KONURALP JOURNAL OF MATHEMATICS
VOLUME 3 NO. 2 PP. 1–16 (2015) ©KJM

EFFECT OF THE CAVITY ANGLE ON FLOW STRUCTURES IN AN ANNULAR WEDGE CAVITY

HALIS BILGIL, ZARIFE DOLEK

ABSTRACT. This paper analyzes the 2-D Stokes flow in annular wedge cavities with different cavity angles. In order to analyze the flow structures, the two dimensional biharmonic equation is solved analytically. The flow is governed by two physical control parameters: the cavity angle α and the ratio of the upper and lower lid speeds ($S = \frac{U_1}{U_2}$). By varying α for each S, the effect of cavity angle on the streamline patterns and their bifurcations are investigated.

1. INTRODUCTION

Stokes flow generated within different shaped cavities is encountered in several manufacturing processes and engineering applications. The list of some of these applications can be found in the references [6,12,14,17]. Flow within the cavities has also been a focus attention for computational fluid dynamic studies since it is a commonly used as a benchmark problem.

There are many works in the literature on cavity flows related to eddy structure and their bifurcations. Gürcan et al. [8] analyzed the generation of eddies in a rectangular cavity. They showed effects of cavity aspect ratio and speed ratio of the moving lids on the streamline topology and the flow bifurcations. Flow bifurcation and eddy generation for steady, viscous flow in an L-shaped cavity, with the lids moving in opposite directions, has been investigated by Deliceoğlu and Aydın [2]. Arun and Satheesh [1] analyzed the effects of aspect ratio and Reynolds number on flow structures in a rectangular cavity.

Most of the these studies in literature related to cavity flow are concerned with the square or rectangular cavity flows, although the cavities may be non-rectangular in applications. Gürcan & Bilgil [9], and Gürcan et al. [10] investigated bifurcations and eddy genesis mechanisms of Stokes flow in a sectorial cavity. Ertürk and Dursun [3] solved 2-D steady and driven skewed cavity flow of an incompressible fluid numerically for skew angles ranging between 15^o and 165^o. Ertürk and Gökçöl [4] studied 2-D,

Date: April 1, 2015 and, in revised form, September 14, 2015.
2010 *Mathematics Subject Classification*. 35Q35, 00A69.
Key words and phrases. Cavity, Bifurcation, Eddy, Flow Structure, Stagnation Point, Cavity Angle.

steady and incompressible flow inside a triangular driven cavity. A sequence of flow structures is illustrated by Gaskell et al. [7] for Stokes flow in a cylindrical cavity. Flow structures in different shaped cavities investigated by Ozalp et al. [15]. They showed effects of cavity shape on flow structure within the cavity in detail.

As can be seen from the literature survey given above and references therein, most of the studies on cavity flows are performed on cavity aspect ratio and speed ratio of moving lids. There is a need to investigate the effect of cavity angle on flow structure in annular wedge cavities. This is aim of this study.

2. Mathematical Formulation

We considered a two-dimensional creeping flow in an annular wedge cavity $r_1 \leq r \leq r_2, -\alpha \leq \theta \leq \alpha$ (Fig. 1). The side walls, $r = r_1$, $r = r_2$ are fixed. The boundaries $\theta = \alpha$ and $\theta = -\alpha$ are two moving lids, which translate with speeds U_1 and U_2 in the radial direction respectively. The equation for the stream function governing the two-

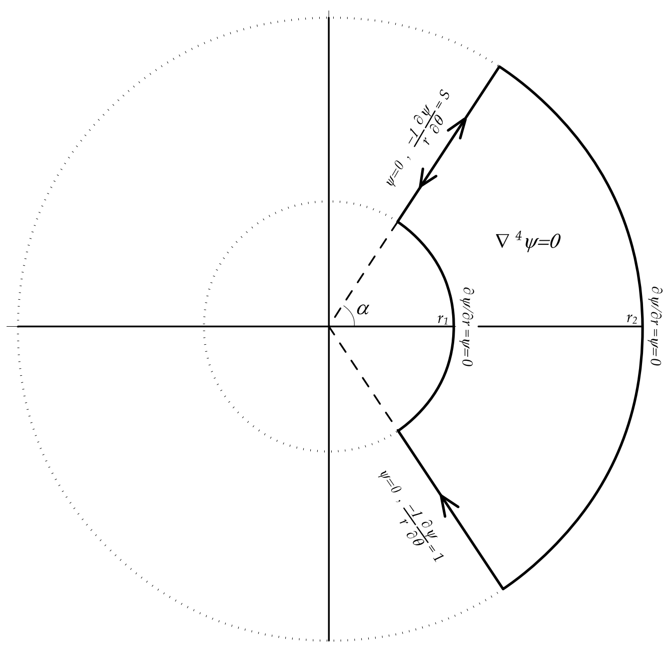


FIGURE 1. Geometry and boundary conditions for the lid driven cavity

dimensional steady flow of a viscous fluid is

$$\nabla^2\nabla^2\psi(r,\theta)=0, \tag{2.1}$$

where ∇^2 stands for the Laplace operator

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \tag{2.2}$$

in polar coordinates.

The derivatives of ψ give the velocity components:

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad , \quad u_\theta = \frac{\partial \psi}{\partial r} \ , \tag{2.3}$$

where u_r and u_θ are the radial and azimuthal components of velocity, respectively.

The streamfunction is constant (taken to be zero) on the boundaries

$$\psi(r_1,\theta) = \psi(r_2,\theta) = 0 \quad , \quad \psi(r,\pm\alpha) = 0 \ . \tag{2.4}$$

The ratio of the radii of the cylinders and speed ratio of the moving lids are our two control parameters which are defined by:

$$A = \frac{r_2}{r_1} \quad , \quad S = \frac{U_1}{U_2} \ . \tag{2.5}$$

In the plane polar coordinate system (r,θ), the other boundary conditions are

$$u_r(r,\alpha) = S \quad , \quad u_r(r,-\alpha) = 1 \ , \tag{2.6}$$

and on the side walls:

$$u_\theta(r_1,\theta) = u_\theta(r_2,\theta) = 0, \tag{2.7}$$

where we fixed $U_1 = S$ and $U_2 = 1$.

2.1. **Eigenfunction solution.** The general solution for the streamfunction can be written [13] in separable form as

$$\psi(r,\theta) = \sum_{-\infty}^{\infty} [E_n \sin(\lambda_n\theta) + F_n \cos(\lambda_n\theta)]\, \phi_1^{(n)}(r), \tag{2.8}$$

where

$$\phi_1^{(n)}(r,\lambda_n) = a_n r^{\lambda_n} + b_n r^{-\lambda_n} + c_n r^{2-\lambda_n} + d_n r^{2+\lambda_n} \ , \tag{2.9}$$

and λ_n are complex eigenvalues given by

$$\sin(\widehat{\lambda}_n) = \pm\beta\widehat{\lambda}_n \ , \tag{2.10}$$

where

$$\widehat{\lambda}_n = (i \ \log\frac{1}{A})\lambda_n \quad \text{and} \quad \beta = \frac{1}{2\log A}(A - \frac{1}{A}). \tag{2.11}$$

These complex eigenvalues are found via a Newton iteration procedure as described by Robbins & Smith [165], Fettis [5] and Khuri [13]; and values of the corresponding eigenvalues λ_n are given in Table 2.1.

The coefficients a_n, b_n, c_n and d_n have to be determined from the sidewall boundary conditions. These coefficients are given by Khuri [13].

The coefficients E_n and F_n in Eq. (2.8) have to be determined from the upper and the lower boundary conditions in (2,4) and (2,6). It is clear that the coefficients and the eigenvalues depends on S and A, respectively (see for details [55,99]).

Table 2.1: The first 30 roots of λ_n for $r_1=1, r_2=4$.

n	λ_n	n	λ_n
1	1.86054+2.99451i	11	3.75801+48.66728i
2	2.46101+7.70825i	12	3.82201+53.203498i
3	2.78163+12.30346i	13	3.88080+57.739178i
4	3.00270+16.86922i	14	3.93515+62.27442i
5	3.17170+21.42265i	15	3.98570+66.80932i
6	3.30856+25.96964i	16	4.03295+71.34393i
7	3.42358+30.51282i	17	4.07728+75.87830i
8	3.52277+35.05353i	18	4.11906+80.41247i
9	3.60998+39.59256i	19	4.15854+84.94646i
10	3.68778+44.13037i	20	4.19598+89.48031i

The infinite series thereby obtained are in practice truncated after N terms, i.e. the lower and upper summation limits are replaced by $-N$ and N, respectively. The convergence of the infinite series in (2.9) are necessary to determine the number N which assures that the truncated series is close enough to the infinite series.

3. RESULTS

We first analyzed flow structures and their bifurcations in a half-annular ring cavity. Then flow structures and their bifurcations in an annular wedge cavity wiht different cavity angle are investigated. Effects of the cavity angle on flow topology are revealed.

3.1. **Flow structures in a half-annular ring.** A half-annular ring cavity, $\alpha = \frac{\pi}{2}$ and $3.2 \leq r \leq 18$, consisting of two stationary side walls and both lids moving is considered; the boundary conditions and solution procedure are as given in section 2. As in Gürcan and Bilgil's work [9], mechanisms for eddy generation are examined via the emergence and the coalesce of corner-eedies or side-eddies as the aspect ratio is decreased; initially for single driven cavity, $S=1$, and then for symmetric flow $S=-1$ and $S=1$. For each case, the flow structures and eddy genesis mechanisms are illustrated in detail with figures.

3.1.1. *Bifurcations for a Single Lid-Driven Sectorial Cavity.* The aim of this section is to consider Stokes flow in a single lid driven annular cavity with three stationary walls (from Fig 1., $S=0$ implies the top lid is stationary ; since the solutions obtained are independent of which lid is stationary and which one moves, to aid visualsation the flow patterns presented are for the case of a stationary bottom lid). Although the analytical solution of this problem was obtained by Khuri [13], he was not interested in the mechanism(s) of eddy generations.

In this case, the boundary conditions are defined as follows:

$$\psi(r_1,\theta)=\psi(r_2,\theta)=0 \quad , \quad \psi(r,\pm\alpha)=0 \tag{3.1}$$

and imposing the no-slip condition on all four walls gives,

$$u_\theta(r_1,\theta)=u_\theta(r_2,\theta)=0 \tag{3.2}$$

$$u_r(r,\alpha)=1 \quad , \quad u_r(r,-\alpha)=0 \tag{3.3}$$

The values of the streamfunction and the radial velocity on top and bottom boundaries are given in Table 3.1. For different values of N in the infinite series in (2.8), the radial velocity on top and bottom boundaries are given in Table 3.2.

To investigate streamline bifurcations and hence a mechanism of eddy generation in this cavity, the aspect ratio was decreased, starting from $A = 18$ where the flow consist of a single main eddy and two smaller ones near the bottom corners (Fig. 2a). As A decreases it is observed that the corner eddies grow in size relative to the large central eddy, to meet each other on the bottom wall at a critical value of $A = 15.49$, see Fig. 2b. At this critical aspect ratio a new eddy is formed with a saddle point and a separatrix with streamfunction value $\psi = 0$. As A is decreased further this new eddy continues to grow (see Fig. 2c-d), the centers of two sub-eddies approach the saddle point and the center near the left-bottom of the cavity (say left center) coalesces with the saddle point to form a cusp bifurcation at $A = 12.05$. Hence these two critical points disappear and only the center near the right-bottom side of the cavity remains (say right center). In fact, at this critical aspect ratio the development of this second eddy is now complete as shown in Fig. 2e, $A = 11.10$. In this figure small corner eddies can be seen once again developing in each of the bottom corners. The process of eddy generation continues as the aspect ratio decreases(see Fig. 2f, where $A = 3.20$).

The same mechanism of eddy generation has similarly been reported by Gurcan [8,11] for the case of a rectangular cavity and Gurcan & Bilgil [9] for a sectorial cavity (for $\alpha = \frac{\pi}{4}$).

Table 3.1: The values of ψ and u_r on boundaries for $r_1 = 1$, $r_2 = 4$ and $N = 30$.

r	$\psi\left(r, \frac{\pi}{2}; 30\right)$	$\psi\left(r, -\frac{\pi}{2}; 30\right)$	$-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\left(r, \frac{\pi}{2}; 30\right)$	$-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\left(r, -\frac{\pi}{2}; 30\right)$
1.0	-9.6458E-17	-3.2719E-020	1.7708E-15	0
1.2	7.1537E-04	-7.8157E-13	0.8817E+01	-0.1332E-10
1.4	-1.1147E-03	3.0229E-12	0.1071E+01	0.3918E-10
1.6	-1.4256E-03	3.7842E-12	0.1122E+01	0.3750E-09
1.8	9.9134E-04	-3.8356E-12	0.9247E+00	-0.3135E-09
2.0	-1.8560E-03	4.8247E-12	0.1163E+01	0.5496E-09
2.2	-1.4157E-03	2.3535E-12	0.1123E+01	0.3588E-09
2.4	-1.9610E-03	3.1865E-12	0.1139E+01	0.3818E-09
2.6	-1.5174E-03	8.4531E-12	0.1047E+01	0.2580E-09
2.8	2.2270E-03	-3.4649E-12	0.9173E+00	-0.3142E-10
3.0	-1.7164E-04	-2.5217E-12	0.1036E+01	-0.4235E-09
3.2	1.6848E-03	2.0594E-11	0.9621E+00	0.5959E-09
3.4	-2.1931E-03	4.9270E-12	0.1170E+01	0.7111E-09
3.6	2.1211E-03	-1.3457E-11	0.9658E+00	-0.1104E-08
3.8	-4.6814E-05	2.9814E-11	0.1195E+01	-0.1758E-09
4.0	-3.9111E-16	2.0768E-19	0.6123E-15	0

Table 3.2: The values of u_r on boundaries for different N and $r_1 = 1$, $r_2 = 4$.

r	$-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\left(r,\frac{\pi}{2};15\right)$	$-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\left(r,-\frac{\pi}{2};15\right)$
1.0	0	0
1.6	0.9127	-0.1003-07
2.2	1.0922	0.8708E-08
2.8	1.0043	0.5807E-08
3.4	0.9439	-0.1674E-7
4.0	0	0
r	$-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\left(r,\frac{\pi}{2};30\right)$	$-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\left(r,-\frac{\pi}{2};30\right)$
1.0	0	0
1.6	1.1227	0.3750E-09
2.2	1.1236	0.3588E-09
2.8	0.9173	-0.3142E-10
3.4	1.1707	0.7111E-09
4.0	0	0
r	$-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\left(r,\frac{\pi}{2};60\right)$	$-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\left(r,-\frac{\pi}{2};60\right)$
1.0	0	0
1.6	1.0484	-0.2170E-10
2.2	1.0698	-0.9644E-11
2.8	1.0628	-0.2684E-10
3.4	0.8381	0.3904E-10
4.0	0	0
r	$-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\left(r,\frac{\pi}{2};90\right)$	$-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\left(r,-\frac{\pi}{2};90\right)$
1.0	0	0
1.6	1.0015	0.3599E-11
2.2	1.0003	-0.4064E-12
2.8	0.9996	-0.1018E-10
3.4	0.9928	-0.4382E-11
4.0	0	0

3.1.2. *Case* $S=-1$. In the case of lids moving in opposite directions with equal speeds, (i.e. $S=-1$), the flow structure is symmetrical about $\theta=0$ for all values of A. For large aspect ratios, a single eddy occupies the cavity, see Fig. 3a for $A=180$. As the aspect ratio is decreased from 180 there are four main stages in the development of the second and third eddies. In the first stage, a 'Pitchfork bifurcation appears at a critical value of $A_1=161.4$. Thus, two additional stagnation points are generated in the cavity, (see Fig. 3b where $A=130$).

As A is decreased further, the separatrix continues to grow and the second critical aspect ratio, $A_2=4.14$, is obtained at which two degenerate critical points appear on the two stationary side walls where side eddies are about to emerge as A is decreased further, see Fig. 3c-h.

In the third stage, at $A_3=3.56$, the heteroclinic connections coalesce with each other at the interior saddle point to produce four heteroclinic connections between the saddle point and the four separation points on the side walls, as shown in Fig. 3i (where $A=3.5$).

At this critical aspect ratio, A_3, there are now two complete eddies within the cavity and between them a third is about to be created. As A decreases, the sub-eddy center lying left of the saddle on $\theta=0$ approach the saddle point on $\theta=0$ and coalesce,

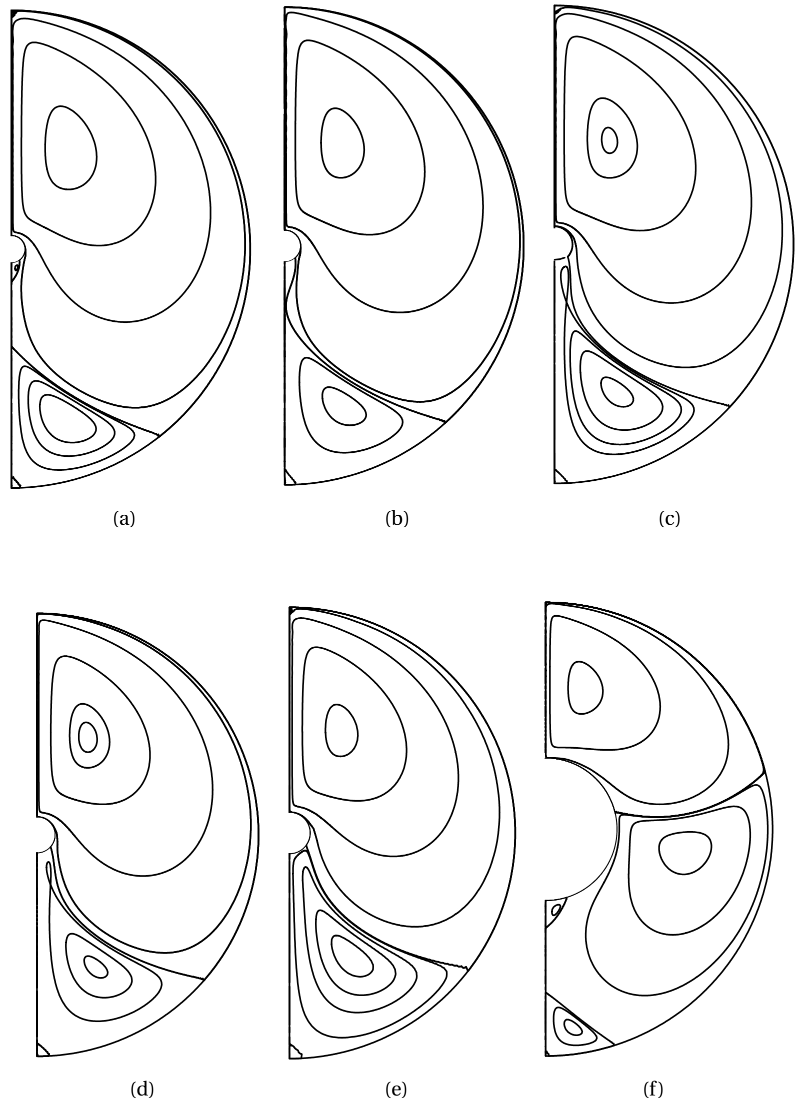


FIGURE 2. Eddy generation with decreasing A and S fixed at $S=0$. a) $A = 18.0$, b) $A = 15.49$, c) $A = 13.20$, d) $A = 12.50$, e) $A = 11.10$, f) $A = 3.20$

disappearing at $A_4 = 3.29$. This is a cusp (saddle-node) bifurcation. At this critical aspect ratio the formation of a third eddy, between the other two, is completed so that three eddies now occupy the cavity (Fig. 3j).

This is a mechanism for eddy generation in which one eddy becomes three. The number of complete eddies increases from 3 to 5 and 5 to 7 etc. via similar eddy genesis mechanism (see Fig. 3k-m).

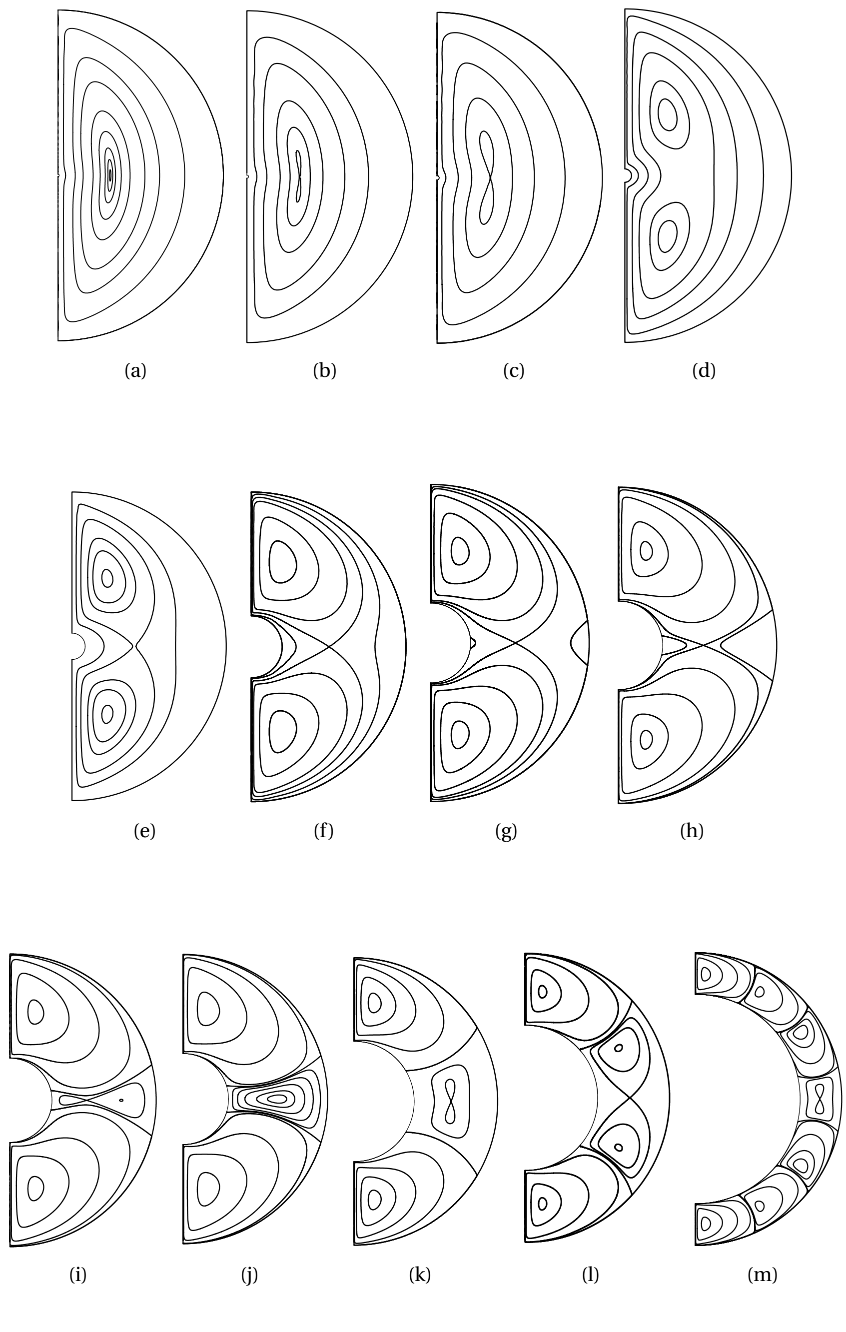
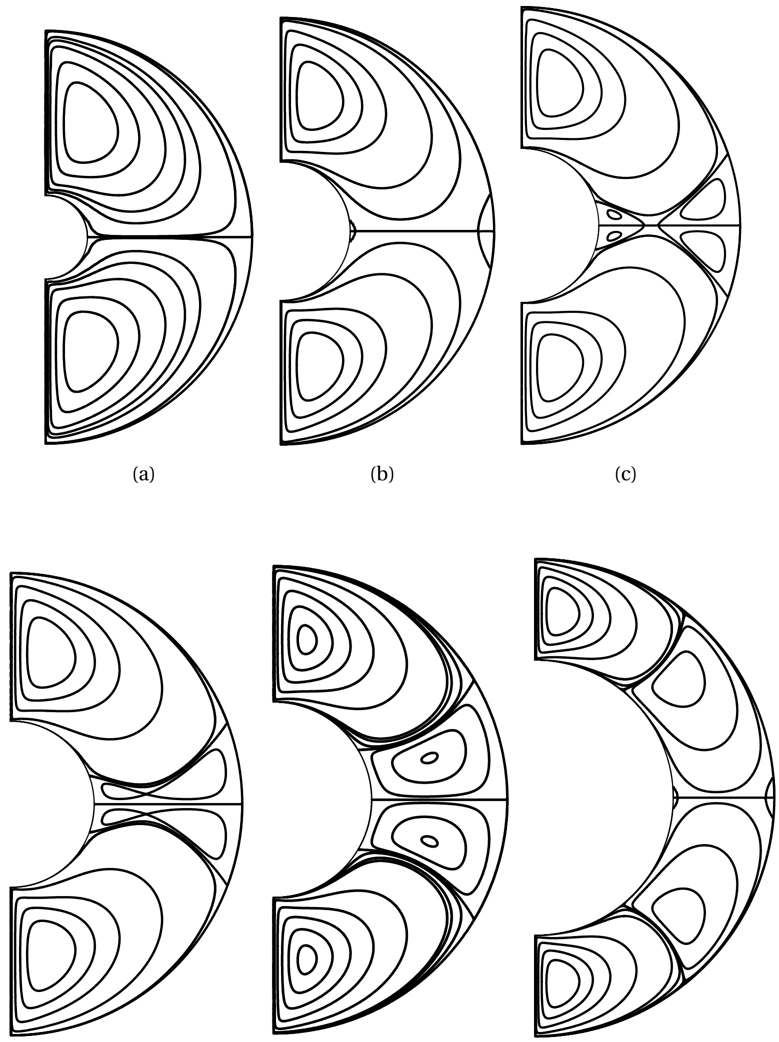


FIGURE 3. Eddy generation via the appearance of sides-eddies with decreasing A and S fixed at $S=-1$ for $\alpha=\frac{\pi}{2}$.
a) $A=180$, b) $A=130$, c) $A=90$, d) $A=25$, e) $A=12$, f) $A=5$, g) $A=4$, h) $A=3.6$, i) $A=3.5$, j) $A=3.2$, k) $A=2.4$, l) $A=2$, m) $A=1.4$



(a) (b) (c)

(d) (e) (f)

FIGURE 4. Eddy generation with decreasing A and S fixed at $S=1$.
a) $A=5$, b) $A=3.1$, c) $A=2.85$, d) $A=2.8$, e) $A=2.4$, f) $A=1.74$

3.1.3. *Case* $S=1$. In the case of lids moving in the same radial direction with equal speed, (i.e. $S=1$), the flow structure is symmetric about $\theta=0$ for all values of A. It is clear that the peripheral velocity is zero on $\theta=0$.

For large aspect ratios, only two symmetric eddies occupy the cavity; see Fig. 4a for $A=5$. As the aspect ratio is decreased from 5, there are three main stages in the simultaneously development of the third and fourth eddies. In the first stage, as A is decreased the first critical aspect ratio, $A_1=3.076$, is reached at which two degenerate critical points appear on each stationary sidewall; see Fig. 4b for $A=3.10$. The side eddies approach each other as A is further decreased, such that, when $A_2=2.831$ is reached, they coalesce on $\theta=0$. Thence a separatrix with a saddle point and two centers (i.e. two sub-eddies) is seen in the cavity (see Fig.4c-d). As A decreases, the sub-eddy centers lying to the left of each of the saddles, approach the saddle point and coalesce, disappearing at $A_3 = 2.664$ which means that this is a cusp (saddle–node) bifurcation. Hence four fully developed eddies are now visible in the cavity (see Fig. 4e-f).

This is a mechanism that consists from three steps for eddy generation from two complete eddies to four. Firstly, side eddies born on each stationary sidewall. Secondly, these side eddies approach and coalesce each other to produce two reflected separatrices enclosing two sub-eddies. In the last step, the cusp bifurcation are seen on the each separatrix and there are now four complete eddies within the cavity. A similar mechanism in a sectorial cavity is given by Gürcan and Bilgil [9].

3.2. **Effect of the cavity angle on flow structures.** It is the aim of this section to track the various flow transformations arising in the cavity as α is gradually increased for $0 < \alpha < \pi$, and to expose the mechanisms by which new eddies emerge and develop within a sectorial cavity.

This work is, which to our knowledge, the first such study in the literature in terms of effect of cavity angle on flow structures and bifurcations.

For $S = 0, A = 3$ and $S = 1, A = 2$, the various flow transformations are tracked as α is increased and hence the means is identified by which new eddies appear and become fully developed.

3.2.1. *Case:* $S = 0$, $A = 3$. Solution of this problem is introduced in above section. For narrow cavity angle, a single eddy occupies the cavity; see Fig.5 for $\alpha = 15$, where the flow consist of a single main eddy and two smaller ones near the bottom corners.

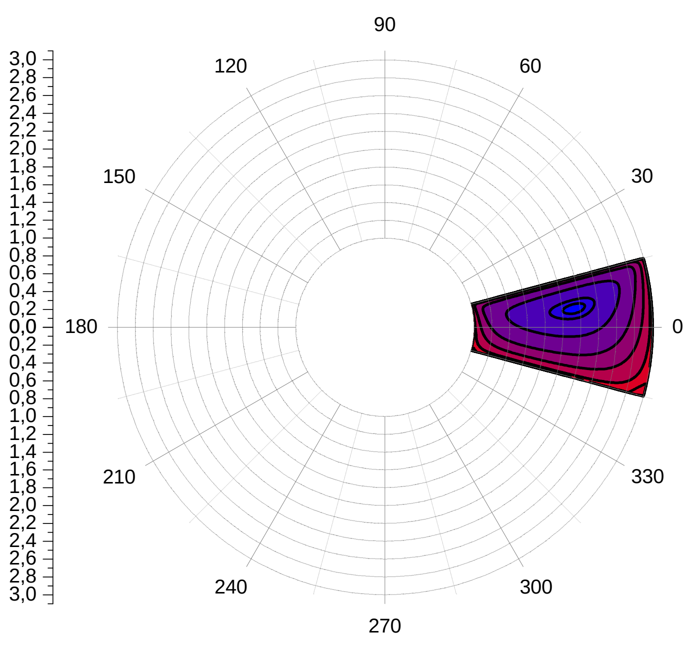


FIGURE 5. The cavity geometry and flow structure for $S = 0$, $A = 3$ and $\alpha = 15$.

As α increases it is observed that the corner eddies grow and meet each other on the bottom wall at a critical value of $\alpha = 47.31$, see Fig. 6a-d. At this critical cavity angle a new eddy is formed with a separatrix. As α is increased further this new eddy continues to grow and the centers of two sub-eddies approach the saddle point and the center near the left-bottom of the cavity coalesces with the saddle point to form a cusp bifurcation at $\alpha = 50.902$. Hence these two critical points disappear and only the center near the right-bottom side of the cavity remains see Fig. 6e-f. Hence the second eddy is completed in the cavity. As α increases the small corner eddies can be seen once again developing in each of the bottom corners. The process of eddy generation continues as the cavity angle increases (see Fig. 6g-p). It is seen that, as $\alpha \to \pi$, the number of completed eddy is five in the cavity for $A = 3$, see Fig.6p. It is clear that, in case of selecting smaller aspect ratio, the number of completed eddy will be more.

3.2.2. *Case:* $S = -1$, $A = 2$. In this special case the flow is symmetric about $\theta = 0$ for all values of α. When $5 \leq \alpha \leq 17.44$ (Fig. 7a-b) the flow in the cavity is in its simplest form: one single eddy with a centre-type stagnation point on $\theta = 0$. As α is gradually increased a sequence of flow transformations unfold, by which two additional eddies are generated in the cavity. For example, at $\alpha = 17.44$ the centre on $\theta = 0$ becomes a saddle point and two new centres appear (see Fig.7b where $\alpha = 22$). As α is increased further, the separatrix continues to grow and the second critical aspect angle is $\alpha = 48.24$, at which two degenerate critical points appear on the two side walls (see Fig. 7c-f).

As α increases, the side eddies expand and approach the saddle point on $\theta = 0$, and at $\alpha = 53.1$ coalesce with each other at the interior saddle point. At this critical cavity angle there are now two complete eddies within the cavity and between them a third is about to be created. As α is increased further, it seen that there are a separatrix between the two complete eddies (see Fig. 7g). As α increases, the sub-eddy center lying left of the saddle approach the saddle point on $\theta = 0$ and coalesce at $\alpha = 56.7$ to produce a centre. At this critical aspect ratio the development of the third eddy, between the other two, is complete so that three eddies now occupy the cavity (see Fig. 7h).

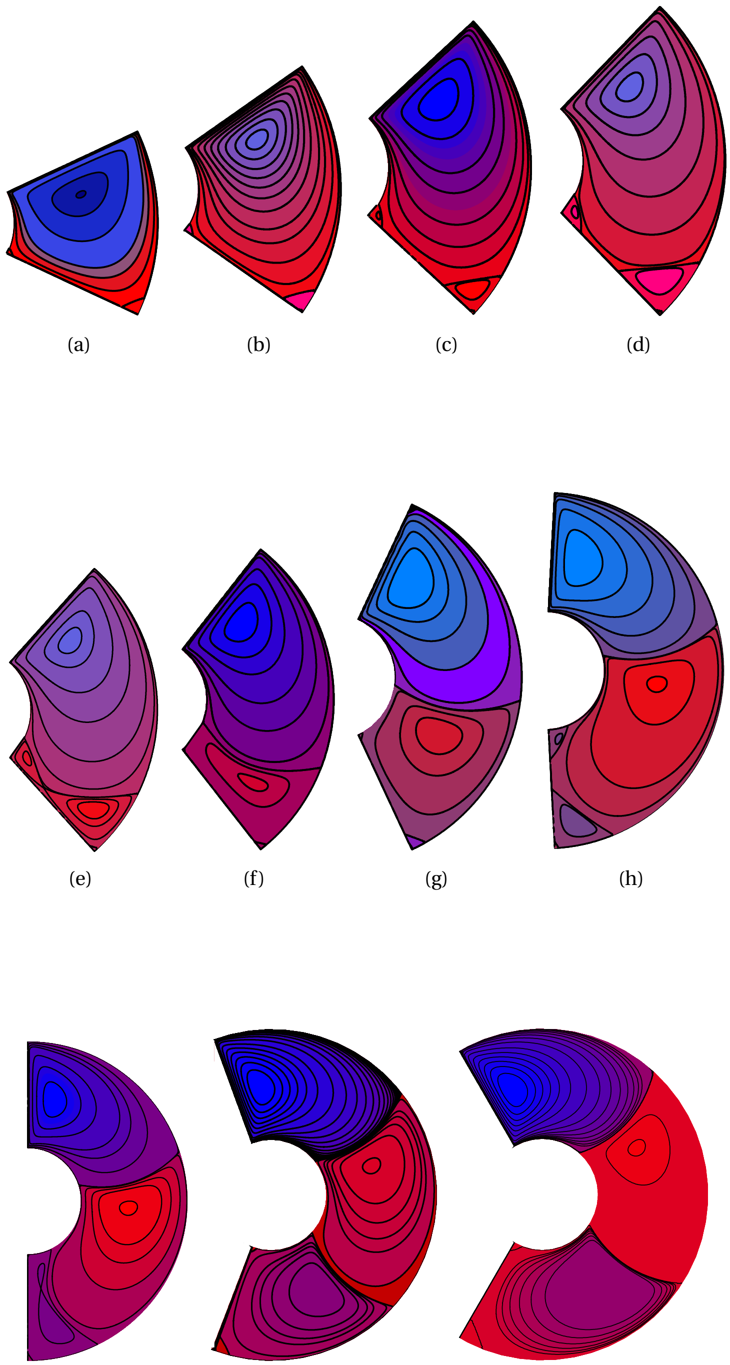
It can be seen from the above that there are four main stages in the development of the flow as the cavity aspect angle is increased: an interior saddle point appears; side eddies appear; the left side eddy and saddle point touch; and the interior substructure disappears.

This mechanism of eddy generation continues as the cavity angle increases (see Fig. 7i-r). It is seen that, as α is increased up to π, there are seven complete eddy and one separatrix in the cavity for $A = 2$ (see Fig. 7r). It is clear that, in the case of selecting smaller aspect ratio than $A = 2$, the number of complete eddy will be more.

In this study, derived from the one of the most important results, decreasing the aspect ratio (A) of cavity with increasing the cavity angle of flow structures cavity shows similar effects on the eddy genesis and their bifurcations.

ACKNOWLEDGEMENT

This research has been supported by Aksaray University Scientific Research Projects Coordination Unit. Project Number: 2014/012. H. Bilgil and Z. Dölek are grateful for this Financial support.



(a) (b) (c) (d)

(e) (f) (g) (h)

(i) (j) (k)

(Continue)

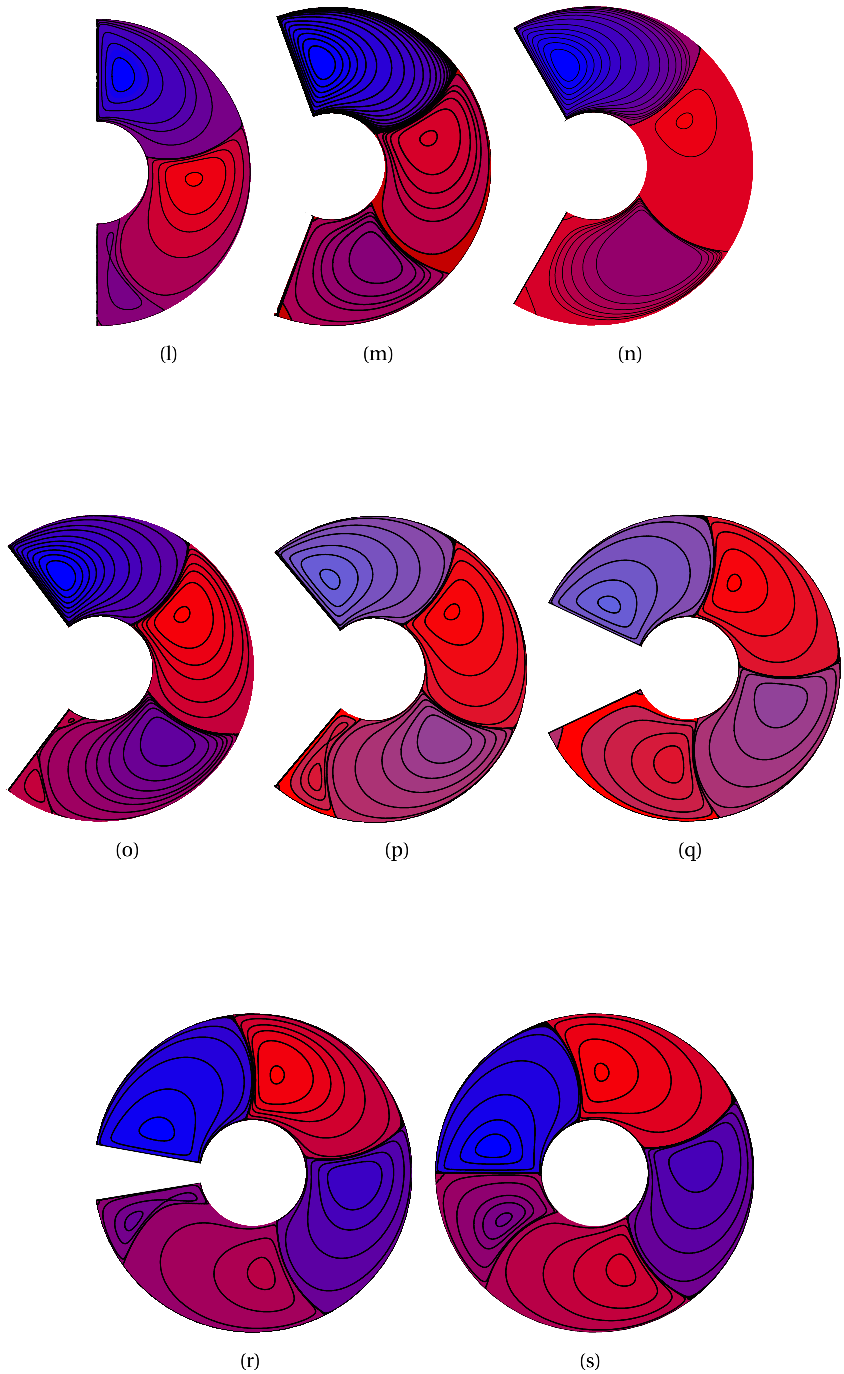
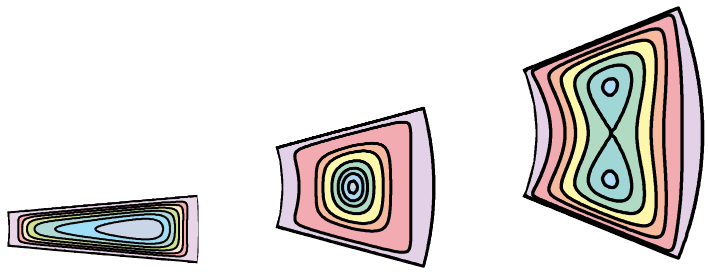
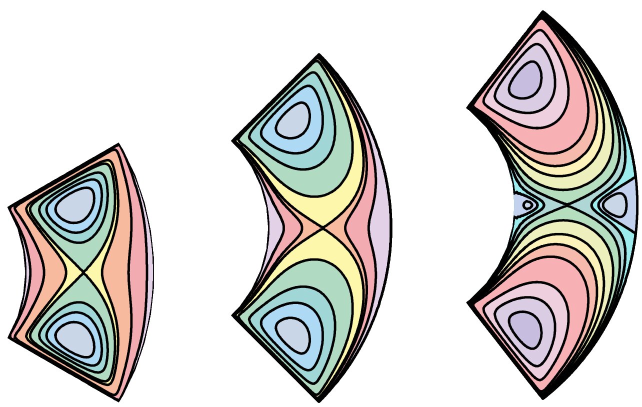


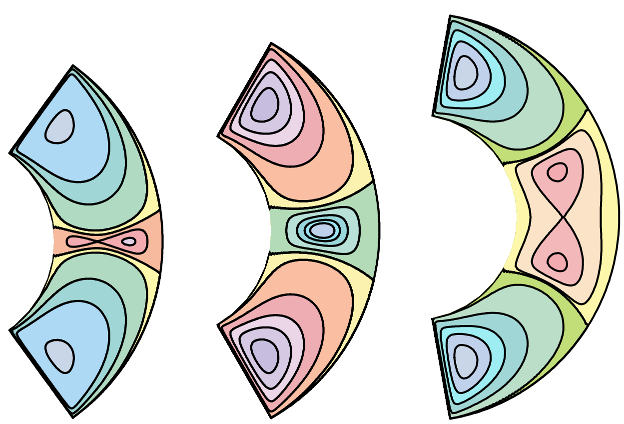
FIGURE 6. Eddy generation with increasing α for $S = 0$ and $A = 3$. a) $\alpha = 25$, b) $\alpha = 35$, c) $\alpha = 43$, d) $\alpha = 46$, e) $\alpha = 48$, f) $\alpha = 52$, g) $\alpha = 65$, h) $\alpha = 87$, i) $\alpha = 90$, j) $\alpha = 110$, k) $\alpha = 120$, l) $\alpha = 127$, m) $\alpha = 130$, n) $\alpha = 155$, o) $\alpha = 170$, p) $\alpha \to 180$



(a) (b) (c)



(d) (e) (f)



(g) (h) (i)

(Continue)

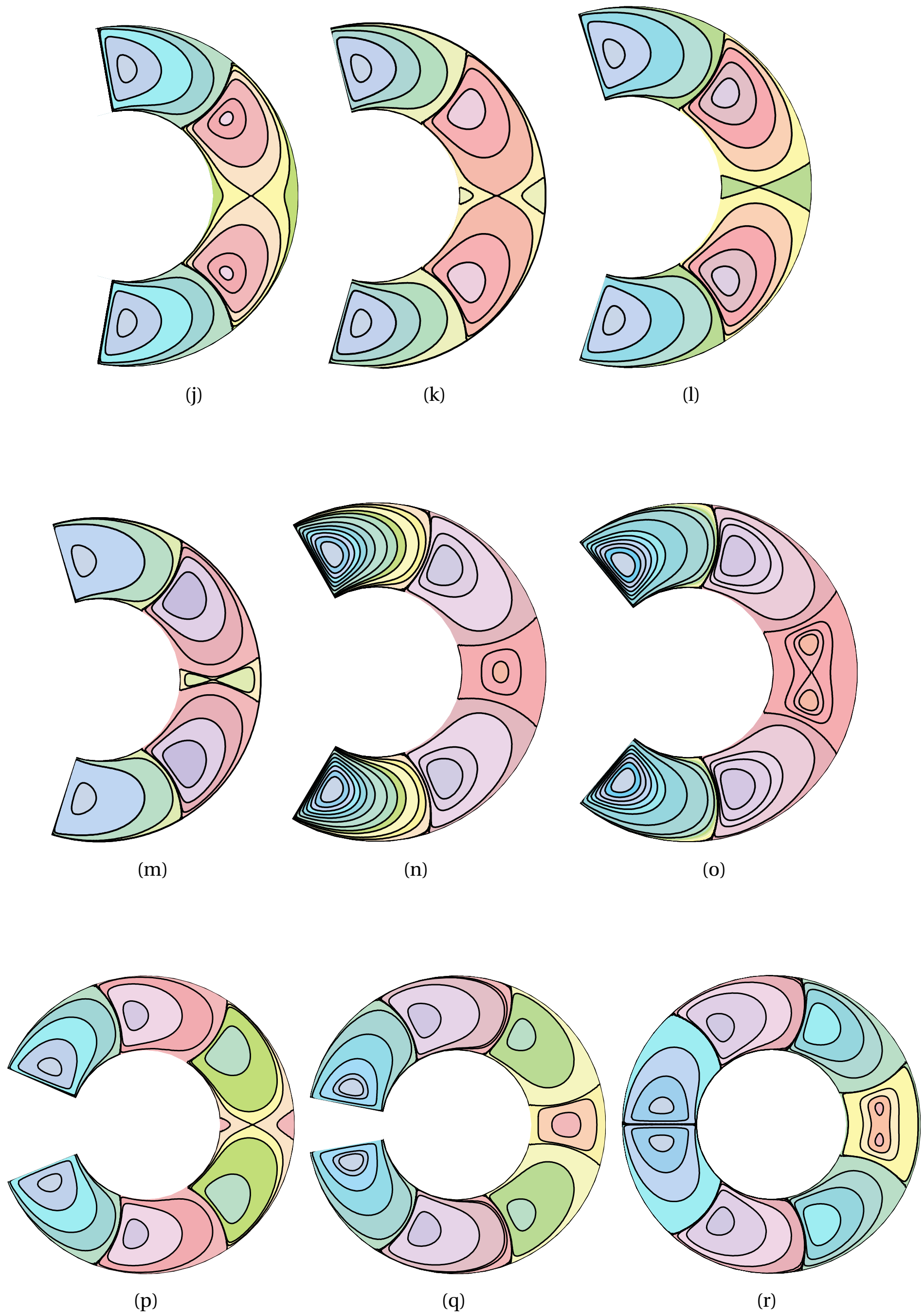


FIGURE 7. Eddy generation with increasing α for $S = -1$ and $A = 4$. a) $\theta = 5$, b) $\theta = 15$, c) $\theta = 22$, d) $\theta = 30$, e) $\theta = 45$, f) $\theta = 52$, g) $\theta = 54$, h) $\theta = 60$, i) $\theta = 80$, j) $\theta = 100$, k) $\theta = 105$, l) $\theta = 106.76$, m) $\theta = 107$, n) $\theta = 120$, o) $\theta = 130$, p) $\theta = 158$, r) $\theta = 170$, s) $\theta \rightarrow 180$

REFERENCES

[1] S. Arun, A. Satheesh, Analysis of flow behaviour in a two sided lid driven cavity using lattice boltzmann technique, Alexandria Engineering Journal, (2015), http://dx.doi.org/10.1016/j.aej.2015.06.005.

[2] A. Deliceoğlu, S.H. Aydın, Flow bifurcation and eddy genesis in an L-shaped cavity, Comput. Fluids 73 (2013), 24âĂŞ46.

[3] E. Erturk, B. Dursun, Numerical solution of 2-D steady incompressible flow in a driven skewed cavity, Z. Angew. Math. Mech. 87 (2007), 377âĂŞ392.

[4] E. Erturk, O. Gokcol, Fine grid numerical solutions of triangular cavity flow, Eur. Phys. J. Appl. Phys. 38 (2007), 97âĂŞ105.

[5] Fettis, H. E., Complex roots of $sinz = az, cosz = az$ and $coshz = az$, Math. of Comp. 30, 135 (1976), 541-545.

[6] Gaskell, P. H., Savage, M. D., Summers, J. L. and Thompson, H. M., Modeling and analysis of meniscus roll coating, J. Fluid Mech. 298 (1995), 113–137.

[7] P.H. Gaskell, M.D. Savage, M. Wilson, Flow structures in a half-filled annulus between rotating co-axial cylinders, J. Fluid Mech. 337 (1997), 263âĂŞ282.

[8] F. Gürcan, P.H. Gaskell, M.D. Savage, M. Wilson, Eddy genesis and transformation of Stokes flow in a double-lid-driven cavity, Proc. Inst. Mech. Eng. C 217, 3 (2003), 353âĂŞ364.

[9] Gürcan, F. and Bilgil, H., Bifurcations and eddy genesis of Stokes flow within a sectorial cavity, European Journal of Mechanics - B/Fluids. 39 (2013), 42-51.

[10] F. Gürcan, H. Bilgil, A. Åđahin, Bifurcations and eddy genesis of Stokes flow within a sectorial cavity PART II: Co-moving lids, European Journal of Mechanics - B/Fluids, (2015), dx.doi.org/10.1016/j.euromechflu.2015.02.008.

[11] Gürcan, F., Flow bifurcations in rectangular, lid-driven cavity flows. PhD Thesis, University of Leeds, 1996.

[12] Hellebrand H., Tape casting. In Materials Science and Technology–Processing of Ceramics, Part I, Vol. 17 A, ed. R. J. Brook. VCH, Weinheim, (1996), 189-260.

[13] S.A. Khuri, Biorthogonal series solution of stokes flow problems in sectorial regions, SIAM J. Appl. Math. 56, 1 (1996), 19âĂŞ39.

[14] Middleman S., Fundamentals of Polymer Processing, McGraw-Hill, New York, 1977.

[15] C. Ozalp, A. Pinarbasi, B. Sahin, Experimental measurement of flow past cavities of different shapes, Experimental Thermal and Fluid Science, 34, 5, (2010), 505-515.

[16] Robbins, C. I. and Smith, R. C. T., A table of roots of $Sinz = -z$, Phil. Mag. 39,7 (1948), 1005.

[17] Scholle M, Haas A, Aksel N, Thompson HM, Hewson RW, Gaskell PH, The effect of locally induced flow structure on global heat transfer for plane laminar shear flow, International Journal of Heat and Fluid Flow. 30, 2 (2009), 175-185.

AKSARAY UNIVERSITY, SCIENCE AND ART FACULTY, DEPARTMENT OF MATHEMATICS, AKSARAY-TURKEY
E-mail address: halis@gmail.com

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 17–32 (2015) ©KJM

NONPOLYNOMIAL CUBIC SPLINE APPROXIMATION FOR THE EQUAL WIDTH EQUATION

ALI SAHIN AND LEVENT AKYUZ

Abstract. In this paper, we investigate the numerical solutions of the equal width (EW) equation via the nonpolynomial cubic spline functions. Crank-Nicolson formulas are used for time discretization of the target equation. A linearization technique is also employed for the numerical purpose. Accuracy of the method is observed by the pointwise rate of convergence. Stability of the suggested method is investigated via the von-Neumann analysis. Six numerical examples related to single solitary wave, interaction of two, three and opposite waves, wave undulation and the Maxwell wave are considered as the test problems. The accuracy and the efficiency of the purposed method are measured by L_∞ and L_2 error norms and conserved constants. The obtained results are compared with the possible analytical values and those in some earlier studies.

1. Introduction

The field of nonlinear dispersive waves is one of the rapidly developed area in science over the last few decades. Because of their attractive solutions such as shallow water and plasma waves, studying on this field has been source of interest. Since the analytical solutions are not available in general and the possible cases are limited, numerical solutions for those equations have importance to understand the nonlinear phenomena.

There are many different models for the nonlinear dispersive waves in the literature. In this paper, we focus on the equal width (EW) equation which is first suggested by Morrison et.al. [2] and it represents an alternative to the well known KdV and RLW equations.

The EW equation has the following form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\mu\frac{\partial^3 u}{\partial x^2\partial t}=0 \tag{1.1}$$

Date: May 19, 2015 and, in revised form, September 2, 2015.
2000 *Mathematics Subject Classification.* 35Q51; 35Q53; 41A15.
Key words and phrases. EW equation; Nonpolynomial spline; Solitary waves.

where μ is a positive parameter and u is a smooth function that represents the wave amplitude on a domain $\Omega \times [0,T]$ with $\Omega \in \mathbb{R}$. The only possible analytical solution of Eq.(1.1) is the single travelling solitary wave solution. Therefore numerical methods have to be used for some other initial conditions such as interactions, undulation or the Maxwell initial condition.

Numerical methods including spectral method[4], least squares finite element method[5], Galerkin method[6][8][9], collocation method[7][10][14], finite difference method[12][13], differential quadrature method, meshless method[14][22] and Petrov-Galerkin method[3][19] have been presented in the literature for the EW equation.

Spline approximation is based upon to divide the solution domain into a collection of subdomains and construct an approximating function on each subdomains. The most known spline approximation is the cubic spline in which piecewise cubic polynomials are used for the approximation. The objective of spline approximation is to obtain an interpolation formula that has continuous derivatives in required order both within the intervals and at the interpolating nodes.

Nonpolynomial spline based methods have been used for some other partial differential equations such as non-linear Schrödinger equation[20], RLW equation[21], Burgers' equation[16], Klein-Gordon equation[17], Bratu's problem[18]. However, with our knowledge, numerical solution of the EW equation has not been published yet. The aim of this paper is to investigate the numerical solution of the EW equation via the nonpolynomial cubic spline method. Crank-Nicolson method and Rubin-Graves technique[1] are also used for the time discretization and the linearization of the governing equation respectively.

This paper is organized as follows: Section 2 is devoted to the numerical method. Truncation error and stability analysis are also given in that section. The numerical testing and the comparisons on the examples are studied in Section 3. Finally, a conclusion is presented in the last section.

2. Numerical method

Let's start the numerical method by partitioning the solution domain $\Omega \in \mathbb{R}$ into subintervals. For this purpose, we consider $N+1$ equally distributed mesh points such that

$$\Omega : x_0 < x_1 < \cdots < x_N$$

where $x_{i+1} = x_i + h$, $i = 0, 1, ..., N-1$ and h is the grid size.

The proposed spline functions in this paper have the form

$$T_3 = \text{span}\{1, x, \sin(\omega x), \cos(\omega x)\}$$

where ω is the frequency of the trigonometric part of the spline. The cubic nonpolynomial spline functions can be constructed over this mesh as follows:

$$\begin{aligned} P_i(x,t_j) = \;& a_i(t_j)\cos[\omega(x-x_i)] + b_i(t_j)\sin[\omega(x-x_i)] \\ & + c_i(t_j)(x-x_i) + d_i(t_j) \end{aligned} \tag{2.1}$$

where i and j are indices for space and time respectively.

Because of the spline properties, it can written that

$$U_i^j = P_i(x_i,t_j), \quad U_{i+1}^j = P_i(x_{i+1},t_j),$$

$$S_i^j = P_i''(x_i,t_j), \quad S_{i+1}^j = P_i''(x_{i+1},t_j).$$

Then the coefficients in Eq.(2.1) are obtained as

$$a_i = -\frac{h^2}{\theta^2} S_i^j \,, \qquad b_i = \frac{h^2 \left(S_i^j \cos\theta - S_{i+1}^j \right)}{\theta^2 \sin\theta},$$

$$c_i = \frac{U_{i+1}^j - U_i^j}{h} + \frac{h \left(S_{i+1}^j - S_i^j \right)}{\theta^2}, \quad d_i = \frac{h^2}{\theta^2} S_i^j + U_i^j$$

where $\theta = \omega h$ and capital U is used for the approximation to the exact function u.

Another useful tool for the purposed method comes from the continuity of the first derivatives. Having first order continuous derivatives at grid points, i.e. $P_i'(x_i, t_j) = P_{i-1}'(x_i, t_j)$, gives the equation

$$b_i\omega + c_i = -a_{i-1}\omega \sin\theta + b_{i-1}\omega \cos\theta + c_{i-1}. \tag{2.2}$$

Substitution of related coefficients in Eq.(2.2) and slight arrangements on it lead to the following relation between the solutions and their second derivatives:

$$U_{i-1}^j - 2U_i^j + U_{i+1}^j = \alpha S_{i-1}^j + \beta S_i^j + \alpha S_{i+1}^j, \quad i = 1, 2, \cdots, N-1 \tag{2.3}$$

where $\alpha = \dfrac{h^2}{\theta \sin\theta} - \dfrac{h^2}{\theta^2}$ and $\beta = -\dfrac{2h^2 \cos\theta}{\theta \sin\theta} + \dfrac{2h^2}{\theta^2}$. Also note here that if $\theta \to 0$ then $\alpha \to \dfrac{h^2}{6}$ and $\beta \to \dfrac{2h^2}{3}$ which means the standard cubic spline case.

Eq.(2.3) can be written between two successive time levels j and $j+1$ so that

$$\begin{aligned} &\left(U_{i-1}^{j+1} - U_{i-1}^j \right) - 2\left(U_i^{j+1} - U_i^j \right) + \left(U_{i+1}^{j+1} - U_{i+1}^j \right) = \\ &\qquad \alpha \left(S_{i-1}^{j+1} - S_{i-1}^j \right) + \beta \left(S_i^{j+1} - S_i^j \right) + \alpha \left(S_{i+1}^{j+1} - S_{i+1}^j \right) \end{aligned} \tag{2.4}$$

where $i = 1, 2, \cdots, N-1$. The present numerical method will be built on Eq.(2.4).

Theorem 2.1. *The difference equation (2.4) has the local truncation error of order*
i) $O(h^2)$ when $2\alpha + \beta \neq h^2$,
ii) $O(h^4)$ when $2\alpha + \beta = h^2$ and $\alpha \neq h^2/12$,
iii) $O(h^6)$ when $2\alpha + \beta = h^2$ and $\alpha = h^2/12$.

Proof. It was proved in [21] by using the Taylor series expansion, see [21]. □

Besides the spline relation (2.4), the EW equation gives some additional facts about the second derivative of the solution. First, Eq.(1.1) may be rearranged as

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2} - \frac{1}{\mu} u \right) = \frac{1}{\mu}\left(u \frac{\partial u}{\partial x} \right).$$

Then following Crank-Nicolson scheme for the time discretization, the EW equation turns into the form:

$$\left[\frac{\partial^2 U}{\partial x^2} - \frac{1}{\mu} U \right]_{x=x_i}^{t=t_{j+1}} - \left[\frac{\partial^2 U}{\partial x^2} - \frac{1}{\mu} U \right]_{x=x_i}^{t=t_j} = \frac{k}{2\mu}\left(\left[U \frac{\partial U}{\partial x} \right]_{x=x_i}^{t=t_{j+1}} + \left[U \frac{\partial U}{\partial x} \right]_{x=x_i}^{t=t_j} \right) \tag{2.5}$$

The nonlinear term in Eq.(2.5) can be linearized with the technique

$$(UU_x)^{j+1} = U^{j+1} U_x^j + U^j U_x^{j+1} - U^j U_x^j$$

which is suggested by Rubin and Graves[1] as

$$S^{j+1}-S^{j}=\frac{1}{\mu}\left(U^{j+1}-U^{j}\right)+\frac{k}{2\mu}\left(U^{j+1}U_x^{j}+U^{j}U_x^{j+1}\right). \tag{2.6}$$

Using difference formulas for the first order space derivatives in Eq.(2.6) leads to

$$\begin{cases} S_{i-1}^{j+1}-S_{i-1}^{j}=\frac{1}{\mu}\left(U_{i-1}^{j+1}-U_{i-1}^{j}\right)+2rU_{i-1}^{j+1}\left(U_{i}^{j}-U_{i-1}^{j}\right)+2rU_{i-1}^{j}\left(U_{i}^{j+1}-U_{i-1}^{j+1}\right), \\ S_{i}^{j+1}-S_{i}^{j}=\frac{1}{\mu}\left(U_{i}^{j+1}-U_{i}^{j}\right)+rU_{i}^{j+1}\left(U_{i+1}^{j}-U_{i-1}^{j}\right)+rU_{i}^{j}\left(U_{i+1}^{j+1}-U_{i-1}^{j+1}\right), \\ S_{i+1}^{j+1}-S_{i+1}^{j}=\frac{1}{\mu}\left(U_{i+1}^{j+1}-U_{i+1}^{j}\right)+2rU_{i+1}^{j+1}\left(U_{i+1}^{j}-U_{i}^{j}\right)+2rU_{i+1}^{j}\left(U_{i+1}^{j+1}-U_{i}^{j+1}\right) \end{cases} \tag{2.7}$$

where $r=k/\left(4\mu h\right)$.

Finally, considering Eq.(2.4) together with Eq.(2.7) gives the recurrence relation

$$A_iU_{i-1}^{j+1}+B_iU_{i}^{j+1}+C_iU_{i+1}^{j+1}=D_iU_{i-1}^{j}+E_iU_{i}^{j}+F_iU_{i+1}^{j} \tag{2.8}$$

where

$$\begin{aligned} A_i &= 1-\alpha/\mu+4\alpha rU_{i-1}^{j}-r\left(2\alpha-\beta\right)U_i^j, \\ B_i &= -2-\beta/\mu+r\left(2\alpha-\beta\right)\left(U_{i+1}^{j}-U_{i-1}^{j}\right), \\ C_i &= 1-\alpha/\mu-4\alpha rU_{i+1}^{j}+r\left(2\alpha-\beta\right)U_i^j, \\ D_i &= 1-\alpha/\mu, \\ E_i &= -2-\beta/\mu, \\ F_i &= 1-\alpha/\mu. \end{aligned}$$

The recurrence relation (2.8) contains $N-1$ equations in $N+1$ unknowns. By adding two equations from the boundary conditions, it will be a solvable linear system. After the initial solutions U^0 computed from the initial condition, all the other solutions at different time levels are calculated from the system (2.8).

2.1. **Stability analysis.** According to von-Neumann analysis, it is assumed that the solution of the governing equation is in the following form:

$$U_i^j=\xi^j e^{q\varphi ih}$$

where q is the imaginary unit, φ is the wave number and ξ is the amplification factor. Substitution of the above expression in Eq.(2.8) yields

$$\begin{aligned} & A_i\xi^{j+1}e^{q\varphi(i-1)h}+B_i\xi^{j+1}e^{q\varphi ih}+C_i\xi^{j+1}e^{q\varphi(i+1)h} \\ = \; & D_i\xi^{j}e^{q\varphi(i-1)h}+E_i\xi^{j}e^{q\varphi ih}+F_i\xi^{j}e^{q\varphi(i+1)h}. \end{aligned}$$

Then

$$\xi=\frac{\left(2-\dfrac{2\alpha}{\mu}\right)\cos\varphi h-2-\dfrac{\beta}{\mu}+q\left(2\alpha+\beta\right)2rd^{*}}{\left(2-\dfrac{2\alpha}{\mu}\right)\cos\varphi h-2-\dfrac{\beta}{\mu}-q\left(2\alpha+\beta\right)2rd^{*}}$$

where d^* is locally constant for U in the nonlinear terms. Hence, the above expression gives $|\xi| = 1$ which means that the numerical method is unconditionally stable.

3. Test problems

In this section, several test problems take part in order to show the accuracy and the efficiency of the numerical method. The accuracy is measured by L_∞ and L_2 error norms that are defined by

$$\begin{aligned} L_\infty &= \max_i \left| u_i^{exact} - U_i^{numeric} \right|, \\ L_2 &= \sqrt{h \sum_{i=0}^{N} \left| u_i^{exact} - U_i^{numeric} \right|^2}. \end{aligned}$$

In all numerical computations except the motion of single solitary wave, the discretization parameters are chosen as $h = 0.1$ and $k = 0.1$. Additionally, similar to the reference [21], the parameters α and β are selected as $2\alpha + \beta = h^2$ and $\alpha = h^2/4$ in all computations.

The EW equation has also the following conserved quantities:

$$\mathrm{C}_1 = \int_a^b u dx, \quad \mathrm{C}_2 = \int_a^b \left(u^2 + \mu \left(u_x \right)^2 \right) dx, \quad \mathrm{C}_3 = \int_a^b u^3 dx$$

which correspond to mass, momentum and energy respectively. These invariants also give an idea about the accuracy of the numerical method especially in cases that the equation does not have an analytical solution. Therefore the invariants are monitored to check the conservation of the numerical algorithms for all test problems.

In order to compute the rate of convergence, the algorithm has been performed for difference space and time steps. Then the results are used in the formula

$$\begin{aligned} \textbf{space order} &= \frac{\log(\|u - u_{h_i}\|_2 \,/\, \|u - u_{h_{i+1}}\|_2)}{\log(h_i/h_{i+1})} \\ \textbf{time order} &= \frac{\log(\|u - u_{k_i}\|_2 \,/\, \|u - u_{k_{i+1}}\|_2)}{\log(k_i/k_{i+1})} \end{aligned}$$

where u is the exact solution and u_{h_i} and u_{k_i} are the numerical solutions for space size h_i and time step k_i respectively.

3.1. Motion of single solitary wave.

A single solitary wave which is initially centered at $\widetilde{x}_s$ and travels with a constant velocity has the following analytical solution

$$u(x,t) = 3c\operatorname{sech}^2 \left[K \left(x - \widetilde{x}_s - ct \right) \right], \tag{3.1}$$

where $K = 1/\sqrt{4\mu}$ is the width of the wave pulse, c is the velocity and $3c$ is the magnitude of the wave.

The initial condition comes from Eq.(3.1) and the boundary conditions are given by

$$u(x_0, t) = 0 \qquad \text{and} \qquad u(x_N, t) = 0.$$

The common parameter choices in the literature are $\mu = 1$ and $x_s = 10$. Although almost all earlier papers use same time increment, i.e. $k = 0.05$, there are some different considerations for the grid size. For instance, $h = 0.15$ in [22] and MM[14], $h = 0.1$ in DQM[14], $h = 0.05$ in [7], [9] and [6]. In this test problem, similar to QBGM[14], the solutions are calculated over $\Omega = [0, 30]$ and $t \in [0, 80]$ with the discretization parameters $h = 0.03$ and $k = 0.05$. The solution profiles are illustrated in Fig.1-2 for $c = 0.1$ and Fig.3-4 for $c = 0.03$ at different times. It is clear from these figures that solutions remain in same profile.

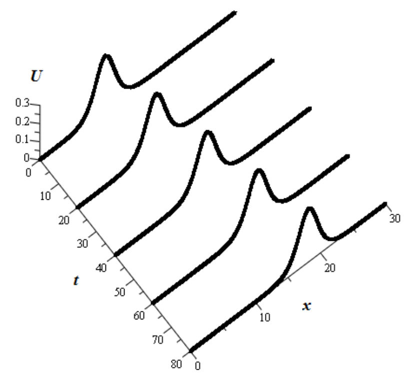


Fig.1: Solitary waves for $c = 0.1$

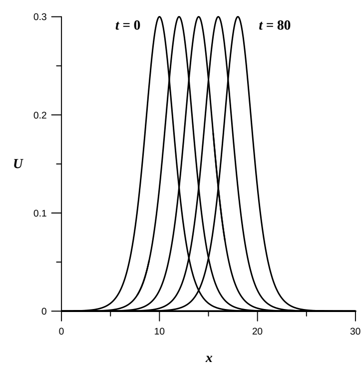


Fig.2: Solitary waves for $c = 0.1$

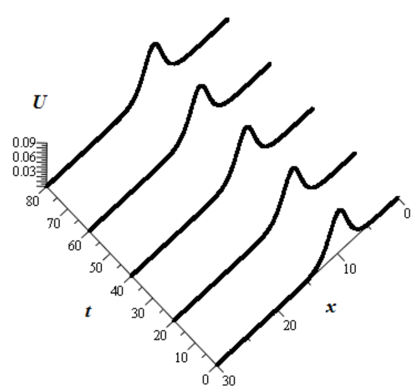


Fig.3: Solitary waves for $c = 0.03$

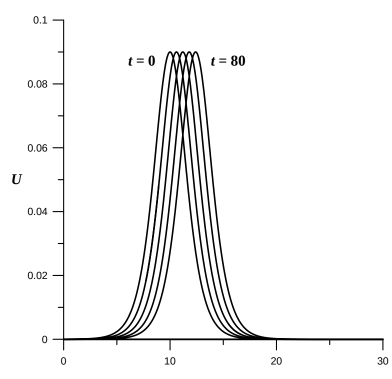


Fig.4: Solitary waves for $c = 0.03$

The analytical values of the invariants are calculated by

$$\mathrm{C}_1 = \frac{6c}{K}, \quad \mathrm{C}_2 = \frac{12c^2}{K} + \frac{48Kc^2\mu}{5}, \quad \mathrm{C}_3 = \frac{144c^3}{5K}$$

that correspond to $\mathrm{C}_1 = 1.2$, $\mathrm{C}_2 = 0.288$ and $\mathrm{C}_3 = 0.0576$ for $c = 0.1$ and $\mathrm{C}_1 = 0.36$, $\mathrm{C}_2 = 0.02592$ and $\mathrm{C}_3 = 0.001555$ for $c = 0.03$. Computed errors and invariants are presented in Table 1 and Table 3 for $c = 0.1$ and $c = 0.03$ respectively. According to Table 1 and 3, the present results are acceptable and the given method is comparable with others.

Table 1
Errors and invariants at time $t = 80$ for $c = 0.1$

Method	$L_\infty \times 10^4$	$L_2 \times 10^4$	C_1	C_2	C_3
Analytic			1.2	0.288	0.05760
Present	0.07372964	0.1289443	1.199985	0.2879897	0.05760
[7]	0.53	0.56	1.19998	0.28798	0.05759
[9]	0.21	0.29	1.19995	0.28798	0.05759
[6]	0.01704	0.03064	1.19999	0.28801	0.05760
QBGM[14]	0.07370	0.06095	1.20000	0.288000	0.05760
DQM[14]	0.07373	0.07035	1.19999	0.288000	0.05760
MM[14]	0.20296	0.31198	1.20003	0.288000	0.05760
W(7,5)[22]	0.03537611	0.03360406	1.19999752	0.28800001	0.05760

Absolute error distributions at $t = 80$ are plotted in Fig.5 and Fig.6. Due to the relatively high velocity, the solution domain is short when $c = 0.1$. Therefore the maximum error is observed at the right hand boundary in Fig.5. To overcome this problem, the solution domain can be extended so that the error at the right hand boundary decreases.

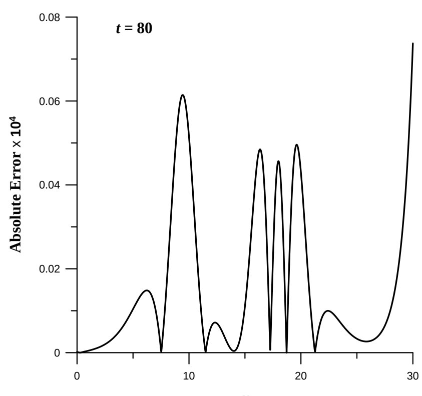


Fig.5: Absolute error for $c = 0.1$

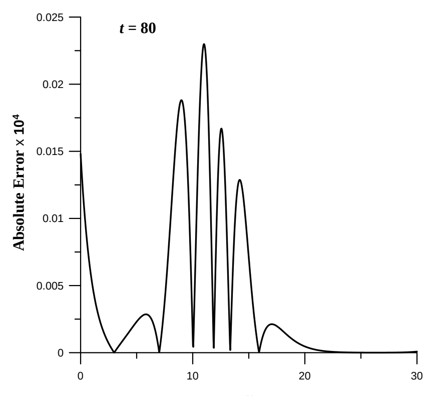


Fig.6: Absolute error for $c = 0.03$

The orders for pointwise rate of convergence are given in Table 2 which shows that the present method has second order accuracy in terms of both space and time.

Table 2: Rate of convergence

Spatial order ($\Delta t = 0.05$)		Temporal order ($h = 0.03$)	
h_i	$t = 80$	Δt_i	$t = 80$
2.00		2.00	
1.00	3.1914841	1.00	1.9942460
0.50	2.1804472	0.50	2.0091253
0.25	2.0403988	0.25	2.0364273
0.125	2.0157560	0.125	1.9910816
0.0625	2.0207153	0.0625	0.9043434

Table 3
Errors and invariants at time $t = 80$ for $c = 0.03$

Method	$L_\infty \times 10^4$	$L_2 \times 10^4$	C_1	C_2	C_3
Analytic			0.36000	0.02592	0.001555
Present	0.02299	0.03812	0.359997	0.025919	0.0015552
[8]	18.36	26.83	0.36665	0.02658	
[9]	0.07	0.13	0.36000	0.02592	0.00156
[6]	0.01483	0.01025	0.36000	0.02592	0.00156
QBGM[14]	0.01483	0.01064	0.36000	0.02592	0.00156
DQM[14]	0.01483	0.00934	0.36000	0.02592	0.00156
MM[14]	0.07598	0.04911	0.36000	0.02592	0.00156
W(7,5)[22]	0.01418041	0.01267701	0.36000055	0.02592	0.0015552

3.2. **Interaction of two solitary waves.** As a second problem, interaction of two solitary waves is considered. The initial condition

$$\left.\begin{aligned} u_0(x) &= U_1 + U_2 \\ U_j &= 3c_j \operatorname{sech}^2\left[K_j\left(x - \widetilde{x}_j - c_j\right)\right], \ j = 1, 2 \end{aligned}\right\} \tag{3.2}$$

yields two waves travelling in same direction and having amplitude $3c_1$ and $3c_2$. These waves are initially positioned at $x = \widetilde{x}_1$ and $x = \widetilde{x}_2$ respectively. The following parameter choices give a complete interaction over the solution domain $x \in [0, 80]$.

$$\mu = 1, \ K_1 = 0.5, \ K_2 = 0.5, \ \widetilde{x}_1 = 10, \ \widetilde{x}_2 = 25, \ c_1 = 1.5, \ c_2 = 0.75.$$

To illustrate the interaction, the solution profiles are figured in Fig.7-8 at three different times. The figures show that there is no decay on the solitary waves after the interaction. However, as seen in Fig.9, there are some changes on magnitudes for both waves at the interaction process.

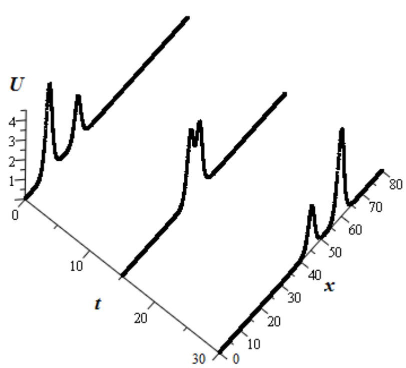


Fig.7: Interaction of two solitary waves

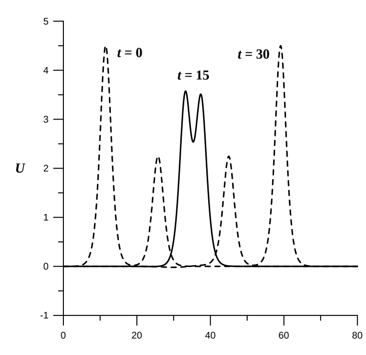


Fig.8: Interaction of two solitary waves

In order to see the results quantitatively and to make a comparison, Table 4 is constructed. Since there is no analytical solution with the considered initial condition (3.2), only the invariants are compared in the table. Analytical values of the invariants are

$$\mathrm{C}_1 = 12\left(c_1 + c_2\right) = 27, \quad \mathrm{C}_2 = 28.8\left(c_1^2 + c_2^2\right) = 81, \quad \mathrm{C}_3 = 57.6\left(c_1^3 + c_2^3\right) = 218.7.$$

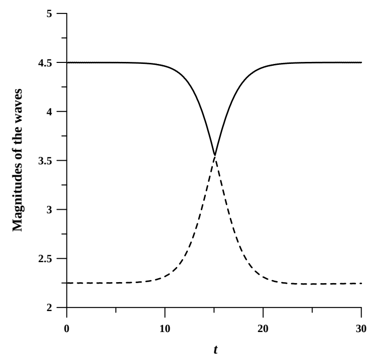


Fig.9: Magnitudes of the waves

Table 4
Invariants for the interaction of two solitary waves at $t = 30$.

Method	C_1	C_2	C_3
Analytic	27	81	218.7
Present	26.999997	80.968402	218.70289
[9]	27.00003	81.01719	218.70650
[6]	27.00068	81.02407	218.73673
[10]	27.12702	80.98988	218.6996
QBGM[14]	26.99973	80.99778	218.69094
DQM[14]	27.00017	81.00044	218.70304
MM[14]	27.00024	81.00140	218.70694
W(7,5)[22]	27.000049	81.000204	218.70186

3.3. **Interaction of three solitary waves.** Interaction of three solitary waves is figured out in this subsection. The initial condition

$$u_0\left(x\right) = \sum_{j=1}^{3} 3c_j \mathrm{sech}^2\left[K_j\left(x - \widetilde{x}_j - c_j\right)\right]$$

where

$$K_1 = K_2 = K_3 = 0.5,\ c_1 = 4.5,\ c_2 = 1.5,\ c_3 = 0.5,\ \widetilde{x}_1 = 10,\ \widetilde{x}_2 = 25,\ \widetilde{x}_3 = 35$$

leads to three waves which interact together. Figs.10-11 shows the complete interaction. The backmost wave passes the others without any decay on its profile. The invariants are tabulated at $t = 15$ for $h = 0.1$ and $k = 0.1$ in Table 5.

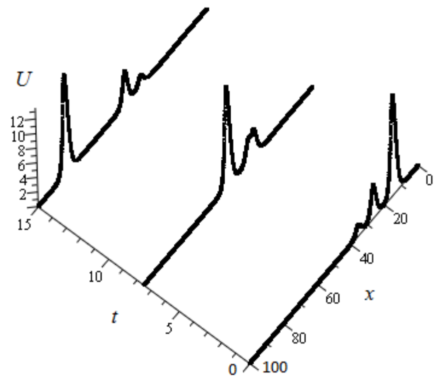


Fig.10: Interaction of three solitary waves

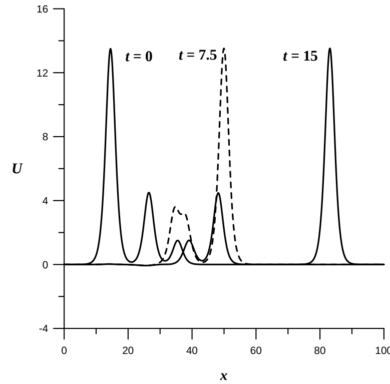


Fig.11: Interaction of three solitary waves

Table 5
Invariants for the interaction of two solitary waves at $t = 15$.

Method	C_1	C_2	C_3
Analytic	78	655.2	5450.4
Present	77.999994	655.069708	5451.895023
[10]	77.99539	652.8104	5411.639
[11]	78.00490	652.3474	5412.232
W(7,5)[22]	78.000004	655.263936	5451.005509

3.4. **Interaction of opposite waves.** The last interaction example is the interaction between two opposite waves that have exactly the same form but different signs. This case is relatively less considered problem in the literature. Although it is stated in [22] that the colliding solitons has never been treated before, it was also studied in [4] and [11].

The initial condition for colliding waves that are initially centered at $x = 40$ and $x = 120$ is given in [4] as

$$u_0(x) = 4.5\text{sech}^2\left[(x-40)/2\right] - 4.5\text{sech}^2\left[(x-120)/2\right]$$

which is also considered here with $h = 0.1$ and $k = 0.1$.

These two opposite waves move towards each other and then a singularity occurs when they meet. The colliding yields trains of smaller waves on both sides, while the singularity gradually vanishes over time, see Figs.12-15.

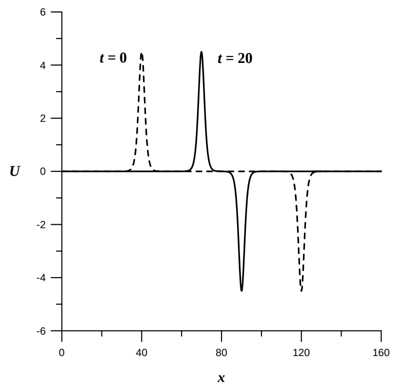


Fig.12: Waves move towards each other

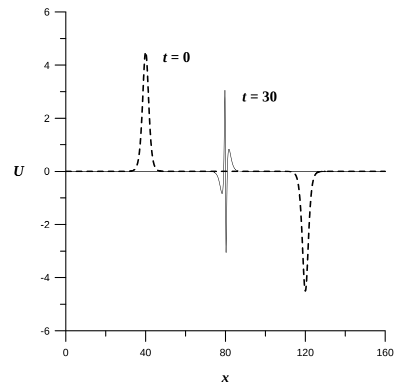


Fig.13: Colliding solitons with singularity

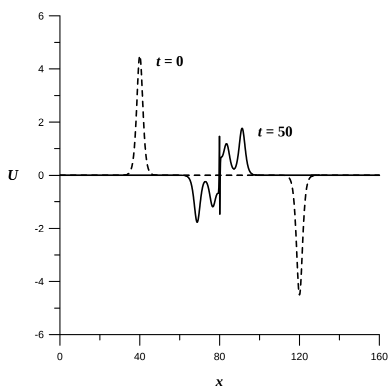


Fig.14: Singularity vanishes over time

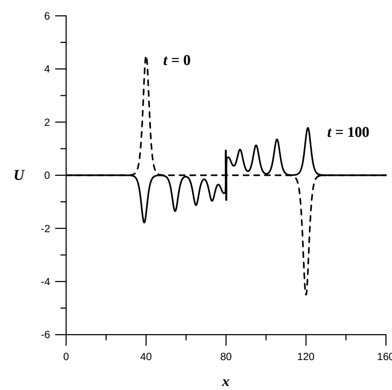


Fig.15: Trains of smaller waves

3.5. **Wave undulation.** Development of an undular bore is studied here by the initial function

$$u_0\left(x\right) = \frac{U_0}{2}\left(1 - \tanh\left(\frac{x - x_c}{d}\right)\right)$$

where d shows the slope between the still and deeper water and x_c is the center of the change in water level of magnitude U_0. The EW equation has not an analytical solution with the mentioned initial condition. So, only the invariants of the EW equation are considered in order to see the efficiency of the method. A comparison on invariants, position and amplitude of the leading undulation is presented in Table 6.

Table 6
Development of undular bore

	Time	C_1	C_2	C_3	x	U
$d=2$	0	2.0000000	0.19027759	0.018500000		
	200	3.0000000	0.32337149	0.033500252	9.4	0.17579731
	400	4.0000000	0.45637134	0.048500614	21.4	0.18142204
	600	5.0000000	0.58937051	0.063500976	33.6	0.18321870
	800	5.9999778	0.72236947	0.078501338	45.8	0.18383578
QBGM[14]	800	0.6002474	0.72386	0.078525	45.85	0.18471
DQM[14]	800	0.6025073	0.72402	0.07853	45.85	0.184713
$d=5$	0	2.0000839	0.17512787	0.01625251521		
	200	3.0000815	0.30837385	0.03125247301	8.8	0.16035721
	400	4.0000815	0.44138580	0.04625256015	20.4	0.17905369
	600	5.0000815	0.57438765	0.06125265022	32.5	0.18242416
	800	6.0000801	0.70738790	0.07625274062	44.7	0.18364070
QBGM[14]	800	6.002578	0.708710	0.076277	44.75	0.18405
DQM[14]	800	6.025306	0.711361	0.076579	44.75	0.17259

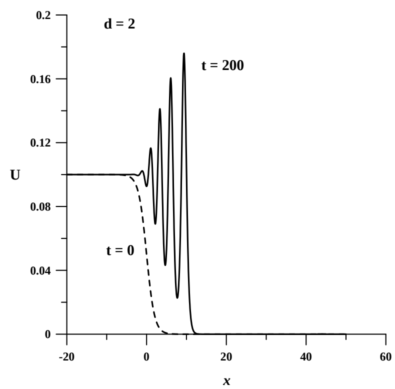


Fig.16: Undulation for $d=2$

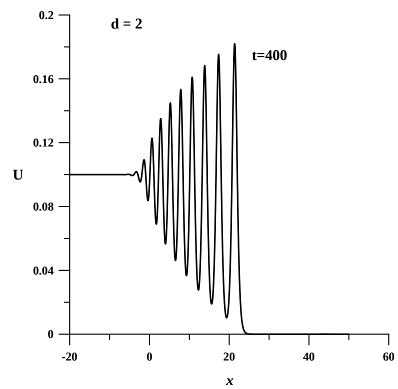


Fig.17: Undulation for $d=2$

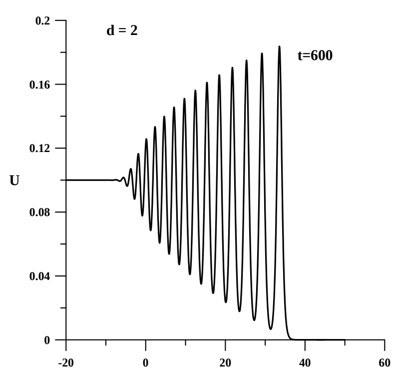


Fig.18: Undulation for $d=2$

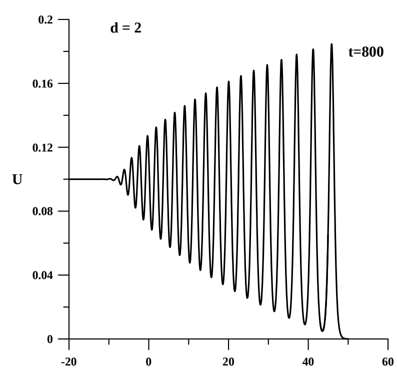


Fig.19: Undulation for $d=2$

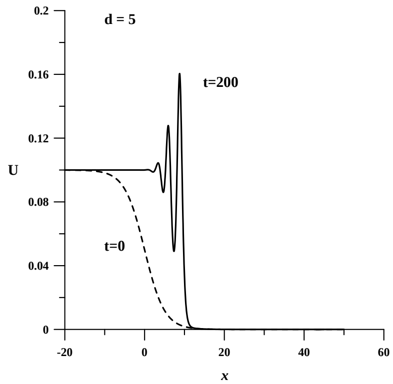


Fig.20: Undulation for $d = 5$

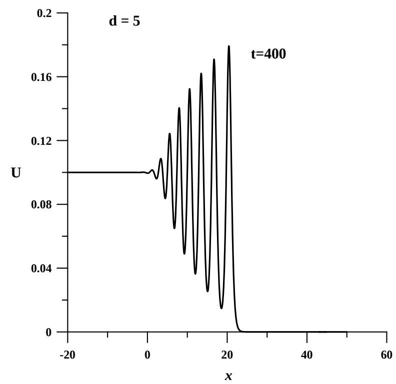


Fig.21: Undulation for $d = 5$

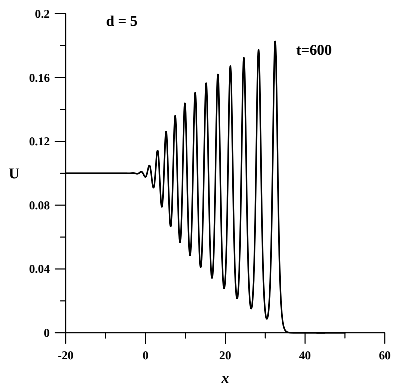


Fig.22: Undulation for $d = 5$

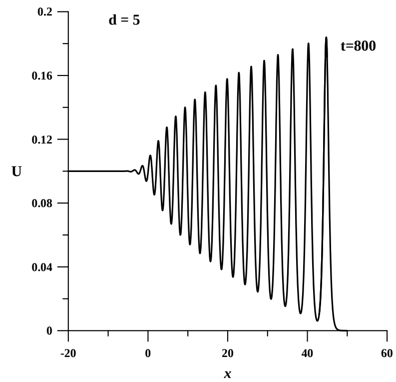


Fig.23: Undulation for $d = 5$

Variations in invariants that are given in Table 7 are calculated numerically with the formula

$$M_i = \frac{C_i\,(\text{at time } t = 800) - C_i\,(\text{at time } t = 0)}{\text{Running time}}$$

and analytically with

$$\begin{aligned} M_1 &= \frac{dC_1}{dt} = \frac{d}{dt}\int_{x_0}^{x_N} u dx = \frac{1}{2}U_0^2 = 5 \times 10^{-3} \\ M_2 &= \frac{dC_2}{dt} = \frac{d}{dt}\int_{x_0}^{x_N} \left(u^2 + \mu u_x^2\right) dx = \frac{2}{3}U_0^3 = 6.66667 \times 10^{-4} \\ M_3 &= \frac{dC_3}{dt} = \frac{d}{dt}\int_{x_0}^{x_N} u^3 dx = \frac{3}{4}U_0^4 = 7.5 \times 10^{-5} \end{aligned}$$

The undulation profiles are illustrated in Figs.16-19 for $d = 2$ and Figs.20-23 for $d = 5$.

Table 7
Variations in invariants

	Method	$M_1 \times 10^{-3}$	$M_2 \times 10^{-4}$	$M_3 \times 10^{-5}$
	Analytical	5	6.66667	7.5
$d = 2$	Present	4.9999723	6.6511485	7.500167
	QBGM[14]	4.99997	6.66665	7.5
	DQM[14]	5	6.669387	7.507
	MM[14]	5	6.669387	7.507
	W(7,5)[22]	4.99937586	6.66667317	7.50000017
$d = 5$	Present	4.9999953	6.6532503	7.5001382
	QBGM[14]	4.99999	6.66665	7.7
	DQM[14]	5	6.671688	7.509
	MM[14]	5	6.671688	7.509

3.6. **The Maxwell wave.** The last problem for testing our method is the Maxwell wave where the starting function is

$$u_0(x) = 0.05 \exp\left(-(x-20)^2/25\right).$$

Again the analytical solution does not exist with this initial condition. The solutions are computed over $\Omega = [0, 50]$ until $T = 1000$. The wave profiles are drawn in Figs.24-25 at four different times to figure out the behavior of the initial wave over time.

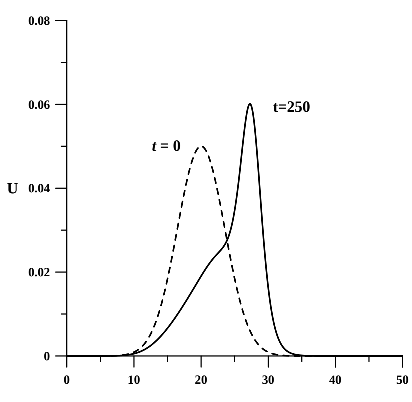


Fig.24: Wave profiles at $t = 0$ and 250

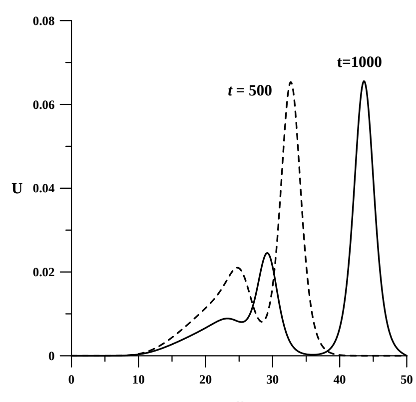


Fig.25: Wave profiles at $t = 500$ and 1000

There are some changes in the initial profile in course of time. It turns into a train such that while its amplitude becomes larger, the wave length becomes smaller and there are tails that will turn to a new small wave.

The invariants are presented at some different times in Table 8. The results show that the method is very conservative in this problem.

Table 8
Invariants for the Maxwell wave

Time	C_1	C_2	C_3
0	0.44311346	0.016292833	0.00063957919
100	0.44311348	0.016292567	0.00063957921
200	0.44311349	0.016291320	0.00063957928
300	0.44311349	0.016289148	0.00063957937
400	0.44311350	0.016287762	0.00063957944
500	0.44311349	0.016287480	0.00063957947
600	0.44311337	0.016287483	0.00063957947
700	0.44311225	0.016287440	0.00063957948
800	0.44310227	0.016287342	0.00063957947
900	0.44301341	0.016287228	0.00063957915
1000	0.44222656	0.016287133	0.00063955380

4. Conclusion

In this study, cubic nonpolynomial spline based numerical method is implemented in order to get the solution of the EW equation. Over the uniform mesh, Crank-Nicolson formulas are employed for time discretization whereas Rubin and Graves[1] technique is used for the linearization. According to pointwise rate of convergence, the present method has second order accuracy for both space and time. Also the von-Neumann stability analysis shows that the purposed method is unconditionally stable. Six problems that related to single solitary wave, interaction of two, three and opposite solitary, the undulation bore and the Maxwell wave are examined for testing the numerical scheme. Comparisons between the obtained results and some earlier papers show that the present results are all acceptable and in agreement with those in the literature. Simple adaptation and yielding band matrices can be stated as the advantages of the method. On the other hand, according to your problem, requiring the determination of two parameters (α and β) is an undesirable situation. In conclusion, cubic nonpolynomial spline method can be considered as a conservative numerical method that leads to reasonable results.

References

[1] Rubin S.G. and Graves R.A., Cubic spline approximation for problems in fluid mechanics, Nasa TR R-436, Washington, DC, (1975).
[2] Morrison P.J., Meiss J.D., Carey J.R., Scattering of RLW solitary waves, Physica 11D (1981) 324–36.
[3] Gardner L.R.T., Gardner G.A., Solitary waves of the equal width wave equation, J Comput Phys 101 (1992) 218–23.
[4] Garcia-Archilla B., A spectral method for the equal width equation, J Comput Phys 125 (1996) 395–402.
[5] Zaki S.I., A least-squares finite element scheme for the EW equation, Comput Meth Appl Mech Eng 189 (2000) 587–94.
[6] Saka B., Dag I., Dogan A., A Galerkin method for the numerical solution of the RLW equation using quadratic B-splines, Int J Comput Math 81 (2004) 727–739.
[7] Dag I., Saka B., A cubic B-spline collocation method for the EW equation. Math Comput Appl 9 (2004) 381–392.

[8] Dogan A., Application of Galerkin's method to equal width wave equation. Appl Math Comput 160 (2005;) 65–76.
[9] Esen A., A numerical solution of the equal width wave equation by a lumped Galerkin method, Appl Math Comput 168 (2005) 270–282.
[10] Raslan K.R., Collocation method using quartic B-spline for the equal width (EW) equation, Int J Comput Math 81 (2004) 63–72.
[11] Raslan K.R., A computational method for the equal width equation, Appl Math Comput 168 (2005) 795–805.
[12] Ramos J.I., Explicit finite difference methods for the EW and RLW equations, Appl Math Comput 179 (2006) 622–638.
[13] Ramos J.I., Solitary waves of the EW and RLW equations, Chaos Solitons and Fractals 34 (2007) 1498–1518.
[14] Saka B., Dag I., Dereli Y., Korkmaz A., Three different methods for numerical solution of the EW equation, Engineering Analysis with Boundary Elements 32 (2008) 556-566.
[15] Rashidinia J., Mohammadi R., Non-polynomial cubic spline methods for the solution of parabolic equations, Int. J. Comput. Math. 85:5 (2008) 843-850.
[16] Griewanka A., El-Danaf T.S., Efficient accurate numerical treatment of the modified Burgers' equation, Applicable Analysis 88 (2009) 75-87.
[17] Rashidinia, J., Mohammadi, R.: Tension spline approach for the numerical solution of nonlinear Klein-Gordon equation, Comput. Phys. Commun. 181 (2010) 78–91.
[18] Jalilian, R.: Non-polynomial spline method for solving Bratu's problem, Comput. Phys. Commun. 181 (2010) 1868–1872.
[19] Roshan T., A Petrov–Galerkin method for equal width equation, Applied Mathematics and Computation 218 (2011) 2730–2739.
[20] El-Danaf T.S., Ramadan M.A., Abd Alaal F.E.I, Numerical studies of the cubic non-linear Schrödinger equation, Nonlinear Dyn. 67 (2012) 619-627.
[21] Chegini N.G., Salaripanah A., Mokhtari R., Isvand D., Numerical solution of the regularized long wave equation using nonpolynomial splines, Nonlinear Dyn. 69 (2012) 459-471.
[22] Dereli Y., Schaback R., The meshless kernel-based method of lines for solving the equal width equation, Applied Mathematics and Computation 219 (2013) 5224-5232.

Department of Mathematics, Aksaray University, 68100, Aksaray,Turkey.
E-mail address: asahinhc@gmail.com

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 33–41 (2015) ©KJM

ON A SUBCLASS OF UNIFORMLY QUASI CONVEX FUNCTIONS OF ORDER α

D. VAMSHEE KRISHNA, B. VENKATESWARLU, AND T. RAMREDDY

Abstract. In this paper, we introduce two new classes of analytic functions namely uniformly quasi convex functions of order α and quasi uniformly convex functions of order α denoted by $UQCV(\alpha)$ and $QUCV(\alpha)$ $(0 \leq \alpha < 1)$ respectively and study certain properties of functions belonging to these two classes. Further, we obtain a necessary and sufficient condition for the function $f(z)$ to be in the class $UQCV(\alpha)$. These results are generalized recent results of Rajalakshmi Rajagopal and Selvaraj [7].

