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SOME INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE SECOND DERIVATIVES ARE $\varphi-$CONVEX BY USING FRACTIONAL INTEGRALS

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Abstract. In this paper, we obtain new estimates on generalization of Hermite-Hadamard type inequalities for functions whose second derivatives is $\varphi-$convex via fractional integrals.

1. Introduction

The following inequality is called the Hermite-Hadamard inequality;

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a convex function and $a,b\in I$ with $a<b$. If f is concave, then both inequalities hold in the reversed direction .

The inequality (1.1) was first discovered by Hermite in 1881 in the Journal Mathesis. This inequality is known as the Hermite-Hadamard inequality, because this inequality was found by Mitrinovic Hermite and Hadamard' note in Mathesis in 1974.

The inequality (1.1) is studied by many authors, see ([1]-[7], [9]-[11], [12], [15]-[21]) where further references are listed.

Firstly, we need to recall some concepts of convexity concerning our work.

Definition 1.1. [6] A function $f:I\subset\mathbb{R}\to\mathbb{R}$ is said to be convex on I if inequality

$$f(ta+(1-t)b)\leq tf(a)+(1-t)f(b), \tag{1.2}$$

holds for all $a,b\in I$ and $t\in[0,1]$.

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Definition 1.2. [8] Let $s \in (0,1]$. A function $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b), \tag{1.3}$$

holds for all $a, b \in I$ and $t \in [0,1]$.

Tunç and Yildirim in [21] introduced the following definition as follows:

Definition 1.3. *A function f:$I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of $MT(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the inequality*;

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y).$$

Dragomir in [3] introduced the following definition as follows:

Definition 1.4. [3] Let $\varphi : (0,1) \to (0,\infty)$ be a measurable function. We say that the function $f : I \to [0,\infty)$ *is a* $\varphi-$convex function on the interval I if for $x, y \in I$, we have

$$f(tx + (1-t)y) \leq t\varphi(t) f(x) + (1-t)\varphi(1-t) f(y).$$

Remark 1.1. According to definition 4, the followings hold for the special choose of φ (t):

For $\varphi(t) \equiv 1$, we obtain the definition of convexness in the classical sense,
for $\varphi(t) = t^{s-1}$, we obtain the definition of $s-$ convexness,
for $\varphi(t) = \dfrac{1}{2\sqrt{t(1-t)}}$, we obtain the definition of $MT-$convexness.

Now, we give some definitions and notations of fractional calculus theory which are used later in this paper. Samko et al. in [14] used the following definitions as follows:

Definition 1.5. [14] The Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \; x > a \tag{1.4}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \; x < b \tag{1.5}$$

where $f \in L_1[a,b]$, respectively. Note that, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{\alpha} f(x) = f(x)$.

Definition 1.6. [14] The Euler Beta function is defined as follows:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \; x, y > 0.$$

The incomplete beta function is defined as follows:

$$\beta(a,x,y) = \int_0^a t^{x-1}(1-t)^{y-1} dt, \; x, y > 0, \; 0 < \alpha < 1.$$

In [13], Jaekeun Park established the following lemma which is necessary to prove our main results:

Lemma 1.1. *Let $f\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$. Then, for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$, we have*

$$\begin{aligned} S_f(x,\lambda,\alpha;a,b) &= \tfrac{(x-a)^{\alpha+2}}{b-a}\int_0^1 t\left(\lambda-t^{\alpha}\right)f''(tx+(1-t)a)\,dt \\ &+\tfrac{(b-x)^{\alpha+2}}{b-a}\int_0^1 t\left(\lambda-t^{\alpha}\right)f''(tx+(1-t)b)\,dt. \end{aligned}$$

2. Main results

Throughout this paper, we use S_f as follows;

$$\begin{aligned} S_f(x,\lambda,\alpha;a,b) &\equiv (1-\lambda)\left\{\tfrac{(b-x)^{\alpha+1}-(x-a)^{\alpha+1}}{b-a}\right\}f'(x) \\ &+(1+\alpha-\lambda)\left\{\tfrac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right\}f(x) \\ &+\lambda\left\{\tfrac{(x-a)^{\alpha}(f(a)+(b-x)^{\alpha}f(b)}{b-a}\right\} \\ &-\tfrac{\Gamma(\alpha+2)}{b-a}\left\{J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right\}, \end{aligned}$$

for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$.

Theorem 2.1. *Let $\varphi:(0,1)\to(0,\infty)$ be a measurable function. Assume also that $f:I\subset[0,\infty)\to\mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I such that $f''\in L_1[a,b]$, where $a,b\in I^0$ with $a<b$. If $|f''|^q$ is $\varphi-$convex on $[a,b]$ for some fixed $q\geq1$, then for any $x\in[a,b]$, $t,\lambda\in[0,1]$ and $\alpha>0$,*

$$\begin{aligned} |S_f(x,\lambda,\alpha,t,\varphi;a,b)| &\leq A_1^{1-\frac{1}{q}}(\alpha,\lambda)\Big[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{A_2(\alpha,\lambda,t,\varphi)\left|f''(x)\right|^q\right. \\ &\left.+A_3(\alpha,\lambda,t,\varphi)\left|f''(a)\right|^q\right\}^{\frac{1}{q}} \qquad (2.1)\\ &+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{A_2(\alpha,\lambda,t,\varphi)\left|f''(x)\right|^q+A_3(\alpha,\lambda,t,\varphi)\left|f''(b)\right|^q\right\}^{\frac{1}{q}}\Big]. \end{aligned}$$

The above inequality for fractional integrals holds, where

$$\begin{aligned} A_1(\alpha,\lambda) &= \tfrac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\tfrac{\lambda}{2}, \\ A_2(\alpha,\lambda,t,\varphi) &= \int_0^1\left|t\left(\lambda-t^{\alpha}\right)\right|t\varphi(t)\,dt, \\ A_3(\alpha,\lambda,t,\varphi) &= \int_0^1\left|t\left(\lambda-t^{\alpha}\right)\right|(1-t)\varphi(1-t)\,dt. \end{aligned}$$

Proof. By using Lemma 1.1, the power mean inequality, we get

(2.2)

$$\begin{aligned}
&|S_f(x,\lambda,\alpha,t,\varphi;a,b)| \\
&\leq \tfrac{(x-a)^{\alpha+2}}{b-a}\left(\int_0^1 |t(\lambda-t^\alpha)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}} \\
&+\tfrac{(b-x)^{\alpha+2}}{b-a}\left(\int_0^1 |t(\lambda-t^\alpha)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)b)|\,dt\right)^{\frac{1}{q}} \\
&= A_1^{1-\frac{1}{q}}(\alpha,\lambda)\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left(\int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left(\int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$A_1(\alpha,\lambda)=\int_0^1 |t(\lambda-t^\alpha)|\,dt=\left(\frac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\frac{\lambda}{2}\right).$$

Since $|f''|^q$ is $\varphi-$convex on $[a,b]$, we have

$$\begin{aligned}
I_1 &= \int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)a)|^q\,dt \\
&\leq \int_0^1 |t(\lambda-t^\alpha)|\left\{t\varphi(t)|f''(x)|^q+(1-t)\varphi(1-t)|f''(a)|^q\right\}dt \qquad (2.3)\\
&= A_2(\alpha,\lambda,t,\varphi)|f''(x)|^q+A_3(\alpha,\lambda,t,\varphi)|f''(a)|^q,
\end{aligned}$$

and similarly, we can obtain

$$\begin{aligned}
I_2 &= \int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)b)|^q\,dt \\
&\leq \int_0^1 |t(\lambda-t^\alpha)|\left\{t\varphi(t)|f''(x)|^q+(1-t)\varphi(1-t)|f''(b)|^q\right\}dt \qquad (2.4)\\
&= A_2(\alpha,\lambda,t,\varphi)|f''(x)|^q+A_3(\alpha,\lambda,t,\varphi)|f''(b)|^q,
\end{aligned}$$

where

$$\begin{aligned}
A_2(\alpha,\lambda,t,\varphi) &= \int_0^1 |t(\lambda-t^\alpha)|\,t\varphi(t)\,dt, \\
A_3(\alpha,\lambda,t,\varphi) &= \int_0^1 |t(\lambda-t^\alpha)|\,(1-t)\varphi(1-t)\,dt.
\end{aligned}$$

By substituting (2.3) and (2.4) in (2.2), we get

$$|S_f(x,\lambda,\alpha,t,\varphi;a,b)|$$

$$\leq \left(\tfrac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\tfrac{\lambda}{2}\right)^{1-\frac{1}{q}} \left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{|f''(x)|^q\int_0^1|t(\lambda-t^\alpha)|\,t\varphi(t)\,dt\right.\right.$$

$$\left.+|f''(a)|^q\int_0^1|t(\lambda-t^\alpha)|(1-t)\,\varphi(1-t)\,dt\right\}^{\frac{1}{q}}$$

$$+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{|f''(x)|^q\int_0^1|t(\lambda-t^\alpha)|\,t\varphi(t)\,dt\right.$$

$$\left.\left.+|f''(b)|^q\int_0^1|t(\lambda-t^\alpha)|(1-t)\,\varphi(1-t)\,dt\right\}^{\frac{1}{q}}\right].$$

Thus the proof is completed. □

Corollary 2.1. *Let $\varphi(t)=1$ in Theorem 2.1, then we get the following inequality:*

$$|S_f(x,\lambda,\alpha;a,b)|$$

$$\leq \left(\tfrac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\tfrac{\lambda}{2}\right)^{1-\frac{1}{q}} \left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{A_2(\alpha,\lambda)|f''(x)|^q+A_3(\alpha,\lambda)|f''(a)|^q\right\}\right.$$

$$\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{A_2(\alpha,\lambda)|f''(x)|^q+A_3(\alpha,\lambda)|f''(b)|^q\right\}\right].$$

Where

$$A_2(\alpha,\lambda) = \int_0^1|t(\lambda-t^\alpha)|\,tdt=\frac{3-(\alpha+3)\lambda+2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}$$

and

$$\begin{aligned} A_3(\alpha,\lambda) &= \int_0^1|t(\lambda-t^\alpha)|(1-t)\,dt \\ &= \frac{\alpha\lambda^{1+\frac{2}{\alpha}}}{\alpha+2}-\frac{2\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}+\frac{\alpha\lambda}{6}-\frac{\alpha}{(\alpha+2)(\alpha+3)}. \end{aligned}$$

Corollary 2.2. *If we choose $\varphi(t)=1$ and $x=\frac{a+b}{2}$ in Theorem 2.1, we can obtain the corollary 2.2, 2.3, 2.4 in* [13]*, respectively for $\lambda=\frac{1}{3}$, $\lambda=0$, $\lambda=1$.*

Corollary 2.3. *Let $\varphi(t)=t^{s-1}$ in Theorem 2.1, then we have*

$$|S_f(x,\lambda,\alpha,t,\varphi;a,b)|$$

$$\leq \left(\tfrac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\tfrac{\lambda}{2}\right)^{1-\frac{1}{q}} \left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{|f''(x)|^q A_4(\alpha,\lambda,s)+|f''(a)|^q A_5(\alpha,\lambda,t,\varphi)\right\}^{\frac{1}{q}}\right.$$

$$\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{|f''(x)|^q A_4(\alpha,\lambda,s)+|f''(b)|^q A_5(\alpha,\lambda,t,\varphi)\right\}^{\frac{1}{q}}\right].$$

Where

$$A_4(\alpha,\lambda,s) = 2\frac{\lambda^{\frac{s+2}{\alpha}+1}}{s+2} - 2\frac{\lambda^{\frac{s+2}{\alpha}+1}}{\alpha+s+2} + \frac{1}{\alpha+s+2}$$

$$A_5(\alpha,\lambda,t,\varphi) = \lambda\beta\left(\lambda^{\frac{1}{\alpha}},2,s+1\right) - \beta\left(\lambda^{\frac{1}{\alpha}},\alpha+2,s+1\right) + \beta\left(1-\lambda^{\frac{1}{\alpha}},\alpha+2,s+1\right) - \lambda\beta\left(1-\lambda^{\frac{1}{\alpha}},2,s+1\right).$$

Theorem 2.2. *Let $\varphi:(0,1)\to(0,\infty)$ be a measurable function. For $f: I\subset[0,\infty)\to\mathbb{R}$ be a twice differentiable function on the interior I^0 assume also that $f''\in L_1[a,b]$, where $a,b\in I^0$ with $a<b$. If $|f''|^q$ is $\varphi-$convex on $[a,b]$ for some fixed $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$ the following inequality holds*

$$\begin{aligned} &|S_f(x,\lambda,\alpha,t,\varphi;a,b)| \\ &\le B^{\frac{1}{p}}(\alpha,\lambda,p)\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{\left(|f''(x)|^q+|f''(a)|^q\right)\int_0^1 t\varphi(t)\,dt\right\}^{\frac{1}{q}}\right. \\ &\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{\left(|f''(x)|^q+|f''(b)|^q\right)\int_0^1 t\varphi(t)\,dt\right\}^{\frac{1}{q}}\right], \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} B(\alpha,\lambda,p) = \tfrac{\lambda^{\frac{1+p+\alpha p}{\alpha}}}{\alpha}&\left\{\Gamma(1+p)\,\Gamma\left(\tfrac{1+p+\alpha}{\alpha}\right)\ \left({}_2F_1\left(1,1+p,2+p+\tfrac{1+p}{\alpha},1\right)\right)\right. \\ &\left.+\beta\left(1+p,-\tfrac{1+p+\alpha p}{\alpha}\right)-\beta\left(\lambda,1+p,-\tfrac{1+p+\alpha p}{\alpha}\right)\right\}, \end{aligned}$$

also, for $0<b<c$ and $|z|<1$, ${}_2F_1$ is hypergeometric function defined by

$${}_2F_1(a,b,c,z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt.$$

Proof. By using Lemma 1.1 and the Hölder inequality, we have the below inequality

$$\begin{aligned} &|S_f(x,\lambda,\alpha,t,\varphi;a,b)| \\ &\le \tfrac{(x-a)^{\alpha+2}}{b-a}\left(\int_0^1|t(\lambda-t^\alpha)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}} \\ &+\tfrac{(b-x)^{\alpha+2}}{b-a}\left(\int_0^1|t(\lambda-t^\alpha)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\ &=\left(\int_0^1|t(\lambda-t^\alpha)|^p\right)^{\frac{1}{p}}\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left(\int_0^1|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right. \\ &\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left(\int_0^1|f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{2.6}$$

Since $|f''|$ is $\varphi-$convex on $[a,b]$, we have

$$\begin{aligned}
\int_0^1 |f''(tx+(1-t)a)|^q\,dt &\leq \int_0^1 t\varphi(t)\,|f''(x)|^q\,dt \\
&+\int_0^1 (1-t)\,\varphi(1-t)\,|f''(a)|^q\,dt \qquad (2.7)\\
&=\left(|f''(x)|^q+|f''(a)|^q\right)\int_0^1 t\varphi(t)\,dt,
\end{aligned}$$

and using same technique, we get

$$\begin{aligned}
\int_0^1 |f''(tx+(1-t)b)|^q\,dt &\leq \int_0^1 t\varphi(t)\,|f''(x)|^q\,dt \\
&+\int_0^1 (1-t)\,\varphi(1-t)\,|f''(b)|^q\,dt \qquad (2.8)\\
&=\left(|f''(x)|^q+|f''(b)|^q\right)\int_0^1 t\varphi(t)\,dt.
\end{aligned}$$

On the other hand, we can obtain the following equality;

$$\begin{aligned}
B(\alpha,\lambda,p) &= \int_0^1 |t(\lambda-t^\alpha)|^p\,dt \\
&= \int_0^{\lambda^{\frac{1}{\alpha}}} \{t(\lambda-t^\alpha)\}^p\,dt+\int_{\lambda^{\frac{1}{\alpha}}}^1 \{t(t^\alpha-\lambda)\}^p\,dt \qquad (2.9)\\
&= C_1(\alpha,\lambda,p)+C_2(\alpha,\lambda,p).
\end{aligned}$$

By letting $\lambda-t^\alpha=u$ and $t^\alpha=u$, respectively, we have
(2.10)

$$\begin{aligned}
C_1(\alpha,\lambda,p) &= \int_0^{\lambda^{\frac{1}{\alpha}}} \{t(\lambda-t^\alpha)\}^p\,dt \\
&= \frac{1}{\alpha}\int_0^\lambda u^p(\lambda-u)^{\frac{1+p-\alpha}{\alpha}}\,du \\
&= \frac{1}{\alpha}\int_0^1 \lambda^p y^p \lambda^{\frac{1+p-\alpha}{\alpha}}(1-y)^{\frac{1-\alpha+p}{\alpha}}\lambda dy \\
&= \frac{\lambda^{\frac{p\alpha+1+p}{\alpha}}}{\alpha}\int_0^1 y^p(1-y)^{\frac{1+p}{\alpha}}(1-y)^{-1}\,dy \\
&= \frac{\lambda^{\frac{1+p+\alpha p}{\alpha}}}{\alpha}\Gamma(1+p)\,\Gamma\left(\frac{1+p+\alpha}{\alpha}\right){}_2F_1\left(1,1+p,2+p+\frac{1+p}{\alpha},1\right),
\end{aligned}$$

and

$$\begin{aligned}
C_2(\alpha,\lambda,p) &= \int_{\lambda^{\frac{1}{\alpha}}}^1 \{t(t^\alpha-\lambda)\}^p\,dt \\
&= \frac{1}{\alpha}\int_{\lambda^u}^1 \frac{1+p-\alpha}{\alpha}(u-\lambda)^p\,du \qquad (2.11)\\
&= \frac{\lambda^{\frac{1+p+\alpha p}{\alpha}}}{\alpha}\left\{\beta\left(1+p,-\frac{1+p+\alpha p}{\alpha}\right)-\beta\left(\lambda,1+p,-\frac{1+p+\alpha p}{\alpha}\right)\right\}.
\end{aligned}$$

By substituting (2.7), (2.8), (2.9), (2.10) and (2.11)in (2.6), we get

$$|S_f(x,\lambda,\alpha,t,\varphi;a,b)|$$

$$\leq B^{\frac{1}{p}}(\alpha,\lambda,p)\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{\left(|f''(x)|^q+|f''(a)|^q\right)\int_0^1 t\varphi(t)\,dt\right\}^{\frac{1}{q}}\right.$$

$$\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{\left(|f''(x)|^q+|f''(b)|^q\right)\int_0^1 t\varphi(t)\,dt\right\}^{\frac{1}{q}}\right],$$

thus, the proof is completed. □

Corollary 2.4. *Let $\varphi(t)=1$ in Theorem 2.2, then we get the following inequality for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$;*

$$|S_f(x,\lambda,\alpha,t,\varphi;a,b)|$$

$$\leq\left(\int_0^1|t(\lambda-t^\alpha)|^p\,dt\right)^{\frac{1}{p}}\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{\tfrac{\left(|f''(x)|^q+|f''(a)|^q\right)}{2}\right\}^{\frac{1}{q}}\right.$$

$$\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{\tfrac{\left(|f''(x)|^q+|f''(b)|^q\right)}{2}\right\}^{\frac{1}{q}}\right].$$

Corollary 2.5. *If we choose $\varphi(t)=1$ and $x=\frac{a+b}{2}$ in Theorem 2.2, we can obtain the corollary 2.6, 2.7, 2.8 in* [13], *respectively for $\lambda=\frac{1}{3}$, $\lambda=0$, $\lambda=1$.*

Corollary 2.6. *Let $\varphi(t)=t^{s-1}$ in Theorem 2.2, then we obtain*

$$|S_f(x,\lambda,\alpha,t,\varphi;a,b)|$$

$$\leq\left(\int_0^1|t(\lambda-t^\alpha)|^p\,dt\right)^{\frac{1}{p}}\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{\tfrac{\left(|f''(x)|^q+|f''(a)|^q\right)}{s+1}\right\}^{\frac{1}{q}}\right.$$

$$\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{\tfrac{\left(|f''(x)|^q+|f''(b)|^q\right)}{s+1}\right\}^{\frac{1}{q}}\right].$$

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PARTIAL DERIVATIVE EFFECTS IN TWO-DIMENSIONAL SPLINE FUNCTION NODES

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Abstract. One of the methods is two-dimensional spline functions for to create geometrical model of surface. In this study Eligibility of partial derivatives values for each node was examined. These nodes are projection of creation aimed surface. Created effects by the chosen values were evaluated. The results of the application example was provided with a computer software developed.

1. Introduction

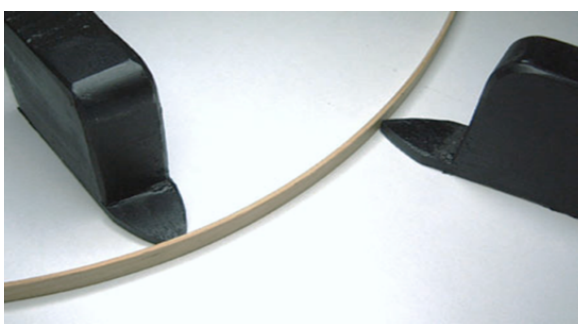

Figure 1. Conversational usage of mechanical spline.

In mathematics, a spline is a numeric function that is piecewise-defined by polynomial functions([5][7]). In dictionary, the word "spline" originally meant a thin wood or metal slat in East Anglian dialect. By 1895 it had come to mean a flexible ruler used to draw curves[10]. These splines were used in the aircraft and shipbuilding industries. The successful design was then plotted on graph paper and the key points of the plot were re-plotted on larger graph paper to full size. The thin wooden strips provided an interpolation of the key points into smooth curves. The

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strips would be held in place at the key points (using lead weights called "ducks" or "dogs" or "rats")([6][7]) as shown in figure1. It is commonly accepted that the first mathematical reference to splines is the 1946 paper by [6], which is probably the first place that the word "spline" is used in connection with smooth, piecewise polynomial approximation([8][7]).

Let T $= (t_0, t_1, \cdots, t_{n-1})$ and U $= (u_0, u_1, \cdots, u_{n-1})$ here, $t_0 < t_1 < \cdots < t_{n-1}$ are distinct ordered real numbers and $u_0, u_1, \cdots, u_{n-1}$ are real numbers that represent each node. It describes a spline function f_{sp}

$$f_{sp}(t) = \begin{cases} f_0(t), & t_0 \le t \le t_1 \\ f_1(t), & t_1 < t \le t_2 \\ \vdots & \\ f_{n-3}(t), & t_{n-3} \le t \le t_{n-2} \\ f_{n-2}(t), & t_{n-2} \le t \le t_{n-1} \end{cases}$$

$$f_j(t_j) = u_j, \qquad f_j(t_{j+1}) = u_{j+1}, \qquad j = 0,\ 1,\ \cdots,\ n-2.$$

$a, b \in R$, $a = t_0 < t_1 < \cdots < t_{n-2} < t_{n-1} = b$ is to be; $f_j : [t_j, t_{j+1}] \to R$, $j = 0, 1, \cdots, n-2$, $f_{sp} : [a, b] \to R$. Each f_j function may have any degree that is polynomial functions. Often the first, second and third order polynomial functions are used in practice([8][1]).

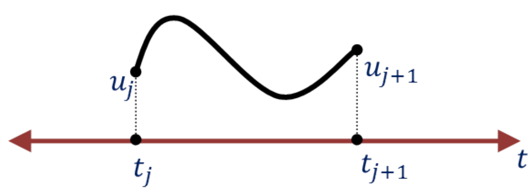

FIGURE 2. f_j piecewise function.

1.1. **Cubic spline functions.** Let T $= (t_0, t_1, \cdots, t_{n-1})$, U $= (u_0, u_1, \cdots, u_{n-1})$ and G $= (g_0, g_1, \cdots, g_{n-1})$. $f_{sp} : [t_0, t_{n-1}] \to R$, $u = f_{sp}(t)$, $t \in [t_0, t_{n-1}]$. $f_j : [t_j, t_{j+1}] \to R$, $f_j(t) = a_j t^3 + b_j t^2 + c_j t + d_j$, $j = 0, 1, \cdots, n-2$ which satisfied the conditions $f'_{sp}(t_i) = g_i$, $i = 0, 1, \cdots, n-1$ is unique [9].

$$f'_j(t_j) = g_j \ and \ f_j(t_j) = u_j$$

$$f'_j(t_{j+1}) = g_{j+1} \ and \ f_j(t_{j+1}) = u_{j+1}$$
$$j = 0,\ 1,\ \cdots,\ n-2$$

Condition can provides, at least third degree spline functions [9]. The cubic spline function $f_{sp}(\ t\ )$ has following representation [1].

$$w_i = \frac{1}{t_i - t_{i-1}} \left(\frac{u_i - u_{i-1}}{t_i - t_{i-1}} - g_{i-1} \right)$$

$$a_i = \frac{1}{t_i - t_{i-1}} \left(\frac{g_i - g_{i-1}}{t_i - t_{i-1}} - 2w_i \right)$$

$$b_i = -(t_i + 2t_{i-1}) a_i + w_i$$

$$c_i = g_{i-1} - 3a_i t_{i-1}^2 - 2b_i t_{i-1}$$

$$d_i = u_{i-1} - a_i t_{i-1}^3 - b_i t_{i-1}^2 - c_i t_{i-1}$$
$$i = 1,\ 2,\ \ldots,\ n-1$$

1.2. **CubicSPL Cubic spline subroutine.** The following subroutine representation have input values that are three vectors establish for cubic spline function and provision sought value of t. The result of this subroutine is a value that $u = f_{sp}(t)$.
double CubicSPL (double T, double* U, double* G, double t)*

Example 1.1. T = (1, 2, 3, 4, 5), U = (−3, 3, 2, −2, 1) and G = (0, 0, 0, 0, 0) are vectors representing the values of nodes.

```
#define TMax 5
T[TMax] = { 1, 2, 3, 4, 5 } ;
U[TMax] = { −3, 3, 2, −2, 1 } ;
G[TMax] = { 0, 0, 0, 0, 0 } ;
double t = 3.7 ;
u = CubicSPL ( T, U, G, t ) ;
```

$$u : -1.1359999999998536$$

```
u = CubicSPL ( T, U, G, 2.07 ) ;
```

$$u : 2.9859860000000111$$

Graphical representation of the results are also observed at figure 3.

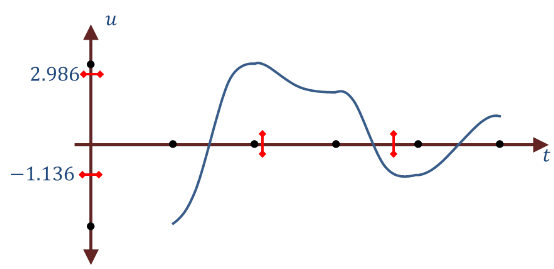

Figure 3. Graphical representation of example 1.1.

2. Two Dimensional Spline

$a, b, c, d \in R$ and $\Omega = [a, b] \times [c, d]$, consider the rectangle on tOx plane as Ω region.

$$a = t_0 \ < \ t_1 \ < \ \cdots \ < t_i \ < \ \cdots \ < t_{m-1} = b;\ m \geq 1$$
$$c = x_0 \ < \ x_1 \ < \ \cdots \ < x_j \ < \ \cdots \ < x_{n-1} = d;\ n \geq 1$$
$$i = 0,\ 1,\ \cdots,\ m-1,\ j = 0,\ 1,\ \cdots,\ n-1$$

Ω region divided into $(n-1) \times (m-1)$sub regions.

$$\Omega_{i,j} = \{(t, x) :\ t_i \ \leq \ t \ \leq \ t_{i+1},\ x_j \ \leq \ x \ \leq \ x_{j+1}\}$$

$i = 0,\ 1,\ \cdots,\ m-2;\ j = 0,\ 1,\ \cdots,\ n-2$. For any $\Omega_{i,j}$ sub region have this edge cardinal points:

$$\zeta_{t_i,\ x_j}, \zeta_{t_{i+1},\ x_j},\ \zeta_{t_{i+1},\ x_{j+1}},\ \zeta_{t_i,\ x_{j+1}}$$

The cardinal points of each $\Omega_{i,j}$ sub region defines a grid Ω_{grd}. Be introduced a function $\lambda : \Omega_{grd} \to R,\ \lambda(t_i, x_j) = u_{(i,j)}$ on the grid extended on the Ω region [8].

$$\mathrm{U} = \left\{u_{(0,0)},\ u_{(0,1)},\ \cdots,\ u_{(0,n-1)},\ u_{(1,0)},\ \cdots,\ u_{(m-1,n-1)}\right\}$$

$$\mathrm{G}_t = \left\{g_{t(0,0)},\ g_{t(0,1)},\ \cdots,\ g_{t(0,n-1)},\ g_{t(1,0)},\ \cdots,\ g_{t(m-1,n-1)}\right\}$$

$$\mathrm{G}_x = \left\{g_{x(0,0)},\ g_{x(0,1)},\ \cdots,\ g_{x(0,n-1)},\ g_{x(1,0)},\ \cdots,\ g_{x(m-1,n-1)}\right\}$$

$$u_{(i,j)} \in R,\ g_{t(i,j)} \in R,\ g_{x(i,j)} \in R$$

$$\lambda(t_i, x_j) = u_{(i,j)},\ \lambda_t^{'}(t_i, x_j) = g_{t(i,j)},\ \lambda_x^{'}(t_i, x_j) = g_{x(i,j)},$$

$$f : \Omega \to R,\ f(t_i, x_j) = u_{(i,j)},\ \lambda(t_i, x_j) = f(t_i, x_j)$$

$$i = 0,\ 1,\ \cdots,\ m-1,\ j = 0,\ 1,\ \cdots,\ n-1$$

The purpose is find $f : \Omega \to R,\ f(t, x)$ derivable real function [8].

$$H(t_0, x),\ H(t_1, x),\ H(t_2, x),\ \ldots,\ H(t_{m-1}, x),\qquad x_0 \ \leq\ x\ \leq\ x_{m-1}$$

$$S(t, x_0),\ S(t, x_1),\ S(t, x_2),\ \ldots,\ S(t, x_{n-1}),\qquad t_0 \ \leq\ t\ \leq\ t_{n-1}$$

$H(t_i, x),\ i = 0,\ 1,\ \cdots,\ m-1,\ x_0 \ \leq\ x\ \leq\ x_{n-1}$ describe direction of x spline functions and $S(t, x_j),\ j = 0,\ 1,\ \cdots,\ n-1,\ t_0\ \leq\ t\ \leq\ t_{m-1}$ describe direction of t spline functions[8].

U, G_x and G_t data sets according with Ω_{grd}. These sets provides m amounts $\mathrm{U}_{\overline{\overline{X}}_i} = \left\{u_{(i,j)} \,\middle|\, j = 0,\ 1,\ \cdots,\ n-1\right\}$ and $\mathrm{G}_{\overline{\overline{X}}_i} = \left\{g_{x(i,j)} \,\middle|\, j = 0,\ 1,\ \cdots,\ n-1\right\}$ vectors for each $H(t_i, x)$ spline functions direction of x and n amounts $\mathrm{U}_{\overline{\overline{T}}_j} = \left\{u_{(i,j)} \,\middle|\, i = 0,\ 1,\ \cdots,\ m-1\right\}$ and $\mathrm{G}_{\overline{\overline{T}}_j} = \left\{g_{t(i,j)} \,\middle|\, i = 0,\ 1,\ \cdots,\ m-1\right\}$ vectors for each $S(t, x_j)$ spline functions direction of t. At the end of the $m+n$ amounts supply one-dimensional spline functions can be calculated.

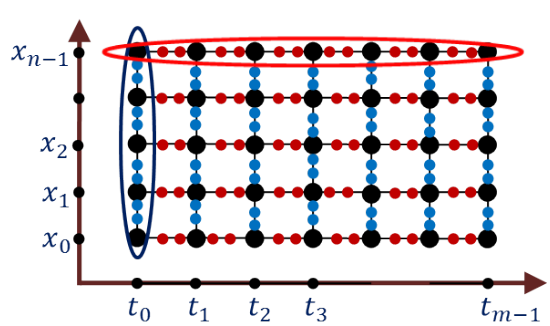

FIGURE 4. $m+n$ amounts one-dimensional spline functions.

FIGURE 5. The demonstration will consist of an auxiliary spline function according to the direction.

3. ANY $f(t,x)$ ON THE Ω

Calculations can be started with the any direction spline functions the direction of t or direction of x arbitrarily chosen. Let $t_0 \le l \le t_{m-1}$ and $x_0 \le k \le x_{n-1}$. If t direction spline functions are chosen, a supplementary spline function can create using these spline functions.The solution is shown below.

Let $k \in (x_0,\ x_{n-1})$ and $l \in (t_0,\ t_{m-1})$. $u_{(t_{sup},\ j)} = S(l,\ x_j),\ \ j = 0,\ 1,\ \cdots,\ n-1,\ f(l,\ k) = H(t_{sup},\ k)$. In detail $u_{(t_{sup},\ j)} = CubicSPL(\mathrm{T},\ \mathrm{U}_{\overline{\overline{T}}_j},\ \mathrm{G}_{\overline{\overline{T}}_j},\ l)$; for $j = 0,\ 1,\ \cdots,\ n-1$ create a new $\mathrm{U}_{\overline{\overline{X}}_{sup}}$ vector for use in x direction. Therefor $CubicSPL$ function need a $\mathrm{G}_{\overline{\overline{X}}_{sup}}$ vector represent x direction derivative values of $H(t_{sup},\ x)$ $t_i \le l \le t_{i+1}$, $\mathrm{G}_{\overline{\overline{X}}_i}$ and $\mathrm{G}_{\overline{\overline{X}}_{i+1}}$ vectors represent partial derivative values relationship $H(t_i,\ x)$ and $H(t_{i+1},\ x)$ spline functions on direction x. Get help these two vectors to determine $\mathrm{G}_{\overline{\overline{X}}_{sup}}$. $\mathrm{U}_{\overline{\overline{X}}_{sup}}$ was obtained. $t_i \le l \le t_{i+1}$ and $j = 0,\ 1,\ \cdots,\ n-1$. As shown in figure 6.

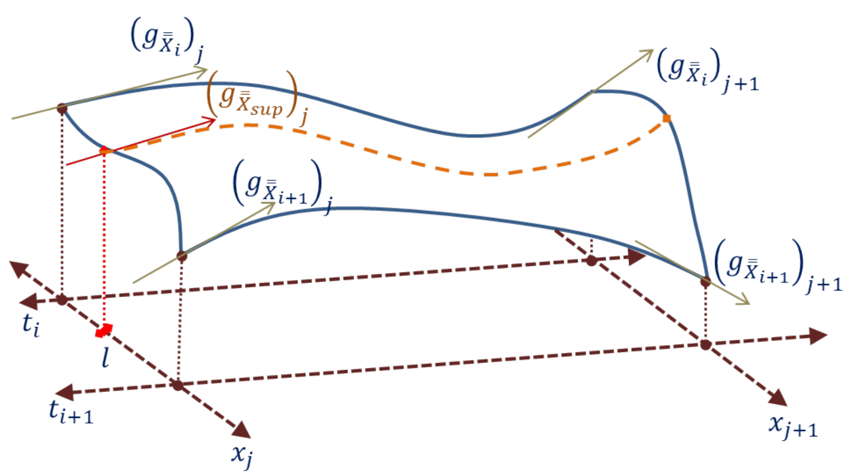

FIGURE 6

$$\left(g_{\overline{\overline{X}}_{sup}}\right)_j = \left(g_{\overline{\overline{X}}_i}\right)_j \frac{|t_{i+1} - l|}{|t_{i+1} - t_i|} + \left(g_{\overline{\overline{X}}_{i+1}}\right)_j \frac{|t_i - l|}{|t_{i+1} - t_i|}$$

$$f(l,\ k) = CubicSPL(\mathrm{X},\ \mathrm{U}_{\overline{\overline{X}}_{sup}},\ \mathrm{G}_{\overline{\overline{X}}_{sup}},\ k);$$

4. Smooth Surface

At the direction of t and the direction of x, partial derivative values can be arbitrarily chosen on the grid nodes. Nevertheless the created surface able to reach somewhat smoothness using some basic rules. For spline functions direction of t :

$$\left(g_{\overline{\overline{T}}_j}\right)_0 = \frac{\left(u_{\overline{\overline{T}}_j}\right)_1 - \left(u_{\overline{\overline{T}}_j}\right)_0}{t_1 - t_0}$$

$$\left(g_{\overline{\overline{T}}_j}\right)_{m-1} = \frac{\left(u_{\overline{\overline{T}}_j}\right)_{m-2} - \left(u_{\overline{\overline{T}}_j}\right)_{m-1}}{t_{m-2} - t_{m-1}}$$

$$\left(g_{\overline{\overline{T}}_j}\right)_i = \left(\frac{\left(u_{\overline{\overline{T}}_j}\right)_i - \left(u_{\overline{\overline{T}}_j}\right)_{i-1}}{t_i - t_{i-1}} \frac{|t_{i+1} - t_i|}{|t_{i+1} - t_{i-1}|} + \frac{\left(u_{\overline{\overline{T}}_j}\right)_{i+1} - \left(u_{\overline{\overline{T}}_j}\right)_i}{t_{i+1} - t_i} \frac{|t_{i-1} - t_i|}{|t_{i+1} - t_{i-1}|}\right)$$

$$i = 1,\ 2,\ \cdots,\ m-2,\ j = 0,\ 1,\ \cdots, n-1.$$

For spline functions direction of x :

$$\left(g_{\overline{\overline{X}}_i}\right)_0 = \frac{\left(u_{\overline{\overline{X}}_i}\right)_1 - \left(u_{\overline{\overline{X}}_i}\right)_0}{x_1 - x_0}$$

$$\left(g_{\overline{\overline{X}}_i}\right)_{n-1} = \frac{\left(u_{\overline{\overline{X}}_i}\right)_{n-2} - \left(u_{\overline{\overline{X}}_i}\right)_{n-1}}{x_{n-2} - x_{n-1}}$$

$$\left(g_{\overline{\overline{X}}_i}\right)_j = \left(\frac{\left(u_{\overline{\overline{X}}_i}\right)_j - \left(u_{\overline{\overline{X}}_i}\right)_{j-1}}{x_j - x_{j-1}} \frac{|x_{j+1} - x_j|}{|x_{j+1} - x_{j-1}|} + \frac{\left(u_{\overline{\overline{X}}_i}\right)_{j+1} - \left(u_{\overline{\overline{X}}_i}\right)_j}{x_{j+1} - x_j} \frac{|x_{j-1} - x_j|}{|x_{j+1} - x_{j-1}|}\right)$$

$$i=0,\ 1,\cdots,m-1,\ j=1, 2,\ \cdots,n-2.$$

5. Results and Discussion

A computer program was developed as a result of this study is. Using the *http://oguzersinan.net.tr* web address that is accessible to this computer program. $\mathrm{U} = \begin{pmatrix} 3 & 4 & 3 \\ 4 & 5 & 4 \\ 3 & 3 & 3 \end{pmatrix}$, $\mathrm{G}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathrm{G}_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ get in that way. Surface appearance is shown in figure 7. Computer software by the method described hereinabove, when it determines partial derivatives of nodes is calculated as $\mathrm{G}_x = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathrm{G}_t = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{pmatrix}$. New surface appearance is shown in figure 7.

Determine the value of partial derivatives with the weighted arithmetic mean method on two-dimensional cubic spline functions reveals appropriate results.

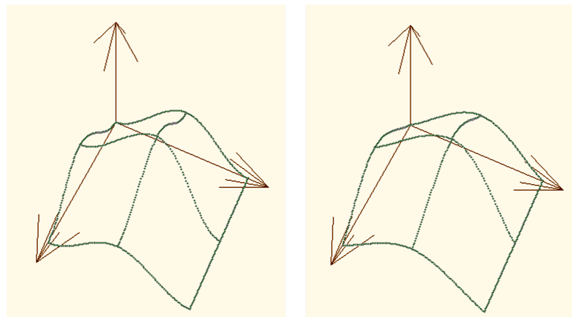

Figure 7. On left side without correction, on right side after smoothness correction.

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ON RECTIFYING SLANT HELICES IN EUCLIDEAN 3-SPACE

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Abstract. In this paper, we study the position vector of rectifying slant helices in E^3. First, we have found the general equations of the curvature and the torsion of rectifying slant helices. After that, we have constructed a second order linear differential equation and by solving the equation, we have obtained a family of rectifying slant helices which lie on cones.

1. Introduction

In classical differential geometry; a general helix in the Euclidean 3-space, is a curve which makes a constant angle with a fixed direction.

The notion of rectifying curve has been introduced by Chen [2, 3]. Chen showed, under which conditions, the position vector of a unit speed curve lies in its rectifying plane. He also stated the importance of rectifying curves in Physics.

On the other hand, the notion of slant helix was introduced by Izuyama and Takeuchi [4, 5]. They showed, under which conditions, a unit speed curve is a slant helix. Later, Ahmet T. Ali published a paper in which position vectors of some slant helices were shown [1]. In [6, 7], L. Kula, et al studied the spherical images under both tangent and binormal indicatrices of slant helices and obtained that the spherical images of a slant helix are spherical helices.

The papers mentioned above led us to study on the notion of rectifying slant helices. We began with finding the equations of curvature and torsion of a rectifying slant helix. After that, we constructed a second order linear differential equation to determine position vector of a rectifying slant helix. By solving this equation for some special cases, we obtained a unit speed family of rectifying slant helices which lie on cones.

2. Preliminaries

The Euclidean 3-space E^3 is the real vector space R^3 with the metric

$$g = dx_1^2 + dx_2^2 + dx_3^2,$$

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where (x_1, x_2, x_3) is a rectangular coordinate system of E^3.

A curve $\alpha : I \subset R \longrightarrow E^3$ is said to be parametrized by the arclength parameter s, if $g\left(\alpha^{'}(s), \alpha^{'}(s)\right) = 1$, where $\alpha'(s) = d\alpha/ds$. Then, we call α unit speed. Consider unit-speed space curve α has at least four continuous derivatives, then α has a natural frame called Frenet Frame with the equations below,

$$\begin{aligned} t^{'} &= \kappa n \\ n^{'} &= -\kappa t + \tau b \\ b^{'} &= -\tau n, \end{aligned}$$

where κ is the curvature, τ is the torsion, and $\{t, n, b\}$ is the Frenet Frame of the curve α. We denote unit tangent vector field with t, unit principal normal vector field with n, and the unit binormal vector field with b. It is possible in general, that $t^{'}(s) = 0$ for some $s \in I$; however, we assume that this never happens.

Definition 2.1. A curve is called a slant helix if its principal normal vector field makes a constant angle with a fixed line in space.

Theorem 2.1. *A unit speed curve α is a slant helix if and only if the geodesic curvature of the spherical image of the principal normal indicatrix of α which is*

$$\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)^{'} \right)(s)$$

is constant [4, 5].

A unit speed curve α is called rectifying curve when the position vector of it always lie in its rectifying plane. So, for a rectifying curve we can write

$$\alpha(s) = \lambda(s)\, t(s) + \mu(s)\, b(s)\,.$$

Theorem 2.2. *A unit speed curve α is congruent to a rectifying curve if and only if*

$$\frac{\tau(s)}{\kappa(s)} = c_1 s + c_2$$

for some constants c_1 and c_2, with $c_1 \neq 0$ [2, 3].

3. Rectifying Slant Helices in E^3

If the position vector of a unit speed slant helix always lies in its rectifying plane we call it a rectifying slant helix. For a rectifying slant helix we have the following theorem.

Theorem 3.1. *Let α be a unit speed curve in E^3. Then, $\alpha(s)$ is a rectifying slant helix if and only if the curvature and torsion of the curve satisfies the equations below;*

$$\kappa(s) = \frac{c_3}{\left(1 + (c_1 s + c_2)^2\right)^{3/2}}, \tau(s) = \frac{c_3\,(c_1 s + c_2)}{\left(1 + (c_1 s + c_2)^2\right)^{3/2}},$$

where $c_1 \neq 0, c_2 \in R$, $\theta \neq 0 + k\pi/2, k \in Z$, and $c_3 \in R^+$.

Proof. Let α be a unit speed rectifying slant helix in E^3, then the equations in Theorem 2.1, and Theorem 2.2 exists. If we combine them then we have

$$m = \frac{c_1}{\kappa\left(1+(c_1 s+c_2)^2\right)^{3/2}}.$$

where m is a constant. So we can write κ as follows

$$\kappa(s) = \frac{c_3}{\left(1+(c_1 s+c_2)^2\right)^{3/2}},$$

then, from Theorem 2.2

$$\tau(s) = \frac{c_3\,(c_1 s+c_2)}{\left(1+(c_1 s+c_2)^2\right)^{3/2}}.$$

where $c_3 = |c_1/m|$.

Conversely, it can be easily seen that, the curvature functions as mentioned above satisfy the equations at Theorem 2.1 and Theorem 2.2. So, α is a rectifying slant helix.

□

Now, we give another Theorem by using the definitions of slant helix and rectifying curve to determine c_3.

Theorem 3.2. *Let α be a unit speed rectifying slant helix whose principal normal vector field makes a constant angle with a unit vector u, then the curvature and torsion of α satisfy the equations below;*

$$\kappa(s) = \frac{|c_1\tan(\theta)|}{\left((c_1 s+c_2)^2+1\right)^{3/2}}, \quad \tau(s) = \frac{|c_1\tan(\theta)|\,(c_1 s+c_2)}{\left((c_1 s+c_2)^2+1\right)^{3/2}}$$

where $c_1 \neq 0, c_2 \in R$.

Proof. Let α be a unit speed rectifying slant helix in E^3. Then, from the definition of slant helix there is a unit fixed vector u with

$$g(n,u) = \cos(\theta),$$

where $\theta \in R^+$. If we differentiate this equation with respect to s, we have,

$$g(-\kappa t+\tau b,u) = 0.$$

If we divide both parts of the equation with κ, we get

$$g(-t+(c_1 s+c_2)b,u) = 0, \tag{3.1}$$

then,

$$g(t,u) = (c_1 s+c_2)g(b,u).$$

While $\{t,n,b\}$ is a orthonormal frame we can write,

$$v = \lambda_1 t+\lambda_2 n+\lambda_3 b,$$

with $\lambda_1^2+\lambda_2^2+\lambda_3^2=+1$. If we make the neccessary calculations we have,

$$\lambda_1=\mp\frac{(c_1s+c_2)\sin(\theta)}{\sqrt{(c_1s+c_2)^2+1}},\qquad \lambda_2=\cos(\theta),\qquad \lambda_3=\pm\frac{\sin(\theta)}{\sqrt{(c_1s+c_2)^2+1}}.$$

By differentiating (3.1) we have,

$$\pm\frac{c_1\sin(\theta)}{\kappa\sqrt{(c_1s+c_2)^2+1}}-(1+(c_1s+c_2)^2)\cos(\theta)=0.$$

Therefore,

$$\kappa(s)=\frac{|c_1\tan(\theta)|}{\left((c_1s+c_2)^2+1\right)^{3/2}},$$

and

$$\tau(s)=\frac{|c_1\tan(\theta)|\,(c_1s+c_2)}{\left((c_1s+c_2)^2+1\right)^{3/2}}.$$

□

Theorem 3.3. *Let $\alpha(s)$ be a unit speed rectifying slant helix. Then, the vector v satisfies the linear vector differential equation of second order as follows;*

$$v''(s)+\frac{(c_1\tan(\theta))^2}{\left(1+(c_1s+c_2)^2\right)^2}v(s)=0,$$

where $v=\frac{n'}{\kappa}$.

Proof. Let α be a unit speed rectifying slant helix then we can write frenet equations as follows,

$$\begin{aligned} t' &= \kappa n \\ n' &= -\kappa t+f\kappa b \\ b' &= -f\kappa n, \end{aligned}\tag{3.2}$$

where $f(s)=c_1s+c_2$. If we divide second equation by κ we have,

$$\frac{n'}{\kappa}=-t+fb.\tag{3.3}$$

By differentiating (3.3), we have

$$c_1b=\left(\frac{n'}{\kappa}\right)'+\kappa(1+f^2)n.\tag{3.4}$$

By differentiating (3.4) and using (3.2) we have

$$\left(\frac{n'}{\kappa}\right)''+\kappa(1+f^2)n'+\left[\left(\kappa(1+f^2)\right)'+c_1f\kappa\right]n=0,\tag{3.5}$$

with the necessary calculations we easily see

$$\left(\kappa(1+f^2)\right)'+c_1f\kappa=0.$$

So we have (3.5) as follows,

$$\left(\frac{n'}{\kappa}\right)''+\kappa(1+f^2)n'=0.\tag{3.6}$$

Let us denote $\frac{n'}{\kappa} = v$. Then (3.6) becomes to

$$v'' + \frac{(c_1 \tan(\theta))^2}{\left(1 + (c_1 s + c_2)^2\right)^2} v = 0, \tag{3.7}$$

this completes the proof. □

As we know every component of vector $v = (v_1, v_2, v_3)$ must satisfy (3.7). We can show

$$\begin{gathered} v_1(s) = -\sqrt{\left(1 + f^2(s)\right)} \sin\left[\sec(\theta) \arctan\left[f(s)\right]\right], \\ v_2(s) = \sqrt{\left(1 + f^2(s)\right)} \cos\left[\sec(\theta) \arctan\left[f(s)\right]\right], \\ v_3(s) = 0. \end{gathered}$$

We can show v is a solution for (3.7). Therefore, we can write $n = (n_1, n_2, n_3)$ as follows,

$$\begin{gathered} n_1(s) = \int \kappa(s) v_1(s) ds = A_1 |c_1| \sin(\theta) \cos\left[\sec(\theta) \arctan\left[f(s)\right]\right], \\ n_2(s) = \int \kappa(s) v_2(s) ds = A_2 |c_1| \sin(\theta) \sin\left[\sec(\theta) \arctan\left[f(s)\right]\right], \\ n_3(s) = \cos(\theta). \end{gathered} \tag{3.8}$$

On the other hand, Let α be a unit speed rectifying slant helix, whose principal normal vector field makes a constant angle θ with e_3. Then, for its principal normal we can write

$$< n, e_3 >= \cos(\theta).$$

While $n = (n_1, n_2, n_3)$ is a unit vector, $n_1^2 + n_2^2 + n_3^2 = 1$. So, $n_1^2 + n_2^2 = 1 - \cos^2(\theta) = \sin^2(\theta)$. Therefore n can be in the form,

$$\begin{gathered} n_1(s) = \sin(\theta) \cos(h(s)) \\ n_2(s) = \sin(\theta) \sin(h(s)) \\ n_3(s) = \cos(\theta), \end{gathered} \tag{3.9}$$

where $h(s)$ is a differentiable function.

If we take, $A_1 = 1/|c_1|, A_2 = 1/|c_1|, h(s) = \sec(\theta) \arctan\left[f(s)\right]$ at (3.8), (3.8) and (3.9) coincides. Thus, a unit speed rectifying slant helix α can be in the form;

$$\begin{gathered} \alpha_1(s) = \sin(\theta) \int \left(\int \kappa(s) \cos\left[\sec(\theta) \arctan(c_1 s + c_2)\right] ds\right) ds, \\ \alpha_2(s) = \sin(\theta) \int \left(\int \kappa(s) \sin\left[\sec(\theta) \arctan(c_1 s + c_2)\right] ds\right) ds, \\ \alpha_3(s) = \int \left(\int \kappa(s) \cos(\theta) ds\right) ds, \end{gathered}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Therefore, we find α as follows.

$$\begin{gathered} \alpha_1(s) = -\tfrac{\cos(\theta)}{c_1} \sqrt{1 + (c_1 s + c_2)^2} \cos\left[\sec(\theta) \arctan(c_1 s + c_2)\right], \\ \alpha_2(s) = -\tfrac{\cos(\theta)}{c_1} \sqrt{1 + (c_1 s + c_2)^2} \sin\left[\sec(\theta) \arctan(c_1 s + c_2)\right], \\ \alpha_3(s) = \tfrac{1}{c_1} \sqrt{1 + (c_1 s + c_2)^2} \sin(\theta). \end{gathered}$$

Now, we can write a new lemma;

Lemma 3.1. *Let $\alpha(s): I \longrightarrow R^3$ be a space curve with the equation below,*

$$\alpha(s) = -\frac{\sqrt{1+(c_1 s + c_2)^2}}{c_1}\big(\cos(\theta)\cos\left[\sec(\theta)\arctan(c_1 s + c_2)\right], \cos(\theta)\sin\left[\sec(\theta)\arctan(c_1 s + c_2)\right], -\sin(\theta)\big), \tag{3.10}$$

where $\theta \neq \frac{\pi}{2} + k\pi, k \in Z$, and $c_1 \neq 0, c_2 \in R$. Then, $\alpha(s)$ is a unit speed rectifying slant helix which lies on the cone

$$\tan^2(\theta)\left(x^2+y^2\right) = z^2. \tag{3.11}$$

Proof. With direct calculations we have $g(\alpha', \alpha') = 1$, $g(n, n) = 1$, and the curvature functions of α as,

$$\kappa(s) = \frac{|c_1 \tan(\theta)|}{((c_1 s+c_2)^2+1)^{3/2}},$$
$$\tau(s) = \frac{|c_1 \tan(\theta)|(c_1 s+c_2)}{((c_1 s+c_2)^2+1)^{3/2}}.$$

with,

$$\frac{\kappa^2(s)}{\left(\kappa^2(s)+\tau^2(s)\right)^{3/2}}\left(\frac{\tau(s)}{\kappa(s)}\right)' = \cot(\theta),$$

and

$$\frac{\tau(s)}{\kappa(s)} = c_1 s + c_2.$$

So, α is a unit speed spacelike rectifying slant helix. We also have

$$\tan^2(\theta)\left({\alpha_1}^2(s)+{\alpha_2}^2(s)\right) - {\alpha_3}^2(s) = 0,$$

then, α lies on the cone above. □

Example 3.1. If we take $c_1 = 1, c_2 = 0$, and $\cos(\theta) = 1/3$ then, $\tan(\theta) = 2\sqrt{2}$. If we put these into (3.10) and (3.11), we have the following equations;

$$\alpha(s) = \left(-\tfrac{1}{3}\sqrt{s^2+1}\cos(3\arctan(s)), -\tfrac{1}{3}\sqrt{s^2+1}\sin(3\arctan(s)), \tfrac{2\sqrt{2}}{3}\sqrt{s^2+1}\right),$$
$$\kappa(s) = \frac{2\sqrt{2}}{(s^2+1)^{3/2}}, \tau(s) = \frac{2\sqrt{2}s}{(s^2+1)^{3/2}},$$
$$8\left(x^2+y^2\right) = z^2.$$

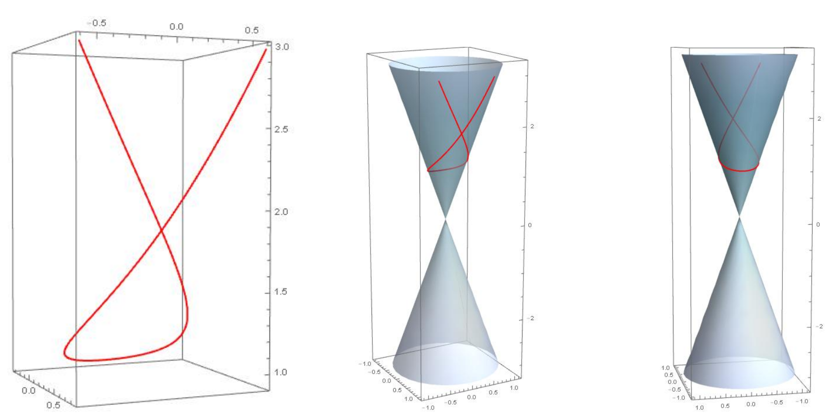

FIGURE 1. Rectifying Slant Helix on $8\left(x^2+y^2\right) = z^2$

Example 3.2. If we take $c_1 = 1/2, c_2 = -1/5$, and $\cos(\theta) = 1/10$ then, $\tan(\theta) = \sqrt{99}$. If we put these into (3.10) and (3.11), we have the following equations;

$$\begin{aligned}\beta(s) = \frac{1}{5}\sqrt{\left(\frac{s}{2}-\frac{1}{5}\right)^2+1}\Bigg(&-\cos\left(10\arctan\left(\frac{s}{2}-\frac{1}{5}\right)\right),\\ &-\sin\left(10\arctan\left(\frac{s}{2}-\frac{1}{5}\right)\right),\\ &\frac{3\sqrt{11}}{5}\Bigg),\end{aligned}$$

$$\kappa(s) = \frac{1500\sqrt{11}}{\left(5s(5s-4)+104\right)^{3/2}}, \tau(s) = \frac{150\sqrt{11}(5s-2)}{\left(5s(5s-4)+104\right)^{3/2}},$$
$$99\left(x^2+y^2\right) = z^2.$$

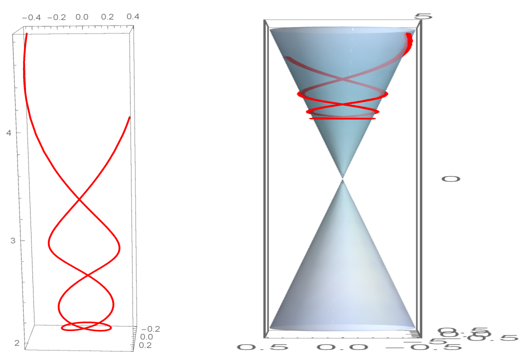

FIGURE 2. Rectifying Slant Helix on $99\left(x^2+y^2\right) = z^2$

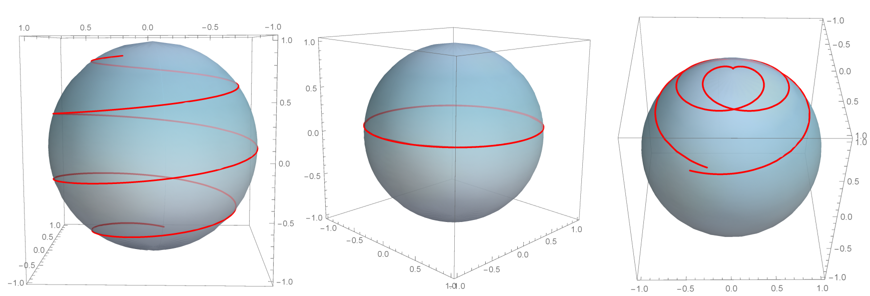

FIGURE 3. Tangent, Normal,and Binormal indicatrix of β resp.

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ON THE FAMILY OF METRICS FOR SOME PLATONIC AND ARCHIMEDEAN POLYHEDRA

ÖZCAN GELIŞGEN AND ZEYNEP CAN

Abstract. Convexity is an important property in mathematics and geometry. In geometry convexity is simply defined as; if every points of a line segment that connects any two points of the set are in the set then this set is convex. A polyhedra, when it is convex, is an extremely important solid in 3-dimensional analytical space. Polyhedra have interesting symmetries. Therefore they have attracted the attention of scientists and artists from past to present. Thus polyhedra are discussed in a lot of scientific and artistic works. There are many relationships between metrics and polyhedra. Some of them are given in previous studies. For example, in [7] the authors have shown that the unit sphere of Chinese Checkers 3-space is the deltoidal icositetrahedron. In this study, we introduce a family of metrics, and show that the spheres of the 3-dimensional analytical space furnished by these metrics are some well-known polyhedra.

1. INTRODUCTION

A polyhedron is a geometric solid bounded by polygons. Polygons form the faces of the solid; an edge of the solid is the intersection of two polygons, and a vertex of the solid is a point where three or more edges intersect. If all faces of a polyhedron are identical regular polygons and at every vertex same number of faces meet then it is called a regular polyhedron. A polyhedron is called semi-regular if all its faces are regular polygons and all its vertices are equal.

Polyhedra have very interesting symmetries. Therefore they have attracted the attention of scientists and artists from past to present. Thus mathematicians, geometers, physicists, chemists, artists have studied and continue to study on polyhedra. Consequently, polyhedra take place in many studies with respect to different fields. As it is stated in [3] and [6], polyhedra have been used for explaining the world around us in philosophical and scientific way. There are only five regular convex polyhedra known as the platonic solids. These regular polyhedra were known by the Ancient Greeks. They are generally known as the "Platonic" or "cosmic" solids

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because Plato mentioned them in his dialogue Timeous, where each is associated with one of the five elements - the cube with earth, the icosahedron with water, the octahedron with air, the tetrahedron with fire and the dodecahedron with universe (or with ether, the material of the heavens). The story of the rediscovery of the Archimedean polyhedra during the Renaissance is not that of the recovery of a 'lost' classical text. Rather, it concerns the rediscovery of actual mathematics, and there is a large component of human muddle in what with hindsight might have been a purely rational process. The pattern of publication indicates very clearly that we do not have a logical progress in which each subsequent text contains all the Archimedean solids found by its author's predecessors. In fact, as far as we know, there was no classical text recovered by Archimedes. The Archimedean solids have that name because in his Collection, Pappus stated that Archimedes had discovered thirteen solids whose faces were regular polygons of more than one kind. Pappus then listed the numbers and types of faces of each solid. Some of these polyhedra have been discovered many times. According to Heron, the third solid on Pappus' list, the cuboctahedron, was known to Plato. During the Renaissance, and especially after the introduction of perspective into art, painters and craftsmen made pictures of platonic solids. To vary their designs they sliced off the corners and edges of these solids, naturally producing some of the Archimedean solids as a result.For more detailed knowledge, see [3] and [6].

The dual polyhedra of the Archimedean solids are called Catalan solids, and they are exactly thirteen just like Archimedean solids. Platonic solids are regular and convex polyhedra and Archimedean solids are semi-regular and convex polyhedra. The Catalan solids are all convex. They are face-transitive when all its faces are the same but not vertex-transitive. Unlike Platonic solids and Archimedean solids, the face of Catalan solids are not regular polygons.

As it is stated in [14], Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions. Here the linear structure is the same as the Euclidean one but distance is not uniform in all directions. That is, the points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. Thus, instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set. Some mathematicians studied and improved metric geometry in plane and space. (Some of these are [1, 4, 5, 8, 9, 10]) According to studies on polyhedra, there are some Minkowski geometries in which unit spheres of these spaces furnished by some metrics are associated with convex solids. For example, unit spheres of maximum space and taxicab space are cubes and octahedrons, respectively, which are Platonic Solids. And unit sphere of CC-space is a deltoidal icositetrahedron which is a Catalan solid. Therefore, there are some metrics in which unit spheres of space furnished by them are convex polyhedra. That is, convex polyhedra are associated with some metrics. When a metric is given we can find its unit sphere. Naturally a question can be asked; "Is it possible to find the metric when a convex polyhedron is given?". In this study, we introduce a family of metrics and show that spheres of 3-dimensional analytical space furnished by these metrics are some polyhedra. Then we give relationships between metrics and some of Platonic and Archimedean solids. Some results for these relationships are already known from previous studies. But we introduce three metrics and give three new relationships for cuboctahedron, truncated cube and truncated octahedron.

2. Archimedean Metric

As it is mentioned in introduction, there are some 3-dimensional Minkowski geometries which have distance function distinct from Euclidean distance and unit spheres of these geometries are convex polyhedrons. That is, convex polyhedra are associated with some metrics. When a metric is given, we can find its unit sphere in related space geometry. This enforce us to the question "Are there some metrics whose unit sphere is a convex polyhedron?". For this goal, firstly, the related polyhedra are placed in the 3-dimensional space in such a way that they are symmetric with respect to the origin. And then the coordinates of vertices are found. Later one can obtain metric which always supply plane equation related with solid's surface. When we started studying on this question, we firstly handled separately convex polyhedra. But we noticed a relationship between the metrics. Now, we introduce a family of distances which include Taxicab distance and maximum distance as special cases in $\mathbb{R}^3$.

Definition 2.1. Let $u \in [0, \infty)$, and $P_1 = (x_1, y_1 z_1)$, $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{AP} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ Archimedean polyhedral distance between P_1 and P_2 is defined by

$$d_{AP}(P_1, P_2) = \max\left\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|, u\left(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\right)\right\}.$$

Clearly, there are infinitely many different distance functions in the family of distance functions defined above, depending on value of u. One can think the definition not to be well-defined since the Archimedean polyhedra distance between two points can also change according to value of u. To remove this confusion, supposing value of u is initially determined and fixed unless otherwise stated. We write $\mathbb{R}^3_{AP} = (\mathbb{R}^3, d_{AP})$ for the 3-dimensional analytical space furnished by Archimedean polyhedral distance defined above.

Since proof is trivial by the definition of maximum function, we give following lemma without proof which is required to show that each of d_{AP} distances gives a metric.

Lemma 2.1. *Let $P_1 = (x_1, y_1 z_1)$ and $P_2 = (x_2, y_2, z_2)$ be any distinct points in $\mathbb{R}^3$. Then*

$$\begin{aligned} &d_{AP}(P_1, P_2) \geq |x_1 - x_2|, \\ &d_{AP}(P_1, P_2) \geq |y_1 - y_2|, \\ &d_{AP}(P_1, P_2) \geq |z_1 - z_2|, \\ &d_{AP}(P_1, P_2) \geq u\left(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\right). \end{aligned}$$

Theorem 2.1. *Every d_{AP} distance determines a metric in $\mathbb{R}^3$.*

Proof. Let $d_{AP} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is Archimedean polyhedral distance function, and $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ and $P_3 = (x_3, y_3, z_3)$ are distinct three points in $\mathbb{R}^3$. We have to show that d_{AP} is positive definite, symmetric, and the triangle inequality holds for d_P.

Absolute value gives always non-negative value and $u \geq 0$, then $d_{AP}(P_1, P_2) \geq 0$. Clearly, $d_{AP}(P_1, P_2) = 0$ iff $P_1 = P_2$. So d_P is positive definite.

Since $|a - b| = |b - a|$ for all a, $b \in \mathbb{R}$, obviously $d_{AP}(P_1, P_2) = d_{AP}(P_2, P_1)$. That is, d_{AP} is symmetric.

Now, we should prove that $d_P(P_1, P_3) \leq d_P(P_1, P_2) + d_P(P_2, P_3)$ for all P_1, P_2, $P_3 \in \mathbb{R}^3$.

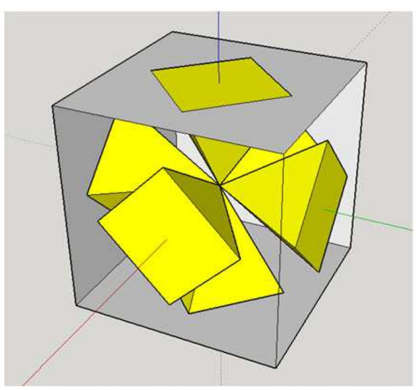
$$\begin{aligned}
&d_P(P_1,P_3)\\
&= \max\left\{|x_1-x_3|,|y_1-y_3|,|z_1-z_3|,u\left(|x_1-x_3|+|y_1-y_3|+|z_1-z_3|\right)\right\}\\
&= \max\left\{\begin{array}{l}|x_3-x_2+x_2-x_1|,|y_3-y_2+y_2-y_1|,|z_3-z_2+z_2-z_1|,\\ u\left(|x_3-x_2+x_2-x_1|+|y_3-y_2+y_2-y_1|+|z_3-z_2+z_2-z_1|\right)\end{array}\right\}\\
&\leq \max\left\{\begin{array}{l}|x_3-x_2|+|x_2-x_1|,|y_3-y_2|+|y_2-y_1|,|z_3-z_2|+|z_2-z_1|\\ u\left(|x_3-x_2|+|x_2-x_1|+|y_3-y_2|+|y_2-y_1|+|z_3-z_2|+|z_2-z_1|\right)\end{array}\right\}\\
&= I
\end{aligned}$$

One can easily find that $I \leq d_{AP}(P_1,P_2)+d_{AP}(P_2,P_3)$ from Lemma 2.1. So $d_{AP}(P_1,P_3) \leq d_{AP}(P_1,P_2)+d_{AP}(P_2,P_3)$. Consequently, Archimedean polyhedral distance is a metric in 3-dimensional analytical space. □

According to Archimedean polyhedral metric, distance is one of quantities $|x_1-x_2|$, $|y_1-y_2|$, $|z_1-z_2|$ or u times sum of quantities $|x_1-x_2|$, $|y_1-y_2|$, $|z_1-z_2|$. Geometrically, there are two different paths between two points in $\mathbb{R}^3_{AP}$. If the line segment $\overline{P_1P_2}$ is out of cones with apex P_1 and square bases which corner points are all permutations of the three axis components and all possible $+/-$ sign change of each axis component of $(\mp 1, \mp(1-u), 0)$, then

$$d_{AP}\left(P_1,P_2\right)=u\left(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\right)$$

,and the path between P_1 and P_2 is union of three line segments which is parallel to a coordinate axis. Otherwise, the path between P_1 and P_2 is a line segment which is parallel to a coordinate axis. Thus Archimedean polyhedral distance between P_1 and P_2 is u times sum of Euclidean lengths of these three line segments or the Euclidean length of line segment (See Figure 1).



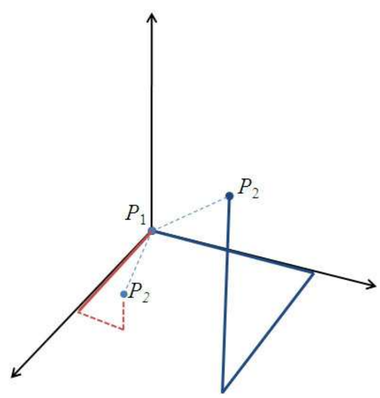


Figure 1: AP ways from P_1 to P_2

The following proposition gives an equation which relates the Euclidean distance to the Archimedean polyhedral distance between the points in $\mathbb{R}^3$:

Proposition 2.1. *Let l be the line through the points $P_1=(x_1,y_1,z_1)$ and $P=(x_2,y_2,z_2)$ in the analytical 3-dimensional space and d_E denote the Euclidean metric. If l has direction vector (p,q,r), then*

$$d_{AP}(A,B)=\mu(AB)d_E(A,B)$$

where

$$\mu(AB)=\frac{\max\{|p|,|q|,|r|,u\left(|p|+|q|+|r|\right)\}}{\sqrt{p^2+q^2+r^2}}.$$

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $\lambda \in \mathbb{R} \setminus \{0\}$. Thus,

$$d_{AP}(A,B) = |\lambda| \left(\max\{|p|, |q|, |r|, u(|p| + |q| + |r|)\}\right)$$

and $d_E(A,B) = |\lambda| \sqrt{p^2 + q^2 + r^2}$ which implies the required result. □

The above lemma says that d_{AP}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 2.1. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_{AP}(P_1, X) = d_{AP}(P_2, X)$.*

Corollary 2.2. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{AP}(X, P_1) \ / \ d_{AP}(X, P_2) = d_E(X, P_1) \ / \ d_E(X, P_2) \ .$$

That is, the ratios of the Euclidean and $d_{AP}-$distances along a line are the same.

3. Some relations about the Archimedean polyhedral distance and Polyhedra

The polyhedral metric gives a family of metrics and unit spheres in 3-dimensional analytical space furnished by Archimedean polyhedral metric which are some polyhedra. Of course, polyhedra varies depending on choice of u. Some results of relations between metrics and polyhedra are already known from previous studies. Here, we especially give three new relations between polyhedra and metrics by using Archimedean polyhedral metric. Now, according to choice of u, we give five cases for Archimedean polyhedral metric.

Case 1. Let $u \geq 1$. So $AP-$metric is u times taxicab metric. In particular, if $u = 1$, then $AP-$metric is taxicab metric. In this case the unit sphere is the octahedron.

Case 2. Set $u \in \left(0, \frac{1}{3}\right)$. Hence, $AP-$metric is the maximum metric. So the unit sphere is the hexahedron.

Case 3. Let $u = \frac{1}{2}$. Then Archimedean polyhedral metric gives a new result. In this case, the unit sphere is cuboctahedron. So we called cuboctahedron metric which is defined by

$$d_{AP}(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|, \frac{1}{2}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)\}.$$

(see Figure 2a).

Case 4. Let $u \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Then Archimedean polyhedral metric gives a new result. In particular, if $u = \sqrt{2} - 1$, then the unit sphere is truncated cube. So we called truncated cube metric which is defined by

$$d_{AP}(P_1, P_2)$$
$$= \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|, \left(\sqrt{2}\text{-}1\right)(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)\}.$$

For $u \in \left(\frac{1}{3}, \frac{1}{2}\right)$ case, the unit sphere is like truncated cube. When $u \to \frac{1}{2}$ and $u \to \frac{1}{3}$, the unit sphere looks like cuboctahedron and cube, respectively. But for all values of u, unit sphere has 8-triangular faces and 6-octagonal faces (see Figure 2b).

Case 5. Let $u \in \left(\frac{1}{2}, 1\right)$. Then Archimedean polyhedral metric gives a new result. In particular, if $u = \frac{2}{3}$, then the unit sphere is truncated octahedron. So we called truncated octahedron metric which is defined by

$$d_{AP}(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|, \frac{2}{3}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)\}.$$

For $u \in \left(\frac{1}{2}, 1\right)$ case, the unit sphere is like truncated octahedron. When $u \to 1$ and $u \to \frac{1}{2}$, the unit sphere looks like octahedron and cuboctahedron, respectively. But for all values of u, unit sphere has 6-square faces and 8-hexagonal faces (see Figure 2c).

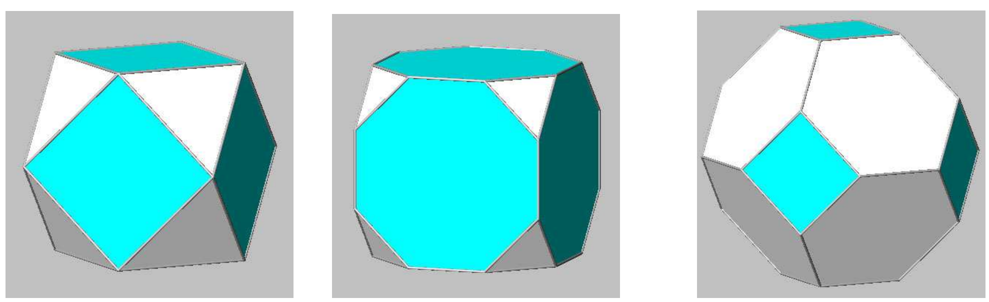


Figure 2a Cuboctahedron Figure 2b Truncated cube Figure 2c Truncated octahedron

One can observe that the Archimedean metric has two parts, one is $\max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$ and the other is $u(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)$. In fact, $\max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$ and $u(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)$ indicate the hexahedron and the octahedron, respectively. Thus sphere of Archimedean polyhedral metric is intersection of hexahedron and octahedron. The cases which defined above are explicated by this way.

One can take $d_{AP}(O, P) = r$. then gets $\max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\} = r$ and $u(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|) = r$. That is, these are the cube with vertices such that all permutations of $(\mp r, \mp r, \mp r)$ and the octahedron with vertices such that all permutations of $\left(\mp \frac{r}{u}, 0, 0\right)$, respectively. The faces of the cube are on the planes with equations $|x| = r$, $|y| = r$ and $|z| = r$, and the faces of octahedron are on the planes with equations $|x| + |y| + |z| = \frac{r}{u}$. The intersection of the faces of the cube and the octahedron are found by solving the systems of linear equations

$$\begin{cases} |x| + |y| + |z| = \dfrac{r}{u} \\ |x| = r \end{cases}, \quad \begin{cases} |x| + |y| + |z| = \dfrac{r}{u} \\ |y| = r \end{cases}, \quad \begin{cases} |x| + |y| + |z| = \dfrac{r}{u} \\ |z| = r \end{cases}.$$

For example, we handle the system of equations $\begin{cases} |x|+|y|+|z| = \dfrac{r}{u} \\ |x| = r \end{cases}$. Since $|x|$=r, it is obtained that $|y|+|z| = \dfrac{r}{u} - r$. The solution is the taxicab circles with the center $(\mp r, 0, 0)$ and radius $\dfrac{r}{u} - r$ on planes $|x| = r$. If $u \in \left[\dfrac{1}{2}, 1\right]$, then the circle is completely on face of the cube. Thus intersection consist of squares and hexagons. If $u \in \left(\dfrac{1}{3}, \dfrac{1}{2}\right)$, then the circle is not completely on face of the cube. Therefore intersection consist of triangles and octagons. If $u = \dfrac{1}{2}$, then intersection consist of squares and triangles. Figure 3a,3b,3c illustrate these cases.

