between the fuzzy versions of classical theories that are very closely related. Also, Kuijken and Van Maldeghem contributed to fuzzy theory by introducing fibered geometries, which is a particular kind of fuzzy geometries [9]. They gave the fibered versions of some classical results in projective planes by using minimum operator. Then the role of the triangular norm in the theory of fibered projective planes and fibered harmonic conjugates and a fibered version of Reidemeister's condition were given in [3] and the fibered version of Menelaus and Ceva's 6-figures was studied in [4]. In these papers, the points and lines of the base geometry mostly have multiple degrees of membership.

Intuitionistic fuzzy set (IFS) was first published by Atanassov [2] and some authors appeared in literature [5], [11]. A model of intuitionistic fuzzy projective geometry and the link between fibered and intuitionistic fuzzy projective geometry were given by Ghassan E. Arif [7].

In the present paper, the intuitionistic fuzzy versions of some classical theorems in projective planes were given.

2. Preliminaries

We first recall the basic notions from the theory of intuitionistic fuzzy geometries and fibered projective geometry. We assume that the reader is familiar with the

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basic notions of fuzzy mathematics, although this is not strictly necessary as the paper is self-contained in this respect.

We denote by $\wedge$ and $\vee$, minimum and maximum operators respectively.

Let $\mathcal{P} = (P, B, \sim)$ be any projective plane with point set P and line set B, i.e., P and B are two disjoint sets endowed with a symmetric relation $\sim$ (called the *incidence relation*) such that the graph $(P \cup B, \sim)$ is a bipartite graph with classes P and B, and such that two distinct points p, q in $\mathcal{P}$ are incident with exactly one line (denoted by $\langle pq \rangle$), every two distinct lines L, M are incident with exactly one point (denoted by $L \cap M$), and every line is incident with at least three points. A set S of *collinear* points is a subset of P each member of which is incident with a common line L. Dually, one defines a set of *concurrent* lines [8].

Definition 2.1. *(see [2])* Let X be a nonempty fixed set. An intuitionistic fuzzy set A on X is an object having the form

$$A = \{\langle x, \lambda(x), \mu(x) \rangle : x \in X\}$$

where the function $\lambda : X \to I$ and $\mu : X \to I$ denote the degree of membership (namely, $\lambda(x)$) and the degree of nonmembership (namely, $\mu(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \lambda(x) + \mu(x) \leq 1$ for each $x \in X$. An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x) \rangle : x \in X\}$ can be written in the $A = \{\langle x, \lambda, \mu \rangle : x \in X\}$, or simply $A = \langle \lambda, \mu \rangle$.

Let $A = \{\langle x, \lambda(x), \mu(x) \rangle : x \in X\}$ and $B = \{\langle x, \delta(x), \gamma(x) \rangle : x \in X\}$ be an intuitionistic fuzzy sets on X . Then,

(a) $\overline{A} = \{\langle x, \mu(x), \lambda(x) \rangle : x \in X\}$ (the complement of A).
(b) $A \cap B = \{\langle x, \lambda(x) \wedge \delta(x),\ \mu(x) \vee \gamma(x) \rangle : x \in X\}$ (the meet of A and B).
(c) $A \cup B = \{\langle x, \lambda(x) \vee \delta(x),\ \mu(x) \wedge \gamma(x) \rangle : x \in X\}$ (the join of A and B).
(d) $A \subseteq B \Leftrightarrow \lambda(x) \leq \delta(x)$ and $\mu(x) \geq \gamma(x)$ for each $x \in X$.
(e) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.
(f) $\widetilde{1} = \{\langle x, 1, 0 \rangle : x \in X\}$, $\widetilde{0} = \{\langle x, 0, 1 \rangle : x \in X\}$.

Definition 2.2. (see [9]) A fibered projective plane FP on the projective plane P consist of a set FP of f- points and a set FB of f-lines , such that every point and line of P is base point and base line of at least one f-point and f-line respectively , (with at least one membership degree different from 1), and such that $\mathcal{FP} = (FP, FB)$ is closed under taking intersections of f-lines and spans of f-points. Finally, a set of f-points are called collinear if each pair of them span the same f-line. Dually, a set of f-lines are called concurrent if each pair of them intersect in the same f-point.

Definition 2.3. (see [9]) Let $\mathcal{P}$ be a projective plane, $a \in P$ and $\alpha \in]0, 1]$. Then an f-point (a, α) is the following fuzzy set on the point set P of $\mathcal{P}$:

$$(a, \alpha) : P \to [0, 1] : \begin{cases} a \to \alpha & \\ x \to 0 & \text{if } x \in P \backslash \{a\}. \end{cases}$$

Dually, one defines in the same way the f-line (L, β) for $L \in B$ and $\beta \in]0, 1]$.

The real number α above is called the membership degree of the f-point (a, α), while the point a is called the base point of it. Similarly for f-lines.

Two f-lines (L, α) and (M, β), with $\alpha \wedge \beta > 0$, intersect in the unique f-point $(L \cap M, \alpha \wedge \beta)$. Dually, the f-points (a, λ) and (b, μ), with $\lambda \wedge \mu > 0$, span the unique f-line $(\langle a, b \rangle, \lambda \wedge \mu)$.

In the above definitions, the $\wedge$ was originally meant to be the minimum operator, but in [3] was considered any triangular norm.

Definition 2.4. (see [7]) An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ on n -dimensional projective space S is an intuitionistic fuzzy n - dimensional projective space on S if $\lambda(p) \geq \lambda(q) \wedge \lambda(r)$ and $\mu(p) \leq \mu(q) \vee \mu(r)$, for any three collinear points p, q, r of A we denoted $[A, S]$.

The projective space S is called the base projective space of $[A, S]$ if $[A, S]$ is an intuitionistic fuzzy point , line , plane , ... ,we use base point , base line , base plane,... , respectively.

Definition 2.5. (see [7]) Consider the projective plane $\mathcal{P} = (P, B, I)$. Suppose $a \in P$ and $\alpha, \beta \in [0, 1]$. The IF-point (a, α, β) is the following intuitionistic fuzzy set on the point set P of $\mathcal{P}$:

$$(a, \alpha, \beta) : P \to [0, 1] : \begin{cases} a \to \alpha,\ a \to \beta & \\ x \to 0 & \text{if } x \in P\backslash\{a\}. \end{cases}$$

The point a is called the base point of the IF-point (a, α, β). An IF-line (L, α, β) with base line L is defined in a similar way .

The IF-lines (L, α, β) and (M, σ, ω) intersect in the unique IF- point $(L \cap M, \alpha \wedge \sigma, \beta \vee \omega)$. The IF-points (a, α, β) and (b, σ, ω) span the unique IF-line $(\langle a, b\rangle, \alpha \wedge \sigma, \beta \vee \omega)$.

Definition 2.6. (see [7]) Suppose $\mathcal{P}$ is a projective plane $\mathcal{P} = (P, B, I)$. The intuitionistic fuzzy set $Z = \langle\lambda, \mu\rangle$ on $P \cup B$ is an intuitionistic fuzzy projective plane on $\mathcal{P}$ if :

1) $\lambda(L) \geq \lambda(p) \wedge \lambda(q)$ and $\mu(L) \leq \mu(p) \vee \mu(q)$, $\forall p, q$: $\langle p, q\rangle = L$
2) $\lambda(p) \geq \lambda(L) \wedge \lambda(M)$ and $\mu(p) \leq \mu(L) \vee \mu(M)$, $\forall L, M$: $L \cap M = p$.

The intuitionistic fuzzy projective plane can be considered as an ordinary projective plane, where to every point (and only to points) one (and only one) degrees of membership and nonmembership are assigned.

3. Some Properties of the $\mathcal{IFP}$ with base plane $\mathcal{P}$

We now consider some classical configurations in $\mathcal{P}$ and extend them to intuitionistic fuzzy projective planes. Firstly, we look at the Desargues configuration in an intuitionistic projective plane $\mathcal{IFP}$ with base plane $\mathcal{P}$ that is Desarguesian.

Theorem 3.1. *Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with base plane $\mathcal{P}$ that is Desarguesian. Choose three IF-points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1, 2, 3\}$with noncollinear base points, and three other f- points (b_i, β_i, β_i'), $i \in \{1, 2, 3\}$with noncollinear base points, such that the f-lines $(\langle a_i, b_i\rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$, for $i \in \{1, 2, 3\}$, meet in an IF-point (p, γ, η) of $\mathcal{IFP}$, with $a_i \neq b_i \neq p \neq a_i$. Then the three IF-points $(c_{\{i,j\}}, \gamma_{\{i,j\}}, \gamma'_{\{i,j\}})$ obtained by intersecting $(\langle a_i, a_j\rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j')$ and $(\langle b_i, b_j\rangle, \beta_i \wedge \beta_j, \beta_i' \vee \beta_j')$, for $i \neq j$ and $i, j \in \{1, 2, 3\}$), are collinear.*

Proof. One calculates that $\gamma = \alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j$ and $\gamma' = \alpha_i' \vee \alpha_j' \vee \beta_i' \vee \beta_j'$ for $\{i, j\} \subseteq \{1, 2, 3\}$, with $i \neq j$. Now, the membership degree of the line spanned by $(c_{\{i,j\}}, \gamma_{\{i,j\}}, \gamma'_{\{i,j\}})$ and $(c_{\{i,k\}}, \gamma_{\{i,k\}}, \gamma'_{\{i,k\}})$, with $\{i, j, k\} = \{1, 2, 3\}$, is equal to

$$\alpha_i \wedge \alpha_i \wedge \alpha_j \wedge \alpha_k \wedge \beta_i \wedge \beta_i \wedge \beta_j \wedge \beta_k = \gamma \wedge \gamma \text{ and } \alpha_i' \vee \alpha_i' \vee \alpha_j' \vee \alpha_k' \vee \beta_i' \vee \beta_i' \vee \beta_j' \vee \beta_k' = \eta \vee \eta$$

which is independent of i. $\square$

The Pappus' theorem was fuzzified using minimum operator in [9]. Now, we give intuitionistic fuzzy version of Pappus theorem as the following:

Theorem 3.2. *Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with Pappian base plane $\mathcal{P}$. Choose two different lines L_1 and L_2 in $\mathcal{P}$. Choose two triples of $IF-$points $(a_i, \alpha_i, \alpha_i')$ and (b_i, β_i, β_i') with a_i on L_1 and b_i on L_2, $i = 1, 2, 3$ and such that no three of the base points a_1, a_2, b_1, b_2 are collinear. Then the three intersection $IF-$points $(c_1, \gamma_1, \gamma_1') = (a_2b_3 \cap a_3b_2, \alpha_2 \wedge \alpha_3 \wedge \beta_2 \wedge \beta_3, \alpha_2' \vee \alpha_3' \vee \beta_2' \vee \beta_3')$, $(c_2, \gamma_2, \gamma_2') = (a_1b_3 \cap a_3b_1, \alpha_1 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_3, \alpha_1' \vee \alpha_3' \vee \beta_1' \vee \beta_3')$ and $(c_3, \gamma_3, \gamma_3') = (a_1b_2 \cap a_2b_1,\ \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2,\ \alpha_1' \vee \alpha_2' \vee \beta_1' \vee \beta_2')$ are collinear.*

Proof. Since $IF-$points $(a_i, \alpha_i, \alpha_i')$ and (b_i, β_i, β_i') are $IF-$collinear, $\alpha_1 \wedge \alpha_2 = \alpha_1 \wedge \alpha_3 = \alpha_2 \wedge \alpha_3, \alpha_1' \vee \alpha_2' = \alpha_1' \vee \alpha_3' = \alpha_2' \vee \alpha_3'$, and $\beta_1 \wedge \beta_2 = \beta_1 \wedge \beta_3 = \beta_2 \wedge \beta_3$, $\beta_1' \vee \beta_2' = \beta_1' \vee \beta_3' = \beta_2' \vee \beta_3'$, $i = 1, 2, 3$. $\gamma_1 \wedge \gamma_2 = \alpha_2 \wedge \alpha_3 \wedge \beta_2 \wedge \beta_3 \wedge \alpha_1 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_3$, $\gamma_1 \wedge \gamma_3 = \alpha_2 \wedge \alpha_3 \wedge \beta_2 \wedge \beta_3 \wedge \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2$ and $\gamma_2 \wedge \gamma_3 = \alpha_1 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_3 \wedge \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2$. Also, $\gamma_1' \vee \gamma_2' = \alpha_2' \vee \alpha_3' \vee \beta_2' \vee \beta_3' \vee \alpha_1' \vee \alpha_3' \vee \beta_1' \vee \beta_3'$, $\gamma_1' \vee \gamma_3' = \alpha_2' \vee \alpha_3' \vee \beta_2' \vee \beta_3' \vee \alpha_1' \vee \alpha_2' \vee \beta_1' \vee \beta_2'$ and $\gamma_2' \vee \gamma_3' = \alpha_1' \vee \alpha_3' \vee \beta_1' \vee \beta_3' \vee \alpha_1' \vee \alpha_2' \vee \beta_1' \vee \beta_2'$. So, it is clear that $\gamma_1 \wedge \gamma_2 = \gamma_1 \wedge \gamma_3 = \gamma_2 \wedge \gamma_3$, $\gamma_1' \vee \gamma_2' = \gamma_1' \vee \gamma_3' = \gamma_2' \vee \gamma_3'$. □

Conclusion: In the present paper, we have considered Desargues and Pappus configurations in projective plane $\mathcal{P}$. We have seen that intuitionistic fuzzy versions of them automatically holds. In further investigation, when using other triangular norms, it will be given contribution to the intuitionistic fuzzy projective geometry and other classical theorems of projective geometry will be extended to the intuitionistic fuzzy projective geometry.

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SOME PARANORMED SEQUENCE SPACES DEFINED BY A MUSIELAK-ORLICZ FUNCTION OVER N-NORMED SPACES

AYHAN ESI AND S. K. SHARMA

Abstract. In this paper we present new classes of sequence spaces using lacunary sequences and a Musielak-Orlicz function over n-normed spaces. We examine some topological properties and prove some interesting inclusion relations between them.

1. Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [5] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [14]. Since then, many others have studied this concept and obtained various results, see Gunawan ([6], [7]) and Gunawan and Mashadi [8]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \; ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly

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independent set in X. Then the following function $||\cdot,\cdots,\cdot||_\infty$ on X^{n-1} defined by

$$||x_1,x_2,\cdots,x_{n-1}||_\infty = \max\{||x_1,x_2,\cdots,x_{n-1},a_i|| : i = 1,2,\cdots,n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1,a_2,\cdots,a_n\}$.

A sequence (x_k) in a n-normed space $(X,||\cdot,\cdots,\cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1,\cdots,z_{n-1}|| = 0 \text{ for every } z_1,\cdots,z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X,||\cdot,\cdots,\cdot||)$ is said to be Cauchy if

$$\lim_{\substack{k\to\infty\\ p\to\infty}} ||x_k - x_p, z_1,\cdots,z_{n-1}|| = 0 \text{ for every } z_1,\cdots,z_{n-1} \in X.$$

If every cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x+y) \leq p(x) + p(y)$ for all $x,y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X,p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [19, Theorem 10.4.2, pp. 183]).

For more details about sequence spaces (see [1], [2], [3], [17], [18]) and references therein.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [10] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function (see [13], [16]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\},\ k = 1,2,\ldots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace h_M are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty}(M_k)(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let ℓ_∞, c and c_0 denotes the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent if all Banach limits of $x = (x_k)$ coincide. In [9], it was shown that

$$\hat{c} = \Big\{x = (x_k) : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_{k+s} \ \text{ exists, uniformly in } \ s\Big\}.$$

In ([11], [12]) Maddox defined strongly almost convergent sequences. Recall that a sequence $x = (x_k)$ is strongly almost convergent if there is a number L such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |x_{k+s} - L| = 0, \ \text{ uniformly in } \ s.$$

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \cdots$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $g_r = (i_r - i_{r-1}) \to \infty \ (r \to \infty)$. The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman [4] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{g_r}\sum_{k\in I_r} |x_k - L| = 0 \ \text{ for some } \ L\Big\}.$$

Mursaleen and Noman [15] introduced the notion of λ-convergent and λ-bounded sequences as follows :

Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \cdots \ \text{ and } \ \lambda_k \to \infty \ \text{ as } \ k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \longrightarrow L$ as $m \to \infty$, where

$$\lambda_m(x) = \frac{1}{\lambda_m}\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1})x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [15] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m |\frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a)| = 0$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space and $w(n - X)$ denotes the space of X-valued sequences. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then we define the following sequence spaces in the present paper:

$[\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta} =$

$$\Big\{ x = (x_k) \in w(n - X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(|| \frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} || \Big) \Big]^{p_k} = 0,$$

$$\text{for some } \ \rho > 0, L \in X \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X \Big\},$$

$[\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^{\theta} =$

$$\Big\{ x = (x_k) \in w(n - X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(|| \frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1} || \Big) \Big]^{p_k} = 0,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X \Big\}$$

and

$[\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_{\infty}^{\theta} =$

$$\Big\{ x = (x_k) \in w(n - X) : \sup_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(|| \frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1} || \Big) \Big]^{p_k} < \infty,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X \Big\}.$$

When, $\mathcal{M}(x) = x$, we get

$[\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta} =$

$$\Big\{ x = (x_k) \in w(n - X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \Big(|| \frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} || \Big)^{p_k} = 0,$$

$$\text{for some } \ \rho > 0, L \in X \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X \Big\},$$

$[\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^{\theta} =$

$$\Big\{ x = (x_k) \in w(n - X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \Big(|| \frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1} || \Big)^{p_k} = 0,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X \Big\}$$

and

$[\ c, p, \Lambda, ||\cdot, \cdots, \cdot||\ ]^{\theta}_{\infty} =$

$$\Big\{x = (x_k) \in w(n-X) : \sup_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} < \infty,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X\Big\}.$$

If we take $p = (p_k) = 1$ for all k, then we get

$[\ c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\ ]^{\theta} =$

$$\Big\{x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0,$$

$$\text{for some } \ \rho > 0, L \in X \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X\Big\},$$

$[\ c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\ ]^{\theta}_{0} =$

$$\Big\{x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X\Big\}$$

and

$[\ c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\ ]^{\theta}_{\infty} =$

$$\Big\{x = (x_k) \in w(n-X) : \sup_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] < \infty,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X\Big\}.$$

The following inequality will be used throughout the paper. If $0 \le \inf_k p_k = H_0 \le p_k \le \sup_k = H < \infty$, $K = \max(1, 2^{H-1})$ and $H = \sup_k p_k < \infty$, then

$$|x_k + y_k|^{p_k} \le K(|x_k|^{p_k} + |y_k|^{p_k}), \tag{1.1}$$

for all $k \in \mathbb{N}$ and $x_k, y_k \in \mathbb{C}$. Also $|x_k|^{p_k} \le \max(1, |x_k|^H)$ for all $x_k \in \mathbb{C}$.

2. Some properties of difference sequence spaces

Theorem 2.1. *Let $M = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $[\ c, M, p, \Lambda, ||., \cdots, \cdot||\ ]^{\theta}$, $[\ c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\ ]^{\theta}_0$ and $[\ c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\ ]^{\theta}_{\infty}$ are linear spaces over the field of complex numbers C.*

Proof. Let $x = (x_k), y = (y_k) \in [\ c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\ ]^{\theta}_0$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\longrightarrow\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0,$$

and

$$\lim_{r\longrightarrow\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, .$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is non-decreasing convex function, by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(\alpha x+\beta y)}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\qquad = \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\alpha\Lambda_k(x)}{\rho_3}, z_1, \cdots, z_{n-1}|| + \frac{\beta\Lambda_k(y)}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\qquad \le K\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\qquad\quad + K\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\Big[M_k\Big(||\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\qquad \le K\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\qquad\quad + K\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(y)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\qquad \longrightarrow 0 \text{ as } r \longrightarrow \infty.
\end{aligned}$$

Thus, we have $\alpha x + \beta y \in [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta$. Hence $[\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta$ is a linear space. Similarly, we can prove that $[\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta$ and $[\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_\infty^\theta$ are linear spaces. □

Theorem 2.2. *For any Musielak-Orlicz function $M = (M_k)$ and a bounded sequence $p = (p_k)$ of positive real numbers, $[\, c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta$ is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r \in \mathbb{N}\Big\},$$

where $H = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \ge 0$ for $x = (x_k) \in [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$\inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r \in \mathbb{N}\Big\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1.$$

Thus

$$\begin{aligned}
\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} &\le \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\le 1,
\end{aligned}$$

for each r. Suppose that $x \neq 0$ for each $k \in N$. This implies that $\Lambda_k(x) \neq 0$, for each $k \in N$. Let $\epsilon \longrightarrow 0$, then $||\frac{\Lambda_k(x)}{\epsilon}, z_1, \cdots, z_{n-1}|| \longrightarrow \infty$. It follows that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \longrightarrow \infty,$$

which is a contradiction. Therefore, $\Lambda_k(x) = 0$ for each k and thus $x = 0$ for each $k \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

and

$$\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

for each r. Let $\rho = \rho_1 + \rho_2$. Then, by Minkowski's inequality, we have

$$\begin{aligned}
&\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x+y)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\quad \leq \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)+\Lambda_k(y)}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\quad \leq \Big(\sum_{k\in I_r}\Big[\frac{\rho_1}{\rho_1+\rho_2}M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big) \\
&\qquad + \frac{\rho_2}{\rho_1+\rho_2}M_k\Big(||\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p^k}\Big)^{\frac{1}{H}} \\
&\quad \leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\qquad + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\quad \leq 1
\end{aligned}$$

Since $\rho's$ are non-negative, so we have

$$\begin{aligned}
g(x+y) &= \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x+y)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r \in \mathbb{N}\Big\}, \\
&\leq \inf\Big\{\rho_1^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\ \Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r \in \mathbb{N}\Big\} \\
&+ \inf\Big\{\rho_2^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r \in \mathbb{N}\Big\}.
\end{aligned}$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition,

$$g(\mu x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(\mu x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r \in \mathbb{N}\Big\}.$$

Then

$$g(\mu x) = \inf\Big\{(|\mu|t)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r \in \mathbb{N}\Big\},$$

where $t = \frac{\rho}{|\mu|}$. Since $|\mu|^{p_r} \le \max(1, |\mu|^{\sup p_r})$, we have

$$g(\mu x) \le \max(1, |\mu|^{\sup p_r}) \inf\Big\{t^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r \in \mathbb{N}\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality.

This completes the proof of the theorem. □

Theorem 2.3. *Let $M = (M_k)$ be a Musielak-Orlicz function. If $\sup_k [M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then $[\, c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta \subset [\, c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_\infty^\theta$.*

Proof. Let $x = (x_k) \in [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta$. There exists some positive ρ_1 such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0.$$

Define $\rho = 2\rho_1$. Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, by using inequality (1.1), we have

$$\begin{aligned}
&\sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&\quad= \sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x) - L + L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&\quad\le K\sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[\frac{1}{2^{p_k}}M_k\Big(||\frac{\Lambda_k(x) - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&\quad+ K\sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[\frac{1}{2^{p_k}}M_k\Big(||\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&\quad\le K\sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x) - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&\quad+ K\sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&\quad< \infty.
\end{aligned}$$

Hence $x = (x_k) \in [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_\infty^\theta$. □

Theorem 2.4. *Let $0 < \inf p_k = g \le p_k \le \sup p_k = H < \infty$ and $M = (M_k)$, $M' = (M'_k)$ are Musielak-Orlicz functions satisfying Δ_2-condition, then we have*

(i) $[\, c, M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta \subset [\, c, M \circ M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta$,

(ii) $[\, c, M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta \subset [\, c, M \circ M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta$,

(iii) $[\, c, M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]_\infty^\theta \subset [\, c, M \circ M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]_\infty^\theta$.

Proof. Let $x = (x_k) \in [c, \mathcal{M}', p, \Lambda, ||\cdot, \cdots, \cdot||]^\theta$. Then we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M'_k\Big(||\frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, \text{ for some } L.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \le t \le \delta$. Let

$$y_k = M'_k\Big(||\frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1}||\Big) \text{ for all } k \in \mathbb{N}.$$

We can write

$$\frac{1}{h_r} \sum_{k\in I_r} [M_k(y_k)]^{p_k} = \frac{1}{h_r} \sum_{k\in I_r, y_k\le\delta} [M_k(y_k)]^{p_k} + \frac{1}{h_r} \sum_{k\in I_r, y_k>\delta} [M_k(y_k)]^{p_k}.$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2-condition, we have

$$\begin{aligned} \frac{1}{h_r} \sum_{k\in I_r, y_k\le\delta} [M_k(y_k)]^{p_k} &\le [M_k(1)]^H \frac{1}{h_r} \sum_{k\in I_r, y_k\le\delta} [M_k(y_k)]^{p_k} \\ &\le [M_k(2)]^H \frac{1}{h_r} \sum_{k\in I_r, y_k\le\delta} [M_k(y_k)]^{p_k} \qquad (2.1) \end{aligned}$$

For $y_k > \delta$

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$

Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, it follows that

$$M_k(y_k) < M_k\Big(1 + \frac{y_k}{\delta}\Big) < \frac{1}{2} M_k(2) + \frac{1}{2} M_k\Big(\frac{2y_k}{\delta}\Big).$$

Since (M_k) satisfies Δ_2-condition, we can write

$$\begin{aligned} M_k(y_k) &< \frac{1}{2} T \frac{y_k}{\delta} M_k(2) + \frac{1}{2} T \frac{y_k}{\delta} M_k(2) \\ &= T \frac{y_k}{\delta} M_k(2). \end{aligned}$$

Hence,

$$\frac{1}{h_r} \sum_{k\in I_r, y_k>\delta} [M_k(y_k)]^{p_k} \le \max\Big(1, \Big(\frac{TM_k(2)}{\delta}\Big)^H\Big) \frac{1}{h_r} \sum_{k\in I_r, y_k>\delta} [(y_k)]^{p_k} \qquad (2.2)$$

from equations (2.1) and (2.2), we have

$$x = (x_k) \in [\, c, \mathcal{M} \circ \mathcal{M}', p, \Lambda, ||\cdot, \cdots, \cdot||]^\theta.$$

This completes the proof of (i). Similarly, we can prove that

$$[\, c, \mathcal{M}'^\theta_0 \subset [\, c, \mathcal{M} \circ \mathcal{M}'^\theta_0$$

and

$$[\, c, \mathcal{M}'^\theta_\infty \subset [\, c, \mathcal{M} \circ \mathcal{M}', p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty.$$

□

Corollary 2.1. *Let $0 < \inf p_k = h \le p_k \le \sup p_k = H < \infty$ and $M = (M_k)$ be a Musielak-Orlicz function satisfying Δ_2-condition, then we have*

$$[\, c, \mathcal{M}', p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0 \subset [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0$$

and

$$[\, c, \mathcal{M}', p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty \subset [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty.$$

Proof. Taking $\mathcal{M}'(x) = x$ in Theorem 2.4, we get the required result. $\square$

Theorem 2.5. *Let $M = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*
(i) $[\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty \subset [c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$,
(ii) $[c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0 \subset [\, c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$,
(iii) $\sup_r \frac{1}{h_r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})]^{p_k} < \infty \;\; (t, \rho > 0)$.

Proof. (i) $\Rightarrow$ (ii) The proof is obvious in view of the fact that $[\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0 \subset [\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$.
(ii) $\Rightarrow$(iii) Let $[\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0 \subset [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$. Suppose that (iii) does not hold. Then for some $t,\ \rho > 0$

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})]^{p_k} = \infty$$

and therefore we can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(j)}} \sum_{k \in I_{r(j)}} \Big[M_k\Big(\frac{j^{-1}}{\rho}\Big)\Big]^{p_k} > j, j = 1, 2, \tag{2.3}$$

Define the sequence $x = (x_k)$ by

$$\Lambda_k(x) = \begin{cases} j^{-1}, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases} \quad \text{for all} \;\; s \in \mathbb{N}.$$

Then $x = (x_k) \in [c, p, \Lambda, ||\cdot, \cdots, \cdot||]^\theta_0$ but by equation (2.3), $x = (x_k) \notin [c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$, which contradicts (ii). Hence (iii) must hold.
(iii) $\Rightarrow$ (i) Let (iii) hold and $x = (x_k) \in [c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$. Suppose that $x = (x_k) \notin [c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$. Then

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = \infty \tag{2.4}$$

Let $t = ||\Lambda_k(x), z_1, \cdots, z_{n-1}||$ for each k, then by equations (2.4)

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(\frac{t}{\rho}\Big)\Big] = \infty,$$

which contradicts (iii). Hence (i) must hold. $\square$

Theorem 2.6. *Let $1 \leq p_k \leq \sup p_k < \infty$ and $M = (M_k)$ be a Musielak Orlicz function. Then the following statements are equivalent:*
(i) $[\, c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0 \subset [c, p, \Lambda, ||\cdot, \cdots, \cdot||]^\theta_0$,
(ii) $[\, c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0 \subset [\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$,
(iii) $\inf_r \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(\frac{t}{\rho}\Big)\Big]^{p_k} > 0 \;\; (t, \rho > 0)$.

Proof. (i) $\Rightarrow$ (ii) It is trivial.
(ii) $\Rightarrow$ (iii) Let (ii) hold. Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(\frac{t}{\rho}\Big)\Big]^{p_k} = 0 \;\; (t, \rho > 0),$$

so we can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(j)}}\sum_{k\in I_{r(j)}}\Big[M_k\Big(\frac{j}{\rho}\Big)\Big]^{p_k} < j^{-1},\ \ j=1,2,. \tag{2.5}$$

Define the sequence $x=(x_k)$ by

$$\Lambda_k(x)=\begin{cases} j, & k\in I_{r(j)}\\ 0, & k\notin I_{r(j)}\end{cases} \text{ for all } \ s\in\mathbb{N}.$$

Thus by equation (2.5), $x=(x_k)\in[\,c,\mathcal{M},p,\Lambda,||\cdot,\cdots,\cdot||]_0^{\theta}$, but by equation (2.3), $x=(x_k)\notin[c,p,\Lambda,||\cdot,\cdots,\cdot||\,]_{\infty}^{\theta}$, which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold and suppose that $x=(x_k)\in[\,c,\mathcal{M},p,\Lambda,||\cdot,\cdots,\cdot||\,]_0^{\theta}$, i.e,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}=0,\ \ \text{for some }\ \rho>0. \tag{2.6}$$

Again, suppose that $x=(x_k)\notin[\,c,p,\Lambda,||\cdot,\cdots,\cdot||\,]_0^{\theta}$. Then, for some number $\epsilon>0$ and a subinterval $I_{r(j)}$ of the set of interval I_r, we have $||\Lambda_k(x),z_1,\cdots,z_{n-1}||\geq\epsilon$ for all $k\in\mathbb{N}$ and some $s\geq s_0$. Then, from the properties of the Orlicz function, we can write

$$M_k\Big(||\frac{\Lambda_k(x)}{\rho},z_1,\cdots,z_{n-1}||\Big)_k^{p}\geq M_k\Big(\frac{\epsilon}{\rho}\Big)^{p_k}$$

and consequently by (2.6)

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\frac{\epsilon}{\rho}\Big)\Big]^{p_k}=0,$$

which contradicts (iii). Hence (i) must hold. $\square$

Theorem 2.7. *Let* $0<p_k\leq q_k$ *for all* $k\in N$ *and* $\Big(\frac{q_k}{p_k}\Big)$ *be bounded. Then,* $[\,c,M,q,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}\subset[\,c,M,p,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}$.

Proof. Let $x\in[\,c,\mathcal{M},q,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}$. Write

$$t_k=\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{q_k}$$

and $\mu_k=\frac{p_k}{q_k}$ for all $k\in\mathbb{N}$. Then $0<\mu_k\leq 1\ \ for\ \ k\in\mathbb{N}$. Take $0<\mu<\mu_k$ for $k\in\mathbb{N}$. Define the sequences (u_k) and (v_k) as follows: For $t_k\geq 1$, let $u_k=t_k$ and $v_k=0$ and for $t_k<1$, let $u_k=0$ and $v_k=t_k$. Then clearly for all $k\in\mathbb{N}$, we have

$$t_k=u_k+v_k,\qquad t_k^{\mu_k}=u_k^{\mu_k}+v_k^{\mu_k}$$

Now it follows that $u_k^{\mu_k}\leq u_k\leq t_k$ and $v_k^{\mu_k}\leq v_k^{\mu}$. Therefore,

$$\frac{1}{h_r}\sum_{k\in I_r}t_k^{\mu_k}=\frac{1}{g_h}\sum_{k\in I_r}(u_k^{\mu_k}+v_k^{\mu_k})\leq\frac{1}{h_r}\sum_{k\in I_r}t_k+\frac{1}{h_r}\sum_{k\in I_r}v_k^{\mu}.$$

Now for each k,

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} v_k^{\mu} &= \sum_{k\in I_r}\Big(\frac{1}{h_r}v_k\Big)^{\mu}\Big(\frac{1}{h_r}\Big)^{1-\mu}\\ &\le \Big(\sum_{k\in I_r}\Big[\Big(\frac{1}{h_r}v_k\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu}\Big(\sum_{k\in I_r}\Big[\Big(\frac{1}{h_r}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu}\\ &= \Big(\frac{1}{h_r}\sum_{k\in I_r} v_k\Big)^{\mu}\end{aligned}$$

and so

$$\frac{1}{h_r}\sum_{k\in I_r} t_k^{\mu_k} \le \frac{1}{h_r}\sum_{k\in I_r} t_k + \Big(\frac{1}{h_r}\sum_{k\in I_r} v_k\Big)^{\mu}.$$

Hence $x \in [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}$. □

Theorem 2.8. *(a) If* $0 < \inf p_k \le p_k \le 1$ *for all* $k \in N$, *then*

$$[\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta} \subset [\, c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}.$$

(b) If $1 \le p_k \le \sup p_k < \infty$ *for all* $k \in N$. *Then*

$$[\, c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta} \subset [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}.$$

Proof. (a) Let $x \in [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}$, then

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0.$$

Since $0 < \inf p_k \le p_k \le 1$. This implies that

$$\begin{aligned}&\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]\\ &\qquad \le \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k},\end{aligned}$$

therefore, $\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0.$

This shows that $x \in [\, c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}\cdot$ Therefore,

$$[\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta} \subset [\, c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}.$$

This completes the proof.