1. Introduction and Preliminaries

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Let S denote the subclass of A which are univalent in U.

Definition 1.1. [3] A function f given in (1.1) is said to be uniformly convex in U, if f is convex and has the property that for every circular arc γ contained in U with centre ξ the arc $f(\gamma)$ is convex. The class of uniformly convex functions is denoted by UCV. The analytical characterization of the function $f \in UCV$ was given by Goodman [3].

Theorem 1.1. [3] *A function f of the form (1.1) is in UCV if and only if $Re\left\{1 + (z-\xi)\frac{f''(z)}{f'(z)}\right\} > 0, \forall\ (z,\xi) \in U \times U$ and $z \neq \xi$.*

Theorem 1.2. [8] *A function f of the form (1.1) is in UCV if and only if $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, \forall\ z \in U$.*

Date: January 1, 2013 and, in revised form, February 2, 2013.
2010 *Mathematics Subject Classification.* 30C45; 30C50.
Key words and phrases. Analytic function, uniformly quasi convex function, quasi uniformly convex function, positive real function.

Definition 1.2. [5] A function f of the form (1.1) is said to be quasi convex in U if there exists a convex function g in U with $g(0) = 0 = g'(0) - 1$ such that $Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > 0, \forall\, z \in U$.

The class of quasi convex functions is denoted by c^*.

Definition 1.3. [6] A function f of the form (1.1) is said to be close-to- uniformly convex if there exists a uniformly convex function g in U such that $Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0, z \ \in U$.

The class of all close-to-uniformly convex functions is denoted by $CUCV$. The subclasses uniformly quasi convex functions and quasi uniformly convex functions denoted by $UQCV$ and $QUCV$ respectively of S are introduced and studied by Rajalakshmi Rajagopal and Selvaraj [7]. The following Definitions are due to them.

Definition 1.4. A function $f(z)$ in A is said to be uniformly quasi convex in U if there exists a uniformly convex function g in U with $g(0) = 0 = g'(0) - 1$ such that $Re\left\{\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\right\} > 0, \ \forall\, z, \xi \in U, z \neq \xi$.

The class of all such functions is denoted by $UQCV$.

Definition 1.5. A function $f(z)$ is A is said to be quasi uniformly convex in U if there exists a uniformly convex function g in U with $g(0) = 0 = g'(0) - 1$ such that $Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > 0, \ \forall\, z \in U$.

The class of all quasi uniformly convex functions is denoted by $QUCV$. From the Definitions 1.4 and 1.5, it is observed that $QUCV \subset UQCV$. Now, we introduce and study certain important properties of the following two classes.

2. Main results:

Definition 2.1. A function $f(z)$ in A is said to be uniformly quasi convex function of order α $(0 \leq \alpha < 1)$ if there exists a uniformly convex function g in U with $g(0) = g'(0) - 1$ such that

$$Re\left\{\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\right\} > \alpha, \ \forall\ (z, \xi) \in U \times U \text{ and } z \neq \xi. \tag{2.1}$$

We denote the class of uniformly quasi convex functions of order α by $UQCV(\alpha)$.

Definition 2.2. A function $f(z)$ in A is said to be uniformly convex of function of order α $(0 \leq \alpha < 1)$ if and only if

$$Re\left\{1 + (z-\xi)\frac{f''(z)}{f'(z)}\right\} > \alpha, \ \forall\ (z, \xi) \in U \times U \text{ and } z \neq \xi. \tag{2.2}$$

The class of such functions is denoted by $UCV(\alpha)$.

Definition 2.3. A function $f(z)$ in A is said to be quasi convex function of order α $(0 \leq \alpha < 1)$ if there exists a convex function g in U with $g(0) = 0 = g'(0) - 1$ such that

$$Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > \alpha, \ \forall\ z \in U. \tag{2.3}$$

The class of such functions is denoted by $c^*(\alpha)$.

Definition 2.4. A function $f(z)$ in A is said to be quasi uniformly convex function of order α $(0 \leq \alpha < 1)$ if there exists a uniformly convex function g in U with $g(0) = 0 = g'(0) - 1$ such that

$$Re\left\{\frac{\{z\ f'(z)\ \}'}{g'(z)}\right\} > \alpha,\ \forall\ z \in U. \tag{2.4}$$

The class of all quasi uniformly convex functions is denoted by $QUCV(\alpha)$.

Definition 2.5. A function $f(z)$ in A is said to be close-to-convex function of order α $(0 \leq \alpha < 1)$ if there exists a convex function g in U such that

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha,\ \forall\ z \in U. \tag{2.5}$$

The class of all close - to- convex functions of order α is denoted by $K(\alpha)$.

Definition 2.6. A function $f(z)$ in A is said to be close-to- uniformly convex function of order α $(0 \leq \alpha < 1)$ if there exists a uniformly convex function g in U such that

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha,\ \forall\ z \in U. \tag{2.6}$$

The class of all close - to- uniformly convex functions of order α is denoted by $CUCV(\alpha)$. From the above Definitions, we observe the following conclusions:

1. Choosing $g(z) = f(z)$ in (2.1), where $g(z) \in UCV$, we obtain

$$Re\left\{\frac{\{(z-\xi)f'(z)\}'}{f'(z)}\right\} = Re\left\{1 + (z-\xi)\frac{f''(z)}{f'(z)}\right\} > \alpha,$$
$$\text{for } z \neq \xi \text{ in } |z| < 1 \text{ for } (0 \leq \alpha < 1).$$

From this result and in view of Definition 2.2, we get

$$UCV(\alpha) \subset UQCV(\alpha) \tag{2.7}$$

2. Taking $\xi = 0$ in (2.1), we obtain

$$Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > \alpha,\ \text{for } z \in\ U,\ \text{for } (0 \leq \alpha < 1). \tag{2.8}$$

From the Definition 2.3, we observe that

$$UQCV(\alpha) \subset c^*(\alpha). \tag{2.9}$$

From the expressions (2.7) and (2.9), we obtain

$$UCV(\alpha) \subset UQCV(\alpha) \subset c^*(\alpha). \tag{2.10}$$

Therefore, an immediate consequence of (2.10) is that every uniformly quasi-convex function of order α is univalent.

3. Choosing $t(z) = zf'(z)$ in (2.3), we get

$$Re\left\{\frac{t'(z)}{g'(z)}\right\} > \alpha,\ \text{for } (0 \leq \alpha < 1).$$

(2.11) From the Definition 2.5, we observe that $c^*(\alpha) \subset K(\alpha)$.

4. Taking $\xi = 0$ in (2.1), we obtain

$$Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > \alpha, \text{ for } z \in U, \tag{2.12}$$

which implies that $QUCV(\alpha) \subset UQCV(\alpha)$.

Lemma 2.1. *If $g(z) \in UCV$, then*

$$|g'(z)| \leq \frac{1}{1-r}, \text{ for } |z| = r < 1, \ z \in U.$$

Proof. Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \Leftrightarrow g'(z) = 1 + \sum_{n=2}^{\infty} n b_n z^{n-1}$. Taking modulus on both sides of $g'(z)$, using the facts $|a+b| \leq |a| + |b|$ and $|ab| = |a||b|$, we get

$$|g'(z)| = \left|1 + \sum_{n=2}^{\infty} n b_n z^{n-1}\right| \Leftrightarrow |g'(z)| \ \leq \left[1 + \sum_{n=2}^{\infty} n|b_n| r^{n-1}\right]. \tag{2.13}$$

For the function $g(z) \in UCV$ (according to Goodman [2]), we have

$$|b_n| \leq \frac{1}{n}, \ \forall n \geq 2. \tag{2.14}$$

Simplifying the expressions (2.13) and (2.14), we obtain

$$|g'(z)| \leq \left[1 + \sum_{n=2}^{\infty} r^{n-1}\right] = \frac{1}{1-r}.$$

Hence the Lemma. □

Theorem 2.1. *Let $f(z)$ be in A. Then f is uniformly quasi-convex function of order α $(0 \leq \alpha < 1)$ if and only if* $Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > \alpha + \left|\frac{zf''(z)}{g'(z)}\right|$.

Proof. Let $f(z) \in UQCV(\alpha)$ $(0 \leq \alpha < 1)$. By virtue of Definition 2.1, there exists uniformly convex function $g(z) \in U$ such that

$$Re\left\{\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\right\} \ > \ \alpha, \ \forall \ (z, \xi) \in U \times U \text{ and } z \neq \xi. \tag{2.15}$$

$$\Leftrightarrow Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > \alpha \ + \ Re\left\{ \ \frac{\xi f''(z)}{g'(z)} \ \right\}. \tag{2.16}$$

If we choose $\xi = ze^{i\beta}$ in a suitable way, for some real β, we get

$$Re\left\{\frac{\xi f''(z)}{g'(z)}\right\} = \left|\frac{ze^{i\beta} f''(z)}{g'(z)}\right| = \left|\frac{zf''(z)}{g'(z)}\right|. \tag{2.17}$$

From the expressions (2.16) and (2.17), we obtain

$$Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} \geq \ \alpha + \left|\frac{zf''(z)}{g'(z)}\right|. \tag{2.18}$$

Hence, the condition is necessary.

Conversely, suppose the condition given by (2.18) is true.
Let ξ be an arbitrary but fixed point in the unit disc U. Since the quotient of two analytic functions, whose real part is harmonic and hence the function $Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\}$ becomes harmonic, provided $g(z) \in UCV$.

Therefore, by the minimum principle it is enough to show that the result is true for $|z| = \rho > |\xi|,\ \rho < 1$. From (2.18), for $|\xi| < |z| = \rho < 1$ and using the fact $Re(z) \leq |z|$, we get

$$Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} \geq \left[\alpha + \left|\frac{zf''(z)}{g'(z)}\right|\right] > \left[\alpha + \left|\frac{\xi f''(z)}{g'(z)}\right|\right] \geq \left[\alpha + Re\left\{\frac{\xi f''(z)}{g'(z)}\right\}\right].$$

$$\Leftrightarrow\ Re\Big\{\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\Big\}\ \geq\ \alpha,$$

which shows that $f(z) \in UQCV(\alpha)$. Hence the condition is sufficient. $\square$

Remark 2.1. Since $\frac{\{zf'(z)\}'}{g'(z)}$ is analytic in $|z| < 1$ and maps 0 to 1, the open mapping theorem implies that equality in (2.18) is not possible.

Theorem 2.2. *If $f(z) \in UQCV(\alpha)$ $(0 \leq \alpha < 1)$ then*

$$|a_n| \leq \frac{1}{n}\left[(1-\alpha) + \frac{\alpha}{(2n-1)}\right],\ n \geq 2.$$

Proof. Let $f(z) \in UQCV(\alpha)$, from the Definition 2.1, there exists uniformly convex function g given in Lemma 2.1, such that

$$Re\left[\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\right] > \alpha,\ \forall\ (z,\xi) \in U \times U \text{ and } z \neq \xi$$

$$\Leftrightarrow Re\left[\frac{(z-\xi)f''(z) + f'(z)}{g'(z)}\right] > \alpha. \tag{2.19}$$

Choosing $\xi = -z$ in (2.19), it takes the form

$$Re\left[\frac{2zf''(z)}{g'(z)} + \frac{f'(z)}{g'(z)}\right] > \alpha. \tag{2.20}$$

Let $p(z) = \frac{2zf''(z)}{g'(z)} + \frac{f'(z)}{g'(z)}$, which is incompatible with $p(z) = \frac{1+(1-2\alpha)w(z)}{1-w(z)}$, where $w(z)$ is schwarz's function in the unit disc U and $p(z) = \sum_{n=0}^{\infty} p_n z^n$ with $p_0 = 1$, then we have

$$2zf''(z) + f'(z) = p(z)g'(z). \tag{2.21}$$

Replacing $f'(z), f''(z), g'(z)$ and $p(z)$ by their equivalent expressions in series in (2.21), after simplifying, we get

$$\begin{aligned} 1 + \sum_{n=2}^{\infty} \{2n(n-1) + n\}\, a_n z^{n-1} &= \big\{1 + p_1 z + p_2 z^2 + \cdots + p_{n-1}z^{n-1} \\ &+ p_n z^n + \cdots\big\} \times \big\{1 + 2b_2 z + 3b_3 z^2 + \cdots + (n-1)b_{n-1}z^{n-2} + nb_n z^{n-1} + \cdots\big\}. \end{aligned} \tag{2.22}$$

Equating the coefficient of z^{n-1} on both sides of (2.22), we have

$$\begin{aligned} \big[2n(n-1) + n\big]a_n = [nb_n &+ p_1(n-1)b_{n-1} + p_2(n-2)b_{n-2} + \\ &\cdots + p_{n-2}2b_2 + p_{n-1}]. \end{aligned} \tag{2.23}$$

Taking the modulus on both sides of (2.23) and using the facts, for the functions with positive real part, $|p_0| = 1,\ |p_n| \leq 2(1-\alpha),\ \forall\ n \geq 1$ with $0 \leq \alpha < 1$ and the result from (2.14), which simplifies to

$$\begin{aligned} n(2n-1)|a_n| &\leq [(1-\alpha)(2n-1)+\alpha] \\ \Leftrightarrow |a_n| &\leq \frac{1}{n}\left[(1-\alpha)+\frac{\alpha}{2n-1}\right], \ \forall\, n \geq 2. \end{aligned}$$

Hence the Theorem. □

Theorem 2.3. *If* $f \in UQCV(\alpha)$ $(0 \leq \alpha < 1)$ *then*

$$|2f'(z) - f(z)| \leq \left[\frac{2(1-\alpha)r}{1-r} + (1-2\alpha)\ \log(1-r)\right],\ \textit{for } |z| = r < 1.$$

Proof. Let $f \in UQCV(\alpha)$, from the Definition 2.1, there exists a uniformly convex function g such that

$$Re\left[\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\right] > \alpha,\ z,\ \xi \in U,\ \text{where } z \neq \xi$$

$$\Rightarrow Re\left[\frac{(z-\xi)f''(z) + f'(z)}{g'(z)}\right] > \alpha \tag{2.24}$$

Choosing $\xi = -z$ in (2.24), we get

$$Re\left[\frac{2zf''(z)}{g'(z)} + \frac{f'(z)}{g'(z)}\right] > \alpha. \tag{2.25}$$

Put $p(z) = \frac{2zf''(z)}{g'(z)} + \frac{f'(z)}{g'(z)}$ in (2.25), which takes the form $\mathrm{Re}(p(z)) > \alpha$ $(0 \leq \alpha < 1)$ so that we can have

$$[2zf''(z) + f'(z)] = p(z)g'(z) \Leftrightarrow [2zf'(z) - f(z)]' = p(z)g'(z).$$

Taking modulus on both sides, we get

$$\left|[2zf'(z) - f(z)]'\right| = \left|p(z)\right|\left|g'(z)\right|.$$

Using the known result for $|p(z)|$ (according to Goodman [2]) and Lemma 2.1, resolving into partial fractions on the right hand side, we obtain

$$\left|[2zf'(z) - f(z)]'\right| \leq \left[\frac{1+(1-2\alpha)r}{(1-r)^2}\right] = \left[\frac{2\alpha-1}{1-r} + \frac{2(1-\alpha)}{(1-r)^2}\right],\ \text{for } |z| = r < 1. \tag{2.26}$$

On integrating along a line segment from 0 to $|z| = r$ in (2.26) and using the fact $|f(z)| \leq \int_0^z |f'(z)||dz|$, which simplifies to give

$$[2zf'(z) - f(z)] \leq \left[\frac{2(1-\alpha)r}{1-r} + (1-2\alpha)\log(1-r)\right], \quad (0 \leq \alpha < 1).$$

Hence the Theorem. □

Theorem 2.4. $f(z) \in QUCV(\alpha) \Leftrightarrow zf' \in CUCV(\alpha)$ $(0 \leq \alpha < 1)$.

Proof. Let $f(z) \in QUCV(\alpha)$, from the Definition 2.4 , we have

$$Re\left[\frac{\{zf'(z)\}'}{g'(z)}\right] > \alpha. \tag{2.27}$$

Choosing $zf'(z) = F(z)$ in (2.27), we get

$$Re\left[\frac{F'(z)}{g'(z)}\right] > \ \alpha, \quad \text{for } |z| < 1.$$

From the Definition 2.6, we conclude that $F = zf' \in CUCV(\alpha)$.

Conversely, let $F = zf' \in CUCV(\alpha$, from the Definition 2.6, we have

$$Re\left[\frac{\{zf'(z)\}'}{g'(z)}\right] > \ \alpha,\ |z| < 1.$$

In view of Definition 2.4, we conclude that $f(z) \in QUCV(\alpha)$.
Hence the Theorem. □

Theorem 2.5. *If $f \in QUCV(\alpha)$ then $f \in CUCV(\alpha)$.*

Proof. Let $f \in QUCV(\alpha)$, then by a result obtained by Libera [4], we have

$$Re\left[\frac{\{zf'(z)\}'}{g'(z)}\right] > \ \alpha \ \Leftrightarrow \left[\frac{zf'(z)}{g(z)}\right] > \ \alpha,\ z \in U, \tag{2.28}$$

where $g \in UCV$, which is also in S_p, denotes the class of parabolic star like functions introduced by Ronning [9]. Geometrically S_p is the class of functions f given (1.1), for which $\frac{zf'(z)}{f(z)}$ takes its value in the interior of the parabola in the right half plane symmetric about the real axis with vertex at $\left(\frac{1}{2}, 0\right)$.

$$\text{Put } h(z) = \int_0^z \frac{g(t)}{t} dt \Leftrightarrow h'(z) = \frac{g(z)}{z} \Leftrightarrow g(z) = zh'(z) \in S_p. \tag{2.29}$$

By the relation between UCV and S_p given in terms of the Alexander type Theorem [1] by Ronning [8], we have

$$zh'(z) \in S_p \Leftrightarrow h(z) \in UCV.$$

Simplifying the relations (2.28) and (2.29), we obtain

$$Re\left[\frac{f'(z)}{h'(z)}\right] \ > \alpha,\ z \in U \text{ for } (0 \leq \alpha < 1).$$

Since $h(z) \in UCV$, from the Definition 2.7, we conclude that $f(z) \in CUCV(\alpha)$.
Hence the Theorem. □

Remark 2.2. From the Theorems 2.4 and 2.5, we conclude that
if $f(z) \in QUCV(\alpha)$ then both $f(z)$ and $zf'(z)$ belongs to $CUCV(\alpha)$.

Theorem 2.6. *If $f \in QUCV(\alpha)$ $(0 \leq \alpha < 1)$ then*

$$|a_n| \leq \frac{1}{n^2}\big[2n(1-\alpha) + (2\alpha - 1)\big],\ \forall\ n \geq 2.$$

Proof. Let $f \in QUCV(\alpha)$, from the Definition 2.4, there exists uniformly convex function g in U such that

$$Re\left[\frac{\{zf'(z)\}'}{g'(z)}\right] \ > \alpha, \quad z \in U. \tag{2.30}$$

Choosing $p(z) = \dfrac{\{zf'(z)\}'}{g'(z)}$ in (2.30), we can have

$$Re(p(z)) > \alpha,\ \text{so that } \{zf'(z)\}' = p(z)g'(z).$$

Applying the same procedure described in Theorem 2.2, we obtain

$$|a_n| \leq \frac{1}{n^2}\big[(1-\alpha)(2n-1)+\alpha\big], \ \forall\ n \geq 2. \tag{2.31}$$

Hence the Theorem. □

Theorem 2.7. *If* $f \in QUCV(\alpha)$ $(0 \leq \alpha < 1)$, *then*

$$|zf'(z)| \leq \left[\frac{2(1-\alpha)r}{1-r} + (1-2\alpha)\log(1-r)\right], for\ |z| \leq r < 1.$$

Proof. Let $f \in QUCV(\alpha)$, from the Definition 2.1, we have

$$Re\left[\frac{\{zf'(z)\}'}{g'(z)}\right] > \alpha \tag{2.32}$$

$$\text{Put} p(z) = \frac{\{zf'(z)\}'}{g'(z)} \text{in (2.32), we get } Re\{p(z)\} > \alpha,$$

so that, we can have

$$\{zf'(z)\}' = p(z)g'(z). \tag{2.33}$$

Taking modulus on both sides of (2.33), which takes the form

$$|zf'(z))'| = |p(z)||g'(z)|.$$

Applying the same procedure described in Theorem 2.3, we obtain

$$|zf'(z)| \leq \left[\frac{2(1-\alpha)r}{(1-r)} + (1-2\alpha)\log(1-r)\right].$$

Hence the Theorem. □

Acknowledgement: The authors would like to express sincere thanks to the esteemed Referee(s) for their careful readings, valuable suggestions and comments, which helped them to improve the presentation of the paper.

References

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Annals. Math.*, 17(1915), 12 -22.
[2] A. W. Goodman, On uniformly convex functions, *Ann. Polon. Math.*, 56(1991), 87 - 92.
[3] A. W. Goodman, *Univalent functions vol. I and vol. II*, Mariner Publishing Comp. Inc., Tampa, Florida, 1983.
[4] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.,* 16(1965), 755 -758.
[5] K. I. Noor, On quasi convex functions and related topics, *Int. J. Math. Sci.,* 10(2)(1987), 241 - 258.
[6] K. S. Padmanabhan, On certain subclasses of Bazilevic functions, *Ind. J. Maths.*, 39(3)(1997).
[7] Rajalakshmi Rajagopal and C. Selvaraj, On a class of uniformly quasi-convex functions, *Bull. Calcutta Math. Soc.,* 95(2003), 199 -206.
[8] F. Ronning, A survey on uniformly convex and uniformly starlike functions, *Ann. Univ. Mariae Curie - Sklodowska Sect. A,* 47(1993), 123 - 134.
[9] F. Ronning, Uniformly convex functions and corresponding class of star like functions, *Proc. Amer. Math. Soc.,* 118(1) (1993), 189 - 196.

Department of Mathematics, GIT, GITAM University, Visakhapatnam- 530 045, A.P., India.
E-mail address: vamsheekrishna1972@gmail.com

Department of Mathematics, GIT, GITAM University, Visakhapatnam- 530 045, A.P., India.
E-mail address: bvlmaths@gmail.com

Department of Mathematics, Kakatiya University, Warangal- 506 009, T. S., India.
E-mail address: reddytr2@gmail.com

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 42–53 (2015) ©KJM

LEFT-HOM-SYMMETRIC AND HOM-POISSON DIALGEBRAS

BAKAYOKO I. AND BANGOURA M.

Abstract. The aim of this paper is to introduce left-Hom-symmetric dialgebras (which contain left-Hom-symmetric algebras or Hom-preLie algebras and Hom-dialgebras as special cases) and Hom-Poisson dialgebras. We give some examples and some construction theorems by using the composition construction. We prove that the commutator bracket of any left-Hom-symmetric dialgebra provides Hom-Leibniz algebra. We also prove that bimodules over Hom-dialgebras are closed under twisting. Next, we show that bimodules over Hom-dendriform algebras D extend to bimodules over the left-Hom-symmetric algebra associated to D. Finally, we give some examples of Hom-Poisson dialgebras and prove that the commutator bracket of any Hom-dialgebra structure map leads to Hom-Poisson dialgebra.

1. Introduction

Leibniz algebras are introduced by J.-L. Loday in [8] as a generalization of Lie algebras where the skew-symmetry of the bracket is dropped and the Jacobi identity is changed by the Leibniz identity. The author showed that the relationship between Lie algebras and associative algebras translates into an analogous relationship between Leibniz algebras and the so-called diassociative algebras or associative dialgebras, which are a generalization of associative algebras possessing two products. In particular, he showed that any dialgebra becomes a Leibniz algebra under the commutator bracket.

Otherwise, left-symmetric dialgebras appear in the work of R. Felipe [10] as an algebraic structure with two products containing dialgebras as particular case, and Poisson dialgebras are introduced in [7] as a vector space endowed with both dialgebra structure and Leibniz structure which are compatible in certain sense.

The purpose of this paper is to study Left-Hom-symmetric dialgebras and Hom-Poisson dialgebras. We define bimodules over Hom-dialgebras and Hom-dendriform algebras [2] and give some construction theorems. Next, we introduce Hom-Poisson

Date: January 13, 2015 .

2000 *Mathematics Subject Classification.* 17A30, 17A32, 17B42, 17D25.

Key words and phrases. Hom-Leibniz algebras, left-Hom-symmetric dialgebras, left-Hom-symmetric algebras, Hom-dendriform algebras.

dialgebras as Hom-type of Poisson dialgebras which are generalization of "non-commutative Poisson algebras".

The paper is organized as follows. In section 2, we recall some basic notions related to Hom-algebras, Hom-Lie algebras and Hom-Leibniz algebras. In Section 3, we show that one can obtain a left-Hom-symmetric algebra from a left-symmetric algebra and an algebra endomorphism. We prove that twisting a Hom-dialgebra module structure map by an endomorphism of Hom-dialgebras, we get another one. Next, we show that any left-Hom-symmetric dialgebra leads to Hom-Leibniz algebra via the Loday commutator. Finally, we introduce affine Hom-Leibniz structure on Hom-Leibniz algebras and point out that one may associate a left-Hom-symmetric algebra to any affine Hom-Leibniz algebra. In section 4, we introduce bimodules over Hom-dendriform algebras and prove that to any bimodule over a Hom-dendriform algebra D corresponds a module over the left-Hom-symmetric algebra associated to D. In section 5, we introduce Hom-Poisson dialgebras ; we give some examples and some construction theorems of Hom-Poisson dialgebras.

Throughout this paper, all vector spaces are assumed to be over a field $\mathbb{K}$ of characteristic different from 2.

2. Preliminaries

In this section, we recall some basic definitions.

Definition 2.1. [1] By a Hom-algebra we mean a triple $(A, [\cdot,\cdot], \alpha)$ in which A is a vector space, $[\cdot,\cdot] : A\otimes A \to A$ is a bilinear map (the multiplication) and $\alpha : A \to A$ is a linear map (the twisting map).

If in addition, $\alpha \circ [\cdot,\cdot] = [\cdot,\cdot] \circ (\alpha \otimes \alpha)$, then the Hom-algebra $(A, [\cdot,\cdot], \alpha)$ is said to be multiplicative.

A morphism $f : (A, [\cdot,\cdot], \alpha) \to (A', [\cdot,\cdot]', \alpha')$ of Hom-algebras is a linear map f of the underlying vector spaces such that $f \circ \alpha = \alpha' \circ f$ and $[\cdot,\cdot]' \circ (f \otimes f) = f \circ [\cdot,\cdot]$.

Remark 2.1. If $(A, [\cdot,\cdot])$ is a non-necessarily associative algebra in the usual sense, we also regard it as the Hom-algebra $(A, [\cdot,\cdot], Id_A)$ with identity twisting map.

Definition 2.2. [1] Let $(A, [\cdot,\cdot], \alpha)$ be a Hom-algebra.

(1) The Hom-associator of A is the trilinear map $as_\alpha : A^{\otimes 3} \to A$ defined as
$$as_\alpha = [\cdot,\cdot] \circ ([\cdot,\cdot] \otimes \alpha - \alpha \otimes [\cdot,\cdot]).$$
(2) The Hom-Jacobian of A is the trilinear map $J_\alpha : A^{\otimes 3} \to A$ defined as
$$J_\alpha = [\cdot,\cdot] \circ ([\cdot,\cdot] \otimes \alpha) \circ (Id_A + \sigma + \sigma^2),$$
where $\sigma : A^{\otimes 3} \to A^{\otimes 3}$ is the cyclic permutation $\sigma(x \otimes y \otimes z) = y \otimes z \otimes x$.
(3) The Hom-Leibnizator of A is a trilinear map $Leib_\alpha : A^{\otimes 3} \to A$ defined as
$$Leib_\alpha = [\cdot,\cdot](\alpha \otimes [\cdot,\cdot]) + [\cdot,\cdot]([\cdot,\cdot] \otimes \alpha) - [\cdot,\cdot]([\cdot,\cdot] \otimes \alpha)(Id_A \otimes \tau),$$
where τ is the twist isomorphism i.e. $\tau(x \otimes y) = y \otimes x$, for any $x, y \in A$.

Definition 2.3. [1] A Hom-associative algebra is a triple $(A, \cdot, \alpha)$ consisting of a linear space A, a $\mathbb{K}$-bilinear map $\cdot : A \times A \longrightarrow A$ and a linear map $\alpha : A \longrightarrow A$ satisfying

$$as_\alpha(x, y, z) = 0 \quad \text{(Hom-associativity)}, \tag{2.1}$$

for all $x, y, z \in A$.

Definition 2.4. [6] A Hom-Lie algebra is a triple $(V, [\cdot,\cdot], \alpha)$ consisting of a linear space V, a bilinear map $[\cdot,\cdot] : V \times V \longrightarrow V$ and a linear map $\alpha : V \longrightarrow V$ satisfying

$$[x, y] = -[y, x] \quad \text{(skew-symmetry)}, \tag{2.2}$$

$$J_\alpha(x, y, z) = 0 \quad \text{(Hom-Jacobi identity)}, \tag{2.3}$$

for all $x, y, z \in V$.

Remark 2.2. When $\alpha = Id_V$, we obtain the definition of Lie algebras.

Definition 2.5. [1] A Hom-algebra $(L, [\cdot,\cdot], \alpha)$ is said to be a Hom-Leibniz algebra if it satisfies the *Hom-Leibniz identity* i.e.

$$Leib_\alpha(x, y, z) = 0, \tag{2.4}$$

for all $x, y, z \in L$.

Remark 2.3.
(1) When $\alpha = Id_L$, we recover the concept of Leibniz algebra.
(2) If the bracket is skew-symmetric, then L is a Hom-Lie algebra. Therefore Hom-Lie algebras are particular cases of Hom-Leibniz algebras.

3. Left-Hom-symmetric dialgebras

We introduce modules over Hom-dialgebras and left-Hom-symmetric dialgebras.

3.1. Left-Hom-symmetric algebras.

Definition 3.1. [1] A left-Hom-symmetric algebra is a vector space S together with a bilinear map $\circ : S \otimes S \to S$ and a linear map $\alpha : S \to S$ such that the following *left-Hom-symmetry identity*

$$\alpha(x) \circ (y \circ z) - (x \circ y) \circ \alpha(z) = \alpha(y) \circ (x \circ z) - (y \circ x) \circ \alpha(z), \tag{3.1}$$

holds.

Remark 3.1.
(1) When $\alpha = Id_S$, we recover the notion of left-symmetric algebras.
(2) In terms of Hom-associators, the *left-Hom-symmetry identity* is
$$as_\alpha(x, y, z) = as_\alpha(y, x, z).$$

Example 3.1. Let $(S, \circ, \alpha_S)$ be a left-Hom-symmetric algebra and $(A, \cdot, \alpha_A)$ a commutative Hom-associative algebra. Then $(S \otimes A, \bullet, \alpha_{S\otimes A})$ is a left-Hom-symmetric algebra, with

$$\begin{aligned} \alpha_{S\otimes A} &= \alpha_S \otimes \alpha_A, \\ (x \otimes a) \bullet (y \otimes b) &= (x \circ y) \otimes (a \cdot b), \end{aligned}$$

for all $x, y \in S, a, b \in A$.

The following theorem allows to obtain left-Hom-symmetric algebras from left-symmetric algebras.

Theorem 3.1. *Let $(S, \bullet)$ be a left-symmetric algebra and $\alpha : S \to S$ be an endomorphism. Then, $S_\alpha = (S, \bullet_\alpha, \alpha)$, where $x \bullet_\alpha y = \alpha(x \bullet y)$, is a left-Hom-symmetric algebra.*

Moreover, suppose that $(S', \bullet')$ is another left-symmetric algebra and $\alpha' : S' \to S'$ is an algebra endomorphism. If $f : S \to S'$ is a left-symmetric algebra morphism that satisfies $f \circ \bullet = \bullet' \circ f$ then $f : S_\alpha \to S'_{\alpha'}$ is a morphism of left-Hom-symmetric algebras.

Proof. For any $x, y, z \in S$, we have

$$\begin{aligned}
\alpha(x) \bullet_\alpha (y \bullet_\alpha z) - (x \bullet_\alpha y) \bullet_\alpha \alpha(z) &= \alpha(x) \bullet_\alpha (\alpha(y) \bullet \alpha(z)) - (\alpha(x) \bullet \alpha(y)) \bullet_\alpha \alpha(z)\\
&= \alpha^2(x) \bullet (\alpha^2(y) \bullet \alpha^2(z)) - (\alpha^2(x) \bullet \alpha^2(y)) \bullet \alpha^2(z)\\
&= (\alpha^2)^{\otimes 3}((x \bullet y) \bullet z) - (x \bullet y) \bullet z)\\
&= (\alpha^2)^{\otimes 3}(y \bullet (x \bullet z) - (y \bullet x) \bullet z)\\
&= \alpha(y) \bullet_\alpha (x \bullet_\alpha z) - (y \bullet_\alpha x) \bullet_\alpha \alpha(z).
\end{aligned}$$

For the second part, we have

$$f \circ \bullet_\alpha = f \circ \alpha \circ \bullet = \alpha' \circ f \circ \bullet = \alpha' \circ \bullet' \circ (f \otimes f) = \bullet'_{\alpha'} \circ (f \otimes f).$$

This completes the proof. □

Example 3.2. : *Left-Hom-symmetric algebra of vector fields*
First we need some definitions. Let M be a differential manifold, and let $\triangledown$ be the covariant operator associated to a connection on the tangent bundle TM. The covariant derivation is a bilinear operator on vector fields (i.e. two sections of the tangent bundle) $(X, Y) \mapsto \triangledown_X Y$ such that the following axioms are fulfilled :

$$\begin{aligned}
\triangledown_{fX} Y &= f \triangledown_X Y,\\
\triangledown_X (fY) &= f \triangledown_X Y + (X \cdot f)Y \quad \text{(Leibniz rule)}.
\end{aligned}$$

The torsion of the connection τ is defined by :

$$\tau(X, Y) = \triangledown_X Y - \triangledown_Y X - [X, Y], \tag{3.2}$$

and the curvature tensor is defined by :

$$R(X, Y) = [\triangledown_X, \triangledown_Y] - \triangledown_{[X,Y]}. \tag{3.3}$$

The connection is flat if the curvature R vanishes identically, and torsion-free if $\tau = 0$.

Now, let M be a smooth manifold endowed with a flat torsion-free connection $\triangledown$, $\chi(M)$ the space of vector fields and $\varphi : M \to M$ a smooth map such that $d\varphi(\triangledown_X Y) = \triangledown_{d\varphi(X)} d\varphi(Y)$. Then $(\chi(M), \circ, d\varphi)$ is a left-Hom-symmetric algebra, with the left-Hom-symmetric product given by :

$$X \circ Y = \triangledown_X Y.$$

3.2. **Modules over Hom-dialgebras.**

Definition 3.2. A Hom-dialgebra is a vector space D equipped with a linear map $\alpha : D \to D$ and two *Hom-associative products*

$$\begin{aligned}
\dashv : D \times D &\to D\\
\vdash : D \times D &\to D.
\end{aligned}$$

satisfying the identities :

$$\alpha(x) \dashv (y \dashv z) = \alpha(x) \dashv (y \vdash z), \tag{3.4}$$

$$(x \vdash y) \dashv \alpha(z) = \alpha(x) \vdash (y \dashv z), \tag{3.5}$$

$$(x \vdash y) \vdash \alpha(z) = (x \dashv y) \vdash \alpha(z). \tag{3.6}$$

If in addition, α is an endomorphism with respect to $\dashv$ and $\vdash$, then D is said to be a multiplicative Hom-dialgebra.

Remark 3.2. For any x, y, z in a Hom-dialgebra, one has

$$\alpha(x) \star (y \star z) = \alpha(x) \star (z \star y) \ \text{(right commutativity)}$$

where $x \star y = x \dashv y + y \vdash x$.

Here are some examples of Hom-dialgebras.

Example 3.3. Any dialgebra is a Hom-dialgebra with $\alpha = Id$.

Example 3.4. If (A, μ, α) is a Hom-associative algebra, then $(D, \dashv, \vdash, \alpha)$ is a Hom-dialgebra in which $\dashv = \mu = \vdash$.

Example 3.5. Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dialgebra. Then $(D, \dashv', \vdash', \alpha)$ is also a Hom-dialgebra, with

$$x \dashv' y := y \vdash x \quad \text{and} \quad x \vdash' y := y \dashv x.$$

Example 3.6. Let $(D, \dashv_D, \vdash_D, \alpha_D)$ and $(D', \dashv_{D'}, \vdash_{D'}, \alpha_{D'})$ be two Hom-dialgebras. The tensor product $D \otimes D'$ is also a Hom-dialgebra with

$$\begin{array}{rcl} \alpha_{D\otimes D'}(x \otimes y) & := & \alpha_D(x) \otimes \alpha_{D'}(x'), \\ (x \otimes x') \dashv (y \otimes y') & := & (x \dashv_D y) \otimes (x' \dashv_{D'} y'), \\ (x \otimes x') \vdash (y \otimes y') & := & (x \vdash_D y) \otimes (x' \vdash_{D'} y'). \end{array}$$

Example 3.7. Let $(A, \cdot, \alpha)$ be a Hom-associative algebra. Then, for any positive integer n, $A^n = A \times A \times \cdots \times A$ (n times) is a Hom-dialgebra, with

$$\begin{array}{rcl} \alpha_{A^n} & := & (\alpha, \alpha, \ldots, \alpha), \\ (x \dashv_{A^n} y)_i & := & x_i \cdot (\sum y_j), \\ (x \vdash_{A^n} y)_i & := & (\sum x_j) \cdot y_i, \end{array}$$

for any $1 \leq i, j \leq n$.

Example 3.8. The Hom-dialgebra arising from a bimodule over Hom-associative algebra and morphism of Hom-bimodules is exposed in [4].

Now, we have the following definitions.

Definition 3.3. [5] A Hom-module is a pair (M, β) in which M is a vector space and $\beta : M \longrightarrow M$ is a linear map.

Definition 3.4. Let $(A, \cdot, \alpha)$ be a Hom-associative algebra and let (M, β) be a Hom-module. A bimodule structure on M consists of :

(1) a left A-action, $\prec: A \otimes M \to M$ $(x \otimes m \mapsto x \prec m)$, and
(2) a right A-action, $\succ: M \otimes A \to M$ $(m \otimes x \mapsto m \succ x)$

such that the following conditions hold for $x, y \in A$ and $m \in M$:

$$\beta(x \prec m) = \alpha(x) \prec \beta(m), \tag{3.7}$$

$$\beta(m \succ x) = \beta(m) \succ \alpha(x), \tag{3.8}$$

$$\alpha(x) \prec (y \prec m) = (x \cdot y) \prec \beta(m), \tag{3.9}$$

$$(m \succ x) \succ \alpha(y) = \beta(m) \succ (x \cdot y), \tag{3.10}$$

$$\alpha(x) \prec (m \succ y) = (x \prec m) \succ \alpha(y). \tag{3.11}$$

Definition 3.5. Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dialgebra and (M, β) be a Hom-module. Assume that M is endowed with two operations $\prec: D \otimes M \to M$ and $\succ: M \otimes D \to M$. We say that $(M, \prec, \succ, \beta)$ is a bimodule over the Hom-dialgebra $(D, \dashv, \vdash, \alpha)$ if, for any $x, y \in D, m \in M$, the following identities are satisfied :

$$\beta(x \prec m) = \alpha(x) \prec \beta(m), \tag{3.12}$$

$$\beta(m \succ x) = \beta(m) \succ \alpha(x) \tag{3.13}$$

$$(x \prec m) \succ \alpha(y) = \alpha(x) \prec (m \succ y), \tag{3.14}$$

$$\beta(m) \succ (x \dashv y) = (m \succ x) \succ \alpha(y) = \beta(m) \succ (x \vdash y) \tag{3.15}$$

$$(x \dashv y) \prec \beta(m) = \alpha(x) \prec (y \prec m) = (x \vdash y) \prec \beta(m) \tag{3.16}$$

Remark 3.3.
(1) (a) Axioms (3.12) and (3.13) can be interpreted as the multiplicativity in the Hom-modules theory.
(b) Axiom (3.15) (resp. (3.16)) is the left-module (resp. right-module) condition.
(c) Axiom (3.14) is the compatibility condition of left and right modules.
(2) Taking $M = D$ (as vector space), $\prec = \dashv$ and $\succ = \vdash$, we see that any Hom-dialgebra is a bimodule over itself.

We have the following result.

Proposition 3.1. *Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dialgebra. Then $(M, \prec, \succ, \beta)$ is a bimodule over $(D, \dashv, \vdash, \alpha)$ if and only if it is a bimodule over (D, μ, α), where $\dashv = \mu = \vdash$.*

Proof. The proof follows from Definition 3.4 and Definition 3.5. □

The following theorem asserts that bimodules over Hom-dialgebras are closed under twisting.

Theorem 3.2. *Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dialgebra and $(M, \prec, \succ, \beta)$ be a bimodule over D. Define the maps*

$$\prec_\alpha := \prec \circ (\alpha^2 \otimes Id_M) : D \otimes M \to M, \quad x \otimes m \mapsto \alpha^2(x) \prec m \tag{3.17}$$

$$\succ_\alpha := \succ \circ (Id_M \otimes \alpha^2) : M \otimes D \to M, \quad m \otimes x \mapsto m \succ \alpha^2(x). \tag{3.18}$$

Then $(M, \prec_\alpha, \succ_\alpha, \beta)$ is a bimodule over D.

Proof. We shall only prove (3.12) and (3.14). For any $x, y \in D, m \in M$,

$$\beta(x \prec_\alpha m) \overset{(3.17)}{=} \beta(\alpha^2(x) \prec m) \overset{(3.12)}{=} \alpha^3(x) \prec \beta(m) \overset{(3.17)}{=} \alpha(x) \prec_\alpha \beta(m),$$

and,

$$\begin{aligned}(x \prec_\alpha m) \succ_\alpha \alpha(y) - \alpha(x) \prec_\alpha (m \succ_\alpha y) \overset{(3.17)}{=} & (\alpha^2(x) \prec m) \succ \alpha^3(y) \\ & -\alpha^3(x) \prec (m \succ \alpha^2(y)) \overset{(3.14)}{=} 0.\end{aligned}$$

All the rest of equalities are proved analogously. □

Proposition 3.2. *Let $(M, \prec, \succ, \beta)$ be a bimodule over the Hom-dialgebra $(D, \dashv, \vdash, \alpha)$. Then, we have the following identities :*

$$[x, y] \prec \beta(m) = \alpha(x) \prec (y \prec m) - \alpha(y) \prec (x \prec m), \tag{3.19}$$

$$\beta(m) \succ [x, y] = (x \prec m) \succ \alpha(y) + \alpha(x) \prec (m \succ y), \tag{3.20}$$

where, $[x, y] = x \dashv y - y \vdash x$.

Proof. The first equality is proved by using (3.16). For the second equality, we have, for any $x, y \in D, m \in M$,

$$\begin{aligned}
&\beta(m) \succ [x,y] - (x \prec m) \succ \alpha(y) - \alpha(x) \prec (m \succ y) = \\
&= \beta(m) \succ (x \dashv y - y \vdash x) - (x \prec m) \succ \alpha(y) - \alpha(x) \prec (m \succ y) = \\
&= \beta(m) \succ (x \dashv y) - \beta(m) \succ (y \vdash x) - (x \prec m) \succ \alpha(y) - \alpha(x) \prec (m \succ y).
\end{aligned}$$

The last line vanishes by (3.14) and (3.15). □

3.3. **Left-Hom-symmetric dialgebras.**

Definition 3.6. A Left-Hom-symmetric dialgebra is a vector space S equipped with two bilinear products

$$\begin{aligned}
\dashv : S \times S &\rightarrow S, \\
\vdash : S \times S &\rightarrow S,
\end{aligned}$$

satisfying the identities

$$\alpha(x) \dashv (y \dashv z) = \alpha(x) \dashv (y \vdash z), \tag{3.21}$$

$$(x \vdash y) \vdash \alpha(z) = (x \dashv y) \vdash \alpha(z), \tag{3.22}$$

$$\alpha(x) \dashv (y \dashv z) - (x \dashv y) \dashv \alpha(z) = \alpha(y) \vdash (x \dashv z) - (y \vdash x) \dashv \alpha(z), \tag{3.23}$$

$$\alpha(x) \vdash (y \vdash z) - (x \vdash y) \vdash \alpha(z) = \alpha(y) \vdash (x \vdash z) - (y \vdash x) \vdash \alpha(z). \tag{3.24}$$

Remark 3.4. The identities (3.23) and (3.24) can be written as

$$\begin{aligned}
L^{\dashv}_{\alpha(x)} L^{\dashv}_{y} - L^{\vdash}_{\alpha(y)} L^{\dashv}_{x} &= L^{\dashv}_{[x,y]} \alpha, \\
L^{\vdash}_{\alpha(x)} L^{\vdash}_{y} - L^{\vdash}_{\alpha(y)} L^{\vdash}_{x} &= L^{\vdash}_{[x,y]} \alpha,
\end{aligned}$$

where, $L^{\dashv}_x$ and $L^{\vdash}_x$ are defined respectively by $L^{\dashv}_x y = x \dashv y$ and $L^{\vdash}_x y = x \vdash y$, and $[x, y] = x \dashv y - y \vdash x$.

Now we give some examples of left-Hom-symmetric dialgebras.

Example 3.9. Any Hom-dialgebra is a left-Hom-symmetric dialgebra.

Example 3.10. Any left-Hom-symmetric algebra is a left-Hom-symmetric dialgebra in which $\vdash = \dashv$.

Example 3.11. Let $(S, \dashv, \vdash, \alpha_S)$ be a left-Hom-symmetric dialgebra and $(A, \cdot, \alpha_A)$ be a left-Hom-symmetric algebra, then $S \times A$ is a left-Hom-symmetric dialgebra with

$$\begin{aligned}
\alpha_{S \times A} &:= (\alpha_S, \alpha_A), \\
(x, a) \dashv_{S \times A} (y, b) &:= (x \dashv y, a \cdot b), \\
(x, a) \vdash_{S \times A} (y, b) &:= (x \vdash y, a \cdot b).
\end{aligned}$$

We have the following result whose ordinary case is Proposition 4 in [10].

Proposition 3.3. *A left-Hom-symmetric dialgebra S is a Hom-dialgebra if and only if both products of S are Hom-associative.*

Proof. If a left-Hom-symmetric dialgebra S is a Hom-dialgebra, then both products $\dashv$ and $\vdash$ defined over S are Hom-associative according to Definition 3.2. Conversely, if each product of a left-Hom-symmetric dialgebra is Hom-associative, then from (3.23), we get (3.5). □

The next statement is one of the main results of this paper ; it states that the commutator bracket of any left-Hom-symmetric dialgebra gives rise to a Hom-Leibniz algebra.

Theorem 3.3. *Let $(S, \dashv, \vdash, \alpha)$ be a left-Hom-symmetric dialgebra. Then the Loday commutator defined by*

$$[x, y] := x \dashv y - y \vdash x, \tag{3.25}$$

defines a structure of Hom-Leibniz algebra on S.

Proof. The proof follows by a straighforward computation in which the identities (3.21) and (3.22) are used once. In fact, for any $x, y, z \in S$, we have

$$\begin{aligned} Leib_\alpha(x, y, z) &= [\alpha(x), [y, z]] - [[x, y], \alpha(z)] + [[x, z], \alpha(y)] \\ &= \alpha(x) \dashv (y \dashv z) - \alpha(x) \dashv (z \vdash y) - (y \dashv z) \vdash \alpha(x) + (z \vdash y) \vdash \alpha(x) \\ &\quad -(x \dashv y) \dashv \alpha(z) + (y \vdash x) \dashv \alpha(z) + \alpha(z) \vdash (x \dashv y) - \alpha(z) \vdash (y \vdash x) \\ &\quad +(x \dashv z) \dashv \alpha(y) - (z \vdash x) \dashv \alpha(y) - \alpha(y) \vdash (x \dashv z) + \alpha(y) \vdash (z \vdash x) \\ &= 0. \end{aligned}$$

Now, by (3.23) and (3.24) it follows that $Leib_\alpha(x, y, z) = 0$. This completes the proof. □

We need the below definition in the next theorem.

Definition 3.7. Let $(S, \dashv, \vdash, \alpha)$ and $(S', \dashv', \vdash', \alpha')$ be two left-Hom-symmetric dialgebras. A map $f : S \to S'$ is said to be a morphism of left-Hom-symmetric dialgebras if

$$\alpha' \circ f = f \circ \alpha, \quad f(x \dashv y) = f(x) \dashv' f(y) \text{ and } f(x \vdash y) = f(x) \vdash' f(y),$$

for any $x, y \in S$.

Twisting a left-symmetric dialgebra by a left-symmetric dialgebras endomorphism, we get a left-Hom-symmetric dialgebra ; this is stated in the following theorem.

Theorem 3.4. *Let $(S, \dashv, \vdash)$ be a left-symmetric dialgebra and $\alpha : S \to S$ be a morphism of left-symmetric dialgebras. Then $(S, \dashv_\alpha, \vdash_\alpha, \alpha)$ is a multiplicative left-Hom-symmetric dialgebra with*

$$\begin{aligned} x \vdash_\alpha y &= \alpha(x \vdash y), \\ x \dashv_\alpha y &= \alpha(x \dashv y). \end{aligned}$$

Proof. The proof is similar to that of Proposition 3.1. □

In the rest of this section, we introduce affine Hom-Leibniz structures on Hom-Leibniz algebras.

Definition 3.8. Let $(L, [-, -], \alpha)$ be a Hom-Leibniz algebra. A pair $(\bigtriangledown_1, \bigtriangledown_2)$ of bilinear maps

$$\bigtriangledown_1 : L \times L \to L$$

and

$$\bigtriangledown_2 : L \times L \to L.$$

is called an affine Hom-Leibniz structure if

$$\bigtriangledown_2(x,y)-\bigtriangledown_1(y,x)=[x,y], \tag{3.26}$$

$$\bigtriangledown_1(\bigtriangledown_1(x,y),\alpha(z))=\bigtriangledown_1(\bigtriangledown_2(x,y),\alpha(z)), \tag{3.27}$$

$$\bigtriangledown_2(\alpha(x),\bigtriangledown_2(y,z)))=\bigtriangledown_2(\alpha(x),\bigtriangledown_1(y,z))), \tag{3.28}$$

$$\bigtriangledown_2(\alpha(x),\bigtriangledown_1(y,z))-\bigtriangledown_1(\alpha(y),\bigtriangledown_2(x,z))=\bigtriangledown_2([x,y],\alpha(z)), \tag{3.29}$$

and

$$\bigtriangledown_1(\alpha(x),\bigtriangledown_1(y,z))-\bigtriangledown_1(\alpha(y),\bigtriangledown_1(x,z))=\bigtriangledown_1([x,y],\alpha(z)), \tag{3.30}$$

for all $x,y,z\in L$.

The next result is the Hom-type of ([10], Theorem 11).

Theorem 3.5. *Let $(L,[-,-],\alpha)$ be a Hom-Leibniz algebra and let $(\bigtriangledown_1,\bigtriangledown_2)$ be an affine Hom-Leibniz structure. Then L is a left-Hom-symmetric dialgebra with $\vdash$ and $\dashv$ defined as*

$$x\vdash y=\bigtriangledown_1(x,y),\quad x\dashv y=\bigtriangledown_2(x,y). \tag{3.31}$$

Proof. Relations (3.27) and (3.28) imply (3.21) and (3.22) respectively. Next, (3.23) follows from (3.26) and (3.29). Finally, (3.24) is established by applying (3.21), (3.26) and (3.30). □

Corollary 3.1. *Let $(\bigtriangledown,\bigtriangledown)$ be an affine structure on the Hom-Leibniz algebra $(L,[-,-],\alpha)$. Then $(L,\bigtriangledown,\alpha)$ is a left-Hom-symmetric algebra.*

4. Hom-dendriform algebras

This section in devoted to modules over Hom-dendriform algebras.

Definition 4.1. [2] A Hom-dendriform algebra is a vector space D together with bilinear maps $\dashv\colon D\otimes D\to D$, $\vdash\colon D\otimes D\to D$ and linear map $\alpha:S\to S$ such that

$$\alpha(x)\vdash(y\dashv z) = (x\vdash y)\dashv\alpha(z), \tag{4.1}$$

$$(x\dashv y)\dashv\alpha(z) = \alpha(x)\dashv(y\dashv z)+\alpha(x)\dashv(y\vdash z), \tag{4.2}$$

$$\alpha(x)\vdash(y\vdash z) = (x\dashv y)\vdash\alpha(z)+(x\vdash y)\vdash\alpha(z). \tag{4.3}$$

Lemma 4.1. [2] *Let $(D,\dashv,\vdash,\alpha)$ be a Hom-dendriform algebra. Defining $x\circ y=x\vdash y-y\dashv x$, one obtains a left-Hom-symmetric algebra structure on D.*

The following result is the Hom-analogue of Proposition 5.3 in [7].

Proposition 4.1. *Let $(D,\dashv,\vdash,\alpha_D)$ and $(\mathcal{D},\prec,\succ,\alpha_{\mathcal{D}})$ be a Hom-dialgebra and a Hom-dendriform algebra respectively. Then, on the tensor product $D\otimes\mathcal{D}$, the bracket*

$$\begin{aligned}[x\otimes a,y\otimes b] \;:=\;& (x\dashv y)\otimes(a\prec b)-(y\vdash x)\otimes(b\succ a)\\ &-(y\dashv x)\otimes(b\prec a)+(x\vdash y)\otimes(a\succ b),\end{aligned}$$

where $x,y\in D, a,b\in\mathcal{D}$, defines a structure of Hom-Lie algebra on $D\otimes\mathcal{D}$, with $\alpha_{D\otimes\mathcal{D}}=\alpha_D\otimes\alpha_{\mathcal{D}}$.

Proof. The bracket is skew-symmetric by definition. Hence, it suffices to show that the Hom-Jacobi identity is fulfilled.

The Hom-Jacobi identity for $x \otimes a$, $y \otimes b$, $z \otimes c$ gives a total of 48 terms, in fact $8 \times 3!$ terms. There are 8 terms for which x, y, z (and also a, b, c) stay in the same order. The other set of 8 terms are permutations of this set which reads :

$$\alpha(x) \dashv (y \dashv z) \otimes \alpha(a) \prec (b \prec c) - (x \dashv y) \dashv \alpha(z) \otimes (a \prec b) \prec \alpha(c),$$
$$\alpha(x) \vdash (y \dashv z) \otimes \alpha(a) \succ (b \prec c) - (x \vdash y) \dashv \alpha(z) \otimes (a \succ b) \prec \alpha(c),$$
$$\alpha(x) \dashv (y \vdash z) \otimes \alpha(a) \prec (b \succ c) - (x \dashv y) \vdash \alpha(z) \otimes (a \prec b) \succ \alpha(c),$$
$$\alpha(x) \vdash (y \vdash z) \otimes \alpha(a) \succ (b \succ c) - (x \vdash y) \vdash \alpha(z) \otimes (a \succ b) \succ \alpha(c).$$

The terms 1 and 3 in column 1 together with the term 1 in column 2 cancel due to Definition 3.2 and (4.2). Similarly, the terms 41, 32 and 42 cancel due to Definition 3.2 and (4.3). Finally the terms 21 and 22 cancel due to Definition 3.2 and (4.1). □

Corollary 4.1. *If D and $\mathcal{D}$ are multiplicative, then $D \otimes \mathcal{D}$ is also a multiplicative Hom-Lie algebra.*

Definition 4.2. Let $(S, \circ, \alpha)$ be a left-Hom-symmetric algebra. An S-bimodule is a vector space M endowed with a linear map $\beta : M \to M$, two bilinear maps $S \otimes M \to M, x \otimes m \mapsto x \prec m$ and $M \otimes S \to M, m \otimes x \mapsto m \succ x$, such that

$$\alpha(x) \prec (y \prec m) - (x \circ y) \prec \beta(m) - \alpha(y) \prec (x \prec m) + (y \circ x) \prec \beta(m) = 0,$$

and,

$$\alpha(x) \prec (m \succ y) - (x \prec m) \succ \alpha(y) - \beta(m) \succ (x \circ y) + (m \succ x) \succ \alpha(y) = 0.$$

Example 4.1. Any left-Hom-symmetric algebra is a bimodule over itself.

The following theorem gives a kind of connection between left-Hom-symmetric algebras and left-Hom-symmetric dialgebras.

Proposition 4.2. *Let $(S, \cdot, \alpha)$ be a left-Hom-symmetric algebra and I be a bimodule over S. Assume that, for all $i, j \in I$ and $a, b, c, d \in S$,*

$$\begin{aligned} \alpha(i) \cdot (a \cdot b) - (i \cdot a) \cdot \alpha(b) &= \alpha(a) \cdot (i \cdot b) - (a \cdot i) \cdot \alpha(b), \\ \alpha(c) \cdot (d \cdot j) - (c \cdot d) \cdot \alpha(j) &= \alpha(d) \cdot (c \cdot j) - (d \cdot c) \cdot \alpha(j). \end{aligned}$$

Then $(S \oplus I, \dashv, \vdash, \alpha_{S \oplus I})$ is a left-Hom-symmetric dialgebra with

$$\begin{aligned} \alpha_{S \oplus I} &= \alpha_S \oplus \alpha_I, \\ (i_1 + a_1) \dashv (i_2 + a_2) &= i_1 a_2 + a_1 a_2, \\ (i_1 + a_1) \vdash (i_2 + a_2) &= a_1 i_2 + a_1 a_2. \end{aligned}$$

Proof. It is straighforward by calculation. □

Corollary 4.2. *Let $(S, \cdot, \alpha)$ be a left-Hom-symmetric algebra and I be an ideal of S. Then $(S \oplus I, \dashv, \vdash, \alpha_{S \oplus I})$ is a left-Hom-symmetric dialgebra.*

Now, we define bimodules over Hom-dendriform algebras which are Hom-analogue of ([9], Definition 5.5).

Definition 4.3. Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dendriform algebra. A D-bimodule is a Hom-module (M, β) together with four bilinear maps

$$D \otimes M \to M, x \otimes m \mapsto x \succ m; \qquad D \otimes M \to M, x \otimes m \mapsto x \prec m;$$
$$M \otimes D \to M, m \otimes x \mapsto m \succ x; \qquad M \otimes D \to M, m \otimes x \mapsto m \prec x$$

such that

$$\begin{aligned}
&(4.4) & \alpha(x)\succ(y\prec m) &= (x\vdash y)\prec\beta(m),\\
&(4.5) & (x\dashv y)\prec\beta(m) &= \alpha(x)\prec(y\prec m)+\alpha(x)\prec(y\succ m),\\
&(4.6) & \alpha(x)\succ(y\succ m) &= (x\dashv y)\succ\beta(m)+(x\vdash y)\succ\beta(m),\\
&(4.7) & \alpha(x)\succ(m\prec y) &= (x\succ m)\prec\alpha(y),\\
&(4.8) & (x\prec m)\prec\alpha(y) &= \alpha(x)\prec(m\prec y)+\alpha(x)\prec(m\succ y),\\
&(4.9) & \alpha(x)\succ(m\succ y) &= (x\prec m)\succ\alpha(y)+(x\succ m)\succ\alpha(y),\\
&(4.10) & \beta(m)\succ(x\dashv y) &= (m\succ x)\prec\alpha(y),\\
&(4.11) & (m\prec x)\prec\alpha(y) &= \beta(m)\prec(x\dashv y)+\beta(m)\prec(x\vdash y),\\
&(4.12) & \beta(m)\succ(x\vdash y) &= (m\prec x)\succ\alpha(y)+(m\succ x)\succ\alpha(y).
\end{aligned}$$

Theorem 4.1. *Let $(D,\dashv,\vdash,\alpha)$ be a Hom-dendriform algebra and $(M,\prec,\succ,\beta)$ be a dendriform bimodule over D. Then $(M,\triangleleft,\triangleright,\beta)$ is a left-symmetric bimodule over the left-Hom-symmetric algebra associated to $(D,\dashv,\vdash,\alpha)$ (i.e. $(D,\circ,\alpha)$, where $x\circ y=x\vdash y-y\dashv x$) by means of*

$$x\triangleleft m=x\succ m-m\prec x \quad \textit{and} \quad m\triangleright x=m\succ x-x\prec m.$$

Proof. The first condition in Definition 4.2 is proved by expanded

$$\alpha(x)\triangleleft(y\triangleleft m)-(x\circ y)\triangleleft\beta(m)-\alpha(y)\triangleleft(x\triangleleft m)+(y\circ x)\triangleleft\beta(m)$$

by means of $\dashv$, $\vdash$, $\prec$ and $\succ$, and using (4.6), (4.7) and (4.11). The second condition is proved similarly by using the rest of relations. □

5. Hom-Poisson dialgebras

In this section, we introduce Hom-Poisson dialgebras and we give some examples and some construction theorems.

Definition 5.1. A Hom-Poisson dialgebra is a quintuple $(P,\dashv,\vdash,[-,-],\alpha)$ in which P is a vector space, $\dashv,\vdash,[-,-]:P\otimes P\to P$ are three bilinear maps and $\alpha:P\to P$ is a linear map such that

$$\begin{aligned}
&(5.1) & [x\dashv y,\alpha(z)] &= \alpha(x)\dashv[y,z]+[x,z]\dashv\alpha(y),\\
&(5.2) & [x\vdash y,\alpha(z)] &= \alpha(x)\vdash[y,z]+[x,z]\vdash\alpha(y),\\
&(5.3) & [\alpha(x),y\dashv z] &= \alpha(y)\vdash[x,z]+[x,y]\dashv\alpha(z)=[\alpha(x),y\vdash z].
\end{aligned}$$

for all $x,y,z\in P$.

Example 5.1. Any Poisson dialgebra is a Hom-Poisson dialgebra with $\alpha=Id$.

Example 5.2. If $(A,\cdot,\alpha)$ is a symmetric Hom-Leibniz algebra [11] i.e. both left and right Hom-Leibniz algebra, then $(A,\dashv,\vdash,[-,-],\alpha)$ is a Hom-Poisson dialgebra, with $[-,-]=\cdot=\dashv=\vdash$.

Example 5.3. Let $(P,\dashv,\vdash,[-,-],\alpha)$ and $(P',\dashv',\vdash',[-,-]',\alpha')$ be two Hom-Poisson dialgebras. Then the direct product $P\times P'$ is also a Hom-Poisson dialgebra with componentwise operation. In particular, for any non-negative integer n, $P^n=P\times P\times\cdots\times P$ (n times) is a Hom-Poisson dialgebra.

The below theorem generalizes Proposition 2.6 in [3].

Theorem 5.1. *Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dialgebra. Then $(D, \dashv, \vdash, [-,-], \alpha)$ is a Hom-Poisson dialgebra, where*

$$[x, y] = x \dashv y - y \vdash x,$$

for any $x, y \in D$.

Proof. It follows from axioms in Definition 3.2. □

Observe that by setting $\dashv = \mu$ and $\vdash = \mu^{op}$, we recover ([3], Proposition 2.6).

Definition 5.2. Let $(P, \dashv, \vdash, [-,-], \alpha)$ and $(P', \dashv', \vdash', [-,-]', \alpha')$ be two Hom-Poisson dialgebras. A linear map $f : P \to P'$ is said to be a morphism of Hom-Poisson dialgebras, if $\alpha' \circ f = f \circ \alpha$ and for any $x, y \in P$,

$$f(x \dashv y) = f(x) \dashv' f(y), \quad f(x \vdash y) = f(x) \vdash' f(y), \quad f([x, y]) = [f(x), f(y)]'.$$

The following theorem allows to obtain a Hom-Poisson dialgebra from Poisson dialgebra and an endomorphism.

Theorem 5.2. *Let $(P, \dashv, \vdash, [-,-])$ be a Poisson dialgebra and $\alpha : P \to P$ an endomorphism of Poisson dialgebras. Then $(P, \dashv_\alpha, \vdash_\alpha, [-,-]_\alpha, \alpha)$ is a Hom-Poisson dialgebra, with*

$$x \dashv_\alpha y = \alpha(x \dashv y), \quad x \vdash_\alpha y = \alpha(x \vdash y), \quad [x, y]_\alpha = \alpha([x, y]),$$

for all $x, y \in P$.

Proof. The proof is analogue to the one of Theorem 3.1. □

References

[1] A. Makhlouf and S. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. Vol.2 (2008), no. 2, 51-64.

[2] A. Makhlouf and D. Yau , *Rota-Baxter Hom-Lie admissible algebras*, Communication in Algebra, **23**, no 3, 1231-1257, 2014.

[3] D. Yau, *Non-commutative Hom-Poisson algebras*, ArXiv : 1010.3408v1, 17 Oct 2010.

[4] D. Yau , *Envelopping algebras of Hom-Lie algebras* , J. Gen. Lie Theory Appl 2(2), 95-108, 2008.

[5] D. Yau, *Module Hom-algebras*, ArXiv:0812.4695v1, 26 Dec 2008.

[6] J. Hartwig, D. Larsson and S. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra 295 (2006), 314-361.

[7] J-L Loday, *Dialgebras*, arXiv : math/0102053v1, 7 Feb 2001.

[8] J. L. Loday, *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Ens. Math., **39** (1993), 269-293.

[9] M. Aguiar, *Infinitesimal bialgebras, pre-Lie and dendriform algebras*, arXiv:math/0211074v3 [Math.QA] 16 Nov 2002.

[10] R. Felipe, *A brief fondation of the left symmetric dialgebras*, Comminicación del CIMAT No I-11-02/18-03-2011 (MB/CIMAT).

[11] S. Benayadi and S. Hidri *Quadratic Leibniz Algebras*, Journal of Lie Theory, 24 (2014) 737-759.

Département de Mathématiques, UJNK/ Centre Universitaire de N'Zérékoré, BP : 50, N'Zérékoré, Guinea.

E-mail address: `ibrahimabakayoko27@gmail.com`

Département de Mathématiques, Université Gamal Abdel Nasser de Conakry (UGANC), BP : 1147, Conakry, Guinea.

E-mail address: `bangm59@yahoo.fr`

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 54–61 (2015) ©KJM

ON SOME INEQUALITIES FOR THE EXPECTATION AND VARIANCE

ZHENG LIU

Abstract. Some elementary inequalities for the expectation and variance of a continuous random variable whose probability density function is defined on a finite interval are obtained by using an identity due to P. Cerone for the Chebyshev functional and some standard results from the theory of inequalities. Thus some mistakes in the literatures are corrected.

1. INTRODUCTION

Let X be a continuous random variable having the probability density function f defined on a finite interval $[a, b]$.

By definition

$$E(X) := \int_a^b t f(t)\, dt \tag{1.1}$$

the expectation of X, and

$$\begin{aligned} \sigma^2(X) &:= \int_a^b [t - E(X)]^2 f(t)\, dt \\ &= \int_a^b t^2 f(t)\, dt - [E(X)]^2 \end{aligned} \tag{1.2}$$

the variance of X.

For two integral functions $f, g : [a, b] \to \mathbf{R}$, define the Chebyshev functional

$$T(f, g) := \frac{1}{b-a}\int_a^b f(t)g(t)\, dt - \frac{1}{b-a}\int_a^b f(t)\, dt \cdot \frac{1}{b-a}\int_a^b g(t)\, dt. \tag{1.3}$$

In [1], P. Cerone has obtained the following identity that involves a Riemann-Stieltjes integral:

Date: December 14, 2014.

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. random variable, expectation, variance, probability density function, Chebyshev functional.

Lemma 1.1. *Let $f, g : [a, b] \to \mathbf{R}$ be such that f is of bounded variation on $[a, b]$ and g is continuous on $[a, b]$. Then*

$$T(f,g) = \frac{1}{(b-a)^2} \int_a^b \Psi(t)\, df(t), \tag{1.4}$$

where

$$\Psi(t) := (t-a)A(t,b) - (b-t)A(a,t), \tag{1.5}$$

with

$$A(c,d) := \int_c^d g(x)\, dx. \tag{1.6}$$

In [1] we can also find the following useful result:

Lemma 1.2. *Let $f, g : [a, b] \to \mathbf{R}$ be such that f is of bounded variation and g is continuous on $[a, b]$. Then*

(1.7)

$$(b-a)^2 |T(f,g)| \le \begin{cases} \sup_{t\in[a,b]} |\Psi(t)| \bigvee_a^b(f), & \\ L \int_a^b |\Psi(t)|\, dt, & \text{for } f \text{ } L\text{-Lipschitzian}, \\ \int_a^b |\Psi(t)|\, df(t), & \text{for } f \text{ monotonic nondecreasing}, \end{cases}$$

where $\bigvee_a^b(f)$ is the total variation of f on $[a, b]$.

The purpose of this paper is to derive some elementary inequalities for the expectation (1.1) and variance (1.2) by using Lemma 1.1 and Lemma 1.2. Thus some mistakes in [1] and [2] are corrected.

2. INEQUALITIES FOR THE EXPECTATION

We prove the following theorem by using the Lemma 1.1.

Theorem 2.1. *Let $f : [a, b] \to \mathbf{R}_+$ be an absolutely continuous probability density function associated with a random variable X, then the expectation $E(X)$ satisfies the inequalities*

$$\begin{aligned} &\left|E(X) - \tfrac{a+b}{2}\right| \\ &\le \begin{cases} \frac{(b-a)^3}{12} \|f'\|_\infty, & f' \in L_\infty[a,b]; \\ \frac{1}{2}(b-a)^{2+\frac{1}{q}} [B(q+1,q+1)]^{\frac{1}{q}} \|f'\|_p, & f' \in L_p[a,b],\ p>1, \\ & \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{8} \|f'\|_1, & f' \in L_1[a,b]. \end{cases} \end{aligned} \tag{2.1}$$

where $\|\cdot\|_p, 1 \le p \le \infty$ are the usual Lebesgue norms on $[a, b]$, i.e.,

$$\|g\|_p := \begin{cases} \left[\int_a^b |g(t)|^p\, dt\right]^{\frac{1}{p}}, & 1 \le p < \infty, \\ ess \sup_{t\in[a,b]} |g(t)|, & p = \infty. \end{cases} \tag{2.2}$$

Proof. Notice that $\int_a^b f(t)\,dt = 1$ and f is absolutely continuous on $[a,b]$, by (1.3) and (1.4)-(1.6) we get

$$E(X) - \frac{a+b}{2} = (b-a)T(t, f(t)) = \frac{1}{2}\int_a^b (t-a)(b-t)f'(t)\,dt,$$

and so

$$|E(X) - \frac{a+b}{2}| \le \frac{1}{2}\int_a^b (t-a)(b-t)|f'(t)|\,dt.$$

Using the Hölder's integral inequality, we have

$$\int_a^b (t-a)(b-t)f'(t)\,dt \le \begin{cases} \frac{1}{2}\|f'\|_\infty \int_a^b (t-a)(b-t)\,dt, & f' \in L_\infty[a,b]; \\ \frac{1}{2}\|f'\|_p [\int_a^b |(t-a)(b-t)|^q\,dt]^{\frac{1}{q}}, & f' \in L_p[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\|f'\|_1 \sup_{t\in[a,b]}(t-a)(b-t), & f' \in L_1[a,b]. \end{cases}$$

Clearly,

$$\int_a^b (t-a)(b-t)\,dt = \frac{(b-a)^3}{6},$$

$$\sup_{t\in[a,b]} (t-a)(b-t) = \frac{(b-a)^2}{4},$$

and it is easy to find by substitution $u = a + (b-a)t$ that

$$\int_a^b [(t-a)(b-t)]^q\,dt = (b-a)^{2q+1}\int_0^1 u^q(1-u)^q\,du = (b-a)^{2q+1}B(q+1, q+1).$$

Thus we have proved the inequalities (2.1).

Remark 2.1. The inequalities (2.1) provide a correction of the inequalities (3.22) in [2].

Theorem 2.2. *Let $f : [a,b] \to \mathbf{R}_+$ be a probability density function associated with a random variable X. Then the expectation $E(X)$ satisfies the inequalities*

$$(2.3)\quad |E(X) - \frac{a+b}{2}| \le \begin{cases} \frac{(b-a)^2}{8}\bigvee_a^b(f), & \textit{for } f \textit{ of bounded variation,} \\ \frac{(b-a)^3}{12}L, & \textit{for } f \textit{ L-Lipschitzian,} \\ \frac{(b-a)^2}{8}[f(b)-f(a)], & \textit{for } f \textit{ monotonic nondecreasing.} \end{cases}$$

Proof. Notice that $\int_a^b f(t)\,dt = 1$, by (1.3), (1.4) and (1.6) we get

$$E(X) - \frac{a+b}{2} = (b-a)T(t, f(t)) = \frac{1}{2}\int_a^b (t-a)(b-t)\,df(t),$$

and so it follows from Lemma 1.2,

$$|E(X) - \frac{a+b}{2}| \le \begin{cases} \frac{1}{2}\sup_{t\in[a,b]}(t-a)(b-t)\bigvee_a^b(f), & \text{for } f \text{ of bounded variation,} \\ \frac{L}{2}\int_a^b (t-a)(b-t)\,dt, & \text{for } f\ L\text{-Lipschitzian,} \\ \frac{1}{2}\int_a^b (t-a)(b-t)\,df(t), & \text{for } f \text{ monotonic nondecreasing.} \end{cases}$$

We need only to calculate and estimate that

$$\begin{aligned}
\int_a^b (t-a)(b-t)\,df(t) &= (t-a)(b-t)f(t)|_a^b + 2\int_a^b (t-\tfrac{a+b}{2})f(t)\,dt \\
&= 2[\int_a^{\frac{a+b}{2}} (t-\tfrac{a+b}{2})f(t)\,dt + \int_{\frac{a+b}{2}}^b (t-\tfrac{a+b}{2})f(t)\,dt] \\
&\leq 2f(a)\int_a^{\frac{a+b}{2}} (t-\tfrac{a+b}{2})\,dt + 2f(b)\int_{\frac{a+b}{2}}^b (t-\tfrac{a+b}{2})\,dt \\
&= \tfrac{(b-a)^2}{4}[f(b)-f(a)].
\end{aligned}$$

Consequently, the inequalities (2.2) are proved.

Remark 2.2. The inequalities (2.2) provide a correction of inequalities (3.14) in [1].

3. INEQUALITIES FOR THE VARIANCE

For convenience in further discussions, we will first to derive some technical results in what follows. Put

$$\phi(t) := (t-\gamma)^3 + \frac{1}{b-a}[(b-t)(\gamma-a)^3 - (t-a)(b-\gamma)^3] \tag{3.1}$$

for $t \in [a,b]$ and $\gamma \in \mathbf{R}$.

It is easy to find that

$$\begin{aligned}
\phi(t) &= t^3 - 3\gamma t^2 - [a^2+ab+b^2-3(a+b)\gamma]t - ab[3\gamma-(a+b)] \\
&= (t-a)(t-b)(t-c),
\end{aligned} \tag{3.2}$$

where $c = 3\gamma - a - b$. This implies that

$$c \begin{cases} > \gamma, & \gamma > \frac{a+b}{2}, \\ = \gamma, & \gamma = \frac{a+b}{2}, \\ < \gamma, & \gamma < \frac{a+b}{2}. \end{cases} \tag{3.3}$$

Moreover, we see that $c < a$ for $\gamma < \frac{2a+b}{3}$, $c > b$ for $\gamma > \frac{a+2b}{3}$ and $a \leq c \leq b$ for $\frac{2a+b}{3} \leq \gamma \leq \frac{a+2b}{3}$. Therefore, by (3.2) we can conclude that $\phi(t) \leq 0$ for $t \in [a,b]$ if $\gamma < \frac{2a+b}{3}$, $\phi(t) \geq 0$ for $t \in [a,b]$ if $\gamma > \frac{a+2b}{3}$ and $\phi(t) > 0$ for $t \in (a,c)$ with $\phi(t) < 0$ for $t \in (c,b)$ if $\frac{2a+b}{3} \leq \gamma \leq \frac{a+2b}{3}$.

Thus we have

$$\int_a^b |\phi(t)|\,dt = -\int_a^b \phi(t)\,dt = \frac{1}{2}(\frac{a+b}{2}-\gamma)(b-a)^3 \tag{3.4}$$

in case $\gamma < \frac{2a+b}{3}$,

$$\int_a^b |\phi(t)|\,dt = \int_a^b \phi(t)\,dt = \frac{1}{2}(\gamma-\frac{a+b}{2})(b-a)^3 \tag{3.5}$$

in case $\frac{a+2b}{3} < \gamma$, and

(3.6)

$$\begin{aligned}
\int_a^b |\phi(t)|\,dt &= \int_a^c \phi(t)\,dt - \int_c^b \phi(t)\,dt \\
&= \tfrac{1}{4}[18(\gamma-a)(b-\gamma)(b-a)^2 - 54(\gamma-a)^2(b-\gamma)^2 - (b-a)^4]
\end{aligned}$$

in case $\frac{2a+b}{3} \le \gamma \le \frac{a+2b}{3}$.

Also, it is not difficult to get by elementary calculus that

$$\sup_{t\in[a,b]} |\phi(t)| = 2\{[(\gamma - \frac{a+b}{2})^2 + \frac{(b-a)^2}{12}]^{\frac{3}{2}} - (\gamma - a)(b-\gamma)|\gamma - \frac{a+b}{2}|\}, \tag{3.7}$$

for $\gamma \in \mathbf{R}$.

Now we would like to give some inequalities for the variance with different bounds.

Theorem 3.1. *Let $f : [a, b] \to \mathbf{R}_+$ be an absolutely continuous probability density function associated with a random variable X. If $f' \in L_\infty[a, b]$, then the variance $\sigma^2(X)$ satisfies the inequalities*

$$\begin{aligned} &|\sigma^2(X) - (\gamma - \tfrac{a+b}{2})^2 - \tfrac{(b-a)^2}{12}| \\ \le \|f'\|_\infty &\begin{cases} \frac{1}{6}(\frac{a+b}{2} - \gamma)(b-a)^3, & a < \gamma < \frac{2a+b}{3} \\ \frac{1}{12}[18(\gamma - a)(b-\gamma)(b-a)^2 - 54(\gamma-a)^2(b-\gamma)^2 - (b-a)^2], & \frac{2a+b}{3} \le \gamma \le \frac{a+2b}{3}, \\ \frac{1}{6}(\gamma - \frac{a+b}{2})(b-a)^3, & \frac{a+2b}{3} < \gamma < b, \end{cases} \end{aligned} \tag{3.8}$$

where $a < \gamma = E(X) < b$.

Proof. It is easy to find from (1.3)-(1.6) that

$$\sigma^2(X) - (\gamma - \frac{a+b}{2})^2 - \frac{(b-a)^2}{12} = -\frac{1}{3}\int_a^b \phi(t) f'(t)\,dt, \tag{3.9}$$

where $\phi(t)$ is as defined in (3.1).

Thus the inequalities (3.8) follow from (3.4), (3.5) and (3.6).

Theorem 3.2. *Let $f : [a, b] \to \mathbf{R}_+$ be an absolutely continuous probability density function associated with a random variable X. If $f' \in L_1[a, b]$, then the variance $\sigma^2(X)$ satisfies the inequality*

$$\begin{aligned} &|\sigma^2(X) - (\gamma - \tfrac{a+b}{2})^2 - \tfrac{(b-a)^2}{12}| \\ \le\ & \tfrac{2}{3}\{[(\gamma - \tfrac{a+b}{2})^2 + \tfrac{(b-a)^2}{12}]^{\frac{3}{2}} - (\gamma - a)(b-\gamma)|\gamma - \tfrac{a+b}{2}|\}\|f'\|_1, \end{aligned} \tag{3.10}$$

where $a < \gamma = E(X) < b$.

Proof. The inequality (3.10) follows immediately from (3.7) and (3.9).

Remark 3.1. The inequalities (3.8) and inequality (3.10) provide a correction of inequalities (3.23) in [2].

Theorem 3.3. *Let $f : [a, b] \to \mathbf{R}_+$ be a probability density function associated with a random variable X which is of bounded variation on $[a, b]$. Then the variance $\sigma^2(X)$ satisfies the inequality*

$$\begin{aligned} &|\sigma^2(X) - (\gamma - \tfrac{a+b}{2})^2 - \tfrac{(b-a)^2}{12}| \\ \le\ & \tfrac{2}{3}\{[(\gamma - \tfrac{a+b}{2})^2 + \tfrac{(b-a)^2}{12}]^{\frac{3}{2}} - (\gamma - a)(b-\gamma)|\gamma - \tfrac{a+b}{2}|\}\bigvee_a^b(f), \end{aligned} \tag{3.11}$$

where $a < \gamma = E(X) < b$ and $\bigvee_a^b(f)$ is the total variation of f on $[a, b]$.

Proof. By Lemma 1.1 and Lemma 1.2 we can conclude that

$$|\sigma^2(X) - (\gamma - \frac{a+b}{2})^2 - \frac{(b-a)^2}{12}| \leq \frac{1}{3} \sup_{t\in[a,b]} |\phi(t)| \bigvee_a^b (f),$$

where $\phi(t)$ is as defined in (3.1).

Thus the inequality (3.11) follows from (3.7).

Theorem 3.4. *Let $f : [a,b] \to \mathbf{R}_+$ be a probability density function associated with a random variable X which is L-Lipschitzian on $[a,b]$. Then the variance $\sigma^2(X)$ satisfies the inequalities*

(3.12)
$$\begin{aligned} &|\sigma^2(X) - (\gamma - \tfrac{a+b}{2})^2 - \tfrac{(b-a)^2}{12}| \\ \leq L &\begin{cases} \frac{1}{6}(\frac{a+b}{2} - \gamma)(b-a)^3, & a < \gamma < \frac{2a+b}{3}, \\ \frac{1}{12}[18(\gamma-a)(b-\gamma)(b-a)^2 - 54(\gamma-a)^2(b-\gamma)^2 - (b-a)^4], & \frac{2a+b}{3} \leq \gamma \leq \frac{a+2b}{3}, \\ \frac{1}{6}(\gamma - \frac{a+b}{2})(b-a)^3, & \frac{a+2b}{3} < \gamma < b, \end{cases} \end{aligned}$$

where $a < \gamma = E(X) < b$.

Proof. By Lemma 1.1 and Lemma 1.2 we can conclude that

$$|\sigma^2(X) - (\gamma - \frac{a+b}{2})^2 - \frac{(b-a)^2}{12}| \leq \frac{L}{3} \int_a^b |\phi(t)|\, dt,$$

where $\phi(t)$ is as defined in (3.1).

Thus the inequalities (3.12) follow from (3.4), (3.5) and (3.6).

Theorem 3.5. *Let $f : [a,b] \to \mathbf{R}_+$ be a probability density function associated with a random variable X which is monotonic nondecreasing on $[a,b]$. Then the variance $\sigma^2(X)$ satisfies the inequality*

(3.13)
$$\begin{aligned} &|\sigma^2(X) - (\gamma - \tfrac{a+b}{2})^2 - \tfrac{(b-a)^2}{12}| \\ \leq &\begin{cases} \frac{5b+4a-9\gamma}{18}(b-a)^2[f(b)-f(a)], & a < \gamma < \frac{2a+b}{3}, \\ \frac{3b-2a-c}{18}(c-a)^2[f(c)-f(a)] + \frac{2b+c-3a}{18}(b-c)^2[f(b)-f(c)], & \frac{2a+b}{3} \leq \gamma \leq \frac{a+2b}{3}, \\ \frac{9\gamma-5a-4b}{18}(b-a)^2[f(b)-f(a)], & \frac{a+2b}{3} < \gamma < b, \end{cases} \end{aligned}$$

where $a < \gamma = E(X) < b$ and $c = 3\gamma - a - b$.

Proof. By Lemma 1.1 and Lemma 1.2 we can conclude that

$$|\sigma^2(X) - (\gamma - \frac{a+b}{2})^2 - \frac{(b-a)^2}{12}| \leq \frac{1}{3} \int_a^b |\phi(t)|\, df(t),$$

where $\phi(t)$ is as defined in (3.1).

Notice that

$$\phi(t) = (t-a)(t-b)(t-c)$$

for $t \in [a,b]$, where $c = 3\gamma - a - b$, it is easy to calculate that

$$\begin{array}{rcl}\int_a^b |\phi(t)|\,df(t) & = & -\int_a^b \phi(t)\,df(t) = \int_a^b \phi'(t)f(t)\,dt \\ & = & \int_a^b [(t-b)(t-c)+(t-a)(t-c)+(t-a)(t-b)]f(t)\,dt \\ & \leq & f(a)\int_a^b (t-b)(t-c)\,dt + f(b)\int_a^b (t-a)(t-c)\,dt + f(a)\int_a^b (t-a)(t-b)\,dt \\ & = & \frac{5b+4a-9\gamma}{6}(b-a)^2[f(b)-f(a)],\end{array}$$

in case $a < \gamma < \frac{2a+b}{3}$,

$$\begin{array}{rcl}\int_a^b |\phi(t)|\,df(t) & = & \int_a^b \phi(t)\,df(t) = -\int_a^b \phi'(t)f(t)\,dt \\ & = & -\int_a^b [(t-b)(t-c)+(t-a)(t-c)+(t-a)(t-b)]f(t)\,dt \\ & \leq & -f(a)\int_a^b (t-b)(t-c)\,dt - f(b)\int_a^b (t-a)(t-c)\,dt - f(b)\int_a^b (t-a)(t-b)\,dt \\ & = & \frac{9\gamma-5a-4b}{6}(b-a)^2[f(b)-f(a)],\end{array}$$

in case $\frac{a+2b}{3} < \gamma < b$, and

$$\begin{array}{rcl}\int_a^b |\phi(t)|\,df(t) & = & \int_a^c \phi(t)\,df(t) - \int_c^b \phi(t)\,df(t) \\ & = & -\int_a^c \phi'(t)f(t)\,dt + \int_c^b \phi'(t)f(t)\,dt \\ & = & -\int_a^c [(t-b)(t-c)+(t-a)(t-c)+(t-a)(t-b)]f(t)\,dt \\ & & +\int_c^b [(t-b)(t-c)+(t-a)(t-c)+(t-a)(t-b)]f(t)\,dt \\ & \leq & -f(a)\int_a^c (t-b)(t-c)\,dt - f(c)\int_a^c (t-a)(t-c)\,dt - f(c)\int_a^c (t-a)(t-b)\,dt \\ & & +f(c)\int_c^b (t-b)(t-c)\,dt + f(b)\int_c^b (t-a)(t-c)\,dt + f(c)\int_c^b (t-a)(t-b)\,dt \\ & = & \frac{3b-2a-c}{6}(c-a)^2[f(c)-f(a)] + \frac{2b+c-3a}{6}(b-c)^2[f(b)-f(c)]\end{array}$$

in case $\frac{2a+b}{3} \leq \gamma \leq \frac{a+2b}{3}$.

Consequently, the inequalities (3.13) are proved.

Corollary 3.1. *Let $f : [a,b] \to \mathbf{R}_+$ be a probability density function associated with a random variable X. If $E(X) = \frac{a+b}{2}$, then the variance $\sigma^2(X)$ satisfies the inequalities*

$$|\sigma^2(X) - \frac{(b-a)^2}{12}| \leq \begin{cases} \frac{(b-a)^3}{36\sqrt{3}}\bigvee_a^b(f), & f \text{ of bounded variation,} \\ \frac{(b-a)^4}{96}L, & f \text{ } L\text{-Lipschitzian,} \\ \frac{5(b-a)^3}{144}[f(b)-f(a)], & f \text{ monotonic nondecreasing.}\end{cases}$$

Proof. It is immediate from the inequalities (3.11), (3.12) and (3.13).

Remark 3.2. The inequalities (3.11), (3.12) and (3.13) provide a correction of inequalities (3.15) in [1].

Remark 3.3. The mistakes of Corollary 8 and Corollary 9 1n [2] as well as the mistakes of Corollary 3.7 and Corollary 3.8 in [1] seemed as if they are originated from having wrongly examined the behaviour of $\phi(t)$ as given by

$$\phi(t) = (t-\gamma)^{n+1} + (\frac{b-t}{b-a})(\gamma-a)^{n+1} - (\frac{t-a}{b-a})(b-\gamma)^{n+1}$$

for $t \in [a,b]$ in case n is even. (See (3.13) of Lemma 2 in [2] and also (3.6) of Lemma 3.3 in [1] and compare them with the assertions expressed at the beginning of this section as a special case of $n = 2$.)

References

[1] P. Cerone, On an identity for the Chebychev functional and some ramifications, J. Inequal. Pure and Appl. Math., 3(1) (2002), Art. 4. (http://jipam.vu.edu.au/).

[2] P. Cerone and S. S. Dragomir, On some inequalities arising from Montgomery's identity, J. Comput. Anal. Applics., 5(4) (2003), 341-368.

Institute of Applied Mathematics, School of Science, University of Science and Technology Liaoning, Anshan 114051, Liaoning, China

E-mail address: `lewzheng@163.net`

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 62–76 (2015) ©KJM

A DIFFERENT LOOK FOR PARANORMED RIESZ SEQUENCE SPACE DERIVED BY FIBONACCI MATRIX

MURAT CANDAN AND GÜLSEN KILINÇ

Abstract. This paper presents the generalized Riesz sequence space $r^q(\widehat{F}_u^p)$ which is formed all sequences whose $R_u^q\widehat{F}$-transforms are in the space $\ell(p)$, where $\widehat{F}$ is a Fibonacci matrix. α- β- and γ-duals of the newly described sequence space have been given in addition to some topological properties of its. Also, it has been established the basis of $r^q(\widehat{F}_u^p)$. Finally, we have been described a matrix class on the sequence space. Results obtained are more general and more comprehensive than presented up to now.

1. Preliminaries

The concept of sequence is widely considered to be one of the important concepts in summability theory, so let us begin by remembering the definition of it. A sequence is a function of which domain set is natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. In other words, an ordered list of numbers $x_0, x_1, ..., x_n, ...$ is a sequence. If it is an infinite sequence, it is illustrated with notation $\{x_n\}_{n=0}^{\infty}$, as a convenience, we write $\{x_n\}$ briefly. A sequence $\{x_n\}$ converges with limit a if each neighborhood of a contains almost all terms of the sequence, i.e., there must be at most only finitely many elements of $\{x_n\}$ outside any neighborhood of a. In this case, we say that $\{x_n\}$ converges to a as n goes to ∞. The set of all real or complex convergent sequences is indicated by c. Let $\{x_n\}$ be a sequence and define a new sequence $\{s_n\}$ called the sequence of partial sums of $\{x_n\}$with relation $s_n = \sum_{k=1}^{n} x_k$. When $\{s_n\}$ is convergent, we say that $\{x_n\}$ is summable and we point out the $\lim_n s_n$ by $\sum_{j=0}^{\infty} x_j$. A real or complex number sequence converges to zero is called null sequence. The set of all real or complex null sequences is denoted by c_0. A sequence is bounded, if all its terms remain between two numbers. The set of all bounded sequences is denoted by l_∞. We denote the family of all $\{x_n\}$ sequences by w, where

Date: January 1, 2015 and, in revised form, June 15, 2015.
2010 *Mathematics Subject Classification.* 46A45, 40C05.
Key words and phrases. Paranormed sequence space, Fibonacci numbers, alpha-, beta- and gamma-duals and matrix mappings..

x_n belongs to real or complex numbers set. Then w is a linear space under the usual pointwise addition and scalar multiplication over $\mathbb{C}$ and $\mathbb{R}$. Since any linear subspace of w is called a sequence space, also c, c_0 and ℓ_∞ are the subspaces of w, we concludes that they are sequence spaces. Further, we symbolizes the spaces of all bounded, convergent, absolutely and $p-$absolutely convergent series by bs, cs, ℓ_1, ℓ_p; respectively.

These spaces are Banach spaces with following norms:

$\|x\|_{\ell_\infty} = \|x\|_c = \|x\|_{c_0} = \sup_k |x_k|, \|x\|_{bs} = \|x\|_{cs} = \sup_n |\sum_{k=1}^n x_k|$, and $\|x\|_{\ell_p} = (\sum_k |x_k|^p)^{\frac{1}{p}}$.

For sake of brevity, here and after the summation without limits runs from 1 to ∞.

Now, let us look at historical information about Fibonacci sequence. Fibonacci sequence consist of $\{f_n\}$ numbers such that each its term is the sum of two terms preceding its. In this sequence, the first two terms are 1. If we write it clearly, it is a sequence of numbers $1, 1, 2, 3, 5, 8, 13, \cdots$. We can define it by the equation $f_n = f_{n-1} + f_{n-2}$, where $n \geq 2$ and $f_1 = f_0 = 1$. Fibonacci numbers were come out by Leonardo Pisano Bogollo (c-1170-c1250), he is known with his nickname Fibonacci. Numbers of the sequence is seen in the book "Liber Abaci "firstly written by Leonardo of Pisa. He helped to replace Roman numerical system with the numbers system used today consists of numbers from 0 to 9 in Europa. Fibonacci sequence has some well-known properties such as Golden Ratio and Cassini Formula. If we take ratio of two successive terms of Fibonacci sequences, limit of the this ratio is famous Golden Ratio which is 1.61803 and written by ϕ.

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \phi \quad \text{(Golden Ratio)}.$$

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad \text{for each } n \in \mathbb{N}.$$

$$\sum_k \frac{1}{f_k} \text{ converges.}$$

$$f_{n-1}.f_{n+1} - f_n^2 = (-1)^{n+1} \qquad \text{for each n} \geq 1 \text{ (Cassini Formula)}.$$

Let $A = (a_{nk})$ be a triangle matrix, that is $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. The equality $A(Bx) = (AB)x$ holds for the triangle matrices A, B and a sequence x. Furthermore, a triangle matrix A has an inverse A^{-1}which is also a triangle matrix and unique such that for each $x \in \omega$, $x = A(A^{-1}x) = A^{-1}(Ax)$.

The domain X_A of an infinite matrix A which is a sequence space is defined as

$$X_A := \{x = (x_k) \in \omega : Ax \in X\}, \tag{1.1}$$

in a sequence space X.

Generally X_A constructed by the limitation matrix A is either the expansion or the contraction of the space X itself, where X is a sequence space. Sometimes they are overlap. The inclusion $X_S \subset X$ is provided strictly for $X \in \{\ell_\infty, c, c_0\}$. From this property, it can be concluded that the inclusion $X \subset X_{\Delta^{(1)}}$ is also provided firmly for $X \in \{\ell_\infty, c, c_0, \ell_p\}$. But, if X is taken as $X := c_0 \oplus span\{z\}$ for each $x \in X$, there exist an $s \in c_0$ and an $\alpha \in \mathbb{C}$ such that $x := s + \alpha z$, where $z = ((-1)^k)$ and it is considered the matrix A with the rows A_n defined by $A_n := (-1)^n e^{(n)}$ for all $n \in \mathbb{N}$, then we obtain $Ae = z \in \lambda$ when $Az = e \notin \lambda$ resulting in the sequences

$z \in X \setminus X_A$ and $e \in X_A \setminus \lambda$, here $e = (1, 1, 1, \ldots)$ and $e^{(n)}$ represents a sequence of which n^{th} term is 1 for each $n \in \mathbb{N}$ and the others are 0. Namely, the sequence spaces X_A and X are overlap when none of them contains the other one [10].

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$, $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha \in \mathbb{R}$ and all $x \in X$, where θ is the zero vector in the linear space X.

Let us suppose that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\,\{1,\ H\}$ and $1/p_k + 1/p_k^{'} = 1$ provided $1 < inf p_k \leq H < \infty$. The linear spaces $\ell_\infty(p)$ and $\ell(p)$ were defined by Maddox in [56, 57] (see also Simons [68] and Nakano [63]) as follows:

$$\ell(p) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty \right\},$$

and

$$\ell_\infty(p) = \left\{ x = (x_k) \in w : \sup_{k \in \mathbb{N}} |x_k|^{p_k} < \infty \right\},$$

which are the complete spaces paranormed by

$$h_1(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M} \quad \text{and} \quad h_2(x) = \sup_{k \in \mathbb{N}} |x_k|^{p_k/M} \quad \text{iff} \quad \inf p_k > 0,$$

respectively. In addition to this, by notation $\mathcal{F}$, we denote the collection consisting of all nonempty and finite subsets of $\mathbb{N}$.