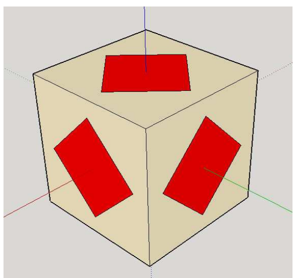


Figure 3a

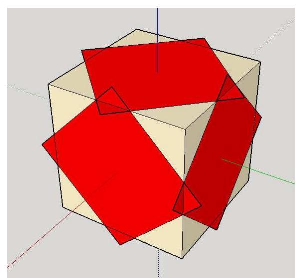


Figure 3b

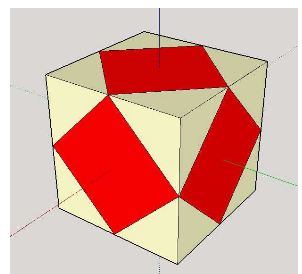


Figure 3c 3

Now, we can give some new results:

The truncated cube, or truncated hexahedron, is an Archimedean solid. It has 14 regular faces (6 octagonal and 8 triangular), 36 edges, and 24 vertices (See [16]).

The cuboctahedron is an archimedean solid with eight triangular faces and six square faces. It has 12 identical vertices, with two triangles and two squares meeting at each, and 24 identical edges, each separating a triangle from a square (See [15]).

The truncated octahedron is an archimedean solid which has 14 faces (8 regular hexagonal and 6 square), 36 edges, and 24 vertices. Since each of its faces has point symmetry the truncated octahedron is a zonohedron (See [17]).

The following corollaries are direct consequences of Proposition 2.1, Corollary 2.1 and Corollary 2.2

Corollary 3.1. *The equations of cuboctahedron, truncated cube and truncated octahedron with center $C = (x_0, y_0, z_0)$ and radius r are*

$$\begin{aligned}
\max\left\{|x-x_0|, |y-y_0|, |z-z_0|, \frac{1}{2}\left(|x-x_0|+|y-y_0|+|z-z_0|\right)\right\} &= r \\
\max\left\{|x-x_0|, |y-y_0|, |z-z_0|, \left(\sqrt{2}-1\right)\left(|x-x_0|+|y-y_0|+|z-z_0|\right)\right\} &= r \\
\max\left\{|x-x_0|, |y-y_0|, |z-z_0|, \frac{2}{3}\left(|x-x_0|+|y-y_0|+|z-z_0|\right)\right\} &= r
\end{aligned}$$

,respectively. The the cuboctahedron, truncated cube and the truncated octahedron have 14- regular faces with vertices such that all permutations of the three axis components and all possible +/- sign changes of each axis component of $\left(r, r, \left(\sqrt{2}-1\right) r\right)$, $(r, r, 0)$ and $(r/2, r, 0)$, respectively (See Figure 4a,4b,4c).

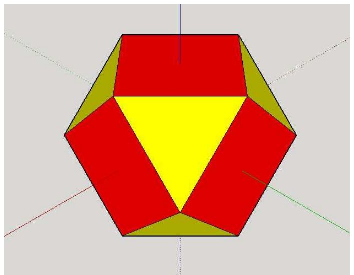


Figure 4a

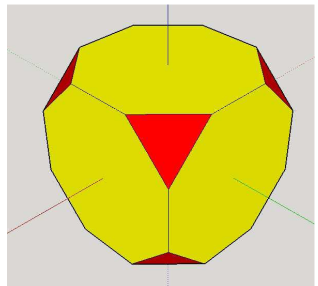


Figure 4b

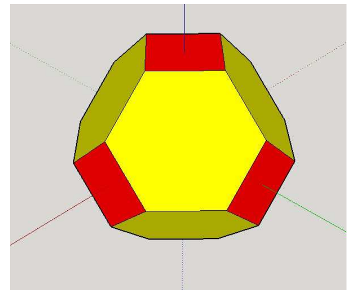


Figure 4c

Lemma 3.1. *Let l be the line through the points $P_1 = (x_1, y_1 z_1)$ and $P_2 = (x_2, y_2, z_2)$ in the analytical $3-$dimensional space and d_E, d_{TC}, d_{CO} and d_{TO} denote the Euclidean metric, the truncated metric, the cuboctahedron metric and the truncated metric respectively. If l has direction vector (p, q, r), then*

$$\begin{aligned} d_{CO}(P_1, P_2) &= \frac{\max\left\{|p|, |q|, |r|, \frac{1}{2}(|p|+|q|+|r|)\right\}}{\sqrt{p^2+q^2+r^2}} d_E(P_1, P_2) \\ d_{TC}(P_1, P_2) &= \frac{\max\left\{|p|, |q|, |r|, \left(\sqrt{2}-1\right)(|p|+|q|+|r|)\right\}}{\sqrt{p^2+q^2+r^2}} d_E(P_1, P_2) \\ d_{TO}(P_1, P_2) &= \frac{\max\left\{|p|, |q|, |r|, \frac{2}{3}(|p|+|q|+|r|)\right\}}{\sqrt{p^2+q^2+r^2}} d_E(P_1, P_2). \end{aligned}$$

Corollary 3.2. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then*

$$\begin{aligned} d_E(P_1, X) &= d_E(P_2, X) \text{ if and only if } d_{CO}(P_1, X) = d_{CO}(P_2, X) \\ d_E(P_1, X) &= d_E(P_2, X) \text{ if and only if } d_{TC}(P_1, X) = d_{TC}(P_2, X) \\ d_E(P_1, X) &= d_E(P_2, X) \text{ if and only if } d_{TO}(P_1, X) = d_{TO}(P_2, X). \end{aligned}$$

Corollary 3.3. *If P_1, P_2 and X are any distinct collinear points in $\mathbb{R}^3$, then*

$$\frac{d_E(P_1, X)}{d_E(P_2, X)} = \frac{d_{CO}(P_1, X)}{d_{CO}(P_2, X)} = \frac{d_{TC}(P_1, X)}{d_{TC}(P_2, X)} = \frac{d_{TO}(P_1, X)}{d_{TO}(P_2, X)}.$$

That is, the ratios of the Euclidean, the cuboctahedron, the truncated cube and the truncated octahedron distances along a line are the same.

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GRAPHS WHICH ARE DETERMINED BY THEIR SPECTRUM

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Abstract. It is well-known that the problem of spectral characterization is related to the Hückel theory from Chemistry. E. R. van Dam and W. H. Haemers [11] conjectured almost all graphs are determined by their spectra. Nevertheless, the set of graphs which are known to be determined by their spectra is small. Hence discovering infinite classes of graphs that are determined by their spectra can be an interesting problem and helps reinforce this conjecture. The main aim of this work is to characterize new classes of graphs that are known as multicone graphs. In this work, it is shown that any graph cospectral with multicone graphs $K_w \bigtriangledown GQ(2,1)$ or $K_w \bigtriangledown GQ(2,2)$ is determined by its adjacency spectra, where $GQ(2,1)$ and $GQ(2,2)$ denote the strongly regular graphs that are known as the generalized quadrangle graphs. Also, we prove that these graphs are determined by their Laplacian spectrum. Moreover, we propose four conjectures for further reseache in this topic.

1. Introduction

All graph considered here are simple and undirected. All notions on graph that are not defined here can be found in [3, 4, 6, 15]. Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, ..., v_n\}$ and edge set $E = E(G) = \{e_1, ..., e_m\}$. Denote by $d(v)$ the degree of vertex v. Let $A(G)$ be the $(0, 1)$-adjacency matrix of graph G. The characteristic polynomial of G is $\det(\lambda I - A(G))$, and it is denoted by $P_G(\lambda)$. The roots of $P_G(\lambda)$ are called the adjaceny eigenvalues of G and since $A(G)$ is real and symmetric, the eigenvalues are real numbers. If G has n vertices, then it has n eigenvalues in descending order as $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. Let λ_1, λ_2, ..., λ_n be the distinct eigenvalues of G with multiplicity m_1, m_2, ... , m_n, respectively. The multi-set of eigenvalues of $A(G)$ is called the adjacency spectrum of G. The matrices $L(G) = D(G) - A(G)$ and $SL(G) = D(G) + A(G)$ are called the Laplacian matrix and signless Laplacian matrix of G, respectively, where $D(G)$ is the diagonal matrix $diag\,\{d(v_1), ..., d(v_n)\}$ and $A(G)$ is the $(0, 1)$ adjacency matrix of G. Two graph with the same spectrum are called cospectral. A graph G is determined by its spectrum (DS for short) if every graph cospectral to it is in fact isomorphic to it. About the

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background of the guestion "which graph are determined by their spectrums?", we refer to [11, 12]. A spectral characterization of multicone graph is studied in [13]. In [13], Wang, Zhao and Huang investigated on the spectral characterization of multicone graph and also they claimed that friendship graph F_n(that are special classes of multicone graph) are DS with respect to their adjacency spectra. In addition, Wang, Belardo, Huang and Borovićanin [14] proposed such conjecture on the adjacency spectrum of F_n. This conjecture caused some activity on the spectral characterization of F_n. Das [5] claims to have a proof, but Abdollahi, Janbaz and Oboudi [2] found a mistake. In addition, these authors give correct proofs in some special cases. Abdian and Mirafzal [1] characterized new classes of multicone graph that were DS with respect to their spectra. In this paper, we present new classes of multicone graph that are DS with respect to their spectra.

This paper is organized as follows. In Section 2, we review some basic information and preliminaries. In Subsection 3.1, we show that any graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,1)$ must be bidegreed (Lemma 3.1). In Subsection 3.2, we prove that any graph cospectral with $K_1 \bigtriangledown GQ(2,1)$ is determined by its adjacency spectra (Lemma 3.2). In Subsection 3.3, we prove that complement of $K_w \bigtriangledown GQ(2,1)$ is DS with respect to their adjacency spectra (Theorem 3.1). In Subsection 3.4, we show that graph $K_w \bigtriangledown GQ(2,1)$ are DS with respect to their Laplacian spectra (Theorem 3.2). In Section 4, we characterize multicone graph $K_w \bigtriangledown GQ(2,2)$ and we show that these graph are DS with respect to their spectra. Subsections 4.1, 4.2 and 4.3 are the similar of Subsections 3.2, 3.3 and 3.4, respectively. We conclude with final remarks and open problems in Section 5.

2. Some definitions and preliminaries

Lemma 2.1. [1, 9] *Let G be a graph. For the adjacency matrix and Laplacian matrix, the following can be obtained from the spectrum:*

(i) *The number of vertices,*
(ii) *The number of edges.*
For the adjacency matrix, the following follows from the spectrum:
(iii) *The number of closed walks of any length,*
(iv) *Being regular or not and the degree of regularity,*
(v) *Being bipartite or not.*
For the Laplacian matrix, the following follows from the spectrum:
(vi) *The number of spanning trees,*
(vii) *The number of components,*
($viii$) *The sum of squares of degrees of vertices.*

Theorem 2.1. [4] *If G_1 is r_1-regular with n_1 vertices, and G_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \bigtriangledown G_2$ is given by:*

$$P_{G_1 \bigtriangledown G_2(y)} = \frac{P_{G_1}(y)P_{G_2}(y)}{(y-r_1)(y-r_2)}((y-r_1)(y-r_2)-n_1n_2).$$

Proposition 2.1. [12, *Proposition* 4] *Let G be a disconnected graph that is determined by the Laplacian spectrum. Then the cone over G, the graph H ; that is, obtained from G by adding one vertex that is adjacent to all vertices of G, is also determined by its Laplacian spectrum.*

Theorem 2.2. [1] *Let G be a simple graph with n vertices and m edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of G and $\varrho(G)$ be the spectral radius of the adjacency matrix of G. Then*

$$\varrho(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}.$$

Equality holds if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$.

Theorem 2.3. [8] *Let G and H be two graphs with the Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq ... \geq \mu_m$, respectively. Then the Laplacian spectrum of $\overline{G}$ and $G \bigtriangledown H$ are $n - \lambda_1, n - \lambda_2, ..., n - \lambda_{n-1}, 0$ and $n+m, m+\lambda_1, ..., m+\lambda_{n-1}, n+\mu_1, ..., n+\mu_{m-1}, 0$, respectively.*

Theorem 2.4. [8] *Let G be a graph on n vertices. Then n is one of the Laplacian eigenvalue of G if and only if G is the join of two graph.*

Theorem 2.5. [7, p.163] *For a graph G, the following statements are equivalent:*
(i) G is d-regular.
(ii) $\varrho(G) = d_G$, the average vertex degree.
(iii) G has $v = (1, 1, ..., 1)^t$ as an eigenvector for $\varrho(G)$.

Proposition 2.2. [4] *Let $G - j$ be the graph obtained from G by deleting the vertex j and all edges containing j. Then $P_{G-j}(y) = P_G(y) \sum_{i=1}^{m} \frac{\alpha_{ij}^2}{y - \mu_i}$, where m is the number of distinct eigenvalues of graph G.*

3. Main Results

In this subsection, we show that any graph cospectral with a multicone graph $K_w \bigtriangledown GQ(2,1)$ must be bidegreed.

3.1. Connected graph cospectral with a multicone graph $K_w \bigtriangledown GQ(2,1)$.

Proposition 3.1. *Let G be a graph cospectral with a multicone graph $K_w \bigtriangledown GQ(2,1)$. Then* $Spec(G) = \left\{ [-1]^{w-1}, [-2]^4, [1]^4, \left[\frac{\Omega + \sqrt{\Omega^2 + 4\Gamma}}{2}\right]^1, \left[\frac{\Omega - \sqrt{\Omega^2 + 4\Gamma}}{2}\right]^1 \right\}$, *where $\Omega = w + 3$ and $\Gamma = 5w + 4$.*

Proof. It is well-known that $Spec(GQ(2,1)) = \left\{ [-2]^4, [1]^4, [4]^1 \right\}$. Now, by Theorem 2.1 the proof is clear. $\square$

Lemma 3.1. *Let G be cospectral with a multicone graph $K_w \bigtriangledown GQ(2,1)$. Then G is bidegreed in which any vertex of G is of degree $w+4$ or $w+8$.*

Proof. It is obvious that G cannot be regular; since regularity of a graph can be determined by its spectrum. By contrary, we suppose that the degrees sequence of graph G consists of at least three number. Hence the equality in Theorem 2.2 cannot happen for any δ. But, if we put $\delta = w + 4$, then the equality in Theorem 2.2 holds. So, G must be bidegreed. Now, we show that $\Delta = \Delta(G) = w + 8$. By contrary, we suppose that $\Delta < w + 8$. Therefore, the equality in Theorem 2.2 cannot hold for any δ. But, if we put $\delta = w + 4$, then this equality holds . This is a contradiction and so $\Delta = w + 8$. Now, $\delta = w + 4$, since G is bidegreed and G has

$w+9$, $\Delta = w+8$ and $w(w+8)+9(w+4) = w\Delta + 9(w+4) = \sum_{i=1}^{w+9} \deg v_i$.
This completes the proof. □

In the following subsection, we prove that the cone of the generalized quadrangle graph $GQ(2,1)$ is DS with respect to its adjacency spectra.

3.2. Connected graph cospectral with the multicone graph $K_1 \bigtriangledown GQ(2,1)$.

Lemma 3.2. *Any graph cospectral with the multicone graph $K_1 \bigtriangledown GQ(2,1)$ is DS with respect to its adjacency spectrum.*

Proof. Let G be cospectral with multicone graph $K_1 \bigtriangledown GQ(2,1)$. By Lemma 3.1, it is easy to see that G has one vertex of degree 9, say j. Now, Proposition 2.2 implies that $P_{G-j}(y) = (y-\mu_3)^3(y-\mu_4)^3[\alpha_{1j}^2 F_1 + \alpha_{2j}^2 F_2 + \alpha_{3j}^2 F_3 + \alpha_{4j}^2 F_4]$, where

$\mu_1 = \dfrac{4+\sqrt{52}}{2}$, $\mu_2 = \dfrac{4-\sqrt{52}}{2}$, $\mu_3 = 1$ and $\mu_4 = -2$.
$F_1 = (y-\mu_2)(y-\mu_3)(y-\mu_4)$,
$F_2 = (y-\mu_1)(y-\mu_3)(y-\mu_4)$,
$F_3 = (y-\mu_1)(y-\mu_2)(y-\mu_4)$,
$F_4 = (y-\mu_1)(y-\mu_2)(y-\mu_3)$.
Now, we have:
$a+b+4 = -(3\mu_3 + 3\mu_4)$,
$a^2+b^2+16 = 36 - (3\mu_3^2 + 3\mu_4^2)$,

where a and b are the eigenvalues of graph $G-j$. If we solve the above equations, then $a=1$ and $b=-2$. Hence $Spec(G-j) = Spec(GQ(2,1))$ and so $G-j \cong GQ(2,1)$.
This follows the result. □

Until now, we have shown the cone of generalized quadrangle graph $K_1 \bigtriangledown GQ(2,1)$ is DS. The natural question is; what happens for multicone graph $K_w \bigtriangledown GQ(2,1)$? we will respond to this question in the following theorem.

3.3. Connected graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,1)$.

Theorem 3.1. *Multicone graph $K_w \bigtriangledown GQ(2,1)$ are DS with respect to their adjacency spectrums.*

Proof. We solve the problem by induction on w. If $w=1$, by Lemma 3.3 there is nothing to prove. Let the claim be true for w; that is, if $Spec(G_1) = Spec(K_w \bigtriangledown GQ(2,1))$, then $G_1 \cong K_w \bigtriangledown GQ(2,1)$, where G_1 is an arbitrary graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,1)$. We show that the claim is true for $w+1$; that is, if $Spec(G) = Spec(K_{w+1} \bigtriangledown GQ(2,1))$, then $G \cong K_{w+1} \bigtriangledown GQ(2,1)$, where G is an arbitrary graph cospectral with multicone graph $K_{w+1} \bigtriangledown GQ(2,1)$. It is clear that G has one vertex and 9 edges more than G_1. Also, By Lemma 3.1 and the spectrums of G and G_1, we can conclude that $G \cong K_1 \bigtriangledown G_1$.
Now, induction hypothesis follows the result. □

In the following subsection, we prove that multicone graph $K_w \bigtriangledown GQ(2,1)$ are DS with respect to their Laplacian spectrum.

3.4. **Connected graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,1)$ with respect to Laplacian spectrum.**

Theorem 3.2. *Multicone graph $K_w \bigtriangledown GQ(2,1)$ are DS with respect to their Laplacian spectrums.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, if $Spec(L(G_1)) = Spec(L(K_w \bigtriangledown GQ(2,1))) = \left\{[w+9]^w,\ [w+3]^4,\ [w+6]^{21},\ [0]^1\right\}$, then $G_1 \cong K_w \bigtriangledown GQ(2,1)$. We show that the problem is true for $w+1$; that is, we show that $Spec(L(G)) = Spec(L(K_{w+1} \bigtriangledown GQ(2,1))) = \left\{[w+10]^{w+1},\ [w+4]^4,\ [w+7]^{21},\ [0]^1\right\}$ follows that $G \cong K_w \bigtriangledown GQ(2,1)$, where G is a graph. Theorem 2.4 implies that G_1 and G are the join of two graph. On the other hand, $Spec(L(K_1 \bigtriangledown G_1)) = Spec(L(G)) = spec(L(K_{w+1} \bigtriangledown GQ(2,1)))$ and also G has one vertex and $w+9$ edges more than G_1. Therefore, we must have $G \cong K_1 \bigtriangledown G_1$. Because, G is the join of two graph and also according to the spectrum of G, must K_1 be joined to G_1 and this is only possibility.

□

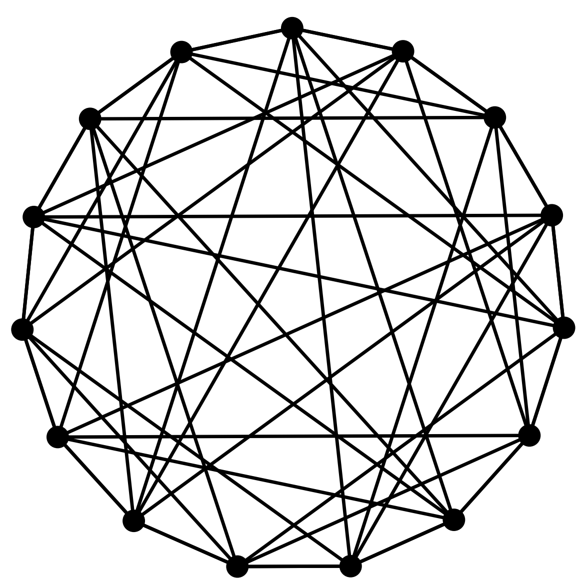

FIGURE 1. Generalized quadrangle $GQ(2,2)$

Hereafter, we characterize another new classes of multicone graph that are DS with respect to their spectra. Our arguments are the similar of the above subsection. So, we will avoid bringing description before each subsection.

4. Connected graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,2)$

Proposition 4.1. *Let G be a graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,2)$. Then*

$$Spec(G) = \left\{ [-1]^{w-1}, [-3]^5, [1]^9, \left[\frac{\vartheta + \sqrt{\vartheta^2 + 4\Upsilon}}{2}\right]^1, \left[\frac{\vartheta - \sqrt{\vartheta^2 + 4\Upsilon}}{2}\right]^1 \right\},$$ *where* $\vartheta = 5 + w$ *and* $\Upsilon = 9w + 6$.

Proof. It is well-known that $Spec(GQ(2,2)) = \left\{ [-3]^5, [1]^9, [6]^1 \right\}$. Now, by Theorem 2.1 the proof is clear. □

In the following lemma, we show that any graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,2)$ must be bidegreed.

Lemma 4.1. *Let G be cospectral with multicone graph $K_w \bigtriangledown GQ(2,2)$. Then G is bidegreed in which any vertex of G is of degree $w+6$ or $w+14$.*

Proof. It is obvious that G cannot be regular; since regularity of a graph can be determined by its spectrum. By contrary, we suppose that the sequence of degrees of vertices of graph G consists of at least three number. Hence the equality in Theorem 2.2 cannot happen for any δ. But, if we put $\delta = w+6$, then the equality in Theorem 2.2 holds. So, G must be bidegreed. Now, we show that $\Delta = \Delta(G) = w + 14$. By contrary, we suppose that $\Delta < w + 14$. Therefore, the equality in Theorem 2.2 cannot hold for any δ. But, if we put $\delta = w + 6$, then this equality holds. This is a contradiction and so $\Delta = w + 14$. Now, $\delta = w + 6$, since G is bidegreed and G has $w+15$ vertices, $\Delta = w+14$ and $w(w+14)+15(w+6) = w\Delta+15(w+6) = \sum_{i=1}^{w+15} \deg v_i$. Therefore, the assertion holds. □

4.1. **Connected graph cospectral with multicone graph $K_1 \bigtriangledown GQ(2,2)$.**

Lemma 4.2. *Any graph cospectral with a multicone graph $K_1 \bigtriangledown GQ(2,2)$ is isomorphic to $K_1 \bigtriangledown GQ(2,2)$.*

Proof. Let G be cospectral with multicone graph $K_1 \bigtriangledown GQ(2,2)$. By Lemma 4.1, it is easy to see that G has one vertex of degree 15, say j. Now, Proposition 2.2 implies that $P_{G-j}(y) = (y-\mu_3)^4(y-\mu_4)^8[\alpha_{1j}^2 N_1 + \alpha_{2j}^2 N_2 + \alpha_{3j}^2 N_3 + \alpha_{4j}^2 N_4]$, where

$\mu_1 = \dfrac{6+\sqrt{96}}{2}$, $\mu_2 = \dfrac{6-\sqrt{96}}{2}$, $\mu_3 = -3$ and $\mu_4 = 1$.
$N_1 = (y-\mu_2)(y-\mu_3)(y-\mu_4)$,
$N_2 = (y-\mu_1)(y-\mu_3)(y-\mu_4)$,
$N_3 = (y-\mu_1)(y-\mu_2)(y-\mu_4)$,
$N_4 = (y-\mu_1)(y-\mu_2)(y-\mu_3)$.

Now, we have:

$\eta + \xi + 6 = -(3\mu_3 + 3\mu_4)$,
$\eta^2 + \xi^2 + 36 = 90 - (3\mu_3^2 + 3\mu_4^2)$,

where η and ξ are the eigenvalues of graph $G-j$. If we solve above equation, then $\eta = 1$ and $\xi = -3$. Hence $Spec(G-j) = Spec(GQ(2,2))$ and so $G-j \cong GQ(2,2)$. Therefore, the assertion holds. □

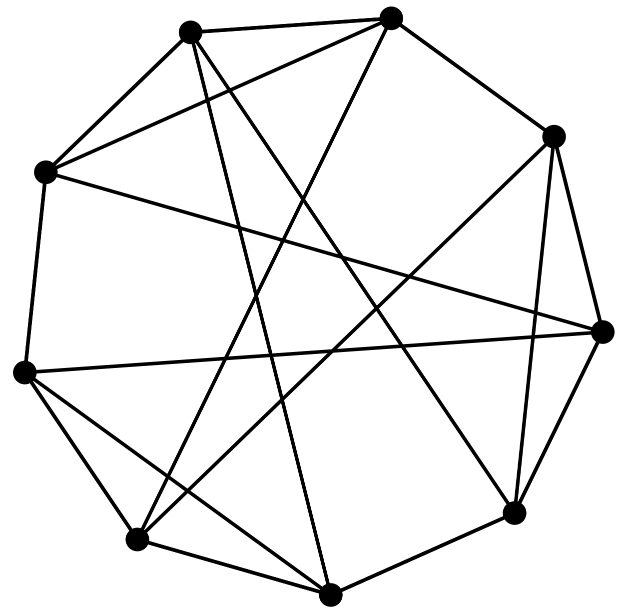

FIGURE 2. Generalized quadrangle $GQ(2,1)$

4.2. **Connected graph cospectral with a multicone graph $K_w \bigtriangledown GQ(2,2)$.**

Theorem 4.1. *Multicone graph $K_w \bigtriangledown GQ(2,2)$ are DS with respect to their adjacency spectra.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, if $Spec(G_1) = Spec(K_w \bigtriangledown GQ(2,2))$, then $G_1 \cong K_w \bigtriangledown GQ(2,2)$, where G_1 is a graph. We show that the claim is true for $w+1$; that is, if $Spec(G) = Spec(K_{w+1} \bigtriangledown GQ(2,2))$, then $G \cong K_{w+1} \bigtriangledown GQ(2,2)$, where G is a graph. By Lemma 4.2, G has one vertex, 15 edges and 280 triangle more than G_1. Hence $G \cong K_1 \bigtriangledown G_1$.
This follows the result. $\square$

4.3. **Multicone graph $K_w \bigtriangledown GQ(2,2)$ are DS with respect to their Laplacian spectrum.**

Theorem 4.2. *Multicone graph $K_w \bigtriangledown GQ(2,2)$ are DS with respect to their Laplacian spectrums.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, $Spec(L(G_1)) = Spec(L(K_w \bigtriangledown GQ(2,2))) = \Big\{[w+15]^w,\ [w+5]^9,\ [w+9]^5,\ [0]^1\Big\}$
follows that $G_1 \cong K_w \bigtriangledown GQ(2,2)$. We show that the claim is true for $w+1$; that is, we show that $Spec(L(G)) = Spec(L(K_{w+1} \bigtriangledown GQ(2,2))) = \Big\{[w+16]^{w+1},\ [w+6]^9,\ [w+10]^5,\ [0]^1\Big\}$
follows that $G \cong K_{w+1} \bigtriangledown GQ(2,2)$, where G is a graph. Theorem 2.4 implies that

G_1 and G are the join of two graph. On the other hand, $Spec(L(K_1 \triangledown G_1)) = Spec(L(G)) = spec(L(K_{w+1} \triangledown GQ(2,2)))$ and also G has one vertex and $w+15$ edges more than G_1. Therefore, we must have $G \cong K_1 \triangledown G_1$. Because, G is the join of two graph and also according to spectrum of G, must K_1 be joined to G_1 and this is only available state. □

5. Conclusion remarks and open problems

In this paper, we have shown multicone graph $K_w \triangledown GQ(2,1)$ and $K_w \triangledown GQ(2,2)$ are DS with respect to their adjacency spectra as well as their Laplacian spectra. Now, in the following, we pose these conjectures.

Conjecture 1. *Graphs $\overline{K_w \triangledown GQ(2,1)}$ are DS with respect to their adjacency spectra.*

Conjecture 2. *Multicone graphs $K_w \triangledown GQ(2,1)$ are DS with respect to their signless Laplacian spectra.*

Conjecture 3. *Graphs $\overline{K_w \triangledown GQ(2,2)}$ are DS with respect to their adjacency spectra.*

Conjecture 4. *Multicone graphs $K_w \triangledown GQ(2,2)$ are DS with respect to their signless Laplacian spectra.*

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ON THE UNIQUENESS OF PRODUCT OF DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we study the uniqueness of product of difference polynomials $f^n[\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}$ and $g^n[\prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)}$, which are sharing a fixed point z and f, g share ∞ IM. The result extends the previous results of Cao and Zhang[1] into product of difference polynomials.

1. Introduction, Definitions and Results

Let $\mathbb{C}$ denote the complex plane and f be a non-constant meromorphic function in $\mathbb{C}$. We shall use the standard notations in the Nevanlinna's value distribution theory of meromorphic functions such as $T(r,f), N(r,f), \overline{N}(r,f)$ and $m(r,f)$, as explained in Yang and Yi[14], L.Yang[12] and Hayman[8]. The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$, as $r \to \infty$ possibly outside a set r of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r,a) = S(r,f)$. A point $z_0 \in \mathbb{C}$ is called as a fixed point of $f(z)$ if $f(z_0) = z_0$.

The following definitions are useful in proving the results.

Definition 1.1. We denote $\rho(f)$ for order of $f(z)$.

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

And $\rho_2(f)$ is to denote hyper order of $f(z)$, defined by

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

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Definition 1.2. Let a be a finite complex number and k be a positive integer. We denote by $N_{k)}(r, 1/(f-a))$ the counting function for the zeros of $f(z)-a$ in $|z| \leq r$ with multiplicity $\leq k$ and by $\overline{N}_{k)}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, 1/(f-a))$ be the counting function for the zeros of $f(z)-a$ in $|z| \leq r$ with multiplicity $\geq k$ and by $\overline{N}_{(k}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Then we have

$$N_k(r, 1/(f-a)) = \overline{N}_{(1}(r, 1/(f-a)) + \overline{N}_{(2}(r, 1/(f-a)) + \ldots + \overline{N}_{(k}(r, 1/(f-a))$$

Definition 1.3. Let $f(z)$ and $g(z)$ be two meromorphic functions in the complex plane $\mathbb{C}$. If $f(z)-a$ and $g(z)-a$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value $'a'$ CM, where $'a'$ is a complex number.

In 2010, J.F.Xu, F.Lu and H.X.Yi obtained the following result on meromorphic function sharing a fixed point.

Theorem A. *([11]) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let n, k be two positive integers with $n > 3k+10$. If $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^nc^2 = -1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Further, Fang and Qiu investigated uniqueness for the same functions as in the theorem A, when $k = 1$.

Theorem B. *([7]) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let $n \geq 11$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

In 2012, Cao and Zhang replaced f' with $f^{(k)}$ and obtained the following theorem.

Theorem C. *([1]) Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, whose zeros are of multiplicities atleast k, where k is a positive integer. Let $n > max\{2k-1, 4+4/k+4\}$ be a positive integer. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and f and g share ∞ IM, then one of the following two conclusions holds.*
(1) $f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z)$
(2) $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

Recently, X.B.Zhang reduced the lower bond of n and relax the condition on multiplicity of zeros in theorem C and proved the below result.

Theorem D. *([15]) Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and n, k two positive integers with $n > k+6$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and f and g share ∞ IM, then one of the following two conclusions holds.*
(1) $f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z)$;
(2) $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

We define a difference product of meromorphic function $f(z)$ as follows.

$$F(z) = f(z)^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j} \right]^{(k)} \tag{1.1}$$

$$F_1(z) = f(z)^n \prod_{j=1}^{d} f(z+c_j)^{s_j} \tag{1.2}$$

Where $c_j \in \mathbb{C} \setminus \{0\}(j = 1, 2, \ldots, d)$ are distinct constants. $n, k, d, s_j(j = 1, 2, \ldots, d)$ are positive integers and $\lambda = \sum_{j=1}^{d} s_j$.
For $j = 1, 2, 3 \ldots d$, $\lambda_1 = \sum_{j=1}^{d} \alpha_j s_j$ and $\lambda_2 = \sum_{j=1}^{d} \beta_j s_j$, where $f(z + c_j)$ and $g(z + c_j)$ have zeros with maximum orders α_j and β_j respectively.

In this article, we prove the theorem on product of difference polynomials sharing a fixed point as follows.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions of hyper order $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let k, n, d, λ be positive integers and $n > \max\{2d(k+2) + \lambda(k+3) + 7, \lambda_1, \lambda_2\}$. If $F(z)$ and $G(z)$ share z CM and f, g share ∞ IM, then one of the following two conclusions holds.*

(1) $F(z) = G(z)$
(2) $\prod_{j=1}^{d} f(z+c_j)s_j = C_1 e^{Cz^2}$, $\prod_{j=1}^{d} g(z+c_j)s_j = C_2 e^{-Cz^2}$, *where C_1, C_2 and C are constants such that $4(C_1C_2)^{n+1}C^2 = -1$.*

2. Lemmas

We need following Lemmas to prove our results.

Lemma 2.1. *([13]) Let f and g be two non-constant meromorphic functions, $'a'$ be a finite non-zero constant. If f and g share $'a'$ CM and ∞ IM, then one of the following cases holds.*

(1) $T(r, f) \leq N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right) + 3\overline{N}(r, f) + S(r, f) + S(r, g)$.
The same inequality holding for $T(r, g)$;
(2) $fg \equiv a^2$;
(3) $f \equiv g$.

Lemma 2.2. *([10]) Let $f(z)$ be a transcendental meromorphic functions of hyper order $\rho_2(f) < 1$, and let c be a non-zero complex constant. Then we have*

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f(z)),$$

$$N(r, f(z+c)) = N(r, f(z)) + S(r, f(z)),$$

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r, f(z)).$$

Lemma 2.3. *([14]) Let f be a non-constant meromorphic function, let $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.4. *([14]) Let f be a non-constant meromorphic function and p, k be positive integers. Then*

$$T\left(r, f^{(k)}\right) \leq T(r,f) + k\overline{N}\,(r,f) + S(r,f), \tag{1}$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r,f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r,f), \tag{2}$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f), \tag{3}$$

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f). \tag{4}$$

Lemma 2.5. *([8]) Suppose that f is a non-constant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)}}\right) = S\left(r, \frac{f'}{f}\right),$$

then $f(z) = e^{az+b}$, where $a \neq 0, b$ are constants.

Lemma 2.6. *([14]) Let f be a transcendental meromorphic function of finite order. Then*

$$m\left(r, \frac{f'}{f}\right) = S(r,f)$$

Lemma 2.7. *Let $f(z)$ be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and $F_1(z)$ be stated as in (1.2). Then*

$$(n-\lambda)T(r,f) + S(r,f) \leq T(r, F_1(z)) \leq (n+\lambda)T(r,f) + S(r,f)$$

Proof: Since f is a meromorphic function with $\rho_2(f) < 1$. From Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} T(r, F_1(z)) &\leq T(r, f(z)^n) + T\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + S(r,f) \\ &\leq (n+\lambda)T(r,f) + S(r,f) \end{aligned}$$

On the other hand, from Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} (n+\lambda)T(r,f) &= T(r, f^n f^\lambda) + S(r,f) \\ &= m(r, f^n f^\lambda) + N(r, f^n f^\lambda) + S(r,f) \\ &\leq m\left(r, \frac{F_1(z) f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + N\left(r, \frac{F_1(z) f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) \\ &\quad + S(r,f) \\ &\leq m(r, F_1(z)) + N(r, F_1(z)) + T\left(r, \frac{f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) \\ &\quad + S(r,f) \\ &\leq T(r, F_1(z)) + 2\lambda T(r,f) + S(r,f) \\ (n-\lambda)T(r,f) &\leq T(r, F_1(z)) + S(r,f) \\ \Rightarrow (n-\lambda)T(r,f) + S(r,f) &\leq T(r, F_1(z)) \end{aligned}$$

Hence we get Lemma 2.7.

3. Proof of theorem

Proof of the theorem 1.1

$$Let, \quad F^* = \frac{F}{z} \qquad and \qquad G^* = \frac{G}{z} \tag{3.1}$$

From the hypothesis of the theorem 1.1, we have F and G share z CM and f, g share ∞ IM. It follows that F^* and G^* share 1 CM and ∞ IM.

By Lemma 2.1, we arrive at 3 cases as follows.

Case 1. Suppose that case (1) of Lemma 2.1 holds.

$$T(r, F^*) \leq N_2\left(r, \frac{1}{F^*}\right) + N_2\left(r, \frac{1}{G^*}\right) + 3\overline{N}(r, F^*) + S(r, F^*) + S(r, G^*) \tag{3.2}$$

We deduce from (3.2) and obtained the following

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 3\overline{N}(r, F) + S(r, F) + S(r, G) \tag{3.3}$$

From Lemma 2.2 and Lemma 2.7, we have $S(r, F) = S(r, f)$ and $S(r, G) = S(r, g)$. From (3.3), we have

$$\begin{aligned} T(r, F) &\leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 3\overline{N}(r, F) + S(r, f) + S(r, g) \\ &\leq N_2\left(r, \frac{1}{f^n}\right) + N_2\left(r, \frac{1}{\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}}\right) + N_2\left(r, \frac{1}{g^n}\right) \\ &\quad + N_2\left(r, \frac{1}{\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}}\right) + 3\overline{N}(r, f^n) + 3\overline{N}\left(r, \left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right) \end{aligned}$$

$$+S(r, f) + S(r, g) \tag{3.4}$$

Using (2) of Lemma 2.4 in (3.4), we have

$$\begin{aligned}
T(r,F) \leq\ & 2\overline{N}_{(2}\left(r,\frac{1}{f^n}\right)+T\left(r,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right)-T\left(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)\\
&+N_{k+2}\left(r,\frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right)+2\overline{N}_{(2}\left(r,\frac{1}{g^n}\right)+T\left(r,\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}\right)\\
&-T\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)+N_{k+2}\left(r,\frac{1}{\prod_{j=1}^{d} g(z+c_j)^{s_j}}\right)+3N(r,f)\\
&+3N\left(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)+S(r,f)+S(r,g)
\end{aligned}$$

$$\begin{aligned}
T(r,F) \leq\ & 2T(r,f)+T\left(r,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right)+T(r,f^n)-T(r,f^n)\\
&-T\left(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)+(k+2)\ d\,T(r,f)+2T(r,g)\\
&+T\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)+k\overline{N}\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)\\
&-T\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)+(k+2)\ d\,T(r,g)\\
&+3T(r,f)+3\lambda T(r,f)+S(r,f)+S(r,g)\\
T(r,F) \leq\ & 2T(r,f)+T(r,F)-T(r,F_1)+(k+2)\ d\,T(r,f)+2T(r,g)+k\lambda T(r,g)\\
&+(k+2)\ d\,T(r,g)+(3+3\lambda)T(r,f)+S(r,f)+S(r,g)\\
T(r,F_1) \leq\ & 2[T(r,f)+T(r,g)]+(k+2)\ d\,[T(r,f)+T(r,g)]+k\lambda T(r,g)\\
&+(3+3\lambda)T(r,f)+S(r,f)+S(r,g)
\end{aligned}$$

From Lemma 2.7, we have

$$(n-\lambda)T(r,f)\leq((k+2)d+2)[T(r,f)+T(r,g)]+k\lambda T(r,g)+(3+3\lambda)T(r,f)+S(r,f)$$

$$+S(r,g) \tag{3.5}$$

Similarly for $T(r,g)$, we obtain the following

$$(n-\lambda)T(r,g)\leq(2+(k+2)d)[T(r,f)+T(r,g)]+k\lambda T(r,f)+(3+3\lambda)T(r,g)+S(r,f)$$

$$+S(r,g) \tag{3.6}$$

From (3.5) and (3.6), we have

$$(n-\lambda)[T(r,f)+T(r,g)] \leq 2(2+(k+2)d))[T(r,f)+T(r,g)]+(k\lambda+3+3\lambda)[T(r,f)+T(r,g)]$$
$$+S(r,f)+S(r,g)$$

Which is contradiction to $n > 2d(k+2)+\lambda(k+3)+7$.

Case 2. Suppose that $FG \equiv z^2$ holds.

$$\text{i.e} \quad f^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} g^n \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)} \equiv z^2 \tag{3.7}$$

Now, (3.7) can be written as

$$f^n g^n = \frac{z^2}{\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}}$$

By using Lemma 2.2, Lemma 2.3 and (4) of Lemma 2.4, we derive

$$n\left[N(r,f)+N(r,g)\right] \leq \lambda \left[N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)\right]$$
$$+kd[N(r,f)+N(r,g)]+S(r,f)+S(r,g) \tag{3.8}$$

From (3.7), we can write

$$\frac{1}{f^n g^n} = \frac{\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}}{z^2}$$

Similarly, as (3.8), we obtain

$$n\left[N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)\right] \leq (\lambda+kd)\left[N(r,f)+N(r,g)\right]+S(r,f)+S(r,g) \tag{3.9}$$

From (3.8) and (3.9), deduce

$$(n-(\lambda+2kd))[N(r,f)+N(r,g)]+(n-\lambda)\left[N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)\right] \leq S(r,f)+S(r,g)$$

Since $n > 2d(k+2)+\lambda(k+3)+7$, we have

$$N(r,f)+N(r,g)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right) < S(r,f)+S(r,g)$$

Hence, we conclude that f and g have finitely many zeros and poles.

Let z_0 be a pole of f of multiplicity p, then z_0 is pole of f^n of multiplicity np, since f and g share ∞ IM, then z_0 is pole of g of multiplicity q.

If z_0 also zero of $\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ then we have from (3.7) that

$$n(p+q) \leq \sum_{j=1}^{d} \alpha_j s_j + \sum_{j=1}^{d} \beta_j s_j - 2k$$

$$\Rightarrow 2n < n(p+q) \leq \sum_{j=1}^{d} \alpha_j s_j + \sum_{j=1}^{d} \beta_j s_j - 2k = \lambda_1 + \lambda_2 - 2k < \lambda_1 + \lambda_2 \leq 2\max\{\lambda_1, \lambda_2\}$$

$\Rightarrow n < \max\{\lambda_1, \lambda_2\}$, which is contradiction to $n > \max\{2d(k+2) + \lambda(k+3) + 7, \lambda_1, \lambda_2\}$. Therefore f has no poles.

Similarly, we can get contradiction for other two cases namely, if z_0 is zero of $\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$, but not zero of $\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ and other way. Therefore f has no poles. Similarly, we get that g also has no poles. By this we conclude that f and g are entire functions and hence $\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ are entire functions.

Then from (3.7), we deduce that f and g have no zeros.
Therefore,

$$f = e^{\alpha(z)},\ g = e^{\beta(z)} \qquad \text{and}$$

$$\prod_{j=1}^{d} f(z+c_j)^{s_j} = \prod_{j=1}^{d} (e^{\alpha(z+c_j)})^{s_j} \quad , \quad \prod_{j=1}^{d} g(z+c_j)^{s_j} = \prod_{j=1}^{d} (e^{\beta(z+c_j)})^{s_j} \tag{3.10}$$

where α, β are entire functions with $\rho_2(f) < 1$. Substitute f and g into (3.7), we get

$$e^{n\alpha(z)} \left[\prod_{j=1}^{d} (e^{\alpha(z+c_j)})^{s_j}\right]^{(k)} e^{n\beta(z)} \left[\prod_{j=1}^{d} (e^{\beta(z+c_j)})^{s_j}\right]^{(k)} \equiv z^2 \tag{3.11}$$

If $k = 1$, then

$$e^{n\alpha(z)} \left[\prod_{j=1}^{d} (e^{\alpha(z+c_j)})^{s_j}\right]' e^{n\beta(z)} \left[\prod_{j=1}^{d} (e^{\beta(z+c_j)})^{s_j}\right]' \equiv z^2 \tag{3.12}$$

$$\Rightarrow e^{n(\alpha+\beta)} e^{\sum_{j=1}^{d}(\alpha(z+c_j)+\beta(z+c_j))s_j} \sum_{j=1}^{d} (\alpha'(z+c_j))s_j \sum_{j=1}^{d} (\beta'(z+c_j))s_j \equiv z^2 \tag{3.13}$$

Since $\alpha(z)$ and $\beta(z)$ are non-constant entire functions, then we have

$$T\left(r, \frac{\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)'}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) = T\left(r, \frac{\left(\prod_{j=1}^{d} e^{\alpha(z+c_j)s_j}\right)'}{\prod_{j=1}^{d} e^{\alpha(z+c_j)s_j}}\right)$$

(3.14)
$$= T\left(r, \frac{\sum_{j=1}^{d} \alpha'(z+c_j)s_j \prod_{j=1}^{d} e^{\alpha(z+c_j)s_j}}{\prod_{j=1}^{d} e^{\alpha(z+c_j)s_j}}\right) = T\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)$$

$$\begin{aligned}
\text{Let} \quad nT(r,f) = T(r,f^n) &= T\left(r, \frac{F}{(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}}\right) \\
&\leq T(r,F) + T\left(r, \left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right) + S(r,f) \\
&\leq T(r,F) + T\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + k\overline{N}\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) \\
&\quad + S(r,f) \\
nT(r,f) &\leq T(r,F) + (\lambda + kd)T(r,f) + S(r,f)
\end{aligned}$$

(3.15) $(n - \lambda - kd)T(r,f) \leq T(r,F) + S(r,f)$

We obtain from (3.15) that

$$T(r,f) = O(T(r,F)) \tag{3.16}$$

as $r \in E$ and $r \to \infty$, where $E \subset (0, +\infty)$ is some subset of finite linear measure.

On the other hand, we have

$$\begin{aligned}
T(r,F) = T\left(r, f^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}\right) &\leq nT(r,f) + \lambda T(r,f) \\
&\quad + k\overline{N}\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + S(r,f) \\
&\leq (n + kd + \lambda)T(r,f) + S(r,f)
\end{aligned}$$

$$\Rightarrow \quad T(r,F) = O(T(r,f)) \tag{3.17}$$

as $r \in E$ and $r \to \infty$, where $E \subset (0, +\infty)$ is some subset of finite linear measure.

Thus from (3.16), (3.17) and the standard reasoning of removing exceptional set we deduce $\rho(f) = \rho(F)$. Similarly, we have $\rho(g) = \rho(G)$. It follows from (3.7) that $\rho(F) = \rho(G)$. Hence we get $\rho(f) = \rho(g)$.

We deduce that either both α and β are polynomials or both α and β are transcendental entire functions. Moreover, we have

$$N\left(r, \frac{1}{(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}}\right) \leq N\left(r, \frac{1}{z^2}\right) = O(\log r) \tag{3.18}$$

From (3.18) and (3.10), we have

$$N\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + N\left(r, \frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right)$$
$$+N\left(r, \frac{1}{(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}}\right) = O(\log r)$$

If $k \geq 2$, then it follows from (3.14),(3.18) and Lemma 2.5 that $\sum_{j=1}^{d} \alpha'(z+c_j)s_j$ is a polynomial and therefore we have $\alpha(z)$ is a non- constant polynomial.

Similarly, we can deduce that $\beta(z)$ is also a non-constant polynomial. From this, we deduce from (3.10) that

$$\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)} = e^{\sum_{j=1}^{d} \alpha(z+c_j)s_j}\left[P_{k-1}(\alpha'(z+c_j)) + \left(\sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)^{k}\right]$$

$$\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)} = e^{\sum_{j=1}^{d} \beta(z+c_j)s_j}\left[Q_{k-1}(\alpha'(z+c_j)) + \left(\sum_{j=1}^{d} \beta'(z+c_j)s_j\right)^{k}\right]$$

Where P_{k-1} and Q_{k-1} are difference-differential polynomials in $\alpha'(z+c_j)$ with degree at most $k-1$.

Then (3.11) becomes

$$e^{n(\alpha+\beta)}e^{\sum_{j=1}^{d}(\alpha(z+c_j)+\beta(z+c_j))s_j}\left[\sum_{j=1}^{d} \alpha^{(k)}(z+c_j)s_j + \left(\sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)^{k}\right]$$

$$\left[\sum_{j=1}^{d} \beta^{(k)}(z+c_j)s_j + \left(\sum_{j=1}^{d} \beta'(z+c_j)s_j\right)^{k}\right] = z^2 \tag{3.19}$$

We deduce from (3.19) that $\alpha(z)+\beta(z) \equiv C$ for a constant C.
If $k = 1$, from (3.13), we have

$$e^{n(\alpha+\beta)+\sum_{j=1}^{d}(\alpha(z+c_j)+\beta(z+c_j))s_j}\left[\sum_{j=1}^{d}(\alpha'(z+c_j))s_j \sum_{j=1}^{d}(\beta'(z+c_j))s_j\right] \equiv z^2 \tag{3.20}$$

Next, we let $\alpha+\beta=\gamma$ and suppose that α, β both are transcendental entire functions.

If γ is a constant, then $\alpha'+\beta'=0$ and $\sum_{j=1}^{d} \alpha'(z+c_j) = -\sum_{j=1}^{d} \beta'(z+c_j)$.

From (3.20) we have

$$e^{n(\alpha+\beta)+\sum_{j=1}^{d}(\alpha(z+c_j)+\beta(z+c_j))s_j}\left\{-\left[\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]^2\right\}=z^2$$

$$e^{n\gamma+d\gamma}\left\{-\left[\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]^2\right\}=z^2 \tag{3.21}$$

Which implies that α' is a non-constant polynomial of degree 1. This together with $\alpha'+\beta'=0$ which implies that β' is also non-constant polynomial of degree 1. Which is contradiction to α, β both are transcendental entire functions.

If γ is not a constant, then we have

$$\alpha+\beta=\gamma \qquad \text{and} \qquad \textstyle\sum_{j=1}^{d}\alpha(z+c_j)s_j+\sum_{j=1}^{d}\beta(z+c_j)s_j=\sum_{j=1}^{d}\gamma(z+c_j)s_j$$

From (3.20) we have

$$\left[\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]\left[\sum_{j=1}^{d}\gamma'(z+c_j)s_j-\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j}=z^2 \tag{3.22}$$

$$\text{Since}\quad T\left(r,\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)=m\left(r,\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)+N\left(r,\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)$$

$$\leq m\left(r,\frac{(e^{\sum_{j=1}^{d}\gamma(z+c_j)s_j})'}{e^{\sum_{j=1}^{d}\gamma(z+c_j)s_j}}\right)+O(1)=S\left(r,e^{\sum_{j=1}^{d}\gamma(z+c_j)s_j}\right) \tag{3.23}$$

And also we have

$$T\left(r,n\gamma'+\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)=m\left(r,n\gamma'+\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)+N\left(r,n\gamma'+\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)$$

$$\leq m\left(r,\frac{(e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j})'}{e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j}}\right)+O(1)=S\left(r,e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j}\right) \tag{3.24}$$

From (3.22), we have

$$T\left(r,e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j}\right)\leq T\left(r,\frac{z^2}{\sum_{j=1}^{d}\alpha'(z+c_j)s_j\left[\sum_{j=1}^{d}\gamma'(z+c_j)s_j-\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]}\right)$$

$$+O(1)$$

$$\leq T(r,z^2)+T\left(r,\sum_{j=1}^{d}\alpha'(z+c_j)s_j\left[\sum_{j=1}^{d}\gamma'(z+c_j)s_j-\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]\right)$$

$$+O(1)$$

$$\leq 2\log r + 2T\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right) + O(1)$$

$$\Rightarrow T\left(r, e^{n\gamma + \sum_{j=1}^{d} \gamma(z+c_j)s_j}\right) \leq O\left(T\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)\right) \tag{3.25}$$

Similarly, we have

$$T\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right) \leq O\left(T\left(r, e^{n\gamma + \sum_{j=1}^{d} \gamma(z+c_j)s_j}\right)\right) \tag{3.26}$$

Thus, from (3.23)-(3.26) we have
$T\left(r, n\gamma' + \sum_{j=1}^{d} \gamma'(z+c_j)s_j\right) = S\left(r, e^{n\gamma + \sum_{j=1}^{d} \gamma(z+c_j)s_j}\right) = S\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)$

By the second fundamental theorem and (3.22), we have

$$T\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right) \leq \overline{N}\left(r, \frac{1}{\sum_{j=1}^{d} \alpha'(z+c_j)s_j}\right)$$

$$+\overline{N}\left(r, \frac{1}{\sum_{j=1}^{d} \alpha'(z+c_j)s_j - \sum_{j=1}^{d} \gamma'(z+c_j)s_j}\right) + S\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)$$

$$\leq O(\log r) + S\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)$$

This implies $\sum_{j=1}^{d} \alpha'(z+c_j)s_j$ is a polynomial, which leads to $\alpha'(z)$ is a polynomial. Which contradicts that $\alpha(z)$ is a trascendental entire function.
Thus α and β are both polynomials and $\alpha(z) + \beta(z) \equiv C$ for a constant C.
Hence, from (3.19) and using $\alpha + \beta = C$ we get

$$(-1)^k \left(\sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)^{2k} = z^2 + P_{2k-1}(\alpha'(z+c_j)s_j) \quad for \;\; j = 1, 2, \ldots, d. \tag{3.27}$$

Where P_{2k-1} is difference-differential polynomial in $\alpha'(z+c_j)s_j$ of degree at most $2k-1$. From (3.27), we have

$$2kT\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right) = 2\log r + S(r, \alpha'(z+c_j)s_j) \tag{3.28}$$

From (3.28), we can see that $\sum_{j=1}^{d} \alpha'(z+c_j)s_j$ is a non-constant polynomial of degree 1 and $k = 1$.

Which implies,

$$\sum_{j=1}^{d} \alpha'(z+c_j)s_j = zl_1$$

Since $\alpha' + \beta' = 0$, we get $\sum_{j=1}^{d} \beta'(z+c_j)s_j = -\sum_{j=1}^{d} \alpha'(z+c_j)s_j$. Which implies $\sum_{j=1}^{d} \beta'(z+c_j)s_j$ is also a non-constant polynomial of degree 1. Hence we have

$$\sum_{j=1}^{d} \beta'(z+c_j)s_j = zl_2$$

Hence, we get

$$\prod_{j=1}^{d} f(z+c_j)s_j = C_1 e^{Cz^2}$$

Similarly, we have

$$\prod_{j=1}^{d} g(z+c_j)s_j = C_2 e^{-Cz^2}$$

where C_1, C_2 and C are constants such that $4(C_1C_2)^{n+1}C^2 = -1$.

This proves the conclusion (2) of theorem 1.1.

Case 3. If $F \equiv G$

i.e $f^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} \equiv g^n \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$

This proves the conclusion (1) of theorem 1.1.

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ON SOME NEW DIFFERENCE SEQUENCE SPACES DERIVED BY USING RIESZ MEAN AND A MUSIELAK-ORLICZ FUNCTION

KULDIP RAJ AND RENU ANAND

Abstract. In this paper we introduce new difference sequence spaces $r^q(\mathcal{M}, \Delta_n^m, u, p)$ by using Riesz mean and Musielak-Orlicz function. We also make an effort to study some topological properties and compute $\alpha-, \beta-$ and $\gamma-$ duals of these spaces. Finally, we study matrix transformations on newly formed spaces.

1. Introduction and Preliminaries

Let w be the vector space of all real or complex sequences. By l_∞, c and c_0; we denote the classes of all bounded, convergent and null sequences; respectively. Also, we write bs, cs and l_p to denote the spaces of all bounded, convergent series and p-absolutely summable sequences, respectively, where $1 \leq p < \infty$. We use the convention that any term with a negative subscript is equal to zero.
Let X and Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, the matrix A defines the $A-$transformation from X into Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk}x_k$.
By $A \in (X : Y)$ we mean the characterizations of matrices $A : X \to Y$. A sequence x is said to be A-summable to l if Ax converges to l which is called the A-limit of x. For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{x = (x_k) \in w : Ax \in X\}. \tag{1.1}$$

The theory of matrix transformations is a wide field in summability theory. It deals with the characterizations of classes of matrix mappings between sequence spaces

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by giving necessary and sufficient conditions on the entries of the infinite matrices. The classical summability theory deals with a generalization of convergence of sequences and series. One original idea was to assign a limit to divergent sequences or series. Toeplitz [29] was the first to study summability methods as a class of transformations of complex sequences by complex infinite matrices.

Let $A = (a_{nk})$ be any matrix. Then a sequence x is said to be summable to l, written $x_k \to l$, if and only if $A_n x = \sum_k a_{nk} x_k$ exists for each n and $A_n x \to l$ $(n \to \infty)$.

For example, if $A_n = I$, the unit matrix for all n, then $x_k \to l(I)$ means precisely that $x_k \to l$ $(k \to \infty)$, in the ordinary sense of convergence.

An infinite matrix $A = (a_{nk})$ is said to be regular ([11], page:165) if and only if the following conditions (or Toplitz conditions) hold:

(i) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1$,

(ii) $\lim_{n\to\infty} a_{nk} = 0, \quad (k = 0, 1, 2, ...)$

(iii) $\sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} |a_{nk}| < \infty$.

Let (q_k) be a sequence of strictly positive numbers and let us write, $Q_n = \sum_{k=0}^{n} q_k$ for $n \in \mathbb{N}$. Then the matrix $R^q = (r^q_{nk})$ of the Riesz mean (R, q_n) is given by

$$r^q_{nk} = \begin{cases} \frac{q_k}{Q_n}, & \text{if } 0 \le k \le n, \\ \\ 0 & \text{if } k > n. \end{cases}$$

The Riesz mean (R, q_n) is regular if and only if $Q_n \to \infty$ as $n \to \infty$ (see, Petersen [22], p.10).

The sequence space $r^q(u, p)$ is introduced by Sheikh and Ganie [26] as:

$$r^q(u,p) = \Big\{x = (x_k) \in w : \sum_k \Big|\frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j\Big|^{p_k} < \infty\Big\},$$

where $0 \le p_k \le D < \infty$.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $\sup_k p_k = D$ and $H = \max\{1, D\}$. Then, the linear spaces $l(p)$ and $l_\infty(p)$ were defined by Maddox [13] (see also, [27],[30]) as follows:

$$l(p) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}$$

and

$$l_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\}$$

which are complete spaces paranormed by

$$g_1(x) = \Big[\sum_k |x_k|^{p_k}\Big]^{\frac{1}{H}} \text{ and } g_2(x) = \sup_k |x_k|^{\frac{p_k}{H}}$$

if and only if $\inf p_k > 0$ for all k.

Throughout the paper we shall assume that ${p_k}^{-1} + \{p_k^{'}\}^{-1} = 1$ provided $1 <$

$\inf p_k \leq D < \infty$ and we denote the collection of all finite subsets of $\mathbb{N}$ by F where $\mathbb{N} = \{0, 1, 2, ...\}$.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [9] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, $k > 0$ and for $L > 1$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [14], [19]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \; k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k)$$

and $x = (x_k) \in t_{\mathcal{M}}$.
We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

The notion of difference sequence spaces was introduced by Kizmaz [8], who studied the difference sequence spaces $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$. The notion was further generalized by Et and Çolak [5] by introducing the spaces $l_\infty(\triangle^m)$, $c(\triangle^m)$ and $c_0(\triangle^m)$. Let n, m be non-negative integers, then for Z a given sequence space, we have

$$Z(\triangle_n^m) = \{x = (x_k) \in w : (\triangle_n^m x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\triangle_n^m x = (\triangle_n^m x_k) = (\triangle_n^{m-1} x_k - \triangle_n^{m-1} x_{k+1})$ and $\triangle^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $n = 1$, we get the spaces $l_\infty(\triangle^m), c(\triangle^m)$ and $c_0(\triangle^m)$ studied by Et and Çolak [5]. Taking $m = n = 1$, we get the spaces $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$ introduced and studied by Kizmaz [8]. Mursaleen et al. ([15], [16], [17], [18]) used the idea of Orilcz function and study different sequence spaces. Esi et al. ([1], [3], [4]) work on these type of sequence spaces. For more details about sequence spaces and matrix transformations (see [2], [7], [12], [20], [21], [23], [24], [25], [28]) and references there in.

2. The Riesz Sequence Space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ of Non-absolute Type

Let X be a linear metric space. A function $g : X \to \mathbb{R}$ is called paranorm, if

(1) $g(x) \geq 0$, for all $x \in X$,
(2) $g(-x) = g(x)$, for all $x \in X$,
(3) $g(x + y) \leq g(x) + g(y)$, for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ *as* $n \to \infty$ and (x_n) is a sequence of vectors with $g(x_n - x) \to 0$ *as* $n \to \infty$, then $g(\lambda_n x_n - \lambda x) \to 0$ *as* $n \to \infty$.

A paranorm g for which $g(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, g) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [31], Theorem 10.4.2, P-183).
Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then we define new difference sequence space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ as follows:

$$r^q(\mathcal{M}, \Delta_n^m, u, p) = \Big\{ x = (x_k) \in w : \sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k} M_j(|u_j q_j \Delta_n^m x_j|) \Big|^{p_k} < \infty \Big\},$$

where $0 < p_k \leq D < \infty$.
With the definition of matrix domain (1.1), the sequence space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ may be redefined as

$$r^q(\mathcal{M}, \Delta_n^m, u, p) = \{l(p)\}_{R^q(\mathcal{M}, \Delta_n^m, u)}$$

where $R^q(\mathcal{M}, \Delta_n^m, u)$ denotes the matrix $R^q(\mathcal{M}, \Delta_n^m, u) = r_{nk}^q(\mathcal{M}, \Delta_n^m, u)$ defined by

$$r_{nk}^q(\mathcal{M}, \Delta_n^m, u) = \begin{cases} \frac{1}{Q_n}(M_k(u_k q_k) - M_{k+1}(u_{k+1} q_{k+1})), & \text{if } 0 \leq k \leq n-1 \\ \frac{M_n(u_n q_n)}{Q_n}, & \text{if } k = n \\ 0, & \text{if } k > n. \end{cases}$$

Define the sequence $y = (y_k)$ which will be used by the $R^q(\mathcal{M}, \Delta_n^m, u)$-transform of a sequence $x = (x_k)$, we have

$$y_k = \frac{1}{Q_k} \sum_{j=0}^{k} M_j(|u_j q_j \Delta_n^m x_j|). \tag{2.1}$$

The main purpose of this paper is to study some new difference sequence spaces generated by Riesz Mean and Musielak-Orlicz function. We shall show that these spaces are complete and paranormed spaces. We have also discuss the $\alpha-, \beta-$duals of these spaces in section third of this paper. Finally, we discuss the matrix transformations on these spaces in the last section of this paper.

Theorem 2.1. *Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $r^q(\mathcal{M}, \Delta_n^m, u, p)$ is a complete linear metric space paranormed by*

$$g(x) = \Big[\sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k-1} (M_j(u_j q_j) - M_{j+1}(u_{j+1} q_{j+1})) x_j + \frac{M_k(u_k q_k)}{Q_k} x_k \Big|^{p_k} \Big]^{\frac{1}{H}}$$

with $0 \le p_k \le D < \infty$ *and* $H = \max\{1, D\}$.

Proof. The linearity of $r^q(\mathcal{M}, \Delta_n^m, u, p)$ follows from the inequality. For $x, y \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ (see [11], p.30)

(2.2)

$$\Big[\sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k-1} (M_j(u_j q_j) - M_{j+1}(u_{j+1} q_{j+1}))(x_j + y_j) + \frac{M_k(u_k q_k)}{Q_k} (x_k + y_k) \Big|^{p_k} \Big]^{\frac{1}{H}}$$

$$\begin{aligned} &\le \Big[\sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k-1} (M_j(u_j q_j) - M_{j+1}(u_{j+1} q_{j+1})) x_j + \frac{M_k(u_k q_k)}{Q_k} x_k \Big|^{p_k} \Big]^{\frac{1}{H}} \\ &+ \Big[\sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k-1} (M_j(u_j q_j) - M_{j+1}(u_{j+1} q_{j+1})) y_j + \frac{M_k(u_k q_k)}{Q_k} y_k \Big|^{p_k} \Big]^{\frac{1}{H}} \end{aligned}$$

and for any $\alpha \in \mathbb{R}$ (See [12])

$$|\alpha|^{p_k} \le \max(1, |\alpha|^H). \tag{2.3}$$

It is clear that $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$. Again the inequality (2.2) and (2.3) yield the subadditivity of g and

$$g(\alpha x) \le \max(1, |\alpha|) g(x).$$

Let $\{x^n\}$ be any sequence of points of the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ such that $g(x^n - x) \to 0$ and (α^n) is a sequence of scalars such that $\alpha^n \to \alpha$. Then since the inequality,

$$g(x^n) \le g(x) + g(x^n - x)$$

holds by subadditivity of g, $\{g(x^n)\}$ is bounded and we thus have

$$\begin{aligned} g(\alpha_n x^n - \alpha x) &= \Big[\sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k} (M_j(u_j q_j) - M_{j+1}(u_{j+1}q_{j+1}))(\alpha_n x_j^n + \alpha x_j) \Big|^{p_k} \Big]^{\frac{1}{H}} \\ &\leq |\alpha_n - \alpha|^{\frac{1}{H}} g(x^n) + |\alpha|^{\frac{1}{H}} g(x^n - x) \end{aligned}$$

which tends to zero as $n \to \infty$. This proves that the scalar multiplication is continuous. Hence g is paranorm on the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$.

Now we prove the completeness of $r^q(\mathcal{M}, \Delta_n^m, u, p)$:

Let $\{x^i\}$ be any Cauchy sequence in the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$, where $x^i = \{x_0^i, x_1^i, ...\}$. Then, for a given $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that

$$g(x^i - x^j) < \epsilon \ \ \forall \ \ i, j \geq n_0(\epsilon). \tag{2.4}$$

Using definition of g and for each fixed $k \in \mathbb{N}$ that

$$\begin{aligned} &|(R^q(\mathcal{M}, \Delta_n^m, u)x^i)_k - (R^q(\mathcal{M}, \Delta_n^m, u)x^j)_k| \\ &\leq \ \Big[\sum_k |(R^q(\mathcal{M}, \Delta_n^m, u)x^i)_k - (R^q(\mathcal{M}, \Delta_n^m, u)x^j)_k|^{p_k} \Big]^{\frac{1}{H}} < \epsilon \ \text{ for } \ i, j \geq n_0(\epsilon) \end{aligned}$$

which yields that $\{(R^q(\mathcal{M}, \Delta_n^m, u)x^0)_k, (R^q(\mathcal{M}, \Delta_n^m, u)x^1)_k, ...\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges say

$$(R^q(\mathcal{M}, \Delta_n^m, u)x^i)_k \to (R^q(\mathcal{M}, \Delta_n^m, u)x)_k \ \text{ as } \ i \to \infty.$$

Using these infinitely many limits $(R^q(\mathcal{M}, \Delta_n^m, u)x)_0, (R^q(\mathcal{M}, \Delta_n^m, u)x)_1, ...$, we define the sequence $\{(R^q(\mathcal{M}, \Delta_n^m, u)x)_0, (R^q(\mathcal{M}, \Delta_n^m, u)x)_1, ...\}$. From (2.4) for each $t \in \mathbb{N}$ and $i, j \geq n_0(\epsilon)$,

$$\begin{aligned} \sum_{k=0}^{t} |(R^q(\mathcal{M}, \Delta_n^m, u)x^i)_k - (R^q(\mathcal{M}, \Delta_n^m, u)x^j)_k|^{p_k} & \\ \leq \ g(x^i - x^j)^H & \\ < \ \epsilon^H. & \end{aligned} \tag{2.5}$$

Take any $i, j \geq n_0(\epsilon)$. First, let $j \to \infty$ in (2.5) and then $t \to \infty$, we obtain

$$g(x^i - x) \leq \epsilon.$$

Finally, taking $\epsilon = 1$ in (2.5) and letting $i \geq n_0(1)$, we have by Minkowski's inequality for each $t \in \mathbb{N}$ that

$$\begin{aligned} \Big[\sum_{k=0}^{t} |(R^q(\mathcal{M}, \Delta_n^m, u)x)_k|^{p_k} \Big]^{\frac{1}{H}} &\leq \ g(x^i - x) + g(x^i) \\ &\leq \ 1 + g(x^i) \end{aligned}$$

which implies that $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$. Since $g(x - x^i) \leq \epsilon$ for all $i \geq n_0(\epsilon)$, it follows that $x^i \to x$ as $i \to \infty$. Hence, the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ is complete. $\square$

Theorem 2.2. *Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the sequence space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ of non-absolute type is linearly isomorphic to the space $l(p)$, where $0 < p_k \leq D < \infty$.*

Proof. To show that the spaces $r^q(\mathcal{M}, \Delta_n^m, u, p)$ and $l(p)$ are linearly isomorphic, we have to prove that there exists a linear bijection between these spaces. Define a linear transformation $T : r^q(\mathcal{M}, \Delta_n^m, u, p) \to l(p)$ by $x \to y = Tx$ by using equation (2.2). The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $T(x) = T(\theta)$ and hence T is injective. Let $y \in l(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{n=0}^{k-1} \Big(\frac{1}{M_n(u_n q_n)} - \frac{1}{M_{n+1}(u_{n+1}q_{n+1})}\Big) Q_k y_k + \frac{Q_k}{M_k(u_k q_k)} y_k$$

for $k \in \mathbb{N}$. Then

$$\begin{aligned} g(x) = & \Big[\sum_k \Big|\frac{1}{Q_k}\sum_{j=0}^{k-1}(M_j(u_j q_j) - M_{j+1}(u_{j+1}q_{j+1}))x_j + \frac{M_k(u_k q_k)}{Q_k} x_k\Big|^{p_k}\Big]^{\frac{1}{H}} \\ = & \Big[\sum_k \Big|\sum_{j=0}^{k} \delta_{kj} y_j\Big|^{p_k}\Big]^{\frac{1}{H}} \\ = & \Big[\sum_k \Big|y_k\Big|^{p_k}\Big]^{\frac{1}{H}} \\ = & g_1(y) < \infty, \end{aligned}$$

where

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{if } k \neq j. \end{cases}$$

Thus, we have $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$. Consequently, T is surjective and paranorm preserving. Hence, T is linear bijection and this shows that the spaces $r^q(\mathcal{M}, \Delta_n^m, u, p)$ and $l(p)$ are linearly isomorphic. $\square$

3. Basis and $\alpha-, \beta-$ and $\gamma-$ duals of the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$

In this section, we compute $\alpha-, \beta-$ and $\gamma-$ duals of the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ and finally we give the basis for the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$.

For the sequence space X and Y, define the set

$$S(X : Y) = \{z = (z_k) : xz = (x_k z_k) \in Y\}.$$

The $\alpha-, \beta-$ and $\gamma-$ duals of a sequence space X, respectively denoted by X^α, X^β and X^γ which are defined by

$$X^\alpha = S(X : l_1), X^\beta = S(X : cs) \text{ and } X^\gamma = S(X : bs).$$

Firstly, we state some lemmas which are required in proving our theorems:

Lemma 3.1. [6] *(i) Let $1 < p_k \leq D < \infty$. Then $A \in (l(p) : l_1)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{k \in F} \sum_k \Big| \sum_{n \in k} \alpha_{nk} B^{-1} \Big|^{p'_k} < \infty.$$

(ii) Let $0 < p_k \leq 1$. Then $A \in (l(p) : l_1)$ if and only if

$$\sup_{k \in F} \sup_k \Big| \sum_{n \in k} \alpha_{nk} B^{-1} \Big|^{p_k} < \infty.$$

Lemma 3.2. [10] *(i) Let $1 < p_k \leq D < \infty$. Then $A \in (l(p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_n \sum_k \Big| \alpha_{nk} B^{-1} \Big|^{p'_k} < \infty. \tag{3.1}$$

(ii) Let $0 < p_k \leq 1$ for every $k \in \mathcal{N}$. Then $A \in (l(p) : l_\infty)$ if and only if

$$\sup_{n,k} \Big| \alpha_{nk} \Big|^{p_k} < \infty. \tag{3.2}$$

Lemma 3.3. [8] *Let $0 < p_k \leq D < \infty$ for every $k \in \mathcal{N}$. Then $A \in (l(p) : c)$ if and only if (3.1) and (3.2) hold along with*

$$\lim_n \alpha_{nk} = \beta_k \text{ for } k \in \mathcal{N} \tag{3.3}$$

also holds.

Theorem 3.1. *Let $\mathcal{M} = (M_j)$ be a Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Define the sets $D_1(\mathcal{M}, \Delta_n^m, u, p)$ and $D_2(\mathcal{M}, \Delta_n^m, u, p)$ as follows:*

$$D_1(\mathcal{M}, \Delta_n^m, u, p) =$$
$$\bigcup_{B>1} \Big\{ \alpha = (\alpha_k) \in w : \sup_{k \in F} \sum_k \Big| \sum_{n \in k} \Big[\Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) Q_k \alpha_n + \frac{Q_n}{M_n(u_n q_n))} \alpha_n \Big] B^{-1} \Big|^{p'_k} < \infty \Big\}$$

and

$$D_2(\mathcal{M}, \Delta_n^m, u, p) =$$
$$\bigcup_{B>1} \Big\{ \alpha = (\alpha_k) \in w : \sum_k \Big| \Big[\Big(\frac{\alpha_k}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{n} \alpha_i \Big) Q_k \Big] B^{-1} \Big|^{p'_k} < \infty \Big\}$$

Then

$$\Big[r^q(\mathcal{M}, \Delta_n^m, u, p) \Big]^\alpha = D_1(\mathcal{M}, \Delta_n^m, u, p)$$

and

$$\Big[r^q(\mathcal{M},\Delta_n^m,u,p)\Big]^{\beta}=D_2(\mathcal{M},\Delta_n^m,u,p)\cap cs.$$

Proof. Let us take any $\alpha=(\alpha_k)\in w$. We can easily derive with (2.1) that

$$\alpha_n x_n=\sum_{k=0}^{n-1}\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\alpha_n Q_k y_k+\frac{\alpha_n}{M_n(u_nq_n)}Q_n y_n = (Cy)_n, \tag{3.4}$$

where $C=(c_{nk})$ is defined as

$$c_{nk}=\begin{cases}\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\alpha_n Q_k, & \text{if } 0\le k\le n-1\\[2ex] \frac{\alpha_n}{M_n(u_nq_n)}Q_n, & \text{if } k=n\\[2ex] 0, & \text{if } k>n,\end{cases}$$

for all $n,k\in\mathcal{N}$. Thus, we observe by combining (3.4) with (i) of lemma (3.1) that $\alpha x=(\alpha_n x_n)\in l_1$ whenever $x=(x_n)\in r^q(\mathcal{M},\Delta_n^m,u,p)$ if and only if $Cy\in l_1$ whenever $y\in l_p$. This gives the result that $\Big[r^q(\mathcal{M},\Delta_n^m,u,p)\Big]^{\alpha}=D_1(\mathcal{M},\Delta_n^m,u,p)$. Further, consider the equation

$$\sum_{k=0}^{n}\alpha_k x_k=\sum_{k=0}^{n}\Big[\Big(\frac{\alpha_k}{M_k(u_kq_k)}+\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\sum_{i=k+1}^{n}\alpha_i\Big)Q_k\Big]y_k = (Dy)_n, \tag{3.5}$$

where $D=(d_{nk})$ is defined as

$$d_{nk}=\begin{cases}\Big(\frac{\alpha_k}{M_k(u_kq_k)}+\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\sum_{i=k+1}^{n}\alpha_i\Big)Q_k, & \text{if } 0\le k\le n\\[2ex] 0, & \text{if } k>n.\end{cases}$$

Thus, we deduce from Lemma (3.3) with (3.5) that $\alpha x=(\alpha_n x_n)\in cs$ whenever $x=(x_n)\in r^q(\mathcal{M},\Delta_n^m,u,p)$ if and only if $Dy\in c$ whenever $y\in l(p)$. Therefore, we derive from (3.1) that

$$\sum_k\Big|\Big[\Big(\frac{\alpha_k}{M_k(u_kq_k)}+\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\sum_{i=k+1}^{n}\alpha_i\Big)Q_k\Big]B^{-1}\Big|^{p'_k}<\infty \tag{3.6}$$

and $\lim_n d_{nk}$ exists and hence shows that $\Big[r^q(\mathcal{M},\Delta_n^m,u,p)\Big]^{\beta}=D_2(\mathcal{M},\Delta_n^m,u,p)\cap cs$. From lemma (3.2) together with (3.5) that $\alpha x=(\alpha_k x_k)\in bs$ whenever $x=(x_n)\in r^q(\mathcal{M},\Delta_n^m,u,p)$ if and only if $Dy\in l_\infty$ whenever $y=(y_k)\in l(p)$. Therefore, we again obtain the condition (3.6) which means that $\Big[r^q(\mathcal{M},\Delta_n^m,u,p)\Big]^{\gamma}=D_2(\mathcal{M},\Delta_n^m,u,p)\cap cs$ and the proof of theorem is complete. □

Theorem 3.2. *Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Define the sets $D_3(\mathcal{M}, \Delta_n^m, u, p)$ and $D_4(\mathcal{M}, \Delta_n^m, u, p)$ as follows:*

$$D_3(\mathcal{M}, \Delta_n^m, u, p) =$$

$$\Big\{\alpha = (\alpha_k) \in w : \sup_{k\in F}\sup_{k}\Big|\sum_{n\in k}\Big[\Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)Q_k\alpha_n + \frac{Q_n}{M_n(u_n q_n)}\alpha_n\Big]\Big|^{p_k} < \infty\Big\}$$

and

$$D_4(\mathcal{M}, \Delta_n^m, u, p) =$$

$$\Big\{\alpha = (\alpha_k) \in w : \sup_{k}\Big|\Big[\Big(\frac{\alpha_k}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\sum_{i=k+1}^{n}\alpha_i\Big)Q_k\Big]\Big|^{p_k} < \infty\Big\}.$$

Then

$$\Big[r^q(\mathcal{M}, \Delta_n^m, u, p)\Big]^{\alpha} = D_3(\mathcal{M}, \Delta_n^m, u, p)$$

and

$$\Big[r^q(\mathcal{M}, \Delta_n^m, u, p)\Big]^{\beta} = D_4(\mathcal{M}, \Delta_n^m, u, p) \cap cs.$$

Proof. This is obtained by proceeding in proof of Theorem (3.1), by using second parts of lemmas (3.1), (3.2) and (3.3) instead of the first parts so we exclude the details. □

Theorem 3.3. *Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Define the sequence $b^{(k)}(q) = \{b_n^{(k)}(q)\}$ of the elements of the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(q) = \begin{cases} \Big(\frac{1}{M_n(u_n q_n)} - \frac{1}{M_{n+1}(u_{n+1}q_{n+1})}\Big)Q_n + u_n^{-1}\frac{Q_k}{M_k(u_k q_k)}, & \text{if } 0 \le n \le k-1 \\ 0, & \text{if } n > k-1. \end{cases}$$

Then the sequence $\{b^{(k)}(q)\}$ is a basis for the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ and any $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ has a unique representation of the form

$$x = \sum_{k} \lambda_k(q) b^{(k)}(q), \tag{3.7}$$

where $\lambda_k(q) = (R^q(\mathcal{M}, \Delta_n^m, u)x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \le D < \infty$.