(b) Let $p_k \ge 1$ for each k and $\sup p_k < \infty$. Let $x \in [\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}$. Then for each $\epsilon > 0$ there exists a positive integer N such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0 < 1.$$

Since $1 \le p_k \le \sup p_k < \infty$, we have

$$
\begin{aligned}
\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}&\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\le \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]\\
&=0\\
&<1.
\end{aligned}
$$

Therefore $x\in[\ c,\mathcal{M},p,\Lambda,||\cdot,\cdots,\cdot||\ ]^{\theta}$. $\square$

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AN INEQUALITY OF GRÜSS LIKE VIA VARIANT OF POMPEIU'S MEAN VALUE THEOREM

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Abstract. The main of this paper is to establish an integral inequality of Grüss type by using a mean value theorem.

1. Introduction

In 1935, G. Grüss [4] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right| \le \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma),$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$\varphi \le f(x) \le \Phi \text{ and } \gamma \le g(x) \le \Gamma \text{ for all } x\in[a,b].$$

The constant $\frac{1}{4}$ is best possible.

In 1882, P. L. Čebyšev [2] gave the following inequality:

$$|T(f,g)| \le \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty ,$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = ess\sup_{t\in[a,b]}|p(t)|$.

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For a differentiable function $f:[a,b]\to\mathbb{R},\ a{\cdot}b>0$, Pachpatte has in [6] proved, using Pompeiu's mean value theorem [9], the following Grüss type inequality:

$$\left|\int_a^b f(t)g(t)dt-\frac{1}{b^2-a^2}\left(\int_a^b f(t)dt.\int_a^b tg(t)dt+\int_a^b g(t)dt.\int_a^b tf(t)dt\right)\right|$$
$$\leq\ \|f-\ell f'\|_\infty\int_a^b |g(t)|\left|\frac{1}{2}-\frac{t}{a+b}\right|dt+\|g-\ell g'\|_\infty\int_a^b |f(t)|\left|\frac{1}{2}-\frac{t}{a+b}\right|dt$$

where $\ell(t)=t,\ t\in[a,b]$.

In [7], Pecaric and Ungar proved a general estimate with the p-norm, $1<p<\infty$, which will for $p=\infty$ give the Pachpatte [6] result.

The interested reader is also referred to ([1], [3], [5]-[11]) for integral inequalities by using Pompeiu's mean value theorem. In this paper, we establish a new integral inequality of Grüss like via Pompeiu's mean value theorem.

2. Main Results

Before starting the main results, we will give the following lemma proved by Pecaric and Ungar in [7]:

Lemma 2.1. *For $\frac{1}{p}+\frac{1}{q}=1,\ 1\leq p,q\leq\infty$, and $0<a\leq x\leq b$, denote*

$$A(x,q):=\left(\int_a^x\left(\int_t^x\frac{t^q du}{u^{2q}}\right)dt\right)^{\frac{1}{q}}+\left(\int_x^b\left(\int_x^t\frac{t^q du}{u^{2q}}\right)dt\right)^{\frac{1}{q}}\tag{2.1}$$

where for $p=1$, i.e. $q=\infty$, the integrals are to be interpreted as the ∞-norms, i.e. as maxima of the function $(u,t)\mapsto\frac{1}{u^2}$ on the corresponding domains of integration. Then,

$$\begin{aligned}A(x,q)\ &=\ \left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}\\&\quad+\left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}},\end{aligned}$$

for $1<p,q<\infty,\ p,q\neq 2$;

$$A(x,2)=\frac{1}{3}\left[\left(\ln\left(\frac{x}{a}\right)^3+\frac{a^3}{x^3}-1\right)^{\frac{1}{2}}+\left(\ln\left(\frac{x}{b}\right)^3+\frac{b^3}{x^3}-1\right)^{\frac{1}{2}}\right]=\lim_{q\to 2}A(x,q);$$

$$A(x,\infty)=\frac{a^2+b^2}{2x}+x-a-b=\lim_{q\to\infty}A(x,q);$$

$$A(x,1)=\frac{1}{a}+\frac{b}{x^2}=\lim_{q\to 1}A(x,1).$$

To prove our theorems, we need the following lemma proved by Sarikaya in [12]:

Lemma 2.2. *$f:[a,b]\to\mathbb{R}$ be continuous function on $[a,b]$ and twice order differentiable function on (a,b) with $0<a<b$. Then for any $t,x\in[a,b]$, we have*

$$tf(x)-xf(t)+xt\frac{f'(t)-f'(x)}{2}=\frac{xt}{2}\int_x^t\left[2f(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}du. \tag{2.2}$$

Theorem 2.1. *$f,g:[a,b]\to\mathbb{R}$ be continuous function on $[a,b]$ and twice order differentiable function on (a,b) with $0<a<b$. Then for $\frac{1}{p}+\frac{1}{q}=1$, with $1<p,q<\infty$ any $t,x\in[a,b]$, we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx\right.\\
&-\left[\frac{3}{b-a}\int f(t)dt-\frac{bf(b)-af(a)}{b-a}\right]\left(\frac{2}{5\left(b^2-a^2\right)}\int_a^b xg(x)dx\right)\\
&-\left[\frac{3}{b-a}\int g(t)dt-\frac{bg(b)-ag(a)}{b-a}\right]\left(\frac{2}{5\left(b^2-a^2\right)}\int_a^b xf(x)dx\right)\\
&\left.-\frac{bf(b)g(b)-af(a)g(a)}{5(b-a)}\right|\\
\leq\ &\frac{2(b-a)^{\frac{1}{p}-2}}{5\left(b+a\right)}\left\{\left\|2f-2lf'+l^2f''\right\|_p\int_a^b xg(x)A(x,q)dx\right.\\
&\left.+\left\|2g-2lg'+l^2g''\right\|_p\int_a^b xf(x)A(x,q)dx\right\}
\end{aligned} \tag{2.3}$$

where $l(t)=t$ for $t\in[a,b]$.

Proof. Applying (2.2) to the function g, we have

$$tg(x)-xg(t)+xt\frac{g'(t)-g'(x)}{2}=\frac{xt}{2}\int_x^t\left[2g(u)-2ug'(u)+u^2g''(u)\right]\frac{1}{u^2}du. \tag{2.4}$$

Multiplying (2.2) by $g(x)$, (2.4) by $f(x)$,summing the resultant equalities, then integrating with respect to t on $[a,b]$, we have

(2.5)
$$\begin{aligned}
&\left(b^2-a^2\right) f(x)g(x) - \frac{3xg(x)}{2}\int_a^b f(t)dt - \frac{3xf(x)}{2}\int_a^b g(t)dt + \frac{xg(x)}{2}\left[bf(b)-af(a)\right] \\
&-\frac{b^2-a^2}{4}xg(x)f'(x) + \frac{xf(x)}{2}\left[bg(b)-ag(a)\right] - \frac{b^2-a^2}{4}xf(x)g'(x) \\
= \ & \frac{xg(x)}{2}\int_a^b t\left[\int_x^t \left[2f(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}du\right]dt \\
&+\frac{xf(x)}{2}\int_a^b t\left[\int_x^t \left[2g(u)-2ug'(u)+u^2g''(u)\right]\frac{1}{u^2}du\right]dt.
\end{aligned}$$

Integrating with respect to x on $[a,b]$ and adding notations $F(u) = 2f(u) - 2uf'(u) + u^2f''(u)$ and $G(u) = 2g(u) - 2ug'(u) + u^2g''(u)$, we obtain

(2.6)
$$\begin{aligned}
&\left(b^2-a^2\right)\int_a^b f(x)g(x)dx \\
&-\frac{3}{2}\left(\int_a^b xg(x)dx\right)\left(\int_a^b f(t)dt\right) - \frac{3}{2}\left(\int_a^b xf(x)dx\right)\left(\int_a^b g(t)dt\right) \\
&+\frac{bf(b)-af(a)}{2}\left(\int_a^b xg(x)dx\right) + \frac{bg(b)-ag(a)}{2}\left(\int_a^b xf(x)dx\right) \\
&-\frac{b^2-a^2}{4}\int_a^b xg(x)f'(x)dx - \frac{b^2-a^2}{4}\int_a^b xf(x)g'(x)dx \\
= \ & \frac{1}{2}\int_a^b xg(x)\left(\int_a^b t\left[\int_x^t F(u)\frac{du}{u^2}\right]dt\right)dx + \frac{1}{2}\int_a^b xf(x)\left(\int_a^b t\left[\int_x^t G(u)\frac{du}{u^2}\right]dt\right)dx.
\end{aligned}$$

$$\int_a^b xg(x)f'(x)dx = bf(b)g(b) - af(a)g(a) - \int_a^b f(x)g(x)dx - \int_a^b xf(x)g'(x)dx. \tag{2.7}$$

Adding (2.7) in (2.6), we have

$$\begin{aligned}
&\frac{5\left(b^2-a^2\right)}{4}\int_a^b f(x)g(x)dx \\
&-\left[\frac{3}{2}\int_a^b f(t)dt-\frac{bf(b)-af(a)}{2}\right]\left(\int_a^b xg(x)dx\right) \\
&-\left[\frac{3}{2}\int_a^b g(t)dt-\frac{bg(b)-ag(a)}{2}\right]\left(\int_a^b xf(x)dx\right) \\
&-\frac{\left(b^2-a^2\right)}{4}\left[bf(b)g(b)-af(a)g(a)\right] \\
= &\ \frac{1}{2}\int_a^b xg(x)\left(\int_a^b t\left[\int_x^t F(u)\frac{du}{u^2}\right]dt\right)dx+\frac{1}{2}\int_a^b xf(x)\left(\int_a^b t\left[\int_x^t G(u)\frac{du}{u^2}\right]dt\right)dx.
\end{aligned}$$

Taking modulus, we have

$$\begin{aligned}
&(2.8) \\
&\left|\frac{5\left(b^2-a^2\right)}{4}\int_a^b f(x)g(x)dx\right. \\
&-\left[\frac{3}{2}\int_a^b f(t)dt-\frac{bf(b)-af(a)}{2}\right]\left(\int_a^b xg(x)dx\right) \\
&-\left[\frac{3}{2}\int_a^b g(t)dt-\frac{bg(b)-ag(a)}{2}\right]\left(\int_a^b xf(x)dx\right) \\
&\left.-\frac{\left(b^2-a^2\right)}{4}\left[bf(b)g(b)-af(a)g(a)\right]\right| \\
\le &\ \frac{1}{2}\left|\int_a^b xg(x)\left(\int_a^b t\left[\int_x^t F(u)\frac{du}{u^2}\right]dt\right)dx\right|+\frac{1}{2}\left|\int_a^b xf(x)\left(\int_a^b t\left[\int_x^t G(u)\frac{du}{u^2}\right]dt\right)\right|dx \\
\le &\ \frac{1}{2}\int_a^b |xg(x)|\left|\int_a^b t\left[\int_x^t F(u)\frac{du}{u^2}\right]dt\right|dx+\frac{1}{2}\int_a^b |xf(x)|\left|\int_a^b t\left[\int_x^t G(u)\frac{du}{u^2}\right]dt\right|dx \\
\le &\ \frac{1}{2}\int_a^b |xg(x)|\left(\int_a^b\left|\int_x^t |F(u)|\frac{t}{u^2}du\right|dt\right)dx+\frac{1}{2}\int_a^b |xf(x)|\left(\int_a^b\left|\int_x^t |G(u)|\frac{t}{u^2}du\right|dt\right)dx.
\end{aligned}$$

In the last line (2.8), we have

$$\int_a^b\left|\int_x^t |F(u)|\frac{t}{u^2}du\right|dt=\int_a^x\int_t^x |F(u)|\frac{t}{u^2}dudt+\int_x^b\int_x^t |F(u)|\frac{t}{u^2}dudt. \tag{2.9}$$

Using Hölder's inequality in (2.9), we obtain

$$
\begin{aligned}
(2.10)\qquad & \int_a^b \left| \int_x^t |F(u)| \frac{t}{u^2} du \right| dt \\
\leq\ & \left(\int_a^x \int_t^x |F(u)|^p \, dudt \right)^{\frac{1}{p}} \left(\int_a^x \int_t^x \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} \\
& + \left(\int_x^b \int_x^t |F(u)|^p \, dudt \right)^{\frac{1}{p}} \left(\int_x^b \int_x^t \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} \\
\leq\ & \left(\int_a^b \int_a^b |F(u)|^p \, dudt \right)^{\frac{1}{p}} \left\{ \left(\int_a^x \int_t^x \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} + \left(\int_x^b \int_x^t \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} \right\} \\
=\ & (b-a)^{\frac{1}{p}} \left\| 2f - 2lf' + l^2 f'' \right\|_p A(x,q).
\end{aligned}
$$

Similarly, we get

$$
(2.11)\qquad \int_a^b \left| \int_x^t |G(u)| \frac{t}{u^2} du \right| dt \leq (b-a)^{\frac{1}{p}} \left\| 2g - 2lg' + l^2 g'' \right\|_p A(x,q).
$$

Adding (2.10) and (2.11) in (2.8), we obtain

$$
\begin{aligned}
(2.12)\qquad & \left| \frac{5\left(b^2-a^2\right)}{4} \int_a^b f(x)g(x)dx \right. \\
& - \left[\frac{3}{2} \int_a^b f(t)dt - \frac{bf(b)-af(a)}{2} \right] \left(\int_a^b xg(x)dx \right) \\
& - \left[\frac{3}{2} \int_a^b g(t)dt - \frac{bg(b)-ag(a)}{2} \right] \left(\int_a^b xf(x)dx \right) \\
& \left. - \frac{\left(b^2-a^2\right)}{4} \left[bf(b)g(b) - af(a)g(a) \right] \right| \\
\leq\ & \frac{1}{2} (b-a)^{\frac{1}{p}} \left\{ \left\| 2f - 2lf' + l^2 f'' \right\|_p \int_a^b |xg(x)| \, A(x,q)dx \right. \\
& \left. + \left\| 2g - 2lg' + l^2 g'' \right\|_p \int_a^b |xf(x)| \, A(x,q)dx \right\}.
\end{aligned}
$$

Dividing (2.12) by $\frac{5\left(b^2-a^2\right)(b-a)}{4}$, we obtain the required inequality (2.3). □

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NEW STRUCTURE TO CONSTRUCT NEW SOLITARY WAVE SOLUTIONS FOR PERTURBED NLSE WITH POWER LAW NONLINEARITY

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Abstract. In this paper we applied new structure to constructing new solitary wave solutions for perturbed nonlinear Schrodinger equation with power law nonlinearity, which describes the effects of quantic nonlinearity on the ultrashort optical solitons pulse propagation in non-Kerr media. These solitary wave solutions demonstrate the fact that solutions to the perturbed nonlinear Schrodinger equation with power law nonlinearity model can exhibit a variety of behaviors.

1. Introduction

Exact solutions can serve as a basis for perfecting and testing computer algebra software packages for solving NLEEs. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions allow researchers to design and run experiments, by creating appropriate natural conditions, to determine these parameters or functions. Therefore, investigation of exact traveling wave solutions is becoming successively attractive in nonlinear sciences day by day. However, not all equations posed of these models are solvable. Exact solutions can serve as a basis for perfecting and testing computer algebra software packages for solving NLEEs. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions allow researchers to design and run experiments, by creating appropriate natural conditions, to determine these parameters or functions. Therefore, investigation of exact traveling wave solutions is becoming successively attractive in nonlinear sciences day by day. Hence it becomes increasingly important to be familiar with all traditional and recently developed methods for solving these models and the implementation of new methods. As a result, many new techniques have been successfully developed by diverse groups of mathematicians and physicists, such as, the trigonometric function series method [5], the modified mapping

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method and the extended mapping method [6], homogeneous balance method [7], tanh function method [8], extended tanh function method [9], hyperbolic function method [10], rational expansion method [11], sine-cosine method [12].
In this present paper we applied the direct algebraic method for finding new exact solitary wave solutions of perturbed NLSE with power law nonlinearity in the following form [13],

$$iq_t + aq_{xx} + b|q|^{2m}q = icq_x - i\gamma q_{xxx} + is(|q|^{2m}q)x + ir(|q|_{2m})_x q, \tag{1.1}$$

Where a, b, c, γ, s and r are all real valued constants. Also, the exponent m represents the power law nonlinearity parameter. For the perturbation terms on the right hand side represents the inter-modal dispersion, γ is the coefficient of third order dispersion, s is the coefficient of self-steepening term while r is the coefficient of nonlinear dispersion. The self-steepening and nonlinear dispersion terms are considered with full nonlinearity, namely their intensities are considered with an exponent m, in order to maintain the problem on a generalized setting [14].

2. Our methodology

For a given partial differential equation

$$G(u, u_x, u_t, u_{xx}, u_{tt}, ...) = 0, \tag{2.1}$$

Our method mainly consists of four steps:
Step 1: We seek complex solutions of Eq. (2.1) as the following form:

$$u = u(\xi), \qquad \xi = ik(x - ct), \tag{2.2}$$

Where k and c are real constants. Under the transformation (2.2), Eq. (2.1) becomes an ordinary differential equation

$$N(u, iku', -ikcu', -k^2u'', ...) = 0, \tag{2.3}$$

Where $u' = \frac{du}{d\xi}$.
Step 2: We assume that the solution of Eq. (2.3) is of the form

$$u(\xi) = \sum_{i=0}^{n} a_i F^i(\xi), \tag{2.4}$$

Where $a_i (i = 1, 2, .., n)$ are real constants to be determined later. $F(\xi)$ expresses the solutions of the auxiliary ordinary differential equation

$$F'(\xi) = b + F^2(\xi), \tag{2.5}$$

Eq. (2.5) admits the following solutions:

$$\begin{array}{ll} F(\xi) = \begin{cases} -\sqrt{-b}\tanh(\sqrt{-b}\xi), & b \prec 0 \quad (a) \\ -\sqrt{-b}\coth(\sqrt{-b}\xi), & b \prec 0 \quad (b) \end{cases} \\ F(\xi) = \begin{cases} \sqrt{b}\tan(\sqrt{b}\xi), & b \succ 0 \quad (c) \\ -\sqrt{b}\cot(\sqrt{b}\xi), & b \succ 0 \quad (d) \end{cases} \\ F(\xi) = -\frac{1}{\xi}, \qquad b = 0 \quad (e) \end{array} \tag{2.6}$$

Integer n in (2.4) can be determined by considering direct algebraic [3] between the nonlinear terms and the highest derivatives of $u(\xi)$ in Eq. (2.3).
Step 3: Substituting (2.4) into (2.3) with (2.5), then the left hand side of Eq. (2.3) is converted into a polynomial in $F(\xi)$, equating each coefficient of the polynomial to zero yields a set of algebraic equations for a_i, k, c.

Step 4: Solving the algebraic equations obtained in step 3, and substituting the results into (2.4), then we obtain the exact traveling wave solutions for Eq. (2.1).

3. Application to the perturbed NLSE with power law nonlinearity

We assume Eq. (2.5) has the traveling wave solution of the form

$$q(x,\ t)=U(\xi)e^{i(\alpha x+\beta t)},\ \xi=i(kx-\omega t), \tag{3.1}$$

where α,β,k and $\omega$are constants, all of them are to be determined. Thus, from the wave transformation (3.1), we have

$$\begin{aligned}
&q_t=i\ \left(\beta U-\omega U'\right)e^{i(\alpha x+\beta t)},\\
&q_x=i\ \left(\alpha U+kU'\right)e^{i(\alpha x+\beta t)},\\
&q_{xx}=-\ \left(\alpha^2U+2\alpha kU'+k^2U''\right)e^{i(\alpha x+\beta t)},\\
&q_{xxx}=-i\ \left(\alpha^3U+3\alpha^2kU'+3\alpha k^2U''+k^3U'''\right)e^{i(\alpha x+\beta t)},\\
&\left(|q|^{2m}q\right)_x=i\ \left(\alpha U^{2m+1}+k\left(U^{2m+1}\right)'\right)e^{i(\alpha x+\beta t)},\\
&\left(|q|^{2m}\right)_x q=ik\ \left(U^{2m}\right)'Ue^{i(\alpha x+\beta t)},
\end{aligned} \tag{3.2}$$

Inserting the expressions (3.2) into Eq. (1.1), we obtain nonlinear ODE in the form

$$\begin{aligned}
&(c\alpha+\gamma\alpha^3-\beta-a\alpha^2)U+(\omega-2a\alpha k+ck+3\alpha^2k\gamma)U'+(3\alpha k^2\gamma-ak^2)U''\\
&+(b+s\alpha)U^{2m+1}+k^3\gamma U'''+sk(U^{2m+1})'+rk(U^{2m})'U=0.
\end{aligned} \tag{3.3}$$

Balancing U''' with $U'U^{2m}$in Eq. (3.3) give

$$N+3=N+1+2mN\Leftrightarrow 3=2mN+1\Leftrightarrow N=\frac{1}{m}.$$

We then assume that Eq. (3.3) has the following formal solutions:

$$U(\xi)=AF^{\frac{1}{m}},\ A\neq 0 \tag{3.4}$$

Substituting Eq (3.4) into Eq. (3.3) and collecting all terms with the same order of F^j together, we convert the left-hand side of Eq. (3.3) into a polynomial in F^j. Setting each coefficient of each polynomial to zero, we derive a set of algebraic equations forα,β,k,ω and A. By solving these algebraic equations we have

$$\begin{aligned}
&A=\Big[-\tfrac{sn(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\gamma}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^2}\times\\
&\left(\tfrac{1}{n^2}+\tfrac{3}{n}+2\right)-\tfrac{s}{2r}\left(\tfrac{1}{n}+2\right)\Big]^{\frac{1}{2n}},\\
&\alpha=\tfrac{a}{3\gamma},\beta=\tfrac{9ca\gamma-2a^3}{27\gamma^2},\\
&k=\pm\tfrac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}.
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\omega=\tfrac{a^2}{3\gamma^2}\tfrac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}-\\
&\tfrac{c\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}-\\
&\tfrac{b\left(2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\right)^{\frac{3}{2}}n^3}{\gamma^2(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^3}\times\\
&\left(\tfrac{1}{n}-1\right)\left(\tfrac{1}{n}-2\right)-\\
&\tfrac{2bn\left(2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\right)^{\frac{3}{2}}}{\gamma^2(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^3},
\end{aligned} \tag{3.6}$$

From Eq. (2.6)(a) and relations (3.5) , (3.6) along with (3.4) we have

$$U(\xi)=\Big[-\frac{sn(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\gamma}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^2}\times \left(\frac{1}{n^2}+\frac{3}{n}+2\right)-\frac{s}{2r}\left(\frac{1}{n}+2\right)\Big]^{\frac{1}{2n}}\left(-\sqrt{-b}\tanh(\sqrt{-b}\xi)\right)^{\frac{1}{n}},$$

So from (3.1) we have solitary wave solutions of Eq. (1.1) as follows

$$q_1(x,\ t)=\Big[-\frac{sn(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\gamma}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^2}\times \left(\frac{1}{n^2}+\frac{3}{n}+2\right)-\frac{s}{2r}\left(\frac{1}{n}+2\right)\Big]^{\frac{1}{2n}}\times\Big[-\sqrt{-b}\tanh(\sqrt{-b}i(\frac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}x- (\frac{a^2}{3\gamma^2}\frac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}- \frac{c\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}- \frac{b\left(2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\right)^{\frac{3}{2}}n^3}{\gamma^2(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^3}\times \left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)- \frac{2bn\left(2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\right)^{\frac{3}{2}}}{\gamma^2(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^3})t\Big]^{\frac{1}{n}}\times exp\left(i(\frac{a}{3\gamma}x+\frac{9ca\gamma-2a^3}{27\gamma^2}t)\right),$$

From (2.6)(b) and relations (3.5) and (3.6) along with (3.1) and (3.4) we obtaion solitary wave solutions of Eq. (1.1) in following form

$$q_2(x,\ t)=\Big[-\frac{sn(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\gamma}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^2}\times \left(\frac{1}{n^2}+\frac{3}{n}+2\right)-\frac{s}{2r}\left(\frac{1}{n}+2\right)\Big]^{\frac{1}{2n}}\times\Big[-\sqrt{-b}\coth(\sqrt{-b}i(\frac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}x- (\frac{a^2}{3\gamma^2}\frac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}- \frac{c\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}- \frac{b\left(2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\right)^{\frac{3}{2}}n^3}{\gamma^2(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^3}\times \left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)- \frac{2bn\left(2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\right)^{\frac{3}{2}}}{\gamma^2(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^3})t\Big]^{\frac{1}{n}}\times exp\left(i(\frac{a}{3\gamma}x+\frac{9ca\gamma-2a^3}{27\gamma^2}t)\right),$$

From (2.6)(c) and relations (3.1),(3.4),(3.5) and (3.6) we obtain solitary wave solutions for nonlinear Schrodinger equation with power law nonlinearity

$$q_3(x,\ t)=\Big[-\tfrac{sn(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\gamma}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^2}\times \left(\tfrac{1}{n^2}+\tfrac{3}{n}+2\right)-\tfrac{s}{2r}\left(\tfrac{1}{n}+2\right)\Big]^{\frac{1}{2n}}\times\Big[\sqrt{b}\tan(\sqrt{b}i(\tfrac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}x- (\tfrac{a^2}{3\gamma^2}\tfrac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}- \tfrac{c\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}- \tfrac{b\left(2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\right)^{\frac{3}{2}}n^3}{\gamma^2(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^3}\times \left(\tfrac{1}{n}-1\right)\left(\tfrac{1}{n}-2\right)- \tfrac{2bn\left(2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\right)^{\frac{3}{2}}}{\gamma^2(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^3})t\Big]^{\frac{1}{n}}\times exp\left(i(\tfrac{a}{3\gamma}x+\tfrac{9ca\gamma-2a^3}{27\gamma^2}t)\right),$$

In this case we obtain solitary wave solution for (1.1) from (2.6)(d) and relations (3.1)-(3.6) as follow

$$q_4(x,\ t)=\Big[-\tfrac{sn(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\gamma}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^2}\times \left(\tfrac{1}{n^2}+\tfrac{3}{n}+2\right)-\tfrac{s}{2r}\left(\tfrac{1}{n}+2\right)\Big]^{\frac{1}{2n}}\times\Big[-\sqrt{b}\cot(\sqrt{b}i(\tfrac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}x- (\tfrac{a^2}{3\gamma^2}\tfrac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}- \tfrac{c\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}- \tfrac{b\left(2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\right)^{\frac{3}{2}}n^3}{\gamma^2(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^3}\times \left(\tfrac{1}{n}-1\right)\left(\tfrac{1}{n}-2\right)- \tfrac{2bn\left(2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\right)^{\frac{3}{2}}}{\gamma^2(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^3})t\Big]^{\frac{1}{n}}\times exp\left(i(\tfrac{a}{3\gamma}x+\tfrac{9ca\gamma-2a^3}{27\gamma^2}t)\right),$$

Finally from (2.6)(e) we obtain solitary wave solutions for perturbed NLSE with power law nonlinearity in following form

$$q_5(x,\ t)=\Big[-\tfrac{sn(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)\gamma}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)^2}\times \left(\tfrac{1}{n^2}+\tfrac{3}{n}+2\right)-\tfrac{s}{2r}\left(\tfrac{1}{n}+2\right)\Big]^{\frac{1}{2n}}\times \sqrt[n]{i}\Big[\tfrac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}\ x- (\tfrac{a^2}{3\gamma^2}\tfrac{\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)}- \tfrac{c\sqrt{2\gamma rs(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)(4sn+4sn^2+2rn+s+4rn^2)n}}{\gamma(s+5sn+8rn+8sn^2+12rn^2+4n^3s+8n^3r)})t\Big]^{-\frac{1}{n}}\times exp\left(i(\tfrac{a}{3\gamma}x+\tfrac{9ca\gamma-2a^3}{27\gamma^2}t)\right),$$

4. Conclusion

In summary we derive many types of optical solitary wave solutions of perturbed nonlinear Schrodinger equation with power law nonlinearity which include the bright and dark optical solitary wave solutions. The results show that the method is reliable and effective and gives more solutions. We hope that the obtained results will be useful for further studies in mathematical physics and engineering.

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ON SOME NEW INEQUALITIES OF HERMITE-HADAMARD TYPE INVOLVING HARMONICALLY CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, some new results related to the left-hand side of the Hermite-Hadamard type inequality for harmonically convex functions using Riemann Liouville fractional integrals are obtained.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In [4], the author introduced the class of harmonically convex functions, defined as follows.

Definition 1.1. Let $I \subseteq \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.1) is reversed, then f is said to be harmonically concave.

We recall the following special functions and inequality

(1) The Beta function:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x, y > 0,$$

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(2) The hypergeometric function:

$$ {}_2F_1\left(a,b;c;z\right)=\frac{1}{\beta\left(b,c-b\right)}\int\limits_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-zt\right)^{-a}dt,\ c>b>0,\ |z|<1. $$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult ([1],[2],[7]).

Definition 1.2. Let $f\in L\left[a,b\right]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of oder $\alpha>0$ with $a\geq 0$ are defined by

$$ J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int\limits_a^x\left(x-t\right)^{\alpha-1}f(t)dt,\ x>a $$

and

$$ J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int\limits_x^b\left(t-x\right)^{\alpha-1}f(t)dt,\ x<b $$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha)=\int\limits_0^{\infty}e^{-t}t^{\alpha-1}dt$ and $J_{a^+}^{0}f(x)=J_{b^-}^{0}f(x)=f(x)$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see ([5],[6],[9],[10],[11]).

In [3], Iscan proved a variant of Hermite-Hadamard inequality which holds for the harmonically convex functions in fractional integral forms as follows:

Theorem 1.1. *Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a function such that $f\in L[a,b]$, where $a,b\in I$ with $a<b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$ \begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J_{1/a-}^{\alpha}\left(f\circ g\right)\left(1/b\right)+J_{1/b+}^{\alpha}\left(f\circ g\right)\left(1/a\right)\right\} \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned} $$

with $\alpha>0$.

Lemma 1.1. *([8],[12]) For $0<\alpha\leq 1$ and $0\leq a<b$, we have*

$$ \left|a^{\alpha}-b^{\alpha}\right|\leq\left(b-a\right)^{\alpha}. $$

Lemma 1.2. *([3]) Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. Then the following equality for fractional integrals holds:*

$$ \begin{aligned} & I_f\left(g;\alpha,a,b\right) \\ =\ & \frac{ab\left(b-a\right)}{2}\int\limits_0^1\frac{\left[t^{\alpha}-(1-t)^{\alpha}\right]}{\left(ta+(1-t)b\right)^2}f'\left(\frac{ab}{ta+(1-t)b}\right)dt. \end{aligned} $$

where

$$\begin{aligned} & I_f\left(g;\alpha,a,b\right) \\ = & \frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J_{1/a-}^{\alpha}\left(f\circ g\right)(1/b)+J_{1/b+}^{\alpha}\left(f\circ g\right)(1/a)\right\}. \end{aligned}$$

with $\alpha>0, g\left(x\right)=1/x$ *and* Γ *is Euler Gamma function.*

In [3], Iscan proved the following theorems using the above Lemma 1.1 and Lemma 1.2.

Theorem 1.2. *Let* $f:I\subseteq(0,\infty)\rightarrow\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$*, where* $a,b\in I^{\circ}$ *with* $a<b$*. If* $|f'|^{q}$ *is harmonically convex on* $[a,b]$ *for some fixed* $q\geq 1$*, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left|I_f\left(g;\alpha,a,b\right)\right| \\ \leq & \frac{ab\left(b-a\right)}{2}C_1^{1-1/q}(\alpha;a,b)\left(C_2(\alpha;a,b)\left|f'(b)\right|^{q}+C_3(\alpha;a,b)\left|f'(a)\right|^{q}\right)^{1/q} \end{aligned}$$

where

$$\begin{aligned} C_1(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)+\ {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\right], \\ C_2(\alpha;a,b) &= \frac{b^{-2}}{\alpha+2}\left[\frac{{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)}{\alpha+1}+\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right)\right], \\ C_3(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right)+\frac{{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right)}{\alpha+1}\right]. \end{aligned}$$

Theorem 1.3. *Let* $f:I\subseteq(0,\infty)\rightarrow\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$*, where* $a,b\in I^{\circ}$ *with* $a<b$*. If* $|f'|^{q}$ *is harmonically convex on* $[a,b]$ *for some fixed* $q\geq 1$*, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left|I_f\left(g;\alpha,a,b\right)\right| \\ \leq & \frac{ab\left(b-a\right)}{2}C_1^{1-1/q}(\alpha;a,b)\left(C_2(\alpha;a,b)\left|f'(b)\right|^{q}+C_3(\alpha;a,b)\left|f'(a)\right|^{q}\right)^{1/q}, \end{aligned}$$

where

$$\begin{aligned} & C_1\left(\alpha;a,b\right) \\ = & \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)-{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)\right. \\ & \left.+{}_2F_1\left(2,1;\alpha+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right], \end{aligned}$$

$$\begin{aligned} & C_2\left(\alpha;a,b\right) \\ = & \frac{b^{-2}}{\alpha+2}\left[{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right)-\frac{1}{\alpha+1}\ {}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)\right. \\ & \left.+\frac{1}{2\left(\alpha+1\right)}\ {}_2F_1\left(2,2;\alpha+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right], \end{aligned}$$

$$
\begin{aligned}
& C_3\left(\alpha; a, b\right) \\
= \; & \frac{b^{-2}}{\alpha+2}\left[\frac{1}{\alpha+1}\,{}_2F_1\left(2, \alpha+1; \alpha+3; 1-\frac{a}{b}\right) -{}_2 F_1\left(2, 1; \alpha+3; 1-\frac{a}{b}\right)\right. \\
& \left. +{}_2F_1\left(2, 1; \alpha+3; \frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right]
\end{aligned}
$$

and $0<\alpha\leq 1$.

Theorem 1.4. *Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for some fixed $q>1$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
& \left|I_f\left(g; \alpha, a, b\right)\right| \\
\leq \; & \frac{a\left(b-a\right)}{2b}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left(\frac{|f'(b)|^q+|f'(a)|^q}{2}\right)^{1/q} \\
& \times\left[{}_2F_1^{1/p}\left(2p, 1; \alpha p+2; 1-\frac{a}{b}\right)+{}_2\,F_1^{1/p}\left(2p, \alpha p+1; \alpha p+2; 1-\frac{a}{b}\right)\right],
\end{aligned}
$$

where $1/p+1/q=1$.