Constructing a new sequence space by means of the matrix domain of a particular triangle has been used in literature as the sequence spaces $X_p = (\ell_p)_{C_1}$ [64], $r^t(p) = (\ell(p))_{R_t}$ [2], $e_p^r = (\ell_p)_{E^r}$ and $e^r(p) = (\ell(p))_{E^r}$ [7, 48, 61]. $Z(u, v, \ell_p) = (\ell_p)_{G(u,v)}$ and $\ell(u, v, p) = (\ell(p))_{G(u,v)}$ [4, 60], $a^r(p) = (\ell_p)_{A^r}$ and $a^r(u, p) = (\ell(p))_{A_u^r}$ [8, 9], $bv_p = (\ell_p)_\Delta$ and $bv(u, p) = (\ell(p))_{A_u}$ [3, 11, 59], $\overline{\ell(p)} = (\ell(p))_S$ [37], $\ell_p^\lambda = (\ell_p)_\Lambda$ in [62], $\lambda_{B(r,s)}$ in [53] $\lambda_{B(\tilde{r},\tilde{s})}$ in [25], $f_0(B)$ and $f(B)$ in [12], $f_0(\widetilde{B})$ and $f(\widetilde{B})$ in [26], where $C_1 = \{c_{nk}\},\ R^t = \{r_{nk}^t\},\ E^r = \{e_{nk}^r\},\ S = \{s_{nk}\},\ \Delta = \{\delta_{nk}\},\ G(u, v) = \{g_{nk}\},\ \Delta^{(m)} = \{\Delta_{nk}^{(m)}\},\ A^r = \{a_{nk}^r\},\ A_u^r = \{a_{nk}(r)\},\ A^u = \{a_{nk}^u\},\ B(r, s) = \{b_{nk}(r, s)\},\ B(\tilde{r}, \tilde{s}) = \{b_{nk}(\tilde{r}, \tilde{s})\},\ \Lambda = \{\lambda_{nk}\}_{n,k=0}^\infty$ and $A(\lambda) = \{a_{nk}(\lambda)\}$ denote the Cesàro, Riesz, Euler, generalized weighted means or factorable matrix, summation matrix, difference matrix, generalized difference matrix and sequential band matrix, respectively [6, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 27, 28, 29, 40, 41, 42, 50, 51, 52, 54, 55, 66]. Let us note here, there are many different ways to construct new sequence spaces from old ones. To get more detailed information, one can look at the articles [24, 30, 35, 36, 69].

Given any infinite matrix $A = (a_{nk})$ of real numbers a_{nk}, where $n, k \in \mathbb{N}$ and let X, Y be sequence spaces. For any sequence x, A-transform of x is written as $Ax = \big((Ax)_n\big)$. If it is A-transform of x, it means that $(Ax)_n = \sum_k a_{nk} x_k$ converges for each $n \in \mathbb{N}$. If $x \in X$ implies that $Ax \in Y$ then A is called a matrix mapping from X into Y and is denoted by $A : X \to Y$. We illustrate the class of all infinite matrices such that $A : X \to Y$ by $(X : Y)$.

The new sequence spaces derived by Riesz mean (R, q_n) and Fibonacci matrix $\widehat{F} = \{\widehat{f}_{nk}\}$ are given in this study.

In this paper, section 2 is dedicated for the spaces of difference sequences and given some historical developments about this subject. In addition, the definition of Fibonacci Matrix and the paranormed sequence space $r^q(\widehat{F}^p_u)$ of non-absolute type which is the set of all sequences whose $R^q_u\widehat{F}$-transforms are in the space $\ell(p)$ are presented. In section 3, alpha-, beta- and gamma-duals of the sequence space $r^q(\widehat{F}^p_u)$ are found. Moreover, the basis of the space $r^q(\widehat{F}^p_u)$ is attained. In the final section, we characterize a matrix class on the sequence space.

2. Difference operator and the Riesz Sequence Space $r^q(\widehat{F}^p_u)$ of Non-absolute Type

Before following non-absolute type the Riesz sequence space $r^q(\widehat{F}^p_u)$, firstly, let us recall some definitions. We remember the idea of difference operator. The difference sequence spaces have been introduced by Kızmaz [49]. For $\lambda \in \{\ell_\infty, c, c_0\}$, $\lambda(\Delta)$ consisting of the sequences $x = (x_k)$ such that $(x_k - x_{k+1}) \in \lambda$ is called the difference sequence spaces [49]. The difference spaces bv_p consisting of the sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ have been studied in the case $0 < p < 1$ by Altay and Başar [5], and in the case $1 \leq p < \infty$ by Başar and Altay [11], and Çolak, et.al. [38].

The concept of difference sequences was generalized by Çolak and Et [39]. They defined and analyzed some property of these sequence spaces

$$\Delta^m\lambda = \Big\{x = (x_k) \in \omega : \Delta^m x \in \lambda\Big\},$$

where $\Delta^1 x = (x_k - x_{k+1})$ and $\Delta^m x = \Delta(\Delta^{m-1}x)$ for $m \in \{1, 2, 3, \ldots\}$. Malkowsky and Parashar [58] introduced the sequence spaces as follows

$$\Delta^{(m)}\lambda = \Big\{x = (x_k) \in \omega : \Delta^{(m)} x \in \lambda\Big\},$$

where $m \in \mathbb{N}$, $\Delta^{(1)}x = (x_k - x_{k-1})$ and $\Delta^{(m)}x = \Delta^{(1)}(\Delta^{(m-1)}x)$. Polat and Başar [65] introduced the spaces $e^r_0(\Delta^{(m)})$, $e^r_c(\Delta^{(m)})$ and $e^r_\infty(\Delta^{(m)})$ consisting of all sequences whose m^{th} order differences are in the Euler spaces e^r_0, e^r_c and e^r_∞, respectively. Altay [1] studied the space $\ell_p(\Delta^{(m)})$ consisting of all sequences whose m^{th} order differences are $p-$absolutely summable which is a generalization of the spaces bv_p [11, 38].

The transformation given by

$$q_n = \frac{q_1 s_1 + \cdots + q_n s_n}{Q_n}$$

is called the Riesz mean (R, q_n) or simply the (R, q_n) mean, where (q_k) is a sequence of positive numbers and $Q_n = q_1 + q_2 + \cdots + q_n$.

The (R, q_n) matrix method is given by

$$r^t_{nk} := \begin{cases} \frac{q_k}{Q_n} & , \quad (0 \leq k \leq n), \\ 0 & , \quad (k > n). \end{cases}$$

The Riesz sequence spaces $r^q(u, p)$ and $r^q(\Delta^p_u)$ of non-absolute type had been studied by Ganie and Sheikh [43, 67]. After then, Candan and Güneş [32] had examined the sequence space $r^q(B^p_u)$.

Many mathematician used Fibonacci numbers to construct new sequence space. Some of them are here. Kara [46] defined $\ell_p(\widehat{F})$ sequence space. After Kara et al. [47] characterized some class of compact operators on the spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$, where $1 \leq p \leq \infty$. Also, Başarır et al. [15] introduced the sequence space $\lambda(\widehat{F})$ and $\lambda(\widehat{F}, p)$. Later, Candan [31] presented the sequence spaces $c_0(\widehat{F}\,(r,s))$ and $c(\widehat{F}\,(r,s))$. After then, Candan and Kayaduman [34] introduced the sequence space $\widehat{c}^{f(r,s)}$ derived by generalized difference Fibonacci matrix. Finally, Candan and Kara [33] studied the space $\ell_p(\widehat{F}(r,s))$, where $1 \leq p \leq \infty$.

Let f_n be the $n-$th Fibonacci number for every $n \in \mathbb{N}$. Then we define the Fibonacci matrix $\widehat{F} = \{\widehat{f}_{nk}\}$ by

$$\widehat{f}_{nk} := \begin{cases} \frac{f_n}{f_{n+1}} & , \quad k = n, \\ -\frac{f_{n+1}}{f_n} & , \quad k = n-1, \\ 0 & , \quad 0 \leq k < n-1 \ \text{ or } \ k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$.

For $0 < p_k \leq H < \infty$, let us define the set $r^q(\widehat{F}_u^p)$ as the set of all sequences whose $R_u^q\widehat{F}$-transforms are in the sequence space $\ell(p)$, that is

$$r^q(\widehat{F}_u^p) = \left\{ x = (x_k) \in w : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j \widehat{F} x_j \right|^{p_k} < \infty \right\}.$$

We can rewrite the set $r^q(\widehat{F}_u^p)$ by means of the notation of (1.1) as follow

$$r^q(\widehat{F}_u^p) = \{\ell(p)\}_{R_u^q \widehat{F}},$$

where $R_u^q F = (r_{nk}^{q_F^u})$ is a matrix defined as follows:

$$r_{nk}^{q_{\widehat{F}}^u} = \begin{cases} \frac{1}{Q_n}\left(\frac{f_k}{f_{k+1}} u_k q_k - \frac{f_{k+2}}{f_{k+1}} u_{k+1} q_{k+1}\right) & , \quad 0 \leq k \leq n-1, \\ \frac{f_n}{f_{n+1}} \frac{q_n u_n}{Q_n} & , \quad k = n, \\ 0 & , \quad k > n. \end{cases}$$

If $y = (y_k)$ is a $R_u^q\widehat{F}$- transform of any given sequence $x = (x_k)$, then it is written as

$$y_k = \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j \widehat{F} x_j. \tag{2.1}$$

Hereafter, when we talk about the sequences $x = (x_k)$ and $y = (y_k)$, we will mean that they are connected with the relation (2.1).

For the sake of simplicity, here and what follows, we shall write

$$\pi_i := \frac{f_{i+1}}{f_i u_i q_i} - \frac{f_{i+1}}{f_{i+2} u_{i+1} q_{i+1}}, \ \varphi_i := \frac{f_i}{f_{i+1}} u_i q_i - \frac{f_{i+2}}{f_{i+1}} u_{i+1} q_{i+1}$$

for every $i \in \mathbb{N}$.

Now, it is time to give the following theorem.

Theorem 2.1. *The set $r^q(\widehat{F}_u^p)$ is a linear space together with coordinatewise addition and scalar multiplication, that is, $r^q(\widehat{F}_u^p)$ is a sequence space.*

Proof. The proof of this theorem is obtained by using elementary calculations of linear algebra. □

Theorem 2.2. *Let $0 < p_k \leq H < \infty$. Then, $r^q(\widehat{F}^p_u)$ is the complete linear metric space with h paronorm defined by the following equality*

$$h_{\widehat{F}}(x) = \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} \varphi_j x_j + \frac{f_k}{f_{k+1}} \frac{u_k q_k}{Q_k} x_k \right|^{p_k} \right]^{\frac{1}{M}}.$$

Proof. According to the definition of paranorm reminded in introduction, it is sufficient to show that the conditions of the paranorm are satisfied. It is easy to see that $h_{\widehat{F}}(\theta) = 0$ for the null element of $r^q(\widehat{F}^p_u)$ and $h_{\widehat{F}}(x) = h_{\widehat{F}}(-x)$ for all $x \in r^q(\widehat{F}^p_u)$.

Now, we shall show the subadditivity of h. By taking $z, x \in r^q(\widehat{F}^p_u)$, we have

$$\begin{aligned}
h_{\widehat{F}}(x+z) &= \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} \varphi_j (x_j + z_j) + \frac{f_k}{f_{k+1}} \frac{u_k q_k}{Q_k} (x_k + z_k) \right|^{p_k} \right]^{\frac{1}{M}} \\
&\leq \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} \varphi_j x_j + \frac{f_k}{f_{k+1}} \frac{u_k q_k}{Q_k} x_k \right|^{p_k} \right]^{\frac{1}{M}} \\
&\quad + \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} \varphi_j z_j + \frac{f_k}{f_{k+1}} \frac{u_k q_k}{Q_k} z_k \right|^{p_k} \right]^{\frac{1}{M}} \\
&= h_{\widehat{F}}(x) + h_{\widehat{F}}(z).
\end{aligned} \tag{2.2}$$

For an arbitrary $\alpha \in \mathbb{R}$ (see [57, p. 30])

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}. \tag{2.3}$$

Again, the inequalities (2.2) and (2.3) are come out by the subadditivity of h and the following inequality clearly holds

$$h_{\widehat{F}}(\alpha x) \leq \max\{1, |\alpha|^M\} h_{\widehat{F}}(x).$$

Finally, we show that the scalar multiplication is continuous. Let α be any complex number and (x^n) be any sequence in $r^q(\widehat{F}^p_u)$ such that $h_{\widehat{F}}(x^n - x) \to 0$. Additionally, let (α_n) be an arbitrary sequence of scalars such that $\alpha_n \to \alpha$, we get

$$\begin{aligned}
h_{\widehat{F}}(\alpha_n x^n - \alpha x) &= \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} \varphi_j (\alpha_n x^n_j - \alpha x_j) \right|^{p_k} \right]^{\frac{1}{M}} \\
&\leq |\alpha_n - \alpha|^{\frac{1}{M}} h_{\widehat{F}}(x^n) + |\alpha|^{\frac{1}{M}} h_{\widehat{F}}(x^n - x),
\end{aligned}$$

tending to zero, for $n \to \infty$, since $\left\{h_{\widehat{F}}(x^n)\right\}$ is bounded due to the inequality

$$h_{\widehat{F}}(x^n) \leq h_{\widehat{F}}(x) + h_{\widehat{F}}(x^n - x).$$

Because of subadditive of $h_{\widehat{F}}$, it is valid. It means that the scalar multiplication is continuous and $h_{\widehat{F}}$ is a paranorm on the space $r^q(\widehat{F}^p_u)$.

Let us suppose that $\{x^i\}$ is an arbitrary Cauchy sequence in the space $r^q(\widehat{F}^p_u)$, where $x^i = \{x^i_0, x^i_1, ...\}$. In that case, there exists a positive integer $n_0(\epsilon)$

$$h_{\widehat{F}}(x^i - x^j) < \infty, \tag{2.4}$$

for all $i, j \geq n_0(\epsilon)$ for a given $\epsilon > 0$. By using definition of $h_{\widehat{F}}$, for each fixed $k \in \mathbb{N}$

$$\left|(R_u^q \widehat{F} x^i)_k - (R_u^q \widehat{F} x^j)_k\right| \leq \left[\sum_k \left|(R_u^q \widehat{F} x^i)_k - (R_u^q \widehat{F} x^j)_k\right|^{p_k}\right]^{\frac{1}{M}} < \infty,$$

for $i, j \geq n_0(\epsilon)$, and $\left\{(R_u^q \widehat{F} x^0)_k, (R_u^q \widehat{F} x^1)_k, ...\right\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges. Therefore, we can write $(R_u^q \widehat{F} x^i)_k \to (R_u^q \widehat{F} x)_k$, for $i \to \infty$. Using these infinitely limits $(R_u^q \widehat{F} x)_0, (R_u^q \widehat{F} x)_1, ...$, we can constitute the sequence $\left\{(R_u^q \widehat{F} x)_0, (R_u^q \widehat{F} x)_1, ...\right\}$. From inequality (2.4) for each $m \in \mathbb{N}$ and $i, j \geq n_0(\epsilon)$, we have

$$\sum_{k=0}^{m} \left|(R_u^q \widehat{F} x^i)_k - (R_u^q \widehat{F} x^j)_k\right|^{p_k} \leq h_{\widehat{F}}(x^i - x^j)^M < \epsilon^M. \tag{2.5}$$

For j and $m \to \infty$ inequality (2.5) becomes

$$h_{\widehat{F}}(x^i - x) < \infty.$$

Taking $\epsilon = 1$, $i \geq n_0(1)$ in inequality (2.5) and using Minkowsky's inequality, for each $m \in \mathbb{N}$, we get

$$\left[\sum_{k=0}^{m} \left|(R_u^q \widehat{F} x)_k\right|^{p_k}\right]^{\frac{1}{M}} \leq h_{\widehat{F}}(x^i - x) + h_{\widehat{F}}(x^i) \leq 1 + h_{\widehat{F}}(x^i),$$

i.e., $x \in r^q(\widehat{F}_u^p)$. Because $h_{\widehat{F}}(x^i - x) \leq \infty$ for all $i \geq n_0(\epsilon)$, $x^i \to x$ as $i \to \infty$, thus it is proved that $r^q(\widehat{F}_u^p)$ is complete. $\square$

It is seen that the absolute property is invalid on the space $r^q(\widehat{F}_u^p)$, in other words $h_{\widehat{F}}(x) \neq h_{\widehat{F}}(|\ x\ |)$ holds for at least one sequence in the space $r^q(\widehat{F}_u^p)$ i.e., $r^q(\widehat{F}_u^p)$ is a sequence space of non-absolute type.

Theorem 2.3. *Let $0 < p_k \leq H < \infty$. Then the sequence space $r^q(\widehat{F}_u^p)$ is linearly isomorphic to the space $\ell(p)$.*

Proof. To prove this theorem's assertion, we firstly have to make sure that there exists a transformation T between the spaces $r^q(\widehat{F}_u^p)$ and $\ell(p)$. Let us take into account the transformation T from $r^q(\widehat{F}_u^p)$ to $\ell(p)$ by $x \to y = Tx$. Since it is obvious to show that T is linear, we omit the details. Now, it is necessary to prove that both T is injective and surjective. If we take $x = \theta$, we obtain that $Tx = \theta$ and this shows that T is injective.

We consider an arbitrary sequence $y \in \ell(p)$ and later define the sequence $x = (x_k)$

$$x_k = \sum_{n=0}^{k-1} \prod_{j=n}^{k-1} \left(\frac{f_{j+2}}{f_{j+1}}\right)^2 \pi_n Q_n y_n + \frac{f_{k+1}}{f_k} \frac{Q_k}{u_k q_k} y_k,$$

for $k \in \mathbb{N}$. If we use the newly defined sequence $x = (x_k)$, then we have

$$\begin{aligned} h_{\widehat{F}}(x) &= \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} \varphi_j x_j + \frac{f_k}{f_{k+1}} \frac{u_k q_k}{Q_k} x_k \right|^{p_k} \right]^{\frac{1}{M}} \\ &= \left[\sum_k \left| \sum_{j=0}^{k} \delta_{kj} y_j \right|^{p_k} \right]^{\frac{1}{M}} \\ &= \left[\sum_k |y_k|^{p_k} \right]^{\frac{1}{M}} \\ &= h_1(y) < \infty \end{aligned}$$

where

$$\delta_{kj} = \begin{cases} 1 & , \quad k = j, \\ 0 & , \quad k \neq j. \end{cases}$$

This shows that $x \in r^q(\widehat{F}_u^p)$. In other words, T is surjective and paranorm preserving. Thus, the transformation T is a linear bijection which means that $r^q(\widehat{F}_u^p)$ and $\ell(p)$ are linearly isomorphic. This completes the proof. □

3. Schauder Basis and $\alpha-, \beta-$ and $\gamma-$ duals of the space $r^q(\widehat{F}_u^p)$

In the present section, firstly, let us recall the definitions of alpha-, beta-, and gamma- dual concepts.

If $\lambda, \mu \subset w$ and z is an arbitrary sequence, we write

$$z^{-1} * \lambda = \{x = (x_k) \in w : xz \in \lambda\},$$

and

$$M(\lambda, \mu) = \cap_{x \in \lambda} x^{-1} * \mu.$$

If we choose $\mu = \ell_1$, cs and bs, then we obtain the $\alpha-, \beta-$ and $\gamma-$ duals of the space λ, respectively as

$$\begin{aligned} \lambda^\alpha &= M(\lambda, \ell_1) = \{a = (a_k) \in w : ax = (a_k x_k) \in \ell_1 \ \text{ for all } x \in \lambda\}, \\ \lambda^\beta &= M(\lambda, cs) = \{a = (a_k) \in w : ax = (a_k x_k) \in cs \ \text{ for all } x \in \lambda\}, \\ \lambda^\gamma &= M(\lambda, bs) = \{a = (a_k) \in w : ax = (a_k x_k) \in bs \ \text{ for all } x \in \lambda\}. \end{aligned}$$

Now, we are going to give the following lemmas necessary to prove the theorems related to the $\alpha-, \beta-$ and $\gamma-$ duals of the space $r^q(\widehat{F}_u^p)$.

Lemma 3.1. [44]

(i) *Let $1 < p_k \leq H < \infty$. Then $A \in (\ell(p) : \ell_1)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} a_{nk} B^{-1} \right|^{p_k'} < \infty.$$

(ii) *Let $0 < p_k \leq 1$. Then $A \in (\ell(p) : \ell_1)$ if and only if*

$$\sup_{K \in \mathcal{F}} \sup_k \left| \sum_{n \in K} a_{nk} \right|^{p_k} < \infty.$$

Lemma 3.2. [45]

(i) *Let $1 < p_k \leq H < \infty$. Then $A \in (\ell(p) : \ell_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_n \sum_k \left| a_{nk} B^{-1} \right|^{p_k'} < \infty. \tag{3.1}$$

(ii) *Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (\ell(p) : \ell_\infty)$ if and only if*

$$\sup_{n,k} |a_{nk}|^{p_k} < \infty. \tag{3.2}$$

Lemma 3.3. [45] *$A \in (\ell(p) : c)$ if and only if there exists an integer $B > 1$ provided that (3.1) and (3.2) hold,*

$$\lim_n a_{nk} = \beta_k \text{ for } k \in \mathbb{N}, \tag{3.3}$$

also holds, where $0 < p_k \leq H < \infty$ for every given $k \in \mathbb{N}$.

Theorem 3.1. *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. The sets $D_1(u,p), D_2(u,p)$ and $D_3(u,p)$ are defined by following equations:*

$$D_1(u,p) = \bigcup_{B>1} \left\{ a = (a_k) \in w : \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \left[\prod_{j=k}^{n-1} \left(\frac{f_{j+2}}{f_{j+1}} \right)^2 \pi_k a_n Q_k + \frac{f_{n+1}}{f_n} \frac{a_n}{u_n q_n} Q_n \right] B^{-1} \right|^{p_k} < \infty \right\},$$

$$D_2(u,p) = \bigcup_{B>1} \left\{ a = (a_k) \in w : \sum_k \left| \left[\left(\frac{f_{k+1}}{f_k} \frac{a_k}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_i \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right) Q_k \right] B^{-1} \right|^{p_k'} < \infty \right\},$$

$$D_3(u,p) = \left\{ a = (a_k) \in w : \sum_{i=k+1}^{\infty} a_i \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \text{ exists} \right\}.$$

In this case,

$$[r^q(B_u^p)]^\alpha = D_1(u,p), \qquad [r^q(B_u^p)]^\beta = D_2(u,p) \cap D_3(u,p), \qquad [r^q(B_u^p)]^\gamma = D_2(u,p).$$

Proof. Let us take any $a = (a_k) \in w$. Then, we obtain

$$\begin{aligned} a_n x_n &= \sum_{k=0}^{n-1} \prod_{j=k}^{n-1} \left(\frac{f_{j+2}}{f_{j+1}} \right)^2 \pi_k a_n Q_k y_k + \frac{f_{n+1}}{f_n} \frac{a_n}{u_n q_n} Q_n y_n \\ &= (Dy)_n, \end{aligned} \tag{3.4}$$

where the matrix $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \prod_{j=k}^{n-1} \left(\frac{f_{j+2}}{f_{j+1}} \right)^2 \pi_k a_n Q_k & , \quad 0 \leq k \leq n-1, \\ \frac{f_{n+1}}{f_n} \frac{a_n}{u_n q_n} Q_n & , \quad k = n, \\ 0 & , \quad k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$. Thus from Eq.(3.4) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_n) \in r^q(F_u^p)$ if and only if $Dy \in \ell_1$ whenever $y \in \ell(p)$. This means that $D \in (\ell(p), \ell_1)$, and Lemma 3.1(ii) gives that $\left[r^q(\widehat{F}_u^p) \right]^\alpha = D_1(u,p)$.

For $\beta-$ dual of space $r^q(F_u^p)$, let us consider following equation,

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n}\left[\left(\frac{f_{k+1}}{f_k}\frac{a_k}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_i \prod_{j=k+1}^{i}\left(\frac{f_{j+1}}{f_j}\right)^2\right) Q_k\right] y_k = (Ey)_n, \tag{3.5}$$

where, $E = (e_{nk})$ is defined as

$$e_{nk} = \begin{cases} \left[\frac{f_{k+1}}{f_k}\frac{a_k}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_i \prod_{j=k+1}^{i}\left(\frac{f_{j+1}}{f_j}\right)^2\right] Q_k & , \quad 0 \le k \le n, \\ 0 & , \quad k > n. \end{cases}$$

From Eq.(3.5), $ax = (a_k x_k) \in cs$ whenever $x \in r^q(\widehat{F}_u^p)$ if and only if $Ey \in c$ whenever $y \in \ell(p)$. In other words, $E \in (\ell(p), c)$. We obtain $\left[r^q(\widehat{F}_u^p)\right]^\beta = D_2(u,p) \cap D_3(u,p)$, using Lemma 3.3.

For $\gamma-$ dual of space $r^q(\widehat{F}_u^p)$, using Eq.(3.5) $ax = (a_k x_k) \in bs$ whenever $x \in r^q(\widehat{F}_u^p)$ iff $Ey \in \ell_\infty$ whenever $y \in \ell(p)$. In other words, $a = (a_k) \in [r^q(\widehat{F}_u^p)]^\gamma$ iff $E \in (\ell(p), \ell_\infty)$. Then from Lemma 3.2 (ii) we obtain $[r^q(\widehat{F}_u^p)]^\gamma = D_2(u,p)$. □

Theorem 3.2. *Let $1 < p_k \le H < \infty$ for every $k \in \mathbb{N}$ and define the sets $D_4(u,p)$ and $D_5(u,p)$ with the following equations:*

$$D_4(u,p) = \left\{ a = (a_k) \in w : \sup_{K \in F} \sup_k \left| \sum_{n \in K} \left[\prod_{j=n+1}^{k} \left(\frac{f_{j+2}}{f_{j+1}}\right)^2 \pi_n a_n Q_n + \frac{f_{k+1}}{f_k}\frac{a_n}{u_k q_k} Q_k \right] \right|^{p_k} < \infty \right\},$$

$$D_5(u,p) = \left\{ a = (a_k) \in w : \sup_k \left| \left(\frac{f_{k+1}}{f_k}\frac{a_k}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_i \prod_{j=k+1}^{i}\left(\frac{f_{j+1}}{f_j}\right)^2 \right) Q_k \right|^{p_k} < \infty \right\}.$$

Then

$$[r^q(F_u^p)]^\alpha = D_4(u,p), \qquad \left[r^q(\widehat{F}_u^p)\right]^\beta = D_3(u,p) \cap D_5(u,p), \qquad \left[r^q(\widehat{F}_u^p)\right]^\gamma = D_5(u,p).$$

Proof. It can be done as that of Theorem 3.1. □

Theorem 3.3. *Let $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Define the sequence $b^{(k)}(q) = \left\{ b_n^{(k)}(q) \right\}$ of the elements of the space $r^q(\widehat{F}_u^p)$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(q) = \begin{cases} \frac{f_{k+1}}{f_k}\frac{Q_k}{u_k q_k} & , \quad n = k, \\ \prod_{j=k+1}^{n}\left(\frac{f_{j+1}}{f_j}\right)^2 \pi_k Q_k & , \quad n > k, \\ 0 & , \quad n < k. \end{cases}$$

Then, the sequence $b^{(k)}(q)$ is a basis for the space $r^q(\widehat{F}_u^p)$ and any $x \in r^q(\widehat{F}_u^p)$ has a unique representation of the form

$$x = \sum_k \lambda_k(q) b^k(q), \tag{3.6}$$

where $\lambda_k(q) = (R_u^q \widehat{F} x)_k$ for all $k \in \mathbb{N}$.

Proof. Let $0 < p_k \le H < \infty$, and for $k \in \mathbb{N}$

$$R_u^q \widehat{F} b^{(k)}(q) = e^{(k)} \in \ell(p), \tag{3.7}$$

where $e^{(k)}$ is a sequence of which k^{th} term is 1 and the others are 0 for each $k \in \mathbb{N}$. Moreover, let $x \in r^q(\widehat{F}_u^p)$. For all non-negative integer m, we get

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) b^{(k)}(q). \tag{3.8}$$

Putting $R_u^q \widehat{F}$ to Eq.(3.8), for $i, m \in \mathbb{N}$, we have

$$R_u^q \widehat{F} x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) R_u^q \widehat{F} b^{(k)}(q) = \sum_{k=0}^{m} (R_u^q \widehat{F} x)_k e^{(k)},$$

and hence

$$\left(R_u^q \widehat{F}(x - x^{[m]})\right)_i = \begin{cases} 0 & , \quad 0 \le i \le m, \\ (R_u^q \widehat{F} x)_i & , \quad i > m. \end{cases}$$

Also, for any given $\epsilon > 0$, there exists an integer m_0 such that for every $m \ge m_0$

$$\left(\sum_{i=m_0}^{\infty} \left| (R_u^q \widehat{F} x)_i \right|^{p_k} \right)^{\frac{1}{M}} < \frac{\epsilon}{2}.$$

Hence, it is obtained that for all $m \ge m_0$

$$\begin{aligned} h_{\widehat{F}}(x - x^m) &= \left(\sum_{i=m}^{\infty} \left| (R_u^q \widehat{F} x)_i \right|^{p_k} \right)^{\frac{1}{M}} \\ &\le \left(\sum_{i=m_0}^{\infty} \left| (R_u^q \widehat{F} x)_i \right|^{p_k} \right)^{\frac{1}{M}} \\ &< \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

By using limit properties, $lim_{m\to\infty} h_{\widehat{F}}(x - x^m) = 0$ is obtained. Thus x is represented as Eq.(3.6).

Let us suppose that it has two representation as $x = \sum_k \mu_k(q) b^{(k)}$ and $x = \sum_k \lambda_k(q) b^{(k)}$. Since the linear transformation from $r^q(\widehat{F}_u^p)$ to $\ell(p)$ is continuous, we get

$$\begin{aligned} (R_u^q \widehat{F} x)_n &= \sum_k \mu_k(q) \left(R_u^q F b^{(k)}(q) \right)_n \\ &= \sum_k \mu_k(q) e_n^{(k)} = \mu_n(q), \end{aligned}$$

for $n \in \mathbb{N}$. Taking $(R_u^q \widehat{F} x)_n = \lambda_n$ for all $n \in \mathbb{N}$, it is obtained $\lambda_n(q) = \mu_n(q)$ thus we get Eq. (3.6). $\square$

4. Matrix Mapping on the Space $r^q(\widehat{F}_u^p)$

In this section, we characterize the matrix class $\left(r^q(\widehat{F}_u^p), \ell_\infty\right)$.

Theorem 4.1.

(i) $A \in \left(r^q(\widehat{F}^p_u), \ell_\infty\right)$ *if and only if there exists an integer $B > 0$ such that*

(4.1)

$$C(B) = \sup_n \sum_k \left| \left[\frac{f_{k+1}}{f_k} \frac{a_{nk}}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_{ni} \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right] Q_k B^{-1} \right|^{p_k'} < \infty,$$

and

$$\{a_{nk}\}_{k\in\mathbb{N}} \in cs \qquad (n \in \mathbb{N}),$$

where $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$.

(ii) $A \in \left(r^q(\widehat{F}^p_u), \ell_\infty\right)$ *if and only if*

$$\sup_{n,k} \left| \left[\frac{f_{k+1}}{f_k} \frac{a_{nk}}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_{ni} \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right] Q_k \right|^{p_k} < \infty, \tag{4.2}$$

and

$$\{a_{nk}\}_{k\in\mathbb{N}} \in cs \qquad (n \in \mathbb{N}),$$

where $0 < p_k \leq 1 < \infty$ for every $k \in \mathbb{N}$.

Proof.

(i) Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$ and $A \in \left(r^q(\widehat{F}^p_u), \ell_\infty\right)$. Then Ax exists for $x \in r^q(\widehat{F}^p_u)$, $\{a_{nk}\}_{k\in\mathbb{N}} \in \left[r^q(\widehat{F}^p_u)\right]^\beta$ for each $n \in \mathbb{N}$. Further, let us consider the following equality obtained by using the relation (3.4) that

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \left[\frac{f_{k+1}}{f_k} \left(\frac{a_{nk}}{u_k q_k} + \pi_k \sum_{i=k+1}^{m} a_{nj} \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right) Q_k \right] y_k. \tag{4.3}$$

From Lemma 3.1 and Eq.(4.3), we obtain the expression.

Conversely, $\{a_{nk}\}_{k\in\mathbb{N}} \in cs$ for each $n \in \mathbb{N}$, $x \in r^q(\widehat{F}^p_u)$. Since $\{a_{nk}\}_{k\in\mathbb{N}} \in \left[r^q(\widehat{F}^p_u)\right]^\beta$ for every fixed $n \in \mathbb{N}$ A-transform of x exists. We derive from Eq.(4.3) as $m \to \infty$ that

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left[\left(\frac{f_{k+1}}{f_k} \frac{a_{nk}}{u_k q_k} + \pi_k \sum_{i=k+1}^{\infty} a_{ni} \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right) Q_k \right] y_k. \tag{4.4}$$

Now, by combining Eq.(4.4) and inequality holding for an arbitrary $B > 0$ and complex numbers a, b

$$|ab| \leq B \left\{ |aB^{-1}|^{p'} + |b|^p \right\},$$

where $p > 1$ and $1/p + 1/p' = 1$. We obtain

$$\begin{aligned} \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{\infty} a_{nk} x_k \right| &\leq \sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} \left| \left[\left(\frac{f_{k+1}}{f_k} \frac{a_{nk}}{u_k q_k} + \pi_k \sum_{i=k+1}^{\infty} a_{ni} \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right) Q_k \right] \right| |y_k| \\ &\leq B[C(B) + h_1^M(y)] < \infty. \end{aligned}$$

This mean that $Ax \in \ell_\infty$ whenever $x \in r^q(\widehat{F}^p_u)$.

(ii) The proof of (ii) can be obtained same way. $\square$

References

[1] B. Altay, *On the space of $p-$summable difference sequences of order m, $(1 \leq p < \infty)$*, Stud. Sci. Math. Hungar., **43**(4)(2006), 387–402.

[2] B. Altay, F. Başar, *On the paranormed Riesz sequence spaces of non-absolute type*, Southeast Asian Bull. Math., **26**(2002), 701–715.

[3] B. Altay, F. Başar, *Some paranormed sequence spaces of non-absolute type derived by weighted mean*, J. Math. Anal. Appl., **319**(2)(2006), 494–508.

[4] B. Altay, F. Başar, *Generalization of the sequence space $\ell(p)$ derived by weighted mean*, J. Math. Anal. Appl., **330**(2007), 174–185.

[5] B. Altay, F. Başar, *The matrix domain and the fine spectrum of the difference operator Δ on the sequence space ℓ_p, $(0 < p < 1)$*, Commun. Math. Anal., **2**(2)(2007), 1–11.

[6] B. Altay, F. Başar, *On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence c_0 and c*, Int. J. Math. Sci., **18**(2008), 3005–3013.

[7] B. Altay, F. Başar, M. Mursaleen, *On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I*, Inform. Sci., **176**(10)(2006), 1450–1462.

[8] C. Aydın, F. Başar, *Some new sequence spaces which include the spaces ℓ_p and ℓ_∞*, Demonstratio Math., **38**(3)(2005), 641-656.

[9] C. Aydın, F. Başar, *Some generalizations of the sequence spaces a_p^r*, Iran J. Sci. Technol. Trans. A. Sci., **30A**(2)(2006), 175–190.

[10] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, e-books, Monographs, xi+405 pp., İstanbul, (2012), ISB:978-1-60805-252-3.

[11] F. Başar, B. Altay, *On the space of sequences of $p-$bounded variation and related matrix mappings*, Ukrainian Math.J., **55**(1)(2003), 136–147.

[12] F. Başar, M. Kirişçi, *Almost convergence and generalized difference matrix*, Comput. Math. Appl., **61**(3) (2011), 602–611.

[13] M. Başarır, *Paranormed Cesàro difference sequence space and related matrix transformation*, Doğa Tr. J. Math., **15**(1991), 14–19.

[14] M. Başarır, *On the generalized Riesz B-difference sequence spaces*, Filomat., **24**(4)(2010), 35–52.

[15] M. Başarır, F. Başar, E. E. Kara, *On the Fibonacci Difference Null and Convergent Sequences*, arXiv:1309.0150.

[16] M. Başarır, E. E. Kara, *On some difference sequence spaces of weighted means and compact operators*, Ann. Funct. Anal., **2**(2)(2011), 116–131.

[17] M. Başarır, E. E. Kara, *On compact operators on the Riesz B^m-difference sequence space*, Iran J. Sci. Technol. Trans., **35A**(4)(2011), 279–285.

[18] M. Başarır, E. E. Kara, *On compact operators on the Riesz B^m-difference sequence space-II*, Iran J. Sci. Technol. Trans., **36A**(3)(2012), 371–376.

[19] M. Başarır, E. E. Kara, *On the $B-$difference sequence space derived by generalized weighted mean and compact operators*, J. Math. Anal. Appl., **391**(2012), 67–81.

[20] M. Başarır, E. E. Kara, *On the m^{th} order difference sequence space of generalized weighted mean and compact operator*, Acta. Math. Sci., **33B**(3)(2013), 1–18.

[21] M. Başarır, M. Kayıkçı, *On the generalized $B^{th}-$Riesz difference sequence space and beta-property*, J. Inequal. Appl., ID 385029, (2009), 18pp.

[22] M. Başarır, M. Öztürk, *On the Riesz diference sequence space*, Rend. Circ. Mat. Palermo., **57**(2008), 377–389.

[23] M. Başarır, M. Öztürk, *On some Generalized B^m-difference Riesz Sequence Spaces and Uniform Opial Property*, J. Inequal. Appl., ID 485730 (2011), 17 pp.

[24] M. Candan, *Some new sequence spaces defined by a modulus function and an infinite matrix in a seminormed space*, J. Math. Anal., **3**(2) (2012), 1–9.

[25] M. Candan, *Domain of the double sequential band matrix in the classical sequence spaces*, J. Inequal. Appl., **281**(2012), 15 pp.

[26] M. Candan, *Almost convergence and double sequential band matrix*, Acta. Math. Sci., **34B**(2)(2014), 354–366.

[27] M. Candan, *A new sequence space isomorphic to the space $\ell(p)$ and compact operators*, J. Math. Comput. Sci., **4**, No: 2(2014), 306–334.

[28] M. Candan, *Domain of the double sequential band matrix in the spaces of convergent and null sequences*, Adv. Difference Edu., (2014)163, 18 pp.

[29] M. Candan, *Some new sequence spaces derived from the spaces of bounded, convergent and null sequences*, Int. J. Mod. Math. Sci., **12**(2)(2014), 74-87.

[30] M. Candan, *Vector-Valued FK-spaces defined by a modulus function and an infinite matrix*, Thai J. Math., **12**(1)(2014),155-165.

[31] M. Candan, *A new aproach on the spaces of generalized Fibonacci difference null and convergent sequences*, Math. Æterna., **1**(5)(2015), 191–210.

[32] M. Candan, A. Güneş, *Paranormed sequence space of non-absolute type founded using generalized difference matrix*, Proc. Natl. Acad. Sci., India Sect. A Phys. Sci., **85**(2)(2015), 269–276.

[33] M. Candan, E. E. Kara, *A study on topological and geometrical characteristics of new Banach sequence spaces*, Gulf J. of Math., 3(4)(2015), 67-84.

[34] M. Candan, K. Kayaduman, *Almost convergent sequence space derived by generalized Fibonacci matrix and Fibonacci core*, Brithish J. Math. Comput. Sci., **7**(2)(2015), 150–167.

[35] M. Candan, İ. Solak, *On some Difference Sequence Spaces Generated by Infinite Matrices*, Int. J. Pure Appl. Math., **25**(1)(2005), 79–85.

[36] M. Candan, İ. Solak, *On New Difference Sequence Spaces Generated by Infinite Matrices*, Int. J. Sci. and Tecnology., **1**(1)(2006), 15–17.

[37] B. Choudhary, S. K Mishra, *On Köthe-Toeplitz duals of certain sequence spaces and their matrix transformations*, Indian J. Pure Appl. Math., **24**5(1993), 291–301.

[38] R. Çolak, M. Et, Malkowsky E, *Some Topics of Sequence Spaces, Lecture Notes in Mathematics*, Fırat Univ, Elazığ, Turkey,(2004), pp. 1–63, Fırat Univ, Press, ISBN: 975-394-038-6.

[39] R. Çolak, M. Et, *On some generalized difference sequence spaces and related matrix transformations*, Hokkaido Math. J., **26**(3)(1997), 483–492.

[40] S. Demiriz, C. Çakan, *Some topolojical and geometrical properties of a new difference sequence space*, Abstr. Appl. Anal., doi:10.1155/2011/213878, 14 pp.

[41] M. Et, *Generalized Cesàro difference sequence spaces of non-absolute type involving lacunary sequence spaces*, Appl. Math. Comput., **219**(17)(2013), 9372–9376.

[42] M. Et, M. Işık, *On pa-dual spaces of generalized difference sequence spaces*, Appl. Math. Lett., **25**(10)(2012), 1486–1489.

[43] A. H. Ganie, N. A. Sheikh, *New type of paranormed sequence space of non-absolute type and a matrix transformation*, Int, J of Mod, Math, Sci., **8**(2)(2013), 196–211.

[44] K. Goswin, G. Erdmann, *Matrix transformations between the sequence spaces of Maddox*, J. Math. Anal. Appl., **180**(1993), 223–238.

[45] C. G. Lascarides, I. J. Maddox, *Matrix transformations between some classes of sequences*, Proc. Cambridge Philos. Soc., **68**(1970), 99–104.

[46] E. E. Kara, *Some topological and geometrical properties of new Banach sequence spaces*, J. Inequal. Appl., **38**(2013).

[47] E. E. Kara, M. Başarır, M. Mursaleen, *Compact operators on the Fibonacci difference sequence spaces* $l_p(\widehat{F})$ *and* $l_\infty(\widehat{F})$, 1st International Eurasian Conf. on Math.Sci.and Appl. Prishtine-Kosovo, (2012), September 3-7.

[48] E. E. Kara, M. Öztürk, M. Başarır, *Some topological and geometric properties of generalized Euler sequence spaces*, Math., Slovaca, **60**(3)(2010), 385–398.

[49] H. Kızmaz, *On certain sequence spaces*, Canad. Math. Bull., **24**(2)(1981), 169–176.

[50] M. Kirişçi, *Almost convergence and generalized weighted mean I*, AIP Conf. Proc. vol, **1470**(2012), pp. 191–194.

[51] M. Kirişçi, *On the spaces of Euler almost null and Euler almost convergent sequences*, Commun. Fac. Sci. Univ., Ankara, **2**(2013), 85–100.

[52] M. Kirişçi, *Almost convergence and generalized weighted mean II*, J. Inequal. Appl., ID 193,(2014), 13pp.

[53] M. Kirişçi, F. Başar, *Some new sequence spaces derived by the domain of generalized difference matrix*, Comput. Math. Appl., **60**(5)(2010), 1299–1309.

[54] Ş. Konca, M. Başarır, *Generalized difference sequence spaces associated with a multiplier sequence on a real* $n-$*normed space*, J. Inequal. Appl., ID 335(2013), 12 pp.

[55] Ş. Konca, M. Başarır, *On some spaces of almost lacunary convergent sequences derived by Riesz mean and weighted almost lacunary statistical convergence in a real* $n-$*normedspace*, J, Inequal. Appl., ID 81(2014), 11 pp.

[56] I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math., Oxford, **18**(2)(1967), 345–355.

[57] I. J. Maddox, *Paranormed sequence spaces generated by infinite matrices*, Proc. Cambridge Philos. Soc., **64**(1968), 335–340.
[58] E. Malkowsky, S. D. Parashar, *Matrix transformations in space of bounded and convergent difference sequence of order m*, Analysis., **17**(1997), 87–97.
[59] M. Malkowsky, V. Rakočević , S. Źivković, *Matrix transformations between the sequence space bv^p and certain BK spaces*, Bull. Cl. Sci. Math. Nat. Sci. Math., **27**(2002), 33–46.
[60] E. Malkowsky, E. Savaş, *Matrix transformations between sequence spaces of generalized weighted means*, Appl. Math. Comput., **147**(2004), 333–345.
[61] M. Mursaleen, F. Başar, B. Altay, *On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ II*, Nonlinear Anal., **65**(3)(2006), 707–717.
[62] M. Mursaleen, A. K. Noman, *On some new sequence spaces of non-absolute type related to the spaces ℓ_p and $\ell_1 I$*, Filomat, **25**(2)(2011), 33–51.
[63] H. Nakano, *Modulared sequence spaces*, Proc. Japan Acad., **27**(2)(1951), 508–512.
[64] P. N. Ng, P. Y. Lee, *Cesàro sequence spaces of non-absolute type*, Comment Math. Prace Mat., **20**, no.2(1978), 429-433.
[65] H. Polat, F. Başar, *Some Euler spaces of difference sequences of order m*, Acta Math. Sci., **27B**(2)(2007), 254–266.
[66] H. Polat, V. Karakaya, N. Şimşek, *Difference sequence spaces derived by generalized weighted mean*, Appl. Math. Lett., **24**(5)(2011), 608–314.
[67] N. A. Sheikh, A. H. Ganie, *A new paranormed sequence space and some matrix transformations*, Acta Math. Acad. Paedago, Nyregy., **28**(2012), 47–58.
[68] S. Simons, *The sequence spaces $\ell(p_v)$ and $m(p_v)$*, Proc. London Math. Soc., **15**(3)(1965), 422–436.
[69] Y. Yılmaz, M. K. Özdemir, İ. Solak, M. Candan, *Operators on some vector-valued Orlicz sequence spaces*, F.Ü. Fen ve Mühendislik Dergisi., **17**(1)(2005), 59–71.

Department of Mathematics, Faculty of Arts and Sciences, İnönü University, The University Campus, 44280-Malatya/TURKEY
E-mail address: murat.candan@inonu.edu.tr

Department of Primary Mathematics, Faculty of Education Adiyaman University, The University Campus, 02040-Adiyaman/TURKEY
E-mail address: gkilinc@adiyaman.edu.tr

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 77–88 (2015) ©KJM

ON SOME ČEBYŠEV TYPE INEQUALITIES FOR FUNCTIONS WHOSE SECOND DERIVATIVES ARE (h_1, h_2)-CONVEX ON THE CO-ORDINATES

B. MEFTAH AND K. BOUKERRIOUA*

Abstract. The aim of this paper is to establish some new Čebyšev type inequalities involving functions whose mixed partial derivatives are (h_1, h_2)-convex on the co-ordinates.

1. Introduction

In 1882, Čebyšev [4] gave the following inequality :

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty \tag{1.1}$$

where $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous functions, whose first derivatives f' and g' are bounded,

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x) g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.2}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|f\|_\infty = ess\sup_{t\in[a,b]} |f(t)|$.

During the past few years many researchers have given considerable attention to the inequality (1.1), various generalizations, extensions and variants of this inequality have appeared in the literature, see $[1,3,6,8,9,10]$. Recently, Guezane-Lakoud and Aissaoui [6] established new Čebyšev type inequalities similar to (1.1) for functions f, g defined on bidimensional intervals $\Delta = [a,b] \times [c,d] \subset [0,\infty)^2$ whose mixed partial derivatives f_{st} and g_{st} are integrable and bounded. The authors of the paper [12] further extend these results in special cases when the mixed partial derivatives belong to certain classes of functions that generalize convex function on the co-ordinates.

2000 *Mathematics Subject Classification.* 26D15, 26D20, 39A12.

Key words and phrases. Čebyšev type inequalities, co-ordinates (h_1, h_2)-convex, integral inequality.

This work has been supported by CNEPRU–MESRS–B01120120103 project grants.

The main purpose of this work is to obtain new Čebyšev type inequalities involving functions whose mixed partial derivatives are (h_1, h_2)-convex on the co-ordinates.

2. Preliminaries

Throughout this paper we denote by Δ the bidimensional interval in $[0,\infty)^2$, $\Delta =: [a,b] \times [c,d]$ with $a < b$ and $c < d$, $k = (b-a)(d-c)$ and $f_{\lambda\alpha}$ for $\frac{\partial^2 f}{\partial\lambda\partial\alpha}$.

Definition 2.1 ([5]). A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ, if the following inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)v) &\leq \lambda\alpha f(x,y) + \lambda(1-\alpha) f(x,v) \\ &\quad + (1-\lambda)\alpha f(t,y) + (1-\lambda)(1-\alpha) f(t,v) \end{aligned} \tag{2.1}$$

holds for all $\lambda, \alpha \in [0,1]$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex.

Definition 2.2 ([2]). A function $f : \Delta \to \mathbb{R}$ is said to be s-convex in the second sense on the co-ordinates on Δ, if the following inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)v) &\leq \lambda^s\alpha^s f(x,y) + \lambda^s (1-\alpha)^s f(x,v) \\ &\quad + (1-\lambda)^s \alpha^s f(t,y) + (1-\lambda)^s (1-\alpha)^s f(t,v) \end{aligned} \tag{2.2}$$

holds for all $\lambda, \alpha \in [0,1]$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$,
for some fixed $s \in (0,1]$.

s-convexity on the co-ordinates does not imply the s-convexity, that is there exist functions which are s-convex on the co-ordinates but are not s-convex.

Definition 2.3 ([7]). Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. A mapping $f : \Delta \to \mathbb{R}$ is said to be h-convex on Δ, if the following inequality

$$f(\alpha x + (1-\alpha)t, \alpha y + (1-\alpha)v) \leq h(\alpha) f(x,y) + h(1-\alpha) f(t,v) \tag{2.3}$$

holds, for all $(x,y), (t,v) \in \Delta$ and $\alpha \in (0,1)$.

Definition 2.4 ([7]). A function $f : \Delta \to \mathbb{R}$ is said to be (h_1, h_2)-convex on the coordinates on Δ, if the following inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)v) &\leq h_1(\lambda)h_2(\alpha) f(x,y) + h_1(\lambda)h_2(1-\alpha) f(x,v) \\ &\quad + h_1(1-\lambda) h_2(\alpha) f(t,y) \\ &\quad + h_1(1-\lambda) h_2(1-\alpha) f(t,v) \end{aligned} \tag{2.4}$$

holds for all $\lambda, \alpha \in]0,1[$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$.

h-convexity on the co-ordinates does not imply the h-convexity, that is there exist functions which are h-convex on the co-ordinates but are not h-convex.

Lemma 2.1 (Lemma 1. [11]). *Let $f : \Delta \to \mathbb{R}$ be a partial differentiable mapping on Δ in $\mathbb{R}^2$. If $f_{\lambda\alpha} \in L_1(\Delta)$, then for any*

$(x,y)\in\Delta$, we have the equality:

$$
\begin{aligned}
f(x,y) &= \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{d-c}\int_c^d f(x,v)dv - \frac{1}{k}\int_a^b\int_c^d f(t,v)dvdt \\
&\quad + \frac{1}{k}\int_a^b\int_c^d (x-t)(y-v) \\
&\quad \times \left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt \qquad (2.5)
\end{aligned}
$$

3. Main result

Theorem 3.1. *Let $h_i : J_i \subseteq \mathbb{R} \to \mathbb{R}$ be positive functions, for $i = 1,2$. $f,g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (h_1,h_2)-convex on the co-ordinates, then we have*

$$
|T(f,g)| \le \frac{49}{3600}k^2\left(\int_0^1 h_1(\lambda)d\lambda\right)^2\left(\int_0^1 h_2(\alpha)d\alpha\right)^2 MN \qquad (3.1)
$$

where

$$
\begin{aligned}
T(f,g) &= \frac{1}{k}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx - \frac{(d-c)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_a^b f(t,y)\,dt\right)dydx \\
&\quad - \frac{(b-a)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_c^d f(x,v)\,dv\right)dydx \\
&\quad + \frac{1}{k^2}\left(\int_a^b\int_c^d f(x,y)\,dydx\right)\left(\int_a^b\int_c^d g(t,v)\,dvdt\right) \qquad (3.2)
\end{aligned}
$$

$$M = \underset{x,t\in[a,b],y,v\in[c,d]}{ess\sup}\left[|f_{\lambda\alpha}(x,y)| + |f_{\lambda\alpha}(x,v)| + |f_{\lambda\alpha}(t,y)| + |f_{\lambda\alpha}(t,v)|\right],$$

$$N = \underset{x,t\in[a,b],y,v\in[c,d]}{ess\sup}\left[|g_{\lambda\alpha}(x,y)| + |g_{\lambda\alpha}(x,v)| + |g_{\lambda\alpha}(t,y)| + |g_{\lambda\alpha}(t,v)|\right]$$

and $k = (b-a)(d-c)$.

Proof. Let F, G, $\widetilde{F}$ and $\widetilde{G}$ be defined as follows

$$F = f(x,y) - \frac{1}{b-a}\int_a^b f(t,y)dt - \frac{1}{d-c}\int_c^d f(x,v)dv + \frac{1}{k}\int_a^b\int_c^d f(t,v)dvdt$$

$$G = g(x,y) - \frac{1}{b-a}\int_a^b g(t,y)dt - \frac{1}{d-c}\int_c^d g(x,v)dv + \frac{1}{k}\int_a^b\int_c^d g(t,v)dvdt$$

$$\widetilde{F} = \frac{1}{k}\int_a^b\int_c^d (x-t)(y-v)\times\left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt$$

$$\widetilde{G}=\frac{1}{k}\int_a^b\int_c^d (x-t)(y-v)\times\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt.$$

By Lemma 2.1, we have

$$F=\widetilde{F}\text{ and }G=\widetilde{G},$$

then

$$FG=\widetilde{F}\widetilde{G}. \tag{3.3}$$

Integrating (3.3) over Δ, with respect to x,y, multiplying the resultant equality by $\frac{1}{k}$, using Fubini's Theoerm and modulus, we get

$$\begin{aligned}
|T(f,g)| &= \frac{1}{k^3}\left|\int_a^b\int_c^d\left[\int_a^b\int_c^d (x-t)(y-v)\right.\right.\\
&\quad\times\left.\left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt\right]\\
&\quad\times\left[\int_a^b\int_c^d (x-t)(y-v)\right.\\
&\quad\times\left.\left.\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt\right]dydx\right|\\
&\le \frac{1}{k^3}\int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-v|\right.\\
&\quad\times\left.\left(\int_0^1\int_0^1 \left|f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v\right)\right|d\alpha d\lambda\right)dvdt\right]\\
&\quad\times\left[\int_a^b\int_c^d |x-t|\,|y-v|\right.\\
&\quad\times\left.\left(\int_0^1\int_0^1 \left|g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v\right)\right|d\alpha d\lambda\right)dvdt\right]dydx. \qquad (3.4)
\end{aligned}$$

Using the (h_1,h_2)-convexity and taking into account that

$$\begin{aligned}
\int_a^b\left(\int_a^b |x-t|\,dt\right)^2 dx &= \frac{7}{60}(b-a)^5,\\
\int_c^d\left(\int_c^d |y-v|\,dv\right)^2 dy &= \frac{7}{60}(d-c)^5,
\end{aligned}$$

$$\int_0^1 h_1(1-\lambda)d\lambda = \int_0^1 h_1(\ \lambda)d\lambda \ \text{ and } \ \int_0^1 h_2(1-\alpha)d\alpha = \int_0^1 h_2(\ \alpha)d\alpha,$$

we obtain

$$\begin{aligned}
|T(f,g)| &\leq \frac{1}{k^3}\left(\int_0^1 h_1(\lambda)d\lambda\right)^2\left(\int_0^1 h_2(\alpha)d\alpha\right)^2 \\
&\quad\times\int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-v|\times[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,v)|\right. \\
&\quad\left.+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,v)|]\,dvdt\right. \\
&\quad\times\left[\int_a^b\int_c^d |x-t|\,|y-v|\times[|g_{\lambda\alpha}(x,y)|+|g_{\lambda\alpha}(x,v)|\right. \\
&\quad+|g_{\lambda\alpha}(t,y)|+|g_{\lambda\alpha}(t,v)|]\,dvdt]\,dydx \\
&\leq \frac{MN}{k^3}\left(\int_0^1 h_1(\lambda)d\lambda\right)^2\left(\int_0^1 h_2(\alpha)d\alpha\right)^2 \\
&\quad\times\int_a^b\int_c^d\left(\int_a^b\int_c^d |x-t|\,|y-v|\,dvdt\right)^2 dydx \\
&= \frac{MN}{k^3}\left(\int_0^1 h_1(\lambda)d\lambda\right)^2\left(\int_0^1 h_2(\alpha)d\alpha\right)^2 \\
&\quad\times\left[\int_a^b\left(\int_a^b |x-t|\ dt\right)^2 dx\right]\left[\int_c^d\left(\int_c^d |y-v|\,dv\right)^2 dy\right] \\
&= \frac{49}{3600}k^2\left(\int_0^1 h_1(\lambda)d\lambda\right)^2\left(\int_0^1 h_2(\alpha)d\alpha\right)^2 MN.
\end{aligned}$$

This completes the proof of Theorem 3.1. □

Corollary 3.1. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be positive function, $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are h-convex on the co-ordinates, then we have*

$$|T(f,g)| \leq \frac{49}{3600}k^2\left(\int_0^1 h(\lambda)d\lambda\right)^4 MN, \tag{3.5}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1.

Proof. Applying Theorem 3.1, for $h_1(v) = h_2(v) = h(v)$, we obtain the desired inequality. □

Corollary 3.2. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are convex on the co-ordinates, then we have*

$$|T(f,g)| \leq \frac{49}{57600} k^2 MN, \tag{3.6}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1.

Proof. In Theorem 3.1, if we replace h_1 and h_2 by the identity, we obtain

$$\begin{aligned}
|T(f,g)| &\leq \frac{49}{3600} k^2 \left(\int_0^1 \lambda d\lambda \right)^2 \left(\int_0^1 \alpha d\alpha \right)^2 MN \\
&= \frac{49}{3600} k^2 \left(\left. \frac{\lambda^2}{2} \right|_{\lambda=0}^{\lambda=1} \right)^2 \left(\left. \frac{\alpha^2}{2} \right|_{\alpha=0}^{\alpha=1} \right)^2 MN \\
&= \frac{49}{3600} k^2 \times \frac{1}{4} \times \frac{1}{4} MN \\
&= \frac{49}{57600} k^2 MN.
\end{aligned}$$

This is the desired inequality in (3.6). The proof is completed. □

Remark 3.1. The result of Corollary 3.2 is similar to the inequality (6) of Theorem 2.1 in [12].

Corollary 3.3. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (s_1, s_2)-convex in the second sense on the co-ordinates, then*

$$|T(f,g)| \leq \frac{49}{3600} k^2 \frac{1}{(1+s_1)^2} \frac{1}{(1+s_2)^2} MN, \tag{3.7}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1 and $s_1, s_2 \in (0,1]$.

Proof. Taking in Theorem 3.1, $h_1(\lambda) = \lambda^{s_1}$ and $h_2(\alpha) = \alpha^{s_2}$, we obtain

$$\begin{aligned}
|T(f,g)| &\leq \frac{49}{3600} k^2 \left(\int_0^1 \lambda^{s_1} d\lambda \right)^2 \left(\int_0^1 \alpha^{s_2} d\alpha \right)^2 MN \\
&= \frac{49}{3600} k^2 \frac{1}{(1+s_1)^2} \frac{1}{(1+s_2)^2} MN.
\end{aligned}$$

This is the desired inequality in (3.7). The proof is completed. □

Corollary 3.4. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are s-convex in the second sense on the co-ordinates, then*

$$|T(f,g)| \leq \frac{49}{3600} k^2 \frac{1}{(1+s)^4} MN, \tag{3.8}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1 and $s \in (0,1]$.

Proof. Putting in Theorem 3.1, $h_1(\lambda) = \lambda^s$ and $h_2(\alpha) = \alpha^s$, we get

$$\begin{aligned} |T(f,g)| &\leq \frac{49}{3600} k^2 \left(\int_0^1 \lambda^s d\lambda \right)^2 \left(\int_0^1 \alpha^s d\alpha \right)^2 MN \\ &= \frac{49}{3600} k^2 \, \frac{1}{(1+s)^4} \; MN. \end{aligned} \tag{3.9}$$

This is the required inequality in (3.8). The proof is completed. □

Theorem 3.2. *Let $h_i : J_i \subseteq \mathbb{R} \to \mathbb{R}$ be positive functions, for $i = 1, 2$, $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (h_1, h_2)-convex on the co-ordinates, then we have*

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{8k^2} \left(\int_0^1 h_1(\lambda) d\lambda \right) \left(\int_0^1 h_2(\alpha) d\alpha \right) \\ &\quad \times \int_a^b \int_c^d \left[M \left| g(x,y) \right| + N \left| f(x,y) \right| \right] \\ &\quad \times \left((x-a)^2 + (b-x)^2 \right) \left((y-c)^2 + (d-y)^2 \right) dy dx. \end{aligned} \tag{3.10}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1.

Proof. By Lemma 2.1, we have

$$\begin{aligned} f(x,y) &= \frac{1}{b-a} \int_a^b f(t,y) dt + \frac{1}{d-c} \int_c^d f(x,s) dv - \frac{1}{k} \int_a^b \int_c^d f(t,v) dv dt \\ &\quad + \frac{1}{k} \int_a^b \int_c^d (x-t)(y-v) \\ &\quad \times \left(\int_0^1 \int_0^1 f_{\lambda\alpha} \left(\lambda x + (1-\lambda)t, \alpha y - (1-\alpha)v \right) d\alpha d\lambda \right) dv dt, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} g(x,y) &= \frac{1}{b-a}\int_a^b g(t,y)dt + \frac{1}{d-c}\int_c^d g(x,v)ds - \frac{1}{k}\int_a^b\int_c^d g(t,v)dvdt \\ &+\frac{1}{k}\int_a^b\int_c^d (x-t)(y-v) \\ &\times\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt. \end{aligned} \tag{3.12}$$

Multiplying (3.11) by $\frac{1}{2k}g(x,y)$ and (3.12) by $\frac{1}{2k}f(x,y)$, summing the resultant equalities, then integrating on Δ, we get

$$\begin{aligned} T(f,g) &= \frac{1}{2k^2}\left[\int_a^b\int_c^d g(x,y)\left[\int_a^b\int_c^d (x-t)(y-v)\right.\right. \\ &\times\left.\left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt\right]dydx \\ &+\int_a^b\int_c^d f(x,y)\left[\int_a^b\int_c^d (x-t)(y-v)\right. \\ &\times\left.\left.\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt\right]dydx\right], \end{aligned} \tag{3.13}$$

using the properties of modulus, (3.13) becomes

$$\begin{aligned} |T(f,g)| &\le \frac{1}{2k^2}\left[\int_a^b\int_c^d |g(x,y)|\left[\int_a^b\int_c^d |x-t|\,|y-v|\right.\right. \\ &\times\left.\left(\int_0^1\int_0^1 \left|f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)\right|d\alpha d\lambda\right)dvdt\right]dydx \\ &+\int_a^b\int_c^d |f(x,y)|\left[\int_a^b\int_c^d |x-t|\,|y-v|\right. \\ &\times\left.\left.\left(\int_0^1\int_0^1 \left|g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)\right|d\alpha d\lambda\right)dvdt\right]dydx\right]. \end{aligned} \tag{3.14}$$

Using the (h_1,h_2)-convexity, (3.14) gives

$$
\begin{aligned}
|T(f,g)| \leq\ & \frac{1}{2k^2}\left[\int_a^b\int_c^d |g(x,y)| \left(\int_0^1 h_1(\lambda)d\lambda\right)\left(\int_0^1 h_2(\alpha)d\alpha\right)\right. \\
& \times\left[\int_a^b\int_c^d |x-t|\,|y-v|\,[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,v)|\right. \\
& \left.+\,|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,v)|]\,dvdt\ \right]dydx \\
& +\int_a^b\int_c^d |f(x,y)| \left(\int_0^1 h_1(\lambda)d\lambda\right)\left(\int_0^1 h_2(\alpha)d\alpha\right) \\
& \times\left[\int_a^b\int_c^d |x-t|\,|y-v|\,[|g_{\lambda\alpha}(x,y)|+|g_{\lambda\alpha}(x,v)|\right. \\
& \left.\left.+\,|g_{\lambda\alpha}(t,y)|+|g_{\lambda\alpha}(t,v)|]\,dvdt\right]dydx\right],
\end{aligned}
\tag{3.15}
$$

By a simple calculation we get

$$
\begin{aligned}
|T(f,g)| &\leq \frac{1}{2k^2}\left(\int_0^1 h_1(\lambda)d\lambda\right)\left(\int_0^1 h_2(\alpha)d\alpha\right) \\
&\quad\times\int_a^b\int_c^d\left[M\,|g(x,y)|\left(\int_a^b\int_c^d |x-t|\,|y-v|\,dvdt\right)\right. \\
&\quad\left.+N\,|f(x,y)|\left(\int_a^b\int_c^d |x-t|\,|y-v|\,dvdt\right)\right]dydx \\
&= \frac{1}{8k^2}\left(\int_0^1 h_1(\lambda)d\lambda\right)\left(\int_0^1 h_2(\alpha)d\alpha\right) \\
&\quad\times\int_a^b\int_c^d [M\,|g(x,y)|+N\,|f(x,y)|] \\
&\quad\times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)dydx.
\end{aligned}
\tag{3.16}
$$

This completes the proof of Theorem 3.2. □

Corollary 3.5. *Let $h: J\subseteq\mathbb{R}\to\mathbb{R}$ be positive function, $f,g:\Delta\ \to\mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable* *on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are h-convex on the co-ordinates, then we have*

$$\begin{aligned}
|T(f,g)| &\leq \frac{1}{8k^2}\left(\int_0^1 h(\lambda)d\lambda\right)^2 \int_a^b\int_c^d \left[(M\,|g(x,y)| + N\,|f(x,y)|)\right. \\
&\quad \times \left.\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)\right] dydx.
\end{aligned}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1.

Proof. Applying Theorem 3.2, for $h_1(\lambda) = h_2(\lambda)$, we obtain the desired inequality. □

Corollary 3.6. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are convex on the co-ordinates, then we have*

$$\begin{aligned}
|T(f,g)| &\leq \frac{1}{32k^2}\int_a^b\int_c^d [M\,|g(x,y)| + N\,|f(x,y)|] \\
&\quad \times \left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right) dydx.
\end{aligned} \tag{3.17}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1.

Proof. In Theorem 3.2, if we replace h_1 and h_2 by the identity, we obtain

$$\begin{aligned}
|T(f,g)| &\leq \frac{1}{8k^2}\left(\int_0^1 \lambda d\lambda\right)\left(\int_0^1 \alpha d\alpha\right) \\
&\quad \times \int_a^b\int_c^d [M\,|g(x,y)| + N\,|f(x,y)|] \\
&\quad \times \left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right) dydx. \\
&= \frac{1}{32k^2}\int_a^b\int_c^d [M\,|g(x,y)| + N\,|f(x,y)|] \\
&\quad \times \left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right) dydx.
\end{aligned}$$

This is the desired inequality in (3.17). The proof is completed. □

Remark 3.2. The result of Corollary 3.6, is similar to the inequality (7) of Theorem 2.1 in [12].

Corollary 3.7. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are*

(s_1,s_2)-convex in the second sense on the co-ordinates, then we have

$$\begin{aligned}|T(f,g)| &\leq \frac{1}{8k^2(1+s_1)(1+s_2)} \\ &\times\int_a^b\int_c^d [M|g(x,y)|+N|f(x,y)|] \\ &\times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)dydx, \end{aligned} \tag{3.18}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1 and $s_1,s_2\in(0,1]$.

Proof. Putting in Theorem 3.2, $h_1(\lambda)=\lambda^{s_1}$ and $h_2(\alpha)=\alpha^{s_2}$, we get

$$\begin{aligned}|T(f,g)| &\leq \frac{1}{8k^2}\left(\int_0^1\lambda^{s_1}d\lambda\right)\left(\int_0^1\alpha^{s_2}d\alpha\right) \\ &\times\int_a^b\int_c^d [M|g(x,y)|+N|f(x,y)|] \\ &\times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)dydx. \\ &= \frac{1}{8(1+s_1)(1+s_2)k^2} \\ &\times\int_a^b\int_c^d [M|g(x,y)|+N|f(x,y)|] \\ &\times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)dydx.\end{aligned}$$

This is the required inequality in (3.18). The proof is completed. □

Corollary 3.8. *Let $f,g:\Delta\to\mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are s-convex in the second sense on the co-ordinates, then we have*

$$\begin{aligned}|T(f,g)| &\leq \frac{1}{8k^2(1+s)^2} \\ &\times\int_a^b\int_c^d [M|g(x,y)|+N|f(x,y)|] \\ &\times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)dydx, \end{aligned} \tag{3.19}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1 and $s\in(0,1]$.

Proof. Taking in Theorem 3.2, $h_1(\lambda) = \lambda^s$ and $h_2(\alpha) = \alpha^s$, we get

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{8k^2}\left(\int_0^1 \lambda^s d\lambda\right)\left(\int_0^1 \alpha^s d\alpha\right) \\ &\quad \times \int_a^b \int_c^d \left[M\,|g(x,y)| + N\,|f(x,y)|\right] \\ &\quad \times \left((x-a)^2 + (b-x)^2\right)\left((y-c)^2 + (d-y)^2\right) dydx. \\ &= \frac{1}{8k^2\,(1+s)^2} \\ &\quad \times \int_a^b \int_c^d \left[M\,|g(x,y)| + N\,|f(x,y)|\right] \\ &\quad \times \left((x-a)^2 + (b-x)^2\right)\left((y-c)^2 + (d-y)^2\right) dydx. \end{aligned}$$

This is the desired inequality in (3.19). The proof is completed. □

4. Acknowledgements

The author would like to thank the anonymous referee for his/her valuable suggestions.

References

[1] Ahmad, F., Barnett, N. S., & Dragomir, S. S. (2009). New weighted Ostrowski and Čebyšev type inequalities. Nonlinear Analysis: Theory, Methods & Applications, 71(12), e1408-e1412.
[2] Alomari, M., & Darus, M. (2008). The Hadamard's inequality for s-convex function of 2-variables on the co-ordinates. International Journal of Math. Analysis, 2(13), 629-638.
[3] Boukerrioua, K., Guezane-Lakoud, A.(2007). On generalization of Čebyšev type inequalities. J. Inequal. Pure Appl. Math. 8,2, Art 55.
[4] Chebyshev, P. L. (1882). Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites. InProc.Math.Soc.Charkov(Vol.2,pp.93-98).
[5] Dragomir, S. S. (2001). On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. Taiwanese J Math. 4, 775–788.
[6] Guazene-Lakoud, A. and Aissaoui, F.2011. New Čebyšev type inequalities for double integrals, J. Math. Inequal, 5(4) , 453–462.
[7] Latif, M. A., & Alomari, M. (2009). On Hadamard-type inequalities for h-convex functions on the co-ordinates. International Journal of Math. Analysis, 3(33), 1645-1656.
[8] Pachpatte, B. G., & Talkies, N. A. (2006). On Čebyšev type inequalities involving functions whose derivatives belong to Lp spaces. J. Inequal. Pure and Appl. Math, 7(2), Art 58.
[9] Pachaptte, B. G. (2003). On some inequalities for convex functions,RGMIA Res.Rep.Coll, 6.
[10] Pachpatte, B. G. (2006). On Čebyšev-Grüss type inequalities via Pečarić's extension of the Montgomery identity. JIPAM. Journal of Inequalities in Pure & Applied Mathematics [electronic only], 7(1), Art 11 .
[11] Sarikaya, M.Z., Budak, H., Yaldiz, H. (2014). Some New Ostrowski Type Inequalities for Co-Ordinated Convex Functions." Turkish Journal of Analysis and Number Theory, vol. 2, no. 5 (2014).
[12] Sarikaya, M.Z., Budak, H., Yaldiz, H. Čebysev type inequalities for co-ordinated convex functions. Pure and Applied Mathematics Letters 2(2014)44-48.

University of Guelma. Guelma, Algeria.
E-mail address: khaledv2004@yahoo.fr

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 89–99 (2015) ©KJM

CONVERGENCE OF MULTI-STEP ITERATIVE SEQUENCE FOR NONLINEAR UNIFORMLY L-LIPSCHITZIAN MAPPINGS

MOGBADEMU, ADESANMI ALAO

Abstract. In this paper, by using the proof method of Xue, Rafiq and Zhou[19] some strong convergence results of multi-step iterative sequence are proved for nearly uniformly $L-$ Lipschitzian mappings in real Banach spaces. Our results generalise and improve some recent known results.

1. **Introduction**

We denote by J the normalized duality mapping from X into 2^{X^*} by

$$J(x) = \{f \in \mathrm{X}^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where X^* denotes the dual space of real Banach space X and $\langle ., .\rangle$ denotes the generalized duality pairing between elements of X and X^*. We first recall and define some concepts as follows (see [4]):
Let K be a nonempty subset of real Banach space X.
The mapping T is said to be uniformly L- Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for any $x, y \in K$ and $\forall n \geq 1$.
The mapping T is said to be asymptotically pseudocontractive if there exists a sequence $(k_n) \subset [1, \infty)$ with $lim_{n\to\infty} k_n = 1$ and for any $x, y \in K$ there exists $j(x - y) \in J(x - y)$ such that

$$< T^n x - T^n y, j(x - y) >\leq k_n\|x - y\|^2, \forall n \geq 1.$$

The concept of asymptotically pseudocontractive mappings was introduced by Schu [17].
A mapping $T : K \to X$ is called Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|,$$

2000 *Mathematics Subject Classification.* 47H10, 46A03.
Key words and phrases. Multistep iteration; asymptotically pseudo-contractive; uniformly $L-$ Lipschitzian; Banach spaces.

for all $x, y \in K$ and is called generalized Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + 1),$$

for all $x, y \in K$.
It is obvious that the class of generalized Lipschitzian map includes the class of Lipschitz map. Moreover, every mapping with a bounded range is a generalized Lipschitzian mapping.
Sahu [18] introduced the following new class of nonlinear map which is more general than the class of generalized Lipschitzian mappings and the class of uniformly L-Lipschitzian mappings. In fact, he introduced the following class of nearly Lipschitzian: Let K be a subset of a normed space X and let $\{a_n\}_{n\geq 0}$ be a sequence in $[0, \infty)$ such that $lim_{n\to\infty} a_n = 0$.
A mapping $T : K \to K$ is called nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in N$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n), \quad \forall \quad x, y \in K. \tag{1.1}$$

Define

$$\mu(T^n) = sup\{\frac{||T^n x - T^n y||}{||x - y|| + a_n} : x, y \in K, x \neq y\}.$$

Observe that for any sequence $\{k_n\}_n \geq 1$ satisfying (1.1) $\mu(T^n) \leq k_n$ $\forall n \in N$ and that

$$\|T^n x - T^n y\| \leq \mu(T^n)(\|x - y\| + a_n), \quad \forall \quad x, y \in K$$

$\mu(T^n)$ is called the nearly Lipschitz constant of the mapping T. A nearly Lipschitzian mapping T is said to be
(i) nearly contraction if $\mu(T^n) < 1$ for all $n \in N$;
(ii) nearly nonexpansive if $\mu(T^n) = 1$ for all $n \in N$;
(iii) nearly asymptotically nonexpansive if $\mu(T^n) \geq 1$ for all $n \in N$ and $\lim_{n\to\infty} \mu(T^n) = 1$;
(iv) nearly uniformly $L-$ Lipschitzian if $\mu(T^n) \leq L$ for all $n \in N$;
(v) nearly uniformly $k-$ contraction if $\mu(T^n) \leq k < 1$ for all $n \in N$.
A nearly Lipschitzian mapping T with sequence $\{a_n\}$ is said to be nearly uniformly $L-$ Lipschitzian if $k_n = L$ for all $n \in N$.
Observe that the class of nearly uniformly $L-$ Lipschitzian mapping is more general than the class of uniformly $L-$ Lipschitzian mappings.
Example 1.1 (see Sahu[18]). Let $E = R,\ K = [0, 1]$.
Define $T : K \to K$ by

$$Tx = \{ \begin{array}{r} 1/2,\ x \in [0, 1/2), \\ 0,\ x \in (1/2, 1]. \end{array}$$

It is obvious that T is not continuous, and thus not Lipschitz. However, T is nearly nonexpansive. Infact, for a real sequence $\{a_n\}_n \geq 1$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$ as $n \to \infty$, we have

$$||Tx - Ty|| \leq ||x - y|| + a_1,\ \forall x, y \in K$$

and

$$||T^n x - T^n y|| \leq ||x - y|| + a_n, \ \forall x, y \in K, \ n \geq 2.$$

This is because $T^n x = \frac{1}{2}, \forall x \in [0,1], \ n \geq 2.$
Remark 1.1: The class of nearly uniformly $L-$ Lipschitzian is not necessarily continuous.
In recent years, many authors have given much attention to iterative methods for approximating fixed points of Lipschitz asymptotically type nonlinear mappings (see [1-4, 6, 9, 17, 18]).
Schu [17] proved the following theorem:
Theorem 1.1 ([17]). Let H be a Hilbert space, K be a nonempty bounded closed convex subset of H and $T : K \to K$ be completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with a sequence $k_n \subset [1, \infty)$ satisfying the following conditions:
(i) $k_n \to 1$ as $n \to \infty$ (ii) $\sum_{n=1}^{\infty} q_n^2 - 1 < \infty$, where $q_n = 2k - 1$.
Suppose further that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences in $[0,1]$ such that $\epsilon < \alpha_n < \beta_n \leq b, \ \ \forall n \geq 1$, where $\epsilon > 0$ and $b \in (0, L^{-2}[(1 + L^2)^{\frac{1}{2}} - 1])$ are some positive numbers. For any $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1.$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T in K.
In [1], Chang extended Theorem 1.3 to a real uniformly smooth Banach space and proved the following theorem:
Theorem 1.2 ([1]). Let E be a real uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E, $T : K \to K$ be an asymptotically pseudocontractive mapping with a sequence $k_n \subset [1, \infty)$ with $k_n \to 1$ and $F(T) \neq \phi$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $[0, 1]$ satisfying the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$ (ii)$\sum_{n=0}^{\infty} \alpha_n = \infty$. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|), \quad n \geq 0$$

where $\rho \in F(T)$ is some fixed point of T in K, then $x_n \to \rho$ as $n \to \infty$.
Ofoedu [13] used the modified Mann iteration process introduced by Schu [17] ,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0, \tag{1.2}$$

to obtain a strong convergence theorem for uniformly Lipschitzian asymptotically pseudocontractive mapping in real Banach space setting. He proved the following theorem:
Theorem 1.3 ([13]). Let E be a real Banach space, K be a nonempty closed convex subset of E, $T : K \to K$, be a uniformly L-Lipschitzian asymptotically mappings with a sequence $k_n \subset [1, \infty)$, $k_n \to 1$ such that $\rho \in F(T)$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $[0, 1]$ satisfying the following conditions: (i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ (iii) $\sum_{n=0}^{\infty} \beta_n < \infty$

(iv) $\sum_{n=0}^{\infty}\alpha_n(k_n-1)<\infty$.
For any $x_0\in K$, let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence defined by (1.1).
If there exists a strictly increasing function $\Phi:[0,\infty)\to[0,\infty)$ with $\Phi(0)=0$ such that

$$<T^nx_n-\rho, j(x_n-\rho)>\leq k_n\|x_n-\rho\|^2-\Phi(\|x_n-\rho\|)$$

for all $x\in K$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to ρ.
Obviously, this result extends Theorem 1.2 of Chang [1] from a real uniformly smooth Banach space to an arbitrary real Banach space and removes the boundedness condition imposed on K.
Chang et al.[3] used an Ishikawa iteration sequence to prove a strong convergence theorem for a pair of $L-$ Lipschitzian mappings instead of a single map used in Ofoedu [13].
Rafiq, Acu and Sofonea [15], improved the results of Chang et al. [3] in a significant more general context. They then gave an open problem whether their results can be extended for the case of three mappings which are more general than the two maps. Indeed, they proved the following theorem.
Theorem 1.3 ([15]). Let K be a nonempty closed convex subset of a real Banach space E, $T_i:K\to K$, $(i=1,2)$ be two uniformly L-Lipschitzian mappings with sequence $k_n\subset[1,\infty)$, $\sum_{n=1}^{\infty}(k_n-1)<\infty$ such that $F(T_1)\cap F(T_2)\neq\phi$, where $F(T_i)$ is the set of fixed points of T_i in K and ρ be a point in $F(T_1)\cap F(T_2)$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences in $[0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n=\infty$, $lim_{n\to\infty}\alpha_n=\beta_n=0$. For any $x_1\in K$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence iteratively defined by

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_nT_1^ny_n$$
$$y_n=(1-\beta_n)x_n+\beta_nT_2^nx_n.$$

Suppose there exists a strictly increasing function $\Phi:[0,\infty)\to[0,\infty)$ with $\Phi(0)=0$ such that

$$<T_i^nx_n-\rho, j(x_n-\rho)>\leq k_n\|x_n-\rho\|^2-\Phi(\|x_n-\rho\|),\forall x\in K(i=1,2),$$

then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\rho\in F(T_1)\cap F(T_2)$.
In [10], the author established a new result on convergence of the modified Noor iteration for three nearly Lipschitzian mappings. His result extends, improves and unifies a host of recent results. Although, Mogbademu and Xue [9] had earlier obtained a strong convergence theorem for asymptotically generalized $\Phi-$ hemicontractive map in real Banach spaces using the iterative sequence generated by this map.
More recently, Xue, Rafiq and Zhou [19] employed an analytical technique to prove the convergent of an Ishikawa and Mann iterations for nonlinear mappings in uniformly smooth real Banach spaces. It is the purpose of this paper, using the style of proof by Xue, Rafiq and Zhou [19] to prove strong convergence theorems of multistep iteration scheme (1.3) for nearly uniformly Lipschitzian mappings in a real Banach space. Our results significantly generalise and improve some recent results of [1-3, 6, 13, 17, 18] in some aspects. For this, we need the following concepts and Lemmas.

The following iteration (see Rhoades and Soltuz [16]):

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n^1,\ n \geq 0,$$
$$y_n^i = (1-\beta_n^i)x_n + \beta_n^i T^n y_n^{k+1},\ i = 1,2,...,p-2,$$
$$y_n^{p-1} = (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T^n x_n,\ n \geq 0, p \geq 2. \tag{1.3}$$

is called the multistep iteration sequence, where $p \geq 2$ is fixed order, $\{\alpha_n\}, \{\beta_n^i\}$ are sequences in $[0,1]$ for $i = 1,2,...,p-1$.
Taking $p = 3$ in (1.3) we obtain the Noor iteration scheme as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n^1,\ n \geq 0,$$
$$y_n^1 = (1-\beta_n^1)x_n + \beta_n^1 T^n y_n^2,$$
$$y_n^2 = (1-\beta_n^2)x_n + \beta_n^2 T^n x_n,\ n \geq 0. \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n^i\}$ are sequences in $[0,1]$ for $i = 1,2$.
Taking $p = 2$ in (1.3) we obtain the Ishikawa iteration scheme as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n^1,\ n \geq 0,$$
$$y_n^1 = (1-\beta_n^1)x_n + \beta_n^1 T^n x_n, \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n^1\}$ are sequences in $[0,1]$.
In particular, if $\beta_n^1 = 0$ for $n \geq 0$ in (1.5) the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 0, \tag{1.6}$$

is called the Mann iteration sequence (see [7]).
We remark that iteration (1.3) generalises the Mann, Ishikawa and Noor iteration sequences. The multistep iteration sequence (1.3) can be viewed as the predictor-corrector methods for solving nonlinear equations in Banach spaces. For the convergence analysis of the predictor-corrector and multistep iteration sequences for solving the variational inequalities and optimization problems (see Noor[11], Noor et al. [12]).
Lemma 1.1 [1, 9]. Let E be real Banach Space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then,for any $x, y \in E$

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, j(x+y) >, \forall j(x+y) \in J(x+y).$$

Lemma 1.2 [8]. Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying

$$\sum_{n=0}^{\infty} b_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n\Phi(a_{n+1}), \quad \forall n \geq N_0$$

where $\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim_{n\to\infty} a_n = 0$.

2. Main results

Theorem 2.1. Let X be a real Banach space, K be a nonempty closed convex subset of X, $T : K \to K$ be a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$. Let $k_n \subset [1, \infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \epsilon_n = 0$ and $F(T) = \{\rho \in K : T\rho = \rho\}$. Let $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n^i\}_{n\geq 0}, (i = 1, 2, ..., p-1)$ be real sequences in $[0, 1]$ satisfying the following conditions: (i)$\sum_{n\geq 0} \alpha_n = \infty$ (ii) $\lim_{n\to\infty} \alpha_n, \beta_n^i = 0, (i = 1, 2, ..., p-1)$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geq 0}$ be iteratively defined by (1.3). If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x - \rho) > \leq k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n$$

for all $x \in K$. Then, $\{x_n\}_{n\geq 0}$ converges strongly to $\rho \in F(T)$.

Proof. Since there exists a strictly increasing continuous function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n \rho, j(x - \rho)\rangle \leq k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n, \tag{2.1}$$

for $x \in K$, $\rho \in F(T)$, that is

$$\epsilon_n + \langle k_n(x - \rho) - (T^n x - \rho), j(x - \rho)\rangle \geq \Phi(\|x - \rho\|). \tag{2.2}$$

Choose some $x_0 \in K$ and $x_0 \neq T x_0$ such that $\epsilon_n + (k_n + L)\|x_0 - \rho\|^2 + L\|x_0 - \rho\|^2 \in R(\Phi)$ and denote that $a_0 = \epsilon_n + (k_n + L)\|x_0 - \rho\|^2 + L\|x_0 - \rho\|^2$, $R(\Phi)$ is the range of Φ. Indeed, if $\Phi(a) \to +\infty$ as $a \to \infty$, then $a_0 \in R(\Phi)$; if $\sup\{\Phi(a) : a \in [0, \infty]\} = a_1 < +\infty$ with $a_1 < a_0$, then for $\rho \in K$, there exists a sequence $\{u_n\}$ in K such that $u_n \to \rho$ as $n \to \infty$ with $u_n \neq \rho$. Clearly, $Tu_n \to T\rho$ as $n \to \infty$ thus $\{u_n - Tu_n\}$ is a bounded sequence. Therefore, there exists a natural number n_0 such that $\epsilon_n + (k_n + L)\|u_n - \rho\|^2 + L\|u_n - \rho\|^2 < \frac{a_1}{2}$ for $n \geq n_0$, then we redefine $x_0 = u_{n_0}$ and $\epsilon_n + (k_n + L)\|x_0 - \rho\|^2 + L\|x_0 - \rho\|^2 \in R(\Phi)$. This is to ensure that $\Phi^{-1}(a_0)$ is well defined.

Step 1. We first show that $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence.

Set $R = \Phi^{-1}(a_0)$, then from above (2.2), we obtain that $\|x_n - \rho\| \leq R$. Denote

$$B_1 = \{x \in K : \|x - \rho\| \leq R\}, \quad B_2 = \{x \in K : \|x - \rho\| \leq 2R\}. \tag{2.3}$$

Now, we want to prove that $x_n \in B_1$. If $n = 0$, then $x_0 \in B_1$. Now assume that it holds for some n, that is, $x_n \in B_1$. Suppose that, it is not the case, then $\|x_{n+1} - \rho\| > R > \frac{R}{2}$.

Since $\{a_n\} \in [0, \infty]$ with $a_n \to 0$, set $M = sup\{a_n : n \in N\}$. Denote

$$\tau_0 \quad = \quad min\{1, \frac{\Phi(\frac{R}{2})}{32R^2}, \frac{\Phi(\frac{R}{2})}{16R[2(L(2R+M)+R)+M]}, \frac{\Phi(\frac{R}{2})}{16R[L(2R+M)+R]}, \frac{\Phi(\frac{R}{2})}{8}\}. \tag{2.4}$$

Since $\lim_{n\to\infty} \alpha_n, \beta_n^i = 0$ for $i = 1, 2, ..., p-1$ and $\lim_{n\to\infty} k_n = 1$. Without loss of generality, let $0 \leq \alpha_n, \beta_n^i, k_n - 1, \epsilon_n \leq \tau_0$ for any $n \geq 0$. Then, we have the following estimates from (2.1) for $i = 1, 2, ..., p-1$.

$$\begin{array}{rcl} \|y_n^{p-1} - \rho\| & \leq & (1-\beta_n^{p-1})\|x_n - \rho\| + \beta_n^{p-1}\|T^n x_n - \rho\| \\ & \leq & R + \tau_0 L(R+M) \\ & \leq & 2R. \end{array}$$

then $y^{p-1} \in B_2$. Similarly,

$$\begin{array}{rcl} \|y_n^{p-2} - \rho\| & \leq & (1-\beta_n^{p-2})\|x_n - \rho\| + \beta_n^{p-2}\|T^n y_n^{p-1} - \rho\| \\ & \leq & R + \tau_0 L(2R+M) \\ & \leq & 2R. \end{array}$$

then $y^{p-2} \in B_2 ...$, we have

$$\begin{array}{rcl} \|y_n^{1} - \rho\| & \leq & (1-\beta_n^{1})\|x_n - \rho\| + \beta_n^{1}\|T^n y_n^{2} - \rho\| \\ & \leq & R + \tau_0 L(2R+M) \\ & \leq & 2R. \end{array}$$

then for $y^1 \in B_2$. We get

$$\begin{array}{rcl} \|x_{n+1} - \rho\| & \leq & (1-\alpha_n)\|x_n - \rho\| + \alpha_n\|T^n y_n^{1} - \rho\| \\ & \leq & R + \tau_0 L(2R+M) \\ & \leq & 2R. \end{array}$$

Therefore, we have

$$\begin{array}{rcl} \|x_{n+1} - x_n\| & \leq & \alpha_n\|T^n y_n^1 - x_n\| \\ & \leq & \alpha_n(\|T^n y_n^1 - \rho\| + \|x_n - \rho\|) \\ & \leq & \tau_0(L(2R+M)+R). \end{array} \tag{2.5}$$

$$\begin{array}{rcl} \|y_n^1 - x_{n+1}\| & \leq & \beta_n\|T^n y_n^2 - x_n\| + \alpha_n\|T^n y_n^1 - x_n\| \\ & \leq & \beta_n(\|T^n y_n^2 - \rho\| + \|x_n - \rho\|) \\ & & +\alpha_n(\|T^n y_n^1 - \rho\| + \|x_n - \rho\|) \\ & \leq & 2\tau_0(L(2R+M)+R). \end{array} \tag{2.6}$$

Using Lemma 1.1 and the above estimates, we have

$$\begin{array}{rcl} \|x_{n+1} - \rho\|^2 & \leq & \|x_n - \rho\|^2 + 2\alpha_n < T^n y_n^1 - x_n, j(x_{n+1} - \rho) > \\ & = & \|x_n - \rho\|^2 + 2\alpha_n < T^n x_{n+1} - x_{n+1}, j(x_{n+1} - \rho) > \\ & & + < x_{n+1} - x_n, j(x_{n+1} - \rho) > \\ & & + < T^n y_n^1 - T^n x_{n+1}, j(x_{n+1} - \rho) > \\ & \leq & \|x_n - \rho\|^2 + 2\alpha_n(k_n\|x_{n+1} - \rho\|^2 - \Phi(\|x_{n+1} - \rho\|) + \epsilon_n) \\ & & -2\alpha_n\|x_{n+1} - \rho\|^2 + 2\alpha_n L(\|y_n^1 - x_{n+1}\| + a_n)\|x_{n+1} - \rho\| \\ & & +2\alpha_n\|x_{n+1} - x_n\|\|x_{n+1} - \rho\| \\ & = & \|x_n - \rho\|^2 + 2\alpha_n(k_n - 1)\|x_{n+1} - \rho\|^2 \\ & & -2\alpha_n\Phi(\|x_{n+1} - \rho\|) + 2\alpha_n\epsilon_n \\ & & +2\alpha_n L(\|y_n^1 - x_{n+1}\| + a_n)\|x_{n+1} - \rho\| \\ & & +2\alpha_n\|x_{n+1} - x_n\|\|x_{n+1} - \rho\| \\ & \leq & \|x_n - \rho\|^2 - 2\alpha_n\Phi(\frac{R}{2}) + 2\alpha_n\frac{\Phi(\frac{R}{2})}{32R^2}4R^2 + 2\alpha_n\frac{\Phi(\frac{R}{2})}{8} \end{array}$$

$$
\begin{aligned}
& +2\alpha_n L\frac{\Phi(\frac{R}{2})}{16R[2(L(2R+M)+R)+M]}2R[2(L(2R+M)+R)+M] \\
& +2\alpha_n\frac{\Phi(\frac{R}{2})}{16R[L(2R+M)+R]}2R[L(2R+M)+R] \\
\leq\ & \|x_n-\rho\|^2-\alpha_n\Phi(\tfrac{R}{2}) \\
\leq\ & R^2.
\end{aligned} \tag{2.7}
$$

which is a contradiction. Hence $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence. So, $\{y_n^1\}$, $\{y^2\}$,..., $\{y_n^{p-1}\}$ are all bounded sequences.
Step 2. We want to prove $\|x_n-\rho\|\to 0$ as $n\to\infty$.
Since $\lim_{n\to\infty}\alpha_n,\beta_n^k=0$, $\lim_{n\to\infty}k_n=1$ and $\{x_n\}_{n=0}^{\infty}$ is bounded. From (2.5) and (2.6), we observed that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=0,\quad \lim_{n\to\infty}L\|y_n^1-x_{n+1}\|=0. \tag{2.8}
$$

So from (2.7), we have

(2.9)

$$
\begin{aligned}
\|x_{n+1}-\rho\|^2 &\leq \|x_n-\rho\|^2+2\alpha_n<T^ny_n^1-x_n,j(x_{n+1}-\rho)> \\
&= \|x_n-\rho\|^2+2\alpha_n<T^nx_{n+1}-x_{n+1},j(x_{n+1}-\rho)> \\
&\quad +<x_{n+1}-x_n,j(x_{n+1}-\rho)> \\
&\quad +<T^ny_n^1-T^nx_{n+1},j(x_{n+1}-\rho)> \\
&\leq \|x_n-\rho\|^2+2\alpha_n(k_n\|x_{n+1}-\rho\|^2-\Phi(\|x_{n+1}-\rho\|)+\epsilon_n) \\
&\quad -2\alpha_n\|x_{n+1}-\rho\|^2+2\alpha_nL(\|y_n^1-x_{n+1}\|+a_n)\|x_{n+1}-\rho\| \\
&\quad +2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
&\leq \|x_n-\rho\|^2+2\alpha_n(k_n-1)\|x_{n+1}-\rho\|^2 \\
&\quad -2\alpha_n\Phi(\|x_{n+1}-\rho\|)+\epsilon_n \\
&\quad +2\alpha_nL(\|y_n^1-x_{n+1}\|+a_n)\|x_{n+1}-\rho\| \\
&\quad +2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
&= \|x_n-\rho\|^2-2\alpha_n\Phi(\|x_{n+1}-\rho\|)+o(\alpha_n),
\end{aligned}
$$

where

$$
\begin{aligned}
&2\alpha_n(k_n-1)\|x_{n+1}-\rho\|^2++2\alpha_nL(\|y_n^1-x_{n+1}\|+a_n)\|x_{n+1}-\rho\| \\
&+2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\|+2\alpha_n\epsilon_n \\
&=o(\alpha_n).
\end{aligned}
$$

By Lemma 1.2, we obtain that

$$
\lim_{n\to\infty}\|x_n-\rho\|=0.
$$

This completes the proof.

Remarks 2.1. Theorem 2.1 improves and extends the corresponding results of [1-3, 6, 13, 17, 18] in some aspects.
(i) The method of proof of Theorem 2.1 is different from the method given in Chang [1], Ofoedu [13] and Chang et al. [3] .
(ii) The control conditions (ii)-(iv) in Theorem 2.1 of Chang [1] and that of Ofoedu[13] are replaced by weaker condition $\lim_{n\to\infty}\alpha_n=0$.
(iii) Under suitable conditions, sequence $\{x_n\}_{n=0}^{\infty}$ defined by (2.1) in Theorem 2.1 can also be generalized to multi-step iterative scheme with errors.
(iv) The assumption that there exists a strictly increasing function $\Phi:[0,\infty)\to$

$[0,\infty)$ with $\Phi(0)=0$ such that

$$< T^n x - T^n \rho, j(x-\rho) > \leq k_n \|x-\rho\|^2 - \Phi(\|x-\rho\|)$$

for all $x \in K$ used by several authors in literature is extended to a more general assumption: there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0)=0$ such that

$$< T^n x - T^n \rho, j(x-\rho) > \leq k_n \|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n$$

for all $x \in K$
(v) The iteration sequences used in Chang [1], Ofoedu [13] Chang et al. [3] and Mogbademu [10] are extended to (1.3).
(vi) The mappings in [1, 3, 9, 13, 15] are extended to a more general class of nearly Lipschitzian mappings.

The following reveals that Theorem 2.1 is applicable.
Application 2.1. Let $X = R, K = [0,1]$ and $T : K \to K$ be a map defined by

$$Tx = \frac{x}{1+x}, \forall x \in [0,1)$$

Clearly, T is nearly uniformly Lipschitzian ($a_n = \frac{1}{2^n}$) with $F(0) = 0$.
Define $\Phi : [0,\infty) \to [0,\infty)$ by

$$\Phi(t) = \frac{t^2}{1+nt}$$

then, Φ is a strictly increasing function with $\Phi(0) = 0$. For all $x \in K, \rho \in F(T)$, we have that operator T in Theorem 2.1 satisfies

$$< T^n x - T^n \rho, j(x-\rho) > \leq k_n \|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n$$

with the sequences $k_n = 1$ and $\epsilon_n = \frac{x^2}{1+nx}$. Set $\alpha_n = \frac{1}{2+n}$ and $\beta_n^i = \frac{1}{3+n}$,$(i = 1, 2, ..., p-1)$ $\forall n \geq 0$.

Remarks 2.2. Our results enrich and develop the theory of multi-step iterative sequence introduced by Rhoades and Soltuz [16].

Taking $p = 3$ in (1.3), Theorem 2.1 leads to the following corllaries.