Proof. It is clear that $\{b^{(k)}(q)\} \subset r^q(\mathcal{M}, \Delta_n^m, u, p)$, since

$$R^q(\mathcal{M}, \Delta_n^m, u)b^{(k)}(q) = e^{(k)} \in l(p) \text{ for } k \in \mathbb{N} \tag{3.8}$$

and $0 < p_k \le D < \infty$, where $e^{(k)}$ is the sequence whose only non-zero term is 1 in kth place for each $k \in \mathbb{N}$.

Let $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ be given. For every non-negative integer t, we put

$$x^{[t]} = \sum_{k=0}^{t} \lambda_k(q) b^{(k)}(q). \tag{3.9}$$

Then, we obtain by applying $R^q(\mathcal{M}, \Delta_n^m, u)$ to (3.9) with (3.8) that

$$R^q(\mathcal{M}, \Delta_n^m, u)x^{[t]} = \sum_{k=0}^{t} \lambda_k(q) R^q(\mathcal{M}, \Delta_n^m, u) b^{(k)}(q) = \sum_{k=0}^{t} (R^q(\mathcal{M}, \Delta_n^m, u)x)_k e^{(k)}$$

and

$$\Big(R^q(\mathcal{M}, \Delta_n^m, u)(x - x^{[t]})\Big)_i = \begin{cases} 0, & \text{if } 0 \le i \le t \\ (R^q(\mathcal{M}, \Delta_n^m, u)x)_i, & \text{if } i > t, \end{cases}$$

where $i, t \in \mathbb{N}$. Given $\epsilon > 0$, there exists an integer t_0 such that

$$\Big(\sum_{i=t}^{\infty} \Big|(R^q(\mathcal{M}, \Delta_n^m, u)x)_i\Big|^{p_k}\Big)^{\frac{1}{H}} < \frac{\epsilon}{2} \ \ \forall \ \ t \ge t_0.$$

Hence,

$$\begin{aligned} g(x - x^{[t]}) &= \Big(\sum_{i=t}^{\infty} \Big|(R^q(\mathcal{M}, \Delta_n^m, u)x)_i\Big|^{p_k}\Big)^{\frac{1}{H}} \\ &\le \Big(\sum_{i=t_0}^{\infty} \Big|(R^q(\mathcal{M}, \Delta_n^m, u)x)_i\Big|^{p_k}\Big)^{\frac{1}{H}} \\ &< \frac{\epsilon}{2} \\ &< \epsilon, \end{aligned}$$

for all $t \ge t_0$ which proves that $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ is represented as equation (3.7).

Let us show that the uniqueness of the representation for $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ given by equation (3.6). Suppose, on the contrary that there exists a representation $x = \sum_k \mu_k(q) b^{(k)}(q)$. Since the linear transformation T from $r^q(\mathcal{M}, \Delta_n^m, u, p)$ to $l(p)$ used in the Theorem (2.2) is continuous, we have

$$(R^q(\mathcal{M}, \Delta_n^m, u)x)_n = \sum_k \mu_k(q)(R^q(\mathcal{M}, \Delta_n^m, u) b^{(k)}(q))_n = \sum_k \mu_k(q) e_n^{(k)} = \mu_n(q)$$

for $n \in \mathbb{N}$, which contradicts the fact that $(R^q(\mathcal{M}, \Delta_n^m, u)x)_n = \lambda_n(q) \ \ \forall \ \ n \in \mathcal{N}$. Hence, the representation (3.7) is unique. $\square$

4. Matrix Mappings on the Space $r^q(\mathcal{M}, \Delta_n^m, u, p)$

In this section, we characterize the matrix mappings from the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ to the space l_∞.

Theorem 4.1. *Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers.*

(i) Let $1 < p_k < D < \infty$ for $k \in \mathbb{N}$. Then $A \in (r^q(\mathcal{M}, \Delta_n^m, u, p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that

(4.1)

$$C(B) = \sup_n \sum_k \Big| \Big[\Big(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{n} \alpha_{ni} \Big) Q_k \Big] B^{-1} \Big|^{p'_k} < \infty$$

and $\{\alpha_{nk}\}_{k\in\mathbb{N}} \in cs$ for each $n \in \mathbb{N}$.

(ii) Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (r^q(\mathcal{M}, \Delta_n^m, u, p) : l_\infty)$ if and only if

$$\sup_{n,k} \Big| \Big[\Big(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{n} \alpha_{ni} \Big) Q_k \Big] \Big|^{p_k} < \infty \tag{4.2}$$

and $\{\alpha_{nk}\}_{k\in\mathbb{N}} \in cs$ for each $n \in \mathbb{N}$.

Proof. We shall prove only (i) and the proof of (ii) will follow on applying similar argument. Let $A \in (r^q(\mathcal{M}, \Delta_n^m, u, p) : l_\infty)$ and $1 < p_k \leq D < \infty$ for every $k \in \mathbb{N}$. Then Ax exists for $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ and implies that $\{\alpha_{nk}\}_{k\in\mathbb{N}} \in \{r^q(\mathcal{M}, \Delta_n^m, u, p)\}^\beta$ for each $n \in \mathbb{N}$. Hence necessity of (4.1) holds. Conversely, suppose that (4.1) holds and $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$, since $\{\alpha_{nk}\}_{k\in\mathbb{N}} \in \{r^q(\mathcal{M}, \Delta_n^m, u, p)\}^\beta$ for every fixed $n \in \mathbb{N}$, so the $A-$ transform of x exists. Consider the following equality obtained by using the relation (3.4) that

(4.3)

$$\sum_{k=0}^{t} \alpha_{nk} x_k = \sum_{k=0}^{t} \Big[\Big(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{t} \alpha_{ni} \Big) Q_k \Big] y_k.$$

Taking into account the assumptions, we derive from (3.3) as $t \to \infty$ that

(4.4)

$$\sum_{k} \alpha_{nk} x_k = \sum_{k} \Big[\Big(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{\infty} \alpha_{ni} \Big) Q_k \Big] y_k.$$

Now by combining (4.4) and the inequality which holds for any $B > 0$ and any complex numbers a, b

$$|ab| \leq B\Big(|aB^{-1}|^{p'} + |b|^p \Big)$$

with $p^{-1} + \{p'\}^{-1} = 1$ [10], we can see that

$$\sup_{n\in\mathcal{N}} \Big| \sum_k \alpha_{nk} x_k \Big| \leq \sup_{n\in\mathbb{N}} \sum_k \Big| \Big[\Big(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{\infty} \alpha_{ni} \Big) Q_k \Big] \Big| |y_k|$$

$$\begin{aligned} &\leq B[C(B) + h_1^B(y)] \\ &< \infty. \end{aligned}$$

This shows that $Ax \in l_\infty$ whenever $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$. The proof is complete. $\square$

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SOME NEW INEQUALITIES OF HERMITE-HADAMARD-FEJÉR TYPE FOR $s-$CONVEX FUNCTIONS

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Abstract. In this paper, we establish some new inequalities for differentiable mappings whose derivatives in absolute value are $s-$convex in the second sense. These results are connected with the celebrated Hermite-Hadamard-Fejér type integral inequality.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I of real numbers, $a, b \in I$ and $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

Many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard inequality, there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example ([3]-[7],[11]-[15],[17]) and the references therein.

Definition 1.1. Let real function f be defined on a nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

The class of functions which are s-convex in the second sense has been given as the following (see [9]).

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Definition 1.2. A function $f : [0, \infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense, if

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y)$$

holds for all $x, y \in [0, \infty)$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Some interesting and important inequalities for $s-$convex (in the second sense) functions can be found in [1],[10],[13]-[16]. It can be easily seen that convexity means just $s-$convexity when $s = 1$.

In [8], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality:

Theorem 1.1. *Let $f : I \to \mathbb{R}$ be convex on I and let $a, b \in I$ with $a < b$. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative and symmetric to $\frac{a+b}{2}$.

If $g = 1$, then we are talking about the Hermite-Hadamard inequalities. More about those inequalities can be found in a number of papers and monographs. For recent results and generalizations concerning Fejér inequality (1.2) see ([2],[18]-[24]).

In [1], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$convex functions in the second sense:

Theorem 1.2. *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1[a, b]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

The main purpose of this paper is to establish new Fejér type inequalities for the class of functions whose derivatives in absolute value at certain powers are $s-$convex in the second sense.

2. MAIN RESULTS

In order to prove our main results, we need the following Lemmas (see [22]):

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$. If $f', g \in L[a, b]$, then the following identity holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b f(t)g(t)dt = \int_a^b p(t)f'(t)dt$$

for each $t \in [a, b]$, where

$$p(t) = \begin{cases} \int_a^t g(s)ds, & t \in \left[a, \frac{a+b}{2}\right) \\ \\ -\int_t^b g(s)ds, & t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Lemma 2.2. *Let $f : I \to \mathbb{R}$ be differentiable on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$. If $f', g \in L[a, b]$, then the following identity holds:*

$$\int_a^b f(u)g(u)du - f(x)\int_a^b g(u)du = (b-a)^2 \int_0^1 k(t)f'(ta + (1-t)b)dt$$

for each $t \in [0, 1]$ and $x, u \in [a, b]$, where

$$k(t) = \begin{cases} \int_0^t g(sa + (1-s)b)ds, & t \in \left[0, \frac{b-x}{b-a}\right) \\ -\int_t^1 g(sa + (1-s)b)ds, & t \in \left[\frac{b-x}{b-a}, 1\right]. \end{cases} \tag{2.1}$$

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$. If $f', g \in L[a, b]$ and $|f'|$ is $s-$convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\ \leq\ & \frac{(b-a)^2}{2^{s+2}(s+1)(s+2)} \Big\{ \|g\|_{[a,\frac{a+b}{2}],\infty} \left[(2^{s+2} - (s+3))\,|f'(a)| + (s+1)\,|f'(b)|\right] \\ &+ \|g\|_{[\frac{a+b}{2},b],\infty} \left[(s+1)\,|f'(a)| + (2^{s+2} - (s+3))\,|f'(b)|)\right] \Big\}. \end{aligned}$$

Proof. By Lemma 2.1 and since $|f'|$ is s-convex on $[a, b]$, then we have

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\ \leq\ & \int_a^{\frac{a+b}{2}} \left|\int_a^t g(s)ds\right| |f'(t)|\,dt + \int_{\frac{a+b}{2}}^b \left|\int_t^b g(s)ds\right| |f'(t)|\,dt \\ \leq\ & \|g\|_{[a,\frac{a+b}{2}],\infty} \int_a^{\frac{a+b}{2}} (t-a)\,|f'(t)|\,dt + \|g\|_{[\frac{a+b}{2},b],\infty} \int_{\frac{a+b}{2}}^b (b-t)\,|f'(t)|\,dt \\ \leq\ & \|g\|_{[a,\frac{a+b}{2}],\infty} \int_a^{\frac{a+b}{2}} (t-a)\left[\left(\frac{b-t}{b-a}\right)^s |f'(a)| + \left(\frac{t-a}{b-a}\right)^s |f'(b)|\right] dt \\ &+ \|g\|_{[\frac{a+b}{2},b],\infty} \int_{\frac{a+b}{2}}^b (b-t)\left[\left(\frac{b-t}{b-a}\right)^s |f'(a)| + \left(\frac{t-a}{b-a}\right)^s |f'(b)|\right] dt \\ =\ & \frac{(b-a)^2}{2^{s+2}(s+1)(s+2)} \Big\{ \|g\|_{[a,\frac{a+b}{2}],\infty} \left[(2^{s+2} - (s+3))\,|f'(a)| + (s+1)\,|f'(b)|\right] \\ &+ \|g\|_{[\frac{a+b}{2},b],\infty} \left[(s+1)\,|f'(a)| + (2^{s+2} - (s+3))\,|f'(b)|)\right] \Big\} \end{aligned}$$

where use the facts that

$$\begin{aligned} \int_a^{\frac{a+b}{2}} (t-a)\left(\frac{b-t}{b-a}\right)^s dt &= \int_{\frac{a+b}{2}}^b (b-t)\left(\frac{t-a}{b-a}\right)^s dt \\ &= \frac{(b-a)^2\left(2^{s+2} - (s+3)\right)}{2^{s+2}(s+1)(s+2)} \end{aligned}$$

and

$$\int_a^{\frac{a+b}{2}} (t-a)\left(\frac{t-a}{b-a}\right)^s dt = \int_{\frac{a+b}{2}}^b (b-t)\left(\frac{b-t}{b-a}\right)^s dt = \frac{(b-a)^2}{2^{s+2}(s+2)}.$$

which completes the proof. □

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$. If $f', g \in L[a, b]$ and $|f'|^q$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $p > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\
\leq\ & \frac{(b-a)^2}{4(p+1)^{1/p}} \left\{ \|g\|_{\left[a,\frac{a+b}{2}\right],\infty} \left(\frac{(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q}{2^s(s+1)}\right)^{\frac{1}{q}} \right. \\
&\left. + \|g\|_{\left[\frac{a+b}{2},b\right],\infty} \left(\frac{|f'(a)|^q + (2^{s+1}-1)|f'(b)|^q}{2^s(s+1)}\right)^{\frac{1}{q}} \right\} \\
\leq\ & \frac{(b-a)^2}{4(p+1)^{1/p}} \left(\frac{1}{2^s(s+1)}\right)^{\frac{1}{q}} \\
&\times \|g\|_{[a,b],\infty} \left\{ \left[1 + (2^{s+1}-1)^{\frac{1}{q}}\right] (|f'(a)| + |f'(b)|) \right\}
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Suppose that $p > 1$. From Lemma 2.1 and using the Hölder inequality, we obtain

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\
\leq\ & \int_a^{\frac{a+b}{2}} \left|\int_a^t g(s)ds\right| |f'(t)|\,dt + \int_{\frac{a+b}{2}}^b \left|\int_t^b g(s)ds\right| |f'(t)|\,dt \\
\leq\ & \left(\int_a^{\frac{a+b}{2}} \left|\int_a^t g(s)ds\right|^p dt\right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q\,dt\right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{a+b}{2}}^b \left|\int_t^b g(s)ds\right|^p dt\right)^{\frac{1}{p}} \left(\int_{\frac{a+b}{2}}^b |f'(t)|^q\,dt\right)^{\frac{1}{q}} \\
\leq\ & \|g\|_{\left[a,\frac{a+b}{2}\right],\infty} \left(\int_a^{\frac{a+b}{2}} |t-a|^p\,dt\right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q\,dt\right)^{\frac{1}{q}} \\
&+ \|g\|_{\left[\frac{a+b}{2},b\right],\infty} \left(\int_{\frac{a+b}{2}}^b |b-t|^p\,dt\right)^{\frac{1}{p}} \left(\int_{\frac{a+b}{2}}^b |f'(t)|^q\,dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Using the s-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b g(t)f(t)dt \right| \\
\leq\ & \|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left[\frac{(b-a)^{p+1}}{2^{p+1}(p+1)}\right]^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}\left[\left(\frac{b-t}{b-a}\right)^s|f'(a)|^q+\left(\frac{t-a}{b-a}\right)^s|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
&+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\left[\frac{(b-a)^{p+1}}{2^{p+1}(p+1)}\right]^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b\left[\left(\frac{b-t}{b-a}\right)^s|f'(a)|^q+\left(\frac{t-a}{b-a}\right)^s|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{(b-a)^2}{4(p+1)^{1/p}}\left\{\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right. \\
&\left.+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\left(\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Let $a_1=(2^{s+1}-1)|f'(a)|^q$, $b_1=|f'(b)|^q$, $a_2=|f'(a)|^q$, $b_2=(2^{s+1}-1)|f'(b)|^q$. Here, $0<\frac{1}{q}<1$ for $q>1$. Using the fact that

$$\sum_{k=1}^n(a_k+b_k)^s\leq\sum_{k=1}^n a_k^s+\sum_{k=1}^n b_k^s$$

for $(0\leq s<1)$, $a_1,a_2,...,a_n\geq 0$, $b_1,b_2,...,b_k$; we obtain

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b g(t)f(t)dt \right| \\
\leq\ & \frac{(b-a)^2}{4(p+1)^{1/p}}\left(\frac{1}{2^s(s+1)}\right)^{\frac{1}{q}} \\
&\times\|g\|_{[a,b],\infty}\left\{(2^{s+1}-1)^{\frac{1}{q}}|f'(a)|+|f'(b)|+|f'(a)|+(2^{s+1}-1)^{\frac{1}{q}}|f'(b)|\right\} \\
=\ & \frac{(b-a)^2}{4(p+1)^{1/p}}\left(\frac{1}{2^s(s+1)}\right)^{\frac{1}{q}} \\
&\times\|g\|_{[a,b],\infty}\left\{\left[1+(2^{s+1}-1)^{\frac{1}{q}}\right](|f'(a)|+|f'(b)|)\right\}.
\end{aligned}$$

Also

$$\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\leq\|g\|_{[a,b],\infty}$$

and

$$\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\leq\|g\|_{[a,b],\infty}.$$

This completes the proof. □

Theorem 2.3. *Let $f: I^\circ\subset\mathbb{R}\to\mathbb{R}$ be differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$ and $g:[a,b]\to[0,\infty)$ be differentiable mapping. If $|f'|$ is s-convex on*

$[a,b]$, for some fixed $s\in(0,1]$, then the following inequality holds:

$$\begin{aligned}
&\left|f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right|\\
\leq\ &\frac{1}{(b-a)^s(s+2)}\\
&\times\left\{\|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty}\left[(b-x)^{s+2}\left|f'(a)\right|+\frac{(b-a)^{s+2}+(x-a)^{s+1}[(x-b)(s+1)-(b-a)]}{s+1}\left|f'(b)\right|\right]\right.\\
&\left.+\|g\|_{\left[\frac{b-x}{b-a},1\right],\infty}\left[\frac{(b-a)^{s+2}+(b-x)^{s+1}[(a-x)(s+1)-(b-a)]}{s+1}\left|f'(a)\right|+(x-a)^{s+2}\left|f'(b)\right|\right]\right\}.
\end{aligned}$$

Proof. Let $x\in[a,b]$. Using Lemma 2.2, we obtain

$$\begin{aligned}
&\left|f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right|\\
\leq\ &(b-a)^2\left\{\int_0^{\frac{b-x}{b-a}}\left|\int_0^t g(sa+(1-s)b)ds\right|\left|f'(ta+(1-t)b)\right|dt\right.\\
&\left.+\int_{\frac{b-x}{b-a}}^1\left|\int_t^1 g(sa+(1-s)b)ds\right|\left|f'(ta+(1-t)b)\right|dt\right\}\\
\leq\ &(b-a)^2\left\{\|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty}\int_0^{\frac{b-x}{b-a}}|t|\left|f'(ta+(1-t)b)\right|dt\right.\\
&\left.+\|g\|_{\left[\frac{b-x}{b-a},1\right],\infty}\int_{\frac{b-x}{b-a}}^1|1-t|\left|f'(ta+(1-t)b)\right|dt\right\}.
\end{aligned}$$

Since $|f'|$ is s-convex on $[a,b]$, we obtain

$$\begin{aligned}
&\left|f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right|\\
\leq\ &(b-a)^2\left\{\|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty}\int_0^{\frac{b-x}{b-a}}t\left[t^s\left|f'(a)\right|+(1-t)^s\left|f'(b)\right|\right]dt\right.\\
&\left.+\|g\|_{\left[\frac{b-x}{b-a},1\right],\infty}\int_{\frac{b-x}{b-a}}^1(1-t)\left[t^s\left|f'(a)\right|+(1-t)^s\left|f'(b)\right|\right]dt\right\}\\
=\ &\frac{1}{(b-a)^s(s+2)}\\
&\times\left\{\|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty}\left[(b-x)^{s+2}\left|f'(a)\right|+\frac{(b-a)^{s+2}+(x-a)^{s+1}[(x-b)(s+1)-(b-a)]}{s+1}\left|f'(b)\right|\right]\right.\\
&\left.+\|g\|_{\left[\frac{b-x}{b-a},1\right],\infty}\left[\frac{(b-a)^{s+2}+(b-x)^{s+1}[(a-x)(s+1)-(b-a)]}{s+1}\left|f'(a)\right|+(x-a)^{s+2}\left|f'(b)\right|\right]\right\}.
\end{aligned}$$

This completes the proof. □

Theorem 2.4. *Let $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$ be differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be differentiable mapping. If $|f'|^q$ is s-convex*

on $[a,b]$, for some fixed $s\in(0,1]$ and $p>1$, then the following inequality holds:

$$
\begin{aligned}
&\left| f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right| \\
\leq\ & \frac{1}{(b-a)^{\frac{s}{q}}(p+1)^{\frac{1}{p}}} \\
&\times\left\{ \|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty}\left[\frac{(b-x)^{2q+s}\left|f'(a)\right|^q+(b-x)^{2q-1}[(b-a)^{s+1}-(x-a)^{s+1}]\left|f'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}\right. \\
&\quad\left. +\|g\|_{\left[\frac{b-x}{b-a},1\right],\infty}\left[\frac{(x-a)^{2q-1}[(b-a)^{s+1}-(b-x)^{s+1}]\left|f'(a)\right|^q+(x-a)^{2q+s}\left|f'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.2, Hölder's inequality and the s-convexity of $|f'|^q$, $\frac{1}{p}+\frac{1}{q}=1$, we have

$$
\begin{aligned}
&\left| f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right| \\
\leq\ & (b-a)^2 \\
&\times\left\{\left(\int_0^{\frac{b-x}{b-a}}\left|\int_0^t g(sa+(1-s)b)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_{\frac{b-x}{b-a}}^1\left|\int_t^1 g(sa+(1-s)b)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & (b-a)^2\left\{\|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty}\left(\int_0^{\frac{b-x}{b-a}}t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\|g\|_{\left[\frac{b-x}{b-a},1\right],\infty}\left(\int_{\frac{b-x}{b-a}}^1(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{1}{(b-a)^{\frac{s}{q}}(p+1)^{\frac{1}{p}}} \\
&\times\left\{ \|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty}\left[\frac{(b-x)^{2q+s}\left|f'(a)\right|^q+(b-x)^{2q-1}[(b-a)^{s+1}-(x-a)^{s+1}]\left|f'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}\right. \\
&\left. +\|g\|_{\left[\frac{b-x}{b-a},1\right],\infty}\left[\frac{(x-a)^{2q-1}[(b-a)^{s+1}-(b-x)^{s+1}]\left|f'(a)\right|^q+(x-a)^{2q+s}\left|f'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

This completes the proof. □

Theorem 2.5. *Let $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$ be differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be differentiable mapping. If $|f'|^q$ is s-convex*

on $[a,b]$, for some fixed $s \in (0,1]$ and $p > 1$, then the following inequality holds:

$$\begin{aligned}
&\left| f(x) \int_a^b g(u)du - \int_a^b f(u)g(u)du \right| \\
\leq\ & \frac{1}{(p+1)^{\frac{1}{p}}} \left\{ \|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty} (b-x)^2 \left[\frac{|f'(x)|^q + |f'(b)|^q}{s+1} \right]^{\frac{1}{q}} \right. \\
&\left. + \|g\|_{\left[\frac{b-x}{b-a},1\right],\infty} (x-a)^2 \left[\frac{|f'(a)|^q + |f'(x)|^q}{s+1} \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.2 and using the Hölder inequality, we have

$$\begin{aligned}
&\left| f(x) \int_a^b g(u)du - \int_a^b f(u)g(u)du \right| \\
\leq\ & (b-a)^2 \\
&\times \left\{ \left(\int_0^{\frac{b-x}{b-a}} \left| \int_0^t g(sa+(1-s)b)ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\int_{\frac{b-x}{b-a}}^1 \left| \int_t^1 g(sa+(1-s)b)ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right\} \\
\leq\ & (b-a)^2 \left\{ \|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty} \left(\int_0^{\frac{b-x}{b-a}} t^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right. \\
&\left. + \|g\|_{\left[\frac{b-x}{b-a},1\right],\infty} \left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Since $|f'|^q$ is s-convex, by (1.3) we have

$$\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q dt \leq \frac{b-x}{b-a} \left(\frac{|f'(x)|^q + |f'(b)|^q}{s+1} \right)$$

and

$$\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q dt \leq \frac{x-a}{b-a} \left(\frac{|f'(a)|^q + |f'(x)|^q}{s+1} \right).$$

Therefore,

$$\begin{aligned}
&\left| f(x) \int_a^b g(u)du - \int_a^b f(u)g(u)du \right| \\
\leq\ & \frac{1}{(p+1)^{\frac{1}{p}}} \left\{ \|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty} (b-x)^2 \left[\frac{|f'(x)|^q + |f'(b)|^q}{s+1} \right]^{\frac{1}{q}} \right. \\
&\left. + \|g\|_{\left[\frac{b-x}{b-a},1\right],\infty} (x-a)^2 \left[\frac{|f'(a)|^q + |f'(x)|^q}{s+1} \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

This completes the proof. $\square$

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ON HADAMARD-TYPE INEQUALITIES FOR k-FRACTIONAL INTEGRALS

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Abstract. In this paper we prove Hadamard-type inequalities for k-fractional Riemann-Liouville integrals and Hadamard-type inequalities for fractional Riemann-Liouville integrals are deduced. Also we deduced some well known results related to Hadamard inequality.

1. introduction

Fractional Calculus is a branch of mathematical study that developed from the established definitions of calculus integral and derived operators [2].

Fractional calculus was mainly a study kept for the finest minds in mathematics. Fourier, Euler, Laplace are among those mathematicians who showed a casual interest by fractional calculus and mathematical consequences. A lot of them established definitions by means of their own notion and style. Most renowned of these definitions are the Grunwald-Letnikov and Riemann-Liouville definition [4].

There are many types of fractional integrals have been defined in literature, the most classical are Riemann-Liouville fractional integrals defined as follows:

Definition 1.1. Let $f \in L_1[a,b]$, then Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as:

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{1.1}$$

and

$$I^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b. \tag{1.2}$$

For further details one may see [3, 6, 7].

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[1] If $k > 0$, then k-Gamma function Γ_k is defined as:

$$\Gamma_k(\alpha) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{\alpha}{k}} - 1}{(\alpha)_{n,k}}.$$

If $\Re(\alpha) > 0$ then k-Gamma function in integral form is defined as

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt,$$

with the property that

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$$

In [5] k-fractional Riemann-Liouville integrals are defined as follows:

Let $f \in L_1[a, b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x - t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a \tag{1.3}$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t - x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b. \tag{1.4}$$

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals.

Besides applications of fractional integrals in applied sciences, now a days many researchers in the field of pure mathematics, for example mathematical analysis have studied them extensively see [2, 3, 4, 6].

In [8], Sarikaya et al. proved the following Hadamard-type inequalities for Riemann-Liouville fractional integrals.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on [a,b], then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{(\frac{a+b}{2})+} f(b) + I^\alpha_{(\frac{a+b}{2})-} f(a)\right] \leq \frac{f(a) + f(b)}{2} \tag{1.5}$$

with $\alpha > 0$.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}[I^\alpha_{(\frac{a+b}{2})+} f(b) + I^\alpha_{(\frac{a+b}{2})-} f(a)] - f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\Big[((\alpha+1)|f'(a)|^q + (\alpha+3)|f'(b)|^q)^{\frac{1}{q}} \\ &+ ((\alpha+3)|f'(a)|^q + (\alpha+1)|f'(b)|^q)^{\frac{1}{q}}\Big]. \end{aligned} \tag{1.6}$$

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q > 1$, then the following inequality for fractional* *integral holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}[I^{\alpha}_{(\frac{a+b}{2})+}f(b)+I^{\alpha}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right| \\
&\le \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \qquad (1.7)\\
&\le \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}[|f'(a)|+|f'(b)|],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In this paper we generalize the fractional Hadamard-type inequalities (1.5), (1.6) and (1.7) via k-fractional integrals and show that these inequalities are special cases of our results. Also we deduced some well known results.

2. Hadamard-type inequalities for k-fractional integrals

Here we give k-fractional Hadamard-type inequalities.

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for k-fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)\right]\le\frac{f(a)+f(b)}{2} \qquad (2.1)$$

with $\alpha,k>0$.

Proof. From convexity of f we have

$$f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2}. \qquad (2.2)$$

Putting $x=\frac{t}{2}a+\frac{(2-t)}{2}b, y=\frac{(2-t)}{2}a+\frac{t}{2}b$ for $t\in[0,1]$. Then $x,y\in[a,b]$ and above equation gives

$$2f\left(\frac{a+b}{2}\right)\le f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right), \qquad (2.3)$$

multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$, and integrating over $[0,1]$ we have

$$\begin{aligned}
&\frac{2k}{\alpha}f\left(\frac{a+b}{2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}dt\\
&\le\int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\
&=\frac{2^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)\right],
\end{aligned}$$

from which one can have

$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)\right]. \qquad (2.4)$$

On the other hand convexity of f gives

$$f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\leq\frac{t}{2}f(a)+\frac{2-t}{2}f(b)+\frac{2-t}{2}f(a)+\frac{t}{2}f(b),$$

multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$, and integrating over $[0,1]$ we have

$$\begin{aligned}&\int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\&\leq[f(a)+f(b)]\int_0^1 t^{\frac{\alpha}{k}-1}dt,\end{aligned}$$

from which one can have

$$\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)\right]\leq\frac{f(a)+f(b)}{2}. \tag{2.5}$$

Combining inequality (2.4) and inequality (2.5) we get inequality (2.1) . □

Remark 2.1. If we take $k=1$, Theorem 2.1 gives inequality (1.5) of Theorem 1.1 and putting $\alpha=1$ along with $k=1$ in Theorem 2.1 we get the classical Hadamard inequality.

3. k-fractional inequalities related to Hadamard inequality

For next results we need the following lemma.

Lemma 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f^{'}\in L[a,b]$, then the following equality for k-fractional integrals holds:*

$$\begin{aligned}&\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\\&=\frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt-\int_0^1 t^{\frac{\alpha}{k}}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right].\end{aligned} \tag{3.1}$$

Proof. One can note that

$$\begin{aligned}&\frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right]\\&=\frac{b-a}{4}\left[t^{\frac{\alpha}{k}}\frac{2}{a-b}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)|_0^1-\int_0^1\frac{\alpha}{k}t^{\frac{\alpha}{k}-1}\frac{2}{a-b}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]\\&=\frac{b-a}{4}\left[-\frac{2}{b-a}f\left(\frac{a+b}{2}\right)-\frac{2\alpha}{k(a-b)}\int_b^{\frac{a+b}{2}}\left(\frac{2}{b-a}(b-x)\right)^{\frac{\alpha}{k}-1}\frac{2}{a-b}f(x)dx\right]\end{aligned}$$

(3.2)

$$=\frac{b-a}{4}\left[-\frac{2}{b-a}f\left(\frac{a+b}{2}\right)+\frac{2^{\frac{\alpha}{k}+1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}+1}}I^{\alpha,k}_{(\frac{a+b}{2})-}f(b)\right].$$

Similarly

$$-\frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}} f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right]$$

$$=-\frac{b-a}{4}\left[\frac{2}{b-a}f\left(\frac{a+b}{2}\right)-\frac{2^{\frac{\alpha}{k}+1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}+1}}I^{\alpha,k}_{(\frac{a+b}{2})+}f(a)\right]. \tag{3.3}$$

Combining (3.2) and (3.3) one can have (3.1). □

Using the above lemma we give the following k-fractional Hadamard-type inequality.

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for $q\geq 1$, then the following inequality for k-fractional integrals holds:*

$$\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{b-a}{4(\frac{\alpha}{k}+1)}\left(\frac{1}{2(\frac{\alpha}{k}+2)}\right)^{\frac{1}{q}}\left[\left(\left(\left(\frac{\alpha}{k}+1\right)\right)|f'(a)|^q+\left(\frac{\alpha}{k}+3\right)|f'(b)|^q\right)^{\frac{1}{q}}\right. \tag{3.4}$$

$$\left.+\left(\left(\frac{\alpha}{k}+3\right)|f'(a)|^q+\left(\frac{\alpha}{k}+1\right)|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

with $\alpha,k>0$.

Proof. From Lemma 3.1 and convexity of $|f'|$ and for $q=1$ we have

$$\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{b-a}{4}\int_0^1 t^{\frac{\alpha}{k}}\left(\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right)dt.$$

$$=\frac{b-a}{4\left(\frac{\alpha}{k}+1\right)}[|f'(a)|+|f'(b)|].$$

For $q>1$ we proceed as follows. Using Lemma (3.1) we have

$$\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right].$$

Using power mean inequality we get

$$\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{b-a}{4}\left(\frac{1}{\frac{\alpha}{k}+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].$$

Convexity of $|f'|^q$ gives

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{b-a}{4}\left(\frac{1}{\frac{\alpha}{k}+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{\frac{\alpha}{k}}\left(\frac{t}{2}|f'(a)|^q+\frac{2-t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right.\\
&\left.+\left[\int_0^1 t^{\frac{\alpha}{k}}\left(\frac{2-t}{2}|f'(a)|^q+\frac{t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right]\\
&=\frac{b-a}{4}\left(\frac{1}{\frac{\alpha}{k}+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f'(a)|^q}{2(\frac{\alpha}{k}+2)}+\frac{|f'(b)|^q}{\frac{\alpha}{k}+1}-\frac{|f'(b)|^q}{2(\frac{\alpha}{k}+2)}\right]^{\frac{1}{q}}+\left[\frac{|f'(a)|^q}{\frac{\alpha}{k}+1}-\frac{|f'(a)|^q}{2(\frac{\alpha}{k}+2)}\right.\right.\\
&\left.\left.+\frac{|f'(b)|^q}{2(\frac{\alpha}{k}+2)}\right]^{\frac{1}{q}}\right],
\end{aligned}$$

which after a little computation gives the required result. □

Remark 3.1. If we take $k=1$ in Theorem 3.1, we get inequality (1.6) of Theorem 1.2 and if we take $\alpha=q=1$ along with $k=1$ in Theorem 3.1, then inequality (3.4) gives inequality the following result.

Corollary 3.1. *With assumptions of Theorem 3.1 we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right).$$

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for $q>1$, then the following inequality for k-fractional integral holds:*

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|\\
&\leq\frac{b-a}{4}\left(\frac{1}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \quad (3.5)\\
&\leq\frac{b-a}{4}\left(\frac{4}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}[|f'(a)|+|f'(b)|],
\end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 3.1 we have

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|\\
&\leq\frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right].
\end{aligned}$$

From $H\ddot{o}lder's$ inequality we get

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{b-a}{4}\left[\left[\int_0^1 t^{\frac{\alpha p}{k}}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.\\
&\left.+\left[\int_0^1 t^{\frac{\alpha p}{k}}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Convexity of $|f'|^q$ gives

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{b-a}{4}\left(\frac{1}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1\left(\frac{t}{2}|f'(a)|^q+\frac{2-t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right.\\
&\left.+\left[\int_0^1\left(\frac{2-t}{2}|f'(a)|^q+\frac{t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right]\\
&=\frac{b-a}{4}\left(\frac{1}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right]^{\frac{1}{q}}\right].
\end{aligned}$$

For second inequality of (3.5) we use Minkowski's inequality as

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{b-a}{16}\left(\frac{4}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[[|f'(a)|^q+3|f'(b)|^q]^{\frac{1}{q}}+[3|f'(a)|^q+|f'(b)|^q]^{\frac{1}{q}}\right]\\
&\leq \frac{b-a}{16}\left(\frac{4}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}(3^{\frac{1}{q}}+1)(|f'(a)|+|f'(b)|)\\
&\leq \frac{b-a}{16}\left(\frac{4}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}4(|f'(a)|+|f'(b)|).
\end{aligned}$$

□

Remark 3.2. For $k=1$ in above theorem we get inequality (1.7). If we take $\alpha=k=1$ we get the following result.

Corollary 3.2. *With assumptions of Theorem 3.2 we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[(|f'(a)|^q+3|f'(b)|^q)^{\frac{1}{q}}+(3|f'(a)|^q+|f'(b)|^q)^{\frac{1}{q}}\right].
\end{aligned}$$

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ON ALMOST IDEAL CONVERGENCE WITH RESPECT TO AN ORLICZ FUNCTION

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Abstract. In this article, we define new classes of ideal convergent and ideal bounded sequence spaces combining an infinite matrix, an Orlicz function and invariant mean. We investigate some linear topological structures and algebraic properties of the resulting spaces. Also we find out some relations related to these spaces.

1. Introduction

By ω and ℓ_∞, we denote the space of all *complex valued sequences* and *bounded sequences*, respectively. $\mathbb{N}$ and $\mathbb{C}$ stand for the set of *natural numbers* and *complex numbers* and $e = (1, 1, 1, ...)$.

The notion of *ideal convergence* which is a generalization of statistical convergence (see [1, 2]) was introduced by Kostyrko et al. [3].

A family $\mathcal{I}$ of subsets of a non-empty set X is called an *ideal* on X if for each $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$ and for each $B \in \mathcal{I}$ and $B \subseteq A$, we have $B \in \mathcal{I}$. If $X \notin \mathcal{I}$, it is called a non-trivial ideal. A non-trivial ideal is said to be admissible if it contains all the finite subsets of X.

A sequence $x = (x_k)$ in $\mathbb{R}$ is called ideal convergent to a real number l if for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ belongs to the ideal [3].

A sequence $x = (x_k)$ of real numbers is said to be ideal bounded if there is a $K > 0$ such that $\{k \in \mathbb{N} : |x_k| > K\} \in \mathcal{I}$ [4].

Later, many authors studied on ideal convergence. See for example [5, 6, 7]. Also, ideal convergence is studied on normed spaces and topological spaces in [8, 9, 10, 11, 12].

Let σ be an injective mapping from the set of the positive integers to itself such that $\sigma^p(n) \neq n$ for all positive integers n and p, where $\sigma^p(n) = \sigma(\sigma^{p-1}(n))$. An invariant mean or a σ-mean is a continuous linear functional defined on the space ℓ_∞ such that for all $x = (x_n) \in \ell_\infty$:

(1) If $x_n \geq 0$ for all n, then $\varphi(x) \geq 0$,

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(2) $\varphi(e) = 1$,
(3) $\varphi(Sx) = \varphi(x)$, where $Sx = (x_{\sigma(n)})$.

V_σ denotes the set of bounded sequences all of whose invariant means are equal which is also called as the space of σ-convergent sequences. In [13], it is defined by

$$V_\sigma = \{x \in \ell_\infty : \lim_k t_{kn}(x) = l, \text{ uniformly in } n, l = \sigma - \lim x\},$$

where $t_{kn}(x) = \frac{x_n + x_{\sigma^1(n)} + ... + x_{\sigma^k(n)}}{k+1}$.

σ-mean is called a Banach limit if σ is the translation mapping $n \to n+1$. In this case, V_σ becomes the set of almost convergent sequences which is denoted by $\hat{c}$ and defined in[14] as

$$\hat{c} = \{x \in \ell_\infty : \lim_k d_{kn}(x) \text{ exists uniformly in } n\},$$

where $d_{kn}(x) = \frac{x_n + x_{n+1} + ... + x_{n+k}}{k+1}$.

The space of strongly almost convergent sequences was introduced by Maddox [15] as follow:

$$[\hat{c}] = \{x \in \ell_\infty : \lim_k d_{kn}(|x - le|) \text{ exists uniformly in } n \text{ for some } l\}.$$

A function $M : [0, \infty) \to [0, \infty)$ is called an *Orlicz function* if M is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. By convexity of M and $M(0) = 0$, we have $M(\lambda x) \le \lambda M(x)$ for all $\lambda \in (0, 1)$.

It is said that M satisfies Δ_2-condition for all $x \in [0, \infty)$ if there exists a constant $K > 0$ such that $M(Lx) \le KLM(x)$, where $L > 1$ (see [16]).

By using the idea of Orlicz function, Lindenstrauss and Tzafriri [17] defined *Orlicz sequence space*

$$\ell_M = \left\{x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}$$

which is a Banach space with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\}.$$

Several authors used the concept of an Orlicz function to define a new sequence space. For some of the related papers, one can see [19, 20, 21, 22].

Let $p = (p_k)$ be a sequence of positive real numbers such that $0 < h = \inf p_k \le p_k \le H = \sup p_k < \infty$. For each $k \in \mathbb{N}$ the inequalities

$$|\alpha_k + \beta_k|^{p_k} \le D\left\{|\alpha_k|^{p_k} + |\beta_k|^{p_k}\right\} \tag{1.1}$$

and

$$|\alpha|^{p_k} \le \max\{1, |\alpha|^H\}$$

hold, where $\alpha, \alpha_k, \beta_k \in \mathbb{C}$ and $D = \max\left\{1, 2^{H-1}\right\}$.

Let $A = (a_{ij})$ be an infinite matrix of complex numbers a_{ij}, where $i, j \in \mathbb{N}$. We write $Ax = (A_i(x))$ if $A_i(x) = \sum_{j=1}^{\infty} a_{ij} x_j$ converges for each $i \in \mathbb{N}$. Throughout the text, by $t_{kn}(Ax)$, we mean

$$t_{kn}(Ax) = \frac{A_n(x) + A_{\sigma^1(n)}(x) + ... + A_{\sigma^k(n)}(x)}{k+1}$$

for all $k, n \in \mathbb{N}$.

A sequence space X is called as *solid (or normal)* if $(\gamma_k x_k) \in X$ whenever $(x_k) \in X$ and (γ_k) is a sequence of scalars such that $|\gamma_k| \leq 1$ for all $k \in \mathbb{N}$.

Let X be a sequence space and $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$. The sequence space $Z_K^X = \{(x_{k_n}) \in \omega : (x_n) \in X\}$ is called K-*step space* of X.

A *canonical preimage* of a sequence $(x_{k_n}) \in Z_K^X$ is a sequence $(y_n) \in \omega$ defined by

$$y_n = \begin{cases} x_n, & \text{if } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

A sequence space X is *monotone* if it contains the canonical preimages of all its step spaces.

Lemma 1.1. ([18],p.53) *If a sequence space X is solid, then X is monotone.*

Recently, strongly almost ideal convergent sequence spaces in 2-normed spaces defined via an Orlicz function was introduced by Esi [23]. Quite recently, Hazarika [24] defined a new class of strongly almost ideal convergent sequence spaces using an infinite matrix, Orlicz functions and a new generalized difference matrix in locally convex spaces and proved some results about this notion. Further in [25, 26, 27], the authors defined new spaces by combining ideal convergence, Orlicz functions and infinite matrices.

The purpose of this paper is to introduce and study some new ideal convergent sequence spaces with respect to an Orlicz function and an infinite matrix.

2. Main results

In this section, by combining ideal convergence, an infinite matrix, an Orlicz function and invariant means, we define some new sequence spaces.

From now on, by $\mathcal{I}$, we denote an admissible ideal of $\mathbb{N}$.

Let M be an Orlicz function, A be an infinite matrix and $p = (p_k)$ be a bounded sequence of positive real numbers.

For every $\varepsilon > 0$ and some $\rho > 0$, we introduce the spaces as follows:

$$\mathcal{I}{-}c_0^\sigma(M, A, p) = \left\{ u \in \omega : \left\{ k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au)|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in \mathcal{I} \text{ for all } n \in \mathbb{N} \right\},$$

$$\mathcal{I}{-}c^\sigma(M, A, p) = \left\{ u \in \omega : \left\{ k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au - le)|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in \mathcal{I} \text{ for all } n \in \mathbb{N} \text{ and some } l \in \mathbb{C} \right\},$$

$$\mathcal{I}{-}\ell_\infty^\sigma(M, A, p) = \left\{ u \in \omega : \exists K > 0 \text{ such that } \left\{ k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au)|}{\rho} \right) \right]^{p_k} > K \right\} \in \mathcal{I} \text{ for all } n \in \mathbb{N} \right\}.$$

If we take $p_k = 1$ for all $k \in \mathbb{N}$, then the above spaces are denoted by $\mathcal{I}{-}c_0^\sigma(M, A)$, $\mathcal{I} - c^\sigma(M, A)$, $\mathcal{I} - \ell_\infty^\sigma(M, A)$, respectively.

Theorem 2.1. *The spaces $\mathcal{I} - c_0^\sigma(M, A, p)$, $\mathcal{I} - c^\sigma(M, A, p)$ and $\mathcal{I} - \ell_\infty^\sigma(M, A, p)$ are linear spaces.*

Proof. The result will be proved only for $\mathcal{I}{-}c_0^\sigma(M, A, p)$. The others follow similarly. Take any $u, v \in \mathcal{I} - c_0^\sigma(M, A, p)$. Then for given $\varepsilon > 0$ the sets

$$S_1 = \left\{ k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au)|}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\}$$

and

$$S_2 = \left\{ k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Av)|}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\}$$

are contained in $\mathcal{I}$ for some $\rho_1, \rho_2 > 0$.

By using the inequality (1.1) and the fact that M is nondecreasing and convex, one can see the following inequality:

$$\begin{aligned} \left[M\left(\frac{|t_{kn}(A(\lambda u + \mu v)|}{\rho} \right) \right]^{p_k} &\leq \left[M\left(\frac{|t_{kn}(A(u)|}{\rho_1} \right) + M\left(\frac{|t_{kn}(A(v)|}{\rho_2} \right) \right]^{p_k} \\ &\leq D\left\{ \left[M\left(\frac{|t_{kn}(A(u)|}{\rho_1} \right) \right]^{p_k} + \left[M\left(\frac{|t_{kn}(A(v)|}{\rho_2} \right) \right]^{p_k} \right\}, \end{aligned}$$

where $\rho = \max\{2|\lambda|\rho_1, 2|\mu|\rho_2\}$ and $\lambda, \mu \in \mathbb{C}$.

If we choose a positive integer k' from $\mathbb{N} \backslash S_1 \cup S_2$, we obtain

$$\left[M\left(\frac{|t_{kn}(A(\lambda u + \mu v)|}{\rho} \right) \right]^{p_k} < \varepsilon.$$

Hence the set

$$\left\{ k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(A(\lambda u + \mu v))|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\}$$

belongs to the ideal which implies $\lambda u + \mu v \in \mathcal{I} - c_0^\sigma(M, A, p)$. This completes the proof. $\square$

Theorem 2.2. *The inclusions*

$$\mathcal{I} - c_0^\sigma(M_1, A, p) \cap \mathcal{I} - c_0^\sigma(M_2, A, p) \subseteq \mathcal{I} - c_0^\sigma(M_1 + M_2, A, p),$$

$$\mathcal{I} - c^\sigma(M_1, A, p) \cap \mathcal{I} - c^\sigma(M_2, A, p) \subseteq \mathcal{I} - c^\sigma(M_1 + M_2, A, p),$$

$$\mathcal{I} - \ell_\infty^\sigma(M_1, A, p) \cap \mathcal{I} - \ell_\infty^\sigma(M_2, A, p) \subseteq \mathcal{I} - \ell_\infty^\sigma(M_1 + M_2, A, p)$$

hold for any Orlicz functions M_1 and M_2.

Proof. Let u belong to the intersection of $\mathcal{I} - c_0^\sigma(M_1, A, p)$ and $\mathcal{I} - c_0^\sigma(M_2, A, p)$. Since the inequality

$$\begin{aligned} \left[(M_1 + M_2)\left(\frac{|t_{kn}(A(u)|}{\rho} \right) \right]^{p_k} &= \left[M_1\left(\frac{|t_{kn}(A(u)|}{\rho} \right) + M_2\left(\frac{|t_{kn}(A(u)|}{\rho} \right) \right]^{p_k} \\ &\leq D\left\{ \left[M_1\left(\frac{|t_{kn}(A(u)|}{\rho} \right) \right]^{p_k} + \left[M_2\left(\frac{|t_{kn}(A(u)|}{\rho} \right) \right]^{p_k} \right\} \end{aligned}$$

holds, the result is obvious.

The other inclusions can be shown similarly. $\square$

Theorem 2.3. *Let M_2 satisfy Δ_2 condition. Then the inclusions*

$$\mathcal{I} - c_0^\sigma(M_1, A, p) \subseteq \mathcal{I} - c_0^\sigma(M_1 \circ M_2, A, p),$$

$$\mathcal{I} - c^\sigma(M_1, A, p) \subseteq \mathcal{I} - c^\sigma(M_1 \circ M_2, A, p),$$

$$\mathcal{I} - \ell_\infty^\sigma(M_1, A, p) \subseteq \mathcal{I} - \ell_\infty^\sigma(M_1 \circ M_2, A, p)$$

hold for any Orlicz functions M_1 and M_2.

Proof. We prove the theorem in two parts. Firstly, let $M_1\left(\frac{|t_{kn}(A(u)|}{\rho}\right) > \delta$. By using the properties of an Orlicz function and the fact that M_2 satisfies Δ_2 condition, we have

$$\begin{aligned}\left[M_2\left(M_1\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right)\right]^{p_k} &\leq (K\delta^{-1}M_2(2))^{p_k}\left[M_1\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k}\\ &\leq \max\left\{1,(K\delta^{-1}M_2(2))^H\right\}\left[M_1\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k},\end{aligned}$$

where $K \geq 1$ and $\delta < 1$. From the last inequality, the inclusion

$$\left\{k\in\mathbb{N}:\left[M_2\left(M_1\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right)\right]^{p_k}\geq\varepsilon\right\}\subseteq\left\{k\in\mathbb{N}:\left[M_1\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k}\geq\frac{\varepsilon}{\max\left\{1,(K\delta^{-1}M_2(2))^H\right\}}\right\}$$

is obtained. If $u \in \mathcal{I} - c_0^\sigma(M_1, A, p)$, then the set in the right side of the above inclusion belongs to the ideal and so $\left\{k\in\mathbb{N}:\left[M_2\left(M_1\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right)\right]^{p_k}\geq\varepsilon\right\}\in\mathcal{I}$.

Secondly, Suppose that $M_1\left(\frac{|t_{kn}(A(u)|}{\rho}\right) \leq \delta$. Since M_2 is continuous, we have $M_2\left(M_1\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right) < \varepsilon$ for all $\varepsilon > 0$ which implies $\mathcal{I}-\lim_k\left[M_2\left(M_1\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right)\right]^{p_k} = 0$ as $\varepsilon \to 0$. This completes the proof.

The other inclusions can be shown similarly. □

Theorem 2.4. *If* $\sup_k[M(t)]^{p_k} < \infty$ *for all* $t > 0$*, then we have*

$$\mathcal{I} - c^\sigma(M, A, p) \subseteq \mathcal{I} - \ell_\infty^\sigma(M, A, p).$$

Proof. Let $x \in \mathcal{I} - c^\sigma(M, A, p)$. The inequality

$$\left[M\left(\frac{|t_{kn}(A(u)|}{\rho}\right)\right]^{p_k}\leq D\left\{\left[M\left(\frac{|t_{kn}(Au-le)|}{\rho_1}\right)\right]^{p_k}+\left[M\left(\frac{|t_{kn}(le)|}{\rho_1}\right)\right]^{p_k}\right\}$$

holds by (1.1), where $\rho = 2\rho_1$. Hence we have

$$\left\{k\in\mathbb{N}:\left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k}\geq K\right\}\subseteq\left\{k\in\mathbb{N}:\left[M\left(\frac{|t_{kn}(Au-le)|}{\rho_1}\right)\right]^{p_k}\geq\varepsilon\right\}$$

for all n and some $K > 0$. Since the set in the right side of the above inclusion belogs to the ideal, all of its subsets are in the ideal. So

$$\left\{k\in\mathbb{N}:\left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k}\geq K\right\}\in\mathcal{I}$$

which completes the proof. □

Theorem 2.5. *Let* $0 < p_k \leq q_k < \infty$ *for each* $k \in \mathbb{N}$ *and* $(\frac{q_k}{p_k})$ *be bounded. Then we have*

$$\mathcal{I} - W(M, A, q) \subseteq \mathcal{I} - W(M, A, p),$$

where $W = c_0^\sigma, c^\sigma$.

Proof. Suppose that $u \in \mathcal{I} - c_0^\sigma(M, A, q)$. Write $\alpha_k = \frac{p_k}{q_k}$. By hypothesis, we have $0 < \alpha \leq \alpha_k \leq 1$. If $\left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{q_k} \geq 1$, the inequality $\left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k} \leq \left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{q_k}$ holds. This implies the inclusion

$$\left\{k\in\mathbb{N}:\left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\subseteq\left\{k\in\mathbb{N}:\left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{q_k}\geq\varepsilon\right\}$$

and so the result is obvious. Conversely, if $\left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{q_k} < 1$, we obtain the following inclusion

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \subseteq \left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{q_k} \geq \varepsilon^{1/\alpha}\right\}$$

since then the inequality $\left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k} \leq \left(\left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{q_k}\right)^{\alpha}$ holds. Hence we conclude that $u \in \mathcal{I} - c_0^{\sigma}(M, A, p)$. □

Theorem 2.6.

(1) *If $0 < \inf p_k \leq p_k \leq 1$ for each $k \in \mathbb{N}$, then $\mathcal{I} - W(M, A, p) \subseteq \mathcal{I} - W(M, A)$, where $W = c_0^{\sigma}, c^{\sigma}$.*
(2) *If $1 \leq p_k \leq \sup p_k < \infty$ for each $k \in \mathbb{N}$, then $\mathcal{I} - W(M, A) \subseteq \mathcal{I} - W(M, A, p)$, where $W = c_0^{\sigma}, c^{\sigma}$.*

Proof.

(1) Let $u \in \mathcal{I} - c_0^{\sigma}(M, A, p)$. Suppose that $k \notin \left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}$ for $0 < \varepsilon < 1$. By hypothesis, the inequality $M\left(\frac{|t_{kn}(Au)|}{\rho}\right) \leq \left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k}$ holds. Then we have $k \notin \left\{k \in \mathbb{N} : M\left(\frac{|t_{kn}(Au)|}{\rho}\right) \geq \varepsilon\right\}$ which implies

$$\left\{k \in \mathbb{N} : M\left(\frac{|t_{kn}(Au)|}{\rho}\right) \geq \varepsilon\right\} \subseteq \left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}.$$

Hence $u \in \mathcal{I} - c_0^{\sigma}(M, A)$ since the set $\left\{k \in \mathbb{N} : M\left(\frac{|t_{kn}(Au)|}{\rho}\right) \geq \varepsilon\right\}$ in $\mathcal{I}$.
(2) The proof is similar to the first part.

□

Theorem 2.7. *The spaces $\mathcal{I} - c_0^{\sigma}(M, A, p)$ and $\mathcal{I} - \ell_{\infty}^{\sigma}(M, A, p)$ are solid.*

Proof. Let $u \in \mathcal{I} - c_0^{\sigma}(M, A, p)$. Then we have $\left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in \mathcal{I}$ for all n. If $\gamma = (\gamma_k)$ is a sequence of scalars such that $|\gamma_k| \leq 1$ for all $k \in \mathbb{N}$, then the following holds:

$$\left[M\left(\frac{|t_{kn}(A\gamma u)|}{\rho}\right)\right]^{p_k} \leq \left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k}.$$

Hence we obtain $\left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(A\gamma u)|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \subseteq \left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au)|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}$ and so $\left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(A\gamma u)|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in \mathcal{I}$ which means $\gamma u \in \mathcal{I} - c_0^{\sigma}(M, A, p)$. We conclude that the space $\mathcal{I} - c_0^{\sigma}(M, A, p)$ is solid.

□

Corollary 2.1. *The spaces $\mathcal{I} - c_0^{\sigma}(M, A, p)$ and $\mathcal{I} - \ell_{\infty}^{\sigma}(M, A, p)$ are monotone.*

Proof. The proof follows from Lemma 1.1. □

Theorem 2.8. *If $\lim_k p_k > 0$ and $u \to u_0(\mathcal{I} - c^{\sigma}(M, A, p))$, then u_0 is unique.*

Proof. Let $\lim_k p_k = p_0 > 0$. We assume that $u \to u_0(\mathcal{I} - c^\sigma(M, A, p))$ and $u \to v_0(\mathcal{I} - c^\sigma(M, A, p))$. Then there exist $\rho_1, \rho_2 > 0$ such that

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au - u_0e)|}{\rho_1}\right)\right]^{p_k} \geq \frac{\varepsilon}{2D}\right\} \in \mathcal{I}$$

and

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au - v_0e)|}{\rho_2}\right)\right]^{p_k} \geq \frac{\varepsilon}{2D}\right\} \in \mathcal{I}$$

for all $n \in \mathbb{N}$. Put $\rho = \max\{2\rho_1, 2\rho_2\}$. Then the inequality

$$\left[M\left(\frac{|u_0 - v_0|}{\rho}\right)\right]^{p_k} \leq D\left\{\left[M\left(\frac{|t_{kn}(Au - u_0e)|}{\rho_1}\right)\right]^{p_k} + \left[M\left(\frac{|t_{kn}(Au - v_0e)|}{\rho_2}\right)\right]^{p_k}\right\}$$

holds. Hence we have for all $n \in \mathbb{N}$

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{|u_0 - v_0|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \subseteq \left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au - u_0e)|}{\rho_1}\right)\right]^{p_k} \geq \frac{\varepsilon}{2D}\right\}$$
$$\cup \left\{k \in \mathbb{N} : \left[M\left(\frac{|t_{kn}(Au - v_0e)|}{\rho_2}\right)\right]^{p_k} \geq \frac{\varepsilon}{2D}\right\}.$$

By this inclusion, we obtain $\left\{k \in \mathbb{N} : \left[M\left(\frac{|u_0 - v_0|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in \mathcal{I}$ which means $\mathcal{I} - \lim \left[M\left(\frac{|u_0 - v_0|}{\rho}\right)\right]^{p_k} = 0$. Also we have

$$\left[M\left(\frac{|u_0 - v_0|}{\rho}\right)\right]^{p_k} \to \left[M\left(\frac{|u_0 - v_0|}{\rho}\right)\right]^{p_0}$$

as $k \to \infty$ since the limit of the sequence (p_k) is p_0 and so $\left[M\left(\frac{|u_0 - v_0|}{\rho}\right)\right]^{p_0} = 0$. This implies that $u_0 = v_0$.

□

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BLASCHKE APPROACH TO EULER-SAVARY FORMULAE FOR ONE PARAMETER DUAL HYPERBOLIC SPHERICAL MOTION

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Abstract. In this paper, we have introduced one parameter dual hyperbolic spherical motions in the dual Lorentzian space. This examination is given using Blaschke frame of axodes corresponding to the curves on the unit dual hyperbolic sphere. By considering Disteli axis on the Blaschke frame we have obtained Euler Savary formulae for one parameter dual hyperbolic spherical motions. At the end of this study, by obtaining orthogonal rotation matrices in the sense of dual Lorentzian type, we have found real and dual invariants of fixed and moving axodes.

1. Introduction

Line trajectories have an important place in the kinematic design and mechanism. In spatial motions, trajectories of directed lines connected in a moving rigid body are ruled surface. Differential geometry of ruled surfaces has been widely used in spatial mechanism, Computer Aided Geometric Design (CAGD), kinematic modeling of analytical tools of robot science and manufacturing of mechanical products. On dual geometry, many applications of ruled surfaces is studied by using transference principle or E. Study mapping. By this transfer, ruled surfaces can be represented by dual spherical curves lying on unit dual sphere of dual space. Then, a motion of a line in the 3-dimensional space can be studied by the motion of a unit dual vector of dual space and the properties of this motion can be obtained [2,3,4,12,19,20,30,32]. On the one parameter spatial motion, instantaneous screw axis ISA which a pair of ruled surface generates moving axode in the moving space and fixed axode in the fixed space. Kinematics and geometry of these axodes with corresponding to dual curves have investigated by some mathematician [2,3,4,14,19]. In the planar kinematics, there exists only one curvature circle and the position of point is given in the moving plane, then the radius and center of this circle can be determined by the famous Euler-Savary formulae. Euler-Savary formulae of a line trajectory were studied. This formula have introduced on the spherical kinematics [2, 3,14,30]. Furthermore, Lorentzian space kinematics is more different and more

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interesting than the Euclidean case. Differential geometry of curves and surfaces in the Lorentzian space are studied [1,13,17,21,23,26,27,28,29]. In this space, the spherical motions are studied according to the Lorentzian casual characters of the lines. Then, the spherical motion is called hyperbolic spherical motion if the motion is determinated by moving and fixed unit hyperbolic spheres and the spherical motion is called Lorentzian spherical motion if it is determinate by moving and fixed unit Lorentzian spheres [16,22]. Similar to the Euclidean case, by considering the E. Study mapping of timelike and spacelike lines, the motions of these lines are studied in dual Lorentzian space and the properties of these motions are obtained [25]. One parameter spherical motion have investigated at reel and dual Lorentzian spaces [5,8,16,24,25]. The purpose of this paper is to introduce one parameter dual hyperbolic spherical motions on the dual Lorentzian space. By considering Disteli axis on the Blaschke frame we have obtained Euler Savary formulae for one parameter dual hyperbolic spherical motions. At the end of this study, we have found real and dual invariants of fixed and moving axodes by using orthogonal rotation matrices in the sense of dual Lorentzian type 3×3.

2. Lorentz Space

Let R_1^3 be a 3-dimensional Minkowski space over the field of real numbers R with the Lorentzian inner product $\langle\, ,\rangle$ given by

$$\left\langle \vec{a}, \vec{b} \right\rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3,$$

where $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in R^3$. A vector $\vec{a} = (a_1, a_2, a_3)$ of IR_1^3 is said to be timelike if $\langle \vec{a}, \vec{a} \rangle < 0$, spacelike if $\langle \vec{a}, \vec{a} \rangle > 0$ or $\vec{a} = 0$, and lightlike (null) if $\langle \vec{a}, \vec{a} \rangle = 0$ and $\vec{a} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha}(s)$ in R_1^3 is spacelike, timelike or lightlike (null), if all of its velocity vectors $\vec{\alpha}'(s)$ are spacelike, timelike or lightlike (null), respectively [15]. The norm of a vector $\vec{a}$ is defined by $\|\vec{a}\| = \sqrt{|\langle \vec{a}, \vec{a} \rangle|}$. Now, let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be two vectors in IR_1^3. Then the Lorentzian cross product of $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2\,,\ a_1 b_3 - a_3 b_1,\ a_2 b_1 - a_1 b_2).$$

The sets of the unit timelike and spacelike vectors are called hyperbolic unit sphere and Lorentzian unit sphere and denoted by

$$H_0^2 = \left\{ \vec{a} = (a_1, a_2, a_3) \in R_1^3 : \ \langle \vec{a}, \vec{a} \rangle = -1 \right\},$$

and

$$S_1^2 = \left\{ \vec{a} = (a_1, a_2, a_3) \in R_1^3 : \ \langle \vec{a}, \vec{a} \rangle = 1 \right\},$$

respectively [28].

3. Dual Space

A dual number has the form $\bar{\lambda} = \lambda + \varepsilon \lambda^*$, where λ and λ^* are real numbers and ε is called dual unit which is subject to following rules:

$$\varepsilon \neq 0\,,\ \varepsilon^2 = 0,\ 0\varepsilon = \varepsilon 0 = 0,\ 1\varepsilon = \varepsilon 1 = \varepsilon\,.$$

We denote the set of all dual numbers by D:

$$\mathrm{D} = \left\{ \bar{\lambda} = \lambda + \varepsilon \lambda^* : \lambda,\ \lambda^* \in R,\ \varepsilon^2 = 0 \right\}.$$

Equality, addition and multiplication are defined in D by

(i) $\lambda + \varepsilon\lambda^* = \beta + \varepsilon\beta^*$ if and only if $\lambda = \beta$ and $\lambda^* = \beta^*$.

(ii)$(\lambda + \varepsilon\lambda^*) + (\beta + \varepsilon\beta^*) = (\lambda + \beta) + \varepsilon(\lambda^* + \beta^*)$.

(iii) $(\lambda + \varepsilon\lambda^*)(\beta + \varepsilon\beta^*) = (\lambda\beta) + \varepsilon(\lambda^*\beta + \beta^*\lambda)$.

respectively. Then it is easy to show that (D , + , .) is a commutative ring with unity [20].

The dual number $\bar{a} = a + \varepsilon a^*$ divide by dual number $\bar{b} = b + \varepsilon b^*$, with $b \neq 0$, is defined by

$$\frac{\bar{a}}{\bar{b}} = \frac{a}{b} + \varepsilon \frac{a^* b - a b^*}{b^2}.$$

Let f be a differentiable function with dual variable $\bar{x} = x + \varepsilon x^*$. Then the Maclaurin series generated by f is

$$f(\bar{x}) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x),$$

where $f'(x)$ is the derivative of f with respect to x.

Let D^3 be the set of all triples of dual numbers, i.e.

$$\mathrm{D}^3 = \{ \tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) \mid \bar{a}_i \in \mathrm{D},\ 1 \leq i \leq 3 \}.$$

The elements of D^3 are called dual vectors. A dual vector $\tilde{a}$ may be expressed in the form $\tilde{a} = \vec{a} + \varepsilon \vec{a}^*$, where $\vec{a}$ and $\vec{a}^*$ are the vectors of R^3. Let $\tilde{a} = \vec{a} + \varepsilon \vec{a}^*$, $\tilde{b} = \vec{b} + \varepsilon \vec{b}^* \in \mathrm{D}^3$ and $\bar{\lambda} = \lambda + \varepsilon \lambda^* \in \mathrm{D}$. Then we define

$$\begin{aligned} \tilde{a} + \tilde{b} &= \vec{a} + \vec{b} + \varepsilon (\vec{a}^* + \vec{b}^*), \\ \bar{\lambda} \tilde{a} &= \lambda \vec{a} + \varepsilon (\lambda \vec{a}^* + \lambda^* \vec{a}). \end{aligned}$$

By these operations, D^3 becomes a unitary module and it is called D-module or dual space (See [7,9]).

For any dual vectors $\tilde{a} = \vec{a} + \varepsilon \vec{a}^*$ and $\tilde{b} = \vec{b} + \varepsilon \vec{b}^*$ in D^3, scalar product and vector product are defined by

$$\left\langle \tilde{a}, \tilde{b} \right\rangle = \left\langle \vec{a}, \vec{b} \right\rangle + \varepsilon \left(\left\langle \vec{a}, \vec{b}^* \right\rangle + \left\langle \vec{a}^*, \vec{b} \right\rangle \right),$$

and

$$\tilde{a} \times \tilde{b} = \vec{a} \times \vec{b} + \varepsilon \left(\vec{a} \times \vec{b}^* + \vec{a}^* \times \vec{b} \right),$$

respectively, where $\left\langle \vec{a}, \vec{b} \right\rangle$ and $\vec{a} \times \vec{b}$ are inner product and vector product of the vectors $\vec{a}$ and $\vec{b}$ in R^3, respectively.

The norm of a dual vector $\tilde{a}$ is given by

$$\|\tilde{a}\| = \sqrt{\langle \tilde{a}, \tilde{a} \rangle} = \|\vec{a}\| + \varepsilon \frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|}, \vec{a} \neq \vec{0}.$$

Definition 3.1 (7,30)**.** The set of all unit dual vectors is called unit dual sphere, and is denoted by $\tilde{S}^2$ and this sphere is defined by

$$\tilde{S}^2 = \left\{ \tilde{a} \in \mathrm{D}^3 \middle| \quad ||\tilde{a}|| = (1, 0) \right\}.$$

Theorem 3.2. *(E. Study's Mapping): There exists a one-to-one correspondence between the points of unit dual sphere $\tilde{S}^2$ and the directed lines of the space R^3 [7].*

4. Dual Lorentzian Space

The Lorentzian inner product of two dual vectors $\tilde{a} = \vec{a} + \varepsilon\,\vec{a}^*$, $\tilde{b} = \vec{b} + \varepsilon\vec{b}^* \in \mathrm{D}^3$ is defined by

$$\left\langle \tilde{a},\ \tilde{b} \right\rangle = \left\langle \vec{a}\,,\, \vec{b} \right\rangle + \varepsilon\left(\left\langle \vec{a}\,,\, \vec{b}^* \right\rangle + \left\langle \vec{a}^*,\, \vec{b} \right\rangle \right),$$

where $\left\langle \vec{a}\,,\, \vec{b} \right\rangle$ is the Lorentzian inner product of the vectors $\vec{a}$ and $\vec{b}$ in the Minkowski 3-space R_1^3. Then, a dual vector $\tilde{a} = \vec{a} + \varepsilon\,\vec{a}^*$ is said to be dual timelike if $\vec{a}$ is timelike, dual spacelike if $\vec{a}$ is spacelike or $\vec{a} = 0$ and dual lightlike (null) if $\vec{a}$ is lightlike (null) and $\vec{a} \neq 0$ [25].

The set of all dual Lorentzian vectors is called dual Lorentzian space and it is denoted by

$$\mathrm{D}_1^3 = \left\{ \tilde{a} = \vec{a} + \varepsilon\,\vec{a}^* :\ \vec{a}\,,\, \vec{a}^* \in R_1^3 \right\}.$$

The Lorentzian cross product of dual vectors $\tilde{a}, \tilde{b} \in \mathrm{D}_1^3$ is defined by

$$\tilde{a} \times \tilde{b} = \vec{a} \times \vec{b} + \varepsilon\,(\vec{a}^* \times \vec{b} + \vec{a} \times \vec{b}^*),$$

where $\vec{a} \times \vec{b}$ is the Lorentzian cross product in R_1^3.

Let $\tilde{a} = \vec{a} + \varepsilon\,\vec{a}^* \in \mathrm{D}_1^3$. Then $\tilde{a}$ is said to be unit dual timelike (resp. spacelike) vector if the vectors $\vec{a}$ and $\vec{a}^*$ satisfy the following equations:

$$< \vec{a}, \vec{a} >= -1\ \ (resp.\ < \vec{a}, \vec{a} >= 1),\ \ < \vec{a}\,,\, \vec{a}^* > = 0.$$

The set of all unit dual timelike vectors is called dual hyperbolic unit sphere, and is denoted by $\tilde{H}_0^2$. Similarly, the set of all unit dual spacelike vectors is called dual Lorentzian unit sphere, and is denoted by $\tilde{S}_1^2$ and these spheres are defined by

$$\tilde{H}_0^2 = \left\{ \tilde{a} \in \mathrm{D}_1^3 :\ \langle \tilde{a}, \tilde{a} \rangle = -1 \right\},\quad \tilde{S}_1^2 = \left\{ \tilde{a} \in \mathrm{D}_1^3 :\ \langle \tilde{a}, \tilde{a} \rangle = 1 \right\}$$

respectively (See [21,25,28]).

Definition 4.1 (18,31)**.** ***(i)*** Dual hyperbolic angle: Let $\tilde{a}$ and $\tilde{b}$ be dual timelike vectors in D_1^3. Then the dual angle between $\tilde{a}$ and $\tilde{b}$ is defined by $< \tilde{a}, \tilde{b} >= -\|\tilde{a}\| \left\|\tilde{b}\right\| \cosh\bar{\theta}$. The dual number $\bar{\theta} = \theta + \varepsilon\theta^*$ is called the *dual hyperbolic angle.* The geometric interpretation of dual hyperbolic angle is that θ is the real hyperbolic angle between timelike lines $L_1,\ L_2$ corresponding to the dual timelike unit vectors $\tilde{a},\ \tilde{b}$, respectively, and θ^* is the shortest distance between those lines.

(ii) Dual central angle: Let $\tilde{a}$ and $\tilde{b}$ be dual spacelike vectors in D_1^3 that span a dual timelike vector subspace. The dual angle between $\tilde{a}$ and $\tilde{b}$ is defined by $\left|< \tilde{a}, \tilde{b} >\right| = \|\tilde{a}\| \left\|\tilde{b}\right\| \cosh\bar{\theta}$. The dual number $\bar{\theta} = \theta + \varepsilon\theta^*$ is called the *dual central angle.* The geometric interpretation of dual central angle is that θ is the real central angle between spacelike lines L_1, L_2 corresponding to the dual spacelike unit vectors $\tilde{a},\ \tilde{b}$ in D_1^3 that span a dual timelike vector subspace, respectively, and θ^* is the shortest distance between those lines.

(iii) Dual spacelike angle: Let $\tilde{a}$ and $\tilde{b}$ be dual spacelike vectors in D_1^3 that span a dual spacelike vector subspace. Then the angle between $\tilde{a}$ and $\tilde{b}$ is defined by $<\tilde{a}, \tilde{b}> = \|\tilde{a}\| \left\|\tilde{b}\right\| \cos\bar{\theta}$. The dual number $\bar{\theta} = \theta + \varepsilon\theta^*$ is called the *dual spacelike angle*. The geometric interpretation of dual spacelike angle is that θ is the real spacelike angle between spacelike lines $L_1,\ L_2$ corresponding to the dual spacelike unit vectors $\tilde{a},\ \tilde{b}$ in D_1^3 that span a dual spacelike vector subspace, respectively, and θ^* is the shortest distance between those lines.

(iv) Dual timelike angle: Let $\tilde{a}$ be a dual spacelike vector and $\tilde{b}$ be a dual timelike vector in D_1^3. Then the angle between $\tilde{a}$ and $\tilde{b}$ is defined by $\left|<\tilde{a}, \tilde{b}>\right| = \|\tilde{a}\| \left\|\tilde{b}\right\| \sinh\bar{\theta}$. The dual number $\bar{\theta} = \theta + \varepsilon\theta^*$ is called the *dual timelike angle*. The geometric interpretation of dual timelike angle is that θ is the real timelike angle between spacelike line L_1 and timelike line L_2 corresponding to the dual spacelike unit vector $\tilde{a}$ and timelike unit vector $\tilde{b}$, respectively, and θ^* is the shortest distance between those lines.