Theorem 1.5. *Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for some fixed $q>1$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
& \left|I_f\left(g; \alpha, a, b\right)\right| \\
\leq \; & \frac{b-a}{2\left(ab\right)^{1-1/p}}L_{2p-2}^{2-2/p}(a,b)\left(\frac{1}{\alpha q+1}\right)^{1/q}\left(\frac{|f'(b)|^q+|f'(a)|^q}{2}\right)^{1/q},
\end{aligned}
$$

where $1/p+1/q=1$ and $L_{2p-2}(a,b)=\left(\frac{b^{2p-1}-a^{2p-1}}{(2p-1)(b-a)}\right)^{1/(2p-2)}$ is $2p-2$-Logarithmic mean.

Theorem 1.6. *Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for some fixed $q>1$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
& \left|I_f\left(g; \alpha, a, b\right)\right| \\
\leq \; & \frac{a\left(b-a\right)}{2b}\left(\frac{1}{\alpha p+1}\right)^{1/p} \\
& \times\left(\frac{{}_2F_1\left(2q, 2; 3; 1-\frac{a}{b}\right)|f'(b)|^q+{}_2\,F_1\left(2q, 1; 3; 1-\frac{a}{b}\right)|f'(a)|^q}{2}\right)^{1/q},
\end{aligned}
$$

where $1/p+1/q=1$.

In this paper, new identity for fractional integrals have been defined. By using of this identity we obtained some new results related to the left-hand side of the Hermite-Hadamard type inequality for harmonically convex functions via Riemann Liouville fractional integral.

2. Main Results

Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I. Throughout this section we will take

$$\begin{aligned} & K_f(g;\alpha,a,b) \\ = \; & f\left(\frac{2ab}{a+b}\right) - \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}-}\left(f\circ g\right)(1/b) + J^{\alpha}_{\frac{1}{b}+}\left(f\circ g\right)(1/a)\right\} \end{aligned}$$

where $a,b \in I$, with $a<b,\ \alpha>0,\ g(x)=\frac{1}{x}$ and Γ is Euler Gamma function.

In order to prove our main results, we need the following Lemma:

Lemma 2.1. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. Then the following equality for fractional integral holds :*

$$\mathcal{K}_f\left(g;\alpha,a,b\right) = \frac{1}{2}\sum_{k=1}^{3} I_k \tag{2.1}$$

where

$$\begin{aligned} I_1 &= ab\left(b-a\right)\int_0^{1/2} f'\left(\tfrac{ab}{A_t}\right)\tfrac{dt}{A_t^2} \\ I_2 &= -ab\left(b-a\right)\int_{1/2}^{1} f'\left(\tfrac{ab}{A_t}\right)\tfrac{dt}{A_t^2} \\ I_3 &= -ab\left(b-a\right)\int_0^1 \left[(1-t)^{\alpha}-t^{\alpha}\right] f'\left(\tfrac{ab}{A_t}\right)\tfrac{dt}{A_t^2} \end{aligned}$$

and $A_t = ta+(1-t)\,b$.

Proof. Calculating I_1, I_2 and I_3 we have

$$\begin{aligned} I_1 &= ab\left(b-a\right)\int_0^{1/2} f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2} \\ &= \int_0^{1/2} df\left(\frac{ab}{A_t}\right) \\ &= f\left(\frac{ab}{A_t}\right)\Bigg|_0^{1/2} \\ &= f\left(\frac{2ab}{a+b}\right) - f\left(a\right), \end{aligned}$$

$$\begin{aligned}
I_2 &= -ab\,(b-a)\int_{1/2}^{1} f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2} \\
&= -\int_{1/2}^{1} df\left(\frac{ab}{A_t}\right) \\
&= -f\left(\frac{ab}{A_t}\right)\Big|_{1/2}^{1} \\
&= f\left(\frac{2ab}{a+b}\right) - f\,(b)
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= -ab\,(b-a)\int_{0}^{1} \left[(1-t)^{\alpha} - t^{\alpha}\right] f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2} \\
&= -\int_{0}^{1} \left[(1-t)^{\alpha} - t^{\alpha}\right] df\left(\frac{ab}{A_t}\right) \\
&= -\int_{0}^{1} (1-t)^{\alpha}\, df\left(\frac{ab}{A_t}\right) + \int_{0}^{1} t^{\alpha} df\left(\frac{ab}{A_t}\right) \\
&= I_3^{*} + I_3^{**}.
\end{aligned}$$

By integrating by part in I_3^{*}, we get

$$\begin{aligned}
I_3^{*} &= -\int_{0}^{1} (1-t)^{\alpha}\, df\left(\frac{ab}{A_t}\right) \\
&= (1-t)^{\alpha} f\left(\frac{ab}{A_t}\right)\Big|_{0}^{1} - \alpha\int_{0}^{1} (1-t)^{\alpha-1} f\left(\frac{ab}{A_t}\right) dt \\
&= f(a) - \alpha\int_{0}^{1} (1-t)^{\alpha-1} f\left(\frac{ab}{A_t}\right) dt.
\end{aligned}$$

Here, by the changes of variables $u = \frac{ab}{A_t}$, we get

$$\begin{aligned}
I_3^{*} &= f(a) - \alpha\int_{a}^{b} \left(\frac{ab}{b-a}\right)^{\alpha-1}\left(\frac{1}{u}-\frac{1}{b}\right)^{\alpha-1} f(u)\frac{ab}{b-a}\frac{1}{u^2}du \\
&= f(a) - \alpha\left(\frac{ab}{b-a}\right)^{\alpha}\int_{a}^{b} \left(\frac{1}{u}-\frac{1}{b}\right)^{\alpha-1}\frac{1}{u^2}f(u)du
\end{aligned}$$

and again by the changes of variables $u = \frac{1}{t}$, we get

$$\begin{aligned} I_3^* &= f(a) - \alpha \left(\frac{ab}{b-a}\right)^{\alpha} \int_{1/a}^{1/b} \left(t - \frac{1}{b}\right)^{\alpha-1} t^2 \left(-\frac{1}{t^2}\right) f(\frac{1}{t})dt \\ &= f(a) + \alpha \left(\frac{ab}{b-a}\right)^{\alpha} \int_{1/a}^{1/b} \left(t - \frac{1}{b}\right)^{\alpha-1} f \circ g(t)dt \\ &= f(a) - \alpha \left(\frac{ab}{b-a}\right)^{\alpha} \Gamma(\alpha)\frac{1}{\Gamma(\alpha)} \int_{1/b}^{1/a} \left(t - \frac{1}{b}\right)^{\alpha-1} f \circ g(t)dt \\ &= f(a) - \Gamma(\alpha+1) \left(\frac{ab}{b-a}\right)^{\alpha} J_{1/a-}^{\alpha} f \circ g(1/b). \end{aligned}$$

Similarly, we get $I_3^{**} = \int\limits_0^1 t^{\alpha} df\left(\frac{ab}{A_t}\right) = f(b) - \Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J_{1/b+}^{\alpha} f \circ g(1/a)$. By adding I_1, I_2 and I_3, the desired result is obtained. □

Using this Lemma, we can obtain the following inequalities.

Theorem 2.1. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ a differentiable increasing function on I° such that $f' \in L[a,b]$, where $a, b \in I^{\circ}$ with $a < b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\ \leq\ & \frac{f(b)-f(a)}{2} + \frac{ab\left(b-a\right)}{2} C_1^{1-1/q}(\alpha;a,b) \\ & \times \left[C_2(\alpha;a,b)\left(f'(b)\right)^q + C_3(\alpha;a,b)\left(f'(a)\right)^q\right]^{1/q}, \end{aligned}$$

where

$$\begin{aligned} C_1(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) + {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\right], \\ C_2(\alpha;a,b) &= \frac{b^{-2}}{\alpha+2}\left[\frac{{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)}{\alpha+1} + {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right)\right], \\ C_3(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) + \frac{{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right)}{\alpha+1}\right]. \end{aligned}$$

Proof. Let $A_t = ta + (1-t)\,b$. From Lemma 2.1, using the property of the modulus , the power mean inequality and harmonically convexity of $(f')^q$, we find

$$\begin{aligned} \left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| &\leq \frac{1}{2}\left\{|I_1| + |I_2| + |I_3|\right\} \\ &= \frac{ab(b-a)}{2}\left[\int\limits_0^{1/2} f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2} + \int\limits_{1/2}^{1} f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2}\right] + \frac{1}{2}\left|I_3\right| \\ &= \frac{f(b)-f(a)}{2} + \frac{1}{2}\left|I_3\right|. \end{aligned}$$

□

As in the proof of the Theorem 1.2, we have

$$\begin{aligned}
\frac{1}{2}|I_3| &= \frac{ab(b-a)}{2}\left|\int_0^1 [(1-t)^{\alpha}-t^{\alpha}]\, f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2}\right| \\
&\le \frac{ab\,(b-a)}{2}C_1^{1-1/q}(\alpha;a,b)\left(C_2(\alpha;a,b)\,(f'(b))^q + C_3(\alpha;a,b)\,(f'(a))^q\right)^{1/q},
\end{aligned}$$

where

$$\begin{aligned}
C_1(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) +{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\right],\\
C_2(\alpha;a,b) &= \frac{b^{-2}}{\alpha+2}\left[\frac{{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)}{\alpha+1} +{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right)\right],\\
C_3(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) + \frac{{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right)}{\alpha+1}\right].
\end{aligned}$$

The proof is competed.

Theorem 2.2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable increasing function on I° such that $f' \in L[a,b]$, where $a,b \in I^{\circ}$ with $a < b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q \ge 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\
\le\ & \frac{f(b)-f(a)}{2} + \frac{ab\,(b-a)}{2}C_1^{1-1/q}(\alpha;a,b) \\
&\times\left[C_2(\alpha;a,b)(f'(b))^q + C_3(\alpha;a,b)(f'(a))^q\right]^{1/q},
\end{aligned}$$

where

$$\begin{aligned}
&C_1(\alpha;a,b)\\
=\ & \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right) -{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)\right.\\
&\left.+{}_2F_1\left(2,1;\alpha+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right],
\end{aligned}$$

$$\begin{aligned}
&C_2(\alpha;a,b)\\
=\ & \frac{b^{-2}}{\alpha+2}\left[{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right) - \frac{1}{\alpha+1}\,{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)\right.\\
&\left.+\frac{1}{2\,(\alpha+1)}\,{}_2F_1\left(2,2;\alpha+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right],
\end{aligned}$$

$$\begin{aligned}
&C_3(\alpha;a,b)\\
=\ & \frac{b^{-2}}{\alpha+2}\left[\frac{1}{\alpha+1}\,{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right) -{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right)\right.\\
&\left.+{}_2F_1\left(2,1;\alpha+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right]
\end{aligned}$$

and $0 < \alpha \le 1$.

Proof. Let $A_t = ta + (1-t)\,b$. From Lemma 2.1, using the property of the modulus , the power mean inequality and the harmonically convexity of $(f')^q$, we find

$$\begin{aligned} |\mathcal{K}_f\,(g;\alpha,a,b)| &\leq \frac{1}{2}\left(|I_1|+|I_2|+|I_3|\right) \\ &\leq \frac{f(b)-f(a)}{2}+\frac{1}{2}\,|I_3| \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}\,|I_3| \\ \leq\;& \frac{ab\,(b-a)}{2}\int\limits_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt \\ \leq\;& \frac{ab\,(b-a)}{2}\left(\int\limits_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} dt\right)^{1-\frac{1}{q}} \left(\int\limits_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2}\left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{\frac{1}{q}} \\ \leq\;& \frac{ab\,(b-a)}{2}K_1^{1-1/q}\left(\int\limits_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2}\left[t(f'(b))^q+(1-t)(f'(a))^q\right] dt\right)^{1/q} \\ \leq\;& \frac{ab\,(b-a)}{2}K_1^{1-1/q}\left(K_2(f'(b))^q+K_3(f'(a))^q\right)^{1/q}, \end{aligned} \tag{2.2}$$

where

$$K_1 = \int\limits_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} dt,$$

$$K_2 = \int\limits_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} t dt,$$

$$K_3 = \int\limits_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2}\,(1-t)\,dt.$$

If K_1, K_2 and K_3 are calculated as in the proof of the Theorem 1.3, and used in the inequality (2.2), the desired result is obtained.

Theorem 2.3. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable increasing function on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &|\mathcal{K}_f\,(g;\alpha,a,b)| \\ \leq\;& \frac{f(b)-f(a)}{2}+\frac{a\,(b-a)}{2b}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left(\frac{(f'(b))^q+(f'(a))^q}{2}\right)^{1/q} \\ &\times\left[{}_2F_1^{1/p}\left(2p,1;\alpha p+2;1-\frac{a}{b}\right)+{}_2F_1^{1/p}\left(2p,\alpha p+1;\alpha p+2;1-\frac{a}{b}\right)\right], \end{aligned} \tag{2.3}$$

where $1/p+1/q=1$.

□

Proof. Let $A_t = ta+(1-t)\,b$. From Lemma 2.1, using the Hölder inequality and the harmonically convexity of $(f')^q$, we get

$$\begin{aligned}
&\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}\left\{\int_0^1 \frac{(1-t)^{\alpha}}{A_t^2} f'\left(\frac{ab}{A_t}\right)dt+\int_0^1 \frac{t^{\alpha}}{A_t^2} f'\left(\frac{ab}{A_t}\right)dt\right\} \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}\left\{\left(\int_0^1 \frac{(1-t)^{\alpha p}}{A_t^{2p}}dt\right)^{1/p}\left(\int_0^1\left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q}\right. \\
&\left.+\left(\int_0^1 \frac{t^{\alpha p}}{A_t^{2p}}dt\right)^{1/p}\left(\int_0^1\left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q}\right\} \qquad (2.4)\\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}\left(K_4^{1/p}+K_5^{1/p}\right)\left(\int_0^1\left[t(f'(b))^q+(1-t)(f'(a))^q\right]dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}\left(K_4^{1/p}+K_5^{1/p}\right)\left(\frac{f'(b)^q+f'(a)^q}{2}\right)^{1/q}.
\end{aligned}$$

As in the proof of Theorem 1.4, calculating K_4 and K_5, we have

$$\begin{aligned}
K_4 &= \int_0^1 \frac{(1-t)^{\alpha p}}{A_t^{2p}}dt \\
&= \frac{b^{-2p}}{\alpha p+1}\cdot{}_2F_1\left(2p,1;\alpha p+2;1-\frac{a}{b}\right),
\end{aligned} \qquad (2.5)$$

$$\begin{aligned}
K_5 &= \int_0^1 \frac{t^{\alpha p}}{A_t^{2p}}dt \\
&= \frac{b^{-2p}}{\alpha p+1}\cdot{}_2F_1\left(2p,\alpha p+1;\alpha p+2;1-\frac{a}{b}\right).
\end{aligned} \qquad (2.6)$$

Thus , if we use (2.5) and (2.6) in (2.4), we obtain the inequality of (2.3).This completes the proof. □

Theorem 2.4. *Let* $f: I\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable increasing function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I^{\circ}$ *with* $a<b$. *If* $(f')^q$ *is harmonically convex on* $[a,b]$ *for some fixed* $q>1$, *then the following inequality for fractional integrals holds:*

(2.7)
$$\begin{aligned}
&\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{b-a}{2\left(ab\right)^{1-1/p}}L_{2p-2}^{2-2/p}(a,b)\left(\frac{1}{\alpha q+1}\right)^{1/q}\left(\frac{(f'(b))^q+(f'(a))^q}{2}\right)^{1/q},
\end{aligned}$$

where $1/p+1/q=1$ and $L_{2p-2}(a,b)=\left(\frac{b^{2p-1}-a^{2p-1}}{(2p-1)(b-a)}\right)^{1/(2p-2)}$ is $2p-2$-Logarithmic mean.

Proof. Let $A_t=ta+(1-t)\,b$. From Lemma 2.1 and Lemma 1.1, using the Hölder inequality and the Harmonically convexity of $(f')^q$, we get

$$\begin{aligned}
&\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}\int_0^1\frac{\left|(1-t)^\alpha-t^\alpha\right|}{A_t^2}\left(f'\left(\frac{ab}{A_t}\right)\right)dt \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}\left(\int_0^1\frac{1}{A_t^{2p}}dt\right)^{1/p} \\
&\times\left(\int_0^1\left|(1-t)^\alpha-t^\alpha\right|^q\left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q} \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}\left(\int_0^1\frac{1}{A_t^{2p}}dt\right)^{1/p} \\
&\times\left(\int_0^1\left|1-2t\right|^{\alpha q}\left[t\left(f'(b)\right)^q+(1-t)\left(f'(a)\right)^q\right]dt\right)^{1/q}
\end{aligned}$$

$$\leq\frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}K_6^{1/p}\left(K_7\left|f'(b)\right|^q+K_8\left|f'(a)\right|^q\right)^{1/q}, \tag{2.8}$$

where as in the proof of Theorem 1.5

$$\begin{aligned}
K_6 &= \int_0^1\frac{1}{A_t^{2p}}dt=b^{-2p}\int_0^1\left(1-t\left(1-\frac{a}{b}\right)\right)^{-2p}dt \\
&= b^{-2p}\cdot{}_2F_1\left(2p,1;2;1-\frac{a}{b}\right)=\frac{L_{2p-2}^{2p-2}(a,b)}{(ab)^{2p-1}},
\end{aligned} \tag{2.9}$$

$$
\begin{aligned}
K_7 &= \int_0^1 |1-2t|^{\alpha q}\, t dt \\
&= \int_0^{1/2} (1-2t)^{\alpha q}\, t dt + \int_{1/2}^1 (2t-1)^{\alpha q}\, t dt \\
&= \frac{1}{2(\alpha q+1)},
\end{aligned} \tag{2.10}
$$

and

$$
\begin{aligned}
K_8 &= \int_0^1 |1-2t|^{\alpha q}\,(1-t) dt \\
&= \frac{1}{2(\alpha q+1)}.
\end{aligned} \tag{2.11}
$$

If we use (2.9),(2.10) and (2.11) in (2.8), we obtain the inequality of (2.7). This completes the proof.

Theorem 2.5. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable increasing function on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
& |\mathcal{K}_f(g;\alpha,a,b)| \\
\leq\ & \frac{f(b)-f(a)}{2} + \frac{a(b-a)}{2b}\left(\frac{1}{\alpha p+1}\right)^{1/p} \\
& \times \left(\frac{{}_2F_1\left(2q,2;3;1-\frac{a}{b}\right)(f'(b))^q + {}_2F_1\left(2q,1;3;1-\frac{a}{b}\right)(f'(a))^q}{2}\right)^{1/q},
\end{aligned} \tag{2.12}
$$

where $1/p + 1/q = 1$.

$\square$

Proof. Let $A_t = ta + (1-t)\,b$. From Lemma 2.1 and Lemma 1.1, using the Hölder inequality and Harmonically convexity of $(f')^q$, we find

$$
\begin{aligned}
&\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\
\leq\ & \frac{f(b)-f(a)}{2} + \frac{ab\,(b-a)}{2}\int_0^1 \frac{\left|(1-t)^\alpha - t^\alpha\right|}{A_t^2}\left(f'\left(\frac{ab}{A_t}\right)\right)dt \\
\leq\ & \frac{f(b)-f(a)}{2} + \frac{ab\,(b-a)}{2}\left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|^p dt\right)^{1/p} \\
& \times\left(\int_0^1 \frac{1}{A_t^{2q}}\left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q} \\
\leq\ & \frac{f(b)-f(a)}{2} + \frac{ab\,(b-a)}{2}\left(\int_0^1 \left|1-2t\right|^{\alpha p} dt\right)^{1/p} \\
& \times\left(\int_0^1 \frac{1}{A_t^{2q}}\left[t\left(f'(b)\right)^q + (1-t)\left(f'(a)\right)^q\right]dt\right)^{1/q}
\end{aligned}
$$

$$
\leq \frac{f(b)-f(a)}{2} + \frac{ab\,(b-a)}{2}K_9^{1/p}\left(K_{10}\left(f'(b)\right)^q + K_{11}\left(f'(a)\right)^q\right)^{1/q}, \tag{2.13}
$$

where as in the proof of Theorem 1.6,

$$
K_9 = \int_0^1 \left|1-2t\right|^{\alpha p} dt = \frac{1}{\alpha p + 1} \tag{2.14}
$$

$$
\begin{aligned}
K_{10} &= \int_0^1 tA_t^{-2q}dt = b^{-2q}\int_0^1 t\left(1-t\left(1-\frac{a}{b}\right)\right)^{-2q} dt \\
&= \frac{1}{2b^{2q}}\ {}_2F_1\left(2q,2;3;1-\frac{a}{b}\right)
\end{aligned} \tag{2.15}
$$

and

$$
K_{11} = \int_0^1 (1-t)A_t^{-2q}dt = \frac{1}{2b^{2q}}\ {}_2F_1\left(2q,1;3;1-\frac{a}{b}\right) \tag{2.16}
$$

Thus, if we use (2.14),(2.15) and (2.16) in (2.13), we obtain the inequality of (2.12). This completes the proof. □

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REFINEMENT OF SOME INEQUALITIES FOR OPERATORS

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Abstract. In this paper, we will use a refinement of the classical Young inequality to improve some inequalities of operators.

1. Introduction

Let $\boldsymbol{H}$ be a complex Hilbert space with inner product $\langle ., .\rangle$ and norm $\|.\|$. Let $\mathcal{B}(\boldsymbol{H})$ denote the algebra of all bounded linear operators on $\boldsymbol{H}$, $\|.\|$ will also denote the operator norm on $\mathcal{B}(\boldsymbol{H})$.
For $A \in \mathcal{B}(\boldsymbol{H})$ the numerical radius is defined as follows,

$$\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in \boldsymbol{H}, \|x\| = 1\}.$$

We recall the following results that were proved in [2, 5].

Lemma 1.1. *Let $A \in \mathcal{B}(\boldsymbol{H})$ and let $\omega(.)$ be the numerical radius. Then*
(i) $\omega(.)$ is a norm on $\mathcal{B}(\boldsymbol{H})$,
(ii) $\omega(UAU^*) = \omega(A)$, for all unitary operators U,
(iii) $\omega(A^k) \leq \omega(A)^k, k = 1, 2, 3, \dots$ (power inequality)
(iv) $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$.

Moreover, $\omega(.)$ is not a unitarily invariant norm and is not submultiplicative. For positive real numbers a, b, the classical Young inequality says that if $p, q > 1$ such that $1/p + 1/q = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.1}$$

Replacing a, b by their squares, we could write (1.1) in the form

$$(ab)^2 \leq \frac{a^{2p}}{p} + \frac{b^{2q}}{q}. \tag{1.2}$$

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A refinement of the scalar Young inequality is as follows [9],

$$ab + r_0(a^{p/2} - b^{q/2})^2 \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{1.3}$$

where $r_0 = \min\{1/p, 1/q\}$.

Some authors considered replacing the numbers a, b by positive operators A, B. But there are some difficulties, for example if A and B are positive operators, the operator AB is not positive in general. Hence the authors studied the singular values and the norms of the operators instead of operators in some inequalities. Let us denote by $\mathbb{M}_n$ the algebra of all $n \times n$ complex matrices. Bhatia and Kittaneh in 1990 [3] established a matrix mean inequality as follows:

$$|||A^*B||| \leq \frac{1}{2} |||A^*A + B^*B||| , \tag{1.4}$$

for matrices $A, B \in \mathbb{M}_n$.

In [2] a generalization of (1.4) was proved, for all $X \in \mathbb{M}_n$,

$$|||A^*XB||| \leq \frac{1}{2} |||AA^*X + XBB^*||| . \tag{1.5}$$

Ando in 1995 [1] established a matrix Young inequality:

$$|||AB||| \leq \left|\left|\left| \frac{A^p}{p} + \frac{B^q}{q} \right|\right|\right| \tag{1.6}$$

for $p, q > 1$ with $1/p + 1/q = 1$ and positive matrices A, B. Also, in [11], we showed that $|||AXB||| \leq \left|\left|\left| \frac{1}{p}A^pX + \frac{1}{q}XB^q \right|\right|\right|$ does not hold in general. In [10] we considered the inequalities (1.4) and (1.6) with the numerical radius norm as follows:

Proposition 1.1. [10, Proposition 1] *If A, B are $n \times n$ matrices, then*

$$\omega(A^*B) \leq \frac{1}{2}\omega(A^*A + B^*B). \tag{1.7}$$

Also if A and B are positive matrices and $p, q > 1$ with $1/p + 1/q = 1$, then

$$\omega(AB) \leq \omega(\frac{A^p}{p} + \frac{B^q}{q}).$$

In this paper we obtain some generalized matrix versions of the inequalities (1.2) and (1.7).

2. MAIN RESULTS

Let $A \in \mathcal{B}(\boldsymbol{H})$. We know that $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$(see Lemma 1.1(iv)). These inequalities were improved in [6, 8] as follows:

$$\omega(A) \leq \frac{1}{2}\||A| + |A^*|\| \leq \frac{1}{2}(\|A\| + \|A^2\|^{1/2}), \tag{2.1}$$

$$\frac{1}{4}\|A^*A + AA^*\| \leq \omega^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|, \tag{2.2}$$

where $|A| := (A^*A)^{\frac{1}{2}}$ is the absolute value of A.

Generalizations of the first inequality in (2.1) and the second inequality in (2.2) have been established in [4]. It has been shown that if $A, B \in \mathcal{B}(\boldsymbol{H})$, for $0 < \alpha < 1$ and $r \geq 1$, then

$$\omega^r(A+B) \leq 2^{r-2} \left\| |A|^{2r\alpha} + |A^*|^{2r(1-\alpha)} + |B|^{2r\alpha} + |B^*|^{2r(1-\alpha)} \right\|, \tag{2.3}$$

$$\omega^r(A) \leq \frac{1}{2} \left\| |A|^{2r\alpha} + |A^*|^{2r(1-\alpha)} \right\|. \tag{2.4}$$

In 2005, Kittaneh extended the above inequalities as follows:

Theorem 2.1. [8, Theorem 2] *If $A, B, C, D, S, T \in \mathcal{B}(\boldsymbol{H})$, then for all $\alpha \in (0,1)$,*

$$\omega(ATB+CSD) \leq \frac{1}{2} \left(\left\| A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2(\alpha)}B + C|S^*|^{2(1-\alpha)}C^* + D^*|S|^{2(\alpha)}D \right\| \right). \tag{2.5}$$

In 2009, Shebrawi and Albadawi extended the inequality (2.5), in the following form:

Theorem 2.2. [12, Theorem 2.5] *Let $A_i, B_i, X_i \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, \dots, n)$, and let f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for all $r \geq 1$,*

$$\omega^r(\sum_{i=1}^{n} A_i^* X_i B_i) \leq \frac{n^{r-1}}{2} \left(\left\| \sum_{i=1}^{n} ([A_i^* g^2(|X_i^*|)A_i]^r + [B_i^* f^2(|X_i|)B_i]^r) \right\| \right). \tag{2.6}$$

In [10] we established a numerical radius inequality that generalizes (2.6) and consequently, generalize (2.3), (2.4), (2.5).

Theorem 2.3. [10, Theorem 5] *Let $A_i, B_i, X_i \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, \dots, n)$, and let f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. If $p \geq q > 1$ with $1/p + 1/q = 1$, then for all $r \geq \frac{2}{q}$,*

$$\omega^r(\sum_{i=1}^{n} A_i^* X_i B_i) \leq n^{r-1} \left\| \sum_{i=1}^{n} \frac{1}{p}(B_i^* f^2(|X_i|)B_i)^{rp/2} + \frac{1}{q}(A_i^* g^2(|X_i^*|)A_i)^{rq/2} \right\|. \tag{2.7}$$

In this section, we refine this inequality by using the inequality (1.3) to improve our results, we need the following basic lemmas.

Lemma 2.1. [7, Theorem 1] *Let A be an operator in $\mathcal{B}(\boldsymbol{H})$, and let f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for all x and y in $\boldsymbol{H}$,*

$$|\langle Ax, y \rangle| \leq \|f(|A|)x\| \|g(|A^*|)y\|. \tag{2.8}$$

The following lemma is a consequence of the spectral theorem for positive operators and Jensen's inequality (see, e.g., [7]).

Lemma 2.2. *Let A be a positive operator in $\mathcal{B}(\boldsymbol{H})$ and let $x \in \boldsymbol{H}$ be any unit vector. Then for all $r \geq 1$,*

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle. \tag{2.9}$$

Now, we state the following theorem which is a refinement of (2.7).

Theorem 2.4. *Let $A_i, B_i, X_i \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, \dots, n)$, and let f and g be nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. If $p \geq q > 1$ with $1/p + 1/q = 1$, then for all $r \geq \frac{2}{q}$,*

(2.10)

$$\omega^r(\sum_{i=1}^n A_i^* X_i B_i) \leq n^{r-1} \left(\left\| \sum_{i=1}^n \frac{1}{p}(B_i^* f^2(|X_i|)B_i)^{rp/2} + \frac{1}{q}(A_i^* g^2(|X_i^*|)A_i)^{rq/2} \right\| - (\frac{1}{p}) \inf_{\|x\|=1} \eta(x) \right),$$

where $\eta(x) := \sum_{i=1}^n \left(\left\langle B_i^* f^2(|X_i|)B_i x, x \right\rangle^{rp/4} - \left\langle A_i^* g^2(|X_i^*|)A_i x, x \right\rangle^{rq/4} \right)^2$.

Proof. For every unit vector $x \in \boldsymbol{H}$, we have

$$\begin{aligned}
\left| \left\langle \left(\sum_{i=1}^n A_i^* X_i B_i \right) x, x \right\rangle \right|^r &\leq \left(\sum_{i=1}^n |\langle X_i B_i x, A_i x \rangle| \right)^r \\
&\underset{(2.8)}{\leq} \left(\sum_{i=1}^n \left\langle f^2(|X_i|)B_i x, B_i x \right\rangle^{1/2} \left\langle g^2(|X_i^*|)A_i x, A_i x \right\rangle^{1/2} \right)^r \\
&\leq n^{r-1} \sum_{i=1}^n \left\langle f^2(|X_i|)B_i x, B_i x \right\rangle^{r/2} \left\langle g^2(|X_i^*|)A_i x, A_i x \right\rangle^{r/2} \\
&= n^{r-1} \sum_{i=1}^n \left\langle B_i^* f^2(|X_i|)B_i x, x \right\rangle^{r/2} \left\langle A_i^* g^2(|X_i^*|)A_i x, x \right\rangle^{r/2} \\
&\leq n^{r-1} \sum_{i=1}^n (\frac{1}{p} \left\langle (B_i^* f^2(|X_i|)B_i)^{rp/2} x, x \right\rangle + \frac{1}{q} \left\langle (A_i^* g^2(|X_i^*|)A_i)^{rq/2} x, x \right\rangle \\
&\quad - \frac{1}{p} \left(\left\langle (B_i^* f^2(|X_i|)B_i) x, x \right\rangle^{rp/4} - \left\langle (A_i^* g^2(|X_i^*|)A_i) x, x \right\rangle^{rq/4} \right)^2) \\
&\underset{(1.3),(2.9)}{=} n^{r-1} (\langle \sum_{i=1}^n (\frac{1}{p}(B_i^* f^2(|X_i|)B_i)^{rp/2} \\
&\quad + \frac{1}{q}(A_i^* g^2(|X_i^*|)A_i)^{rq/2}) x, x \rangle - (\frac{1}{p}) \eta(x)).
\end{aligned}$$

Now, the result follows by taking the supremum over all unit vectors in $\boldsymbol{H}$. $\square$

Remark 2.1. Let $p = q = r = 2$. Then $\eta(x) \equiv 0$ if and only if $\omega(B_i^* f^2(|X_i|)B_i - A_i^* g^2(|X_i^*|)A_i) = 0$, for all $i = 1, \dots, n$. In general, $\eta(x) = 0$ if and only if $\left\langle B_i^* f^2(|X_i|)B_i x, x \right\rangle^{rp/4} = \left\langle A_i^* g^2(|X_i^*|)A_i x, x \right\rangle^{rq/4}$, for all $i = 1, \dots, n$. Moreover, in the refinement of the Kittaneh's inequalitity, $\inf \eta(x) = 0$.
Because $0 \in \sigma_{app}(|A| - |A^*|)$ (approximate point spectrum) and the approximate point spectrum is a subset of the closure of the numerical range. Then $\inf \langle |A| - |A^*| x, x \rangle = 0$, where $\langle x, x \rangle = 1$ and hence $\inf \eta(x) = 0$.

In the following example we show that (2.10) is a refinement of the inequality (2.7) and $\inf_{\|x\|=1} \eta(x) > 0$.

Example 2.1. Let $X = I, n = 1, f(t) = g(t) = t^{1/2}, r = p = q = 2$ and $|A|^2 = diag(5, 1), |B|^2 = diag(2, 0)$ in the inequality (2.10). Then $\inf_{\|x\|=1} \eta(x) > 0$ and (2.10) is a refinement of the inequality (2.7).

The inequality (2.10) includes several numerical radius inequalities as special cases. Examples of inequalities are shown in the following.

Corollary 2.1. *Let $A_i, B_i, \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, \ldots, n)$. If $p \geq q > 1$ with $1/p + 1/q = 1$ and $r \geq \frac{2}{q}$, then*

$$\omega^r(\sum_{i=1}^{n} A_i^* B_i) \leq n^{r-1} \left(\left\| \sum_{i=1}^{n} (\frac{1}{p}|B_i|^{rp} + \frac{1}{q}|A_i|^{rq}) \right\| - (\frac{1}{p}) \inf_{\|x\|=1} \eta(x) \right),$$

where $\eta(x) := \sum_{i=1}^{n} \left(\langle |B_i|^2 x, x\rangle^{rp/4} - \langle |A_i|^2 x, x\rangle^{rq/4} \right)^2$.
In particular, if $n = 1$, then

$$\omega^r(A^* B) \leq \left\| \frac{1}{p}|B|^{rp} + \frac{1}{q}|A|^{rq}) \right\| - (\frac{1}{p}) \inf_{\|x\|=1} \eta(x),$$

where $\eta(x) := \left(\langle |B| x, x\rangle^{rp/4} - \langle |A| x, x\rangle^{rq/4} \right)^2$.