Corollary 2.2. Let X be a real Banach space, K be a nonempty closed convex subset of X, $T : K \to K$ be a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$. Let $k_n \subset [1,\infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \epsilon_n = 0$ and $F(T) = \{\rho \in K : T\rho = \rho\}$. Let $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n^i\}_{n\geq 0}, (i = 1, 2)$ be real sequences in $[0,1]$ satisfying the following conditions: (i)$\sum_{n\geq 0} \alpha_n = \infty$ (ii) $\lim_{n\to\infty} \alpha_n, \beta_n^i = 0, (i = 1, 2)$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geq 0}$ be iteratively defined by (1.4). If there exists a strictly increasing function
$\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x-\rho) > \leq k_n \|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n$$

for all $x \in K$. Then, $\{x_n\}_{n\geq 0}$ converges strongly to $\rho \in F(T)$.

Taking $p = 2$ in (1.3), Theorem 2.1 leads to the following results.

Corollary 2.3. Let X be a real Banach space, K be a nonempty closed convex subset of X, $T : K \to K$ be a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$. Let $k_n \subset [1, \infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1,\ \lim_{n\to\infty} \epsilon_n = 0$ and $F(T) = \{\rho \in K : T\rho = \rho\}$. Let $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n^i\}_{n\geq 0}, (i = 1)$ be real sequences in $[0, 1]$ satisfying the following conditions: (i)$\sum_{n\geq 0} \alpha_n = \infty$ (ii) $\lim_{n\to\infty} \alpha_n, \beta_n^i = 0, (i = 1)$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geq 0}$ be iteratively defined by (1.5). If there exists a strictly increasing function
$\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x - \rho) >\leq k_n\|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n$$

for all $x \in K$. Then, $\{x_n\}_{n\geq 0}$ converges strongly to $\rho \in F(T)$.

Corollary 2.4. Let X be a real Banach space, K be a nonempty closed convex subset of X, $T : K \to K$ be a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$. Let $k_n \subset [1, \infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1,\ \lim_{n\to\infty} \epsilon_n = 0$ and $F(T) = \{\rho \in K : T\rho = \rho\}$. Let $\{\alpha_n\}_{n\geq 0}$ be a real sequence in $[0, 1]$ satisfying the following conditions: (i)$\sum_{n\geq 0} \alpha_n = \infty$ (ii) $\lim_{n\to\infty} \alpha_n = 0$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geq 0}$ be iteratively defined by (1.6). If there exists a strictly increasing function
$\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x - \rho) >\leq k_n\|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n$$

for all $x \in K$. Then, $\{x_n\}_{n\geq 0}$ converges strongly to $\rho \in F(T)$.

Acknowlegment: The author would like to express his thanks to Professor Z. Xue and Professor Chika Moore for their unique style of mentoring and supports toward the completion of this work.

References

[1] Chang, S. S. *Some results for asymptotically pseudocontractive mappings and asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., 129 , **99** (2000), 845-853.

[2] Chang, S. S., Cho, Y. J., Lee, B. S. and Kang, S. H. *Iterative approximation of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces* , J. Math. Anal. Appl., 224 (1998), 165-194.

[3] Chang, S. S., Cho, Y. J., and Kim, J. K. *Some results for uniformly L-Lipschitzian mappings in Banach spaces* , Appl. Math. Lett., 22, (2009), 121-125.

[4] Goebel, K. and Kirk, W. A. *A fixed point theorem for asymptotically nonexpansive mappings* , Proc. Amer. Math. Soc., Vol. 35 , (1972), 171-174.

[5] Ishikawa, S. *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., 44 (1974), 147-150.

[6] Kim, J. K., Sahu, D. R. and Nam, Y. M.*Convergence theorem for fixed points of nearly uniformly $L-$ Lipschitzian asymptotically generalized $\Phi-$ hemicontractive mappings*, Nonl. Anal., 71, **99**(2009), e2833- e2838.

[7] Mann, W.R. *Mean value methods in iteration*, Proc. Amer. Math. Soc., 4, **99** (1953), 506-610 .

[8] Moore, C. and Nnoli, B. V. C. *Iterative solution of nonlinear equations involving set-valued uniformly accretive operators* , Comput. Math. Anal. Appl., 42, (2001), 131-140 .

[9] Mogbademu, A.A. and Xue, Z. *Some convergence results for nonlinear maps in Banach spaces*, Int. J. Open Problems Compt. Math., Vol. 6, (2013), 1- 10.

[10] Mogbademu, A.A. *Convergence theorem of modified Noor iteration for nonlinear maps in Banach spaces*, J. Adv. Math. Stud., Vol. 7 (2014), nos. 1,56-64.

[11] Noor, M.A. *Three-step iterative algorithms for multi-valued quasi variational inclusions* J. Math. Anal. Appl., 225 (2001), 589-604.

[12] Noor, M.A., Kassias, T. M. and Huang, Z. *Three-step iterations for nonlinear accretive operator equations* , J. Math. Anal. Appl., 274 (2002), 59-68.

[13] Ofoedu, E.U. *Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space* , J. Math. Anal. Appl., 321 (2006), 722-728.

[14] Olaleru, J.O. and Mogbademu, A.A. *Modified Noor iterative procedure for uniformly continuous mappings in Banach spaces*, Boletin de la Asociacion Matematica Venezolana, Vol. XVIII, No. 2 (2011), 127- 135.

[15] Rafiq, A., Acu, A. M. and Sofonea, F. *An iterative algorithm for two asymptotically pseudocontractive mappings* , Int. J. Open Problems Compt. Math., Vol. 2 (2009), 371- 382.

[16] Rhoades, B.E. and Soltuz, S.M. *The equivalence between Mann-Ishikawa iterations and multistep iteration* , Nonl. Anal.: Theory, Methods and Applications, Vol. 58 (2004), 218-228.

[17] Schu, J. *Iterative construction of fixed points of asymptotically nonexpansive mappings* , J. Math. Anal. Appl., 158 (1999), 407-413.

[18] Sahu, D. R. *Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces* , Comment. Math. Univ. Carolin 46 (4) (2005), 653-666.

[19] Xue, Z., Rafiq, A. and Zhou, H. *On the convergence of multi-step iteration for uniformly continuous* $\Phi-$ *Hemicontractive mappings*, Abstract and Applied Analysis, Vol. 2012, Article ID 386983, (2012), 1-9.

Department of Mathematics, Faculty of Science, University of Lagos, Yaba- Lagos, Nigeria

E-mail address: amogbademu@unilag.edu.ng

KONURALP JOURNAL OF MATHEMATICS
VOLUME 3 NO. 2 PP. 100–109 (2015) ©KJM

L^p LOCAL UNCERTAINTY PRINCIPLE FOR THE DUNKL TRANSFORM

FETHI SOLTANI

ABSTRACT. In this paper, we establish L^p local uncertainty principle for the Dunkl transform on $\mathbb{R}^d$; and we deduce L^p version of the Heisenberg-Pauli-Weyl uncertainty principle for this transform. We use also the L^p local uncertainty principle for the Dunkl transform and the techniques of Donoho-Stark, we obtain uncertainty principles of concentration type in the L^p theory, when $1 < p \leq 2$.

1. INTRODUCTION

In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle .,. \rangle$ and norm $|y| := \sqrt{\langle y, y \rangle}$. For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha y := y - \frac{2\langle \alpha, y \rangle}{|\alpha|^2}\alpha.$$

A finite set $\Re \subset \mathbb{R}^d \backslash \{0\}$ is called a root system, if $\Re \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \Re = \Re$ for all $\alpha \in \Re$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in \Re$. For a root system $\Re$, the reflections σ_α, $\alpha \in \Re$, generate a finite group G. The Coxeter group G is a subgroup of the orthogonal group $O(d)$. All reflections in G, correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \backslash \bigcup_{\alpha \in \Re} H_\alpha$, we fix the positive subsystem $\Re_+ := \{\alpha \in \Re : \langle \alpha, \beta \rangle > 0\}$. Then for each $\alpha \in \Re$ either $\alpha \in \Re_+$ or $-\alpha \in \Re_+$.

Let $k : \Re \to \mathbb{C}$ be a multiplicity function on $\Re$ (a function which is constant on the orbits under the action of G). As an abbreviation, we introduce the index $\gamma = \gamma_k := \sum_{\alpha \in \Re_+} k(\alpha)$.

Date: February 19, 2014 and, in revised form, April 21, 2015.
2000 *Mathematics Subject Classification.* 42B10; 42B30; 33C45.
Key words and phrases. Dunkl transform; local uncertainty principle; Heisenberg-Pauli-Weyl uncertainty principle; Donoho-Stark's uncertainty principles.
Author partially supported by the DGRST research project LR11ES11 and CMCU program 10G/1503.

Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in \Re$. Moreover, let w_k denote the weight function $w_k(y) := \prod_{\alpha\in\Re_+} |\langle\alpha, y\rangle|^{2k(\alpha)}$, for all $y \in \mathbb{R}^d$, which is G-invariant and homogeneous of degree 2γ.

Let c_k be the Mehta-type constant given by $c_k := (\int_{\mathbb{R}^d} e^{-|y|^2/2} w_k(y)\mathrm{d}y)^{-1}$. We denote by μ_k the measure on $\mathbb{R}^d$ given by $\mathrm{d}\mu_k(y) := c_k w_k(y)\mathrm{d}y$; and by $L^p(\mu_k)$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\begin{aligned} \|f\|_{L^p(\mu_k)} &:= \Big(\int_{\mathbb{R}^d} |f(y)|^p \mathrm{d}\mu_k(y)\Big)^{1/p} < \infty, \quad 1 \leq p < \infty, \\ \|f\|_{L^\infty(\mu_k)} &:= \operatorname*{ess\,sup}_{y\in\mathbb{R}^d} |f(y)| < \infty, \end{aligned}$$

and by $L^p_{rad}(\mu_k)$ the subspace of $L^p(\mu_k)$ consisting of radial functions.

For $f \in L^1(\mu_k)$ the Dunkl transform is defined (see [4]) by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y) f(y)\mathrm{d}\mu_k(y), \quad x \in \mathbb{R}^d,$$

where $E_k(-ix, y)$ denotes the Dunkl kernel (for more details, see the next section).

Many uncertainty principles have already been proved for the Dunkl transform, namely by Rösler [10] and Shimeno [11] who established the Heisenberg-Pauli-Weyl inequality for the Dunkl transform, by showing that for every $f \in L^2(\mu_k)$,

$$\|f\|^2_{L^2(\mu_k)} \leq \frac{2}{2\gamma + d} \| |x|f\|_{L^2(\mu_k)} \| |y|\mathcal{F}_k(f)\|_{L^2(\mu_k)}. \tag{1.1}$$

Recently the author [17] proved the following L^p version of the Heisenberg-Pauli-Weyl inequality for the Dunkl transform $\mathcal{F}_k$. Let $0 < a < (2\gamma + d)/q$, $b > 0$, if $1 < p \leq 2$, $q = p/(p - 1)$ and $f \in L^p(\mu_k)$, then

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq C(a, b)\| |x|^a f\|^{\frac{b}{a+b}}_{L^p(\mu_k)} \| |y|^b \mathcal{F}_k(f)\|^{\frac{a}{a+b}}_{L^q(\mu_k)}, \tag{1.2}$$

where $C(a, b)$ is a positive constant.

Building on the ideas of Faris [5] and Price [8, 9] for the Fourier transform, we show a local uncertainty principles for the Dunkl transform $\mathcal{F}_k$. More precisely, we will show the following results. Let E be a measurable subset of $\mathbb{R}^d$ such that $0 < \mu_k(E) < \infty$, and $a > 0$. If $1 < p \leq 2$, $q = p/(p - 1)$ and $f \in L^p(\mu_k)$, then

$$\|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \begin{cases} K_1(a)(\mu_k(E))^{\frac{a}{2\gamma+d}} \| |x|^a f\|_{L^p(\mu_k)}, & 0 < a < \frac{2\gamma+d}{q}, \\ K_2(a)(\mu_k(E))^{1/q} \|f\|^{1-\frac{2\gamma+d}{qa}}_{L^p(\mu_k)} \| |x|^a f\|^{\frac{2\gamma+d}{qa}}_{L^p(\mu_k)}, & a > \frac{2\gamma+d}{q}, \\ 2K_1(\frac{a}{2})(\mu_k(E))^{\frac{1}{2q}} \|f\|^{1/2}_{L^p(\mu_k)} \| |x|^a f\|^{1/2}_{L^p(\mu_k)}, & a = \frac{2\gamma+d}{q}, \end{cases}$$

where χ_E is the characteristic function of the set E and $K_1(a)$, $K_2(a)$ are positive constants given explicitly by Theorem 2.1.

We shall use the L^p local uncertainty principle to show L^p version of the Heisenberg-Pauli-Weyl uncertainty principle for the Dunkl transform $\mathcal{F}_k$. Let $a, b > 0$, if $1 < p \leq 2$, $q = p/(p - 1)$ and $f \in L^p(\mu_k)$, then

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \begin{cases} K_1(a, b)\| |x|^a f\|^{\frac{b}{a+b}}_{L^p(\mu_k)} \| |y|^b \mathcal{F}_k(f)\|^{\frac{a}{a+b}}_{L^q(\mu_k)}, & 0 < a < \frac{2\gamma+d}{q}, \\ K_2(a, b)\|f\|^{\frac{b(qa-2\gamma-d)}{a(qb+2\gamma+d)}}_{L^p(\mu_k)} \| |x|^a f\|^{\frac{b(2\gamma+d)}{a(2\gamma+d+qb)}}_{L^p(\mu_k)} \| |y|^b \mathcal{F}_k(f)\|^{\frac{2\gamma+d}{2\gamma+d+qb}}_{L^q(\mu_k)}, & a > \frac{2\gamma+d}{q}, \\ K_3(a, b)\|f\|^{\frac{b}{a+2b}}_{L^p(\mu_k)} \| |x|^a f\|^{\frac{b}{a+2b}}_{L^p(\mu_k)} \| |y|^b \mathcal{F}_k(f)\|^{\frac{a}{a+2b}}_{L^q(\mu_k)}, & a = \frac{2\gamma+d}{q}, \end{cases}$$

where $K_1(a,b)$, $K_2(a,b)$ and $K_3(a,b)$ are positive constants given explicitly by Theorem 2.2. The inequalities which generalize the Heisenberg-Pauli-Weyl inequalities given by (1.1) and (1.2). In the case $k=0$ and $q=2$, these inequalities are due to Cowling-Price [1] and Hirschman [6].

We shall use also the local uncertainty principle, and building on the techniques of Donoho-Stark [2, 14, 15, 16, 18], we show uncertainty principles of concentration type in the L^p theory, when $1<p\le 2$.

This paper is organized as follows. In Section 2 we show a local uncertainty principle for the Dunkl transform $\mathcal{F}_k$; and we deduce L^p version of the Heisenberg-Pauli-Weyl uncertainty principle for this transform. The last section is devoted to present uncertainty principles of concentration type in the L^p theory, when $1<p\le 2$.

2. L^p uncertainty principles

The Dunkl operators $\mathcal{D}_j$; $j=1,...,d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given, for a function f of class C^1 on $\mathbb{R}^d$, by

$$\mathcal{D}_j f(y) := \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha\in\Re_+} k(\alpha)\alpha_j \frac{f(y)-f(\sigma_\alpha y)}{\langle \alpha, y\rangle}.$$

For $y\in\mathbb{R}^d$, the initial problem $\mathcal{D}_j u(.,y)(x) = y_j u(x,y)$, $j=1,...,d$, with $u(0,y)=1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x,y)$ and called Dunkl kernel [3, 7]. This kernel has a unique analytic extension to $\mathbb{C}^d\times\mathbb{C}^d$. In our case (see [3, 4]),

$$|E_k(-ix,y)|\le 1, \quad x,y\in\mathbb{R}^d. \tag{2.1}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and was introduced by Dunkl in [4], where already many basic properties were established. Dunkl's results were completed and extended later by De Jeu [7]. The Dunkl transform of a function f in $L^1(\mu_k)$, is defined by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix,y) f(y)\mathrm{d}\mu_k(y), \quad x\in\mathbb{R}^d.$$

We notice that $\mathcal{F}_0$ agrees with the Fourier transform $\mathcal{F}$ that is given by

$$\mathcal{F}(f)(x) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} f(y)\mathrm{d}y, \quad x\in\mathbb{R}^d.$$

Some of the properties of Dunkl transform $\mathcal{F}_k$ are collected bellow (see [4, 7]).

(a) L^1-L^∞-boundedness. *For all $f\in L^1(\mu_k)$, $\mathcal{F}_k(f)\in L^\infty(\mu_k)$ and*

$$\|\mathcal{F}_k(f)\|_{L^\infty(\mu_k)} \le \|f\|_{L^1(\mu_k)}. \tag{2.2}$$

(b) Inversion theorem. *Let $f\in L^1(\mu_k)$, such that $\mathcal{F}_k(f)\in L^1(\mu_k)$. Then*

$$f(x) = \mathcal{F}_k(\mathcal{F}_k(f))(-x), \quad \text{a.e.} \quad x\in\mathbb{R}^d. \tag{2.3}$$

(c) Plancherel theorem. *The Dunkl transform $\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2(\mu_k)$ onto itself. In particular,*

$$\|f\|_{L^2(\mu_k)} = \|\mathcal{F}_k(f)\|_{L^2(\mu_k)}. \tag{2.4}$$

Using relations (2.2) and (2.4) with Marcinkiewicz's interpolation theorem [19, 20], we deduce that for every $1 \le p \le 2$, and for every $f \in L^p(\mu_k)$, the function $\mathcal{F}_k(f)$ belongs to the space $L^q(\mu_k)$, $q = p/(p-1)$, and

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \le \|f\|_{L^p(\mu_k)}. \tag{2.5}$$

If $f \in L^1_{rad}(\mu_k)$ with $f(x) = F(|x|)$, then

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_k(x) = \frac{1}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})}\int_0^\infty F(r)r^{2\gamma+d-1}\mathrm{d}r. \tag{2.6}$$

In the following we use the inequality (2.5) to establish L^p local uncertainty principle for the Dunkl transform $\mathcal{F}_k$, more precisely, we will show the following theorem.

Theorem 2.1. *Let E be a measurable subset of $\mathbb{R}^d$ such that $0 < \mu_k(E) < \infty$, and $a > 0$. If $1 < p \le 2$, $q = p/(p-1)$ and $f \in L^p(\mu_k)$, then*

$$\|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} \le \begin{cases} K_1(a)(\mu_k(E))^{\frac{a}{2\gamma+d}}\|\,|x|^a f\|_{L^p(\mu_k)}, & 0 < a < \frac{2\gamma+d}{q}, \\ K_2(a)(\mu_k(E))^{1/q}\|f\|_{L^p(\mu_k)}^{1-\frac{2\gamma+d}{qa}}\|\,|x|^a f\|_{L^p(\mu_k)}^{\frac{2\gamma+d}{qa}}, & a > \frac{2\gamma+d}{q}, \\ 2K_1(\frac{a}{2})(\mu_k(E))^{\frac{1}{2q}}\|f\|_{L^p(\mu_k)}^{1/2}\|\,|x|^a f\|_{L^p(\mu_k)}^{1/2}, & a = \frac{2\gamma+d}{q}, \end{cases}$$

where

$$K_1(a) = \frac{2\gamma+d}{2\gamma+d-qa}\left[\frac{(2\gamma+d-qa)^{q-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})(qa)^q}\right]^{\frac{a}{2\gamma+d}},$$

$$K_2(a) = \frac{qa}{qa-2\gamma-d}\left(\frac{qa}{2\gamma+d}-1\right)^{\frac{2\gamma+d}{pqa}}\left[\frac{(qa-2\gamma-d)\Gamma(\frac{qa-2\gamma-d}{pa})\Gamma(\frac{2\gamma+d}{pa})}{2^{\gamma+\frac{d}{2}-1}pqa^2\Gamma(\gamma+\frac{d}{2})\Gamma(\frac{q}{p})}\right]^{\frac{1}{q}}.$$

Proof. (i) The first inequality holds if $\|\,|x|^a f\|_{L^p(\mu_k)} = \infty$. Assume that $\|\,|x|^a f\|_{L^p(\mu_k)} < \infty$. For $r > 0$, let $B_r = \{x : |x| < r\}$ and $B_r^c = \mathbb{R}^d \backslash B_r$. Denote by χ_E, χ_{B_r} and $\chi_{B_r^c}$ the characteristic functions. Let $f \in L^p(\mu_k)$, $1 < p \le 2$ and let $q = p/(p-1)$. By Minkowski's inequality, for all $r > 0$,

$$\begin{aligned} \|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} &\le \|\chi_E \mathcal{F}_k(\chi_{B_r} f)\|_{L^q(\mu_k)} + \|\chi_E \mathcal{F}_k(\chi_{B_r^c} f)\|_{L^q(\mu_k)} \\ &\le (\mu_k(E))^{1/q}\|\mathcal{F}_k(\chi_{B_r} f)\|_{L^\infty(\mu_k)} + \|\mathcal{F}_k(\chi_{B_r^c} f)\|_{L^q(\mu_k)}; \end{aligned}$$

hence it follows from (2.2) and (2.5) that

$$\|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} \le (\mu_k(E))^{1/q}\|\chi_{B_r} f\|_{L^1(\mu_k)} + \|\chi_{B_r^c} f\|_{L^p(\mu_k)}. \tag{2.7}$$

On the other hand, by Hölder's inequality,

$$\|\chi_{B_r} f\|_{L^1(\mu_k)} \le \|\,|x|^{-a}\chi_{B_r}\|_{L^q(\mu_k)}\|\,|x|^a f\|_{L^p(\mu_k)}.$$

By (2.6) and hypothesis $a < (2\gamma+d)/q$,

$$\|\,|x|^{-a}\chi_{B_r}\|_{L^q(\mu_k)} = a_k r^{-a+(2\gamma+d)/q},$$

where

$$a_k = \left[(2\gamma+d-qa)2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})\right]^{-1/q},$$

and therefore,

$$\|\chi_{B_r} f\|_{L^1(\mu_k)} \le a_k r^{-a+(2\gamma+d)/q}\|\,|x|^a f\|_{L^p(\mu_k)}. \tag{2.8}$$

Moreover,

$$\|\chi_{B_r^c} f\|_{L^p(\mu_k)} \leq \| \, |x|^{-a}\chi_{B_r^c}\|_{L^\infty(\mu_k)} \| \, |x|^a f\|_{L^p(\mu_k)} \leq r^{-a} \| \, |x|^a f\|_{L^p(\mu_k)}. \tag{2.9}$$

Combining the relations (2.7), (2.8) and (2.9), we deduce that

$$\|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \Big[r^{-a} + a_k (\mu_k(E))^{1/q} r^{-a+(2\gamma+d)/q} \Big] \| \, |x|^a f\|_{L^p(\mu_k)}.$$

We choose $r = \left(\frac{qa}{(2\gamma+d-qa)a_k}\right)^{\frac{q}{2\gamma+d}} (\mu_k(E))^{-\frac{1}{2\gamma+d}}$, we obtain the first inequality.

(ii) The second inequality holds if $\|f\|_{L^p(\mu_k)} = \infty$ or $\| \, |x|^a f\|_{L^p(\mu_k)} = \infty$. Assume that $\|f\|_{L^p(\mu_k)} + \| \, |x|^a f\|_{L^p(\mu_k)} < \infty$. From the hypothesis $a > (2\gamma+d)/q$, we deduce that the function $x \to (1+|x|^{pa})^{-1/p}$ belongs to $L^q(\mu_k)$, and by Hölder's inequality,

$$\begin{aligned} \|f\|^p_{L^1(\mu_k)} &= \left(\int_{\mathbb{R}^d} (1+|x|^{pa})^{1/p} |f(x)| (1+|x|^{pa})^{-1/p} \mathrm{d}\mu_k(x) \right)^p \\ &= \left(\int_{\mathbb{R}^d} \frac{\mathrm{d}\mu_k(x)}{(1+|x|^{pa})^{q/p}} \right)^{p/q} \Big[\|f\|^p_{L^p(\mu_k)} + \| \, |x|^a f\|^p_{L^p(\mu_k)} \Big]. \end{aligned}$$

Then the function f belongs to $L^1(\mu_k)$. Replacing $f(x)$ by $f(rx)$, $r > 0$, in the last inequality gives

$$\|f\|^p_{L^1(\mu_k)} \leq \left(\int_{\mathbb{R}^d} \frac{\mathrm{d}\mu_k(x)}{(1+|x|^{pa})^{q/p}} \right)^{p/q} \Big[r^{(2\gamma+d)(p-1)} \|f\|^p_{L^p(\mu_k)} + r^{(2\gamma+d)(p-1)-pa} \| \, |x|^a f\|^p_{L^p(\mu_k)} \Big].$$

We choose $r = \left(\frac{qa}{2\gamma+d} - 1\right)^{\frac{1}{pa}} \left(\frac{\| \, |x|^a f\|_{L^p(\mu_k)}}{\|f\|_{L^p(\mu_k)}} \right)^{1/a}$ and the fact that

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}\mu_k(x)}{(1+|x|^{pa})^{q/p}} = \frac{1}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})} \int_0^\infty \frac{r^{2\gamma+d-1}\mathrm{d}r}{(1+r^{pa})^{q/p}} = \frac{\Gamma(\frac{qa-2\gamma-d}{pa})\Gamma(\frac{2\gamma+d}{pa})}{2^{\gamma+\frac{d}{2}-1} pa \Gamma(\gamma+\frac{d}{2})\Gamma(\frac{q}{p})},$$

we deduce that

$$\|f\|_{L^1(\mu_k)} \leq K_2(a) \|f\|^{1-\frac{2\gamma+d}{qa}}_{L^p(\mu_k)} \| \, |x|^a f\|^{\frac{2\gamma+d}{qa}}_{L^p(\mu_k)}.$$

Thus,

$$\begin{aligned} \|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} &\leq (\mu_k(E))^{1/q} \|\mathcal{F}_k(f)\|_{L^\infty(\mu_k)} \\ &\leq (\mu_k(E))^{1/q} \|f\|_{L^1(\mu_k)} \\ &\leq K_2(a) (\mu_k(E))^{1/q} \|f\|^{1-\frac{2\gamma+d}{qa}}_{L^p(\mu_k)} \| \, |x|^a f\|^{\frac{2\gamma+d}{qa}}_{L^p(\mu_k)}, \end{aligned}$$

which gives the second inequality.

(iii) Let $r > 0$. From the inequality $\left(\frac{|x|}{r}\right)^{\frac{2\gamma+d}{2q}} \leq 1 + \left(\frac{|x|}{r}\right)^{\frac{2\gamma+d}{q}}$, it follows that

$$\| \, |x|^{\frac{2\gamma+d}{2q}} f\|_{L^p(\mu)} \leq r^{\frac{2\gamma+d}{2q}} \|f\|_{L^p(\mu)} + r^{-\frac{2\gamma+d}{2q}} \| \, |x|^{\frac{2\gamma+d}{q}} f\|_{L^p(\mu)}.$$

Optimizing in r, we get

$$\| \, |x|^{\frac{2\gamma+d}{2q}} f\|_{L^p(\mu)} \leq 2 \|f\|^{1/2}_{L^p(\mu)} \| \, |x|^{\frac{2\gamma+d}{q}} f\|^{1/2}_{L^p(\mu)}.$$

Thus, we deduce that

$$\begin{aligned} \|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} &\leq K_1(\frac{2\gamma+d}{2q}) (\mu_k(E))^{\frac{1}{2q}} \| \, |x|^{\frac{2\gamma+d}{2q}} f\|_{L^p(\mu)} \\ &\leq 2K_1(\frac{2\gamma+d}{2q}) (\mu_k(E))^{\frac{1}{2q}} \|f\|^{1/2}_{L^p(\mu)} \| \, |x|^{\frac{2\gamma+d}{q}} f\|^{1/2}_{L^p(\mu)}, \end{aligned}$$

which gives the result for $a = (2\gamma + d)/q$. $\square$

Remark 2.1. Let $a > 0$. If $1 < p \leq 2$, $q = p/(p-1)$ and $f \in L^p(\mu_k)$, then

$$\|f\|_{L^{\frac{q(2\gamma+d)}{2\gamma+d-qa},q}(\mu_k)} \leq K_1(a)\| \, |x|^a f\|_{L^p(\mu_k)}, \quad 0 < a < (2\gamma+d)/q,$$

$$\|f\|_{L^{\infty,q}(\mu_k)} \leq K_2(a)\|f\|^{1-\frac{2\gamma+d}{2a}}_{L^p(\mu_k)} \| \, |x|^a f\|^{\frac{2\gamma+d}{2a}}_{L^p(\mu_k)}, \quad a > (2\gamma+d)/q,$$

$$\|f\|_{L^{2q,q}(\mu_k)} \leq 2K_1(\frac{a}{2})\|f\|^{1/2}_{L^p(\mu_k)}\| \, |x|^a f\|^{1/2}_{L^p(\mu_k)}, \quad a = (2\gamma+d)/q,$$

where $L^{s,q}(\mu_k)$ is the Lorentz-space defined by the norm

$$\|f\|_{L^{s,q}(\mu_k)} := \sup_{\substack{E \subset \mathbb{R}^d \\ 0<\mu_k(E)<\infty}} \Big((\mu_k(E))^{\frac{1}{s}-\frac{1}{q}} \|\chi_E f\|_{L^q(\mu_k)} \Big).$$

In the next part of this section, we shall use the L^p local uncertainty principle (Theorem 2.1) to extend the Heisenberg-Pauli-Weyl uncertainty principles (1.1) and (1.2) to more general case.

Theorem 2.2. *Let $a, b > 0$,If $1 < p \leq 2$, $q = p/(p-1)$ and $f \in L^p(\mu_k)$, then*

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \begin{cases} K_1(a,b)\| \, |x|^a f\|^{\frac{b}{a+b}}_{L^p(\mu_k)} \| \, |y|^b \mathcal{F}_k(f)\|^{\frac{a}{a+b}}_{L^q(\mu_k)}, & 0 < a < \frac{2\gamma+d}{q}, \\ K_2(a,b)\|f\|^{\frac{b(qa-2\gamma-d)}{a(qb+2\gamma+d)}}_{L^p(\mu_k)} \| \, |x|^a f\|^{\frac{b(2\gamma+d)}{a(2\gamma+d+qb)}}_{L^p(\mu_k)} \| \, |y|^b \mathcal{F}_k(f)\|^{\frac{2\gamma+d}{2\gamma+d+qb}}_{L^q(\mu_k)}, & a > \frac{2\gamma+d}{q}, \\ K_3(a,b)\|f\|^{\frac{b}{a+2b}}_{L^p(\mu_k)}\| \, |x|^a f\|^{\frac{b}{a+2b}}_{L^p(\mu_k)} \| \, |y|^b \mathcal{F}_k(f)\|^{\frac{a}{a+2b}}_{L^q(\mu_k)}, & a = \frac{2\gamma+d}{q}, \end{cases}$$

where

$$K_1(a,b) = \frac{\left[\left(\frac{b}{a}\right)^{\frac{a}{a+b}} + \left(\frac{a}{b}\right)^{\frac{b}{a+b}} \right]^{1/q}}{\left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{\frac{ab}{(2\gamma+d)(a+b)}}} (K_1(a))^{\frac{b}{a+b}},$$

$$K_2(a,b) = \frac{\left[\left(\frac{qb}{2\gamma+d}\right)^{\frac{2\gamma+d}{2\gamma+d+qb}} + \left(\frac{2\gamma+d}{qb}\right)^{\frac{qb}{2\gamma+d+qb}} \right]^{1/q}}{\left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{\frac{b}{2\gamma+d+qb}}} (K_2(a))^{\frac{qb}{2\gamma+d+qb}},$$

and

$$K_3(a,b) = \frac{\left[\left(\frac{2b}{a}\right)^{\frac{a}{a+2b}} + \left(\frac{a}{2b}\right)^{\frac{2b}{a+2b}} \right]^{1/q}}{\left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{\frac{b}{2\gamma+d+2qb}}} (2K_1(\frac{a}{2}))^{\frac{2b}{a+2b}}.$$

Proof. (i) Let $0 < a < (2\gamma+d)/q$, $b > 0$ and $r > 0$. Then

$$\|\mathcal{F}_k(f)\|^q_{L^q(\mu_k)} = \|\chi_{B_r}\mathcal{F}_k(f)\|^q_{L^q(\mu_k)} + \|\chi_{B_r^c}\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}. \tag{2.10}$$

Firstly,

$$\|\chi_{B_r^c}\mathcal{F}_k(f)\|^q_{L^q(\mu_k)} \leq r^{-qb}\| \, |y|^b \mathcal{F}_k(f)\|^q_{L^q(\mu_k)}. \tag{2.11}$$

By (2.6) and Theorem 2.1, we get

$$\|\chi_{B_r}\mathcal{F}_k(f)\|^q_{L^q(\mu_k)} \leq K_1 r^{qa}\| \, |x|^a f\|^q_{L^p(\mu_k)}, \tag{2.12}$$

where

$$K_1=\Big(K_1(a)\Big)^q\left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{-\frac{qa}{2\gamma+d}}.$$

Combining the relations (2.10), (2.11) and (2.12), we obtain

$$\|\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}\le K_1r^{qa}\|\,|x|^af\|^q_{L^p(\mu_k)}+r^{-qb}\|\,|y|^b\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}.$$

We choose $r=\left(\frac{b}{aK_1}\right)^{\frac{1}{q(a+b)}}\left(\frac{\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}}{\|\,|x|^af\|_{L^p(\mu_k)}}\right)^{\frac{1}{a+b}}$, we get the first inequality.

(ii) Let $a>(2\gamma+d)/q$, $b>0$ and $r>0$. By (2.6) and Theorem 2.1, we get

$$\|\chi_{B_r}\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}\le K_2r^{2\gamma+d}\|f\|^{q-\frac{2\gamma+d}{a}}_{L^p(\mu_k)}\|\,|x|^af\|^{\frac{2\gamma+d}{a}}_{L^p(\mu_k)},\tag{2.13}$$

where

$$K_2=(K_2(a))^q\left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{-1}.$$

Combining the relations (2.10), (2.11) and (2.13), we obtain

$$\|\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}\le K_2r^{2\gamma+d}\|f\|^{q-\frac{2\gamma+d}{a}}_{L^p(\mu_k)}\|\,|x|^af\|^{\frac{2\gamma+d}{a}}_{L^p(\mu_k)}+r^{-qb}\|\,|y|^b\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}.$$

We choose $r=\left(\frac{qb}{(2\gamma+d)K_2}\right)^{\frac{1}{2\gamma+d+qb}}\left(\frac{\|\,|y|^b\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}}{\|f\|^{q-\frac{2\gamma+d}{a}}_{L^p(\mu_k)}\|\,|x|^af\|^{\frac{2\gamma+d}{a}}_{L^p(\mu_k)}}\right)^{\frac{1}{2\gamma+d+qb}}$, we get the second inequality.

(iii) Let $a=(2\gamma+d)/q$, $b>0$ and $r>0$. From Theorem 2.1, we get

$$\int_{B_r}|\mathcal{F}_k(f)(y)|^q\mathrm{d}\mu_k(y)\le K_3r^{\gamma+\frac{d}{2}}\|f\|^{q/2}_{L^p(\mu_k)}\|\,|x|^{\frac{2\gamma+d}{q}}f\|^{q/2}_{L^p(\mu_k)},$$

where

$$K_3=(K_1(\frac{2\gamma+d}{2q}))^q\left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{-1/2}.$$

Therefore,

$$\|\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}\le K_3r^{\gamma+\frac{d}{2}}\|f\|^{q/2}_{L^p(\mu_k)}\|\,|x|^{\frac{2\gamma+d}{q}}f\|^{q/2}_{L^p(\mu_k)}+r^{-qb}\|\,|y|^b\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}.$$

We choose $r=\left(\frac{2qb}{(2\gamma+d)K_3}\right)^{\frac{2}{2\gamma+d+2qb}}\left(\frac{\|\,|y|^b\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}}{\|f\|^{1/2}_{L^p(\mu_k)}\|\,|x|^{\frac{2\gamma+d}{q}}f\|^{1/2}_{L^p(\mu_k)}}\right)^{\frac{2q}{2\gamma+d+2qb}}$, we get the third inequality. □

Remark 2.2. The inequalities of Theorem 2.2 generalize the results of the papers [12, 13, 17]. Furthermore, we have explicitly given the values of the constants $K_1(a,b)$, $K_2(a,b)$ and $K_3(a,b)$. In particular case, if $q=2$, the inequalities of Theorem 2.2 are given by

$$\|f\|_{L^2(\mu_k)}\le K(a,b)\|\,|x|^af\|^{\frac{b}{a+b}}_{L^2(\mu_k)}\|\,|y|^b\mathcal{F}_k(f)\|^{\frac{a}{a+b}}_{L^2(\mu_k)},$$

where

$$K(a,b)=\begin{cases}K_1(a,b), & 0<a<(2\gamma+d)/2,\ b>0,\\ (K_2(a,b))^{\frac{a(2\gamma+d+2b)}{(2\gamma+d)(a+b)}}, & a>(2\gamma+d)/2,\ b>0,\\ (K_3(a,b))^{\frac{a+2b}{a+b}}, & a=(2\gamma+d)/2,\ b>0.\end{cases}$$

Here $K_1(a,b)$, $K_2(a,b)$ and $K_3(a,b)$ are the constants given by Theorem 2.2 with $p=q=2$.

3. L^p Donoho-Stark uncertainty principles

Let T and E be a measurable subsets of $\mathbb{R}^d$. We introduce the time-limiting operator P_T by

$$P_T f := \chi_T f,$$

and, we introduce the partial Dunkl integral $S_E f$ by

$$\mathcal{F}_k(S_E f) = \chi_E \mathcal{F}_k(f). \tag{3.1}$$

We shall use the L^p local uncertainty principle (Theorem 2.1) to obtain the following results for the partial Dunkl integral $S_E f$.

Lemma 3.1. *(i) If $\mu_k(E) < \infty$ and $f \in L^p(\mu_k)$, $1 \le p \le 2$,*

$$S_E f(x) = \mathcal{F}_k^{-1}(\chi_E \mathcal{F}_k(f))(x).$$

(ii) If $0 < \mu_k(E) < \infty$, $a > 0$, $1 < p \le 2$, $q = p/(p-1)$ and $f \in L^p(\mu_k)$, then

$$\|S_E f\|_{L^q(\mu_k)} \le \begin{cases} K_1(a)(\mu_k(E))^{\frac{2}{p}+\frac{a}{2\gamma+d}-1}\| \, |x|^a f\|_{L^p(\mu_k)}, & 0 < a < \frac{2\gamma+d}{q}, \\ K_2(a)(\mu_k(E))^{1/p}\|f\|_{L^p(\mu_k)}^{1-\frac{2\gamma+d}{qa}} \| \, |x|^a f\|_{L^p(\mu_k)}^{\frac{2\gamma+d}{qa}}, & a > \frac{2\gamma+d}{q}, \\ 2K_1(\frac{a}{2})(\mu_k(E))^{\frac{3}{2p}-\frac{1}{2}}\|f\|_{L^p(\mu_k)}^{1/2}\| \, |x|^a f\|_{L^p(\mu_k)}^{1/2}, & a = \frac{2\gamma+d}{q}, \end{cases}$$

where $K_1(a)$ and $K_2(a)$ are the constants given by Theorem 2.1.

Proof. (i) Let $f \in L^p(\mu_k)$, $1 \le p \le 2$ and let $q = p/(p-1)$. Then by Hölder's inequality and (2.5), we have

$$\|\chi_E \mathcal{F}_k(f)\|_{L^1(\mu_k)} \le (\mu_k(E))^{1/p}\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \le (\mu_k(E))^{1/p}\|f\|_{L^p(\mu_k)},$$

and

$$\|\chi_E \mathcal{F}_k(f)\|_{L^2(\mu_k)} \le (\mu_k(E))^{\frac{q-2}{2q}}\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \le (\mu_k(E))^{\frac{q-2}{2q}}\|f\|_{L^p(\mu_k)}.$$

Thus $\chi_E \mathcal{F}_k(f) \in L^1(\mu_k) \cap L^2(\mu_k)$. Then by (2.3) and (3.1), we obtain

$$S_E f = \mathcal{F}_k^{-1}(\chi_E \mathcal{F}_k(f)).$$

(ii) Let $f \in L^p(\mu_k)$, $1 < p \le 2$ and let $q = p/(p-1)$. By (2.5) and Hölder's inequality, we have

$$\|S_E f\|_{L^q(\mu_k)} \le \|\chi_E \mathcal{F}_k(f)\|_{L^p(\mu_k)} \le (\mu_k(E))^{\frac{2}{p}-1}\|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

Then we obtain the results from Theorem 2.1. □

Let T be a measurable subset of $\mathbb{R}^d$. We say that a function $f \in L^p(\mu_k)$, $1 \le p \le 2$, is ε-concentrated to T in $L^p(\mu_k)$-norm, if

$$\|f - P_T f\|_{L^p(\mu_k)} \le \varepsilon_T \|f\|_{L^p(\mu_k)}. \tag{3.2}$$

Let E be a measurable subset of $\mathbb{R}^d$, and $f \in L^p(\mu_k)$, $1 \le p \le 2$. We say that $\mathcal{F}_k(f)$ is ε_E-concentrated to E in $L^q(\mu_k)$-norm, $q = p/(p-1)$, if

$$\|\mathcal{F}_k(f) - \mathcal{F}_k(S_E f)\|_{L^q(\mu_k)} \le \varepsilon_E \|\mathcal{F}_k(f)\|_{L^q(\mu_k)}. \tag{3.3}$$

Let $B_p(E)$, $1 \le p \le 2$, be the set of functions $f \in L^p(\mu_k)$ that are bandlimited to E (i.e. $f \in B_p(E)$ implies $S_E f = f$).

Then, the space $B_p(E)$ satisfies the following property.

Lemma 3.2. *Let T and E be a measurable subsets of $\mathbb{R}^d$ such that $0<\mu_k(E)<\infty$, and $a>0$. If $1<p\le 2$, $q=p/(p-1)$ and $f\in B_p(E)$, then*

$$\|P_T f\|_{L^p(\mu_k)} \le \begin{cases} K_1(a)(\mu_k(T))^{1/p}(\mu_k(E))^{\frac{1}{p}+\frac{a}{2\gamma+d}}\|\,|x|^a f\|_{L^p(\mu_k)}, & 0<a<\frac{2\gamma+d}{q}, \\ K_2(a)(\mu_k(T))^{1/p}\mu_k(E)\|f\|_{L^p(\mu_k)}^{1-\frac{2\gamma+d}{qa}}\|\,|x|^a f\|_{L^p(\mu_k)}^{\frac{2\gamma+d}{qa}}, & a>\frac{2\gamma+d}{q}, \\ 2K_1(\frac{a}{2})(\mu_k(T))^{1/p}(\mu_k(E))^{\frac{1}{2p}+\frac{1}{2}}\|f\|_{L^p(\mu_k)}^{1/2}\|\,|x|^a f\|_{L^p(\mu_k)}^{1/2}, & a=\frac{2\gamma+d}{q}, \end{cases}$$

Proof. If $\mu_k(T)=\infty$, the inequality is clear. Assume that $\mu_k(T)<\infty$.
For $f\in B_p(E)$, $1<p\le 2$, from Lemma 3.1 (i), we have

$$S_E f(x)=\mathcal{F}_k^{-1}(\chi_E\mathcal{F}_k(f))(x).$$

By (2.1) and Hölder's inequality, we obtain

$$|f(x)|\le(\mu_k(E))^{1/p}\|\chi_E\mathcal{F}_k(f)\|_{L^q(\mu_k)},\quad q=p/(p-1).$$

Hence,

$$\|P_T f\|_{L^p(\mu_k)}\le(\mu_k(T))^{1/p}(\mu_k(E))^{1/p}\|\chi_E\mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

Then we obtain the results Theorem 2.1. □

The following theorem, states an uncertainty principle of concentration type for the L^p theory.

Theorem 3.1. *Let T and E be a measurable subsets of $\mathbb{R}^d$ such that $0<\mu_k(E)<\infty$, and $a>0$. If $1<p\le 2$, $q=p/(p-1)$, $f\in B_p(E)$ and f is ε_T-concentrated to T in $L^p(\mu_k)$-norm, then*

$$\|f\|_{L^p(\mu_k)} \le \begin{cases} \frac{K_1(a)}{1-\varepsilon_T}(\mu_k(T))^{1/p}(\mu_k(E))^{\frac{1}{p}+\frac{a}{2\gamma+d}}\|\,|x|^a f\|_{L^p(\mu_k)}, & 0<a<\frac{2\gamma+d}{q}, \\ \left(\frac{K_2(a)}{1-\varepsilon_T}\right)^{\frac{qa}{2\gamma+d}}(\mu_k(T))^{\frac{qa}{p(2\gamma+d)}}(\mu_k(E))^{\frac{qa}{2\gamma+d}}\|\,|x|^a f\|_{L^p(\mu_k)}, & a>\frac{2\gamma+d}{q}, \\ \left(\frac{2K_1(\frac{a}{2})}{1-\varepsilon_T}\right)^2(\mu_k(T))^{2/p}(\mu_k(E))^{\frac{1}{p}+1}\|\,|x|^a f\|_{L^p(\mu_k)}, & a=\frac{2\gamma+d}{q}, \end{cases}$$

Proof. Let $f\in B_p(E)$, $1<p\le 2$. Since f is ε_T-concentrated to T in $L^p(\mu_k)$-norm, then by (3.2), we have

$$\|f\|_{L^p(\mu_k)}\le\varepsilon_T\|f\|_{L^p(\mu_k)}+\|P_T f\|_{L^p(\mu_k)}.$$

Thus,

$$\|f\|_{L^p(\mu_k)}\le\frac{1}{1-\varepsilon_T}\|P_T f\|_{L^p(\mu_k)}.$$

Then we obtain the results from Lemma 3.2. □

Another uncertainty principle of concentration type for the L^p theory is given by the following theorem.

Theorem 3.2. *Let E be a measurable subset of $\mathbb{R}^d$ such that $0<\mu_k(E)<\infty$, and $a>0$. If $1<p\le 2$, $q=p/(p-1)$, $f\in L^p(\mu_k)$ and $\mathcal{F}_k(f)$ is ε_E-concentrated to E in $L^q(\mu_k)$-norm, then*

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \le \begin{cases} \frac{K_1(a)}{1-\varepsilon_E}(\mu_k(E))^{\frac{a}{2\gamma+d}}\|\,|x|^a f\|_{L^p(\mu_k)}, & 0<a<\frac{2\gamma+d}{q}, \\ \frac{K_2(a)}{1-\varepsilon_E}(\mu_k(E))^{1/q}\|f\|_{L^p(\mu_k)}^{1-\frac{2\gamma+d}{qa}}\|\,|x|^a f\|_{L^p(\mu_k)}^{\frac{2\gamma+d}{qa}}, & a>\frac{2\gamma+d}{q}, \\ \frac{2K_1(\frac{a}{2})}{1-\varepsilon_E}(\mu_k(E))^{\frac{1}{2q}}\|f\|_{L^p(\mu_k)}^{1/2}\|\,|x|^a f\|_{L^p(\mu_k)}^{1/2}, & a=\frac{2\gamma+d}{q}, \end{cases}$$

Proof. Let $f \in L^p(\mu_k)$, $1 < p \leq 2$. Since $\mathcal{F}_k(f)$ is ε_E-concentrated to E in $L^q(\mu_k)$-norm, $q = p/(p-1)$, then by (3.3), we deduce that

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \varepsilon_E \|\mathcal{F}_k(f)\|_{L^q(\mu_k)} + \|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

Thus,

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \frac{1}{1-\varepsilon_E} \|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

Then we obtain the results from Theorem 2.1. □

References

[1] M. Cowling and J.F. Price, Bandwidth versus time concentration: the Heisenberg-Pauli-Weyl inequality, SIAM J. Math. Anal. Vol:15 (1984), 151-165.

[2] D.L. Donoho and P.B. Stark, Uncertainty principles and signal recovery, SIAM J. Appl. Math. Vol:49, No.3 (1989), 906-931.

[3] C.F. Dunkl, Integral kernels with reflection group invariance, Canad. J. Math. Vol:43 (1991), 1213-1227.

[4] C.F. Dunkl, Hankel transforms associated to finite reflection groups, Contemp. Math. Vol:138 (1992), 123-138.

[5] W.G. Faris, Inequalities and uncertainty inequalities, Math. Phys. Vol:19 (1978), 461-466.

[6] I.I. Hirschman, A note on entropy, Amer. J. Math. Vol:79 (1957), 152-156.

[7] M.F.E.de Jeu, The Dunkl transform, Invent. Math. Vol:113 (1993), 147-162.

[8] J.F. Price, Inequalities and local uncertainty principles, J. Math. Phys. Vol:24 (1983), 1711-1714.

[9] J.F. Price, Sharp local uncertainty principles, Studia Math. Vol:85 (1987), 37-45.

[10] M. Rösler, An uncertainty principle for the Dunkl transform, Bull. Austral. Math. Soc. Vol:59 (1999), 353-360.

[11] N. Shimeno, A note on the uncertainty principle for the Dunkl transform, J. Math. Sci. Univ. Tokyo Vol:8 (2001), 33-42.

[12] F. Soltani, Heisenberg-Pauli-Weyl uncertainty inequality for the Dunkl transform on R^d, Bull. Austral. Math. Soc. Vol:87 (2013), 316-325.

[13] F. Soltani, A general form of Heisenberg-Pauli-Weyl uncertainty inequality for the Dunkl transform, Int. Trans. Spec. Funct. Vol:24, No.5 (2013), 401-409.

[14] F. Soltani, Donoho-Stark uncertainty principle associated with a singular secondorder differential operator, Int. J. Anal. Appl. Vol:4, No.1 (2014), 1-10.

[15] F. Soltani, L^p uncertainty principles on Sturm-Liouville hypergroups, Acta Math. Hungar. Vol:142, No.2 (2014), 433-443.

[16] F. Soltani, L^p local uncertainty inequality for the Sturm-Liouville transform, CUBO Math. J. Vol:16, No.1 (2014), 95-104.

[17] F. Soltani, An L^p Heisenberg-Pauli-Weyl uncertainty principle for the Dunkl transform, Konuralp J. Math. Vol:2, No.1 (2014), 1-6.

[18] F. Soltani, L^p Donoho-Stark uncertainty principles for the Dunkl transform on $\mathbb{R}^d$, J. Phys. Math. Vol:5, No.1 (2014), 4 pages.

[19] E.M. Stein, Interpolation of linear operators, Trans. Amer. Math. Soc. Vol:83 (1956), 482-492.

[20] E.M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press., Princeton, N.J, 1971.

Department of Mathematics, Faculty of Science, Jazan University, P.O.Box 277, Jazan 45142, Saudi Arabia

E-mail address: fethisoltani10@yahoo.com

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 110–121 (2015) ©KJM

MATRICES OF GENERALIZED DUAL QUATERNIONS

MEHDI JAFARI

Abstract. After a brief review of some algebraic properties of a generalized dual quaternion, we investigate properties of matrix associated with a generalized dual quaternion and examine De Moivre's formula for this matrix, from which the n-th power of such a matrix can be determined. We give the relation between the powers of these matrices.

1. Introduaction

Mathematically, quaternions represent the natural extension of complex numbers, forming an associative algebra under addition and multiplication. Dual numbers and dual quaternions were introduced in the 19th century by W.K. Clifford [5], as a tool for his geometrical investigation. Study [17] and Kotel'nikov [12] systematically applied the dual number and dual vector in their studies of line geometry and kinematics and independently discovered the transfer principle.

The use of dual numbers, dual numbers matrix and dual quaternions in instantaneous spatial kinematics are investigated in [15,18]. The Euler's and De-Moivre's formulas for the complex numbers are generalized for quaternions in [4]. These formulas are also investigated for the cases of split and dual quaternions in [11,14]. Some algebraic properties of Hamilton operators are considered in [1,2] where dual quaternions have been expressed in terms of 4×4 matrices by means of these operators. Properties of these matrices have applications in mechanics, quantum physics and computer-aided geometric design [3,20]. Recently, we have derived the De-Moivre's and Euler's formulas for matrices associated with real, dual quaternions and every power of these matrices are immediately obtained [9,10].

A generalization of real and dual quaternions are also investigated by author and et al. [6,7]. Here, after a review of some algebraic properties of generalized dual quaternions, we study the Euler's and De-Moivre's formulas for generalized dual quaternions and for the matrices associated with them. Also, the n-th roots of these matrices are obtained. Finally, we give some examples for more clarification.

Date: January 1, 2013 and, in revised form, February 2, 2013.

2000 *Mathematics Subject Classification.* 30G35.

Key words and phrases. De-Moiver's formula, Hamilton operator, Generalized dual quaternion, Dual quasi-orthogonal.

2. Preliminaries

In this section, we give a brief summary algebra of generalized dual quaternions. For detailed information about this concept, we refer the reader to [7, 8].

Definition 2.1. A generalized dual quaternion Q is written as

$$Q = A_\circ 1 + A_1 i + A_2 j + A_3 k,$$

where $A., A_1, A_2$ and A_3 are dual numbers and i, j, k are quaternionic units which satisfy the equalities

$$\begin{aligned} i^2 &= -\alpha,\ j^2 = -\beta,\ k^2 = -\alpha\beta, \\ ij &= k = -ji,\ jk = \beta i = -kj, \end{aligned}$$

and

$$ki = \alpha j = -ik, \quad \alpha, \beta \in \mathbb{R}.$$

As a consequence of this definition, a generalized dual quaternion Q can also be written as;

$$Q = q + \varepsilon q^*,\ q, q^* \in H_{\alpha\beta}$$

where q and q^*, real and pure generalized dual quaternion components, respectively. A quaternion $Q = A_0 1 + A_1 i + A_2 j + A_3 k$ is pieced into two parts with scalar piece $S_Q = A.$ and vectorial piece $\overrightarrow{V}_Q = A_1 i + A_2 j + A_3 k$. We also write $Q = S_Q + \overrightarrow{V}_Q$. The conjugate of $Q = S_Q + \overrightarrow{V}_Q$ is then defined as $\overline{Q} = S_Q - \overrightarrow{V}_Q$. If $S_Q = 0$, then Q is called pure generalized dual quaternion, we may be called its generalized dual vector. The set of all generalized dual vectors denoted by $D^3_{\alpha\beta}$ [15].

Dual quaternionic multiplication of two dual quaternions $Q = S_Q + \overrightarrow{V}_Q$ and $P = S_P + \overrightarrow{V}_P$ is defined;

$$\begin{aligned} QP &= S_Q S_P - g(\overrightarrow{V}_Q, \overrightarrow{V}_P) + S_P \overrightarrow{V}_Q + S_Q \overrightarrow{V}_P + \overrightarrow{V}_Q \wedge \overrightarrow{V}_P \\ &= A_\circ B_\circ - (\alpha A_1 B_1 + \beta A_2 B_2 + \alpha\beta A_3 B_3) + A_\circ (B_1, B_2, B_3) + B_\circ (A_1, A_2, A_3) \\ &\quad + (\beta(A_2 B_3 - A_3 B_2), \alpha(A_3 B_1 - A_1 B_3), (A_1 B_2 - A_2 B_1)). \end{aligned}$$

Also, It could be written

$$QP = \begin{bmatrix} A_\circ & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_\circ & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_\circ & -\alpha\ A_1 \\ A_3 & -A_2 & A_1 & A_\circ \end{bmatrix} \begin{bmatrix} B_\circ \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

So, the multiplication of dual quaternions as matrix-by-vector product. The norm of Q is defined as $N_Q = Q\overline{Q} = \overline{Q}Q = A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2$. If $N_Q = 1$, then Q is called a unit generalized dual quaternion.The set of all generalized dual quaternions (abbreviated GDQ) are denoted by $\widetilde{H}_{\alpha\beta}$.

Theorem 2.1. *Every unit generalized dual quaternion is a screw operator* [8].

We investigate the properties of the generalized dual quaternions in two different cases.

Case 1: Let α, β be positive numbers.

Definition 2.2. Let $\widehat{S}^3_D$ be the set of all unit generalized dual quaternions and $\widehat{S}^2_D$ the set of unit generalized dual vector, that is,

$$\begin{aligned}
\widehat{S}^3_D &= \left\{Q \in \widetilde{H}_{\alpha\beta} : N_Q = 1\right\} \subset \widetilde{H}_{\alpha\beta}, \\
\widehat{S}^2_D &= \left\{\overrightarrow{V}_Q = (A_1, A_2, A_3) : g(\overrightarrow{V}_Q, \overrightarrow{V}_Q) = \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 = 1\right\}.
\end{aligned}$$

Definition 2.3. Every nonzero unit generalized dual quaternion can be written in the polar form

$$\begin{aligned}
Q &= A_0 + A_1 i + A_2 j + A_3 k \\
&= \cos\phi + \overrightarrow{W}\sin\phi,
\end{aligned}$$

where $\cos\phi = A_0$, $\sin\phi = \sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}$. $\phi = \varphi + \varepsilon\varphi^*$ is a dual angle and the unit generalized dual vector $\overrightarrow{W}$ is given by

$$\overrightarrow{W} = \frac{A_1 i + A_2 j + A_3 j}{\sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}} = \frac{A_1 i + A_2 j + A_3 j}{\sqrt{1 - A_0^2}},$$

with $\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 \neq 0$.

Note that $\overrightarrow{W}$ is a unit generalized dual vector to which a directed line in $\mathbb{R}^3_{\alpha\beta}$ corresponds by means of the generalized E. Study map [16].

Theorem 2.2. *(De-Moivre's formula) Let* $Q = e^{\overrightarrow{W}\phi} = \cos\phi + \overrightarrow{W}\sin\phi \in \widehat{S}^3_D$, *where* $\phi = \varphi + \varepsilon\varphi^*$ *is dual angle and* $\overrightarrow{W} \in \widehat{S}^2_D$. *Then for every integer* n;

$$Q^n = \cos n\phi + \overrightarrow{W}\sin n\phi.$$

Proof. The proof follows immediately from the induction (see [13]). □

Every generalized dual qauetrnion can be separated into two cases:
1) Generalized dual quaternions with dual angles ($\phi = \varphi + \varepsilon\varphi^*$); *i.e.*

$$Q = \sqrt{N_Q}(\cos\phi + \overrightarrow{W}\sin\phi).$$

2) Generalized dual quaternions with real angles ($\phi = \varphi$, $\varphi^* = 0$); *i.e.*

$$Q = \sqrt{N_Q}(\cos\varphi + \overrightarrow{W}\sin\varphi).$$

Theorem 2.3. *Let* $Q = \cos\varphi + \overrightarrow{W}\sin\varphi \in \widehat{S}^3_D$.*De-Moivre's formula implies that there are uncountably many unit dual generalized quaternions* Q *satisfying* $Q^n = 1$ *for* $n > 2$ [13].

Case 2: Let α be a positive and β a negative numbers.

In this case, for a generalized dual quaternion $Q = A_0 + A_1 i + A_2 j + A_3 k$, we can consider three different subcases.

Subcase (i): The norm of generalized dual quaternion is negative, *i.e.*

$$N_Q = A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 < 0,$$

since $0 < A_0^2 < -\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2$ thus $\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 < 0$. In this case, the polar form of Q is defined as

$$Q = r(\sinh\Psi + \overrightarrow{W}\cosh\Psi)$$

where we assume

$$\begin{aligned} r &= \sqrt{|N_Q|} = \sqrt{|A_\circ^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2|}, \\ \sinh\Psi &= \frac{A_0}{\sqrt{|N_Q|}}, \quad \cosh\Psi = \frac{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}{\sqrt{|N_Q|}}. \end{aligned}$$

The unit dual vector $\overrightarrow{W}$ (axis of quaternion) is defined as

$$\overrightarrow{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}(A_1, A_2, A_3).$$

Theorem 2.4. *(De-Moivre's formula) Let* $Q = \sinh\Psi + \overrightarrow{W}\cosh\Psi$ *be a unit generalized dual quaternion with* $N_Q < 0$. *Then for every integer* n;

$$Q^n = \sinh n\Psi + \overrightarrow{W}\cosh n\Psi.$$

Proof. The proof follows immediately from the induction [13]. □

Subcase (ii): The norm of generalized dual quaternion is positive and the norm of its vector part to be negative, *i.e.*

$$N_Q > 0, \quad N_{\overrightarrow{V}_Q} = \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 < 0,$$

In this case, the polar form of Q is defined as

$$Q = r(\cosh\Phi + \overrightarrow{W}\sinh\Phi)$$

where we assume

$$\begin{aligned} r &= \sqrt{N_Q} = \sqrt{A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}, \\ \cosh\Phi &= \frac{A_0}{\sqrt{N_Q}}, \quad \sinh\Phi = \frac{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}{\sqrt{N_Q}}. \end{aligned}$$

The unit dual vector $\overrightarrow{W}$ (axis of quaternion) is defined as

$$\overrightarrow{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}(A_1, A_2, A_3).$$

Theorem 2.5. *Let* $Q = \cosh\Phi + \overrightarrow{W}\sinh\Phi$ *be a unit generalized dual quaternion with* $N_Q > 0$ *and* $N_{\overrightarrow{V}_Q} < 0$*. Then for every integer* n*;*

$$Q^n = \cosh n\Phi + \overrightarrow{W}\sinh n\Phi.$$

Proof. The proof follows immediately from the induction [13]. □

Subcase (iii): The norm of generalized dual quaternion is positive and the norm of its vector part to be positive, *i.e.*

$$N_Q > 0, \quad N_{\overrightarrow{V}_Q} = \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 > 0,$$

In this case, the polar form of Q is defined as

$$Q = r(\cos\Theta + \overrightarrow{W}\sin\Theta)$$

where we assume

$$\begin{aligned} r &= \sqrt{N_Q} = \sqrt{A_\circ^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}, \\ \cos\Theta &= \frac{A_0}{\sqrt{N_Q}}, \quad \sin\Theta = \frac{\sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}}{\sqrt{N_Q}}. \end{aligned}$$

The unit dual vector $\overrightarrow{W}$ (axis of quaternion) is defined as

$$\overrightarrow{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}}(A_1, A_2, A_3).$$

Theorem 2.6. *Let* $Q = \cos\Theta + \overrightarrow{W}\sin\Phi$ *be a unit generalized dual quaternion with* $N_Q > 0$ *and* $N_{\overrightarrow{V}_Q} > 0$*. Then for every integer* n*;*

$$Q^n = \cos n\Theta + \overrightarrow{W}\sin n\Theta.$$

Proof. The proof follows immediately from the induction. □

2.1. 4×4 **Dual Matrix representation of GDQ.**

In this section, we introduce the $\mathbb{R}$-linear transformations representing left multiplication in $\widetilde{H}_{\alpha\beta}$ and look for also the De-Moiver's formula for corresponding matrix representation. Let Q be a generalized dual quaternion, then the linear map $\overset{+}{h}_Q : \widetilde{H}_{\alpha\beta} \to \widetilde{H}_{\alpha\beta}$ defined as follows;

$$\overset{+}{h}_Q(P) = QP, \quad P \in \widetilde{H}_{\alpha\beta}.$$

The Hamilton's operator $\overset{+}{H}$, could be represented as the matrix

$$\overset{+}{H}(Q) = \begin{bmatrix} A_0 & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_0 & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & -\alpha A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}.$$

Theorem 2.7. *If Q and P are two generalized dual quaternions, λ is a real number, then the following identities hold;*

i. $Q = P \Leftrightarrow \overset{+}{H}(Q) = \overset{+}{H}(P)$

ii. $\overset{+}{H}(Q+P) = \overset{+}{H}(Q) + \overset{+}{H}(P)$

iii. $\overset{+}{H}(\lambda Q) = \lambda \overset{+}{H}(Q)$

iv. $\overset{+}{H}(QP) = \overset{+}{H}(Q)\overset{+}{H}(P)$

v. $\overset{+}{H}(Q^{-1}) = [\overset{+}{H}(Q)]^{-1}$, $N_Q \neq 0$.

vi. $\overset{+}{H}(\overline{Q}) = [\overset{+}{H}(Q)]^T$

vii. $\det[\overset{+}{H}(Q)] = (N_Q)^2$

viii. $tr[\overset{+}{H}(Q)] = 4A_\circ$

Proof. The proof can be found in [7]. □

Following the usual matrix nomenclature, a matrix $\hat{A}$ is called a dual quasi-orthogonal matrix if $\hat{A}^T \epsilon \hat{A} = A\epsilon$, where A is a dual number and ϵ is a 4×4 diagonal matrix. A matrix $\hat{A}$ is called dual quasi-orthonormal matrix if $A = 1$ [8].

Theorem 2.8. *Matrices generated by operators by $\overset{+}{H}$ is a dual quasi-orthogonal matrices; i.e. $[\overset{+}{H}(Q)]^T \epsilon \overset{+}{H}(Q) = N_Q \epsilon$ where*

$$\epsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix}.$$

Also, $\overset{+}{H}(Q)$ is a dual quasi-orthonormal matrices if Q is a unit generalized dual quaternion [8].

Theorem 2.9. *The ϕ map defined as*

$$\phi : (\widetilde{H}_{\alpha\beta}, +, .) \rightarrow (M_{(4,D)}, \oplus, \otimes)$$

$$\phi(A_0 + A_1 i + A_2 j + A_3 k) \rightarrow \begin{bmatrix} A_0 & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_0 & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & -\alpha A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix},$$

is an isomorphism of algebras.

Proof. We first demonstrate its homomorphic properties. If $Q = A_0 1 + A_1 i + A_2 j + A_3 k$ and $P = B_0 1 + B_1 i + B_2 j + B_3 k$ are any two GDQ, then

$$\begin{aligned}
\phi\{Q+P\} &= \phi\{(A_0+B_0)+(A_1+B_1)i+(A_2+B_2)j+(A_3+B_3)k\} \\
&= \begin{bmatrix} A_0+B_0 & -\alpha(A_1+B_1) & -\beta(A_2+B_2) & -\alpha\beta(A_3+B_3) \\ (A_1+B_1) & A_0+B_0 & -\beta(A_3+B_3) & \beta(A_2+B_2) \\ (A_2+B_2) & \alpha(A_3+B_3) & A_0+B_0 & -\alpha(A_1+B_1) \\ (A_3+B_3) & -(A_2+B_2) & (A_1+B_1) & A_0+B_0 \end{bmatrix} \\
&= \begin{bmatrix} A_0 & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_0 & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & -\alpha A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} + \begin{bmatrix} B_0 & -\alpha B_1 & -\beta B_2 & -\alpha\beta B_3 \\ B_1 & B_0 & -\beta B_3 & \beta B_2 \\ B_2 & \alpha B_3 & B_0 & -\alpha B_1 \\ B_3 & -B_2 & B_1 & B_0 \end{bmatrix} \\
&= \phi\{Q\} \oplus \phi\{P\}, \\
\phi\{QP\} &= \phi\{A_0B_0 - (\alpha A_1B_1 + \beta A_2B_2 + \alpha\beta A_3B_3) + A_\circ(B_1,B_2,B_3) + B_\circ(A_1,A_2,A_3) \\
&\quad (\beta(A_2B_3 - A_3B_2), \alpha(A_3B_1 - A_1B_3), (A_1B_2 - A_2B_1))\} \\
&= \begin{bmatrix} A_0 & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_0 & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & -\alpha A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} \otimes \begin{bmatrix} B_0 & -\alpha B_1 & -\beta B_2 & -\alpha\beta B_3 \\ B_1 & B_0 & -\beta B_3 & \beta B_2 \\ B_2 & \alpha B_3 & B_0 & -\alpha B_1 \\ B_3 & -B_2 & B_1 & B_0 \end{bmatrix}
\end{aligned}$$

□

We can express the matrix $\overset{+}{H}(Q)$ in polar form. Let Q be a unit generalized dual quaternion and $\alpha, \beta > 0$. Since

$$\begin{aligned}
Q &= A_0 + A_1e_1 + A_2e_2 + A_3e_3 \\
&= \cos\phi + \overrightarrow{W}\sin\phi \\
&= \cos\phi + (w_1, w_2, w_3)\sin\phi
\end{aligned}$$

so we have

$$\begin{bmatrix} A_0 & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_0 & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & -\alpha A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\alpha w_1\sin\phi & -\beta w_2\sin\phi & -\alpha\beta w_3\sin\phi \\ w_1\sin\phi & \cos\phi & -\beta w_3\sin\phi & \beta w_2\sin\phi \\ w_2\sin\phi & \alpha w_3\sin\phi & \cos\phi & -\alpha w_1\sin\phi \\ w_3\sin\phi & -w_2\sin\phi & w_1\sin\phi & \cos\phi \end{bmatrix}.$$

Theorem 2.10. *(De-Moivre's formula) For an integer n and matrix*

$$A = \begin{bmatrix} \cos\phi & -\alpha w_1\sin\phi & -\beta w_2\sin\phi & -\alpha\beta w_3\sin\phi \\ w_1\sin\phi & \cos\phi & -\beta w_3\sin\phi & \beta w_2\sin\phi \\ w_2\sin\phi & \alpha w_3\sin\phi & \cos\phi & -\alpha w_1\sin\phi \\ w_3\sin\phi & -w_2\sin\phi & w_1\sin\phi & \cos\phi \end{bmatrix} \quad (1.1)$$

the n-th power of the matrix A reads

$$A^n = \begin{bmatrix} \cos n\phi & -\alpha w_1\sin n\phi & -\beta w_2\sin n\phi & -\alpha\beta w_3\sin n\phi \\ w_1\sin n\phi & \cos n\phi & -\beta w_3\sin n\phi & \beta w_2\sin n\phi \\ w_2\sin n\phi & \alpha w_3\sin n\phi & \cos n\phi & -\alpha w_1\sin n\phi \\ w_3\sin n\phi & -w_2\sin n\phi & w_1\sin n\phi & \cos n\phi \end{bmatrix}.$$

Proof. The proof follows immediately from the induction. □

Special cases:

1) If ϕ, w_1, w_2 and w_3 be real numbers, then Theorem 3.4 holds for real quaternions (see [10]).
2) If $\alpha = \beta = 1$, then Theorem 3.4 holds for dual quaternions (see [9]).

Example 2.1. Let $Q = \frac{1}{\sqrt{2}} + \frac{1}{2}(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}}, \varepsilon)$ be a unit generalized dual quaternion. The matrix corresponding to this quaternion is

$$\begin{aligned} A &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{\alpha}}{2} & -\frac{\sqrt{\beta}}{2} & -\frac{\alpha\beta\varepsilon}{2} \\ \frac{1}{\sqrt{2\alpha}} & \frac{1}{\sqrt{2}} & -\frac{\beta\varepsilon}{2} & \frac{\sqrt{\beta}}{2} \\ \frac{1}{\sqrt{2\beta}} & \frac{\alpha\varepsilon}{2} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{\alpha}}{2} \\ \frac{\varepsilon}{2} & -\frac{1}{\sqrt{2\beta}} & \frac{1}{\sqrt{2\alpha}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \\ &= \begin{bmatrix} \cos\frac{\pi}{4} & -\alpha w_1 \sin\frac{\pi}{4} & -\beta w_2 \sin\frac{\pi}{4} & -\alpha\beta w_3 \sin\frac{\pi}{4} \\ w_1 \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & -\beta w_3 \sin\frac{\pi}{4} & \beta w_2 \sin\frac{\pi}{4} \\ w_2 \sin\frac{\pi}{4} & \alpha w_3 \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & -\alpha w_1 \sin\frac{\pi}{4} \\ w_3 \sin\frac{\pi}{4} & -w_2 \sin\frac{\pi}{4} & w_1 \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \end{aligned}$$

every powers of this matix are found to be with the aid of Theorem *3.4* , for example, 6−th and 15-th power is

$$\begin{aligned} A^6 &= \begin{bmatrix} 0 & \sqrt{\frac{\alpha}{2}} & \sqrt{\frac{\beta}{2}} & \frac{1}{\sqrt{2}}\varepsilon\alpha\beta \\ -\frac{1}{\sqrt{2\alpha}} & 0 & \frac{1}{\sqrt{2}}\varepsilon\beta & -\sqrt{\frac{\beta}{2}} \\ -\frac{1}{\sqrt{2\beta}} & -\frac{1}{\sqrt{2}}\varepsilon\alpha & 0 & \sqrt{\frac{\alpha}{2}} \\ -\frac{1}{\sqrt{2}}\varepsilon & \frac{1}{\sqrt{2\beta}} & -\frac{1}{\sqrt{2\alpha}} & 0 \end{bmatrix}, \\ A^{15} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{\alpha}}{2} & \frac{\sqrt{\beta}}{2} & \alpha\beta\frac{\varepsilon}{2} \\ -\frac{1}{\sqrt{2\alpha}} & \frac{1}{\sqrt{2}} & \beta\frac{\varepsilon}{2} & -\frac{\sqrt{\beta}}{2} \\ \frac{1}{2\sqrt{\beta}} & -\alpha\frac{\varepsilon}{2} & \frac{1}{\sqrt{2}} & \frac{\sqrt{\alpha}}{2} \\ -\frac{\varepsilon}{2} & \frac{1}{2\sqrt{\beta}} & -\frac{1}{\sqrt{2\alpha}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \end{aligned}$$

2.2. **Euler's Formula for Matrices of GDQ.**

Definition 2.4. Let A be a dual matrix. We choose

$$A = \begin{bmatrix} 0 & -\alpha w_1 & -\beta w_2 & -\alpha\beta w_3 \\ w_1 & 0 & -\beta w_3 & \beta w_2 \\ w_2 & \alpha w_3 & 0 & -\alpha w_1 \\ w_3 & -w_2 & w_1 & 0 \end{bmatrix}$$

then one immediately finds $A^2 = -I_4$. We have a netural generalization of Euler's formula for matrix A;

$$\begin{aligned} e^{A\phi} &= I_4 + A\phi + \frac{(A\phi)^2}{2!} + \frac{(A\phi)^3}{3!} + \frac{(A\phi)^4}{4!} + ... \\ &= I_4(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} -)... + A(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - ...) \\ &= \cos\phi + A\sin\phi, \\ &= \begin{bmatrix} \cos\phi & -\alpha w_1 \sin\phi & -\beta w_2 \sin\phi & -\alpha\beta w_3 \sin\phi \\ w_1 \sin\phi & \cos\phi & -\beta w_3 \sin\phi & \beta w_2 \sin\phi \\ w_2 \sin\phi & \alpha w_3 \sin\phi & \cos\phi & -\alpha w_1 \sin\phi \\ w_3 \sin\phi & -w_2 \sin\phi & w_1 \sin\phi & \cos\phi \end{bmatrix}. \end{aligned}$$

2.3. *n*-th Roots of Matrices of GDQ.

Let Q be a unit generalized dual quaternion with real angle, *i.e.* $\phi = \varphi$ and $\varphi^* = 0$. The matrix associated with the quaternion Q is of the form (*1.1*). In a more general case, we assume for the matrix of (*1.1*)

$$A = \begin{bmatrix} \cos(\varphi + 2k\pi) & -\alpha w_1 \sin(\varphi + 2k\pi) & -\beta w_2 \sin(\varphi + 2k\pi) & -\alpha\beta w_3 \sin(\varphi + 2k\pi) \\ w_1 \sin(\varphi + 2k\pi) & \cos(\varphi + 2k\pi) & -\beta w_3 \sin(\varphi + 2k\pi) & \beta w_2 \sin(\varphi + 2k\pi) \\ w_2 \sin(\varphi + 2k\pi) & \alpha w_3 \sin(\varphi + 2k\pi) & \cos(\varphi + 2k\pi) & -\alpha w_1 \sin(\varphi + 2k\pi) \\ w_3 \sin(\varphi + 2k\pi) & -w_2 \sin(\varphi + 2k\pi) & w_1 \sin(\varphi + 2k\pi) & \cos(\varphi + 2k\pi) \end{bmatrix},$$

where $k \in \mathbb{Z}$.

The equation $X^n = A$ has n roots. Thus

$$A_k^{\frac{1}{n}} = \begin{bmatrix} \cos(\frac{\varphi+2k\pi}{n}) & -\alpha w_1 \sin(\frac{\varphi+2k\pi}{n}) & -\beta w_2 \sin(\frac{\varphi+2k\pi}{n}) & -\alpha\beta w_3 \sin(\frac{\varphi+2k\pi}{n}) \\ w_1 \sin(\frac{\varphi+2k\pi}{n}) & \cos(\frac{\varphi+2k\pi}{n}) & -\beta w_3 \sin(\frac{\varphi+2k\pi}{n}) & \beta w_2 \sin(\frac{\varphi+2k\pi}{n}) \\ w_2 \sin(\frac{\varphi+2k\pi}{n}) & \alpha w_3 \sin(\frac{\varphi+2k\pi}{n}) & \cos(\frac{\varphi+2k\pi}{n}) & -\alpha w_1 \sin(\frac{\varphi+2k\pi}{n}) \\ w_3 \sin(\frac{\varphi+2k\pi}{n}) & -w_2 \sin(\frac{\varphi+2k\pi}{n}) & w_1 \sin(\frac{\varphi+2k\pi}{n}) & \cos(\frac{\varphi+2k\pi}{n}) \end{bmatrix}.$$

For $k = 0$, the first root is

$$A_0^{\frac{1}{n}} = \begin{bmatrix} \cos(\frac{\varphi}{n}) & -\alpha w_1 \sin(\frac{\varphi}{n}) & -\beta w_2 \sin(\frac{\varphi}{n}) & -\alpha\beta w_3 \sin(\frac{\varphi}{n}) \\ w_1 \sin(\frac{\varphi}{n}) & \cos(\frac{\varphi}{n}) & -\beta w_3 \sin(\frac{\varphi}{n}) & \beta w_2 \sin(\frac{\varphi}{n}) \\ w_2 \sin(\frac{\varphi}{n}) & \alpha w_3 \sin(\frac{\varphi}{n}) & \cos(\frac{\varphi}{n}) & -\alpha w_1 \sin(\frac{\theta}{n}) \\ w_3 \sin(\frac{\varphi}{n}) & -w_2 \sin(\frac{\varphi}{n}) & w_1 \sin(\frac{\varphi}{n}) & \cos(\frac{\varphi}{n}) \end{bmatrix},$$

for $k = 1$, the second root is

$$A_1^{\frac{1}{n}} = \begin{bmatrix} \cos(\frac{\varphi+2\pi}{n}) & -\alpha w_1 \sin(\frac{\varphi+2\pi}{n}) & -\beta w_2 \sin(\frac{\varphi+2\pi}{n}) & -\alpha\beta w_3 \sin(\frac{\varphi+2\pi}{n}) \\ w_1 \sin(\frac{\varphi+2\pi}{n}) & \cos(\frac{\varphi+2\pi}{n}) & -\beta w_3 \sin(\frac{\varphi+2\pi}{n}) & \beta w_2 \sin(\frac{\varphi+2\pi}{n}) \\ w_2 \sin(\frac{\varphi+2\pi}{n}) & \alpha w_3 \sin(\frac{\varphi+2\pi}{n}) & \cos(\frac{\varphi+2\pi}{n}) & -\alpha w_1 \sin(\frac{\varphi+2\pi}{n}) \\ w_3 \sin(\frac{\varphi+2\pi}{n}) & -w_2 \sin(\frac{\varphi+2\pi}{n}) & w_1 \sin(\frac{\varphi+2\pi}{n}) & \cos(\frac{\varphi+2\pi}{n}) \end{bmatrix}.$$

Similarly, for $k = n - 1$, we obtain the n-th root.

2.4. **Relation Between Power of Matrices.** The relations between the powers of matrices associated with a generalized dual quaternion can be realized by the following Theorem.

Theorem 2.11. *Q be a unit generalized dual quaternion with the polar form $Q = \cos\varphi + \overrightarrow{W}\sin\varphi$. If $m = \frac{2\pi}{\varphi} \in \mathbb{Z}^+ - \{1\}$, then $n \equiv p(mod\ m)$ is possible if and only if $Q^n = Q^p$.*

Proof. Let $n \equiv p(mod\ m)$. Then we have $n = a.m + p$, where $a \in \mathbb{Z}$.

$$\begin{aligned} Q^n &= \cos n\varphi + \overrightarrow{W}\sin n\varphi \\ &= \cos(am+p)\varphi + \overrightarrow{W}\sin(am+p)\varphi \\ &= \cos(a\frac{2\pi}{\varphi}+p)\varphi + \overrightarrow{W}\sin(a\frac{2\pi}{\varphi}+p)\varphi \\ &= \cos(p\varphi + a2\pi) + \overrightarrow{W}\sin(p\varphi + a2\pi) \\ &= \cos(p\varphi) + \overrightarrow{W}\sin(p\varphi) \\ &= Q^p. \end{aligned}$$

Now suppose $Q^n = \cos n\varphi + \overrightarrow{W}\sin n\varphi$ and $Q^p = \cos p\varphi + \overrightarrow{W}\sin p\varphi$. Since $Q^n = Q^p$, we have $\cos n\varphi = \cos p\varphi$ and $\sin n\varphi = \sin p\varphi$, which means $n\varphi = p\varphi + 2\pi a$, $a \in \mathbb{Z}$. Thus $n = a\frac{2\pi}{\varphi} + p$, $n \equiv p(mod\ m)$.

□

Example 2.2. Let $Q = \frac{1}{\sqrt{2}} + \frac{1}{2}(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}}, \varepsilon)$ be a unit generalized dual quaternion. From the theorem 6.1, $m = \frac{2\pi}{\pi/4} = 8$, we have

$$\begin{aligned} Q &= Q^9 = Q^{17} = ... \\ Q^2 &= Q^{10} = Q^{18} = ... \\ Q^3 &= Q^{11} = Q^{19} = ... \\ Q^4 &= Q^{12} = Q^{20} = ... = -1 \\ &... \\ Q^8 &= Q^{16} = Q^{24} = ... = 1. \end{aligned}$$

Theorem 2.12. *Q be a unit dual quaternion with the polar form $Q = \cos\varphi + \overrightarrow{W}\sin\varphi$. Let $m = \frac{2\pi}{\varphi} \in \mathbb{Z}^+ - \{1\}$ and the matrix A corresponds to Q. Then $n \equiv p(mod\ m)$ is possible if and only if $A^n = A^p$.*

Proof. Proof is same as above. □

Example 2.3. Let $Q = -\frac{1}{2} + (\varepsilon, \frac{1}{\sqrt{2\beta}}, \frac{1}{2\sqrt{\alpha\beta}}) = \cos\frac{2\pi}{3} + \overrightarrow{W}\sin\frac{2\pi}{3}$ be a unit generalized dual quaternion. The matrix corresponding to this dual quaternion is

$$A = \begin{bmatrix} -\frac{1}{2} & -\alpha\varepsilon & -\sqrt{\frac{\beta}{2}} & -\frac{1}{2}\sqrt{\alpha\beta} \\ \varepsilon & -\frac{1}{2} & -\frac{1}{2}\sqrt{\frac{\beta}{\alpha}} & \frac{\sqrt{\beta}}{2} \\ \frac{1}{\sqrt{2\beta}} & \frac{1}{2}\sqrt{\frac{\alpha}{\beta}} & -\frac{1}{2} & -\alpha\varepsilon \\ \frac{1}{2\sqrt{\alpha\beta}} & -\frac{1}{\sqrt{2\beta}} & \varepsilon & -\frac{1}{2} \end{bmatrix},$$

From the Theorem 6.2, $m=\frac{2\pi}{2\pi/3}=3$, we have

$$\begin{aligned} A &= A^4=A^7=... \\ A^2 &= A^5=A^8=... \\ A^3 &= A^6=A^9=...=I_4 \end{aligned}$$

The square roots of the matrix A can be achieved too;

$$A_k^{\frac{1}{2}}=\begin{bmatrix} \cos\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -\alpha w_1\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -\beta w_2\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -\alpha\beta w_3\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) \\ w_1\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & \cos\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -\beta w_3\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & \beta w_2\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) \\ w_2\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & \alpha w_3\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & \cos\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -\alpha w_1\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) \\ w_3\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -w_2\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & w_1\sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & \cos\left(\frac{2k\pi+120^{\cdot}}{2}\right) \end{bmatrix}$$

The first root for $k=0$ reads

$$A_0^{\frac{1}{2}}=\begin{bmatrix} \frac{1}{2} & -\alpha\varepsilon & -\sqrt{\frac{\beta}{2}} & -\frac{1}{2}\sqrt{\alpha\beta} \\ \varepsilon & \frac{1}{2} & -\frac{1}{2}\sqrt{\frac{\beta}{\alpha}} & \frac{\sqrt{\beta}}{2} \\ \frac{1}{\sqrt{2\beta}} & \frac{1}{2}\sqrt{\frac{\alpha}{\beta}} & \frac{1}{2} & -\alpha\varepsilon \\ \frac{1}{2\sqrt{\alpha\beta}} & -\frac{1}{\sqrt{2\beta}} & \varepsilon & \frac{1}{2} \end{bmatrix}$$

and the second one for $k=1$ becomes

$$A_1^{\frac{1}{2}}=\begin{bmatrix} -\frac{1}{2} & \alpha\varepsilon & \sqrt{\frac{\beta}{2}} & \frac{1}{2}\sqrt{\alpha\beta} \\ -\varepsilon & -\frac{1}{2} & \frac{1}{2}\sqrt{\frac{\beta}{\alpha}} & -\frac{\sqrt{\beta}}{2} \\ -\frac{1}{\sqrt{2\beta}} & -\frac{1}{2}\sqrt{\frac{\alpha}{\beta}} & -\frac{1}{2} & \alpha\varepsilon \\ -\frac{1}{2\sqrt{\alpha\beta}} & \frac{1}{\sqrt{2\beta}} & -\varepsilon & -\frac{1}{2} \end{bmatrix}.$$

Also, it is easy to see that $A_0^{\frac{1}{2}}+A_1^{\frac{1}{2}}=0$.

Remark 2.1. Let α be a positive number and β be a negative number, the Theorem 3.4 holds.

References

[1] Agrawal O. P., Hamilton operators and dual-number-quaternions in spatial kinematics, Mechanism and machine theory, 22, no.6 (1987) 569-575.
[2] Akyar B., Dual Quaternions in Spatial Kinematics in an Algebraic Sense, Turk jornal of mathemathics, 32 (2008) 373-391.
[3] Ata E., Yayli y., Dual unitary matrices and unit dual quaternions, Differential geometry-dynamical system, 10 (2008) 1-12.
[4] Cho E., De-Moivre Formula for Quaternions, Applied mathematics letters, Vol. 11(6) (1998) 33-35.
[5] Clifford W., Preliminary sketch of biquaternions. Proc. of london Math. Soc. No.10, (1873) 381-395.
[6] Jafari M., Yayli Y., Hamilton operators and generalized quaternions, 8. Geometri Sempozyumu, 29 Apr.-2 May 2010, Antaliya, Turkey.
[7] Jafari M., Yayli Y., Dual generalized quaternions in spatial kinematics. 41^{st} Annual Iranian Math. Conference, 12-15 Sep. 2010, Urmia, Iran.

[8] Jafari M., Generalized Screw Transformation and Its Applications in Robotics, Journal of Advanced Technology Sciences, Vol. 4 (2) (2015) 34-46.
[9] Jafari M., Meral M., Yayli Y., Matrix reperesentaion of dual quaternions, Gazi university journal of science, 26(4):535-542 (2013).
[10] Jafari M., Mortazaasl H., Yayli Y., De Moivre's Formula for Matrices of Quaternions, JP journal of algebra, number theory and applications, Vol.21(1) (2011)57-67.
[11] Kabadayi H., Yayli y., De-Moivre's Formula for Dual Quaternions, Kuwait journal of science & technology, Vol. 38(1) 1(2011) 15-23.
[12] Kotel nikov A.P., Vintovoe Schislenie i Nikotoriya Prilozheniye evo k geometrie i mechaniki, Kazan, 1895.
[13] Mortazaasl H., Jafari M., Yayli Y., Some Algebraic properties of dual generalized quaternions algebra, Far east journal of Mathematical science, Vol. 69 (2), (2012) 307-318.
[14] Ozdemir M., The Roots of a Split Quaternion, Applied mathematics letters, 22(2009) 258-263.
[15] Pennestri E., Stefanelli R., Linear algebra and numerical algorithms using dual numbers, University of Roma, Italy.
[16] Rashidi M., Shahsavari M., Jafari M., The E. Study mapping for directed lines in 3-space, International Research journal of applied and basic sciences, Vol. 5(11) 1374-1379 (2013).
[17] Study e., Von Den bewegungen und umlegungen, Mathematische Annalen 39 (1891) 441-564.
[18] Veldkamp G.R., On the use of dual numbers, vectors and matrices in instantaneous spatial kinematics, Mechanism and machine theory, 11 (1976) 141-156.
[19] Ward J. P., Quaternions and Cayley numbers algebra and applications, Kluwer Academic Publishers, London, 1997.
[20] Yang A.T., Freudenstein F., Application of dual-number quaternion algebra to the analysis of spatial mechanisms. ASME Journal of applied Mechnics 86E (2)(1964) 300-308.

Department of Mathematics, University College of Science and Technology Elm o Fan, Urmia-IRAN

E-mail address: mj_msc@yahoo.com

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 122–130 (2015) ©KJM

SYMMETRY REDUCTIONS AND EXACT SOLUTIONS TO THE SEVENTH-ORDER KDV TYPES OF EQUATION

YOUWEI ZHANG

Abstract. In present paper, the seventh-order KdV types of equation is considered by the Lie symmetry analysis. All of the geometric vector fields of the KdV equation are obtained, then the symmetry reductions and exact solutions to the KdV equation are investigated by the dynamical system and the power series method.

1. Introduction

Recently, mathematics and physics field have devoted considerable effort to the study of solutions to ordinary and partial differential equations (ODEs and PDEs). Among many powerful methods for solving the equation, Lie symmetry analysis provides an effective procedure for integrability, conservation laws, reducing equations and exact solutions of a wide and general class of differential systems representing real physical problems [12, 15]. Sinkala et al [14] have performed the group classification of a bond-pricing PDE of mathematical finance to discover the combinations of arbitrary parameters that allow the PDE to admit a nontrivial symmetry Lie algebra, and computed the admitted Lie point symmetries, identify the corresponding symmetry Lie algebra and solve the PDE. Under the condition of the symmetry group of the PDE is nontrivial, it contains a standard integral transform of the fundamental solution for PDEs, and fundamental solution can be reduced to inverting a Laplace transform or some other classical transform in [1]. In [7], by the direct construction method, all of the first-order multipliers of the the generalized nonlinear second-order equation are obtained, and the corresponding complete conservation laws of such equations are provided. Furthermore, Lie symmetry analysis helps to study their group theoretical properties, and effectively assists to derive several mathematical characteristics related with their complete integrability [10]. Also, Lie symmetry analysis and dynamical system method is a feasible approach to dealing with exact explicit solutions to nonlinear PDEs and systems, (see, e.g.,

Date: March 31, 2015 and, in revised form, April 22, 2015.
2010 *Mathematics Subject Classification.* 37L20, 35C05.
Key words and phrases. KdV equation, Lie symmetry analysis, Dynamical system method, Power series method, Exact solution.

[2, 3, 8, 11]). Liu et al have derived the symmetries, bifurcations and exact explicit solutions to the KdV equation by using Lie symmetry analysis and the dynamical system method [5, 6]. The KdV equation models the dust-ion-acoustic waves in such cosmic environments as those in the supernova shells and Saturn's F-ring [4], etc., In present paper, we will investigate the vector fields, symmetry reductions and exact solutions to the KdV equation with power law nonlinearity and linear damping with dispersion

$$u_t + u^2u_x + uu_{4x} + 2u_xu_{3x} + u_{xx}^2 + u_{7x} = 0, \tag{1.1}$$

where $u = u(x,t)$ is the unknown functions, x is the spatial coordinate in the propagation direction and t is the temporal coordinates, which occur in different contexts in mathematical physics.

The rest of this paper is organized as follows: in Section 2, the vector fields of Eqs. (1.1) are presented by using Lie symmetry analysis method. Based on the optimal system method, all the similarity reductions to the Eqs. (1.1) are obtained. In Section 3, the exact analytic solutions to the equations are investigated by means of the power series method. Finally, the conclusions will be given in Section 4.

2. Lie symmetry analysis and similarity reductions

Recall that the geometric vector field of a PDE equation is as follows:

$$V = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u, \tag{2.1}$$

where the coefficient functions $\xi(x,t,u)$, $\tau(x,t,u)$, $\eta(x,t,u)$ of the vector field are to be determined later.

If the vector field (2.1) generates a symmetry of the equation (1.1), then V must satisfy the Lie symmetry condition

$$\mathrm{Pr}V(\Delta)|_{\Delta=0} = 0,$$

where $\mathrm{Pr}V$ denotes the 7-th prolongation of V , and $\Delta = u_t + u^2u_x + uu_{4x} + 2u_xu_{3x} + u_{xx}^2 + u_{7x}$. Moreover, the prolongation $\mathrm{Pr}V$ depends on the equation

$$\mathrm{Pr}V = \eta\partial_u + \eta^x\partial_{u_x} + \eta^{xx}\partial_{u_{xx}} + \eta^{3x}\partial_{u_{3x}} + \eta^{4x}\partial_{u_{4x}} + \eta^{7x}\partial_{u_{7x}},$$

where the coefficient functions $\eta^{kx}(k = 1,2,3,4,7)$ are given as

$$\eta^{kx} = D_x^k(\eta - \tau u_t - \xi u_x) + \tau u_{kxt} + \xi u_{(k+1)x}, \quad k = 1,2,3,4,7,$$

here symbol D_x denotes the total differentiation operator and is defined as

$$D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \dots.$$

Then, in terms of the Lie symmetry analysis method, we obtain that all of the geometric vector fields of Eq. (1.1) are as follows:

$$V_1 = x\partial_x + 7t\partial_t - 3u\partial_u, \quad V_2 = \partial_x, \quad V_3 = \partial_t.$$

Moreover, it is necessary to show that the vector fields of Eq. (1.1) are closed under the Lie bracket, we have

$$[V_i, V_i] = 0, \quad i = 1,2,3,$$
$$[V_1,V_2] = -[V_2,V_1] = V_2, \quad [V_1,V_3] = -[V_3,V_1] = 7V_3, \quad [V_2,V_3] = -[V_3,V_2] = 0.$$

In the preceding section, we obtained the vector fields and the optimal systems of Eq. (1.1). Now, we deal with the symmetry reductions and exact solutions to the equations. We will consider the following similarity reductions and group-invariant solutions based on the optimal system method. From an optimal system of group-invariant solutions to an equation, every other such solution to the equation can be derived.

For the generator V_1, we have

$$u = t^{-\frac{3}{7}} f(z), \tag{2.2}$$

where $z = xt^{-\frac{1}{7}}$. Substituting (2.2) into Eq. (1.1), we reduce it into the following ODE

$$f^{(7)} + f^2 f'^{(4)} + 2f'^{(3)} + f''^2 - \frac{1}{7} z f' - \frac{3}{7} f = 0, \tag{2.3}$$

where $f' = \frac{df}{dz}$.

For the generator V_2, we get the trivial solution to Eq. (1.1) is $u(x,t) = c$, where c is an arbitrary constant.

For the generator V_3, we have

$$u = f(z), \tag{2.4}$$

where $z = x$. Substituting (2.4) into Eq. (1.1), we reduce it into the following ODE

$$f^{(7)} + f^2 f'^{(4)} + 2f'^{(3)} + f''^2 = 0, \tag{2.5}$$

where $f' = \frac{df}{dz}$.

For the generator $V_3 + \upsilon V_2$, υ is an arbitrary constant, we have

$$u = f(z), \tag{2.6}$$

where $z = x - \upsilon t$. Substituting (2.6) into Eq. (1.1), we reduce it into the following ODE

$$f^{(7)} + f^2 f'^{(4)} + 2f'^{(3)} + f''^2 - \upsilon f' = 0, \tag{2.7}$$

where $f' = \frac{df}{dz}$.

3. The exact power series solutions

By seeking for exact solutions of the PDEs, we mean that those can be obtained from some ODEs or, in general, from PDEs of lower order than the original PDE. In terms of this definition, the exact solutions to Eq. (1.1) are obtained actually in both of the preceding Sections 2. In spite of this, we still want to detect the explicit solutions expressed in terms of elementary or, at least, known functions of mathematical physics, in terms of quadratures, and so on. But this is not always the case, even for simple semilinear PDEs. However, we know that the power series can be used to solve differential equations, including many complicated differential equations [9, 13]. In this section, we will consider the exact analytic solutions to the reduced equations by using the power series method. Once we get the exact analytic solutions of the reduced ODEs, the exact power series solutions to the original PDEs are obtained, now we consider the solutions of ODEs (2.3), (2.5) and (2.7).

3.1 Exact analytic solutions to Eq. (2.3)

In view of (2.3), we seek a solution in a power series of the form

$$f(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{3.1}$$

Substituting (3.1) into (2.3), and comparing coefficients, then we obtain the following recursion formula:

$$\begin{aligned} c_{n+7} = &-\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)} \\ &\times \Bigg(\sum_{k=0}^{n} \sum_{i=0}^{k} (n-k+1) c_i c_{k-i} c_{n-k+1} \\ &+ \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(n-k+4) c_k c_{n-k+4} \\ &+ 2\sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(k+1) c_{k+1} c_{n-k+3} \\ &+ \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1)(k+2) c_{k+2} c_{n-k+2} \\ &- \frac{1}{7} n c_n - \frac{3}{7} c_n \Bigg), \end{aligned} \tag{3.2}$$

for all $n = 0, 1, 2, \ldots$.