Theorem 4.2 (E. Study's Mapping for Lorentzian Space)**.** *: The dual timelike (respectively spacelike) unit vectors of the dual hyperbolic (respectively Lorentzian) unit sphere $\tilde{H}_0^2$ (respectively $\tilde{S}_1^2$) are in one-to-one correspondence with the directed timelike (respectively spacelike) lines of the Minkowski 3-space IR_1^3 [25].*

5. Differential Geometry of Dual Hyperbolic Spherical Curves

$\tilde{q} = \vec{q}(t) + \varepsilon\vec{q}^*(t)$ be a unit dual timelike vector is connected to a real parameter t , this vector draws a curve on the unit dual hyperbolic sphere $\tilde{H}_0^2$. Applying Study's map, this curve represents a timelike ruled surface M. If the ruling $\vec{q}$ is timelike, then the ruled surface M is said to be of type M_-^1 [11]. Therefore, differential geometry of dual hyperbolic spherical curves corresponds to differential geometry of timelike ruled surface M_-^1.

Let $d\bar{\theta} = d\theta + \varepsilon d\theta^*$ dual arc-length of dual hyperbolic spherical curve $\tilde{q} = \tilde{q}(t)$. Thus, we have

$$d\bar{\theta}^2 = \langle d\vec{q}, d\vec{q}\rangle + 2\varepsilon\,\langle d\vec{q}, d\vec{q}^*\rangle \tag{5.1}$$

Hence we obtain

$$d\theta^2 = \langle d\vec{q}, d\vec{q}\rangle\,,\ \ d\theta d\theta^* = \langle d\vec{q}, d\vec{q}^*\rangle\,. \tag{5.2}$$

Therefore, differential invariant of timelike ruled surface M_-^1 given by

$$\delta_q = \frac{d\theta^*}{d\theta} = \frac{\langle d\vec{q}, d\vec{q}^*\rangle}{\langle d\vec{q}, d\vec{q}\rangle} = \frac{\left\langle \vec{q}', \vec{q*}'\right\rangle}{\langle \vec{q}', \vec{q}'\rangle}. \tag{5.3}$$

The invariant δ_q is said to be distribution parameter (or drall) of the timelike ruled surface. If $\left\langle \vec{q'}, \vec{q'}\right\rangle = 0$, the ruled surface is said to be timelike cylindrical and we except this case [17,21].

We now give an orthonormal moving frame of a dual hyperbolic spherical curve as follows:

$$(5.4) \qquad \tilde{q} = \tilde{q}(t), \quad \tilde{h} = \frac{\tilde{q}'}{\|\tilde{q}'\|} \quad , \quad \tilde{a} = -\tilde{q} \times \tilde{h}.$$

This frame is called the Blaschke frame, and the corresponding lines intersect at the striction point of timelike ruled surface M_-^1. The set of the striction points constitute a curve $C = C(t)$ lying on the timelike ruled surface M_-^1 and is called striction curve. $\tilde{h}$ and $\tilde{a}$ are known as the central tangent and the central normal of the timelike ruled surface M_-^1. So, Blaschke formula is given by

$$(5.5) \qquad \begin{cases} \tilde{q}' = \bar{k}_1 \tilde{h}, \\ \tilde{h}' = \bar{k}_1 \tilde{q} + \bar{k}_2 \tilde{a}, \\ \tilde{a}' = -\bar{k}_2 \tilde{h} \end{cases} \qquad \begin{array}{l} \bar{k}_1 = \sqrt{\langle \tilde{q}', \tilde{q}' \rangle} \\ \bar{k}_2 = -\frac{(\tilde{q}, \tilde{q}', \tilde{q}'')}{\langle \tilde{q}', \tilde{q}' \rangle} \end{array}$$

and

$$(5.6) \qquad \frac{dC}{dt} = \cosh \bar{\phi} \tilde{q} + \sinh \bar{\phi} \tilde{a}$$

where $\bar{k}_1,\ \bar{k}_2$ are called the Blaschke's invariants. From (5.5) for dual vector $\tilde{\psi} = \vec{\psi} + \varepsilon \vec{\psi}^* = -\bar{k}_2 \tilde{q} - \bar{k}_1 \tilde{a}$ we can write

$$\tilde{q}' = \tilde{\psi} \times \tilde{q}, \quad \tilde{h}' = \tilde{\psi} \times \tilde{h}, \quad \tilde{a}' = \tilde{\psi} \times \tilde{a},$$

where dual vector $\tilde{\psi} = \vec{\psi} + \varepsilon \vec{\psi}^* = -\bar{k}_2 \tilde{q} - \bar{k}_1 \tilde{a}$ is called the dual instantaneous Pfaffian vector. The pole vector and the Steiner vector of the motion are given by

$$(5.7) \qquad \tilde{\psi} = \left\| \tilde{\psi} \right\| \tilde{P}\,, \quad \tilde{d} = \oint \tilde{\psi},$$

respectively [17,21].

6. One Parameter Dual Hyperbolic Spherical Motions

Let two coordinate systems $\left\{O'; \vec{q}_f, \vec{h}_f, \vec{a}_f\right\}$ and $\left\{O; \vec{q}_m, \vec{h}_m, \vec{a}_m\right\}$ be orthonormal coordinate systems which one represents fixed space L_2 and which one represents moving space L_3 in R_1^3 , respectively, where $\vec{q}_f$ and $\vec{q}_m$ are assumed as timelike vectors. In order to introduce the motion L_3/L_2 let take the coordinate system $\left\{Q; \vec{q}, \vec{h}, \vec{a}\right\}$ as an orthonormal relative system which represent the relative space L_1. Let Σ_1, Σ_2 and Σ_3 be unit dual hyperbolic spheres with same center O. According to the E. Study mapping, the points of unit dual hyperbolic spheres Σ_1, Σ_2 and Σ_3 can be represented by dual orthogonal systems $\left\{O; \tilde{q}, \tilde{h}, \tilde{a}\right\}$, $\left\{O; \tilde{q}_f, \tilde{h}_f, \tilde{a}_f\right\}$ and $\left\{O; \tilde{q}_m, \tilde{h}_m, \tilde{a}_m\right\}$, respectively. Therefore, the motions $L_1/L_2,\ L_1/L_3$ and L_3/L_2 can be considered as dual hyperbolic spherical motions $\Sigma_1/\Sigma_2,\ \Sigma_1/\Sigma_3$ and Σ_3/Σ_2, respectively.

Let A_f and A_m be a unit dual Lorentzian orthogonal matrices of type 3×3 and we can write

$$(6.1) \qquad \Sigma_1 = A_f \Sigma_2, \qquad \Sigma_1 = A_m \Sigma_3,$$

where

$$\Sigma_1 = \begin{bmatrix} \tilde{q} \\ \tilde{h} \\ \tilde{a} \end{bmatrix}, \ \Sigma_2 = \begin{bmatrix} \tilde{q}_f \\ \tilde{h}_f \\ \tilde{a}_f \end{bmatrix}, \ \Sigma_3 = \begin{bmatrix} \tilde{q}_m \\ \tilde{h}_m \\ \tilde{a}_m \end{bmatrix}$$

are dual column matrices. The elements of the matrices A_f and A_m are continuous and differentiable functions of dual parameter $\bar{t} = t + \varepsilon t^*$. In order to introduce one parameter hyperbolic motion we assume that $t^* = 0$.

Differential of the relative orthonormal coordinate frame Σ_1 with respect to unit dual fixed and moving hyperbolic spheres Σ_2 and Σ_3 are

$$d\Sigma_{1f} = dA_f \Sigma_2 = dA_f (A_f)^{-1} \Sigma_1, \quad d\Sigma_{1m} = dA_m \Sigma_3 = dA_m (A_m)^{-1} \Sigma_1. \tag{6.2}$$

By choosing $\tilde{\Omega}_f = dA_f (A_f)^{-1}$, $\quad \tilde{\Omega}_m = dA_m (A_m)^{-1}$ Eq. (6.2) can be rewritten as follows

$$d\Sigma_{1f} = \tilde{\Omega}_f \Sigma_1, \quad d\Sigma_{1m} = \tilde{\Omega}_m \Sigma_1 \tag{6.3}$$

where $\tilde{\Omega}_f$ and $\tilde{\Omega}_m$ matrices are anti-symmetric in the sense of Lorentzian.

During the one parameter dual hyperbolic motion Σ_3/Σ_2 the differential velocity vector of a fixed dual hyperbolic point $\tilde{X}_i = \vec{x}_i + \varepsilon \vec{x}_i^*$ $(1 \leq i \leq 3)$ on Σ_3 is

$$\frac{d\tilde{X}}{dt} = \tilde{\Omega} \times \tilde{X} \tag{6.4}$$

where $\tilde{\Omega} = \vec{\omega} + \varepsilon \vec{\omega}^*$ is called the instantaneous dual hyperbolic Pfaffian vector of the motion Σ_3/Σ_2. The Pfaffian dual vector $\tilde{\Omega}$ of the motion Σ_3/Σ_2, at the instant t, is like to the Darboux vector of space curves in the differential geometry. In this case ω and ω^* correspond to instantaneous rotational differential velocity vector and the instantaneous translational differential velocity vector of corresponding hyperbolic motion L_3/L_2, respectively. The dual number $\left\| \tilde{\Omega} \right\| = \bar{\Omega} = \omega + \varepsilon \omega^*$ is said to be dual angular speed of the one parameter dual hyperbolic motion Σ_3/Σ_2.

We consider the following identification

$$\bar{\Omega} = \begin{bmatrix} 0 & \bar{\Omega}_3 & -\bar{\Omega}_2 \\ \bar{\Omega}_3 & 0 & -\bar{\Omega}_1 \\ -\bar{\Omega}_2 & \bar{\Omega}_1 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \bar{\Omega}_1 \\ \bar{\Omega}_2 \\ \bar{\Omega}_3 \end{bmatrix} = \tilde{\Omega}. \tag{6.5}$$

Lemma 6.1. *For a one parameter dual hyperbolic spherical motion the following conditions are provided:*

(i) *The skew-symmetric in the sense of Lorentzian matrix of type* 3×3 *determined by* $\tilde{\Omega}_m(t) = A^{-1}A'$ *is called the moving polode.*

(ii) *The skew-symmetric in the sense of Lorentzian matrix of type* 3×3 *determined by* $\tilde{\Omega}_f(t) = A'A^{-1}$ *is called the fixed polode.*

(iii) *The moving and fixed polodes are related by* $\tilde{\Omega}_f(t) = adA(t)\tilde{\Omega}_m(t)$*, where* $adA\tilde{\Omega}_m = A\tilde{\Omega}_m A^{-1}$.

(iv) $\left\| \tilde{\Omega}_f \right\| = \left\| \tilde{\Omega}_m \right\|$.

(v) $\tilde{q}_f(t) = \frac{\tilde{\Omega}_f(t)}{\|\tilde{\Omega}_f(t)\|}$ *and* $\tilde{q}_m(t) = \frac{\tilde{\Omega}_m(t)}{\|\tilde{\Omega}_m(t)\|}$ *are called the fixed axode and moving axodes of the one parameter dual hyperbolic motion* Σ_3/Σ_2*, respectively.*

(vi) $\frac{d\tilde{q}_f}{dt} = adA\frac{d\tilde{q}_m}{dt} \Leftrightarrow \frac{d\tilde{q}_f}{dt} = A\frac{d\tilde{q}_m}{dt}A^{-1}$ *[5].*

During the dual hyperbolic motion Σ_3/Σ_2, the differentiable curve

$$t \in \mathrm{R} \rightarrow \tilde{q}_m(t) \in \Sigma_3 \tag{6.6}$$

states a differentiable family of straight lines on the moving axode. Now give an orthonormal moving frame along curve $\tilde{q}_m(t)$;

$$\tilde{q}_m = \tilde{q}_m(t)\,(timelike), \tilde{h}_m = \left(\frac{d\tilde{q}_m}{dt}\right)\left\|\frac{d\tilde{q}_m}{dt}\right\|^{-1}, \quad \tilde{a}_m = -\tilde{q}_m \times \tilde{h}_m. \tag{6.7}$$

This frame is called the Blaschke frame, and the corresponding lines intersect at the striction point of the axode $\tilde{q}_m = \tilde{q}_m(t)$. $\tilde{a}_m$ and $\tilde{h}_m$ are described as the central tangent and central normal of the timelike ruled surface $\tilde{q}_m = \tilde{q}_m(t)$, respectively. Let Σ_1^m be a dual unit hyperbolic sphere generated by the set $\left\{O; \tilde{q}_m, \tilde{h}_m, \tilde{a}_m\right\}$. Therefore, the motion Σ_1^m/Σ_3 is given by

$$\begin{bmatrix} d\tilde{q}_m \\ d\tilde{h}_m \\ d\tilde{a}_m \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_{1m} & 0 \\ \bar{k}_{1m} & 0 & \bar{k}_{2m} \\ 0 & -\bar{k}_{2m} & 0 \end{bmatrix} \begin{bmatrix} \tilde{q}_m \\ \tilde{h}_m \\ \tilde{a}_m \end{bmatrix} \tag{6.8}$$

where dual functions

$$\bar{k}_{1m} = k_{1m} + \varepsilon k_{1m}^* = \left\|\frac{d\tilde{q}_m}{dt}\right\|, \quad \bar{k}_{2m} = k_{2m} + \varepsilon k_{2m}^* = -\frac{\det\left(\tilde{q}_m, \frac{d\tilde{q}_m}{dt}, \frac{d^2\tilde{q}_m}{dt^2}\right)}{\bar{k}_{1m}^2} \tag{6.9}$$

are called Blaschke invarians of the moving axode. Striction curve is given by

$$\frac{dC^m}{dt} = \bar{k}_{2m}^*\tilde{q}_m + \bar{k}_{1m}^*\tilde{a}_m. \tag{6.10}$$

In this case dual functions in Eq. (6.9) abide by

$$\bar{k}_{1m} = k_{1m} + \varepsilon \sinh\bar{\sigma}_m, \quad \bar{k}_{2m} = k_{2m} + \varepsilon \cosh\bar{\sigma}_m \tag{6.11}$$

where $\bar{\sigma}_m$ is the striction angle measuring the derivation of the generating lines of $\tilde{q}_m(t)$ from the striciton curve. The distribution of timelike moving axode is

$$\lambda_m = \frac{k_{1m}^*}{k_{1m}} = \frac{\sinh\bar{\sigma}_m}{k_{1m}}. \tag{6.12}$$

During the one parameter dual hyperbolic motion Σ_3/Σ_2, the ISA on fixed hyperbolic sphere Σ_2 generates the fixed polode which accepts the Blaschke frame

$$\tilde{q}_f = \tilde{q}_f(t)(timelike), \; \tilde{h}_f = \left(\frac{d\tilde{q}_f(t)}{dt}\right)\left\|\frac{d\tilde{q}_f}{dt}\right\|^{-1}, \quad \tilde{a}_f = -\tilde{q}_f \times \tilde{h}_f. \tag{6.13}$$

Similarly, the set $\left\{O; \tilde{q}_f, \tilde{h}_f, \tilde{a}_f\right\}$ describes a unit dual hyperbolic sphere Σ_1^f, and the hyperbolic spherical motion Σ_1^f/Σ_2 is given by

$$\begin{bmatrix} d\tilde{q}_f \\ d\tilde{h}_f \\ d\tilde{a}_f \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_{1f} & 0 \\ \bar{k}_{1f} & 0 & \bar{k}_{2f} \\ 0 & -\bar{k}_{2f} & 0 \end{bmatrix} \begin{bmatrix} \tilde{q}_f \\ \tilde{h}_f \\ \tilde{a}_f \end{bmatrix} \tag{6.14}$$

where the dual functions

$$\bar{k}_{1f} = k_{1f} + \varepsilon k_{1f}^* = \left\| \frac{d\tilde{q}_f}{dt} \right\|, \quad \bar{k}_{2f} = k_{2f} + \varepsilon k_{2f}^* = -\frac{\det\left(\tilde{q}_f, \frac{d\tilde{q}_f}{dt}, \frac{d^2\tilde{q}_f}{dt^2}\right)}{\bar{k}_{1f}^2} \tag{6.15}$$

are the Blaschke invariants of fixed polode. Striction curve is given by

$$\frac{dC^f}{dt} = \bar{k}_{2f}^* \tilde{q}_f + \bar{k}_{1f}^* \tilde{a}_f. \tag{6.16}$$

Likewise the dual functions in (6.15) are

$$\bar{k}_{1f} = k_{1f} + \varepsilon \sinh \bar{\sigma}_f, \quad \bar{k}_{2f} = k_{2f} + \varepsilon \cosh \bar{\sigma}_f, \tag{6.17}$$

where $\bar{\sigma}_f$ is the striction angle between the lines of $\tilde{q}_f(t)$ and the striction curve. Therefore, the distribituon parameter of the fixed axode is

$$\lambda_f = \frac{k_{1f}^*}{k_{1f}} = \frac{\sinh \bar{\sigma}_f}{k_{1f}}. \tag{6.18}$$

Theorem 6.2. *Relations between Blaschke invariants of the timelike axodes are given by the equalities*

$$\bar{k}_{1m} = \bar{k}_{1f}, \quad \bar{k}_{2m} - \bar{k}_{2f} = \left\| \tilde{\Omega} \right\|. \tag{6.19}$$

Proof. Using (6.8) and (6.14) and Lemma (6.1) can be easily proved. □

Consequently, the following corollary can be given.

Corollary 6.3. During the one parameter hyperbolic spherical motion Σ_3/Σ_2, the moving polode is contact with the fixed polode along ISA in the first order at any instant t. The common distribution parameter of timelike axodes is

$$\lambda := \lambda_m = \lambda_f = \frac{k_1^*}{k_1}. \tag{6.20}$$

Let Σ_1 be unit dual hyperbolic sphere generated by the system $\left\{O; \tilde{q}(timelike), \tilde{h}, \tilde{a}\right\}$. In this system, $\tilde{a}(t) = a(t) + \varepsilon a^*(t)$ is the common perpendicular of $\tilde{q}(t)$ and $\tilde{q}(t + dt)$ and $\tilde{a}(t) = a(t) + \varepsilon a^*(t) = -\tilde{q} \times \tilde{h}$ and; $\tilde{q}, \tilde{h}$ and $\tilde{a}$ correspond to orthogonal lines in the Minkowski 3-space R_1^3. Then, the derivative equations of the one parameter dual hyperbolic spherical motions Σ_1/Σ_3 and Σ_1/Σ_2 are

$$\left.\frac{d\tilde{q}}{dt}\right|_m = C(M)\tilde{q}(t), \quad \tilde{q}(t) = \begin{bmatrix} \tilde{q} \\ \tilde{h} \\ \tilde{a} \end{bmatrix}, \quad C(M) = \begin{bmatrix} 0 & \bar{k}_1 & 0 \\ \bar{k}_1 & 0 & \bar{k}_{2m} \\ 0 & -\bar{k}_{2m} & 0 \end{bmatrix}, \tag{6.21}$$

and

$$\left.\frac{d\tilde{q}}{dt}\right|_f = C(F)\tilde{q}(t), \qquad \tilde{q}(t) = \begin{bmatrix} \tilde{q} \\ \tilde{h} \\ \tilde{a} \end{bmatrix}, \quad C(F) = \begin{bmatrix} 0 & \bar{k}_1 & 0 \\ \bar{k}_1 & 0 & \bar{k}_{2f} \\ 0 & -\bar{k}_{2f} & 0 \end{bmatrix}, \tag{6.22}$$

respectively,where

$$\bar{k}_1 = k_1 + \varepsilon k_1^*, \ \ \bar{k}_{2m} = k_{2m} + \varepsilon k_{2m}^*, \ \ \bar{k}_{2f} = k_{2f} + \varepsilon k_{2f}^* \tag{6.23}$$

are the Blaschke invariants of the one parameter dual hyperbolic spherical motion.

7. The approach to a timelike ruled surface with axodes

In this section, we introduce geometrical and kinematic meanings of dual invariants of hyperbolic polodes. In order to this analysis we consider a timelike point $\tilde{X}$ on the unit dual hyperbolic sphere such that its coordinates are

$$-\bar{X}_1^2 + \bar{X}_2^2 + \bar{X}_3^2 = -1, \quad \tilde{X} = X^T\tilde{q} \quad \tilde{X} = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \end{bmatrix}. \tag{7.1}$$

If $\tilde{X}$ is a function of t, the velocity of $\tilde{X}$ at the instant t with according to the moving unit dual hyperbolic sphere Σ_3 and fixed unit dual hyperbolic sphere Σ_2 are

$$\left.\frac{d\tilde{X}}{dt}\right|_m = \frac{d\tilde{X}^T}{dt}\tilde{q} + \tilde{X}^T \left.\frac{d\tilde{q}}{dt}\right|_m \tag{7.2}$$

and

$$\left.\frac{d\tilde{X}}{dt}\right|_f = \frac{d\tilde{X}^T}{dt}\tilde{q} + \tilde{X}^T \left.\frac{d\tilde{q}}{dt}\right|_f \tag{7.3}$$

respectively. From (6.21) and (6.22), we get

$$\left.\frac{d\tilde{X}}{dt}\right|_m = \left(\frac{d\tilde{X}^T}{dt} + \tilde{X}^T C(M)\right)\tilde{q} \tag{7.4}$$

and

$$\left.\frac{d\tilde{X}}{dt}\right|_f = \left(\frac{d\tilde{X}^T}{dt} + \tilde{X}^T C(F)\right)\tilde{q}\,. \tag{7.5}$$

If the line $\tilde{X}$ is fixed relative to the moving unit dual hyperbolic sphere, then the derivative $\left.\frac{d\tilde{X}}{dt}\right|_m = 0$. That is we have

$$\frac{d\tilde{X}^T}{dt} = -\tilde{X}^T C(M). \tag{7.6}$$

Now, assume that $\tilde{X}$ is fixed according to the moving unit dual hyperbolic sphere Σ_3 and let us compute its velocity according to the fixed unit dual hyperbolic sphere Σ_2. Then we obtain equation

$$\frac{d\tilde{X}}{dt} = \tilde{X}^T(C(F) - C(M))\tilde{q}. \tag{7.7}$$

Let us define a matrix $C(R)$ by

$$C(R) = C(F) - C(M). \tag{7.8}$$

Then (7.7) can be rewritten as

$$\frac{d\tilde{X}}{dt} = \tilde{X}^T(C(R))\tilde{q}. \tag{7.9}$$

We have an axial dual vector $\tilde{D}_r = d + \varepsilon d^*$ such that

$$C(R)\tilde{X} = \tilde{D}_r \times \tilde{X}. \tag{7.10}$$

Therefore (7.9) can be stated as

$$\frac{d\tilde{X}}{dt} = \tilde{D}_r \times \tilde{X}, \quad \tilde{D}_r = \tilde{D}_f - \tilde{D}_m = -\bar{\Omega}\tilde{q}, \tag{7.11}$$

where $\left\|\tilde{\Omega}\right\| = \bar{\Omega} = \omega + \varepsilon\omega^*$. Then from Theorem 6.2 and (7.11) we have

$$\frac{d\tilde{X}}{dt} = (-\bar{X}_3\bar{\Omega})\tilde{h} + (\bar{X}_2\bar{\Omega})\tilde{a}. \tag{7.12}$$

From (7.11) and (7.12), it follows that the acceleration of $\tilde{X}$ is given by

$$\frac{d^2\tilde{X}}{dt^2} = (-\bar{\Omega}\bar{k}_1\bar{X}_3)\tilde{q} + (-\bar{\Omega}'\bar{X}_3 - \bar{\Omega}^2\bar{X}_2)\tilde{h} + (-\bar{\Omega}\bar{k}_1\bar{X}_1 + \bar{\Omega}'\bar{X}_2 - \bar{\Omega}^2\bar{X}_3)\tilde{a}. \tag{7.13}$$

8. Line complex during one parameter hyperbolic spherical motion

In this section, we investigate timelike ruled surface generated by the timelike line $\tilde{X}$. Now we describe a frame moving along the curve $\tilde{X}(t)$ on the unit hyberbolic sphere Σ_2. According to transference principle, this curve corresponds to a timelike ruled surface in the fixed Lorentzian space L_2. The Blaschke frame along $\tilde{X}(t)$ is defined as follows:

$$\tilde{E}_1 = \tilde{X} = \bar{X}_1\tilde{q} + \bar{X}_2\tilde{h} + \bar{X}_3\tilde{a}, (time) \tag{8.1}$$

$$\tilde{E}_2 = \frac{\tilde{X}'}{\left\|\tilde{X}'\right\|} = \frac{-\bar{X}_3\tilde{h} + \bar{X}_2\tilde{a}}{\sqrt{\bar{X}_2^2 + \bar{X}_3^2}} \tag{8.2}$$

$$\tilde{E}_3 = -(\tilde{E}_1 \times \tilde{E}_2) = -\left(\frac{(1 + \bar{X}_1^2)\tilde{q} + \bar{X}_1\bar{X}_2\tilde{h} + \bar{X}_1\bar{X}_3\tilde{a}}{\sqrt{\bar{X}_2^2 + \bar{X}_3^2}}\right). \tag{8.3}$$

The unit dual timelike vector $\tilde{E}_1$ is one-to-one correspondence with the directed timelike line of the Minkowski 3-space IR_1^3 and dual spacelike unit vectors $\tilde{E}_2$, $\tilde{E}_3$ are one-to-one correspondence with the directed spacelike lines of the Minkowski 3-space. The Blaschke derivative formulas are

$$\frac{d}{dt}\begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \\ \tilde{E}_3 \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_{1x} & 0 \\ \bar{k}_{1x} & 0 & \bar{k}_{2x} \\ 0 & -\bar{k}_{2x} & 0 \end{bmatrix} \begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \\ \tilde{E}_3 \end{bmatrix} \tag{8.4}$$

where

$$\begin{aligned} \bar{k}_{1x} &= k_{1x} + \varepsilon k_{1x}^* = \left\| \tfrac{d\tilde{X}}{dt} \right\| = \bar{\Omega}\sqrt{\bar{X}_2^2 + \bar{X}_3^2}, \\ \bar{k}_{2x} &= k_{2x} + \varepsilon k_{2x}^* = -\tfrac{\det(\tilde{X}, \tilde{X}', \tilde{X}'')}{(\bar{k}_{1x})^2} = -(\bar{\Omega}\bar{X}_1 + \tfrac{\bar{k}_{1x}\bar{X}_3}{\bar{X}_2^2 + \bar{X}_3^2}) \end{aligned} \tag{8.5}$$

are Blaschke invariants of the timelike curve $\tilde{X}(t)$.

Theorem 8.1. *During the one parameter dual hyperbolic spherical motion Σ_3/Σ_2, consider a set of lines are contact with the timelike moving axode and these lines are generators of timelike ruled surfaces having the same distribution parameter in the fixed Lorentzian space L_2. Therefore this set of lines belongs to a quadratic line complex.*

Proof. The distribution parameter of the timelike ruled surface generated by the line $\tilde{X}$ from (8.5) can be stated by

$$\lambda_x = \frac{\bar{k}_{1x}^*}{\bar{k}_{1x}} = \frac{x_2 x_2^* + x_3 x_3^* + h(x_2^2 + x_3^2)}{(x_2^2 + x_3^2)}. \tag{8.6}$$

This equation can be applied to determine those lines of timelike moving axode that trace timelike ruled surfaces having the same distribution parameter. This set of timelike lines is called a line complex and is stated by the equation

$$x_2 x_2^* + x_3 x_3^* + (h - \lambda_x)(x_2^2 + x_3^2) = 0. \tag{8.7}$$

This equation shows a quadratic line complex. □

Now let $p(x, y, z)$ be the position vector of an arbitrary point on the timelike line $\tilde{X}$. In order to introduce (8.7) If we use Lorentzian cross product then,

$$x^* = p \times x$$

$$(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} \vec{e}_1 & -\vec{e}_2 & -\vec{e}_3 \\ x & y & z \\ x_1 & x_2 & x_3 \end{bmatrix} = (yx_3 - zx_2, xx_3 - zx_1, yx_1 - xx_2). \tag{8.8}$$

After that, substituting (8.8) into (8.7) we have

$$x_1 x_3 y - x_1 x_2 z + (h - \lambda_x)(x_2^2 + x_3^2) = 0. \tag{8.9}$$

This equation represent that the timelike lines $\tilde{X}$ of timelike moving axode that trace timelike ruled surfaces with the same distribution parameter lie on a plane parallel to the ISA of the one parameter Lorentzian spatial motion L_3/L_2.

From (8.9), we have two different cases: In the case of $\lambda_x = h$ the distribution parameter is associated with the lines in planes passing through the ISA. In the case of $\lambda_x = 0$, the timelike line $\tilde{X}$ of the timelike moving axode, generate a developable timelike ruled surface, (8.9) reduces to

$$x_1 x_3 y - x_1 x_2 z + h(x_2^2 + x_3^2) = 0. \tag{8.10}$$

Now, kinematic investigation of Blaschke frame is given by using Blaschke invariants $\bar{k}_{1x} = k_{1x} + \varepsilon k_{1x}^*$ and $\bar{k}_{2x} = k_{2x} + \varepsilon k_{2x}^*$. To realize this, we define dual vector

$$\tilde{D}_x = -\bar{k}_{2x}\tilde{E}_1 - \bar{k}_{1x}\tilde{E}_3 \tag{8.11}$$

known as Darboux's vector. $\left\|\tilde{D}\right\| = \sqrt{\bar{k}_{1x}^2 - \bar{k}_{2x}^2} = \omega_x + \varepsilon\omega_x^*$ is the angular speed of timelike line $\tilde{E}_1$ about the Darboux vector.

$$\omega_x = \sqrt{|k_{1x}^2 - k_{2x}^2|}\,, \;\; \omega_x^* = \frac{k_{1x}k_{1x}^* - k_{2x}k_{2x}^*}{\sqrt{|k_{1x}^2 - k_{2x}^2|}} \tag{8.12}$$

are the rotational angular speed and translational angular speed of timelike line $\tilde{E}_1$, respectively. The pitch of $\tilde{E}_1$ along the Darboux vector is

$$h_x = \frac{\omega_x^*}{\omega_x} = \frac{k_{1x}k_{1x}^* - k_{2x}k_{2x}^*}{k_{1x}^2 - k_{2x}^2}. \tag{8.13}$$

Disteli axis is axis of hyperbolic motion of the timelike line $\tilde{E}_1$ and it's defined by

$$\tilde{U} = \frac{\tilde{D}_x}{\left\|\tilde{D}_x\right\|} = \frac{-\bar{k}_{2x}\tilde{E}_1 - \bar{k}_{1x}\tilde{E}_3}{\sqrt{\bar{k}_{1x}^2 - \bar{k}_{2x}^2}} \quad . \tag{8.14}$$

From (8.14), the Disteli axis is parallel to tangent plane of timelike ruled surface $\tilde{X} = \tilde{X}(t)$, and is unit dual timelike vector. Then the ISA of one parameter hyperbolic spherical motion Σ_3/Σ_2 and the Disteli axis lie on a single great dual hyperbolic circle determined by the intersection of $\tilde{E}_1\tilde{E}_3$-plane and the unit dual hyperbolic sphere Σ_2. Now let $\Delta = \delta + \varepsilon\delta^*$ be the dual hyperbolic angle between the Disteli axis and the timelike line $\tilde{X}$; then we have

$$\tilde{U} = -\cosh\Delta\,\tilde{E}_1 - \sinh\Delta\,\tilde{E}_3, \tag{8.15}$$

where $\Delta = \delta + \varepsilon\delta^*$ is dual hyperbolic spherical radius of curvature. For differential of (8.15) we have

$$\tilde{U}' = (-sinh\Delta\,\tilde{E}_1 - \cosh\Delta\,\tilde{E}_3)\Delta' + (\bar{k}_{2x}\sinh\Delta - \bar{k}_{1x}\cosh\Delta)\tilde{E}_2 \tag{8.16}$$

and

$$\coth\Delta = \frac{\bar{k}_{2x}}{\bar{k}_{1x}}. \tag{8.17}$$

This equation shows that the relationship between the dual hyperbolic spherical curvature $\bar{\rho}$ and the dual hyperbolic spherical radius of curvature is

$$\bar{\rho} = \rho + \varepsilon\rho^* = \coth\Delta. \tag{8.18}$$

9. During one parameter hyperbolic spherical motion line trajectories and Euler Savary formulae

In this section, by using dual hyperbolic angle we give a different method for deriving a new Euler-Savary formula of Lorentzian spatial kinematics. This means that we investigate an oriented timelike line in the moving Lorentzian space L_3 with a fixed hyperbolic angle with respect to a given timelike line in the fixed Lorentzian space L_2.

Theorem 9.1. *Let Σ_3/Σ_2 be the one parameter dual hyperbolic motion. In this case, the relation between the spherical radii of curvature of the pole curves is given by*

$$(\coth\bar{\theta}_c - \coth\bar{\theta})\sin\bar{\phi} = \bar{\rho} = \frac{\bar{\Omega}}{\bar{k}_1} = \coth\bar{\gamma}_f - \coth\bar{\gamma}_m, \tag{9.1}$$

where $\bar{\gamma}_f$ and $\bar{\gamma}_m$ are the dual hyperbolic spherical curvatures, $\bar{\Omega}$ is the dual screw velocity and $\bar{k}_{1m} = \bar{k}_{1f}$ are dual invariants.

Proof. For instantaneous fixed timelike line $\tilde{X}$ of the hyperbolic motion Σ_3/Σ_2, we present the dual hyperbolic angle $\bar{\theta} = \theta + \varepsilon\theta^*$ and dual spacelike angle $\bar{\phi} = \phi + \varepsilon\phi^*$ to determine the direction of timelike line $\tilde{X}$. Because $\tilde{X}$ is a unit dual timelike vector, we can give the components of $\tilde{X}$ in the following form:

$$\tilde{X} = \cosh\bar{\theta}\tilde{q} + \sinh\bar{\theta}\tilde{L}, \quad \tilde{L} = \cos\bar{\phi}\tilde{h} + \sin\bar{\phi}\tilde{a}. \tag{9.2}$$

The dual hyperbolic angle $\bar{\theta} = \theta + \varepsilon\theta^*$ describes the position of timelike line $\tilde{X}$ relative to the ISA of the one parameter dual hyperbolic spherical motion Σ_3/Σ_2.

A similar set of coordinates may be used to determine the timelike Disteli axis $\tilde{U}$ of the timelike ruled surface $\tilde{X} = \tilde{X}(t)$. Since central normal $\tilde{E}_2$ is also normal to the timelike Disteli axis, it is determined by the same dual central angle $\bar{\varphi}$ about the ISA of the hyperbolic motion Σ_3/Σ_2. Describing its dual hyperbolic angle with the ISA by $\bar{\theta}_c = \theta_c + \varepsilon\theta_c^*$, we can write

$$\tilde{U} = \cosh\bar{\theta}_c\,\tilde{q} + \sinh\bar{\theta}_c\cos\bar{\varphi}\,\tilde{h} + \sinh\bar{\theta}_c\sin\bar{\varphi}\,\tilde{a}. \tag{9.3}$$

From (9.2) and (9.3) we have

$$\left\langle \tilde{X}, \tilde{U} \right\rangle = -\cosh(\bar{\theta}_c - \bar{\theta}). \tag{9.4}$$

This equation describes a hyperbolic circle on the dual hyperbolic unit sphere Σ_2 where $(\bar{\theta}_c - \bar{\theta})$ a given dual hyperbolic spherical radius is and $\tilde{U}$ is a fixed dual

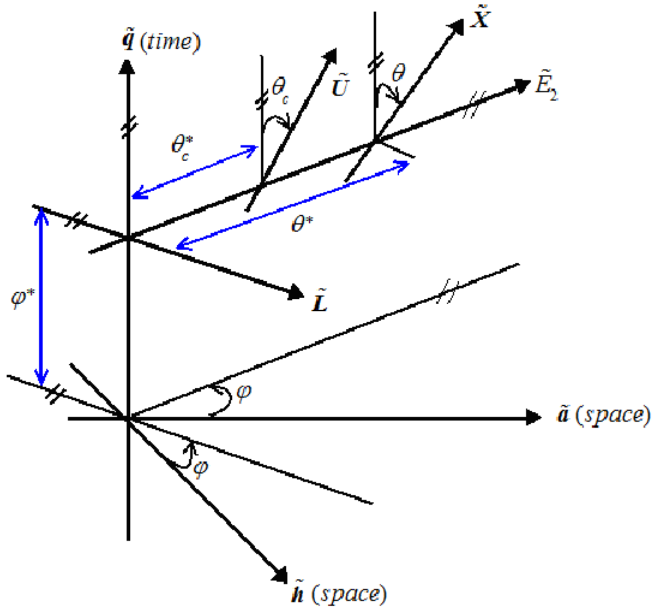

FIGURE 1. The moved timelike line $\tilde{X}$ and its timelike Disteli axis $\tilde{U}$

unit timelike vector which identifies the hyperbolic circle's center. According to E. Study's map (9.4) defines the set of all oriented timelike lines $\tilde{X}$. Like this a set of timelike lines depends on two parameters and is called linear timelike line congruence. Since osculating hyperbolic circle should have contact of at least second order with the curve, timelike Disteli axis $\tilde{U}$ and $(\bar{\theta}_c - \bar{\theta})$ remain constant up to second order at $t = t_0$, that is

$$\left.\frac{d(\bar{\theta}_c - \bar{\theta})}{dt}\right|_{t=t_0} = 0, \qquad \left.\frac{d\tilde{U}}{dt}\right|_{t=t_0} = 0 \tag{9.5}$$

and

$$\left.\frac{d^2(\bar{\theta}_c - \bar{\theta})}{dt^2}\right|_{t=t_0} = 0, \qquad \left.\frac{d^2\tilde{U}}{dt^2}\right|_{t=t_0} = 0. \tag{9.6}$$

From differentiation of (9.4) and equation (9.5) we have

$$\left\langle \frac{d\tilde{X}}{dt}, \tilde{U} \right\rangle = 0. \tag{9.7}$$

We have second order

$$\left\langle \frac{d^2\tilde{X}}{dt^2}, \tilde{U} \right\rangle = 0. \tag{9.8}$$

We substitute from (7.13) and (9.3) into (9.8) and obtain:

$$(\coth\bar{\theta}_c - \coth\bar{\theta})\sin\bar{\phi} = \frac{\bar{\Omega}}{\bar{k}_1}. \tag{9.9}$$

This equation is dual hyperbolic Euler-Savary equation of one parameter dual hyperbolic spherical motion Σ_3/Σ_2 [24]. By using (8.18) we can rewrite Euler-Savary equation the form as desired

$$(\coth\bar{\theta}_c - \coth\bar{\theta})\sin\bar{\phi} = \bar{\rho}. \tag{9.10}$$

If this equation separate real and dual part then we have

$$(\coth\theta_c - \coth\theta)\sin\phi = \rho \tag{9.11}$$

and

$$(\coth\theta_c - \coth\theta)\phi^*\cos\phi - \left(\frac{\theta_c^*}{\sinh^2\theta_c} - \frac{\theta^*}{\sinh^2\theta}\right)\sin\phi = \rho^*. \tag{9.12}$$

Lorentzian Euler-Savary Eq. (9.11) together with (9.12) is called Disteli formulae of axode of dual hyperbolic spherical motion. (9.11) is Euler-Savary equation for axode of real hyperbolic spherical motion in the Lorentzian space. In order to Eq. (9.12) simplified to reduce by using (9.11) we have

$$\rho\phi^*\cot\phi - \left(\frac{\theta_c^*}{\sinh^2\theta_c} - \frac{\theta^*}{\sinh^2\theta}\right)\sin\phi = \rho^*. \tag{9.13}$$

□

10. Example

In this section we display the use of dual Lorentzian vectors for denoting the ISA of the one parameter dual hyperbolic spherical motion Σ_3/Σ_2. The one parameter dual hyperbolic spherical motion Σ_3/Σ_2 can be denoted analytically by the matrix equation

$$\tilde{x}_f(t) = A(t)\tilde{x}_m(t) + \tilde{m}_f(t) \quad , \quad \tilde{x}_m(t) = A^{-1}(t)\tilde{x}_f(t) + \tilde{m}_m(t) \tag{10.1}$$

where $\tilde{x}_f$, $\tilde{x}_m$ are vectors of a same point, with respect to the orthonormal frames of the moving space and fixed space, respectively, and $\tilde{m}_f$, $\tilde{m}_m$ and A are differentiable functions of a dual parameter $\bar{t} = t + \varepsilon t^*$, since we study one parameter hyperbolic spherical motion we consider the case $t^* = 0$. Also we know that

$$\tilde{m}_f = -A\,\tilde{m}_m \quad , \quad \tilde{m}_m = -A^{-1}\,\tilde{m}_f \tag{10.2}$$

where A and A^{-1} matrices are anti-symmetric in the sense of Lorentzian.

The velocity of a fixed point $\tilde{x}_m \in \Sigma_3$ is

$$\tilde{x}'_f = A'\tilde{x}_m + \tilde{m}'_f\,. \tag{10.3}$$

From (10.1) we get

$$\tilde{x}'_f = A'A^{-1}\tilde{x}_f + (\tilde{m}'_f - A'A^{-1}\tilde{m}_f). \tag{10.4}$$

If we consider matrix $\omega = A'A^{-1}$ is anti-symmetric in the sense of Lorentzian, then Eq. (10.4) can be rewritten in the form

$$\tilde{x}'_f = \omega\, \tilde{x}_f + (\tilde{m}'_f - \omega\, \tilde{m}_f). \tag{10.5}$$

As a consequence of this equation, there is a dual vector

$$\tilde{\Omega}(t) = \omega(t) + \varepsilon\, \omega^*(t) \tag{10.6}$$

such that

$$\omega x_f = \omega \times x_f; \quad \omega^* = (m' - \omega \times m). \tag{10.7}$$

Now we give a simple example using by above statement. First we consider the one parameter dual hyperbolic spherical motion Σ_3/Σ_2 denoting by the dual Lorentzian orthogonal matrix

$$A = R_1 \cdot R_2 = \begin{pmatrix} \cosh^2\phi & -\sinh\phi & -\cosh\phi\sinh\phi \\ -\sinh\phi\cosh\phi & \cosh\phi & \sinh^2\phi \\ -\sinh\phi & 0 & \cosh\phi \end{pmatrix} \tag{10.8}$$

such that

$$R_1 = \begin{pmatrix} \cosh\bar{\theta} & -\sinh\bar{\theta} & 0 \\ -\sinh\bar{\theta} & \cosh\bar{\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} \cosh\bar{\phi} & 0 & -\sinh\bar{\phi} \\ 0 & 1 & 0 \\ -\sinh\bar{\phi} & 0 & \cosh\bar{\phi} \end{pmatrix} \tag{10.9}$$

where we assume that $\bar{\theta} = \bar{\phi}$, $\theta^* = \phi^* = 0$. Also we consider an anti-symmetric in the sense of Lorentzian matrix

$$m(\phi) = \begin{pmatrix} 0 & 0 & \mu\sinh\phi \\ 0 & 0 & -\mu\cosh\phi \\ \mu\sinh\phi & \mu\cosh\phi & 0 \end{pmatrix}, \tag{10.10}$$

where we assume that $\mu > 1$. Since $\tilde{q},\ \tilde{q}_m,\ \tilde{q}_f$ are timelike vectors we can write

$$m(\phi) = \begin{pmatrix} \mu\cosh\phi \\ \mu\sinh\phi \\ 0 \end{pmatrix}. \tag{10.11}$$

If we substitute the (10.8) and (10.10) in (10.7), we have

$$\omega(\phi) = \begin{pmatrix} -\sinh\phi \\ -\cosh\phi \\ 1 \end{pmatrix}, \quad \omega^*(\phi) = \begin{pmatrix} 2\mu\sinh\phi \\ 2\mu\cosh\phi \\ \mu \end{pmatrix}. \tag{10.12}$$

Therefore the dual hyperbolic Pfaffian dual vector $\tilde{\Omega}$ at the instant ϕ of the one parameter dual hyperbolic spherical motion Σ_3/Σ_2 is

$$\tilde{\Omega}(\phi) = \omega(\phi) + \varepsilon\omega^*(\phi) = \begin{pmatrix} -\sinh\phi + 2\varepsilon\mu\sinh\phi \\ -\cosh\phi + 2\varepsilon\mu\cosh\phi \\ 1+\varepsilon\mu \end{pmatrix}. \tag{10.13}$$

Fixed axode is given by

$$\tilde{q}_f(\phi) = \frac{\tilde{\Omega}}{\left\|\tilde{\Omega}\right\|} = \frac{1}{\sqrt{2-2\varepsilon\mu}} \begin{pmatrix} -\sinh\phi + 2\varepsilon\mu\sinh\phi \\ -\cosh\phi + 2\varepsilon\mu\cosh\phi \\ 1+\varepsilon\mu \end{pmatrix}. \tag{10.14}$$

Moving polode on Σ_3 is denoted by

$$\Omega_m = \frac{dM^{-1}}{d\phi}\cdot M;\ M = (A + \varepsilon m A) \tag{10.15}$$

where

$$M = \begin{pmatrix} \cosh^2\phi + \varepsilon\mu(-\sinh^2\phi) & -\sinh\phi & -\sinh\phi\cosh\phi + \varepsilon\mu(\sinh\phi\cosh\phi) \\ -\sinh\phi\cosh\phi + \varepsilon\mu(\sinh\phi\cosh\phi) & \cosh\phi & \sinh^2\phi - \varepsilon\mu(\cosh^2\phi) \\ -\sinh\phi & \varepsilon\mu & \cosh\phi \end{pmatrix}.$$

Therefore the moving axode is given by

$$\tilde{q}_m(\phi) = \frac{\tilde{\Omega}_m}{\left\|\tilde{\Omega}_m\right\|} = \frac{1}{\sqrt{2-2\varepsilon\mu}} \begin{pmatrix} \sinh\phi \\ 1-\varepsilon\mu \\ -\cosh\phi \end{pmatrix}. \tag{10.16}$$

Now we introduce the Blaschke invariants of the fixed axode $\tilde{q} = \tilde{q}_f(\phi)$. For the one parameter hyperbolic spherical motion Σ_3/Σ_2, from (10.14), we can give

$$\tilde{\Omega}_f(\phi) = \bar{\Omega}\,\tilde{q}(\phi);\quad \bar{\Omega} = \sqrt{2-2\varepsilon\mu}. \tag{10.17}$$

For differential of (10.17) with respect to ϕ, we have

$$\frac{d\tilde{\Omega}_f}{d\phi} = \tilde{\Omega}'_f = \bar{\Omega}'\tilde{q} + \bar{k}_1\bar{\Omega}\tilde{h} \tag{10.18}$$

and by writing the (6.22) in the differentiation of (10.18)we obtain

$$\tilde{\Omega}''_f = (\bar{\Omega}'' + \bar{k}_1^2\bar{\Omega})\tilde{q} + (2\bar{k}_1\bar{\Omega}' + \bar{k}'_1\bar{\Omega})\tilde{h} + (\bar{k}_1\bar{\Omega}\bar{k}_{2f})\tilde{a}. \tag{10.19}$$

Further, if we consider Lorentzian vectorial product of (10.18) and (10.19) we find

$$\tilde{\Omega}_f(\phi)\times\tilde{\Omega}'_f(\phi) = -\bar{k}_1\bar{\Omega}^2\tilde{a}. \tag{10.20}$$

And then by using following Lorentzian property

$$\left\|\tilde{\Omega}_f(\phi)\times\tilde{\Omega}'_f(\phi)\right\| = -\left\langle\tilde{\Omega}_f(\phi),\tilde{\Omega}_f(\phi)\right\rangle\left\langle\tilde{\Omega}'_f(\phi),\tilde{\Omega}'_f(\phi)\right\rangle + \left(\left\langle\tilde{\Omega}_f(\phi),\tilde{\Omega}'_f(\phi)\right\rangle\right)^2 \tag{10.21}$$

we find that

$$-\left\langle \bar{\Omega}\tilde{q}, \bar{\Omega}\tilde{q} \right\rangle \left\langle \bar{\Omega}'\tilde{q} + \bar{k}_1\bar{\Omega}\tilde{h}\,,\, \bar{\Omega}'\tilde{q} + \bar{k}_1\bar{\Omega}\tilde{h} \right\rangle + \left(\left\langle \bar{\Omega}\tilde{q}, \bar{\Omega}'\tilde{q} + \bar{k}_1\bar{\Omega}\tilde{h} \right\rangle \right)^2 = \bar{k}_1^2\bar{\Omega}^4. \tag{10.22}$$

Finally, we have

$$\det(\tilde{\Omega}_f, \tilde{\Omega}'_f, \tilde{\Omega}''_f) = \bar{k}_1^2\bar{\Omega}^3\bar{k}_{2f}. \tag{10.23}$$

From (10.13) we can give

$$\tilde{\Omega}'_f(\phi) = \begin{pmatrix} -\cosh\phi + 2\varepsilon\mu\cosh\phi \\ -\sinh\phi + 2\varepsilon\mu\sinh\phi \\ 0 \end{pmatrix} \tag{10.24}$$

and

$$\tilde{\Omega}''_f(\phi) = \begin{pmatrix} -\sinh\phi + 2\varepsilon\mu\sinh\phi \\ -\cosh\phi + 2\varepsilon\mu\cosh\phi \\ 0 \end{pmatrix}. \tag{10.25}$$

From (10.13) and (10.14) we obtain

$$\left\langle \tilde{\Omega}_f(\phi), \tilde{\Omega}'_f(\phi) \right\rangle = 0 \tag{10.26}$$

and so

$$\left(\left\langle \tilde{\Omega}_f(\phi), \tilde{\Omega}'_f(\phi) \right\rangle \right)^2 = 0. \tag{10.27}$$

Besides, we have

$$\left\langle \tilde{\Omega}'_f(\phi), \tilde{\Omega}'_f(\phi) \right\rangle = -1 + 4\varepsilon\mu. \tag{10.28}$$

Substituting the (10.13), (10.27) and (10.28) in (10.22), we find

$$-(2 - 2\varepsilon\mu)(-1 + 4\varepsilon\mu) = \bar{k}_1^2\bar{\Omega}^4. \tag{10.29}$$

If we separate the real and dual parts the (10.29), we have

$$k_1 = \pm\frac{1}{\sqrt{2}}, \;\; k_1^* = -\frac{3\sqrt{2}\mu}{4}. \tag{10.30}$$

By using (6.20) we find that the common distribution parameter of the axodes is given by

$$\lambda = \frac{3\mu}{2}. \tag{10.31}$$

From (10.13), (10.23), (10.24) and (10.25), we find that

$$\det(\tilde{\Omega}_f, \tilde{\Omega}'_f, \tilde{\Omega}''_f) = 1 - 3\varepsilon\mu = \bar{k}_1^2\bar{\Omega}^3\bar{k}_{2f}. \tag{10.32}$$

If we separate that the real and dual parts of above equations, we have

$$k_{2f} = \frac{\sqrt{2}}{2}, \quad k_{2f}^{*} = \mu\frac{3\sqrt{2}}{4}. \tag{10.33}$$

By means of (6.19) and (10.33) we get

$$k_{2m} = \frac{3\sqrt{2}}{2}, \; k_{2m}^{*} = \mu\frac{\sqrt{2}}{4}. \tag{10.34}$$

Therefore we obtain real and dual parts of the integral invariants of the axodes.

11. Conclusion

In this paper, we have introduced one parameter dual hyperbolic spherical motions in the dual Lorentzian space. By considering Disteli axis on the Blaschke frame we have obtained Euler Savary formulae of dual hyperbolic spherical motions. At the end of study, for given orthogonal rotation matrices in the sense of dual Lorentzian type 3×3, we have found real and dual invariants of fixed and moving axodes.

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THE ANALYSIS OF THE EFFECT OF THE NORMS IN THE STEP SIZE SELECTION FOR THE NUMERICAL INTEGRATION

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Abstract. In scientific studies involving norm calculations, the choice of the norm affects the obtained results. We have aimed to examine the behavior of the step sizes using different norms and norm inequalities in step size strategy obtained in [1] for linear Cauchy problems.

1. Introduction

Selection of step size is an important concept for the convergence of the numerical solution to exact solution in numerical integration of differential equation systems. For the use constant step size, it must be investigated how should be selected the step size in the first step of numerical integration. Also, if the solution is changing slowly in some regions and it is changing rapidly in some another regions then it is not practical to use constant step size in numerical integration. So, we should use small step sizes in the region where the solution changes rapidly and we should choose larger step size in the region where the solution changes slowly. In literature, step size strategies have been given for the numerical integration. Consider the Cauchy problem

$$X' = F(t, X), X(t_0) = X_0$$

on the region $D = \{(t, X) : |t - t_0| \leq T, |x_j - x_{j0}| \leq b_j\}$, where $X(t) = (x_j(t))$, $X_0 = (x_{j0})$; $x_{j0} = x_j(t_0)$, $F(t, X) = (f_j)$; $f_j = f_j(t, x_1, x_2, ..., x_N)$, $F(t, X) \in C^1([t_0 - T, t_0 + T] \times R^N)$, $X(t)$, X_0 and $b = (b_j) \in R^N$. In [1, 2] a step size strategy for $F(t, X) = AX$ is proposed by

$$h_i \leq \frac{1}{\alpha \sqrt[4]{N^5}} \left(\frac{2\delta_L}{\beta_{i-1}}\right)^{\frac{1}{2}} \tag{1.1}$$

such that the local error $||LE_i|| \leq \delta_L$. Strategy given in (1.1) is the generalization of the strategy in [3, 4].

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The above-mentioned step size strategies are based on matrix and vector norms. As in all the scientific studies involving norm calculations, the choice of the norm affects the obtained results in step size strategies.

The aim of this paper to examine the behavior of the step sizes using different norms and norm inequalities in step size strategy obtained in [1] for linear Cauchy problems. In section 2, we have introduced the step size strategy based on error analysis for the linear systems (SSS). We have reminded commonly used vector and matrix norms. In section 3, we have investigated the effects of choice of the norms on step size strategy. Finally, we have analyzed the all strategies with numerical examples.

2. The step size strategy and norms

2.1. **The Step Size Strategy (SSS).** Let us consider the Cauchy problem

$$X' = AX, X(t_0) = X_0. \tag{2.1}$$

Following inequality is given

$$||LE_i|| \leq \frac{h_i^2}{2}||A||^2||Z(\tau_i)||, \tau_i \in [t_{i-1}, t_i) \tag{2.2}$$

for the local error of the Cauchy problem (2.1) in i-th step of the numerical integration. According to equation (2.2), the upper bound of local error for the system (2.1) is given by

$$||LE_i|| \leq (\frac{1}{2}\alpha^2\beta_{i-1})\sqrt{N^5}h_i^2, \tag{2.3}$$

where

$$||A|| \leq N \max_{i,j}|a_{ij}| = N\alpha,$$
$$||Z|| \leq \sqrt{N}\max_j|z_j| \leq \sqrt{N}\max_j(\sup_{\tau_i}|z_j(\tau_i)|) \leq \sqrt{N}\beta_{i-1}.$$

From the inequality(2.3)in the step$i$, the step size is calculated by

$$h_i \leq (\frac{1}{\alpha\sqrt[4]{N^5}})(\frac{2\delta_L}{\beta_{i-1}})^{\frac{1}{2}} \tag{2.4}$$

such that the local error $||LE_i|| \leq \delta_L$ where δ_L is the error level that is determined by user ([1, 2]).

While formulating the step sizes (2.4), a more practical way is obtained for calculations by using the upper bound (2.3) instead of the upper bound (2.2) of the local error. The effects of the calculation errors resulting from floating point arithmetic are reduced in doing so.

2.2. **Vector and Matrix Norms and Relations between Matrix Norms.** A norm is a real valued function that provides a measure of the size of vectors and matrices. For $X = (x_j) \in R^N$, some commonly used norms are given below. The l_2 norm (Euclidean norm) is defined by

$$||X||_2 = (\sum_{j=1}^N x_j^2)^{\frac{1}{2}}.$$

The l_1 norm (sum norm) is given as

$$||X||_1 = \sum_{j=1}^N |x_j|.$$

Another norm is formulated by

$$||X||_\infty = \max_j|x_j|,$$

which is called as l_∞ norm (maximum norm).

For $A = (a_{ij}) \in R^{M\times N}$, the most frequently used matrix norms are the l_1 (maximum column) norm

$$||A||_1 = \max_j \sum_{i=1}^{M} |a_j|,$$

the l_∞ (maximum row) norm

$$||A||_\infty = \max_i \sum_{j=1}^{N} |a_j|,$$

the l_2 (spectral) norm

$$||A||_2 = \sqrt{\lambda_{max}(A^T A)},$$

where $\lambda_{max}(A^T A)$ is the maximum eigenvalue of the matrix $A^T A$, Frobenius norm

$$||A||_F = (\sum_{i=1}^{M} \sum_{j=1}^{N} |a_{ij}|^2)^{\frac{1}{2}},$$

and the maximum norm

$$||A||_{\max} = \max_{i,j} |a_{ij}|.$$

We have used in our study the relations

$$||A||_2 \leq ||A||_F,\ ||A||_F \leq \sqrt{N}||A||_2,\ ||A||_2 \leq N||A||_{\max},\ ||A||_2 \leq \sqrt{||A||_1||A||_\infty}$$

which hold for all matrices $A = (a_{ij}) \in R^{N\times N}$. And we have also used the compatible norms in this study.

For all information about norms in this section, you can see for example [5, 6, 7, 8, 9, 10].

3. An Analysis on the Effect of the Norms in the Step Size Selection

3.1. **The Effect of Choice of Norm to Step Size Strategy.** The inequality (2.4) given in [1, 2] gives step sizes based on matrix and vector norms in the i-th step of numerical integration of the Cauchy problem (2.1) such that local error is smaller than δ_L error level. Different formulations are obtained for the step size according to the choice of the norms in the inequality (2.2). Changes that occur in step sizes may be significant. Now, let investigate the effect of the norms to step sizes. In calculations consider that

$$||Z(\tau_i)||_k \leq sup_{\tau_i}||Z(\tau_i)||_k \leq \beta_{k,i-1},\ k = 1, 2, \infty.$$

Strategy 1 (SSS1) The step sizes given by

$$h_i \leq \frac{1}{||A||_2}(\frac{2\delta_L}{\beta_{2,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.1}$$

are obtained from the inequality (2.2) according to l_2 norm.

Strategy 2 (SSS2) The step sizes given by

$$h_i \leq \frac{1}{||A||_1}(\frac{2\delta_L}{\beta_{1,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.2}$$

are obtained from the inequality (2.2) according to l_1 norm.

Strategy 3 (SSS3) The step sizes given by

$$h_i \leq \frac{1}{||A||_\infty}(\frac{2\delta_L}{\beta_{\infty,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.3}$$

from the inequality (2.2) according to l_∞ norm.

Strategy 4 (SSS4) The step sizes given by

$$h_i \leq \frac{1}{||A||_F}(\frac{2\delta_L}{\beta_{2,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.4}$$

Ex.	INPUT		Step number with SSSLS and SSS*k* (*k*=1,2,3,4,5,6,7)							
	A	*T*	SSS	SSS1	SSS2	SSS3	SSS4	SSS5	SSS6	SSS7
1	$\begin{pmatrix} 0.2 & 1 \\ 0 & -0.1 \end{pmatrix}$	10	168	72	81	84	72	103	142	93
2	$\begin{pmatrix} 50 & 1 \\ -0.01 & 0 \end{pmatrix}$	0.1	134	54	55	55	54	78	112	2986
3	$\begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$	25	34	26	31	24	27	28	35	43

TABLE 1. Step number with the strategies in numerical integration

from the inequality (2.2) according to l_2 norm.

Strategy 5 (SSS5) By using inequality $||A||_2 \leq ||A||_F \leq \sqrt{N}||A||_2$, the step sizes are calculated by

$$h_i \leq \frac{1}{||A||_2}(\frac{2\delta_L}{N\beta_{2,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.5}$$

from the inequality (2.2) according to l_2 norm.

Strategy 6 (SSS6) By using inequality $||A||_2 \leq N||A||_{\max}$, the step sizes obtained by

$$h_i \leq \frac{1}{N||A||_{\max}}(\frac{2\delta_L}{\beta_{2,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.6}$$

from the inequality (2.2).

Strategy 7 (SSS7) The step sizes are given as follows

$$h_i \leq \frac{1}{||A||_1||A||_\infty}(\frac{2\delta_L}{\beta_{2,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.7}$$

by considering the inequality $||A||_2 \leq \sqrt{||A||_1||A||_\infty}$.

3.2. **Analysis of the Strategies with Numerical Examples.** Consider $X'(t) = AX(t)$, $X(t_0) = X_0$ on the region $D = \{(t, X) : |t - t_0| \leq T, |x_j - x_{j0}| \leq b_j\}$. Let $t_0 = 0$, $b_j = 5$, $x_{j0} = 1$ and $\delta_L = 10^{-1}$.

Following figures give us an idea about the step sizes obtained from strategies. The values and numbers of the step sizes depend on the choice of norm.

The main strategy SSS usually generates little step sizes which cause an expensive computation as shown in Figure 1 and Figure 3. However, no matter how the matrix, SSS provides ease of calculation for the step sizes. Because, calculation the parameters α and β_{i-1} of SSS in inequalities (2.4) is easier to obtain the parameters of the other strategies.

As we can see from Figure 1, Figure 2 and Figure 3, SSS1 gives the largest step sizes than other strategies. But, in this case local errors may occur very close to error level δ_L in calculations (see, Figure 4.(b)). The calculation errors may cause to be $||LE_i|| > \delta_L$ on some steps in numerical integration because of the effects of the floating-point arithmetic (Remark 3.1. in [2], and Remark 1. in [1]). If the situation that the occurred errors exceeds desired error level is not so important, then SSS1 is the most suitable strategy for the numerical integration. Because it always provides quite cheap computations.

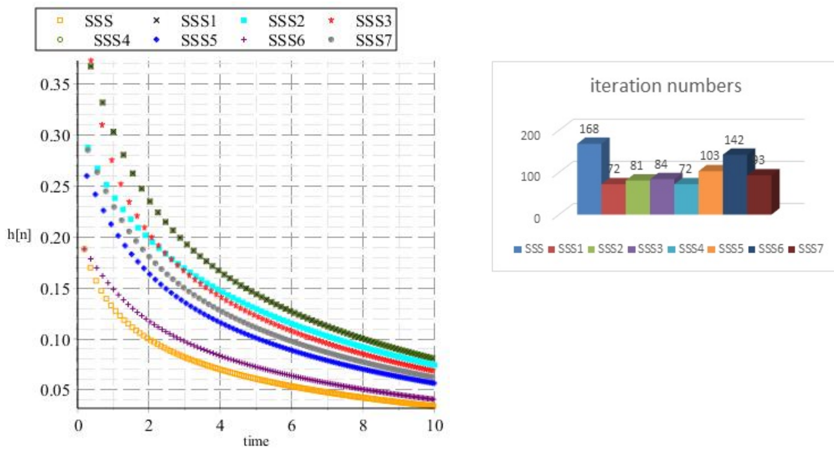


FIGURE 1. Step sizes and iteration numbers for Example 1.

For SSSk (k=2,3,4,5), almost similar results have been obtained as SSS1. So, we think that it will be enough to comment only SSS1.

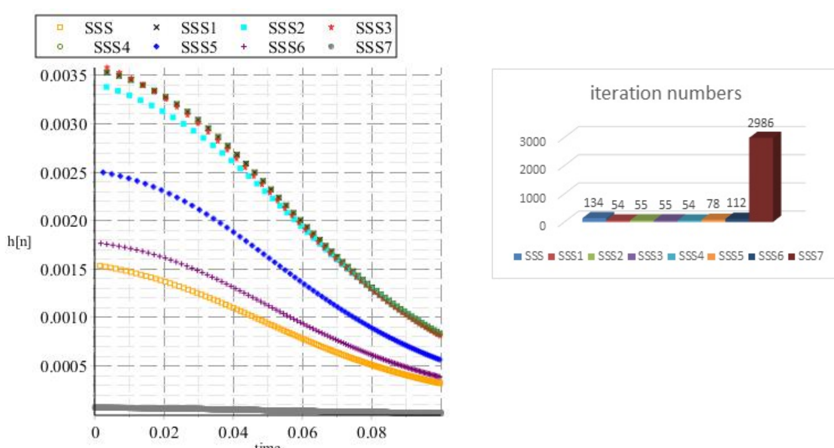


FIGURE 2. Step sizes and iteration numbers for Example 2.

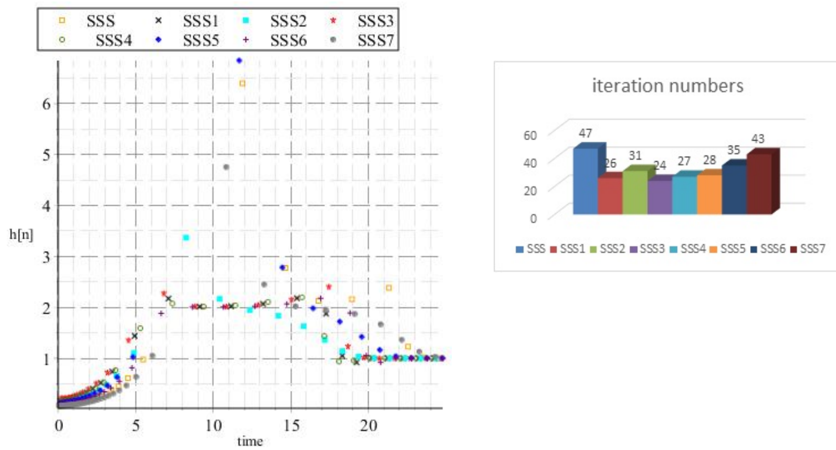


FIGURE 3. Step sizes and iteration numbers for Example 3.

SSS6 completes the calculation process a little less step when compared with SSS. The step sizes are partially calculated more easily with SSS6 than SSSk (k=1,2,3,4,5,7) because of the term $||A||_{max}$. But the calculation of the step sizes with SSS is easiest of among all the strategies.

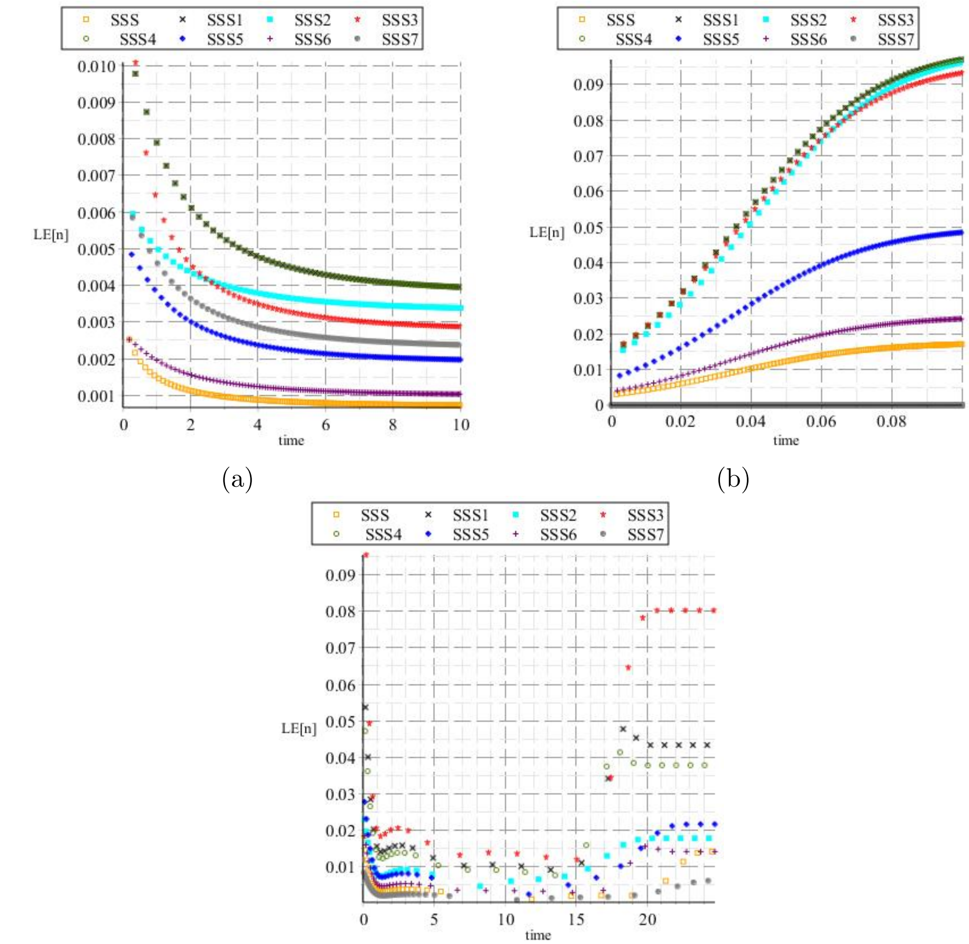
It is not practical to compare SSS7 directly with the other strategies regarding largeness of calculated step sizes and the number of iterations. For instance, iteration has taken 2986 steps in Example 2, but it has taken 43 steps in Example 3 as we can see in Figure 2 and Figure 3. That is, it may calculate the largest or the smallest step sizes according to given coefficient matrix. Even one of the elements of the coefficient matrix is large, the number of iterations increases in the calculation. The term $||A||_1||A||_\infty$ in SSS7 causes the becoming smaller of the step sizes. So, if the elements of the matrix is not very large, SSS7 should be used.

Figure 4 shows the local errors calculated by the strategies for Example 1, Example 2 and Example 3.

4. Conclusion

In this paper, the effects of choice of the norms have been examined in the calculation of the step sizes. It has been seen that some norms and norm inequalities provide ease of calculation for step size.

SSS1 gives the larger step sizes than other strategies. So, SSS1 completes the numerical integration in less time and fewer steps. It provides quite cheap computations. Although it is advantageous with this aspect, local errors may occur very close to error level δ_L in calculations. As all computations are done with floating-point arithmetic on computer, the calculation errors may cause to be $||LE_i|| > \delta_L$ on some steps in numerical integration. If this situation is unimportant, users



(a) (b)

(c)

FIGURE 4. Local errors for Example 1, Example 2 and Example 3.

should prefer SSS1 (or SSSk (k=2,3,4,5) that have similar properties) for cheap computations.

However the effects of floating point arithmetic does not considered in this study, it has emphasized that SSS is given to reduce these effects. SSS usually generates little step sizes which cause an expensive computation, but even so, it allows us to easier calculations for the step sizes. SSS should be used for ease of calculations.

SSS4 may be suggested if the elements of the matrix is not very large. If the coefficient matrix has at least one large element, it may calculate too small step sizes according to coefficient matrix.

Consequently, the choice of the norm should be considered as an important part of the step size strategy.

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ON FANO CONFIGURATIONS OF THE LEFT HALL PLANE OF ORDER 9

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Abstract. In this paper, we introduce Fano subplanes of the projective plane of order 9 coordinatized by elements of a left nearfield of order 9. We give an algorithm for checking Fano subplanes of this projective plane and apply the algorithm (implemented in C#) to determine and classify Fano subplanes.

1. Introduction

It is shown that the projective plane of order 2, 3, 4, 5, 7 and 8 are unique and projective plane of order 9 is not unique. There are four known projective planes of order 9: the Desarguesian plane, a nearfield plane, the dual of the nearfield plane and the Hughes plane of order 9, [4]. The last three planes of order 9 are called "miniquaternion planes" because they can be coordinatized by the miniquaternion near field. O. Veblen and J. M. Wedderburn discovered these miniquaternion planes in 1907, [6].

The regular near field of order q^2, for q an odd prime power, are defined taking the elements of $GF(q^2)$, using the field addition and definition a new multiplication on the elements in terms of the field multiplication. This gives an algebraic system in which the non-zero elements form a group under the multiplication and the right or left distributive laws hold. The near field can be used to define and coordinatize the near field plane of order 9.

In the first section, we give the left near field of order 9 by taking the elements of $GF(3)$ and using the field addition and a new multiplication on the elements in terms of the left near field multiplication, . In the second section, we identify the non-homegeneous coordinates of the points and lines and then homegeneous coordinates of the points and lines in this left near field plane of order 9. In third section, we investigate the Fano subplanes imbedded in this projective plane. It is shown that there are 18 complete quadrangles which generate Fano plane. Finally,

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we write a computer program $C\#$ that determine the complete quadrangles which generate Fano plane in this plane.

2. The left nearfield system of order 9

We give a near field of order 9 which is not left disributive.

Definition 2.1. A left nearfield is a system $(S, \oplus, \odot)$, where $\oplus$ and $\odot$ are binary operations on the set S and

(1) S is finite
(2) $(S, \oplus)$ is a group, with identity 0
(3) $(S \backslash \{0\}, \odot)$ is a group, with identity 1
(4) $0 \odot x = 0$ for all $x \in S$
(5) $\odot$ is left distributive over $\oplus$, that is $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ for all $x, y, z \in S$
(6) Given $m, n, k \in S$ with $m \neq n$, there exists a unique $x \in S$ such that

$$-m \odot x \oplus n \odot x = k.$$

Let $(F_3, +, .)$ be the Galois field of order 3.We now construct $(S, \oplus, \odot)$, using F_3, a left nearfield of order 9.

The nine elements of S are $a + \lambda b$, a,$b \in F_3$, $\lambda \notin F_3$. Addition in S is defined by the rule

$$(a + \lambda b) \oplus (c + \lambda d) = (a + c) + \lambda(b + d) \tag{1}$$

and multiplication by

$$(a + \lambda b) \odot (c + \lambda d) = \begin{cases} ac + \lambda(ad), & \text{if} \quad b = 0 \\ ac - b^{-1}df(a) + \lambda(bc - (a-1)d), & \text{if} \quad b \neq 0 \end{cases} \tag{2}$$

where, $a, b, c, d \in F_3$, $\lambda \notin F_3$ and $f(t) = t^2 + 1$ is a irreducible polynom on F_3, [5].

For the sake of sorthness if we use ab instead of $a + \lambda b$ in equation (1) and (2), then addition and multiplication tables are obtained as follows:

$\oplus$	00	01	02	10	11	12	20	21	22
00	00	01	02	10	11	12	20	21	22
01	01	02	00	11	12	10	21	22	20
02	02	00	01	12	10	11	22	20	21
10	10	11	12	20	21	22	00	01	02
11	11	12	10	21	22	20	01	02	00
12	12	10	11	22	20	21	02	00	01
20	20	21	22	00	01	02	10	11	12
21	21	22	20	01	02	00	11	12	10
22	22	20	21	02	00	01	12	10	11

Table 1.

$\odot$	00	01	02	10	11	12	20	21	22
00	00	00	00	00	00	00	00	00	00
01	00	20	10	01	21	11	02	22	12
02	00	10	20	02	12	22	01	11	21
10	00	01	02	10	11	12	20	21	22
11	00	12	21	11	20	02	22	01	10
12	00	22	11	12	01	20	21	10	02
20	00	02	01	20	22	21	10	12	11
21	00	11	22	21	02	10	12	20	01
22	00	21	12	22	10	01	11	02	20

Table 2.

If we use the following equlities

$$\begin{aligned}0&=(0,0)\\1&=(1,0)\\2&=(2,0)\\3&=(0,1)\\4&=(1,1)\\5&=(2,1)\\6&=(0,2)\\7&=(1,2)\\8&=(2,2),\end{aligned}$$

the addition and multiplication tables in $(S,\oplus,\odot)$ can be arranged as follows :

$\oplus$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	3
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	7	8	0	1	2
4	4	5	3	7	8	6	1	2	0
5	5	3	4	8	6	7	2	0	1
6	6	7	8	0	1	2	3	4	5
7	7	8	6	1	2	0	4	5	3
8	8	6	7	2	0	1	5	3	4

$\odot$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	1	6	8	7	3	5	4
3	0	3	6	2	5	8	1	4	7
4	0	4	8	7	2	3	5	6	1
5	0	5	7	4	6	2	8	1	3
6	0	6	3	1	7	4	2	8	5
7	0	7	5	8	3	1	4	2	6
8	0	8	4	5	1	6	7	3	2

The system $(S,\oplus,\odot)$ satisfies the conditions of Definition 2.1 and therefore a left nearfield of order 9.

3. The Projective Plane P_2S

Definition 3.1. While N and D are two distinct sets whose elements are called as the points and the lines, respectively and o is the incidence relation between N and D ; then the ordered triple (N,D,o) is called as geometrical structure. (N,D,o) satisfying the following three axioms is called a projective plane and denoted by P. If N is finite, projective plane P is called as finite projective plane. P1. Any distinct two points are incident with just one line.

P2. Any two lines are incident with at least one point.

P3. There exists four points of which no three are collinear.