Remark 2.2. By replacing $n = 1$ in Theorem 2.4, we obtain the following

$$\omega^r(A^* X B) \leq \left\| \frac{1}{p}(B^*|X|B)^{rp/2} + \frac{1}{q}(A^*|X^*|A)^{rq/2} \right\| - (\frac{1}{p}) \inf_{\|x\|=1} \eta(x), \tag{2.11}$$

where $\eta(x) := \left(\langle (B^*|X|B)x, x\rangle^{rp/4} - \langle (A^*|X^*|)A)x, x\rangle^{rq/4} \right)^2$.
Furthermore, by Lemma 1.1, for all $A, B, X \in \mathcal{B}(\boldsymbol{H})$, we obtain the following inequalities:

$$\omega((A^* X B)^2) \leq \omega(\frac{1}{p}(A^*|X^*|A)^p + \frac{1}{q}(B^*|X|B)^q) - (\frac{1}{p}) \inf_{\|x\|=1} \eta(x), \tag{2.12}$$

where $\eta(x) := \left(\langle (B^*|X|B)x, x\rangle^{p/2} - \langle A^*|X^*|Ax, x\rangle^{q/2} \right)^2$, and

$$\omega(A^* X B) \leq \frac{1}{2}\omega(A^*|X^*|A + B^*|X|B) - (\frac{1}{2}) \inf_{\|x\|=1} \eta(x), \tag{2.13}$$

where $\eta(x) := \left(\langle (B^*|X|B)x, x\rangle^{1/2} - \langle (A^*|X^*|)A)x, x\rangle^{1/2} \right)^2$.
The inequalities (2.12) and (2.13) are generalized matrix versions of the inequalities (1.2) and (1.7), respectively.

Remark 2.3. By the Example 2.1 we can show that $\inf_{\|x\|=1} \eta(x) > 0$, in Corollary 2.1 and the inequalities (2.11), (2.12), (2.13).

3. Additional results

Some of usual operator norm inequalities for summation of operators have been proved. It has been shown in [4] that if A and B are normal and $r \geq 1$, then

$$\|A + B\|^r \leq 2^{r-1} \| |A|^r + |B|^r \|. \tag{3.1}$$

In this section, we get a norm inequality for Hilbert space operators, so that new inequalities for operators and generalizations of earlier results will be obtained. By the same method as in the proof of Theorem 2.4 we obtain the following:

Proposition 3.1. *Let $A_i, B_i, X_i \in \mathcal{B}(\boldsymbol{H})(i=1,2,...,n)$, and let f and g be as in (2.1) and $p \geq q > 1$ with $1/p + 1/q = 1$. Then for all $r \geq \frac{2}{q}$,*

$$\left\| \sum_{i=1}^{n} A_i^* X_i B_i \right\|^r \leq n^{r-1} \Big(\frac{1}{p} \left\| \sum_{i=1}^{n} (B_i^* f^2(|X_i|) B_i)^{rp/2} \right\| + \frac{1}{q} \left\| \sum_{i=1}^{n} (A_i^* g^2(|X_i^*|) A_i)^{rq/2} \right\| - (\frac{1}{p}) \inf_{\|x\|=\|y\|=1} \eta(x,y) \Big), \tag{3.2}$$

where $\eta(x,y) := \sum_{i=1}^{n} \left(\langle (B_i^* f^2(|X_i|) B_i)x, x\rangle^{rp/4} - \langle (A_i^* g^2(|X_i^*|) A_i)y, y\rangle^{rq/4} \right)^2$.

Inequality (3.2) yields several norm inequalities as special cases. Samples of these inequalities are demonstrated below.

Corollary 3.1. *Let $A_i, B_i, X_i \in \mathcal{B}(\boldsymbol{H})(i=1,2,...,n), r \geq \frac{2}{q}$ and $p \geq q > 1$ with $1/p + 1/q = 1$ and $\alpha \in (0,1)$. Then*

$$\left\| \sum_{i=1}^{n} A_i^* X_i B_i \right\|^r \leq n^{r-1} \Big(\frac{1}{p} \left\| \sum_{i=1}^{n} (B_i^* |X_i|^{2\alpha} B_i)^{rp/2} \right\| + \frac{1}{q} \left\| \sum_{i=1}^{n} (A_i^* |X_i^*|^{2(1-\alpha)} A_i)^{rq/2} \right\| - (\frac{1}{p}) \inf_{\|x\|=\|y\|=1} \eta(x,y) \Big), \tag{3.3}$$

where $\eta(x,y) := \sum_{i=1}^{n} \left(\langle (B_i^* |X_i|^{2\alpha} B_i)x, x\rangle^{rp/4} - \langle (A_i^* |X_i^*|^{2(1-\alpha)} A_i)y, y\rangle^{rq/4} \right)^2$.

In particular,

$$\|A^* X B\|^r \leq \frac{1}{p} \left\| (B^* |X| B)^{rp/2} \right\| + \frac{1}{q} \left\| (A^* |X^*| A)^{rq/2} \right\| - (\frac{1}{p}) \inf_{\|x\|=\|y\|=1} \eta(x,y)$$

where $\eta(x,y) := \left(\langle (B^* |X| B)x, x\rangle^{rp/4} - \langle (A^* |X^*| A)y, y\rangle^{rq/4} \right)^2$.

For $X_i = I(i=1,2,...,n)$ in inequality (3.3), we get norm inequalities for products of operators.

Corollary 3.2. *Let $A_i, B_i \in \mathcal{B}(\boldsymbol{H})(i=1,2,...,n), r \geq \frac{2}{q}$. Then*

$$\left\| \sum_{i=1}^{n} A_i^* B_i \right\|^r \leq n^{r-1} \left(\frac{1}{p} \left\| \sum_{i=1}^{n} |B_i|^{rp} \right\| + \frac{1}{q} \left\| \sum_{i=1}^{n} |A_i|^{rq} \right\| - (\frac{1}{p}) \inf_{\|x\|=\|y\|=1} \eta(x,y) \right),$$

where $\eta(x,y) := \sum_{i=1}^{n} \left(\langle |B_i|^2 x, x\rangle^{rp/4} - \langle |A_i|^2 y, y\rangle^{rq/4} \right)^2$. In particular,

$$\left\| \sum_{i=1}^{n} A_i^* B_i \right\|^2 \leq n \Big(\frac{1}{p} \left\| \sum_{i=1}^{n} |B_i|^{2p} \right\| + \frac{1}{q} \left\| \sum_{i=1}^{n} |A_i|^{2q} \right\| - (\frac{1}{p}) \inf_{\|x\|=\|y\|=1} \sum_{i=1}^{n} \left(\langle |B_i|^2 x, x\rangle^{p/2} - \langle |A_i|^2 y, y\rangle^{q/2} \right)^2 \Big).$$

Example 3.1. Let $X = I, n = 1, f(t) = t^{\alpha}, g(t) = t^{1-\alpha}, \alpha = 1/2, r = p = q = 2$ and $|A|^2 = diag(5,7), |B|^2 = diag(2,3)$ in the inequalities (3.2) and (3.3) and Corollary 3.2 if needed. Then $\eta(x,y) := \left(\langle |B|^2 x, x\rangle - \langle |A|^2 y, y\rangle \right)^2$ and hence

$$\inf_{\|x\|=\|y\|=1} \eta(x,y) \geq 4 > 0.$$

For $n = 2$ in inequality (3.3), we get the interesting norm inequalities that give an
estimate for the operator norm of commutators. Also for $A_i = B_i = I(i = 1, 2, ..., n)$ in the inequality (3.3), we get the following operator inequalities for summation of operators.

Corollary 3.3. *Let $X_i \in \mathcal{B}(\boldsymbol{H})(i = 1, 2, ..., n), r \geq \frac{2}{q}$ and $\alpha \in (0, 1)$. Then*

$$\left\|\sum_{i=1}^{n} X_i\right\|^r \leq n^{r-1}\left(\frac{1}{p}\left\|\sum_{i=1}^{n} |X_i|^{\alpha r p}\right\| + \frac{1}{q}\left\|\sum_{i=1}^{n} |X_i^*|^{(1-\alpha) r q}\right\|\right),$$

In particular, if $X_i(i = 1, 2, ..., n)$ are normal, then

$$\left\|\sum_{i=1}^{n} X_i\right\|^r \leq n^{r-1}\left(\frac{1}{p}\left\|\sum_{i=1}^{n} |X_i|^{\alpha r p}\right\| + \frac{1}{q}\left\|\sum_{i=1}^{n} |X_i|^{(1-\alpha) r q}\right\|\right), \tag{3.4}$$

The inequality (3.4) is a generalized form of (3.1) and this inequality is not true for arbitrary operators.
The following example shows that in the inequality (3.4) normality of X_i is necessary,

Example 3.2. Let $X_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, (non normal) and $X_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let $p = q = 2, \alpha = 1/2$ and $r = 1$. Then $\|X_1 + X_2\| = \sqrt{2}$ as $|X_1| + |X_2| = I$, consequently $\||X_1| + |X_2|\| = 1$, that is a contradiction with $\sqrt{2} \leq 1$.

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OSTROWSKI TYPE INEQUALITIES FOR HARMONICALLY s-CONVEX FUNCTIONS

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ABSTRACT. The author introduces the concept of harmonically s-convex functions and establishes some Ostrowski type inequalities and a variant of Hermite-Hadamard inequality for these classes of functions.

1. INTRODUCTION

Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in I° (the interior of I) and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, for all $x \in [a, b]$, then the following inequality holds

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \tag{1.1}$$

for all $x \in [a, b]$. This inequality is known in the literature as the Ostrowski inequality (see [13]), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a} \int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$. For some results which generalize, improve and extend the inequalities(1.1) we refer the reader to the recent papers (see [2, 12]).

In [7], Hudzik and Maligranda considered the following class of functions:

Definition 1.1. A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ where $\mathbb{R}_+ = [0, \infty)$, is said to be s-convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in I$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and s fixed in $(0, 1]$. They denoted this by K_s^2.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

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In [5], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the s-convex functions.

Theorem 1.1. *Suppose that $f:\mathbb{R}_+\to\mathbb{R}_+$ is an s-convex function in the second sense, where $s\in[0,1)$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L\,[a,b]$, then the following inequalities hold*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int\limits_a^b f(x)dx\leq\frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

the constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.2).

The above inequalities are sharp. For some recent results and generalizations concerning s-convex functions see [3, 4, 5, 6, 8, 10, 11].

In [9], the author gave harmonically convex and established Hermite-Hadamard's inequality for harmonically convex functions as follows:

Definition 1.2. Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\to\mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq tf(y)+(1-t)f(x) \tag{1.3}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality in (1.3) is reversed, then f is said to be harmonically concave.

Theorem 1.2. *Let $f:I\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a harmonically convex function and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{2}. \tag{1.4}$$

The above inequalities are sharp.

The goal of this paper is to introduce the concept of the harmonically s-convex functions, obtain the similar the inequalities (1.4) for harmonically s-convex functions and establish some new inequalities of Ostrowski type for harmonically s-convex functions.

2. Main Results

Definition 2.1. Let $I\subset(0,\infty)$ be an real interval. A function $f:I\to\mathbb{R}$ is said to be harmonically s-convex (concave), if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq(\geq)\,t^sf(y)+(1-t)^sf(x) \tag{2.1}$$

for all $x,y\in I$, $t\in[0,1]$ and for some fixed $s\in(0,1]$.

Proposition 2.1. *Let $I\subset(0,\infty)$ be an real interval and $f:I\to\mathbb{R}$ is a function, then ;*

1. *if f is s-convex and nondecreasing function then f is harmonically s-convex.*
2. *if f is harmonically s-convex and nonincreasing function then f is s-convex.*

Proof. Since $f:(0,\infty)\to\mathbb{R},\ f(x)=x$, harmonically convex function, we have

$$\frac{xy}{tx+(1-t)y}\leq ty+(1-t)x \tag{2.2}$$

for all $x,y\in(0,\infty)$, $t\in[0,1]$. The proposition (1) and (2) is easily obtained from the inequality (2.2). $\square$

Example 2.1. Let $s\in(0,1]$ and $f:(0,1]\to(0,1]\,,\ f(x)=x^s$. Since f is s-convex (see [7]) and nondecreasing function, f is harmonically $s-$convex.

Proposition 2.2. *Let $s\in(0,1]$, $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a function and $g:[a,b]\to[a,b]$, $g(x)=\frac{ab}{a+b-x}$. Then f is harmonically s-convex on $[a,b]$ if and only if $f\circ g$ is s-convex on $[a,b]$.*

Proof. Since

$$(f\circ g)(ta+(1-t)b)=f\left(\frac{ab}{tb+(1-t)a}\right) \tag{2.3}$$

for all $t\in[0,1]$. The proof is obvious from equality (2.3). $\square$

The following result of the Hermite-Hadamard type holds.

Theorem 2.1. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be an harmonically s-convex function, $s\in(0,1]$ and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequalities hold:*

$$2^{s-1}f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{s+1}. \tag{2.4}$$

Proof. Since $f:I\to\mathbb{R}$ is an harmonically s-convex function, we have, for all $x,y\in I$ (with $t=\frac{1}{2}$ in the inequality (2.1))

$$f\left(\frac{2xy}{x+y}\right)\leq\frac{f(y)+f(x)}{2^s}$$

Choosing $x=\frac{ab}{ta+(1-t)b},\ y=\frac{ab}{tb+(1-t)a}$, we get

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{f\left(\frac{ab}{tb+(1-t)a}\right)+f\left(\frac{ab}{ta+(1-t)b}\right)}{2^s}$$

Further, integrating for $t\in[0,1]$, we have

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{1}{2^s}\left[\int\limits_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt+\int\limits_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt\right] \tag{2.5}$$

Since each of the integrals is equal to $\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}dx$, we obtain the left-hand side of the inequality (2.4) from (2.5).

The proof of the second inequality follows by using (2.1) with $x=a$ and $y=b$ and integrating with respect to t over $[0,1]$. $\square$

In order to prove our main theorems, we need the following lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a,b]$ then*

$$\begin{aligned} & f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \\ = & \frac{ab}{b-a}\left\{ (x-a)^2 \int_0^1 \frac{t}{(ta+(1-t)x)^2} f'\left(\frac{ax}{ta+(1-t)x}\right)dt \right. \\ & \left. - (b-x)^2 \int_0^1 \frac{t}{(tb+(1-t)x)^2} f'\left(\frac{bx}{tb+(1-t)x}\right)dt \right\} \end{aligned}$$

Proof. Integrating by part and changing variables of integration yields

$$\begin{aligned} & \frac{ab}{b-a}\left\{ (x-a)^2 \int_0^1 \frac{t}{(ta+(1-t)x)^2} f'\left(\frac{ax}{ta+(1-t)x}\right)dt \right. \\ & \left. - (b-x)^2 \int_0^1 \frac{t}{(tb+(1-t)x)^2} f'\left(\frac{bx}{tb+(1-t)x}\right)dt \right\} \\ = & \frac{1}{x(b-a)}\left[b(x-a)\int_0^1 tdf\left(\frac{ax}{ta+(1-t)x}\right) + a(b-x)\int_0^1 tdf\left(\frac{bx}{tb+(1-t)x}\right)\right] \\ = & \frac{1}{x(b-a)}\left[b(x-a)\left\{ tf\left(\frac{ax}{ta+(1-t)x}\right)\Big|_0^1 - \int_0^1 f\left(\frac{ax}{ta+(1-t)x}\right)dt\right\}\right] \\ & + \frac{1}{x(b-a)}\left[a(b-x)\left\{ tf\left(\frac{bx}{tb+(1-t)x}\right)\Big|_0^1 - \int_0^1 f\left(\frac{bx}{tb+(1-t)x}\right)dt\right\}\right] \\ = & f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du. \end{aligned}$$

□

Theorem 2.2. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically s- convex on $[a,b]$ for $q \geq 1$, then for all $x \in [a,b]$, we have*

$$\left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \right| \tag{2.6}$$

$$\begin{aligned} \leq & \frac{ab}{b-a}\left\{ (x-a)^2 \left(\lambda_1(a,x,s,q,q)\left|f'(x)\right|^q + \lambda_2((a,x,s,q,q)\left|f'(a)\right|^q\right)^{\frac{1}{q}} \right. \\ & \left. + (b-x)^2 \left(\lambda_3(b,x,s,q,q)\left|f'(x)\right|^q + \lambda_4(b,x,s,q,q)\left|f'(b)\right|^q\right)^{\frac{1}{q}} \right\}, \end{aligned}$$

where

$$\lambda_1(a,x,s,\vartheta,\rho)=\frac{\beta\left(\rho+s+1,1\right)}{x^{2\vartheta}}\cdot{}_2F_1\left(2\vartheta,\rho+s+1;\rho+s+2;1-\frac{a}{x}\right),$$

$$\lambda_2(a,x,s,\vartheta,\rho)=\frac{\beta\left(\rho+1,1\right)}{x^{2\vartheta}}\cdot{}_2F_1\left(2\vartheta,\rho+1;\rho+s+2;1-\frac{a}{x}\right),$$

$$\lambda_3(b,x,s,\vartheta,\rho)=\frac{\beta\left(1,\rho+s+1\right)}{b^{2\vartheta}}\cdot{}_2F_1\left(2\vartheta,1;\rho+s+2;1-\frac{x}{b}\right),$$

$$\lambda_4(b,x,s,\vartheta,\rho)=\frac{\beta\left(s+1,\rho+1\right)}{b^{2\vartheta}}\cdot{}_2F_1\left(2\vartheta,s+1;\rho+s+2;1-\frac{x}{b}\right),$$

β *is Euler Beta function defined by*

$$\beta\left(x,y\right)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}\left(1-t\right)^{y-1}dt,\quad x,y>0,$$

and ${}_2F_1$ *is hypergeometric function defined by*

$${}_2F_1\left(a,b;c;z\right)=\frac{1}{\beta\left(b,c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-zt\right)^{-a}dt,\ c>b>0,\ |z|<1 \text{ (see [1]).}$$

Proof. From Lemma 2.1, Power mean inequality and the harmonically s-convexity of $|f'|^q$ on $[a,b]$,we have

$$\begin{aligned}
&\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right| \\
\leq\ &\frac{ab}{b-a}\left\{\left(x-a\right)^2\int_0^1\frac{t}{\left(ta+(1-t)x\right)^2}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|dt\right. \\
&\left.+\left(b-x\right)^2\int_0^1\frac{t}{\left(tb+(1-t)x\right)^2}\left|f'\left(\frac{bx}{tb+(1-t)x}\right)\right|dt\right\}
\end{aligned}$$

$$\begin{aligned}
\leq\ &\frac{ab\left(x-a\right)^2}{b-a}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\frac{t^q}{\left(ta+(1-t)x\right)^{2q}}\left[t^s\left|f'\left(x\right)\right|^q+(1-t)^s\left|f'\left(a\right)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&+\frac{ab\left(b-x\right)^2}{b-a}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\frac{t^q}{\left(tb+(1-t)x\right)^{2q}}\left[t^s\left|f'\left(x\right)\right|^q+(1-t)^s\left|f'\left(b\right)\right|^q\right]dt\right)^{\frac{1}{q}},
\end{aligned} \tag{2.7}$$

where an easy calculation gives

(2.8)
$$\int_0^1 \frac{t^{q+s}}{(ta+(1-t)x)^{2q}}dt = \frac{\beta(q+s+1,1)}{x^{2q}}.{}_2F_1\left(2q, q+s+1; q+s+2; 1-\frac{a}{x}\right),$$

$$\int_0^1 \frac{t^{q+s}}{(tb+(1-t)x)^{2q}}dt = \frac{\beta(1,q+s+1)}{b^{2q}}.{}_2F_1\left(2q, 1; q+s+2; 1-\frac{x}{b}\right),$$

$$\int_0^1 \frac{t^{q}(1-t)^s}{(ta+(1-t)x)^{2q}}dt = \frac{\beta(q+1,s+1)}{x^{2q}}.{}_2F_1\left(2q, q+1; s+q+2; 1-\frac{a}{x}\right),$$

$$\int_0^1 \frac{t^{q}(1-t)^s}{(tb+(1-t)x)^{2q}}dt = \frac{\beta(s+1,q+1)}{b^{2q}}.{}_2F_1\left(2q, s+1; s+q+2; 1-\frac{x}{b}\right). \tag{2.9}$$

Hence, If we use (2.8)-(2.9) in (2.7), we obtain the desired result. This completes the proof. □

Corollary 2.1. *In Theorem 2.2, additionally, if* $|f'(x)| \leq M$, $x \in [a,b]$, *then inequality*

$$\begin{aligned} &\left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \right| \\ \leq\ & \frac{ab}{b-a}M\left\{(x-a)^2\left(\lambda_1(a,x,s,q,q)+\lambda_2((a,x,s,q,q)\right)^{\frac{1}{q}} \right. \\ & \left. + (b-x)^2\left(\lambda_3(b,x,s,q,q)+\lambda_4(b,x,s,q,q)\right)^{\frac{1}{q}}\right\} \end{aligned}$$

holds.

Theorem 2.3. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I°, $a, b \in I^\circ$ *with* $a < b$, *and* $f' \in L[a,b]$. *If* $|f'|^q$ *is harmonically s- convex on* $[a,b]$ *for* $q \geq 1$, *then for all* $x \in [a,b]$, *we have*

$$\left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \right| \tag{2.10}$$

$$\begin{aligned} \leq\ & \frac{ab}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,1)\left|f'(x)\right|^q+\lambda_2(a,x,s,q,1)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right. \\ & \left. + (b-x)^2\left(\lambda_3(b,x,s,q,1)\left|f'(x)\right|^q+\lambda_4(b,x,s,q,1)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\} \end{aligned}$$

where λ_1, λ_2, λ_3 *and* λ_4 *are defined as in Theorem 2.2.*

Proof. From Lemma 2.1, Power mean inequality and the harmonically s-convexity of $|f'|^q$ on $[a,b]$,we have

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right| \tag{2.11}$$

$$\leq \frac{ab\,(x-a)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\frac{t}{(ta+(1-t)x)^{2q}}\left[t^s\left|f'\left(x\right)\right|^q+(1-t)^s\left|f'\left(a\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{ab\,(b-x)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\frac{t}{(tb+(1-t)x)^{2q}}\left[t^s\left|f'\left(x\right)\right|^q+(1-t)^s\left|f'\left(b\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$\leq \frac{ab}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,1)\left|f'\left(x\right)\right|^q+\lambda_2(a,x,s,q,1)\left|f'\left(a\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\,(b-x)^2\left(\lambda_3(b,x,s,q,1)\left|f'\left(x\right)\right|^q+\lambda_4(b,x,s,q,1)\left|f'\left(b\right)\right|^q\right)^{\frac{1}{q}}\right\}$$

This completes the proof. $\square$

Corollary 2.2. *In Theorem 2.3, additionally, if* $|f'(x)|\leq M$, $x\in[a,b]$, *then inequality*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right|$$

$$\leq \frac{ab}{b-a}M\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,1)+\lambda_2((a,x,s,q,1)\right)^{\frac{1}{q}}\right.$$
$$\left.+\,(b-x)^2\left(\lambda_3(b,x,s,q,1)+\lambda_4(b,x,s,q,1)\right)^{\frac{1}{q}}\right\}$$

holds.

Theorem 2.4. *Let* $f:I\subset(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I°, $a,b\in I^\circ$ *with* $a<b$, *and* $f'\in L[a,b]$. *If* $|f'|^q$ *is harmonically* s*- convex on* $[a,b]$ *for* $q\geq 1$, *then for all* $x\in[a,b]$, *we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right| \tag{2.12}$$

$$\leq \frac{ab}{b-a}\left\{\lambda_5^{1-\frac{1}{q}}(a,x)\,(x-a)^2\left(\lambda_1(a,x,s,1,1)\left|f'(x)\right|^q+\lambda_2(a,x,s,1,1)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\,\lambda_5^{1-\frac{1}{q}}(b,x)\,(b-x)^2\left(\lambda_3(b,x,s,1,1)\left|f'(x)\right|^q+\lambda_4(b,x,s,1,1)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}$$

where

$$\lambda_5(\theta,x)=\frac{1}{x-\theta}\left\{\frac{1}{\theta}-\frac{\ln x-\ln\theta}{x-\theta}\right\},$$

and λ_1, λ_2, λ_3 and λ_4 are defined as in Theorem 2.2.

Proof. From Lemma 2.1, Power mean inequality and the harmonically s-convexity of $|f'|^q$ on $[a,b]$,we have

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right| \tag{2.13}$$
$$\leq \frac{ab\,(x-a)^2}{b-a}\left(\int_0^1\frac{t}{(ta+(1-t)x)^2}dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\frac{t}{(ta+(1-t)x)^2}\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{ab\,(b-x)^2}{b-a}\left(\int_0^1\frac{t}{(tb+(1-t)x)^2}dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\frac{t}{(tb+(1-t)x)^2}\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}.$$

It is easily check that

$$\int_0^1\frac{t}{(ta+(1-t)x)^2}dt=\frac{1}{x-a}\left\{\frac{1}{a}-\frac{\ln x-\ln a}{x-a}\right\}, \tag{2.14}$$

$$\int_0^1\frac{t}{(tb+(1-t)x)^2}dt=\frac{1}{b-x}\left\{\frac{\ln b-\ln x}{b-x}-\frac{1}{b}\right\},$$

Hence, If we use (2.8)-(2.9) for $q=1$ and (2.14) in (2.13), we obtain the desired result. This completes the proof. □

Corollary 2.3. *In Theorem 2.4, additionally, if* $|f'(x)| \leq M$, $x \in [a,b]$, *then inequality*

$$\left|f(x)-\frac{ab}{b-a}\int\limits_a^b \frac{f(u)}{u^2}du\right|$$
$$\leq \frac{ab}{b-a}M\left\{\lambda_5^{1-\frac{1}{q}}(a,x)(x-a)^2(\lambda_1(a,x,s,1,1)+\lambda_2(a,x,s,1,1))^{\frac{1}{q}}\right.$$
$$\left.+\lambda_5^{1-\frac{1}{q}}(b,x)(b-x)^2(\lambda_3(b,x,s,1,1)+\lambda_4(b,x,s,1,1))^{\frac{1}{q}}\right\}$$

holds.

Theorem 2.5. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I°, $a,b \in I^\circ$ *with* $a < b$, *and* $f' \in L[a,b]$. *If* $|f'|^q$ *is harmonically* s*-convex on* $[a,b]$ *for* $q > 1$, $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|f(x)-\frac{ab}{b-a}\int\limits_a^b \frac{f(u)}{u^2}du\right| \tag{2.15}$$

$$\leq \frac{ab}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,0)\left|f'(x)\right|^q+\lambda_2(a,x,s,q,0)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^2\left(\lambda_3(b,x,s,q,0)\left|f'(x)\right|^q+\lambda_4(b,x,s,q,0)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}.$$

where λ_1, λ_2, λ_3 *and* λ_4 *are defined as in Theorem 2.2.*

Proof. From Lemma 2.1, Hölder's inequality and the harmonically convexity of $|f'|^q$ on $[a,b]$,we have

$$\left|f(x)-\frac{ab}{b-a}\int\limits_a^b \frac{f(u)}{u^2}du\right|$$
$$\leq \frac{ab(x-a)^2}{b-a}\left(\int\limits_0^1 t^p dt\right)^{\frac{1}{p}}$$
$$\times\left(\int\limits_0^1 \frac{1}{(ta+(1-t)x)^{2q}}\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{ab(b-x)^2}{b-a}\left(\int\limits_0^1 t^p dt\right)^{\frac{1}{p}}$$
$$\times\left(\int\limits_0^1 \frac{1}{(tb+(1-t)x)^{2q}}\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$\leq \frac{ab}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,0)\left|f'(x)\right|^q+\lambda_2(a,x,s,q,0)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^2\left(\lambda_3(b,x,s,q,0)\left|f'(x)\right|^q+\lambda_4(b,x,s,q,0)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

Corollary 2.4. *In Theorem 2.5, additionally, if* $|f'(x)| \leq M$, $x \in [a,b]$, *then inequality*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right|$$
$$\leq \frac{ab}{b-a}M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,0)+\lambda_2(a,x,s,q,0)\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^2\left(\lambda_3(b,x,s,q,0)+\lambda_4(b,x,s,q,0)\right)^{\frac{1}{q}}\right\}$$

holds.

Theorem 2.6. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I°, $a,b \in I^\circ$ *with* $a < b$, *and* $f' \in L[a,b]$. *If* $|f'|^q$ *is harmonically s-convex on* $[a,b]$ *for* $q > 1$, $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right|$$

$$\leq \frac{ab}{b-a}\left\{\left(\lambda_1(a,x,0,p,p)\right)^{\frac{1}{p}}(x-a)^2\left(\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\lambda_3(b,x,0,p,p)\right)^{\frac{1}{p}}(b-x)^2\left(\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right\}.$$

where λ_1, λ_2, λ_3 *and* λ_4 *are defined as in Theorem 2.2.*

Proof. From Lemma 2.1, Hölder's inequality and the harmonically convexity of $|f'|^q$ on $[a,b]$,we have

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right|$$
$$\leq \frac{ab(x-a)^2}{b-a}\left(\int_0^1 \frac{t^p}{(ta+(1-t)x)^{2p}}dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&+\frac{ab\left(b-x\right)^2}{b-a}\left(\int_0^1 \frac{t^p}{\left(tb+(1-t)x\right)^{2p}}dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 \left[t^s\left|f'\left(x\right)\right|^q+(1-t)^s\left|f'\left(b\right)\right|^q\right]dt\right)^{\frac{1}{q}} \\
\leq\quad & \frac{ab}{b-a}\left\{\left(\lambda_1(a,x,0,p,p)\right)^{\frac{1}{p}}\left(x-a\right)^2\left(\frac{\left|f'\left(x\right)\right|^q+\left|f'\left(a\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\lambda_3(b,x,0,p,p)\right)^{\frac{1}{p}}\left(b-x\right)^2\left(\frac{\left|f'\left(x\right)\right|^q+\left|f'\left(b\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

This completes the proof. $\square$

Corollary 2.5. *In Theorem 2.6, additionally, if* $|f'(x)| \leq M$, $x \in [a,b]$, *then inequality*

$$
\begin{aligned}
&\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| \\
\leq\quad & \frac{ab}{b-a}M\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left\{\left(\lambda_1(a,x,0,p,p)\right)^{\frac{1}{p}}\left(x-a\right)^2\right. \\
&\left.+\left(\lambda_3(b,x,0,p,p)\right)^{\frac{1}{p}}\left(b-x\right)^2\right\}
\end{aligned}
$$

holds.

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ON THE DETERMINATION OF A DEVELOPABLE SPHERICAL ORTHOTOMIC TIMELIKE RULED SURFACE

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Abstract. In this paper, a method for determination of developable spherical orthotomic ruled surfaces generated by a spacelike curve on dual hyperbolic unit sphere is given by using dual vector calculus in $\mathbb{R}^3_1$. We show that dual vectorial expression of a developable spherical orthotomic timelike ruled surface can be obtained from coordinates and the first derivatives of the base curve. The paper concludes with an example related to this method.

1. Introduction

In geometry, a surface is a called ruled surface if it is swept out by a moving line. The theory of ruled surfaces is a classical subject in differential geometry. Ruled surface, espicially developable ruled surface have been widely investigated in mathematics, engineering and architecture [13]. In today's manufacturing industries, the developable ruled surface desing and its application are extensively used in CAD, CAM and CNC. Also it has been popular in architecture such as saddle roofs, cooling towers, gridshell etc.

Dual numbers were first introduced by W.K. Clifford (1849–79) as a tool for his geometrical investigations. After him E. Study has done fundamental research with dual numbers and dual vectors on the geometry of lines and kinematics [2] which is so-called E. Study mapping. This mapping constitutes a one to one correpondence between the dual points of dual unit sphere S^2 and the directed lines of space of lines $\mathbb{R}^3$ [15]. If we consider the Minkowski 3-space $\mathbb{R}^3_1$ instead of $\mathbb{R}^3$ the E. Study mapping can be stated as follows. The dual timelike and spacelike unit vectors of dual hyperbolic and Lorentzian unit spheres $\mathbb{H}^2_0$ and $\mathbb{S}^2_1$ at the dual Lorentzian space $\mathbb{D}^3_1$ are in a one to one correspondence with the directed timelike and spacelike lines of the space of Lorentzian lines $\mathbb{R}^3_1$, respectively. Then a differentiable curve on $\mathbb{H}^2_0$ corresponds to a timelike ruled surface in $\mathbb{R}^3_1$. Similarly the timelike (resp.

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spacelike) curve on $\mathbb{S}_1^2$ corresponds to any spacelike (resp. timelike) ruled surface at $\mathbb{R}^3$[19].

Let α be a regular curve and $\overrightarrow{T}$ be its tangent, and let u be a source. An orthotomic of α with respect to the source (u) is defined as a locus of reflection of u about tangents $\overrightarrow{T}$ [7]. Bruce and Giblin applied the unfolding theory to the study of the evolutes and orthotomics of plane and space curves [3], [4] and [5]. Georgiou, Hasanis and Koutroufiotis investigated the orthotomics in the Euclidean (n + 1)-space [6]. Alamo and Criado studied the antiorthotomics in the Euclidean (n+1)-space [1]. Xiong defined the spherical orthotomic and the spherical antiorthotomic [18]. Yıldız and Hacısalihoğlu examined the Study of spherical orthotomic of a circle [9]. Also, orthotomic concept can be apply to surface. For a given surface S and a fixed point (source) P, orthotomic surface of S relative to P is defined as a locus of reflection of P about all tangent planes of S [8].

Köse introduced a new method for determination of developable ruled surfaces [11]. Ekici and Özüsağlam [12] study this method in $\mathbb{R}_1^3$. And also, Yıldız et al. applied this method in $\mathbb{R}^3$ by using orthotomic concept [10]. For all these the following question is interesting: Can we obtain a remarkable method for determination of developable spherical orthotomic timelike ruled surface in $\mathbb{R}_1^3$. The answer is positive. In this article, we construct a method for determination of developable orthotomic timelike ruled surfaces by using dual vector calculus.

2. Basic concept

A dual number has the form $\widetilde{a} = a + \varepsilon a^*$ where a and a^* are real numbers and $\varepsilon = (0,1)$ stands for the dual unit which $\varepsilon^2 = 0$.

The set of all dual numbers is denoted by $\mathbb{D}$ which is a commutative ring over $\mathbb{R}$.