Thus, for arbitrarily chosen constants c_i $(i = 0, 1, \ldots, 6)$, we obtain

$$c_7 = -\frac{1}{5040}\big(c_0^2 c_1 + 24 c_0 c_4 + 12 c_1 c_3 + 4 c_2^2 - \frac{3}{7} c_1\big). \tag{3.3}$$

Furthermore, from (3.2), it yield

$$\begin{aligned} c_8 &= -\frac{1}{20160}\big(c_0 c_1^2 + c_0^2 c_2 + 60 c_0 c_5 + 36 c_1 c_4 + 24 c_2 c_3 - \frac{2}{7} c_1\big), \\ c_9 &= -\frac{1}{181440}\big(3 c_0^2 c_3 + c_1^3 + 6 c_0 c_1 c_2 + 360 c_0 c_6 + 240 c_1 c_5 + 144 c_2 c_4 \\ &\quad + 72 c_3^2 + 24 c_0 c_4 - \frac{5}{7} c_2\big), \end{aligned} \tag{3.4}$$

and so on.

Thus, for arbitrary chosen constant numbers c_i $(i = 0, 1, \ldots)$, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (3.3) and (3.4) in a unique manner. This implies that for Eq. (2.3), there exists a power series solution (3.1) with the coefficients given by (3.3) and (3.4). Furthermore, it is easy to prove the convergence of the power series (3.1) with the coefficients given by (3.3) and (3.4). Therefore, this power series solution (3.1) to Eq. (2.3) is an exact analytic solution.

Hence, the power series solution of Eq. (2.3) can be written as

$$\begin{aligned}
f(z) &= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 + c_7 z^7 + \sum_{n=1}^{\infty} c_{n+7} z^{n+7} \\
&= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 \\
&\quad - \frac{1}{5040}\big(c_0^2 c_1 + 24 c_0 c_4 + 12 c_1 c_3 + 4 c_2^2 - \frac{3}{7} c_1\big) z^7 \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)} \\
&\quad \times \Bigg(\sum_{k=0}^{n} \sum_{i=0}^{k} (n-k+1) c_i c_{k-i} c_{n-k+1} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(n-k+4) c_k c_{n-k+4} \\
&\quad + 2 \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(k+1) c_{k+1} c_{n-k+3} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1)(k+2) c_{k+2} c_{n-k+2} - \frac{1}{7} n c_n - \frac{3}{7} c_n \Bigg) z^{n+7}.
\end{aligned}$$

Thus, the exact power series solution of Eq. (1.1) is

$$\begin{aligned}
u(x,t) &= c_0 t^{-\frac{3}{7}} + c_1 x t^{-\frac{4}{7}} + c_2 x^2 t^{-\frac{5}{7}} + c_3 x^3 t^{-\frac{6}{7}} + c_4 x^4 t^{-1} + c_5 x^5 t^{-\frac{8}{7}} \\
&\quad + c_6 x^6 t^{-\frac{9}{7}} - \frac{1}{5040}\big(c_0^2 c_1 + 24 c_0 c_4 + 12 c_1 c_3 + 4 c_2^2 - \frac{3}{7} c_1\big) x^7 t^{-\frac{10}{7}} \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)} \\
&\quad \times \Bigg(\sum_{k=0}^{n} \sum_{i=0}^{k} (n-k+1) c_i c_{k-i} c_{n-k+1} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(n-k+4) c_k c_{n-k+4} \\
&\quad + 2 \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(k+1) c_{k+1} c_{n-k+3} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1)(k+2) c_{k+2} c_{n-k+2} - \frac{1}{7} n c_n \\
&\quad - \frac{3}{7} c_n \Bigg) \times x^{n+7} t^{-\frac{n+10}{7}}.
\end{aligned} \tag{3.5}$$

3.2 Exact analytic solutions to Eq. (2.5)

In view of (2.5), we have

$$\frac{1}{3} f^3 + f f''' + f' f''^{(6)} + c = 0, \tag{3.6}$$

where c is an integration constant.

We seek a solution of Eq. (3.6) in a power series of the form (3.1). Substituting (3.1) into (3.6), and comparing coefficients, we obtain

$$\begin{aligned} c_{n+6} = -&\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}\Big(\frac{1}{3}\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\ &+\sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} \\ &+\sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)c_{k+1}c_{n-k+2}\Big), \end{aligned} \tag{3.7}$$

for all $n = 1, 2, \ldots$.

Thus, for arbitrarily chosen constants c_i $(i = 0, 1, \ldots, 5)$, we have

$$c_6 = -\frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2 + c\big).$$

Furthermore, from (3.7), we have

$$\begin{aligned} c_7 &= -\frac{1}{5040}\Big(c_0^2c_1 + 24c_0c_4 + 12c_1c_3 + 4c_1c_2\Big), \\ c_8 &= -\frac{1}{20160}\Big(c_0^2c_2 + c_0c_1^2 + 60c_0c_5 + 36c_1c_4 + 24c_2c_3\Big), \end{aligned}$$

and so on.

Hence, the power series solution of Eq. (2.5) can be written as

$$\begin{aligned} f(z) &= c_0 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + c_5z^5 + c_6z^6 + \sum_{n=1}^{\infty} c_{n+6}z^{n+6} \\ &= c_0 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + c_5z^5 - \frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2 + c\big)z^6 \\ &\quad - \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}\Big(\frac{1}{3}\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\ &\quad + \sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} \\ &\quad + \sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)c_{k+1}c_{n-k+2}\Big)z^{n+6}. \end{aligned}$$

Thus, the exact power series solution of Eq. (1.1) is

$$
\begin{aligned}
u(x,t) = {} & c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 \\
& - \frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2 + c\big)x^6 \\
& - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \\
& \times \bigg(\frac{1}{3}\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\
& + \sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} \\
& + \sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)c_{k+1}c_{n-k+2}\bigg)x^{n+6}.
\end{aligned} \tag{3.8}
$$

3.3 Exact analytic solutions to Eq. (2.7)

In view of (2.7), we have

$$\frac{1}{3}f^3 + ff''' + f'f''^{(6)} - \upsilon f + c = 0, \tag{3.9}$$

where c is an integration constant.

Similarly, we seek a solution of Eq. (3.9) in a power series of the form (3.1). Substituting (3.1) into (3.9), and comparing coefficients, we obtain

$$
\begin{aligned}
c_{n+6} = {} & -\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \\
& \times \bigg(\frac{1}{3}\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\
& + \sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} \\
& + \sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)c_{k+1}c_{n-k+2} - \upsilon c_n\bigg),
\end{aligned} \tag{3.10}
$$

for all $n = 1, 2, \ldots$.

Thus, for arbitrarily chosen constants c_i $(i = 0, 1, \ldots, 5)$, we can get

$$c_6 = -\frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2 - \upsilon c_0 + c\big).$$

Furthermore, from (3.10), we have

$$
\begin{aligned}
c_7 &= -\frac{1}{5040}\Big(c_0^2c_1 + 24c_0c_4 + 12c_1c_3 + 4c_1c_2 - \upsilon c_1\Big), \\
c_8 &= -\frac{1}{20160}\Big(c_0^2c_2 + c_0c_1^2 + 60c_0c_5 + 36c_1c_4 + 24c_2c_3 - \upsilon c_2\Big),
\end{aligned}
$$

and so on.

Hence, the power series solution of Eq. (2.7) can be written as

$$\begin{aligned}
f(z) &= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 + \sum_{n=1}^{\infty} c_{n+6} z^{n+6} \\
&= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 - \frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2 - \upsilon c_0\big) z^6 \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \Bigg(\frac{1}{3} \sum_{k=0}^{n} \sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3) c_k c_{n-k+3} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1) c_{k+1} c_{n-k+2} - \upsilon c_n \Bigg) z^{n+6}.
\end{aligned}$$

Thus, we obtain the traveling wave solution to Eq. (1.1) as follows

$$\begin{aligned}
u(x,t) &= c_0 + c_1(x-\upsilon t) + c_2(x-\upsilon t)^2 + c_3(x-\upsilon t)^3 + c_4(x-\upsilon t)^4 \\
&\quad + c_5(x-\upsilon t)^5 - \frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2\big)(x-\upsilon t)^6 \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \\
&\quad \times \Bigg(\frac{1}{3} \sum_{k=0}^{n} \sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3) c_k c_{n-k+3} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1) c_{k+1} c_{n-k+2} - \upsilon c_n \Bigg) \\
&\quad \times (x-\upsilon t)^{n+6}.
\end{aligned} \tag{3.11}$$

Remark 3.1. We would like to reiterate that the power series solutions which have been obtained in this section are exact analytic solutions to the equations. Moreover, we can see that these power series solutions converge for the given chosen constants c_i $(i = 0, 1, \dots, 6)$ of (3.5), c_i $(i = 0, 1, \dots, 5)$ of (3.8) and (3.11), respectively, it is actual value for mathematical and physical applications.

4. Summary and discussion

In this paper, we have obtained the symmetries and similarity reductions of the seventh-order KdV types of equations by using Lie symmetry analysis method. All the group-invariant solutions to the equations are considered based on the optimal system method. Then the exact analytic solutions are investigated by using the power series method, and we can see that these power series solutions converge.

Acknowledgements

The author is very grateful to the anonymous referees for their carefully reading the paper and for constructive comments and suggestions which have improved this paper.

References

[1] M. Craddock, K. Lennox, Lie group symmetries as integral transforms of fundamental solutions, *J. Differential Equations*, 232 (2007), 652-674.

[2] F. Güngör, C. Ōzemir, Lie symmetries of a generalized Kuznetsov-Zabolotskaya-Khokhlov equation, *J. Math. Anal. Appl.*, 423 (2015), 623-638.

[3] M. Lakshmanan, P. Kaliappan, Lie transformations, nonlinear evolution equations and Painlevé forms, *J. Math. Phys.*, 24 (1983), 795-806.

[4] H. Liu, J. Li, Lie symmetry analysis and exact solutions for the extended mKdV equation, *Acta Appl. Math.*, 109 (2010), 1107-1119.

[5] H. Liu, J. Li, F. Chen, Exact periodic wave solutions for the hKdV equation, *Nonlinear Anal.*, 70 (2009), 2376-2381.

[6] H. Liu, J. Li, L. Liu, Lie symmetry analysis, optimal systems and exact solutions to the fifth-order KdV types of equations, *J. Math. Anal. Appl.*, 368 (2010), 551-558.

[7] H. Liu, J. Li, L. Liu, Conservation law classification and integrability of generalized nonlinear second-order equation, *Commun. Theor. Phys. (Beijing)*, 56 (2011), 987-991.

[8] H. Liu, Y. Geng, Symmetry reductions and exact solutions to the systems of carbon nanotubes conveying fluid, *J. Differential Equations*, 254 (2013), 2289-2303.

[9] W. Balser, Multisummability of formal power series solutions of partial differential equations with constant coefficients, *J. Differential Equations*, 201 (2004), 63-74.

[10] A.B. Mikhailov, A.B. Shabat, V.V. Sokolov, *The symmetry approach to classification of integrable equation, in: What is Integrability?*, Springer Series on Nonlinear Dynamics, Berlin, 1991.

[11] B. Muatjetjeja, C.M. Khalique, Symmetry analysis and conservation laws for a coupled (2+1)-dimensional hyperbolic system, *Commun. Nonlinear Sci. Numer. Simul.*, 22 (2015), 1252-1262.

[12] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Grad. Texts in Math., vol. 107, Springer, New York, 1993.

[13] P. Razborova, A.H. Kara, A. Biswas, Additional conservation laws for Rosenau-KdV-RLW equation with power law nonlinearity by Lie symmetry, *Nonlinear Dynam.*, 79 (2015), 743-748.

[14] W. Sinkala, P. Leach, J. O'Hara, Invariance properties of a general-pricing equation, *J. Differential Equations*, 244 (2008), 2820-2835.

[15] P. Winternitz, *Lie groups and solutions of nonlinear partial differential equations*, in: Lecture Notes in Physics, CRM-1841, Canada 1993.

School of Mathematics and Statistics, Hexi University, Gansu 734000, China
E-mail address: ywzhang0288@163.com

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 131–139 (2015) ©KJM

SOME GENERATING RELATIONS INVOLVING 2-VARIABLE LAGUERRE AND EXTENDED SRIVASTAVA POLYNOMIALS

Ahmed Ali Al Gonah [1]

ABSTRACT: In this paper, we derive families of bilateral and mixed multilateral generating relations involving 2-variable Laguerre and extended Srivastava polynomials. Further, several bilateral and trilateral generating functions involving 2-variable Laguerre polynomials and other classical polynomials are obtained as applications of main results.

1. INTRODUCTION

Srivastava [9] introduced the Srivastava polynomials (SP) $S_n^N(w)$ by the following series definition:

$$S_n^N(w) = \sum_{k=0}^{[\frac{n}{N}]} \frac{(-n)_{Nk}}{k!} A_{n,k}\ w^k \ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; N \in \mathbb{N}), \tag{1.1}$$

where $\mathbb{N}$ is the set of positive integers, $\{A_{n,k}\}_{n,k=0}^{\infty}$ is a bounded double sequence of real or complex numbers, $[a]$ denotes the greatest integer in $a \in \mathbb{R}$ and $(\lambda)_\nu$, $(\lambda)_0 \equiv 1$, denotes the Pochhammer symbol defined by [10]

$$(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}, \tag{1.2}$$

in terms of familiar Gamma function.

Afterward, González *et al*. [3] extended the SP $S_n^N(w)$ as follows:

$$S_{n,q}^N(w) = \sum_{k=0}^{[\frac{n}{N}]} \frac{(-n)_{Nk}}{k!} A_{n+q,k}\ w^k \ (q, n \in \mathbb{N}_0; N \in \mathbb{N}), \tag{1.3}$$

which were investigated rather extensively in [3] and more recently in [6]. The polynomials $S_{n,q}^N(w)$ called as extended Srivastava polynomials (ESP), since $S_{n,0}^N(w) = S_n^N(w)$.

It is important that, appropriate choices of the double sequence $\{A_{n,k}\}$ in equation (1.3) give many well known polynomials such as Laguerre, Jacobi and Bessel polynomials (see [3]). Here, we will recall them and add further new particular cases as the following remarks:

---

[1]2010 *Mathematics Subject Classification*: 33B10, 33C45, 33E30.

Keywords: Generating relations, Laguerre polynomials, Extended Srivastava polynomials.

Remark 1.1. ([3; p.147] see also [6]) Choosing $A_{q,n} = (-\alpha - q)_n$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S^1_{n,q}\left(\frac{-1}{w}\right) = \frac{n!}{(-w)^n} L_n^{(\alpha+q)}(w), \tag{1.4}$$

where $L_n^{(\alpha)}(w)$ denotes the associated Laguerre polynomials defined by [10; p.42]

$$L_n^{(\alpha)}(w) = \frac{(-w)^n}{n!}\, {}_2F_0\left(-n, -\alpha - n; -; \frac{-1}{w}\right) \tag{1.5}$$

and ${}_pF_q$ is the generalized hypergeometric function defined by [10; p.42]:

$${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}, \tag{1.6}$$

where $p, q \in \mathbb{N}_0$ and for $p = q = 1$ reduces to the confluent hypergeometric function ${}_1F_1$.

Remark 1.2. ([3; p.146]) Choosing $A_{q,n} = \frac{(\alpha+\beta+1)_{2q}(-\beta-q)_n}{(\alpha+\beta+1)_q(-\alpha-\beta-2q)_n}$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S^1_{n,q}\left(\frac{2}{1+w}\right) = n!(\alpha + \beta + q + n + 1)_q \left(\frac{2}{1+w}\right)^n P_n^{(\alpha+q,\beta+q)}(w), \tag{1.7}$$

where $P_n^{(\alpha,\beta)}(w)$ denotes the classical Jacobi polynomials defined by [8; p.255]

$$P_n^{(\alpha,\beta)}(w) = \binom{\alpha+\beta+2n}{n} \left(\frac{1+w}{2}\right)^n {}_2F_1\left(-n, -\beta - n; -\alpha - \beta - 2n; \frac{2}{1+w}\right). \tag{1.8}$$

Remark 1.3. ([3; p.148]) Choosing $A_{q,n} = (-\alpha - q)_n$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S^1_{n,q}\left(\frac{-w}{\beta}\right) = y_n(w, 1 - \alpha - q - 2n, \beta) \ (\beta \neq 0), \tag{1.9}$$

where $y_n(w, \alpha, \beta)$ denotes the Bessel polynomials defined by [10; p.75]

$$y_n(w, \alpha, \beta) = {}_2F_0\left(-n, \alpha + n - 1; -; \frac{-w}{\beta}\right). \tag{1.10}$$

Now, we add the following new particular cases as remarks:

Remark 1.4. Choosing $A_{q,n} = \frac{2^q(\nu)_q(\frac{1}{2}-\nu-q)_n}{(1-2\nu-2q)_n}$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S^1_{n,q}\left(\frac{2}{1-w}\right) = \frac{n! 2^q(\nu)_q}{(w-1)^n} C_n^{\nu+q}(w), \tag{1.11}$$

where $C_n^{\nu}(w)$ denotes the classical Gegenbauer polynomials defined by [8; p.279]

$$C_n^{\nu}(w) = \frac{2^{2n}(\nu)_n}{n!} \left(\frac{w-1}{2}\right)^n {}_2F_1\left(-n, \frac{1}{2} - \nu - n; 1 - 2\nu - 2n; \frac{2}{1-w}\right). \tag{1.12}$$

Remark 1.5. Choosing $A_{q,n} = \frac{n!(p+1)_q}{(-p-q)_n}$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S^1_{n,q}(w) = n!(p+1)_q \ g_n^{(p+q)}(w), \tag{1.13}$$

where $g_n^{(p)}(w)$ denotes the Cesaro polynomials defined by [10; p.449]

$$g_n^{(p)}(w) = \binom{p+n}{n} {}_2F_1(-n, 1; -p-n; w). \tag{1.14}$$

Remark 1.6. Choosing $A_{q,n} = \frac{(a)_{2q}}{(a)_q (a+q)_n}$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S^1_{n,q}(w) = n!(a+q+2n)_q \; R_n(a+q, w), \tag{1.15}$$

where $R_n(a, w)$ denotes the Shively's pseudo Laguerre polynomials defined by [8; p.298]

$$R_n(a, w) = \frac{(a)_{2n}}{n!(a)_n} {}_1F_1(-n; a+n; w). \tag{1.16}$$

Next, we recall that the 2-variable Laguerre polynomials (2VLP) $L_n(x, y)$ are defined by the series definition (see[1,2])

$$L_n(x, y) \;=\; n! \sum_{k=0}^{n} \frac{(-1)^k \; x^k \; y^{n-k}}{(r!)^2 \; (n-k)!} \tag{1.17}$$

and specified by the following generating functions:

$$\exp(yt)\, C_0(xt) = \sum_{n=0}^{\infty} L_n(x, y) \frac{t^n}{n!}, \tag{1.18}$$

or, equivalently

$$\frac{1}{(1-yt)} \exp\left(\frac{-xt}{1-yt}\right) = \sum_{n=0}^{\infty} L_n(x, y) t^n \; \big(|yt| < 1\big), \tag{1.19}$$

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined by [10]

$$C_n(x) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k \; x^k}{k! \; (n+k)!}. \tag{1.20}$$

Also, we note that the 2VLP $L_n(x, y)$ satisfy the following generating function:

$$\frac{1}{(1-yt)^a} \; {}_1F_1\Big(a; 1; \frac{-xt}{1-yt}\Big) = \sum_{n=0}^{\infty} (a)_n \; L_n(x, y) \frac{t^n}{n!} \; \big(|yt| < 1\big), \tag{1.21}$$

which for $a = 1$ reduces to Eq. (1.19).

The 2VLP $L_n(x, y)$ are linked to the classical Laguerre polynomials $L_n(x)$ by the relations

$$L_n(x, y) = y^n \; L_n\left(\frac{x}{y}\right), \tag{1.22}$$

$$L_n(x, 1) = L_n(x), \tag{1.23}$$

where $L_n(x)$ are defined by [8]

$$L_n(x) \;=\; \sum_{k=0}^{n} \frac{(-1)^k \, n! \, x^k}{(k!)^2 \, (n-k)!}. \tag{1.24}$$

The aim of this paper is to derive some families of bilateral and mixed multilateral generating relations involving the 2VLP $L_n(x, y)$ and the ESP $S^N_{n,q}(w)$ by using series rearrangement techniques. Also, the above mentioned remarks will be used to obtain some illustrative bilateral and trilateral generating functions involving the 2VLP $L_n(x,y)$ and many classical polynomials in terms of the confluent hypergeometric function.

2. BILATERAL GENERATING RELATIONS

We prove the following results:

Theorem 2.1. *The following family of bilateral generating relation involving the 2VLP $L_n(x,y)$ and the ESP $S^N_{n,q}(w)$ holds true:*

$$\sum_{q,n=0}^{\infty} L_{q+n}(x,y)\ S^N_{n,q}(w)\frac{t^q}{q!}\frac{u^n}{n!} = \sum_{q,n=0}^{\infty} L_{q+Nn}(x,y)\ A_{q+Nn,n}\frac{(t+u)^q}{q!}\frac{\left(w(-u)^N\right)^n}{n!}. \tag{2.1}$$

Proof. Denoting the l.h.s. of Eq. (2.1) by Δ_1 and using definition (1.3), we have

$$\Delta_1 = \sum_{q,n=0}^{\infty} L_{q+n}(x,y)\sum_{k=0}^{[\frac{n}{N}]}\frac{(-1)^{Nk}}{k!(n-Nk)!}A_{q+n,k}\ w^k\frac{t^q}{q!}\ u^n. \tag{2.2}$$

Replacing n by $n+Nk$ in the above equation and using the lemma [10; p.101]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+mk), \tag{2.3}$$

in the resultant equation, we find

$$\Delta_1 = \sum_{q,n,k=0}^{\infty} L_{q+n+Nk}(x,y)\frac{(-1)^{Nk}}{k!}A_{q+n+Nk,k}\ w^k\frac{t^q}{q!}\frac{u^{n+Nk}}{n!}. \tag{2.4}$$

Again, replacing q by $q-n$ in the r.h.s. of Eq. (2.4) and using the lemma [10; p.100]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k), \tag{2.5}$$

in the resultant equation, we get

$$\Delta_1 = \sum_{q,k=0}^{\infty} L_{q+Nk}(x,y)A_{q+Nk,k}\ \frac{t^q}{q!}\frac{\left(w(-u)^N\right)^k}{k!}\sum_{n=0}^{q}\frac{(-q)_n}{n!}\left(\frac{-u}{t}\right)^n, \tag{2.6}$$

which on using the binomial expansion [10]

$$(1-x)^{-\lambda} = \sum_{n=0}^{\infty}(\lambda)_n\frac{x^n}{n!}, \tag{2.7}$$

in the r.h.s., yields the r.h.s. of Eq. (2.1), then the proof of Theorem (2.1) is completed.

Remark 2.1. Taking $u=-t$ in assertion (2.1) of Theorem 2.1, we deduce the following consequence of Theorem 2.1.

Corollary 2.1. *The following family of bilateral generating relation involving the 2VLP $L_n(x,y)$ and the ESP $S^N_{n,q}(w)$ holds true:*

$$\sum_{q,n=0}^{\infty} L_{q+n}(x,y)\ S^N_{n,q}(w)\ \frac{t^q}{q!}\frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} L_{Nn}(x,y)\ A_{Nn,n}\ \frac{\left(wt^N\right)^n}{n!}. \tag{2.8}$$

Remark 2.2. Taking $t=0$ in assertion (2.1) of Theorem 2.1 and using the relation $S^N_{n,0}(w) = S^N_n(w)$, we deduce the following consequence of Theorem 2.1.

Corollary 2.2. *The following family of bilateral generating relation involving the 2VLP $L_n(x,y)$ and the SP $S^N_n(w)$ holds true:*

$$\sum_{n=0}^{\infty} L_n(x,y)\ S^N_n(w)\frac{u^n}{n!} = \sum_{q,n=0}^{\infty} L_{q+Nn}(x,y)\ A_{q+Nn,n}\ \frac{u^q}{q!}\frac{\left(w(-u)^N\right)^n}{n!}. \tag{2.9}$$

In the next section, Corollaries 2.1 and 2.2 will be exploited to get families of mixed multilateral generating relations involving the 2VLP $L_n(x,y)$, ESP $S^N_{n,q}(w)$ and SP $S^N_n(w)$ with the help of the method considered in [10,5,7].

3. MULTILATERAL GENERATING RELATIONS

First, we prove the following theorem by using Corollary 2.1:

Theorem 3.1. *Corresponding to an identically non-vanishing function $\Omega_\mu(\xi_1,\ldots,\xi_l)$ of complex variables $\xi_1,\ldots,\xi_l$ $(l\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(\xi_1,\ldots,\xi_l;\eta) := \sum_{k=0}^{\infty} a_k\ \Omega_{\mu+\psi k}(\xi_1,\ldots,\xi_l)\eta^k,\ \ (a_k\neq 0, \psi\in\mathbb{C}). \tag{3.1}$$

Then we have, for $n,p\in\mathbb{N}$,

$$\sum_{q,n=0}^{\infty}\sum_{k=0}^{[\frac{q}{p}]} a_k\ L_{q+n-pk}(x,y) S^N_{n,q-pk}(w)\ \Omega_{\mu+\psi k}(\xi_1,\ldots,\xi_l)\ \eta^k\ \frac{(-1)^n t^{n+q-pk}}{(q-pk)!n!}$$

$$= \Lambda_{\mu,\psi}(\xi_1,\ldots,\xi_l;\eta)\sum_{n=0}^{\infty} L_{Nn}(x,y)\ A_{Nn,n}\ \frac{\left(wt^N\right)^n}{n!}. \tag{3.2}$$

provided that each member of assertion (3.2) exists.

Proof. Denoting the l.h.s. of Eq. (3.2) by Δ_2 and using relation (2.3) , we find

$$\Delta_2 = \sum_{k=0}^{\infty} a_k\ \Omega_{\mu+\psi k}(\xi_1,\ldots,\xi_l)\eta^k \sum_{q,n=0}^{\infty} L_{q+n}(x,y)\ S^N_{n,q}(w)\ \frac{t^q}{q!}\frac{(-t)^n}{n!}. \tag{3.3}$$

Using Eqs. (3.1) and (2.8) in the r.h.s. of Eq. (3.3), we get the r.h.s. of Eq. (3.2), then the proof of Theorem 3.1 is completed.

Next, proceeding on the same lines of proof of Theorem 3.1 and using Corollary 2.2, we get the following result:

Theorem 3.2. *Corresponding to an identically non-vanishing function $\Omega_\mu(\xi_1,\ldots,\xi_l)$ of complex variables $\xi_1,\ldots,\xi_l$ $(l\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(\xi_1,\ldots,\xi_l;\eta):=\sum_{k=0}^{\infty}a_k\ \Omega_{\mu+\psi k}(\xi_1,\ldots,\xi_l)\eta^k,\ \ (a_k\neq 0,\psi\in\mathbb{C}),$$

$$\Theta_{n,p}^{\mu,\psi}(x,y,z,w;\xi_1,\ldots,\xi_l;\tau)=\sum_{k=0}^{[\frac{n}{p}]}a_k\ L_{n-pk}(x,y)\ S_{n-pk}^{N}(w)\ \Omega_{\mu+\psi k}(\xi_1,\ldots,\xi_l)\frac{\tau^k}{(n-pk)!},\tag{3.4}$$

where $n,p\in\mathbb{N}$. Then, we have

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x,y,z,w;\xi_1,\ldots,\xi_l;\frac{\eta}{t^p}\right)\ t^n$$
$$=\Lambda_{\mu,\psi}(\xi_1,\ldots,\xi_l;\eta)\sum_{q,n=0}^{\infty}L_{q+Nn}(x,y)\ A_{q+Nn,n}\ \frac{t^q}{q!}\frac{\left(w(-t)^N\right)^n}{n!}.\tag{3.5}$$

provided that each member of assertion (3.5) exists.

Notice that, for every suitable choice of the coefficients a_k $(k\in\mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\psi k}(\xi_1,\ldots,\xi_l)$, $(l\in\mathbb{N})$, is expressed in terms of simpler function of one and more variables, the assertions of Theorems 3.1 and 3.2 can be applied in order to derive various families of multilateral generating relations involving the 2VLP $L_n(x,y)$ and the ESP $S_{n,q}^N(w)$.

For example, if we set $l=1$, $\xi_1=v$, $\psi=1$, $\Omega_{\mu+k}(v)=y_j(v,\mu+k,\beta)$ and $a_k=\binom{\mu+j+k-2}{k}$, $(k,j\in\mathbb{N}_0,\mu\in\mathbb{C})$ in assertion (3.2) of Theorem 3.1 and making use of the following generating relation [4; p.270]:

$$\sum_{n=0}^{\infty}\binom{\mu+j+n-2}{k}\ y_j(x,\mu+n,\beta)\ t^n=(1-t)^{1-\mu-j}\ y_j\left(\frac{x}{1-t},\mu,\beta\right),\tag{3.6}$$

we readily obtain the following mixed trilateral generating function:

$$\sum_{q,n=0}^{\infty}\sum_{k=0}^{[\frac{q}{p}]}\binom{\mu+j+k-2}{k}\ L_{q+n-pk}(x,y)S_{n,q-pk}^{N}(w)\ y_j(v,\mu+k,\beta)\frac{(-1)^n\eta^k}{(q-pk)!}\frac{t^{n+q-pk}}{n!}$$
$$=(1-\eta)^{1-\mu-j}\ y_j\left(\frac{v}{1-\eta},\mu,\beta\right)\sum_{n=0}^{\infty}L_{Nn}(x,y)A_{Nn,n}\frac{\left(wt^N\right)^n}{n!}.\tag{3.7}$$

In the next section, we derive some bilateral and trilateral generating functions for the 2VLP $L_n(x,y)$ in terms of the confluent hypergeometric function as applications of the results derived in Sections 2 and 3 with the help of generating function (1.21) and the remarks introduced in Section 1.

4. APPLICATIONS

First, the following bilateral generating functions are obtained as applications of Corollary 2.1:

I. Taking $N=1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.1 and using relation (1.4) in Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty} L_{q+n}(x,y)\ L_n^{(\alpha+q)}(w)\frac{t^q}{q!}\left(\frac{t}{w}\right)^n=\sum_{n=0}^{\infty}(\alpha+1)_n\ L_n(x,y)\frac{\left(\frac{t}{w}\right)^n}{n!}, \tag{4.1}$$

which on using relation (1.21) in the r.h.s. gives

$$\sum_{n,q=0}^{\infty} L_{q+n}(x,y)\ L_n^{(\alpha+q)}(w)\frac{t^q}{q!}\left(\frac{t}{w}\right)^n=\left(\frac{w}{w-yt}\right)^{\alpha+1}\ {}_1F_1\Big(\alpha+1;1;\frac{-xt}{w-yt}\Big). \tag{4.2}$$

II. Taking $N=1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.2 and using relation (1.7) in Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty}(\alpha+\beta+q+n+1)_q\ L_{q+n}(x,y)\ P_n^{(\alpha+q,\beta+q)}(w)\frac{t^q}{q!}\left(\frac{-2t}{1+w}\right)^n$$

$$=\sum_{n=0}^{\infty}(\beta+1)_n\ L_n(x,y)\frac{\left(\frac{2t}{1+w}\right)^n}{n!}, \tag{4.3}$$

which on using relation (1.21) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty}(\alpha+\beta+q+n+1)_q\ L_{q+n}(x,y)\ P_n^{(\alpha+q,\beta+q)}(w)\frac{t^q}{q!}\left(\frac{-2t}{1+w}\right)^n$$

$$=\left(\frac{1+w}{1+w-2yt}\right)^{\beta+1}\ {}_1F_1\Big(\beta+1;1;\frac{-2xt}{1+w-2yt}\Big). \tag{4.4}$$

III. Taking $N=1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.3 and using relation (1.9) in Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty} L_{q+n}(x,y)\ y_n(w,1-\alpha-q-2n,\beta)\frac{t^q}{q!}\frac{(-t)^n}{n!}=\sum_{n=0}^{\infty}(\alpha+1)_n\ L_n(x,y)\frac{\left(\frac{wt}{\beta}\right)^n}{n!}, \tag{4.5}$$

which on using relation (1.21) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty} L_{q+n}(x,y)\, y_n(w,1-\alpha_q-2n,\beta)\frac{t^q}{q!}\frac{(-t)^n}{n!}=\left(\frac{\beta}{\beta-ywt}\right)^{\alpha+1}\ {}_1F_1\Big(\alpha+1;1;\frac{-xwt}{\beta-ywt}\Big). \tag{4.6}$$

IV. Taking $N=1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.4 and using relation (1.11) in

Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty} (\nu)_q \; L_{q+n}(x,y) \; C_n^{\nu+q}(w) \frac{2t^q}{q!} \left(\frac{t}{1-w}\right)^n = \sum_{n=0}^{\infty} (2\nu)_n \; L_n(x,y) \frac{\left(\frac{t}{1-w}\right)^n}{n!}, \quad (4.7)$$

which on using relation (1.21) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty} (\nu)_q \; L_{q+n}(x,y) \; C_n^{\nu+q}(w) \frac{2t^q}{q!} \left(\frac{t}{1-w}\right)^n = \left(\frac{1-w}{1-w-yt}\right)^{2\nu} \; {}_1F_1\Big(2\nu; 1; \frac{-xt}{1-w-yt}\Big). \quad (4.8)$$

V. Taking $N=1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.5 and using relation (1.13) in Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty} (p+1)_q \; L_{q+n}(x,y) \; g_n^{(p+q)}(w) \frac{t^q}{q!} \; (-t)^n = \sum_{n=0}^{\infty} L_n(x,y) \; (-wt)^n, \quad (4.9)$$

which on using relation (1.19) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty} (p+1)_q \; L_{q+n}(x,y) \; g_n^{(p+q)}(w) \frac{t^q}{q!} \; (-t)^n = \frac{1}{1+ywt} \; \exp\Big(\frac{xwt}{1+ywt}\Big). \quad (4.10)$$

VI. Taking $N=1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.6 and using relation (1.15) in Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty} (a+q+2n)_q \; L_{q+n}(x,y) \; R_n(a+q,w) \frac{t^q}{q!} \; (-t)^n = \sum_{n=0}^{\infty} L_n(x,y) \frac{(wt)^n}{n!}, \quad (4.11)$$

which on using relation (1.18) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty} (a+q+2n)_q \; L_{q+n}(x,y) \; R_n(a+q,w) \frac{t^q}{q!} \; (-t)^n = \exp(ywt) \; C_0(xwt). \quad (4.12)$$

Next, the following trilateral generating function is obtained as applications of result (3.7):

VII. Taking $N=1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.1 and using relation (1.4) in Eq. (3.7), we get

$$\sum_{q,n=0}^{\infty} \sum_{k=0}^{[\frac{q}{p}]} \binom{\mu+j+k-2}{k} L_{q+n-pk}(x,y) \; L_n^{(\alpha+q-pk)}(w) \; y_j(v,\mu+k,\beta) \frac{t^{q-pk}\eta^k}{(q-pk)!} \left(\frac{t}{w}\right)^n$$
$$= (1-\eta)^{1-\mu-j} \; y_j\left(\frac{v}{1-\eta},\mu,\beta\right) \sum_{n=0}^{\infty} (\alpha+1)_n \; L_n(x,y) \frac{\left(\frac{t}{w}\right)^n}{n!}, \quad (4.13)$$

which on using relation (1.21) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty} \sum_{k=0}^{[\frac{q}{p}]} \binom{\mu+j+k-2}{k} L_{q+n-pk}(x,y) \; L_n^{(\alpha+q-pk)}(w) \; y_j(v,\mu+k,\beta) \frac{t^{q-pk}\eta^k}{(q-pk)!} \left(\frac{t}{w}\right)^n$$

$$= (1-\eta)^{1-\mu-j} \left(\frac{w}{w-yt}\right)^{\alpha+1} \; y_j\left(\frac{v}{1-\eta}, \mu, \beta\right) \; {}_1F_1\Big(\alpha+1; 1; \frac{-xt}{w-yt}\Big). \quad (4.14)$$

Similarly other trilateral generating functions can be obtained as applications of result (3.7) with the help of Remarks 1.2–1.6 and relation (1.21).

Finally, it is worthy to note that, by taking $y = 1$ and using relation (1.23) the results obtained in this section give many bilateral and trilateral generating functions for the classical Laguerre polynomials $L_n(x)$ associated with other classical polynomials.

REFERENCES

[1] G. Dattoli, A. Torre, Operational methods and two variable Laguerre polynomials, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **132** (1998) 1-7.

[2] G. Dattoli, A. Torre, Exponential operators, quasi-monomials and generalized polynomials, *Radiat. Phys. Chem.* **57** (2000) 21-26.

[3] B. González, J. Matera, H.M. Srivastava, Some q-generating functions and associated generalized hypergeometric polynomials, *Math. Comput. Modelling* **34**(2001) 133–175.

[4] S.-D. Lin, I.-C. Chen, H.M. Srivastava, Certain classes of finite-series relationships and generating functions involving the generalized Bessel polynomials, *Appl. Math. Comput.* **137** (2003) 261-275.

[5] S.-D. Lin, H.M. Srivastava, P.Y. Wang, Some families of hypergeometric transformations and generating relations, *Math. Comput. Model.* **36** (2002) 445-459.

[6] M.A. Özarslan, Some families of generating functions for the extended Srivastava polynomials, *Appl. Math. Comput.* **218** (2011) 959-964.

[7] M.A. Özarslan, A. Altin, Some families of generating functions for the multiple orthogonal polynomials associated with modified Bessel K-functions, *J. Math. Anal. Appl.* **297** (2004) 186–193.

[8] E.D. Rainville, *Special Functions*, Macmillan, New York, 1960, reprinted by Chelsea Publ. Co., Bronx, New York, 1971.

[9] H.M. Srivastava, A contour integral involving Fox's H-function, *Indian J. Math.* **14** (1972) 1–6.

[10] H.M. Srivastava, H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press, New York, 1984.

DEPARTMENT OF MATHEMATICS, ADEN UNIVERSITY, ADEN, YEMEN

E-mail address: gonah1977@yahoo.com

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 140–151 (2015) ©KJM

ON RIGHT INVERSE Γ-SEMIGROUP

SUMANTA CHATTOPADHYAY

Abstract. Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. S is called a Γ-semigroup if $a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. An element $e \in S$ is said to be α-idempotent for some $\alpha \in \Gamma$ if $e\alpha e = e$. A Γ- semigroup S is called regular Γ-semigroup if each element of S is regular i.e, for each $a \in S$ there exists an element $x \in S$ and there exist $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. A regular Γ-semigroup S is called a right inverse Γ-semigroup if for any α-idempotent e and β-idempotent f of S, $e\alpha f\beta e = f\beta e$. In this paper we introduce ip - congruence on regular Γ-semigroup and ip - congruence pair on right inverse Γ-semigroup and investigate some results relating this pair.

1. Introduction

Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. S is called a Γ-semigroup if
(i)$a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and
(ii)$(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

A semigroup can be considered to be a Γ-semigroup in the following sense. Let S be an arbitrary semigroup. Let 1 be a symbol not representing any element of S. Let us extend the binary operation defined on S to $S \cup \{1\}$ by defining $11 = 1$ and $1a = a1$ for all $a \in S$. It can be shown that $S \cup \{1\}$ is a semigroup with identity element 1. Let $\Gamma = \{1\}$. If we take $ab = a1b$, it can be shown that the semigroup S is a $\Gamma-$semigroup where $\Gamma = \{1\}$.

In [8] we introduced right inverse Γ-semigroup. In [2] Gomes introduced the notion of congruence pair on inverse semigroup and studied some of its properties. In this paper we introduce the notion of ip - congruence on regular Γ-semigroup, ip - congruence pair on right inverse Γ-semigroup and studied some of its properties. We now recall some definition and results.

Date: July 25, 2014 and, in revised form, April 28, 2015.
2000 *Mathematics Subject Classification.* 20M17.
Key words and phrases. Γ-Semigroup, right orthodox Γ-Semigroup, right inverse Γ - semigroup, left partial congruence, ip - congruence, normal subsemigroup, ip - congruence pair.

Definition 1.1. *Let S be a Γ-semigroup. An element $a \in S$ is said to be regular if $a \in a\Gamma S\Gamma a$ where $a\Gamma S\Gamma a = \{a\alpha b\beta a : b \in S, \alpha, \beta \in \Gamma\}$. S is said to be regular if every element of S is regular.*

Example 1.1. [8] Let M be the set of all 3×2 matrices and Γ be the set of all 2×3 matrices over a field. Then M is a regular Γ semigroup.

Example 1.2. Let S be a set of all negative rational numbers. Obviously S is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} : p$ is prime $\}$. Let $a, b, c \in S$ and $\alpha \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence S is a Γ-semigroup. Let $a = \frac{m}{n} \in S$ where $m > 0$ and $n < 0$. Suppose $m = p_1 p_2 ....p_k$ where p_i's are prime. Now $\frac{p_1 p_2 ....p_k}{n}(-\frac{1}{p_1})\frac{n}{p_2 ....p_{k-1}}(-\frac{1}{p_k})\frac{m}{n} = \frac{p_1 p_2 ....p_k}{n}$. Thus taking $b = \frac{n}{p_2 ....p_{k-1}}$, $\alpha = (-\frac{1}{p_1})$ and $\beta = (-\frac{1}{p_k})$ we can say that a is regular. Hence S is a regular Γ-semigroup.

Definition 1.2. *Let S be a Γ-semigroup and $\alpha \in \Gamma$. Then $e \in S$ is said to be an α-idempotent if $e\alpha e = e$. The set of all α-idempotents is denoted by E_α and we denote $\bigcup_{\alpha \in \Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent element of S.*

Definition 1.3. *Let S be a Γ-semigroup and $a, b \in S$, $\alpha, \beta \in \Gamma$. b is said to be an (α, β)-inverse of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. This is denoted by $b \in V_\alpha^\beta(a)$.*

Theorem 1.1. Let S be a regular Γ-semigroup and $a \in S$. Then $V_\alpha^\beta(a)$ is non-empty for some $\alpha, \beta \in \Gamma$.

Proof: Since S is regular there exist $b \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha b\beta a$. Now we consider the element $b\beta a\alpha b$. $a\alpha(b\beta a\alpha b)\beta a = (a\alpha b\beta a)\alpha b\beta a = a\alpha b\beta a = a$ and $(b\beta a\alpha b)\beta a\alpha(b\beta a\alpha b) = b\beta(a\alpha b)\beta a)\alpha b\beta a\alpha b = b\beta a\alpha b\beta a\alpha b = b\beta a\alpha b$. Hence $b\beta a\alpha b \in V_\alpha^\beta(a)$.

Definition 1.4. *Let S be a Γ-semigroup. An equivalence relation ρ on S is said to be a right (left) congruence on S if $(a, b) \in \rho$ implies $(a\alpha c, b\alpha c) \in \rho$, $((c\alpha a, c\alpha b) \in \rho)$ for all $a, b, c \in S$ and for all $\alpha \in \Gamma$. An equivalence relation which is both left and right congruence on S is called* congruence *on S.*

Definition 1.5. *A regular Γ-semigroup S is called a right orthodox Γ-semigroup if for any α-idempotent e and β-idempotent f of S, $e\alpha f$ is a β-idempotent.*

Definition 1.6. *A regular Γ-semigroup M is a right orthodox Γ-semigroup if and only if for $a, b \in S$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$, $a' \in V_{\alpha_1}^{\alpha_2}(a)$ and $b' \in V_{\beta_1}^{\beta_2}(b)$, we have $b'\beta_2 a' \in V_{\beta_1}^{\alpha_2}(a\alpha_1 b)$.*

Definition 1.7. *A regular Γ-semigroup S is called a right inverse Γ-semigroup if for any α-idempotent e and β-idempotent f of S, $e\alpha f\beta e = f\beta e$.*

Theorem 1.2. Every right inverse Γ-semigroup is a right orthodox Γ-semigroup.

Theorem 1.3. Let S be a regular Γ-semigroup and E_α be the set of all α-idempotents in S. Let $e \in E_\alpha$ and $f \in E_\beta$. Then

$$RS(e, f) = \Big\{g \in V_\beta^\alpha(e\alpha f) \cap E_\alpha : g\alpha e = f\beta g = g\Big\}$$

is non-empty.

Proof: Since S is regular, there exist $b \in S$ and $\gamma, \delta \in \Gamma$ such that $e\alpha f\gamma b\delta e\alpha f = e\alpha f$ and $b\delta e\alpha f\gamma b = b$. Now $(e\alpha f)\beta(f\gamma b\delta e)\alpha(e\alpha f) = e\alpha f\gamma b\delta e\alpha f = e\alpha f$ and $(f\gamma b\delta e)\alpha(e\alpha f)\beta(f\gamma b\delta e) = f\gamma b\delta e\alpha f\gamma b\delta e = f\gamma b\delta e$. Hence $f\gamma b\delta e \in V_\beta^\alpha(e\alpha f)$. Thus $V_\beta^\alpha(e\alpha f) \neq \phi$. Now let $x \in V_\beta^\alpha(e\alpha f)$ and setting $g = f\beta x\alpha e$ we have $g\alpha g = (f\beta x\alpha e)\alpha(f\beta x\alpha e) = f\beta(x\alpha e)\alpha f\beta x)\alpha e = f\beta x\alpha e = g$. Thus $g \in E_\alpha$.

Again $g\alpha e\alpha f\beta g = f\beta x\alpha e\alpha e\alpha f\beta f\beta x\alpha e = f\beta x\alpha e\alpha f\beta x\alpha e = f\beta x\alpha e = g$ and $e\alpha f\beta g\alpha e\alpha f = e\alpha f\beta f\beta x\alpha e\alpha e\alpha f = e\alpha f\beta x\alpha e\alpha f = e\alpha f$ implies that $g \in V_\beta^\alpha(e\alpha f)$. Hence $g\alpha e = f\beta x\alpha e\alpha e = f\beta x\alpha e = g$ and $f\beta g = f\beta f\beta x\alpha e = f\beta x\alpha e = g$. Therefore $RS(e, f) \neq \emptyset$.

Definition 1.8. *Let S be a regular Γ- semigroup and e and f be α and β- idempotents respectively. Then the set $RS(e, f)$ described in the above Theorem is called the right sandwich set of e and f.*

Theorem 1.4. Let S be a regular Γ-semigroup and e and f be α and β-idempotents respectively. Then the set $RS(e, f) = \{g \in V_\beta^\alpha(e\alpha f) : g\alpha e = g = f\beta g \text{ and } e\alpha g\alpha f = e\alpha f\}$.

Proof: Let $P = \{g \in V_\beta^\alpha(e\alpha f) : g\alpha e = g = f\beta g \text{ and } e\alpha g\alpha f = e\alpha f\}$ and let $g \in RS(e, f)$. Then $g \in E_\alpha, g\alpha e = g = f\beta g$ and $g \in V_\beta^\alpha(e\alpha f)$. Now $e\alpha g\alpha f = e\alpha g\alpha e\alpha f\beta g\alpha f = e\alpha f\beta g\alpha e\alpha f\beta g\alpha e\alpha f = e\alpha f\beta g\alpha e\alpha f = e\alpha f$. Hence $RS(e, f) \subseteq P$. Next let $g \in P$. Now $g\alpha g = g\alpha e\alpha f\beta g = g$. Hence $g \in E_\alpha$, which shows that $P \subseteq RS(e, f)$ and hence the proof.

Theorem 1.5. Let S be a regular Γ- semigroup and $a, b \in S$.If $a' \in V_\alpha^\beta(a), b' \in V_\gamma^\delta(b)$ and $g \in RS(a'\beta a, b\gamma b')$ then $b'\delta g\alpha a' \in V_\gamma^\beta(a\alpha b)$.

Proof: Let $e = a'\beta a$ and $f = b\gamma b'$. Then e is an α-idempotent and f is a δ-idempotent and also g is an α-idempotent. Now $(a\alpha b)\gamma(b'\delta g\alpha a')\beta(a\alpha b) = a\alpha f\delta g\alpha e\alpha b = a\alpha g\alpha b = a\alpha a'\beta a\alpha g\alpha b\gamma b'\delta b = a\alpha e\alpha g\alpha e\alpha b = a\alpha e\alpha f\delta b = a\alpha a'\beta a\alpha b \gamma b'\delta b = a\alpha b$. Again $(b'\delta g\alpha a')\beta(a\alpha b)\gamma(b'\delta g\alpha a') = b'\delta g\alpha e\alpha f\delta g\alpha a' = b'\delta g\alpha g\alpha a' = b'\delta g\alpha a'$. Hence $b'\delta g\alpha a' \in V_\gamma^\beta(a\alpha b)$.

Corollary 1.1. For $a, b \in S$, if $V_\alpha^\beta(a)$ and $V_\gamma^\delta(b)$ are nonempty then $V_\gamma^\beta(a\alpha b)$ is nonempty.

Proof: Let $a' \in V_\alpha^\beta(a)$ and $b' \in V_\gamma^\delta(b)$ then we know that $RS(a'\beta a, b\gamma b') \neq \phi$. For $g \in RS(a'\beta a, b\gamma b')$ and hence we get $b'\delta g\alpha a' \in V_\gamma^\beta(a\alpha b)$. Hence the proof.

2. IP- CONGRUENCE PAIR ON RIGHT INVERSE Γ-SEMIGROUP

In this section we characterize some congruences on a right inverse Γ - semigroup S.

Definition 2.1. *Let S be a Γ-semigroup. A nonempty subset K of S is said to be partial Γ-subsemigroup if for $a, b \in K, a\alpha b \in K$, whenever $V_\alpha^\beta(a) \neq \phi$. for $\alpha, \beta \in \Gamma$.*

Definition 2.2. *A partial Γ-subsemigroup K of S is said to be regular if $V_\alpha^\beta(k) \subseteq K$ for all $k \in K$ and $\alpha, \beta \in \Gamma$.*

Definition 2.3. *A partial Γ-subsemigroup K is said to be full if $E(S) \subseteq K$ where $E(S)$ is the set of all idempotent elements of S.*

Definition 2.4. *A partial Γ-subsemigroup K of S is said to be self conjugate if for all $a \in S, k \in K$ and $a' \in V_\alpha^\beta(a), a'\beta k\gamma a \in K$ whenever $V_\gamma^\delta(k) \neq \phi$ for some $\delta \in \Gamma$.*

Definition 2.5. *A partial Γ-subsemigroup K of S is said to be normal if it is regular, full and self conjugate.*

Definition 2.6. *An equivalence relation ρ on S is said to be left partial congruence if $(a,b) \in \rho$ implies $(c\alpha_3 a, c\alpha_3 b) \in \rho$ whenever $V_{\alpha_3}^{\beta_3}(c)$ is nonempty. Note that every left congruence is a left partial congruence.*

Here we consider these left partial congruence which satisfy the following condition:
$(a,b) \in \rho$ implies $(a\alpha_1 c, b\alpha_2 c) \in \rho$ whenever each of the sets $V_{\alpha_1}^{\beta_1}(a), V_{\alpha_2}^{\beta_2}(b)$ is nonempty for $\alpha_i, \beta_i \in \Gamma, i = 1,2$. We call this left partial congruence as inverse related partial congruence (ip - congruence).

Example 2.1. *Let $A = \{1,2,3\}$ and $B = \{4,5\}$. S denotes the set of all mappings from A to B. Here members of S will be described by the images of the elements 1, 2, 3. For example the map $1 \to 4, 2 \to 5, 3 \to 4$ will be written as $(4,5,4)$ and $(5,5,4)$ denotes the map $1 \to 5, 2 \to 5, 3 \to 4$. A map from B to A will be described in the same fashion. For example $(1,2)$ denotes $4 \to 1, 5 \to 2$. Now $S = \Big\{(4,4,4),(4,4,5),(4,5,4),(4,5,5),(5,5,5),(5,4,5),\ (5,4,4),\ (5,5,4)\Big\}$ and let $\Gamma = \{(1,1),(1,2),(2,3),(3,1)\}$. Let $f,g \in S$ and $\alpha \in \Gamma$. We define $f\alpha g$ by $(f\alpha g)(a) = f\alpha\Big(g(a)\Big)$ for all $a \in A$. So $f\alpha g$ is a mapping from A to B and hence $f\alpha g \in S$ and we can show that $(f\alpha g)\beta h = f\alpha(g\beta h)$ for all $f,g,h \in S$ and $\alpha,\beta \in \Gamma$. Hence S is a Γ - semigroup.*

We can also show that it is right inverse. We now give a partition $S = \bigcup_{1 \le i \le 5} S_i$ and let ρ be the equivalence relation yielded by the partition where each S_i is given by:
$S_1 = \{(4,4,4)\}$,
$S_2 = \{(5,5,5)\}$,
$S_3 = \{(4,5,4),(5,4,5)\}$,
$S_4 = \{(4,5,5),(5,4,4)\}$,
$S_5 = \{(4,4,5),(5,5,4)\}$.
Here we see that $(4,5,4)\rho(5,4,5)$ but $(4,5,4)(3,1)(4,4,4) = (4,4,4)$ and $(5,4,5)$ $(3,1)(4,4,4) = (5,5,5)$ i.e ρ is not a congruence.

Now for $f \in S$ we observe the following cases:
(a) $(4,4,4)\alpha f = (4,4,4)$ *for all* $\alpha \in \Gamma$,
(b) $(5,5,5)\alpha f = (5,5,5)$ *for all* $\alpha \in \Gamma$,
(c) $(4,5,4)(1,2)f = f$ *and* $(4,5,4)(2,3)f = f'$,
$(5,4,5)(2,3)f = f$ *and* $(5,4,5)(1,2)f = f'$,
(d) $(4,4,5)(2,3)f = f$ *and* $(4,4,5)(3,1)f = f'$,
$(5,5,4)(3,1)f = f$ *and* $(5,5,4)(2,3)f = f'$,
(e) $(4,5,5)(1,2)f = f$ *and* $(4,5,5)(3,1)f = f'$,
$(5,4,4)(3,1)f = f$ *and* $(5,4,4)(1,2)f = f'$,

From the above cases we can easily verify that ρ is a ip - congruence on S.

Definition 2.7. *An ip - congruence ξ on $E(S)$ of S is said to be normal if for any α-idempotent e and β-idempotent $f, a \in S$ and $a' \in V_\gamma^\delta(a), (e,f) \in \xi$ implies $(a'\delta e\alpha a, a'\delta f\beta a) \in \xi$ whenever $a'\delta e\alpha a, a'\delta f\beta a \in E(S)$.*

Let ρ be an ip - congruence on a regular Γ - semigroup S then we can define a binary operation on S/ρ as $(a\rho)(b\rho) = (a\alpha b)\rho$ whenever $V_\alpha^\beta(a)$ exists for some $\beta \in \Gamma$. This is well defined because if $a\rho = a'\rho$ and $b\rho = b'\rho$ then

$$\begin{aligned}(a\rho)(b\rho) &= (a\alpha b)\rho \text{ (Since } V_\alpha^\beta(a) \neq \phi \text{ for some } \alpha, \beta \in \Gamma) \\ &= (a\alpha b')\rho \\ &= (a'\alpha_1 b')\rho (\text{Since } V_{\alpha_1}^{\beta_1}(a') \neq \phi \text{ for some } \alpha_1, \beta_1 \in \Gamma) \\ &= (a'\rho)(b'\rho).\end{aligned}$$

The operation is easily seen to be associative, and so S/ρ is a semigroup.

Definition 2.8. *Let ρ be an ip - congruence on a regular Γ-semigroup S. Let $\alpha \in \Gamma$, then the subset $\{a \in S : a\rho \in E(S/\rho)\}$ of S is called kernel of ρ and it is denoted by K.*

Definition 2.9. *Let ρ be an ip - congruence on a regular Γ-semigroup S. Then the restriction of ρ to the subset $E(S)$ is called the trace of ρ and it is denoted by $tr\rho$.*

We now treat S as a right inverse Γ-semigroup throughout the paper.

Definition 2.10. *A pair (ξ, K) consisting of a normal ip - congruence ξ on $E(S)$ and a normal partial Γ- subsemigroup K of S is said to be ip - congruence pair for S if for all $a, b \in S, a' \in V_\alpha^\beta(a)$ and $e \in E_\gamma$*
(i) $e\gamma a \in K, (e, a\alpha a') \in \xi \Rightarrow a \in K$
(ii) $a \in K \Rightarrow (a\alpha e\gamma a', e\gamma a\alpha a') \in \xi$

Given a pair (ξ, K) we define a relation $\rho_{(\xi,K)}$ on S by $(a, b) \in \rho_{(\xi,K)}$ if and only if there exist $a' \in V_\alpha^\beta(a)$ and $b' \in V_\gamma^\delta(b)$ such that $a\alpha b' \in K, (a'\beta a, b'\delta b) \in \xi$.

Theorem 2.1. Let S be a right inverse Γ-semigroup. Then for an ip - congruence pair (ξ, K) and a μ-idempotent $e, a\alpha b \in K$ implies $a\alpha e\mu b \in K$ for all $a, b \in S$ and $V_\alpha^\beta(a) \neq \phi$ for some $\beta \in \Gamma$.

Proof: Let $a\alpha b \in K$. Since S is regular there exist $\gamma, \delta \in \Gamma$ such that $V_\gamma^\delta(b) \neq \phi$. Then by Corollary 1.1 , $V_\gamma^\beta(a\alpha b) \neq \phi$. Let $b' \in V_\gamma^\delta(b)$. Then $b\gamma b'$ is a δ-idempotent and since S is a right inverse Γ-semigroup $(b\gamma b')\delta e\mu(b\gamma b') = e\mu(b\gamma b')$. Now $a\alpha e\mu b = a\alpha e\mu b\gamma b'\delta b = a\alpha(b\gamma b')\delta e\mu(b\gamma b')\delta b = (a\alpha b)\gamma(b'\delta e\mu b)$. Since S is right inverse Γ-semigroup $b'\delta e\mu b \in E_\gamma \subseteq K$. Since K is a partial Γ-subsemigroup and $a\alpha b \in K$, $(a\alpha b)\gamma(b'\delta e\mu b) \in K$. So $a\alpha e\mu b \in K$.

Theorem 2.2. Let (ξ, K) be an ip - congruence pair for S and $a, b \in S$ are such that $(a, b) \in \rho_{(\xi,K)}$, then there exist $a' \in V_\alpha^\beta(a)$ and $b' \in V_\gamma^\delta(b)$ such that
(i) $a\alpha b' \in K$ and $(a'\beta a, b'\delta b) \in \xi$
(ii) $b\gamma a' \in K$ and so $(b, a) \in \rho_{(\xi,K)}$
(iii) $(b\gamma b', a\alpha a'\beta b\gamma b') \in \xi$ and $(a\alpha a', b\gamma b'\delta a\alpha a') \in \xi$

Proof: (i) Let $a, b \in S$ and $(a, b) \in \rho_{(\xi,K)}$. Then (i) follows from definition of $\rho_{(\xi,K)}$. Now from (i) we have $a\alpha b' \in K$ and $(a'\beta a, b'\delta b) \in \xi$. Let $g \in RS(b'\delta b, a'\beta a)$, then g is a γ-idempotent. So by Theorem 1.5 we have $a\alpha g\gamma b' \in V_\beta^\delta(b\gamma a')$. Also by Theorem 2.1 $a\alpha g\gamma b' \in K$ since $a\alpha b' \in K$ and $g \in E_\gamma$. On the other hand $b\gamma a' \in V_\delta^\beta(a\alpha g\gamma b')$ and so $b\gamma a' \in K$, since K is a normal subsemigroup of S. Therefore $(b, a) \in \rho_{(\xi,K)}$ since ξ is symmetric. Hence (ii) follows.
Again for $g \in RS(b'\delta b, a'\beta a)$, $g = g\gamma b'\delta b = a'\beta a\alpha g$ and $(b'\delta b)\gamma g\gamma(a'\beta a) = (b'\delta b)\gamma$ $(a'\beta a)$ by Theorem 1.4. Hence $b\gamma g\gamma b' \in E_\delta$. Now $b'\delta b = (b'\delta b)\gamma(b'\delta b)\ \xi\ (b'\delta b)\gamma$

$(a'\beta a) = (b'\delta b)\gamma g\gamma(a'\beta a)\ \xi\ (b'\delta b)\gamma g\gamma(b'\delta b)$ and so by normality of ξ we have $b\gamma(b'\delta b)\gamma b'\ \xi\ b\gamma(b'\delta b\gamma g\gamma b'\delta b)\gamma b'$ i.e $b\gamma b'\ \xi\ b\gamma g\gamma b'$. Now $a\alpha g\gamma b' \in V_\beta^\delta(b\gamma a')$ and so we have

$$\begin{aligned}
b\gamma b' &\ \xi\ b\gamma g\gamma b'\\
&= b\gamma(a'\beta a\alpha g)\gamma b' \text{ (Since } g\in RS(b'\delta b, a'\beta a))\\
&= (b\gamma a')\beta(a\alpha a'\beta a)\alpha g\gamma b'\\
&= (b\gamma a')\beta(a\alpha a')\beta(a\alpha g\gamma b') \text{ (Since } a\alpha a'\in E_\beta \text{ and } b\gamma a'\in K)\\
&\ \xi\ (a\alpha a')\beta(b\gamma a')\beta(a\alpha g\gamma b') \text{ (by Definition 2.6 and } a\alpha g\gamma b'\in V_\beta^\delta(b\gamma a'))\\
&= a\alpha a'\beta b\gamma g\gamma b'\\
&\ \xi\ (a\alpha a')\beta(b\gamma b').
\end{aligned}$$

Similarly interchanging the role of a and b we can get the second relation.

Theorem 2.3. Let (ξ, K) be an ip - congruence pair for S and $a, b \in S$ are such that $a, b \in \rho_{(\xi,K)}$, then for all $a^* \in V_\alpha^\beta(a)$ and $b^* \in V_\gamma^\delta(b)$, $a\alpha b^* \in K$ and $(a^*\beta a, b^*\delta b) \in \xi$

Proof: Since $(a, b) \in \rho_{(\xi,K)}$, there exist $a' \in V_{\alpha_1}^{\beta_1}(a)$ and $b' \in V_{\gamma_1}^{\delta_1}(b)$ such that all the three conditions of Theorem 2.2 are satisfied. Now

$$\begin{aligned}
a'\beta_1 a &= a'\beta_1 a\alpha a^*\beta a\\
&= a'\beta_1 a\alpha a^*\beta a\alpha_1 a'\beta_1 a\\
&\ \xi\ a'\beta_1 a\alpha_1 a^*\beta a\alpha a'\beta_1 a \text{ (Since } \xi \text{ is an ip - congruence and } V_\alpha^\beta(a) \text{ and } V_{\alpha_1}^{\beta_1}(a) \text{ are nonempty.)}\\
&= (a'\beta_1 a)\alpha_1(a^*\beta a)\alpha(a'\beta_1 a)\\
&= (a^*\beta a)\alpha(a'\beta a)\\
&\ \xi\ a^*\beta a\alpha_1 a'\beta a \text{ (Since } \xi \text{ is an ip - congruence and } V_\alpha^\beta(a) \text{ and } V_{\alpha_1}^{\beta_1}(a) \text{ are nonempty.)}\\
&= a^*\beta a.
\end{aligned}$$

Similarly we can show that $(b'\delta_1 b, b^*\delta b) \in \xi$. Hence we have $a^*\beta a\ \xi\ a'\beta_1 a\ \xi\ b'\delta_1 b\ \xi\ b^*\delta b$. Hence $(a^*\beta a, b^*\delta b) \in \xi$. We now prove that $a\alpha b^* \in K$. To prove this we proceed by five steps.

Step1: $b\gamma_1 a' \in K$.
Step2: $b'\delta_1 a \in K$.
Step3: $b^*\delta a \in K$.
Step4: $(b\gamma b^*, a\alpha a^*\beta b\gamma b^*) \in \xi$.
Step5: $a\alpha b^* \in K$.

Let $g \in RS(b'\delta_1 b, a'\beta_1 a)$, then g is a γ_1-idempotent and we have $a\alpha_1 g\gamma_1 b' \in V_{\beta_1}^{\delta_1}(b\gamma_1 a')$. Also since $a\alpha_1 b' \in K$ and $g \in E_{\gamma_1}$, by Theorem 2.1 $a\alpha_1 g\gamma_1 b' \in K$. On the other hand $b\gamma_1 a' \in V_{\delta_1}^{\beta_1}(a\alpha_1 g\gamma_1 b')$. Since K is regular we have $b\gamma_1 a' \in K$.

Let $h \in RS(b\gamma_1 b', a\alpha_1 a')$. Then $a'\beta_1 h\delta_1 b \in V_{\alpha_1}^{\gamma_1}(b'\delta_1 a)$ i.e, $b'\delta_1 a \in V_{\gamma_1}^{\alpha_1}(a'\beta_1 h \delta_1 b)$. Now since $b\gamma_1 a' \in K$ and K is full self conjugate partial Γ-subsemigroup of S, we have

$(b'\delta_1 b)\gamma_1(a'\beta_1 a)\alpha_1(a'\beta_1 h\delta_1 b) = b'\delta_1((b\gamma_1 a')\beta_1 h)\delta_1 b \in K$.

Now

$$\begin{aligned}
h\delta_1(a\alpha_1 a') &= (a\alpha_1 a')\beta_1 h\delta_1(a\alpha_1 a')\\
&\ \xi\ (b\gamma_1 b')\delta_1(a\alpha_1 a')\beta_1 h\delta_1(a\alpha_1 a') \text{(By Theorem 2.2)}\\
&= (b\gamma b')\delta_1 h\delta_1(a\alpha a') \text{ (Since } S \text{ is right inverse)}\\
&= (b\gamma b')\delta_1(a\alpha a') \text{ (Since } h\in RS(b\gamma_1 b', a\alpha_1 a').\\
&\ \xi\ a\alpha_1 a' \text{ (By Theorem 2.2).}
\end{aligned}$$

Again

$$\begin{array}{rcl}(a'\beta_1 h\delta_1 b)\gamma_1(b'\delta_1 a) & = & a'\beta_1 h\delta_1 a \\ & \xi & a\alpha_1 a' \\ & \xi & (b'\delta_1 b)\gamma_1(a'\beta_1 a) \text{ (By Theorem 2.2).}\end{array}$$

Now since S is a right inverse Γ-semigroup, it is right orthodox and hence $(b'\delta_1 b)\gamma_1$ $(a'\beta_1 a)$ is an α_1-idempotent. Thus by Definition 2.10 $a'\beta_1 h\delta_1 b \in K$ and since K is regular, $b'\delta_1 a \in K$.

Now we have $b'\delta_1 a \in K$. Hence we get $b'\delta_1(b\gamma b^*)\delta a \in K$ by Theorem 2.1. Again $b^*\delta a = b^*\delta b\gamma b^*\delta a = b^*\delta(b\gamma_1 b'\delta_1 b)\gamma b^*\delta a = (b^*\delta b)\gamma_1(b'\delta b\gamma b^*\delta a) \in K$ since $b^*\delta b \in E_\gamma \subseteq K$, $V_{\gamma_1}^{\delta_1}(b)$ is nonempty and K is a partial Γ-subsemigroup.

We now prove step 4.

$$\begin{array}{rcl}b\gamma b^* & = & (b\gamma_1 b')\delta_1(b\gamma b^*) \\ & \xi & (a\alpha_1 a')\beta_1(b\gamma_1 b')\delta_1(b\gamma b^*) \\ & = & (a\alpha a^*)\beta(a\alpha_1 a')\beta_1(b\gamma_1 b')\delta_1(b\gamma b^*) \\ & \xi & (a\alpha a^*)\beta(b\gamma_1 b')\delta_1(b\gamma b^*) \\ & = & (a\alpha a^*)\beta(b\gamma b^*).\end{array}$$

Finally we show the last step. Now we have $b^*\delta a \in K$. Since $a^* \in V_\alpha^\beta(a)$ and $b^* \in V_\gamma^\delta(b)$, we have $(a^*\beta b) \in V_\alpha^\gamma(b^*\delta a)$ and hence $a^*\beta b \in K$, since K is regular. Let $x \in RS(a^*\beta a, b^*\delta b)$. Then $b\gamma x\alpha a^* \in V_\delta^\beta(a\alpha b^*)$. Now $((a\alpha a^*)\beta(b\gamma b^*))\delta(b\gamma x\alpha a^*) = a\alpha a^*\beta b\gamma x\alpha a^* = a\alpha((a^*\beta b)\gamma x)\alpha a^* \in K$, since $a^*\beta b \in K, x \in E_\alpha \subseteq K$ and hence $(a^*\beta b)\gamma x \in K$ and also K is self conjugate. Again

$$\begin{array}{rcl}x\alpha(b^*\delta b) & = & (b^*\delta b)\gamma x\alpha(b^*\delta b) \text{ (Since } S \text{ is right inverse)} \\ & \xi & ((b^*\delta b\gamma(a^*\beta a))\alpha x\alpha(b^*\delta b) \text{ (Since } (a^*\beta a, b^*\delta b) \in \xi \\ & = & (b^*\delta b)\gamma(a^*\beta a)\alpha(b^*\delta b) \text{ (Since } x \in RS(a^*\beta a, b^*\delta b).) \\ & \xi & ((b^*\delta b)\gamma(b^*\delta b)\gamma(b^*\delta b) \text{(Since } \xi \text{ is an ip - congruence and} \\ & & \qquad (a^*\beta a, b^*\delta b) \in \xi) \\ & = & b^*\delta b.\end{array}$$

Thus

$$\begin{array}{rcl}b\gamma x\alpha b^* & = & b\gamma(x\alpha(b^*\delta b))\gamma b^* \\ & \xi & b\gamma(b^*b)\gamma b^* \\ & = & b\gamma b^*.\end{array}$$

Now

$$\begin{array}{rcl}(b\gamma x\alpha a^*)\beta(a\alpha b^*) & = & b\gamma(x\alpha(a^*\beta a))\alpha b^* \\ & = & b\gamma x\alpha b^* \\ & \xi & b\gamma b^* \\ & \xi & (a\alpha a^*)\beta(b\gamma b^*).\end{array}$$

Again since S is a right inverse Γ-semigroup, $(a\alpha a^*)\beta(b\gamma b^*)$ is a δ-idempotent and by Definition 2.10(i) $b\gamma x\alpha a^* \in K$ and hence $a\alpha b^* \in K$ since K is regular. Hence the Theorem.

Remark 2.1. *From the previous Theorem, we can say that in the definition 3.11 of $\rho_{(\xi,K)}$ and in the Theorem 2.2 "there exist" can be substituted by "for all".*

Theorem 2.4. Let (ξ, K) be an ip - congruence pair for S and $a, b, c \in S$ and let $a' \in V_{\alpha_1}^{\beta_1}(a)$, $b' \in V_{\alpha_2}^{\beta_2}(b), c' \in V_{\alpha_3}^{\beta_3}(c), g \in RS(c'\beta_3 c, a\alpha_1 a'), h \in RS(c'\beta_3 c, b\alpha_2 b')$. Then $(a'\beta_1 a, b'\beta_2 b) \in \xi$, $a\alpha_1 b' \in K$ implies $(a'\beta_1 g\alpha_3 a, b'\beta_2 h\alpha_3 b) \in \xi$.

Proof: Let (ξ, K) be an ip - congruence pair for S and $a, b \in S$ are such that for some $a' \in V_{\alpha_1}^{\beta_1}(a)$, $b' \in V_{\alpha_2}^{\beta_2}(b), (a'\beta_1 a, b'\beta_2 b) \in \xi$ and $a\alpha_1 b' \in K$. Given $c \in S$

and $c' \in V_{\alpha_3}^{\beta_3}(c)$, let $g \in RS(c'\beta_3 c, a\alpha_1 a')$ and $h \in RS(c'\beta_3 c, b\alpha_2 b')$. Then g and h are α_3-idempotents. Choose an arbitrary element $x \in RS(a'\beta_1 a, b'\beta_2 b)$. Then $b\alpha_2 x\alpha_1 a' \in V_{\beta_2}^{\beta_1}(a\alpha_1 b')$. So $a\alpha_1 b'\beta_2 b\alpha_2 x\alpha_1 a' \in E_{\beta_1}$. Also let $t \in RS(g, a\alpha_1 b'\beta_2 b\alpha_2 x\alpha_1 a')$ then $t \in E_{\alpha_3}$ and $t = t\alpha_3 g$ and hence $b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 g \in V_{\beta_2}^{\alpha_3}(g\alpha_3 a\alpha_1 b')$ and $b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b' = (b\alpha_2 x\alpha_1 a')\beta_1(t\alpha_3 g)\alpha_3 a\alpha_1 b' = (b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 g)\alpha_3(g\alpha_3 a\alpha_1 b') \in E_{\beta_2}$. On the other hand $b\alpha_2 x\alpha_1 a' \in K$, since it is an (β_2, β_1)-inverse of $a\alpha_1 b'$ which belongs to K. Now since (ξ, K) is an ip - congruence pair for S, by definition we have $((b\alpha_2 x\alpha_1 a')\beta_1 t\alpha_3(a\alpha_1 b')\ ,\ t\alpha_3 b\alpha_2 x\alpha_1 a'\beta_1 a\alpha_1 b') \in \xi$. Again since $x\alpha_1(a'\beta_1 a) = x$ we get

$$(b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b', t\alpha_3 b\alpha_2 x\alpha_1 b') \in \xi \tag{2.1}$$

for all $x \in RS(a'\beta_1 a, b'\beta_2 b)$

Now since ξ is an ip - congruence and $(a'\beta_1 a, b'\beta_2 b) \in \xi$, we have $b'\beta_2 b\alpha_2 x\alpha_1 b'\beta_2 b\ \xi\ a'\beta_1 a\alpha_1 x\alpha_1 b'\beta_2 b = a'\beta_1 a\alpha_1 b'\beta_2 b\ \xi\ b'\beta_2 b\alpha_2 b'\beta_2 b = b'\beta_2 b$. Again and hence $(b\alpha_2 x\alpha_1 b')\beta_2(b\alpha_2 x\alpha_1 b') = b\alpha_2 x\alpha_1(b'\beta_2 b\alpha_2 x)\alpha_1 b' = b\alpha_2 x\alpha_1 b'$ and hence $b\alpha_2 x\alpha_1 b' \in E_{\beta_2}$. Hence ξ is normal, we have $(b\alpha_2(b'\beta_2 b\alpha_2 x\alpha_1 b'\beta_2 b)\alpha_2 b', b\alpha_2(b'\beta_2 b)\alpha_2 b') \in \xi$ which implies

$$(b\alpha_2 x\alpha_1 b', b\alpha_2 b') \in \xi \tag{2.2}$$

Similarly we can show that

$$(a\alpha_1 x\alpha_1 a', a\alpha_1 a') \in \xi \tag{2.3}$$

Using (2.1)and(2.2) we get

$$(b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b', t\alpha_3 b\alpha_1 b') \in \xi \tag{2.4}$$

Since $a\alpha_1 a'\beta_1 t = a\alpha_1 a'\beta_1((a\alpha_1 b'\beta_2 b\alpha_2 x\alpha_1 a')\beta_1 t) = a\alpha_1 b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t = t$, we have $a'\beta_1 t\alpha_3 a \in E_{\alpha_1}$. Since $(b'\beta_2 b, a'\beta_1 a) \in \xi$, we have

$$\begin{array}{rcl}
b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b & \xi & a'\beta_1 a\alpha_1 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 a'\beta_1 a \\
& = & a'\beta_1 a\alpha_1 x\alpha_1 a'\beta_1 t\alpha_3 a \\
& = & a'\beta_1 a\alpha_1(x\alpha_1 a'\beta_1 a)\alpha_1 a'\beta_1 t\alpha_3 a \\
& \xi & a'\beta_1 a\alpha_1 x\alpha_1(b'\beta_2 b)\alpha_2 a'\beta_1 t\alpha_3 a \text{ (Since } \xi \text{ is an} \\
& = & \text{ip - congruence)} \\
& = & a'\beta_1 a\alpha_1 b'\beta_2 b\alpha_2 a'\beta_1 t\alpha_3 a \text{ (Since } x \in \\
& & RS(a'\beta_1 a, b'\beta_2 b)) \\
& \xi & a'\beta_1 a\alpha_1 a'\beta_1 a\alpha_1 a'\beta_1 t\alpha_3 a \\
& = & a'\beta_1 t\alpha_3 a.
\end{array}$$

Hence

$$(b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b, a'\beta_1 t\alpha_3 a) \in \xi \tag{2.5}$$

Next since $g \in RS(c'\beta_3 c, a\alpha_1 a')$, $a\alpha_1 a'\beta_1 g = g$ and hence we have $a'\beta_1 g\alpha_3 a \in E_{\alpha_1}$. Now since $x \in RS(a'\beta_1 a, b'\beta_2 b), a\alpha_1 b'\beta_2 b\alpha_2 x\alpha_1 a' = a\alpha_1 x\alpha_1 a' \in E_{\beta_1}$ and hence $t \in RS(g, a\alpha_1 x\alpha_1 a')$. Thus we have $g\alpha_3 t\alpha_3 a\alpha_1 x\alpha_1 a' = g\alpha_3 a\alpha_1 x\alpha_1 a'$. Now by (2.3) we have $((g\alpha_3 t)\alpha_3 a\alpha_1 x\alpha_1 a', (g\alpha_3 t)\alpha_3 a\alpha_1 a') \in \xi$ i.e, $(g\alpha_3 a\alpha_1 x\alpha_1 a', g\alpha_3 t\alpha_3 a\alpha_1 a') \in \xi$ since $t \in RS(ga\alpha_1 x\alpha_1 a')$ and again using (2.3)we have $g\alpha_3 a\alpha_1 a'\ \xi\ g\alpha_3 a\alpha_1 x\alpha_1 a'\ \xi$

$g\alpha_3 t\alpha_3 a\alpha_1 a'$ i.e, we get $(g\alpha_3 a\alpha_1 a', g\alpha_3 t\alpha_3 a\alpha_1 a') \in \xi$. Now since S is a right inverse Γ-semigroup $t\alpha_3 g\alpha_3 t = g\alpha_3 t$ and hence we have $g\alpha_3 t\alpha_3 a\alpha_1 a' = t\alpha_3 g\alpha_3 t\alpha_3 a\alpha_1 a' = t\alpha_3 a\alpha_1 a'$ since $t\alpha_3 g = t$. Thus $(g\alpha_3 a\alpha_1 a', t\alpha_3 a\alpha_1 a') \in \xi$ by transitivity of ξ. Now since ξ is normal, we have $(a'\beta_1(g\alpha_3 a\alpha_1 a')\beta_1 a\ ,\ a'\beta_1(t\alpha_3 a\alpha_1 a')\beta_1 a) \in \xi$. i.e,

$$(a'\beta_1 g\alpha_3 a, a'\beta_1 t\alpha_3 a) \in \xi \tag{2.6}$$

Again since S is a right inverse Γ-semigroup and the fact that $t \in RS(g, a\alpha_1 x\alpha_1 a')$ and $g \in RS(c'\beta_3 c, a\alpha_1 a')$ we see that

$$\begin{aligned} t\alpha_3 b\alpha_2 b' &= b\alpha_2 b'\beta_2 t\alpha_3 b\alpha_2 b' \text{ (Since } S \text{ is right inverse } \Gamma\text{-semigroup)} \\ &= b\alpha_2 b'\beta_2 (t\alpha_3 g)\alpha_3 (b\alpha_2 b') \\ &= b\alpha_2 b'\beta_2 (t\alpha_3 g\alpha_3 c'\beta_3 c)\alpha_3 b\alpha_2 b'. \end{aligned}$$

Now since $(a'\beta_1 a, b'\beta_2 b) \in \xi$ and $a\alpha_1 b' \in K$, proceeding the same way of Theorem 2.2 we have $(b\alpha_2 b', a\alpha_1 a'\beta_1 b\alpha_2 b') \in \xi$. Now

$$\begin{aligned} t\alpha_3 b\alpha_2 b' &= b\alpha_2 b'\beta_2 t\alpha_3 g\alpha_3 c'\beta_3 c\alpha_3 b\alpha_2 b' \\ &\ \xi\ b\alpha_2 b'\beta_2 t\alpha_3 g\alpha_3 c'\beta_3 c\alpha_3 (a\alpha_1 a'\beta_1 b\alpha_2 b') \text{ (Since } (b\alpha_2 b', a\alpha_1 a'\beta_1 b\alpha_2 b') \in \xi) \\ &= b\alpha_2 b'\beta_2 (g\alpha_3 t\alpha_3 g)\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (since } S \text{ is right inverse)} \\ &= b\alpha_2 b'\beta_2 g\alpha_3 t\alpha_3 (a\alpha_1 a'\beta_1 g)\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (Since } g \in RS(c'\beta_3 c, a\alpha_1 a')) \\ &\ \xi\ b\alpha_2 b'\beta_2 g\alpha_3 t\alpha_3 (a\alpha_1 x\alpha_1 a')\beta_1 g\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (by (2.3))} \\ &= b\alpha_2 b'\beta_2 (g\alpha_3 (a\alpha_1 x\alpha_1 a')\beta_1 g)\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (since } t \in RS(g, a\alpha_1 x\alpha_1 a')) \\ &\ \xi\ b\alpha_2 b'\beta_2 (g\alpha_3 (a\alpha_1 a')\beta_1 g)\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (By (2.3))} \\ &= b\alpha_2 b'\beta_2 g\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (Since } (a\alpha_1 a')\beta_1 g = g) \\ &= b\alpha_2 b'\beta_2 (c'\beta_3 c\alpha_3 g\alpha_3 c'\beta_3 c)\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (since } S \text{ is right inverse)} \\ &= b\alpha_2 b'\beta_2 c'\beta_3 c\alpha_3 g\alpha_3 (a\alpha_1 a'\beta_1 c'\beta_3 c\alpha_3 a\alpha_1 a')\beta_1 b\alpha_2 b' \text{ (Since } S \text{ is right inverse)} \\ &= b\alpha_2 b'\beta_2 (c'\beta_3 c\alpha_3 a\alpha_1 a')\beta_1 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{(since } g \in RS(c'\beta_3 c, a\alpha_1 a')) \\ &= b\alpha_2 b'\beta_2 a\alpha_1 a'\beta_1 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{(since } S \text{ is right inverse)} \\ &= b\alpha_2 b'\beta_2 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b \\ &= b\alpha_2 b'\beta_2 (c'\beta_3 c\alpha_3 a\alpha_1 a')\beta_1 b\alpha_2 b' \\ &= c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (Since } S \text{ is right inverse and hence right orthodox)} \\ &\ \xi\ c'\beta_3 c\alpha_3 b\alpha_2 b' \\ &= c'\beta_3 \alpha_3 h\alpha_3 b\alpha_2 b' \text{(since } h \in RS(c'\beta_3 c, b\alpha_2 b')) \\ &= h\alpha_3 c'\beta_3 c\alpha_3 h\alpha_3 b\alpha_2 b' \text{ (since } S \text{ is right inverse)} \\ &= h\alpha_3 b\alpha_2 b' \text{ (Since } h \in RS(c'\beta_3 c, b\alpha_2 b')) \end{aligned}$$

Hence we have

$$(t\alpha_3 b\alpha_2 b', h\alpha_3 b\alpha_2 b') \in \xi \tag{2.7}$$

Finally from (2.4) and (2.7) we have $(b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b', h\alpha_3 b\alpha_2 b') \in \xi$ and by normality of ξ we have $(b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b, b'\beta_2 h\alpha_3 b\alpha_2 b'\beta_2 b) \in \xi$ i.e, $(b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b, b'\beta_2 h\alpha_3 b) \in \xi$. It is to be noted that both the elements belong to E_{α_2}. Also by normality of ξ together with (2.5) and (2.6) we have $(a'\beta_1 g\alpha_3 a, b'\beta_2 h\alpha_3 b) \in \xi$. Hence the proof.

Theorem 2.5. If (ξ, K) is an ip - congruence pair for S,then $\rho_{(\xi,K)}$ is an ip - congruence with trace ξ and kernel K. Conversely if ρ is an ip - congruence on S then $(tr\rho, Ker\rho)$is an ip - congruence pair and $\rho = \rho_{(tr\rho,Ker\rho)}$.

Proof. Let (ξ, K) be an ip - congruence pair for S and $\rho_{(\xi,K)}$ and let $\rho = \rho_{(\xi,K)}$. Since $E(S) \subseteq K$ and ξ is reflexive, ρ is also reflexive. Again from Theorem 2.2 and Remark 2.1, we see that ρ is symmetric. We now show that ρ is transitive. For this let us suppose that $(a, b) \in \rho$ and $(b, c) \in \rho$ and let $a' \in V^{\beta_1}_{\alpha_1}(a)$, $b' \in V^{\beta_2}_{\alpha_2}(b), c' \in V^{\beta_3}_{\alpha_3}(c)$. Then we have $(a'\beta_1 a, b'\beta_2 b) \in \xi, (b'\beta_2 b, c'\beta_3 c) \in \xi, a\alpha_1 b' \in K, b\alpha_2 c' \in K$. Since ξ is transitive we have $(a'\beta_1 a, c'\beta_3 c) \in \xi$. We now show that $a\alpha_1 c' \in K$. Now by Theorem 2.2, $b\alpha_2 a' \in K$ and $c\alpha_3 b' \in K$. Hence $c\alpha_3 b'\beta_2 b\alpha_2 a' \in K$, Since K is a Γ-subsemigroup. Let $g \in RS(c'\beta_3 c, b'\beta_2 b)$ and $h \in RS(c'\beta_3 c, a'\beta_1 a)$. By Theorem 2.1 and since $g = g\alpha_3 c'\beta_3 c \in E_{\alpha_3}$, we have,

$$(c\alpha_3 b'\beta_2 b)\alpha_2 (g\alpha_3 c'\beta_3 c)\alpha_3 a' \in K \tag{2.8}$$

Again since $b\alpha_2 g\alpha_3 c' \in V^{\beta_3}_{\beta_2}(c\alpha_3 b'), c\alpha_3 b'\beta_2 b\alpha_2 g\alpha_3 c' \in E_{\beta_3}$. Now $c'\beta_3 c = c'\beta_3 c\alpha_3 c'\beta_3 c \ \xi \ c'\beta_3 c\alpha_3 b'\beta_2 b = c'\beta_3 c\alpha_3 g\alpha_3 b'\beta_2 b \ \xi \ c'\beta_3 c\alpha_3 g\alpha_3 c'\beta_3 c = c'\beta_3 c\alpha_3 g$, since $(b'\beta_2 b, c'\beta_3 c) \in \xi$ and $g \in RS(c'\beta_3 c, b'\beta_2 b)$. Also since $c\alpha_3 g\alpha_3 c' \in E_{\beta_3}$ and ξ is normal, it follows that $(c\alpha_3 (c'\beta_3 c)\alpha_3 c, c\alpha_3 (c'\beta_3 c\alpha_3 g)\alpha_3 c') \in \xi$ i.e,$(c\alpha_3 c', c\alpha_3 g\alpha_3 c') \in \xi$. Similarly since $(c'\beta_3 c, a'\beta_1 a) \in \xi$ and $c\alpha_3 h\alpha_3 c' \in E_{\beta_3}$ we have $(c\alpha_3 c, c\alpha_3 h\alpha_3 c') \in \xi$. By transitivity of ξ , $(c\alpha_3 g\alpha_3 c', c\alpha_3 h\alpha_3 c') \in \xi$. Again $c\alpha_3 (b'\beta_2 b\alpha_2 g)\alpha_3 c' = c\alpha_3 g\alpha_3 c' \ \xi \ c\alpha_3 h\alpha_3 c' = c\alpha_3 (a'\beta_1 a\alpha_1 h)\alpha_3 c'$. i.e,
$(c\alpha_3 b'\beta_2 b\alpha_2 g\alpha_3 c', c\alpha_3 a'\beta_1 a\alpha_1 h\alpha_3 c') \in \xi$. Again since $b\alpha_2 g\alpha_3 c' \in V^{\beta_3}_{\beta_2}(c\alpha_3 b')$, $c\alpha_3 b'\beta_2 b\alpha_2 g\alpha_3 c' \in E_{\beta_3}$ and since $a\alpha_1 h\alpha_3 c' \in V^{\beta_3}_{\beta_1}(c\alpha_3 a')$, from (2.8) and Definition 2.10 we can say that $c\alpha_3 a' \in K$ and by Theorem 2.2 we have $a\alpha_1 c' \in K$. Hence ρ is transitive. Hence ρ is an equivalence relation.
We now prove that ρ is an ip - congruence. Let us suppose that $(a, b) \in \rho$. Then for all $a' \in V^{\beta_1}_{\alpha_1}(a), b' \in V^{\beta_2}_{\alpha_2}(b), (a'\beta_1 a, b'\beta_2 b) \in \xi$ and $a\alpha_1 b' \in K$. Let $c \in S$ and $c' \in V^{\beta_3}_{\alpha_3}(c)$. We now prove that $(c\alpha_3 a, c\alpha_3 b) \in \rho$. Let $g \in RS(c'\beta_3 c, a\alpha_1 a')$ and $h \in RS(c'\beta_3 c, b\alpha_2 b')$. Then $a'\beta_1 g\alpha_3 c' \in V^{\beta_3}_{\alpha_1}(c\alpha_3 a)$ and $b'\beta_2 h\alpha_3 c' \in V^{\beta_3}_{\alpha_2}(c\alpha_3 b)$ and by Theorem 2.4 we have $a'\beta_1 g\alpha_3 c'\beta_3 c\alpha_3 a = a'\beta_1 g\alpha_3 a \ \xi \ b'\beta_2 h\alpha_3 b = b'\beta_2 h\alpha_3 c'\beta_3 c\alpha_3 b$. Also $(c\alpha_3 a)\alpha_1 (b'\beta_2 h\alpha_3 c') = c\alpha_3 (a\alpha_1 b')\beta_2 h\alpha_3 c' \in K$ since $a\alpha_1 b' \in K$ and $h \in E_{\alpha_3}$ and K is self conjugate. Hence by definition of ρ we have $(c\alpha_3 a, c\alpha_3 b) \in \rho$. Next we prove that $(a\alpha_1 c, b\beta_1 c) \in \rho$. For this let $g \in RS(a'\beta_1 a, c\alpha_3 c')$ and $h \in RS(b'\beta_2 b, c\alpha_3 c')$. Then $c'\beta_3 g\alpha_1 a' \in V^{\beta_1}_{\alpha_3}(a\alpha_1 c)$ and $c'\beta_3 h\alpha_2 b' \in V^{\beta_2}_{\alpha_3}(b\alpha_2 c)$. Now

$$
\begin{aligned}
g\alpha_1 c\alpha_3 c' &= g\alpha_1 a'\beta_1 a\alpha_1 c\alpha_3 c' \text{ (Since } g \in RS(a'\beta_1 a, c\alpha_3 c'))\\
&\ \xi\ \ g\alpha_1 b'\beta_2 b\alpha_2 c\alpha_3 c'\\
&= g\alpha_1 b'\beta_2 b\alpha_2 h\alpha_2 c\alpha_3 c' \text{ (Since } h \in RS(b'\beta_2 b, c\alpha_3 c'))\\
&\ \xi\ \ g\alpha_1 (a'\beta_1 a)\alpha_1 h\alpha_2 c\alpha_3 c' \text{ (Since } \xi \text{ is an ip - congruence and } (a'\beta_1 a, b'\beta_2 b) \in \xi)\\
&= (a'\beta_1 a\alpha_1 g\alpha_1 a'\beta_1 a)\alpha_1 h\alpha_2 c\alpha_3 c' \text{ (Since } S \text{ is right inverse)}\\
&= a'\beta_1 a\alpha_1 g\alpha_1 a'\beta_1 a\alpha_1 (c\alpha_3 c'\beta_3 h)\alpha_2 c\alpha_3 c' \text{(Since } h \in RS(b'\beta_2 b, c\alpha_3 c'))\\
&= a'\beta_1 a\alpha_1 g\alpha_1 (a'\beta_1 a\alpha_1 c\alpha_3 c')\beta_3 h\alpha_2 c\alpha_3 c'\\
&= a'\beta_1 a\alpha_1 g\alpha_1 (c\alpha_3 c'\beta_3 a'\beta_1 a\alpha_1 c\alpha_3 c')\beta_3 h\alpha_2 c\alpha_3 c' \text{(Since } S \text{ is right inverse)}\\
&= a'\beta_1 a\alpha_1 g\alpha_1 c\alpha_3 c'\beta_3 a'\beta_1 a\alpha_1 h\alpha_2 c\alpha_3 c' \text{(Since } h \in RS(b'\beta_2 b, c\alpha_3 c'))\\
&= (a'\beta_1 a\alpha_1 c\alpha_3 c'\beta_3 a'\beta_1 a)\alpha_1 h\alpha_2 c\alpha_3 c' \text{(Since } g \in RS(a'\beta_1 a, c\alpha_3 c'))\\
&= c\alpha_3 c'\beta_3 (a'\beta_1 a\alpha_1 h)\alpha_2 c\alpha_3 c' \text{(Since } S \text{ is right inverse)}\\
&= a'\beta_1 a\alpha_1 h\alpha_2 c\alpha_3 c' \text{(Since } S \text{ is right inverse and hence right orthodox)}\\
&\ \xi\ \ b'\beta_2 b\alpha_2 h\alpha_2 c\alpha_3 c'\\
&= b'\beta_2 b\alpha_2 h\alpha_2 b'\beta_2 b\alpha_2 c\alpha_3 c' \text{(Since } h \in RS(b'\beta_2 b, c\alpha_3 c'))\\
&\ \xi\ \ h\alpha_2 b'\beta_2 b\alpha_2 c\alpha_3 c' \text{(Since } S \text{ is right inverse)}\\
&= h\alpha_2 c\alpha_3 c'.
\end{aligned}
$$

Hence

$$(g\alpha_1 c\alpha_3 c', h\alpha_2 c\alpha_3 c') \in \xi \tag{2.9}$$

Now since $g \in RS(a'\beta_1 a, c\alpha_3 c')$ and $h \in RS(b'\beta_2 b, c\alpha_3 c'), c'\beta_3 h\alpha_2 c \in E_{\alpha_3}$ and $c'\beta_3 g\alpha_1 c \in E_{\alpha_3}$. Again by normality of ξ and by (2.9) we have $(c'\beta_3(g\alpha_1 c\alpha_3 c')\beta_3 c,$ $c'\beta_3(h\alpha_2 c\alpha_3 c')\beta_3 c) \in \xi$. i.e, $(c'\beta_3 g\alpha_1 c, c'\beta_3 h\alpha_3 c) \in \xi$. Thus $(c'\beta_3 g\alpha_1 a')\beta_1(a\alpha_1 c)\ \xi$ $(c'\beta_3 h\alpha_2 b')\beta_2(b\alpha_2 c)$. Finally $(a\alpha_1 c)\alpha_3(c'\beta_3 h\alpha_2 b') = a\alpha_1(c\alpha_3 c'\beta_3 h)\alpha_2 b' \in K$ since $a\alpha_1 b' \in K$. Hence $(a\alpha_1 c, b\alpha_2 c) \in \rho$ by definition of ρ.

Let us now show that $tr\rho = \xi$. Let us suppose that e be an α-idempotent and f be a β-idempotent are such that $(e, f) \in \rho$. Then by definition of ρ we have $(e, f) \in \xi$, since $e \in V_\alpha^\alpha(e)$ and $f \in V_\beta^\beta(f)$. Hence $tr\rho \subseteq \xi$. Conversely let $e \in E_\alpha$ and $f \in E_\beta$ and $(e, f) \in \xi$. We now show that $(e, f) \in \rho$. Since S is right inverse Γ-semigroup, $e\alpha f \in E_\beta \subseteq K$. Again considering $e \in V_\alpha^\alpha(e)$ and $f \in V_\beta^\beta(f)$ we can say that $(e, f) \in \rho$. Hence $\xi = tr\rho$.

Let us now show that $K = ker\rho$. For that let $a \in Ker\rho$. Then there exists an α-idempotent $e \in S$ such that $(a, e) \in \rho$ and hence $(a'\delta a, e) \in \xi$ for all $a' \in V_\gamma^\delta(a)$ and $a\gamma e \in K$. Then by Theorem 2.2 and Remark 2.1 $e\alpha a' \in K$ and so by definition of (ξ, K) we have $a' \in K$ and hence from regularity of $K, a \in K$.

Conversely suppose that $a \in K$. Let $a' \in V_\alpha^\beta(a)$ then $(a'\beta a, a'\beta a\alpha a'\beta a) \in \xi$ and $a\alpha a'\beta a \in K$ i.e, $(a, a'\beta a) \in \rho$ by definition of ρ. Thus $a \in Ker\rho$. Hence $K = Ker\rho$.

We now prove the converse part of the Theorem. Let us suppose that ρ is a ip - congruence on S. We show that $(tr\rho, Ker\rho)$ is an ip - congruence pair and $\rho = \rho_{(tr\rho, Ker\rho)}$. Let $a, b \in ker\rho$ and let $V_\alpha^\beta(a) \neq \phi$. Hence $a\rho = e\rho$ and $b\rho = f\rho$ for some γ-idempotent e and δ-idempotent f. Now $a\rho e$ implies $a\alpha b\ \rho\ e\gamma b\ \rho\ e\gamma f$. Since S is a right inverse Γ-semigroup $e\gamma f \in E_\delta$ and hence $a\alpha b \in Ker\rho$. Thus $Ker\rho$ is a partial Γ-subsemigroup of S. Clearly $Ker\rho$ contains $E(S)$. Let $a \in Ker\rho$ and $a' \in V_\alpha^\beta(a)$. We show that $a' \in Ker\rho$. Since $a \in Ker\rho$, $a\rho = e\rho$ for some $e \in E_\gamma$.