The order of P is defined to be the number of points on any line of projective plane $P = (N, D, o)$. If the order of a finite projective plane is q, total number of its points and lines is equal and $q^2 + q + 1$.

It is well known that every projective plane has also an algebraic structure obtained by coordinazation. Conversely, certain algebraic structures can be used to construct projective planes.

In this section, we will construct projective plane order 9. The projective plane whose the points and the lines are coordinatized by the elements of $(S, \oplus, \odot)$.

The 91 points of P_2S are the elements of the set

$$\{\,(x, y)|\ x, y \in S\} \cup \{(m) : m \in S\} \cup \{(\infty)\}.$$

The points of the form (x, y) are called *proper points*, and the unique point (∞) and the points of the form (m) are called *ideal points*. The 91 lines of P_2S are defined to be set of points satisfying one of the three conditions:

$$\begin{aligned}
&[m, k] = \{\,(x, y) \in S^2 \;\big|\; y = m \odot x \oplus k\} \cup \{(m)\}\\
&[a] = \{\,(x, y) \in S^2 \;\big|\; \ x = a\} \cup \{(\infty)\}\\
&[\infty] = \{(m) \in S\} \cup \{(\infty)\}
\end{aligned}$$

The 81 lines having form $y = m \odot x \oplus k$ and 9 lines having equation of the form $x = \lambda$ are called the *proper lines* and the unique line $[\infty]$ is called *the ideal line.*

The system of points, lines and incidence relation given above defines a projective plane of order 9, which is the left nearfiled plane.

Now, we are considering the projective plane of order 9 homogeneous coordinatized by elements of the above left nearfield. We notice that the homogeneous coordinates of a point are not unique. Two triples that are multiples of each other specify are the same point. Thus the same point has many sets of homogeneous coordinates: (x, y, z) and (x', y', z') represent the same point if and only if there is some $\lambda \neq 0$, $\lambda \in S$ such that $x' = \lambda \odot x$, $y' = \lambda \odot y$, $z' = \lambda \odot z$. We convert a point expressed in Cartesian coordinates to homogeneous coordinates in left nearfield plane of order 9. We have seen that a point (x, y) in the P_2S has homogeneous coordinates $\lambda \odot (x, y, 1) = (\lambda \odot x, \lambda \odot y, \lambda \odot 1)$, $\lambda \neq 0$, $\lambda \in S$. Homogeneous coordinates of the form $\lambda \odot (m, 1, 0)$ do correspond to all ideal points (m), m, $\lambda \in S^*$ in the P_2S. Homogeneous coordinates of the form $(\lambda, 0, 0)$ do correspond to the unique point at infinity in the P_2S .

We have seen that a line $[m, k]$ in the P_2S has homogeneous coordinates $\mu \odot [m, -1, k] = [\mu \odot m, \mu \odot (-1), \mu \odot k]$, $\mu \neq 0$, $\mu \in S$. Homogeneous coordinates of the form $\mu \odot [x, 0, 1]$ do correspond to all lines $[a]$, $a \neq 0$, $a \in S$ in the P_2S. Homogeneous coordinates of the form $[0, 0, \mu]$ do correspond to the unique line $[\infty]$ at infinity in the P_2S .

A line in the P_2S has general equation $y = m \odot x \oplus k$. Suppose (x_1, x_2, x_3), $x_3 \neq 0$ are the homogeneous coordinates of a point (x, y) on the line; hence $x = x_3^{-1} \odot x_1$ and $y = x_3^{-1} \odot x_2$. Substituting for x and y in the line equation and multiplying through by x_3, yields the conditions for (x_1, x_2, x_3) to be the homogeneous coordinates of a point on the line:

$$m \odot x_1 \oplus (-1) \odot x_2 \oplus k \odot x_3 = 0.$$

The following table lists all homogeneous coordinates of the 91 points and lines in the projective plane of order 9 coordinatized by elements of the above left nearfield.

i	p_i					l_i					
1	(1,0,0)	2	11	20	29	38	47	56	65	74	83
2	(0,1,0)	1	11	12	13	14	15	16	17	18	19
3	(1,1,0)	4	11	22	30	44	55	63	68	79	87
4	(2,1,0)	3	11	21	31	41	51	61	71	81	91
5	(3,1,0)	5	11	23	35	40	54	60	66	82	88
6	(4,1,0)	6	11	24	37	43	49	62	72	77	84
7	(5,1,0)	7	11	25	36	46	50	58	69	75	89
8	(6,1,0)	8	11	26	32	39	52	64	67	78	90
9	(7,1,0)	9	11	27	34	42	53	57	73	76	86
10	(8,1,0)	10	11	28	33	45	48	59	70	80	85
11	(0,0,1)	1	2	3	4	5	6	7	8	9	10
12	(1,0,1)	2	13	22	31	40	49	58	67	76	85
13	(2,0,1)	2	12	21	30	39	48	57	66	75	84
14	(3,0,1)	2	14	23	32	41	50	59	68	77	86
15	(4,0,1)	2	15	24	33	42	51	60	69	78	87
16	(5,0,1)	2	16	25	34	43	52	61	70	79	88
17	(6,0,1)	2	17	26	35	44	53	62	71	80	89
18	(7,0,1)	2	18	27	36	45	54	63	72	81	90
19	(8,0,1)	2	19	28	37	46	55	64	73	82	91
20	(0,1,1)	1	29	30	31	32	33	34	35	36	37
21	(1,1,1)	4	13	21	29	46	54	62	70	78	86
22	(2,1,1)	3	12	22	29	42	52	59	72	82	89
23	(3,1,1)	5	14	26	29	45	51	58	73	79	84
24	(4,1,1)	6	15	28	29	40	53	61	68	75	90
25	(5,1,1)	7	16	27	29	41	49	64	66	80	87
26	(6,1,1)	8	17	23	29	43	55	57	69	81	85
27	(7,1,1)	9	18	25	29	44	48	60	67	77	91
28	(8,1,1)	10	19	24	29	39	50	63	71	76	88
29	(0,2,1)	1	20	21	22	23	24	25	26	27	28
30	(1,2,1)	3	13	20	30	43	50	60	73	80	90
31	(2,2,1)	4	12	20	31	45	53	64	69	77	88
32	(3,2,1)	8	14	20	35	46	48	61	72	76	87
33	(4,2,1)	10	15	20	37	41	54	57	67	79	89
34	(5,2,1)	9	16	20	36	39	51	62	68	82	85
35	(6,2,1)	5	17	20	32	42	49	63	70	75	91
36	(7,2,1)	7	18	20	34	40	55	59	71	78	84
37	(8,2,1)	6	19	20	33	44	52	58	66	81	86
38	(0,3,1)	1	38	39	40	41	42	43	44	45	46
39	(1,3,1)	8	13	28	34	38	51	63	66	77	89
40	(2,3,1)	5	12	24	36	38	55	61	67	80	86
41	(3,3,1)	4	14	27	37	38	52	60	71	75	85
42	(4,3,1)	7	15	22	32	38	48	62	73	81	88
43	(5,3,1)	10	16	23	30	38	53	58	72	78	91
44	(6,3,1)	3	17	25	33	38	54	64	68	76	84
45	(7,3,1)	6	18	26	31	38	50	57	70	82	87
46	(8,3,1)	9	19	21	35	38	49	59	69	79	90
47	(0,4,1)	1	47	48	49	50	51	52	53	54	55
48	(1,4,1)	10	13	27	32	44	47	61	69	82	84
49	(2,4,1)	6	12	25	35	41	47	63	73	78	85
50	(3,4,1)	9	14	24	30	40	47	64	70	81	89
51	(4,4,1)	4	15	26	36	43	47	59	66	76	91
52	(5,4,1)	5	16	22	33	46	47	57	71	77	90
53	(6,4,1)	7	17	28	31	39	47	60	72	79	86
54	(7,4,1)	8	18	21	37	42	47	58	68	80	88
55	(8,4,1)	3	19	23	34	45	47	62	67	75	87
56	(0,5,1)	1	56	57	58	59	60	61	62	63	64
57	(1,5,1)	9	13	26	33	41	55	56	72	75	88
58	(2,5,1)	7	12	23	37	44	51	56	70	76	90
59	(3,5,1)	6	14	22	34	39	54	56	69	80	91
60	(4,5,1)	8	15	25	30	45	49	56	71	82	86
61	(5,5,1)	4	16	28	35	42	50	56	67	81	84
62	(6,5,1)	10	17	21	36	40	52	56	73	77	87
63	(7,5,1)	3	18	24	32	46	53	56	66	79	85
64	(8,5,1)	5	19	27	31	43	48	56	68	78	89
65	(0,6,1)	1	65	66	67	68	69	70	71	72	73
66	(1,6,1)	5	13	25	37	39	53	59	65	81	87
67	(2,6,1)	8	12	27	33	40	50	62	65	79	91
68	(3,6,1)	3	14	28	36	44	49	57	65	78	88
69	(4,6,1)	9	15	23	31	46	52	63	65	80	84
70	(5,6,1)	6	16	21	32	45	55	60	65	76	89
71	(6,6,1)	4	17	24	34	41	48	58	65	82	90
72	(7,6,1)	10	18	22	35	43	51	64	65	75	86
73	(8,6,1)	7	19	26	30	42	54	61	65	77	85
74	(0,7,1)	1	74	75	76	77	78	79	80	81	82
75	(1,7,1)	7	13	24	35	45	52	57	68	74	91
76	(2,7,1)	9	12	28	32	43	54	58	71	74	87
77	(3,7,1)	10	14	25	31	42	55	62	66	74	90
78	(4,7,1)	5	15	21	34	44	50	64	72	74	85
79	(5,7,1)	3	16	26	37	40	48	63	69	74	86
80	(6,7,1)	6	17	27	30	46	51	59	67	74	88
81	(7,7,1)	4	18	23	33	39	49	61	73	74	89
82	(8,7,1)	8	19	22	36	41	53	60	70	74	84
83	(0,8,1)	1	83	84	85	86	87	88	89	90	91
84	(1,8,1)	6	13	23	36	42	48	64	71	79	83
85	(2,8,1)	10	12	26	34	46	49	60	68	81	83
86	(3,8,1)	7	14	21	33	43	53	63	67	82	83
87	(4,8,1)	3	15	27	35	39	55	58	70	77	83
88	(5,8,1)	8	16	24	31	44	54	59	73	75	83
89	(6,8,1)	9	17	22	37	45	50	61	66	78	83
90	(7,8,1)	5	18	28	30	41	52	62	69	76	83
91	(8,8,1)	4	19	25	32	40	51	57	72	80	83

3.1. **Fano Subplanes of P_2S.** The completion of a regular quadrangle has got the important role in many investigations of the structure of projective planes. In a projective plane of order 9, the non-projective subplanes can have orders 2 or 3. An order 2 affine subplane is a quadrangle, so projective affine subplane of order 2 are quadrangles which generate Fano configurations. We search for the Fano subplanes in P_2S by starting with a quadrangle.

Definition 3.2. A regular quadrangle in a projective plane is a set of four points of which no three are collinear. If $OIXP$ is a regular quadrangle, the six lines OX, OI, OP, IX, PX, PI are called the sides of the quadrangle, and the three points $OP \cap IX = U$, $OI \cap XP = V$, $OX \cap IP = W$ are called the diagonal points of the quadrangle.

The Fano plane occurs as a subplane of many larger planes. Therefore, the discovery of the Fano plane has played an important role in the improvement of the theory of finite geometries. Fano subplanes in some projective planes have been examined by many authors. For instance, Taş [7]Room-Kirpatrick [5], Calişkan and Moorhouse [2], Çifçi-Kaya [3], Akça-Günaltılı-Güney [1] ext. A Fano plane is a configuration of 7 points and 7 lines with 3 points on a line and 3 lines through a point. In Fano plane the diagonal points of any regular quadrangle are collinear.

Now, in this part of the study we will determine all Fano planes in P_2S by choosing a regular quadrangle $OIXP_i$ with $O = 11 = (0,0,1)$, $I = 21 = (1,1,1)$, $X = 1 = (1,0,0)$ and $P_i = (a,b,1)$, $a,b \in S$.

A regular quadrangle $OIXP_i$ can be completed to a Fano plane if and only if the diagonal points $OI \cap XP_i = V_i$, $OP_i \cap IX = U_i$, $OX \cap IP_i = W_i$ are collinear.

Theorem 3.1. *There are exactly six Fano subplanes of P_2S which are completions of the regular quadrangles $OIXP_i$ with $P_i = (0,b,1)$, $b \in S$.*

Proof. If $b \in F_3$ then $OIXP_i$ do not the regular quadrangles. Consider the quadrangles $OIXP_i$ with O,I, X and $P_i = (0,b,1)$, $b \in S \backslash F_3$. Then $OIXP_i$ is a regular quadrangle with the diagonal points $(0,1,1)$, $(b,b,1)$, and $(c,0,1)$. If $b,c \in S \backslash F_3$ and $b \oplus c = 2$ then the diagonal points are collinear and the completion of $OIXP_i$ is a Fano plane. There are six classes of Fano subplanes which are completions of $OIXP_i$. These are represented by:
$\{11,21,1,38,41,20,19\}$,
$\{11,21,1,47,51,20,18\}$,
$\{11,21,1,56,61,20,17\}$,
$\{11,21,1,65,71,20,16\}$,
$\{11,21,1,74,81,20,15\}$
and
$\{11,21,1,83,91,20,14\}$ □

Theorem 3.2. *There are exactly six Fano subplanes of P_2S which are completions of the regular quadrangles $OIXR_i$ with $R_i = (1,b,1)$, $b \in S$.*

Proof. If $b \in F_3$ then $OIXR_i$ do not the regular quadrangles. Consider the quadrangles $OIXR_i$ with O, I, X and $R_i = (1,b,1)$, $b \in S \backslash F_3$. Then $OIXR_i$ is a regular quadrangle with the diagonal points $(1,0,1)$, $(b,b,1)$, and $(c,1,1)$. If $b,c \in S \backslash F_3$ and $b \oplus c = 0$ then the diagonal points are collinear and the completion of $OIXR_i$ is a Fano plane. There are six classes of Fano subplanes which are completions of $OIXR_i$. These are represented by:
$\{11,21,1,39,41,26,12\}$,
$\{11,21,1,48,51,28,12\}$,
$\{11,21,1,57,61,27,12\}$,
$\{11,21,1,66,71,23,12\}$,
$\{11,21,1,75,81,25,12\}$
and

$\{11, 21, 1, 84, 91, 24, 12\}$ □

Theorem 3.3. *There are exactly six Fano subplanes of P_2S which are completions of the regular quadrangles $OIXS_i$ with $S_i = (2, b, 1)$, $b \in S$.*

Proof. If $b \in F_3$ then $OIXR_i$ do not the regular quadrangles. Consider the quadrangles $OIXS_i$ with O, I, X and $S_i = (2, b, 1)$, $b \in S \backslash F_3$. Then $OIXS_i$ is a regular quadrangle with the diagonal points $(b, 0, 1)$, $(b, b, 1)$, and $(b, 1, 1)$. If $b \in S \backslash F_3$ then the diagonal points are collinear and the completion of $OIXS_i$ is a Fano plane. There are six classes of Fano subplanes which are completions of $OIXS_i$. These are represented by:
$\{11, 21, 1, 40, 41, 23, 14\}$,
$\{11, 21, 1, 49, 51, 24, 15\}$,
$\{11, 21, 1, 58, 61, 25, 16\}$,
$\{11, 21, 1, 67, 71, 26, 17\}$,
$\{11, 21, 1, 76, 81, 27, 18\}$
and
$\{11, 21, 1, 85, 91, 28, 19\}$ □

4. Algorithm

In this section, we will give an algorithm for checking Fano subplanes in projective plane P_2S.

Steps of algorithm

Read the Incidence matrice of projective plane P_2S from Excell File of table 5 and assign to array variable

Input the points A_i, $i = 1, 2, 3, 4$ and $A_i \in \{1, 2, ..., 91\}$
Begin
$S_1 \leftarrow$ the row on A_1, A_2
$S_2 \leftarrow$ the row on A_3, A_4
$D_1 \leftarrow$ the same point on $S_1$and S_2
$S_3 \leftarrow$ the row on A_1, A_3
$S_4 \leftarrow$ the row on A_2, A_4
$D_2 \leftarrow$ the same point on $S_3$and S_4
$S_5 \leftarrow$ the row on A_1, A_4
$S_6 \leftarrow$ the row on A_2, A_3
$D_3 \leftarrow$ the same point on $S_5$and S_6
$S_7 \leftarrow$ the row on D_1, D_2
if D_3 on S_7 then
print "the set of points $\{A_1, A_2 A_3, A_4, D_1, D_2, D_3\}$ is Fano plane"
else
print "it is not Fano plane"
go to Begin
end

Conclusion: We attempted to construct Fano subplanes to contain a regular quadrangle with one ideal point X. There are just 18 Fano subplanes containing O, I, X namely the completions of the regular quadrangles O, I, X, $(a, b, 1)$, with $a \in F_3$, $b \in S \backslash F_3$. Every Fano subplane of P_2S contains precisely diagonal point $(b, b, 1)$, $b \in S \backslash F_3$. There are 18 Fano pairs determined by taking from two different classes which have contained one comman diagonal point. These are represented by three classes. Two classes have one comman diagonal point. These are checked once again, with computer program in C#, the same results are obtained.

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ON THE PARANORMED TAYLOR SEQUENCE SPACES

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Abstract. In this paper, the sequence spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ of non-absolute type which are the generalization of the Maddox sequence spaces have been introduced and it is proved that the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ are linearly isomorphic to spaces $c_0(p)$, $c(p)$ and $\ell(p)$, respectively. Furthermore, the $\alpha-,\beta-$ and $\gamma-$duals of the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ have been computed and their bases have been constructed and some topological properties of these spaces have been investigated. Besides this, the class of matrices $(t_0^r(p):\mu)$ has been characterized, where μ is one of the sequence spaces ℓ_∞, c and c_0 and derives the other characterizations for the special cases of μ.

1. Introduction

By w, we shall denote the space of all real-valued sequences. Any vector subspace of w is called a sequence space. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and $p-$absolutely convergent series, respectively, where $1 < p < \infty$.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g: X \to \mathbb{R}$ such that $g(\theta) = 0, g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that (p_k) be a bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $L = \max\{1, H\}$. Then, the linear spaces $\ell_\infty(p), c(p), c_0(p)$ and $\ell(p)$ were defined by Maddox [12] (see also Simons [14] and

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Nakano [13]) as follows:

$$\begin{aligned}
&\ell_\infty(p) = \{x = (x_k) \in w : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\},\\
&c(p) = \{x = (x_k) \in w : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R}\},\\
&c_0(p) = \{x = (x_k) \in w : \lim_{k\to\infty} |x_k|^{p_k} = 0\},\\
&\ell(p) = \left\{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\right\},
\end{aligned}$$

which are the complete spaces paranormed by

$$g_1(x) = \sup_{k\in\mathbb{N}} |x_k|^{p_k/L} \Longleftrightarrow \inf p_k > 0 \text{ and } g_2(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/L},$$

respectively. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $\mathcal{F}$ and $\mathbb{N}_k$, we shall denote the collection of all finite subsets of $\mathbb{N}$ and the set of all $n \in \mathbb{N}$ such that $n \geq k$, respectively. We shall assume throughout that $p_k^{-1} + (p'_k)^{-1} = 1$ provided $1 < \inf p_k \leq H < \infty$.

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the $A-$transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk}x_k, \ (n \in \mathbb{N}). \tag{1.1}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \mu$. A sequence x is said to be $A-$summable to α if Ax converges to α which is called the $A-$limit of x.

2. The Sequence Spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ of Non-Absolute Type

In this section, we define the sequence spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$, and prove that $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ are the complete paranormed linear spaces.

For a sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\}. \tag{2.1}$$

In [5], Choudhary and Mishra have defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that $S-$transforms are in $\ell_{(p)}$, where $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1 \quad , & 0 \leq k \leq n,\\ 0 \quad , & k > n.\end{cases}$$

Başar and Altay [3] have studied the space $bs(p)$ which is formerly defined by Başar in [4] as the set of all series whose sequences of partial sums are in $\ell_\infty(p)$.

More recently, Altay and Başar have studied the sequence spaces $r^t(p), r^t_\infty(p)$ in [1] and $r^t_c(p), r^t_0(p)$ in [2] which are derived by the Riesz means from the sequence spaces $\ell(p), \ell_\infty(p), c(p)$ and $c_0(p)$ of Maddox, respectively.

With the notation of (2.1), the spaces $\overline{\ell(p)}, bs(p), r^t(p), r^t_\infty(p), r^t_c(p)$ and $r^t_0(p)$ may be redefined by

$$\overline{\ell(p)} = [\ell(p)]_S, bs(p) = [\ell_\infty(p)]_S, r^t(p) = [\ell(p)]^t_R$$
$$r^t_\infty(p) = [\ell_\infty(p)]^t_R, r^t_c(p) = [c(p)]^t_R, r^t_0(p) = [c_0(p)]^t_R.$$

In [6], Demiriz and Çakan have defined the sequence spaces $e^r_0(u,p)$ and $e^r_c(u,p)$ which consists of all sequences such that $E^{r,u}$- transforms are in $c_0(p)$ and $c(p)$, respectively $E^{r,u} = \{e^r_{nk}(u)\}$ is defined by

$$e^r_{nk}(u) = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k u_k & , \quad (0 \le k \le n), \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$ and $0 < r < 1$.

In [9], the Taylor sequence spaces t^r_0 and t^r_c of non-absolute type, which are the matrix domains of Taylor mean T^r of order r in the sequence spaces c_0 and c, respectively, are introduced, some inclusion relations and Schauder basis for the spaces t^r_0 and t^r_c are given, and the $\alpha-, \beta-$ and $\gamma-$ duals of those spaces are determined. The main purpose of this paper is to introduce the sequence spaces $t^r_0(p)$, $t^r_c(p)$ and $t^r(p)$ of nonabsolute type which are the set of all sequences whose T^r-transforms are in the spaces $c_0(p)$, $c(p)$ and $\ell(p)$, respectively; where T^r denotes the matrix $T^r = \{t^r_{nk}\}$ defined by

$$t^r_{nk} = \begin{cases} \binom{k}{n}(1-r)^{n+1}r^{k-n} & , \quad (k \ge n), \\ 0 & , \quad (0 \le k < n) \end{cases}$$

where $0 < r < 1$. Also, we have constructed the basis and computed the $\alpha-, \beta-$ and $\gamma-$duals and investigated some topological properties of the spaces $t^r_0(p)$, $t^r_c(p)$ and $t^r(p)$.

Following Choudhary and Mishra [5], Başar and Altay [3], Altay and Başar [1, 2], Demiriz [6], Kirişçi [9], we define the sequence spaces $t^r_0(p)$, $t^r_c(p)$ and $t^r(p)$, as the sets of all sequences such that T^r-transforms of them are in the spaces $c_0(p)$,$c(p)$ and $\ell(p)$, respectively, that is,

$$t^r_0(p) = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left| \sum_{k=n}^{\infty} \binom{k}{n}(1-r)^{n+1}r^{k-n}x_k \right|^{p_n} = 0 \right\},$$

$$t^r_c(p) = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left| \sum_{k=n}^{\infty} \binom{k}{n}(1-r)^{n+1}r^{k-n}x_k - l \right|^{p_n} = 0 \text{ for some } l \in \mathbb{R} \right\}$$

and

$$t^r(p) = \left\{ x = (x_k) \in w : \sum_{n} \left| \sum_{k=n}^{\infty} \binom{k}{n}(1-r)^{n+1}r^{k-n}x_k \right|^{p_n} < \infty \right\}.$$

In the case $(p_n) = e = (1,1,1,...)$, the sequence spaces $t^r_0(p)$, $t^r_c(p)$ and $t^r(p)$ are, respectively, reduced to the sequence spaces t^r_0 and t^r_c which are introduced by Kirişçi [9] and $t^r(p)$ is reduced to the sequence space t^r_p. With the notation of (2.1), we may redefine the spaces $t^r_0(p)$, $t^r_c(p)$ and $t^r(p)$ as follows:

$$t^r_0(p) = [c_0(p)]_{T^r}, \; t^r_c(p) = [c(p)]_{T^r} \text{ and } t^r(p) = [\ell(p)]_{T^r}. \tag{2.2}$$

Define the sequence $y = \{y_k(r)\}$, which will be frequently used, as the T^r-transform of a sequence $x = (x_k)$, i.e.,

$$y_k(r) := \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k \text{ for all } k \in \mathbb{N}. \tag{2.3}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *$t_0^r(p)$ and $t_c^r(p)$ are the complete linear metric space paranormed by g, defined by*

$$g(x) = \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k/L}.$$

Also, $t_p^r(p)$ is the complete linear metric space paranormed by h, defined by

$$h(x) = \left(\sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k} \right)^{1/M}. \tag{2.4}$$

Proof. Since the proof is similar for $t_0^r(p)$ and $t_c^r(p)$, we give the proof only for the space $t_0^r(p)$. The linearity of $t_0^r(p)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $x, z \in t_0^r(p)$ (see Maddox [11, p.30])

$$\begin{aligned} & \sup_{n \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} (x_j + z_j) \right|^{p_k/L} \\ \leq & \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k/L} + \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} z_j \right|^{p_k/L} \end{aligned} \tag{2.5}$$

and for any $\alpha \in \mathbb{R}$ (see [14])

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^L\}. \tag{2.6}$$

It is clear that $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in t_0^r(p)$. Again the inequalities (2.5) and (2.6) yield the subadditivity of g and

$$g(\alpha x) \leq \max\{1, |\alpha|^L\} g(x).$$

Let $\{x^n\}$ be any sequence of the points $x^n \in t_0^r(p)$ such that $g(x^n - x) \to 0$ and (α_n) also be any sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality

$$g(x^n) \leq g(x) + g(x^n - x)$$

holds by the subadditivity of g, $\{g(x^n)\}$ is bounded and we thus have

$$\begin{aligned} g(\alpha^n x^n - \alpha x) &= \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} (\alpha^n x_j^n - \alpha x_j) \right|^{p_k/L} \\ &\leq |\alpha_n - \alpha| g(x^n) + |\alpha| g(x^n - x), \end{aligned}$$

which tends to zero as $n \to \infty$. This means that the scalar multiplication is continuous. Hence, g is paranorm on the space $t_0^r(p)$.

It remains to prove the completeness of the space $t_0^r(p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $t_0^r(p)$, where $x^i = \{x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, \ldots\}$. Then, for a given $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that

$$g(x^i - x^j) < \frac{\epsilon}{2}$$

for all $i, j > n_0(\epsilon)$. Using the definition of g we obtain for each fixed $k \in \mathbb{N}$ that

$$|(T^r x^i)_k - (T^r x^j)_k|^{p_k/L} \leq \sup_{k\in\mathbb{N}} |(T^r x^i)_k - (T^r x^j)_k|^{p_k/L} < \frac{\epsilon}{2} \tag{2.7}$$

for every $i, j > n_0(\epsilon)$ which leads to the fact that $\{(T^r x^0)_k, (T^r x^1)_k, (T^r x^2)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(T^r x^i)_k \to (T^r x)_k$ as $i \to \infty$. Using these infinitely many limits $(T^r x)_0, (T^r x)_1, \ldots$, we define the sequence $\{(T^r x)_0, (T^r x)_1, \ldots\}$. From (2.7) with $j \to \infty$, we have

$$|(T^r x^i)_k - (T^r x)_k|^{p_k/L} \leq \frac{\epsilon}{2} \ (i, j > n_0(\epsilon)) \tag{2.8}$$

for every fixed $k \in \mathbb{N}$. Since $x^i = \{x_k^{(i)}\} \in t_0^r(p)$ for each $i \in \mathbb{N}$, there exists $k_0(\epsilon) \in \mathbb{N}$ such that

$$|(T^r x^i)_k|^{p_k/L} < \frac{\epsilon}{2} \tag{2.9}$$

for every $k \geq k_0(\epsilon)$ and for each fixed $i \in \mathbb{N}$. Therefore, taking a fixed $i > n_0(\epsilon)$ we obtain by (2.8) and (2.9) that

$$|(T^r x)_k|^{p_k/L} \leq |(T^r x)_k - (T^r x^i)_k|^{p_k/L} + |(T^r x^i)_k|^{p_k/L} < \frac{\epsilon}{2}$$

for every $k > k_0(\epsilon)$. This shows that $x \in t_0^r(p)$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $t_0^r(p)$ is complete and this concludes the proof.

Now, $t^r(p)$ is the complete linear metric space paranormed by h defined by (2.4). It is easy to see that the space $t^r(p)$ is linear with respect to the coordinate-wise addition and scalar multiplication. Therefore, we first show that it is a paranormed space with the paranorm h defined by (2.4).

It is clear that $h(\theta) = 0$ where $\theta = (0, 0, 0, ...)$ and $h(x) = h(-x)$ for all $x \in t^r(p)$.

Let $x, y \in t^r(p)$; then by Minkowski's inequality we have

$$\begin{aligned} h(x+y) &= \left(\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}(x_j+y_j)\right|^{p_k}\right)^{1/M} \\ &= \left(\sum_{k=0}^{\infty}\left[\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}(x_j+y_j)\right|^{p_k/M}\right]^{M}\right)^{1/M} \\ &\le \left(\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j\right|^{p_k}\right)^{1/M} \\ &\qquad + \left(\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}y_j\right|^{p_k}\right)^{1/M} \\ &= h(x)+h(y) \end{aligned} \tag{2.10}$$

Let $\{x^n\}$ be any sequence of the points $x^n \in t^r(p)$ such that $h(x^n - x) \to 0$ and (λ_n) also be any sequence of scalars such that $\lambda_n \to \lambda$. We observe that

$$\begin{aligned} h(\lambda^n x^n - \lambda x) &\le h[(\lambda^n - \lambda)(x^n - x)] \\ &+ h[\lambda(x^n - x)] \\ &+ h[(\lambda^n - \lambda)x]. \end{aligned} \tag{2.11}$$

It follows from $\lambda^n \to \lambda (n \to \infty)$ that $|\lambda^n - \lambda| < 1$ for all sufficiently large n; hence

$$\lim_{n\to\infty} h[(\lambda_n - \lambda)(x^n - x)] \le \lim_{n\to\infty} h(x^n - x) = 0. \tag{2.12}$$

Furthermore, we have

$$\lim_{n\to\infty} h[\lambda(x^n - x)] \le \max\{1, |\lambda|^M\} \lim_{n\to\infty} h(x^n - x) = 0. \tag{2.13}$$

Also, we have

$$\lim_{n\to\infty} h[(\lambda_n - \lambda)x)] \le \lim_{n\to\infty} |\lambda_n - \lambda| h(x) = 0. \tag{2.14}$$

Then, we obtain from (2.11), (2.12), (2.13) and (2.14) that $h(\lambda^n x^n - \lambda x) \to 0$, as $n \to \infty$. This shows that h is a paranorm on $t^r(p)$.

Furthermore, if $h(x) = 0$, then $\left(\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j\right|^{p_k}\right)^{1/M} = 0$. Therefore $\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j\right|^{p_k} = 0$ for each $k \in \mathbb{N}$. Since $0 < r < 1$, we have $\binom{j}{k}(1-r)^{k+1}r^{j-k} > 0$. Then, we obtain $x_k = 0$ for all $k \in \mathbb{N}$. That is, $x = \theta$. This shows that h is a total paranorm.

Now, we show that $t^r(p)$ is complete. Let $\{x^n\}$ be any Cauchy sequence in the space $t^r(p)$, where $x^n = \{x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, ...\}$. Then, for a given $\epsilon > 0$, there exists a positive integer $n_0(\epsilon)$ such that $h(x^n - x^m) < \epsilon$ for all $n, m > n_0(\epsilon)$. Since for each fixed $k \in \mathbb{N}$ that

$$\begin{aligned} |(T^r x^n)_k - (T^r x^m)_k| &\leq \left[\sum_k |(T^r x^n)_k - (T^r x^m)_k|^{p_k} \right]^{\frac{1}{M}} \\ &= h(x^n - x^m) < \epsilon \end{aligned} \tag{2.15}$$

for every $n, m > n_0(\epsilon)$, $\{(T^r x^0)_k, (T^r x^1)_k, (T^r x^2)_k, ...\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(T^r x^n)_k \to (T^r x)_k$ as $n \to \infty$. Using these infinitely many limits $(T^r x)_0, (T^r x)_1, ...$, we define the sequence $\{(T^r x)_0, (T^r x)_1, ...\}$. For each $K \in \mathbb{N}$ and $n, m > n_0(\epsilon)$

$$\left[\sum_{k=0}^{K} |(T^r x^n)_k - (T^r x^m)_k|^{p_k} \right]^{\frac{1}{M}} \leq h(x^n - x^m) < \epsilon. \tag{2.16}$$

By letting $m, K \to \infty$, we have for $n > n_0(\epsilon)$ that

$$h(x^n - x) = \left[\sum_k |(T^r x^n)_k - (T^r x)_k|^{p_k} \right]^{\frac{1}{M}} < \epsilon. \tag{2.17}$$

This shows that $x^n - x \in t^r(p)$. Since $t^r(p)$ is a linear space, we conclude that $x \in t^r(p)$; it follows that $x^n \to x$, as $n \to \infty$ in $t^r(p)$, thus we have shown that $t^r(p)$ is complete. □

Note that the absolute property does not hold on the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$, since there exists at least one sequence in the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ and such that $g(x) \neq g(|x|)$, where $|x| = (|x_k|)$. This says that $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ are the sequence spaces of non-absolute type.

Theorem 2.2. *The sequence spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ of non-absolute type are linearly isomorphic to the spaces $c_0(p)$, $c(p)$ and $\ell(p)$, respectively, where $0 < p_k \leq H < \infty$.*

Proof. To avoid repetition of similar statements, we give the proof only for $t_0^r(p)$. We should show the existence of a linear bijection between the spaces $t_0^r(p)$ and $c_0(p)$. With the notation of (2.3), define the transformation T from $t_0^r(p)$ and $c_0(p)$ by $x \mapsto y = Tx$. The linearity of T is trivial. Furthermore, it is obvious that $x = \theta$ whenever $Tx = \theta$, and hence T is injective.

Let $y \in c_0(p)$ and define the sequence

$$x_k(r) := \sum_{j=k}^{\infty} \binom{j}{k} (-r)^{j-k} (1-r)^{-(j+1)} y_j; \quad k \in \mathbb{N}.$$

Then, we have

$$g(x) = \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k/L} = \sup_{k \in \mathbb{N}} |y_k|^{p_k/L} = g_1(y) < \infty.$$

Thus, we have that $x \in t_0^r(p)$ and consequently T is surjective. Hence, T is a linear bijection and this says that the spaces $t_0^r(p)$ and $c_0(p)$ are linearly isomorphic, as was desired.

□

Theorem 2.3. *Convergence in $t^r(p)$ is stronger than coordinate-wise convergence.*

Proof. First we show that $h(x^n - x) \to 0$, as $n \to \infty$ implies $x_k^n \to x_k$; as $n \to \infty$ for every $k \in \mathbb{N}$. We fix k, then we have

$$\begin{aligned} &\lim_{n\to\infty} \left| \sum_{n=k}^{\infty} \binom{n}{k} (1-r)^{k+1} r^{n-k} [x_k^{(n)} - x_k] \right|^{p_k} \\ &\leq \lim_{n\to\infty} \sum_k \left| \sum_{n=k}^{\infty} \binom{n}{k} (1-r)^{k+1} r^{n-k} [x_k^{(n)} - x_k] \right|^{p_k} \\ &= \lim_{n\to\infty} [h(x^n - x)]^M = 0. \end{aligned} \tag{2.18}$$

Hence, we have for $k = 0$ that

$$\lim_{n\to\infty} \left| \sum_{n=0}^{\infty} (1-r) r^n [x_0^{(n)} - x_0] \right| = 0$$

which gives the fact that $|x_0^{(n)} - x_0| \to 0$, as $n \to \infty$. Similarly, for each $k \in \mathbb{N}$, we have $x_k^n \to x_k$; as $n \to \infty$.

A sequence space λ with a linear topology is called a K-space provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A K-space λ is called an FK-space provided λ is complete linear metric space. An FK-space whose topology is normable is called a BK-space. Given a BK-space $\lambda \supset \phi$, we denote the n th section of a sequence $x = (x_k) \in \lambda$ by $x^{[n]} := \sum_{k=0}^{n} x_k e^{(k)}$, and we say that $x = (x_k)$ has the property AK if $\lim_{n\to\infty} ||x - x^{[n]}||_\lambda = 0$. If AK property holds for every $x \in \lambda$, then we say that the space λ is called AK-space (cf. [7]). Now, we may give the following. □

Theorem 2.4. *The space $t^r(p)$ has AK.*

Proof. For each $x = (x_k) \in t^r(p)$, we put

$$x^{<m>} = \sum_{k=0}^{m} x_k e^{(k)}, \forall m \in \{1, 2, \ldots\}. \tag{2.19}$$

Let $\epsilon > 0$ and $x \in t^r(p)$ be given. Then, there is $N = N(\epsilon) \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k} < \epsilon^M. \tag{2.20}$$

Then we have for all $m \geq N$,

$$\begin{aligned} h(x - x^{<m>}) &= h\left(x - \sum_{k=0}^{m} x_k e^{(k)} \right) \\ &= \left(\sum_{k=m+1}^{\infty} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k} \right)^{1/M} \\ &\leq \left(\sum_{k=N}^{\infty} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k} \right)^{1/M} < \epsilon. \end{aligned} \tag{2.21}$$

This shows that $x = \sum_k x_k e^{(k)}$.

Now, we have to show that this representation is unique. We assume that $x = \sum_k \lambda_k e^{(k)}$. Then for each k,

$$\left(\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}\lambda_j - \sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j\right|^{p_k}\right)^{1/M}$$

$$\begin{aligned}
&\leq \left(\sum_k\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}\lambda_j - \sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j\right|^{p_k}\right)^{1/M} \\
&= h(x-x) = 0 \qquad (2.22)
\end{aligned}$$

Hence, $\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}\lambda_j = \sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j$ for each j. Then, $\lambda_j = x_j$ for each j. Therefore, the representation is unique. $\square$

3. The Basis for the Spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$

Let (λ, h) be a paranormed space. Recall that a sequence (b_k) of the elements of λ is called a basis for λ if and only if, for each $x \in \lambda$, there exists a unique sequence (α_k) of scalars such that

$$h\left(x - \sum_{k=0}^{n}\alpha_k b_k\right) \to 0 \text{ as } n \to \infty.$$

The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$. Since it is known that the matrix domain λ_A of a sequence space λ has a basis if and only if λ has a basis whenever $A = (a_{nk})$ is a triangle (cf. [8, Remark 2.4]), we have the following. Because of the isomorphism T is onto, defined in the proof of Theorem 2.2, the inverse image of the basis of those spaces $c_0(p)$, $c(p)$ and $\ell(p)$ are the basis of the new spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$, respectively. Therefore, we have the following:

Theorem 3.1. *Let $\lambda_k(r) = (T^r x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \leq H < \infty$. Define the sequence $b^{(k)}(r) = \{b^{(k)}(r)\}_{k\in\mathbb{N}}$ of the elements of the space $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ by*

$$b^{(k)}(r) = \begin{cases} \binom{k}{n}(1-r)^{-(k+1)}(-r)^{k-n} & , \quad k \geq n \\ 0 & , \quad 0 \leq k < n \end{cases}$$

for every fixed $k \in \mathbb{N}$. Then

(a): *The sequence $\{b^{(k)}(r)\}_{k\in\mathbb{N}}$ is a basis for the space $t_0^r(p)$, and any $x \in t_0^r(p)$ has a unique representation of the form*

$$x = \sum_k \lambda_k(r) b^{(k)}(r),$$

(b): *The set $e, b^{(1)}(r), b^{(2)}(r), ...$ is a basis for the space $t_c^r(p)$, and any $x \in t_c^r(p)$ has a unique representation of the form*

$$x = le + \sum_k [\lambda_k(r) - l] b^{(k)}(r),$$

where $l = \lim_{k\to\infty}(T^r x)_k$.

(c): *The sequence* $\{b^{(k)}(r)\}_{k\in\mathbb{N}}$ *is a basis for the space* $t^r(p)$*, and any* $x \in t^r(p)$ *has a unique representation of the form*

$$x = \sum_k \lambda_k(r) b^{(k)}(r).$$

4. The $\alpha-, \beta-$ and $\gamma-$Duals of the Spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$

In this section, we state and prove the theorems determining the $\alpha-, \beta-$ and $\gamma-$duals of the sequence spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ of non-absolute type.

We shall firstly give the definition of $\alpha-, \beta-$ and $\gamma-$duals of sequence spaces and after quoting the lemmas which are needed in proving the theorems given in Section 4.

The set $S(\lambda, \mu)$ defined by

$$S(\lambda,\mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x = (x_k) \in \lambda\} \tag{4.1}$$

is called the multiplier space of the sequence spaces λ and μ. One can easily observe for a sequence space ν with $\lambda \supset \nu \supset \mu$ that the inclusions

$$S(\lambda,\mu) \subset S(\nu,\mu) \text{ and } S(\lambda,\mu) \subset S(\lambda,\nu)$$

hold. With the notation of (4.1), the alpha-, beta- and gamma-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \lambda^\beta = S(\lambda, cs) \text{ and } \lambda^\gamma = S(\lambda, bs).$$

The alpha-, beta- and gamma-duals of a sequence space are also referred as Köthe-Toeplitz dual, generalized Köthe-Toeplitz dual and Garling dual of a sequence space, respectively.

For to give the alpha-, beta- and gamma-duals of the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ of non-absolute type, we need the following Lemma:

Lemma 4.1. [7] *Let* $A = (a_{nk})$ *be an infinite matrix. Then, the following statements hold*

(i): $A \in (c_o(p) : \ell(q))$ *if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} M^{-1/p_k} \right|^{q_n} < \infty, \quad \exists M \in \mathbb{N}_2. \tag{4.2}$$

(ii): $A \in (c(p) : \ell(q))$ *if and only if (4.2) holds and*

$$\sum_n \left| \sum_k a_{nk} \right|^{q_n} < \infty. \tag{4.3}$$

(iii): $A \in (c_0(p) : c(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| M^{-1/p_k} < \infty, \ \exists M \in \mathbb{N}_2, \tag{4.4}$$

$$\exists (\alpha_k) \subset \mathbb{R} \ni \lim_{n\to\infty} |a_{nk} - \alpha_k|^{q_n} = 0 \text{ for all } k \in \mathbb{N}, \tag{4.5}$$

$$\exists (\alpha_k) \subset \mathbb{R} \ni \sup_{n\in\mathbb{N}} N^{1/q_n} \sum_k |a_{nk} - \alpha_k| M^{-1/p_k} < \infty, \ \exists M \in \mathbb{N}_2 \text{ and } \forall N \in \mathbb{N}_1. \tag{4.6}$$

(iv): *$A \in (c(p) : c(q))$ if and only if (4.4), (4.5), (4.6) hold and*

$$\exists \alpha \in \mathbb{R} \ni \lim_{n\to\infty} |\sum_k a_{nk} - \alpha|^{q_n} = 0. \tag{4.7}$$

(v): *$A \in (c_o(p) : \ell_\infty(q))$ if and only if*

$$\sup_{n\in\mathbb{N}} \left(\sum_k |a_{nk}| M^{-1/p_k} \right)^{q_n} < \infty, \ \exists M \in \mathbb{N}_2. \tag{4.8}$$

(vi): *$A \in (\ell(p) : \ell_1)$ if and only if*

(a): *Let $0 < p_k \le 1$ for all $k \in \mathbb{N}$. Then*

$$\sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in N} a_{nk} \right|^{p_k} < \infty. \tag{4.9}$$

(b): *Let $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, there exists an integer $M > 1$ such that*

$$\sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in N} a_{nk} M^{-1} \right|^{p_k'} < \infty. \tag{4.10}$$

Lemma 4.2. [10] *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold*

(i): *$A \in (\ell(p) : \ell_\infty)$ if and only if*

(a): *Let $0 < p_k \le 1$ for all $k \in \mathbb{N}$. Then,*

$$\sup_{n,k\in\mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{4.11}$$

(b): *Let $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, there exists an integer $M > 1$ such that*

$$\sup_{n\in\mathbb{N}} \sum_k \left| a_{nk} M^{-1} \right|^{p_k'} < \infty. \tag{4.12}$$

(ii): *Let $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(p) : c)$ if and only if (4.11) and (4.12) hold, and*

$$\lim_{n\to\infty} a_{nk} = \beta_k, \ \forall k \in \mathbb{N}. \tag{4.13}$$

Theorem 4.1. *Let $K \in \mathcal{F}$ and $K^* = \{k \in \mathbb{N} : n \ge k\} \cap K$ for $K \in \mathcal{F}$. Define the sets $T_1^r(p)$, T_2^r, $T_3(p)$ and $T_4(p)$ as follows:*

$$\begin{aligned}
T_1^r(p) &= \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K^*} c_{nk} M^{-1/p_k} \right|^{q_n} < \infty \right\}, \\
T_2^r &= \left\{ a = (a_k) \in w : \sum_n \left| \sum_{k=0}^{n} c_{nk} \right| \text{ exists for each } n \in \mathbb{N} \right\}, \\
T_3(p) &= \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in N} c_{nk} M^{-1} \right|^{p_k'} < \infty, \right\}, \\
T_4(p) &= \left\{ a = (a_k) \in w : \sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in N} c_{nk} \right|^{p_k} < \infty \right\},
\end{aligned}$$

where the matrix $C(r) = (c^r_{nk})$ defined by

$$c^r_{nk} = \begin{cases} \binom{k}{n}(-r)^{k-n}(1-r)^{-(k+1)}a_n & , \quad (k \geq n), \\ 0 & , \quad (0 \leq k < n). \end{cases} \tag{4.14}$$

Then, $[t^r_0(p)]^\alpha = T^r_1(p)$, $[t^r_c(p)]^\alpha = T^r_1(p) \cap T^r_2$ *and*

$$[t^r(p)]^\alpha = \begin{cases} T_3(p) & , \quad 1 < p_k \leq H < \infty, \forall k \in \mathbb{N}, \\ T_4(p) & , \quad 0 < p_k \leq 1, \forall k \in \mathbb{N}. \end{cases} \tag{4.15}$$

Proof. We chose the sequence $a = (a_k) \in w$. We can easily derive that with the (2.3) that

$$a_n x_n = \sum_{k=n}^{\infty} \binom{k}{n} (-r)^{k-n}(1-r)^{-(k+1)} a_n y_k = (C^r y)_n, \quad (n \in \mathbb{N}). \tag{4.16}$$

for all $k, n \in \mathbb{N}$, where $C^r = (c^r_{nk})$ defined by (4.14). It follows from (4.16) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in t^r_0(p)$ if and only if $Cy \in \ell_1$ whenever $y \in c_0(p)$. This means that $a = (a_n) \in [t^r_0(p)]^\alpha$ if and only if $C \in (c_0(p) : \ell_1)$. Then, we derive by (4.2) with $q_n = 1$ for all $n \in \mathbb{N}$ that $[t^r_0(p)]^\alpha = T^r_1(p)$.

Using the (4.3) with $q_n = 1$ for all $n \in \mathbb{N}$ and (4.16), the proof of the $[t^r_c(p)]^\alpha = T^r_1(p) \cap T_2$ can also be obtained in a similar way. Also, using the (4.9),(4.10) and (4.16), the proof of the

$$[t^r(p)]^\alpha = \begin{cases} T_3(p) & , \quad 1 < p_k \leq H < \infty, \forall k \in \mathbb{N}, \\ T_4(p) & , \quad 0 < p_k \leq 1, \forall k \in \mathbb{N}, \end{cases}$$

can also be obtained in a similar way. □

Theorem 4.2. *The matrix $D(r) = (d^r_{nk})$ is defined by*

$$d^r_{nk} = \begin{cases} \sum_{k=0}^{n} \binom{n}{k}(-r)^{n-k}(1-r)^{-(n+1)}a_k & , \quad (0 \leq k \leq n) \\ 0 & , \quad (k > n) \end{cases} \tag{4.17}$$

for all $k, n \in \mathbb{N}$. Define the sets $T^r_5(p)$, T^r_6, T^r_7, $T_8(p)$, $T_9(p)$ and $T_{10}(p)$ as follows:

$$\begin{aligned}
T^r_5(p) &= \bigcup_{M>1} \left\{ a = (a_k) \in w : \sum_k \left| d^r_{nk} M^{-1/p_k} \right| < \infty \right\}, \\
T^r_6 &= \left\{ a = (a_k) \in w : \lim_{n\to\infty} |d^r_{nk}| \ \text{exists for each } k \in \mathbb{N} \right\}, \\
T^r_7 &= \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} |d^r_{nk}| \ \text{exists} \right\}, \\
T_8(p) &= \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_k \left| d_{nk} M^{-1} \right|^{p'_k} < \infty \right\}, \\
T_9(p) &= \left\{ a = (a_k) \in w : d_{nk} < \infty \right\}, \\
T_{10}(p) &= \left\{ a = (a_k) \in w : \sup_{n,k\in\mathbb{N}} |d_{nk}|^{p_k} < \infty \right\}.
\end{aligned}$$

Then, $[t^r_0(p)]^\beta = T^r_5(p) \cap T^r_6$, $[t^r_c(p)]^\beta = [t^r_0(p)]^\beta \cap T^r_7$ *and*

$$[t^r(p)]^\beta = \begin{cases} T_8(p) \cap T_9(p) & , \quad 1 < p_k \leq H < \infty, \forall k \in \mathbb{N}, \\ T_9(p) \cap T_{10}(p) & , \quad 0 < p_k \leq 1, \forall k \in \mathbb{N}. \end{cases} \tag{4.18}$$

Proof. We give the proof again only for the space $t_0^r(p)$. Consider the equation

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left[\sum_{k=j}^{\infty}\binom{k}{j}(-r)^{k-j}(1-r)^{-(k+1)}y_k\right]a_k \\ &= \sum_{k=0}^{n}\left[\sum_{j=0}^{k}\binom{k}{j}(-r)^{k-j}(1-r)^{-(k+1)}a_j\right]y_k = (D^r y)_n, \qquad (4.19)\end{aligned}$$

where $D^r = (d_{nk}^r)$ defined by (4.17). Thus, we decude from (4.19) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in t_0^r(p)$ if and only if $D^r y \in c$ whenever $y = (y_k) \in c_0(p)$. That is to say that $a = (a_k) \in [t_0^r(p)]^\beta$ if and only if $D^r \in (c_0(p) : c)$. Therefore, we derive from (4.4),(4.5) and (4.6) with $q_n = 1$ for all $n \in \mathbb{N}$ that $[t_0^r(p)]^\beta = T_5^r(u,p) \cap T_6^r(u)$.

Using the (4.4),(4.5), (4.6) and (4.7) with $q_n = 1$ for all $n \in \mathbb{N}$ and (4.19), the proofs of the $[t_c^r(p)]^\beta = [t_0^r(p)]^\beta \cap T_7^r$ can also be obtained in a similar way. Also, using the (4.11),(4.12), (4.13) and (4.19), the proofs of the

$$[t^r(p)]^\beta = \begin{cases} T_8(p) \cap T_9(p) & , \quad 1 < p_k \le H < \infty, \forall k \in \mathbb{N}, \\ T_9(p) \cap T_{10}(p) & , \quad 0 < p_k \le 1, \forall k \in \mathbb{N}. \end{cases}$$

can also be obtained in a similar way. □

Theorem 4.3. *Define the set* $T_6^r(u)$ *by*

$$T_{11}^r(u) = \left\{ a = (a_k) \in w : \left\{ \sum_{j=0}^{k}\binom{k}{j}(-r)^{k-j}(1-r)^{-(k+1)}a_j \right\} \in bs \right\}.$$

Then, $[t_0^r(p)]^\gamma = T_5^r(p) \cap T_6^r$, $[t_c^r(p)]^\gamma = [t_0^r(p)]^\gamma \cap T_{11}^r$ *and*

$$[t^r(p)]^\gamma = \begin{cases} T_8(p) & , \quad 1 < p_k \le H < \infty, \forall k \in \mathbb{N}, \\ T_{10}(p) & , \quad 0 < p_k \le 1, \forall k \in \mathbb{N}. \end{cases}$$

Proof. This is obtained in the similar way used in the proof of Theorem 4.2. □

5. Certain Matrix Mappings on the Sequence Spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$

In this section, we characterize some matrix mappings on the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$.

We known that, if $t_0^r(p) \cong c_0(p)$, $t_c^r(p) \cong c(p)$ and $t^r(p) \cong \ell(p)$, we can say: The equivalence "$x \in t_0^r(p), t_c^r(p)$ or $t^r(p)$ if and only if $y \in c_0(p), c(p)$ or $\ell(p)$" holds.

In what follows, for brevity, we write,

$$\tilde{a}_{nk} := \sum_{k=0}^{n}\binom{n}{k}(-r)^{n-k}(1-r)^{-(n+1)}a_{nk}$$

for all $k, n \in \mathbb{N}$.

Theorem 5.1. *Suppose that the entries of the infinite matrices* $A = (a_{nk})$ *and* $E = (e_{nk})$ *are connected with the relation*

$$e_{nk} := \tilde{a}_{nk} \qquad (5.1)$$

for all $k, n \in \mathbb{N}$ *and* μ *be any given sequence space. Then,*

(i): *$A \in (t_0^r(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_0^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (c_0(p) : \mu)$.*
(ii): *$A \in (t_c^r(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_c^r(0)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (c(p) : \mu)$.*
(iii): *$A \in (t^r(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (\ell(p) : \mu)$.*

Proof. We prove only part of (i). Let μ be any given sequence space. Suppose that (5.1) holds between $A = (a_{nk})$ and $E = (e_{nk})$, and take into account that the spaces $t_0^r(p)$ and $c_0(p)$ are linearly isomorphic.

Let $A \in (t_0^r(p) : \mu)$ and take any $y = (y_k) \in c_0(p)$. Then $ET(r)$ exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in T_5^r(p) \cap T_6^r$ which yields that $\{e_{nk}\}_{k\in\mathbb{N}} \in c_0(p)$ for each $n \in \mathbb{N}$. Hence, Ey exists and thus

$$\sum_k e_{nk} y_k = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$.

We have that $Ey = Ax$ which leads us to the consequence $E \in (c_0(p) : \mu)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_0^r(p)\}^\beta$ for each $n \in \mathbb{N}$ and $E \in (c_0(p) : \mu)$ hold, and take any $x = (x_k) \in t_0^r(p)$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} \binom{j}{k} (-r)^{j-k} (1-r)^{-(j+1)} a_{nj} \right] y_k$$

for all $n \in \mathbb{N}$, that $Ey = Ax$ and this shows that $A \in (t_0^r(p) : \mu)$. This completes the proof of part of (i). $\square$

Theorem 5.2. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} := \sum_{j=n}^{\infty} \binom{j}{n} (1-r)^{n+1} r^{(j-n)} a_{jk} \text{ for all } k, n \in \mathbb{N}. \tag{5.2}$$

Let μ be any given sequence space. Then,
(i): *$A \in (\mu : t_0^r(p))$ if and only if $B \in (\mu : c_0(p))$.*
(ii): *$A \in (\mu : t_c^r(p))$ if and only if $B \in (\mu : c(p))$.*
(iii): *$A \in (\mu : t^r(p))$ if and only if $B \in (\mu : \ell(p))$.*

Proof. We prove only part of (i). Let $z = (z_k) \in \mu$ and consider the following equality.

$$\sum_{k=0}^{m} b_{nk} z_k = \sum_{j=n}^{\infty} \binom{j}{n} (1-r)^{n+1} r^{j-n} \left(\sum_{k=0}^{m} a_{jk} z_k \right) \quad \text{for all } m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $(Bz)_n = \{T(r)(Az)\}_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Az \in t_0^r(p)$ whenever $z \in \mu$ if and only if $Bz \in c_0(p)$ whenever $z \in \mu$. This completes the proof of part of (i). $\square$

Of course, Theorems 5.1 and 5.2 have several consequences depending on the choice of the sequence space μ. Whence by Theorem 5.1 and Theorem 5.2, the necessary and sufficient conditions for $(t_0^r(p) : \mu)$, $(\mu : t_0^r(p))$, $(t_c^r(p) : \mu)$, $(\mu : t_c^r(p))$ and $(t^r(p) : \mu)$, $(\mu : t^r(p))$ may be derived by replacing the entries of C and A by those of the entries of $E = C\{T(r)\}^{-1}$ and $B = T(r)A$, respectively; where the necessary and sufficient conditions on the matrices E and B are read from the concerning results in the existing literature.

The necessary and sufficient conditions characterizing the matrix mappings between the sequence spaces of Maddox are determined by Grosse-Erdmann [7]. Let N and K denote the finite subset of $\mathbb{N}$, L and M also denote the natural numbers. Prior to giving the theorems, let us suppose that (q_n) is a non-decreasing bounded sequence of positive numbers and consider the following conditions:

$$\lim_n |a_{nk}|^{q_n} = 0, \text{ for all } k \tag{5.3}$$

$$\forall L, \exists M \ni \sup_n L^{1/q_n} \sum_k |a_{nk}| M^{-1/p_k} < \infty, \tag{5.4}$$

$$\sup_n |\sum_k a_{nk}|^{q_n} < \infty, \tag{5.5}$$

$$\lim_n |\sum_k a_{nk}|^{q_n} = 0, \tag{5.6}$$

$$\forall L, \sup_n \sup_{k\in K_1} |a_{nk} L^{1/q_n}|^{p_k} < \infty, \tag{5.7}$$

$$\forall L, \exists M \ni \sup_n \sum_{k\in K_2} |a_{nk} L^{1/q_n} M^{-1}|^{p'_k} < \infty, \tag{5.8}$$

$$\forall M, \lim_n (\sum_k |a_{nk} M^{1/p_k})^{q_n} = 0, \tag{5.9}$$

$$\forall M, \sup_n \sum_k |a_{nk}| M^{1/p_k} < \infty, \tag{5.10}$$

$$\forall M, \exists (\alpha_k) \ni \lim_n (\sum_k |a_{nk} - \alpha_k| M^{1/p_k})^{q_n} = 0, \tag{5.11}$$

$$\forall M, \sup_K \sum_n |\sum_{k\in K} a_{nk} M^{1/p_k}|^{q_n} < \infty. \tag{5.12}$$

Lemma 5.1. *Let $A = (a_{nk})$ be an infinite matrix. Then*

- **(i):** *$A = (a_{nk}) \in (c_0(p) : \ell_\infty(q))$ if and only if (4.8) holds.*
- **(ii):** *$A = (a_{nk}) \in (c(p) : \ell_\infty(q))$ if and only if (4.8) and (5.5) hold.*
- **(iii):** *$A = (a_{nk}) \in (\ell(p) : \ell_\infty)$ if and only if (4.11) and (4.12) hold.*
- **(iv):** *$A = (a_{nk}) \in (c_0(p) : c(q))$ if and only if (4.4), (4.5) and (4.6) hold.*
- **(v):** *$A = (a_{nk}) \in (c(p) : c(q))$ if and only if (4.4), (4.5), (4.6) and (4.7) hold.*
- **(vi):** *$A = (a_{nk}) \in (\ell(p) : c)$ if and only if (4.11), (4.12) and (4.13) hold.*
- **(vii):** *$A = (a_{nk}) \in (c_0(p) : c_0(q))$ if and only if (5.3) and (5.4) hold.*
- **(viii):** *$A = (a_{nk}) \in (c(p) : c_0(q))$ if and only if (5.3), (5.4) and (5.6) hold.*
- **(ix):** *$A = (a_{nk}) \in (\ell(p) : c_0(q))$ if and only if (5.3), (5.7) and (5.8) hold.*
- **(x):** *$A = (a_{nk}) \in (\ell_\infty(p) : c_0(q))$ if and only if (5.9) holds.*
- **(xi):** *$A = (a_{nk}) \in (\ell_\infty(p) : c(q))$ if and only if (5.10) and (5.11) hold.*
- **(xii):** *$A = (a_{nk}) \in (\ell_\infty(p) : \ell(q))$ if and only if (5.12) holds.*
- **(xiii):** *$A = (a_{nk}) \in (c_0(p) : \ell(q))$ if and only if (4.2) holds.*
- **(xiv):** *$A = (a_{nk}) \in (c(p) : \ell(q))$ if and only if (4.2) and (4.4) hold.*

Corollary 5.1. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i): *$A \in (t_0^r(p) : \ell_\infty(q))$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_0^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and (4.8) holds with $\tilde{a}_{nk}$ instead of a_{nk} with $q = 1$.*

(ii): *$A \in (t_0^r(p) : c_0(q))$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_0^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and (5.3) and (5.4) hold with $\tilde{a}_{nk}$ instead of a_{nk} with $q = 1$.*

(iii): *$A \in (t_0^r(p) : c(q))$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_0^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and (4.4), (4.5) and (4.6) hold with $\tilde{a}_{nk}$ instead of a_{nk} with $q = 1$.*

Corollary 5.2. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i): *$A \in (t_c^r(p) : \ell_\infty(q))$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_c^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and (4.8) and (5.5) hold with $\tilde{a}_{nk}$ instead of a_{nk} with $q = 1$.*

(ii): *$A \in (t_c^r(p) : c_0(q))$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_c^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and (5.3), (5.4) and (5.6) hold with $\tilde{a}_{nk}$ instead of a_{nk} with $q = 1$.*

(iii): *$A \in (t_c^r(p) : c(q))$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_c^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and (4.4), (4.5), (4.6) and (4.7) hold with $\tilde{a}_{nk}$ instead of a_{nk} with $q = 1$.*

Corollary 5.3. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i): *$A \in (t^r(p) : \ell_\infty)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and (4.11) and (4.12) hold with $\tilde{a}_{nk}$ instead of a_{nk}.*

(ii): *$A \in (t^r(p) : c_0(q))$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and (5.3), (5.7) and (5.8) hold with $\tilde{a}_{nk}$ instead of a_{nk} with $q = 1$.*

(iii): *$A \in (t^r(p) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and (4.11), (4.12) and (4.13) hold with $\tilde{a}_{nk}$ instead of a_{nk}.*

Corollary 5.4. *Let $A = (a_{nk})$ be an infinite matrix and b_{nk} be defined by (5.2). Then, following statements hold:*

(i): *$A \in (\ell_\infty(q) : t_0^r(p))$ if and only if (5.9) holds with b_{nk} instead of a_{nk} with $q = 1$.*

(ii): *$A \in (c_0(q) : t_0^r(p))$ if and only if (5.3) and (5.4) hold with b_{nk} instead of a_{nk} with $q = 1$.*

(iii): *$A \in (c(q) : t_0^r(p))$ if and only if (5.3), (5.4) and (5.6) holds with b_{nk} instead of a_{nk} with $q = 1$.*

Corollary 5.5. *Let $A = (a_{nk})$ be an infinite matrix and b_{nk} be defined by (5.2). Then, following statements hold:*

(i): *$A \in (\ell_\infty(q) : t_c^r(p))$ if and only if (5.10) and (5.11) hold with b_{nk} instead of a_{nk} with $q = 1$.*

(ii): *$A \in (c_0(q) : t_c^r(p))$ if and only if (4.4), (4.5) and (4.6) hold with b_{nk} instead of a_{nk} with $q = 1$.*

(iii): *$A \in (c(q) : t_c^r(p))$ if and only if (4.4), (4.5), (4.6) and (4.7) hold with b_{nk} instead of a_{nk} with $q = 1$.*

Corollary 5.6. *Let $A = (a_{nk})$ be an infinite matrix and b_{nk} be defined by (5.2). Then, following statements hold:*

(i): *$A \in (\ell_\infty(q) : t^r(p))$ if and only if (5.12) holds with b_{nk} instead of a_{nk} with $q = 1$.*

(ii): *$A \in (c_0(q) : t^r(p))$ if and only if (4.2) holds with b_{nk} instead of a_{nk} with $q = 1$.*

(iii): *$A \in (c(q) : t^r(p))$ if and only if (4.2) and (4.4) hold with b_{nk} instead of a_{nk} with $q = 1$.*

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A MESH-FREE TECHNIQUE OF NUMERICAL SOLUTION OF NEWLY DEFINED CONFORMABLE DIFFERENTIAL EQUATIONS

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Abstract. Motivated by the recently defined conformable derivatives proposed in [2], we introduced a new approach of solving the conformable ordinary differential equation with the mesh-free numerical method. Since radial basis function collocation technique has outstanding feature in comparison with the other numerical methods, we use it to solve non-integer order of differential equation. We subsequently present the results of numerical experimentation to show that our algorithm provide successful consequences.

1. Introduction

Until quite recently, the question of how to take non-integer order of derivative or integration was phenomenon among the mathematicans. However together with the development of mathematics knowledge, this question was answered via fractional differentiation and integration [8], [9], [11], [12]. Although there are a number of different type of definition of fractional derivatives or integrations, Riemann-Liouville and Caputo are the most popular ones among them. Then Abdeljawad [1] and Khalil et. al. [7] defined the limit based conformable derivative which is another type of fractional derivative and integrations. In more recent times, Anderson and Ulness [2] have described another precise definition of conformable derivatives motivated by a proportional derivative controller. As a result of this new definition of conformable derivatives, its differential equations need to be handled.

In this paper, we develop a meshless algorithm for the numerical solution of the conformable differential equations by taking advantageous of radial basis function (RBF) interpolation [3], [5], [10]. The goal of this approach is to acquire approximate solution of conformable differential equations with RBF collocation method. Of course this approach would provide an insight the solution of more complex cases.

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The remainder of this work is organized as follows: In Section 2, the conformable derivatives are summarised, along with the newly defined type. In Section 3, the RBF interpolation method is reviewed while in Section 4 the numerical scheme of solving conformable ordinary differential equation using mesh-free method is introduced and we also reviewed the RBF collocation technique. Numerical examples are given in Section 5, while some conclusions and further directions of research are discussed in Section 6.

2. A class of conformable derivatives

In [7] and [1], a new version of limit based fractional derivative called conformable derivative have been defined via

$$D^{\alpha}u(x) = \lim_{\xi\to 0} \frac{u(x+\xi x^{1-\alpha}) - u(x)}{\xi}, \tag{2.1}$$

on condition that limit exists. Another proposed limit based fractional derivative is

$$D^{\alpha}u(x) = \lim_{\xi\to 0} \frac{u(xe^{\xi x^{-\alpha}}) - u(x)}{\xi}, \tag{2.2}$$

in [6]. For both approaches the conformable derivative can be summarised via

$$D^{\alpha}u(x) = x^{1-\alpha}\frac{d}{dx}u(x), \tag{2.3}$$

where $\frac{d}{dx}$ denotes the classical derivative operators. In addition to this, Anderson and Ulness [2] introduced a new class of conformable derivatives via proportional-derivative controller.

Definition 2.1. [2] Let $\alpha \in [0,1]$. The conformable derivative operator D^{α} describe as

$$D^{\alpha}u(x) = \kappa_1(\alpha,x)u(x) + \kappa_0(\alpha,x)\frac{d}{dx}u(x) \tag{2.4}$$

where $\kappa_1, \kappa_0 : [0,1]\times\mathbb{R} \to [0,\infty)$ are continuous function such that

$$\begin{aligned}
&\lim_{\alpha\to 0^+}\kappa_1(\alpha,x) = 1, \qquad \lim_{\alpha\to 0^+}\kappa_0(\alpha,x) = 0, \qquad \text{for all x} \in \mathbb{R},\\
&\lim_{\alpha\to 1^-}\kappa_1(\alpha,x) = 0, \qquad \lim_{\alpha\to 1^-}\kappa_0(\alpha,x) = 1, \qquad \text{for all x} \in \mathbb{R},\\
&\kappa_1(\alpha,x), \kappa_0(\alpha,x) \neq 0, \qquad \alpha \in (0,1], \qquad \text{for all x} \in \mathbb{R}.
\end{aligned}$$

So, for instance, one can define the conformable derivative operator

$$D^{\alpha}u(x) = (1-\alpha)e^{\alpha}u(x) + \alpha e^{1-\alpha}\frac{d}{dx}u(x), \tag{2.5}$$

or

$$D^{\alpha}u(x) = \cos(\alpha\pi/2)e^{\alpha}u(x) + \sin(\alpha\pi/2)e^{1-\alpha}\frac{d}{dx}u(x). \tag{2.6}$$

This new definition of conformable derivative enables to compute the non-integer order of derivatives via classical derivative operator. Thus, conformable differential equations can be solved with the numerical methods after this transformation has been applied. In next section, we will summarised the RBF methods which is one of the mesh-free techniques and then applied it to solve conformable differential equations.

3. Radial basis function interpolation method

The history of the RBF approximation goes back to 1968 with Hardy who introduced the multiquadric RBFs in academia [4]. Thereafter RBF method become increasingly popular interpolation technique as it provides us delicately and accurately results with no mesh. Not only interpolation or quadrature of any function, but also solving partial differential equations is also an application area of RBFs technique.

One can define the RBF interpolation as follows:

Definition 3.1. Consider a given data set $\mathbf{f} = (f_1, ..., f_N)^T \in \mathbb{R}^N$ of function values, taken from an unknown function $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}$ at scattered data points $\mathbf{x}_k \in \mathbb{R}^d$, $k = 1, ..., N$ such that $\mathbf{f}_k = \mathbf{f}(x_k)$ and $d \geq 1$. The RBF interpolation is given by

$$P_f(\mathbf{x}) = \sum_{k=1}^{N} a_k \varphi(\|\mathbf{x} - \mathbf{x_k}\|), \tag{3.1}$$

where $\varphi(\cdot)$ is a radial function and $\|\cdot\|$ is the Euclidean distance. The coefficient a_j can be determined from interpolation requirements $P_f(\mathbf{x}_j) = \mathbf{f}_j$ by solving the following symmetric linear system:

$$\mathbf{A}\mathbf{a} = \mathbf{f}, \tag{3.2}$$

where the matrix $A_{(N\times N)}$ is constructed for φ_{jk} such that $\varphi_{jk} = \varphi(\|x_j - x_k\|)$, $j, k = 1, \ldots, N$.

Here the basis function φ must be choose as a positive definite function. Additionally, radial basis functions can be divided into two major groups: piecewise smooth and infinitely smooth which are given in Table 1 and Table 2. The rate of convergence in the infinitely smooth RBFs is quicker in comparison with the piecewise smooth RBFs which cause to an algebraical rate of convergence.

Piecewise Smooth RBFs	$\varphi(r)$
Piecewise Polynomial (R_n)	$\|r\|^n$, n odd
Thin Plate Spline (TPS_n)	$\|r\|^n ln\|r\|$, n even

Table 1. Piecewise Smooth

Additionally, RBFs can be expressed by using a scaling parameter named the shape parameter ε. This can be done in the manner that $\varphi(r)$ is replaced by $\varphi(\varepsilon r)$.

Infinitely Smooth RBFs	$\varphi(r)$
Multiquadric (MQ)	$\sqrt{1+r^2}$
Inverse Multiquadric (IMQ)	$\frac{1}{\sqrt{1+r^2}}$
Inverse Quadratic (IQ)	$\frac{1}{1+r^2}$
Gaussian (GA)	e^{-r^2}
Bessel (BE)	$J_0(2r)$

TABLE 2. Infinitely Smooth

In general shape parameter have been chosen arbitrarily since there are no exact results about how to choose best shape parameter.

4. Numerical scheme using mesh-free technique

Together with the development of derivative concept, the question of how to solve non-integer order differential equations have arisen in the scientific area. One of the similar problem has been faced for the conformable differential equations since it contains the non-integer order derivative terms. However through the definition of conformable derivative operator one can transform it to classical ordinary differential equations that there are huge amount of literature about it. Thus by applying the mesh-free numerical methods, we can find an approximation results of conformable differential equations. The conformable ordinary differential equation can be expressed via

$$D^\alpha u(x) + \vartheta(x)u(x) = v(x), \qquad u_0(x) = u(x_0). \tag{4.1}$$

Then by substituting of equation (2.4) into equation (4.1), we get

$$\kappa_1(\alpha, x)u(x) + \kappa_0(\alpha, x)\frac{d}{dx}u(x) + \vartheta(x)u(x) = v(x). \tag{4.2}$$

Then by rearranging of equation (4.2), we obtain the below classical ordinary differential equation, that is

$$\frac{d}{dx}u(x) + A(\alpha, x)u(x) = B(\alpha, x), \qquad u_0(x) = u(x_0), \tag{4.3}$$

where

$$A(\alpha, x)u(x) = \frac{\kappa_1(\alpha, x) + \vartheta(x)}{\kappa_0(\alpha, x)} \qquad \text{and} \qquad B(\alpha, x) = \frac{v(x)}{\kappa_0(\alpha, x)}. \tag{4.4}$$

Now the above equation can be solved easily by applying the RBF collocation method which will present next section.

4.1. **RBF collocation technique.** In order to solve equation (4.3) by numerically we use the RBF collocation method which is quite popular method in the engineering and applied mathematics. Let $\mathbf{x}_{k=1}^{N}$ be the collocation points for interior and boundary region. Then by using definition of RBF interpolation, we get

$$\sum_{k=1}^{N} a_k \left[\frac{d}{dx}u(x) + A(\alpha, x)\right] \varphi(\|\mathbf{x} - \mathbf{x_k}\|) = B(\alpha, x), \tag{4.5}$$

with the boundary condition

$$\sum_{k=1}^{N} a_k \varphi(\|\mathbf{x_0} - \mathbf{x_k}\|) = u(x_0). \tag{4.6}$$

Then by using the points $\mathbf{x}_{k=1}^{N}$, we can collocate the equations (4.5) and (4.6) to determine the unknown coefficients a_k's. Thus the unknown function value $u(x)$ can be calculated by using the determined coefficients with collocation method.

An algorithm for RBF collocation of conformable differential equation is as follows:

Algorithm 1: RBF collocation method for conformable differential equation

Require: Equally spaced grid data decomposition for $0, M$.
1: Initialize the matrix $\mathbf{A}$ and $\mathbf{f}$ via collocation points $\mathbf{x}_{k=1}^{N}$.
2: Construct and solve the matrix equality $\mathbf{Aa} = \mathbf{f}$ to determine the unknown values of a_k's.
3: By using the value of a_k's, calculate the solution of equation for each collocation points.
4: **return** Approximation value

5. Numerical experiments

In this section, we presents some numerical results to verify proposed algorithm. To do that, we take the first order conformable ODE which is solved by RBF collocation technique.

5.1. **Numerical solution of conformable ODE.** For this example, we take the below conformable ODE [2] to solve it via RBF method,

$$D^{\alpha}u(x) + u(x) = v(x) \tag{5.1}$$

with the boundary condition

$$u_0(x) = u(x_0) \tag{5.2}$$

Let x_i be equally spaced grid points in the interval $0 \leq x_i \leq M$ such that $1 \leq i \leq N$, $x_1 = 0$ and $x_N = M$. Additionally, because collocation approach has been used we not only require an expression for the value of the function

$$u(x) = \sum_{k=1}^{N} a_j \varphi(\|x - x_k\|) \tag{5.3}$$

but also for the conformal derivative given in (5.1). Thus, by conformal differentiating (5.3), we get

$$D^\alpha u(x) = \sum_{k=1}^{N} a_j D^\alpha \varphi(\|x - x_k\|) \tag{5.4}$$

where D^αdenotes the conformable derivative the with respect to x. In a particular case of Multiquadric and Gaussian basis functions, we have

$$D^\alpha \varphi(\|x - x_k\|) = \quad \kappa_1(\alpha, x)\sqrt{\|x - x_k\|^2 + \varepsilon^2} + \kappa_0(\alpha, x)\frac{x - x_k}{\sqrt{\|x - x_k\|^2 + \varepsilon^2}}$$

$$D^\alpha \varphi(\|x - x_k\|) = \quad \kappa_1(\alpha, x)e^{-\|x-x_k\|^2/\varepsilon^2} - \kappa_0(\alpha, x)\frac{2(x - x_k)}{\varepsilon^2}e^{-\|x-x_k\|^2/\varepsilon^2} \tag{5.5}$$

where κ_0 and κ_1 are given in Definition 3.1. So in order to determine the value of a_j's in equation (5.3), we need to solve

$$\sum_{k=1}^{N} a_j D^\alpha \varphi(\|x_j - x_k\|) + \sum_{k=1}^{N} a_j \varphi(\|x_j - x_k\|) = v(x) \tag{5.6}$$

by using

$$\sum_{k=1}^{N} a_j \varphi(\|x_1 - x_k\|) = u(x_0) \tag{5.7}$$

where $j = 2, \ldots, N$. If we put the equations (5.5) into equation (5.6), we get the classical ODE which can be solved easily. In other words, one need to solve below algebraic systems

$$\phi_{[N\times N]} a_{[N\times 1]} = \nu_{[N\times 1]} \tag{5.8}$$

where

$$\phi = \begin{pmatrix} D^\alpha \varphi_{1,1} + \varphi_{1,1} & \dots & D^\alpha \varphi_{1,N} + \varphi_{1,N} \\ D^\alpha \varphi_{2,1} + \varphi_{2,1} & \dots & D^\alpha \varphi_{2,N} + \varphi_{2,N} \\ \vdots & \ddots & \vdots \\ D^\alpha \varphi_{N,1} + \varphi_{N,1} & \dots & D^\alpha \varphi_{N,N} + \varphi_{N,N} \end{pmatrix}, a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}, \nu = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

to determine a_i's. Then one can obtain the numerical solution using a_i's into RBF method. The numerical experiment results has been presented for different left hand side functions such as $v_1(x) = x\sqrt{x} + 1/2x^2\sqrt{x} + x^2$, $v_2(x) = e^{-x}(x + \sqrt{x}/2)$ and $v_3(x) = (1 - \sqrt{x}/2)\cos(4\sqrt{x}) - \sin(4\sqrt{x})$ in Figures 1, 2 and 3 respectively. These results confirm that RBF method converge the solution of ordinary conformable differential equations.