$\mathbb{D}^3$ is the set of all triples of dual numbers. $\mathbb{D}^3$ can be written as

$$\mathbb{D}^3 = \left\{ \overrightarrow{\widetilde{a}} = (\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3) \, | \widetilde{a}_i \in \mathbb{D}, 1 \leq i \leq 3 \right\}.$$

A dual vector has the form $\overrightarrow{\widetilde{a}} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$, where $\overrightarrow{a}$ and $\overrightarrow{a}^*$ are real vectors in $\mathbb{R}^3$. The set $\mathbb{D}^3$ becomes a modul under addition and scalar multiplication on the set $\mathbb{D}$ [17].

For any dual Lorentzian vector $\overrightarrow{\widetilde{a}} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$ and $\overrightarrow{\widetilde{b}} = \overrightarrow{b} + \varepsilon \overrightarrow{b}^*$, inner product is defined by

$$\left\langle \overrightarrow{\widetilde{a}}, \overrightarrow{\widetilde{b}} \right\rangle = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \varepsilon \left(\left\langle \overrightarrow{a}, \overrightarrow{b}^* \right\rangle + \left\langle \overrightarrow{a}^*, \overrightarrow{b} \right\rangle \right)$$

where $\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle$ is the Lorentzian inner product with signature $(+,+,-)$ of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ in the Minkowski 3-Space $\mathbb{R}_1^3$.

A dual vector $\overrightarrow{\widetilde{a}}$ is said to be time-like if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle < 0$, space-like if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle > 0$ and light-like (or null) if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle = 0$ and $\overrightarrow{a} \neq 0$. The set of all dual Lorentzian vector is called dual Lorentzian space and is denoted by

$$\mathbb{D}_1^3 = \left\{ \overrightarrow{\widetilde{a}} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \, | \overrightarrow{a}, \overrightarrow{a}^* \in \mathbb{R}_1^3 \right\}$$

For any vector $\overrightarrow{\widetilde{a}} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$ and $\overrightarrow{\widetilde{b}} = \overrightarrow{b} + \varepsilon \overrightarrow{b}^*$, vector product is defined by

$$\overrightarrow{\widetilde{a}} \wedge \overrightarrow{\widetilde{b}} = \overrightarrow{a} \wedge \overrightarrow{b} + \varepsilon \left(\overrightarrow{a} \wedge \overrightarrow{b}^* + \overrightarrow{a}^* \wedge \overrightarrow{b} \right),$$

where $\overrightarrow{a} \wedge \overrightarrow{b}$ is the Lorentzian vector product.

The norm $\left\| \overrightarrow{\widetilde{a}} \right\|$ of $\overrightarrow{\widetilde{a}} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$ is defined as

$$\left\| \overrightarrow{\widetilde{a}} \right\| = \| \overrightarrow{a} \| + \varepsilon \frac{\langle \overrightarrow{a}, \overrightarrow{a}^* \rangle}{\| \overrightarrow{a} \|}, \quad \overrightarrow{a} \neq 0.$$

The dual vector $\overrightarrow{\widetilde{a}}$ with norm 1 is called a dual unit vector.

The dual Lorentzian unit sphere and the dual hyperbolic unit sphere are

$$\mathbb{S}_1^2 = \left\{ \overrightarrow{\widetilde{x}} = x + \varepsilon x^* \in \mathbb{D}_1^3 \mid < \widetilde{x}, \widetilde{x} >= 1; x, x^* \in \mathbb{R}_1^3 \right\}$$

and

$$\mathbb{H}_0^2 = \left\{ \overrightarrow{\widetilde{x}} = x + \varepsilon x^* \in \mathbb{D}_1^3 \mid < \widetilde{x}, \widetilde{x} >= -1; x, x^* \in \mathbb{R}_1^3 \right\}$$

respectively. The dual spacelike unit vectors of dual Lorentzian sphere $\mathbb{S}_1^2$ represent oriented spacelike lines is $\mathbb{R}_1^3$. The dual timelike unit vectors of dual hyperbolic unit sphere $\mathbb{H}_0^2$ represent oriented timelike lines in $\mathbb{R}_1^3$.

For $\mathbb{R}_1^3$, the Study Mapping is defined as follows: "There are one-to-one correspondence between the directed timelike (resp. spacelike) lines in three dimensional Minkowski space and the dual point on the surface of a dual hyperbolic (resp. dual Lorentzian) unit sphere (resp.) in three dimensional dual Lorentzian space " [16].

Let $\mathbb{S}_1^2$ (resp. $\mathbb{H}_0^2$), O and $\left\{ O; \overrightarrow{\widetilde{e}}_1, \overrightarrow{\widetilde{e}}_2, \overrightarrow{\widetilde{e}}_3 \right\}$ denote the dual hyperbolic (resp. Lorentzian) unit sphere , the center of $\mathbb{S}_1^2$ (resp. $\mathbb{H}_0^2$) and dual orthonormal system at O, respectively, where

$$\overrightarrow{\widetilde{e}}_i = \overrightarrow{e}_i + \varepsilon \overrightarrow{e}_i^*; \quad 1 \leq i \leq 3.$$

Let S_3 be the group of all the permutations of the set $\{1, 2, 3\}$, then it can be written as

$$\left. \begin{array}{c} \overrightarrow{\widetilde{e}}_{\sigma(1)} = sgn(\sigma) \overrightarrow{\widetilde{e}}_{\sigma(2)} \wedge \overrightarrow{\widetilde{e}}_{\sigma(3)}, sgn(\sigma) = \pm 1, \\ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix} \end{array} \right\}.$$

In the case that the orthonormal system

$$\{O; \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3\}$$

is the system of $\mathbb{R}_1^3$.

By using the Study Mapping, we can conclude that there exists a one to one correspondence between the dual orthonormal system and the real orthonormal system.

Now, define of spherical normal, spherical tangent and spherical orthotomic of a spherical curve α. Let $\left\{ \overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B} \right\}$ be the Frenet frame of α. The spherical normal of α is the great circle, passing through $\alpha(s)$, that is normal to α at $\alpha(s)$ and given by

$$\begin{cases} \langle \overrightarrow{x}, \overrightarrow{x} \rangle = 1, \\ \left\langle \overrightarrow{x}, \overrightarrow{T} \right\rangle = 0, \end{cases}$$

where x is an arbitrary point of the spherical normal. The spherical tangent of α is the great circle which tangent to α at $\alpha(s)$ and given by

$$\begin{cases} \langle \overrightarrow{y}, \overrightarrow{y} \rangle = 1, \\ \left\langle \overrightarrow{y}, (\overrightarrow{\alpha} \wedge \overrightarrow{T}) \right\rangle = 0, \end{cases} \tag{2.1}$$

where y is an arbitrary point of the spherical tangent.

Let u be a source on a sphere. Then, Xiong defined the spherical orthotomic of α relative to u as to be the set of reflections of u about the planes, lying on the above great circles (2.1) for all $s \in I$ and given by

$$\overrightarrow{\tilde{u}} = 2 \left\langle (\overrightarrow{\alpha} - \overrightarrow{u}), \overrightarrow{v} \right\rangle \overrightarrow{v} + \overrightarrow{u} \tag{2.2}$$

where $\overrightarrow{v} = \frac{\overrightarrow{B} - \left\langle \overrightarrow{B}, \overrightarrow{\alpha} \right\rangle \overrightarrow{\alpha}}{\left\| \overrightarrow{B} - \left\langle \overrightarrow{B}, \overrightarrow{\alpha} \right\rangle \overrightarrow{\alpha} \right\|}$ [18].

3. The dual vector formulation

Let L be a line and x denotes the direction and p be the position vector of any point on L. Dual vector representation allows us the Plucker vectors x and $p \wedge x$. Thus, dual Lorentzian vector $\widetilde{x}(t)$ can be written as

$$\widetilde{x}(t) = x + \varepsilon(p \wedge x) = x + \varepsilon x^*,$$

where ε is the dual unit and $\varepsilon^2 = 0$.

By using the dual Lorentzian vector function $\widetilde{x}(t) = x(t) + \varepsilon(p(t) \wedge x(t)) = x(t) + \varepsilon x^*(t)$, a ruled surface can be given as

$$m(u, t) = p(t) + ux(t).$$

It is known that the dual unit Lorentzian vector $\widetilde{x}(t)$ is a differentiable curve on the dual hyperbolic unit sphere and also having unit magnitude [14].

$$\begin{aligned} \langle \widetilde{x}, \widetilde{x} \rangle &= \langle x + \varepsilon p \wedge x, x + \varepsilon p \wedge x \rangle \\ &= \langle x, x \rangle + \langle 2\varepsilon x, p \wedge x \rangle + \varepsilon^2 \langle p \wedge x, p \wedge x \rangle \\ &= \langle x, x \rangle \\ &= -1. \end{aligned}$$

The dual arc-length of the dual Lorentzian curve $\widetilde{x}(t)$ is defined as

$$\hat{s}\,(t) = \int_0^t \left\| \frac{d\hat{x}}{dt} \right\| dt. \tag{3.1}$$

The integrant of (3.1) is the dual speed, $\widetilde{\delta}$ of $\widetilde{x}(t)$ and is

$$\widetilde{\delta} = \left\| \frac{d\widehat{x}}{dt} \right\| = \left\| \frac{dx}{dt} \right\| \left(1 + \varepsilon \frac{\left\langle \frac{dx}{dt}, \frac{dp}{dt} \wedge x \right\rangle}{\left\| \frac{dx}{dt} \right\|^2} \right) = \left\| \frac{dx}{dt} \right\| (1 + \varepsilon \Delta).$$

The curvature function

$$\Delta = \frac{\left\langle \frac{dx}{dt}, \frac{dp}{dt} \wedge x \right\rangle}{\left\| \frac{dx}{dt} \right\|^2} = \frac{\left\langle \frac{dx}{dt}, \frac{dx^*}{dt} \right\rangle}{\left\| \frac{dx}{dt} \right\|^2}$$

is the well-known distribution parameter (drall) of the ruled surface.

4. The Determination of Timelike Developable Spherical Orthotomic Ruled Surface

Let $\widetilde{x}(t)$ be a point on hyperbolic unit sphere, centered at the origin. The dual coordinates of $\widetilde{x}(t) = x_i + \varepsilon x_i^*$ can be expressed as

$$\begin{aligned} \widetilde{x_1} &= x_1 + \varepsilon x_1^* = \sinh \widetilde{\varphi} \cos \widetilde{\psi}, \\ \widetilde{x_2} &= x_2 + \varepsilon x_2^* = \sinh \widetilde{\varphi} \sin \widetilde{\psi}, \\ \widetilde{x_3} &= x_3 + \varepsilon x_3^* = \cosh \widetilde{\varphi}. \end{aligned} \tag{4.1}$$

where $\widetilde{\varphi} = \varphi + \varepsilon \varphi^*$and $\widetilde{\psi} = \psi + \varepsilon \psi^*$ are dual hyperbolic angle and dual angle respectively. Since $\varepsilon^2 = \varepsilon^3 = ... = 0$ according to the Taylor series expansion from (4.1), we obtain the real parts of $\widetilde{x}(t)$ as

$$\begin{aligned} x_1 &= \sinh \varphi \cos \psi, \\ x_2 &= \sinh \varphi \sin \psi, \\ x_3 &= \cosh \varphi, \end{aligned}$$

and the dual parts of $\widetilde{x}(t)$ as

$$\begin{aligned} x_1^* &= \varphi^* \cosh \varphi \cos \psi - \psi^* \sinh \varphi \sin \psi, \\ x_2^* &= \varphi^* \cosh \varphi \sin \psi + \psi^* \sinh \varphi \cos \psi, \\ x_3^* &= \varphi^* \sinh \varphi. \end{aligned}$$

Hence, the dual Lorentzian curve $\widetilde{x}(t) = x(t) + \varepsilon x^*(t)$ may be represented by

$$\begin{aligned} \widetilde{x}(t) &= (\sinh \varphi(t) \cos \psi(t), \sinh \varphi(t) \sin \psi(t), \cosh \varphi(t)) \\ &\quad + \varepsilon \begin{pmatrix} \varphi^*(t) \cosh \varphi(t) \cos \psi(t) - \psi^*(t) \sinh \varphi(t) \sin \psi(t), \\ \varphi^*(t) \cosh \varphi(t) \sin \psi(t) + \psi^*(t) \sinh \varphi(t) \cos \psi(t), \\ \varphi^*(t) \sinh \varphi(t) \end{pmatrix}. \end{aligned}$$

Let $\widetilde{\sigma}(t) = \sigma(t) + \varepsilon \sigma^*(t)$ be spherical orthotomic of the great circle, which lies on the $\overrightarrow{e}_2 \overrightarrow{e}_3$ plane, relative to the dual curve $\widetilde{x}(t)$. By (2.2), we get $\widetilde{\sigma}(t) = (-\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ where $\widetilde{x}_i$'s are the coordinates of $\widetilde{x}(t)$ for $i = 1, 2, 3$. By considering the spherical orthotomic dual curve, we have;

$$\begin{aligned} \widetilde{\sigma}(t) &= (-\sinh \varphi(t) \cos \psi(t), \sinh \varphi(t) \sin \psi(t), \cosh \varphi(t)) \\ &\quad + \varepsilon \begin{pmatrix} -\varphi^*(t) \cosh \varphi(t) \cos \psi(t) + \psi^*(t) \sinh \varphi(t) \sin \psi(t), \\ \varphi^*(t) \cosh \varphi(t) \sin \psi(t) + \psi^*(t) \sinh \varphi(t) \cos \psi(t), \\ \varphi^*(t) \sinh \varphi(t) \end{pmatrix} \end{aligned} \tag{4.2}$$

on the hyperbolic unit sphere corresponding to a timelike developable spherical orthotomic ruled surface $m(t, u) = p(t) + u\sigma(t)$. Because of two timelike vectors are never ortogonal, then a base curve, $p(t)$, must be a spacelike.

Since $\sigma^* = p \wedge \sigma$, we have the following system of linear equations in variables p_1, p_2, p_3;

$$\begin{aligned}
-\varphi^* \cosh\varphi\cos\psi + \psi^* \sinh\varphi\sin\psi &= p_2\cosh\varphi - p_3\sinh\varphi\sin\psi \\
\varphi^* \cosh\varphi\sin\psi + \psi^* \sinh\varphi\cos\psi &= -p_1\cosh\varphi - p_3\sinh\varphi\cos\psi \\
\varphi^* \sinh\varphi &= -p_1\sinh\varphi\sin\psi - p_2\sinh\varphi\cos\psi.
\end{aligned}$$

where p_i's are the coordinates of $p(t)$ for $i = 1, 2, 3$.

The matrix of coefficients of unknowns p_1, p_2 and p_3 is

$$\begin{bmatrix} 0 & \cosh\varphi & -\sinh\varphi\sin\psi \\ -\cosh\varphi & 0 & -\sinh\varphi\cos\psi \\ -\sinh\varphi\sin\psi & -\sinh\varphi\cos\psi & 0 \end{bmatrix}$$

and therefore its rank is 2 .

$$\begin{aligned}
p_1 &= -(p_3+\psi^*)\cos\psi\tanh\varphi - \varphi^*\sin\psi, \\
p_2 &= (p_3+\psi^*)\sin\psi\tanh\varphi - \varphi^*\cos\psi, \\
p_3 &= p_3.
\end{aligned} \tag{4.3}$$

Since $p_3(t)$ can be chosen arbitrarily, then we may take $p_3(t) = -\psi^*(t)$. In this case, (4.3) reduces to

$$\begin{aligned}
p_1 &= -\varphi^*\sin\psi, \\
p_2 &= -\varphi^*\cos\psi, \\
p_3 &= -\psi^*.
\end{aligned} \tag{4.4}$$

The distribution parameter of the timelike spherical orthotomic ruled surface given by (4.2) is obtained as follows

$$\begin{aligned}
\Delta &= \frac{< \frac{dx}{dt}, \frac{dx^*}{dt} >}{\| \frac{dx}{dt} \|^2} \\
&= \frac{\frac{d\psi}{dt}\frac{d\psi^*}{dt}\sinh^2\varphi(t) + \varphi^*\left(\frac{d\psi}{dt}\right)^2\cosh\varphi(t)\sinh\varphi(t) + \frac{d\varphi^*}{dt}\frac{d\varphi}{dt}}{\left(\frac{d\psi}{dt}\right)^2\sinh^2\varphi(t) + \left(\frac{d\varphi}{dt}\right)^2}.
\end{aligned} \tag{4.5}$$

If this timelike spherical orthotomic ruled surface is a developable, then $\Delta = 0$ and by (4.5) becomes

$$\frac{d\varphi^*}{dt}\frac{d}{dt}(\coth\varphi(t)) - \varphi^*\left(\frac{d\psi}{dt}\right)^2\coth\varphi(t) - \frac{d\psi}{dt}\frac{d\psi^*}{dt} = 0.$$

Setting

$$y(t) = \coth\varphi(t), A(t) = -\frac{\varphi^*\left(\frac{d\psi}{dt}\right)^2}{\frac{d\varphi^*}{dt}}, B(t) = -\frac{\frac{d\psi}{dt}\frac{d\psi^*}{dt}}{\frac{d\varphi^*}{dt}}$$

we are lead to a linear differential equation of first degree

$$\frac{dy}{dt} + A(t)y + B(t) = 0. \tag{4.6}$$

Let $p(t)$ be a curve. Then we can find a developable spherical orthotomic ruled surface such that its base curve is the curve $p(t)$ and by (4.4), we have

$$\begin{aligned}\tan\psi &= \frac{p_1}{p_2},\\ \varphi^* &= \sqrt{p_1^2+p_2^2},\\ \psi^* &= -p_3.\end{aligned}$$

Now only $\varphi(t)$ remains to be determined. The solution of the linear differential (4.6) gives $\coth\varphi(t)$. This solution includes an integral constant therefore we have infinitely many timelike developable spherical orthotomic ruled surface such that its base curve is $p(t)$.

Moreover, it is to be noted that $\varphi^*(t)$ has two values; by using the minus sign we obtain the reciprocal of the timelike developable spherical orthotomic ruled surface $\widetilde{x}(t)$ obtained by using the plus sign for a given integral constant.

Example 4.1. Consider $p\left(t\right)=\left(t,t,2t^3+1\right)$. If $-\frac{1}{\sqrt[4]{18}}<t<\frac{1}{\sqrt[4]{18}}$, then the ruled surface is timelike. Then we have,

$$\tan\psi=1, \varphi^*=\sqrt{2}t,\ \frac{d\psi^*}{dt}=-6t^2,\ \frac{d\psi}{dt}=0 \text{ and } \frac{d\varphi^*}{dt}=\sqrt{2}.$$

Substituting these values into (4.6) we obtain the linear differential equation of first degree

$$\frac{dy}{dt}=0.$$

The solution of this differential equation gives

$$\coth\varphi(t)=c.$$

Hence, the family of the developable timelike ruled surface is given by

$$m(t,u)=p(t)+u\sigma(t)$$

where $\sigma\left(t\right)=\left(\frac{p_2}{\varphi^*}\sinh\varphi,-\frac{p_1}{\varphi^*}\sinh\varphi,\cosh\varphi\right)$.

The graph of the developable timelike ruled surface given by this equation for $c=2$ in domain

$$D:\begin{cases}-\frac{1}{\sqrt[4]{18}}<t<\frac{1}{\sqrt[4]{18}}\\ -1<u<1\end{cases}$$

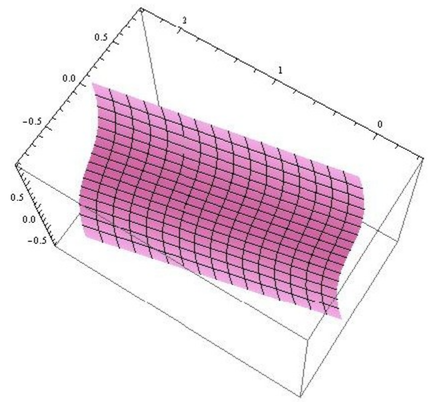

Figure 1. Spherical Orthotomic Timelike Ruled Surface

is given in Fig. 1.

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COSINE ENTROPY AND SIMILARITY MEASURES FOR FUZZY SETS

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Abstract. In the present paper, based on the cosine function, a new fuzzy entropy measure is defined. Some interesting properties of this measure are analyzed. Furthermore, a new fuzzy similarity measure has been proposed with its elegant properties. A relation between the proposed fuzzy entropy and fuzzy similarity measure has also been proved.

1. Introduction

The notion of fuzzy sets was introduced by Zadeh [19] in order to provide a scheme for handling non-statistical vague concepts. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines that include engineering, medical science, social science, artificial intelligence, signal processing, multi-agent systems, robotics, computer networks, and expert systems. Fuzzy entropy and similarity measures are as two important topics in fuzzy set theory, which have been investigated widely by many researchers from different points of view.

The first attempt to quantify fuzziness of a fuzzy set was made by Zadeh [20] in 1968, he proposed a probabilistic frame work and defined the entropy of a fuzzy set as weighted Shannon [10] entropy. In 1972, De Luca and Termini [3] first provided an axiomatic framework for the entropy of fuzzy sets based on the concept of Shannon's entropy. Kaufmann [5] introduced a fuzzy entropy measure based on a metric distance between a fuzzy set and its nearest crisp set. Yager [15] defined entropy of a fuzzy set in terms of a lack of distinction between the fuzzy set and its negation, a kind of 'norm'. Pal and Pal [7] proposed fuzzy entropy based on exponential function to measure the fuzziness called exponential fuzzy entropy. Bhandari and Pal [1] proposed generalized order-α fuzzy entropy to measure the fuzziness. In 2008, Parkash et al. [9] defined two new fuzzy entropy measures based on trigonometric functions and proved entropy maximization principle corresponding to these fuzzy entropies. Besides these, there exists quite a body of research work on applications of these theoretical studies [12, 16, 17 and 18].

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Similarity measures between two fuzzy sets, in particular, have found widespread applications in diverse fields like decision making, pattern recognition, machine learning, market prediction etc.

Talking of 'similarity measures', first, Wang [13] proposed a measure of similarity between two fuzzy sets. Salton and McGill [11] introduced a cosine similarity measure between fuzzy sets, which in essence is a kind of 'coefficient or a quotient' and applied it to information retrieval of words. Zwick et al. [21] used geometric distance and Huasdorff metrics for presenting similarity measures among fuzzy sets. Pappis and Karacapilidis [8] proposed three similarity measures for fuzzy sets based on union and intersection operations, the maximum difference, and the difference and sum of membership grades. Chen et al. [2] extended the work of Pappis and Karacapilidis [8], and defined some similarity measures on fuzzy sets based on the geometric model, the set theoretic approach, and matching function. Wang [14] proposed two similarity measures between fuzzy sets and between the elements of sets. Liu [6] as well as Fan and Xie [4] provided an axiomatic definition of similarity measure for fuzzy sets.

In the present paper two new measures called '*cosine fuzzy entropy*' and '*cosine fuzzy similarity*' are proposed. This paper is organized as follows:

In Section 2, some basic definitions related to probability theory and fuzzy sets are briefly discussed. In Section 3 cosine fuzzy entropy measure is proposed and there we verify its axiomatic requirement [3]. Some mathematical properties of the proposed entropy are also proved there. In Section 4 the cosine fuzzy similarity measure is introduced along with some of its properties. A relation between cosine fuzzy entropy and cosine fuzzy similarity is also established here.

2. Preliminaries

We start with probabilistic background. Let us denote the set of n-complete probability distributions by

$$\Gamma_n = \left\{ P = (p_1, p_2, ..., p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}, \quad n \geq 2. \tag{2.1}$$

For a probability distribution $P = (p_1, p_2, ..., p_n) \in \Gamma_n$, Shannon's entropy [14], is defined as

$$H(P) = -\sum_{i=1}^{n} p(x_i) \log_2 p(x_i). \tag{2.2}$$

Definition 2.1. Fuzzy Set [19]: A fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, ..., x_n\}$ is given by

$$A = \{ \langle x, \ \mu_A(x) \rangle \mid x \in X \}, \tag{2.3}$$

where $\mu_A(x) : X \to [0, 1]$ is the membership function of A. The number $\mu_A(x)$ describes the degree of membership of $x \in X$ in A.

Definition 2.2. A fuzzy set A^* is called a sharpened version of fuzzy set A if the following conditions are satisfied:

$$\begin{aligned} \mu_{A^*}(x_i) &\leq \mu_A(x_i) \qquad \text{if } \mu_A(x) \leq 0.5 \ \forall \ i, \\ \mu_{A^*}(x_i) &\geq \mu_A(x_i) \qquad \text{if } \mu_A(x) \geq 0.5 \ \forall \ i. \end{aligned}$$

Note: Throughout this paper, we shall denote the set of all fuzzy sets defined in X by$FS\,(X)$.

Definition 2.3. Set Operations on FSs [19]: Let $A, B \in FS\,(X)$ be given by

$$A = \{\langle x,\, \mu_A\,(x)\rangle \mid x \in X\}\,,$$
$$B = \{\langle x,\, \mu_B\,(x)\rangle \mid x \in X\}\,,$$

then usually set operations are defined as follows:

[(i)]

(1) $A \subseteq B$ iff $\mu_A\,(x) \leq \mu_B\,(x) \quad \forall\, x \in X$;
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
(3) $A^C = \{\,\langle x,\ 1 - \mu_A\,(x)\rangle \mid x \in X\}$;
(4) $A \cap B = \{\langle x, \mu_A\,(x) \wedge \mu_B\,(x)\rangle \mid x \in X\}$;
(5) $A \cup B = \{\langle x, \mu_A\,(x) \vee \mu_B\,(x)\rangle \mid x \in X\}$;

where $\vee$, $\wedge$ stand for max. and min. operators, respectively.

In fuzzy set theory, a measure of fuzziness is the 'fuzzy entropy' which expresses the amount of aggregated ambiguity of a fuzzy set A. The first attempt to quantify the fuzziness was made in 1968 by Zadeh [20], who defined the entropy of a fuzzy set A with respect to (X, P) as

$$H\,(A, P) = -\sum_{i=1}^{n} \mu_A\,(x_i)\, p\,(x_i) \log_2 p\,(x_i)\,. \tag{2.4}$$

De Luca and Termini [3] defined fuzzy entropy for a fuzzy set A corresponding (2.2) as

(2.5)

$$H_{DT}\,(A) = -\frac{1}{n}\sum_{i=1}^{n}\left[\mu_A\,(x_i)\ \log_2\,(\mu_A\,(x_i)) + (1 - \mu_A\,(x_i)) \log_2\,(1 - \mu_A\,(x_i))\right].$$

Based on exponential function, Pal and Pal [7] introduced exponential fuzzy entropy for fuzzy set A as

$$_{e}H\,(A) = \frac{1}{n\,(\sqrt{e} - 1)}\sum_{i=1}^{n}\left[\mu_A\,(x_i)\ e^{1-\mu_A(x_i)} + (1 - \mu_A\,(x_i))\ e^{\mu_A(x_i)} - 1\right]. \tag{2.6}$$

Later, Bhandari and Pal [1]made a survey on entropy measures on fuzzy sets and introduced the following parametric fuzzy entropy for fuzzy set A as

$$H_\alpha\,(A) = \frac{1}{n\,(1 - \alpha)}\sum_{i=1}^{n} \log\left[\mu_A^\alpha\,(x_i) \ + (1 - \mu_A\,(x_i))^\alpha\right]. \tag{2.7}$$

Parkash et al. [9] defined two fuzzy entropy measures for fuzzy set A based on trigonometric functions (sine and cosine) given by

(2.8)

$$H_{OPR1}\,(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\sin\frac{\pi\mu_A\,(x_i)}{2} + \sin\frac{\pi\,(1 - \mu_A\,(x_i))}{2} - 1\right\} \times \frac{1}{(\sqrt{2} - 1)}\right],$$

(2.9)

$$H_{OPR2}\,(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\cos\frac{\pi\mu_A\,(x_i)}{2} + \cos\frac{\pi\,(1 - \mu_A\,(x_i))}{2} - 1\right\} \times \frac{1}{(\sqrt{2} - 1)}\right].$$

Definition 2.4. Similarity Measure of FSs [6]: A real function $S : FS(X) \times FS(X) \rightarrow [0,1]$ is called the similarity measure of the fuzzy sets, if S satisfies the following properties:

[S1.]

(1) $0 \leq S(A,B) \leq 1 \;\forall\; A,B \in FS(X)$.
(2) $S(A,B) = S(B,A) \;\forall\; A,B \in FS(X)$.
(3) $S(A,B) = 1$ if and only if $A = B$, i.e. $\mu_A(x_i) = \mu_B(x_i)$ for all $i = 1, 2, ..., n$.
(4) For all $A,B,C \in FS(X)$, if $A \subseteq B \subseteq C$, then $S(A,C) \leq S(A,B)$, $S(A,C) \leq S(B,C)$.

In the next section, we introduce a new entropy measure on fuzzy sets called '*cosine fuzzy entropy*' and verify its axiomatic validity.

3. Cosine Fuzzy Entropy

We submit following formal definition of a new measure of 'fuzzy entropy':

Definition 3.1. Cosine Fuzzy Entropy: Let A be a fuzzy set defined on $X = \{x_1, x_2, ..., x_n\}$ having the membership values $\mu_A(x_i)$, $i = 1, 2, ..., n$. We define the cosine fuzzy entropy for fuzzy set A, $H_{\cos}(A)$ as:

$$H_{\cos}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\cos\left(\frac{(2\mu_A(x_i)-1)}{2}\pi\right)\right]. \tag{3.1}$$

As a first step, in the next theorem, we establish properties that according to De Luca and Termini [3] justify the above proposed measure to be a valid 'fuzzy entropy'.

Theorem 3.1. *The $H_{\cos}(A)$ measure in (3.1) of the cosine fuzzy entropy satisfies the following propositions:*

*[P1.]**(Sharpness)**: $H(A)$ is minimum if and only if A is a crisp set, i.e. $\mu_A(x_i) = 0$ or $1 \;\forall\; x_i \in X$.* ***(Maximality)**: $H(A)$ is maximum if and only if A is a most fuzzy set, i.e. $\mu_A(x_i) = 0.5 \;\forall\; x_i \in X$.* ***(Resolution)**: $H(A^*) \leq H(A)$, where A^* is a sharpened version of the set A.* ***(Symmetry)**: $H(A) = H(A^C)$, where A^C is the complement set of the fuzzy set A.*

(3) *Proof.* Let $\Delta_A = \left(\frac{(2\mu_A(x_i)-1)}{2}\pi\right)$ and then from $0 \leq \mu_A(x_i) \leq 1$, we note that

$$-\frac{\pi}{2} \leq \Delta_A \leq \frac{\pi}{2} \Rightarrow 0 \leq \cos\frac{(2\mu_A(x_i)-1)}{2}\pi \leq 1 \Rightarrow 0 \leq H_{\cos}(A) \leq 1.$$

P1. ***(Sharpness)*:** First, let A be a crisp set with membership values either 0 or 1 for all $x_i \in X$. Then from (3.1) we simply obtain

$$H_{\cos}(A) = 0. \tag{3.2}$$

This proves 'if' part of the statement. Next let us suppose that $H_{\cos}(A) = 0$, i.e.

$$\sum_{i=1}^{n}\left[\cos\frac{(2\mu_A(x_i)-1)}{2}\pi\right] = 0. \tag{3.3}$$

Then, this being the sum of n terms and each term in the summation is non negative, then for all i,

$$\cos\frac{\left(2\mu_A\left(x_i\right)-1\right)}{2}\pi=0. \tag{3.4}$$

From (3.4), it is easy to deduce that $\mu_A\left(x_i\right)=0$ or 1 for all $x_i\in X$, that is A is crisp.

P2. ***(Maximality)***: Let $\mu_A\left(x_i\right)=0.5$ for all $x_i\in X$. From (3.1) we obtain $H_{\cos}\left(A\right)=1$.

Now, let $H_{\cos}\left(A\right)=1$, and then also from (3.1), we have

$$\cos\Delta_A=1\Rightarrow\Delta_A=0\Rightarrow\mu_A\left(x_i\right)=0.5\,\forall\,x_i\in X.$$

P3. ***(Resolution):*** Let

$$f\left(\mu_A\left(x_i\right)\right)=\cos\frac{\left(2\mu_A\left(x_i\right)-1\right)}{2}\pi\qquad\forall\,x_i\in X. \tag{3.5}$$

Since $f\left(\mu_A\left(x_i\right)\right)$ is an increasing function of $\mu_A\left(x_i\right)$ in the range $[0,0.5)$ and is a decreasing function of $\mu_A\left(x_i\right)$ in the range $(0.5,1]$, therefore

$$\begin{aligned}\mu_{A^*}\left(x_i\right)\le\mu_A\left(x_i\right)&\Rightarrow\frac{\left(2\mu_{A^*}\left(x_i\right)-1\right)}{2}\pi\le\frac{\left(2\mu_A\left(x_i\right)-1\right)}{2}\pi\\&\Rightarrow f\left(\mu_{A^*}\left(x_i\right)\right)\le f\left(\mu_A\left(x_i\right)\right)\forall\,x_i\in[0\ ,\ 0.5)\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\mu_{A^*}\left(x_i\right)\ge\mu_A\left(x_i\right)&\Rightarrow\frac{\left(2\mu_{A^*}\left(x_i\right)-1\right)}{2}\pi\ge\frac{\left(2\mu_A\left(x_i\right)-1\right)}{2}\pi\\&\Rightarrow f\left(\mu_{A^*}\left(x_i\right)\right)\ge f\left(\mu_A\left(x_i\right)\right)\forall\,x_i\in(0.5\ ,\ 1]\,.\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$f\left(\mu_{A^*}\left(x_i\right)\right)\le f\left(\mu_A\left(x_i\right)\right). \tag{3.8}$$

Since $H_{\cos}\left(A\right)=\frac{1}{n}\sum_{i=1}^{n}\left(f\left(\mu_A\left(x_i\right)\right)\right)$ and $H_{\cos}\left(A^*\right)=\frac{1}{n}\sum_{i=1}^{n}\left(f\left(\mu_{A^*}\left(x_i\right)\right)\right)$, then we obtain

$$H_{\cos}\left(A^*\right)\le H_{\cos}\left(A\right). \tag{3.9}$$

P4. ***(Symmetry):*** It is clear from definition of $H_{\cos}\left(A\right)$ and with $\mu_{A^C}\left(x_i\right)=1-\mu_A\left(x_i\right)$, we conclude that

$$H\left(A\right)=H\left(A^C\right). \tag{3.10}$$

Hence $H_{\cos}\left(A\right)$ is an axiomatically valid measure of fuzzy entropy.
This proves the theorem. □

We now turn to study of properties of $H_{\cos}\left(A\right)$. The proposed cosine fuzzy entropy $H_{\cos}\left(A\right)$, satisfies the following interesting properties.