Now $a' = a'\beta a\alpha a' \ \rho \ a'\beta e\gamma a' = a'\beta e\gamma e\gamma a' \ \rho \ a'\beta a\alpha e\gamma a' \ \rho \ a'\beta a\alpha a\alpha a'$. Since $(a'\beta a)\alpha$ $(a\alpha a') \in E_\beta, a' \in Ker\rho$. Thus $Ker\rho$ is regular. Next let $a \in S$ and $a' \in V_\alpha^\beta(a)$ and $k \in Ker\rho$ where $V_\gamma^\delta(k) \neq \phi$. Since $k \in Ker\rho, k\rho = e\rho$ for some μ-idempotent e. Now since S is a right inverse Γ-semigroup, $(a'\beta e\mu a)\alpha(a'\beta e\mu a) = a'\beta(e\mu a\alpha a'\beta e)\mu a = a'\beta(a\alpha a'\beta e)\mu a = a'\beta e\mu a$ i.e,$a'\beta e\mu a \in E_\alpha$.
Now $a'\beta k\gamma a \ \rho \ a'\beta e\mu a$ and hence $a'\beta k\gamma a \in Ker\rho$ i.e, $Ker\rho$ is self conjugate. Thus $Ker\rho$ is a normal partial Γ-subsemigroup of S. We now prove that $(tr\rho, Ker\rho)$ is an ip - congruence pair for S. Since ρ is a ip - congruence and for $a' \in V_\alpha^\beta(a)$ and $e \in E_\gamma, a'\beta e\gamma a \in E_\alpha, tr\rho$ is a normal ip - congruence. Now let $a \in S$ and $a' \in V_\alpha^\beta(a)$ and $e \in E_\gamma$ be such that $e\gamma a \in ker\rho$ and $(e, a\alpha a') \in tr\rho$. Now $a \ \rho$ $(a\alpha a')\beta a \ \rho \ e\gamma a \ \rho \ f$ for some $f \in E(S)$ since $e\gamma a \in Ker\rho$. Hence condition (i) of Definition 2.10 is satisfied. Next let $a \in Ker\rho$ and $e \in E_\gamma$ and let $a' \in V_\alpha^\beta(a)$. Now since $a \in Ker\rho, a\rho = f\rho$ for some δ-idempotent f and $a'\rho = g\rho$ for some μ-idempotent g.
Now $a\alpha e\gamma a' = a\alpha e\gamma a'\beta a\alpha a' \ \rho \ f\delta e\gamma g\mu f\delta g \ \rho \ f\delta e\gamma f\delta g \ \rho \ e\gamma f\delta g \ \rho \ e\gamma a\alpha a'$. Now since $a\alpha e\gamma a', e\gamma a\alpha a' \in E_\beta$,we have $(a\alpha e\gamma a', e\gamma a\alpha a') \in tr\rho$. Thus condition (ii) of definition 2.10 is also satisfied. Finally we show that $\rho = \rho_{(tr\rho, Ker\rho)}$ i.e, we prove $(a, b) \in \rho$ if and only if for all $a' \in V_{\alpha_1}^{\beta_1}(a)$ and for all $b' \in V_{\alpha_2}^{\beta_2}(b), a\alpha_1 b' \in Ker\rho$ and $(a'\beta_1 a, b'\beta_2 b) \in tr\rho$. Suppose $(a, b) \in \rho$ and $a' \in V_{\alpha_1}^{\beta_1}(a), b' \in V_{\alpha_2}^{\beta_2}(b)$. Now $a\alpha_1 b' \ \rho \ b\alpha_2 b'$ since ρ is an ip - congruence. Again since $b\alpha_2 b'$ is a β_2-idempotent we can say that $a\alpha_1 b' \in Ker\rho$. Now $a'\beta_1 a \ \rho \ a'\beta_1 b = a'\beta_1 b\alpha_2 b'\beta_2 b \ \rho \ a'\beta_1 a\alpha_1 b'\beta_2 b \ \rho$ $(a'\beta_1 a)\alpha_1(b'\beta_2 a) = (a'\beta_1 a)\alpha_1 b'\beta_2 a\alpha_1 a'\beta_1 a \ \rho \ (a'\beta_1 a)\alpha_1(b'\beta_2 b)\alpha_2(a'\beta_1 a) = (b'\beta_2 b)\alpha_2$ $(a'\beta_1 a) = b'\beta_2 b\alpha_2(a'\beta_1 a) \ \rho \ b'\beta_2(a\alpha_1 a'\beta_1 a) = b'\beta_2 a \ \rho \ b'\beta_2 b$. Now since $a'\beta_1 a$ and $b'\beta_2 b$ are α_1-idempotent and α_2-idempotent respectively, we have $(a'\beta_1 a, b'\beta_2 b) \in tr\rho$. Hence $\rho \subseteq \rho_{(tr\rho, Ker\rho)}$.
Conversely let $(a, b) \in S$ such that for all $a' \in V_{\alpha_1}^{\beta_1}(a), b' \in V_{\alpha_2}^{\beta_2}(b), (a'\beta_1 a, b'\beta_2 b) \in tr\rho$ and $a\alpha_1 b' \in Ker\rho$.
Now

$$\begin{aligned}(a\alpha_1 b')\beta_2(b\alpha_2 a')\beta_1(a\alpha_1 b') &= a\alpha_1(b'\beta_2 b)\alpha_2(a'\beta_1 a)\alpha_1(b'\beta_2 b)\alpha_2 b' \\ &= a\alpha_1(a'\beta_1 a)\alpha_1(b'\beta_2 b)\alpha_2 b' \\ &= a\alpha_1 b'\end{aligned}$$

and

$$\begin{aligned}(b\alpha_2 a')\beta_1(a\alpha_1 b')\beta_2(b\alpha_2 a') &= b\alpha_2(a'\beta_1 a)\alpha_1(b'\beta_2 b)\alpha_2(a'\beta_1 a)\alpha_1 a' \\ &= b\alpha_2(b'\beta_2 b)\alpha_2(a'\beta_1 a)\alpha_1 a' \\ &= b\alpha_2 a'\end{aligned}$$

Hence $a\alpha_1 b' \in V_{\beta_1}^{\beta_2}(b\alpha_2 a')$. Again since $a\alpha_1 b' \in Ker\rho, b\alpha_2 a' \in Ker\rho$ and let $(a\alpha_1 b') \ \rho \ e$ and $(b\alpha_2 a') \ \rho \ f$ for γ-idempotent e and δ-idempotent f. Now $a = a\alpha_1(a'\beta_1 a)\alpha_1(a'\beta_1 a) \ \rho \ a\alpha_1(b'\beta_2 b)\alpha_2(a'\beta_1 a) \ \rho \ (a\alpha_1 b')\beta_2(b\alpha_2 a')\beta_1 a \ \rho \ e\gamma f\delta a = f\delta e\gamma f$ $\delta a \ \rho \ (b\alpha_2 a')\beta_1(a\alpha_1 b')\beta_2(b\alpha_2 a')\beta_1 a = b\alpha_2(a'\beta_1 a)\alpha_1(b'\beta_2 b)\alpha_2(a'\beta_1 a) = b\alpha_2(b'\beta_2 b)\alpha_2$ $(a'\beta_1 a) \ \rho \ b\alpha_2(b'\beta_2 b)\alpha_2(b'\beta_2 b = b$. i.e, $(a, b) \in \rho$. Hence the proof.

References

[1] F. Pastijn and M. Petrich., *Congruences on regular semigroups*, Trans. Amer. Math. Soc., 295(1986), 607-633.
[2] G.M.S. Gomes., *R-unipotent congruences on regular semigroups*, emigroup Forum, 31(1985), 265-280.
[3] J.M. Howie, *An introduction to semigroup Theory*, Clarendon Press, Oxford, 1995,

[4] K.S.S. Nambooripad, *Structure of regular semigroups I*, Mem. Amer. Math. Soc. 22 (1979), no.224.
[5] M.K. Sen, M.K. and N.K. Saha., *On* Γ*-semigroup I* , Bull. Cal. Math. Soc., 78(1986), 180-186.
[6] N.K. Saha., *On* Γ*-semigroup II*, Bull. Cal. Math. Soc, 79(1987), 331-335.
[7] N.K. Saha., *On* Γ*-semigroup III*, Bull. Cal. Math. Soc., 80(1988), 1-12.
[8] S. Chattopadhyay., *Right inverse* Γ*-semigroup*, Bull. Cal. Math. Soc., 93(6),(2001), 435-442.
[9] S. Chattopadhyay., *Right orthodox* Γ*-semigroup*, Southeast Asian Bull. of Mathematics,(2005)29, 1-18.
[10] S. Chattopadhyay., *Sandwich sets on regular* Γ*-semigroup*, Communicated.

Sovarani Memorial College, Jagatballavpur, Howrah -711408, West Bengal, INDIA
E-mail address: chatterjees04@yahoo.co.in

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 165–175 (2015) ©KJM

ON $\mathcal{I}_2$-ASYMPTOTICALLY λ^2-STATISTICAL EQUIVALENT DOUBLE SEQUENCES

ÖMER KİŞİ

Abstract. In this paper, we introduce the concept of $\mathcal{I}_2-$asymptotically λ^2-statistically equivalence of multiple L for the double sequences (x_{kl}) and (y_{kl}). Also we give some inclusion relations.

1. Introduction

Pobyvanets [14] introduced the concept of asymptotically regular matrices which preserve the asymptotic equivalence of two nonnegative numbers sequences. In 1993, Marouf [9] presented definitions for asymptotically equivalent and asymptotic regular matrices. In 2003, Patterson extended these concepts by prensenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Later these definitions extended to λ-sequences by Savas and Başarır in [18]. Esi and Acıkgöz [1] extended the definitions prensented in [18] to double λ^2-sequence.

2. Preliminaries and Background

In this section, we recall some definitions and notations, which form the base for the present study.

The notion of statistical convergence depends on the density (asymptotic or natural) of subsets of natural numbers $\mathbb{N}$. A subsets of natural numbers $\mathbb{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in E\}| \text{ exists.}$$

Definition 2.1. [4] A real or complex number sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

1991 *Mathematics Subject Classification.* 40A05, 40A35.
Key words and phrases. Double Statistical convergence, $\mathcal{I}_2-$convergence , λ^2-convergence, asymptotically equivalence, double sequences.

In this case, we write $S-\lim x=L$ or $x_k \to L\,(S)$, and S denotes the statistically convergent sequences.

Definition 2.2. [7] A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$,
(ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,
(iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

Definition 2.3. [7] A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if

(i) $\emptyset \notin \mathcal{F}$,
(ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,
(iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

Lemma 2.1. [7] *If $\mathcal{I}$ is proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets*

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$ and it is called the filter associated with the ideal.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 2.4. [7] A sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be $\mathcal{I}-$convergent to a real number L if

$$\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I},$$

for every $\varepsilon > 0$. In this case we write $\mathcal{I}$-$\lim x = L$.

Definition 2.5. [10] Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinity such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. A sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent or $S_\lambda-$convergent to L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |k \in I_n : |x_k - L| \geq \varepsilon| = 0$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$

In 1900 Pringsheim presented the following definition for the convergence of double sequences.

Definition 2.6. [15] A double sequence $x = (x_{kl})$ has a Pringsheim limit L (denoted by $P-\lim x = L$) provided that for given $\varepsilon > 0$, there exists a $n \in \mathbb{N}$ such that $|x_{kl} - L| < \varepsilon$, whenever $k, l > n$. We describe such an $x = (x_{kl})$ more briefly as "P-convergent".

The double sequence $(x_{k,l})$ is bounded if there exists a positive integer M such that $|x_{kl}| < M$ for all k and l. We denote all bounded double sequence by l_∞^2.

Definition 2.7. [11] A real double sequence $x = (x_{kl})$ is to be statistically convergent to L provided that for every $\varepsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{mn} |\{(k,l) : k \leq m \text{ and } l \leq n : |x_{k,l} - L| \geq \varepsilon\}| = 0,$$

denoted by $S^L - \lim x = L$.

Now we give a brief history for asymptotical equivalence for single and double sequences.

Definition 2.8. [15] Two non-negative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be $P-$asymptotically double equivalent of multiple L provided that for every $\varepsilon > 0$,

$$P - \lim_{k,l} \frac{x_{kl}}{y_{kl}} = L,$$

denoted by $(x_{kl}) \sim^P (y_{kl})$ and simply asymptotically double equivalent if $L = 1$.

Definition 2.9. [1] Two non-negative double sequences (x_{kl}) and (y_{kl}) are said to be asymptotically double statistical equivalent of multiple L provided that for every $\varepsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \left| \left\{ k \le m, l \le n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \ge \varepsilon \right\} \right| = 0,$$

denoted by $(x_{kl}) \sim^{S^L} (y_{kl})$ and simply asymptotically double statistical equivalent if $L = 1$.

Definition 2.10. [6] Two non-negative double sequences (x_{kl}) and (y_{kl}) are said to be asymptotically $\mathcal{I}-$equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \ge \varepsilon \right\} \in \mathcal{I},$$

denoted by $(x_{kl}) \sim^{\mathcal{I}^L} (y_{kl})$ and simply asymptotically $\mathcal{I}-$equivalent if $L = 1$.

Definition 2.11. [6] Two non-negative double sequences (x_{kl}) and (y_{kl}) are said to be asymptotically $\mathcal{I}$-statistically equivalent of multiple L provided that for every $\varepsilon > 0$, and for every $\delta > 0$,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ k \le m,\ l \le n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \ge \varepsilon \right\} \right| \ge \delta \right\} \in \mathcal{I},$$

denoted by $(x_{kl}) \overset{S^L(\mathcal{I})}{\sim} (y_{kl})$ and simply asymptotically $\mathcal{I}-$statistical equivalent if $L = 1$.

Definition 2.12. [5] Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinity such that $\lambda_1 = 1$ and $\lambda_{n+1} \le \lambda_n + 1$. Two non-negative sequences (x_k) and (y_k) are $S_\lambda-$asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \right| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$

Definition 2.13. [5] Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinity such that $\lambda_1 = 1$ and $\lambda_{n+1} \le \lambda_n + 1$. Two non-negative sequences (x_k) and (y_k) are strong $\lambda-$asymptotically equivalent of multiple L provided that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} \left| \frac{x_k}{y_k} - L \right| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$

The double sequence (λ_{mn}) of positive real numbers tending to infinity such that

$$\lambda_{m+1,n} \leq \lambda_{mn} + 1, \lambda_{m,n+1} \leq \lambda_{mn} + 1,$$

$$\lambda_{mn} - \lambda_{m+1,n} \leq \lambda_{m,n+1} - \lambda_{m+1,n+1}, \lambda_{1,1} = 1$$

and

$$I_{mn} = \{(k,l) : m - \lambda_{mn} + 1 \leq k \leq m,\ n - \lambda_{mn} + 1 \leq l \leq n\}.$$

Definition 2.14. [1] For double λ^2-sequence; two non-negative double sequences (x_{kl}) and $(y_{k,l})$ are said to be λ^2-asymptotically double statistical equivalent of multiple L if for every $\varepsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \left| k \in I_n,\ l \in I_m : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right| = 0,$$

$\left(\text{denoted by } (x_{kl}) \overset{S^L_{\lambda^2}}{\sim} (y_{kl}) \right).$

Definition 2.15. [1] For double λ^2-sequence; two non-negative double sequences (x_{kl}) and $(y_{k,l})$ are said to be strong λ^2-asymptotically double equivalent of multiple L provided that

$$P - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{(k,l)\in I_{mn}} \left| \frac{x_{kl}}{y_{kl}} - L \right| = 0,$$

$\left(\text{denoted by } (x_{kl}) \overset{N^L_{\lambda^2}}{\sim} (y_{kl}) \right).$

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$. A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times i$ belongs to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

3. Main results

In this section we define $\mathcal{I}_2-$asymptotically λ^2-statistically equivalent, strongly $\lambda^2_{\mathcal{I}_2}-$asymptotically equivalent, strongly Cesaro asymptotically $\mathcal{I}_2-$equivalent of double sequences and obtain some analogous results from these new definitons point of views.

Definition 3.1. For double $\lambda^2 = (\lambda_{mn})$-sequence; two nonnegative sequences (x_{kl}) and (y_{kl}) are said to be $\mathcal{I}_2-$asymptotically λ^2-statistically equivalent of multiple L if for every $\varepsilon, \delta > 0$,

$$\left\{ (m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}} \left| \left\{ (k,l) \in I_{mn} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}_2,$$

denoted by $(x_{kl}) \overset{S^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{k,l})$.

Definition 3.2. For double $\lambda^2 = (\lambda_{mn})$-sequence; two non-negative double sequences (x_{kl}) and (y_{kl}) are said to be strongly $\lambda^2_{\mathcal{I}_2}-$asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\left\{ (m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}} \sum_{(k,l)\in I_{mn}} \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2,$$

$\left(\text{denoted by } x_{kl} \overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim} y_{kl}\right)$.

Definition 3.3. Two non-negative double sequences (x_{kl}) and (y_{kl}) are said to be strongly Cesaro asymptotically $\mathcal{I}_2-$equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}{}_{k,l=1,1}^{m,n} \left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon\right\} \in \mathcal{I}_2,$$

$\left(\text{denoted by } x_{kl} \overset{\mathcal{I}_2[C,1]^L}{\sim} y_{kl}\right)$.

Theorem 3.1. *Let $\lambda^2 = (\lambda_{mn})$ be a double sequence and $\mathcal{I}_2$ is strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. If $(x_{kl}) \overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$ then $(x_{kl}) \overset{S^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$.*

Proof. Assume that $(x_{k,l}) \overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$ and $\varepsilon > 0$. Then,

$$\begin{aligned} {}_{(k,l)\in I_{mn}} \left|\frac{x_{kl}}{y_{kl}} - L\right| &\geq {}_{\substack{(k,l)\in I_{mn} \\ \left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon}} \left|\frac{x_{kl}}{y_{kl}} - L\right| \\ &\geq \varepsilon . \left|\left\{(k,l) \in I_{m,n} : \left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon\right\}\right|, \end{aligned}$$

and so,

$$\frac{1}{\varepsilon.\lambda_{mn}}{}_{(k,l)\in I_{mn}} \left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \frac{1}{\lambda_{mn}} \left|\left\{(k,l) \in I_{mn} : \left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon\right\}\right|.$$

Then for any $\delta > 0$,

$$\begin{aligned} &\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}} \left|\left\{(k,l) \in I_{mn} : \left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\qquad \subseteq \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}}{}_{(k,l)\in I_{mn}} \left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon.\delta\right\}. \end{aligned}$$

Since right hand belongs to $\mathcal{I}_2$, then left hand also belongs to $\mathcal{I}_2$ and this completes the proof. $\square$

Theorem 3.2. *Let $\lambda^2 = (\lambda_{mn})$ be a double sequence and $\mathcal{I}_2$ is a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. If (x_{kl}) and (y_{kl}) are bounded sequences and $(x_{kl}) \overset{S^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$ then $(x_{kl}) \overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$.*

Proof. Let (x_{kl}) and (y_{kl}) are bounded sequences and let $(x_{kl}) \overset{S^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$. Then there is a M such that

$$\left|\frac{x_{kl}}{y_{kl}} - L\right| \leq M$$

for all $(k,l)\in\mathbb{N}\times\mathbb{N}$. For each $\varepsilon>0$,

$$\begin{aligned}\frac{1}{\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\frac{x_{kl}}{y_{kl}}-L\right| &= \frac{1}{\lambda_{mn}}\sum_{\substack{(k,l)\in I_{mn}\\ \left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon}}\left|\frac{x_{kl}}{y_{kl}}-L\right| \\ &\quad+\frac{1}{\lambda_{mn}}\sum_{\substack{(k,l)\in I_{mn}\\ \left|\frac{x_{kl}}{y_{kl}}-L\right|<\varepsilon}}\left|\frac{x_{kl}}{y_{kl}}-L\right| \\ &\leq M.\frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{mn}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\frac{\varepsilon}{2}\right\}\right|+\frac{\varepsilon}{2}.\end{aligned}$$

And define the sets

$$D_1=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}$$

and

$$D_2=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{mn}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\frac{\varepsilon}{2}\right\}\right|\geq\frac{\varepsilon}{2M}\right\}.$$

If $(m,n)\notin D_2$, then $\frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{mn}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\frac{\varepsilon}{2}\right\}\right|<\frac{\varepsilon}{2M}$. Also we can get

$$\begin{aligned}\frac{1}{\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\frac{x_{kl}}{y_{kl}}-L\right| &\leq \frac{M}{\lambda_{mn}}\left|\left\{(k,l)\in I_{mn}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\frac{\varepsilon}{2}\right\}\right|+\frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Thus $(m,n)\notin D_1$. Consequently, we have

$$\begin{aligned}&\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\} \\ &\qquad\subseteq\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{mn}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\frac{\varepsilon}{2}\right\}\right|\geq\frac{\varepsilon}{2M}\right\}\in\mathcal{I}_2.\end{aligned}$$

Therefore $(x_{kl})\overset{V_{\lambda^2}^L(\mathcal{I}_2)}{\sim}(y_{kl})$. □

The following example shows that if (x_{kl}), (y_{kl}) are not bounded, then theorem 2 can not be true.

Example 3.1. Let (x_{kl}) and (y_{kl}) be two double sequences as follows:

$$(x_{kl})=\begin{cases} kl, & \text{if } k_{m-1}<k\leq k_{m-1}+\left[\sqrt{\lambda_m}\right],\ l_{n-1}<l\leq l_{n-1}+\left[\sqrt{\lambda_n}\right],\ m,n=1,2,3,...;\\ 0, & \text{otherwise.}\end{cases}$$

and $(y_{k,l})=1$ for all $k,l\in\mathbb{N}$.

It is clear that $(x_{kl})\notin l_\infty^2$ and for $\varepsilon>0$,

$$\frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{mn}:\left|\frac{x_{kl}}{y_{kl}}-1\right|\geq\varepsilon\right\}\right|\leq\frac{\left[\sqrt{\lambda_{mn}}\right]}{\lambda_{mn}}\to 0 \text{ as } m,n\to\infty, \tag{1.1}$$

This implies that

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{mn}:\left|\frac{x_{kl}}{y_{kl}}-1\right|\geq\varepsilon\right\}\right|\geq\delta\right\}$$

$$\subseteq\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{\left[\sqrt{\lambda_{mn}}\right]}{\lambda_{mn}}\geq\delta\right\}.$$

By virtue of last part (1.1), the set on the right side is a finite set, and so it belongs to $\mathcal{I}_2$. Consequently, we have

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{m,n}}\left|\left\{(k,l)\in I_{mn}:\left|\frac{x_{kl}}{y_{kl}}-1\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}_2.$$

Therefore, $(x_{kl})\overset{S^L_{\lambda^2}(\mathcal{I}_2)}{\sim}(y_{kl})$, On the other hand, we shall show that $(x_{kl})\overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim}(y_{kl})$ is not satisfied. Suppose that $(x_{kl})\overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim}(y_{kl})$. Then for every $\delta>0$, we have

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\frac{x_{kl}}{y_{kl}}-1\right|\geq\delta\right\}\in\mathcal{I}_2. \tag{1.2}$$

Now,

$$\lim_{m,n\to\infty}\frac{1}{\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\frac{x_{kl}}{y_{kl}}-1\right|=\lim_{m,n\to\infty}\frac{1}{\lambda_{mn}}\left(\frac{\left[\sqrt{\lambda_{mn}}\right].\left(\left[\sqrt{\lambda_{mn}}\right]-1\right)}{2}\right)=\frac{1}{2}.$$

It follows for the particular choice $\delta=\frac{1}{4}$ that

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\frac{x_{kl}}{y_{kl}}-1\right|\geq\frac{1}{4}\right\}$$

$$=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\left(\frac{\left[\sqrt{\lambda_{mn}}\right].\left(\left[\sqrt{\lambda_{mn}}\right]-1\right)}{\lambda_{m,n}}\right)\geq\frac{1}{4}\right\}$$

$$=\{(r,s)\,,\ (r+1,s+1)\,,\ (r+2,s+2)\,,....\}$$

for some $r,s\in\mathbb{N}$ which belongs to $\mathcal{F}(\mathcal{I}_2)$ as $\mathcal{I}_2$ is admissible. This contradicts (1.2) for the choice $\delta=\frac{1}{4}$. Therefore $(x_{kl})\,{V^L_{\lambda^2}(\mathcal{I}_2)}\,(y_{kl})$.

Theorem 3.3. *Let $\lambda^2=(\lambda_{mn})$ be a double sequence and $\mathcal{I}_2$ is a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. If $(x_{kl})\overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim}(y_{kl})$ is then $(x_{kl})\overset{\mathcal{I}_2[C,1]^L}{\sim}(y_{kl})$.*

Proof. Assume that $(x_{kl})\overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim}(y_{kl})$ and $\varepsilon>0$. Then,

$$\begin{aligned}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{kl}}{y_{kl}}-L\right| &= \frac{1}{mn}\sum_{k,l=1,1}^{m-\lambda_m,\ n-\lambda_n}\left|\frac{x_{kl}}{y_{kl}}-L\right|+\frac{1}{mn}\sum_{(k,l)\in I_{m,n}}\left|\frac{x_{kl}}{y_{kl}}-L\right| \\ &\leq \frac{1}{\lambda_{mn}}\sum_{k,l=1,1}^{m-\lambda_m,\ n-\lambda_n}\left|\frac{x_{kl}}{y_{kl}}-L\right|+\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}}\left|\frac{x_{kl}}{y_{kl}}-L\right| \\ &\leq \frac{2}{\lambda_{mn}}\sum_{(k,l)\in I_{m,n}}\left|\frac{x_{kl}}{y_{kl}}-L\right|\end{aligned}$$

and so,

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\subseteq\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:_{(k,l)\in I_{m,n}}\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\frac{\varepsilon}{2}\right\}\in\mathcal{I}_2.$$

Hence $(x_{kl})\overset{\mathcal{I}_2[C,1]^L}{\sim}(y_{kl})$. □

Theorem 3.4. *If* $\liminf\frac{\lambda_{mn}}{mn}>0$ *then* $(x_{kl})\overset{S^L(\mathcal{I}_2)}{\sim}(y_{kl})$ *implies* $(x_{kl})\overset{S^L_{\lambda^2}(\mathcal{I}_2)}{\sim}(y_{kl})$.

Proof. Assume that $\liminf\frac{\lambda_{mn}}{mn}>0$. Then, there exists a $\delta>0$ such that $\frac{\lambda_{mn}}{mn}\geq\delta$ for sufficiently large m, n. For given $\varepsilon>0$ we have,

$$\frac{1}{mn}\left\{0\leq k\leq m;0\leq l\leq n,\ (m,n)\in\mathbb{N}\times\mathbb{N}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}$$
$$\supseteq\frac{1}{mn}\left\{(k,l)\in I_{m,n},\ (m,n)\in\mathbb{N}\times\mathbb{N}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}.$$

Therefore,

$$\frac{1}{mn}\left|\left\{0\leq k\leq m;0\leq l\leq n:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right|$$
$$\geq\frac{1}{mn}\left|\left\{(k,l)\in I_{m,n}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right|$$
$$\geq\frac{\lambda_{mn}}{mn}.\frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{m,n}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right|$$
$$\geq\delta.\frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{m,n}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right|,$$

then for any $\eta>0$ we get

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{m,n}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right|\geq\eta\right\}$$
$$\subseteq\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}\left|\left\{0\leq k\leq m;0\leq l\leq n:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right|\geq\eta\delta\right\}\in\mathcal{I}_2,$$

and this completes the proof. □

Theorem 3.5. *Let* $\lambda^2=(\lambda_{mn})$ *be a double sequence and* $\mathcal{I}_2$ *is a strongly admissible ideal in* $\mathbb{N}\times\mathbb{N}$*, and* (x_{kl}) *and* (y_{kl}) *are two non-negative double sequences. Then*

(i) *If* $(x_{kl})\overset{\mathcal{I}_2[C,1]^L}{\sim}(y_{kl})$ *then* $(x_{kl})\overset{S^L(\mathcal{I}_2)}{\sim}(y_{kl})$,

(ii) *Let* $(x_{kl}),\ (y_{kl})\in l^2_\infty$ *and* $(x_{kl})\overset{S^L(\mathcal{I}_2)}{\sim}(y_{kl})$, *then* $(x_{kl})\overset{\mathcal{I}_2[C,1]^L}{\sim}(y_{kl})$.

Proof. (i) Let $\varepsilon > 0$ and $(x_{kl}) \overset{\mathcal{I}_2[C,1]^L}{\sim} (y_{kl})$. Then we can write

$$\begin{aligned}
\sum_{k,l=1,1}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| &\geq \sum_{\substack{k,l=1,1 \\ \left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon}}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| \\
&\geq \varepsilon . \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right|
\end{aligned}$$

$$\Longrightarrow \frac{1}{\varepsilon . mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \frac{1}{mn} \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right| .$$

Thus, for any $\delta > 0$,

$$\frac{1}{mn} \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right| \geq \delta$$

implies that

$$\frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon . \delta .$$

Therefore, we have

$$\begin{aligned}
&\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \\
&\qquad \subset \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon . \delta \right\} .
\end{aligned}$$

Since $(x_{kl}) \overset{\mathcal{I}_2[C,1]^L}{\sim} (y_{kl})$, so that

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon . \delta \right\} \in \mathcal{I}_2$$

which implies that

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}_2 .$$

This shows that $(x_{kl}) \overset{S^L(\mathcal{I}_2)}{\sim} (y_{kl})$.

(ii) Suppose that $(x_{kl}), (y_{kl}) \in l_\infty^2$ and $(x_{k,l}) \overset{S^L(\mathcal{I}_2)}{\sim} (y_{kl})$. Then there is an M such that

$$\left| \frac{x_{kl}}{y_{kl}} - L \right| \leq M$$

for all $(k,l) \in \mathbb{N} \times \mathbb{N}$. Given $\varepsilon > 0$, we get

$$\begin{aligned}
\frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| &= \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon}}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| + \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left|\frac{x_{kl}}{y_{kl}} - L\right| < \varepsilon}}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| \\
&\leq \frac{M}{mn} \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right| + \varepsilon .
\end{aligned}$$

If we put

$$A(\varepsilon)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}$$

and

$$B(\varepsilon_1)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}\left|\left\{1\leq k\leq m,\ 1\leq l\leq n:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right|\geq\frac{\varepsilon_1}{M}\right\},$$

where $\varepsilon_1=\delta-\varepsilon>0$, (and δ and ε are independent), then we have $A(\varepsilon)\subset B(\varepsilon_1)$, and so $A(\varepsilon)\in\mathcal{I}_2$. This shows that $(x_{kl})\overset{\mathcal{I}_2[C,1]^L}{\sim}(y_{kl})$. $\square$

4. REFERENCES.

[1] Esi, A., Acikgoz, M., (2014). *On λ^2-Asymptotically Double Statistical Equivalent Sequences*, Int. J. Nonlinear Anal. Appl. 5. No. 2, 16-21 ISNN:2008-6822.

[2] Fast, H. (1951). *Sur la convergence statistique,* Coll. Math., 2, 241-244.

[3] Freedman, A. R. and Sember, J. J. (1981) *Densities and Summability*, Pacific Journal of Mathematics, 95, 239- 305.

[4] Fridy, J. A. (1985). *On statistical convergence.* Analysis, 5, 301, 313.

[5] Gumus, H., Savas, E. (2012) *On $S_\lambda^L(\mathcal{I})-$asymptotically statistical equivalent sequences,* Numerical Analysis and Applied Mathematics Icnaam Aıp Conf. Proc. 1479, pp.936-941

[6] Hazarika, B., Kumar V., (2013), *On asymptotically double lacunary statistical equivalent sequences in ideal context*, J. Ineq. Appl. 2013:543

[7] Kostyrko P. , Salat T. , Wilczynski W., *$\mathcal{I}-$convergence*, Real Anal. Exchange, 26 (2) (2000/2001), 669-686.

[8] Kostyrko P. , Macaj M. , Šalát T. , and Sleziak M. , "*$\mathcal{I}-$convergence and extremal $\mathcal{I}-$limit points*,"Mathematica Slovaca, vol. 55, no. 4, pp. 443–464, 2005.

[9] Marouf, M. (1993) *Asymptotic equivalence and summability. Internat. J. Math. Sci., 16 (4)*

[10] Mursaleen, *(2000), $\lambda-$Statistical Convergence,* Math. Slovaca, 50, No. 1, pp. 111-115.

[11] Mursaleen M., Edely O.H.H. (2003), *Statistical convergence of double sequences*, J. Math. Anal. Appl., 288,223-231.

[12] Patterson, R.F. (2003). *On asymptotically statistically equivalent sequences. Demostratio Math.*, (1), 149-153.

[13] Patterson, R.F. *Some characterization of asymptotic equivalent double sequences*, (in press).

[14] Pobyvanets I. P. (1980). *Asymptotic equivalence of some linear transformations, defined by a nonnegative matrix and reduced to generalized equivalence in the sense of Cesàro and Abel.* Mat. Fiz., no. **28**, 83–87, 123. MR 632482 (83h:40004).

[15] Pringsheim A. (1900). *Zur theorie der zweifach unendlichen Zahlenfolgen*, Mathematische Annalen 53 289-321

[16] Savaş, E., Das P. (2011). *A generalized statistical convergence via ideals.* Appl.Math. Lett., **24** 826–830.

[17] Savaş, E. (2012). *On generalized double statistical convergence via ideals.* The Fifth Saudi Science Conference. 16-18 April, 2012.

[18] Savaş, R., Başarır M., (2006). (σ,λ)-*Asymptotically Statistically Equivalent* Sequences, Filomat 20 (1), 35-42.

[19] Schoenberg, I. J., (1959). *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, 66, 361-375.

FACULTY OF SCİENCE, MATHEMATICS DEPERTMANT , BARTIN UNIVERSITY, BARTIN, TURKEY
E-mail address: okisi@bartin.edu.tr

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 176–184 (2015) ©KJM

LACUNARY STATISTICAL SUMMABILITY OF SEQUENCES OF SETS

UĞUR ULUSU AND FATİH NURAY

Abstract. In this paper we define the $WS_\theta-$analog of the Cauchy criterion for convergence and show that it is equivalent to Wijsman lacunary statistical convergence. Also, Wijsman lacunary statistical convergence is compared to other summability methods which are defined in this paper. After giving new definitions for convergence, we prove a result comparing them. In addition, we give the relationship between Wijsman lacunary statistical convergence and Hausdorf lacunary statistical convergence.

1. INTRODUCTION AND BACKGROUND

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [5] and Schoenberg [11]. The concept of lacunary statistical convergence and summability were defined by Fridy and Orhan in [7, 8].

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence (see, [1],[2],[3],[4],[9],[12],[13],[14]). Nuray and Rhoades [9] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [12] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades.

In this paper, we shall define the concept of Wijsman lacunary statistical Cauchy sequences for sequences of sets and show that this concept is equivalent to the concept of Wijsman lacunary statistically convergence. Also, Wijsman lacunary statistical convergence will be compared to newly defined Wijsman lacunary summability methods. Further, the definition of Wijsman lacunary almost convergence for sequences of sets is introduced and some comparison theorems are given.

Date: April 30, 2015 and, in revised form, July 1, 2015.

2010 *Mathematics Subject Classification.* 40A35, 40G15.

Key words and phrases. Statistical convergence, lacunary sequence, sequence of sets, Wijsman convergence, Hausdorff convergence, Cesàro summability, almost convergence.

2. DEFINITIONS AND NOTATIONS

Now, we recall the concept of statistical, lacunary statistical, Wijsman, Hausdorff, Wijsman statistiscal, Hausdorff statistical, Wijsman strongly almost, Wijsman almost statistical, Wijsman lacunary statistical convergence, Wijsman lacunary summability, Wijsman strongly lacunary summability and Wijsman Cesàro summability of the sequences of sets (see, [2],[6],[7],[9],[12])

Definition 2.1. A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : |x_k - L| \ge \varepsilon\}| = 0.$$

In this case, we write $st - \lim x_k = L$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Definition 2.2. A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{r} \frac{1}{h_r}|\{k \in I_r : |x_k - L| \ge \varepsilon\}| = 0.$$

In this case, we write $S_\theta - \lim x_k = L$ or $x_k \to L(S_\theta)$.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a\in A} \rho(x, a).$$

Definition 2.3. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k\to\infty} d(x, A_k) = d(x, A),$$

for each $x \in X$. In this case, we write $W - \lim A_k = A$.

As an example, consider the following sequence of circles in the (x, y)-plane:

$$A_k = \{(x, y) : x^2 + y^2 + 2kx = 0\}.$$

As $k \to \infty$ the sequence is Wijsman convergent to the y-axis $A = \{(x, y) : x = 0\}$.

Definition 2.4. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Hausdorff convergent to A if

$$\lim_{k\to\infty} \sup_{x\in X} |d(x, A_k) - d(x, A)| = 0.$$

In this case, we write $H - \lim A_k = A$.

The concepts of Wijsman statistical convergence and Hausdorff statistical convergence were given by Nuray and Rhoades [9] as follows:

Definition 2.5. Let (X,ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A if $\{d(x,A_k)\}$ is statistically convergent to $d(x,A)$; i.e., for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : |d(x,A_k) - d(x,A)| \ge \varepsilon\}| = 0.$$

In this case, we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.

Definition 2.6. Let (X,ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Hausdorff statistical convergent to A if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : \sup_{x\in X} |d(x,A_k) - d(x,A)| \ge \varepsilon\}| = 0.$$

In this case, we write $st - \lim_H A_k = A$ or $A_k \to A(HS)$.

Let (X,ρ) be a metric space. For any non-empty closed subsets A_k of X, we say that the sequence $\{A_k\}$ is bounded if $\sup_k d(x,A_k) < \infty$, for each $x \in X$.

Also, the concepts of Wijsman Cesàro Summability, Wijsman strongly almost convergence and Wijsman almost statistical convergence for sequences of sets were given by Nuray and Rhoades [9] as follows:

Definition 2.7. Let (X,ρ) be a metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman Cesàro summable to A if $\{d(x,A_k)\}$ is Cesàro summable to $d(x,A)$; i.e., for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x,A_k) = d(x,A).$$

Definition 2.8. Let (X,ρ) be a metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman strongly almost convergent to A if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x,A_{k+i}) - d(x,A)| = 0,$$

uniformly in i.

Definition 2.9. Let (X,ρ) be a metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman almost statistically convergent to A if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |d(x,A_{k+i}) - d(x,A)| \ge \varepsilon\}| = 0,$$

uniformly in i.

The concepts of Wijsman lacunary summability, Wijsman strongly lacunary Summability and Wijsman lacunary statistical convergence of sequences of sets were given by Ulusu and Nuray [12] as follows:

Definition 2.10. Let (X,ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman lacunary summable to A, if $\{d(x,A_k)\}$ is lacunary summable to $d(x,A)$; i.e., for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{I_r} d(x,A_k) = d(x,A).$$

Definition 2.11. Let (X,ρ) be a metric space and $\theta=\{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k\subseteq X$, we say that $\{A_k\}$ is Wijsman strongly lacunary Summable to A, if $\{d(x,A_k)\}$ is strongly lacunary summable to $d(x,A)$; i.e., for each $x\in X$,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{I_r}|d(x,A_k)-d(x,A)|=0.$$

Definition 2.12. Let (X,ρ) be a metric space and $\theta=\{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k\subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary statistical convergent to A, if $\{d(x,A_k)\}$ is lacunary statistically convergent to $d(x,A)$; i.e., for every $\varepsilon>0$ and for each $x\in X$,

$$\lim_{r}\frac{1}{h_r}|k\in I_r:|d(x,A_k)-d(x,A)|\geq\varepsilon|=0.$$

In this case, we write $S_\theta-\lim_W A_k=A$ or $A_k\to A(WS_\theta)$.

Example 2.1. Let $X=\mathbb{R}$ and we define a sequence $\{A_k\}$ as follows:

$$A_k:=\begin{cases}\{x\in\mathbb{R}:2\leq x\leq k\} & , \begin{array}{l}\text{if } k\geq 2,\quad k_{r-1}<k\leq k_r\\ \text{and } k \text{ is a square integer,}\end{array}\\ \\ \{1\} & ,\quad \text{otherwise.}\end{cases}$$

As $k\to\infty$ this sequence is Wijsman lacunary statistical converget to the set $A=\{1\}$.

3. MAIN RESULTS

Definition 3.1. Let (X,ρ) be a metric space and $\theta=\{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k\subseteq X$, we say that the sequence $\{A_k\}$ is said to be a Wijsman lacunary statistical Cauchy sequence if there is a subsequence $\{A_{k'(r)}\}$ of $\{A_k\}$ such that $k'(r)\in I_r$ for each r, $W-\lim_r A_{k'(r)}=A$, and for every $\varepsilon>0$ and $x\in X$,

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r:|d(x,A_k)-d(x,A_{k'(r)})|\geq\varepsilon\}|=0. \tag{3.1}$$

Theorem 3.1. *Let (X,ρ) be a metric space and $\theta=\{k_r\}$ be a lacunary sequence. The sequence $\{A_k\}$ is Wijsman lacunary statistical convergent if and only if $\{A_k\}$ is a Wijsman lacunary statistical Cauchy sequence.*

Proof. $(\Rightarrow)$ Let $A_k\to A(WS_\theta)$ and write

$$K^{(j)}:=\left\{k\in\mathbb{N}:|d(x,A_k)-d(x,A)|<\frac{1}{j}\right\},$$

for each $x\in X$ and each $j\in\mathbb{N}$. Hence, for each $j, K^{(j)}\supseteq K^{(j+1)}$ and

$$\lim_{r\to\infty}\frac{|K^{(j)}\cap I_r|}{h_r}=1.$$

Choose $m(1)$ such that $r\geq m(1)$ implies $\dfrac{|K^{(1)}\cap I_r|}{h_r}>0$, i.e., $K^{(1)}\cap I_r\neq\emptyset$. Next choose $m(2)>m(1)$ so that $r\geq m(2)$ implies $K^{(2)}\cap I_r\neq\emptyset$. Then, for each r satisfying $m(1)\leq r<m(2)$, choose $k'(r)\in I_r$ such that $k'(r)\in I_r\cap K^{(1)}$, i.e., $|d(x,A_{k'(r)})-d(x,A)|<1$. In general, choose $m(p+1)>m(p)$ such that

$r > m(p+1)$ implies $I_r \cap K^{(p+1)} \neq \emptyset$. Then, for all r satisfying $m(p) \leq r < m(p+1)$, choose $k'(r) \in I_r \cap K^{(p)}$, i.e.,

$$|d(x, A_{k'(r)}) - d(A, x)| < \frac{1}{p}. \tag{3.2}$$

Hence, we get $k'(r) \in I_r$ for every r and (3.2) implies that

$$W - \lim_r d(x, A_{k'(r)}) = d(x, A).$$

Furthermore, for every $\varepsilon > 0$ we have,

$$\begin{aligned}
&\frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A_{k'(r)})| \geq \varepsilon\}| \\
&\qquad \leq \frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \frac{\varepsilon}{2}\}| \\
&\qquad\quad + \frac{1}{h_r}|\{k \in I_r : |d(x, A_{k'(r)}) - d(x, A)| \geq \frac{\varepsilon}{2}\}|.
\end{aligned}$$

Using the assumptions that $A_k \to A(WS_\theta)$ and $W - \lim_r d(x, A_{k'(r)}) = d(x, A)$, we infer (3.1), whence A_k is a Wijsman lacunary statistical Cauchy sequence.

($\Leftarrow$) Conversely, suppose that $\{A_k\}$ is a Wijsman lacunary statistical Cauchy sequence. For every $\varepsilon > 0$, we have

$$\begin{aligned}
|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \quad &\leq \quad \left|\{k \in I_r : |d(x, A_k) - d(x, A_{k'(r)})| \geq \frac{\varepsilon}{2}\}\right| \\
&\quad + \left|\{k \in I_r : |d(x, A_{k'(r)}) - d(x, A)| \geq \frac{\varepsilon}{2}\}\right|
\end{aligned}$$

from which it follows that $A_k \to A(WS_\theta)$. □

Now we give following theorem where Δ denotes the forward difference operator defined by $\Delta d(x, A_i) = d(x, A_i) - d(x, A_{i+1})$.

Theorem 3.2. *Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. If $A_k \to A(WS_\theta)$ and for each $x \in X$*

$$\max\{|\Delta d(x, A_i)| : i \in I_r\} = o(\frac{1}{h_r}) \quad \textit{as } r \to \infty,$$

then $W - \lim A_k = A$.

Proof. Assume that $A_k \to A(WS_\theta)$ and by Theorem (3.1), choose a subsequence $\{A_{k'(r)}\}$ of $\{A_k\}$ as in Definition (3.1). Since $k'(r) \in I_r$, for each $x \in X$ we have

$$\begin{aligned}
|d(x, A_k) - d(x, A_{k'(r)})| \quad &\leq \quad \sum_{i=k}^{k'(r)-1} |\Delta d(x, A_i)| \\
&\leq \quad h_r.\,(\max_{i \in I_r}\{|\Delta d(x, A_i)| : i \in I_r\}) \\
&= \quad o(1)
\end{aligned}$$

and therefore $A_{k'(r)} \to A(WS)$ implies that $A_k \to A(WS)$. □

Theorem 3.3. *Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. If $\{A_k\}$ is a bounded sequence and $A_k \to A(WS_\theta)$, then $\{A_k\}$ is Wijsman Cesàro summable to A.*

Proof. Let (X,ρ) be a metric space, $\theta=\{k_r\}$ be a lacunary sequence and let n be a positive integer with $n\in I_r$; then

$$\frac{1}{n}\sum_{k=1}^{n}\Big(d(x,A_k)-d(x,A)\Big) = \frac{1}{n}\sum_{p=1}^{r-1}\sum_{k\in I_p}\Big(d(x,A_k)-d(x,A)\Big) + \frac{1}{n}\sum_{k=1+k_{r-1}}^{n}\Big(d(x,A_k)-d(x,A)\Big). \tag{3.3}$$

Consider the first term on the right in (3.3),

$$\begin{aligned}\frac{1}{n}\sum_{p=1}^{r-1}\sum_{k\in I_p}\Big(d(x,A_k)-d(x,A)\Big) &\le \frac{1}{k_{r-1}}\sum_{p=1}^{r-1}\sum_{k\in I_p}\Big|d(x,A_k)-d(x,A)\Big|\\ &= \frac{1}{k_{r-1}}\sum_{p=1}^{r-1}h_p.t_p=(Ht)_r,\end{aligned}\tag{3.4}$$

say, where

$$t_p=\frac{1}{h_p}\sum_{k\in I_p}|d(x,A_k)-d(x,A)|\,.$$

Since $\{A_k\}$ is bounded and $A_k\to A(WS_\theta)$, it follows from Theorem 1 (ii) of [12] that $t_p\to 0$. Moreover

$$k_{r-1}=\sum_{p=1}^{r-1}h_p\to\infty \qquad \text{as } r\to\infty,$$

because θ is a lacunary sequence, which implies that (3.4) is a regular weighted mean matrix transform of t in [10]; hence,

$$(Ht)_r\to 0. \tag{3.5}$$

Now consider the second term on the right in (3.3). Since $\{A_k\}$ is bounded, there is a constant $M>0$ such that $|d(x,A_k)-d(x,A)|\le M$, for all k. Therefore, for every $\varepsilon>0$ we have,

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1+k_{r-1}}^{n}\Big(d(x,A_k)-d(x,A)\Big)\right| &\le \frac{1}{n}\sum_{\substack{k_{r-1}<k\le n\\ |d(x,A_k)-d(x,A)|\ge\varepsilon}}|d(x,A_k)-d(x,A)|\\ &\quad+\frac{1}{n}\sum_{\substack{k_{r-1}<k\le n\\ |d(x,A_k)-d(x,A)|<\varepsilon}}|d(x,A_k)-d(x,A)|\\ &\le \frac{M}{h_r}|\{k\in I_r:|d(x,A_k)-d(x,A)|\ge\varepsilon\}|+\varepsilon.\end{aligned}\tag{3.6}$$

Since $A_k\to A(WS_\theta)$ and ε is an arbitrary, the expression on the left side of (3.6) tends to zero as $r\to\infty$. Hence, (3.3), (3.5) and (3.6) imply that $\{A_k\}$ is Wijsman Cesàro summable to A. □

Definition 3.2. Let (X,ρ) be a metric space and $\theta=\{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k\subseteq X$, we say that $\{A_k\}$ is Wijsman strongly $p-$lacunary summable to A if $\{d(x,A_k)\}$ is strongly $p-$lacunary summable to $d(x,A)$; i.e., for each p positive real number and for each $x\in X$

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{I_r}|d(x,A_k)-d(x,A)|^p=0.$$

Theorem 3.4. *Let (X,ρ) be a metric space, $\theta=\{k_r\}$ be a lacunary sequence and let p positive real number. Then, for any non-empty closed subsets A, $A_k\subseteq X$;*

(i) *$\{A_k\}$ is Wijsman lacunary statistical convergent to A if it is Wijsman strongly $p-$lacunary summable to A.*

(ii) *If $\{A_k\}$ is bounded and Wijsman lacunary statistical convergent to A then it is Wijsman strongly $p-$lacunary summable to A.*

Proof. (i) For any $\{A_k\}$, fix an $\varepsilon>0$. Then

$$\sum_{I_r}|d(x,A_k)-d(x,A)|^p\geq\varepsilon^p|\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\varepsilon\}|,$$

and it follows that if $\{A_k\}$ is Wijsman strongly $p-$lacunary summable to A then $\{A_k\}$ is Wijsman lacunary statistical convergent to A.

(ii) Let $\{A_k\}$ be bounded and Wijsman lacunary statistical convergent to A. Since $\{A_k\}$ is bounded set

$$\sup_k\{d(x,A_k)\}+d(x,A)=M.$$

Since $\{A_k\}$ is Wijsman lacunary statistically convergent to A, for given $\varepsilon>0$ we can select N_ε such that for each $x\in X$

$$\frac{1}{h_r}\left|\left\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}\right|<\frac{\varepsilon}{2M^p},$$

for all $r>N_\varepsilon$ and we let the set

$$L_r=\left\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}.$$

Then, for each $x\in X$

$$\begin{aligned}\frac{1}{h_r}\sum_{I_r}|d(x,A_k)-d(x,A)|^p &= \frac{1}{h_r}\Bigg(\sum_{\substack{k\in I_r\\ k\in L_r}}|d(x,A_k)-d(x,A)|^p\\ &\qquad+\sum_{\substack{k\in I_r\\ k\notin L_r}}|d(x,A_k)-d(x,A)|^p\Bigg)\\ &< \frac{1}{h_r}.\frac{h_r.\varepsilon}{2M^p}M^p+\frac{1}{h_r}.\frac{h_r.\varepsilon}{2}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Hence, $\{A_k\}$ is Wijsman strongly $p-$lacunary summable to A. $\square$

Definition 3.3. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman lacunary almost convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{I_r} d(x, A_{k+i}) = d(x, A),$$

uniformly in i.

Definition 3.4. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman lacunary strongly almost convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{I_r} |d(x, A_{k+i}) - d(x, A)| = 0,$$

uniformly in i.

Example 3.1. Let $X = \mathbb{R}^2$ and we define a sequence $\{A_k\}$ as follows:

$$A_k := \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y+1)^2 = \dfrac{1}{k}\right\} & , \quad \text{if} \;\; k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ \\ \{(1,0)\} & , \quad \text{otherwise.} \end{cases}$$

As $k \to \infty$ this sequence is Wijsman lacunary strongly almost convergent to the set $A = \{(1, 0)\}$.

Definition 3.5. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman lacunary strongly $p-$almost convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{I_r} |d(x, A_{k+i}) - d(x, A)|^p = 0,$$

uniformly in i, where p is a positive real number.

Definition 3.6. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary almost statistically convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |d(x, A_{k+i}) - d(x, A)| \geq \varepsilon| = 0,$$

uniformly in i.

Let L_∞, C, $(WAC)_\theta$ and $|WAC|_\theta$, respectively, denote the sets of the all bounded, Wijsman convergent, Wijsman lacunary almost convergent and Wijsman lacunary strongly almost convergent sequences of sets. It is easy to see that

$$C \subset (WAC)_\theta \subset |WAC|_\theta \subset L_\infty.$$

Theorem 3.5. *Let (X, ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary sequence and p be a positive number. Then, for any non-empty closed subsets A, $A_k \subseteq X$,*

(i) *$\{A_k\}$ is Wijsman lacunary almost statistically convergent to A, if it is Wijsman lacunary strongly $p-$almost converget to A,*

(ii) *If $\{A_k\}$ is bounded and Wijsman lacunary almost statistically convergent to A, then it is Wijsman lacunary strongly $p-$almost convergent to A.*

Proof. The proof is similar to the proof of Theorem (3.4). □

Definition 3.7. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Hausdorff lacunary statistically convergent to A, if for each $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \sup_{x\in X} |d(x, A_k) - d(x, A)| \geq \varepsilon \right\} \right| = 0$$

i.e.,

$$\sup_{x\in X} |d(x, A_k) - d(x, A)| < \varepsilon \qquad \text{a.a.k.}$$

in this case, we write $HS_\theta - \lim A_k = A$, $S_\theta - \lim_H A_k = A$, $A_k \to A(HS_\theta)$.

Theorem 3.6. *Let (X, ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary and $\{A_k\}$ be a sequence of non-empty closed subsets of X. If $\{A_k\}$ is Hausdorff lacunary statistical converget, then $\{A_k\}$ is Wijsman lacunary statistical convergent.*

Proof. For any sequence $\{A_k\}$ and for every $\varepsilon > 0$, since

$$|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \leq \left| \left\{ k \in I_r : \sup_{x\in X} |d(x, A_k) - d(x, A)| \geq \varepsilon \right\} \right|$$

we get the result. □

References

[1] J.-P. Aubin and H. Frankowska, Set-valued analysis, Birkhauser, Boston, 1990.

[2] M. Baronti and P. Papini, Convergence of sequences of sets. In: Methods of Functional Analysis in Approximation Theory, ISNM 76, Birkhauser, Basel, 133-155, 1986.

[3] G. Beer, On convergence of closed sets in a metric space and distance functions, *Bull. Austral. Math. Soc.,* **31** (1985), 421-432.

[4] G. Beer, Wijsman convergence: A survey. *Set-Valued Var. Anal.*, **2** (1994), 77-94.

[5] H. Fast, Sur la convergence statistique, *Collog. Math.*, **2** (1951), 241-244.

[6] J.A. Fridy, On statistical convergence, *Analysis*, **5** (1985), 301-313.

[7] J.A. Fridy and C. Orhan, Lacunary statistical convergence, *Pacific J. Math.,* **160**(1) (1993), 43-51.

[8] J.A. Fridy and C. Orhan, Lacunary statistical summability, *J. Math. Anal. Appl.,* **173** (1993), 497-504.

[9] F. Nuray and B.E. Rhoades, Statistical convergence of sequences of sets, *Fasc. Math.,* **49** (2012), 87-99.

[10] R.E. Powel and S.M. Shah, Summability theory and its applications, Van Nostrand-Rheinhold, London, 1972.

[11] I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly,* **66** (1959), 361-375.

[12] U. Ulusu and F. Nuray, Lacunary statistical convergence of sequences of sets, *Progress in Applied Mathematics,* **4** (2012), 99-109.

[13] R.A. Wijsman, Convergence of sequences of convex sets, cones and functions, *Bull. Amer. Math. Soc.,* **70** (1964), 186-188.

[14] R. A. Wijsman, Convergence of sequences of Convex sets, Cones and Functions II, *Trans. Amer. Math. Soc.,* **123** (1) (1966), 32-45.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND LITERATURE, AFYON KOCATEPE UNIVERSITY, AFYONKARAHİSAR, TURKEY

E-mail address: `ulusu@aku.edu.tr, fnuray@aku.edu.tr`

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 185–189 (2015) ©KJM

A NOTE ON FIBERED QUADRANGLES

S. EKMEKÇİ & A. BAYAR

Abstract. In this work, the fibered versions of the diagonal triangle and the quadrangular set of a complete quadrangle in fibered projective planes are introduced. And then some related theorems with them are given.

1. Introduction

Fuzzy set theory was introduced by Zadeh [10] and this theory has been applied in many areas. One of them is projective geometry, see for instance [1,2,5,7,8,9]. A first model of fuzzy projective geometries was introduced by Kuijken, Van Maldeghem and Kerre [7,8]. Also, Kuijken and Van Maldeghem contributed to fuzzy theory by introducing fibered geometries, which is a particular kind of fuzzy geometries [6]. They gave the fibered versions of some classical results in projective planes by using minimum operator. Then the role of the triangular norm in the theory of fibered projective planes and fibered harmonic conjugates and a fibered version of Reidemeister's condition were given in [3]. The fibered version of Menelaus and Ceva's 6-figures was studied in [4].

It is well known that triangles and quadrangles have an important role in projective geometry. A complete quadrangle is a system of geometric objects consisting of any four points in a plane, no three of which are on a common line, and of the six lines connecting each pair of points. The free completion of a configuration containing either a quadrangle or a quadrilateral is a projective plane. In contrast, the free completion of a (non-empty) configuration which does not contain either a quadrangle or a quadrilateral is not a projective plane. Notice that the existence of a quadrangle and the associated diagonal triangle forces any projective plane to have at least seven points. If, in fact, the points of the diagonal triangle are collinear, we obtain a projective plane with seven points. This projective plane is known as the Fano plane.

In the present paper, we consider the fibered versions of classical theorems related to complete quadrangles. We start by defining the fiber version of the diagonal

Date: January 1, 2015 and, in revised form, February 2, 2015.
2000 *Mathematics Subject Classification.* 20N25.
Key words and phrases. Fibered projective plane; complete quadrangle, quadrangular set.

triangle of a complete quadrangle in fibered projective planes. And then some related theorems are given between them. It is shown that four f-points intersecting two opposite sides of two f-complete quadrangles are f-collinear when four f-lines spanned by two opposite vertices of two f-complete quadrangles are f-concurrent. Finally, the fiber version of a quadrangular set is defined and related theorems with f-quadrangular sets are given in fibered projective planes.

2. Preliminaries

We first recall some basic notions from fuzzy set theory and fibered geometry. We denote by $\wedge$ a triangular norm on the (real) unit interval $[0,1]$, i.e., a symmetric and associative binary operator satisfying $(a \wedge b) \leq (c \wedge d)$ whenever $a \leq c$ and $b \leq d$, and $a \wedge 1 = a$, for all $a, b, c, d \in [0,1]$.

Definition 2.1. (*see* [6]) Let $\mathcal{P} = (P, B, \circ)$ be any projective plane with point set P and line set B, i.e., P and B are two disjoint sets endowed with a symmetric relation $\circ$ (called the *incidence relation*) such that the graph $(P \cup B, \circ)$ is a bipartite graph with classes P and B, and such that two distinct points p, q in $\mathcal{P}$ are incident with exactly one line, every two distinct lines L, M are incident with exactly one point, and every line is incident with at least three points. A set S of *collinear* points is a subset of P each member of which is incident with a common line L. Dually, one defines a set of *concurrent* lines. We now define fibered points and fibered lines, briefly called f-points and f-lines.

Definition 2.2. *(see* [6]*) Suppose $a \in P$ and $\alpha \in]0,1]$. Then an f-*point *(a, α) is the following fuzzy set on the point set P of $\mathcal{P}$:*

$$(a, \alpha) : P \to [0,1] : \begin{cases} a \to \alpha, \\ x \to 0 \quad if \quad x \in P \backslash \{a\}. \end{cases}$$

Dually, one defines in the same way the f-line (L, β) for $L \in B$ and $\beta \in]0,1]$. The real number α above is called the membership degree *of the f-point (a, α), while the point a is called the* base point *of it. Similarly for f-lines.*

Definition 2.3. *(see* [6]*) Two f-lines (L, α) and (M, β), with $\alpha \wedge \beta > 0$,* intersect *in the unique f-point $(L \cap M, \alpha \wedge \beta)$. Dually, the f-points (a, λ) and (b, μ), with $\lambda \wedge \mu > 0$,* span *the unique f- line $(\langle a, b \rangle, \lambda \wedge \mu)$.*

Definition 2.4. *(see* [6]*) A (nontrivial) fibered projective plane $\mathcal{FP}$ consists of a set FP of f-points of P and a set FB of f-lines of P such that every point and every line of P is the base point and base line of at least one f-point and f-line, respectively (with at least one membership degree different from 1), and such that $\mathcal{FP} = (FP, FB)$ is closed under taking intersections of f-lines and spans of f-points. Finally, a set of f-points are called collinear if each pair of them span the same f-line. Dually, a set of f-lines are called concurrent if each pair of them intersect in the same f-point.*

3. Fibered Version of quadrangles

Definition 3.1. *(see [3]) Suppose we have a fibered projective plane $\mathcal{FP}$ with base projective plane $\mathcal{P}$. Choose four f-points $(a_1, \alpha_1), (a_2, \alpha_2), (a_3, \alpha_3)$, and (a_4, α_4) in $\mathcal{FP}$ no three base points of which are collinear. These f-points are called f-vertices. The configuration that consists of these four f-points, the six f-lines $(A_{\{i,j\}}, \beta_{\{i,j\}}) =: (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$, for $i \neq j$, $i, j \in \{1, 2, 3, 4\}$ (which we call f-sides), and three f-points $(A_{\{i,j\}} \wedge A_{\{k,l\}}, \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4)$, with $\{i, j, k, l\} = \{1, 2, 3, 4\}$ (the f-diagonal points), is called an* f-complete quadrangle.

Definition 3.2. *Suppose we have a fibered projective plane $\mathcal{FP}$ with base projective plane $\mathcal{P}$. If the vertices of an f-triangle are the f-diagonal points of an f-complete quadrangle, it this f-triangle is called an* f-diagonal triangle.

To simplify notation, we will sometimes omit the binary operator $\wedge$ and write $\alpha\beta$ for $\alpha \wedge \beta$. In this notation, we will also abbreviate $\alpha \wedge \alpha$ to α^2.

Theorem 3.1. *Suppose we have a fibered projective plane $\mathcal{FP}$ with base plane $\mathcal{P}$ that is Desarguesian. Let four f-points $(a_1, \alpha_1), (a_2, \alpha_2), (a_3, \alpha_3)$ and (a_4, α_4) form an f-complete quadrangle and let the three f- points (b_i, β_i), $i = \{1, 2, 3\}$, with $b_1 = a_2a_3 \cap a_1a_4$, $b_2 = a_2a_4 \cap a_1a_3$ and $b_3 = a_1a_2 \cap a_3a_4$, be the associated f-diagonal triangle in $\mathcal{FP}$. The three intersection f-points (c_k, γ_k) of the f-lines $(a_ia_j, \alpha_i \wedge \alpha_j)$ and $(b_ib_j, \beta_i \wedge \beta_j)$, with $\{i, j, k\} = \{1, 2, 3\}$, are f-collinear if $\alpha_1^2\alpha_2\alpha_3 = \alpha_1\alpha_2^2\alpha_3 = \alpha_1\alpha_2\alpha_3^2$.*

Proof. Note that by Definition 3.1 $\beta_i = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 =: \beta$, $i = 1, 2, 3$. The lines $a_ib_i, i = 1, 2, 3$ are incident with the point a_4 in $\mathcal{P}$. If Desargues' theorem is applied to the triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, we see that the points $c_k = a_ia_j \wedge b_ib_j$, with $\{i, j, k\} = \{1, 2, 3\}$, are collinear in $\mathcal{P}$. Also, using $\alpha_1^2\alpha_2\alpha_3 = \alpha_1\alpha_2^2\alpha_3 = \alpha_1\alpha_2\alpha_3^2$, the equality

$$\alpha_1^2\alpha_2\alpha_3\beta^4 = \alpha_1\alpha_2^2\alpha_3\beta^4 = \alpha_1\alpha_2\alpha_3^2\beta^4$$

is obtained. So, the three f-points $(c_k, \gamma_k) = (a_ia_j \wedge b_ib_j, \alpha_i\alpha_j\beta^2)$, with $\{i, j, k\} = \{1, 2, 3\}$, are f-collinear. □

Corollary 3.1. *Suppose we have a fibered projective plane $\mathcal{FP}$ with base plane $\mathcal{P}$ that is Desarguesian and let $\wedge$ be the minimum triangular norm. Let four f-points $(a_1, \alpha_1), (a_2, \alpha_2), (a_3, \alpha_3)$ and (a_4, α_4) form an f-complete quadrangle and let (b_i, β_i), $i = \{1, 2, 3\}$ be the corresponding f-diagonal triangle, with $b_1 = a_2a_3 \cap a_1a_4$, $b_2 = a_2a_4 \cap a_1a_3$ and $b_3 = a_1a_2 \cap a_3a_4$. Then the three f-points $(c_k, \gamma_k) = (a_ia_j \cap b_ib_j, \alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j)$, with $\{i, j, k\} = \{1, 2, 3\}$, are f-collinear.*

Theorem 3.2. *Suppose we have a fibered projective plane $\mathcal{FP}$ with Desarguesian base plane $\mathcal{P}$. Choose two different f-quadrangles (a_i, α_i) and (b_i, β_i), $i = 1, 2, 3, 4$ in $\mathcal{FP}$. Let the f-lines $(\langle a_i, b_i \rangle, \alpha_i\beta_i)$, for $i \in \{1, 2, 3, 4\}$, be concurrent with intersection points (p, γ) in $\mathcal{FP}$, $a_i \neq b_i \neq p_i \neq a_i$. Let the f-lines $(\langle a_1, a_2 \rangle, \alpha_1\alpha_2)$, $(\langle a_3, a_4 \rangle, \alpha_3\alpha_4)$, $(\langle b_1, b_2 \rangle, \beta_1\beta_2)$ and $(\langle b_3, b_4 \rangle, \beta_3\beta_4)$ meet in the f-point (c_1, γ_1), the f-lines $(\langle a_1, a_4 \rangle, \alpha_1\alpha_4)$, $(\langle a_2, a_3 \rangle, \alpha_2\alpha_3)$, $(\langle b_1, b_4 \rangle, \beta_1\beta_4)$ and $(\langle b_2, b_3 \rangle, \beta_2\beta_3)$ meet in the f-point (c_2, γ_2), the f-lines $(\langle a_2, a_4 \rangle, \alpha_2\alpha_4)$ and $(\langle b_2, b_4 \rangle, \beta_2\beta_4)$ meet in the f-point (c_3, γ_3) and let the f-point (c_4, γ_4) be the intersection point of the f-lines $(\langle a_1, a_3 \rangle, \alpha_1\alpha_3)$ and $(\langle b_1, b_3 \rangle, \beta_1\beta_3)$. Then (c_i, γ_i), $i \in \{1, 2, 3, 4\}$ are collinear in $\mathcal{FP}$ (in particular, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4$).*

Proof. One calculates $\gamma = \alpha_i\alpha_j\beta_i\beta_j$, for $\{i,j\} \subseteq \{1,2,3,4\}$,with $i \neq j$. Since the f-lines $(\langle a_1, a_2\rangle, \alpha_1\alpha_2), (\langle a_3, a_4\rangle, \alpha_3\alpha_4), (\langle b_1, b_2\rangle, \beta_1\beta_2)$ and $(\langle b_3, b_4\rangle, \beta_3\beta_4)$ are f-concurrent in (c_1, γ_1), $\gamma_1 = \alpha_1\alpha_2\alpha_3\alpha_4 = \beta_1\beta_2\beta_3\beta_4 = \alpha_1\alpha_2\beta_1\beta_2 = \alpha_1\alpha_2\beta_3\beta_4 = \alpha_3\alpha_4\beta_1\beta_2 = \alpha_3\alpha_4\beta_3\beta_4 = \gamma$. Similarly, it is seen that $\gamma_2 = \alpha_1\alpha_2\alpha_3\alpha_4 = \beta_1\beta_2\beta_3\beta_4 = \alpha_1\alpha_4\beta_1\beta_4 = \alpha_1\alpha_4\beta_2\beta_4 = \alpha_2\alpha_3\beta_1\beta_4 = \alpha_2\alpha_3\beta_2\beta_3 = \gamma$, $\gamma_3 = \alpha_2\alpha_4\beta_2\beta_4 = \gamma$, $\gamma_4 = \alpha_1\alpha_3\beta_1\beta_3 = \gamma$. Since $\mathcal{P}$ is a Desarguesian plane, the c_i, $i \in \{1,2,3,4\}$, are collinear and the memberships degrees of them are equal to γ. Hence the f-points (c_i, γ_i), $i \in \{1,2,3,4\}$ are collinear in $\mathcal{FP}$. □

Although the assumptions of the previous Theorem imply a lot of equalities between expressions in the membership degrees of the points a_i and b_j, $i,j \in \{1,2,3,4\}$, they do not imply that all membership degrees should be equal. For instance, if the minimum operator is used, then $\alpha_1 = \alpha_2 = \beta_1 = \beta_3 \leq \alpha_i, \beta_j$, for $i = 1,2$ and $j = 2,4$, satisfies the assumptions.

Definition 3.3. *Suppose we have a fibered projective plane $\mathcal{FP}$ with base projective plane $\mathcal{P}$. Let (a_i, α_i), $i = 1,2,3,4$, be the vertices any f-quadrangle in $\mathcal{FP}$ and let (L, α) be any f-line such that the base line L is not incident with any of the points a_i, $i = 1,2,3,4$. Let (p_1, β_1), (p_2, β_2), (p_3, β_3) be the f-intersection point of the f-line (L, α) with the f-line $(a_1a_2, \alpha_1\alpha_2)$, $(a_1a_3, \alpha_1\alpha_3)$, $(a_1a_4, \alpha_1, \alpha_4)$, respectively, and let (q_1, γ_1), (q_2, γ_2), (q_3, γ_3) be the f-intersection point of the f-line (L, α) with the f-line $(a_3a_4, \alpha_3\alpha_4)$, $(a_2a_4, \alpha_2\alpha_4)$, $(a_2a_3, \alpha_2, \alpha_3)$, respectively. Then these six (not necessarily distinct) points are called an* f-quadrangular set.

The f-quadrangular set may be consist of five or four f-points if the f-line (L, α) happens to pass through one or two f-diagonal points.

Although the six base points p_1, p_2, p_3, q_1, q_2 and q_3 of the f-quadrangular set are collinear in the base plane $\mathcal{P}$, the six f-points (p_1, β_1), (p_2, β_2), (p_3, β_3), (q_1, γ_1), (q_2, γ_2) and (q_3, γ_3) are not necessarily f-collinear in $\mathcal{FP}$. But we do have the following property.

Theorem 3.3. *Suppose we have a fibered projective plane $\mathcal{FP}$ with Desarguesian base plane $\mathcal{P}$. Let, with the notation of Definition 3.3, $\{(p_i, \beta_i), (q_i, \gamma_i)\}$, $i = 1,2,3$, be the f- quadrangular set determined by the f-quadrangle (a_i, α_i), $i = 1,2,3,4$, and the f-line (L, α) in $\mathcal{FP}$. Then the three pairs of f-points $\{(p_i, \beta_i), (q_i, \gamma_i)\}$, $i = 1,2,3$, span the same f-line, namely $(L, \alpha^2\alpha_1\alpha_2\alpha_3\alpha_4)$.*

Proof. Easy calculation. □

References

[1] Akça Z, Bayar A, Ekmekçi S, Van Maldeghem H., *Fuzzy projective spreads of fuzzy projective spaces*, Fuzzy Sets and Systems 157 (2006) 3237-3247.

[2] Akça Z, Bayar A, Ekmekçi S., *On the classification of Fuzzy projective lines of Fuzzy 3-dimensional projective spaces*, Communications Mathematics and Statistics, Vol. 55(2) (2007) 17-23.

[3] Bayar A, Akça Z, Ekmekçi S., *A Note on Fibered Projective Plane Geometry*, Information Science 178 (2008) 1257-1262.

[4] Bayar A, Ekmekçi S., *On the Menelaus and Ceva 6-Figures in The Fibered Projective Planes*, Abstract and Applied Analysis, Vol 2014, Article ID 803173, http://dx.doi.org/10.1155/2014/803173.

[5] Ekmekçi S, Bayar A, Akça Z., *On the classification of Fuzzy projective planes of Fuzzy 3-dimensional projective spaces*, Chaos, Solitons and Fractals 40 (2009) 2146-2151.

[6] Kuijken L., Van Maldeghem H., *Fibered Geometries*, Discrete Mathematics 255 (2002) 259-274.
[7] Kuijken L., Van Maldeghem H, Kerre E., *Fuzzy projective geometries from fuzzy vector spaces*, in: A. Billot et al. (Eds.), Information Processing and Management of Uncertainty in Knowledge-based Systems, Editions Medicales et Scientifiques, Paris, La Sorbonne, (1998), 1331-1338.
[8] Kuijken L., Van Maldeghem H, Kerre E., *Fuzzy projective geometries from fuzzy groups*, Tatra Mt. Math. Publ., 16 (1999), 95-108.
[9] Lubczonok P., *Fuzzy Vector Spaces*, Fuzzy Sets and Systems 38 (1990), 329-343.
[10] L. Zadeh, *Fuzzy sets*, Information control 8 (1965) 338-353.

Department of Mathematics - Computer, Eskişehir Osmangazi University, Eskişehir, Turkey
E-mail address: sekmekci@ogu.edu.tr, akorkmaz@ogu.edu.tr

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 190–201 (2015) ©KJM

ON THE INVOLUTES FOR DUAL SPLIT QUATERNIONIC CURVES

CUMALI EKICI AND HATICE TOZAK

Abstract. In this study, definition of involute-evolute curves for semi-dual quaternionic curves in semi-dual spaces $\mathbb{D}_2^4$ known as dual split quaternion and $\mathbb{D}_1^3$ are given and also some well-known theorems for involute-evolute dual split quaternionic curves are obtained.

1. Introduction

The idea of a string involute is due to C. Huygens (1658) who is also known with his work in optics. He discovered involutes while trying to build a more accurate clock [4]. Later, the relations Frenet frame of involute-evolute couple in the space E^3 were given in [10].

In recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to Lorentz manifold. For instance, in [23], the authors extended and studied the spacelike involute-evolute curves in Minkowski space-time ([2], [5], [23]).

The quaternions were first defined in 1843 by Hamilton. The dual quaternions are extension of the real quaternions by means of the dual numbers [3], [22], and they were first introduced by Clifford [6]. In $\mathbb{D}^3$ and $\mathbb{D}^4$ dual spaces, Serret Frenet Formulas had been defined by Sivridağ [21]. Inclined curves and characterization of quaternionic Lorentz manifolds were given in 1999 by Karadağ. In 2002, Serret Frenet Formulas for quaternionic curves in Semi-Euclidean space were defined by Tuna. The quaternionic inclined curves in the Semi-Euclidean space E_2^4 were given

Date: January 1, 2013 and, in revised form, February 2, 2013.

1991 *Mathematics Subject Classification.* 53A04, 53A17, 53A25.

Key words and phrases. semi-dual quaternions, semi-dual space, Serret-Frenet formula, involute-evolute curve couple.

The author is supported by ...

in 2004 by Çöken and Tuna [8]. The split quaternions were identified with Semi-Euclidean space E_2^4, while the vector part of split quaternions were identified with Minkowski 3-space [11]. In 2009, Serret Frenet Formulas for split quaternionic curves in Semi-Euclidean space E_2^4 were given in [7].

In this paper, we firstly define involute-evolute curve couples in definition of involute-evolute curves on $\mathbb{D}_1^3$ and $\mathbb{D}_2^4$. Later, we calculate Frenet frame of the evolute curve by the help of the frame of the involute curve. We use the methods expressed in [7]. (In this paper, we consider non-null curves, and a version of this adapted to null curves can be studied.)

2. Preliminaries

In this section, we will give basic definitions of the dual spaces $\mathbb{D}^3$ and $\mathbb{D}^4$ and then the semi-dual spaces $\mathbb{D}_1^3$ and $\mathbb{D}_2^4$.

A dual number has the form $a + \xi a^*$ where a and a^* are real numbers and $\xi = (0, 1)$ is the dual unit with the property that $\xi^2 = 0$. The set of all dual numbers form a comutative ring over the real number field and denoted by $\mathbb{D}$ [25]. $\mathbb{D}^3$ dual vector space ($\mathbb{D}$ - Module) can be written as

$$\mathbb{D}^3 = \{(A_1, A_2, A_3) : A_1, A_2, A_3 \in \mathbb{D}\ \}.$$

The Euclidean inner-product of two dual vectors A, $B \in \mathbb{D}^3$ is defined as

$$\begin{array}{lccl} \langle,\rangle: & \mathbb{D}^3 \times \mathbb{D}^3 & \longrightarrow & \mathbb{D} \\ & (A, B) & \longrightarrow & \langle A, B\rangle = \langle a, b\rangle + \xi(\langle a^*, b\rangle + \langle a, b^*\rangle). \end{array}$$

Given a dual vector $A = a + \xi a^*$, the norm of A is

$$\|A\| = (\langle A, A\rangle)^{\frac{1}{2}} = \|a\| + \xi\frac{\langle a, a^*\rangle}{\|a\|}, \qquad a \neq 0.$$

The cross-product of two dual vectors $A, B \in \mathbb{D}^3$ is defined as,

$$A \wedge B = a \wedge b + \xi\,(a \wedge b^* + a^* \wedge b)\,.$$

Similarly, $\mathbb{D}^4$ dual vector space can be written as

$$\mathbb{D}^4 = \{(A_1, A_2, A_3, A_4) : A_1, A_2, A_3, A_4 \in \mathbb{D}\ \}.$$

The same definitions of inner-product, norm and cross-product are hold for $\mathbb{D}^4$.

The Lorentzian inner-product of two dual vectors $A = a + \xi a^*$ and $B = b + \xi b^*$, a, $b \in \mathbb{R}_1^3$ is given as

$$\langle A, B\rangle = \langle a, b\rangle + \xi(\langle a^*, b\rangle + \langle a, b^*\rangle)$$

with the signature $(-, +, +)$ in $\mathbb{R}_1^3$. The $\mathbb{D}-$module $\mathbb{D}^3$ with the Lorentzian inner-product is called the semi-dual space $\mathbb{D}_1^3$ [24].

On the other hand, a semi-Euclidean inner-product of two dual vectors in $\mathbb{D}^4$, $A = a + \xi a^*$ and $B = b + \xi b^*$, a, $b \in \mathbb{R}_2^4$, can be defined as

$$\langle A, B\rangle = \langle a, b\rangle + \xi(\langle a^*, b\rangle + \langle a, b^*\rangle)$$

with the signature $(-, -, +, +)$ in $\mathbb{R}_2^4$. The dual space $\mathbb{D}^4$ with the semi-Euclidean inner-product is called the semi-dual space $\mathbb{D}_2^4$ or dual-split quaternion [12].

Let A be a dual vector in $\mathbb{D}_1^3$. If $\langle a, a\rangle < 0$, then A is called timelike, if $\langle a, a\rangle > 0$, then A is called spacelike and if $\langle a, a\rangle = 0$, then A is called lightlike (or null) vector. A smooth curve on the semi-dual space $\mathbb{D}_1^3$ is said to be timelike, spacelike or null if its tangent vectors are timelike, spacelike or null, respectively. Observe that, a timelike curve corresponds to the path of an observer moving at less than the speed of light while the spacelike curves are faster and the null curves are equal to the speed of light [17].

A real quaternion consists of a set of four ordered real numbers a, b, c, d associated with four units e_1, e_2, e_3 and 1, respectively. The three units e_1, e_2 and e_3 have the following properties:

$$\begin{array}{lll} e_1^2 = e_2^2 = e_3^2 = -1, & & \\ e_1 \times e_2 = e_3, & e_2 \times e_3 = e_1, & e_3 \times e_1 = e_2 \\ e_2 \times e_1 = -e_3, & e_3 \times e_2 = -e_1, & e_1 \times e_3 = -e_2 \end{array} \tag{1}$$

A real quaternion q may be written as $q = ae_1 + be_2 + ce_3 + d$.

Clearly, a quaternion q consists of two parts: the scalar part $S_q = d$ and the vector part $V_q = ae_1 + be_2 + ce_3$. The set of all real quaternions is denoted by $Q_{\mathbb{R}}$.

The multiplication of two real quaternions p and q is defined as

$$p \times q = V_p \wedge V_q - \langle V_p, V_q \rangle + S_p S_q + S_p V_q + S_q V_p \tag{2}$$

where $\langle , \rangle$ and $\wedge$ are the inner-product and the cross-product on $\mathbb{R}^3$, respectively. The conjugate of the quaternion q is denoted by αq and defined as $\alpha q = S_q - V_q$.

The $h-$inner-product of two quaternions is defined as

$$h(p, q) = \frac{1}{2}(p \times \alpha q + q \times \alpha p), \quad p, q \in Q_{\mathbb{R}} \tag{3}$$

The real number $[h(p, p)]^{1/2}$ is called the norm of the real quaternion p and is denoted by $\|p\|$. Hence we obtain that

$$\|p\|^2 = h(p, q) = a^2 + b^2 + c^2 + d^2. \tag{4}$$

It is easy to see that, if $p = a_1e_1 + b_1e_2 + c_1e_3 + d_1$ and $q = a_2e_1 + b_2e_2 + c_2e_3 + d_2$, then

$$h(p, q) = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 \text{ [1].} \tag{5}$$

Given two real quaternions p and p^*, we define the dual quaternion as P=p+ξ p^* and denote the set of dual quaternions by $Q_{\mathbb{D}}$. For given $A, B, C, D \in \mathbb{D}$, we can write $P = Ae_1 + Be_2 + Ce_3 + D$. Here $S_P = D$ is called the scalar part of P and $V_P = Ae_1 + Be_2 + Ce_3$ is called the vector part of P.

The multiplication of two dual quaternions P and Q is defined as

$$P \times Q = p \times q + \xi\, (p \times q^* + p^* \times q) \tag{6}$$

where $P = p + \xi\, p^*$ and $Q = q + \xi q^*$ and $\times$ shows the real quaternion multiplication. It is clear that

$$P \times Q = S_P S_Q + S_P V_Q + S_Q V_P - \langle V_P, V_Q \rangle + V_P \wedge V_Q \tag{7}$$

where $\langle , \rangle$ is the inner product and $\wedge$ is the cross-product on $\mathbb{D}^3$. If $P = S_P + V_P$, then the conjugate of P is defined by $\alpha P = S_P - V_P$. By using this definition, the following properties can be easily proved:

$$\begin{array}{ll} (i) & \alpha(\alpha P) = P, \\ (ii) & \alpha(P \times Q) = \alpha Q \times \alpha P. \end{array}$$

The symmetric dual-valued bilinear form H is defined as

$$H(P, Q) = \frac{1}{2}(P \times \alpha Q + Q \times \alpha P). \tag{8}$$

As a result, we obtain the followings :

1- For all elements P, Q of $Q_{\mathbb{D}}$, we have

$$H(P,Q) = h(p,q) + \xi[h(p,q^*) + h(p^*,q)]$$

where h is the symmetric real-valued bilinear form.

2- If $P = Ae_1 + Be_2 + Ce_3 + D$, then we have

$$H(P,P) = A^2 + B^2 + C^2 + D^2.$$

3- $\forall\ P \in Q_{\mathbb{D}}$, the norm of P is defined by

$$\|P\| = \|p\| + \xi \frac{h(p,p^*)}{\|p\|},$$

and so

$$\|P\|^2 = H(P,P) = P \times \alpha P. \tag{9}$$

4- $\forall\ P \in Q_{\mathbb{D}}$, the scalar part and the vector part of P is

$$S_P = \frac{1}{2}(P + \alpha P), \qquad V_P = \frac{1}{2}(P - \alpha P).$$

As a result,

$$\begin{array}{ll} (i) & \text{if } P + \alpha P = 0, \text{ then } P \in \mathbb{D} - module, \\ & \text{in this case, } P \text{ is called dual-spatial quaternion} \\ (ii) & \text{if } P - \alpha P = 0, \text{ then } P \in \mathbb{D}, \\ & \text{in this case, } P \text{ is called dual-temporal quaternion.} \end{array}$$

Let P and Q be two dual-spatial quaternion. If $H(P,Q) = 0$, we say that P and Q are $H-$orthogonal[19].

A semi-real quaternion consists of a set of four ordered real numbers a, b, c, d associated with four units e_1, e_2, e_3 and 1, respectively. The three units e_1, e_2 and e_3 have the following properties:

$$\begin{array}{llll} i) & & e_i \times e_i = -\varepsilon(e_i), & 1 \leq i \leq 3 \\ ii) & \text{in } \mathbb{R}_1^3, & e_i \times e_j = \varepsilon(e_i)\varepsilon(e_j)e_k & 1 \leq i,j,k \leq 3, \\ iii) & \text{in } \mathbb{R}_2^4, & e_i \times e_j = -\varepsilon(e_i)\varepsilon(e_j)e_k, & 1 \leq i,j,k \leq 3, \end{array} \tag{10}$$

where (ijk) is the even permutation of (123).

Notice here that,

$$\varepsilon(e_i) = \begin{cases} -1 & , \quad e_i \text{ timelike} \\ +1 & , \quad e_i \text{ spacelike} \end{cases}.$$

As a notation, we denote the semi-real quaternions by Q_ν with an index $\nu = 1, 2$ such that

$$Q_\nu = \left\{ \begin{array}{lll} q & | & q = ae_1 + be_2 + ce_3 + d, \quad a,b,c,d \in \mathbb{R} \\ & & e_1, e_2, e_3 \in \mathbb{R}_1^3, \quad h_\nu(e_i,e_i) = \varepsilon(e_i), \quad 1 \leq i \leq 3 \end{array} \right\}.$$

The multiplication of two semi-real quaternions p and q is defined as

$$p \times q = V_p \wedge V_q - \langle V_p, V_q \rangle + S_p S_q + S_p V_q + S_q V_p$$

where $\langle , \rangle$ and $\wedge$ are the inner-product and the cross-product on $\mathbb{R}_1^3$, respectively. The conjugate of the quaternion q is denoted by αq and defined as $\alpha q = S_q - V_q$.

For every $p, q \in Q_\nu$, the $h-$inner-product $h_\nu : Q_\nu \times Q_\nu \longrightarrow \mathbb{D}$ of p and q is defined as:

$$h_1(p,q)=\frac{1}{2}[\varepsilon(p)\varepsilon(\alpha q)(p\times\alpha q)+\varepsilon(q)\varepsilon(\alpha p)(q\times\alpha p)]\qquad\text{for }\mathbb{R}^3_1$$

and

$$h_2(p,q)=\frac{-1}{2}[\varepsilon(p)\varepsilon(\alpha q)(p\times\alpha q)+\varepsilon(q)\varepsilon(\alpha p)(q\times\alpha p)]\qquad\text{for }\mathbb{R}^4_2.$$

The real number $[h_\nu(p,p)]^{1/2}$ is called the norm of semi-real quaternion p and is denoted by $\|p\|$. Hence we see that

$$\|p\|^2=|h_\nu(p,p)|=|\varepsilon(p)(p\times\alpha p)|.$$

Given $q\in Q_\nu$, if $q+\alpha q=0$, then q is called semi-real spatial quaternion. If $q-\alpha q=0$, q is called semi-real temporal quaternion. The set of semi-real spatial quaternions is isomorphic to $\mathbb{R}^3_1$.

In general, we can write that

$$q=\frac{1}{2}[q+\alpha q]+\frac{1}{2}[q-\alpha q].$$

For $p,q\in Q_\nu$, if $h(p,q)=0$, p and q are called $h-$orthogonal. If the norm of q is unit, then it is called unit semi-real quaternion and denoted by q_0. So,

$$N_q=\sqrt{|q\times\alpha q|}=\sqrt{|-a^2-b^2+c^2+d^2|}$$

and

$$q_0=\frac{q}{N_q}=\frac{ae_1+be_2+ce_3+d}{\sqrt{|-a^2-b^2+c^2+d^2|}}$$

([8],[20]).

Let p and p^* be two semi-real quaternions. We define the semi-dual quaternion as $P=p+\xi\ p^*$ and denote the set of semi-dual quaternions by $Q_{\mathbb{D},\nu}$ with an index $\nu=1,2$ such that

$$Q_{\mathbb{D},\nu}=\left\{\ P\mid\quad P=Ae_1+Be_2+Ce_3+D,\quad A,B,C,D\in\mathbb{D},\ e_1,e_2,e_3\in\mathbb{R}^3_1\ \right\}.$$

We will use $H_1(e_i,e_i)=\varepsilon_i$, $i=0,1,2$ for $\mathbb{D}^3_1$ and $H_2(e_i,e_i)=\varepsilon(e_i)$, $i=0,1,2,3$ for $\mathbb{D}^4_2$. The multiplication of two dual quaternions P and Q is defined as

$P\times Q=p\times q+\xi\,(p\times q^*+p^*\times q)$ where $P=p+\xi\ p^*$ and $Q=q+\xi q^*$ and $\times$ shows the quaternion multiplication. It is clear that

$$P\times Q=S_PS_Q+S_PV_Q+S_QV_P-\langle V_P,V_Q\rangle+V_P\wedge V_Q \tag{12}$$

where $\langle,\rangle$ is the inner product and $\wedge$ is the cross-product on $\mathbb{D}^3_1$. If $P=S_P+V_P$, then the conjugate of P is defined by $\alpha P=S_P-V_P$. By using this, the following properties can be easily proved:

$$\begin{array}{ll}(i) & \alpha(\alpha P)=P,\\ (ii) & \alpha(P\times Q)=\alpha Q\times\alpha P.\end{array}$$

For every $P,Q\in Q_{\mathbb{D},\nu}$, we define the symmetric dual-valued bilinear form $H_\nu:Q_{\mathbb{D},\nu}\times Q_{\mathbb{D},\nu}\longrightarrow\mathbb{D}$ as

$$H_1(P,Q)=\frac{1}{2}[\varepsilon(P)\varepsilon(\alpha Q)(P\times\alpha Q)+\varepsilon(Q)\varepsilon(\alpha P)(Q\times\alpha P)]\qquad\text{for }\mathbb{D}^3_1 \tag{13}$$

and

$$H_2(P,Q) = \frac{-1}{2}[\varepsilon(P)\varepsilon(\alpha Q)(P\times\alpha Q) + \varepsilon(Q)\varepsilon(\alpha P)(Q\times\alpha P)] \qquad \text{for } \mathbb{D}_2^4. \tag{14}$$

The following results may be obtained:

1- For all elements P, Q of $Q_{\mathbb{D},\nu}$, we have

$$H_\nu(P,Q) = h_\nu(p,q) + \xi[h_\nu(p,q^*) + h_\nu(p^*,q)]$$

where h is the symmetric real-valued bilinear form.

2- If $P = Ae_1 + Be_2 + Ce_3 + D$, then we have

$$H_\nu(P,P) = -A^2 - B^2 + C^2 + D^2.$$

3- $\forall\ P \in Q_{\mathbb{D},\nu}$, the norm of P is defined by

$$\|P\| = \|p\| + \xi\frac{h_\nu(p,p^*)}{\|p\|}$$

and so

$$\|P\|^2 = |H_\nu(P,P)| = |\varepsilon(P)(P\times\alpha P)|. \tag{15}$$

4- $\forall\ P \in Q_{\mathbb{D},\nu}$, the scalar part and the vector part of P are

$$S_P = \frac{1}{2}(P+\alpha P), \qquad V_P = \frac{1}{2}(P-\alpha P).$$

As a result,

(i) if $P + \alpha P = 0$, then $P \in \mathbb{D} - module$,
in this case, P is called semi-dual-spatial quaternion

(ii) if $P - \alpha P = 0$, then $P \in \mathbb{D}$,
in this case, P is called semi-dual-temporal quaternion.

Let P and Q be two semi-dual spatial quaternion. If $H_\nu(P,Q) = 0$, we say that P and Q are $H_\nu-$orthogonal.

Now, we give the Serret-Frenet formulas for a non-null semi-dual quaternionic curve in $\mathbb{D}_1^3$.

Consider the smooth curve $\beta \subset \mathbb{D}_1^3$, $\{\beta \in Q_\nu \mid \beta + \alpha\beta = 0\}$ given by

$$\begin{array}{ccccl} \beta & : & I \subset \mathbb{R} & \longrightarrow & Q_\nu \subset \mathbb{D}_1^3 \\ & & s & \longrightarrow & \beta(s) = \sum\limits_{i=1}^{3} \beta_i(s)e_i. \end{array}$$

Let s be the parameter along β. For any $s \in I$, if $\{t(s),\, n_1(s),\, n_2(s)\}$ is the Serret-Frenet frame and $k(s)$, $r(s)$ are the curvatures, then we have the following formulas

$$\begin{array}{rcl} t' & = & \varepsilon(n_1)kn_1 \\ n_1' & = & \varepsilon(t)[\varepsilon(t)\varepsilon(n_1)rn_2 - kt] \\ n_2' & = & -\varepsilon(n_2)rn_1 \end{array} \tag{16}$$

where $t(s) = t + \xi t^*$, $n_1(s) = n_1 + \xi n_1^*$ and $n_2(s) = n_2 + \xi n_2^*$ with the Serret-Frenet frame $\{t(s), n_1(s), n_2(s)\}$ of $\mathbb{R}_1^3$.

If a curve is a non-null semi-dual quaternionic curve, then the Serret-Frenet formulas in $\mathbb{D}_2^4$ are defined as following :

Consider the smooth curve $\gamma \subset \mathbb{D}_2^4$,

$$\begin{array}{rcccl} \gamma & : & I & \longrightarrow & Q_{\mathbb{D},\nu} \subset \mathbb{D}_2^4 \\ & & s & \longrightarrow & \gamma(s) = \sum\limits_{i=1}^{4} \beta_i(s) e_i, \quad e_4 = 1. \end{array}$$

with $\beta_4(s)e_4 = D(s)$, $D(s) = d(s) + \xi d^*(s)$. For any $s \in I$, if $\{T(s),\ N_1(s),\ N_2(s),\ N_3(s)\}$ is the Serret-Frenet frame of dual-split quaternionic curve, then

$$\begin{array}{rcl} T' & = & \varepsilon(N_1)KN_1 \\ N_1' & = & \varepsilon(n_1)kN_2 - \varepsilon(N_1)\varepsilon(t)KT \\ N_2' & = & -\varepsilon(t)kN_1 + \varepsilon(n_1)[r - \varepsilon(T)\varepsilon(t)\varepsilon(N_1)K]N_3 \\ N_3' & = & -\varepsilon(n_2)[r - \varepsilon(T)\varepsilon(t)\varepsilon(N_1)K]N_2 \end{array} \tag{17}$$

where $T(s) = T + \xi T^*$, $N_1(s) = N_1 + \xi N_1^*$, $N_2(s) = N_2 + \xi N_2^*$ and $N_3(s) = N_3 + \xi N_3^*$ with the Serret-Frenet frame $\{T(s),\ N_1(s),$
$N_2(s),\ N_3(s)\}$ of $\mathbb{R}_2^4$ and $K = \varepsilon(N_1)\left\| T' \right\|$ [7].

3. THE INVOLUTES OF THE SEMI-DUAL CURVES IN $\mathbf{D}_1^3$

Definition 3.1. Let $M_1,\ M_2 \subset \mathbb{D}_1^3$ be two curves which are given by (I, β) and (I, β^*) coordinate neighbourhoods, respectively. Let Frenet frame of M_1 and M_2 be $\{t, n_1, n_2\}$ and $\{t^*, n_1^*, n_2^*\}$, respectively. M_2 is called the involute of M_1(M_1 is called the evolute of M_2) if

$$H_1(t, t^*) = 0. \tag{18}$$

Theorem 3.1. *Let $(M_1,\ M_2)$ be the involute-evolute curve couple which are given by (I, β) and (I, β^*) coordinate neighbourhoods, respectively. The distance between the points $\beta(s) \in M_1$ and $\beta^*(s^*) \in M_2$ is given by*

$$d(\beta(s), \beta^*(s)) = \varepsilon_0 \left| c - s \right|, \quad c\text{=}\textit{dual constant.}$$

Proof. If M_2 is the involute of M_1, we have

$$\beta^*(s) = \beta(s) + \lambda(s)t(s) \tag{19}$$

Let us derivate both side with respect to s:

$$\frac{d\beta^*}{ds} = \frac{d\beta}{ds} + \frac{d\lambda}{ds}t + \lambda\frac{dt}{ds} \tag{20}$$

Because of $\dfrac{dt}{ds} = t' = \varepsilon_1 k n_1$,

$$\frac{d\beta^*}{ds} = (1 + \frac{d\lambda}{ds})t + \lambda\varepsilon_1 k n_1 \tag{21}$$

where s and $s^*$are arc parameters of M_1 and M_2, respectively.