Function	**Alpha**	ε	**Number of Nodes**	**Max-Error**	**RMS-Error**
$v_1(x)$	0.5	5	500	3.829195e-006	5.527372e-008
$v_2(x)$	0.5	5	500	2.352912e-005	3.757950e-007
$v_3(x)$	0.5	5	500	2.267579e-004	3.500312e-006

TABLE 3. Numerical results of conformable ordinary differential equation via RBF using Multiquadric on the domain $[0, 10]$.

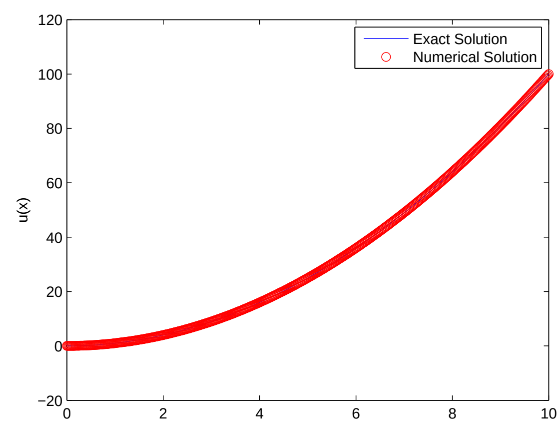

FIGURE 1. $u(x)$ versus x using Multiquadric basis function with $\varepsilon = 5$ for $v_1(x) = x\sqrt{x} + 1/2x^2\sqrt{x} + x^2$: Exact solution (Blue) and Numerical solution (Red circle) on equally spaced evaluation grid.

In the numerical experiments, *Max-Error* represents the maximum modulus error, i.e., $\|f - g\|_\infty$ and *Rms-Error* represents the standard root mean squared error, i.e.

$$\sqrt{\frac{\sum_{i=1}^{Neval} |f_i - g_i|^2}{Neval}}, \tag{5.9}$$

where f is the exact solution, g is the approximate solution, and $Neval$ is the number of the test points.

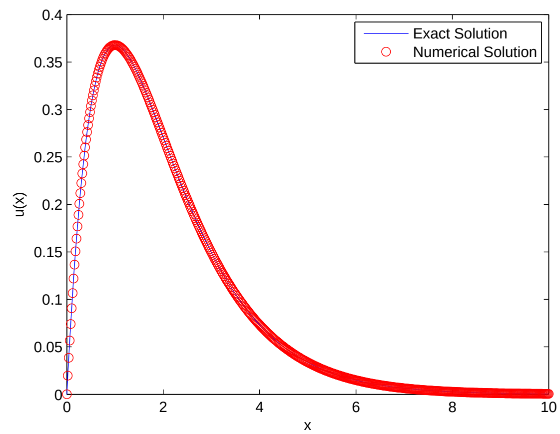

FIGURE 2. $u(x)$ versus x using Multiquadric basis function with $\varepsilon = 5$ for $v_2(x) = e^{-x}(x + \sqrt{x}/2)$: Exact solution (Blue) and Numerical solution (Red circle) on equally spaced evaluation grid.

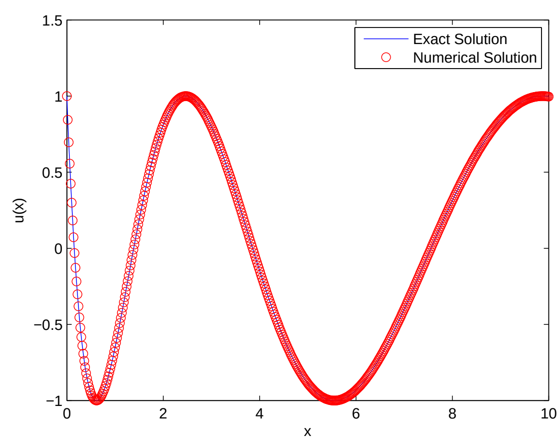

FIGURE 3. $u(x)$ versus x using Multiquadric basis function with $\varepsilon = 5$ for $v_3(x) = (1 - \sqrt{x}/2)\cos(4\sqrt{x}) - \sin(4\sqrt{x})$: Exact solution (Blue) and Numerical solution (Red circle) on equally spaced evaluation grid.

6. Concluding remark

A new radial basis function collocation technique to solve conformable ordinary differential equation is proposed and tested in this paper. To do that Gaussian or Multiquadric basis functions can be used. In order to verify this methods stability, we have presented some numerical results. Thus this study would help to solve modelled non-integer order of differential equations.

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PERIODIC SOLUTIONS FOR THIRD ORDER DELAY DIFFERENTIAL EQUATION IMPULSES WITH FREDHOLM OPERATOR OF INDEX ZERO

S.BALAMURALITHARAN

Abstract. In this paper the periodic solutions for third order delay differential equation of the form

$$x'''(t) + f(t, x''(t)) + g(t, x'(t)) + h(x(t - \tau(t)) = p(t), t \geq 0, t \neq t_k,$$

is investigated. We derive a third order delay differential equation with Fredholm operator of index zero and periodic solution. We obtain the existence of periodic solution and Mawhin's continuation theorem. The delay conditions for the Schwarz inequality of the periodic solutions are also obtained. An example is also furnished which demonstrates validity of main result. Some new positive periodic criteria are given. Therefore it has at least one 2π-periodic solution.

1. Introduction

The theory of impulsive delay differential equations is promising as an important role of investigation, since it is better than the corresponding theory of delay differential equation without impulse effects. Furthermore, such equations may demonstrate several real-world phenomena in physics,chemistry, biology, engineering, etc. In the last few years the theory of periodic solutions and delay differential equations with impulses has been studied by many authors, respectively [3, 5, 7, 8]. There are several books and a lot of papers dealing with the periodic solution of delay differential equations [1, 2, 4, 6, 9]. Periodic solutions of impulsive delay differential equations is a new research area and there are many publications in this field. The paper deals with impulsive equations with constant delay and Fredholm operator of index zero. We obtain the theorems of existence of periodic solution based on the following Mawhin's continuation theorem.

Let $PC(\mathbb{R}, \mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R}, x(t)$ be continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)\}$,

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$PC^1(\mathbb{R},\mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R}, x(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k^-) = x'(t_k)\}$.
$PC^2(\mathbb{R},\mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R}, x(t)$ is continuous everywhere except for some t_k at which $x''(t_k^+)$ and $x''(t_k^-)$ exist and $x''(t_k^-) = x''(t_k)\}$.
Let $X = \{x(t) \in PC^1(\mathbb{R},\mathbb{R}), x(t+T) = x(t)\}$ with norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$, where $|x|_\infty = \sup_{t\in[0,T]} |x(t)|$,
$Y = PC(\mathbb{R},\mathbb{R})\times\mathbb{R}^n\times\mathbb{R}^n$, with norm $\|y\| = \max\{|u|_\infty, |c|\}$, where $u \in PC(\mathbb{R},\mathbb{R}), c = (c_1, \dots c_{2n}) \in R^n \times \mathbb{R}^n$, $|c| = \max_{1\le k\le 2n}\{|c_k|\}$.
$Z = PC(\mathbb{R},\mathbb{R})\times\mathbb{R}^n\times\mathbb{R}^n$, with norm $\|z\| = \max\{|v|_\infty, |d|\}$, where $v \in PC(\mathbb{R},\mathbb{R}), d = (d_1, \dots d_{2n}) \in R^n \times \mathbb{R}^n$, $|d| = \max_{1\le k\le 2n}\{|d_k|\}$.
Then X, Y and Z are Banach spaces. $L : D(L) \subset X \to Y$ and $L : D(L) \subset Y \to Z$ are a Fredholm operator of index zero, where $D(L)$ denotes the domain of L. $P : X \to X, Q : Y \to Y, R : Z \to Z$ are projectors such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad \ker R = \operatorname{Im} L,$$

$$X = \ker L \oplus \ker P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q, \quad Z = \operatorname{Im} L \oplus \operatorname{Im} R.$$

It continues that

$$L|_{D(L)\cap\ker P} : D(L) \cap \ker P \to \operatorname{Im} L$$

is invertible and we assume the inverse of that map by K_p. Let Ω be an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact.
Similarly it follows that

$$L|_{D(L)\cap\ker Q} : D(L) \cap \ker Q \to \operatorname{Im} L$$

is invertible and we assume the inverse of that map by K_q. Let Ω be an open bounded subset of Y, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : Y \to Z$ will be called L-compact in $\overline{\Omega}$, if $RN(\overline{\Omega})$ is bounded and $K_q(I - R)N : \overline{\Omega} \to Y$ is compact.

2. Preliminaries

This paper obtains the existence of periodic solutions for the third-order delay differential equations with impulses

$$\begin{aligned} &x'''(t) + f(t, x''(t)) + g(t, x'(t)) + h(x(t - \tau(t)) = p(t), t \ge 0, t \neq t_k,\\ &\Delta x(t_k) = I_k,\\ &\Delta x'(t_k) = J_k,\\ &\Delta x''(t_k) = K_k. \end{aligned} \tag{2.1}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{t\to t_k^+} x(t)$, $x(t_k^-) = \lim_{t\to t_k^-} x(t)$, $x(t_k^-) = x(t_k)$;
$\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$, $x'(t_k^+) = \lim_{t\to t_k^+} x'(t)$, $x'(t_k^-) = \lim_{t\to t_k^-} x'(t)$, $x'(t_k^-) = x'(t_k)$;
$\Delta x''(t_k) = x''(t_k^+) - x''(t_k^-)$, $x''(t_k^+) = \lim_{t\to t_k^+} x''(t)$, $x''(t_k^-) = \lim_{t\to t_k^-} x''(t)$, $x''(t_k^-) = x''(t_k)$.

We assume that the following conditions:

(H1) $f \in C(\mathbb{R}^2,\mathbb{R})$ and $g(t + T, x) = g(t, x)$, $h \in C(\mathbb{R},\mathbb{R})$, $p, \tau \in C(\mathbb{R},\mathbb{R})$ with $\tau(t + T) = \tau(t)$, $p(t + T) = p(t)$;

(H2) $\{t_k\}$ satisfies $t_k < t_{k+1}$ and $\lim_{k\to\pm\infty} t_k = \pm\infty$, $k \in Z$, $I_k(x,y), J_k(x,y), K_k(x,y) \in C(\mathbb{R}^2,\mathbb{R})$, and there is a positive n such that $\{t_k\} \cap [0,T] = \{t_1, t_2, \dots, t_n\}, t_{k+n} = t_k + T$, $I_{k+n}(x,y) = I_k(x,y), J_{k+n}(x,y) = J_k(x,y), K_{k+n}(x,y) = K_k(x,y)$.

(H3) There are constants $\sigma, \beta \geq 0$ such that

$$|f(t,x)| \leq \sigma|x|, \quad \forall (t,x) \in [0,T] \times \mathbb{R}, \tag{2.2}$$

$$xf(t,x) \geq \beta|x|^2, \quad \forall (t,x) \in [0,T] \times \mathbb{R}; \tag{2.3}$$

(H4) There are constants $\sigma, \beta \geq 0$ such that

$$|g(t,x)| \leq \sigma|x|, \quad \forall (t,x) \in [0,T] \times \mathbb{R}, \tag{2.4}$$

$$x^2 g(t,x) \geq \beta|x|^2, \quad \forall (t,x) \in [0,T] \times \mathbb{R}; \tag{2.5}$$

(H5) there are constants $\beta_i \geq 0$ $(i = 1,2,3)$ such that

$$|h(x)| \geq \beta_1 + \beta_2|x|, \tag{2.6}$$

$$|h(x) - h(y)| \leq \beta_3|x - y|; \tag{2.7}$$

(H6) there are constants $\gamma_i > 0$ $(i = 1,2,3)$, such that $|\int_x^{x+\lambda J_k(x,y)} h(s)ds| \leq |J_k(x,y)|(\gamma_1 + \gamma_2|x| + \gamma_3|J_k(x,y)|), \quad \forall \lambda \in (0,1)$;

(H7) there are constants $a_k, a'_k, a''_k \geq 0$ such that $|K_k(x,y)| \leq a_k|x|^2 + a'_k|x| + a''_k$;

(H8) $zK_k(x,y) \leq 0$ and there are constants $b_k \geq 0$ such that $|K_k(x,y)| \leq b_k$.

Lemma 2.1. *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. We assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;
(ii) $RNx \neq 0$, *for all* $x \in \partial\Omega \cap \ker L$;
(iii) $\deg\{KRNx, \Omega \bigcap \ker L, 0\} \neq 0$, *where* $K : \operatorname{Im} R \to \ker L$ *is an isomorphism.*

Then the abstract equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \bigcap D(L)$.

We assume the operators $L : D(L) \subset X \to Y$ and $L : D(L) \subset Y \to Z$ by

$$Lx = (x''', \Delta x(t_1), \dots, \Delta x(t_n), \Delta x'(t_1), \dots, \Delta x'(t_n), \Delta x''(t_1), \dots, \Delta x''(t_n)), \tag{2.8}$$

and $N : X \to Y$, $N : Y \to Z$ by

$$\begin{aligned} Nx = {} & (-f(t, x''(t)) - g(t, x'(t)) - h(x(t - \tau(t))) + p(t), \\ & I_1(x(t_1)), \dots, I_n(x(t_n)), J_1(x'(t_1)), \dots, J_n(x'(t_n)), K_1(x''(t_1)), \dots, K_n(x''(t_n))). \end{aligned} \tag{2.9}$$

Lemma 2.2. *L is a Fredholm operator of index zero with*

$$\ker L = \{x(t) = c, t \in \mathbb{R}\}, \tag{2.10}$$

and

$$\begin{aligned} & \operatorname{Im} L(y, z, a_1, \dots, a_n, b_1, \dots, b_n) \\ = {} & \int_0^T (y(s) + z(s))ds + \sum_{k=1}^n b_k(T - t_k) + \sum_{k=1}^n a_k + x'(0)T = 0. \end{aligned} \tag{2.11}$$

Let the linear operators $P : X \to X$, $Q : Y \to Y$ and $R : Z \to Z$ be defined by

$$Px = x(0), \tag{2.12}$$

$$Q(y,a_1,\dots,a_n,b_1,\dots,b_n) = \frac{2}{T^2}[\int_0^T (T-s)y(s)ds + \sum_{k=1}^n b_k(T-t_k) + \sum_{k=1}^n a_k + x'(0)T], 0,\dots,0), \tag{2.13}$$

and

$$R(z,a_1,\dots,a_n,b_1,\dots,b_n) = \frac{2}{T^2}[\int_0^T (T-s)z(s)ds + \sum_{k=1}^n b_k(T-t_k) + \sum_{k=1}^n a_k + x'(0)T], 0,\dots,0). \tag{2.14}$$

Lemma 2.3. *If $\alpha > 0$, $x(t) \in PC^2(\mathbb{R},\mathbb{R})$ with $x(t+T) = x(t)$, then*

$$\int_0^T \int_{t-\alpha}^t |x'(s)|^2\, ds\, dt = \alpha \int_0^T |x'(t)|^2 dt \tag{2.15}$$

and

$$\int_0^T \int_t^{t+\alpha} |x'(s)|^2\, ds\, dt = \alpha \int_0^T |x'(t)|^2 dt. \tag{2.16}$$

Let

$$A_1(t,\alpha) = \sum_{t-\alpha\le t_k\le t} a_k, \quad A_2(t,\alpha) = \sum_{t\le t_k\le t+\alpha} a_k,$$
$$B_1(t,\alpha) = \sum_{t-\alpha\le t_k\le t} a'_k, \quad B_2(t,\alpha) = \sum_{t\le t_k\le t+\alpha} a'_k,$$
$$C_1(t,\alpha) = \sum_{t-\alpha\le t_k\le t} a''_k, \quad C_2(t,\alpha) = \sum_{t\le t_k\le t+\alpha} a''_k,$$
$$I_1 = \Big(\int_0^T A_1^2(t,\alpha)dt\Big)^{1/2} + \Big(\int_0^T A_2^2(t,\alpha)dt\Big)^{1/2},$$
$$I_2 = \Big(\int_0^T B_1^2(t,\alpha)dt\Big)^{1/2} + \Big(\int_0^T B_2^2(t,\alpha)dt\Big)^{1/2},$$
$$I_3 = \int_0^T A_1^2(t,\alpha)dt + \int_0^T A_2^2(t,\alpha)dt,$$
$$I_4 = \int_0^T A_1(t,\alpha)B_1(t)dt + \int_0^T A_2(t,\alpha)B_2(t)dt,$$
$$I_5 = \int_0^T B_1^2(t,\alpha)dt + \int_0^T B_2^2(t,\alpha)dt$$

The following Lemma is important for us to the delay $\tau(t)$.

Lemma 2.4. *Suppose $\tau(t) \in C(\mathbb{R},\mathbb{R})$ with $\tau(t+T) = \tau(t)$ and $\tau(t) \in [-\alpha,\alpha]$ for all $t \in [0,T]$, $x(t) \in PC^1(\mathbb{R},\mathbb{R})$ with $x(t+T) = x(t)$ and there is a positive n such that $\{t_k\}\cap[0,T] = \{t_1,t_2,\dots,t_n\}$, $\Delta x(t_k) = \lambda I_k(x(t_k),x'(t_k))$ for all $\lambda \in (0,1)$ and $t_{k+n} = t_k + T, I_{k+n}(x,y) = I_k(x,y)$. Furthermore there exist nonnegative constants*

a_k, a_k *such that* $|I_k(x, y)| \leq a_k|x| + a'_k$. *Then*

$$\begin{aligned} &\int_0^T |x(t) - x(t - \tau(t))|^2 dt \\ &\leq 2\alpha^2 \int_0^T |x'(t)|^2 dt + 2\alpha I_1 |x(t)|_\infty \Big(\int_0^T |x'(t)|^2 dt \Big)^{1/2} \\ &\quad + 2\alpha I_2 \Big(\int_0^T |x'(t)|^2 dt \Big)^{1/2} + I_3 |x(t)|_\infty^2 + I_4 |x(t)|_\infty + I_5. \end{aligned} \tag{2.17}$$

3. Main results

We establish the theorems of existence of periodic solution based on the following two conditions.

Theorem 3.1. *We assume that* (H1)–(H8) *hold. Then* (3.3) *has at least one* T-*periodic solution and*

$$\sum_{k=1}^n a_k < 1, \tag{3.1}$$

$$\begin{aligned} &\Big[\gamma_2 (\sum_{k=1}^n a_k) + \gamma_3 (\sum_{k=1}^n a_k^2)\Big] M^2 + \beta_3 \Big[2|\tau(t)|_\infty^2 \\ &+ 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty) M + I_3(|\tau(t)|_\infty) M^2 \Big]^{1/2} < \beta, \end{aligned} \tag{3.2}$$

where

$$M = \frac{1}{1 - \sum_{k=1}^n a_k} \big(\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2} \big).$$

Proof. Consider the abstract equation $Lx = \lambda Nx$, with $\lambda \in (0, 1)$, where L and N are given by (2.8) and (2.9). Let

$$\Omega_1 = \{x \in D(L) : \ker L, Lx = \lambda Nx \text{ for some } \lambda \in (0, 1)\}.$$

For $x \in \Omega_1$, we get

$$x'''(t) + f(t, x''(t)) + g(t, x'(t)) + h(x(t - \tau(t)) = p(t), t \geq 0, t \neq t_k, \tag{3.3}$$

Integrating the interval on $[0, T]$, using Schwarz inequality, we get

$$\begin{aligned} &|\int_0^T h(x(t - \tau(t)) dt| \\ &= |\int_0^T p(t) dt - \int_0^T f(t, x''(t)) dt - \int_0^T g(t, x'(t)) dt + \sum_{k=1}^n K_k(x(t_k), x''(t_k))| \\ &\leq T|p(t)|_\infty + \sigma \int_0^T |x''(t)| dt + \sum_{k=1}^n b_k \\ &\leq \sigma T^{1/2} \Big(\int_0^T |x''(t)|^2 dt \Big)^{1/2} + T|p(t)|_\infty + \sum_{k=1}^n b_k. \end{aligned}$$

From the above formula, there is a interval on $t_0 \in [0, T]$ such that

$$|h(x(t_0 - \tau(t_0))| \leq \frac{\sigma}{T^{1/2}} \big(\int_0^T |x''(t)|^2 dt \big)^{1/2} + |p(t)|_\infty + \frac{1}{T} \sum_{k=1}^n b_k.$$

From (2.6),we get

$$\beta_1+\beta_2|x(t_0-\tau(t_0))| \le \frac{\sigma}{T^{1/2}}(\int_0^T |x''(t)|^2dt)^{1/2}+|p(t)|_\infty+\frac{1}{T}\sum_{k=1}^{n}b_k.$$

Then

$$|x(t_0-\tau(t_0))| \le \frac{\sigma}{\beta_2 T^{1/2}}\Big(\int_0^T |x''(t)|^2dt\Big)^{1/2}+d,$$

where $d=\big(||p(t)|_\infty+\frac{1}{T}\sum_{k=1}^{n}b_k-\beta_1|\big)/\beta_2$. So there is an integer m and an interval $t_1\in[0,T]$ such that $t_0-\tau(t_0)=mT+t_1$. Therefore

$$|x(t_1)|=|x(t_0-\tau(t_0))| \le \frac{\sigma}{\beta_2 T^{1/2}}\Big(\int_0^T |x''(t)|^2dt\Big)^{1/2}+d,$$

$$x(t)=x(t_1)+\int_{t_1}^{t}x''(s)ds+\sum_{t_1\le t_k<t}K_k(x(t_k),x''(t_k)).$$

Thus

$$\begin{aligned}|x(t)|_\infty &\le |x(t_1)|+\int_{t_1}^{t}|x''(s)|ds+\sum_{t_1\le t_k<t}|K_k(x(t_k))|\\ &\le \frac{\sigma}{\beta_2 T^{1/2}}(\int_0^T |x''(t)|^2dt)^{1/2}+d+\int_0^T|x''(t)|dt+\sum_{k=1}^{n}a_k|x|_\infty+\sum_{k=1}^{n}a'_k+\sum_{k=1}^{n}a''_k\\ &\le |x|_\infty\sum_{k=1}^{n}a_k+(\frac{\sigma}{\beta_2 T^{1/2}}+T^{1/2})\Big(\int_0^T |x''(t)|^2dt\Big)^{1/2}+d+\sum_{k=1}^{n}a'_k+\sum_{k=1}^{n}a''_k.\end{aligned}$$

It continues that

$$\begin{aligned}|x(t)|_\infty &\le \frac{d+\sum_{k=1}^{n}a''_k}{1-\sum_{k=1}^{n}a_k}+\frac{1}{1-\sum_{k=1}^{n}a_k}(\frac{\sigma}{\beta_2 T^{1/2}}+T^{1/2})(\int_0^T |x''(t)|^2dt)^{1/2}\\ &=c_1+M(\int_0^T |x''(t)|^2dt)^{1/2},\end{aligned}\tag{3.4}$$

where c_1 is a positive constant. On the other hand, multiplying both side of (3.3) by $x'(t)$, we have

$$\int_0^T x'''(t)x''(t)dt+\lambda\int_0^T f(t,x''(t))x'(t)dt \quad +\lambda\int_0^T g(t,x'(t))x'(t)dt+\lambda\int_0^T h(t,x(t-\tau(t))x'(t)dt$$
$$=\lambda\int_0^T p(t)x'(t)dt.$$

Since

$$\int_0^T x'''(t)x''(t)dt=-\frac{1}{2}\sum_{i=1}^{n}[(x''(t_k^+))^2-(x''(t_k))^2],$$

Our assumption (H7) that

$$
\begin{aligned}
&(x'(t_k^+))^2 - (x'(t_k))^2 \\
&= (x'(t_k^+) + x'(t_k))(x'(t_k^+) - (x'(t_k)) \\
&= \Delta x'(t_k)(2x'(t_k) + \Delta x'(t_k)) \\
&= \lambda K_k(x(t_k), x'(t_k))(2x'(t_k) + \lambda K_k(x(t_k), x'(t_k)) \\
&= 2\lambda K_k(x(t_k), x'(t_k))x'(t_k) + [\lambda K_k(x(t_k), x'(t_k))]^2 \leq b_k^2.
\end{aligned}
$$

In (2.5), by use Schwarz inequality

(3.5)

$$
\begin{aligned}
&\beta \int_0^T |x''(t)|^2 dt \\
&\leq -\int_0^T h(x(t-\tau(t))x'(t)dt + \int_0^T p(t)x'(t)dt + \frac{1}{2}\sum_{k=1}^{n} b_k^2 \\
&= \int_0^T [h(x(t) - h(x(t-\tau(t))]x'(t)dt - \int_0^T h(x(t))x'(t)dt \\
&\quad + \int_0^T p(t)x'(t)dt + \frac{1}{2}\sum_{i=1}^{n} b_k^2 \\
&\leq \int_0^T |h(x(t)) - h(x(t-\tau(t))||x'(t)|dt + |p(t)|_\infty \int_0^T |x'(t)|dt \\
&\quad + |\int_0^T h(x(t))x'(t)dt| + \frac{1}{2}\sum_{i=1}^{n} b_k^2 \\
&\leq \Big[\Big(\int_0^T |h(x(t)) - h(x(t-\tau(t)))|^2 dt\Big)^{1/2} + |p(t)|_\infty T^{1/2}\Big]\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} \\
&\quad + |\int_0^T h(x(t))x'(t)dt| + \frac{1}{2}\sum_{i=1}^{n} b_k^2.
\end{aligned}
$$

From (H5) and (H6), we get

$$
\begin{aligned}
&|\int_0^T h(x(t))x'(t)dt| \\
&= |\int_{x(0)}^{x(t_1)} h(s)ds + \int_{x(t_1^+)}^{x(t_2)} h(s)ds + \cdots + \int_{x(t_n^+)}^{x(T)} h(s)ds| \\
&= |\int_{x(0)}^{x(T)} h(s)ds - \sum_{k=1}^{n} \int_{x(t_k)}^{x(t_k^+)} h(s)ds| \\
&\le \sum_{k=1}^{n} |\int_{x(t_k)}^{x(t_k)+\lambda K_k(x(t_k),x'(t_k))} h(s)ds| \\
&\le \sum_{k=1}^{n} [|K_k(x(t_k),x'(t_k))|(\gamma_1 + \gamma_2|x(t_k)| + \gamma_3|K_k(x(t_k),x'(t_k))|)] \\
&\le [\gamma_2(\sum_{k=1}^{n} a_k) + \gamma_3(\sum_{k=1}^{n} a_k^2)]|x(t)|_\infty^2 + c_2|x(t)|_\infty + c_3,
\end{aligned}
$$

where c_2, c_3 are constants. From (3.4), we get

$$
\begin{aligned}
&|\int_0^T h(x(t))x'(t)dt| \\
&\le [\gamma_2(\sum_{k=1}^{n} a_k) + \gamma_3(\sum_{k=1}^{n} a_k^2)]M^2 \int_0^T |x'(t)|^2 dt + c_4(\int_0^T |x'(t)|^2 dt)^{1/2} + c_5,
\end{aligned}
\tag{3.6}
$$

where c_4, c_5 are constants. From Lemma 2.4, we get

$$
\begin{aligned}
&\int_0^T |h(x(t) - h(x(t-\tau(t)))|^2 dt \\
&\le \beta_3^2 \int_0^T |x(t) - x(t-\tau(t))|^2 dt \\
&\le \beta_3^2 [2|\tau(t)|_\infty^2 \int_0^T |x'(t)|^2 dt + 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)|x(t)|_\infty \Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} \\
&\quad + 2|\tau(t)|_\infty I_2(|\tau(t)|_\infty)\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} + I_3(|\tau(t)|_\infty)|x(t)|_\infty^2 \\
&\quad + I_4(|\tau(t)|_\infty)|x(t)|_\infty + I_5(|\tau(t)|_\infty)].
\end{aligned}
$$

Substituting (3.4) into the above inequality, we get

$$
\begin{aligned}
&\int_0^T |h(x(t) - h(x(t-\tau(t)))|^2 dt \\
&\le \beta_3^2 [2|\tau(t)|_\infty^2 + 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)M \\
&\quad + I_3(|\tau(t)|_\infty)M^2] \int_0^T |x'(t)|^2 dt + c_6\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} + c_7,
\end{aligned}
$$

where c_6, c_7 are constants. From above inequality

$$
(a+b)^{1/2} \le a^{1/2} + b^{1/2} \quad for \quad a \ge 0, b \ge 0,
\tag{3.7}
$$

we get

$$\Big(\int_0^T |h(x(t)) - h(x(t-\tau(t)))|^2 dt\Big)^{1/2}$$
$$\le \beta_3[2|\tau(t)|_\infty^2 + 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)M$$
$$+ I_3(|\tau(t)|_\infty)M^2]^{1/2}\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} + c_6^{1/2}\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/4} + c_7^{1/2}.$$

Substituting the above formula and (3.6) in (3.5), we get

$$\big\{\beta - [\gamma_2(\sum_{k=1}^n a_k) + \gamma_3(\sum_{k=1}^n a_k^2)]M^2 - \beta_3[2|\tau(t)|_\infty^2$$
$$+ 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)M + I_3(|\tau(t)|_\infty)M^2]^{1/2}\big\}\int_0^T |x'(t)|^2 dt$$
$$\le c_8(\int_0^T |x'(t)|^2 dt)^{\frac{3}{4}} + c_9(\int_0^T |x'(t)|^2 dt)^{1/2} + c_{10},$$

where c_8, c_9, c_{10} are constants. There is a constant $M_1 > 0$ such that

$$\int_0^T |x'(t)|^2 dt \le M_1. \tag{3.8}$$

From (3.4), we get

$$|x(t)|_\infty \le d + M(\int_0^T |x'(t)|^2 dt)^{1/2} \le d + M(M_1)^{1/2}.$$

Then there is a constant $M_2 > 0$ such that $|x(t)|_\infty \le M_2$. Therefore, integrating (3.3) on the interval $[0,T]$, using Schwarz inequality, we get

$$\int_0^T |x'''(t)|dt = \int_0^T |-f(t,x''(t)) - g(t,x'(t)) - h(x(t-\tau(t))) + p(t)|dt$$
$$\le \int_0^T |f(t,x''(t))|dt + \int_0^T |g(t,x''(t))|dt + \int_0^T |h(x(t-\tau(t)))|dt + \int_0^T |p(t)|dt$$
$$\le \sigma\int_0^T |x''(t)|dt + h_\delta T + T|p(t)|_\infty$$
$$\le \sigma T^{1/2}(\int_0^T |x''(t)|^2 dt)^{1/2} + h_\delta T + T|p(t)|_\infty$$
$$\le \sigma T^{1/2}(M_1)^{1/2} + h_\delta T + T|p(t)|_\infty,$$

where $h_\delta = \max_{|x|\le\delta}|g(x)|$. Then there is a constant $M_3 > 0$ such that

$$\int_0^T |x''(t)|dt \le M_3. \tag{3.9}$$

From (3.8),then there are $t_2 \in [0,T]$ and $c > 0$ such that $|x'(t_2)| \le c$ for $t \in [0,T]$

$$|x'(t)|_\infty \le |x'(t_2)| + \int_0^T |x''(t)|dt + \sum_{k=1}^n b_k. \tag{3.10}$$

Then there is a constant $M_4 > 0$ such that

$$|x'(t)|_\infty \le M_4. \tag{3.11}$$

It follows that there is a constant $I_2 > \max\{M_2, M_4\}$ such that $\|x\| \le I_2$, Thus Ω_1 is bounded.

Let $\Omega_2 = \{x \in \ker L, RNx = 0\}$. If $x \in \Omega_2$, then $x(t) = c \in R$ and satisfies

$$RN(x,0) = (-\frac{2}{T^2}\int_0^T [f(t,0) + g(t,0) + h(c) - p(t)]dt, 0, \dots, 0) = 0. \tag{3.12}$$

we get

$$\int_0^T [f(t,0) + g(t,0) + h(c) - p(t)]dt = 0. \tag{3.13}$$

In (3.13),there must be a interval $t_0 \in [0,T]$ such that

$$h(c) = -f(t_0,0) - g(t_0,0) + p(t_0). \tag{3.14}$$

From (3.14) and assumption (H3), (H4), we get

$$\beta_1 + \beta_2|c| \le |h(c)| \le |f(t_0,0)| + |g(t_0,0)| + |p(t_0)| \le \sigma \times 0 + |p(t)|_\infty. \tag{3.15}$$

Then

$$|c| \le \frac{||p(t)|_\infty - \beta_1|}{\beta_2} \tag{3.16}$$

which implies Ω_2 is bounded. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3}$, where $\Omega_3 = \{x \in X : |x| < ||p(t)|_\infty - \beta_1|/\beta_2 + 1\}$. By Lemmas 2.2, we can see that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Then by the above argument,

(i) $Lx \ne \lambda Nx$ for all $x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;
(ii) $RNx \ne 0$ for all $x \in \partial\Omega \cap \ker L$.

Finally we prove that (iii) of Lemma 2.1 is satisfied. We take $H(x,\mu) : \Omega \times [0,1] \to X$,

$$H(x,\mu) = \mu x + \frac{2(1-\mu)}{T^2}\int_0^T [-f(t,x''(t)) - g(t,x'(t)) + h(x(t-\tau(t)) + p(t)]dt.$$

From assumptions (H3) and (H4), we can easily verify $H(x,\mu) \ne 0$, for all $(x,\mu) \in \partial\Omega \cap \ker L \times [0,1]$, which results in

$$\begin{aligned}\deg\{KRNx, \Omega \cap \ker L, 0\} &= \deg\{H(x,0), \Omega \cap \ker L, 0\}\\ &= \deg\{H(x,1), \Omega \cap \ker L, 0\} \ne 0,\end{aligned}$$

where $K(x,0,\dots,0) = x$. Therefore, by Lemma 2.1, Equation (3.3) has at least one T-periodic solution. □

Example 1. Consider the third order delay differential equation with impulses

$$\begin{aligned} x'''(t) + \frac{1}{3}x''(t) + \frac{1}{6}x'(t) + \frac{1}{21}x(t - \frac{1}{10}\cos t) &= \sin t, \quad t \ne k,\\ I_k(x,y) &= \frac{\sin\frac{k\pi}{3}}{120}x + \frac{y}{1+y^2},\\ J_k(x,y) &= -\frac{2x^2y}{1+x^4y^2},\\ K_k(x,y) &= -\frac{4x^4y}{1+x^8y^2}, \end{aligned} \tag{3.17}$$

where $t_k = k$, $f(t,x) = \frac{1}{3}x^2$,$g(t,x) = \frac{1}{6}x$, $h(y) = \frac{1}{21}y$, $p(t) = \sin t$, $\tau(t) = \frac{1}{10}\cos t$, it is easy to see that $|\tau(t)|_\infty = \frac{1}{10}$, $T = 2\pi, \{k\} \cap [0, 2\pi] = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\sigma = \beta = \frac{1}{3}$, $\beta_1 = 0$, $\beta_2 = \beta_3 = \frac{1}{21}$. Since $|I_k(x,y)| \leq \frac{1}{120}|x| + \frac{1}{2}$,
$|J_k(x,y)| \leq 1$,$|\int_x^{x+I_k(x,y)} h(s)ds| \leq |I_k(x,y)|(\frac{1}{21}|x| + \frac{1}{42}|I_k(x,y)|)$,
$|K_k(x,y)| \leq 1$,$|\int_x^{x+J_k(x,y)} h(s)ds| \leq |J_k(x,y)|(\frac{1}{21}|x| + \frac{1}{42}|J_k(x,y)|)$,
then we take $a_k = \frac{1}{120}$, $a'_k = \frac{1}{2}$, $b'_k = 1$ $(k = 1, 2, 3, 4, 5, 6, 7, 8)$, $\gamma_1 = 0$, $\gamma_2 = 1/21$, $\gamma_3 = 1/42$.

$$\sum_{k=1}^{8} a_k = \frac{1}{20} < 1,$$

$$M = \frac{1}{1 - \sum_{k=1}^{n} a_k}\left(\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2}\right) = \frac{1}{1 - \frac{1}{20}}\left(\frac{\frac{1}{3}}{\frac{1}{21}(2\pi)^{1/2}} + (2\pi)^{1/2}\right) < 8.$$

By Theorem 3.1, Equation (3.17) has at least one 2π-periodic solution.

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SOME SPACES OF A-IDEAL CONVERGENT SEQUENCES DEFINED BY MUSIELAK-ORLICZ FUNCTION

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Abstract. We introduce basic properties of some sequence spaces using ideal convergent and Musielak Orlicz function $\mathcal{M} = (M_k)$. Including relations related to these spaces are investigated in this paper.

1. Introduction, Definitions and Notations

Throughout this article w, c, c_0, l_∞, l_p denote the spaces of all, convergent, null, bounded and p -absolutely summable sequences, where $1 \leq p < \infty$.

Firstly, the notion of I -convergence was introduced by Kostryrko et all [1] and it is the generalization of statistical convergence.
$A = (a_{nk})$ be an infinite matrix of complex entries a_{nk} and $x = (x_k)$ be a sequence in w. If $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$ converges for each, then we write $n \in \mathbb{N}$.

Definition 1.1. If X is a non-empty set then a family of sets $I \subseteq 2^X$ is ideal if and only if for each $A, B \in I$ we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$ we have $B \in I$.[1]

Definition 1.2. A non-empty family of sets $F \subset 2^X$ is said to be a filter on X if and only if $\emptyset \notin F$, for each $A, B \in F$ we have $A \cap B \in F$ and for each $A \in F$ and each $B \supset A$ we have $B \in F$.[1]

Definition 1.3. An ideal $I \neq \emptyset$ is called non-trivial if $I \neq \emptyset$ and $X \notin I$.[1]

Definition 1.4. A non-trivial $I \subseteq 2^X$ is called admissible ideal if and only if $\{\{x\} : x \in X\} \subset I$.[1]

Definition 1.5. A sequence $x = (x_n) \in w$ is said to be *I-convergent* to L if there exists $L \in \mathbb{C}$ such that for all $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in I$. We say x, $I-$ *convergent* to L and we write $I - \lim x = L$. The number L is called $I-$ *limit* of x.[2]

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Definition 1.6. An Orlicz function M is a function which is continuous, nondecreasing, and convex with $M(0)=0$, for $x>0$ and $M(x)\to\infty$ as $x\to\infty$.
Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to construct the sequence space

$$l_M=\left\{x\in w:\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)<\infty \text{ for some } \rho>0\right\}$$

which is called an Orlicz sequence space. The space l_M becomes a Banach space with the norm

$$\|x\|=\inf\left\{\rho>0:\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)\leq 1\right\}.$$

The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x)=x^p$ for $1\leq p<\infty$. Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [5], Bhardwaj and Singh [6] and many others. It is well known that since M is a convex function and $M(0)=0$ then $M(tx)\leq tM(x)$ for all t with $0<t<1$. Dutta and Başar [18] have recently introduced and studied the Orlicz sequence spaces $l_M^{'}(C,\Lambda)$ and $h_M(C,\Lambda)$ generated by Cesàro mean of order one associated with a fixed multiplier sequence of non-zero scalars. The readers may refer to [17] for relevant terminology and details on the algebraic and topological properties on sequence spaces. An Orlicz function M is said to satisfy Δ_2- *condition* for all values of u, if there exists constant $K>0$ such that $M(2u)\leq KM(u)\quad(u\geq 0)$. The Δ_2- *condition* is equivalent to the inequality $M(Lu)\leq KLM(u)$ satisfying for all values of u and for $L>1$ [7]. A sequence $\mathcal{M}=(M_k)$ of Orlicz function is called a Musielak-Orlicz function see [8], [9]. The sequence $N=(N_k)$ defined by

$$N_k(v)=\sup\left\{|v|\,u-(M_k):u\geq 0\right\},\quad k=1,2,...$$

is called the complementary function of a Musileak-Orlicz function $\mathcal{M}=(M_k)$. For a given Musileak-Orlicz function $\mathcal{M}=(M_k)$, the Musileak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}}=\left\{x\in\omega:I_{\mathcal{M}}(cx)<\infty \text{ for some c}>0\right\},$$
$$h_{\mathcal{M}}=\left\{x\in\omega:I_{\mathcal{M}}(cx)<\infty \text{ for all c}>0\right\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x)=\sum_{k=1}^{\infty}M_k(x_k),\quad x=(x_k)\in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\|=\inf\left\{\rho>0:I_{\mathcal{M}}\left(\frac{x}{\rho}\right)\leq 1\right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{\rho} \left(1 + I_{\mathcal{M}} \left(\rho x\right)\right) : \rho > 0 \right\}.$$

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < h = \inf p_n \leq p_n \leq H = \sup p_n < \infty$ and let $D = \max \left\{1, 2^{H-1}\right\}$. Then for $a_k, b_k \in \mathbb{C}$, the set of complex numbers for all $k \in \mathbb{N}$, we have

$$|a_k + b_k|^{p_k} \leq D \left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}. \tag{1.1}$$

Also, $|a|^{p_k} \leq \max \left\{1, |a|^H\right\}$ for all $a \in \mathbb{C}$.

The notion of paranormed space was introduced by Nakano [10] and Simons [11] and many others.

Definition 1.7. Let X be a linear metric space. A function $g : X \to \mathbb{R}$ is called paranorm if

(1) $g(x) \geq 0$, for all $x \in X$,

(2) $g(-x) = g(x)$, for all $x \in X$,

(3) $g(x + y) \leq g(x) + g(y)$, for all $x, y \in X$,

(4) if (λ_n) be a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $g(x_n - x) \to 0$ as $n \to \infty$, then $g(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

Definition 1.8. A sequence space X is solid (or normal) if $(\alpha_n x_n) \in X$ whenever $(x_n) \in X$ for all sequences (α_n) of scalars with $|\alpha_n| \leq 1$ for all $n \in \mathbb{N}$.

Definition 1.9. A sequence space X is said to be monotone if it contains the canonical preimages of its step spaces.[19]

Lemma 1.1. *If a sequence space X is solid, then X is monotone.*[12]

Definition 1.10. A sequence space X is sequence algebra if $xy = (x_n y_n) \in X$ whenever $x = (x_n), y = (y_n) \in X$.

We define the following sequence spaces in this article,

$$c^I(M, A, p) = \left\{ x \in w : I - \lim_k \left[M_k \left(\frac{|A_k(x) - L|}{\rho} \right) \right]^{p_k} = 0 \quad \text{for some } L \text{ and } \rho > 0 \right\},$$

$$c_0^I(\mathcal{M}, A, p) = \left\{ x \in w : I - \lim_k \left[M_k \left(\frac{|A_k(x)|}{\rho} \right) \right]^{p_k} = 0 \quad \text{for some } \rho > 0 \right\},$$

$$l_\infty(\mathcal{M}, A, p) = \left\{ x \in w : \sup_k \left[M_k \left(\frac{|A_k(x)|}{\rho} \right) \right]^{p_k} < \infty \quad \text{for some } \rho > 0 \right\}.$$

Also we write

$$m^I(\mathcal{M}, A, p) = c^I(\mathcal{M}, A, p) \cap l_\infty(\mathcal{M}, A, p)$$
$$m_0^I(\mathcal{M}, A, p) = c_0^I(\mathcal{M}, A, p) \cap l_\infty(\mathcal{M}, A, p).$$

If we take $A = \lambda$, these spaces are respectively reduced to the spaces $c_0^I(\mathcal{M}, \lambda, p)$, $c^I(\mathcal{M}, \lambda, p)$, $l_\infty(\mathcal{M}, \lambda, p)$, $m_0^I(\mathcal{M}, \lambda, p)$, $m^I(\mathcal{M}, \lambda, p)$ defined by Mursaleen and Sharma [19]. If we take $p_k = 1$ for all k, $\mathcal{M}(x) = M(x)$ and $A = I$, we get the spaces $c_0^I(\mathcal{M})$, $c^I(\mathcal{M})$, $l_\infty(\mathcal{M})$, $m_0^I(\mathcal{M})$, $m^I(\mathcal{M})$ which were studied by Tripathy and Hazarika [14].

Our aim is to define the paranormed space of ideal convergent sequence space with matrix transformation and Musielak-Orlicz function.

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then, the spaces $c^I(\mathcal{M}, A, p)$, $c_0^I(\mathcal{M}, A, p)$, $m^I(\mathcal{M}, A, p)$ and $m_0^I(\mathcal{M}, A, p)$ are linear.*

Proof. Let $x, y \in c^I(\mathcal{M}, A, p)$ and α, β be scalars. So, there exist positive numbers ρ_1, ρ_2 and for given $\varepsilon > 0$, we have

$$A_1 = \left\{k \in \mathbb{N} : \left[M_k\left(\frac{|A_k(x) - L_1|}{\rho_1}\right)\right]^{p_k} \geq \frac{\varepsilon}{2D}\right\} \in I,$$

$$A_2 = \left\{k \in \mathbb{N} : \left[M_k\left(\frac{|A_k(x) - L_2|}{\rho_2}\right)\right]^{p_k} \geq \frac{\varepsilon}{2D}\right\} \in I.$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since $\mathcal{M} = (M_k)$ is nondecreasing and convex function, we can obtain

$$M_k\left(\frac{|A_k(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right) < M_k\left(\frac{|A_k(x) - L_1|}{\rho_1}\right) + M_k\left(\frac{|A_k(y) - L_2|}{\rho_2}\right).$$

So, we have

$$\left[M_k\left(\frac{|A_k(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right)\right]^{p_k} < D\left\{\left[M_k\left(\frac{|A_k(x) - L_1|}{\rho_1}\right)\right]^{p_k} + \left[M_k\left(\frac{|A_k(y) - L_2|}{\rho_2}\right)\right]^{p_k}\right\}.$$

Suppose that $k \notin A_1 \cup A_2$. So, $\left[M_k\left(\frac{|A_k(\alpha x+\beta y)-(\alpha L_1+\beta L_2)|}{\rho_3}\right)\right]^{p_k} < \varepsilon$ and hence

$$k \notin \left\{k \in \mathbb{N} : \left[M_k\left(\frac{|A_k(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right)\right]^{p_k} \geq \varepsilon\right\} \subset A_1 \cup A_2.$$

Therefore, $I\text{-}\lim_k \left[M_k\left(\frac{|A_k(\alpha x+\beta y)-(\alpha L_1+\beta L_2)|}{\rho_3}\right)\right]^{p_k} = 0$. Hence $\alpha x + \beta y \in c^I(\mathcal{M}, A, p)$ and so $c^I(\mathcal{M}, A, p)$ is a linear space. Similarly, we can prove that $c_0^I(\mathcal{M}, A, p)$, $m_0^I(\mathcal{M}, A, p)$ and $m^I(\mathcal{M}, A, p)$ are linear spaces. □

Theorem 2.2. *$l_\infty(\mathcal{M}, A, p)$ is a paranormed space with the paranorm g defined by*

$$g(x) = \inf\left\{\rho^{\frac{p_k}{S}} : \sup_k \left[M_k\left(\frac{|A_k(x)|}{\rho}\right)\right]^{\frac{p_k}{S}} \leq 1,\ k = 1, 2, ...\right\},$$

where $S = \max\{1, H\}$.

Proof. It is clear that $g(x) = g(-x)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Let us take $x = (x_k)$ and $y = (y_k)$ in $l_\infty(\mathcal{M}, A, p)$. We denote,

$$B(x) = \left\{\rho_1 : \sup_k \left[M_k\left(\frac{|A_k(x)|}{\rho_1}\right)\right]^{\frac{p_k}{S}} \leq 1\right\}$$

$$B(y) = \left\{\rho_2 : \sup_k \left[M_k\left(\frac{|A_k(y)|}{\rho_2}\right)\right]^{\frac{p_k}{S}} \leq 1\right\}.$$

Let $\rho = \rho_1 + \rho_2$. Then using the convexity of Mursielak-Orlicz function $\mathcal{M} = (M_k)$, we obtain

$$M_k\left(\frac{|A_k(x+y)|}{\rho}\right) \leq \frac{\rho_1}{\rho} M_k\left(\frac{|A_k(x)|}{\rho_1}\right) + \frac{\rho_2}{\rho} M_k\left(\frac{|A_k(x)|}{\rho_2}\right) \leq \frac{\rho_1}{\rho} + \frac{\rho_2}{\rho} = 1.$$

Therefore,

$$\sup_k \left[M_k \left(\frac{|A_k (x+y)|}{\rho} \right) \right]^{\frac{p_k}{S}} \leq 1.$$

We can see that

$$\begin{aligned} g(x+y) &= \inf \left\{ (\rho_1 + \rho_2)^{\frac{p_k}{S}} : \rho_1 \in B(x), \rho_2 \in B(y) \right\} \\ &\leq \inf \left\{ (\rho_1)^{\frac{p_k}{S}} : \rho_1 \in B(x) \right\} + \inf \left\{ (\rho_2)^{\frac{p_k}{S}} : \rho_2 \in B(y) \right\} = g(x) + g(y). \end{aligned}$$

Let $B(x^n) = \left\{ \rho : \sup_k \left[M_k \left(\frac{|A_k(x^n)|}{\rho} \right) \right]^{\frac{p_k}{S}} \leq 1 \right\}$, $B(x^n - x) = \left\{ \rho : \sup_k \left[M_k \left(\frac{|A_k(x^n - x)|}{\rho} \right) \right]^{\frac{p_k}{S}} \leq 1 \right\}$ and $\rho_n \in B(x^n)$, $\rho_n' \in B(x^n - x)$. We can obtain,

$$\begin{aligned} M_k \left(\tfrac{|A_k(\gamma_n x^n - \gamma x)|}{\rho_n |\gamma_n - \gamma| + \rho_n' |\gamma|} \right) &\leq \tfrac{|\gamma_n - \gamma| \rho_n}{\rho_n |\gamma_n - \gamma| + \rho_n' |\gamma|} M_k \left(\tfrac{|A_k(x^n)|}{\rho_n} \right) + \tfrac{|\gamma| \rho_n'}{\rho_n |\gamma_n - \gamma| + \rho_n' |\gamma|} M_k \left(\tfrac{|A_k(x^n - x)|}{\rho_n'} \right) \\ &\leq \tfrac{|\gamma_n - \gamma| \rho_n}{\rho_n |\gamma_n - \gamma| + \rho_n' |\gamma|} + \tfrac{|\gamma| \rho_n'}{\rho_n |\gamma_n - \gamma| + \rho_n' |\gamma|} = 1. \end{aligned}$$

Taking supremum over k on both sides,

$$\sup_k \left[M_k \left(\frac{|A_k (\gamma_n x^n - \gamma x)|}{\rho_n |\gamma_n - \gamma| + \rho_n' |\gamma|} \right) \right]^{\frac{p_k}{S}} \leq 1$$

and so,

$$\{\rho_n |\gamma_n - \gamma| + \rho_n' |\gamma| : \rho_n \in B(x^n),\ \rho_n' \in B(x^n - x)\} \subset \left\{ \rho > 0 : \sup_k \left[M_k \left(\frac{|A_k (\gamma_n x^n - \gamma x)|}{\rho} \right) \right]^{p_k} \leq 1 \right\}.$$

Therefore,

$$\begin{aligned} g(\gamma_n x^n - \gamma x) &= \inf \left\{ (\rho_n |\gamma_n - \gamma| + \rho_n' |\gamma|)^{\frac{p_k}{S}} : \rho_n \in B(x^n),\ \rho_n' \in B(x^n - x) \right\} \\ &\leq |\gamma_n - \gamma|^{\frac{p_k}{S}} \inf \left\{ (p_n)^{\frac{p_k}{S}} : \rho_n \in B(x^n),\ k = 1, 2, ... \right\} \\ &\quad + \max \{1, |\gamma|^s\} \inf \left\{ (\rho_n')^{\frac{p_k}{S}} : \rho_n' \in B(x^n - x),\ k = 1, 2, ... \right\} \end{aligned}$$

where $s = \sup_k \left(\frac{p_k}{S} \right) = \min \{1, H\}$. Since $|\gamma_n - \gamma| \to 0$ and $g(x^n - x) \to 0$ as $n \to \infty$, we obtain that $g(\gamma_n x^n - \gamma x) \to 0$ as $n \to \infty$. $\square$

Theorem 2.3. *Let (M_k) and (M_k') be Musielak-Orlicz functions that Δ_2-condition satisfies. Then,*
(i) $W(M_k, A, p) \subseteq W(M_k' \circ M_k, A, p)$
(ii) $W(M_k, A, p) \cap W(M_k', A, p) \subseteq W(M_k + M_k', A, p)$
where $W = c_0^I, c^I, m_0^I, m^I$.

Proof. (i) Since $W \in \left\{ c^I, m_0^I, m^I \right\}$ can be proved similarly, we give the prove only for $W = c_0^I$. Let $x \in c_0^I(\mathcal{M}, A, p)$. So, we have $\rho > 0$ for every $\varepsilon > 0$,

$$B = \left\{ k \in \mathbb{N} : \left(M_k \left(\frac{|A_k (x)|}{\rho} \right) \right)^{p_k} \geq \varepsilon \right\} \in I.$$

Since (M_k') is continuous, given for $\varepsilon > 0$ chosen δ with $0 < \delta < 1$ such that $M_k'(t) < \varepsilon$ for $0 \leq t \leq \delta$. We define $y_k = M_k \left(\frac{|A_k(x)|}{\rho} \right)$. For $y_k > \delta$,

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$$

Therefore;

$$M_k'(y_k) < M_k' \left(1 + \frac{y_k}{\delta} \right) = M_k' \left(\frac{1}{2} 2 + \frac{1}{2} \frac{y_k}{\delta} 2 \right) \leq \frac{1}{2} M_k'(2) + \frac{1}{2} M_k' \left(\frac{y_k}{\delta} 2 \right) \tag{2.1}$$

Since (M'_k) satisfies $\Delta_2 - condition$, we can write that

$$M'_k\left(\frac{y_k}{\delta}2\right) \leq K\frac{y_k}{\delta}M'_k(2)\, for K \geq 1. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned} M'_k(y_k) &< \tfrac{1}{2}M'_k(2) + \tfrac{1}{2}K\tfrac{y_k}{\delta}M'_k(2) \\ &\leq \tfrac{1}{2}K\tfrac{y_k}{\delta}M'_k(2) + \tfrac{1}{2}K\tfrac{y_k}{\delta}M'_k(2) \\ &= K\tfrac{y_k}{\delta}M'_k(2). \end{aligned}$$

Hence;$[M'_k(y_k)]^{p_k} < \left[K\frac{1}{\delta}M'_k(2)\right]^{p_k}(y_k)^{p_k} \leq \max\left\{1, \left(K\frac{1}{\delta}M'_k(2)\right)^H\right\}(y_k)^{p_k}$. Since $y_k = M_k\left(\frac{|A_k(x)|}{\rho}\right)$, we have $I - \lim_k (y_k)^{p_k} = 0$. So,

$$C = \left\{k : (y_k)^{p_k} \geq \frac{\varepsilon}{\max\left\{1, \left(K\frac{y_k}{\delta}M'_k(2)\right)^H\right\}}\right\} \in I.$$

Suppose that $k \notin C$. Then, $(y_k)^{p_k} < \frac{\varepsilon}{\max\left\{1,\left(K\frac{y_k}{\delta}M'_k(2)\right)^H\right\}}$. Hence,

$$(M'_k(y_k))^{p_k} < \max\left\{1, \left(K\frac{y_k}{\delta}M'_k(2)\right)^H\right\}\frac{\varepsilon}{\max\left\{1, \left(K\frac{y_k}{\delta}M'_k(2)\right)^H\right\}} = \varepsilon.$$

Therefore, $k \notin \left\{k : (M'_k(y_k))^{p_k} \geq \varepsilon,\ y_k > \delta\right\} = D$. Thus $D \subseteq C$ and $D \in I$. Since $M'_k(y_k) < \varepsilon$ for $y_k \leq \delta$, we have

$$[M_k(y_k)]^{p_k} < \varepsilon^{p_k} \leq \max\left\{\varepsilon^h, \varepsilon^H\right\}.$$

From this inequality, we have $I - \lim\,[M'_k(y_k)]^{p_k} = 0$ for $y_k \leq \delta$. Therefore $E = \left\{k : (M'_k(y_k))^{p_k} \geq \varepsilon,\ y_k \leq \delta\right\} \in I$. So $D \cup E \in I$ and $x \in c_0^I(M'_k \circ M_k, A, p)$.

(ii) Let $x \in c_0^I(M_k, A, p) \cap c_0^I(M'_k, A, p)$. So, there exists $\rho > 0$ such that

$$B = \left\{k \in \mathbb{N} : \left(M_k\left(\frac{|A_k(x)|}{\rho}\right)\right)^{p_k} \geq \frac{\varepsilon}{2D}\right\} \in I,$$

$$C = \left\{k \in: \left(M'_k\left(\frac{|A_k(x)|}{\rho}\right)\right)^{p_k} \geq \frac{\varepsilon}{2D}\right\} \in I.$$

Let $k \notin B \cup C$. Hence $k \notin \left\{k : \left((M_k + M'_k)\left(\frac{|A_k(x)|}{\rho}\right)\right)^{p_k} \geq \varepsilon\right\}$. Therefore $\left\{k : \left((M_k + M'_k)\left(\frac{|A_k(x)|}{\rho}\right)\right)^{p_k} \geq \varepsilon\right\} \in I$. This completes the proof. □

Corollary 2.1. *Let* $\mathcal{M}{=}(M_k)$ *be a Musielak-Orlicz functions which satisfies* $\Delta_2 -$ *condition. Then* $W(A,p) \subseteq W(\mathcal{M}, A, p)$ *where* $W = c_0^I, c^I, m_0^I, m^I$.

Proof. We can obtain $W(A,p) \subseteq W(\mathcal{M}, A, p)$ from Theorem 2.3 by taking $M_k(x) = x$ and $M'_k(x) = M_k(x)$ for all $x \in [0, \infty)$. □

Theorem 2.4. *The spaces* $c_0^I(\mathcal{M}, A, p)$ *and* $m_0^I(\mathcal{M}, A, p)$ *are solid for* $A = I$.

Proof. We will prove for the space $c_0^I(\mathcal{M},A,p)$.
Let $x\in c_0^I(\mathcal{M},A,p)$. So, for every $\varepsilon>0$

$$B=\left\{k\in\mathbb{N}:\left(M_k\left(\frac{|A_k(x)|}{\rho}\right)\right)^{p_k}\geq\varepsilon\right\}\in I(\rho>0).$$

Let $\alpha=(\alpha_k)$ be a sequence of scalars with $|\alpha_k|\leq 1$ for all $k\in\mathbb{N}$. Suppose that $k\notin B$. Therefore, we obtain

$$\begin{aligned}\left[M_k\left(\tfrac{|A_k(\alpha x)|}{\rho}\right)\right]^{p_k}&=\left[M_k\left(\tfrac{|I_k(\alpha x)|}{\rho}\right)\right]^{p_k}=\left[M_k\left(\tfrac{|\alpha_k x_k|}{\rho}\right)\right]^{p_k}\\&\leq\left[M_k\left(\tfrac{|x_k|}{\rho}\right)\right]^{p_k}=\left[M_k\left(\tfrac{|I_k(x)|}{\rho}\right)\right]^{p_k}=\left[M_k\left(\tfrac{|A_k(x)|}{\rho}\right)\right]^{p_k}.\end{aligned}$$

Hence, $k\notin\left\{k\in\mathbb{N}:\left(M_k\left(\frac{|A_k(\alpha x)|}{\rho}\right)\right)^{p_k}\geq\varepsilon\right\}$. Therefore, we obtain
$I-\lim\limits_k\left(M_k\left(\frac{|A_k(\alpha x)|}{\rho}\right)\right)^{p_k}=0.$ □

Corollary 2.2. *The spaces $c_0^I(\mathcal{M},A,p)$ and $m_0^I(\mathcal{M},A,p)$ are monotone for $A=I$.*

Proof. This is clear from Lemma 1.1. □

Theorem 2.5. *The spaces $c_0^I(\mathcal{M},A,p)$ and $c^I(\mathcal{M},A,p)$ are sequence algebra for $A=I$.*

Proof. Let $x,y\in c_0^I(\mathcal{M},A,p)$. Then there exists $\rho_1,\rho_2>0$ such that for every $\varepsilon>0$, we have

$$A_1=\left\{k\in\mathbb{N}:\left[M_k\left(\frac{|x_k|}{\rho_1}\right)\right]^{p_k}\geq\frac{\varepsilon}{2D}\right\}\in I,$$
$$A_2=\left\{k\in\mathbb{N}:\left[M_k\left(\frac{|y_k|}{\rho_2}\right)\right]^{p_k}\geq\frac{\varepsilon}{2D}\right\}\in I.$$

Let $\rho=\rho_2|x_k|+\rho_1|y_k|>0$. By using this fact one can see that

$$M_k\left(\frac{|x_ky_k|}{\rho}\right)\leq\frac{\rho_2|x_k|}{2\rho}M_k\left(\frac{|y_k|}{\rho_2}\right)+\frac{\rho_1|y_k|}{2\rho}M_k\left(\frac{|y_k|}{\rho_2}\right)<M_k\left(\frac{|y_k|}{\rho_2}\right)+M_k\left(\frac{|y_k|}{\rho_2}\right),$$

which shows that $A_3=\left\{k\in\mathbb{N}:\left[M_k\left(\frac{|x_ky_k|}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\in I.$
Thus $(x_ky_k)\in c_0^I(M,A,p)$ for $A=I$. □

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SOME ESTIMATES FOR THE GENERALIZED FOURIER-DUNKL TRANSFORM IN THE SPACE $L^2_{\alpha,n}$

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Abstract. Some estimates are proved for the generalized Fourier-Dunkl transform in the space $L^2_{\alpha,n}$ on certain classes of functions characterized by the generalized continuity modulus.

1. Introduction

In [5], Abilov et al. proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator.
In this paper, we consider a first-order singular differential-difference operator Λ on $\mathbb{R}$ which generalizes the Dunkl operator Λ_α, we prove some estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the generalized Fourier-Dunkl transform associated to Λ in $L^2_{\alpha,n}$ analogs of the statements proved in [5]. For this purpose, we use a generalized translation operator.
In section 2, we give some definitions and preliminaries concerning the generalized Fourier-Dunkl transform. The some estimates are proved in section 3.

2. Preliminaries

In this section, we develop some results from harmonic analysis related to the differential-difference operator Λ. Further details can be found in [1] and [6]. In all what follows assume where $\alpha > -1/2$ and n a non-negative integer.
Consider the first-order singular differential-difference operator on $\mathbb{R}$ defined by

$$\Lambda f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x} - 2n\frac{f(-x)}{x}.$$

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For $n = 0$, we regain the differential-difference operator

$$\Lambda_\alpha f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x},$$

which is referred to as the Dunkl operator of index $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Such operators have been introduced by Dunkl (see [3], [4]) in connection with a generalization of the classical theory of spherical harmonics. Let M be the map defined by

$$Mf(x) = x^{2n} f(x), \quad n = 0, 1, ...$$

Let $L^p_{\alpha,n}$, $1 \leq p < \infty$, be the class of measurable functions f on $\mathbb{R}$ for which

$$\|f\|_{p,\alpha,n} = \|M^{-1} f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx \right)^{1/p}.$$

If $p = 2$, then we have $L^2_{\alpha,n} = L^2(\mathbb{R}, |x|^{2\alpha+1})$.
The one-dimensional Dunkl kernel is defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz), z \in \mathbb{C}, \tag{2.1}$$

where

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m! \Gamma(m+\alpha+1)}, z \in \mathbb{C}, \tag{2.2}$$

is the normalized spherical Bessel function of index α. It is well-known that the functions $e_\alpha(\lambda.)$, $\lambda \in \mathbb{C}$, are solutions of the differential-difference equation

$$\Lambda_\alpha u = \lambda u, u(0) = 1.$$

In the terms of $j_\alpha(x)$, we have (see [2])

$$1 - j_\alpha(x) = O(1), x \geq 1, \tag{2.3}$$

$$1 - j_\alpha(x) = O(x^2), 0 \leq x \leq 1, \tag{2.4}$$

$$\sqrt{hx} J_\alpha(hx) = O(1), hx \geq 0, \tag{2.5}$$

where $J_\alpha(x)$ is Bessel function of the first kind, which is related to $j_\alpha(x)$ by the formula

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1)}{x^\alpha} J_\alpha(x), x \in \mathbb{R}^+. \tag{2.6}$$

For $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n} e_{\alpha+2n}(i\lambda x),$$

where $e_{\alpha+2n}$ is the Dunkl kernel of index $\alpha + 2n$ given by (1).

Proposition 2.1. *(i) φ_λ satisfies the differential equation*

$$\Lambda \varphi_\lambda = i\lambda \varphi_\lambda.$$

(ii) For all $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$

$$|\varphi_\lambda(x)| \leq |x|^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Dunkl transform we call the integral transform

$$\mathcal{F}_\Lambda f(\lambda) = \int_{\mathbb{R}} f(x)\varphi_{-\lambda}(x)|x|^{2\alpha+1}dx, \lambda \in \mathbb{R}, f \in L^1_{\alpha,n}.$$

Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_\Lambda(f) \in L^1_{\alpha+2n} = L^1(\mathbb{R}, |x|^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Dunkl transform is given by the formula

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n}|\lambda|^{2\alpha+4n+1}d\lambda, \quad a_\alpha = \frac{1}{2^{2\alpha+2}(\Gamma(\alpha+1))^2}.$$

Proposition 2.2. *(i) For every $f \in L^2_{\alpha,n}$,*

$$\mathcal{F}_\Lambda(\Lambda f)(\lambda) = i\lambda\mathcal{F}_\Lambda(f)(\lambda).$$

(ii) For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2|x|^{2\alpha+1}dx = \int_{\mathbb{R}} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

(iii) The generalized Fourier-Dunkl transform $\mathcal{F}_\Lambda$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2(\mathbb{R}, \mu_{\alpha+2n})$.

The generalized translation operators τ^x, $x \in \mathbb{R}$, tied to Λ are defined by

$$\begin{aligned}\tau^x f(y) &= \frac{(xy)^{2n}}{2}\int_{-1}^{1} \frac{f(\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n}\left(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt \\ &+ \frac{(xy)^{2n}}{2}\int_{-1}^{1} \frac{f(-\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n}\left(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt,\end{aligned}$$

where

$$A(t) = \frac{\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+1/2)}(1+t)(1-t^2)^{\alpha+2n-1/2}.$$

Proposition 2.3. *Let $x \in \mathbb{R}$ and $f \in L^2_{\alpha,n}$. Then $\tau^x f \in L^2_{\alpha,n}$ and*

$$\|\tau^x f\|_{2,\alpha,n} \le 2x^{2n}\|f\|_{2,\alpha,n}.$$

Furthermore,

$$\mathcal{F}_\Lambda(\tau^x f)(\lambda) = x^{2n}e_{\alpha+2n}(i\lambda x)\mathcal{F}_\Lambda(f)(\lambda). \tag{2.7}$$

The generalized modulus of continuity of function $f \in L^2_{\alpha,n}$ is defined as

$$w(f,\delta)_{2,\alpha,n} = \sup_{0<h\le\delta} \|\tau^h f(x) + \tau^{-h}f(x) - 2h^{2n}f(x)\|_{2,\alpha,n}, \delta > 0.$$

3. Main Results

The goal of this work is to prove some estimates for the integral

$$J_N^2(f)=\int_{|\lambda|\geq N}|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

in certain classes of functions in $L^2_{\alpha,n}$.

Lemma 3.1. *For $f\in L^2_{\alpha,n}$, we have,*

$$\|\tau^h f(x)+\tau^{-h}f(x)-2h^{2n}f(x)\|^2_{2,\alpha,n}=4h^{4n}\int_{\mathbb{R}}|j_{\alpha+2n}(\lambda h)-1|^2|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

where $r=0,1,2,...$

Proof. By using the formulas (2.1), (2.2) and (2.7), we conclude that

$$\mathcal{F}_\Lambda(\tau^h f+\tau^{-h}f-2h^{2n}f)(\lambda)=2h^{2n}(j_{\alpha+2n}(\lambda h)-1)\mathcal{F}_\Lambda f(\lambda). \tag{3.1}$$

Now by formula (3.1) and Plancherel equality, we have the result. □

Theorem 3.1. *Given $f\in L^2_{\alpha,n}$. Then there exist a constant $C>0$ such that, for all $N>0$,*

$$J_N(f)=O(N^{2n}\omega(f,CN^{-1})_{2,\alpha,n}).$$

Proof. Firstly, we have

$$J_N^2(f)\leq\int_{|\lambda|\geq N}|j|d\mu+\int_{|\lambda|\geq N}|1-j|d\mu, \tag{3.2}$$

with $j=j_p(\lambda h)$, $p=\alpha+2n$ and $d\mu=|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$. The parameter $h>0$ will be chosen in an instant.
In view of formulas (2.5) and (2.6), there exist a constant $C_1>0$ such that

$$|j|\leq C_1(|\lambda|h)^{-p-\frac{1}{2}}.$$

Then

$$\int_{|\lambda|\geq N}|j|d\mu\leq C_1(hN)^{-p-\frac{1}{2}}J_N^2(f).$$

Choose a constant C_2 such that the number $C_3=1-C_1C_2^{-p-\frac{1}{2}}$ is positif.
Setting $h=C_2/N$ in the inequality (3.2), we have

$$C_3J_N^2(f)\leq\int_{|\lambda|\geq N}|1-j|d\mu. \tag{3.3}$$

By Hölder inequality the second term in (3.3) satisfies

$$\begin{aligned}\int_{|\lambda|\geq N}|1-j|d\mu &= \int_{|\lambda|\geq N}|1-j|.1.d\mu\\ &\leq \left(\int_{|\lambda|\geq N}|1-j|^2d\mu\right)^{1/2}\left(\int_{|\lambda|\geq N}d\mu\right)^{1/2}\\ &\leq \left(\int_{|\lambda|\geq N}|1-j|^2d\mu\right)^{1/2}J_N(f).\end{aligned}$$

From Lemma 3.1, we conclude that

$$\int_{|\lambda|\geq N}|1-j|^2 d\mu \leq h^{-4n}\|\tau^h f(x)+\tau^{-h}f(x)-2h^{2n}f(x)\|^2_{2,\alpha,n}.$$

Therefore

$$\int_{|\lambda|\geq N}|1-j| d\mu \leq h^{-2n}\|\tau^h f(x)+\tau^{-h}f(x)-2h^{2n}f(x)\|_{2,\alpha,n}J_N(f).$$

For $h=C_2/N$, we obtain

$$C_3 J_N^2(f) \leq C_2^{-2n}N^{2n}w(f,C_2/N)_{2,\alpha,n}J_N(f).$$

Consequently

$$C_2^{2n}C_3 J_N(f) \leq N^{2n}w(f,C_2/N)_{2,\alpha,n}.$$

for all $N>0$. The theorem is proved with $C=C_2$. □

Theorem 3.2. *Let $f\in L^2_{\alpha,n}$. Then, for all $N>0$,*

$$\omega(f,N^{-1})_{2,\alpha,n}=O\left(N^{-2(n+1)}\left(\sum_{l=0}^{N-1}(l+1)^3J_l^2(f)\right)^{\frac{1}{2}}\right).$$

Proof. From Lemma 3.1, we have

$$\|\tau^h f(x)+\tau^{-h}f(x)-2h^{2n}f(x)\|^2_{2,\alpha,n}=4h^{4n}\int_{\mathbb{R}}|j_{\alpha+2n}(\lambda h)-1|^2|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

This integral is divided into two

$$\int_{\mathbb{R}}=\int_{|\lambda|\leq N}+\int_{|\lambda|\geq N}=I_1+I_2,$$

where $N=[h^{-1}]$. We estimate them separately.
From (2.3), we have the estimate

$$I_2\leq C_4\int_{|\lambda|\geq N}|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)=C_4J_N^2(f).$$

Now, we estimate I_1. From formula (2.4), we have

$$\begin{aligned} I_1 &\leq C_5h^4\int_{|\lambda|\leq N}\lambda^4|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)=C_5h^4\sum_{l=0}^{N-1}\int_{l\leq|\lambda|\leq l+1}\lambda^4|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\ &= C_5h^4\sum_{l=0}^{N-1}a_l\left(J_l^2(f)-J_{l+1}^2(f)\right),\end{aligned}$$

with $a_l=(l+1)^4$.
For all integers $m\geq 1$, the Abel transformation shows

$$\begin{aligned}\sum_{l=0}^{m}a_l\left(J_l^2(f)-J_{l+1}^2(f)\right) &= a_0J_0^2(f)+\sum_{l=1}^{m}\left(a_l-a_{l-1}\right)J_l^2(f)-a_mJ_{m+1}^2(f)\\ &\leq a_0J_0^2(f)+\sum_{l=1}^{m}\left(a_l-a_{l-1}\right)J_l^2(f),\end{aligned}$$

because $a_m J_{m+1}^2(f) \geq 0$.
Hence

$$I_1 \leq C_5 h^4 \left(J_0^2(f) + \sum_{l=1}^{N-1} \left((l+1)^4 - l^4\right) J_l^2(f) - N^4 J_N^2(f) \right).$$

Moreover by the finite increments theorem, we have $(l+1)^4 - l^4 \leq 4(l+1)^3$. Then

$$I_1 \leq C_5 N^{-4} \left(J_0^2(f) + 4 \sum_{l=1}^{N-1} (l+1)^3 J_l^2(f) - N^4 J_N^2(f) \right),$$

since $N \leq \frac{1}{h}$. Combining the estimates for I_1 and I_2 gives

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha,n}^2 = O\left(N^{-4-4n} \sum_{l=0}^{N-1} (l+1)^3 J_l^2(f) \right),$$

which implies

$$\omega(f, N^{-1})_{2,\alpha,n} = O\left(N^{-2(n+1)} \left(\sum_{l=0}^{N-1} (l+1)^3 J_l^2(f) \right)^{\frac{1}{2}} \right),$$

and this ends the proof. □

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$\mathcal{I}$-LIMIT SUPERIOR AND $\mathcal{I}$-LIMIT INFERIOR FOR SEQUENCES OF FUZZY NUMBERS

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Abstract. The statistical limit inferior and limit superior for sequences of fuzzy numbers have been introduced by Aytar, Pehlivan and Mammadov [Statistical limit inferior and limit superior for sequences of fuzzy numbers, Fuzzy Sets and Systems, 157(7) (2006) 976–985]. In this paper, we extend concepts of statistical limit superior and inferior to $\mathcal{I}$-limit superior and $\mathcal{I}$-inferior for a sequence of fuzzy numbers. Also, we prove some basic properties.

1. Introduction

The definition of convergence for sequences of fuzzy numbers has been firstly presented by Matloka [21] and the Cauchy Criterion for sequences of fuzzy numbers is defined by Nanda [22].

The notions of limit superior and limit inferior for a bounded sequence of fuzzy numbers is introduced by Aytar et al. [4]. Afterwards, some properties of these concepts have been obtained by Hong et al. [15], Talo and Çakan [29], Talo [30].

The notion of statistical convergence was defined by Nuray and Savaş [23] for sequences of of fuzzy numbers. Also, Aytar et al. [5] introduced the characterization of statistical limit superior and limit inferior for statistically bounded sequences of fuzzy numbers and proved some fuzzy-analogues of properties of statistical limit superior and limit inferior.