Theorem 3.2. *Let $A,B\in FS\left(X\right)$ be given by*

$$A=\left\{\left\langle x,\,\mu_A\left(x\right)\right\rangle\,|\,x\in X\right\},$$

$$B=\left\{\left\langle x,\,\mu_B\left(x\right)\right\rangle\,|\,x\in X\right\},$$

such that they satisfy for any x_i either $A\subseteq B$ or$A\supset B$, then we have

$$H_{\cos}\left(A\cup B\right)+H_{\cos}\left(A\cap B\right)=H_{\cos}\left(A\right)+H_{\cos}\left(B\right).$$

Proof. Let us separate X into two parts X_1 and X_2, where

$$X_1 = \{x_i \in X : A \subseteq B\},$$

and

$$X_2 = \{x_i \in X : A \supset B\}.$$

That is, for all $x_i \in X_1$

$$\mu_A(x_i) \leq \mu_B(x_i), \tag{3.11}$$

and for all $x_i \in X_2$

$$\mu_A(x_i) > \mu_B(x_i). \tag{3.12}$$

From definition in (3.1), we have

$$H_{\cos}(A \cup B) = \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(2\mu_{A\cup B}(x_i) - 1)}{2}\pi\right]$$

$$= \frac{1}{n}\left[\left\{\sum_{x_i \in X_1}\cos\frac{(2\mu_B(x_i) - 1)}{2}\pi\right\} + \left\{\sum_{x_i \in X_2}\cos\frac{(2\mu_A(x_i) - 1)}{2}\pi\right\}\right]. \tag{3.13}$$

Again from definition in (3.1), we have

$$H_{\cos}(A \cap B) = \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(2\mu_{A\cap B}(x_i) - 1)}{2}\pi\right]$$

$$= \frac{1}{n}\left[\left\{\sum_{x_i \in X_1}\cos\frac{(2\mu_A(x_i) - 1)}{2}\pi\right\} + \left\{\sum_{x_i \in X_2}\cos\frac{(2\mu_B(x_i) - 1)}{2}\pi\right\}\right]. \tag{3.14}$$

Now adding (3.13) and (3.14), we get

$$H_{\cos}(A \cup B) + H_{\cos}(A \cap B) = H_{\cos}(A) + H_{\cos}(B).$$

This proves the theorem. □

Corollary 3.1. *For any $A \in FS(X)$, and A^C the complement of fuzzy setA, then*

$$H_{\cos}(A) = H_{\cos}\left(A^C\right) = H_{\cos}\left(A \cup A^C\right) = H_{\cos}\left(A \cap A^C\right). \tag{3.15}$$

Proof. This follows from the result $H(A) = H\left(A^C\right)$ and the above theorem. □

In the next section, we propose a new similarity measure between fuzzy sets called '*cosine fuzzy similarity*' and study their properties. We have also given a relation between cosine fuzzy entropy and cosine fuzzy similarity here.

4. Cosine Fuzzy Similarity Measure

In this section, we propose a new similarity measure for FSs. The formal definition is as follows:

Definition 4.1. Cosine Fuzzy Similarity Measure: Given two fuzzy sets A and B defined in $X = \{x_1, x_2, ..., x_n\}$ having the membership values $\mu_A(x_i)$, $i = 1, 2, ..., n$ and $\mu_B(x_i)$, $i = 1, 2, ..., n$ respectively, we define the measure of cosine fuzzy similarity, $S_{FS}(A, B)$, between FSs A and B, as

$$S_{FS}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi\right]. \tag{4.1}$$

In the next theorem, we establish properties that according to Liu [6], justify our proposed measure to be a valid 'fuzzy similarity':

Theorem 4.1. *The $S_{FS}(A,B)$ measure in* (4.1) *of the fuzzy similarity satisfies the following properties:*

[S1.] $0 \leq S_{FS}(A,B) \leq 1$; $\quad S_{FS}(A,B) = S_{FS}(B,A)$; $\quad S_{FS}(A,B) = 1$ *if and only if* $A = B$, *i.e.* $\mu_A(x_i) = \mu_B(x_i)$ *for all* $i = 1, 2, ..., n$. *For all* $A, B, C \in FS(X)$, *if* $A \subseteq B \subseteq C$, *then* $S_{FS}(A,C) \leq S_{FS}(A,B)$, $S_{FS}(A,C) \leq S_{FS}(B,C)$.

(3) *Proof.* *S1.* Let $\Delta_{(A,B)} = \frac{(\mu_A(x_i)-\mu_B(x_i))}{2}\pi$, then from $0 \leq \mu_A(x_i), \mu_B(x_i) \leq 1$, we have

$$-\frac{\pi}{2} \leq \Delta_{(A,B)} \leq \frac{\pi}{2} \Rightarrow 0 \leq \cos\frac{(\mu_A(x_i)-\mu_B(x_i))}{2}\pi \leq 1 \Rightarrow 0 \leq S_{FS}(A,B) \leq 1. \tag{4.2}$$

S2. This simply follows from symmetric expression of $S_{FS}(A,B)$.

S3. Let $A = B$, i.e. $\mu_A(x_i) = \mu_B(x_i)$ for all $i = 1, 2, ..., n$. Then from (4.1) we obtain that

$$S_{FS}(A,B) = 1. \tag{4.3}$$

This proves 'if' part of the statement. Next suppose that $S_{FS}(A,B) = 1$, i.e.

$$\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i)-\mu_B(x_i))}{2}\pi\right] = n. \tag{4.4}$$

Then, this being the sum of n terms, each term in the summation being less than or equal to 1, then for all i,

$$\cos\frac{(\mu_A(x_i)-\mu_B(x_i))}{2}\pi = 1 \tag{4.5}$$

or

$$\mu_A(x_i) - \mu_B(x_i) = 0. \tag{4.6}$$

From (4.6), it immediately follows that $\mu_A(x_i) = \mu_B(x_i)$ for any $x_i \in X$, i.e. $A = B$.

S4. Since

$$A \subseteq B \subseteq C \Rightarrow \mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i), \tag{4.7}$$

then

$$\left.\begin{array}{l} \frac{(\mu_A(x_i)-\mu_B(x_i))}{2}\pi \geq \frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi \\ \frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi \geq \frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi \end{array}\right\}. \tag{4.8}$$

From (4.8) and the nature of cosine function, we get

$$\cos\frac{(\mu_A(x_i)-\mu_B(x_i))}{2}\pi \geq \cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi \Rightarrow S_{FS}(A,C) \leq S_{FS}(A,B), \tag{4.9}$$

$$\cos\frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi \geq \cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi \Rightarrow S_{FS}(A,C) \leq S_{FS}(B,C). \tag{4.10}$$

This proves the theorem. $\square$

The importance and strength of this measure lies in its properties that we study in the following theorems.
For proofs of the properties, we will consider separation of X into two parts X_1 and X_2, such that

$$X_1 = \{x_i \in X : A \subseteq B\},$$

and

$$X_2 = \{x_i \in X : A \supset B\}.$$

That is, for all $x_i \in X_1$

$$\mu_A(x_i) \leq \mu_B(x_i), \tag{4.11}$$

and for all $x_i \in X_2$

$$\mu_A(x_i) > \mu_B(x_i). \tag{4.12}$$

Theorem 4.2. *For $A, B \in FS(X)$, and if they satisfy that for any $x_i \in X$, either $A \subseteq B$or $A \supset B$, then*

$$S_{FS}(A \cup B, A \cap B) = S_{FS}(A\,, B).$$

Proof. Using Definition 4.1, we have

$$\begin{aligned}
&S_{FS}(A \cup B, A \cap B) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_{A\cup B}(x_i) - \mu_{A\cap B}(x_i))}{2}\pi\right] \\
&= \frac{1}{n}\left[\sum_{x_i \in X_1}\left\{\cos\frac{(\mu_B(x_i) - \mu_A(x_i))}{2}\pi\right\} + \sum_{x_2 \in X_2}\left\{\cos\frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi\right\}\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi\right] \\
&= S_{FS}(A, B).
\end{aligned}$$

This proves the theorem. □

Theorem 4.3. *For $A, B, C \in FS(X)$,*
[(i).]$S_{FS}(A \cup B, C) \leq S_{FS}(A, C) + S_{FS}(B, C)$, $\quad S_{FS}(A \cap B, C) \leq S_{FS}(A, C) + S_{FS}(B, C)$.

(2) *Proof.* We prove (i) only, (ii) can be proved analogously.
(i) Let us consider the expressions for

$$S_{FS}(A, C) + S_{FS}(B, C) - S_{FS}(A \cup B, C) \tag{4.13}$$

$$\begin{aligned}
= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i) - \mu_C(x_i))}{2}\pi\right] + \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_B(x_i) - \mu_C(x_i))}{2}\pi\right] \\
- \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_{A\cup B}(x_i) - \mu_C(x_i))}{2}\pi\right]
\end{aligned}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi\right] + \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi\right]$$
$$-\frac{1}{n}\left[\sum_{x_i\in X_1}\left\{\cos\frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi\right\} + \sum_{x_i\in X_2}\left\{\cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi\right\}\right]$$

$$= \frac{1}{n}\left[\sum_{x_i\in X_1}\left\{\cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi\right\} + \sum_{x_i\in X_2}\left\{\cos\frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi\right\}\right]$$
$$\geq 0.$$

This proves the theorem. □

Theorem 4.4. *For $A, B, C \in FS(X)$,*

$$S_{FS}(A\cup B, C) + S_{FS}(A\cap B, C) = S_{FS}(A, C) + S_{FS}(B, C).$$

Proof. From Definition 4.1, we first have:

$$S_{FS}(A\cup B, C)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_{A\cup B}(x_i)-\mu_C(x_i))}{2}\pi\right]$$

(4.14)

$$= \frac{1}{n}\left[\sum_{x_i\in X_1}\left\{\cos\frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi\right\} + \sum_{x_i\in X_2}\left\{\cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi\right\}\right].$$

Next, again from Definition 4.1, we have

$$S_{FS}(A\cap B, C)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_{A\cap B}(x_i)-\mu_C(x_i))}{2}\pi\right]$$

(4.15)

$$= \frac{1}{n}\left[\sum_{x_i\in X_1}\left\{\cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi\right\} + \sum_{x_i\in X_2}\left\{\cos\frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi\right\}\right].$$

After adding (4.14) and (4.14), we get the result.
This proves the theorem. □

Theorem 4.5. *For $A, B \in FS(X)$,*
[(i).]$S_{FS}(A, B) = S_{FS}\left(A^C, B^C\right)$; $S_{FS}\left(A, B^C\right) = S_{FS}\left(A^C, B\right)$; $S_{FS}(A, B) + S_{FS}\left(A^C, B\right) = S_{FS}\left(A^C, B^C\right) + S_{FS}\left(A, B^C\right)$;

where A^C and B^C represent complements of the fuzzy sets A and B, respectively.

(3) *Proof.* *(i).* It simply follows from the relation that membership of an element in a set has with its complement.
(ii). Let us consider the expressions for

$$S_{FS}\left(A, B^C\right) - S_{FS}\left(A^C, B\right) \tag{4.16}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i)-(1-\mu_B(x_i)))}{2}\pi\right]-\frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{((1-\mu_A(x_i))-\mu_B(x_i))}{2}\pi\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(1-\mu_A(x_i)-\mu_B(x_i))}{2}\pi\right]-\frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(1-\mu_A(x_i)-\mu_B(x_i))}{2}\pi\right]$$
$$= 0.$$

(iii). It is obvious from *(i)* and *(ii)*.

This proves the theorem. □

Interestingly, the cosine fuzzy similarity measure given in (4.1) leads to interesting situations when it is consider between a set and its complement. The measure (4.1) reduces to cosine fuzzy entropy (3.1), as shown in the next theorem.

Theorem 4.6. *For each $A \in FS(X)$,*

$$S_{FS}\left(A, A^C\right) = H_{\cos}(A). \tag{4.17}$$

Proof. The proof follows directly from the Definitions 2.3, 3.1 and 4.1. □

5. Conclusions

We have introduced two measures using cosine function. These measures having elegant properties, present a new vista for applications and further considerations.

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PARALLEL AND SEMIPARALLEL LIGHTLIKE HYPERSURFACES OF SEMI-RIEMANNIAN SPACE FORMS

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Abstract. In this paper, some properties of lightlike hypersurfaces with parallel and semiparallel second fundamental forms are investigated in semi-Riemannian space forms. Then some generalizations of these conditions are performed.

1. Introduction

The interest on submanifolds with parallel second fundamental forms increased in 1970s. The study on submanifolds with parallel second fundamental form of Euclidean spaces was started by J. Vilms [21] and similar case for hypersurfaces was studied by U. Simon and A. Weinstein [19]. A classification to the submanifolds with parallel second fundamental form of space forms was carried by Takeuchi [20] who makes the term parallel submanifolds more popular, especially from the local point of view. After then parallel submanifolds of Riemannian space forms and non-degenerate ones of semi-Riemannian space forms have been studied in many papers [1], [12, 13, 14], [16]. Later the condition for parallelity was generalized to higher orders and k-parallel submanifolds were introduced [4], [5], [15].

Parallel submanifolds were also extended to a more general class of submanifolds called semiparallel submanifolds. These wider class of submanifolds in Euclidean space was introduced and classified by J. Deprez [2], [3]. F. Dillen has given a classification of semiparallel hypersurfaces of a real space form [6]. Ü. Lumiste has written a book on this subject and its generalization including many of the old and recent studies [11].

Here some conditions related to parallel and semiparallel hypersurfaces are investigated for the degenerate case which is mostly ignored in the mentioned studies. We will use the screen distribution approach of a lightlike hypersurface explained as in the books [7],[9].

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2. Preliminaries

Let (M,g) be a hypersurface of an $(m+2)-$dimensional semi-Riemannian manifold $(\bar{M},\bar{g})$ of index $q\in 1,...,m+1$. As for any $p\in M, T_pM$ is a hyperplane of the semi-Euclidean space $(T_p\bar{M},\bar{g}_p)$, we consider

$$T_pM^{\perp}=\{V_p\in T_p\bar{M};\bar{g}_p(V_p,W_p)=0,\forall W_p\in T_pM\},$$

and

$$RadT_pM=T_pM\cap T_pM^{\perp}.$$

Then M is called a lightlike hypersurface of $\bar{M}$ if $RadT_pM\neq\{0\}$ at any $p\in M$. The semi-Riemannian metric $\bar{g}$ on $\bar{M}$ induces on M a symmetric tensor field g of type $(0,2)$, i.e., $g_p(X_p,Y_p)$, for any $p\in M$. Also we know that g has a constant rank m on M and $RadT_pM=TM^{\perp}$ [7].

The tangent bundle space TM of a lightlike hypersurface has the decomposition

$$TM=RadTM\perp S(TM) \tag{2.1}$$

where the complementary vector bundle $S(TM)$ is called the screen distribution on M. So, a lightlike hypersurface (M,g) of a semi-Riemannian manifold $\left(\bar{M},\bar{g}\right)$ is generally shown by $(M,g,S(TM))$. By [7, Theorem 1.1] there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non-zero null section $\xi\in RadTM$ on a coordinate neighborhood $U\subset M$, there exists a unique null section N of $tr(TM)$ on U satisfying

$$\bar{g}\left(\xi,N\right)=1,\bar{g}\left(N,N\right)=\bar{g}\left(N,X\right)=0,\forall X\in\Gamma\left(S\left(TM\right)_{|U}\right)$$

where $tr\left(TM\right)$ and N are called the lightlike transversal vector bundle and the null transversal vector field of M with respect to $S\left(TM\right)$ respectively. Then we have the following decomposition of $T\bar{M}_{|M}$:

$$T\bar{M}|_M=S(TM)\perp\left(RadTM\oplus tr\left(TM\right)\right)=TM\oplus tr\left(TM\right).$$

Let ∇ be the induced connection on the lightlike hypersurface $(M,g,S(TM))$ and P be the projection morphism of TM on $S\left(TM\right)$ with respect to the decomposition (2.1). Then the local Gauss and Weingarten formulas are given by

$$\begin{aligned}\bar{\nabla}_XY &= \nabla_XY+h\left(X,Y\right),\\ \bar{\nabla}_XN &= -A_NX+\nabla^t_XN,\\ \nabla_XPY &= \nabla^*_XPY+h^*\left(X,PY\right),\\ \nabla_X\xi &= -A^*_{\xi}X-\nabla^{*t}_X\xi,\end{aligned} \tag{2.2}$$

for any $X,Y\in\Gamma(TM)$, where ∇^*,∇^t and ∇^{*t} are the linear connections on $S(TM),tr(TM)$ and $RadTM$, h and h^* are the second fundamental forms of M and $S(TM)$, A_N and A^*_{ξ} are the shape operators of M and $S(TM)$ respectively. Locally, suppose ξ,N is a pair of sections on $U\subset M$ satisfying (2). Then define a symmetric $slF(U)-$ bilinear form which is called the local second fundamental form of M and a 1-form τ on $U\subset M$ defined by

$$\begin{aligned}B(X,Y) &= \bar{g}(h(X,Y),\xi),\\ \tau(X) &= \bar{g}(\nabla^t_XN,\xi)\end{aligned}$$

for any $X, Y \in \Gamma(TM_{|U})$. It follows that

$$\begin{aligned} h(X,Y) &= B(X,Y)N, \\ \nabla^t_X N &= \tau(X)N, \\ \nabla^{*t}_X \xi &= \bar{g}(\nabla_X \xi, N) = -\bar{g}(\xi, \bar{\nabla}_X N) = -\tau(X)\xi. \end{aligned}$$

Also we define the local screen fundamental form of $S(TM)$ as

$$C(X, PY) = \bar{g}(h^*(X, PY), N).$$

Hence, on U the Gauss and Weingarten equations become

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + B(X,Y)N, \\ \bar{\nabla}_X N &= -A_N X + \tau(X)N, \\ \nabla_X PY &= \nabla^*_X PY + C(X,PY)\xi, \\ \nabla_X \xi &= -A^*_\xi X + \tau(X)\xi, \end{aligned} \tag{2.3}$$

h is independent of the choice of $S(TM)$ and it satisfies the equation

$$h(X,\xi) = 0, \ \forall X \in \Gamma(TM). \tag{2.4}$$

The linear connection ∇ of M is not metric and satisfies the equation

$$(\nabla_X \ g)(Y,Z) = \bar{g}(h(X,Y), Z) + \bar{g}(h(X,Z), Y) \tag{2.5}$$

for any $X, Y, Z \in \Gamma(TM)$. But the connection ∇^* of $S(TM)$ is metric.

The second fundamental forms h and h^* are related to their shape operators with the equations

$$\bar{g}(h(X,Y), \xi) = B(X,Y) = g\left(A^*_\xi X, Y\right), \ \bar{g}\left(A^*_\xi X, N\right) = 0, \tag{2.6}$$

$$\bar{g}(h^*(X,PY), N) = C(X,PY) = g(A_N X, PY), \ \bar{g}(A_N X, N) = 0. \tag{2.7}$$

From (2.6), A^*_ξ is $S(TM)-$valued and self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{2.8}$$

Covariant derivatives of h and A_N with respect to the connection ∇ are defined as

$$(\nabla_X h)(Y,Z) = \nabla^t_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.9}$$

$$\nabla_X (A_N Y) = (\nabla_X A_N) Y + A_N (\nabla_X Y). \tag{2.10}$$

The Riemann curvature tensor of a lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $\left(\bar{M}, \bar{g}\right)$ is given at [10] by

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ &\quad + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z). \end{aligned} \tag{2.11}$$

Then for a lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ we get the Gauss curvature equation as

$$R(X,Y)Z = c\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} - A_{h(X,Z)}Y + A_{h(Y,Z)}X \tag{2.12}$$

and the Codazzi equation as

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z).$$

For a lightlike hypersurface M of a semi-Euclidean space $\bar{M}$, using the equality $h(X,Y) = B(X,Y)N$ the equation (2.12) becomes

$$R(X,Y)Z = B(X,Z)A_N Y + B(Y,Z)A_N X. \tag{2.13}$$

The Ricci tensor of a lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ is given at [8] by

$$R^{(0,2)}(X,Y) = mcg(X,Y) + B(X,Y)\, trA_N - B(Y, A_N X). \tag{2.14}$$

Let $(M, g, S(TM))$ be lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. M is totally umbilical, if and only if, locally, on each $U \subset M$ there exists a smooth function ρ such that

$$B(X,Y) = \rho g(X,Y), \quad \forall X, Y \in \Gamma(TM|_U) \tag{2.15}$$

is satisfied [7].

For a $(r,s)-$ tensor field T we define the second covariant derivative $(\nabla^2 T)$ as the $(r, s+2)-$tensor field [17]

$$\begin{aligned}(\nabla^2 T)(W_1, ..., W_s; U, V) &= (\nabla^2_{U,V} T)(W_1, ..., W_s)\\ &= \nabla_U((\nabla_V T)(W_1, ..., W_s))\\ &- (\nabla_{\nabla_U V} T)(W_1, ..., W_s)\\ &- (\nabla_V T)(\nabla_U W_1, ..., W_s)\\ &- ... - (\nabla_V T)(W_1, ..., \nabla_U W_s).\end{aligned} \tag{2.16}$$

3. Parallel and 2-Parallel Lightlike Hypersurfaces

A tensor field is said to be parallel if its covariant derivative vanishes. A hypersurface whose second fundamental form h is parallel, that is $\nabla h = 0$, is called a parallel hypersurface. In general if the second fundamental form h of a hypersurface satisfies the condition

$$\nabla^k h = 0, \quad \nabla^s h \neq 0 \ (s < k),$$

then the hypersurface is said to be $k-$parallel [11]. Thus, a $0-$parallel hypersurface is simply a totally geodesic one and a $1-$paralel hypersurface is parallel that is not totally geodesic.

We already have the following theorem for parallel lightlike hypersurfaces:

Theorem 3.1. [18] *Let M be a lightlike hypersurface of a Lorentzian manifold $\bar{M}$. Then the second fundamental form of M is parallel if and only if M is totally geodesic.*

For the general case the following theorem can be proved.

Theorem 3.2. *There exists no proper totally umbilical $2-$parallel lightlike hypersurface of a semi-Riemannian space form.*

Proof. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$. Using the identity (3.1) the second order covariant derivative of the second fundamental form h of M can be found as

$$\begin{aligned}\left(\nabla^2_{V,W} h\right)(X,Y) &= \nabla^t_V((\nabla_W h)(X,Y)) - (\nabla_W h)(\nabla_V X, Y)\\ &\quad - (\nabla_W h)(X, \nabla_V Y) - (\nabla_{\nabla_V W} h)(X,Y)\end{aligned}$$

for any $X, Y, V, W \in \Gamma(TM)$. If we assume that M is $2-$ parallel, setting $W = X = \xi$, we get

$$\begin{aligned}0 &= \nabla^t_V((\nabla_\xi h)(\xi, Y)) - (\nabla_\xi h)(\nabla_V \xi, Y)\\ &\quad - (\nabla_\xi h)(\xi, \nabla_V Y) - (\nabla_{\nabla_V \xi} h)(\xi, Y).\end{aligned} \tag{3.1}$$

Substituting (2.4) into (2.9) and using the last equation of (2.2) with (2.8) we obtain $(\nabla_\xi h)(\xi, Y) = 0$ and $(\nabla_\xi h)(\xi, \nabla_V Y) = 0$. Since h is symmetric, from the equation (2.9) we see that ∇h is also symmetric. Then by the Codazzi equation we can write

$$(\nabla_\xi h)(\nabla_V \xi, Y) = (\nabla_{\nabla_V \xi} h)(\xi, Y) = (\nabla_Y h)(\xi, \nabla_V \xi).$$

So the equation (3.1) becomes

$$0 = -2(\nabla_Y h)(\xi, \nabla_V \xi).$$

Again by the equations (2.4),(2.6),(2.8),(2.9),(2.15), the last equation of (2.2) and since the lightlike hypersurface is totally umbilical we obtain the result

$$0 = h(A^*_\xi Y, A^*_\xi V) = B(A^*_\xi Y, A^*_\xi V)N = \rho^2 h(Y, V).$$

Since the second fundamental form of a $2-$parallel lightlike hypersurface can not vanish and $\rho \neq 0$, we get a contradiction and the theorem is proved. □

4. Semiparallel and 2-Semiparallel Lightlike Hypersurfaces

The integrability condition of the differential system $\nabla h = 0$ is given by the equation

$$R(X, Y) \cdot h = 0$$

where $R(X, Y)$ is the curvature operator and h is the second fundamental form. This equation characterizes the semiparallel hypersurfaces. Equivalently, for $X, Y, Z, W \in \Gamma(TM)$ any hypersurface satisfying the equation

$$h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0 \tag{4.1}$$

is called a semiparallel hypersurface [18]. As a generalization of this, we consider the following integrability condition of the system $\nabla^k h = 0$:

$$R(X, Y) \cdot \nabla^{k-1} h = 0. \tag{4.2}$$

Hypersurfaces with this condition are said to be $k-$semiparallel. $1-$semiparallel is simply a semiparallel one. We know that non-degenerate parallel hypersurfaces of semi-Riemannian spaces are semiparallel [11]. It is clear that the converse of this is not true. We know the following theorem for the lightlike hypersurfaces of semi-Euclidean spaces:

Theorem 4.1. *Let $(M, g, S(TM))$ be a semiparallel lightlike hypersurface of semi-Euclidean $(n+2)-$space. Then either M is totally geodesic or $C\left(\xi, A^*_\xi U\right) = 0$ for any $U \in (S(TM))$ and $\xi \in \Gamma\left(TM^\perp\right)$, where C and A^*_ξ are the second fundamental form and the shape operator of the screen distribution $S(TM)$, respectively* [18].

This theorem can be extended as to be valid also for Lorentzian space forms:

Theorem 4.2. *Let $(M, g, S(TM))$ be a semiparallel lightlike hypersurface of a Lorentzian space form $(\bar{M}(c), \bar{g})$. Then, for any $Z \in \Gamma(TM)$, either M is totally geodesic or the equation $R^{(0,2)}(\xi, Z) = 0$ is satisfied.*

Proof. Substituting (2.12) in (4.1) we get

$$\begin{aligned} & h\left(R\left(X,Y\right)Z,W\right)+h\left(Z,R\left(X,Y\right)W\right) \\ = \; & c\{g\left(Y,Z\right)B\left(X,W\right)-g\left(X,Z\right)B\left(Y,W\right) \\ & +g\left(Y,W\right)B\left(X,Z\right)-g\left(X,W\right)B\left(Y,Z\right)\} \\ & -B\left(X,Z\right)h\left(A_NY,W\right)+B\left(Y,Z\right)h\left(A_NX,W\right) \\ & -B\left(X,W\right)h\left(A_NY,Z\right)+B\left(Y,W\right)h\left(A_NX,Z\right). \end{aligned} \tag{4.3}$$

Since the lightlike hypersurface is semiparallel, setting $X=\xi$ and $Z=W$ in the equation above, with (2.4) and (2.6) we find

$$0=B\left(Y,Z\right)g\left(A_N\xi,A_\xi^*Z\right).$$

Then by the definition of Ricci tensor (2.14) we obtain $g\left(A_N\xi,A_\xi^*Z\right)=R^{(0,2)}\left(\xi,Z\right)$. Hence, either $B=0$, that is M is totally geodesic, or $R^{(0,2)}\left(\xi,Z\right)=0$. □

Corollary 4.1. *Let $(M,g,S(TM))$ be a totally umbilical lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c),\bar{g})$. M is semiparallel if and only if M is semiparallel as a lightlike hypersurface of the ambient semi-Euclidean space.*

Proof. Since M is totally umbilical, substituting (2.15) in (4.3) we get

$$\begin{aligned} & h\left(R\left(X,Y\right)Z,W\right)+h\left(Z,R\left(X,Y\right)W\right)= \\ = \; & c\rho\{g\left(Y,Z\right)g\left(X,W\right)-g\left(X,Z\right)g\left(Y,W\right) \\ & +g\left(Y,W\right)g\left(X,Z\right)-g\left(X,W\right)g\left(Y,Z\right)\} \\ & -B\left(X,Z\right)h\left(A_NY,W\right)+B\left(Y,Z\right)h\left(A_NX,W\right) \\ & -B\left(X,W\right)h\left(A_NY,Z\right)+B\left(Y,W\right)h\left(A_NX,Z\right) \\ = \; & -h\left(B\left(X,Z\right)A_NY-B\left(Y,Z\right)A_NX,W\right) \\ & -h\left(Z,B\left(X,W\right)A_NY-B\left(Y,W\right)A_NX\right) \end{aligned} \tag{4.4}$$

The result is obvious by the equation above. □

Example 4.1. In Minkowski space $\mathbb{R}_1^{m+2}$ the lightlike cone $\wedge_0^{m+1}$ is given by the equations

$$-\left(x^0\right)^2+\sum_{a=1}^{m+1}\left(x^a\right)^2=0,\; x=\sum_{A=0}^{m+1}x^A\frac{\partial}{\partial x^A}\neq 0.$$

The radical space and the lightlike transversal vector bundle of M are spanned by the lightlike vector fields

$$\xi=\sum_{A=0}^{m+1}x^A\frac{\partial}{\partial x^A}$$

and

$$N=\frac{1}{2\left(x^0\right)^2}\left\{-x^0\frac{\partial}{\partial x^0}+\sum_{a=1}^{m+1}x^a\frac{\partial}{\partial x^a}\right\}$$

respectively, for any $X\in S\left(T\wedge_0^{m+1}\right)$, $X=\sum_{a=1}^{m+1}X^a\frac{\partial}{\partial x^a}$. The lightcone and its screen ditribution $S\left(T\Lambda_0^{m+1}\right)$ are totally umbilical as the equations

$$B\left(X,Y\right)=-g\left(X,Y\right)$$

ve

$$C(X,Y) = -\frac{1}{2(x^0)^2} g(X,Y)$$

are satisfied for any $X, Y \in S\left(T\wedge_0^{m+1}\right)$. Also the Riemann curvature tensor of Λ_0^{m+1} is calculated as

$$R(X,Y)Z = -\frac{1}{2(x^0)^2}\{g(Y,Z)X - g(X,Z)Y\}$$

similar to the given in [7]. Since Λ_0^{m+1} is not totally geodesic, it is not parallel. But using the definition of semiparallelity for any $X, Y, Z, W \in \Gamma(TM)$ we get

$$\begin{aligned}(R(X,Y)\cdot h)(Z,W) &= -h(R(X,Y)Z,W) - h(Z,R(X,Y)W)\\ &= \frac{1}{2(x^0)^2}\{g(Y,Z)h(X,W) - g(X,Z)h(Y,W)\}\\ &\quad + \frac{1}{2(x^0)^2}\{g(Y,W)h(Z,X) - g(X,W)h(Z,Y)\}\end{aligned}$$

and with $B(X,Y) = -g(X,Y)$ we have

$$(R(X,Y)\cdot h)(Z,W) = 0.$$

So Λ_0^{m+1} is semiparallel.

Theorem 4.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$. If M is $2-$semiparallel, then either M is totally geodesic or it satisfies the equation $R^{(0,2)}\left(A_\xi^* W, \xi\right) = 0$.*

Proof. From (4.2), if M is $2-$semiparallel, then we get

$$\begin{aligned}0 &= (R(X,Y)\cdot\nabla h)(U,V,W) = (R(X,Y)\cdot\nabla_W h)(U,V)\\ &= -(\nabla_W h)(R(X,Y)U,V) - (\nabla_W h)(U,R(X,Y)V)\\ &\quad -\left(\nabla_{R(X,Y)W}h\right)(U,V)\\ &= -B(Y,U)(\nabla_W h)(A_N X,V) + B(X,U)(\nabla_W h)(A_N Y,V)\\ &\quad -B(Y,V)(\nabla_W h)(U,A_N X) + B(X,V)(\nabla_W h)(U,A_N Y)\\ &\quad -B(Y,W)(\nabla_{A_N X}h)(U,V) + B(X,W)(\nabla_{A_N Y}h)(U,V).\end{aligned}$$

Setting $U = X = \xi$ we have

$$\begin{aligned}0 &= B(Y,V)h(\nabla_W\xi, A_N\xi) + B(Y,W)h(\nabla_{A_N\xi}\xi, V)\\ &= -B(Y,V)h\left(A_\xi^* W, A_N\xi\right) - B(Y,W)h\left(A_\xi^*(A_N\xi), V\right)\end{aligned}$$

and taking $V = W$ it becomes

$$0 = -2B(Y,W)h\left(A_\xi^* W, A_N\xi\right).$$

Hence, either $B = 0$, that is M is totally geodesic, or using (2.14) we see that it satisfies $g\left(A_\xi^* A_\xi^* W, A_N\xi\right) = R^{(0,2)}\left(A_\xi^* W, \xi\right) = 0.$ □

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ON THE ISOMETRIES OF 3-DIMENSIONAL MAXIMUM SPACE

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Abstract. In this article, the hexahedron associated to metric geometry fullfiled by the metric of which unit sphere is hexahedron. We have analytically proved that the isometry group of the space with respect to this metric is the semi direct product of the Euclidean symmetry group of the cube and $T(3)$ which is all translations of analytical $3-$space.

1. Introduction

Many geometric studies and investigations are concerned with transformations of geometric objects on various spaces. Some of the transformations form group. Many of these groups consist simply of the symmetries of those spaces. The Platonic solids provide an excellent model for the investigation of symmetries. Also, Platonic solids are very important in the sense that they can be used not only in studies on properties of geometric structures, but also investigations on physical and chemical properties of the system under consideration. The isometry group have extensive applications in the theory of molecular and crystalline structure [1], [6]. The importance of isometries is that they preserve some of geometric properties; distance, angle measure, congruence, betweenness, and incidence [4], [5], [7], [8]. The isometry group is a fundamental concept in art as well as science. To develop this concept, it must be given a precise mathematical formulation.

Through the article we will use the definitions, explanations, propositions and the methods of proofs in the main reference [3].

2. The Maximum Metric

It is important to work on concepts related to the distance in geometric studies, because change of metric can reveals interesting results. What appears to be essential here is the way in which the lengths are to be measured. The present study aims to present isometry group of $\mathbb{R}^3$ by achieving the measuring process via the maximum metric $d_{\mathbf{M}}$ in preference to the usual Euclidean metric $d_{\mathbf{E}}$.