Thus we have

$$t^*\frac{ds^*}{ds} = (1 + \frac{d\lambda}{ds})t + \lambda\varepsilon_1 k n_1. \tag{22}$$

By using the equation (22), we have

$$H_1(t, t^*)\frac{ds^*}{ds} = (1 + \frac{d\lambda}{ds})H_1(t, t) + \lambda\varepsilon_1 k H_1(t, n_1). \tag{23}$$

From the definition of the involute-evolute curve couple, $H_1(t, t^*) = 0$. Thus we obtain

$$1 + \frac{d\lambda}{ds} = 0 \text{ and } \lambda = c - s, \quad c = \text{dual constant.} \tag{24}$$

From the definition of the distance on Lorentzian space, we easily find

$$\begin{aligned} d(\beta(s),\, \beta^*(s)) &= \|\beta^*(s) - \beta(s)\| \\ &= \varepsilon_0 \, |c - s| \, . \end{aligned} \tag{25}$$

□

Theorem 3.2. *Let $(M_1,\, M_2)$ be the involute-evolute curve couple which are given by (I, β) and (I, β^*) coordinate neighbourhoods, respectively. Let Frenet frames of M_1 and M_2 in the points $\beta(s) \in M_1$ and $\beta^*(s^*) \in M_2$ be $\{t, n_1, n_2\}$ and $\{t^*, n_1^*, n_2^*\}$, respectively. For the curvature and torsion of curve M_2, we have*

$$k^* = \frac{\varepsilon_1^*}{(c - s)k} \sqrt{|\varepsilon_0 k^2 + \varepsilon_2 r^2|}.$$

Proof. If M_2 is the involute of M_1, we have

$$\beta^*(s) = \beta(s) + \lambda(s) t(s).$$

Let us derivate both side with respect to s. From equations (22) and (24), we obtain

$$t^* \frac{ds^*}{ds} = (c - s)\, \varepsilon_1 k n_1 \tag{26}$$

where s and $s^*$are arc parameters of M_1 and M_2, respectively. We can find

$$\frac{ds^*}{ds} = \varepsilon_0^* \varepsilon_1 \, (c - s)\, k. \tag{27}$$

Thus we have

$$t^* = \varepsilon_0^* n_1. \tag{28}$$

Hence $\{t^*(s),\, n_1(s)\}$ is linear dependent. That's why we consider that

$$t^*(s) = n_1(s). \tag{29}$$

By derivating t^* and using equations (16), (27) and (29), then we get

$$\varepsilon_1^* k^* n_1^* = \frac{\varepsilon_0^* \varepsilon_1}{(c - s)\, k} \left[\varepsilon_0 [\varepsilon_0 \varepsilon_1 r n_2 - k t]\right]. \tag{30}$$

Then, by the norm of the both side of the equation (30), we have

$$k^* = \frac{\varepsilon_1^*}{(c - s)k} \sqrt{|\varepsilon_0 k^2 + \varepsilon_2 r^2|}. \tag{31}$$

□

Theorem 3.3. *Let $(M_1,\, M_2)$ be the involute-evolute curve couple which are given by (I, α) and (I, β) coordinate neighbourhoods,respectively. Let Frenet frames of M_1 and M_2 in the points $\beta(s) \in M_1$ and $\beta^*(s^*) \in M_2$ be $\{t, n_1, n_2\}$ and $\{t^*, n_1^*, n_2^*\}$,* *respectively, and let the curvature and torsion of curves M_1 and M_2 be k, r and k^*, r^*, respectively. We have*

$$\begin{aligned} n_1^* &= \frac{\varepsilon_1^*\varepsilon_0^*}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2+\varepsilon_0 k^2)|}}\left[\varepsilon_0(\varepsilon_0\varepsilon_1 r n_2 - kt)\right] \\ n_2^* &= \frac{1}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2+\varepsilon_0 k^2)|}}(\varepsilon_2 rt+\varepsilon_1 k n_2) \\ r^* &= \frac{\varepsilon_2(k'r-kr')}{(\varepsilon_2 r^2+\varepsilon_0 k^2)(c-s)k}. \end{aligned}$$

Proof. By using equation (30) and (31), we get

$$n_1^* = \frac{\varepsilon_1^*\varepsilon_0^*}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2+\varepsilon_0 k^2)|}}\left[\varepsilon_0(\varepsilon_0\varepsilon_1 r n_2 - kt)\right]. \tag{32}$$

From $n_2 = \varepsilon_0\varepsilon_1\,(t\times n_1)$, we find

$$\begin{aligned} n_2^* &= \varepsilon_0^*\varepsilon_1^*\,(t^*\times n_1^*) \\ n_2^* &= \varepsilon_0^*\varepsilon_1^*\left(n_1 \times \frac{\varepsilon_1^*\varepsilon_0^*}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2+\varepsilon_0 k^2)|}}\left[\varepsilon_0(\varepsilon_0\varepsilon_1 r n_2 - kt)\right]\right) \\ n_2^* &= \frac{1}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2+\varepsilon_0 k^2)|}}(\varepsilon_2 rt+\varepsilon_1 k n_2). \qquad (3.1) \end{aligned}$$

By derivating n_2^* and using this result in equation (16), we obtain

$$r^* = \frac{\varepsilon_0^*\varepsilon_1^*\varepsilon_2^*\varepsilon_2(k'r-kr')}{(\varepsilon_2 r^2+\varepsilon_0 k^2)(c-s)k}. \tag{34}$$

□

4. THE INVOLUTES OF THE SEMI-DUAL CURVES IN $\mathbb{D}_2^4$

Definition 4.1. Let M_1, $M_2 \subset \mathbb{D}_2^4$ be two curves which are given by (I,β) and (I,β^*) coordinate neighbourhoods, respectively. Let Frenet frame of M_1 and M_2 be $\{T, N_1, N_2, N_3\}$ and $\{T^*, N_1^*, N_2^*, N_3^*\}$, respectively. M_2 is called the involute of M_1(M_1 is called the evolute of M_2) if

$$H_2(T,T^*) = 0. \tag{35}$$

Theorem 4.1. *Let $(M_1,\, M_2)$ be the involute-evolute curve couple which are given by (I,γ) and (I,γ^*) coordinate neighbourhoods, respectively. The distance between the points $\gamma(s)\in M_1$ and $\gamma^*(s^*)\in M_2$ is given by*

$$d(\gamma(s),\, \gamma^*(s)) = |c-s|\,, \quad c = \text{dual constant}.$$

Proof. If M_2 is the involute of M_1, we have

$$\gamma^*(s) = \gamma(s) + \lambda(s)T(s). \tag{36}$$

Let us derivate both side with respect to s:

$$\frac{d\gamma^*}{ds} = \frac{d\gamma}{ds} + \frac{d\lambda}{ds}T + \lambda\frac{dT}{ds}. \tag{37}$$

Because of $\dfrac{dT}{ds} = T^{'} = \varepsilon(N_1)KN_1$,

$$\frac{d\gamma^*}{ds} = (1 + \frac{d\lambda}{ds})T + \lambda\varepsilon(N_1)KN_1 \tag{38}$$

where s and $s^*$are arc parameters of M_1 and M_2, respectively. Thus we have

$$T^*\frac{ds^*}{ds} = (1 + \frac{d\lambda}{ds})T + \lambda\varepsilon(N_1)KN_1. \tag{39}$$

Taking inner product with t this equation's both side, we have

$$H_2(T,T^*)\frac{ds^*}{ds} = (1 + \frac{d\lambda}{ds})H_2(T,T^*) + \lambda\varepsilon_1 KH(T,N_1). \tag{40}$$

From the definition of the involute-evolute curve couple, $H_2(T,T^*) = 0$. Thus we obtain

$$1 + \frac{d\lambda}{ds} = 0 \text{ and } \lambda = c - s, \quad \text{c=dual constant.} \tag{41}$$

From the definition of the distance on Lorentzian space, we easily find

$$\begin{aligned} d(\gamma(s),\, \gamma^*(s)) \quad &= \|\gamma^*(s) - \gamma(s)\| \\ &= |c - s|\,. \end{aligned} \tag{42}$$

□

Theorem 4.2. *Let $(M_1,\, M_2)$ be the involute-evolute curve couple which are given by (I,γ) and (I,γ^*) coordinate neighbourhoods, respectively. Let Frenet frames of M_1 and M_2 in the points $\gamma(s) \in M_1$ and $\gamma^*(s^*) \in M_2$ be $\{T,N_1,N_2,N_3\}$ and $\{T^*,N_1^*,N_2^*,N_3^*\}$, respectively. For the curvature and torsion of curve M_2, we have*

$$K^*(s^*) = \frac{|\varepsilon(N_1^*)|\,\sqrt{|\varepsilon(N_2)k^2 + \varepsilon(T)K^2|}}{(c-s)K}.$$

Proof. If M_2 is the involute of M_1, we have

$$\gamma^*(s) = \gamma(s) + \lambda(s)T(s).$$

Let us derivate both side with respect to s. From equations (39) and (41), we obtain

$$T^*\frac{ds^*}{ds} = \varepsilon(N_1)\,(c-s)\,KN_1 \tag{43}$$

where s and $s^*$are arc parameter of M_1 and M_2, respectively. We can find

$$\frac{ds^*}{ds} = |\varepsilon(T^*)|\,|(c-s)|\,K. \tag{44}$$

Thus we have

$$T^* = |\varepsilon(T^*)|\,\varepsilon(N_1)N_1. \tag{45}$$

Hence $\{T^*(s),\, N_1(s)\}$ is linear dependent. We consider that

$$T^*(s) = N_1(s). \tag{46}$$

By derivating T^* and using equations (17), (44) and (46), then we get

$$\varepsilon(N_1^*)K^*N_1^* = \frac{|\varepsilon(T^*)|}{|(c-s)|\,k}\left(\varepsilon_1 kN_2 - \varepsilon(N_1)\varepsilon_1 KT\right). \tag{47}$$

Then, by the norm of the both side of equation (47), we have

$$K^*(s^*) = \frac{|\varepsilon(N_1^*)| \sqrt{|\varepsilon(N_2)k^2 + \varepsilon(T)K^2|}}{(c-s)K}. \tag{48}$$

□

Theorem 4.3. *Let $(M_1,\ M_2)$ be the involute-evolute curve couple which are given by (I,γ) and (I,γ^*) coordinate neighbourhoods, respectively. Let Frenet frames of M_1 and M_2 in the points $\gamma(s) \in M_1$ and $\gamma^*(s^*) \in M_2$ be $\{T, N_1, N_2, N_3\}$ and $\{T^*, N_1^*, N_2^*, N_3^*\}$, respectively, and let the curvature and torsion of curves M_1 and M_2 be $K,\ r, k$ and $K^*,\ r^*,\ k$,respectively. we have*

$$\begin{aligned}
N_1^* &= \frac{\varepsilon(N_1^*)\,|\varepsilon(N_1)|}{\sqrt{|\varepsilon(N_2)k^2 + \varepsilon(T)K^2|}}\left(\varepsilon_1 k N_2 - \varepsilon(N_1)\varepsilon_1 K T\right)\\
N_2^* &= \frac{\varepsilon(T^*)}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2 + \varepsilon_0 k^2)|}}(\varepsilon_2\varepsilon_0 r N_2 + \varepsilon(T)kT)\\
N_3^* &= \frac{\varepsilon(T^*)\varepsilon_2\varepsilon_0}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2 + \varepsilon_0 k^2)|}}(-\varepsilon(T)T + \varepsilon_1 k N_2).
\end{aligned}$$

Proof. By using equations (47) and (48), we get

$$N_1^* = \frac{\varepsilon(N_1^*)\,|\varepsilon(N_1)|}{\sqrt{||\varepsilon(N_2)|\,k^2 + |\varepsilon(T)|\,K^2|}}\left(\varepsilon_1 k N_2 - \varepsilon(N_1)\varepsilon_1 K T\right). \tag{49}$$

From equalities $N_2 = \varepsilon(T)\,(n_1 \times T)$, $n_2 \times N_1 = \varepsilon_1\varepsilon_2 N_2$ and $t \times N_1 = -\varepsilon(T)T$, we find

$$\begin{aligned}
N_2^* &= \varepsilon(T^*)\,(n_1^* \times T^*)\\
N_2^* &= \frac{\varepsilon(T^*)\varepsilon_1^*\varepsilon_0^*}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2 + \varepsilon_0 k^2)|}}(\varepsilon_2\varepsilon_0 r N_2 + \varepsilon(T)kT).50
\end{aligned} \tag{4.1}$$

If we use similar step as equation (50) and equality $N_3 = \varepsilon(T)\,(n_2 \times T)$, then

$$\begin{aligned}
N_3^* &= \varepsilon(T^*)\,(n_2^* \times T^*)\\
N_3^* &= \frac{\varepsilon(T^*)\varepsilon_2\varepsilon_0}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2 + \varepsilon_0 k^2)|}}(-\varepsilon(T)T + \varepsilon_1 k N_2).
\end{aligned}$$

□

References

[1] Bharathi, K. and Nagaraj, M. Quaternion valued function of a real variable Serret-Frenet formula, Indian Journal of Pure and Applied Mathematics 18: (1987), 507-511.

[2] Bilici, M. and Çalışkan, M., On the Involutes of the Spacelike Curve with a Timelike Binormal in Minkowski 3-Space, International Mathematical Forum, 4 no 31 (2009), 1497-1509.

[3] Blaschke, W., Diferensiyel Geometri Dersleri, İstanbul Üniversitesi Yayınları, 1949.

[4] Boyer, C., A History of Mathematics, New York: Wiley, 1968.

[5] Bükcü, B. and Karacan, M.K., On the Involute and Evolute Curves of the Spacelike Curve with a Spacelike Binormal in Minkowski 3-space, Int. J. Math. Sciences, 2(5): (2007), 221-232.

[6] Clifford, W. K., Preliminary skecth of biquaternions, Proceedings of London Math. Soc. 4, (1873), 361-395.

[7] Çöken, A.C., Ekici, C., Kocayusufoğlu, İ. and Görgülü, A., Formulas for dual split quaternionic curves, Kuwait J. Sci. Eng.1A(36): (2009), 1-14

[8] Çöken, A.C. and Tuna, A., On the quaternionic inclined curves in the semi-Euclidean space $\mathbb{E}_2^4$, Applied Mathematics and Computation 155(2): (2004), 373-389.

[9] do Carmo, M.P., Differential Geometry of Curves and Surfaces, 1976.
[10] Hacısalihoğlu, H. H., Hareket Geometrisi ve Kuaterniyonlar Teorisi, Gazi Üniversitesi, Fen-Edebiyat Fakultesi Yayinlari 2, 1983.
[11] Inoguchi, J., Timelike surfaces of constant mean curvature in Minkowski 3-space, Tokyo Journal of Mathematics 21(1): (1998), 141-152.
[12] Keçilioğlu, O. and Gündoğan, H., Dual split quaternions and motions in Lorentz space $\mathbb{R}^3_1$, Far East Journal of Mathematical Sciences (FJMS) 24(3): (2007), 425-437.
[13] Kobayashi, S. and Nomizu, K., Foundations of differential geometry, Vol. I, John Wiley Sons Inc. Lcccn: (1963), 63-19209.
[14] Kühnel, W., Differential Geometry, Curves-Surfaces-Manifolds, American Mathematical Society, 2002.
[15] Lopez, R., Differential geometry of curves and surfaces in Lorentz-Minkowski space, Mini-Course taught at the Instituto de Matematica e Estatistica (IME-USP), University of Sao Paulo, Brasil, 2008.
[16] Nizamoglu, Ş., Surfaces réglées parallèles, Ege Üniv. Fen Fak. Derg., 9 (Ser. A), (1986), 37-48.
[17] O'Neill, B., Semi Riemannian Geometry with Applications to Relativity, Academic Press, Inc. New York, 1983.
[18] O'Neill, B., Elementary Differential Geometry, Academic Press, Inc. New York, 2006.
[19] Özyılmaz, E. and Yılmaz, S., Involute-Evolute Curve Couples in the Euclidean 4-Space, Int. J. Open Problems Compt.Math., vol.2 No.2, (2009).
[20] Özdemir, M. and Ergin, A. A., Rotations with unit timelike quaternions in Minkowski 3-space, Journal of Geometry and Physics 56: (2006), 322-336.
[21] Sivridağ, A.İ., Güneş, R. and Keleş, S., The Serret-Frenet formulae for dual-valued functions of a single real variable, Mechanism and Machine Theory 29: (1994), 749-754.
[22] Study, E., Geometrie der Dynamen, Leipzig, Teubner, 1903.
[23] Turgut, M. and Yilmaz,S., On The Frenet Frame and A Characterization of space-like Involute-Evolute Curve Couple in Minkowski Space-time, Int. Math. Forum 3(16): (2008), 793-801.
[24] Uğurlu, H.H. and Çalışkan , A., The study mapping for directed space-like and time-like line in Minkowski 3-space $\mathbb{R}^3_1$, Mathematical and ComputationalApplications 1(2): (1996), 142-148.
[25] Veldkamp, G. R., On the use of dual numbers, vectors and matrices in instantaneous spatial kinematics, Mechanism and Machine Theory 11: (1976), 141-156.
[26] Willmore, T.J., Riemannian Geometry, Published in the United States by Oxford University Press Inc., Newyork, 1993.

Eskişehir Osmangazi University, Department of Mathematics-Computer, 26480, Eskişehir - Turkey, e-mail: cekici@ogu.edu.tr

Eskişehir Osmangazi University, Department of Mathematics-Computer, 26480, Eskişehir - Turkey, e-mail: hatice.tozak@gmail.com

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 202–210 (2015) ©KJM

WEAK SOLUTIONS VIA LAGRANGE MULTIPLIERS FOR CONTACT MODELS WITH NORMAL COMPLIANCE

ANDALUZIA CRISTINA MATEI

Abstract. We consider a 3D elastostatic frictional contact problem with normal compliance, which consists of a systems of partial differential equations associated with a displacement boundary condition, a traction boundary condition and a frictional contact boundary condition. The frictional contact is modeled by means of a normal compliance condition and a version of Coulomb's law of dry friction. After we state the problem and the hypotheses, we deliver a variational formulation as a mixed variational problem with solution-dependent Lagrange multipliers set. Next, we prove the existence and the boundedness of the weak solutions.

1. Introduction

The present work focuses on a 3D elastostatic frictional contact problem with normal compliance. A normal compliance condition was firstly proposed in [11].

Then, the contact with normal compliance was involved in many models, see e.g. the papers [2, 7, 8, 9, 18].

The model we discuss herein consists of a system of partial differential equations associated with a displacement boundary condition, a traction boundary condition and a frictional contact boundary condition. The frictional contact is modeled by means of a normal compliance condition and a version of Coulomb's law of dry friction. This model was already analyzed in the frame of quasivariational inequalities,

$$a(u, v-u) + j(u, v) - j(u, u) \geq (f, v-u)_X;$$

for details see [19] and the references therein. The novelty in the present paper consists in the variational approach we adopt; herein, we propose a mixed variational formulation in a form of a generalized saddle point problem with solution

Date: January 1, 2013 and, in revised form, February 2, 2013.
2000 *Mathematics Subject Classification.* 74M10, 74M15,49J40.
Key words and phrases. normal compliance contact condition, Coulomb's law of dry friction, Lagrange multiplier.
This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-RU-TE-2011-3-0223.

dependent Lagrange multipliers set $\Lambda = \Lambda(u)$,

$$\begin{aligned} a(u,v) + b(v,\lambda) &= (f,\, v)_X && \text{for all } \ v \in X, \\ b(u, \mu - \lambda) &\leq 0 && \text{for all } \ \mu \in \Lambda(u); \end{aligned}$$

Let us refer to [3, 4] for basic elements on the saddle point theory. For recent papers related to mixed variational formulations in contact mechanics see e.g. [6, 13, 14, 15].

The mixed variational formulations are related to modern numerical techniques in order to approximate the weak solutions of contact models and this motivates the research on this direction. Referring to numerical techniques for approximating weak solutions of contact problems via saddle point technique, we send the reader to, e.g., [5, 20, 21].

The main goal of the present paper is to prove the existence and the boundedness of the weak solutions of the considered model, via Lagrange multipliers technique. The results we obtain rely on the abstract results in [12] which, for the convenience of the reader, will be recalled below, in Section 2.

The problem we analyze in the present paper can be viewed as a new application to the abstract results in [12]. A first application was delivered in the antiplane framework, see [12]. A second application was presented in the conference paper [17], for a 3D bilateral contact model with slip-dependent friction (see also [16] for an extended and improved version of the conference paper [17]).

The structure of the paper is as follows. In Section 2 we present abstract auxiliary results. In Section 3 we state the problem and we fix the hypotheses. Then, in Section 4 we prove the existence and the boundedness of the weak solutions of the frictional contact model with normal compliance.

2. Abstract auxiliary results

Let us consider the following abstract mixed variational problem.

Problem 1. *Given $f \in X$, $f \neq 0_X$, find $(u,\lambda) \in X \times Y$ such that $\lambda \in \Lambda(u) \subset Y$ and*

$$a(u,v) + b(v,\lambda) = (f,\, v)_X \qquad \textit{for all } \ v \in X, \tag{2.1}$$

$$b(u, \mu - \lambda) \leq 0 \qquad \textit{for all } \ \mu \in \Lambda(u). \tag{2.2}$$

We made the following assumptions.

Assumption 1. *$(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ are two Hilbert spaces.*

Assumption 2. *$a(\cdot,\,\cdot) : X \times X \to \mathbb{R}$ is a symmetric bilinear form such that*
(i_1) there exists $M_a > 0 :\ |a(u,v)| \leq M_a \|u\|_X \|v\|_X$ for all $u, v \in X$,
(i_2) there exists $m_a > 0 :\ a(v,v) \geq m_a \, \|v\|_X^2$ for all $v \in X$.

Assumption 3. *$b(\cdot,\,\cdot) : X \times Y \to \mathbb{R}$ is a bilinear form such that*
(j_1) there exists $M_b > 0 :\ |b(v,\mu)| \leq M_b \|v\|_X \|\mu\|_Y$ for all $v \in X$, $\mu \in Y$,
(j_2) there exists $\alpha > 0 :$ $\displaystyle \inf_{\mu \in Y, \mu \neq 0_Y} \ \sup_{v \in X, v \neq 0_X} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha.$

Assumption 4. *For each $\varphi \in X$, $\Lambda(\varphi)$ is a closed convex subset of Y such that $0_Y \in \Lambda(\varphi)$.*

Assumption 5. *Let $(\eta_n)_n \subset X$ and $(u_n)_n \subset X$ be two weakly convergent sequences, $\eta_n \rightharpoonup \eta$ in X and $u_n \rightharpoonup u$ in X, as $n \to \infty$.*

(k_1) *For each $\mu \in \Lambda(\eta)$, there exists a sequence $(\mu_n)_n \subset Y$ such that $\mu_n \in \Lambda(\eta_n)$ and* $\liminf_{n\to\infty} b(u_n, \mu_n - \mu) \geq 0$.

(k_2) *For each subsequence $(\Lambda(\eta_{n'}))_{n'}$ of the sequence $(\Lambda(\eta_n))_n$, if $(\mu_{n'})_{n'} \subset Y$ such that $\mu_{n'} \in \Lambda(\eta_{n'})$ and $\mu_{n'} \rightharpoonup \mu$ in Y as $n' \to \infty$, then $\mu \in \Lambda(\eta)$.*

Theorem 2.1. *If Assumptions* 1-5 *hold true, then Problem 1 has a solution. In addition, if $(u, \lambda) \in X \times \Lambda(u)$ is a solution of Problem 1, then*

$$(u, \lambda) \in K_1 \times \big(\Lambda(u) \cap K_2\big),$$

where

$$\begin{aligned} K_1 &= \{v \in X \,|\, \|v\|_X \leq \frac{1}{m_a}\|f\|_X\}; \\ K_2 &= \{\mu \in Y \,|\, \|\mu\|_Y \leq \frac{m_a + M_a}{\alpha\, m_a}\|f\|_X\}, \end{aligned}$$

m_a, α and M_a being the constants in Assumptions 2-3.

For the proof of this theorem we refer to [12].

3. The model and hypotheses

3.1. **The statement of the problem.** We consider the following 3D frictional contact model with normal compliance.

Problem 2. *Find* : $\overline{\Omega} \to \mathbb{R}^3$ *and* : $\overline{\Omega} \to \mathbb{S}^3$ *such that*

(3.1) $\mathrm{Div}() +_0 () = \mathbf{0}$ *in* Ω,

(3.2) $() = \mathcal{E}(())$ *in* Ω,

(3.3) $() =$ *on* Γ_1,

(3.4) $() =_2 ()$ *on* Γ_2,

(3.5) $-\sigma_\nu() = p_\nu(u_\nu() - g_a)$ *on* Γ_3,

(3.6) $\|_\tau()\| \leq p_\tau(, u_\nu() - g_a)$,

$$_\tau() = -p_\tau(, u_\nu() - g_a)\,\frac{_\tau()}{\|_\tau()\|} \ \textit{if}\ _\tau() \neq \ \textit{on}\ \Gamma_3.$$

Herein Ω is a bounded domain in $\mathbb{R}^3$, Γ_1, Γ_2, Γ_3 is a partition of the boundary $\partial\Omega := \Gamma$, $\overline{\Omega} = \Omega \cup \Gamma$, $_0 : \Omega \to \mathbb{R}$ denotes the density of the volume forces, $_2 : \Gamma_2 \to \mathbb{R}$ represents the density of the tractions, $= ()$ denotes the infinitesimal strain tensor ($\varepsilon_{i\,j} = (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ for all $i, j \in \{1, 2, 3\}$) and $\mathcal{E}$ denotes the elastic operator. Here and everywhere below $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^3$ or $\mathbb{S}^3$. Finally, $u_\nu = \cdot$, $_\tau = -u_\nu$, $\sigma_\nu = ()\cdot$, $_\tau = -\sigma_\nu$, where " $\cdot$ " denotes the inner product of two vectors and is the unit outward normal vector.

Problem 2 has the following structure: (3.1) represents the equilibrium equation, (3.2) represents a constitutive law for linearly elastic materials, (3.3) represents the displacements boundary condition, (3.4) represents the traction boundary condition and (3.5)-(3.6) models the frictional contact with normal compliance, the friction law in (3.6) being a version of the Coulomb law of dry friction, where p_τ is a given nonnegative function. In the normal compliance contact condition (3.5) p_ν is a nonnegative prescribed function which vanishes for negative argument and $g_a > 0$

denotes the gap. When $u_\nu < g_a$ there is no contact and the normal pressure vanishes. When there is contact then $u_\nu - g_a$ is positive and represents a measure of the interpenetration of the asperities. Then, condition (3.5) shows that the foundation exerts a pressure on the body which depends on the penetration.

3.2. **Assumptions.** In order to weakly solve Problem 2 we make the following assumptions.

Assumption 6. $\mathcal{E} = (\mathcal{E}_{ijls}) : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3$,

- $\mathcal{E}_{ijls} = \mathcal{E}_{ijsl} = \mathcal{E}_{lsij} \in L^\infty(\Omega)$,
- *There exists* $m_\mathcal{E} > 0$ *such that* $\mathcal{E}_{ijls}\varepsilon_{ij}\varepsilon_{ls} \geq m_\mathcal{E}\,\|\|^2,\ \in \mathbb{S}^3,$ *a.e. in* Ω.

Assumption 7. $_0 \in L^2(\Omega)^3, \quad _2 \in L^2(\Gamma_2)^3$.

Assumption 8. $p_\nu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$;

- *there exists* $L_\nu > 0 : |p_\nu(, r_1) - p_\nu(, r_2)| \leq L_\nu\,|r_1 - r_2| \quad r_1, r_2 \in \mathbb{R}_+,$ *a.e.* $\in \Gamma_3$;
- *the mapping* $\mapsto p_\nu(, r)$ *is Lebesgue measurable on* Γ_3, *for all* $r \in \mathbb{R}_+$;
- $p_\nu(, r) = 0$ *for all* $r \leq 0$ *a.e.* $\in \Gamma_3$.

Assumption 9. $p_\tau : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$;

- *there exists* $L_\tau > 0 : |p_\tau(, r_1) - p_\tau(, r_2)| \leq L_\tau\,|r_1 - r_2| \quad r_1, r_2 \in \mathbb{R}_+,$ *a.e.* $\in \Gamma_3$;
- *the mapping* $\mapsto p_\tau(, r)$ *is Lebesgue measurable on* Γ_3, *for all* $r \in \mathbb{R}_+$;
- $p_\tau(, r) = 0$ *for all* $r \leq 0$ *a.e.* $\in \Gamma_3$.

3.3. **Weak formulation.** Let us introduce the following functional space.

$$V = \{ \in H^1(\Omega)^3 \mid\ = 0 \text{ a.e. on } \Gamma_1\}. \tag{3.7}$$

This is a Hilbert space endowed with the following inner product

$$(,)_V = \int_\Omega (()) : (())\,dx,$$

where " : " denotes the inner product of two tensors.

Everywhere in this paper, for each $\in V$, we denote $w_\nu = \cdot$ and $_\tau = -w_\nu$ a.e. on Γ, where denotes the Sobolev trace operator for vectors.

Define $\in V$ using Riesz's representation theorem,

$$(,)_V = \int_\Omega {}_0() \cdot ()\,dx + \int_{\Gamma_2} {}_2() \cdot ()\,d\Gamma \quad \text{ for all } v \in V. \tag{3.8}$$

Let be a sufficiently regular solution of Problem 2. By a Green formula we get

$$a(,) = (,)_V + \int_{\Gamma_3} () \cdot ()\,d\Gamma \quad \text{ for all } \in V \tag{3.9}$$

where

$$a(\cdot,\cdot) : V \times V \to \mathbb{R} \quad a(,) = \int_\Omega \mathcal{E}(()) : (())\,dx. \tag{3.10}$$

Let us introduce the spaces

$$S = \{|_{\Gamma_3} \quad \in V\}; \tag{3.11}$$

$$D = S'. \tag{3.12}$$

For each $\in S$, we denote $\zeta_\nu = \cdot$ and $_\tau = -\zeta_\nu$ a.e. on Γ_3.

Notice that $|_{\Gamma_3}$ denotes the restriction of the trace of the element $\in V$ to Γ_3. Thus, $S \subset H^{1/2}(\Gamma_3; \mathbb{R}^3)$ where $H^{1/2}(\Gamma_3; \mathbb{R}^3)$ is the space of the restrictions on Γ_3 of traces on Γ of functions of $H^1(\Omega)^3$. On S we consider the Sobolev-Slobodeckii norm

$$\| \|_S = \Big(\int_{\Gamma_3} \int_{\Gamma_3} \frac{\|() - ()\|^2}{\| - \|^3}\, ds_x\, ds_y \Big)^{1/2};$$

see e.g. [1, 10].

For each $\in V$ we define

$$\Lambda() = \{ \in D \,|\, \langle , |_{\Gamma_3} \rangle \leq \int_{\Gamma_3} (p_\nu(, \varphi_\nu() - g_a)|v_\nu()| + p_\tau(, \varphi_\nu() - g_a)\|_\tau()\|) d\Gamma \quad \in V \}; \tag{3.13}$$

here and below $\langle \cdot, \cdot \rangle$ denotes the duality pairing between D and S.

Let us define a Lagrange multiplier $\in S$,

$$\langle , \rangle = - \int_{\Gamma_3} () \cdot ()\, d\Gamma. \tag{3.14}$$

Thus, for all $\in V$,

$$\langle , |_{\Gamma_3} \rangle = - \int_{\Gamma_3} (\sigma_\nu() v_\nu() +_\tau () \cdot_\tau ()) d\Gamma.$$

By (3.14) and (3.13) we deduce that $\in \Lambda()$.

We also define

$$b : V \times D \to \mathbb{R} \quad b(,) = \langle , |_{\Gamma_3} \rangle. \tag{3.15}$$

Let us rewrite (3.9) as

$$a(,) = (,)_V \; - \langle , |_{\Gamma_3} \rangle \quad \text{for all } \in V.$$

By the definition of the form $b(\cdot, \cdot)$, we obtain

$$a(,) + b(,) = (,)_V \quad \text{for all } \in V. \tag{3.16}$$

On the other hand, the normal compliance condition (3.5) leads us to the identity

$$\int_{\Gamma_3} \sigma_\nu() u_\nu()\, d\Gamma = - \int_{\Gamma_3} p_\nu(, u_\nu() - g_a)|u_\nu()|\, d\Gamma$$

while the friction law (3.6) leads us to the identity

$$\int_{\Gamma_3} {}_\tau() \cdot_\tau ()\, d\Gamma = - \int_{\Gamma_3} p_\tau(, u_\nu() - g_a)\|_\tau()\|\, d\Gamma.$$

Thus,

$$b(,) = \int_{\Gamma_3} (p_\nu(, u_\nu() - g_a)|u_\nu(x)| + p_\tau(, u_\nu() - g_a)\|_\tau()\|) d\Gamma. \tag{3.17}$$

By (3.13) with = we are led to

$$b(,) \leq \int_{\Gamma_3} (p_\nu(, u_\nu() - g_a)|u_\nu()| + p_\tau(, u_\nu() - g_a)\|_\tau()\|) d\Gamma \quad \text{for all } \in \Lambda(). \tag{3.18}$$

Subtract now (3.17) from (3.18) to obtain the inequality

$$b(, -) \leq 0 \quad \text{for all } \in \Lambda(). \tag{3.19}$$

Therefore, Problem 2 has the following weak formulation.

Problem 3. *Find* $\in V$ *and* $\in \Lambda() \subset S$ *such that* (3.16) *and* (3.19) *hold true.*

Each solution of Problem 3 is called *weak solution* of Problem 2.

3.4. **Existence and boundedness results.**

Theorem 3.1 (An existence result)**.** *If Assumptions 6 -9 hold true, then Problem 2 has a weak solution.*

Proof. As the spaces V and D, see (3.7) and (3.12) are real Hilbert spaces, then Assumption 1 is fulfilled with $X = V$ and $Y = D$.

The form $a(\cdot,\cdot)$ defined in (3.10) verifies Assumption 2 with

$$M_a = \|\mathcal{E}\|_\infty \text{ and } m_a = m_{\mathcal{E}}, \tag{3.20}$$

where

$$\|\mathcal{E}\|_\infty = \max_{0\le i,j,k,l\le d} \|E_{ijkl}\|_{L^\infty(\Omega)}.$$

Let us prove (j_1) in Assumption 3. We have

$$|b(,)| \le \|\|_D\|\|_{H_\Gamma}.$$

We recall that $H_\Gamma = \{\quad \in V\}$ and the Sobolev trace operator $: H^1(\Omega)^3 \to H_\Gamma$ is a linear and continuous operator. Since $\|\cdot\|_V$ and $\|\cdot\|_{H^1(\Omega)^3}$ are equivalent norms, we deduce that there exists $M_b > 0$ such that (j_1) holds true.

We also recall that there exists a linear and continuous operator $\mathcal{Z}$ such that

$$\mathcal{Z} : H_\Gamma \to H^1(\Omega)^3 \quad (\mathcal{Z}()) = \quad \text{for all } \in H_\Gamma.$$

The operator $\mathcal{Z}$ is called *the right inverse* of the operator . Obviously,

$$(\mathcal{Z}()) = \qquad \text{for all } \in V.$$

For every $\in V$, we denote by * an element of V such that $=^*$ a.e. on Γ_3 and $^* = 0$ a.e. on Γ_2. Therefore, $\||_{\Gamma_3}\|_S = \|^*\|_{H_\Gamma}$.

Since, for each $^* \in V$, $\mathcal{Z}(^*)$ has the same trace as *, we deduce that for each $^* \in V$, $\mathcal{Z}(^*) \in V$.

Let us prove now (j_2) in Assumption 3.

$$\begin{aligned}
\|\|_D &= \sup_{|_{\Gamma_3}\in S,\, |_{\Gamma_3}\neq 0_S} \frac{\langle, |_{\Gamma_3}\rangle}{\||_{\Gamma_3}\|_S} \\
&= \sup_{|_{\Gamma_3}\in S,\, |_{\Gamma_3}\neq 0_S} \frac{\langle,^* |_{\Gamma_3}\rangle}{\|^*\|_{H_\Gamma}} \\
&\le c \sup_{|_{\Gamma_3}\in S,\, |_{\Gamma_3}\neq 0_S} \frac{b(\mathcal{Z}(^*),)}{\|\mathcal{Z}(^*)\|_V} \\
&\le c \sup_{\in V,\, \neq_V} \frac{b(,)}{\|\|_V},
\end{aligned}$$

where $c > 0$. We can take

$$\alpha = \frac{1}{c}. \tag{3.21}$$

Obviously, $0_D \in \Lambda()$. Also, $\Lambda()$ is a closed convex subset of the space D. Hence, Assumption 4 is fulfilled.

Let us verify Assumption 5. To start, let $(_n)_n \subset V$ and $(_n)_n \subset V$ be two weakly convergent sequences, $_n \rightharpoonup$ in V and $_n \rightharpoonup$ in V, as $n \to \infty$. Let us take $\in \Lambda()$.

In order to check (k_1) in Assumption 5, we define $({}_n)_n$ as follows: for each $n \geq 1$,

$$
\begin{aligned}
\langle {}_n, \rangle \quad = \quad & \int_{\Gamma_3} p_\nu(, \eta_{n\,\nu}() - g_a)\, \mathrm{sgn}\, u_{n\,\nu}()\, \zeta_\nu()\, d\Gamma \\
& + \int_{\Gamma_3} p_\tau(, \eta_{n\,\nu}() - g_a)({}_{n\,\tau}()) \cdot_\tau ()\, d\Gamma \\
& - \int_{\Gamma_3} p_\nu(, \eta_\nu() - g_a)|u_{n\,\nu}()|\, d\Gamma \\
& - \int_{\Gamma_3} p_\tau(, \eta_\nu() - g_a)\|{}_{n\,\tau}()\|\, d\Gamma \\
& + \langle, {}_n|_{\Gamma_3}\rangle, \qquad \in S,
\end{aligned}
$$

where

$$
() = \begin{cases} \frac{}{\|\|} & \text{if } \neq; \\ & \text{if } =, \end{cases}
$$

and, as usually,

$$
\mathrm{sgn}(r) = \begin{cases} 1 & \text{if } r > 0; \\ 0 & \text{if } r = 0; \\ -1 & \text{if } r < 0. \end{cases}
$$

Taking into account (3.13), we deduce that, for each positive integer n, we have ${}_n \in \Lambda({}_n)$.

Since ${}_n \rightharpoonup$ in V and ${}_n \rightharpoonup$ in V as $n \to \infty$, we deduce that

$$
{}_{n\,\tau}() \to_\tau () \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty,
$$

$$
u_{n\,\nu}() \to u_\nu() \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty,
$$

$$
p_\nu(, \eta_{n\,\nu}() - g_a) \to p_\nu(, \eta_\nu() - g_a) \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty
$$

and

$$
p_\tau(, \eta_{n\,\nu}() - g_a) \to p_\tau(, \eta_\nu() - g_a) \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty.
$$

Setting $={}_n|_{\Gamma_3}$ we can write

$$
\begin{aligned}
\langle {}_n - , {}_n|_{\Gamma_3}\rangle = & \int_{\Gamma_3} (p_\nu(, \eta_{n\,\nu}() - g_a) - p_\nu(, \eta_\nu() - g_a))|u_{n\,\nu}()|\, d\Gamma \\
& + \int_{\Gamma_3} (p_\tau(, \eta_{n\,\nu}() - g_a) - p_\tau(, \eta_\nu() - g_a))\|_{\tau\, n}()\|\, d\Gamma.
\end{aligned}
$$

Hence, passing to the limit as $n \to \infty$, we get

$$
\begin{aligned}
& \liminf_{n\to\infty} b({}_n, {}_n -) = \lim_{n\to\infty} \int_{\Gamma_3} (p_\nu(, \eta_{n\,\nu}() - g_a) - p_\nu(, \eta_\nu() - g_a))|u_{n\,\nu}()|\, d\Gamma \\
& + \lim_{n\to\infty} \int_{\Gamma_3} (p_\tau(, \eta_{n\,\nu}() - g_a) - p_\tau(, \eta_\nu() - g_a))\|{}_{n\,\tau}()\|)d\Gamma \\
& = 0.
\end{aligned}
$$

Using again the properties of the trace operator and the assumptions on the friction bound we deduce that (k_2) in Assumption 5 is also verified.

We apply now Theorem 2.1. $\square$

Let us introduce the notation:

$$K_1 \quad = \quad \{\in V \mid \|\|_V \leq \frac{1}{m_a}\|\|_V\}; \tag{3.22}$$

$$K_2 \quad = \quad \{\in D \mid \|\|_D \leq \frac{m_a + M_a}{\alpha\, m_a}\|\|_V\}. \tag{3.23}$$

Theorem 3.2 (A boundedness result)**.** *If* $(,)$ *is a weak solution of Problem 2, then*

$$(,) \in K_1 \times \big(\Lambda() \cap K_2\big)$$

where K_1 *and* K_2 *are given by* (3.22)-(3.23)*,* V *given by* (3.7)*,* D *given by* (3.12)*, given by* (3.8)*,* m_a *and* M_a *being the constants in* (3.20) *and* α *being the constant in* (3.21)*.*

The proof is a straightforward consequence of Theorem 2.1.

References

[1] R. A. Adams. *Sobolev spaces*, Academic Press, 1975.

[2] L.-E. Andersson, A quasistatic frictional problem with normal compliance, *Nonlinear Analysis TMA* **16** (1991), 347-370.

[3] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, Classics in Applied Mathematics **28** SIAM, Philadelphia, PA, 1999.

[4] J. Haslinger, I. Hlaváček and J. Nečas, *Numerical Methods for Unilateral Problems in Solid Mechanics*, in "Handbook of Numerical Analysis", J.-L. L. P. Ciarlet, ed., **IV**, North-Holland, Amsterdam, 1996, 313–485.

[5] P. Hild, Y. Renard, A stabilized Lagrange multiplier method for the finite element approximation of contact problems in elastostatics. *Numer. Math.* **115** 101–129, 2010.

[6] S. Hüeber, A. Matei, B. Wohlmuth, A contact problem for electro-elastic materials, Journal of Applied Mathematics and Mechanics (ZAMM), *Z. Angew. Math. Mech.*, DOI: 10.1002/zamm.201200235, 93 (10-11) (2013), 789-800. Special Issue: Mathematical Modeling: Contact Mechanics, Phase Transitions, Multiscale Problems.

[7] N. Kikuchi and J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, 1988.

[8] A. Klarbring, A. Mikelič and M. Shillor, Frictional contact problems with normal compliance, *Int. J. Engng. Sci.* **26** (1988), 811–832.

[9] A. Klarbring, A. Mikelič and M. Shillor, A global existence result for the quasistatic frictional contact problem with normal compliance, in G. del Piero and F. Maceri, eds., *Unilateral Problems in Structural Analysis* Vol. **4**, Birkhäuser, Boston, 1991, 85-111.

[10] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes*, Dunod, Paris, 1968.

[11] J.A.C. Martins and J.T. Oden, Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws, *Nonlinear Analysis TMA*, **11** (1987), 407-428.

[12] A. Matei, On the solvability of mixed variational problems with solution-dependent sets of Lagrange multipliers, Proceedings of The Royal Society of Edinburgh, Section: A Mathematics, **143**(05) (2013), 1047-1059.

[13] A. Matei, Weak solvability via Lagrange multipliers for contact problems involving multi-contact zones, Mathematics and Mechanics of Solids, DOI: 10.1177/1081286514541577.

[14] A. Matei, An existence result for a mixed variational problem arising from Contact Mechanics, Nonlinear Analysis Series B: Real World Application, 20 (2014), 74-81.

[15] A. Matei, An evolutionary mixed variational problem arising from frictional contact mechanics, Mathematics and Mechanics of Solids, DOI: 10.1177/1081286512462168, 19(3) (2014), 225 - 241.

[16] A. Matei, Weak Solutions via Lagrange Multipliers for a Slip-dependent Frictional Contact Model, IAENG International Journal of Applied Mathematics, 44 (3), 2014, 151-156 (special issue WCE 2014-ICAEM).

[17] A. Matei, A mixed variational formulation for a slip-dependent frictional contact model, Lecture Notes in Engineering and Computer Science: *Proceedings of The World Congress*

on Engineering 2014, 2-4 July, 2014, London, U.K., pp 750-754 (ISBN: 978-988-19253-5-0, ISSN: 2078-0958).

[18] M. Rochdi, M. Shillor and M. Sofonea, Quasistatic viscoelastic contact with normal compliance and friction, *Journal of Elasticity* **51** (1998), 105–126.

[19] M. Sofonea and A. Matei, Mathematical Models in Contact Mechanics, London Mathematical Society, Lecture Note Series 398, 280 pages, Cambridge University Press, 2012.

[20] B. Wohlmuth, A Mortar Finite Element Method Using Dual Spaces for the Lagrange Multiplier, *SIAM Journal on Numerical Analysis*, **38**(2000), 989-1012.

[21] B. Wohlmuth, *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, in "Lecture Notes in Computational Science and Engineering", 17, Springer, 2001.

University of Craiova, Department of Mathematics, A.I. Cuza 13, 200585, Craiova, Romania

E-mail address: andaluziamatei@inf.ucv.ro

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 211–218 (2015) ©KJM

OSCILLATION OF A CLASS OF NONLINEAR DIFFERENCE EQUATIONS OF SECOND ORDER WITH OSCILLATING COEFFICIENTS

MUSTAFA KEMAL YILDIZ

Abstract. In this paper, we study asymptotic behaviour of solutions of the following second-order difference equation:

$$\Delta\big[a(n)\Delta\big[x(n)+r(n)F(x(n-\rho))\big]\big]+p(n)G\left(x(n-\tau)\right)-q(n)G\left(x(n-\sigma)\right)=s(n),$$

where $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $\{r(n)\}_{n\in\mathbb{N}_0}$ and $\{s(n)\}_{n\in\mathbb{N}_0}$ are sequences of real numbers, $\{p(n)\}_{n\in\mathbb{N}_0}$ and $\{q(n)\}_{n\in\mathbb{N}_0}$ are nonnegative sequences of real numbers, $\{a(n)\}_{n\in\mathbb{N}_0}$ is positive, $\rho,\tau,\sigma\geq 0$ are integers and F,G are continuous functions satisfying the usual sign condition; i.e., $F(u)/u, G(u)/u>0$ for $u\in\mathbb{R}\backslash\{0\}$. Various ranges of the sequence $\{r(n)\}_{n\in\mathbb{N}_0}$ are considered, and illustrating examples are provided to show applicability of the results.

1. Introduction

In the literature, all the papers concerning second-order equations deal with asymptotic behaviour of all solutions of delay difference equations have the following form:

$$\Delta\left[a(n)\Delta\left[x(n)+r(n)x(n-\rho)\right]\right]+p(n)x(n-\tau)=f(n),$$

where $n\in\mathbb{N}_0$, $\{r(n)\}_{n\in\mathbb{N}_0}$ is of single sign, $\{a(n)\}_{n\in\mathbb{N}_0}$ and $\{p(n)\}_{n\in\mathbb{N}_0}$ are nonnegative sequences of real numbers, $\rho,\tau\geq 0$ are integers and $\{f(n)\}_{n\in\mathbb{N}_0}$ is a sequence of real numbers (see [1, 2]). Here, the forward difference operator Δ is defined as $\Delta x(n):=x(n+1)-x(n)$ and $\Delta^2x(n):=\Delta\left[\Delta x(n)\right]$ for $n\in\mathbb{N}_0$.

In this paper, depending on the sign of the sequence $\{r(n)\}_{n\in\mathbb{N}_0}$, we investigate the oscillatory and asymptotic behavior of solutions of the second-order neutral

Date: **Address**: Department of Mathematics, Faculty of Science and Arts, ANS Campus, Afyon Kocatepe University, 03200 Afyonkarahisar, Turkey
Email: myildiz@aku.edu.tr
AMS 2000 Subject Classification: 39A10.
Keywords: Asymptotic behaviour, neutral difference equations, oscillating coefficients, positive and negative coefficients, second-order.

nonlinear difference equation with positive and negative coefficients having the following form:

(1.1)
$$\Delta\left[a(n)\Delta\left[x(n)+r(n)F\left(x(n-\rho)\right)\right]\right]+p(n)G\left(x(n-\tau)\right)-q(n)G\left(x(n-\sigma)\right)=s(n),$$

where $n\in\mathbb{N}_0$, $\{r(n)\}_{n\in\mathbb{N}_0}$ and $\{s(n)\}_{n\in\mathbb{N}_0}$ are allowed to oscillate, $\{p(n)\}_{n\in\mathbb{N}_0}$ and $\{q(n)\}_{n\in\mathbb{N}_0}$ are nonnegative, $\{a(n)\}_{n\in\mathbb{N}_0}$ is positive, $\rho,\tau,\sigma\geq 0$ are integers. To the best of our knowledge, in the literature, there is no work done on second-order difference equations involving oscillating coefficients inside the neutral part, and positive and negative coefficients outside the neutral part. Moreover, some of our results are not restricted with boundedness of the solutions. Also the readers are referred to the paper [3] which introduces a new method for

$$\Delta\left[a(n)\Delta\left[x(n)+r(n)x(n-\rho)\right]\right]+p(n)x(n-\tau)-q(n)x(n-\sigma)=s(n).$$

In [4], the authors study the following difference equation

$$\Delta\left[a(n)\Delta\left[x(t)+\sum_{i\in R}r_i(n)x(n-\rho_i)\right]\right]+\sum_{i\in P}p_i(n)x(n-\tau_i)-\sum_{i\in Q}q_i(n)x(n-\sigma_i)=f(n),$$

and state new results depending on three different ranges of the sequence $\left\{\sum_{i\in R}r_i(n)\right\}_{n\in\mathbb{N}_0}$. Our results here extend the results of [4] for nonlinear equations, also see the results in the paper [5] where the author gives results for the existence of positive solutions.

For the fundamentals on the oscillation theory, the readers are referred to the books [6, 7, 8].

Let $\delta:=\max\{\rho,\tau,\sigma\}$. As is usual, a solution x of (1.1) is a sequence of real numbers defined for all integers satisfying $n\geq-\sigma$, and satisfies (1.1) identically for all $n\in\mathbb{N}_0$. It is also known that (1.1) has a unique solution x if an initial sequence x_0 is given to hold $x(n)=x_0(n)$ for $n=-\delta,-\delta+1,\dots,1$. Throughout the paper, for convenience, we do not consider eventually null solutions of (1.1).

2. Main Results

In this section, we give sufficient conditions for (1.1) to be almost oscillatory, that is every solution of (1.1) oscillates or tends to zero at infinity. We state our primary assumptions as follows:

(H1) $0<F(u)/u\leq M$ and $N_1\leq G(u)/u\leq N_2$ for all $u\neq 0$ and some positive constants M, N_1, N_2,

(H2) There exists a pair of nonnegative real numbers r^-,r^+ such that either one the followings are true:
- {i} $-r^-\leq r(n)\leq r^+$ holds for all sufficiently large n , and that $[r^-+r^+]\,M<1$ holds,
- {ii} $r^-\leq r(n)\leq r^+$ holds for all sufficiently large n and, satisfying $Mr^->1$,
- {iii} $-r^-\leq r(n)\leq -r^+$ holds for all sufficiently large n and satisfying $Mr^+>1$,

(H3) $\sum_n^\infty(1/a(n))$ is divergent,

(H4) {i} $\delta\geq 1$ holds, where $\delta=\tau-\sigma$,
- {ii} $\{h(n)\}_{n\in\mathbb{N}_0}$ defined by $h(n):=p(n)-q(n-\delta)$ is an eventually positive sequence of reals,
- {iii} $\sum_n^\infty h(n)$ is divergent,

{iv} $\sum_n^\infty (1/a(n)) \sum_{k=n-\delta}^{n-1} q(k)$ is convergent,

(H5) There exists a sequence $\{S(n)\}_{n\in\mathbb{N}_0}$ such that $\lim_{n\to\infty} S(n)$ exists and $\Delta\left((1/a(n))\Delta S(n)\right) = s(n)$ holds for all $n \in \mathbb{N}_0$.

Theorem 2.1. *Assume that* (H1)*,* (H2){i}*,* (H3)*,* (H4){i–iv} *and* (H5) *hold, then every solution of* (1.1) *oscillates or tends to zero at infinity.*

Proof. Let (1.1) have a nonoscillatory solution x, which does not tend to zero at infinity. Without loss in the generality, we may suppose that x is eventually positive, the case where x is eventually negative is very similar and thus we omit. There exists $n_1 \in \mathbb{N}_0$ such that $x(n) > 0$ for all $n \geq n_1$. From (H2){i} and (H4){iv}, we may find $n_2 \geq n_1 + \delta$ such that

$$N_2 \sum_{n=n_2}^{\infty} \frac{1}{a(n)} \sum_{k=n-\delta}^{n-1} q(k) < \frac{1}{2}\left(1 - r^-\right) \tag{2.1}$$

holds. For $n \geq n_2$, set

$$y(n) := x(n) + r(n)F\left(x(n-\rho)\right) \tag{2.2}$$

and

$$z(n) := y(n) - \sum_{k=n_2}^{n-1} \frac{1}{a(k)} \sum_{l=k-\delta}^{k-1} q(l)G\left(x(l-\sigma)\right) - S(n). \tag{2.3}$$

Using the fact that x is a solution of (1.1) and (H4){i,ii}, we have

$$\begin{aligned}
\Delta w(n) =& \Delta\left[a(n)\Delta y(n)\right] - \left[q(n)G\left(x(n-\sigma)\right) - q(n-\delta)G\left(x(n-\tau)\right)\right] - s(n)\\
=& - p(n)F(x(n-\tau)) + q(n-\delta)G(x(n-\tau))\\
\leq& - p(n)G(x(n-\tau)) + q(n-\delta)G(x(n-\tau))\\
=& - \left[p(n) - q(n-\delta)\right]G(x(n-\tau))\\
=& - h(n)G\left(x(n-\tau)\right) \leq 0
\end{aligned} \tag{2.4}$$

for all $n \geq n_2$, where w is defined by $w(n) := a(n)\Delta z(n)$ for $n \geq n_2$. Clearly, w is eventually nonincreasing. Then, from (H4){ii,iii} and (2.4), we have either $w < 0$ or $w > 0$ for all $n \geq n_3$ for some $n_3 \geq n_2$. Consider the following possible ranges:

(C1) $w(n) < 0$ holds for all $n \geq n_3$. We first claim that

$$\lim_{n\to\infty} z(n) = -\infty \tag{2.5}$$

holds. Considering the definition of w, we may write

$$\Delta z(n) \leq \frac{w(n_3)}{a(n)} < 0 \tag{2.6}$$

for all $n \geq n_3$, which proves that (2.5) is true by summing up from n_3 to ∞ because of (H3). Hence, (H5) and (2.5) implies that

$$\lim_{n\to\infty}[z(n) + S(n)] = -\infty \tag{2.7}$$

holds. Next, we claim that x is bounded. If it is not the case, from (2.7), there exists $T \geq n_3$,

$$x(T) = \max\{x(n) : n_3 \leq n \leq T\} \text{ and } z(T) + S(T) < 0. \tag{2.8}$$

Therefore, considering (H2){i}, (2.1), (2.3) and (2.8), we obtain the following contradiction:

$$\begin{aligned}
0 >& z(T) + S(T) = y(T) - \sum_{k=n_2}^{N-1} \frac{1}{a(k)} \sum_{l=k-\delta}^{k-1} q(l) G\left(x(l-\sigma)\right) \\
\geq& x(T) - r^{-} x(T-\rho) - N_2 \sum_{k=n_2}^{N-1} \frac{1}{a(k)} \sum_{l=k-\delta}^{k-1} q(l) x(l-\sigma) \\
\geq& \left(1 - r^{-} - N_2 \sum_{k=n_2}^{N-1} \frac{1}{a(k)} \sum_{l=k-\delta}^{k-1} q(l)\right) x(T) \\
\geq& \frac{1}{2}\left(1 - r^{-}\right) x(T) \geq 0.
\end{aligned}$$

Thus, by (H2){i}, (H5), (2.1)–(2.3), we see that z is bounded. This is a contradiction to (2.5). Hence, this case is not possible.

(C2) $w(n) > 0$ for all $n \geq n_3$. In this case, we see that L is a nonnegative constant, where $L := \lim_{n\to\infty} w(n)$. Considering (H4){iii} and summing up (2.4) from n_3 to ∞, we obtain

$$\infty > w(n_3) - L = N_1 \sum_{n=n_3}^{\infty} h(n) x(n-\tau), \tag{2.9}$$

which implies that $\liminf_{n\to\infty} x(n) = 0$ and $\ell \in (0,\infty)$ are true, where $\ell := \limsup_{n\to\infty} x(n)$. Note that, z has limit at infinity because $\Delta z > 0$ holds since $a > 0$ holds. Because of the boundedness of x, monotonicity of z, (H4){iv}, (H5) and (2.3), we infer that y has a finite limit at infinity. Now, we prove the contradiction that $\ell = 0$ holds. For this purpose, pick two increasing divergent sequences of integers $\{\zeta_n\}_{n\in\mathbb{N}_0}, \{\xi_n\}_{n\in\mathbb{N}_0}$ such that $\lim_{n\to\infty} x(\zeta_n) = \ell$ and $\lim_{n\to\infty} x(\xi_n) = 0$ hold. Without loss in the generality, we may suppose that $\lim_{n\to\infty} x(\zeta_n - \rho)$ and $\lim_{n\to\infty} x(\xi_n - \rho)$ exist because of the boundedness of x, and it is trivial that all these limits are not greater than ℓ. From (2.2), we can estimate that

$$\begin{aligned}
y(\zeta_n) - y(\xi_n) =& x(\zeta_n) + r(\zeta_n) F\left(x(\zeta_n - \rho)\right) - \left[x(\xi_n) + r(\xi_n) F\left(x(\xi_n - \rho)\right)\right] \\
\geq& x(\zeta_n) - r^{-} F\left(x(\zeta_n - \rho)\right) - \left[x(\xi_n) + r^{+} F\left(x(\xi_n - \rho)\right)\right] \\
\geq& x(\zeta_n) - r^{-} M x(\zeta_n - \rho) - x(\xi_n) - r^{+} M x(\xi_n - \rho)
\end{aligned}$$

is true for all $n \in \mathbb{N}_0$, which yields the inequality

$$0 \geq \left(1 - \left(r^{-} + r^{+}\right) M\right) \ell$$

by letting n tend to infinity, and this implies that $\ell = 0$ holds by (H2){i}. A contradiction.

Contradictions appear in both possible distinct cases. Hence, every solution of (1.1) oscillates or tends to zero at infinity. □

Theorem 2.2. *Assume that* (H1), (H2){ii}, (H3), (H4){i–iv}, *and* (H5) *hold, then every solution of* (1.1) *oscillates or tends to zero at infinity.*

Proof. Assume that (1.1) has an eventually positive solution x, which does not tend to zero at infinity. Pick $n_1 \in \mathbb{N}_0$ such that $x(n) > 0$ for all $n \geq n_1$. From (H2){ii} and (H4){iv}, we may find $n_2 \geq n_1 + \delta$ such

$$M \sum_{n=n_2}^{\infty} \frac{1}{a(n)} \sum_{k=n-\delta}^{n-1} q(k) < \frac{1}{2} \tag{2.10}$$

holds. Set y, z and w as in the proof of Theorem 2.1, then we have (2.4) for all $n \geq n_2$ for some $n_2 \geq n_1$. It is not hard to prove that $w < 0$ is not possible by following the steps in (C1) of the proof of Theorem 2.1. Then, by following the steps in (C2) of the proof of Theorem 2.1, we learn that $\liminf_{n\to\infty} x(n) = 0$ and $\ell \in (0, \infty)$ are true, where ℓ is the superior limit of x, and y has a finite limit at infinity. Now, we show that $\ell = 0$ holds. Pick two increasing divergent sequences of integers $\{\xi_n\}_{n\in\mathbb{N}_0}, \{\zeta_n\}_{n\in\mathbb{N}_0}$ as in the proof of Theorem 2.1. Without loss in the generality, we may suppose that $\lim_{n\to\infty} x(\xi_n + \rho)$ and $\lim_{n\to\infty} x(\zeta_n + \rho)$ exist. We can estimate that

$$\begin{aligned} y(\xi_n + \rho) - y(\zeta_n + \rho) =& x(\xi_n + \rho) + r(\xi_n + \rho)F\left(x(\xi_n)\right) - \left[x(\zeta_n + \rho) + r(\zeta_n + \rho)F\left(x(\zeta_n)\right)\right] \\ \leq& x(\xi_n + \rho) + r^+ M x(\xi_n) - r^- M x(\zeta_n) \end{aligned}$$

is true for all $n \in \mathbb{N}_0$, which yields to the inequality

$$0 \leq \left(1 + r^- M\right) \ell$$

by letting n tend to infinity, and this implies that $\ell = 0$ by (H2){ii}. This is a contradiction. Hence, every solution of (1.1) oscillates or tends to zero at infinity. $\square$

The proof of the following theorem is very similar to that of Theorem 2.2, and thus we omit.

Theorem 2.3. *Assume that* (H3)*,* (H2){iii}*,* (H3)*,* (H4){i–iv} *and* (H5) *hold, then every bounded solution of* (1.1) *oscillates or tends to zero at infinity.*

3. Applications

To illustrate the applicability of our main results in § 2, we give the following examples.

Example 3.1. Consider the following neutral nonlinear difference equation:

$$\begin{aligned} \Delta^2 \left[x(n) + \frac{2}{5}(-1)^n \frac{x(n-2)|x^3(n-2)|}{|x^3(n-2)|+1}\right] &+ \frac{n}{n^2+1} \frac{x(n-3)\left(|x^3(n-3)|+1\right)}{|x^3(n-3)|+3} \\ &- \frac{1}{3^n} \frac{x(n-1)\left(|x^3(n-1)|+1\right)}{|x^3(n-1)|+3} = \frac{2}{(n+1)(n+2)(n+3)}. \end{aligned} \tag{3.1}$$

For this equation, we see that $a(n) \equiv 1$, $r(n) = 2(-1)^n/5$, $\rho = 2$, $p(n) = 1/(n+1)$, $\tau = 3$, $q(n) = 1/3^n$, $\sigma = 1$, Hence, we have $r^- = r^+ = 2/5$, $r^- + r^+ = 4/5 < 1$, $\delta = \tau - \sigma = 3 - 1 = 2$, $h(n) = p(n) - q(n-\delta) = 1/(n+1) - 1/3^{n-2} \to 0^+$ as $n \to \infty$ and $S(n) = 1/\left(n+1\right)$ for $n \in \mathbb{N}_0$. It is not hard to see that

$$\sum_{n}^{\infty} h(n) = \sum_{n}^{\infty} \left(\frac{n}{n^2+1} - \frac{1}{3^{n-2}}\right) = \infty,$$

and

$$\sum_{n=2}^{\infty}\left(\sum_{k=n-\delta}^{n-1} q(k)\right) = \sum_{n=2}^{\infty}\left(\sum_{k=n-2}^{n-1} \frac{1}{3^k}\right) = 2$$

are true. Therefore, all conditions of Theorem 2.1 are satisfied, and thus every solution of (3.1) oscillates or tends to zero at infinity. The following graphic belongs to the solution with the initial conditions $x(-3) = x(-2) = x(-1) = x(0) = x(1) = 1$ and of 70 iterates:

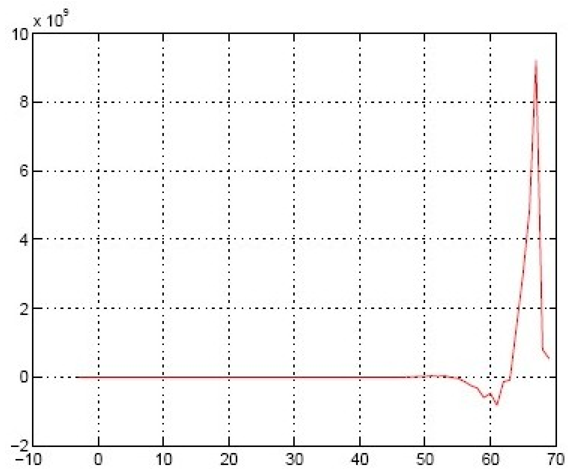


Figure 1. Graphic of $(n, x(n))$

Next, we give another example.

Example 3.2. Consider the following neutral nonlinear difference equation:

$$\begin{aligned}\Delta\left[\frac{1}{n}\Delta\left[x(n) + 3\frac{x(n-3)|x(n-3)|}{|x(n-3)|+1}\right]\right] &+ \frac{n^2}{n^3+1}\frac{x(n-2)\left(|x(n-2)|+3\right)}{|x(n-2)|+5} \\ &- \frac{1}{7^n}\frac{x(n-1)\left(|x(n-1)|+3\right)}{|x(n-1)|+5} = \frac{2}{(n+2)(n+3)(n+4)}.\end{aligned} \tag{3.2}$$

For this equation, we see that $a(n) = 1/n$, $r(n) = 3$, $\rho = 3$, $p(n) = n^2/(n^3+1)$, $\tau = 2$, $q(n) = 1/7^n$, $\sigma = 1$. Hence, we have $r^- = r^+ = 3 > 1$, $\delta = \tau - \sigma = 2 - 1 = 1$, $h(n) = p(n) - q(n-\delta) = n^2/(n^3+1) - 1/7^{n-1} \to 0^+$ as $n \to \infty$ and $S(n) = 1/(n+2)$ for $n \in \mathbb{N}_0$. It is not hard to see that

$$\sum_{n}^{\infty} h(n) = \sum_{n}^{\infty}\left(\frac{n^2}{n^3+1} - \frac{1}{7^{n-1}}\right) = \infty,$$

and

$$\sum_{n=1}^{\infty}\left(\frac{1}{a(n)}\sum_{k=n-\delta}^{n-1} q(k)\right) = \sum_{n=1}^{\infty}\frac{n}{7^{n-1}} = \frac{49}{36}$$

are true. Therefore, all conditions of Theorem 2.2 are satisfied, and thus every solution of (3.2) oscillates or tends to zero at infinity. The following graphic belongs to the solution with the initial conditions $x(-3) = x(-2) = x(-1) = x(0) = x(1) = 1$ and of 70 iterates:

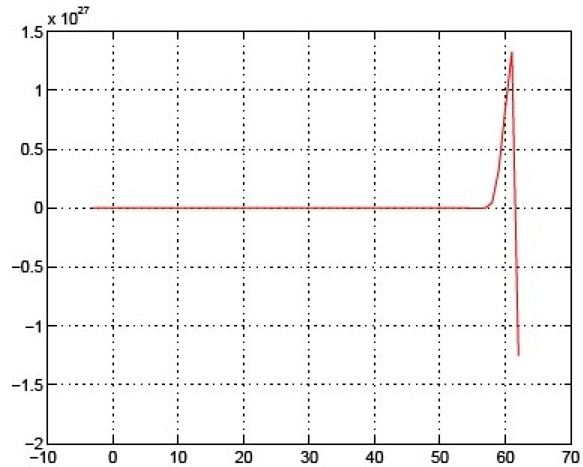


Figure 2. Graphic of $(n, x(n))$

Next, we give another example.

Example 3.3. Consider the following neutral nonlinear difference equation:

$$\Delta\left[\frac{1}{n}\Delta\left[x(n)-2\frac{x^3(n-1)}{x^2(n-1)+1}\right]\right] + \frac{n^2}{n^3+1}\frac{x(n-3)(x^2(n-3)+2)}{x^2(n-3)+3} - \frac{1}{5^n}\frac{x(n-2)(x^2(n-2)+2)}{x^2(n-2)+3} = 0. \tag{3.3}$$

For this equation, we see that $a(n)=1/n$, $r(n)=1/4$, $p(n)=n^2/(n^3+1)$, $\tau=3$, $q(n)=1/5^n$, $\sigma=1$. Hence, we have $r^+=2$, $r_1^-=2$, $\delta=\tau-\sigma=3-2=1$, $h(n)=p(n)-q(n-\delta)=n^2/(n^3+1)-1/5^{n-1}\to 0^+$ as $n\to\infty$ and $S(n)\equiv 0$ for $n\in\mathbb{N}_0$. Also, one can shown that

$$\sum_{n}^{\infty} h(n) = \sum_{n}^{\infty}\left(\frac{n^2}{n^3+1}-\frac{1}{5^{n-1}}\right)=\infty,$$

and

$$\sum_{n=1}^{\infty}\left(\frac{1}{a(n)}\sum_{k=n-\delta}^{n-1} q(k)\right)=\sum_{n=1}^{\infty}\frac{n}{5^{n-1}}=\frac{25}{16}$$

hold. Therefore, all bounded solutions of (3.3) oscillate or tend to zero at infinity by Theorem 2.3.

The following graphic probably belongs to an unbounded solution with the initial conditions $x(-3)=x(-2)=x(-1)=x(0)=x(1)=1$ and of 75 iterates:

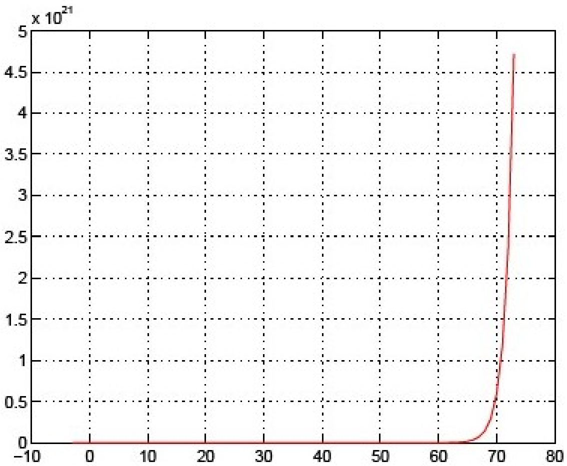


Figure 3. Graphic of $(n, x(n))$

Thus, the equation may also admit unbounded solutions.

References

[1] M. Budinčević, Oscillations and the asymptotic behaviour of certain second order neutral difference equations, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., vol. 21, no. 1, pp. 165–172, (1991).

[2] B. G. Zhang, Oscillation and asymptotic behavior of second order difference equations, J. Math. Anal. Appl., vol. 173, no. 1, pp. 58–68, (1993).

[3] H. A. El-Morshedy, New oscillation criteria for second order linear difference equations with positive and negative coefficients, (submitted).

[4] B. Karpuz, Ö. Öcalan, M. K. Yıldız, Oscillation of a class of difference equations of second order, Math. Comput. Model., vol. 49, no. 5-6, pp. 912–917, (2009).

[5] J. Cheng, Existence of nonoscillatory solution of second order linear neutral difference equation, Appl. Math. Lett., vol. 20, no. 8, pp. 892–899, (2007).

[6] R. P. Agarwal, Difference equations and inequalities, Marcel Dekker, New York, 1992.

[7] L. H. Erbe, Q. Kong and B. G. Zhang, Oscillation theory for functional differential equations, Marcel Dekker, 1995.

[8] I. Gyori and G. Ladas, Oscillation theory of delay differential equations with applications, Clarendon Press, Oxford, 1991.

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 219–244 (2015) ©KJM

OPTIMAL SURPLUS, MINIMUM PENSION BENEFITS AND CONSUMPTION PLANS IN A MEAN-VARIANCE PORTFOLIO APPROACH FOR A DEFINED CONTRIBUTION PENSION SCHEME

CHARLES I. NKEKI

Abstract. In this paper, we study the problem of simultaneous maximization of the value of expected terminal surplus and, minimization of risks associated with the terminal surplus in a defined contribution (DC) pension scheme. The surplus, which is discounted, is solved with dynamic programming techniques. The pension plan member (PPM) makes a flow of contributions from his or her stochastic salary into the scheme. The flow of contributions are invested into a market that is characterized by a cash account, an index bond and a stock. The efficient frontier for the discounted and real surplus are obtained. Optimal consumption of the PPM was found to depend on the terminal wealth, random evolution of minimum pension benefit and "variance minimizing" parameter. It was found that as the variance minimizing parameter, tends to zero, the optimal consumption tends to negative infinity. The optimal expected discounted and real surplus, optimal total expected pension benefits and expected minimum pension benefits were obtained. We found that the optimal portfolio depends linearly on the random evolution of PPM's minimum benefits. Some numerical examples of the results are established.

Keywords. pension scheme, mean-variance, stochastic funding, defined contribution, efficient frontier, surplus, minimum pension benefits, optimal consumption

AMS subject classifications. 91B28, 91B30, 91B70, 93E20.

1. Introduction

In this paper, we consider a mean variance portfolio selection problem for a defined contribution pension scheme. We study the optimal surplus process, minimum pension benefit and optimal total benefit that will accrued to a PPM at terminal time. The salary process of the PPM is assumed to be stochastic. The flow of contribution by by the PPM are invested into a market that is composed of cash

Date: Department of Mathematics, Faculty of Physical Sciences, University of Benin, P. M. B. 1154, Benin City, Edo State, Nigeria.
email: nkekicharles2003@yahoo.com.

account, index bond and stock. The real and nominal surplus for the stakeholders (i.e., PPM and PFA) are obtained. The consumption process of the PPM at time, t is examined in this paper. The optimal investment allocation strategy can be found by solving a mean and variance optimization problem, see Nkeki (2012). Optimal surplus, optimal pension benefits, minimum pension benefits and optimal consumption plan in a mean-variance portfolio selection approach for a defined contribution pension scheme are considered in this paper.

Haberman and Sung (1994), considered a defined benefit (DB) plans and modeled it as linear-quadratic optimal control problems. Markowitz (1952) studied a meanvariance optimization model and used it to compare securities and portfolios based in a tradeoff between their expected return and its variance. Colombo and Haberman (2005) and Huang and Cairns (2005) considered a mean-variance portfolio problem in pension plans from a static point. Chiu and Li (2006) studied a dynamic case of the model for asset and liability management under the meanvariance criteria. Josa-Fombellida and Rincon-Zapatero (2008) studied the benefits of the DB plan by assuming that the benefits are stochastic, modeled by a geometric Brownian motion. They assumed that benefit is a non-tradeable asset. They also considered the existence of correlation between the sources of uncertainty in the benefits and in the asset returns.

Our paper follows the work of Josa-Fombellida and Rincon-Zapatero (2008). In our own case, we study optimal surplus, minimum pension benefit, optimal total benefit and optimal consumption plan under the context of a defined contribution pension plan. We assume that the salary process of a PPM is stochastic and modeled by a geometric Brownian motion.

There are extensive literature that exist on the area of accumulation phase of DC pension plan and optimal investment strategies. This can be found in Cains *et.al* (2006), Deelstra *et.al* (2000), Korn and Krekel (2001), Blake *et.al* (2008), Battocchio and Menoncin (2004), Boulier *et.al* (2001), Di Giacinto *et.al* (2010), Haberman and Vigna (2002), Vigna (2010), Gao (2008), Devolder *et.al* (2003), Nkeki and Nwozo (2012), Nkeki (2013). For optimal portfolio and life-cycle of a PPM consumption plan, see Nkeki (2011), Nwozo and Nkeki (2011), Merton (1971).

In the context of DC pension plans, the problem of finding the optimal surplus, minimum pension benefits, total pension benefits, and optimal consumption plan, with stochastic funding in a DC pension scheme under mean-variance efficient approach has not been reported in published articles. Hϕjgaard and Vigna (2007) and Vigna (2010) assumed a constant flow of contributions into the pension scheme which will not be applicable to a time-dependent salary earners in pension scheme. We assume that the contribution of the PPM grows as the salary grows over time. In the literature, the problem of determining the minimum variance on trading strategy in continuous-time framework has been studied by Richardson (1989) via the Martingale approach. Bajeux-Besnainou and Portait (1998) used the same approach in a more general framework. Li and Ng (2000) solved a mean-variance optimization problem in a discrete-time multi-period framework. Zhou and Li (2000) considered a mean-variance in a continuous-time framework. They show the possibility of transforming the difficult problem of mean-variance optimization problem into a tractable one, by embedding the original problem into a stochastic linear-quadratic control problem, that can be solved using standard methods. These approaches have been extended and used by many in the financial literature, see for instance, Vigna (2010), Bielecky *et.al* (2005), Hϕjgaard and Vigna (2007), Chiu and Li (2006), Josa-Fombellida and Rincon-Zapatero (2008).

In this paper, we study a mean-variance approach to portfolio selection problem for optimal surplus, minimum pension benefits, total pension benefits and optimal consumption plan with stochastic salary of a PPM in accumulation phase of a DC pension scheme. Nkeki (2012) considered a mean-variance portfolio selection problem with inflation hedging strategy for a defined contributory pension scheme. The efficient frontier was obtained for three asset classes which include cash account, stock and index bond. The paper assumed that the flow of contributions of the PPM is stochastic. In this paper, we assumed that the salary of the PPM is stochastic.

The remainder of this paper is organized as follows. In section 2, we present financial market models. In section 3, we presents the pension benefits that will accrued to PPM. In section 4, we present the expected discounted flow of contributions, discounted wealth, discounted minimum pension benefit and discounted surplus. Section 5 presents the problem formulation of the paper. In section 6, we present the optimal portfolio and optimal consumption plan of a PPM. Section 7 presents the efficient frontier of the optimal terminal expected surplus. In section 8, we presents optimal pension benefit for a PPM at retirement. Section 9 presents the numerical examples of our models. Finally, section 10 concludes the paper.

2. Financial Market

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathbf{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0, T]\}$, where $\mathcal{F}_t = \sigma(W^I(s), W^S(s) : s \leq t)$, where $W^S(t)$ and $W^I(t)$ are Brownian motions with respect to stock and index bond at time t. The Brownian motions $W(t) = (W^I(t), W^S(t))'$, $0 \leq t \leq T$ is a 2-dimensional process, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}(\mathcal{F}), \mathbf{P})$, where $\mathbf{P}$ is the real world probability measure.

In this paper, we assume that the pension fund administrator (PFA) manage the fund contributed by the PPMs through pension fund custodians during the planning interval $[0, T]$ by means of a portfolio characterized by a cash account with price process, $B(t)$, index bond with price process, $Z(t, I(t))$ which is correlated geometric Brownian motion, generated by source of inflation risks, $W^I(t)$, where $I(t)$ is the price index at time t and has the dynamics: $dI(t) = j(t)I(t)dt + \sigma_1(t)I(t)dW^I(t)$, $j(t)$ the expected inflation index, which is the difference between nominal interest rate, $r(t)$, real interest rate $R(t)$ (i.e. $j(t) = r(t) - R(t) + \sigma_1(t)\theta_I(t)$) and $\sigma_I(t) = (\sigma_1(t), 0)$. $Z(t, I(t))$ is a zero-coupon bond which pays the price index at maturity, with a payoff

$$Z(t, I(t)) = E_t\left[I(T)\frac{\Lambda(T)}{\Lambda(t)}\right],$$

where

$$\Lambda(t) = B(t)^{-1}H(t)$$

and $H(t)$ satisfies the process

$$H(t) = \exp(-\theta'(t)W(t) - \frac{1}{2}\|\theta(t)\|^2), \tag{2.1}$$

which we assume to be martingale in $\mathbf{P}$, and a stock with price process, $S(t)$ correlated to geometric Brownian motions, $W^I(t)$ and $W^S(t)$, whose evolutions are respectively given by the equations:

$$dB(t) = r(t)B(t)dt, B(0) = 1, \tag{2.2}$$

$$(2.3)\quad dZ(t,I(t)) = Z(t,I(t))((r(t)+\sigma_1(t)\theta_I(t))dt+\sigma_1(t)dW^I(t)), Z(0)=z\in\mathcal{R}_+,$$

$$(2.4)\quad dS(t) = S(t)(\mu(t)dt+\sigma_S(t)qdW^I(t)+\sigma_S(t)\sqrt{1-q^2}dW^S(t)), S(0)=s\in\mathcal{R}_+.$$

Here $r(t)\in\mathcal{R}_+$ denotes the short risk-free interest rate, $\mu(t)\in\mathcal{R}_+$ the mean rate of return of the stock, $\sigma_S(t)\in\mathcal{R}$ the volatility of stock, $\sigma_1(t)\in\mathcal{R}$ the volatility of index bond, $q\in(-1,1)$ correlation coefficient of sources of risks from inflation, $W^I(t)$ and stock, $W^S(t)$ and $\theta_I(t)\in\mathcal{R}$ the inflation price of risk. Moreover, $\sigma_S(t)$ and $\sigma_1(t)$ are the volatilities for the stock and index bond respectively, referred to as the coefficients of the market and are progressively measurable with respect to the filtration $\mathcal{F}$.

The proportion of fund invested in stock, $S(t)$ at time, t is denoted by $\Delta^S(t)$ and proposition fund invested in index bond is $\Delta^I(t)$. The remainder, $1-\Delta^I(t)-\Delta^S(t)$ is invested in cash account at time, t. We suppose the trading strategy $\{\Delta(t):t\geq 0\}$, with $\Delta(t)=(\Delta^I(t),\Delta^S(t))$ is a control process adapted to filtration $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{F}_t$-measurable, Markovian, and stationary processes, satisfying

$$E\int_0^T \Delta(t)\Delta'(t)dt<\infty, \tag{2.5}$$

where E is the expectation operator. Let $C(t)$ be the consumption rate process at time t. Then $C(t)$ is an adapted process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, satisfying

$$E\int_0^T C(t)^2dt<\infty. \tag{2.6}$$

Let $Y(t)$ be the salary process of a PPM at time t, then $Y(t)$ satisfies the following stochastic differential equation:

$$dY(t) = Y(t)(\beta(t)dt+\sigma_{Y_1}(t)dW^I(t)+\sigma_{Y_2}(t)dW^S(t)), Y(0)=y_0\in\mathcal{R}. \tag{2.7}$$

where $\beta(t)\in\mathcal{R}_+$ is the expected growth rate of the salary, $\sigma_{Y_1}(t)$ is volatility of the salary of a PPM arising from the uncertainty of inflation, $W^I(t)$ and $\sigma_{Y_2}(t)$ is volatility of the salary of a PPM arising from the uncertainty of stock market, $W^S(t)$. We can express (2.3), (2.4) and (2.7) in compact form respectively, as follows:

$$(2.8)\quad dZ(t,I(t)) = Z(t,I(t))((r(t)+\sigma_1(t)\theta_I(t))dt+\sigma_Z(t)dW(t)), Z(0)=z\in\mathcal{R}_+,$$

$$dS(t) = S(t)(\mu(t)dt+\sigma(t)dW(t)), S(0)=s\in\mathcal{R}_+, \tag{2.9}$$

$$dY(t) = Y(t)(\beta(t)dt+\sigma_Y(t)dW(t)), Y(0)=y_0\in\mathcal{R}, \tag{2.10}$$

where $\sigma_Z(t)=(\sigma_1(t),0)$, $\sigma(t)=(\sigma_S(t)q,\sigma_S(t)\sqrt{1-q^2})$, $\sigma_Y(t)=(\sigma_{Y_1}(t),\sigma_{Y_2}(t))$ and $W(t)=(W(^I(t),W(^S(t))'$. Suppose the proportion $c\in\mathcal{R}_+$ of the salary process is a contribution of the PPM into the scheme, then $cY(t)$ is the gross amount of fund contributed into the scheme at time t.

Remark 2.1. If the pension PPM's salary is deterministic, then (2.10) becomes $dY(t)=\beta(t)Y(t)dt$.

Then, the volatility matrix

$$\Sigma(t):=\begin{pmatrix}\sigma_1(t) & 0\\ q\sigma_S(t) & \sigma_S(t)\sqrt{1-q^2}\end{pmatrix} \tag{2.11}$$

corresponding to the two risky assets and satisfies $det(\Sigma(t)) = \sigma_S(t)\sigma_1(t)\sqrt{1-q^2} \neq 0$. Therefore, the market is complete and there exists a unique market price of risks vector, $\theta(t)$ satisfying

$$\theta(t) := \begin{pmatrix} \theta_I(t) \\ \theta_S(t) \end{pmatrix} = \begin{pmatrix} \theta_I(t) \\ \frac{\mu(t)-r(t)-\theta_I(t)q\sigma_S(t)}{\sigma_S(t)\sqrt{(1-q^2)}} \end{pmatrix} \tag{2.12}$$

where $\theta_S(t)$ is the market price of stock risks. In this paper, we assume that $r(t)$, $\mu(t)$, $\sigma(t)$, $\sigma_S(t)$, $\sigma_1(t)$, $\sigma_Y(t)$, $\sigma_{Y_1}(t)$, $\sigma_{Y_2}(t)$, $\theta_I(t)$, $\theta_S(t)$, $q(t)$, $\beta(t)$, $\sigma_Z(t)$ are constants in time.

Therefore, the fund, $X(t)$ dynamic evolution under the investment policy Δ is (2.13)

$$\begin{aligned} &dX(t) = (X(t)(r+\Delta(t)\lambda) + c(1-\eta)Y(t) - C(t))dt + X(t)(\Sigma'\Delta'(t))'dW(t), \\ &X(0) = x_0 \in \mathcal{R}_+, \end{aligned}$$

where $\lambda = (\sigma_1\theta_I, \mu - r)'$, η denotes the proportion of PPM's contribution that is set aside for administrative cost (AC). It implies that $\eta cY(t)$ is the AC at time t and the net contribution is $c(1-\eta)Y(t)$ at time t. We observe that when $\eta = 0$, it implies that the PFA do not charge any management costs. If $\eta = 1$, it implies that the entire contributions by the PPM is taken as management costs, which is unrealistic. Since the PFA may not (or may) charge management costs for the operation, we assume that $0 \leq \eta < 1$.

3. Pension Benefits

In this section, we consider the minimum pension benefits, $P^m(t)$ at time t that will accrued to a PPM up to the final time, T. Let $P(t)$ be the total pension benefits of the contributor at time, $t \in [0, T]$. It is assumed that the value of minimum benefits a PPM can get at retirement should not be less than the value of contributions made into the scheme.

The PPM makes a flow of contribution to the pension fund. This flow consists of a lump sum at time 0, denoted by x_0, and a continuously paid premium, at a rate denoted by $cY(t)(t), t \in [0, T]$. The value at time 0 of the cash given by the contributor (i.e., PPM) to the pension scheme is equal to:

$$\bar{X}_0 = x_0 + c(1-\eta)E\left[\int_0^T \Lambda(s)Y(s)ds\right].$$

At time T, the PFA will provide a benefit which consists of two parts: The first part $P^m(T)$ is the minimum pension benefit, which means that the total benefit will be greater than $P^m(T)$ almost surely. The minimum pension benefit is not a constant (it is a stochastic minimum pension benefit), but a nonnegative random variable that is F_T-measurable, which is L^p integrable with $p > 2$. The second part of the benefit is a fixed fraction of the surplus $\Theta_T(P^m(T))$ (the difference between the terminal wealth $X(T)$ of the managed portfolio and the minimum pension benefit $P^m(T)$. Indeed, we suppose that the PFA receives a fixed fraction of the surplus, as a way to encourage him/her (see Jensen and Sϕrensen, 1999). Let h denotes the fixed fraction of the surplus that will be kept by the PFA. Then, the total benefit of the PPM at time T equals:

$$\begin{aligned} P(T) &= P^m(T) + (1-h)(X(T) - P^m(T)) \\ &= P^m(T) + \Theta_T(P^m(T)), \end{aligned}$$

where $\Theta_T(P^m(T)) = (1-h)V(T)$ is the surplus function at the final time, T and $V(T) = X(T) - P^m(T)$. For $h = 0$, it implies that the PFA does not keep any profit from the surplus, so introduction of the minimum pension benefit is more an obstacle for the PPM, since minimum pension benefits may induce a significant utility loss for quadratic risk tolerant investors (see Jensen and Sϕrensen, 1999 for relative risk averse investor). On the other hand, if $h = 1$, it implies that the PPM will receive only the minimum pension benefit, no matter the final surplus, which is not reasonable. In order to avoid these trivial cases, we therefore assume that $h \in (0,1)$. One of the aims of this paper is to find the optimal discounted benefit that will accrued to the PPM at the final time, T. This is obtained from the discounted surplus and discounted minimum pension benefit at the final time, T.

Definition 3.1. The flow of expected discounted minimum pension benefits for $t \leq T$ is defined by

$$P^m(t) = E_t\left[\int_0^T \frac{\Lambda(u)}{\Lambda(t)} cY(u)du\right], t \geq 0. \tag{3.1}$$

where $E_t = E(\cdot|\mathcal{F}_t)$ is the conditional expectation with respect to the Brownian filtration $\{\mathcal{F}\}_{t\geq 0}$.

Definition 3.2. The flow of expected pension benefits, $P(t)$ is defined by

$$P(t) = \begin{cases} P^m(t), \text{ if } 0 \leq t \leq T_0 < T, \\ P^m(t) + \Theta(t, V(t)|_{t-T,t}), \text{ if } t \geq T, \end{cases} \tag{3.2}$$

where T_0 is the time of voluntary retirement and $\Theta(\cdot,\cdot)$ is the surplus function. At time $t \geq T$ the surplus depends on the fund wealth level in time period $[t-T, t]$.

Proposition 3.1. *Let $P^m(t)$ be the value of flow of the minimum pension benefits that will accrued to PPM at time t, then*

$$P^m(t) = \frac{cY(t)}{\delta}(e^{\delta T} - 1), \tag{3.3}$$

where $\delta = \beta - \xi - \sigma_Y\theta$, $\xi \in [0, r]$ is the instantaneous guaranteed rate of return and $cY(t)$ is flow of contributions of PPM at time t.

Proof: By definition 3.1, we have that

$$\begin{aligned} P^m(t) &= E_t\left[\int_0^T \frac{\Lambda(u)}{\Lambda(t)} cY(u)du\right] \\ &= cY(t)E_t\left[\int_0^T \frac{\Lambda(u)}{\Lambda(t)}\frac{Y(u)}{Y(t)}du\right]. \end{aligned}$$

Applying change of variable and Markovian rule on the above equation, we have

$$P^m(t) = cY(t)E\left[\int_0^T \frac{\Lambda(\tau)}{\Lambda(0)}\frac{Y(\tau)}{Y(0)}d\tau\right], \tag{3.4}$$

Applying parallelogram law and martingale principles on (3.4), we have

$$P^m(t) = cY(t)E\int_0^T e^{(\beta-\xi-\sigma_Y\theta)\tau}d\tau. \tag{3.5}$$

Integrating, we have

$$P^m(t) = \frac{cY(t)(e^{(\beta-\xi-\sigma_Y\theta)T} - 1)}{\beta - \xi - \sigma_Y\theta}. \tag{3.6}$$

Therefore,

$$P^m(t) = \frac{cY(t)(e^{\delta T} - 1)}{\delta}. \tag{3.7}$$

where $\delta = \beta - \xi - \sigma_Y\theta$.

This implies that the final minimum pension benefits for a PPM is

$$P^m(T) = \frac{cY(T)(e^{\delta T} - 1)}{\delta}, \tag{3.8}$$

and the present value of a PPM's future minimum pension benefit is

$$P^m(0) = P_0^m = \frac{cy_0(e^{\delta T} - 1)}{\delta}. \tag{3.9}$$

Taking the differential of both sides of (3.7), we have

$$dP^m(t) = P^m(t)(\beta dt + \sigma_Y dW(t)). \tag{3.10}$$

Corollary 3.1. *Let $P^m(T_0)$ be the minimum pension benefits for a PPM who retired voluntarily from the scheme and T_0 the time of voluntary retirement, then*

$$P^m(T_0) = \frac{cY(T_0)(e^{\delta T_0} - 1)}{\delta}, 0 < T_0 < T. \tag{3.11}$$

4. Expected Discounted Flow of Contributions

In this section, we presents the expected discounted flow of PPM's contributions at time t.

Definition 4.1. The expected value of flow of a PPM's net contribution is defined as

$$\Phi(t) = E_t\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)} c(1-\eta)Y(u)du\right]. \tag{4.1}$$

Theorem 4.1. *Suppose $\Phi(t)$ is the expected value of a PPM's net contributions, then*

$$\Phi(t) = \frac{c(1-\eta)Y(t)(e^{\alpha(T-t)} - 1)}{\alpha}, \tag{4.2}$$

where $\alpha = \beta - r - \sigma_Y\theta$.

Proof: By definition 4.1, we have

$$\Phi(t) = c(1-\eta)Y(t)E_t\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)}\frac{Y(u)}{Y(t)}du\right]. \tag{4.3}$$

Applying change of variable and Markovian rule on (4.3), we have

$$\Phi(t) = c(1-\eta)Y(t)E\left[\int_0^T \frac{\Lambda(\tau)}{\Lambda(0)}\frac{Y(\tau)}{Y(0)}d\tau\right], \tag{4.4}$$

Applying parallelogram law and martingale principles on (3.4) and then integrate, we have

$$\Phi(t) = \frac{c(1-\eta)Y(t)(e^{\alpha(T-t)}-1)}{\alpha}, \tag{4.5}$$

where $\alpha = \beta - r - \sigma_Y\theta$. The present value of a PPM's future contribution is obtain as

$$\Phi(0) = \frac{c(1-\eta)y_0(e^{\alpha T}-1)}{\alpha}. \tag{4.6}$$

Taking the differential of both sides of (4.5), we have

$$d\Phi(t) = \Phi(t)((r+\sigma_Y\theta)dt + \sigma_Y dW(t)) - c(1-\eta)Y(t)dt. \tag{4.7}$$

Corollary 4.1. *Let $\Phi(T_0)$ be the value of the contributions of a PPM who will retired voluntarily at time period T_0, then*

$$\Phi(t) = \frac{c(1-\eta)Y(t)(e^{\alpha(T_0-t)}-1)}{\alpha}, 0 \leq t \leq T_0. \tag{4.8}$$

It implies that the present value of the PPM's contributions that retired voluntarily from the scheme is

$$\Phi(0) = \frac{c(1-\eta)y_0(e^{\alpha T_0}-1)}{\alpha}. \tag{4.9}$$

4.1. **Discounted Wealth, Contribution, Minimum Pension Benefit and Surplus Process.** In this subsection, we consider the discounted wealth, discounted contributions and discounted minimum pension benefit of a PPM at time t. The discounted surplus process of the stakeholder is also established in this subsection. The discounted wealth of a PPM is given by (4.10).

$$d(\Lambda(t)X(t)) = \Lambda(t)X(t)(\Sigma'\Delta'(t)-\theta)'dW(t) + (c(1-\eta)\Lambda(t)Y(t) - \Lambda(t)C(t))dt \tag{4.10}$$

(4.11) gives the discounted contributions of a PPM at time t and is given by

$$d(\Lambda(t)\Phi(t)) = \Lambda(t)\Phi(t)(\sigma_Y' - \theta)'dW(t) - (c(1-\eta)\Lambda(t)Y(t)dt. \tag{4.11}$$

The discounted minimum pension benefits is given by

$$d(\Lambda(t)P^m(t)) = \Lambda(t)P^m(t)(\sigma_Y' - \theta)'dW(t). \tag{4.12}$$

Setting $\tilde{X}(t) = \Lambda(t)X(t)$, $\tilde{Y}(t) = \Lambda(t)Y(t)$, $\tilde{C}(t) = \Lambda(t)C(t)$, $\tilde{\Phi}(t) = \Lambda(t)\Phi(t)$, $\tilde{P}^m(t) = \Lambda(t)P^m(t)$, (4.10)-(4.12) become:

$$d\tilde{X}(t) = \tilde{X}(t)(\Sigma'\Delta'(t)-\theta)'dW(t) + (c(1-\eta)\tilde{Y}(t) - \tilde{C}(t))dt \tag{4.13}$$

$$d\tilde{\Phi}(t) = \tilde{\Phi}(t)(\sigma_Y' - \theta)'dW(t) - (c(1-\eta)\tilde{Y}(t)dt \tag{4.14}$$

$$d\tilde{P}^m(t) = \tilde{P}^m(t)(\sigma_Y' - \theta)'dW(t)dt \tag{4.15}$$

Remark 4.1.

$$d(\Lambda(t)Y(t)) = \Lambda(t)Y(t)(\beta - r - \sigma_Y\theta)dt + \Lambda(t)Y(t)(\sigma_Y' - \theta)'dW(t). \tag{4.16}$$

Solving (4.16), we have

$$E(\Lambda(t)Y(t)) = y_0e^{(\beta - r - \sigma_Y\theta)t}. \tag{4.17}$$

Solving (4.14), we have

$$E\tilde{\Phi}(t) = \Phi_0 - c(1-\eta)\int_0^t E(\Lambda(s)Y(s))ds. \tag{4.18}$$

Using (4.17) on (4.18), we have

$$E\tilde{\Phi}(t) = \Phi_0 - \frac{cy_0(1-\eta)}{\beta - r - \sigma_Y\theta}\left(e^{(\beta - r - \sigma_Y\theta)t} - 1\right). \tag{4.19}$$

Hence, the value of a PPM's surplus is given as

$$V(t) = X(t) + \Phi(t) - P^m(t). \tag{4.20}$$

Therefore, the value of a PPM's discounted surplus process is given as

$$\tilde{V}(t) = \tilde{X}(t) + \tilde{\Phi}(t) - \tilde{P}^m(t). \tag{4.21}$$

Proposition 4.1. *Suppose $\tilde{X}(t)$ satisfies (4.13), $\tilde{P}^m(t)$ (4.15) and $\tilde{\Phi}(t)$ satisfies (4.14), then the discounted surplus process, $\tilde{V}(t)$ has the following dynamics*

$$\begin{aligned} d\tilde{V}(t) &= [\tilde{X}(t)(\Sigma'\Delta'(t) - \theta)' + (\tilde{\Phi}(t) - \tilde{P}^m(t))(\sigma_Y' - \theta)']dW(t) - \tilde{C}(t)dt, \\ \tilde{V}(0) &= v_0. \end{aligned} \tag{4.22}$$

5. The Mean-Variance Formulation

The objective of the PFA is double. The first objective is to maximize the expected value of fund's (and discounted) assets. The second objective is aim at to minimize the variance of the terminal discounted surplus (and real surplus or simply surplus), $Var(\tilde{V}^*(T))$ (and $Var(V^*(T))$) and the consumption risk, $C^*(t)$ on the interval $[0, T]$. This dual-objective problem reflects the major concern of the stakeholders (in this paper, stakeholders represents the PFA and the PPM only) to increase fund assets in order to pay due pension benefits as at when due, but at the same time not exposed the pension fund to large variations in other to provide stability to the scheme. According to Josa-Fombellida and Rincon-Zapatero (2008), minimization of the contribution risk (though, in this paper, we consider consumption risk) has been considered in other works as Haberman and Sung (1994), Haberman *et al.* (2000) and Josa-Fombellida and Rincon-Zapatero (2001, 2004).