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [16]. Kostyrko et al. [17] and Aytar et al. [6] proved some of basic properties of $\mathcal{I}$-convergence. Also, Demirci [10] presented the notions of $\mathcal{I}$-limit superior and inferior of a real sequence and gave some properties.

Kumar and Kumar [18] studied the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence and $\mathcal{I}$-Cauchy sequence for sequences of fuzzy numbers. Kumar et al. [19] introduced the concepts of $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points for sequences of fuzzy numbers. Dündar and Talo [11] presented the notions of $\mathcal{I}_2$-convergence, $\mathcal{I}_2^*$-convergence

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for double sequences of fuzzy numbers and proved their some properties and relations. Recently, various types of $\mathcal{I}$-convergence for sequences of fuzzy numbers have been studied by many authors [13, 14, 25, 27, 33]

In this paper, we extend the concepts of $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior to fuzzy numbers space and prove several basic properties.

2. Preliminaries, Background and Notation

First, we recall basics of fuzzy numbers.

Let E^1 denote the set of fuzzy subsets of the real line, if $u : \mathbb{R} \to [0,1]$, satisfying the following properties:

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, i.e.,
$u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$;

(iii) u is upper semi-continuous;

(iv) The set $[u]_0 := cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact.

Then u is called a fuzzy number and E^1 is called fuzzy number space. λ-level set $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda := \begin{cases} \{x \in \mathbb{R} : u(x) \geq \lambda\} & , \quad (0 < \lambda \leq 1), \\ \overline{\{x \in \mathbb{R} : u(x) > 0\}} & , \quad (\lambda = 0). \end{cases}$$

Obviously, $[u]_\lambda$ is closed, bounded and non-empty interval for each $\lambda \in [0,1]$ and denoted as $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$. For any $r \in \mathbb{R}$, define a fuzzy number $\hat{r}$ by

$$\widehat{r}(x) := \begin{cases} 1 & , \quad (x = r), \\ 0 & , \quad (x \neq r), \end{cases}$$

for any $x \in \mathbb{R}$.

Let $u, v, w \in E^1$ and $k \in \mathbb{R}$, the addition, scalar multiplication and product are defined by

$$u + v = w \iff [w]_\lambda = [u]_\lambda + [v]_\lambda \text{ for all } \lambda \in [0,1]$$

$$[ku]_\lambda = k[u]_\lambda \text{ for all } \lambda \in [0,1]$$

and

$$uv = w \iff [w]_\lambda = [u]_\lambda [v]_\lambda \text{ for all } \lambda \in [0,1].$$

Let $W = \{A = [A^-, A^+] : A \text{ is closed bounded intervals on the real line } \mathbb{R}\}$. Define

$$d(A, B) := \max\{|A^- - B^-|, |A^+ - B^+|\}$$

as the metric on W.

Hausdorff metric D between fuzzy numbers defined by

$$D(u, v) = \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda) = \sup_{\lambda \in [0,1]} \max\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}.$$

The partial ordering relation on E^1 is defined as follows:

$$u \preceq v \iff [u]_\lambda \preceq [v]_\lambda \iff u^-(\lambda) \leq v^-(\lambda) \text{ and } u^+(\lambda) \leq v^+(\lambda) \text{ for all } \lambda \in [0,1].$$

$u \prec v$ means $u \preceq v$ and at least one of $u^-(\alpha) < v^-(\alpha)$ and $u^+(\alpha) < v^+(\alpha)$ holds for some $\alpha \in [0,1]$.

Two fuzzy numbers u and v are said to be incomparable if neither $u \preceq v$ nor $v \preceq u$ holds. In this case we write $u \not\sim v$.

Combining the results of Lemma 6 in [5], Lemma 5 in [3], Lemma 3.4, Theorem 4.9 in [20] and Lemma 14 in [31], following Lemma is obtained.

Lemma 2.1. *Let $u, v, w, e \in E^1$ and $\hat{\varepsilon} > 0$. The following statements hold:*

(i) $D(u, v) \leq \varepsilon$ *if and only if* $u - \hat{\varepsilon} \preceq v \preceq u + \hat{\varepsilon}$
(ii) *If* $u \preceq v + \hat{\varepsilon}$ *for every* $\varepsilon > 0$, *then* $u \preceq v$.
(iii) *If* $u \preceq v$ *and* $v \preceq w$, *then* $u \preceq w$
(iv) *If* $u \prec v$, $v \preceq w$, *then* $u \prec w$.
(v) *If* $u \preceq w$ *and* $v \preceq e$, *then* $u + v \preceq w + e$.
(vi) *if* $u \prec w$ *and* $v \preceq e$, *then* $u + v \prec w + e$.
(vii) *If* $u \succeq \overline{0}$ *and* $v \succ w$, *then* $uv \succeq uw$.
(viii) *If* $u + w \preceq v + w$ *then* $u \preceq v$.

Wu and Wu [28] defined boundness of a set of fuzzy numbers according to relation $\preceq$ and proved that if a set A of E^1 is bounded, then supremum and infimum of A exist.

We denote the set of all sequences of fuzzy numbers by $w(F)$.

A sequence $(u_n) \in w(F)$ is called convergent with limit $u \in E^1$, if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$D(u_n, u) < \varepsilon \text{ for all } n \geq n_0.$$

A sequence (u_n) of fuzzy numbers is said to be bounded if there exists $M > 0$ such that $D(u_n, \hat{0}) \leq M$ for all $n \in \mathbb{N}$. By $\ell_\infty(F)$, we denote the set of all bounded sequences of fuzzy numbers.

The statistical convergence of sequences of fuzzy numbers defined as follows:

For a subset K of natural numbers $\mathbb{N}$, the natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|$$

if this limit exists, where $|A|$ denotes the number of elements in A.

A sequence $u = (u_k)$ of fuzzy numbers is said to be statistically convergent to some fuzzy number μ_0, if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{n} \big|\{k \leq n : D(u_k, \mu_0) \geq \varepsilon\}\big| = 0.$$

The statistical boundedness of a sequence of fuzzy numbers was introduced and studied by Aytar and Pehlivan [3]. The sequence $u = (u_k)$ is said to be statistically bounded if there exists a real number M such that the set $\{k \in \mathbb{N} : D(u_k, \overline{0}) > M\}$ has natural density zero.

Aytar et al. [5] defined the concepts of statistical limit superior and limit inferior of statistically bounded sequences of fuzzy numbers.

Let $u = (u_k)$ be statistically bounded and let us define the following sets:

$$A_u = \left\{\mu \in E^1 : \delta\left(\{k \in \mathbb{N} : u_k \prec \mu\}\right) \neq 0\right\},$$

$$\overline{A}_u = \left\{\mu \in E^1 : \delta\left(\{k \in \mathbb{N} : u_k \succ \mu\}\right) = 1\right\},$$

$$B_u = \left\{\mu \in E^1 : \delta(\{k \in \mathbb{N} : u_k \succ \mu\}) \neq 0\right\},$$

$$\overline{B}_u = \left\{\mu \in E^1 : \delta(\{k \in \mathbb{N} : u_k \prec \mu\}) = 1\right\}.$$

The statistical limit superior and limit inferior are defined as follows:

$$\begin{aligned} \text{st}-\liminf u_k &= \inf A_u = \sup \overline{A}_u, \\ \text{st}-\limsup u_k &= \sup B_u = \inf \overline{B}_u. \end{aligned}$$

For more result on sequences of fuzzy numbers we refer to [1, 2, 7, 9, 26, 32] and [8, Section 8].

Now, we recall the concept of ideal and ideal convergence of sequences of fuzzy numbers.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.2. [16] *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Lemma 2.3. [24, Lemma 2.5] *$K \in F(\mathcal{I})$ and $M \subseteq \mathbb{N}$. If $M \notin \mathcal{I}$ then $M \cap K \notin \mathcal{I}$.*

Throughout this paper we take $\mathcal{I}$ as a nontrivial admissible ideal in $\mathbb{N}$.

Definition 2.1. Let $u = (u_n)$ be a sequences of fuzzy numbers.

(i)[18] $u = (u_n)$ is said to be $\mathcal{I}$-convergent to a fuzzy number u_0, if for any $\varepsilon > 0$ we have

$$A(\varepsilon) = \{n \in \mathbb{N} : D(u_n, u_0) \geq \varepsilon\} \in \mathcal{I}.$$

In this case we say that u is $\mathcal{I}$-convergent and we write $\mathcal{I} - \lim_{n\to\infty} u_n = u_0$.

(ii)[19] The fuzzy number μ is said to be $\mathcal{I}$-limit point of $u = (u_n)$ if there exits a subset $K = \{k_1 < k_2 < k_3 < \cdots\} \notin \mathcal{I}$ such that $\lim_{n\to\infty} u_{k_n} = \mu$. The set of all $\mathcal{I}$-limit points of the sequence $u = (u_n)$ will be denoted by $\mathcal{I}(\Lambda_u)$.

(iii)[19] The fuzzy number μ is said to be the $\mathcal{I}$-cluster point of $u = (u_n)$ if for each $\varepsilon > 0$, $\{n \in \mathbb{N} : D(u_n, \mu) < \varepsilon\} \notin \mathcal{I}$. The set of all $\mathcal{I}$-cluster points of the fuzzy number sequence $u = (u_n)$ will be denoted by $\mathcal{I}(\Gamma_u)$.

The propose of this paper is to present the notions of ideal limit superior and inferior for a sequence of fuzzy numbers and give some ideal analogues of properties of the statistical limit superior and inferior of sequences of fuzzy numbers.

3. The Main Results

Definition 3.1. $u = (u_k) \in w(F)$ is said to be $\mathcal{I}$-*bounded above* if there exists a fuzzy number μ such that

$$\{k \in \mathbb{N} : u_k \succ \mu\} \cup \{k \in \mathbb{N} : u_k \nsim \mu\} \in \mathcal{I}.$$

Similarly, $u = (u_k)$ is said to be $\mathcal{I}$-*bounded below* if there exists a fuzzy number ν such that

$$\{k \in \mathbb{N} : u_k \prec \nu\} \cup \{k \in \mathbb{N} : u_k \not\sim \nu\} \in \mathcal{I}.$$

If $u = (u_k)$ is both $\mathcal{I}$-bounded above and below, then it is said to be $\mathcal{I}$-bounded.

This definition can be stated as follows:
$u = (u_k) \in w(F)$ is said to be $\mathcal{I}$-bounded if there is a real number M such that

$$\{k \in \mathbb{N} : D(u_k, \hat{0}) > M\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$, if $u = (u_k)$ is bounded, then u is $\mathcal{I}$-bounded.

We give a generalization of notions of $st-\liminf u$ and $st-\limsup u$ of a sequence $u = (u_k)$ of [5]. Given $\mathcal{I}$-bounded sequence $u = (u_k) \in w(F)$, we define the following sets:

$$\begin{aligned}
A_u &= \left\{\mu \in E^1 : \{k \in \mathbb{N} : u_k \prec \mu\} \notin \mathcal{I}\right\}, \\
\overline{A}_u &= \left\{\mu \in E^1 : \{k \in \mathbb{N} : u_k \succ \mu\} \in \mathcal{F}(\mathcal{I})\right\}, \\
B_u &= \left\{\mu \in E^1 : \{k \in \mathbb{N} : u_k \succ \mu\} \notin \mathcal{I}\right\}, \\
\overline{B}_u &= \left\{\mu \in E^1 : \{k \in \mathbb{N} : u_k \prec \mu\} \in \mathcal{F}(\mathcal{I})\right\}.
\end{aligned}$$

It is evident that if the sequence $u = (u_k)$ is $\mathcal{I}$-bounded, then the sets $A_u, \overline{A}_u, B_u$ and $\overline{B}_u$ are non-empty. It is also evident that the sets A_u and $\overline{B}_u$ have lower bounds, and the sets $\overline{A}_u$ and B_u have upper bounds. Hence, we obtain that $\inf A_u$, $\sup \overline{A}_u$, $\sup B_u$ and $\inf \overline{B}_u$ exist.

Now, we prove the main results in line of Theorem 2, Theorem 3, Theorem 5 and Theorem 7 in [5]. Our proofs are similar to those in [5].

Theorem 3.1. *If* $u = (u_k) \in w(F)$ *is* $\mathcal{I}$*-bounded, then* $\inf A_u = \sup \overline{A}_u$ *and* $\sup B_u = \inf \overline{B}_u$.

Proof. We prove only for $\inf A_u = \sup \overline{A}_u$. Denote $\nu := \inf A_u$ and $\mu := \sup \overline{A}_u$. Then, we have $\nu \preceq \widetilde{\nu}$ for all $\widetilde{\nu} \in A_u$, and $\mu \succeq \widetilde{\mu}$ for all $\widetilde{\mu} \in \overline{A}_u$. Since $\widetilde{\nu} \in A_u$, $\{k \in \mathbb{N} : u_k \prec \widetilde{\nu}\} \notin \mathcal{I}$. On the other hand, from $\widetilde{\mu} \in \overline{A}_u$, we have $\{k \in \mathbb{N} : u_k \succ \widetilde{\mu}\} \in \mathcal{F}(\mathcal{I})$. Therefore,

$$\{k \in \mathbb{N} : u_k \prec \widetilde{\nu}\} \cap \{k \in \mathbb{N} : u_k \succ \widetilde{\mu}\} \notin \mathcal{I}$$

that is, $\{k \in \mathbb{N} : u_k \prec \widetilde{\nu}\} \cap \{k \in \mathbb{N} : u_k \succ \widetilde{\mu}\} \neq \emptyset$. Then, there is a number $k \in \mathbb{N}$ such that $\widetilde{\mu} \prec u_k \prec \widetilde{\nu}$. This implies that

$$\widetilde{\mu} \prec \widetilde{\nu} \quad \text{for all} \quad \widetilde{\nu} \in A_u, \ \widetilde{\mu} \in \overline{A}_u. \tag{3.1}$$

From (3.1), it is immediate that $\widetilde{\mu}$ is a lower bound of the set A_u. Then, we have $\widetilde{\mu} \preceq \nu = \inf A_u$. This inequality is valid for all $\widetilde{\mu} \in \overline{A}_u$. Then, we get $\mu \preceq \nu$. Now, we show that the case $\mu \prec \nu$ is impossible.

To the contrary, assume that $\mu \prec \nu$. This means that, there is a number $\alpha \in [0, 1]$ such that

$$\mu^-(\alpha) < \nu^-(\alpha) \quad \text{or} \quad \mu^+(\alpha) < \nu^+(\alpha).$$

Without of loss of generality, we take into account the case $\mu^-(\alpha) < \nu^-(\alpha)$ and show that it leads to a contradiction.

Denote $b := \nu(\mu^-(\alpha))$. It is obvious that $b < \alpha$ (b may be zero). Furthermore, the inequality $\mu^-(\lambda) < \nu^-(\lambda)$ holds, for all $\lambda \in (b, \alpha]$. Since the functions $\mu(x)$ and

$\nu(x)$ are upper semi-continuous, there is a point (z,β) such that $z\in(\mu^-(\alpha),\nu^-(\alpha))$, $\beta\in(b,\alpha)$ and

$$\mu^-(\lambda)<z,\ \nu^-(\lambda)>z \ \text{ for all } \lambda\in[\beta,\alpha]. \tag{3.2}$$

We define the numbers $\gamma_1,\gamma_2\in E^1$ by

$$\gamma_1(t):=\begin{cases}0 & , \quad t<t^-(0),\\ \beta & , \quad t\in[t^-(0),z],\\ 1 & , \quad t=z,\\ 0 & , \quad t>z,\end{cases} \quad \text{and} \quad \gamma_2(t):=\begin{cases}0 & , \quad t<z,\\ \beta & , \quad t\in[z,t^+(0)],\\ 1 & , \quad t=t^+(0),\\ 0 & , \quad t>t^+(0),\end{cases}$$

where the numbers $t^-(0)=\mathcal{I}-\liminf u_k^-(0)-1$ and $t^+(0)=\mathcal{I}-\limsup u_k^+(0)+1$ are finite.

From (3.2), it is easily seen that

$$\mu^-(\beta) \geq \mathcal{I}-\liminf u_k^-(\beta)\geq \mathcal{I}-\liminf u_k^-(0)>t^-(0)=\gamma_1^-(\beta),$$
$$\mu^-(\alpha)<z=\gamma_1^-(\alpha)$$

and

$$\nu^-(b)\leq\mu^-(\alpha)<z=\gamma_2^-(b),\ \nu^-(\beta)>z=\gamma_2^-(\beta).$$

This means that

$$\mu\not\succ\gamma_1 \ \text{ and } \ \nu\not\prec\gamma_2. \tag{3.3}$$

Let

$$C_1:=\left\{k\in\mathbb{N}:u_k^-(\lambda)\leq z \ \text{ for some } \lambda\in(\beta,\alpha]\right\},$$

$$C_2:=\left\{k\in\mathbb{N}:u_k^-(\lambda)\geq z \ \text{ for some } \lambda\in(\beta,\alpha]\right\}.$$

Clearly, we have

$$C_1\cup C_2=\mathbb{N}. \tag{3.4}$$

First we assume that $C_1\notin\mathcal{I}$. Considering γ_2 and $t^+(0)$, we have

$$u_k\prec\gamma_2, \quad \text{for all} \quad k\in C_1\backslash K_1,$$

where $K_1:=\left\{k\in\mathbb{N}:u_k^+(\lambda)>t^+(0), \text{ for some } \lambda\in[0,1]\right\}$. This means that

$$\{k\in\mathbb{N}:u_k\prec\gamma_2\}\supseteq C_1\setminus K_1.$$

It is evident that $K_1\in\mathcal{I}$ and $C_1\setminus K_1\notin\mathcal{I}$. For this reason, $\{k\in\mathbb{N}:u_k\prec\gamma_2\}\notin\mathcal{I}$. This means that $\gamma_2\in A_u$ and therefore, from the definition of $\inf A_u$ we get $\gamma_2\succeq\nu=\inf A_u$. This contradicts to (3.3), that is, $\nu\not\prec\gamma_2$.

Hence, we have shown that $C_1\in\mathcal{I}$. In this case, from (3.4), it follows that the set $C_2\in\mathcal{F}(\mathcal{I})$. Considering γ_1 and $t^-(0)$, we have

$$u_k\succ\gamma_1 \ \text{ for all } \ k\in C_2\backslash(C_1\cup K_2),$$

where $K_2:=\left\{k\in\mathbb{N}:u_k^-(\lambda)<t^-(0) \ \text{ for some } \ \lambda\in[0,\beta]\right\}$. This means that

$$\{k\in\mathbb{N}:u_k\succ\gamma_1\}\supseteq C_2\backslash(C_1\cup K_2).$$

It is obvious that the set $K_2\in\mathcal{I}$ and consequently we have $C_2\backslash(C_1\cup K_2)\in\mathcal{F}(\mathcal{I})$. Therefore

$$\{k\in\mathbb{N}:u_k\succ\gamma_1\}\in\mathcal{F}(\mathcal{I}).$$

This implies that $\gamma_1\in\overline{A}_u$. Thus, $\gamma_1\preceq\mu=\sup\overline{A}_u$. This contradicts to (3.3), that is, $\mu\not\succ\gamma_1$. This completes the proof. $\square$

Definition 3.2. If $u = (u_k)$ is a $\mathcal{I}$-bounded sequence of fuzzy numbers, then

$$\mathcal{I} - \liminf u_k := \inf A_u,$$

and

$$\mathcal{I} - \limsup u_k := \sup B_u.$$

Example 3.1. We will give some example of ideals.

1. Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$ limit superior and inferior coincides with the ordinary limit superior and inferior of sequences of fuzzy numbers [4],[15].
2. Let $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denotes the natural density of the set A. Then $\mathcal{I}_\delta$ is a non-trivial admissible ideal and $\mathcal{I}_\delta$ limit superior and inferior coincides with the statistical limit superior and inferior of sequences of fuzzy numbers [5].
3. A set $K \subset \mathbb{N}$ has C-density if $\delta_C(K) := \lim_{n\to\infty} \sum_{k\in K} c_{nk}$ exists, where $C = (c_{nk})$ is a non-negative regular matrix [12]. If $\mathcal{I}_{\delta_C} = \{A \subset \mathbb{N} : \delta_C(A) = 0\}$, then $\mathcal{I}_{\delta_C}$ is a non-trivial admissible ideal and $\mathcal{I}_{\delta_C}$ limit superior and inferior coincides with the C-statistical limit superior and inferior of sequences of fuzzy numbers, which is also mentioned in [5].

Theorem 3.2. *For any $\mathcal{I}$-bounded sequence of fuzzy numbers $u = (u_k)$,*

$$\mathcal{I} - \liminf u \preceq \mathcal{I} - \limsup u.$$

Proof. Let $\mu \in \overline{A}_u$. Then $\{k : u_k \succ \mu\} \in \mathcal{F}(\mathcal{I})$. Since $\mathcal{I}$ is a nontirvial ideal of $\mathbb{N}$, we get $\{k : u_k \succ \mu\} \notin \mathcal{I}$. Therefore $\mu \in B_u$. This implies $\overline{A}_u \subseteq B_u$. Hence $\sup \overline{A}_u \preceq \sup B_u$. This means that $\mathcal{I} - \liminf u \preceq \mathcal{I} - \limsup u$. □

Since $\mathcal{I}$ is an admissible ideal, the inclusion $\mathcal{I}_f \subset \mathcal{I}$ holds. Therefore, the inequalities

$$\operatorname{Lim\,inf} u \preceq \mathcal{I} - \liminf u \preceq \mathcal{I} - \limsup u \preceq \operatorname{Lim\,sup} u$$

hold for every bounded sequence (u_k) of fuzzy numbers.

Theorem 3.3. *Let $u = (u_k)$ be a $\mathcal{I}$-bounded sequence of fuzzy numbers.*

(i) *If $\nu := \mathcal{I} - \liminf u_k$, then*

$$\{k \in \mathbb{N} : u_k \prec \nu - \hat{\varepsilon}\} \in \mathcal{I},\ \{k \in \mathbb{N} : u_k \prec \nu + \hat{\varepsilon}\} \cup \{k \in \mathbb{N} : u_k \not\sim \nu + \hat{\varepsilon}\} \notin \mathcal{I} \tag{3.5}$$

for every $\varepsilon > 0$.

(ii) *If $\mu := \mathcal{I} - \limsup u_k$, then*

$$\{k \in \mathbb{N} : u_k \succ \mu + \hat{\varepsilon}\} \in \mathcal{I} \text{ and } \{k \in \mathbb{N} : u_k \succ \mu - \hat{\varepsilon}\} \cup \{k \in \mathbb{N} : u_k \not\sim \mu - \hat{\varepsilon}\} \notin \mathcal{I}$$

for every $\varepsilon > 0$.

Proof. We prove (i). To the contrary, we assume that there exists $\varepsilon > 0$ such that $\{k \in \mathbb{N} : u_k \prec \nu - \hat{\varepsilon}\} \notin \mathcal{I}$. This means that $\nu - \hat{\varepsilon} \in A_u$. Since $\nu = \inf A_u$, we get $\nu \preceq \nu - \hat{\varepsilon}$ which is a contradiction.

Now, let us show that (3.5) holds. Suppose that it is not true, that is, there exists $\varepsilon > 0$ such that

$$\{k \in \mathbb{N} : u_k \prec \nu + \hat{\varepsilon}\} \in \mathcal{I} \text{ and } \{k \in \mathbb{N} : u_k \not\sim \nu + \hat{\varepsilon}\} \in \mathcal{I}.$$

For each $k \in \mathbb{N}$, only the following three cases are possible: $u_k \prec \nu + \hat{\varepsilon}$, $u_k \not\sim \nu + \hat{\varepsilon}$ and $u_k \succeq \nu + \hat{\varepsilon}$. Then,

$$\{k \in \mathbb{N} : u_k \prec \nu + \hat{\varepsilon}\} \cup \{k \in \mathbb{N} : u_k \not\sim \nu + \hat{\varepsilon}\} \cup \{k \in \mathbb{N} : u_k \succeq \nu + \hat{\varepsilon}\} = \mathbb{N}.$$

Thus, from (3.6), we have $\{k \in \mathbb{N} : u_k \succeq \nu + \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$. This means that $\nu + \hat{\varepsilon} \in \overline{A}_u$. Hence, we can write $\nu + \hat{\varepsilon} \preceq \sup \overline{A}_u = \nu$, which is a contradiction. □

Theorem 3.4. *If $u = (u_k) \in w(F)$ is $\mathcal{I}$ convergent to μ, then*

$$\mathcal{I} - \liminf u_k = \mathcal{I} - \limsup u_k = \mu.$$

Proof. First suppose that $\mathcal{I} - \lim u_k = \mu$ and $\varepsilon > 0$. Then, $\{k \in \mathbb{N} : D(x_k, \mu) \geq \varepsilon\} \in \mathcal{I}$, so we have $\{k \in \mathbb{N} : D(x_k, \mu) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$. By Lemma 2.1, we get $\{k \in \mathbb{N} : \mu - \hat{\varepsilon} \prec u_k \prec \mu + \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$,
$\{k \in \mathbb{N} : \mu - \hat{\varepsilon} \prec u_k\} \cap \{k \in \mathbb{N} : u_k \prec \mu + \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$. Therefore,

1) $\{k \in \mathbb{N} : \mu - \hat{\varepsilon} \prec u_k\} \in \mathcal{F}(\mathcal{I})$. This means that $\mu - \hat{\varepsilon} \in \overline{A}_u$. Then, $\mathcal{I} - \liminf u_k = \sup \overline{A}_u \succeq \mu - \hat{\varepsilon}$.
2) $\{k \in \mathbb{N} : u_k \prec \mu + \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$. This means that $\mu + \hat{\varepsilon} \in \overline{B}_u$. Then, $\mathcal{I} - \limsup u_k = \inf \overline{B}_u \preceq \mu + \hat{\varepsilon}$.

By these inequalities and Theorem 3.4, we obtain

$$\mu - \hat{\varepsilon} \preceq \mathcal{I} - \liminf u_k \preceq \mathcal{I} - \limsup u_k \preceq \mu + \hat{\varepsilon}. \tag{3.6}$$

Since $\varepsilon > 0$ is an arbitrary, we obtain $\mathcal{I} - \liminf u_k = \mathcal{I} - \limsup u_k = \mu$. □

Example 3.2. We decompose the set $\mathbb{N}$ into countably many disjoint sets

$$N_p = \{2^{p-1}(2k-1) : k \in \mathbb{N}\}, \ (j = 1, 2, 3, ...).$$

It is obvious that $\mathbb{N} = \bigcup_{p=1}^{\infty} N_p$ and $N_i \cap N_j = \emptyset$ for $i \neq j$. Denote by $\mathcal{I}$ the class of all $A \subseteq \mathbb{N}$ such that A intersects only a finite number of N_p. It is easy to see that $\mathcal{I}$ is an admissible ideal. Define (u_n) as follows: for $n \in N_p$ we put $u_n = v_p$ $(p = 1, 2, 3, ...)$, where

$$v_p(x) := \begin{cases} 1 - px & , \quad \text{if } 0 \leq x \leq \frac{1}{p}, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then, for $n \in N_p$, $D(u_n, \hat{0}) = 1/p$ $(p = 1, 2, 3, \dots)$. Then, obviously $\mathcal{I} - \lim D(u_n, \hat{0}) = 0$ that is $\mathcal{I} - \lim u_n = \hat{0}$.

Now, consider the ideal $\mathcal{I}_\delta$. It can be easily shown that the natural density of N_p is $\delta(N_p) = 1/2^p$ $(p = 1, 2, 3, ...)$. Then, it is clear that $a \in \overline{A_u}$ for each $a \in E^1$ with $a \preceq \hat{0}$ and $b \in \overline{B_u}$ for each with $b \in E^1$ with $b \succ v_1$. So, we obtain

$$\mathcal{I}_\delta - \liminf u = \hat{0} \text{ and } \mathcal{I}_\delta - \limsup u = v_1.$$

The converse of Theorem 3.4 is not valid in general as shown Example 2 in [5]. The following theorem gives a sufficient condition for a sequence of fuzzy numbers to be $\mathcal{I}-$onvergent.

Theorem 3.5. *Assume that $\mathcal{I} - \limsup u_k = \mathcal{I} - \liminf u_k = \mu$ and there is a number $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the sets $\{k \in \mathbb{N} : u_k \not\sim \mu + \hat{\varepsilon}\}$ and $\{k \in \mathbb{N} : u_k \not\sim \mu - \hat{\varepsilon}\}$ belong to $\mathcal{I}$. Then, we have $\mathcal{I} - \lim u_k = \mu$.*

Proof. Take any number $\varepsilon \in (0, \varepsilon_0)$. Since $\mathcal{I} - \liminf x_k = \mathcal{I} - \limsup x_k = \mu$, by Theorem 3.3 we have

$$\{k \in \mathbb{N} : u_k \prec \mu - \hat{\varepsilon}\} \in \mathcal{I} \text{ and } \{k \in \mathbb{N} : u_k \succ \mu + \hat{\varepsilon}\} \in \mathcal{I},$$

for all $\varepsilon > 0$. From $\{k \in \mathbb{N} : u_k \not\prec \mu - \hat{\varepsilon}\} \in \mathcal{I}$ and $\{k \in \mathbb{N} : u_k \not\prec \mu + \hat{\varepsilon}\} \in \mathcal{I}$, we conclude that

$$\{k \in \mathbb{N} : u_k \preceq \mu + \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I}) \text{ and } \{k \in \mathbb{N} : u_k \succeq \mu - \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I}).$$

By Lemma 2.1, we obtain $\{k \in \mathbb{N} : u_k \preceq \mu + \hat{\varepsilon}\} \cap \{k \in \mathbb{N} : u_k \succeq \mu - \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$, $\{k \in \mathbb{N} : \mu - \hat{\varepsilon} \preceq u_k \preceq \mu + \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$, $\{k \in \mathbb{N} : D(u_k, \mu) \leq \varepsilon\} \in \mathcal{F}(\mathcal{I})$. Therefore, $\{k \in \mathbb{N} : D(u_k, \mu) > \varepsilon\} \in \mathcal{I}$. Since $\varepsilon > 0$ is an arbitrary number, we conclude that $\mathcal{I} - \lim u_k = \mu$. $\square$

The proofs of following theorems are clear and omitted.

Theorem 3.6. *If $u = (u_k)$ and $v = (v_k)$ are $\mathcal{I}$-bounded sequences of fuzzy numbers such that $\{k \in \mathbb{N} : u_k \neq v_k\} \in \mathcal{I}$, then we have:*

(i) $\mathcal{I} - \limsup u_k = \mathcal{I} - \limsup v_k$,
(ii) $\mathcal{I} - \liminf u_k = \mathcal{I} - \liminf v_k$.

Theorem 3.7. *Let $u = (u_k) \in w(F)$ be $\mathcal{I}$-bounded from above. Assume that $\mathcal{I} - \limsup u_k = \mu$ and there is a number $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, the sets*

$$\{k \in \mathbb{N} : u_k \not\prec \mu + \hat{\varepsilon}\} \text{ and } \{k \in \mathbb{N} : u_k \not\prec \mu - \hat{\varepsilon}\}$$

belong to $\mathcal{I}$. Then, $\mu \in \mathcal{I}(\Gamma_u)$.

Theorem 3.8. *Let $u = (u_k) \in w(F)$ be $\mathcal{I}$-bounded from below. Assume that $\mathcal{I} - \liminf u_k = \nu$ and there exists a number $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, the sets*

$$\{k \in \mathbb{N} : u_k \not\prec \nu + \hat{\varepsilon}\} \text{ and } \{k \in \mathbb{N} : u_k \not\prec \nu - \hat{\varepsilon}\}$$

belong to $\mathcal{I}$. Then, $\nu \in \mathcal{I}(\Gamma_u)$.

Theorem 3.9. *Let $u = (u_k) \in w(F)$ be $\mathcal{I}$-bounded. If $\gamma \in \mathcal{I}(\Gamma_u)$, then $\mathcal{I}$-$\liminf u \preceq \gamma \preceq \mathcal{I} - \limsup u$.*

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BIOPERATIONS ON α-SEMIOPEN SETS

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Abstract. The aim of this paper is to introduce and study the concept of $\alpha_{[\gamma,\gamma']}$-semiopen sets. Using this set, we introduce and study the concept of $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous and $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute functions.

1. Introduction

The notion of semiopen sets is an important concept in general topology. In 1963, Levine [4] defined semiopen sets in a space X and discussed many of its properties. Njastad [3] introduced α-open sets in a topological space and studied some of its properties. Ibrahim [2] defined the concept of an operation on $\alpha O(X, \tau)$ and introduced α_γ-open sets in topological spaces and studied some of their basic properties. Khalaf, et. al. [1] introduced the notion of $\alpha O(X, \tau)_{[\gamma,\gamma']}$, which is the collection of all $\alpha_{[\gamma,\gamma']}$-open sets in a topological space (X, τ). In this paper, we introduce and study the notion of $\alpha SO(X, \tau)_{[\gamma,\gamma']}$ which is the collection of all $\alpha_{[\gamma,\gamma']}$-semiopen by using operations γ and $\gamma^{'}$ on a topological space $\alpha O(X, \tau)$. We also introduce $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous and $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute functions and investigate some important properties of these functions.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. The closure and the interior of a subset A of X are denoted by $Cl(A)$ and $Int(A)$, respectively.

Definition 2.1. A subset A of a topological space (X, τ) is called α-open [3] (resp., semiopen [4]) if $A \subseteq Int(Cl(Int(A)))$ (resp., $A \subseteq Cl(Int(A))$). The complement of an α-open (resp., semiopen) set is called α-closed (resp., semiclosed) set.

The family of all α-open (resp., semiopen) sets in a topological space (X, τ) is denoted by $\alpha O(X, \tau)$ or $\alpha O(X)$ (resp.,$SO(X, \tau)$ or $SO(X)$).

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Definition 2.2. [2] Let X be a topological space. An operation γ on the topology $\alpha O(X)$ is a mapping from $\alpha O(X)$ into the power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \alpha O(X)$, where V^γ denotes the value of γ at V. It is denoted by $\gamma : \alpha O(X) \to P(X)$.

Definition 2.3. [2] An operation γ on $\alpha O(X)$ is said to be α-regular if for every α-open sets U and V containing $x \in X$, there exists an α-open set W of X containing x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$.

Definition 2.4. [1] A subset A of X is said to be $\alpha_{[\gamma,\gamma']}$-open if for each $x \in A$, there exist α-open sets U and V of X containing x such that $U^\gamma \cap V^{\gamma'} \subseteq A$. A subset F of (X, τ) is said to be $\alpha_{[\gamma,\gamma']}$-closed if its complement $X \setminus F$ is $\alpha_{[\gamma,\gamma']}$-open.

The family of all $\alpha_{[\gamma,\gamma']}$-open sets of (X, τ) is denoted by $\alpha O(X, \tau)_{[\gamma,\gamma']}$.

Definition 2.5. [1] Let (X, τ) be a topological space and A be a subset of X, then:

(1) The intersection of all $\alpha_{[\gamma,\gamma']}$-closed sets containing A is called the $\alpha_{[\gamma,\gamma']}$-closure of A and denoted by $\alpha_{[\gamma,\gamma']}$-$Cl(A)$.
(2) The union of all $\alpha_{[\gamma,\gamma']}$-open sets contained in A is called the $\alpha_{[\gamma,\gamma']}$-interior of A and denoted by $\alpha_{[\gamma,\gamma']}$-$Int(A)$.

Definition 2.6. [5] A nonempty subset A of (X, τ) is said to be $[\gamma, \gamma']$-open if for each $x \in A$ there exist open sets U and V of X containing x such that $U^\gamma \cap V^{\gamma'} \subseteq A$.

The family of all $[\gamma, \gamma']$-open sets of (X, τ) is denoted by $\tau_{[\gamma,\gamma']}$.

Definition 2.7. [1] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-closed if for $\alpha_{[\gamma,\gamma']}$-closed set A of X, $f(A)$ is $\alpha_{[\beta,\beta']}$-closed in Y.

3. $\alpha_{[\gamma,\gamma']}$-Semiopen Sets

Definition 3.1. Let (X, τ) be a topological space and γ, γ' be two operations on $\alpha O(X, \tau)$. A subset A of X is said to be $\alpha_{[\gamma,\gamma']}$-semiopen, if there exists an $\alpha_{[\gamma,\gamma']}$-open set U of X such that $U \subseteq A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(U)$.

The family of all $\alpha_{[\gamma,\gamma']}$-semiopen sets of a topological space (X, τ) is denoted by $\alpha SO(X, \tau)_{[\gamma,\gamma']}$. Also, the family of all $\alpha_{[\gamma,\gamma']}$-semiopen sets of (X, τ) containing x is denoted by $\alpha SO(X, x)_{[\gamma,\gamma']}$.

Theorem 3.1. *If A is an $\alpha_{[\gamma,\gamma']}$-open set in (X, τ), then it is $\alpha_{[\gamma,\gamma']}$-semiopen set.*

Proof. The proof follows from the definition. □

The following example shows that the converse of the above theorem is not true in general.

Example 3.1. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ be a topology on X. For each $A \in \alpha O(X, \tau)$, we define two operations γ and γ', respectively, by

$$A^\gamma = A^{\gamma'} = \begin{cases} X & \text{if } c \in A \\ A & \text{if } c \notin A. \end{cases}$$

Now, $\alpha O(X, \tau)_{[\gamma,\gamma']} = \{\phi, X, \{a\}, \{a, b\}\}$. Let $A = \{a, c\}$, then there exists an $\alpha_{[\gamma,\gamma']}$-open set $\{a\}$ such that $\{a\} \subseteq A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\{a\}) = X$. Thus, A is $\alpha_{[\gamma,\gamma']}$-semiopen but not $\alpha_{[\gamma,\gamma']}$-open.

Theorem 3.2. *If A is a $[\gamma,\gamma^{'}]$-open set in (X,τ), then it is $\alpha_{[\gamma,\gamma']}$-semiopen set.*

Proof. The proof follows from [[1], Proposition 3.14] and Theorem 3.1. □

The converse of the above theorem need not be true. The subset $\{a,b\}$ in [[1], Example 3.15.], is an $\alpha_{[\gamma,\gamma']}$-semiopen set but it is not $[\gamma,\gamma^{'}]$-open.

Also by Theorem 3.1 and [[1], Proposition 3.14], we obtain the following inclusion
$\tau_{[\gamma,\gamma']} \subseteq \alpha O(X,\tau)_{[\gamma,\gamma']} \subseteq \alpha SO(X,\tau)_{[\gamma,\gamma']}$.

The following examples show that the concept of semiopen and $\alpha_{[\gamma,\gamma']}$-semiopen sets are independent.

Example 3.2. Let $X=\{a,b,c\}$ and $\tau=\{\phi,X,\{a\},\{b\},\{a,b\},\{b,c\}\}$ be a topology on X. For each $A\in\alpha O(X,\tau)$, we define two operations γ and $\gamma^{'}$, respectively, by

$$A^{\gamma}=A^{\gamma^{'}}=\begin{cases} A & \text{if } a\in A \\ Cl(A) & \text{if } a\notin A. \end{cases}$$

Calculations give $\alpha O(X,\tau)_{[\gamma,\gamma']}=\{\phi,X,\{a\},\{a,b\}\}$. Then, $A=\{a,c\}$ is $\alpha_{[\gamma,\gamma']}$-semiopen but not a semiopen set.

Example 3.3. Let $X=\{a,b,c\}$ and $\tau=\{\phi,X,\{a\},\{b\},\{a,b\},\{a,c\}\}$ be a topology on X. For each $A\in\alpha O(X,\tau)$, we define two operations γ and $\gamma^{'}$, respectively, by

$$A^{\gamma}=A^{\gamma^{'}}=\begin{cases} A & \text{if } b\in A \\ Cl(A) & \text{if } b\notin A. \end{cases}$$

Calculations give $\alpha O(X,\tau)_{[\gamma,\gamma']}=\{\phi,X,\{b\},\{a,b\},\{a,c\}\}$. Then, $A=\{a\}$ is semiopen but not an $\alpha_{[\gamma,\gamma']}$-semiopen set.

Theorem 3.3. *A subset A is $\alpha_{[\gamma,\gamma']}$-semiopen if and only if $A\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$.*

Proof. Let $A\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$. Take $U=\alpha_{[\gamma,\gamma']}$-$Int(A)$. Then, by [[1], Proposition 3.44 (1)], U is $\alpha_{[\gamma,\gamma']}$-open and we have $U=\alpha_{[\gamma,\gamma']}$-$Int(A)\subseteq A\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(U)$. Hence, A is $\alpha_{[\gamma,\gamma']}$-semiopen.

Conversely, suppose that A is an $\alpha_{[\gamma,\gamma']}$-semiopen set in X. Then, $U\subseteq A\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(U)$, for some $\alpha_{[\gamma,\gamma']}$-open sets U in X. Since $U\subseteq\alpha_{[\gamma,\gamma']}$-$Int(A)$. Thus, we have $\alpha_{[\gamma,\gamma']}$-$Cl(U)\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$. Hence, $A\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$. □

Theorem 3.4. *Let A be an $\alpha_{[\gamma,\gamma']}$-semiopen set in a space X and B a subset of X. If $A\subseteq B\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(A)$, then B is $\alpha_{[\gamma,\gamma']}$-semiopen.*

Proof. Since A is an $\alpha_{[\gamma,\gamma']}$-semiopen set in X, then there exists an $\alpha_{[\gamma,\gamma']}$-open set U of X such that $U\subseteq A\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(U)$. Since $A\subseteq B$, so $U\subseteq B$. But $\alpha_{[\gamma,\gamma']}$-$Cl(A)\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(U)$, then $B\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(U)$. Hence $U\subseteq B\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(U)$. Thus, B is $\alpha_{[\gamma,\gamma']}$-semiopen. □

Theorem 3.5. *If A_i is $\alpha_{[\gamma,\gamma']}$-semiopen for every $i\in I$, then $\cup\{A_i : i\in I\}$ is $\alpha_{[\gamma,\gamma']}$-semiopen.*

Proof. Since A_i is an $\alpha_{[\gamma,\gamma']}$-semiopen set for every $i \in I$, so there exist an $\alpha_{[\gamma,\gamma']}$-open set U_i of X such that $U_i \subseteq A_i \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(U_i)$ this impies that $\bigcup_{i\in I} U_i \subseteq \bigcup_{i\in I} A_i \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(\bigcup_{i\in I} U_i)$. By [[1], Proposition 3.2], $\bigcup_{i\in I} U_i$ is $\alpha_{[\gamma,\gamma']}$-open. Therefore, $\cup_{i\in I} A_i$ is an $\alpha_{[\gamma,\gamma']}$-semiopen set of (X, τ). □

If A and B are two $\alpha_{[\gamma,\gamma']}$-semiopen sets in (X, τ), then the following example shows that $A \cap B$ need not be $\alpha_{[\gamma,\gamma']}$-semiopen.

Example 3.4. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ be a topology on X. For each $A \in \alpha O(X, \tau)$, we define two operations γ and $\gamma^{'}$, by

$$A^{\gamma} = \begin{cases} Cl(A) & \text{if } c \in A, \\ X & \text{if } c \notin A, \end{cases}$$

and

$$A^{\gamma^{'}} = \begin{cases} A & \text{if } A \neq \{a\}, \\ X & \text{if } A = \{a\}. \end{cases}$$

Then, it is obvious that the sets $\{a, b\}$ and $\{a, c\}$ are $\alpha_{[\gamma,\gamma']}$-semiopen, however their intersection $\{a\}$ is not $\alpha_{[\gamma,\gamma']}$-semiopen.

Remark 3.1. From the above example we notice that the family of all $\alpha_{[\gamma,\gamma']}$-semiopen subsets of a space X is a supratopology and need not be a topology in general.

Theorem 3.6. *Let γ and $\gamma^{'}$ be α-regular operations on $\alpha O(X)$. If A is a subset of X, then for every $\alpha_{[\gamma,\gamma']}$-open set G of X, we have:*

(1) $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap G \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(A \cap G)$.
(2) $\alpha_{[\gamma,\gamma']}\text{-}Cl(A \cap G) = \alpha_{[\gamma,\gamma']}\text{-}Cl(\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap G)$.

Proof.
(1) Let $x \in \alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap G$ and V be any $\alpha_{[\gamma,\gamma']}$-open set containing x. Then by [[1], Proposition 3.4], $V \cap G$ is also an $\alpha_{[\gamma,\gamma']}$-open set containing x. Since $x \in \alpha_{[\gamma,\gamma']}\text{-}Cl(A)$, implies that $(V \cap G) \cap A \neq \phi$, this implies that $V \cap (A \cap G) \neq \phi$ and hence by [[1], Proposition 3.31], $x \in \alpha_{[\gamma,\gamma']}\text{-}Cl(A \cap G)$. Therefore $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap G \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(A \cap G)$.
(2) By (1), $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap G \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(A \cap G)$ and so $\alpha_{[\gamma,\gamma']}\text{-}Cl(\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap G) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(A \cap G)$. But $A \cap G \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap G$ implies that $\alpha_{[\gamma,\gamma']}\text{-}Cl(A \cap G) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap G)$. Therefore, $\alpha_{[\gamma,\gamma']}\text{-}Cl(A \cap G) = \alpha_{[\gamma,\gamma']}\text{-}Cl(\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \cap G)$.

□

Theorem 3.7. *Let γ and $\gamma^{'}$ be α-regular operations on $\alpha O(X)$. If A is $\alpha_{[\gamma,\gamma']}$-open and B is $\alpha_{[\gamma,\gamma']}$-semiopen, then $A \cap B$ is $\alpha_{[\gamma,\gamma']}$-semiopen.*

Proof. Since B is $\alpha_{[\gamma,\gamma']}$-semiopen, there exists an $\alpha_{[\gamma,\gamma']}$-open set G such that $G \subseteq B \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(G)$ and so $A \cap G \subseteq A \cap B \subseteq A \cap \alpha_{[\gamma,\gamma']}\text{-}Cl(G)$. By [[1], Proposition 3.4], $A \cap G$ is $\alpha_{[\gamma,\gamma']}$-open and so $A \cap G = \alpha_{[\gamma,\gamma']}\text{-}Int(A \cap G)$. By Theorem 3.6 (1), $A \cap \alpha_{[\gamma,\gamma']}\text{-}Cl(G) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(A \cap G)$. Therefore, $A \cap B \subseteq A \cap \alpha_{[\gamma,\gamma']}\text{-}Cl(G) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(A \cap G) = \alpha_{[\gamma,\gamma']}\text{-}Cl(\alpha_{[\gamma,\gamma']}\text{-}Int(A \cap G)) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(\alpha_{[\gamma,\gamma']}\text{-}Int(A \cap B))$. By Theorem 3.3, $A \cap B$ is $\alpha_{[\gamma,\gamma']}$-semiopen.

□

Proposition 3.1. *The set A is $\alpha_{[\gamma,\gamma']}$-semiopen in X if and only if for each $x \in A$, there exists an $\alpha_{[\gamma,\gamma']}$-semiopen set B such that $x \in B \subseteq A$.*

Proof. Suppose that A is an $\alpha_{[\gamma,\gamma']}$-semiopen set in the space X. Then for each $x \in A$, put $B = A$ which is an $\alpha_{[\gamma,\gamma']}$-semiopen set such that $x \in B \subseteq A$.

Conversely, suppose that for each $x \in A$, there exists an $\alpha_{[\gamma,\gamma']}$-semiopen set B such that $x \in B \subseteq A$. Thus $A = \cup_{x\in A} B_x$, where $B_x \in \alpha SO(X,\tau)_{[\gamma,\gamma']}$. Therefore, by Theorem 3.5, A is an $\alpha_{[\gamma,\gamma']}$-semiopen set. □

Proposition 3.2. *Let (X,τ) be a topological space and $\gamma, \gamma^{'}$ be operations on $\alpha O(X)$. A subset A of X is $\alpha_{[\gamma,\gamma']}$-semiopen if and only if $\alpha_{[\gamma,\gamma']}$-$Cl(A) = \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$.*

Proof. Let $A \in \alpha SO(X)_{[\gamma,\gamma']}$. Then, we have $A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$, which implies that $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A)$ and hence $\alpha_{[\gamma,\gamma']}$-$Cl(A) = \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$.

Conversely, since by [[1], Proposition 3.44 (1)] and Theorem 3.1, $\alpha_{[\gamma,\gamma']}$-$Int(A)$ is an $\alpha_{[\gamma,\gamma']}$-semiopen set such that $\alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A) = \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$ and hence A is $\alpha_{[\gamma,\gamma']}$-semiopen.

□

Proposition 3.3. *If A is a nonempty $\alpha_{[\gamma,\gamma']}$-semiopen set in X, then $\alpha_{[\gamma,\gamma']}$-$Int(A) \neq \phi$.*

Proof. Since A is $\alpha_{[\gamma,\gamma']}$-semiopen, by Proposition 3.2, we have $\alpha_{[\gamma,\gamma']}$-$Cl(A) = \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$. Suppose that $\alpha_{[\gamma,\gamma']}$-$Int(A) = \phi$. Then, we have $\alpha_{[\gamma,\gamma']}$-$Cl(A) = \phi$ and hence $A = \phi$. This contradicts the hypothesis. Therefore, $\alpha_{[\gamma,\gamma']}$-$Int(A) \neq \phi$. □

Proposition 3.4. *Let (X,τ) be a topological space and $\gamma, \gamma^{'}$ be operations on $\alpha O(X)$. Then a subset A of X is $\alpha_{[\gamma,\gamma']}$-semiopen if and only if $A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)))$ and $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$.*

Proof. Let A be an $\alpha_{[\gamma,\gamma']}$-semiopen set. Then, we have $A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)))$. Moreover, $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$.

Conversely, since $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$. Thus, we obtain that $\alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A))) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$. By hypothesis, we have $A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A))) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$. Hence, A is an $\alpha_{[\gamma,\gamma']}$-semiopen set. □

Definition 3.2. Let A be a subset of a topological space (X,τ) and $\gamma, \gamma^{'}$ be operations on $\alpha O(X)$. Then, a subset A of X is said to be $\alpha_{[\gamma,\gamma']}$-semiclosed if and only if $X \setminus A$ is $\alpha_{[\gamma,\gamma']}$-semiopen. The family of all $\alpha_{[\gamma,\gamma']}$-semiclosed sets of a topological space (X,τ) is denoted by $\alpha SC(X,\tau)_{[\gamma,\gamma']}$.

The following theorem gives characterizations of $\alpha_{[\gamma,\gamma']}$-semiclosed sets.

Theorem 3.8. *Let A be a subset of X and $\gamma, \gamma^{'}$ be operations on $\alpha O(X)$. Then, the following statements are equivalent:*

(1) *A is $\alpha_{[\gamma,\gamma']}$-semiclosed.*
(2) $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq A$.
(3) $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) = \alpha_{[\gamma,\gamma']}$-$Int(A)$.
(4) *There exists an $\alpha_{[\gamma,\gamma']}$-closed set F such that $\alpha_{[\gamma,\gamma']}$-$Int(F) \subseteq A \subseteq F$.*

Proof. (1) $\Rightarrow$ (2): Since $A \in \alpha SC(X,\tau)_{[\gamma,\gamma']}$, then we have $X \setminus A \in \alpha SO(X,\tau)_{[\gamma,\gamma']}$. Hence, by Theorem 3.3 and [[1], Proposition 3.45], $X \setminus A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(X \setminus A)) = X \setminus (\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)))$. Therefore, we obtain $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq A$.

(2) $\Rightarrow$ (3): Since $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq A$ implies that $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq \alpha_{[\gamma,\gamma']}$-$Int(A)$ but $\alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq \alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A))$ and so $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) = \alpha_{[\gamma,\gamma']}$-$Int(A)$.

(3) $\Rightarrow$ (4): Let $F = \alpha_{[\gamma,\gamma']}$-$Cl(A)$, then F is an $\alpha_{[\gamma,\gamma']}$-closed set such that $\alpha_{[\gamma,\gamma']}$-$Int(F) = \alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) = \alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq A \subseteq F$, which proves (4).

(4) $\Rightarrow$ (1): If there exists an $\alpha_{[\gamma,\gamma']}$-closed set F such that $\alpha_{[\gamma,\gamma']}$-$Int(F) \subseteq A \subseteq F$, then $X \setminus F \subseteq X \setminus A \subseteq X \setminus \alpha_{[\gamma,\gamma']}$-$Int(F) = \alpha_{[\gamma,\gamma']}$-$Cl(X \setminus F)$. Since $X \setminus F$ is $\alpha_{[\gamma,\gamma']}$-open, then $X \setminus A$ is $\alpha_{[\gamma,\gamma']}$-semiopen and so A is $\alpha_{[\gamma,\gamma']}$-semiclosed. □

Theorem 3.9. *Let (X,τ) be a topological space and $\gamma, \gamma^{'}$ be operations on $\alpha O(X)$. Arbitrary intersection of $\alpha_{[\gamma,\gamma']}$-semiclosed sets is always $\alpha_{[\gamma,\gamma']}$-semiclosed.*

Proof. Follows from Theorem 3.5. □

Lemma 3.1. *Let $A \in \alpha SC(X,\tau)_{[\gamma,\gamma']}$ and suppose that $\alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq B \subseteq A$. Then, $B \in \alpha SC(X,\tau)_{[\gamma,\gamma']}$.*

Proof. Let $A \in \alpha SC(X,\tau)_{[\gamma,\gamma']}$, then by Theorem 3.8, there exists an $\alpha_{[\gamma,\gamma']}$-closed set F such that $\alpha_{[\gamma,\gamma']}$-$Int(F) \subseteq A \subseteq F$. Since $B \subseteq A$ and $A \subseteq F$. Thus, $B \subseteq F$ also $\alpha_{[\gamma,\gamma']}$-$Int(F) \subseteq \alpha_{[\gamma,\gamma']}$-$Int(A)$ and $\alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq B$. This implies that $\alpha_{[\gamma,\gamma']}$-$Int(F) \subseteq B$. Hence, $\alpha_{[\gamma,\gamma']}$-$Int(F) \subseteq B \subseteq F$, where F is $\alpha_{[\gamma,\gamma']}$-closed in X. This proves that $B \in \alpha SC(X,\tau)_{[\gamma,\gamma']}$.
□

Proposition 3.5. *Let (X,τ) be a topological space and $\gamma, \gamma^{'}$ be operations on $\alpha O(X)$. Then, a subset A of X is $\alpha_{[\gamma,\gamma']}$-semiclosed if and only if $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))) \subseteq A$ and $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$.*

Proof. Let A be an $\alpha_{[\gamma,\gamma']}$-semiclosed set. Then, by Theorem 3.8 (2), we have $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))) \subseteq \alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq A$. Moreover, by Theorem 3.8 (3), $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) = \alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$.

Conversely, since $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))$. Thus, we obtain that $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq \alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A)))$. By hypothesis, we have $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq \alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(A))) \subseteq A$. Hence, by Theorem 3.8, A is an $\alpha_{[\gamma,\gamma']}$-semiclosed set. □

Definition 3.3. Let A be a subset of a topological space (X,τ) and $\gamma,\gamma^{'}$ be operations on $\alpha O(X)$. Then:

(1) The $\alpha_{[\gamma,\gamma']}$-semiclosure of A is defined as the intersection of all $\alpha_{[\gamma,\gamma']}$-semiclosed sets containing A. That is, $\alpha_{[\gamma,\gamma']}$-$sCl(A) = \bigcap\{F : F$ is $\alpha_{[\gamma,\gamma']}$-semiclosed and $A \subseteq F\}$.
(2) The $\alpha_{[\gamma,\gamma']}$-semiinterior of A is defined as the union of all $\alpha_{[\gamma,\gamma']}$-semiopen sets contained in A. That is, $\alpha_{[\gamma,\gamma']}$-$sInt(A) = \bigcup\{U : U$ is $\alpha_{[\gamma,\gamma']}$-semiopen and $U \subseteq A\}$.
(3) The $\alpha_{[\gamma,\gamma']}$-semiboundary of A, denoted by $\alpha_{[\gamma,\gamma']}$-$sBd(A)$ is defined as $\alpha_{[\gamma,\gamma']}$-$sCl(A) \setminus \alpha_{[\gamma,\gamma']}$-$sInt(A)$.
(4) The set denoted by $\alpha_{[\gamma,\gamma']}$-$sD(A)$ and defined by $\{x$: for every $\alpha_{[\gamma,\gamma']}$-semiopen set U containing x, $U \cap (A \setminus \{x\}) \neq \phi\}$ is called the $\alpha_{[\gamma,\gamma']}$-semiderived set of A.

The proofs of the following theorems are obvious and therefore are omitted.

Theorem 3.10. *Let A, B be subsets of a topological space (X,τ) and $\gamma,\gamma^{'}$ be operations on $\alpha O(X)$. Then:*

(1) $\alpha_{[\gamma,\gamma']}$-$sCl(A)$ *is the smallest* $\alpha_{[\gamma,\gamma']}$-*semiclosed subset of* X *containing* A.
(2) $A \in \alpha SC(X,\tau)_{[\gamma,\gamma']}$ *if and only if* $\alpha_{[\gamma,\gamma']}$-$sCl(A) = A$.
(3) $\alpha_{[\gamma,\gamma']}$-$sCl(\alpha_{[\gamma,\gamma']}$-$sCl(A)) = \alpha_{[\gamma,\gamma']}$-$sCl(A)$.
(4) $A \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(A)$.
(5) *If* $A \subseteq B$, *then* $\alpha_{[\gamma,\gamma']}$-$sCl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(B)$.
(6) $\alpha_{[\gamma,\gamma']}$-$sCl(A \cap B) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(A) \cap \alpha_{[\gamma,\gamma']}$-$sCl(B)$.
(7) $\alpha_{[\gamma,\gamma']}$-$sCl(A \cup B) \supseteq \alpha_{[\gamma,\gamma']}$-$sCl(A) \cup \alpha_{[\gamma,\gamma']}$-$sCl(B)$.
(8) $x \in \alpha_{[\gamma,\gamma']}$-$sCl(A)$ *if and only if* $V \cap A \neq \phi$ *for every* $V \in \alpha SO(X,x)_{[\gamma,\gamma']}$.

Theorem 3.11. *Let A, B be subsets of a topological space (X,τ) and $\gamma,\gamma^{'}$ be operations on $\alpha O(X)$. Then:*

(1) $\alpha_{[\gamma,\gamma']}$-$sInt(A)$ *is the largest* $\alpha_{[\gamma,\gamma']}$-*semiopen subset of* X *contained in* A.
(2) A *is* $\alpha_{[\gamma,\gamma']}$-*semiopen if and only if* $A = \alpha_{[\gamma,\gamma']}$-$sInt(A)$.
(3) $\alpha_{[\gamma,\gamma']}$-$sInt(\alpha_{[\gamma,\gamma']}$-$sInt(A)) = \alpha_{[\gamma,\gamma']}$-$sInt(A)$.
(4) $\alpha_{[\gamma,\gamma']}$-$sInt(A) \subseteq A$.
(5) *If* $A \subseteq B$, *then* $\alpha_{[\gamma,\gamma']}$-$sInt(A) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(B)$.
(6) $\alpha_{[\gamma,\gamma']}$-$sInt(A \cup B) \supseteq \alpha_{[\gamma,\gamma']}$-$sInt(A) \cup \alpha_{[\gamma,\gamma']}$-$sInt(B)$.
(7) $\alpha_{[\gamma,\gamma']}$-$sInt(A \cap B) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(A) \cap \alpha_{[\gamma,\gamma']}$-$sInt(B)$.
(8) $X \setminus \alpha_{[\gamma,\gamma']}$-$sInt(A) = \alpha_{[\gamma,\gamma']}$-$sCl(X \setminus A)$.
(9) $X \setminus \alpha_{[\gamma,\gamma']}$-$sCl(A) = \alpha_{[\gamma,\gamma']}$-$sInt(X \setminus A)$.
(10) $\alpha_{[\gamma,\gamma']}$-$sInt(A) = X \setminus \alpha_{[\gamma,\gamma']}$-$sCl(X \setminus A)$.
(11) $\alpha_{[\gamma,\gamma']}$-$sCl(A) = X \setminus \alpha_{[\gamma,\gamma']}$-$sInt(X \setminus A)$.

Theorem 3.12. *Let A, B be subsets of a topological space (X,τ) and $\gamma,\gamma^{'}$ be operations on $\alpha O(X)$. Then:*

(1) $\alpha_{[\gamma,\gamma']}$-$sCl(A) = \alpha_{[\gamma,\gamma']}$-$sInt(A) \cup \alpha_{[\gamma,\gamma']}$-$sBd(A)$.
(2) $\alpha_{[\gamma,\gamma']}$-$sInt(A) \cap \alpha_{[\gamma,\gamma']}$-$sBd(A) = \phi$.
(3) $\alpha_{[\gamma,\gamma']}$-$sBd(A) = \alpha_{[\gamma,\gamma']}$-$sCl(A) \cap \alpha_{[\gamma,\gamma']}$-$sCl(X \setminus A)$.
(4) $\alpha_{[\gamma,\gamma']}$-$sBd(A) = \alpha_{[\gamma,\gamma']}$-$sBd(X \setminus A)$.

(5) *$\alpha_{[\gamma,\gamma']}$-$sBd(A)$ is an $\alpha_{[\gamma,\gamma']}$-semiclosed set.*

Theorem 3.13. *Let A, B be subsets of a topological space (X, τ) and $\gamma, \gamma^{'}$ be operations on $\alpha O(X)$. Then:*

(1) *If $x \in \alpha_{[\gamma,\gamma']}$-$sD(A)$, then $x \in \alpha_{[\gamma,\gamma']}$-$sD(A \setminus \{x\})$.*
(2) *$\alpha_{[\gamma,\gamma']}$-$sD(A \cup B) \supseteq \alpha_{[\gamma,\gamma']}$-$sD(A) \cup \alpha_{[\gamma,\gamma']}$-$sD(B)$.*
(3) *$\alpha_{[\gamma,\gamma']}$-$sD(A \cap B) \subseteq \alpha_{[\gamma,\gamma']}$-$sD(A) \cap \alpha_{[\gamma,\gamma']}$-$sD(B)$.*
(4) *$\alpha_{[\gamma,\gamma']}$-$sD(\alpha_{[\gamma,\gamma']}$-$sD(A)) \setminus A \subseteq \alpha_{[\gamma,\gamma']}$-$sD(A)$.*
(5) *$\alpha_{[\gamma,\gamma']}$-$sD(A \cup \alpha_{[\gamma,\gamma']}$-$sD(A)) \subseteq A \cup \alpha_{[\gamma,\gamma']}$-$sD(A)$.*
(6) *$\alpha_{[\gamma,\gamma']}$-$sCl(A) = A \cup \alpha_{[\gamma,\gamma']}$-$sD(A)$.*
(7) *A is $\alpha_{[\gamma,\gamma']}$-semiclosed if and only if $\alpha_{[\gamma,\gamma']}$-$sD(A) \subseteq A$.*

Remark 3.2. Let A be subset of a topological space (X, τ) and $\gamma, \gamma^{'}$ be operations on $\alpha O(X)$. Then:
$\alpha_{[\gamma,\gamma']}$-$Int(A) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(A) \subseteq A \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A)$.

Theorem 3.14. *Let (X, τ) be a topological space, $\gamma, \gamma^{'}$ operations on $\alpha O(X)$ and A a subset of X. Then, the following statements are equivalent:*

(1) *$A = \alpha_{[\gamma,\gamma']}$-$sCl(A)$.*
(2) *$\alpha_{[\gamma,\gamma']}$-$sInt(\alpha_{[\gamma,\gamma']}$-$sCl(A)) \subseteq A$.*
(3) *$(\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A))) \setminus (X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A)))) \supseteq (\alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus A)$.*

Proof. (1) $\Rightarrow$ (2): If $A = \alpha_{[\gamma,\gamma']}$-$sCl(A)$, then $\alpha_{[\gamma,\gamma']}$-$sInt(\alpha_{[\gamma,\gamma']}$-$sCl(A)) = \alpha_{[\gamma,\gamma']}$-$sInt(A) \subseteq A$.

(2) $\Rightarrow$ (1): Suppose that $\alpha_{[\gamma,\gamma']}$-$sInt(\alpha_{[\gamma,\gamma']}$-$sCl(A)) \subseteq A$. Now, by Theorem 3.10 (1), $\alpha_{[\gamma,\gamma']}$-$sCl(A)$ is an $\alpha_{[\gamma,\gamma']}$-semiclosed set and so, by Theorem 3.8, there is an $\alpha_{[\gamma,\gamma']}$-closed set F such that $\alpha_{[\gamma,\gamma']}$-$Int(F) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(A) \subseteq F$. Since $\alpha_{[\gamma,\gamma']}$-$Int(F)$ is $\alpha_{[\gamma,\gamma']}$-semiopen, then $\alpha_{[\gamma,\gamma']}$-$sInt(\alpha_{[\gamma,\gamma']}$-$Int(F)) = \alpha_{[\gamma,\gamma']}$-$Int(F)$. Therefore, $\alpha_{[\gamma,\gamma']}$-$Int(F) = \alpha_{[\gamma,\gamma']}$-$sInt(\alpha_{[\gamma,\gamma']}$-$Int(F)) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(\alpha_{[\gamma,\gamma']}$-$sCl(A))$ and hence $\alpha_{[\gamma,\gamma']}$-$Int(F) \subseteq A$. But $A \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(A) \subseteq F$. Thus, $\alpha_{[\gamma,\gamma']}$-$Int(F) \subseteq A \subseteq F$, where F is $\alpha_{[\gamma,\gamma']}$-closed. Hence by Theorem 3.8, A is $\alpha_{[\gamma,\gamma']}$-semiclosed and by Theorem 3.10 (2), $A = \alpha_{[\gamma,\gamma']}$-$sCl(A)$.

(3) $\Leftrightarrow$ (1): We have $(\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A))) \setminus (X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A)))) \supseteq (\alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus A)$
$\Leftrightarrow \alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A))) \setminus (X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A)))) \subseteq A$
$\Leftrightarrow \alpha_{[\gamma,\gamma']}$-$Cl(A) \cap [X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A))) \setminus (X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A))))] \subseteq A$
$\Leftrightarrow \alpha_{[\gamma,\gamma']}$-$Cl(A) \cap [X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A))) \cap (\alpha_{[\gamma,\gamma']}$-$Cl(A)))] \subseteq A$
$\Leftrightarrow \alpha_{[\gamma,\gamma']}$-$Cl(A) \cap [(X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A))))) \cup (X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A)))] \subseteq A$
$\Leftrightarrow [\alpha_{[\gamma,\gamma']}$-$Cl(A) \cap (X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A)))))] \cup [\alpha_{[\gamma,\gamma']}$-$Cl(A) \cap (X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(A)))] \subseteq A$
$\Leftrightarrow \alpha_{[\gamma,\gamma']}$-$Cl(A) \cap \alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq A$
$\Leftrightarrow \alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq A$
$\Leftrightarrow A$ is $\alpha_{[\gamma,\gamma']}$-semiclosed
$\Leftrightarrow A = \alpha_{[\gamma,\gamma']}$-$sCl(A)$. $\square$

Theorem 3.15. *If A is a subset of a nonempty space X and $\gamma, \gamma^{'}$ are operations on $\alpha O(X)$, then the following statements are equivalent:*

(1) $\alpha_{[\gamma,\gamma']}$-$Cl(A) = X$.
(2) $\alpha_{[\gamma,\gamma']}$-$sCl(A) = X$.
(3) *If B is any $\alpha_{[\gamma,\gamma']}$-semiclosed subset of X such that $A \subseteq B$, then $B = X$.*
(4) *Every nonempty $\alpha_{[\gamma,\gamma']}$-semiopen set has a nonempty intersection with A.*
(5) $\alpha_{[\gamma,\gamma']}$-$sInt(X \setminus A) = \phi$.

Proof. (1) $\Rightarrow$ (2): Suppose $x \notin \alpha_{[\gamma,\gamma']}$-$sCl(A)$. Then, by Theorem 3.10 (8), there exists an $\alpha_{[\gamma,\gamma']}$-semiopen set G containing x such that $G \cap A = \phi$. Since G is a nonempty $\alpha_{[\gamma,\gamma']}$-semiopen set, then there is a nonempty $\alpha_{[\gamma,\gamma']}$-open set H such that $H \subseteq G$ and so $H \cap A = \phi$ which implies that $\alpha_{[\gamma,\gamma']}$-$Cl(A) \neq X$, a contradiction. Hence $\alpha_{[\gamma,\gamma']}$-$sCl(A) = X$.

(2) $\Rightarrow$ (3): If B is any $\alpha_{[\gamma,\gamma']}$-semiclosed set such that $A \subseteq B$, then $X = \alpha_{[\gamma,\gamma']}$-$sCl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(B) = B$ and so $B = X$.

(3) $\Rightarrow$ (4): If G is any nonempty $\alpha_{[\gamma,\gamma']}$-semiopen set such that $G \cap A = \phi$, then $A \subseteq X \setminus G$ and $X \setminus G$ is $\alpha_{[\gamma,\gamma']}$-semiclosed. By hypothesis, $X \setminus G = X$ and so $G = \phi$, a contradiction. Therefore, $G \cap A \neq \phi$.

(4) $\Rightarrow$ (5): Suppose that $\alpha_{[\gamma,\gamma']}$-$sInt(X \setminus A) \neq \phi$. Then, by Theorem 3.11 (1), $\alpha_{[\gamma,\gamma']}$-$sInt(X \setminus A)$ is a nonempty $\alpha_{[\gamma,\gamma']}$-semiopen set such that $\alpha_{[\gamma,\gamma']}$-$sInt(X \setminus A) \cap A = \phi$, a contradiction. Therefore, $\alpha_{[\gamma,\gamma']}$-$sInt(X \setminus A) = \phi$.

(5) $\Rightarrow$ (1): Since $\alpha_{[\gamma,\gamma']}$-$sInt(X \setminus A) = \phi$ implies that $X \setminus \alpha_{[\gamma,\gamma']}$-$sInt(X \setminus A) = X$ by Theorem 3.11 (11), implies that $\alpha_{[\gamma,\gamma']}$-$sCl(A) = X$. By Remark 3.2, $\alpha_{[\gamma,\gamma']}$-$sCl(B) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(B)$ for every subset B of X. Therefore, $\alpha_{[\gamma,\gamma']}$-$sCl(A) = X$ implies that $\alpha_{[\gamma,\gamma']}$-$Cl(A) = X$. □

Proposition 3.6. *Let γ and $\gamma^{'}$ be α-regular operations on $\alpha O(X)$. If A is a subset of X and $\alpha_{[\gamma,\gamma']}$-$sCl(A) = X$, then for every $\alpha_{[\gamma,\gamma']}$-open set G of X, we have $\alpha_{[\gamma,\gamma']}$-$Cl(A \cap G) = \alpha_{[\gamma,\gamma']}$-$Cl(G)$.*

Proof. The proof follows from Theorem 3.15 and Theorem 3.6 (2). □

Definition 3.4. Let (X, τ) be a topological space and $\gamma, \gamma^{'}$ be operations on $\alpha O(X)$. A subset B_x of X is said to be an $\alpha_{[\gamma,\gamma']}$-semineighborhood (resp. $\alpha_{[\gamma,\gamma']}$-neighborhood) of a point $x \in X$ if there exists an $\alpha_{[\gamma,\gamma']}$-semiopen (resp. $\alpha_{[\gamma,\gamma']}$-open) set U such that $x \in U \subseteq B_x$.

Theorem 3.16. *Let (X, τ) be a topological space and $\gamma, \gamma^{'}$ be operations on $\alpha O(X)$. A subset G of X is $\alpha_{[\gamma,\gamma']}$-semiopen if and only if it is an $\alpha_{[\gamma,\gamma']}$-semineighborhood of each of its points.*

Proof. Let G be an $\alpha_{[\gamma,\gamma']}$-semiopen set of X. Then, by Definition 3.4, it is clear that G is an $\alpha_{[\gamma,\gamma']}$-semineighborhood of each of its points, since for every $x \in G, x \in G \subseteq G$ and G is $\alpha_{[\gamma,\gamma']}$-semiopen.

Conversely, suppose that G is an $\alpha_{[\gamma,\gamma']}$-semineighborhood of each of its points. Then, for each $x \in G$, there exists $S_x \in \alpha SO(X,x)_{[\gamma,\gamma']}$ such that $S_x \subseteq G$. Then, $G = \bigcup\{S_x : x \in G\}$. Since each S_x is $\alpha_{[\gamma,\gamma']}$-semiopen, hence by Theorem 3.5, G is $\alpha_{[\gamma,\gamma']}$-semiopen in (X,τ). $\square$

Proposition 3.7. *For any two subsets A, B of a topological space (X,τ) and $A \subseteq B$, if A is an $\alpha_{[\gamma,\gamma']}$-semineighborhood of a point $x \in X$, Then, B is also $\alpha_{[\gamma,\gamma']}$-semineighborhood of the same point x.*

Proof. Obvious. $\square$

4. Some New Functions

Throughout this section, let $\gamma, \gamma^{'} : \alpha O(X) \to P(X)$ and $\beta, \beta^{'} : \alpha O(Y) \to P(Y)$ be operations on $\alpha O(X)$ and $\alpha O(Y)$, respectively.

Definition 4.1. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous if for each $x \in X$ and each $\alpha_{[\beta,\beta']}$-open set V of Y containing $f(x)$, there exists an $\alpha_{[\gamma,\gamma']}$-semiopen set U of X such that $x \in U$ and $f(U) \subseteq V$.

Theorem 4.1. *For a function $f : (X,\tau) \to (Y,\sigma)$ the following statements are equivalent:*

(1) *f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous.*
(2) *The inverse image of each $\alpha_{[\beta,\beta']}$-open set in Y is $\alpha_{[\gamma,\gamma']}$-semiopen in X.*
(3) *The inverse image of each $\alpha_{[\beta,\beta']}$-closed set in Y is $\alpha_{[\gamma,\gamma']}$-semiclosed in X.*
(4) *For each subset A of X, $f(\alpha_{[\gamma,\gamma']}\text{-}sCl(A)) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(f(A))$.*
(5) *For each subset B of Y, $\alpha_{[\gamma,\gamma']}\text{-}sCl(f^{-1}(B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}Cl(B))$.*
(6) *For each subset B of Y, $f^{-1}(\alpha_{[\beta,\beta']}\text{-}Int(B)) \subseteq \alpha_{[\gamma,\gamma']}\text{-}sInt(f^{-1}(B))$.*

Proof. (1) $\Rightarrow$ (2): Let f be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous. Let V be any $\alpha_{[\beta,\beta']}$-open set in Y. To show that $f^{-1}(V)$ is an $\alpha_{[\gamma,\gamma']}$-semiopen set in X, if $f^{-1}(V) = \phi$, then $f^{-1}(V)$ is an $\alpha_{[\gamma,\gamma']}$-semiopen set in X, if $f^{-1}(V) \neq \phi$, then there exists $x \in f^{-1}(V)$ which implies $f(x) \in V$. Since f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous, there exists an $\alpha_{[\gamma,\gamma']}$-semiopen set U in X containing x such that $f(U) \subseteq V$. This implies that $x \in U \subseteq f^{-1}(V)$. This shows $f^{-1}(V)$ is $\alpha_{[\gamma,\gamma']}$-semiopen.

(2) $\Rightarrow$ (3): Let F be any $\alpha_{[\beta,\beta']}$-closed set of Y. Then $Y \setminus F$ is an $\alpha_{[\beta,\beta']}$-open set of Y. By (2), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is an $\alpha_{[\gamma,\gamma']}$-semiopen set in X and hence $f^{-1}(F)$ is an $\alpha_{[\gamma,\gamma']}$-semiclosed set in X.

(3) $\Rightarrow$ (4): Let A be any subset of X. Then, $f(A) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(f(A))$ and $\alpha_{[\beta,\beta']}\text{-}Cl(f(A))$ is an $\alpha_{[\beta,\beta']}$-closed set in Y. Hence $A \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}Cl(f(A)))$. By (3), we have $f^{-1}(\alpha_{[\beta,\beta']}\text{-}Cl(f(A)))$ is an $\alpha_{[\gamma,\gamma']}$-semiclosed set in X. Therefore, $\alpha_{[\gamma,\gamma']}\text{-}sCl(A) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}Cl(f(A)))$. Hence, $f(\alpha_{[\gamma,\gamma']}\text{-}sCl(A)) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(f(A))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then $f^{-1}(B)$ is a subset of X. By (4), we have $f(\alpha_{[\gamma,\gamma']}\text{-}sCl(f^{-1}(B))) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(f(f^{-1}(B))) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(B)$. Hence, $\alpha_{[\gamma,\gamma']}\text{-}sCl(f^{-1}(B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}Cl(B))$.