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For the sake of simple, $\mathbb{R}^3$ fullfiled by maximum metric is denoted $\mathbb{R}^3_{\mathbf{M}}$ in the rest of the article. Linear structure except distance function in the $\mathbb{R}^3_{\mathbf{M}}$ is the same as Euclidean analytical space [9]. This distance function $d_{\mathbf{M}}$ is defined as following.

Definition 2.1. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{\mathbf{M}} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow [0, \infty)$ defined by

$$d_{\mathbf{M}}(P_1, P_2) := \max\{|x_2 - x_1|, |y_2 - y_1|, |z_2 - z_1|\}$$

is called maximum distance function.

According to this distance function, the unit sphere is a hexahedron in the $\mathbb{R}^3_{\mathbf{M}}$.

Proposition 2.1. *The distance function $d_{\mathbf{M}}$ is a metric of which unit sphere is cube in $\mathbb{R}^3$ (see Figure 2.1).*

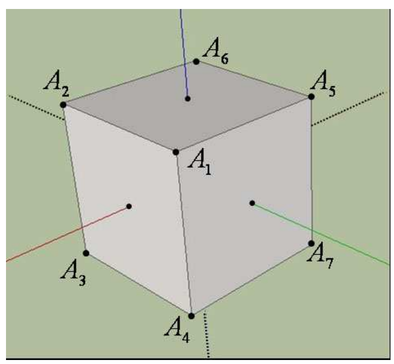

Figure 2.1

Proposition 2.2. *Given any two points A and B in $\mathbb{R}^3_{\mathbf{M}}$. Let direction vector of the line l through A and B be (p, q, r). Then,*

$$d_{\mathbf{E}}(A, B) = \mu(AB)\, d_{\mathbf{M}}(A, B)$$

where

$$\mu(AB) = \frac{\max\{|p|, |q|, |r|\}}{\sqrt{p^2 + q^2 + r^2}}.$$

Proof. Let $A=(x_1, y_1, z_1)$ and $B=(x_2, y_2, z_2)$ be two points in $\mathbb{R}^3_{\mathbf{M}}$. If line l with direction vector (p, q, r) passes through the points A and B, then $\overleftrightarrow{AB} \parallel (p, q, r)$. Therefore $\overrightarrow{AB} = \lambda(p, q, r)$ such that $\lambda \in \mathbb{R} \setminus \{0\}$. So

$$d_{\mathbf{M}}(A, B) = |\lambda| \max\{|p|, |q|, |r|\}$$

and similarly,

$$d_{\mathbf{E}}(A, B) = |\lambda| \sqrt{(p)^2 + (q)^2 + (r)^2}.$$

Consequently $\dfrac{d_{\mathbf{M}}(A, B)}{d_{\mathbf{E}}(A, B)} = \dfrac{\max\{|p|, |q|, |r|\}}{\sqrt{p^2 + q^2 + r^2}}$ is obtained. $\square$

3. Isometries of the $\mathbb{R}^3_{\mathbf{M}}$

We want to show that isometry group of the maximum space $\mathbb{R}^3_{\mathbf{M}}$ in this section. At the end of this section we are going to show isometry group of $\mathbb{R}^3_{\mathbf{M}}$ is the semi direct product of " Euclidean symmetry group of cube " and " all translations of $\mathbb{R}^3$ ". Also, O_h consist of identity, reflections, rotations, inversion, rotary reflection and rotary inversions. Before we give isometries of $\mathbb{R}^3_{\mathbf{M}}$, we introduce elements of the set O_h.

A transformation is any function mapping a set to itself in $\mathbb{R}^3$. A figure in $\mathbb{R}^3$ is any subset of $\mathbb{R}^3$. An isometry of $\mathbb{R}^3_{\mathbf{M}}$ is a transformation from $\mathbb{R}^3$ onto $\mathbb{R}^3$ that preserves distance. This means $d_{\mathbf{M}}(X,Y) = d_{\mathbf{M}}(\alpha(X),\alpha(Y))$ for each points X and Y in $\mathbb{R}^3_{\mathbf{M}}$. A symmetry of a figure F in $\mathbb{R}^3$ is an isometry mapping F onto itself—that is, an isometry $f:\mathbb{R}^3\to\mathbb{R}^3$ such that $f(F)=F$. The identity function I is a transformation is given $I(X)=X$ for each point X in $\mathbb{R}^3$. If Δ represents a plane, then the reflection σ_Δ across the plane Δ fixes every point on Δ, and takes every point X not on Δ to Y, where Δ is the perpendicular bisector of X and Y. A rotation is an isometric transformation which can be written as the composition of two distinc reflections. That is, a rotation about axis l is defined by $\sigma_\Delta\sigma_\Gamma$ where two planes Γ and Δ intersect at line l. A rotary reflection is an transformation which is the combination of a rotation about an axis and a reflection in a plane. That is, a rotary reflection is defined by $\sigma_\Pi\sigma_\Delta\sigma_\Gamma$ such that Γ and Δ are two intersecting planes each perpendicular to plane Π. A inversion according to a point P can be written as the $\sigma_P(X)=Y$ such that P is the midpoint of X and Y for X , $Y\in\mathbb{R}^3$. Rotary inversion is the combination of a rotation and an inversion in a point.

Proposition 3.1. *All Euclidean translation in $\mathbb{R}^3$ is an isometry of $\mathbb{R}^3_{\mathbf{M}}$.*

Proof. Given a points $A=(a_1,a_2,a_3)$ in $\mathbb{R}^3_{\mathbf{M}}$. The translation $T_A:\mathbb{R}^3_{\mathbf{M}}\to\mathbb{R}^3_{\mathbf{M}}$ is a mapping such that $T_A(X)=A+X$.

Let $X=(x_1,y_1,z_1)$ and $Y=(x_2,y_2,z_2)$ be any two points in $\mathbb{R}^3_{\mathbf{M}}$, then

$$\begin{aligned} d_{\mathbf{M}}(T_A(X),T_A(Y)) = &\ \max\left\{ \begin{array}{r} |(a_1+x_2)-(a_1+x_1)|, |(a_2+y_2)-(a_2+y_1)| \\ , |(a_3+z_2)-(a_3+z_1)| \end{array} \right\} \\ = &\ d_{\mathbf{M}}(X,Y). \end{aligned}$$

This means that translation T_A is an isometry. □

Therefore, we now consider planes passing through the origin for all calculations in the rest of the article.

The following proposition gives reflections which preserve distance $\mathbb{R}^3_{\mathbf{M}}$.

Proposition 3.2. *Given the plane Δ having equation $ax+by+cz=0$ in $\mathbb{R}^3_{\mathbf{M}}$. Reflection σ_Δ is a isometry iff unit normal vector of the plane Δ is written as $\lambda.\overrightarrow{V}$ where λ is a scalar and $\overrightarrow{V}\in D$ such that*

$$D=\{(1,0,0),(0,1,0),(0,0,1),(\pm1,1,0),(\pm1,0,1),(0,\pm1,1)\}.$$

Proof. Euclidean reflection $\sigma_\Delta:\mathbb{R}^3\to\mathbb{R}^3$ can be defined by

$$\sigma_\Delta(x,y,z) = \left(\begin{array}{r} \left(1-2a^2\right)x-2aby-2acz, -2abx+\left(1-2b^2\right)y-2bcz, \\ , -2acx-2bcy+\left(1-2c^2\right)z \end{array} \right)$$

such that (a,b,c) is the unit normal vector of the plane Δ.

We know that the reflection preserving a base of $\mathbb{R}^3$ is a isometry. If we take vector set $T=\{A_1=(1,1,1), A_2=(1,-1,1), A_3=(1,-1,-1)\}$ as base of $\mathbb{R}^3$, we shall find that reflections which preserve vectors of this base. To find reflections, we shall calculate $a,\ b,\ c$. If we calculate image of set T under Euclidean reflection, we get

$$\begin{aligned}
\sigma_\Delta(A_1) &= \left(1-2a^2-2ab-2ac, -2ab+1-2b^2-2bc, -2ac-2bc+1-2c^2\right),\\
\sigma_\Delta(A_2) &= \left(1-2a^2+2ab-2ac, -2ab-1+2b^2-2bc, -2ac+2bc+1-2c^2\right),\\
\sigma_\Delta(A_3) &= \left(1-2a^2+2ab+2ac, -2ab-1+2b^2+2bc, -2ac+2bc-1+2c^2\right).
\end{aligned}$$

If reflection preserves $d_{\mathbf{M}}-$distance, we have three equations;

$$\begin{aligned}
d_{\mathbf{M}}(O,A_1) &= d_{\mathbf{M}}(\sigma_\Delta(O), \sigma_\Delta(A_1)) = 1\\
d_{\mathbf{M}}(O,A_2) &= d_{\mathbf{M}}(\sigma_\Delta(O), \sigma_\Delta(A_2)) = 1\\
d_{\mathbf{M}}(O,A_3) &= d_{\mathbf{M}}(\sigma_\Delta(O), \sigma_\Delta(A_3)) = 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\max\left\{\left|1\text{-}2a^2\text{-}2ab\text{-}2ac\right|, \left|\text{-}2ab+1\text{-}2b^2\text{-}2bc\right|, \left|\text{-}2ac\text{-}2bc+1\text{-}2c^2\right|\right\}=1\\
&\max\left\{\left|1\text{-}2a^2+2aby\text{-}2ac\right|, \left|\text{-}2ab\text{-}1+2b^2y\text{-}2bc\right|, \left|\text{-}2ac+2bc+1\text{-}2c^2\right|\right\}=1\\
&\max\left\{\left|1\text{-}2a^2+2ab+2ac\right|, \left|\text{-}2ab\text{-}1+2b^2+2bc\right|, \left|\text{-}2ac+2bc\text{-}1+2c^2\right|\right\}=1
\end{aligned}$$

is obtained. Consequently, we have the system of equations with three unknows $a,\ b$ and c. Solving these system of equations for $a,\ b$ and c, we get

$$(\mp 1,0,0),\ (0,\mp 1,0),\ (0,0,\mp 1),$$
$$\left(0,\mp\frac{\sqrt{2}}{2},\mp\frac{\sqrt{2}}{2}\right),\ \left(\mp\frac{\sqrt{2}}{2},0,\mp\frac{\sqrt{2}}{2}\right),\ \left(\mp\frac{\sqrt{2}}{2},\mp\frac{\sqrt{2}}{2},0\right).$$

Conversely, we shall show that reflections σ_Δ preserve distance $d_{\mathbf{M}}$. Given reflection σ_Δ such that $\sigma_\Delta(X)=Y$ for $X,\ Y\in\mathbb{R}^3_{\mathbf{M}}$. Let (p_1,q_1,r_1) and (p_2,q_2,r_2) be the direction vectors of the lines OX and OY, respectively. If $\mu(OX)=\mu(OY)$, then $d_{\mathbf{M}}(O,X)=d_{\mathbf{M}}(O,Y)$ is obtained by Proposition 2. 2.

To show $d_{\mathbf{M}}(O,X)=d_{\mathbf{M}}(O,Y)$, we must check for all possible;

Δ	(p_2,q_2,r_2)
$x=0$	$(-p_1,q_1,r_1)$
$y=0$	$(p_1,-q_1,r_1)$
$z=0$	$(p_1,q_1,-r_1)$
$x+y=0$	$(-q_1,-p_1,r_1)$
$x-y=0$	(q_1,p_1,r_1)

Δ	(p_2,q_2,r_2)
$x+z=0$	$(-r_1,q_1,-p_1)$
$x-z=0$	(r_1,q_1,p_1)
$y+z=0$	$(p_1,-r_1,-q_1)$
$y-z=0$	(p_1,r_1,q_1)

□

Corollary 3.1. *In $\mathbb{R}^3_{\mathbf{M}}$, nine Euclidean reflections according to the planes having equations $x=0,\ y=0,\ z=0,\ x+y=0,\ x-y=0,\ x+z=0,\ x-z=0,\ y+z=0,\ y-z=0$ are isometric reflections.*

Following proposition tell us isometric rotations in $\mathbb{R}^3_{\mathbf{M}}$.

Proposition 3.3. *Given a rotation $r_\theta:\mathbb{R}^3_{\mathbf{M}}\to\mathbb{R}^3_{\mathbf{M}}$ according to l having equation $\frac{x}{p}=\frac{y}{q}=\frac{z}{r}$. Rotation r_θ is an isometry iff $r_\theta\in R_M=R_1\cup R_2\cup R_3$ such that*

$$R_1=\left\{r_\theta:\theta\in\left\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\right\},\ \textit{rotation axis has a direction vector in } D_1\right\},$$

$R_2 = \left\{ r_\theta : \theta \in \left\{ \frac{2\pi}{3}, \frac{4\pi}{3} \right\},\ \textit{rotation axis has a direction vector in } D_2 \right\},$

$R_3 = \{ r_\theta : \theta \in \{\pi\},\ \textit{rotation axis has a direction vector in } D_3 \},$

where

$D_1 = \{(1,0,0), (0,1,0), (0,0,1)\},$

$D_2 = \{(1,1,1), (-1,1,1), (1,-1,1), (1,1,-1)\},$

and

$D_3 = \{(1,1,0), (1,-1,0), (1,0,1), (1,0,-1), (0,1,1), (0,1,-1)\}.$

Proof. Recall that if $r_\theta : \mathbb{R}^3_{\mathbf{M}} \to \mathbb{R}^3_{\mathbf{M}}$ according to l having equation $\frac{x}{p} = \frac{y}{q} = \frac{z}{r}$ where (p,q,r) is a unit vector is a Euclidean rotation, then r_θ has following matrix representation:

$$\begin{bmatrix} \cos\theta + p^2(1-\cos\theta) & pq(1-\cos\theta) - r\sin\theta & pr(1-\cos\theta) + q\sin\theta \\ pq(1-\cos\theta) + r\sin\theta & \cos\theta + q^2(1-\cos\theta) & qr(1-\cos\theta) - p\sin\theta \\ pr(1-\cos\theta) - q\sin\theta & qr(1-\cos\theta) + p\sin\theta & \cos\theta + r^2(1-\cos\theta) \end{bmatrix}$$

A rotation can be written as the combination of two distinc reflections. So, a rotation with axis l can be defined by $\sigma_\Delta\sigma_\Gamma$ where l is line of intersection between planes Γ and Δ. Consequently, vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,1)$ $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$, $(1,\pm1,0)$, $(1,0,\pm1)$, $(0,1,\pm1)$, $(0,1,-1)$ can be take as direction vector of the line l by Corollary 3.1. To show isometric rotations in $\mathbb{R}^3_{\mathbf{M}}$, our next step is to give that rotations which preserve the lenghts of the edges of the unit sphere. To do this it will be enough to find isometric rotations. Let $A_1 = (1,1,1)$ and $A_2 = (1,-1,1)$ be points on the unit sphere. If we find image of A_1 and A_2 under r_θ, we get

$$r_\theta(A_1) = \begin{pmatrix} \cos\theta + p^2(1-\cos\theta) + pq(1-\cos\theta) - r\sin\theta + pr(1-\cos\theta) + q\cos\theta \\ ,\ pq(1-\cos\theta) + r\sin\theta + \cos\theta + q^2(1-\cos\theta) + qr(1-\cos\theta) - p\sin\theta \\ ,\ pr(1-\cos\theta) - q\sin\theta + qr(1-\cos\theta) + p\sin\theta + \cos\theta + r^2(1-\cos\theta) \end{pmatrix}$$

$$r_\theta(A_2) = \begin{pmatrix} \cos\theta + p^2(1-\cos\theta) - pq(1-\cos\theta) + r\sin\theta + pr(1-\cos\theta) + q\cos\theta \\ ,\ pq(1-\cos\theta) + r\sin\theta - \cos\theta - q^2(1-\cos\theta) + qr(1-\cos\theta) - p\sin\theta \\ ,\ pr(1-\cos\theta) - q\sin\theta - qr(1-\cos\theta) - p\sin\theta + \cos\theta + r^2(1-\cos\theta) \end{pmatrix}$$

If r_θ preserves $d_{\mathbf{M}}-$distance, we have equation;

$$d_{\mathbf{M}}(A_1, A_2) = d_{\mathbf{M}}(r_\theta(A_1), r_\theta(A_2)) = 2.$$

Let $(1,0,0)$ can be taken the direction vector of l in D_1. Then $(p,q,r) = (1,0,0)$. Setting $p = 1$, $q = 0$ and $r = 0$ in the equation $d_{\mathbf{M}}(r_\theta(A_1), r_\theta(A_2)) = 2$, we get $\max\{|\cos\theta|, |\sin\theta|\} = 1$. Solving this equation for $\theta \neq 0$, we obtain $\theta = \pi/2$, π or $3\pi/2$. Consequently, all Euclidean rotation about the $x-$axis with $\theta = \pi/2$, π or $3\pi/2$ is an isometry of $\mathbb{R}^3_{\mathbf{M}}$. Similarly, if the direction vector of l is one of $(0,1,0)$, $(0,0,1)$, then $\theta = \pi/2$, π or $3\pi/2$.

Let $(1,1,1)$ can be taken the direction vector of l in D_2. Then $(p,q,r) = \frac{1}{\sqrt{3}}(1,1,1)$.

Setting p, q and $r{=}1/\sqrt{3}$ in the equation $d_{\mathbf{M}}\left(r_\theta\left(A_1\right), r_\theta\left(A_2\right)\right)=2$, we get:

$$\begin{aligned} d_{\mathbf{M}}\left(r_\theta\left(A_1\right), r_\theta\left(A_2\right)\right) &= \left\{ \begin{array}{r} \left|\frac{1}{3}(1-\cos\theta)-\frac{1}{\sqrt{3}}\sin\theta\right|, \left|\cos\theta+\frac{1}{3}(1-\cos\theta)\right| \\ , \left|\frac{1}{3}(1-\cos\theta)+\frac{1}{\sqrt{3}}\sin\theta\right| \end{array} \right\} \\ &= 1. \end{aligned}$$

Solving above equation for $\theta \neq 0$, we get $\theta=2\pi/3$ or $4\pi/3$. Consequently, rotations r_θ according to the line l having direction $(1,1,1)$ with $\theta=2\pi/3$ or $4\pi/3$ is an isometry of $\mathbb{R}^3_{\mathbf{M}}$. Similarly, if the direction vector of l is one of $(\text{-}1,1,1)$, $(1,\text{-}1,1)$, $(1,1,\text{-}1)$, then $\theta=2\pi/3$ or $4\pi/3$.

Let $(1,1,0)$ can be taken the direction vector of l in D_3. Then $(p,q,r){=}\frac{1}{\sqrt{2}}(1,1,0)$. Setting $p=1/\sqrt{2}$, $q=1/\sqrt{2}$ and $r=0$ in the equation $d_{\mathbf{M}}\left(r_\theta\left(A_1\right), r_\theta\left(A_2\right)\right){=}2$, we get

$$\begin{aligned} d_{\mathbf{M}}\left(r_\theta\left(A_1\right), r_\theta\left(A_2\right)\right) &= \max\left\{\left|1-\cos\theta\right|, \left|\cos\theta\right|, \left|\sin\theta\right|\right\} \\ &= 1. \end{aligned}$$

Solving above equation for $\theta \neq 0$, we get $\theta=\pi$. That is, every Euclidean rotation about the line l that has the direction $(1,1,0)$ with $\theta=\pi$ is an isometry of $\mathbb{R}^3_{\mathbf{M}}$. Similarly, if the direction vector of l is one of $(1,-1,0)$, $(1,0,1)$, $(1,0,-1)$, $(0,1,1)$, $(0,1,-1)$, then $\theta=\pi$.

Conversely, we must show that rotations $r_\theta \in R_M = R_1 \cup R_2 \cup R_3$ preserve distance $d_{\mathbf{M}}$. To show $d_{\mathbf{M}}(O,X) = d_{\mathbf{M}}(O,Y)$, we shall consider the following cases to check $\mu(OX)=\mu(OY)$. One can easily calculate $\mu(OX)=\mu(OY)$ for all possible cases as in Proposition 3. 2. For example:

rotation	$(1,0,0)$ $\theta=\pi/2$	$\frac{1}{\sqrt{3}}(1,1,1)$ $\theta=2\pi/3$	$\frac{1}{\sqrt{2}}(1,1,0)$ $\theta=\pi$	$\cdots$
(p_2,q_2,r_2)	$(p_1,-r_1,q_1)$	(r_1,p_1,q_1)	(q_1,p_1,r_1)	$\cdots$

□

Corollary 3.2. *Twenty three Euclidean rotations about the lines passing through origin are isometric rotations in* $\mathbb{R}^3_{\mathbf{M}}$.

Note that the inversion σ_O about $O=(0,0,0)$ is the transformation such that $\sigma_O(x,y,z)=(-x,-y,-z)$ for each point (x,y,z) in $\mathbb{R}^3_{\mathbf{M}}$. Also, inversion σ_O is a isometry in $\mathbb{R}^3_{\mathbf{M}}$. We use σ_O to prove following propositions.

Proposition 3.4. *There are only six rotary reflections about O that preserve the $d_{\mathbf{M}}-$distances.*

Proof. We know that the composition of a reflection in a plane and a rotation about an axis orthogonal to the plane is called a rotary reflection. A rotary reflection is determined by the reflection and an angle of rotation [2]. So, rotary reflection can be written briefly as $\rho := \sigma_\Pi \sigma_\Delta \sigma_\Gamma = \sigma_\Pi r_\theta$ such that $r_\theta \in R_M$, Γ and Δ perdendicular to Π [7]. This means that 9 rotation axes can be selected from 13 rotation axes are given in Proposition 3. 3, because vectors of the set D_2 are not normal vectors of the planes is given Corrollary 3.1. Let $A_1 = (1,1,1)$ and $A_2 = (1,-1,1)$ be two points in $\mathbb{R}^3_{\mathbf{M}}$. Then $d_{\mathbf{M}}(A_1, A_2) = 2$.

If Π is the plane having equation $x = 0$, then $(1,0,0)$ is unit direction vector of r_θ and $\rho(x,y,z) = \sigma_\Pi r_\theta(x,y,z) = (-x, y\cos\theta - z\sin\theta, y\sin\theta + z\cos\theta)$.

$$\begin{aligned}
\rho(A_1) &= (-1, \cos\theta - \sin\theta, \sin\theta + z\cos\theta) \\
\rho(A_2) &= (-1, -\cos\theta - \sin\theta, -\sin\theta + z\cos\theta).
\end{aligned}$$

Therefore,

$$d_{\mathbf{M}}(\rho(A_1), \rho(A_2)) = 2 \Leftrightarrow |2\cos\theta| + |2\sin\theta| = 2 \Leftrightarrow \theta \in \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\},$$

but one can easily obtain that $\sigma_\Pi r_\pi$ is equals to the inversiyon σ_O about $O = (0,0,0)$. Therefore, there are only two rotary reflections according to the plane $x = 0$. Similarly, two rotary reflections are obtained using the planes $y = 0$ and $z = 0$.

If Π is the plane having equation $x+y = 0$, then $\left(1/\sqrt{2}, 1/\sqrt{2}, 0\right)$ is unit direction vector of r_θ and

$$\rho(x,y,z) = \sigma_\Pi r_\theta(x,y,z) = \left(\begin{array}{r}
\left(\frac{\text{-}1+\cos\theta}{2}\right)x\text{-}\left(\frac{1+\cos\theta}{2}\right)y+\left(\frac{\sin\theta}{\sqrt{2}}\right)z, \\
,\left(\frac{\text{-}1\text{-}\cos\theta}{2}\right)x+\left(\frac{\text{-}1+\cos\theta}{2}\right)y\text{-}\left(\frac{\sin\theta}{\sqrt{2}}\right)z, \\
,\frac{\text{-}\sin\theta}{\sqrt{2}}x+\frac{\sin\theta}{\sqrt{2}}y+(\cos\theta)z
\end{array}\right).$$

Clearly

$$\begin{aligned}
\rho(A_1) &= \left(-1+\frac{\sin\theta}{\sqrt{2}}, -1-\frac{\sin\theta}{\sqrt{2}}, \cos\theta\right) \\
\rho(A_2) &= \left(\cos\theta+\frac{\sin\theta}{\sqrt{2}}, -\cos\theta-\frac{\sin\theta}{\sqrt{2}}, \cos\theta-\sqrt{2}\sin\theta\right)
\end{aligned}$$

and

$$d_{\mathbf{M}}(\rho(A_1), \rho(A_2)) {=} 2 \Leftrightarrow \max\left\{|1+\cos\theta|, |\text{-}1+\cos\theta|, \left|\sqrt{2}\sin\theta\right|\right\} {=} 2 \Leftrightarrow \theta \in \{0, \pi\},$$

but one can easily obtain that $\sigma_\Pi r_\pi$ is equals to the inversiyon σ_O about $O = (0,0,0)$. This means that if $\theta = \pi$, then there is no new rotary reflection. Similarly, if Π are the planes having equations $x-y{=}0$, $x+z{=}0$, $x-z{=}0$, $y+z{=}0$, $y-z{=}0$, there is no new rotary reflection. □

Proposition 3.5. *There are only eight rotary inversions about O that preserve the $d_{\mathbf{M}}-$distances.*

Proof. We know that a rotary inversions is defined by $\rho := \sigma_O\sigma_\Delta\sigma_\Gamma = \sigma_O r_\theta$ such that $r_\theta \in R_M$. To show isometric rotary inversions, we have to consider 13 axes of rotations is given in Proposition 3. 3.

If r_θ represents the rotations about the $x-$axis, then $(1,0,0)$ is the unit direction vector of r_θ and

$$\rho(x,y,z) = \sigma_O r_\theta(x,y,z) = (-x, -y\cos\theta + z\sin\theta, -y\sin\theta - z\cos\theta).$$

Consequently,

$$\begin{aligned}\rho(A_1) &= (-1, -\cos\theta + \sin\theta, -\sin\theta - \cos\theta)\\ \rho(A_2) &= (-1, \cos\theta + \sin\theta, \sin\theta - \cos\theta)\end{aligned}$$

and

$$d_{\mathbf{M}}(\rho(A_1), \rho(A_2)) = 2 \Leftrightarrow \max\{|2\cos\theta|, |2\sin\theta|\} = 2 \Leftrightarrow \theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}.$$

One can easily obtain that $\sigma_{\Pi} r_{\pi}$ is equals to a rotary reflection or a reflection. This means that if $\theta = \frac{\pi}{2}$, π and $\frac{3\pi}{2}$, then there is no new rotary inversion. Similarly, if r_θ represents the rotations about the y, $z-$axis, then there is no new rotary inversion.

If r_θ represents the rotations about the parallel to $(1,1,0)$, then $\left(1/\sqrt{2}, 1/\sqrt{2}, 0\right)$ is unit direction vector r_θ and

$$\rho(x,y,z) = \sigma_O r_\theta(x,y,z) = \begin{pmatrix} \left(\frac{\text{-1-}\cos\theta}{2}\right)x + \left(\frac{\text{-1+}\cos\theta}{2}\right)y\text{-}\left(\frac{\sin\theta}{\sqrt{2}}\right)z, \\ ,\left(\frac{\text{-1+}\cos\theta}{2}\right)x + \left(\frac{\text{-1-}\cos\theta}{2}\right)y + \left(\frac{\sin\theta}{\sqrt{2}}\right)z, \\ ,\frac{\sin\theta}{\sqrt{2}}x\text{-}\frac{\sin\theta}{\sqrt{2}}y\text{-}\cos\theta z \end{pmatrix}.$$

Clearly

$$\begin{aligned}\rho(A_1) &= \left(-1 - \frac{\sin\theta}{\sqrt{2}}, -1 + \frac{\sin\theta}{\sqrt{2}}, -\cos\theta\right)\\ \rho(A_2) &= \left(-\cos\theta - \frac{\sin\theta}{\sqrt{2}}, \cos\theta + \frac{\sin\theta}{\sqrt{2}}, -\cos\theta + \sqrt{2}\sin\theta\right)\end{aligned}$$

and

$$d_{\mathbf{M}}(\rho(A_1), \rho(A_2)) = 2 \Leftrightarrow \max\left\{|-1+\cos\theta|, |1+\cos\theta|, \left|\sqrt{2}\sin\theta\right|\right\} = 2 \Leftrightarrow \theta \in \{0, \pi\}.$$

but one can easily obtain that $\sigma_O r_\pi$ is equals to a reflection. This means that if $\theta = \pi$, then there is no new rotary inversion. Similarly, it is easily seen that there is no new rotary inversion if r_θ is any of the remaining rotation axes parallel to $(1,-1,0)$, $(1,0,1)$, $(1,0,-1)$, $(0,1,1)$, $(0,1,-1)$.

If r_θ represents the rotations about the parallel to $(1,1,1)$, then $\frac{1}{\sqrt{3}}(1,1,1)$ is the unit direction vector of r_θ and $\rho(x,y,z) = \sigma_O r_\theta(x,y,z)$ is equals to

$$\begin{pmatrix} \left(\frac{\text{-1-2}\cos\theta}{3}\right)x + \left(\frac{\text{-1+}\cos\theta\text{+}\sqrt{3}\sin\theta}{3}\right)y + \left(\frac{\text{-1+}\cos\theta\text{-}\sqrt{3}\sin\theta}{3}\right)z, \\ \left(\frac{\text{-1+}\cos\theta\text{-}\sqrt{3}\sin\theta}{3}\right)x + \left(\frac{\text{-1-2}\cos\theta}{3}\right)y + \left(\frac{\text{-1+}\cos\theta\text{+}\sqrt{3}\sin\theta}{3}\right)z, \\ \left(\frac{\text{-1+}\cos\theta\text{+}\sqrt{3}\sin\theta}{3}\right)x + \left(\frac{\text{-1+}\cos\theta\text{-}\sqrt{3}\sin\theta}{3}\right)y + \left(\frac{\text{-1-2}\cos\theta}{3}\right)z \end{pmatrix}$$

Clearly

$$\rho(A_1) = (-1,-1,-1)$$
$$\rho(A_2) = \left(\frac{-1-2\cos\theta-2\sqrt{3}\sin\theta}{3}, \frac{-1+4\cos\theta}{3}, \frac{-1-2\cos\theta+2\sqrt{3}\sin\theta}{3}\right)$$

and

$$\begin{aligned} d_{\mathbf{M}}(\rho(A_1),\rho(A_2))=2 \quad &\Leftrightarrow \max\left\{\left|\frac{-2+2\cos\theta+2\sqrt{3}\sin\theta}{3}\right|, \left|\frac{4-4\cos\theta}{3}\right|,\right. \\ &\qquad\left.\left|\frac{4+2\cos\theta-2\sqrt{3}\sin\theta}{3}\right|\right\}=2 \\ &\Leftrightarrow \theta \in \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\} \end{aligned}$$

Therefore, we obtain that only two rotary inversion according to rotation about the axis parallel to $(1,1,1)$. Similarly, it is easily obtained that there are two new rotary inversions each of the remaining rotation axes parallel to $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$. That is, there are eight rotary inversions that preserve $d_{\mathbf{M}}-$distances. □

It can be easily check that $\sigma_O\sigma_\Delta = r_\pi$, $r_\pi \in R_1 \cup R_3$. Thus we have the **octahedral group**, O_h, consisting of nine reflections about planes, twenty-three rotations, six rotary reflections, eight rotary inversions, one inversion and the identity. That is, the Euclidean symmetry group of the cube.

Now, let us show that all isometries of $\mathbb{R}^3_{\mathbf{M}}$ are in $T(3).O_h$.

Definition 3.1. Given $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ points in $\mathbb{R}^3_{\mathbf{M}}$. The minimum distance set of A, B is defined by

$$\{X : d_{\mathbf{M}}(A,X) + d_{\mathbf{M}}(X,B) = d_{\mathbf{M}}(A,B)\}$$

and is denoted by $[AB]$.

Proposition 3.6. *If $\phi : \mathbb{R}^3_{\mathbf{M}} \to \mathbb{R}^3_{\mathbf{M}}$ is an isometry, then*

$$\phi([AB]) = [\phi(A)\,\phi(B)]\ .$$

Proof. Let $Y \in \phi([AB])$. Then,

$$\begin{aligned} Y \in \phi([AB]) \quad &\Leftrightarrow \exists X \ni Y = \phi(X) \\ &\Leftrightarrow d_{\mathbf{M}}(A,X) + d_{\mathbf{M}}(X,B) = d_{\mathbf{M}}(A,B) \\ &\Leftrightarrow d_{\mathbf{M}}(\phi(A),\phi(X)) + d_{\mathbf{M}}(\phi(X),\phi(B)) = d_{\mathbf{M}}(\phi(A),\phi(B)) \\ &\Leftrightarrow Y = \phi(X) \in [\phi(A)\,\phi(B)]\ . \end{aligned}$$

□

Corollary 3.3. *Let $\phi : \mathbb{R}^3_{\mathbf{M}} \to \mathbb{R}^3_{\mathbf{M}}$ be an isometry. Then ϕ maps vertices to vertices and preserves the lengths of edges of $[AB]$.*

Proposition 3.7. *Let $f : \mathbb{R}^3_{\mathbf{M}} \to \mathbb{R}^3_{\mathbf{M}}$ be an isometry such that $f(O) = O$. Then f is in O_h.*

Proof. Let $A_1 = (1,1,1)$, $A_2 = (1,-1,1)$, $A_5 = (-1,1,1)$, $A_6 = (-1,-1,1)$ and $D = (0,0,2)$. Consider the minimum distance set $[OD]$ with corner point D (see

Figure 3.1).