Therefore, this paper is considering a multi-objective optimization problem involving two criteria

$$\min_{(\Delta,C)\in\mathcal{A}}(L_1(\Delta,C), L_2(\Delta,C)) = \min_{(\Delta,C)\in\mathcal{A}}\left(-E(\tilde{V}(T)), E\int_0^T e^{\rho t}\tilde{C}^2(t)dt + Var(\tilde{V}(T))\right) \tag{5.1}$$

subject to (4.22). Here $\mathcal{A}$ is the set of measurable processes (Δ, C), where Δ satisfies (2.5), C satisfies (2.6), and such that (4.22) admit a unique solution that is $\mathcal{F}_t$-measurable adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

An admissible control process (Δ^*, C^*) is Pareto efficient if there exists no admissible (Δ, C) such that

$$L_1(\Delta, C) \leq L_1(\Delta^*, C^*), L_2(\Delta, C) \leq L_2(\Delta^*, C^*)$$

with at least one of the inequalities hold strictly. The pairs $(L_1(\Delta^*, C^*); L_2(\Delta^*, C^*)) \in \mathcal{R}^2$ form the Pareto frontier. We will refer to C^* an efficient consumption rate and

Δ^* an efficient portfolio. The aim of this paper is to find $(E(\tilde{V}^*(T)), Var(\tilde{V}^*(T)))$ (and then deduce $(E(V^*(T)), Var(V^*(T))))$ refer to as an efficient frontier.

According to Da Cunha and Polak (1967) (in Josa-Fombellida and Rincon-Zapatero (2008)), when the objective functionals defining the multi-objective program are convex, the Pareto efficient points can be found by solving a scalar optimal control problem where the dynamics is fixed and the objective functional is a convex combination of the original cost functionals. In our case (4.15) and (4.22) are linear, so both L_1 and L_2 are indeed convex. Hence, the original problem (4.15), (4.22) and (5.1) are equivalent to the scalar problem

$$\min_{(\Delta,C)\in\mathcal{A}} L_1(\Delta,C)+\psi L_2(\Delta,C) = \min_{(\Delta,C)\in\mathcal{A}} -E(\tilde{V}(T))+\psi\left(E\int_0^T e^{\rho t}\tilde{C}^2(t)dt + Var(\tilde{V}(T))\right) \tag{5.2}$$

subject to (4.22) and (4.15), with $\psi > 0$ a weight parameter. As ψ varies within the interval $(0,\infty)$, the solutions of (5.2) describe the Pareto frontier (see Josa-Fombellida and Rincon-Zapatero (2008)). Observe that ψ serves the PFA the opportunity to transfer linearly units of risk to units of expected return, and vice versa. The size of ψ shows which one of the objectives is of major concern for the PFA, to reduce risk or to maximize return.

Problem (4.15), (4.22) and (5.2) are not standard stochastic optimal problem due to the presents of the term $(E(\tilde{V}(T)))^2$ in the variance term, and the dynamic programming approach cannot be applied at this point. Following Zhou and Li (2000), Li and Ng (2000) and Josa-Fombellida and Rincon-Zapatero (2008), we propose an auxiliary problem that transform into a stochastic problem of linear-quadratic case:

$$\min_{(\Delta,C)\in\mathcal{A}} J(\Delta,C) = \min_{(\Delta,C)\in\mathcal{A}}\left(E\int_0^T e^{\rho t}\tilde{C}^2(t)dt + E(\tilde{V}^2(T) - 2\varphi\tilde{V}(T))\right) \tag{5.3}$$

subject to (4.22) and (4.15) and $\varphi \in \mathcal{R}$.

The relationship between problems (4.15), (4.22), (5.2) and (4.15), (4.22), (5.3) is shown in the following result.

Theorem 5.1. *For any $\varphi > 0$, if (Δ^*, C^*) is an optimal control of (4.15), (4.22), (5.2) with associated optimal surplus, V^*, then it is an optimal control of (4.15), (4.22), (5.3) for $\varphi = \frac{1}{2\psi} + E(\tilde{V}^*(T))$.*

Proof: see Josa-Fombellida and Rincon-Zapatero (2008).

6. Optimal Portfolio and Optimal Consumption

In this section, we find the optimal portfolio and optimal consumption rate for a PPM. First, we determine the Hamilton-Jacobi-Bellman equation for our surplus process. We define the follows differential operator:

$$\mathcal{L} = \left(\frac{1}{2}\tilde{\Phi}^2\frac{\partial^2}{\partial\tilde{\Phi}^2} - \tilde{\Phi}\tilde{P}\frac{\partial^2}{\partial\tilde{\Phi}\partial\tilde{P}} + \frac{1}{2}\tilde{P}^2\frac{\partial^2}{\partial\tilde{P}^2}\right)(\sigma_Y\sigma_Y' - 2\sigma_Y\theta + \theta'\theta). \tag{6.1}$$

We define the general value function

$$L(t,\tilde{x},\tilde{\Phi},\tilde{P}) = E[V(T,\tilde{X},\tilde{\Phi},\tilde{P})|\tilde{X}(t) = \tilde{x}, \tilde{\Phi}(t) = \tilde{\Phi}, \tilde{P}^m(t) = \tilde{P}]$$

where $L(t,\tilde{x},\tilde{\Phi},\tilde{P})$ is the path of $V(t)$ given the portfolio strategy $\Delta(t)=(\Delta^I(t),\Delta^S(t))$. Let $L(t,\tilde{x},\tilde{\Phi},\tilde{P})$ be a convex function in $V(t)$ such that

$$\begin{aligned} &U(t,\tilde{x},\tilde{\Phi},\tilde{P}) = \min_{\Delta,C} L(t,\tilde{x},\tilde{\Phi},\tilde{P}),\\ &\text{subject to (3.8).}\end{aligned} \tag{6.2}$$

Then $U(t,\tilde{x},\tilde{\Phi},\tilde{P})$ satisfies the HJB equation

$$\begin{aligned} &U_t - \tilde{C}(t)U_{\tilde{x}} + \tilde{C}^2(t)e^{-\rho t} + \tfrac{1}{2}\tilde{x}^2(\Sigma\Delta(t)\Sigma'\Delta'(t) - 2\Sigma\Delta(t)\theta + \theta'\theta)U_{\tilde{x}\tilde{x}}\\ &+2(\tilde{x}\tilde{\Phi}U_{\tilde{x}\tilde{\Phi}} - \tilde{x}\tilde{P}U_{\tilde{x}\tilde{P}})(\Sigma\Delta(t)\sigma_Y' - 2\Sigma\Delta(t)\theta + \theta'\theta) + \mathcal{L}U = 0,\\ &\text{subject to: } U(T,\tilde{x},\tilde{\Phi},\tilde{P}) = (\tilde{x}-\tilde{P})^2 - 2\varphi(\tilde{x}-\tilde{P}).\end{aligned} \tag{6.3}$$

Proposition 6.1. *The optimal rate of consumption and the optimal investment in the risky assets (index bond and stock) are respectively given by*

$$\Delta^{'*}(t) = \frac{(\Sigma\Sigma')^{-1}(\tilde{x}\Sigma\theta U_{\tilde{x}\tilde{x}} - 2(\tilde{\Phi}U_{\tilde{x}\tilde{\Phi}} - \tilde{P}U_{\tilde{x}\tilde{P}})(\Sigma\sigma_Y' - 2\Sigma\theta))}{\tilde{x}U_{\tilde{x}\tilde{x}}}. \tag{6.4}$$

$$\tilde{C}^*(t) = \frac{1}{2}U_{\tilde{x}}e^{\rho t}. \tag{6.5}$$

Substituting (6.4) and (6.5) into (6.3), we have

$$\begin{aligned} &U_t - \tfrac{1}{2}U_{\tilde{x}}^2e^{\rho t} - \tfrac{1}{2}\theta'\theta(\tilde{x}^2-1)U_{\tilde{x}\tilde{x}} - 2[\theta'\Sigma'M(\Sigma\sigma_Y' - 2\Sigma\theta)\tilde{\Phi}\\ &-2\tilde{x}\tilde{\Phi}(\sigma_Y\theta - 2\theta'\theta) - \tilde{x}\tilde{\Phi}\theta'\theta]U_{\tilde{x}\tilde{\Phi}} + 2\tilde{P}[\theta'\Sigma'M(\Sigma\sigma_Y' - 2\Sigma\theta)\\ &+2\tilde{x}(2\theta'\theta - \sigma_Y\theta) - \tilde{x}\theta'\theta]U_{\tilde{x}\tilde{P}} - 4(\sigma_Y\sigma_Y' - 4\theta'\theta)\tilde{P}\tilde{\Phi}\tfrac{U_{\tilde{x}\tilde{\Phi}}U_{\tilde{x}\tilde{P}}}{U_{\tilde{x}\tilde{x}}}\\ &+[2(\sigma_Y\sigma_Y' - 4\theta'\theta)\tilde{P}^2 - 4\tilde{P}^2(3\sigma_Y\theta - 2\theta'\theta - \sigma_Y\sigma_Y')]\tfrac{U_{\tilde{x}\tilde{P}}^2}{U_{\tilde{x}\tilde{x}}}\\ &+[2(\sigma_Y\sigma_Y' - 4\theta'\theta)\tilde{\Phi}^2 + 4\tilde{\Phi}^2(3\sigma_Y\theta - 2\theta'\theta - \sigma_Y\sigma_Y')]\tfrac{U_{\tilde{x}\tilde{\Phi}}^2}{U_{\tilde{x}\tilde{x}}} + \mathcal{L}U = 0,\\ &\text{subject to: } U(T,\tilde{x},\tilde{\Phi},\tilde{P}) = (\tilde{x}-\tilde{P})^2 - 2\varphi(\tilde{x}-\tilde{P}).\end{aligned} \tag{6.6}$$

We assume a quadratic solution of the form:

$$\begin{aligned} &U(t,\tilde{x},\tilde{\Phi},\tilde{P}) = \phi_0(t) + \tilde{P}\phi_{\tilde{P}}(t) + \tilde{\Phi}\phi_{\tilde{\Phi}}(t) + \tilde{x}\phi_{\tilde{x}}(t) + \tilde{x}\tilde{\Phi}\phi_{\tilde{x}\tilde{\Phi}}(t)\\ &+\tilde{x}\tilde{P}\phi_{\tilde{x}\tilde{P}}(t) + \tilde{\Phi}\tilde{P}\phi_{\tilde{\Phi}\tilde{P}}(t) + \tilde{\Phi}^2\phi_{\tilde{\Phi}\tilde{\Phi}}(t) + \tilde{x}^2\phi_{\tilde{x}\tilde{x}}(t) + \tilde{P}^2\phi_{\tilde{P}\tilde{P}}(t).\end{aligned} \tag{6.7}$$

Finding the partial derivatives of (6.7) with respect to t, $\tilde{x}$, $\tilde{P}$, $\tilde{\Phi}$, $\tilde{x}\tilde{x}$, $\tilde{x}\tilde{\Phi}$, $\tilde{x}\tilde{P}$, $\tilde{\Phi}\tilde{P}$, $\tilde{P}\tilde{P}$, $\tilde{\Phi}\tilde{\Phi}$ as follows:

$$\begin{aligned} &U_t = \dot{\phi}_0(t) + \tilde{P}\dot{\phi}_{\tilde{P}}(t) + \tilde{\Phi}\dot{\phi}_{\tilde{\Phi}}(t) + \tilde{x}\dot{\phi}_{\tilde{x}}(t) + \tilde{x}\tilde{\Phi}\dot{\phi}_{\tilde{x}\tilde{\Phi}}(t)\\ &+\tilde{x}\tilde{P}\dot{\phi}_{\tilde{x}\tilde{P}}(t) + \tilde{\Phi}\tilde{P}\dot{\phi}_{\tilde{\Phi}\tilde{P}}(t) + \tilde{\Phi}^2\dot{\phi}_{\tilde{\Phi}\tilde{\Phi}}(t) + \tilde{x}^2\dot{\phi}_{\tilde{x}\tilde{x}}(t) + \tilde{P}^2\dot{\phi}_{\tilde{P}\tilde{P}}(t),\end{aligned} \tag{6.8}$$

$$U_{\tilde{x}} = \phi_{\tilde{x}}(t) + \tilde{\Phi}\phi_{\tilde{x}\tilde{\Phi}}(t) + \tilde{P}\phi_{\tilde{x}\tilde{P}}(t) + 2\tilde{x}\phi_{\tilde{x}\tilde{x}}(t), \tag{6.9}$$

$$U_{\tilde{\Phi}} = \phi_{\tilde{\Phi}}(t) + \tilde{x}\phi_{\tilde{x}\tilde{\Phi}}(t) + \tilde{P}\phi_{\tilde{x}\tilde{P}}(t) + 2\tilde{\Phi}\phi_{\tilde{\Phi}\tilde{\Phi}}(t), \tag{6.10}$$

$$U_{\tilde{P}} = \phi_{\tilde{P}}(t) + \tilde{x}\phi_{\tilde{x}\tilde{P}}(t) + \tilde{\Phi}\phi_{\tilde{\Phi}\tilde{P}}(t) + 2\tilde{P}\phi_{\tilde{P}\tilde{P}}(t), \tag{6.11}$$

$$\begin{aligned} &U_{\tilde{x}\tilde{x}} = 2\phi_{\tilde{x}\tilde{x}}(t), U_{\tilde{\Phi}\tilde{\Phi}} = 2\phi_{\tilde{\Phi}\tilde{\Phi}}(t), U_{\tilde{P}\tilde{P}} = 2\phi_{\tilde{P}\tilde{P}}(t),\\ &U_{\tilde{x}\tilde{\Phi}} = 2\phi_{\tilde{x}\tilde{\Phi}}(t), U_{\tilde{x}\tilde{P}} = 2\phi_{\tilde{x}\tilde{P}}(t), U_{\tilde{\Phi}\tilde{P}} = 2\phi_{\tilde{\Phi}\tilde{P}}(t).\end{aligned} \tag{6.12}$$

The following ordinary differential equations are obtained for the above coefficients of $\tilde{x}$, $\tilde{P}$, $\tilde{\Phi}$, $\tilde{x}\tilde{x}$, $\tilde{x}\tilde{\Phi}$, $\tilde{x}\tilde{P}$, $\tilde{\Phi}\tilde{P}$, $\tilde{P}\tilde{P}$, $\tilde{\Phi}\tilde{\Phi}$ in (6.6):

$$\dot{\phi}_0(t) = \tfrac{1}{4}e^{\rho t}\phi_{\tilde{x}}^2(t) - \theta'\theta\phi_{\tilde{x}\tilde{x}}(t), \phi_0(T) = 0,$$

$$\dot{\phi}_{\tilde{P}}(t) = \tfrac{1}{2}e^{\rho t}\phi_{\tilde{x}}(t)\phi_{\tilde{x}\tilde{P}}(t) - 4\theta'\Sigma' M(\Sigma\sigma_Y' - 2\Sigma\theta)\phi_{\tilde{x}\tilde{P}}(t), \phi_{\tilde{P}}(T) = 2\varphi,$$

$$\dot{\phi}_{\tilde{\Phi}}(t) = \tfrac{1}{2}e^{\rho t}\phi_{\tilde{x}}(t)\phi_{\tilde{x}\tilde{\Phi}}(t) + 4\theta'\Sigma' M(\Sigma\sigma_Y' - 2\Sigma\theta)\phi_{\tilde{x}\tilde{\Phi}}(t), \phi_{\tilde{\Phi}}(T) = 0,$$

$$\dot{\phi}_{\tilde{\Phi}\tilde{P}}(t) = \tfrac{1}{2}e^{\rho t}\phi_{\tilde{x}\tilde{\Phi}}(t)\phi_{\tilde{x}\tilde{P}}(t) + 2(\sigma_Y\sigma_Y' - 2\sigma_Y\theta + \theta'\theta)\phi_{\tilde{P}\tilde{\Phi}}$$
$$+8(\sigma_Y\sigma_Y' - 4\theta'\theta)\tfrac{\phi_{\tilde{x}\tilde{P}}(t)\phi_{\tilde{x}\tilde{\Phi}}(t)}{\phi_{\tilde{x}\tilde{x}}(t)}, \phi_{\tilde{\Phi}\tilde{P}}(T) = 0,$$

$$\dot{\phi}_{\tilde{\Phi}\tilde{\Phi}}(t) = \tfrac{1}{4}e^{\rho t}\phi^2_{\tilde{x}\tilde{\Phi}}(t) - (\sigma_Y\sigma_Y' - 2\sigma_Y\theta + \theta'\theta)\phi_{\tilde{\Phi}\tilde{\Phi}}$$
$$+4(\sigma_Y\sigma_Y' + 3\sigma_Y\theta + 8\theta'\theta)\tfrac{\phi^2_{\tilde{x}\tilde{\Phi}}}{\phi_{\tilde{x}\tilde{x}}}, \phi_{\tilde{\Phi}\tilde{\Phi}}(T) = 0,$$

$$\dot{\phi}_{\tilde{P}\tilde{P}}(t) = \tfrac{1}{4}e^{\rho t}\phi^2_{\tilde{x}\tilde{P}}(t) - (\sigma_Y\sigma_Y' - 2\sigma_Y\theta + \theta'\theta)\phi_{\tilde{P}\tilde{P}}$$
$$+12(\sigma_Y\sigma_Y' - 2\sigma_Y\theta)\tfrac{\phi^2_{\tilde{x}\tilde{P}}}{\phi_{\tilde{x}\tilde{x}}}, \phi_{\tilde{P}\tilde{P}}(T) = 0,$$

$$\dot{\phi}_{\tilde{x}}(t) = e^{\rho t}\phi_{\tilde{x}}(t)\phi_{\tilde{x}\tilde{x}}(t), \phi_{\tilde{x}}(T) = -2\varphi, \tag{6.13}$$

$$\dot{\phi}_{\tilde{x}\tilde{\Phi}}(t) = e^{\rho t}\phi_{\tilde{x}\tilde{\Phi}}(t)\phi_{\tilde{x}\tilde{x}}(t) - 4(2\sigma_Y\theta - 3\theta'\theta)\phi_{\tilde{x}\tilde{\Phi}}(t), \phi_{\tilde{x}\tilde{\Phi}}(T) = 0, \tag{6.14}$$

$$\dot{\phi}_{\tilde{x}\tilde{P}}(t) = e^{\rho t}\phi_{\tilde{x}\tilde{P}}(t)\phi_{\tilde{x}\tilde{x}}(t) - 4(3\theta'\theta - 2\sigma_Y\theta)\phi_{\tilde{x}\tilde{P}}(t), \phi_{\tilde{x}\tilde{P}}(T) = -2. \tag{6.15}$$

$$\dot{\phi}_{\tilde{x}\tilde{x}}(t) = e^{\rho t}\phi^2_{\tilde{x}\tilde{x}}(t) + \theta'\theta\phi_{\tilde{x}\tilde{x}}(t), \phi_{\tilde{x}\tilde{x}}(T) = 1. \tag{6.16}$$

Therefore, from (6.6) the optimal controls must be

$$\Delta'^*(t) = (\Sigma\Sigma')^{-1}\Sigma\theta - 2\left(\tilde{\Phi}\tfrac{\phi_{\tilde{x}\tilde{\Phi}}(t)}{\tilde{x}\phi_{\tilde{x}\tilde{x}}(t)} - \tilde{P}\tfrac{\phi_{\tilde{x}\tilde{P}}(t)}{\tilde{x}\phi_{\tilde{x}\tilde{x}}(t)}\right)(\Sigma\Sigma')^{-1}(\Sigma\sigma_Y' - 2\Sigma\theta). \tag{6.17}$$

$$\tilde{C}^*(t) = (\phi_{\tilde{x}}(t) + \tilde{\Phi}\phi_{\tilde{x}\tilde{\Phi}}(t) + \tilde{P}\phi_{\tilde{x}\tilde{P}}(t) + 2\tilde{x}\phi_{\tilde{x}\tilde{x}}(t))e^{\rho t}. \tag{6.18}$$

Solving (6.13), we have

$$\phi_{\tilde{x}}(t) = \frac{2\varphi(\theta'\theta + \rho)}{e^{-\theta'\theta(T-t)+\rho t} - e^{\rho T} - (\theta'\theta + \rho)}, \tag{6.19}$$

Solving (6.14), we have

$$\phi_{\tilde{x}\tilde{\Phi}}(t) = 0, \tag{6.20}$$

Solving (6.15), we have

$$\phi_{\tilde{x}\tilde{P}}(t) = \frac{2e^{4(3\theta'\theta - 2\sigma_Y\theta)(T-t)[e^{-\theta'\theta(T-t)+\rho t} - e^{\rho T} - (\theta'\theta+\rho)]}}{\theta'\theta + \rho}. \tag{6.21}$$

Solving (6.16), we have

$$\phi_{\tilde{x}\tilde{x}}(t) = \frac{(\theta'\theta + \rho)e^{-\theta'\theta(T-t)}}{(\theta'\theta + \rho) - e^{-\theta'\theta(T-t)+\rho t} + e^{\rho T}}. \tag{6.22}$$

Substituting (6.19)-(6.22) into (6.17) and (6.18), we have the following optimal portfolio and optimal discounted consumption for the PPM at time t:

$$\Delta'^*(t) = M\Sigma\theta - \tfrac{4P^m(t)M(\Sigma\sigma_Y' - 2\Sigma\theta)}{X^*(t)} \times \tfrac{e^{(13\theta'\theta + 8\sigma_Y\theta)(T-t)}(\theta'\theta + \rho - e^{-\theta'\theta(T-t)+\rho t} + e^{\rho T})^2}{(\theta'\theta+\rho)^2}. \tag{6.23}$$

The first part of (6.23) is the Merton portfolio process. The second part is the variational part which is the intertemporal hedging term that offset any shock to the stochastic salary of a quadratic risk PPM in the scheme.

$$\begin{aligned}\tilde{C}^*(t) = {}& \tfrac{2\tilde{P}^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)+\rho t}[e^{-\theta'\theta(T-t)+\rho t}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho} \\ &+\tfrac{2(\theta'\theta+\rho)e^{\rho t}(\tilde{X}^*(t)e^{-\theta'\theta(T-t)}-\varphi)}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}.\end{aligned} \tag{6.24}$$

At time $t=0$, we have

$$\Delta'^*(0) = M\Sigma\theta - \tfrac{4P^m(0)M(\Sigma\sigma_Y'-2\Sigma\theta)}{x_0}\,\tfrac{e^{(13\theta'\theta+8\sigma_Y\theta)T}(\theta'\theta+\rho-e^{-\theta'\theta T}+e^{\rho T})^2}{(\theta'\theta+\rho)^2}. \tag{6.25}$$

$$\tilde{C}^*(0) = \tfrac{2\tilde{P}^m(0)e^{4(3\theta'\theta-2\sigma_Y\theta)T}[e^{-\theta'\theta T}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho} + \tfrac{2(\theta'\theta+\rho)(\tilde{x}_0e^{-\theta'\theta T}-\varphi)}{\theta'\theta+\rho-e^{-\theta'\theta T}+e^{\rho T}}. \tag{6.26}$$

The terminal discounted consumption can be obtained by setting $t=T$ as follows:

$$\tilde{C}^*(T) = -2\tilde{P}^m(T)e^{\rho T} + \frac{2(\theta'\theta+\rho)e^{\rho T}(\tilde{X}^*(T)-\varphi)}{\theta'\theta+\rho}. \tag{6.27}$$

6.1. **Optimal Consumption of a PPM.** In this subsection, we consider the optimal consumption process of a PPM at time t. It is given by

$$C^*(t) = \Lambda(t)^{-1}\tilde{C}^*(t) = \tilde{C}^*(t)e^{(r+\|\theta\|^2)t+\theta'W(t)}. \tag{6.28}$$

It implies that

$$\begin{aligned}C^*(t) = {}& \tfrac{2P^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)+\rho t}[e^{-\theta'\theta(T-t)+\rho t}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho} \\ &+\tfrac{2(\theta'\theta+\rho)X^*(t)e^{\rho t-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}} - \tfrac{2(\theta'\theta+\rho)\varphi e^{(\rho+r+\|\theta\|^2)t+\theta'W(t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}.\end{aligned} \tag{6.29}$$

The positive term $\theta'\theta$ captures the uncertainty of the financial markets. (6.29) shows that when the market become bearish, it induces the PPM not make more contributions into the pension fund and consume more, and vice versa. It is also observed that when the preference consumption rate, ρ increases, the consumption process increases over time, for all other parameters remain fixed.

At time $t=0$, (6.29) becomes

$$\begin{aligned}C^*(0) = {}& \tfrac{2P_0^m e^{4(3\theta'\theta-2\sigma_Y\theta)T}[e^{-\theta'\theta T}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho} \\ &+\tfrac{2(\theta'\theta+\rho)x_0e^{-\theta'\theta T}}{\theta'\theta+\rho-e^{-\theta'\theta T}+e^{\rho T}} - \tfrac{2(\theta'\theta+\rho)\varphi}{\theta'\theta+\rho-e^{-\theta'\theta T}+e^{\rho T}}.\end{aligned} \tag{6.30}$$

At time $t=T$, (6.29) becomes

$$\begin{aligned}C^*(T) = {}& -\tfrac{2P^m(T)e^{\rho T}[(\theta'\theta+\rho)]}{\theta'\theta+\rho} + \tfrac{2(\theta'\theta+\rho)X^*(T)e^{\rho T}}{\theta'\theta+\rho} \\ &-\tfrac{2(\theta'\theta+\rho)\varphi e^{(\rho+r+\|\theta\|^2)T+\theta'W(T)}}{\theta'\theta+\rho}.\end{aligned} \tag{6.31}$$

We can express (6.29) in terms of the parameter ψ (which represents the variance minimizer) as follows:

$$\begin{aligned}C^*(t) = {}& \tfrac{2P^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)+\rho t}[e^{-\theta'\theta(T-t)+\rho t}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho} \\ &+\tfrac{2(\theta'\theta+\rho)X^*(t)e^{\rho t-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}} - \tfrac{2(\theta'\theta+\rho)(1+2\psi E(\tilde{V}^*(T)))e^{(\rho+r+\|\theta\|^2)t+\theta'W(t)}}{2\psi(\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T})}.\end{aligned} \tag{6.32}$$

It is observe that as ψ becomes smaller and smaller for all other parameters remain constant, consumption rate reduces and vice versa. It imply that

$$\lim_{\psi\to\infty} C^*(t) = \frac{2P^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)+\rho t}[e^{-\theta'\theta(T-t)+\rho t}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho} + \frac{2(\theta'\theta+\rho)X^*(t)e^{\rho t-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}, \tag{6.33}$$

and

$$\lim_{\psi\to 0} C^*(t) = -\infty. \tag{6.34}$$

This is an intuitive result, since the PPM will consume more over time when the market is volatile and consume less when the market is not volatile. Observe that the taste of consumption will be negative if the market is absolutely riskless.

6.2. **Special Cases: $\theta = (0,0)'$, $\rho \neq 0$; $\rho = 0$, $\theta \in \mathcal{R}^2_+$. Special Case I:** Suppose $\theta = (0,0)'$ and $\rho \neq 0$, then (6.29) becomes

$$C^*(t) = \frac{2P^m(t)e^{\rho t}[e^{\rho t}-e^{\rho T}-\rho]}{\rho} + \frac{2\rho X^*(t)e^{\rho t}}{\rho-e^{\rho t}+e^{\rho T}} - \frac{2\rho\varphi e^{(\rho+r)t}}{\rho-e^{\rho t}+e^{\rho T}}. \tag{6.35}$$

(6.35) shows the consumption level when the investment is not in the risky assets. It implies that consumption level of the investor do not depends on the uncertainty of the market over time, but upon the riskless asset. In this case, the initial consumption level and the terminal consumption level are given respectively in (6.36) and (6.37). At $t = 0$, (6.35) becomes

$$C^*(0) = \frac{2P^m_0[1-e^{\rho T}-\rho]}{\rho} + \frac{2\rho x_0}{\rho-1+e^{\rho T}} - \frac{2\rho\varphi}{\rho-1+e^{\rho T}}. \tag{6.36}$$

At $t = T$, (6.35) becomes

$$C^*(T) = -2P^m(T)e^{\rho T} + 2X^*(T)e^{\rho T} - 2\varphi e^{(\rho+r)T}. \tag{6.37}$$

Special Case II: Suppose $\rho = 0$, and $\theta \in \mathcal{R}^2_+$, then (6.29) becomes

$$C^*(t) = \frac{2P^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)}[e^{-\theta'\theta(T-t)}-1-\theta'\theta]}{\theta'\theta} + \frac{2\theta'\theta X^*(t)e^{-\theta'\theta(T-t)}}{\theta'\theta-e^{-\theta'\theta(T-t)}+1} - \frac{2\theta'\theta\varphi e^{(r+\|\theta\|^2)t+\theta'W(t)}}{\theta'\theta-e^{-\theta'\theta(T-t)}+1}. \tag{6.38}$$

(6.38) shows the consumption level when the sharpe ratio $\theta'\theta$ is not zero and the discount factor, ρ is zero. It is observe that consumption level strictly depend on the risky assets with respect to the riskless one. We observe that the market is booming, consumption reduces, and vice versa. In this case, the initial and terminal consumption level are given respectively in (6.39) and (6.40).

Similarly, at $t = 0$, (6.38) becomes

$$C^*(0) = \frac{2P^m_0 e^{4(3\theta'\theta-2\sigma_Y\theta)T}[e^{-\theta'\theta T}-1-\theta'\theta]}{\theta'\theta} + \frac{2\theta'\theta x_0 e^{-\theta'\theta T}}{\theta'\theta-e^{-\theta'\theta T}+1} - \frac{2\theta'\theta\varphi}{\theta'\theta-e^{-\theta'\theta T}+1}. \tag{6.39}$$

At $t = T$, (6.38) becomes

$$C^*(T) = -2P^m(T) + 2X^*(T) - 2\varphi e^{(r+\|\theta\|^2)T+\theta'W(T)}. \tag{6.40}$$

We therefore have the following propositions.

Proposition 6.2. *Let*

$$C^*_\rho(t) = \frac{2P^m(t)e^{\rho t}[e^{\rho t}-e^{\rho T}-\rho]}{\rho} + \frac{2\rho X^*(t)e^{\rho t}}{\rho-e^{\rho t}+e^{\rho T}} - \frac{2\rho\varphi e^{(\rho+r)t}}{\rho-e^{\rho t}+e^{\rho T}}$$

and

$$C^*_\theta(t) = \tfrac{2P^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)}[e^{-\theta'\theta(T-t)}-1-\theta'\theta]}{\theta'\theta} + \tfrac{2\theta'\theta X^*(t)e^{-\theta'\theta(T-t)}}{\theta'\theta-e^{-\theta'\theta(T-t)}+1} - \tfrac{2\theta'\theta\varphi e^{(r+\|\theta\|^2)t+\theta' W(t)}}{\theta'\theta-e^{-\theta'\theta(T-t)}+1},$$

then

$$C^*(t) = \begin{cases} C^*_\rho(t), \text{ if } \theta'\theta = 0,\ \rho \neq 0, \\ C^*_\theta(t), \text{ if } \theta \in \mathcal{R}^2_+,\ \rho = 0. \end{cases} \tag{6.41}$$

7. The Efficient Frontier

In this section, we determine the efficient frontier of the surplus process. Substituting (6.23) and (6.24) into (4.22), we have the dynamics of the surplus as follows:

$$\begin{aligned} d\tilde{V}^*(t) &= [(\tilde{P}^m(t)(1+f(t)) + \tilde{\Phi}(t))(\sigma'_Y - \theta)']dW(t) \\ &-2e^{\rho t}[\tilde{P}^m(t)g(t) + \tfrac{(\theta'\theta+\rho)(\tilde{P}^m(t)-\tilde{\Phi}(t))e^{-\theta'\theta(T-t)}-(\theta'\theta+\rho)\varphi}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}} \\ &+\tfrac{(\theta'\theta+\rho)\tilde{V}^*(t)e^{-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}]dt, \tilde{V}^*(0) = v_0, \end{aligned} \tag{7.1}$$

where

$$f(t) = -4e^{(13\theta'\theta+8\sigma_Y\theta)(T-t)}\frac{(\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T})^2}{(\theta'\theta+\rho)^2},$$

$$g(t) = e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)}\frac{e^{-\theta'\theta(T-t)+\rho t}-e^{\rho T}-(\theta'\theta+\rho)}{\theta'\theta+\rho}.$$

Re-writing (7.1) in a more compact form, we have

$$d\tilde{V}^*(t) = (K(t)\tilde{V}^*(t) + G(t) + \varphi\alpha(t))dt + F(t)'dW(t), \tilde{V}^*(0) = v_0, \tag{7.2}$$

where

$$F(t) = (\tilde{P}^m(t)(1+f(t)) + \tilde{\Phi}(t))(\sigma'_Y - \theta)$$

$$G(t) = -2e^{\rho t}[\tilde{P}^m(t)g(t) + \frac{(\theta'\theta+\rho)(\tilde{P}^m(t)-\tilde{\Phi}(t))e^{-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}]$$

$$\alpha(t) = \frac{2(\theta'\theta+\rho)e^{\rho t}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}$$

$$K(t) = -\frac{2(\theta'\theta+\rho)e^{\rho t-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}. \tag{7.3}$$

Applying Ito Lemma on (7.2), we have

$$\begin{aligned} d\tilde{V}^{*2}(t) &= (2K(t)\tilde{V}^{*2}(t) + 2\tilde{V}^*(t)G(t) + 2\varphi\tilde{V}^*(t)\alpha(t) \\ &+F(t)'F(t))dt + F(t)'dW(t), \tilde{V}^{*2}(0) = v_0^2, \end{aligned} \tag{7.4}$$

Taking the mathematical expectation of (7.2) and (7.4), we have

$$dE(\tilde{V}^*)(t) = (K(t)E(\tilde{V}^*(t)) + E(G(t)) + \varphi\alpha(t))dt, E(\tilde{V}^*)(0) = v_0, \tag{7.5}$$

$$\begin{aligned} dE(\tilde{V}^{*2})(t) &= (2K(t)E(\tilde{V}^{*2})(t) + 2E(\tilde{V}^*)(t)E(G(t)) + 2\varphi E(\tilde{V}^*)(t)\alpha(t) \\ &+E(F(t))'E(F(t)))dt, E(\tilde{V}^{*2})(0) = v_0^2, \end{aligned} \tag{7.6}$$

Solving the ordinary differential equations (ODEs), (7.5) and (7.6), we have followings:

$$E(\tilde{V}^*)(t) = A(t)e^{\int_0^t K(s)ds} + \varphi e^{\int_0^t K(s)ds}\int_0^t e^{-\int_0^t K(s)ds}\alpha(s)ds, \tag{7.7}$$

where

$$A(t) = v_0 + \int_0^t e^{-\int_0^t K(s)ds} G(s)ds.$$

(7.8)

$$\begin{aligned}&E(\tilde{V}^{*2})(t) = v_0^2 e^{2\int_0^t K(\tau)d\tau} + e^{2\int_0^t K(\tau)d\tau} \int_0^t E(F(\tau))' E(F(\tau)) d\tau + \\ &2e^{2\int_0^t K(\tau)d\tau} \int_0^t E(G(\tau)) \left(A(\tau)e^{\int_0^\tau K(u)du} + \varphi e^{\int_0^\tau K(u)du} \int_0^\tau e^{-\int_0^s K(u)du} \alpha(s)ds\right) d\tau + \\ &2\varphi e^{2\int_0^t K(\tau)d\tau} \int_0^t \alpha(\tau) \left(A(\tau)e^{\int_0^\tau K(u)du} + \varphi e^{\int_0^\tau K(u)du} \int_0^\tau e^{-\int_0^s K(u)du} \alpha(s)ds\right) d\tau\end{aligned}$$

Simplifying (7.8), we have

(7.9)

$$\begin{aligned}&E(\tilde{V}^{*2})(t) = v_0^2 e^{2\int_0^t K(\tau)d\tau} + e^{2\int_0^t K(\tau)d\tau} \int_0^t E(F(\tau))' E(F(\tau)) d\tau \\ &+2e^{2\int_0^t K(\tau)d\tau} \int_0^t E(G(\tau))A(\tau)e^{\int_0^\tau K(u)du} d\tau + 2\varphi e^{2\int_0^t K(\tau)d\tau} \int_0^t \alpha(\tau)A(\tau)e^{\int_0^\tau K(u)du} d\tau \\ &+2\varphi e^{2\int_0^t K(\tau)d\tau} \int_0^t \int_0^\tau e^{\int_0^\tau K(u)du} e^{-\int_0^s K(u)du} E(G(\tau))\alpha(s)dsd\tau \\ &+2\varphi^2 e^{2\int_0^t K(\tau)d\tau} \int_0^t \int_0^\tau e^{\int_0^\tau K(u)du} e^{-\int_0^s K(u)du} \alpha(\tau)\alpha(s)dsd\tau\end{aligned}$$

Re-writing (7.9) in compact form, we have

$$\begin{aligned}&E(\tilde{V}^{*2})(t) = v_0^2 e^{2\int_0^t K(\tau)d\tau} + D_1(t)e^{2\int_0^t K(\tau)d\tau} \\ &+2D_2(t)\varphi e^{2\int_0^t K(\tau)d\tau} + 2D_3(t)\varphi^2 e^{2\int_0^t K(\tau)d\tau}\end{aligned} \tag{7.10}$$

where

$$D_1(t) = \int_0^t E(F(\tau))' E(F(\tau)) d\tau + 2\int_0^t E(G(\tau))A(\tau)d\tau$$

$$D_2(t) = \int_0^t \int_0^\tau e^{\int_0^\tau K(u)du} e^{-\int_0^s K(u)du} E(G(\tau))\alpha(s)dsd\tau + \int_0^t \alpha(\tau)A(\tau)d\tau$$

$$D_3(t) = 2\int_0^t \int_0^\tau e^{\int_0^\tau K(u)du} e^{-\int_0^s K(u)du} \alpha(\tau)\alpha(s)dsd\tau$$

At $t = T$, (7.7) and (7.10) becomes:

$$E(\tilde{V}^*)(T) = A(T)\gamma + \varphi\gamma\omega, \tag{7.11}$$

where, $\gamma = e^{\int_0^T K(u)du}$, $\omega = \int_0^T e^{-\int_0^T K(s)ds} \alpha(s)ds$.

Lemma 7.1. *Suppose that $K(t)$ satisfies (7.14), then*

$$\gamma = \left(\frac{(\theta'\theta + \rho)e^{\theta'\theta T}}{1 - e^{\theta'\theta T}(\theta'\theta + \rho + e^{\rho T})}\right)^2.$$

Proof: Using (7.14), we have that

$$\int_0^t K(u)du = 2\log_e \left(\frac{e^{(\theta'\theta+\rho)t} - e^{(\theta'\theta+\rho)T} - (\theta'\theta + \rho)e^{\theta'\theta T}}{1 - e^{\theta'\theta T}(\theta'\theta + \rho + e^{\rho T})}\right).$$

It implies that

$$e^{\int_0^t K(u)du} = \left(\frac{e^{(\theta'\theta+\rho)t} - e^{(\theta'\theta+\rho)T} - (\theta'\theta + \rho)e^{\theta'\theta T}}{1 - e^{\theta'\theta T}(\theta'\theta + \rho + e^{\rho T})}\right)^2.$$

Therefore, setting $t = T$, we have

$$\gamma = \left(\frac{(\theta'\theta + \rho)e^{\theta'\theta T}}{1 - e^{\theta'\theta T}(\theta'\theta + \rho + e^{\rho T})}\right)^2.$$

Using Lemma 7.1, the second moments of the surplus process becomes

$$E(\tilde{V}^{*2})(T) = v_0^2\gamma^2 + D_1(T)\gamma^2 + 2D_2(T)\varphi\gamma^2 + D_3(T)\varphi^2\gamma^2 \tag{7.12}$$

Substituting (7.11) into (7.12), we have

$$\begin{aligned} E(\tilde{V}^{*2})(T) &= v_0^2\gamma^2 + D_1(T)\gamma^2 + \tfrac{2\gamma D_2(T)}{\omega}(E(\tilde{V}^*(T)) - A(T)\gamma) \\ &+ \tfrac{D_3(T)}{\omega^2}(E(\tilde{V}^*(T)) - A(T)\gamma)^2 \end{aligned} \tag{7.13}$$

The variance of the discounted surplus process for the stakeholders is

$$\begin{aligned}
Var(\tilde{V}^*(T)) &= E(\tilde{V}^{*2})(T) - (E(\tilde{V}^*)(T))^2 \\
&= v_0^2\gamma^2 + D_1(T)\gamma^2 + \frac{2\gamma D_2(T)}{\omega}(E(\tilde{V}^*(T)) - A(T)\gamma) \\
&+ \frac{D_3(T)}{\omega^2}(E(\tilde{V}^*(T)) - A(T)\gamma)^2 - (E(\tilde{V}^*)(T))^2 \\
&= v_0^2\gamma^2 - A(T)^2\gamma^2 + D_1(T)\gamma^2 \\
&+ 2\gamma\left(\frac{D_2(T)}{\omega} - A(T)\right)(E(\tilde{V}^*(T)) - A(T)\gamma) \\
&+ \left(\frac{D_3(T)}{\omega^2} - 1\right)(E(\tilde{V}^*(T)) - A(T)\gamma)^2 \\
&= v_0^2\gamma^2 - A(T)^2\gamma^2 + D_1(T)\gamma^2 - \frac{\gamma^2\left(\frac{D_2(T)}{\omega} - A(T)\right)^2}{\left(\frac{D_3(T)}{\omega^2} - 1\right)} \\
&+ \left(\frac{D_3(T)}{\omega^2} - 1\right)[\gamma^2\frac{\left(\frac{D_2(T)}{\omega} - A(T)\right)^2}{\left(\frac{D_3(T)}{\omega^2} - 1\right)^2} + 2\gamma\frac{\left(\frac{D_2(T)}{\omega} - A(T)\right)}{\left(\frac{D_3(T)}{\omega^2} - 1\right)} \\
&\times (E(\tilde{V}^*(T)) - A(T)\gamma) + (E(\tilde{V}^*(T)) - A(T)\gamma)^2] \\
&= \gamma^2 Q + \left(\frac{D_3(T)}{\omega^2} - 1\right)\left[\gamma\frac{\left(\frac{D_2(T)}{\omega} - A(T)\right)}{\left(\frac{D_3(T)}{\omega^2} - 1\right)} + E(\tilde{V}^*(T)) - A(T)\gamma\right]^2
\end{aligned}$$

where

$$Q = v_0^2 - A(T)^2 + D_1(T) - \frac{\left(\frac{D_2(T)}{\omega} - A(T)\right)^2}{\left(\frac{D_3(T)}{\omega^2} - 1\right)}.$$

Therefore, the efficient frontier of discounted surplus is obtain as

$$E(\tilde{V}^*(T)) = \frac{\gamma(2\omega A(T) - D_2(T))}{\omega\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}} + \frac{\sqrt{\sigma^2_{V^*(T)} - \gamma^2 Q}}{\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}}. \tag{7.14}$$

From (7.14), shows a kind quadratic relation between optimal discounted surplus and its variance. The minimum possible variance, $Var(\tilde{V}(^*(T)) = \gamma^2 Q \geq 0$, could be attained when the stakeholder borrows money from the total amount of surplus at time $t = 0$ for T years, so that

$$E(\tilde{V}(^*(T)) = \frac{\gamma(2\omega A(T) - D_2(T))}{\omega\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}}.$$

We now establish the efficient frontier of the optimal terminal surplus of the stakeholders. The expected surplus and the variance, $\sigma^2_{V^*(T)}$, at time T are related by the following (7.15).

Proposition 7.1. *Suppose (7.14) holds and* $E(\Lambda(T)) = e^{-(r+2\|\theta\|^2)T}$, *then*

$$E(V^*(T)) = \frac{\gamma(2\omega A(T) - D_2(T))e^{(r+2\|\theta\|^2)T}}{\omega\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}} + \frac{e^{(r+2\|\theta\|^2)T}\sqrt{\sigma^2_{V^*(T)} - \gamma^2 Q}}{\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}}. \tag{7.15}$$

From (7.15), shows the quadratic relation between surplus and its variance. The minimum possible variance, $Var(V^*(T)) = \gamma^2 Q \geq 0$, could be attained when the stakeholder borrows money from the total amount of surplus at time $t = 0$ for T years, so that

$$E(V^*(T)) = \frac{\gamma(2\omega A(T) - D_2(T))e^{(r+2\|\theta\|^2)T}}{\omega\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}}.$$

We observe that if $\frac{D_3(T)}{\omega^2} = 1$, we have infinite slope, if $\frac{D_3(T)}{\omega^2} > 1$, we have real slope and complex slope if $\frac{D_3(T)}{\omega^2} < 1$.

8. Optimal Pension Benefit for a PPM at Retirement

In this section, we consider the optimal benefit that will accrued to the PPM at retirement. By definition, the benefit that will accrued to a PPM at the final time, T is given by

$$P(T) = P^m(T) + \Theta_T(P^m(T)). \tag{8.1}$$

Proposition 8.1. *: Let* $\tilde{\Theta}^*_T(\tilde{P}^m(T))$ *be the optimal discounted surplus function at the final time,* T, *then*

$$\tilde{P}^*(T) = \tilde{P}^m(T) + \tilde{\Theta}^*_T(\tilde{P}^m(T)), \tag{8.2}$$

with

$$\tilde{V}^*(T) = v_0 + \int_0^T (K(t)\tilde{V}^*(t) + G(t) + \varphi\alpha(t))dt + \int_0^T F(t)'dW(t). \tag{8.3}$$

Corollary 8.1. *: Let* $\Theta^*_T(P^m(T))$ *be the optimal surplus function at the final time,* T, *then*

$$P^*(T) = P^m(T) + \Theta^*_T(P^m(T)), \tag{8.4}$$

with

(8.5)

$$V^*(T) = \frac{1}{\Lambda(T)}\left(v_0 + \int_0^T (K(t)\tilde{V}^*(t) + G(t) + \varphi\alpha(t))dt + \int_0^T F(t)'dW(t)\right).$$

Corollary 8.2. *: Let $E(\tilde{\Theta}^*_T(\tilde{P}^m(T)))$ be the expected optimal discounted surplus function at the final time, T, then*

$$E(\tilde{P}^*(T)) = E(\tilde{P}^m(T)) + E(\tilde{\Theta}^*_T(\tilde{P}^m(T))), \tag{8.6}$$

with

$$E(\tilde{V}^*(T)) = v_0 + \int_0^T (K(t)E(\tilde{V}^*(t)) + E(G(t)) + \varphi\alpha(t))dt. \tag{8.7}$$

Corollary 8.3. *: Let $E(\Theta^*_T(P^m(T)))$ be the optimal expected surplus function at the final time, T, then*

$$E(P^*(T)) = E(P^m(T)) + E(\Theta^*_T(P^m(T))), \tag{8.8}$$

with

$$E(V^*(T)) = \frac{1}{E(\Lambda(T))} E\left[v_0 + \int_0^T (K(t)\tilde{V}^*(t) + G(t) + \varphi\alpha(t))dt\right]. \tag{8.9}$$

9. Numerical Illustration

In this section, we give numerical illustration of our results in the previous sections. The aim of this numerical illustration is to observe the nature of the expected final optimal surplus (both discounted case and the real case) as against the final standard deviation, the initial optimal consumption, optimal final pension benefits, minimum pension benefits, with respect to the terminal time to retirement, the expected final surplus and parameter, ψ given to the minimization of the variance. The values of parameters that we consider are as followings.

$c = 0.15$, $\eta = 0.01$, $r = 0.04$, $\psi = 10$, $\xi = 0.3$, $\rho = 0.01$, $\mu = 0.09$, $\sigma_Y = (0.25, 0.32)$, $\theta_I = 0.02$, $\beta = 0.0292$, $y_0 = 0.8$, $x_0 = 1$, $\sigma_S = 0.35$, $\sigma_1 = 0.23$.

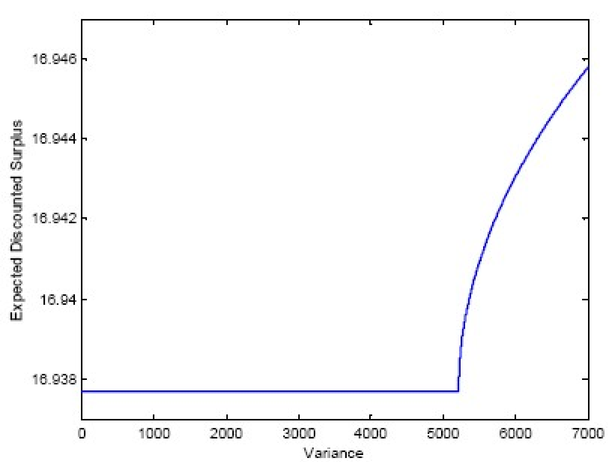


Figure 1. Efficient Frontier

Table 1: Initial Optimal Consumption, C_0, with $\tilde{z} = E(\tilde{V}^*(T))$

$\tilde{z}$	$\psi = 0.1$ $T = 1$	$\psi = 0.1$ $T = 2$	$\psi = 0.1$ $T = 10$	$\psi = 0.1$ $T = 20$	$\psi = 1$ $T = 1$	$\psi = 1$ $T = 2$	$\psi = 1$ $T = 10$	$\psi = 1$ $T = 20$
6	-10.4172	-7.8403	-13.3382	-21.0730	-5.9279	-4.8594	-12.5551	-20.6823
10	-14.4077	-10.4901	-14.0343	-21.4202	-9.9184	-7.5091	-13.2512	-21.0296
20	-24.3838	-17.1144	-15.7745	-22.2883	-19.8945	-14.1334	-14.9914	-21.8877
30	-34.3600	-23.7387	-17.5148	-23.1564	-29.8707	-20.7577	-16.7317	-22.7658
35	-39.3481	-27.0509	-18.3849	-23.5905	-34.8588	-24.0699	-17.6018	-23.1998

Continuation of Table 1

$\tilde{z}$	$\psi = 10$ $T = 1$	$\psi = 10$ $T = 2$	$\psi = 10$ $T = 10$	$\psi = 10$ $T = 20$	$\psi = \infty$ $T = 1$	$\psi = \infty$ $T = 2$	$\psi = \infty$ $T = 10$	$\psi = \infty$ $T = 10$
6	-5.4790	-4.5613	-2.4767	-20.6433	-5.4266	-4.2949	-3.36116	-1.4281
10	-9.4695	-7.2110	-13.1728	-20.9905	-9.4443	-6.9929	-4.1453	-1.8692
20	-19.4456	-13.8353	-14.8131	-21.8586	-19.4886	-13.7380	-6.1057	-2.9719
30	-29.4200	-20.4597	-16.6534	-22.7267	-29.5328	-20.4830	-8.0661	-4.0747
35	-34.4099	-23.7718	-17.5235	-23.1608	-34.5550	-23.8556	-9.0463	-4.6260

Table 2: EODS, EODPB and Minimum Pension Benefit for a PPM

T	EV^*	$E\tilde{V}^*$	$E\tilde{P}^m$	EP^m	$E\tilde{P}^*$ $h = 0.2$	EP^* $h = 0.2$	$E\tilde{P}^*$ $h = 0.3$	EP^* $h = 0.3$	$E\tilde{P}^*$ $h = 0.4$	EP^* $h = 0.4$
1	-2.8513	-2.7400	0.1202	0.1189	-2.1676	-2.2573	-1.8816	-1.9603	-1.5956	-1.6632
2	-0.7245	0.2409	0.2409	0.2357	-0.5314	-0.5807	-0.4349	-0.4786	-0.3383	-0.3766
3	0.2708	0.2402	0.3622	0.3567	0.2646	0.2868	0.2768	0.2948	0.2890	0.3027
4	1.4329	1.2211	0.4844	0.4640	1.0737	1.2391	1.0001	1.1422	0.9264	1.0453
5	2.7673	2.2657	0.6077	0.5758	1.9341	2.3290	1.7683	2.1099	1.6025	1.8907
6	4.2720	3.3605	0.7322	0.6853	2.8348	3.5548	2.5720	3.1963	2.3091	2.8377
10	11.7552	7.8797	1.2463	1.1187	6.5531	9.6279	5.8897	8.5643	5.2264	7.5006
20	40.9465	18.3985	2.7168	2.1890	15.2621	33.1950	13.6940	29.3193	12.1258	25.4435

Continuation of Table 2

T	$E\tilde{P}^*$ $h = 0.5$	EP^* $h = 0.5$	$E\tilde{P}^*$ $h = 0.6$	EP^* $h = 0.6$	$E\tilde{P}^*$ $h = 0.7$	EP^* $h = 0.7$	$E\tilde{P}^*$ $h = 0.8$	EP^* $h = 0.8$
1	-1.3097	-1.3662	-1.0237	-1.0692	-0.7377	-0.7722	-0.4518	-0.4751
2	-0.2418	-0.2745	-0.1453	-0.1725	-0.0487	-0.0704	0.0478	0.03161
3	0.3012	0.3107	0.3134	0.3187	0.3256	0.3267	0.3378	0.3347
4	0.8527	0.9484	0.7791	0.8515	0.7054	0.7546	0.6318	0.6577
5	1.4367	1.6715	1.2709	1.4524	1.1051	1.2332	0.9393	1.0141
6	2.0463	2.4791	1.7835	2.1205	1.5207	1.7620	1.2578	1.4034
10	4.5630	6.4370	3.8997	5.3733	3.2363	4.3097	2.5730	3.2460
20	10.5576	21.5678	8.9895	17.6920	7.4213	13.8163	5.8531	9.9405

EODS denotes Expected Optimal Discounted Surplus, EODPB denotes Expected Optimal Discounted Pension Benefit.

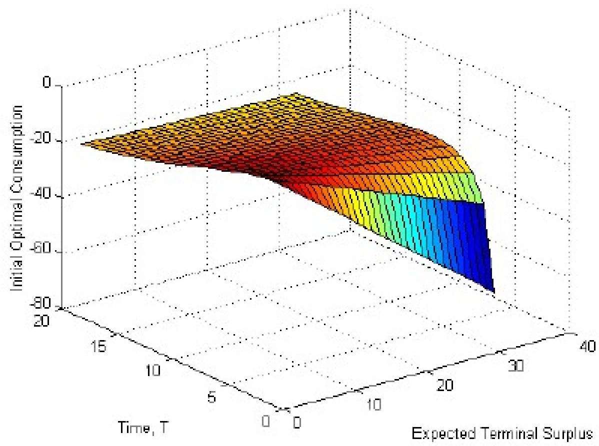


Figure 2. Initial Optimal Consumption

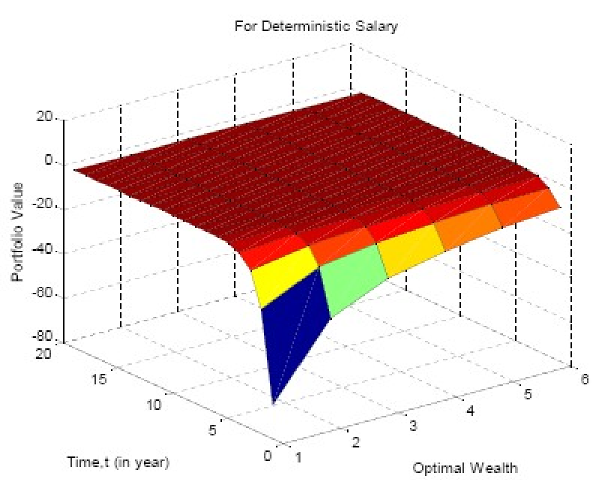


Figure 3. Portfolio Value in Cash Account

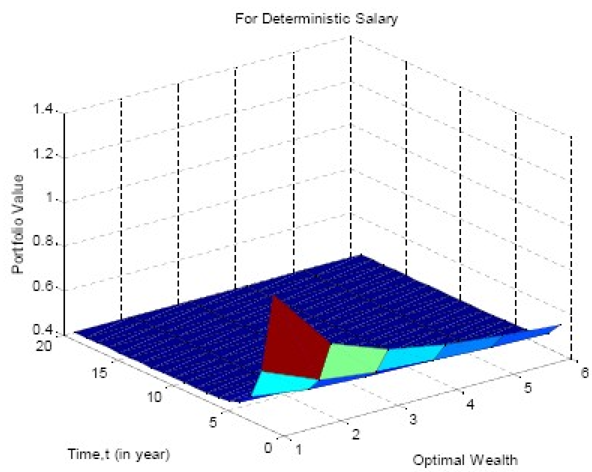


Figure 4. Portfolio Value in Index Bond

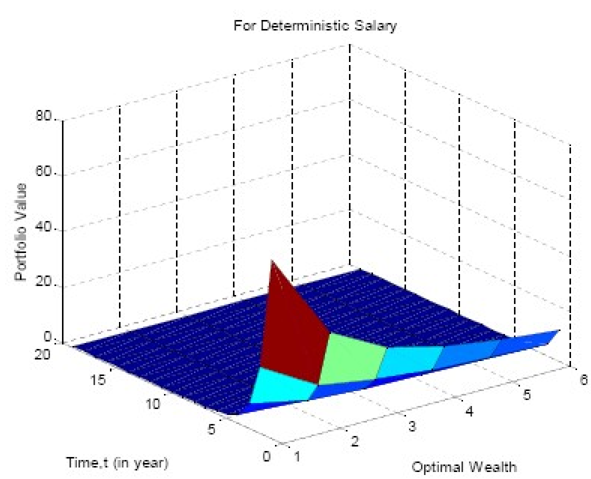


Figure 5. Portfolio Value in Stock

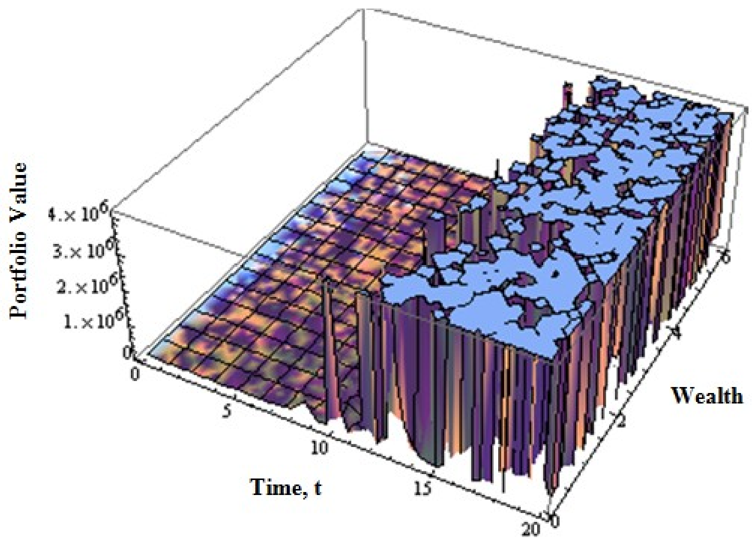


Figure 6. Portfolio Value in Index bond for Stochastic Salary

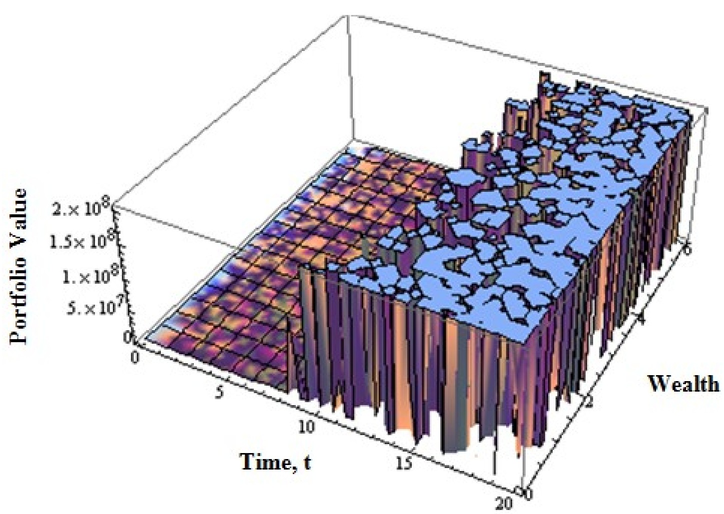


Figure 7. Portfolio Value in Stock for Stochastic Salary

Table 1 shows the initial optimal consumption of a PPM at various values of expected discounted surplus in a varying value of variance minimizer parameter, ψ at time $T = 1, 2, 10$ and 20 years. We observed at different values of ψ and T, that as the expected optimal surplus increases, the initial optimal consumption decreases and vice versa. This shows that the positive growth of the surplus (resulting from the positive growth of the financial market and effective management on the part of the PFA) is capable of discouraging consumption. It was also observed that as ψ increases, the initial optimal consumption increases. This shows that as the market continues to be volatile, consumption rate will continue increase and vice versa.

Table 2 show the expected optimal surplus, expected discounted optimal surplus, expected discounted minimum pension benefit, expected minimum pension benefit, expected discounted total benefit and expected total pension benefit for a PPM, at time $T = 1, 2, 3, 4, 5, 6, 10$ and 20 years, and at different value of h (i.e., the proportion of the surplus that will accrued to the PFA). It was observed that the optimal surplus increases in time, T. It also observed that as the value of h increases, the benefit of the PPM will decrease, and vice versa, which is an obvious result. Hence, what ever the bargain between the PPM and PFA on the sharing of the surplus will be, the values of the PPM minimum and total pension benefit are given in table 2.

Figure **??** shows the efficient frontier of the surplus process. We observed that the shape of the efficient frontier is parabolic in nature. It is also observed that the minimum possible variance is attained when the PFA borrows the amount of about 5200 from the total surplus at time $t = 0$ for time $t = T$. Figure **??** shows the initial optimal consumption of a PPM with $0 - 40$ optimal surplus at time period $0 - 20$ years. It is observed that the initial optimal consumption of a PPM remain negative over the time period, T. This confirm the results obtained in Table 1. Figure **??** shows the portfolio value of the investment in cash account and figure **??** and figure **??** show the portfolio values of a PPM in index bond and stock respectively, under deterministic salary of a PPM. Figure **??** and figure **??** show the portfolio values of a PPM in index bond and stock under stochastic salary. Figure **??**, figure **??** and figure **??** tell us that the fund should be invested in index bond and stock only and that cash account should be shorten and then invest in the risky assets (which include index bond and stock), with high proportion of it being invested in stock over time in order to attain the required target.

10. Conclusion

This paper have studied the management of a stochastic pension funding process of a defined contributory pension scheme. The objectives are to determine the minimum pension benefits, total pension benefits, optimal consumption and optimal investment strategies maximizing the expected terminal surplus and simultaneously minimizing the variance of the terminal surplus. The financial market is made up of cash account, index bond and stock. The salary of the pension plan member is stochastic. The problem was formulated as a modified mean-variance optimization problem and was solved using dynamic programming approach.

The efficient frontier which was found to be nonlinear (i.e., possess a parabolic shape). The optimal investment strategies have two components. The first component depends ultimately on the risky assets and its correlation. The second component is proportional to the ratio of the present expected value of PPM's minimum benefit to the optimal wealth. The second component is the inter-temporal hedging terms that offset any shock to the stochastic finding overtime.

The optimal consumption (both for real and discounted cases) plan have three components. First component depends on the current level of minimum pension benefit, with a coefficient involving the instantaneous variance of salary, preference rate of consumption and risky assets. The second component depends on the optimal wealth, preference rate of consumption and risky assets. The third component is proportional to the present expected value of discounted surplus planned, with coefficient involving preference rate of consumption, variance minimizer, short term interest rate, risky assets and the Brownian motion term, which shows that the consumption of the PPM is stochastic. We found the as the variance minimizing parameter tends to zero, the consumption level tends to negative infinity. This shows that PPM will consume more over time when the market is volatile and consume less when the market is less volatile. Also, the taste of consumption will be negative, if the market is absolutely riskless.

The optimal terminal surplus for the stakeholders was determined in this paper. The pension fund administrator (PFA) was encouraged by sharing the surplus arising from the investment with the PPM. This strategy will go a long way increasing the final benefit that will accrued to the PPM at retirement. The PFA charge propositional administrative costs (AC) for the management of the fund. This costs is on the PPM stochastic contributions into the scheme.

The minimum pension benefit is taken not to be less than the gross contributions of the PPM. It implies that the total benefit must be greater or equal to the minimum pension benefit.

A numerical illustrations show the analytical results and models established in the paper.

References

[1] Bajeux-Besnainou I. and Portait, R. (1998). Dynamic asset allocation in a mean-variance framework. *Management Science*, 44, S79-S95.

[2] Battocchio, P. and Menoncin, F. (2004). Optimal pension management in a stochastic framework. *Insurance: Mathematics and Economics*, 34, 79-95.

[3] Bielecky, T., Jim, H., Pliska, S. and Zhou, X. (2005). Continuous-time mean-variance portfolio selection with bankruptcy prohibition. *Mathematical Finance*, 15, 213-244.

[4] Blake, D., Wright, D. and Zhang, Y. (2008). Optimal funding and investment strategies in defined contribution pension plans under Epstein-Zin utility. Discussion paper, the pensions Institute, Cass Business School, City University, UK.

[5] Boulier, J. F., Huang, S. J. and Taillard, G. (2001). Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund. *Insurance: Mathematics and Economics*, 28, 173-189.

[6] Cairns, A. J. G., Blake, D. and Dowd, K. (2006). Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans. *Journal of Economic Dynamic and Control*, 30, 843-377.

[7] Chiu, M. and Li, D. (2006). Asset and liability management under a continuous-time mean-variance optimization framework. *Insurance: Mathematics and Economics*, 39, 330-355.

[8] Colombo, L. and Haberman, S. (2005). Optimal contributions in a defined benefit pension scheme with stochastic new entrants. *Insurance: Mathematics and Economics* 37, 335354.

[9] Da Cunha, N.O., Polak, E., (1967). Constrained minimization under vector-valued criteria in finite dimensional spaces. *Journal of Mathematical Analysis and Applications* 19, 103124.

[10] Deelstra, G., Grasselli, M. and Koehl, P. (2000). Optimal investment strategies in a CIR framework. *Journal of Applied Probability*, 37, 936-946.

[11] Devolder, P. Bosch Princep, M.and Fabian, I. D. (2003). Stochastic optimal control of annuity contracts. *Insurance: Mathematics and Economics*, 33, 227-238.

[12] Di Giacinto, M., Federico, S. and Gozzi, F. (2010). Pension funds with a minimum guarantee: a stochastic control approach. *Finance and Stochastic*.

[13] Gao, J. (2008). Stochastic optimal control of DC pension funds. *Insurance: Mathematics and Economics*, 42, pp. 1159-1164.

[14] Gerrard, R., Haberman S. and Vigna, E. (2004). Optimal investment choices post retirement in a defined contribution pension scheme. *Insurance: Mathematics and Economics*, 35, 321-342.

[15] Haberman, S., Sung, J.H., 1994. Dynamics approaches to pension funding. *Insurance: Mathematics and Economics* 15, 151162.

[16] Haberman, S., Butt, Z., Megaloudi, C., (2000). Contribution and solvency risk in a defined benefit pension scheme. *Insurance: Mathematics and Economics* 27, 237259.

[17] Haberman, S. and Vigna, E. (2002). Optimal investment strategies and risk measures in defined contribution pension schemes. *Insurance: Mathematics and Economics*, 31, 35-69.

[18] Hϕjgaard, B. and Vigna, E. (2007). Mean-variance portfolio selection and efficient frontier for defined contribution pension schemes. technical report R-2007-13, Department of Mathematical Sciences, Aalborg University.

[19] Huang, H.C., Cairns, A.J.G., (2005). On the control of defined-benefit pension plans. *Insurance: Mathematics and Economics* 38, 113131.

[20] Jensen, B.A. and Sϕrensen, C. (1999). Paying for minimum interest guarantees. Who should compensate who? *European Financial Management* 7: 183-211.

[21] Josa-Fombellida, R., Rincon-Zapatero, J.P., (2001). Minimization of risks in pension funding by means of contribution and portfolio selection. *Insurance: Mathematics and Economics* 29, 3545.

[22] Josa-Fombellida, R., Rincon-Zapatero, J.P., (2004). Optimal risk management in defined benefit stochastic pension funds. *Insurance: Mathematics and Economics* 34, 489503.

[23] Josa-Fombellida, R. and Rincón-Zapatero, J. (2008). Mean-variance portfolio and contribution selection in stochastic pension funding. *European Journal of Operational Research*, 187, 120-137.

[24] Korn, R, and Krekel, M. (2001). Optimal portfolios with fixed consumption or income streams. Working paper, University of Kaiserslautern.

[25] Li, D. and Ng, W. -L. (2000). Optimal dynamic portfolio selection: multiperiod mean-variance formulation. *Mathematical Finance*, 10, 387-406.

[26] Markowitz, H. (1952). Portfolio selection. *Journal of finance*, 7, 77-91.

[27] Markowitz, H. (1959). Portfolio selection: efficient diversification of investments, New York, Wiley.

[28] Merton, R.C. (1971). Optimal consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* 3, 373413.

[29] Nkeki, C. I. (2011). On optimal portfolio management of the accumulation phase of a defined contributory pension scheme. Ph.D thesis, Department of Mathematics, University of Ibadan, Ibadan, Nigeria.

[30] Nkeki, C. I. and Nwozo, C. R. (2012), Variational form of classical portfolio strategy and expected wealth for a defined contributory pension scheme. *Journal of Mathematical Finance* 2(1): 132-139.

[31] Nkeki, C. I, (2013). Optimal portfolio strategy with discounted stochastic cash inflows. *Journal of Mathematical Finance,* 3: 130-137.

[32] Nkeki, C. I. (2012). Mean-variance portfolio selection with inflation hedging strategy: a case of a defined contributory pension scheme. *Theory and Applications of Mathematics and Computer Science*, 2(2), 67-82.

[33] Nkeki, C. I. and Nwozo, C. R. (2013). Optimal investment under inflation protection and optimal portfolios with stochastic cash flows strategy. To appear in *IAENG Journal of Applied Mathematics*.

[34] Nwozo, C. R. and Nkeki, C. I. (2011) Optimal portfolio and strategic consumption planning in a life-cycle of a pension plan member in a defined contributory pension scheme, *IAENG International Journal of Applied Mathematics*, 41(4), 299-309.

[35] Vigna, E. (2010). On efficiency of mean-variance based portfolio selection in DC pension schemes, Collegio Carlo Alberto Notebook, 154.

[36] Richardson, H. (1989). A minimum variance result in continuous trading portfolio optimization. *Management Science*, 35, 1045-1055.

[37] Zhou, X and Li, D. (2000). Continuous-time mean-variance portfolio selection: A stochastic LQ framework. *Applied Mathematics and Optimization*, 42, 19-33.

[38] Zhang, A., Korn, R. and Ewald, C. O. (2007). Optimal management and inflation protection for defined contribution pension plans, Working paper, University of St. Andrews.

Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 245–253 (2015) ©KJM

ON INVARIANT SUBMANIFOLDS OF ALMOST α-COSYMPLECTIC f-MANIFOLDS

SELAHATTIN BEYENDI, NESIP AKTAN, AND ALI IHSAN SIVRIDAĞ

Abstract. In this paper, we investigate some properties of invariant submanifolds of almost α-cosymplectic f- manifolds. We show that every invariant submanifold of an almost α-cosymplectic f- manifold with Kaehlerian leaves is also an almost α-cosymplectic f- manifold with Kaehlerian leaves. Moreover, we give a theorem on minimal invariant submanifold and obtain a necessary condition on a invariant submanifold to be totally geodesic. Finally, we study some properties of the curvature tensors of M and $\widetilde{M}$.

1. Introduction

In 1963, Yano [13] introducted an f-structure on a C^{∞} m-dimensional manifold M, defined by a non-vanishing tensor field φ of type $(1,1)$ which satisfies $\varphi^3+\varphi=0$ and has constant rank r. It is know that in this case r is even, $r=2n$. Moreover. TM splits into two complemantary subbundles $Im\varphi$ and $ker\varphi$ and the restriction of φ to $Im\varphi$ determines a complex structure on such subbundle. It is know that the exixtence of an f-structure on M is equivalent to a reduction of the structure group to $U(n)\times O(s)$ [2], where $s=m-2n$. The geometry of invariant submanifolds of a Riemannian manifold was studied by many geometers (see [3], [4], [6], [7], [8], [9], [10]). In general, the geometry an invariant submanifold inherits almost all properties of the ambient manifold. In 2014, Öztürk et.al. introduced and studied almost α-cosymplectic f-manifold [7] defined for any real number α which is defined a metric f-manifold with f-structure (φ,ξ_i,η^i,g) satisfying the condition $d\eta^i=0$, $d\Omega=2\alpha\overline{\eta}\wedge\Omega$.

In this paper, we introduce properties of invariant submanifolds of an almost α-cosymplectic f-manifold. In Section 2, we review basic formulas and definitions for almost α-cosymplectic f-manifolds. In Section 3, we show that every invariant submanifold of an almost α-cosymplectic f- manifold with Kaehlerian leaves is also an almost α-cosymplectic f- manifold with Kaehlerian leaves. Further, we give

Date: January 1, 2013 and, in revised form, February 2, 2013.
2000 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.
Key words and phrases. Almost α-cosypmlectic f-manifold, invariant submanifold, semi parallel and 2-semi parallel submanifold.

a theorem on minimal invariant submanifold and obtain a necessary condition on a invariant submanifold to be totally geodesic. In last section, we obtain some relations of curvature tensors M and $\widetilde{M}$.

2. Preliminaries

Let $\widetilde{M}$ be a real $(2n+s)$-dimensional framed metric manifold [12] with a framed $(\varphi, \xi_i, \eta^i, g)$, $i \in \{1, ..., s\}$, that is, φ is a non-vanishing tensor field of type (1,1) on $\widetilde{M}$ which satisfies $\varphi^3 + \varphi = 0$ and has constant rank $r = 2n$; $\xi_1, ...\xi_s$ are s vector fields; $\eta^1, ..., \eta^s$ are 1-forms and g is a Riemannian metric on $\widetilde{M}$ such that

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \tag{2.1}$$

$$\eta^i(\xi_j) = \delta^i_j,\ \varphi(\xi_i) = 0,\ \eta^i o\varphi = 0, \tag{2.2}$$

$$\eta^i(X) = g(X, \xi_i), \tag{2.3}$$

$$g(X, \varphi Y) + g(\varphi X, Y) = 0, \tag{2.4}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y) \tag{2.5}$$

for all $X, Y \in \Gamma(T\widetilde{M})$ and $i, j \in \{1, ..., s\}$. In above case, we say that $\widetilde{M}$ is a metric f-manifold and its associated structure will be denoted by $\widetilde{M}(\varphi, \xi_i, \eta^i, g)$ [12].
A 2-form Ω is defined by $\Omega(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(T\widetilde{M})$, is called the fundamental 2-form. A framed metric structure is called normal [12] if

$$[\varphi, \varphi] + 2d\eta^i \otimes \xi_i = 0$$

where $[\varphi, \varphi]$ is denoting the Nijenhuis tensor field associated to φ. Throughout this paper we denote by $\overline{\eta} = \eta^1+\eta^2+...+\eta^s$, $\overline{\xi} = \xi_1+\xi_2+...+\xi_s$ and $\overline{\delta}^j_i = \delta^1_i+\delta^2_i+...+\delta^s_i$. In the sequel, from [7] we give the following definition.

Definition 2.1. Let $\widetilde{M}(\varphi, \xi_i, \eta^i, g)$ be a $(2n+s)$-dimensional a metric f-manifold for each $\eta^i, (1 \leq i \leq s)$ 1-forms and each 2-form Ω, if $d\eta^i = 0$ and $d\Omega = 2\alpha\overline{\eta} \wedge \Omega$ satisfy, then $\widetilde{M}$ is called almost α-cosymplectic f-manifold [7].

Let $\widetilde{M}$ be an almost α-cosypmlectic f-manifold. Since the distribution D is integrable, we have $L_{\xi_i}\eta^j = 0$, $[\xi_i, \xi_j] \in D$ and $[X, \xi_j] \in D$ for any $X \in \Gamma(D)$. Then the Levi-Civita connection is given by [7]:

$$\begin{aligned} 2g((\widetilde{\nabla}_X\varphi)Y, Z) = 2\alpha g\left(\sum_{i=1}^{s}(g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X), Z\right) \\ + g(N(Y, Z), \varphi X) \end{aligned} \tag{2.6}$$

for any $X, Y \in \Gamma(T\widetilde{M})$. Putting $X = \xi_i$ we obtain $\widetilde{\nabla}_{\xi_i}\varphi = 0$ which implies $\widetilde{\nabla}_{\xi_i}\xi_j \in D^{\perp}$ and then $\widetilde{\nabla}_{\xi_i}\xi_j = \widetilde{\nabla}_{\xi_j}\xi_i$, since $[\xi_i, \xi_j] = 0$. We put $A_iX = -\widetilde{\nabla}_X\xi_i$ and $h_i =$

$\frac{1}{2}(L_{\xi_i}\varphi)$, where L denotes the Lie derivative operator. If $\widetilde{M}$ is almost α-cosymplectic f-manifold with Kaehlerian leaves [6], we have

$$(\widetilde{\nabla}_X\varphi)Y = \sum_{i=1}^{s}\left[-g(\varphi A_i X, Y)\xi_i + \eta^i(Y)\varphi A_i X\right]$$

or

$$(\widetilde{\nabla}_X\varphi)Y = \sum_{i=1}^{s}\left[\alpha\left(g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\right) + g(h_i X, Y)\xi_i - \eta^i(Y)h_i X\right]. \tag{2.7}$$

Proposition 2.1. *([7]) For any $i \in \{1, ..., s\}$ the tensor field A_i is a symmetric operator such that*

(i) $A_i(\xi_j) = 0$, *for any* $j \in \{1, ..., s\}$
(ii) $A_i o\varphi + \varphi o A_i = -2\alpha\varphi$
(iii) $tr(A_i) = -2\alpha n$
(iv) $\widetilde{\nabla}_X\xi_i = -\alpha\varphi^2 X - \varphi h_i X.$

for any $X \in \Gamma(T\widetilde{M})$.

Proposition 2.2. *([2]) For any $i \in \{1, ..., s\}$ the tensor field h_i is a symmetric operator and satisfies*

(i) $h_i(\xi_j) = 0$, *for any* $j \in \{1, ..., s\}$
(ii) $h_i o\varphi + \varphi o h_i = 0$
(iii) $trh_i = 0$
(iv) $tr(\varphi h_i) = 0.$

Let $\widetilde{M}$ be an almost α-cosymplectic f-manifold with respect to the curvature tensor field $\widetilde{R}$ of $\widetilde{\nabla}$, the following formulas are proved in [7], for all $X, Y \in \Gamma(T\widetilde{M}),\ i, j \in \{1, ..., s\}$.

$$\begin{aligned}\widetilde{R}(X,Y)\xi_i = {} & \alpha^2\sum_{k=1}^{s}(\eta^k(Y)\varphi^2 X - \eta^k(X)\varphi^2 Y) \\ & - \alpha\sum_{k=1}^{s}(\eta^k(X)\varphi h_k Y - \eta^k(Y)\varphi h_k X) \\ & + (\widetilde{\nabla}_Y\varphi h_i)X - (\widetilde{\nabla}_X\varphi h_i)Y,\end{aligned} \tag{2.8}$$

$$\begin{aligned}\widetilde{R}(X,\xi_j)\xi_i = {} & \sum_{k=1}^{s}\delta_j^k(\alpha^2\varphi^2 X + \alpha\varphi h_k X) \\ & + \alpha\varphi h_i X - h_i h_j X + \varphi(\widetilde{\nabla}_{\xi_j}h_i)X,\end{aligned} \tag{2.9}$$

$$\widetilde{R}(\xi_j, X)\xi_i - \varphi\widetilde{R}(\xi_j, \varphi X)\xi_i = 2(-\alpha^2\varphi^2 X + h_i h_j X). \tag{2.10}$$

Moreover, by using the above formulas, in [7] it is obtained that

$$\widetilde{S}(X, \xi_i) = -2n\alpha^2\sum_{k=1}^{s}\eta^k(X) - (div\varphi h_i)X, \tag{2.11}$$

$$\widetilde{S}(\xi_i, \xi_j) = -2n\alpha^2 - tr(h_j h_i) \tag{2.12}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, $i, j \in \{1, ..., s\}$, where $\widetilde{S}$ denote, the Ricci tensor field of the Riemannian connection.

From [7], we have the following result.

Proposition 2.3. *Let $\widetilde{M}$ be an almost α-cosymplectic f-manifold and M be an integral manifold of D. Then*

(i) *when $\alpha = 0$, M is totally geodesic if and only if all the operators h_i vanish;*
(ii) *when $\alpha \neq 0$, M is totally umbilic if and only if all the operators h_i vanish.*

Theorem 2.1. [2] *A C-manifold $\widetilde{M}^{2n+s}$ is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold $\widetilde{M}_1^{2n}$ and an Abelian Lie group $\widetilde{M}_2^s$.*

3. On Invariant Submanifold Of Almost α -Cosymplectic f-Manifolds

Let M be a submanifold of the a $(2n+s)$-dimensional almost α-cosymplectic f-manifold $\widetilde{M}$. If $\varphi(T_pM) \subset T_pM$, for any point $p \in M$ and ξ_i are tangent to M for all $i \in \{1, ..., s\}$, the M is called an invariant submanifold of $\widetilde{M}$.
Let ∇ be the Levi-Civita connection of M with respect to the induced metric g. Then Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y) \tag{3.1}$$

$$\widetilde{\nabla}_X N = \nabla_X^{\perp} N - A_N X \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^{\perp}$. $\nabla^{\perp}$ is the connection in the normal bundle, B is the second fundamental form of M and A_N is the Weingarten endomorfhism associated with N. The second fundamental form B and the shape operator A related by

$$g(B(X, Y), N) = g(A_N X, Y). \tag{3.3}$$

The curvature transformattion of M and $\widetilde{M}$ will be denote by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{3.4}$$

and

$$\widetilde{R}(X, Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z, \tag{3.5}$$

respectively. Using (3.1) and (3.2) in (3.4) and (3.5), we obtain

$$\begin{aligned} \widetilde{R}(X, Y)Z = R(X, Y)Z &- A_{B(Y,Z)} X + A_{B(X,Z)} Y \\ &+ (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \end{aligned} \tag{3.6}$$

for any $X, Y, Z \in \Gamma(TM)$. Then, if W is tangent to M, then using (3.6), we get

$$\begin{aligned} g(\widetilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) &+ g(B(Y, W), B(X, Z)) \\ &- g(B(X, W), B(Y, Z)). \end{aligned} \tag{3.7}$$

Proposition 3.1. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. Then we have*

$$(\widetilde{\nabla}_X \varphi)Y = (\nabla_X \varphi)Y \tag{3.8}$$

and

$$B(X, \varphi Y) = \varphi B(X, Y) = B(\varphi X, Y) \tag{3.9}$$

for any $X, Y \in \Gamma(TM)$.

Proof. For any $X, Y \in \Gamma(TM)$, using (3.1) we get

$$\begin{aligned}(\widetilde{\nabla}_X\varphi)Y &= \widetilde{\nabla}_X\varphi Y - \varphi\widetilde{\nabla}_X Y \\ &= \nabla_X\varphi Y + B(X,\varphi Y) - \varphi\nabla_X Y - \varphi B(X,Y) \\ &= (\nabla_X\varphi)Y + B(X,\varphi Y) - \varphi B(X,Y)\end{aligned}$$

In above equation, comparing the tangential and normal part of last equation, we obtain $B(X,\varphi Y) = \varphi B(X,Y)$. Then (3.9) follows in both cases by the symmetry of B. $\square$

From (3.9) and using symmetry of B, we have the following result.

Corollary 3.1. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. Then we get*

$$B(\varphi X,\varphi Y) = -B(X,Y) \tag{3.10}$$

for any $X, Y \in \Gamma(TM)$.

Definition 3.1. A submanifold of an almost α-cosymplectic f-manifold is called totally geodesic if $B(X,Y)$=0, for any $X, Y \in \Gamma(TM)$.

Proposition 3.2. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold. Then we have*

$$\widetilde{\nabla}_X\xi_j = \nabla_X\xi_j \tag{3.11}$$

and

$$B(X,\xi_j) = 0 \tag{3.12}$$

for any $X \in \Gamma(TM)$.

Proof. From (3.8), we obtain

$$\begin{aligned}(\widetilde{\nabla}_X\varphi)\xi_j = (\nabla_X\varphi)\xi_j \quad &\Rightarrow \varphi\widetilde{\nabla}_X\xi_j = \varphi\nabla_X\xi_j \\ &\Rightarrow \widetilde{\nabla}_X\xi_j = \nabla_X\xi_j.\end{aligned}$$

Then, using (3.11) we have

$$\begin{aligned}\widetilde{\nabla}_X\xi_j \quad &= \nabla_X\xi_j + B(X,\xi_j) \\ &\Rightarrow B(X,\xi_j) = 0.\end{aligned}$$

$\square$

Proposition 3.3. *An invariant submanifold of an almost α-cosymplectic f-manifold with Kaehlerian leaves is also almost α-cosymplectic f-manifold with Kaehlerian leaves.*

Proof. For any $X, Y \in \Gamma(TM)$, using (3.1) we get

$$\begin{aligned}(\widetilde{\nabla}_X\varphi)Y &= \widetilde{\nabla}_X\varphi Y - \varphi(\widetilde{\nabla}_X Y) \\ &= \nabla_X\varphi Y + B(X,\varphi Y) - \varphi(\nabla_X Y) - \varphi B(X,Y).\end{aligned}$$

From (2.7) and the above equation, we get by considering the submanifold as invariant and comparing tangential and normal compenents, we obtain

$$(\nabla_X \varphi)Y = \sum_{i=1}^{s} \left[\alpha \left(g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\right) + g(h_i X, Y)\xi_i - \eta^i(Y)h_i X\right]. \tag{3.13}$$

From (3.13), we get the proof. □

Theorem 3.1. *Each invariant submanifold of almost α-cosymplectic f- manifold is minimal.*

Proof. Suppose that M minimal submanifold of an almost α-cosymplectic f-manifold and dim$M = 2m + s(m < n)$. From (3.3), one can write,

$$\begin{aligned}(2m+s)tr(A_N) &= \sum_{i=1}^{m} g(B(e_i, e_i), N)\\ &+ \sum_{i=1}^{m} g(B(\varphi e_i, \varphi e_i), N)\\ &+ \sum_{i=1}^{s} g(B(\xi_i, \xi_i), N)\\ &= 0.\end{aligned}$$

Hence from above calculations, mean curvature of M, so $tr(A_N) = 0$. □

4. Curvature properties

Proposition 4.1. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. Then we*

$$\widetilde{R}(X, Y)\xi_i = R(X, Y)\xi_i \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$.

Proof. For any $X, Y \in \Gamma(TM)$, using (3.1) in (3.6) we get

$$\begin{aligned}\widetilde{R}(X, Y)\xi_i &= R(X, Y)\xi_i - A_{B(Y,\xi_i)}X + A_{B(X,\xi_i)}Y + (\nabla_X B)(Y, \xi_i) - (\nabla_Y B)(X, \xi_i)\\ &= R(X, Y)\xi_i - B(Y, \nabla_X \xi_i) + B(X, \nabla_Y \xi_i)\\ &= R(X, Y)\xi_i + \alpha\varphi^2 B(Y, X) - \alpha\varphi^2 B(X, Y) + \varphi B(Y, h_i X) - \varphi B(X, h_i, Y)\\ &= R(X, Y)\xi_i.\end{aligned}$$

□

Corollary 4.1. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. Then $\widetilde{R}(X, Y)\xi_i$ is tangent to M for any $X, Y \in \Gamma(TM)$ and $i = 1, ..., s$.*

Proposition 4.2. *Let M be an invariant submanifold of the almost α-cosymlectic f- manifold $\widetilde{M}$. Then, we have*

$$\widetilde{R}(\xi_j, X)\xi_i = R(\xi_j, X)\xi_i, \tag{4.2}$$

$$\widetilde{R}(X, \xi_j)\xi_i = R(X, \xi_j)\xi_i, \tag{4.3}$$

$$\widetilde{R}(\xi_k, \xi_j)\xi_i = R(\xi_k, \xi_j)\xi_i = 0, \tag{4.4}$$

$$\widetilde{R}(\xi_j, X)Y = R(\xi_j, X)Y \tag{4.5}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Using (4.1), we obtain (4.2), (4.3), (4.4) and (4.5). □

Proposition 4.3. *Let M be an invariant submanifold of the almost α-cosymlectic f-manifold $\widetilde{M}$. Then, following relations hold*

$$\varphi(A_N X) = A_{\varphi N} X = -A_N \varphi X \tag{4.6}$$

for any $X \in \Gamma(TM)$,

Proof. For any $X, Y \in \Gamma(TM)$, using (3.3) and (3.9) we have

$$\begin{aligned} g(\varphi(A_N X), Y) &= -g(A_N X, \varphi Y) \\ &= -g(B(X, \varphi Y), N) \\ &= -g(B(\varphi X, Y), N) \\ &= -g(A_N \varphi X, Y) \end{aligned}$$

and then,

$$\varphi(A_N X) = -A_N \varphi X.$$

Moreover, we have

$$\begin{aligned} g(\varphi(A_N X), Y) &= -g(B(X, \varphi Y), N) \\ &= -g(\varphi B(X, Y), N). \end{aligned}$$

On the other hand, using (3.3) we have

$$g(A_{\varphi N} X, Y) = g(B(X, Y), \varphi N) = -g(\varphi B(X, Y), N)$$

and then we get

$$\varphi(A_N X) = A_{\varphi N} X.$$

□

Proposition 4.4. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. Then we have*

$$\begin{aligned} g(R(X, \varphi X)\varphi X, X) &= g(\widetilde{R}(X, \varphi X)\varphi X, X) \\ &\quad - 2g(B(X, X), B(X, X)). \end{aligned} \tag{4.7}$$

for any $X \in \Gamma(TM)$.

Proof. In (3.6), if we take $Z = Y = \varphi X$ and $W = X$, then we obtain (4.7). □

Proposition 4.5. *Let M be an invariant submanifold of the almost α-cosymplectic f-manifold $\widetilde{M}$. And let $\widetilde{M}$ be of constant φ sectional curvature* [2]. *Then M is totally geodesic if only if M has constant φ sectional curvature.*

Proof. Let M be totally geodesic then from (4.7), the sectional curvature of M is the same as $\widetilde{M}$. Vice versa we suppose that the sectional φ-curvature determined by $\{X, \varphi X\}$ is the same for M and $\widetilde{M}$ for any $X \in \Gamma(TM)$. Hence from (4.7), we get that $B(X, X) = 0$ and $B = 0$. □

Proposition 4.6. *Let M be an invariant submanifold of the almost α cosymplectic f-manifold $\widetilde{M}$ and let $\alpha = 0$. Then B is parallel if only if M is totally geodesic.*

Proof. An easy calculation, we get

$$(\nabla_X B)(Y, \xi_i) = -\alpha B(Y, X) + h_i \varphi B(Y, X).$$

for any $X, Y \in \Gamma(TM)$. Hence, if B is parallel, then $B(Y, X) = 0$, for any $X, Y \in \Gamma(TM)$. Vice versa, it is clear that if $B = 0$, then $\nabla B = 0$, so B is parallel. □

Let M be a submanifold of a Rieamannian manifold $\widetilde{M}$. An isometric immersion $i : M \longrightarrow \widetilde{M}$ is semi- parallel if

$$\widetilde{R}(X, Y)B = \widetilde{\widetilde{\nabla}}_X(\widetilde{\widetilde{\nabla}}_Y B) - \widetilde{\widetilde{\nabla}}_Y(\widetilde{\widetilde{\nabla}}_X B) - \widetilde{\widetilde{\nabla}}_{[X,Y]} B = 0$$

where $\widetilde{R}$ is the curvature tensor of $\widetilde{\widetilde{\nabla}}$ [3], where $\widetilde{R}$ curvature tensor of the Van der Waerden-Bortolotti connection $\widetilde{\widetilde{\nabla}}$ and B the second fundamental from. In([1]), K. Arslan et al. defined and studied 2-semi parallel submanifold if

$$R(X, Y)\nabla B = 0$$

for any $X, Y \in \Gamma(TM)$.

Theorem 4.1. *Let M be an invariant submanifold of the α-cosymplectic f-manifold. If $\widetilde{M}$ is semi-parallel, then*
1) *When $\alpha = 0$, M totally geodesic and $\widetilde{M}$ is a locally decomposable Riemannian manifold which is locally the product of a kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s.*
2) *When $\alpha \neq 0$, M totally geodesic.*

Proof. $\widetilde{\widetilde{\nabla}}$ is the connection in $TM \oplus TM^{\perp}$ built with ∇ and $\nabla^{\perp}$, where R $(resp. R^{\perp})$ denotes curvature tensor of the connection $\nabla(resp. \nabla^{\perp})$. If $R^{\perp}$ denotes the curvature tensor of $\nabla^{\perp}$ then we have

$$(\widetilde{R}(X, Y)B)(Z, U) = R^{\perp}(X, Y)B(Z, U) - B(R(X, Y)Z, U) - B(Z, R(X, Y)U) \tag{4.8}$$

for any $X, Y, Z, U \in \Gamma(TM)$. Now, we suppose that M is semi-parallel. Then $\widetilde{R}(X, Y)B = 0$ for any $X, Y \in \Gamma(TM)$. Using (4.8), we get

$$R^{\perp}(X, Y)B(Z, K) - B(R(X, Y)Z, K) - B(Z, R(X, Y)K) = 0.$$

If we take $X = \xi_i$, $K = \xi_j$, then we obtain,

$$R^{\perp}(\xi_i, Y)B(Z, \xi_j) - B(R(\xi_i, Y)Z, \xi_j) - B(Z, R(\xi_i, Y)\xi_j) = 0.$$

From (3.12), we have

$$B(Z, R(\xi_i, Y)\xi_j) = 0$$

and from the above equation, we arrive

$$\alpha^2 B(Z, Y) = 0.$$

So, we get $\alpha = 0$ or $B = 0$. □

References

[1] Arslan K., Lumiste C., Murathan C. and Özgür C., *2- semiparallel Surfaces in Space Forms.* I. Two Particular Cases, Proc. Estonian Acad. Sci Phys. Math., 49(3), (2000), 139-148.

[2] Blair D.E., *Geometry of manifolds with structural group* $U(n) \times O(s)$, J. Differential Geometry, 4(1970), 155-167.

[3] Chen B.Y., *Geometry of submanifolds*, Marcel Dekker Inc., New York, (1973).

[4] Chinea D., Prestelo P.S., *Invariant submanifolds of a trans-Sasakian manifolds.* Publ. Mat. Debrecen, 38/1-2 (1991), 103-109.

[5] Endo H., *Invariant submanifolds in contact metric manifolds*, Tensor (N.S.) 43 (1) (1886), pp. 193-202.

[6] Erken K.I, Dacko P. and Murathan C., *Almost α-paracosymplectic manifolds*, arxiv: 1402.6930v1 [$Math.DG$] 27 Feb 2014.

[7] Öztürk H., Murathan C., Aktan N., Vanli A.T., *Almost α-cosymplectic f-manifolds* Analele stııntıfice ale unıversıtatıı 'AI.I Cuza' Dı ıaşı (S.N.) Matematica, Tomul LX, f.1., (2014).

[8] Kon M., *Invariant submanifolds of normal contact metric manifolds*, Kodai Math. Sem. Rep., 27, (1973), 330-336.

[9] Terlizi L. D., *On invariant submanifolds of C and S-manifolds.* Acta Math. Hungar. 85(3), (1999), 229-239.

[10] Sarkar A. and Sen M., *On invariant submanifold of trans- sasakian manifolds*, Proceedings of the Estonian Academy of Sciences, 61(1), (2012), 29-37.

[11] De A., *Totally geodesic submanifolds of a trans-Sasakian manifold*, Proceedings of the Estonian Academy of Sciences, 62(4), (2013), 249-257.

[12] Yano K. and Kon M., *Structures on manifolds.* World Scientific, Singapore (1984).

[13] Yano K., *On a structure defined by a tensor f of type $(1,1)$ satisfying* $\varphi^3 + \varphi = 0$, tensor N S., 14, (1963), 99-109.

Inönü University, Faculty of Arts and Sciences, Deparment of Mathematics, 44000, Malatya, Turkey

E-mail address: selahattinbeyendi@gmail.com

Konya Necmettin Erbakan University, Faculty of Arts and Sciences, Department of Mathematics and Computer Sciences, 42090, Konya, Turkey

E-mail address: nesipaktan@gmail.com

Inönü University, Faculty of Arts and Sciences, Deparment of Mathematics, 44000, Malatya, Turkey

E-mail address: ali.sivridag@inonu.edu.tr