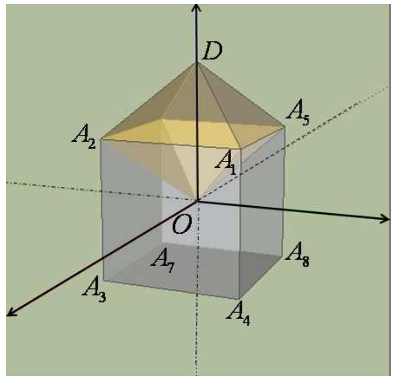

Figure 3.1

So, $f(A_1) \in A_iA_j,\ i \neq j,\ i,j \in \{1,2,3,4,5,6,7,8\}$. Here the points A_i and A_j are not on the same coordinate axis. Since f is an isometry by Corollary 3.3, $f(A_1),\ f(A_2),\ f(A_5)$ and $f(A_6)$ must be the vertices of the minimum distance set with corner point D and origin. Therefore, if $f(A_1) \in A_iA_j$, then $f(A_1) = A_i$ or $f(A_1) = A_j$. Similarly $f(A_2) = A_i$ or $f(A_2) = A_j,\ f(A_5) = A_i$ or $f(A_5) = A_j$ and $f(A_6) = A_i$ or $f(A_6) = A_6$. Also any three of $f(A_1),\ f(A_2),\ f(A_5)$ or $f(A_6)$ is not on the same coordinate axis. Now the following eight cases are possible;

$$f(A_1) = A_1 \Rightarrow \begin{cases} f(A_2) = A_2 & , f(A_5) = A_5 & , f(A_6) = A_6 \\ f(A_2) = A_4 & , f(A_5) = A_5 & , f(A_6) = A_6 \\ f(A_2) = A_4 & , f(A_5) = A_2 & , f(A_6) = A_3 \\ f(A_2) = A_5 & , f(A_5) = A_2 & , f(A_6) = A_6 \\ f(A_2) = A_5 & , f(A_5) = A_4 & , f(A_6) = A_8 \end{cases}$$

$$f(A_1) = A_2 \Rightarrow \begin{cases} f(A_2) = A_1 & , f(A_5) = A_6 & , f(A_6) = A_5 \\ f(A_2) = A_1 & , f(A_5) = A_3 & , f(A_6) = A_4 \\ f(A_2) = A_3 & , f(A_5) = A_1 & , f(A_6) = A_4 \\ f(A_2) = A_3 & , f(A_5) = A_6 & , f(A_6) = A_7 \\ f(A_2) = A_6 & , f(A_5) = A_1 & , f(A_6) = A_5 \\ f(A_2) = A_6 & , f(A_5) = A_3 & , f(A_6) = A_7 \end{cases}$$

$$f(A_1) = A_3 \Rightarrow \begin{cases} f(A_2) = A_2 & , f(A_5) = A_4 & , f(A_6) = A_1 \\ f(A_2) = A_2 & , f(A_5) = A_7 & , f(A_6) = A_6 \\ f(A_2) = A_4 & , f(A_5) = A_2 & , f(A_6) = A_1 \\ f(A_2) = A_4 & , f(A_5) = A_7 & , f(A_6) = A_8 \\ f(A_2) = A_7 & , f(A_5) = A_2 & , f(A_6) = A_6 \\ f(A_2) = A_7 & , f(A_5) = A_4 & , f(A_6) = A_8 \end{cases}$$

$$f(A_1) = A_4 \Rightarrow \begin{cases} f(A_2) = A_1 & , f(A_5) = A_3 & , f(A_6) = A_2 \\ f(A_2) = A_1 & , f(A_5) = A_8 & , f(A_6) = A_5 \\ f(A_2) = A_3 & , f(A_5) = A_1 & , f(A_6) = A_2 \\ f(A_2) = A_3 & , f(A_5) = A_8 & , f(A_6) = A_7 \\ f(A_2) = A_8 & , f(A_5) = A_1 & , f(A_6) = A_5 \\ f(A_2) = A_8 & , f(A_5) = A_3 & , f(A_6) = A_7 \end{cases}$$

$$f(A_1)=A_5 \Rightarrow \begin{cases} f(A_2)=A_1 & , f(A_5)=A_6 & , f(A_6)=A_2 \\ f(A_2)=A_1 & , f(A_5)=A_8 & , f(A_6)=A_4 \\ f(A_2)=A_6 & , f(A_5)=A_1 & , f(A_6)=A_2 \\ f(A_2)=A_6 & , f(A_5)=A_8 & , f(A_6)=A_7 \\ f(A_2)=A_8 & , f(A_5)=A_6 & , f(A_6)=A_7 \\ f(A_2)=A_8 & , f(A_5)=A_1 & , f(A_6)=A_4 \end{cases}$$

$$f(A_1)=A_6 \Rightarrow \begin{cases} f(A_2)=A_2 & , f(A_5)=A_5 & , f(A_6)=A_1 \\ f(A_2)=A_2 & , f(A_5)=A_7 & , f(A_6)=A_3 \\ f(A_2)=A_5 & , f(A_5)=A_2 & , f(A_6)=A_1 \\ f(A_2)=A_5 & , f(A_5)=A_7 & , f(A_6)=A_8 \\ f(A_2)=A_7 & , f(A_5)=A_2 & , f(A_6)=A_3 \\ f(A_2)=A_7 & , f(A_5)=A_5 & , f(A_6)=A_8 \end{cases}$$

$$f(A_1)=A_7 \Rightarrow \begin{cases} f(A_2)=A_3 & , f(A_5)=A_6 & , f(A_6)=A_2 \\ f(A_2)=A_3 & , f(A_5)=A_8 & , f(A_6)=A_4 \\ f(A_2)=A_6 & , f(A_5)=A_3 & , f(A_6)=A_2 \\ f(A_2)=A_6 & , f(A_5)=A_8 & , f(A_6)=A_5 \\ f(A_2)=A_8 & , f(A_5)=A_6 & , f(A_6)=A_5 \\ f(A_2)=A_8 & , f(A_5)=A_3 & , f(A_6)=A_4 \end{cases}$$

$$f(A_1)=A_8 \Rightarrow \begin{cases} f(A_2)=A_4 & , f(A_5)=A_5 & , f(A_6)=A_1 \\ f(A_2)=A_4 & , f(A_5)=A_7 & , f(A_6)=A_3 \\ f(A_2)=A_5 & , f(A_5)=A_4 & , f(A_6)=A_1 \\ f(A_2)=A_5 & , f(A_5)=A_7 & , f(A_6)=A_6 \\ f(A_2)=A_7 & , f(A_5)=A_5 & , f(A_6)=A_6 \\ f(A_2)=A_7 & , f(A_5)=A_4 & , f(A_6)=A_3 \end{cases}$$

In each case it is easy to show that f is unique and is O_h. For instance in the first case:

If $f(A_1)=A_1,\ f(A_2)=A_2,\ f(A_5)=A_5,\ f(A_6)=A_6$, then f is the identity.

If $f(A_1)=A_1,\ f(A_2)=A_4,\ f(A_5)=A_5,\ f(A_6)=A_6$, then $f=\sigma_\Delta$ such that $\Delta: y-z=0$.

If $f(A_1)=A_1,\ f(A_2)=A_4,\ f(A_5)=A_2,\ f(A_6)=A_3$, then $f=r_{2\pi/3}$ with rotation axis $\parallel (1,1,1)$.

If $f(A_1)=A_1,\ f(A_2)=A_5,\ f(A_5)=A_2,\ f(A_6)=A_6$, then $f=\sigma_\Delta$ such that $\Delta: x-y=0$.

If $f(A_1)=A_1,\ f(A_2)=A_5,\ f(A_5)=A_4,\ f(A_6)=A_8$, then $f=r_{4\pi/3}$ with rotation axis $\parallel (1,1,1)$.

The proofs of the remaining cases are quite similar to that of the first case. □

Theorem 3.1. *Let $f:\mathbb{R}^3_{\mathbf{M}}\to\mathbb{R}^3_{\mathbf{M}}$ be an isometry. Then there exists a unique $T_A\in T(3)$ and $g\in O_h$ such that $f=T_A\circ g$.*

Proof. Let $f(O)=A$ where $A=(a_1,a_2,a_3)$. Define $g=T_{-A}\circ f$. We know that g is an isometry and $g(O)=O$. Thus, $g\in O_h$ and $f=T_A\circ g$ by Proposition 3.7. The proof of uniqueness is trivial. □

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ON THE BINOMIAL SUMS OF HORADAM SEQUENCE

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Abstract. The main purpose of this paper is to establish some new properties of Horadam numbers in terms of binomial sums. By that, we can obtain these special numbers in a new and direct way. Moreover, some connections between Horadam and generalized Lucas numbers are revealed to get a more strong result.

1. Introduction

For $a, b, p, q \in \mathbb{Z}$, Horadam [1] considered the sequence $W_n(a, b\ ;\ p, q)$, shortly W_n, which was defined by the recursive equation

$$W_n(a, b\ ;\ p, q) = pW_{n-1} + qW_{n-2} \qquad (n \geq 2), \tag{1.1}$$

where initial conditions are $W_0 = a$, $W_1 = b$ and $n \in \mathbb{N}$.

In equation (1.1), for special choices of a, b, p and q, the following recurrence relations can be obtained.

- For $a = 0$, $b = 1$, it is obtained generalized Fibonacci numbers:

$$U_n = pU_{n-1} + qU_{n-2}. \tag{1.2}$$

- For $a = 2$, $b = p$, it is obtained generalized Lucas numbers:

$$V_n = pV_{n-1} + qV_{n-2}. \tag{1.3}$$

- Finally, we should note that choosing suitable values on p, q, a and b in equation (1.1), it is actually obtained others second order sequences such as Fibonacci, Pell, Jacobsthal, Horadam and etc. (for example, see [16] and references therein).

Considering [1] (or [4]), one can clearly obtain the characteristic equation of (1.1) as the form $t^2 - pt - q = 0$ with the roots

$$\alpha = \frac{p + \sqrt{p^2 + 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 + 4q}}{2}. \tag{1.4}$$

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Hence the Binet formula

$$W_n = W_n\left(a, b\ ;\ p, q\right) = A\alpha^n + B\beta^n, \tag{1.5}$$

where $A = \frac{b - a\beta}{\alpha - \beta},\ B = \frac{a\alpha - b}{\alpha - \beta}$, can be thought as a solution of the recursive equation in (1.1).

The number sequences have been interested by the researchers for a long time. Recently, there have been so many studies in the literature that concern about subsequences of Horadam numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers. They were widely used in many research areas as Physics, Engineering, Architecture, Nature and Art (see [1-16]). For example, in [7], Taskara et al. examined the properties of Lucas numbers with binomial coefficients.

In [3], they also computed the sums of products of the terms of the Lucas sequence $\{V_{kn}\}$. In addition in [2], the authors established identities involving sums of products of binomial coefficients.

And, in [8], we obtained Horadam numbers with positive and negative indices by using determinants of some special tridiagonal matrices.

In this study, we are mainly interested in some new properties of the binomial sums of Horadam numbers.

2. Main Results

Let us first consider the following lemma which will be needed later in this section. In fact, this lemma enables us to construct a relation between Horadam numbers and generalized Lucas numbers by using their subscripts.

Lemma 2.1. *[3]For $n \geq 1$, we have*

$$W_{ni+i} = V_i W_{ni} - \left(-q\right)^i W_{ni-i}. \tag{2.1}$$

Theorem 2.1. *For $n \geq 2$, the following equalities are hold:*

$$W_{ni+i} = V_i^{n-1} W_{2i} - \left(-q\right)^i \sum_{j=1}^{n-1} V_i^{n-1-j} W_{ij}.$$

Proof. Let us show this by induction, for $n = 2$, we can write

$$W_{3i} = V_i W_{2i} - \left(-q\right)^i W_i,$$

which coincides with equation (2.1). Now, assume that, it is true for all positive integers m, i.e.

$$W_{mi+i} = V_i^{m-1} W_{2i} - \left(-q\right)^i \sum_{j=1}^{m-1} V_i^{m-j-1} W_{ij}. \tag{2.2}$$

Then, we need to show that above equality holds for $n = m + 1$, that is,

$$W_{(m+1)i+1} = V_i^m W_{2i} - \left(-q\right)^i \sum_{j=1}^{m} V_i^{m-j} W_{ij}. \tag{2.3}$$

By considering the right hand side of equation (2.3), we can expand the summation as

$$\begin{aligned} V_i^m W_{2i} - (-q)^i \sum_{j=1}^{m} V_i^{m-j} W_{ij} &= V_i^m W_{2i} - (-q)^i \sum_{j=1}^{m-1} V_i^{m-j} W_{ij} - (-q)^i W_{mi} \\ &= V_i \left(V_i^{m-1} W_{2i} - (-q)^i \sum_{j=1}^{m-1} V_i^{m-j-1} W_{ij} \right) - (-q)^i W_{mi}. \end{aligned}$$

Then, using equation (2.2), we have

$$V_i^m W_{2i} - (-q)^i \sum_{j=1}^{m} V_i^{m-j} W_{ij} = V_i W_{mi+i} - (-q)^i W_{mi}$$

Finally, by considering (2.1), we obtain

$$V_i^m W_{2i} - (-q)^i \sum_{j=1}^{m} V_i^{m-j} W_{ij} = W_{(m+1)i+i}$$

which ends up the induction. □

Choosing some suitable values on $a,\ b,\ p$ and q, one can also obtain the sums of the well known Fibonacci, Lucas and etc. in terms of the sum in Theorem 2.1.

Corollary 2.1. *In Theorem 2.1, for special choices of $a,\ b,\ p$ and q, the following results can be obtained for well-known number sequences in literature.*

- For $a = 0,\ b = 1$, it is obtained generalized Fibonacci numbers:
$$U_{ni+i} = V_i^{n-1} U_{2i} - (-q)^i \sum_{j=1}^{n-1} V_i^{n-1-j} U_{ij}.$$
- For $a = 2,\ b = p$, it is obtained generalized Lucas numbers:
$$V_{ni+i} = V_i^{n-1} V_{2i} - (-q)^i \sum_{j=1}^{n-1} V_i^{n-1-j} V_{ij}.$$
- By choosing other suitable values on $a,\ b,\ p$ and q, almost all other special numbers can also be obtained in terms of the sum in Theorem 2.1.

Now, we will show the relation between Horadam numbers and generalized Lucas numbers using binomial sums as follows.

Theorem 2.2. *For $n \geq 2$, the following equalities are satisfied:*

$$W_{ni+i} = \begin{cases} \sum\limits_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} V_i^{n-2j} q^{ij} W_i + q^i a \sum\limits_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} V_i^{n-2j-1} q^{ij}, & i \text{ is odd} \\ \sum\limits_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} (-1)^j V_i^{n-2j} q^{ij} W_i - q^i a \sum\limits_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} (-1)^j V_i^{n-2j-1} q^{ij}, & i \text{ is even.} \end{cases}$$

Proof. There are two cases of subscript i.

Case 1: Let be i is odd. Then, by Theorem 2.1, we can write

$$\begin{aligned} W_{ni+i} &= V_i^{n-1}W_{2i} + q^i \sum_{j=1}^{n-1} V_i^{n-1-j} W_{ij} \\ &= V_i^{n-1}W_{2i} + q^i V_i^{n-2} W_i + q^i V_i^{n-3} W_{2i} + \cdots + q^i W_{(n-1)i}. \end{aligned}$$

We must note that the proof should be investigated for both cases of n.

If n is odd, then we have

$$\begin{aligned} W_{ni+i} &= V_i^{n-2}\left(V_i W_{2i} + q^i W_i\right) + q^i V_i^{n-4}\left(V_i W_{2i} + W_{3i}\right) \\ &\quad + \cdots + q^i V_i \left(V_i W_{(n-3)i} + W_{(n-2)i}\right) + q^i W_{(n-1)i}. \end{aligned} \tag{2.4}$$

Hence, it is given the binomial summation, when the recursive substitutions equation (2.4) by using (2.1),

$$W_{ni+i} = \sum_{j=0}^{\frac{n-1}{2}} \binom{n-j}{j} V_i^{n-2j} q^{ij} W_i + q^i a \sum_{j=0}^{\frac{n-1}{2}} \binom{n-j-1}{j} V_i^{n-2j-1} q^{ij}. \tag{2.5}$$

If n is even, then similar approach can be applied to obtain

$$\begin{aligned} W_{ni+i} &= V_i^{n-2}\left(V_i W_{2i} + q^i W_i\right) + q^i V_i^{n-4}\left(V_i W_{2i} + W_{3i}\right) \\ &\quad + \cdots + q^i V_i^0 \left(V_i W_{(n-2)i} + W_{(n-1)i}\right). \end{aligned}$$

and

$$W_{ni+i} = \sum_{j=0}^{\frac{n}{2}} \binom{n-j}{j} V_i^{n-2j} q^{ij} W_i + q^i a \sum_{j=0}^{\frac{n-2}{2}} \binom{n-j-1}{j} V_i^{n-2j-1} q^{ij}. \tag{2.6}$$

For the final step, we combine (2.5) and (2.6) to see the equality

$$W_{ni+i} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} V_i^{n-2j} q^{ij} W_i + q^i a \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} V_i^{n-2j-1} q^{ij},$$

as required. Now, for the next case, consider

Case 2: Let be i is even. Then, by Theorem 2.1, we know

$$\begin{aligned} W_{ni+i} &= V_i^{n-1}W_{2i} - q^i \sum_{j=1}^{n-1} V_i^{n-1-j} W_{ij} \\ &= V_i^{n-1}W_{2i} - q^i V_i^{n-2} W_i - q^i V_i^{n-3} W_{2i} - \cdots - q^i W_{(n-1)i}. \end{aligned}$$

and therefore, we write

(2.7)

$$W_{ni+i} = \sum_{j=0}^{\frac{n-1}{2}} \binom{n-j}{j} (-1)^j V_i^{n-2j} q^{ij} W_i - q^i a \sum_{j=0}^{\frac{n-1}{2}} \binom{n-j-1}{j} (-1)^j V_i^{n-2j-1} q^{ij}$$

if n is odd. And we get

(2.8)

$$W_{ni+i} = \sum_{j=0}^{\frac{n}{2}} \binom{n-j}{j} (-1)^j V_i^{n-2j} q^{ij} W_i - q^i a \sum_{j=0}^{\frac{n-2}{2}} \binom{n-j-1}{j} (-1)^j V_i^{n-2j-1} q^{ij}$$

if n is even. Thus, by combining (2.7) and (2.8), we obtain

$$W_{ni+i}=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-j}{j}(-1)^j V_i^{n-2j}q^{ij}W_i-q^i a\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-j-1}{j}(-1)^j V_i^{n-2j-1}q^{ij}.$$

Hence the result follows. □

Choosing some suitable values on $i,\ a,\ b,\ p$ and q, one can also obtain the binomial sums of the well known Fibonacci, Lucas, Pell, Jacobsthal numbers, etc. in terms of binomial sums in Theorem 2.2.

Corollary 2.2. *In Theorem 2.2, for special choices of* $i,\ a,\ b,\ p,\ q$, *the following result can be obtained.*

- For $i=1$,
  * For $a=0$ and $b,p,q=1$, Fibonacci number

$$F_{n+1}=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-j}{j},$$

  * For $a=2$ and $b,p,q=1$, Lucas number

$$L_{n+1}=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-j}{j}+2\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-j-1}{j}.$$

  * For $a=0,\ b=1,\ p=2$ and $q=1$, Pell number

$$P_{n+1}=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-j}{j}2^{n-2j}.$$

  * For $a=0,\ b=1,\ p=1$ and $q=2$, Jacobsthal number

$$J_{n+1}=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-j}{j}2^{j}.$$

- For $i=2$,
  * For $a=0$ and $b,p,q=1$, Fibonacci number

$$F_{2n+2}=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-j}{j}(-1)^j\,3^{n-2j}.$$

  * For $a=2$ and $b,p,q=1$, Lucas number

$$L_{2n+2}=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-j}{j}(-1)^j\,3^{n+1-2j}-2\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-j-1}{j}(-1)^j\,3^{n-1-2j}.$$

  * For $a=0,\ b=1,\ p=2$ and $q=1$, Pell number

$$P_{2n+2}=2\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-j}{j}(-1)^j\,6^{n-2j}.$$

* For $a=0,\ b=1,\ p=1$ and $q=2$, Jacobsthal number

$$J_{2n+2}=\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{n-j}{j}(-1)^{j}\,2^{j}.$$

- By choosing other suitable values on $i,\ a,\ b,\ p$ and q, almost all other special numbers can also be obtained in terms of the binomial sum in Theorem 2.2.

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EMBEDDING THE COMPLEMENT OF A COMPLETE GRAPH IN A FINITE PROJECTIVE PLANE

İBRAHIM GÜNALTILI

Abstract. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a non-trivial regular finite linear space with v points, $v + k$ lines, $k \geq 3$. We show that if $\mathcal{S}$ contains at least $\binom{k}{2}$ lines of size $b(p) - 2$ and one line size $b(p)$ for some point p, then S is embeddable in a unique projective plane π of order $b(p) - 1$ and $\pi - s$ is a complete graph of order k , where $b(p) \geq 4$ for some point p.

Key Words: linear space, projective plane, complete graph.
AMS Subject Classification: 51E20 , 51A45.

1. Introduction

Linear spaces lie at the foundation of incidence geometry, and more in particular, of finite geometry. A lot of characterizations of projective and affine spaces use linear spaces. Also, many important diagram geometries related to classes of simple groups are build with linear spaces. Linear spaces with constant block size are called Steiner systems and also play a prominent role in finite geometry. But there are also linear spaces that are not Steiner systems, and yet they appear often naturally. One such class of linear spaces is the class of A-affine linear spaces Let us first recall some definitions and results. For more details, (see [1] , [2]).

A *finite linear space* is a pair $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a finite set of *points* and $\mathcal{L}$ is a family of proper subsets of $\mathcal{P}$, which are called *lines*, such that

(L1) Any two distinct points lie on exactly one line,
(L2) Any line contains at least two points,
(L3) There exist at least two lines.

It is clear that (L3) could be replaced by an axiom $(L3)^{'}$: There are three lines of $\mathcal{S}$ not incident with a common point. In any case, (L3) and $(L3)^{'}$ are 'non-triviality' conditions. Systems satisfying (L1) and (L2) but not (L3) are called *trivial linear spaces.*

In a finite linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, v and b denote the total number of points and lines, respectively. The degree $b(p)$ of a point p is the total number of lines through p, and the size $v(l)$ of a line l is the total number of points on l. Thus; if $v(l) = k$ then l is called a $k-$line. The total number of $k-$lines is denoted by b_k.

The integer n defined by $n + 1 = \max\{b(p) : p \in \mathcal{P}\ \}$ is the *order* of a linear space. It is clear that any line of size $n+1$ meets every other line in a linear space of order n.The linear spaces with constant point degree is called regular linear spaces.

The numbers v, b, $v(l)$ and $b(p)$ will be called the *parameters* of $\mathcal{S}$.

A *projective plane* π is a linear space in which all lines meet and in which all points are on $n + 1$ lines, $n \geq 2$. The number n is called the order of π.

An *affine plane* $\mathbb{A}$ is a linear space in which, for any point p not on a line l, there is a unique line on p missing l, and in which all points are on $n + 1$ lines, $n \geq 2$.

A *k-arc* in a projective plane of order n is a set of k points no three of which are colinear. A k-arc can be thought of as a *complete graph* embedded in the projective plane.

An *hyperoval* is an $(n + 2)$ $-$arc in a projective plane of even order n.

For any line l of a linear space $\mathcal{S}$ of order n, the difference $n + 1 - v(l)$ is called a *deficiency* of l, denoted $d(l)$. Since the size of any line cannot exceed $n + 1$, the deficiency of any line is non-negative.

Let μ and λ be the respective minimum and maximum deficiencies among those lines of $\mathcal{S}$ which have size less than n.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a linear space and let $\mathcal{X}$ be a subset of $\mathcal{P}$. Then we can define the linear space $\mathcal{S}' = (\mathcal{X}, \{l \cap \mathcal{X} : l \in \mathcal{L}, |l \cap \mathcal{X}| \geq 2\})$. If $\mathcal{C} = \mathcal{P} - \mathcal{X}$, then $\mathcal{S}'$ is called the *complement* of $\mathcal{C}$ in $\mathcal{S}$ and we say that $\mathcal{S}'$ is obtained by removing $\mathcal{C}$ from $\mathcal{S}$. We denote the complement of $\mathcal{C}$ in $\mathcal{S}$ by $\mathcal{S} - \mathcal{C}$.

Let $\mathcal{X}$ be a set of points in a projective plane π of order n. Suppose that we remove $\mathcal{X}$ from π. We obtain a linear space $\pi - \mathcal{X}$ having certain parameters (i.e., the number of points, the number of lines, the point-degrees and line-degrees) (see [1]).

We call any linear space, which has the same parameters as $\pi - \mathcal{X}$, a *pseudo-complement* of $\mathcal{X}$ in π.

We have already encountered the notation of a pseudo-complement, namely the *pseudo-complement of one line*. This is a linear space with n^2 points, $n^2 + n$ lines in which any point has degree $n + 1$ and any line has degree n. We know that this is an affine plane, which is a structure embeddable in a projective plane of order n.

A linear space with $n^2 + n - m^2 - m$ points, $b = n^2 + n + 1$ lines, constant point-degree $n + 1$ and containing at least $m^2 + m + 1$ lines of size $n - m$ will be called the *pseudo-complement of a projective subplane of order* m in a projective plane of order n. It is clear that $m < n$.

A linear space with $n^2 + n + 1 - m^2$ points, $b = n^2 + n + 1$ lines, constant point-degree $n + 1$ and containing at least $m^2 + m$ lines of size $n + 1 - m$ will be called the *pseudo-complement of an affine subplane order* m in a projective plane of order n. It is clear that $m < n$.

A linear space with $n^2 + n + 1 - k$ points, $b = n^2 + n + 1$ lines, constant point-degree $n+1$ and lines of size $n+1$, n and $n-1$ will be called the *pseudo-complement of a k-arc* in a projective plane of order n.

Two lines l and l' are *parallel* if $l = l'$ or $l \cap l' = \phi$. Two lines l and l' are *disjoint* if $l \cap l' = \phi$.

A *parallel class* in the linear space $(\mathcal{P}, \mathcal{L})$ is a subset of $\mathcal{L}$ with the property that each point of $\mathcal{P}$ is on a unique element of this subset.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ and $\mathcal{S}' = (\mathcal{P}', \mathcal{L}')$ be two finite linear spaces. We say that $\mathcal{S}$ can be *embedded* in $\mathcal{S}'$ if $\mathcal{P} \subseteq \mathcal{P}'$ and $\mathcal{L} = \{l' \cap \mathcal{P} : l' \in \mathcal{L}' \text{ and } |l' \cap \mathcal{P}| \geq 2\}$. Hall proved in [10] that every finite linear space can be embedded in an infinite projective plane.

The complementation problem with respect to a projective plane is the following: Remove a certain subset of points and lines from the projective plane. Determine the parameters of the resulting space. Now assume that you are starting with a space having these parameters. Does this somehow force this subset to reappear, thus giving an embedding in the original projective plane? A number of people have considered complementation problems $([1], [2], [3], ..., [13])$. In 1970, Dickey solved the problem for the case where the configuration removed was a unital [7]. Batten [2] gave characterizations of linear spaces which are the complement of affine or projective subplanes of finite projective planes.

In this paper, We show that if $\mathcal{S}$ contains at least $\binom{k}{2}$ lines of size $b(p) - 2$ and one line size $b(p)$ for some point p, then S is embeddable in a projective plane π of order $b(p) - 1$ and $\pi - s$ is a complete graph of order k , where $b(p) \geq 4$ for some point p.

2. Main Results

Theorem 2.1. *If S is a $(n+1)-$regular linear space with $v = n^2 + n + 1 - k$ points, $b = n^2 + n + 1$ lines and contains exactly $k(n + 2 - k) > 0$ lines of size n, S is uniquely embeddable in a projective plane π of order n*

Proof. Fix an $n-$line l. Then the number induces a parallel class of $n + 1$ lines. Let a be the number of $n-$lines in a fixed parallel class. Then

$$an + (n + 1 - a)(n - 1) = n^2 + n + 1 - k$$

It requires that the number of $n-$lines in a parallel class is $n + 2 - k$. Since $b_n = k(n + 2 - k)$, the number of distinct parallel classes is k. Consider the structure $S^\star = (P^\star, L^\star)$ where $P^\star$ is P along with the parallel classes and $L^\star$ consist the lines of L extended by those parallel classes to which they belong. We shall prove that $S^\star$ is a linear space: It is clear that two old points (points of $P^\star$) or an old and a new point are one unique line of $L^\star$, since S is a linear space. Let p and q be two new distinct points. We must show that thet determine a unique line of $L^\star$. Let l_p and l_q be $n-$lines which determine the parallel classes corresponding to p and q, respectively. If $l_p \cap l_q = \emptyset$, $p = q$ which is a contradiction. So l_p and l_q meet. Each point of l_q is on a unique line of the parallel classes determined by l_p. Thus l_q does not meet precisely one line to the parallel class determined by l_p. This leaves precisely one line d to parallel to both l_p and l_q such that $p, q \in d$. Thus $S^\star$ is a projective plane of order n. Therefore, S can be embedded in a projective plane π of order n

□

Theorem 2.2. *Let $S = (P, L)$ be a non-trivial regular finite linear space with v points and b lines, $3 \leq b - v = k$. If S contains at least $\binom{k}{2}$ lines of size $b(p) - 2$,then S can be embedded in a projective plane π of order $b(p) - 1$ and $\pi - S$ is a complete graph of order k embedded a finite projective plane π of order $b(p) - 1$ for some point $p \in P$.*

Proof. Let $b(p) = n+1$, $b - v = k \geq 3$ for some point p of S. By all hypothesis of theorem, $n \geq 2$ and S is a non trivial $(n+1)-$regular linear space with $n^2+n+1-k$ points and n^2+n+1 lines. Let b_i be the number of all $i-$lines of S. Then also, by simple counting methots,

i) $\sum\limits_i b_i = n^2+n+1$

ii) $\sum\limits_i ib_i = (n+1)(n^2+n+1-k)$

iii) $\sum\limits_i i(i-1)b_i = (n^2+n-k)(n^2+n+1-k)$

iv) $\sum (n-i)\,(n+i-1)b_i = \binom{k}{2}$

Hence However, S has at least $\binom{k}{2}$ lines of size $n-1$, and each of them contributes 2 to the left hand side of the equality iv). Thus $b_i = 0,\ i \neq n+1,\ n,\ n-1$. Therefore, by i)-iv), the lines of S consist of $\binom{k}{2}$ lines of size $n-1$, $k(n+2-k)$ lines of size n and $n^2+n+1+k^2-\binom{k}{2}-(n+2)k$ lines of size $n+1$.

Case 1. Let $k < n+2$. In this case, S is the pseudo-complement of a $k-arc$ in a finite peojective plane of order n and $k \leq n+2$ since $b_n \geq 0$, $k \leq n+2$. Therefore by theorem 1, S can be embeded in a projective plane of order n. Then $k \leq n+2$

Case 2. Let $k = n+2$. In this case, every point is contained in $\frac{n+2}{2}$ lines of size $n+1$ and in $\frac{n}{2}$ lines of size $n-1$. The number of lines size $n-1$ is $\frac{1}{2}(n+2)(n+1)$ and the number of lines of size $n+1$ is $\frac{1}{2}n(n-1)$. Further more a short line of size $n-1$ is parallel to $2n$ other $(n-1)-$lines and $a(n+1)-$lines meets ever other line Fix $a(n-1)-$line l and denote by $\pi(l)$ the set of the $2n$ lines parallel to l. It follows from proposition 1.1 that if $\pi(l)$ were to contain a triangle then $n \leq 6$ this case contradiction to $n > 6$. Let l_1 and l_2 be intersecting lines of $\pi(l)$; denote by M_1 the set of lines of $\pi(l)$ which meet l_2 and by M_2 the set of lines of $\pi(l)$ which meet l_i since $\pi(l)$ contains no triangle, M_1 and M_2 consists of mutually parallel lines. We have $|M_j| = n-1$ and $l_j \in M_j$. Furthermore $M_1 \cap M_2 =$ (because $\pi(l)$ contains no triangle). Let d_1 and d_2 be two lines of $\pi(l) - (M_1 \cup M_2)$. We claim that each line of M_1 is parallel to at $n-1$ other lines of $\pi(l)$. Then every line of M_1 meets at least $n-3$ lines of M_2. Therefore, $\pi(l) = (M_1 \cup M_2) \cup \{d_1 \cup d_2\}$ and M_1 consist of mutually parallel lines.

$a(n-2)+(n+1-a)n = n^2-2$ and $a = \frac{n+2}{2}$.

*S*ince $a \in \mathbb{Z}$, n is even integer.

The line d_1 meets $n-1$ other lines of $\pi(l)$. One of these lines may be d_2 but at least $n-2$ of them are in $M_1 \cup M_2$. Therefore without ....of generality, d_1 meets at least $\frac{1}{2}(n-2) > 2$, lines of M_2. Hence, İf h is an arbitrary line of M_1, then h meets a line of M_2, which also meets d_1. Since $\pi(l)$ has no trianles, two implies that d_1 is parallel to h. So d_1 is parallel to every line of M_1. Consequently, π_i $M_1 \cup \{l, d_1\}$ is a set of mutually parallel lines with $|\pi_i| = n+1$. In wiew of $v = n^2-1 = |\pi_i| \,.\, (n-1)$, π_i is a parallel class. Therefore, $\pi_1 \cap \pi_2 = \{l\}$. If α is the totall number of parallel classes, $\alpha = \frac{2b_{n-1}}{n+1} = n+2$. Thus extension of S is a projective planes of order n and S can be embedded into a projective plane of order n as the complement of a hyperoval.

□

In fact, this case was originally proved by R. C Bose and S. S. Shrikhandle (1973) and then generalized greatly, allowing $n \geq 2$ by P.de Witte (1977)

Corollary 2.1. *If S is a non-trivial regular linear space with b lines, v points, $b-v \geq 3$ and at least $\binom{b-v}{2}$ lines of size $\frac{\sqrt{4b-3}-3}{2}$ and at least one point of degree $\frac{\sqrt{4b-3}+1}{2}$, S can be embedded in a projective plane π of order $\frac{\sqrt{4b-3}-1}{2}$ and is the pseudo-complement of a $(b-v)-arc$ in a projective plane of order $\frac{\sqrt{4b-3}-1}{2}$*

Corollary 2.2. *If S is a non -trivial regular linear space with v points, b lines, $b-v \geq 3$, at least $\binom{b-v}{2}$ lines of size $b(p)-2$, S can be embedded in a projective plane π of order $b(p)-1$ and is the pseudo-complement of a $(b-v)-arc$ in a projective plane of order $b(p)-1$*

Theorem 2.3. *Let $S=(P,L)$ be a non-trivial $n+1-$regular linear space having properties follows:*

i) $|P| = n^2+n+1-k,\ |L| = n^2+n+1,\ k \geq 3,\ n \geq 2$
ii) $v(l) \in \{n+1, n, n-1\}$ for each line l.
Then S can be embeded in a finite projective plane π of order n and $\pi - S$ is the $k-arc$

Proof. The proof of this theorem is completely similar to theorem 2.2. □

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