(5) $\Leftrightarrow$ (6): Let B be any subset of Y. Then apply (5) to $Y \setminus B$ we obtain $\alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(Y \setminus B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(Y \setminus B)) \Leftrightarrow \alpha_{[\gamma,\gamma']}$-$sCl(X \setminus f^{-1}(B)) \subseteq f^{-1}(Y \setminus \alpha_{[\beta,\beta']}$-$Int(B)) \Leftrightarrow X \setminus \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(B)) \subseteq X \setminus f^{-1}(\alpha_{[\beta,\beta']}$-$Int(B)) \Leftrightarrow f^{-1}(\alpha_{[\beta,\beta']}$-$Int(B)) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(B))$. Therefore, $f^{-1}(\alpha_{[\beta,\beta']}$-$Int(B)) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(B))$.

(6) $\Rightarrow$ (1): Let $x \in X$ and V be any $\alpha_{[\beta,\beta']}$-open set of Y containing $f(x)$. Then, $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X. By (6), we have $f^{-1}(\alpha_{[\beta,\beta']}$-$Int(V)) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(V))$. Since V is an $\alpha_{[\beta,\beta']}$-open set, then $f^{-1}(V) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is an $\alpha_{[\gamma,\gamma']}$-semiopen set in X which contains x and clearly $f(f^{-1}(V)) \subseteq V$. Hence, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous. $\square$

Theorem 4.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous function. Then, for each subset B of Y, $f^{-1}(\alpha_{[\beta,\beta']}$-$Int(B)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(B)))$.*

Proof. Let B be any subset of Y. Then, $\alpha_{[\beta,\beta']}$-$Int(B)$ is $\alpha_{[\beta,\beta']}$-open in Y and so by Theorem 4.1, $f^{-1}(\alpha_{[\beta,\beta']}$-$Int(B))$ is $\alpha_{[\gamma,\gamma']}$-semiopen in X. Hence, Theorem 3.3, we have $f^{-1}(\alpha_{[\beta,\beta']}$-$Int(B)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(\alpha_{[\beta,\beta']}$-$Int(B)))) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(\alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(B)))$. $\square$

Corollary 4.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous function. Then, for each subset B of Y, $\alpha_{[\gamma,\gamma']}$-$Int(\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(B))) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(B))$.*

Proof. The proof is obvious. $\square$

Theorem 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ a bijective function. Then, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous if and only if $\alpha_{[\beta,\beta']}$-$Int(f(A)) \subseteq f(\alpha_{[\gamma,\gamma']}$-$sInt(A))$ for each subset A of X.*

Proof. Let A be any subset of X. Then, by Theorem 4.1, $f^{-1}(\alpha_{[\beta,\beta']}$-$Int(f(A))) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(f(A)))$. Since f is a bijective function, then $\alpha_{[\beta,\beta']}$-$Int(f(A)) = f(f^{-1}(\alpha_{[\beta,\beta']}$-$Int(f(A)))) \subseteq f(\alpha_{[\gamma,\gamma']}$-$sInt(A))$.

Conversely, let B be any subset of Y. Then, $\alpha_{[\beta,\beta']}$-$Int(f(f^{-1}(B))) \subseteq f(\alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(B)))$. Since f is a bijection, so, $\alpha_{[\beta,\beta']}$-$Int(B) = \alpha_{[\beta,\beta']}$-$Int(f(f^{-1}(B))) \subseteq f(\alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(B)))$. Hence, $f^{-1}(\alpha_{[\beta,\beta']}$-$Int(B)) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(B))$. Therefore, by Theorem 4.1, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous. $\square$

Proposition 4.1. *A function $f : (X, \tau) \to (Y, \sigma)$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous if and only if $\alpha_{[\gamma,\gamma']}$-$sBd(f^{-1}(B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(B) \setminus \alpha_{[\beta,\beta']}$-$Int(B))$, for each subset B in Y.*

Proof. Let B be any subset of Y. By Theorem 4.1 (2) and (5), we have $f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(B) \setminus \alpha_{[\beta,\beta']}$-$Int(B)) = f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(B)) \setminus f^{-1}(\alpha_{[\beta,\beta']}$-$Int(B)) \supseteq \alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(B)) \setminus f^{-1}(\alpha_{[\beta,\beta']}$-$Int(B)) = \alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(B)) \setminus \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(\alpha_{[\beta,\beta']}$-$Int(B))) \supseteq \alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(B)) \setminus \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(B)) = \alpha_{[\gamma,\gamma']}$-$sBd(f^{-1}(B))$, and hence $f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(B) \setminus \alpha_{[\beta,\beta']}$-$Int(B)) \supseteq \alpha_{[\gamma,\gamma']}$-$sBd(f^{-1}(B))$.

Conversely, let V be $\alpha_{[\beta,\beta']}$-open in Y and $F = Y\backslash V$. Then by (2), we obtain $\alpha_{[\gamma,\gamma']}$-$sBd(f^{-1}(F)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(F)\backslash\alpha_{[\beta,\beta']}$-$Int(F)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(F)) = f^{-1}(F)$ and hence by Theorem 3.12 (1), $\alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(F)) = \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(F)) \cup \alpha_{[\gamma,\gamma']}$-$sBd(f^{-1}(F)) \subseteq f^{-1}(F)$. Thus, $f^{-1}(F)$ is $\alpha_{[\gamma,\gamma']}$-semiclosed and hence $f^{-1}(V)$ is $\alpha_{[\gamma,\gamma']}$-semiopen in X. Therefore, by Theorem 4.1 (2), f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous. □

Proposition 4.2. *A function* $f : (X,\tau) \to (Y,\sigma)$ *is* $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$*-semicontinuous if and only if* $f(\alpha_{[\gamma,\gamma']}$-$sD(A)) \subseteq \alpha_{[\beta,\beta']}$-$Cl(f(A))$, *for any subset* A *of* X.

Proof. Let A be any subset of X. By Theorem 4.1 (4), and by the fact that $\alpha_{[\gamma,\gamma']}$-$sCl(A) = A \cup \alpha_{[\gamma,\gamma']}$-$sD(A)$, we get $f(\alpha_{[\gamma,\gamma']}$-$sD(A)) \subseteq f(\alpha_{[\gamma,\gamma']}$-$sCl(A)) \subseteq \alpha_{[\beta,\beta']}$-$Cl(f(A))$.

Conversely, let F be any $\alpha_{[\beta,\beta']}$-closed set in Y. By (2), we obtain $f(\alpha_{[\gamma,\gamma']}$-$sD(f^{-1}(F))) \subseteq \alpha_{[\beta,\beta']}$-$Cl(f(f^{-1}(F))) \subseteq \alpha_{[\beta,\beta']}$-$Cl(F) = F$. This implies $\alpha_{[\gamma,\gamma']}$-$sD(f^{-1}(F)) \subseteq f^{-1}(F)$. Hence, by Theorem 3.13 (7), $f^{-1}(F)$ is $\alpha_{[\gamma,\gamma']}$-semiclosed in X. Therefore, by Theorem 4.1 (3), f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous. □

Definition 4.2. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semiopen if and only if for each $\alpha_{[\gamma,\gamma']}$-open set U in X, $f(U)$ is $\alpha_{[\beta,\beta']}$-semiopen set in Y.

Theorem 4.4. *A function* $f : (X,\tau) \to (Y,\sigma)$ *is* $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$*-semiopen if and only if for every subset* $E \subseteq X$, *we have* $f(\alpha_{[\gamma,\gamma']}$-$Int(E)) \subseteq \alpha_{[\beta,\beta']}$-$Cl(\alpha_{[\beta,\beta']}$-$Int(f(E)))$.

Proof. Let f be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semiopen. Since $f(\alpha_{[\gamma,\gamma']}$-$Int(E)) \subseteq f(E)$, and $f(\alpha_{[\gamma,\gamma']}$-$Int(E))$ is $\alpha_{[\beta,\beta']}$-semiopen. Then, $f(\alpha_{[\gamma,\gamma']}$-$Int(E)) \subseteq \alpha_{[\beta,\beta']}$-$Cl(\alpha_{[\beta,\beta']}$-$Int(f(\alpha_{[\gamma,\gamma']}$-$Int(E)))) \subseteq \alpha_{[\beta,\beta']}$-$Cl(\alpha_{[\beta,\beta']}$-$Int(f(E)))$.

Conversely, let G be any $\alpha_{[\gamma,\gamma']}$-open set in X. Then, $\alpha_{[\beta,\beta']}$-$Int(f(G)) \subseteq f(G) \subseteq f(\alpha_{[\gamma,\gamma']}$-$Int(G)) \subseteq \alpha_{[\beta,\beta']}$-$Cl(\alpha_{[\beta,\beta']}$-$Int(f(G)))$. Therefore, $f(G)$ is $\alpha_{[\beta,\beta']}$-semiopen and consequently f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semiopen. □

Theorem 4.5. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be an* $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$*-semiopen function, then for every subset* G *of* Y, $\alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(G)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(G)))$.

Proof. Let f be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semiopen. By Theorem 4.4, we have $f(\alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(G))) \subseteq \alpha_{[\beta,\beta']}$-$Cl(\alpha_{[\beta,\beta']}$-$Int(f(f^{-1}(G)))) \subseteq \alpha_{[\beta,\beta']}$-$Cl(\alpha_{[\beta,\beta']}$-$Int(G)) \subseteq \alpha_{[\beta,\beta']}$-$Cl(G)$ implies that $\alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(G)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(G)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(\alpha_{[\beta,\beta']}$-$Cl(G)))$. □

Theorem 4.6. *A function* $f : (X,\tau) \to (Y,\sigma)$ *is* $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$*-semiopen if and only if for every* $x \in X$ *and for every* $\alpha_{[\gamma,\gamma']}$*-neighborhood* U *of* x, *there exists an* $\alpha_{[\beta,\beta']}$*-semineighborhood* V *of* $f(x)$ *such that* $V \subseteq f(U)$.

Proof. Let U be an $\alpha_{[\gamma,\gamma']}$-neighborhood of $x \in X$. Then, there exists an $\alpha_{[\gamma,\gamma']}$-open set O such that $x \in O \subseteq U$. By hypothesis, $f(O)$ is $\alpha_{[\beta,\beta']}$-semineighborhood in Y such that $f(x) \in f(O) \subseteq f(U)$.

Conversely, let U be any $\alpha_{[\gamma,\gamma']}$-open set in X. For each $y \in f(U)$, by hypothesis there exists an $\alpha_{[\beta,\beta']}$-semineighborhood V_y of y in Y such that $V_y \subseteq f(U)$. Since V_y is $\alpha_{[\beta,\beta']}$-semineighbourhood of y, there exists an $\alpha_{[\beta,\beta']}$-semiopen set A_y in Y such that $y \in A_y \subseteq V_y$. Therefore, $f(U) = \cup\{A_y : y \in f(U)\}$ is an $\alpha_{[\beta,\beta']}$-semiopen in Y. This shows that f is an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semiopen function. □

Theorem 4.7. *The following statements are equivalent for a bijective function $f : (X,\tau) \to (Y,\sigma)$:*

(1) *f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semiopen.*
(2) *$f(\alpha_{[\gamma,\gamma']}\text{-}Int(A)) \subseteq \alpha_{[\beta,\beta']}\text{-}sInt(f(A))$, for every $A \subseteq X$.*
(3) *$\alpha_{[\gamma,\gamma']}\text{-}Int(f^{-1}(B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(B))$, for every $B \subseteq Y$.*
(4) *$f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(B)) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(B))$, for every $B \subseteq Y$.*
(5) *$\alpha_{[\beta,\beta']}\text{-}sCl(f(A)) \subseteq f(\alpha_{[\gamma,\gamma']}\text{-}Cl(A))$, for every $A \subseteq X$.*
(6) *$\alpha_{[\beta,\beta']}\text{-}sD(f(A)) \subseteq f(\alpha_{[\gamma,\gamma']}\text{-}Cl(A))$, for every $A \subseteq X$.*

Proof. (1) ⇒ (2): Let A be any subset of X. Since $f(\alpha_{[\gamma,\gamma']}\text{-}Int(A))$ is $\alpha_{[\beta,\beta']}$-semiopen and $f(\alpha_{[\gamma,\gamma']}\text{-}Int(A)) \subseteq f(A)$, and thus $f(\alpha_{[\gamma,\gamma']}\text{-}Int(A)) \subseteq \alpha_{[\beta,\beta']}\text{-}sInt(f(A))$.
The proof of the other implications are obvious. □

Theorem 4.8. *Let $f : (X,\tau) \to (Y,\sigma)$ be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous and $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semiopen and let $A \in \alpha SO(X)_{[\gamma,\gamma']}$. Then, $f(A) \in \alpha SO(Y)_{[\beta,\beta']}$.*

Proof. Since A is $\alpha_{[\gamma,\gamma']}$-semiopen, then there exists an $\alpha_{[\gamma,\gamma']}$-open set O in X such that $O \subseteq A \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(O)$. Therefore, $f(O) \subseteq f(A) \subseteq f(\alpha_{[\gamma,\gamma']}\text{-}Cl(O)) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(f(O))$. Thus, by Theorem 3.4, $f(A) \in \alpha SO(Y)_{[\beta,\beta']}$. □

Theorem 4.9. *Let π and π' be operations on $\alpha O(Z)$. If $f : X \to Y$ is a function, $g : Y \to Z$ is $(\alpha_{[\beta,\beta']}, \alpha_{[\pi,\pi']})$-semiopen and injective, and $gof : X \to Z$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\pi,\pi']})$-semicontinuous. Then, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous.*

Proof. Let V be an $\alpha_{[\beta,\beta']}$-open subset of Y. Since g is $(\alpha_{[\beta,\beta']}, \alpha_{[\pi,\pi']})$-semiopen, $g(V)$ is $\alpha_{[\pi,\pi']}$-semiopen subset of Z. Since gof is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\pi,\pi']})$-semicontinuous and g is injective, then $f^{-1}(V) = f^{-1}(g^{-1}(g(V))) = (gof)^{-1}(g(V))$ is $\alpha_{[\gamma,\gamma']}$-semiopen in X, which proves that f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous. □

Definition 4.3. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute if the inverse image of every $\alpha_{[\beta,\beta']}$-semiopen set of Y is $\alpha_{[\gamma,\gamma']}$-semiopen in X.

Proposition 4.3. *Every $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute function is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous.*

Proof. Straightforward. □

The converse of the above proposition need not be true in general as it is shown below.

Example 4.1. Let $X = \{a, b, c\}$ and $\tau = \sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ be a topology on X. For each $A \in \alpha O(X)$, define the operations $\gamma : \alpha O(X,\tau) \to P(X)$, γ' :

$\alpha O(X,\tau) \to P(X)$, $\beta : \alpha O(X,\sigma) \to P(X)$ and $\beta' : \alpha O(X,\sigma) \to P(X)$, respectively, by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } A = \{a,b\} \\ X & \text{if } A \neq \{a,b\} \end{cases}$$

and

$$A^{\beta} = A^{\beta'} = \begin{cases} A & \text{if } A = \{b\} \\ X & \text{if } A \neq \{b\}. \end{cases}$$

Define a function $f : (X,\tau) \to (X,\sigma)$ as follows:

$$f(x) = \begin{cases} a & \text{if } x = a \\ a & \text{if } x = b \\ c & \text{if } x = c. \end{cases}$$

Then, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous, but not $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute because $\{b,c\}$ is an $\alpha_{[\beta,\beta']}$-semiopen set of Y but $f^{-1}(\{b,c\}) = \{c\}$ is not $\alpha_{[\gamma,\gamma']}$-semiopen in X.

Theorem 4.10. *If $f : (X,\tau) \to (Y,\sigma)$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous and $f^{-1}(\alpha_{[\beta,\beta']}\text{-}Cl(V)) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(V))$ for each subset $V \in \alpha O(Y)_{[\beta,\beta']}$, then f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute.*

Proof. Let B be any $\alpha_{[\beta,\beta']}$-semiopen subset of Y. Then, there exists $V \in \alpha O(Y)_{[\beta,\beta']}$ such that $V \subseteq B \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(V)$. Therefore, we have $f^{-1}(V) \subseteq f^{-1}(B) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}Cl(V)) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(V))$. Since f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semicontinuous and $V \in \alpha O(Y)_{[\beta,\beta']}$, then $f^{-1}(V)$ is an $\alpha_{[\gamma,\gamma']}$-semiopen set of X. Hence, by Theorem 3.4, $f^{-1}(B)$ is an $\alpha_{[\gamma,\gamma']}$-semiopen set of X. This shows that f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute. □

Theorem 4.11. *A function $f : (X,\tau) \to (Y,\sigma)$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute if and only if for each $x \in X$ and each $\alpha_{[\beta,\beta']}$-semiopen set V of Y containing $f(x)$, there exists an $\alpha_{[\gamma,\gamma']}$-semiopen set U of X containing x such that $f(U) \subseteq V$.*

Proof. Let $x \in X$ and V be any $\alpha_{[\beta,\beta']}$-semiopen set of Y containing $f(x)$. Set $U = f^{-1}(V)$, then by f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute, U is an $\alpha_{[\gamma,\gamma']}$-semiopen subset of X containing x and $f(U) \subseteq V$.

Conversely, let V be any $\alpha_{[\beta,\beta']}$-semiopen set of Y and $x \in f^{-1}(V)$. By hypothesis, there exists an $\alpha_{[\gamma,\gamma']}$-semiopen set U of X containing x such that $f(U) \subseteq V$. Thus, we have $x \in U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(V)$. By Proposition 3.1, $f^{-1}(V)$ is $\alpha_{[\gamma,\gamma']}$-semiopen of X. Therefore, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute. □

Theorem 4.12. *A function $f : (X,\tau) \to (Y,\sigma)$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute if and only if for every $\alpha_{[\beta,\beta']}$-semiclosed subset H of Y, $f^{-1}(H)$ is $\alpha_{[\gamma,\gamma']}$-semiclosed in X.*

Proof. Let f be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute, then for every $\alpha_{[\beta,\beta']}$-semiopen subset Q of Y, $f^{-1}(Q)$ is $\alpha_{[\gamma,\gamma']}$-semiopen in X. Let H be any $\alpha_{[\beta,\beta']}$-semiclosed subset of Y, then $Y \setminus H$ is $\alpha_{[\beta,\beta']}$-semiopen. Thus, $f^{-1}(Y \setminus H)$ is $\alpha_{[\gamma,\gamma']}$-semiopen, but $f^{-1}(Y \setminus H) = X \setminus f^{-1}(H)$ so that $f^{-1}(H)$ is $\alpha_{[\gamma,\gamma']}$-semiclosed.

Conversely, suppose that for all $\alpha_{[\beta,\beta']}$-semiclosed subset H of Y, $f^{-1}(H)$ is $\alpha_{[\gamma,\gamma']}$-semiclosed in X and let Q be any $\alpha_{[\beta,\beta']}$-semiopen subset of Y, then $Y \setminus Q$ is $\alpha_{[\beta,\beta']}$-semiclosed. By hypothesis, $X \setminus f^{-1}(Q) = f^{-1}(Y \setminus Q)$ is $\alpha_{[\gamma,\gamma']}$-semiclosed. Thus, $f^{-1}(Q)$ is $\alpha_{[\gamma,\gamma']}$-semiopen. $\square$

Theorem 4.13. *Let $f : (X, \tau) \to (Y, \sigma)$ be function. Then, the following statements are equivalent:*

(1) *f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute.*
(2) *$\alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(B))$, for each subset B of Y.*
(3) *$f(\alpha_{[\gamma,\gamma']}$-$sCl(A)) \subseteq \alpha_{[\beta,\beta']}$-$sCl(f(A))$, for each subset A of X.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Then, $B \subseteq \alpha_{[\beta,\beta']}$-$sCl(B)$ and $f^{-1}(B) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(B))$. Since f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute, so, $f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(B))$ is an $\alpha_{[\gamma,\gamma']}$-semiclosed subset of X. Hence, $\alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(B)) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(B))) = f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(B))$

(2) $\Rightarrow$ (3): Let A be any subset of X. Then, $f(A) \subseteq \alpha_{[\beta,\beta']}$-$sCl(f(A))$ and $\alpha_{[\gamma,\gamma']}$-$sCl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(f(A))) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(f(A)))$. This implies that $f(\alpha_{[\gamma,\gamma']}$-$sCl(A)) \subseteq f(f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(f(A)))) \subseteq \alpha_{[\beta,\beta']}$-$sCl(f(A))$.

(3) $\Rightarrow$ (1): Let V be an $\alpha_{[\beta,\beta']}$-semiclosed subset of Y. Then, $f(\alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(V))) \subseteq \alpha_{[\beta,\beta']}$-$sCl(f(f^{-1}(V))) \subseteq \alpha_{[\beta,\beta']}$-$sCl(V) = V$. This implies that $\alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(V)) \subseteq f^{-1}(f(\alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(V)))) \subseteq f^{-1}(V)$. Thus, $f^{-1}(V)$ is an $\alpha_{[\gamma,\gamma']}$-semiclosed subset of X and consequently f is an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute function. $\square$

Theorem 4.14. *A function $f : (X, \tau) \to (Y, \sigma)$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute if and only if $f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(B)) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(B))$ for each subset B of Y.*

Proof. Let B be any subset of Y. Then, $\alpha_{[\beta,\beta']}$-$sInt(B) \subseteq B$. Since f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute, $f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(B))$ is an $\alpha_{[\gamma,\gamma']}$-semiopen subset of X. Hence, $f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(B)) = \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(B))) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(B))$.

Conversely, let V be an $\alpha_{[\beta,\beta']}$-semiopen subset of Y. Then, $f^{-1}(V) = f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(V)) \subseteq \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is an $\alpha_{[\gamma,\gamma']}$-semiopen subset of X and consequently f is an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute function. $\square$

Proposition 4.4. *A function $f : (X, \tau) \to (Y, \sigma)$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute if and only if $\alpha_{[\gamma,\gamma']}$-$sBd(f^{-1}(B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sBd(B))$, for each subset B of Y.*

Proof. Let B be any subset of Y. Then, $\alpha_{[\gamma,\gamma']}$-$sBd(f^{-1}(B)) = \alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(B)) \setminus \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(B)) \setminus \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(B))$ used Theorem 4.13. Therefore, by Theorem 4.14, we have $\alpha_{[\gamma,\gamma']}$-$sBd(f^{-1}(B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(B)) \setminus f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(B)) = f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(B)) \setminus \alpha_{[\beta,\beta']}$-$sInt(B)) = f^{-1}(\alpha_{[\beta,\beta']}$-$sBd(B))$.

Conversely, let V be $\alpha_{[\beta,\beta']}$-semiopen in Y and $F = Y \setminus V$. Then, by hypothesis, we obtain $\alpha_{[\gamma,\gamma']}$-$sBd(f^{-1}(F)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sBd(F)) = f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(F) \setminus \alpha_{[\beta,\beta']}$-$sInt(F)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(F)) = f^{-1}(F)$ and hence by Theorem 3.12 (1), $\alpha_{[\gamma,\gamma']}$-$sCl(f^{-1}(F)) = \alpha_{[\gamma,\gamma']}$-$sInt(f^{-1}(F)) \cup \alpha_{[\gamma,\gamma']}$-$sBd(f^{-1}(F)) \subseteq$

$f^{-1}(F)$. Thus, $f^{-1}(F)$ is $\alpha_{[\gamma,\gamma']}$-semiclosed and hence $f^{-1}(V)$ is $\alpha_{[\gamma,\gamma']}$-semiopen in X. Therefore, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute. □

Corollary 4.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-closed and $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute, then $f(\alpha_{[\gamma,\gamma']}\text{-}sCl(A)) = \alpha_{[\beta,\beta']}\text{-}sCl(f(A))$ for every subset A of X.*

Proof. Since for any subset A of X, $A \subseteq \alpha_{[\gamma,\gamma']}\text{-}sCl(A)$. Therefore, $f(A) \subseteq f(\alpha_{[\gamma,\gamma']}\text{-}sCl(A))$. Since f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-closed, then $\alpha_{[\beta,\beta']}\text{-}sCl(f(A)) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl(f(\alpha_{[\gamma,\gamma']}\text{-}sCl(A))) = f(\alpha_{[\gamma,\gamma']}\text{-}sCl(A))$. Hence, $f(\alpha_{[\gamma,\gamma']}\text{-}sCl(A)) \supseteq \alpha_{[\beta,\beta']}\text{-}sCl(f(A))$ and by Theorem 4.13, we have $f(\alpha_{[\gamma,\gamma']}\text{-}sCl(A)) = \alpha_{[\beta,\beta']}\text{-}sCl(f(A))$. □

Corollary 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective function. Then, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-semiopen and $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute if $f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(V)) = \alpha_{[\gamma,\gamma']}\text{-}sCl(f^{-1}(V))$ for every subset V of Y.*

Proof. The proof is follows from Remark 3.2, Theorems 4.7 and 4.13. □

Theorem 4.15. *If $f : X \to Y$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-irresolute and $g : Y \to Z$ is $(\alpha_{[\beta,\beta']}, \alpha_{[\delta,\delta']})$-irresolute, then $g(f) : X \to Z$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\delta,\delta']})$-irresolute.*

Proof. If $A \subseteq Z$ is $\alpha_{[\delta,\delta']}$-semiopen, then $g^{-1}(A)$ is $\alpha_{[\beta,\beta']}$-semiopen and $f^{-1}(g^{-1}(A))$ is $\alpha_{[\gamma,\gamma']}$-semiopen. Thus, $(g(f))^{-1}(A) = f^{-1}(g^{-1}(A))$ is $\alpha_{[\gamma,\gamma']}$-semiopen and hence $g(f)$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\delta,\delta']})$-irresolute. □

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COVARIENT DERIVATIVES OF ALMOST CONTACT STRUCTURE AND ALMOST PARACONTACT STRUCTURE WITH RESPECT TO X^C AND X^V ON TANGENT BUNDLE $T(M)$

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Abstract. The differential geometry of tangent bundles was studied by several authors, for example: D. E. Blair [1], V. Oproiu [4], A. Salimov [5], Yano and Ishihara [8] and among others. It is well known that differant structures deffined on a manifold M can be lifted to the same type of structures on its tangent bundle. Several authors cited here in obtained result in this direction. Our goal is to study covarient derivatives of almost contact structure and almost paracontact structure with respect to X^C and X^V on tangent bundle $T(M)$. In addition, this covarient derivatives which obtained shall be studied for some special values in almost contact structure and almost paracontact structure.

1. Introduction

Let M be an $n-$dimensional differentiable manifold of class C^∞ and let $T_p(M)$ be the tangent space of M at a point p of M. Then the set [8]

$$T(M) = \underset{p\in M}{\cup} T_p(M) \tag{1.1}$$

is called the tangent bundle over the manifold M. For any point $\tilde{p}$ of $T(M)$, the correspondence $\tilde{p} \to p$ determines the bundle projection $\pi : T(M) \to M$, Thus $\pi(\tilde{p}) = p$, where $\pi : T(M) \to M$ defines the bundle projection of $T(M)$ over M. The set $\pi^{-1}(p)$ is called the fibre over $p \in M$ and M the base space.

Suppose that the base space M is covered by a system of coordinate neighbourhoods $\{U; x^h\}$, where (x^h) is a system of local coordinates defined in the neighbourhood U of M. The open set $\pi^{-1}(U) \subset T(M)$ is naturally differentiably homeomorphic to the direct product $U \times R^n$, R^n being the $n-$dimensional vector space over the real field R, in such a way that a point $\tilde{p} \in T_p(M)(p \in U)$ is represented by an ordered pair (P, X) of the point $p \in U$, and a vector $X \in R^n$,whose components are given by the cartesian coordinates (y^h) of $\tilde{p}$ in the tangent space $T_p(M)$ with respect

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to the natural base $\{\partial_h\}$, where $\partial_h = \frac{\partial}{\partial x^h}$. Denoting by (x^h) the coordinates of $p = \pi(\tilde{p})$ in U and establishing the correspondence $(x^h, y^h) \to \tilde{p} \in \pi^{-1}(U)$, we can introduce a system of local coordinates (x^h, y^h) in the open set $\pi^{-1}(U) \subset T(M)$. Here we cal (x^h, y^h) the coordinates in $\pi^{-1}(U)$ induced from (x^h) or simply, the induced coordinates in $\pi^{-1}(U)$.

We denote by $\Im_s^r(M)$ the set of all tensor fields of class $C^\infty$and of type (r,s) in M. We now put $\Im(M) = \sum\limits_{r,s=0}^{\infty} \Im_s^r(M)$, which is the set of all tensor fields in M. Similarly, we denote by $\Im_s^r(T(M))$ and $\Im(T(M))$ respectively the corresponding sets of tensor fields in the tangent bundle $T(M)$.

1.1. **Vertical lifts.** If f is a function in M, we write f^v for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \to M$ and $f : M \to R$, so that

$$f^v = f o \pi \tag{1.2}$$

Thus, if a point $\tilde{p} \in \pi^{-1}(U)$ has induced coordinates (x^h, y^h), then

$$f^v(\tilde{p}) = f^v(x, y) = f o \pi(\tilde{p}) = f(p) = f(x). \tag{1.3}$$

Thus the value of $f^v(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^v the vertical lift of the function f [8].

Let $\tilde{X} \in \Im_0^1(T(M))$ be such that $\tilde{X} f^v = 0$ for all $f \in \Im_0^0(M)$. Then we say that $\tilde{X}$ is a vertical vector field. Let $\binom{\tilde{X}^h}{\tilde{X}^{\bar{h}}}$ be components of $\tilde{X}$ with respect to the induced coordinates. Then $\tilde{X}$ is vertical if and only if its components in $\pi^{-1}(U)$ satisfy

$$\begin{pmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{pmatrix} = \begin{pmatrix} 0 \\ X^{\bar{h}} \end{pmatrix}. \tag{1.4}$$

Suppose that $X \in \Im_0^1(M)$, so that is a vector field in M. We define a vector field X^v in $T(M)$ by

$$X^v(\iota\, \omega) = (\omega X)^v \tag{1.5}$$

ω being an arbitrary $1-$form in M. We cal X^v the vertical lift of X [8].

Let $\tilde{\omega} \in \Im_1^0(T(M))$ be such that $\tilde{\omega}(X)^v = 0$ for all $X \in \Im_0^1(M)$. Then we say that $\tilde{\omega}$ is a vertical $1-$form in $T(M)$. We define the vertical lift ω^v of the $1-$form ω by

$$\omega^v = (\omega_i)^v (dx^i)^v \tag{1.6}$$

in each open set $\pi^{-1}(U)$, where $(U; x^h)$ is coordinate neighbourhood in M and ω is given by $\omega = \omega_i dx^i$ in U. The vertical lift ω^v of ω with lokal expression $\omega = \omega_i dx^i$ has components of the form

$$\omega^v : (\omega^i, 0) \tag{1.7}$$

with respect to the induced coordinates in $T(M)$.

Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients by the conditions

$$(P \otimes Q)^V = P^V \otimes Q^V, \ (P + R)^V = P^V + R^V \tag{1.8}$$

P, Q and R being arbitrary elements of $T(M)$. The vertical lifts F^V of an element $F \in \Im_1^1(M)$ with lokal components F_i^h has components of the form [8]

$$F^V : \begin{pmatrix} 0 & 0 \\ F_i^h & 0 \end{pmatrix}. \tag{1.9}$$

Vertical lift has the following formulas [3, 8]:

$$\begin{aligned} (fX)^v &= f^v X^v, I^v X^v = 0, \eta^v (X^v) = 0 \\ (f\eta)^v &= f^v \eta^v, [X^v, Y^v] = 0, \varphi^v X^v = 0 \\ X^v f^v &= 0, X^v f^v = 0 \end{aligned} \tag{1.10}$$

hold good, where $f \in \Im_0^0(M_n)$, $X, Y \in \Im_0^1(M_n)$, $\eta \in \Im_1^0(M_n)$, $\varphi \in \Im_1^1(M_n)$, $I = id_{M_n}$.

1.2. **Complete lifts.** If f is a function in M, we write f^c for the function in $T(M)$ defined by

$$f^c = \iota(df)$$

and call f^c the comple lift of the function f. The complete lift f^c of a function f has the lokal expression

$$f^c = y^i \partial_i f = \partial f \tag{1.11}$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X \in \Im_0^1(M)$. Then we define a vector field X^c in $T(M)$ by

$$X^c f^c = (Xf)^c, \tag{1.12}$$

f being an arbitrary function in M and call X^c the complete lift of X in $T(M)$ [2, 8]. The complete lift X^c of X with components x^h in M has components

$$X^c = \begin{pmatrix} X^h \\ \partial X^h \end{pmatrix} \tag{1.13}$$

with respect to the induced coordinates in $T(M)$.

Suppose that $\omega \in \Im_1^0(M)$, then a $1-$form ω^c in $T(M)$ defined by

$$\omega^c(X^c) = (\omega X)^c \tag{1.14}$$

X being an arbitrary vector field in M. We call ω^c the complete lift of ω. The complete lift ω^c of ω with components ω_i in M has components of the form

$$\omega^c : (\partial \omega_{i,} \omega_i) \tag{1.15}$$

with respect to the induced coordinates in $T(M)$ [2].

The complete lifts to a unique algebra isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients, is given by the conditions

$$(P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C,\ (P + R)^C = P^C + R^C, \tag{1.16}$$

where P, Q and R being arbitrary elements of $T(M)$. The complete lifts F^C of an element $F \in \Im_1^1(M)$ with lokal components F_i^h has components of the form

$$F^C : \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}. \tag{1.17}$$

In addition, we know that the complete lifts are defined by [3, 8]:

$$
\begin{aligned}
(fX)^c &= f^cX^v + f^vX^c = (Xf)^c, \\
X^cf^v &= (Xf)^v, \eta^v(x^c) = (\eta(x))^v, \\
X^vf^c &= (Xf)^v, \varphi^vX^c = (\varphi X)^v, \\
\varphi^cX^v &= (\varphi X)^v, (\varphi X)^c = \varphi^cX^c, \\
\eta^v(X^c) &= (\eta(X))^c, \eta^c(X^v) = (\eta(X))^v, \\
[X^v, Y^c] &= [X,Y]^v, I^c = I, I^vX^c = X^v, [X^c, Y^c] = [X,Y]^c.
\end{aligned} \tag{1.18}
$$

Let M_n be an $n-$dimensional diferentiable manifold. Differantial transformation of algebra $T(M_n)$, defined by

$$D = \nabla_X : T(M_n) \to T(M_n), X \in \Im_0^1(M_n),$$

is called as covariant derivation with respect to vector field X if

$$
\begin{aligned}
\nabla_{fX+gY}t &= f\nabla_Xt + g\nabla_Yt, \\
\nabla_Xf &= Xf,
\end{aligned}
$$

where $\forall f, g \in \Im_0^0(M_n), \forall X, Y \in \Im_0^1(M_n), \forall t \in \Im(M_n)$.

On the other hand, a transformation defined by

$$\nabla : \Im_0^1(M_n) \times \Im_0^1(M_n) \to \Im_0^1(M_n),$$

is called as affin connection [5, 8].

We now assume that M_n is a manifold with an affine connection ∇. Then there exist a unique affine connection ∇^c in $\Im(M_n)$ which satisfies

$$\nabla^c_{X^c}Y^c = (\nabla_XY)^c \tag{1.19}$$

for any $X, Y \in \Im_0^1(M_n)$. This affine connection is called the complete lift of the affine connection ∇ to $T(M_n)$ and denoted by ∇^c [8].

Proposition 1.1. *For any $X \in \Im_0^1(M_n)$, $f \in \Im_0^0(M_n)$ and ∇^c is the complete lift of the affine connection ∇ to $T(M_n)$* [8]

i) $\nabla^c_{X^v}f^v = 0$,
ii) $\nabla^c_{X^v}f^c = (\nabla_Xf)^v$,
iii) $\nabla^c_{X^c}f^v = (\nabla_Xf)^v$,
iv) $\nabla^c_{X^c}f^c = (\nabla_Xf)^c$.

Proposition 1.2. *For any $X, Y \in \Im_0^1(M_n)$ and ∇^c is the complete lift of the affine connection ∇ to $T(M_n)$* [8]

i) $\nabla^c_{X^v}Y^v = 0$,
ii) $\nabla^c_{X^v}Y^c = (\nabla_XY)^v$,
iii) $\nabla^c_{X^c}Y^v = (\nabla_XY)^v$,
iv) $\nabla^c_{X^c}Y^c = (\nabla_XY)^c$.

2. Main Results

Let an $n-$dimensional differentiable manifold M_n be endowed with a tensor field φ of type $(1,1)$, a vector field ξ, a $1-$form η, I the identity and let them satisfy

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \varphi(\xi) = 0, \qquad \eta o \varphi = 0, \qquad \eta(\xi) = 1. \tag{2.1}$$

Then (φ, ξ, η) define almost contact structure on M_n [3, 6, 8]. From (2.1), we get on taking complete and vertical lifts

$$\begin{aligned} (\varphi^c)^2 &= -I + \eta^v \otimes \xi^c + \eta^c \otimes \xi^v, \\ \varphi^c \xi^v &= 0, \varphi^c \xi^c = 0, \eta^v o \varphi^c = 0, \\ \eta^c o \varphi^c &= 0, \eta^v(\xi^v) = 0, \eta^v(\xi^c) = 1, \\ \eta^c(\xi^v) &= 1, \eta^c(\xi^c) = 0. \end{aligned} \tag{2.2}$$

We now define a $(1,1)$ tensor field J on $T(M_n)$ by

$$J = \varphi^c - \xi^v \otimes \eta^v + \xi^c \otimes \eta^c. \tag{2.3}$$

Then it is easy to show that $J^2 X^v = -X^v$ and $J^2 X^c = -X^c$, which give that J is an almost contact structure on $T(M_n)$. We get from (2.3)

$$\begin{aligned} JX^v &= (\varphi X)^v + (\eta(X))^v \xi^c, \\ JX^c &= (\varphi X)^c - (\eta(X))^v \xi^v + (\eta(X))^c \xi^c \end{aligned}$$

for any $X \in \Im_0^1(M_n)$ [3].

Theorem 2.1. *For ∇_X the operator covarient derivation with respect to X, $J \in \Im_1^1(T(M_n))$ defined by (2.3) and $\eta(Y) = 0$, we have*

$$\begin{aligned} i)\ (\nabla^c_{X^v} J)Y^v &= 0, \\ ii)\ (\nabla^c_{X^v} J)Y^c &= ((\nabla_X \varphi)Y)^v + ((\nabla_X \eta)Y)^v \xi^c, \\ iii)\ (\nabla^c_{X^c} J)Y^v &= ((\nabla_X \varphi)Y)^v + ((\nabla_X \eta)Y)^v \xi^c, \\ iv)\ (\nabla^c_{X^c} J)Y^c &= ((\nabla_X \varphi)Y)^c - ((\nabla_X \eta)Y)^v \xi^v + ((\nabla_X \eta)Y)^c \xi^c, \end{aligned}$$

where $X, Y \in \Im_0^1(M_n)$, a tensor field $\varphi \in \Im_1^1(M_n)$, a vector field ξ and a $1-$form $\eta \in \Im_1^0(M_n)$.

Proof. For $J = \varphi^c - \xi^v \otimes \eta^v + \xi^c \otimes \eta^c$ and $\eta(Y) = 0$, we get

$$\begin{aligned} i)\ (\nabla^c_{X^v} J)Y^v &= \nabla^c_{X^v}(\varphi^c - \xi^v \otimes \eta^v + \xi^c \otimes \eta^c)Y^v - (\varphi^c - \xi^v \otimes \eta^v + \xi^c \otimes \eta^c)\nabla^c_{X^v} Y^v \\ &= \nabla^c_{X^v}(\varphi Y)^v - \nabla^c_{X^v}(\eta^v(Y)^v)\xi^v + \nabla^c_{X^v}(\eta(Y))^v \xi^c \\ &= 0, \end{aligned}$$

$$\begin{aligned} ii)\ (\nabla^c_{X^v} J)Y^c &= \nabla^c_{X^v}(\varphi^c - \xi^v \otimes \eta^v + \xi^c \otimes \eta^c)Y^c - (\varphi^c - \xi^v \otimes \eta^v + \xi^c \otimes \eta^c)\nabla^c_{X^v} Y^c \\ &= \nabla^c_{X^v}\varphi^c Y^c - \nabla^c_{X^v}(\eta Y)^v \xi^v + \nabla^c_{X^v}(\eta(Y))^c \xi^c - \varphi^c \nabla^c_{X^v} Y^c \\ &\quad + \eta^v(\nabla_X Y)^v \xi^v - (\eta(\nabla_X Y))^v \xi^c \\ &= (\nabla^c_{X^v}\varphi^c)Y^c + \varphi^c(\nabla^c_{X^v} Y^c) - \varphi^c \nabla^c_{X^v} Y^c - (\nabla_X(\eta(Y)))^v \xi^c \\ &\quad + ((\nabla_X \eta)Y)^v \xi^c \\ &= (\nabla_X \varphi)Y)^v + ((\nabla_X \eta)Y)^v \xi^c, \end{aligned}$$

$$\begin{aligned} iii)\ (\nabla^c_{X^c} J)Y^v &= \nabla^c_{X^c}(\varphi^c - \xi^v \otimes \eta^v + \xi^c \otimes \eta^c)Y^v - (\varphi^c - \xi^v \otimes \eta^v + \xi^c \otimes \eta^c)\nabla^c_{X^c} Y^v \\ &= \nabla^c_{X^c}\varphi^c Y^v - \nabla^c_{X^c}(\eta^v(Y)^v)\xi^v + \nabla^c_{X^c}(\eta(Y))^v \xi^c - \varphi^c \nabla^c_{X^c} Y^v \\ &\quad + \eta^v(\nabla_X Y)^v \xi^v - (\eta(\nabla_X Y))^v \xi^c \\ &= (\nabla^c_{X^c}\varphi^c)Y^v + \varphi^c(\nabla^c_{X^c} Y^v) - \varphi^c \nabla^c_{X^c} Y^v - (\nabla_X(\eta(Y)))^v \xi^c \\ &\quad + (\nabla_X \eta)Y)^v \xi^c \\ &= (\nabla_X \varphi)Y)^v + ((\nabla_X \eta)Y)^v \xi^c, \end{aligned}$$

$$
\begin{aligned}
iv)\ (\nabla^c_{X^c}J)Y^c &= \nabla^c_{X^c}(\varphi^c - \xi^v\otimes\eta^v + \xi^c\otimes\eta^c)Y^c - (\varphi^c - \xi^v\otimes\eta^v + \xi^c\otimes\eta^c)\nabla^c_{X^c}Y^c \\
&= \nabla^c_{X^c}\varphi^c Y^c - \nabla^c_{X^c}((\eta Y)^v)\xi^v + \nabla^c_{X^c}(\eta(Y))^c\xi^c - \varphi^c\nabla^c_{X^c}Y^c \\
&\quad +(\eta(\nabla_X Y))^v\xi^v - (\eta(\nabla_X Y))^c\xi^c \\
&= (\nabla^c_{X^c}\varphi^c)Y^c + \varphi^c(\nabla^c_{X^c}Y^c) - \varphi^c\nabla^c_{X^c}Y^c + (\nabla_X(\eta(Y)))^v\xi^v \\
&\quad -((\nabla_X\eta)Y)^v\xi^v - (\nabla_X(\eta(Y)))^c\xi^c + ((\nabla_X\eta)Y)^c\xi^c \\
&= (\nabla_X\varphi)Y)^c - ((\nabla_X\eta)Y)^v\xi^v + ((\nabla_X\eta)Y)^c\xi^c.
\end{aligned}
$$

□

Corollary 2.1. *If we put $Y = \xi$, i.e. $\eta(\xi) = 1$ and ξ has the conditions of (2.1), then we get different results*

$$
\begin{aligned}
i)\ (\nabla^c_{X^v}J)\xi^v &= (\nabla_X\xi)^v, \\
ii)\ (\nabla^c_{X^v}J)\xi^c &= ((\nabla_X\varphi)\xi)^v + (((\nabla_X\eta))\xi)^v\xi^c, \\
iii)\ (\nabla^c_{X^c}J)\xi^v &= ((\nabla_X\varphi)\xi)^v + (\nabla_X\xi)^c + ((\nabla_X\eta)\xi)^v\xi^c, \\
iv)\ (\nabla^c_{X^c}J)\xi^c &= (\nabla_X\varphi)\xi)^c - (\nabla_X\xi)^v - ((\nabla_X\eta)\xi)^v\xi^v + ((\nabla_X\eta)\xi)^c\xi^c.
\end{aligned}
$$

Let an $n-$dimensional differentiable manifold M_n be endowed with a tensor field φ of type $(1,1)$, a vector field ξ, a $1-$form η, I the identity and let them satisfy

$$\varphi^2 = I - \eta\otimes\xi, \qquad \varphi(\xi) = 0, \qquad \eta o\varphi = 0, \qquad \eta(\xi) = 1. \tag{2.4}$$

Then (φ, ξ, η) define almost paracontact structure on M_n [3, 6]. From (2.4), we get on taking complete and vertical lifts

$$
\begin{aligned}
(\varphi^c)^2 &= I - \eta^v\otimes\xi^c - \eta^c\otimes\xi^v, \\
\varphi^c\xi^v &= 0, \varphi^c\xi^c = 0, \eta^v o\varphi^c = 0, \\
\eta^c o\varphi^c &= 0, \eta^v(\xi^v) = 0, \eta^v(\xi^c) = 1, \\
\eta^c(\xi^v) &= 1, \eta^c(\xi^c) = 0.
\end{aligned} \tag{2.5}
$$

We now define a $(1,1)$ tensor field $\widetilde{J}$ on $T(M_n)$ by

$$\widetilde{J} = \varphi^c - \xi^v\otimes\eta^v - \xi^c\otimes\eta^c. \tag{2.6}$$

Then it is easy to show that $\widetilde{J}^2X^v = X^v$ and $\widetilde{J}^2X^c = X^c$, which give that $\widetilde{J}$ is an almost product structure on $T(M_n)$. We get from (2.6)

$$
\begin{aligned}
\widetilde{J}X^v &= (\varphi X)^v - (\eta(X))^v\xi^c, \\
\widetilde{J}X^c &= (\varphi X)^v - (\eta(X))^v\xi^v - (\eta(X))^c\xi^c
\end{aligned}
$$

for any $X \in \Im^1_0(M_n)$.

Theorem 2.2. *For ∇_X the operator covarient derivation with respect to X, $\widetilde{J} \in \Im^1_1(T(M_n))$ defined by (2.6) and $\eta(Y) = 0$, we have*

$$
\begin{aligned}
i)\ (\nabla^c_{X^v}\widetilde{J})Y^v &= 0, \\
ii)\ (\nabla^c_{X^v}\widetilde{J})Y^c &= ((\nabla_X\varphi)Y)^v - ((\nabla_X\eta)Y)^v\xi^c, \\
iii)\ (\nabla^c_{X^c}\widetilde{J})Y^v &= ((\nabla_X\varphi)Y)^v - ((\nabla_X\eta)Y)^v\xi^c, \\
iv)\ (\nabla^c_{X^c}\widetilde{J})Y^c &= ((\nabla_X\varphi)Y)^c - ((\nabla_X\eta)Y)^v\xi^v - ((\nabla_X\eta)Y)^c\xi^c,
\end{aligned}
$$

where $X, Y \in \Im^1_0(M_n)$, a tensor field $\varphi \in \Im^1_1(M_n)$, a vector field $\xi \in \Im^1_0(M_n)$ and a $1-$form $\eta \in \Im^0_1(M_n)$.

Proof. For $\widetilde{J} = \varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c$ and $\eta(Y) = 0$, we get

$$\begin{aligned}
i)\ (\nabla^c_{X^v}\widetilde{J})Y^v &= \nabla^c_{X^v}(\varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c)Y^v - (\varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c)\nabla^c_{X^v}Y^v \\
&= \nabla^c_{X^v}(\varphi Y)^v - \nabla^c_{X^v}(\eta^v(Y)^v)\xi^v - \nabla^c_{X^v}(\eta(Y))^v\xi^c \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
ii)\ (\nabla^c_{X^v}\widetilde{J})Y^c &= \nabla^c_{X^v}(\varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c)Y^c - (\varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c)\nabla^c_{X^v}Y^c \\
&= \nabla^c_{X^v}\varphi^c Y^c - \nabla^c_{X^v}(\eta Y)^v\xi^v - \nabla^c_{X^v}(\eta(Y))^c\xi^c - \varphi^c\nabla^c_{X^v}Y^c \\
&\quad +\eta^v(\nabla_X Y)^v\xi^v + (\eta(\nabla_X Y))^v\xi^c \\
&= (\nabla^c_{X^v}\varphi^c)Y^c + \varphi^c(\nabla^c_{X^v}Y^c) - \varphi^c\nabla^c_{X^v}Y^c + (\nabla_X(\eta(Y)))^v\xi^c \\
&\quad -((\nabla_X\eta)Y)^v\xi^c \\
&= (\nabla_X\varphi)Y)^v - ((\nabla_X\eta)Y)^v\xi^c,
\end{aligned}$$

$$\begin{aligned}
iii)\ (\nabla^c_{X^c}\widetilde{J})Y^v &= \nabla^c_{X^c}(\varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c)Y^v - (\varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c)\nabla^c_{X^c}Y^v \\
&= \nabla^c_{X^c}\varphi^c Y^v - \nabla^c_{X^c}(\eta^v(Y)^v)\xi^v - \nabla^c_{X^c}(\eta(Y))^v\xi^c - \varphi^c\nabla^c_{X^c}Y^v \\
&\quad +\eta^v(\nabla_X Y)^v\xi^v + (\eta(\nabla_X Y))^v\xi^c \\
&= (\nabla^c_{X^c}\varphi^c)Y^v + \varphi^c(\nabla^c_{X^c}Y^v) - \varphi^c\nabla^c_{X^c}Y^v + (\nabla_X(\eta(Y)))^v\xi^c \\
&\quad -(\nabla_X\eta)Y)^v\xi^c \\
&= (\nabla_X\varphi)Y)^v - ((\nabla_X\eta)Y)^v\xi^c,
\end{aligned}$$

$$\begin{aligned}
iv)\ (\nabla^c_{X^c}\widetilde{J})Y^c &= \nabla^c_{X^c}(\varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c)Y^c - (\varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c)\nabla^c_{X^c}Y^c \\
&= \nabla^c_{X^c}\varphi^c Y^c - \nabla^c_{X^c}((\eta Y)^v)\xi^v - \nabla^c_{X^c}(\eta(Y))^c\xi^c - \varphi^c\nabla^c_{X^c}Y^c \\
&\quad +(\eta(\nabla_X Y))^v\xi^v + (\eta(\nabla_X Y))^c\xi^c \\
&= (\nabla^c_{X^c}\varphi^c)Y^c + \varphi^c(\nabla^c_{X^c}Y^c) - \varphi^c\nabla^c_{X^c}Y^c + (\nabla_X(\eta(Y)))^v\xi^v \\
&\quad -((\nabla_X\eta)Y)^v\xi^v + (\nabla_X(\eta(Y)))^c\xi^c - ((\nabla_X\eta)Y)^c\xi^c \\
&= (\nabla_X\varphi)Y)^c - ((\nabla_X\eta)Y)^v\xi^v - ((\nabla_X\eta)Y)^c\xi^c.
\end{aligned}$$

□

Corollary 2.2. *If we put $Y = \xi$, i.e. $\eta(\xi) = 1$ and ξ has the conditions of (2.4), then we have*

$$\begin{aligned}
i)\ (\nabla^c_{X^v}\widetilde{J})\xi^v &= -(\nabla_X\xi)^v, \\
ii)\ (\nabla^c_{X^v}\widetilde{J})\xi^c &= ((\nabla_X\varphi)\xi)^v - (((\nabla_X\eta))\xi)^v\xi^c, \\
iii)\ (\nabla^c_{X^c}\widetilde{J})\xi^v &= ((\nabla_X\varphi)\xi)^v - (\nabla_X\xi)^c - ((\nabla_X\eta)\xi)^v\xi^c, \\
iv)\ (\nabla^c_{X^c}\widetilde{J})\xi^c &= (\nabla_X\varphi)\xi)^c - (\nabla_X\xi)^v - ((\nabla_X\eta)\xi)^v\xi^v - ((\nabla_X\eta)\xi)^c\xi^c.
\end{aligned}$$

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GRONWALL TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRALS

MEHMET ZEKI SARIKAYA

Abstract. In this paper, some new generalized Gronwall-type inequalities are investigated for conformable differential equations. The established results are extensions of some existing Gronwall-type inequalities in the literature.

1. Introduction

Fractional Calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. The subject is as old as the calculus of differentiation and goes back to times when Leibniz, Gauss, and Newton invented this kind of calculation. During three centuries, the theory of fractional calculus developed as a pure theoretical field, useful only for mathematicians, we refer to [10], see also [11]. Recently a new local, limit-based definition of a conformable derivative has been formulated [1], [4], [8], with several follow-up papers [2], [3], [5]-[9]. In this paper, we use the Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$and $t \in [0,\infty)$ given by

$$D^{\alpha}\left(f\right)\left(t\right)=\lim_{\varepsilon\to 0}\frac{f\left(te^{\varepsilon t^{-\alpha}}\right)-f\left(t\right)}{\varepsilon},\ D^{\alpha}\left(f\right)\left(0\right)=\lim_{t\to 0}D^{\alpha}\left(f\right)\left(t\right), \tag{1.1}$$

provided the limits exist (for detail see, [8]). If f is fully differentiable at t, then

$$D^{\alpha}\left(f\right)\left(t\right)=t^{1-\alpha}\frac{df}{dt}\left(t\right). \tag{1.2}$$

A function f is $\alpha-$differentiable at a point $t\geq 0$ if the limit in (1.1) exists and is finite. This definition yields the following results;

Theorem 1.1. *Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then*
i. $D^{\alpha}\left(af+bg\right)=aD^{\alpha}\left(f\right)+bD^{\alpha}\left(g\right)$, *for all* $a,b\in\mathbb{R}$,
ii. $D^{\alpha}\left(\lambda\right)=0$, *for all constant functions* $f\left(t\right)=\lambda$,
iii. $D^{\alpha}\left(fg\right)=fD^{\alpha}\left(g\right)+gD^{\alpha}\left(f\right)$,

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iv. $D^{\alpha}\left(\frac{f}{g}\right)=\frac{fD^{\alpha}(g)-gD^{\alpha}(f)}{g^{2}}$
v. $D^{\alpha}(t^{n})=nt^{n-\alpha}$ *for all* $n\in\mathbb{R}$
vi. $D^{\alpha}(f\circ g)(t)=f'(g(t))\,D^{\alpha}(g)(t)$ *for* f *is differentiable at* $g(t)$.

Definition 1.1 (Conformable fractional integral)**.** Let $\alpha\in(0,1]$ and $0\leq a<b$. A function $f:[a,b]\to\mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral

$$\int_{a}^{b}f(x)\,d_{\alpha}x:=\int_{a}^{b}f(x)\,x^{\alpha-1}dx$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_{\alpha}^{1}([a,b])$

Remark 1.1.

$$I_{\alpha}^{a}(f)(t)=I_{1}^{a}\left(t^{\alpha-1}f\right)=\int_{a}^{t}\frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

We will also use the following important results, which can be derived from the results above.

Lemma 1.1. *Let the conformable differential operator* D^{α} *be given as in (1.1), where* $\alpha\in(0,1]$ *and* $t\geq0$*, and assume the functions* f *and* g *are* α*-differentiable as needed. Then*
i. $D^{\alpha}(\ln t)=t^{-\alpha}$ *for* $t>0$
ii. $D^{\alpha}\left[\int_{a}^{t}f(t,s)\,d_{\alpha}s\right]=f(t,t)+\int_{a}^{t}D^{\alpha}[f(t,s)]\,d_{\alpha}s$
iii. $\int_{a}^{b}f(x)\,D^{\alpha}(g)(x)\,d_{\alpha}x=fg|_{a}^{b}-\int_{a}^{b}g(x)\,D^{\alpha}(f)(x)\,d_{\alpha}x.$

In this paper, some new generalized Gronwall-type inequalities are investigated for conformable differential equations. The established results are extensions of some existing Gronwall-type inequalities in the literature.

2. Main Results

Troughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals involved exist on the respective domains of their definitions, and $C(M,S)$ and $C^{1}(M,S)$denote the class of all continuous functions and the first order conformable derivative, respectively, defined on set M with range in the set S.

Firstly, we start with the following definition, which is a generalization of the limit definition of the derivative for the case of a function with many variables.

Definition 2.1. Let f be a function with n variables $t_{1},...,t_{n}$ and the conformable partial derivative of f of order $\alpha\in(0,1]$ in x_{i} is defined as follows

$$\frac{\partial^{\alpha}}{\partial t_{i}^{\alpha}}f(t_{1},...,t_{n})=\lim_{\varepsilon\to0}\frac{f(t_{1},...,t_{i-1},t_{i}e^{\varepsilon t_{i}^{-\alpha}},...,t_{n})-f(t_{1},...,t_{n})}{\varepsilon}. \tag{2.1}$$

The first result is the generalization of Theorem 2.10 of [3].

Theorem 2.1. *Assume that* $f(t,s)$ *is function for which* $\partial_{t}^{\alpha}\left[\partial_{s}^{\beta}f(t,s)\right]$ *and* $\partial_{s}^{\beta}\left[\partial_{t}^{\alpha}f(t,s)\right]$ *exist and are continuos over the domain* $D\subset\mathbb{R}^{2}$*, then*

$$\partial_{t}^{\alpha}\left[\partial_{s}^{\beta}f(t,s)\right]=\partial_{s}^{\beta}\left[\partial_{t}^{\alpha}f(t,s)\right]. \tag{2.2}$$

Proof. By using the (1.1), it follows that

$$\begin{aligned}\partial_t^\alpha\left[\partial_s^\beta f(t,s)\right] &= \partial_t^\alpha\left[\lim_{\varepsilon\to 0}\frac{f\left(t, se^{\varepsilon s^{-\beta}}\right)-f(t,s)}{\varepsilon}\right]\\ &= \partial_t^\alpha\left[\lim_{\varepsilon\to 0}\frac{f\left(t, s+\varepsilon s^{1-\beta}+O(\varepsilon^2)\right)-f(t,s)}{\varepsilon}\right].\end{aligned}$$

Making the change of variable $k=\varepsilon s^{1-\beta}\left(1+O(\varepsilon)\right)$, we get

$$\partial_t^\alpha\left[\partial_s^\beta f(t,s)\right]=\partial_t^\alpha\left[\lim_{k\to 0}\frac{f(t,s+k)-f(t,s)}{\frac{ks^{\beta-1}}{1+O(\varepsilon)}}\right].$$

Since f is diffentiable in s-direction, we obtain

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right]=s^{1-\beta}\partial_t^\alpha\left[\frac{\partial}{\partial s}f(t,s)\right]. \tag{2.3}$$

Again by definition (1.1), it follows that

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right]=s^{1-\beta}\lim_{\varepsilon\to 0}\frac{\frac{\partial}{\partial s}f\left(te^{\varepsilon t^{-\alpha}},s\right)-\frac{\partial}{\partial s}f(t,s)}{\varepsilon}.$$

Similarly, after making the change of variable, we have

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right]=s^{1-\beta}t^{1-\alpha}\lim_{h\to 0}\frac{\frac{\partial}{\partial s}f(t+h,s)-\frac{\partial}{\partial s}f(t,s)}{\varepsilon}.$$

Since f is diffentiable in t-direction, we obtain

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right]=s^{1-\beta}t^{1-\alpha}\frac{\partial^2}{\partial t\partial s}f(t,s). \tag{2.4}$$

Since f is continuous, by using the Clairaut's theorem for partial derivatives, it follows that

$$\frac{\partial^2}{\partial s\partial t}f(t,s)=\frac{\partial^2}{\partial t\partial s}f(t,s).$$

Therefore the equation (2.4) becomes

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right]=s^{1-\beta}t^{1-\alpha}\frac{\partial^2}{\partial t\partial s}f(t,s)=s^{1-\beta}t^{1-\alpha}\lim_{k\to 0}\frac{\frac{\partial}{\partial t}f(t,s+k)-\frac{\partial}{\partial t}f(t,s)}{k}.$$

Thus, taking $k=\varepsilon s^{1-\beta}\left(1+O(\varepsilon)\right)$ and laler $h=\varepsilon t^{1-\alpha}\left(1+O(\varepsilon)\right)$ we arrive at

$$\partial_t^\alpha\left[\partial_s^\alpha f(t,s)\right]=\partial_s^\alpha\left[\lim_{k\to 0}\frac{\frac{\partial}{\partial t}f(t,s+k)-\frac{\partial}{\partial t}f(t,s)}{k}\right]=\partial_s^\alpha\left[\partial_t^\alpha f(t,s)\right]$$

which completes the proof. □

Theorem 2.2. *Let $k\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$, $y\in C\left(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+\right)$, $r\in C^1\left(\mathbb{R}^+,\mathbb{R}^+\right)$ with $(t,s)\to\partial_t^\alpha y(t,s)\in C\left(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+\right)$. Assume in additional that r is nondecreasing and $r(t)\le t$ for $t\ge 0$. If $u\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$ satisfies*

$$u(t)\le k(t)+\int_0^{r(t)}y(t,s)\,u(s)d_\alpha s,\quad t\ge 0, \tag{2.5}$$

then
(2.6)

$$u(t) \leq k(t)+e^{\int_0^{r(t)} y(t,s)d_\alpha s} \int_0^t e^{-\int_0^{r(\tau)} y(\tau,s)d_\alpha s} \frac{\partial^\alpha}{\partial \tau^\alpha}\left(\int_0^{r(\tau)} y(\tau,s)\, k(s) d_\alpha s\right) d_\alpha \tau, \quad t \geq 0.$$

Proof. If we set

$$z(t)=\int_0^{r(t)} y(t,s)\, u(s) d_\alpha s$$

then our assumptions on y and r imply that z is nondecreasing on $\mathbb{R}^+$. Thus, for $t \geq 0$, by using Lemma 1.1 (ii), we get

$$\begin{aligned} D^\alpha z(t) &= y(t, r(t))\, u(r(t)) D^\alpha r(t)+\int_0^{r(t)}\left[\frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\right] u(s) d_\alpha s \\ &\leq y(t, r(t))\,[k(r(t))+z(r(t))]\, D^\alpha r(t)+\int_0^{r(t)}\left[\frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\right][k(s)+z(s)]\, d_\alpha s \\ &\leq y(t, r(t))\,[k(r(t))+z(t)]\, D^\alpha r(t)+\int_0^{r(t)}\left[\frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\right] k(s) d_\alpha s+z(t) \int_0^{r(t)} \frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\, d_\alpha s \end{aligned}$$

or, equivalently

$$D^\alpha z(t)-z(t) \frac{\partial^\alpha}{\partial t^\alpha}\left(\int_0^{r(t)} y(t,s)\, d_\alpha s\right) \leq \frac{\partial^\alpha}{\partial t^\alpha}\left(\int_0^{r(t)} y(t,s)\, k(s) d_\alpha s\right).$$

Multiplying the above inequality by $e^{-\int_0^{r(t)} y(t,s)d_\alpha s}$, we obtain that

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(z(t) e^{-\int_0^{r(t)} y(t,s)d_\alpha s}\right) \leq e^{-\int_0^{r(t)} y(t,s)d_\alpha s} \frac{\partial^\alpha}{\partial t^\alpha}\left(\int_0^{r(t)} y(t,s)\, k(s) d_\alpha s\right).$$

Integrating this from 0 to t yields

$$z(t) \leq e^{\int_0^{r(t)} y(t,s)d_\alpha s} \int_0^t e^{-\int_0^{r(\tau)} y(\tau,s)d_\alpha s} \frac{\partial^\alpha}{\partial \tau^\alpha}\left(\int_0^{r(\tau)} y(\tau,s)\, k(s) d_\alpha s\right) d_\alpha \tau.$$

Combine the above inequality with $u(t) \leq k(t)+z(t)$ this imply (2.4). The proof is complete. □

Corollary 2.1. *Assume y, r are as in Theorem 2.2 and $k(t)=k>0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (2.5), then*

$$u(t) \leq k e^{\int_0^{r(t)} y(t,s)d_\alpha s}, \quad t \geq 0$$

Proof. Applying Theorem 2.2 for $k(t)=k$ and , we arrive at

$$\begin{aligned} u(t) &\leq k+k e^{\int_0^{r(t)} y(t,s)d_\alpha s} \int_0^t e^{-\int_0^{r(\tau)} y(\tau,s)d_\alpha s} \frac{\partial^\alpha}{\partial \tau^\alpha}\left(\int_0^{r(\tau)} y(\tau,s)\, d_\alpha s\right) d_\alpha \tau \\ &= k+k e^{\int_0^{r(t)} y(t,s)d_\alpha s}\left(1-e^{-\int_0^{r(t)} y(t,s)d_\alpha s}\right) \\ &= k e^{\int_0^{r(t)} y(t,s)d_\alpha s}, \ t \geq 0. \end{aligned}$$

□

Remark 2.1. If we take $r(t) = t$ in Corollary 2.1, then the inequality given by Corollary 2.1 reduces to Gronwall's inequality for conformable integrals in [1].

Theorem 2.3. *Let $k, y, x \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and assume that r is nondecreasing with $r(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$u(t) \leq k(t) + y(t) \int_0^{r(t)} x(s)u(s)d_\alpha s, \quad t \geq 0, \tag{2.7}$$

then

$$u(t) \leq k(t) + y(t) \int_0^t e^{\int_{r(\tau)}^{r(t)} x(s)y(s)d_\alpha s} x(r(\tau)) k(r(\tau)) D^\alpha r(\tau) d_\alpha \tau, \quad t \geq 0. \tag{2.8}$$

Proof. If we set

$$z(t) = \int_0^{r(t)} x(s)u(s)d_\alpha s$$

then, by using conformable rules we see that

$$\begin{aligned} D^\alpha z(t) &= x(r(t)) u(r(t)) D^\alpha r(t) \\ &\leq x(r(t)) [k(r(t)) + y(r(t)) z(r(t))] D^\alpha r(t) \\ &\leq x(r(t)) [k(r(t)) + y(r(t)) z(t)] D^\alpha r(t). \end{aligned}$$

Thus, we have

$$D^\alpha z(t) - x(r(t)) y(r(t)) z(t) D^\alpha r(t) \leq x(r(t)) k(r(t)) D^\alpha r(t).$$

Multiplying the above inequality by $e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s}$, we obtain that

$$\frac{\partial^\alpha}{\partial t^\alpha} \left(z(t) e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s} \right) \leq e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s} x(r(t)) k(r(t)) D^\alpha r(t).$$

Integrating this from 0 to t yields

$$\begin{aligned} z(t) &\leq e^{\int_0^{r(t)} x(s)y(s)d_\alpha s} \int_0^t e^{-\int_0^{r(\tau)} x(s)y(s)d_\alpha s} x(r(\tau)) k(r(\tau)) D^\alpha r(\tau) d_\alpha \tau \\ &= \int_0^t e^{\int_{r(\tau)}^{r(t)} x(s)y(s)d_\alpha s} x(r(\tau)) k(r(\tau)) D^\alpha r(\tau) d_\alpha \tau \end{aligned}$$

and hence the claim follows because of $u(t) \leq k(t) + y(t)z(t)$. The proof is complete. □

Corollary 2.2. *Assume y, x, k are as in Theorem 2.3 and $r(t) = t$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (2.7), then*

$$u(t) \leq k(t) + y(t) \int_0^t e^{\int_\tau^t x(s)y(s)d_\alpha s} x(\tau) k(\tau) d_\alpha \tau, \quad t \geq 0.$$

Remark 2.2. If we take $y(t) = t$ in Corollary 2.2, then the inequality given by Corollary 2.2 reduces to Gronwall's inequality for conformable integrals in [2].

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AN EXAMINATION ON THE *MANNHEIM FRENET RULED SURFACE* BASED ON NORMAL VECTOR FIELDS IN $\mathbf{E}^3$

ŞEYDA KILIÇOĞLU

Abstract. In this paper we consider six special Frenet ruled surfaces along to the Mannheim pairs $\{\alpha^*, \alpha\}$. First we define and find the parametric equations of Frenet ruled surfaces which are called $Mannheim\ Frenet\ ruled\ surface$, along Mannheim curve α, in terms of the Frenet apparatus of Mannheim curve α. Later, we find only one matrix gives us all nine positions of normal vector fields of these six Frenet ruled surfaces and $Mannheim\ Frenet\ ruled\ surface$ in terms of Frenet apparatus of Mannheim curve α too. Further using that matrix we have some results such as; normal ruled surface and $Mannheim$ $normal\ ruled\ surface$ of Mannheim curve α have perpendicular normal vector fields along the curve $\varphi_2(s) = \alpha + \frac{\tan\theta}{k_1 \tan\theta - k_2} V_2$, under the condition $\tan\theta \neq \frac{k_2}{k_1}$.

1. Introduction and Preliminaries

Mannheim curve was firstly defined by A. Mannheim in 1878. A curve is called a Mannheim curve if and only if $\frac{k_1}{\left(k_1^2+k_2^2\right)}$ is a non-zero constant, k_1 is the curvature and k_2 is the torsion, respectively. Recently, a new definition of the associated curves was given by Liu and Wang in [7]. According to this new definition, if the principal normal vector of first curve and binormal vector of second curve are linearly dependent, then first curve is called Mannheim curve, and the second curve is called Mannheim partner curve. As a result they called these new curves as Mannheim partner curves.

The quantities $\{V_1, V_2, V_3, k_1, k_2\}$ are collectively Frenet-Serret apparatus of the curve $\alpha : I \to E^3$. The Frenet formulae are also well known as

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Let $\alpha : I \to E^3$ be the C^2 differentiable unit speed and $\alpha^* : I \to E^3$ be second curve and let $V_1(s), V_2(s), V_3(s)$ and $V_1^*(s^*), V_2^*(s^*), V_3^*(s^*)$ be the Frenet frames of

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the curves α and α^*, respectively. If the principal normal vector V_2 of the curve α is linearly dependent on the binormal vector V_3^* of the curve α^*, then the pair $\{\alpha, \alpha^*\}$ is said to be Mannheim pair, then α is called a Mannheim curve and α^* is called Mannheim partner curve of α where $\langle V_1, V_1^* \rangle = \cos\theta$ and besides the equality $\frac{k_1}{k_1^2+k_2^2} =$ constant; is known the offset property, for some non-zero constant. Mannheim partner curve of α can be represented $\alpha(s^*) = \alpha^*(s^*) + \lambda(s^*)V_3^*(s^*)$ for some function λ, since V_2 and V_3 are linearly dependent, Equation can be rewritten as [8]

$$\alpha^*(s) = \alpha(s) - \lambda V_2(s)$$

where

$$\lambda = \frac{-k_1}{k_1^2 + k_2^2}.$$

Frenet-Serret apparatus of Mannheim partner curve α^*, based in Frenet-Serret vectors of Mannheim curve α are

$$\begin{aligned} V_1^* &= cos\theta\ V_1 - sin\theta\ V_3 \\ V_2^* &= sin\theta\ V_1 + cos\theta\ V_3 \\ V_3^* &= V_2. \end{aligned}$$

The curvature and the torsion have the following equalities,

$$\begin{aligned} k_1^* &= -\frac{d\theta}{ds^*} = \frac{\dot{\theta}}{\cos\theta} \\ & \quad and \\ k_2^* &= \frac{k_1}{\lambda k_2}. \end{aligned}$$

we use dot to denote the derivative with respect to the arc-length parameter of the curve α. For more detail see in [8]

Also we can write

$$\frac{ds}{ds^*} = \frac{1}{\sqrt{1 + \lambda k_2}}$$

or

$$\frac{ds}{ds^*} = \frac{1}{\cos\theta}$$

and since $d(\alpha(s), \alpha^*(s)) = \|\alpha(s) - \alpha^*(s)\| = \|\lambda V_2(s)\| = |\lambda|$ we have $|\lambda|$ is the distance between the curves α and α^*.

By using the similiar method we produce a new ruled surface based on the other ruled surface. A ruled surface is one which can be generated by the motion of a straight line in Euclidean $3 - space$, ([1]). To illustrate the current situation, we bring here the famous example of L. K. Graves, so called the $B - scroll$, in [3]. A Frenet ruled surface is a ruled surfaces generated by Frenet vectors of the base curve. $Involute\ B - scroll$ is defined in [5]˙The differential geometric elements of the $involute\ \tilde{D}\ scroll$ are examined in [10]. The positions of Frenet ruled surfaces along Bertrand pairs are examined based on their normal vector fields in [6]. Also in [9] Mannheim offsets of ruled surfaces are defined and characterized

Definition 1.1. In the Euclidean $3-space$, let $\alpha(s)$ be the arc-length of a parametrized curve. The equations

$$\begin{cases} \varphi_1(s,u_1) = \alpha(s) + u_1 V_1(s) \\ \varphi_2(s,u_2) = \alpha(s) + u_2 V_2(s) \\ \varphi_3(s,u_3) = \alpha(s) + u_3 V_3(s) \end{cases}$$

are the parametrization of Frenet ruled surfaces which are called $V_1 - scroll$ (tangent ruled surface), $V_2 - scroll$ (normal ruled surface), $V_3 - scroll$ (binormal ruled surface), respectively in [2].

Theorem 1.1. *In the Euclidean $3-space$, let η_1, η_2, η_3 be the normal vector fields of ruled surfaces $\varphi_1, \varphi_2, \varphi_3$ recpectively, along the curve α. They can be expressed by the following matrix;*

$$\begin{aligned} [\eta] &= [A]\,[V] \\ [\eta] &= \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a & 0 & b \\ c & d & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} a &= \frac{-u_2k_2}{\sqrt{(u_2k_2)^2 + (1-u_2k_1)^2}}, \quad c = \frac{-u_3k_2}{\sqrt{(u_3k_2)^2+1}} \\ b &= \frac{(1-u_2k_1)}{\sqrt{(u_2k_2)^2 + (1-u_2k_1)^2}}, \quad d = \frac{-1}{\sqrt{(u_3k_2)^2+1}}. \end{aligned}$$

Proof. The normal vector fields η_1, η_2, η_3 of ruled surfaces $\varphi_1, \varphi_2, \varphi_3$ can be expressed as in the following four equalities

$$\begin{aligned} \eta_1 &= -V_3 \\ \eta_2 &= \frac{-u_2k_2V_1 + (1-u_2k_1)V_3}{\sqrt{(u_2k_2)^2 + (1-u_2k_1)^2}} \\ \eta_3 &= \frac{-u_3k_2V_1 - V_2}{\sqrt{(u_3k_2)^2+1}} \end{aligned}$$

for more detail see in [4]. Same way some results on Frenet Ruled Surfaces along the *evolute-involute* curves, based on normal vector fields are given in [4]. □

2. Mannheim Frenet ruled surfaces

In this section, we found eight special Frenet ruled surfaces along to the Bertrand pairs $\{\alpha^*, \alpha\}$. First we define and find the parametric equations of Frenet ruled surfaces which are called $Bertrandian\ Frenet\ ruled\ surface$, along Bertrand curve α, in terms of the Frenet apparatus of of Bertrand curve α. Later we found only one matrix gives us all sixteen positions of normal vector fields of eight Frenet ruled surfaces and $Bertrandian\ Frenet\ ruled\ surface$ in terms of Frenet apparatus of Bertrand curve α too. Further using that matrix we have some results such as; normal ruled surface and $Bertrandian\ tangent\ ruled\ surface$ have perpendicular normal vector fields along the curve.

Definition 2.1. Let $\{\alpha^*, \alpha\}$ be Mannheim curve pair with $k_1 \neq 0$ and $k_2 \neq 0$. The equations of the ruled surfaces

$$\begin{cases} \varphi_1^*(s, v_1) = \alpha^*(s) + v_1 V_1^*(s), \\ \varphi_2^*(s, v_2) = \alpha^*(s) + v_2 V_2^*(s), \\ \varphi_3^*(s, v_3) = \alpha^*(s) + v_3 V_3^*(s), \end{cases}$$

are the parametrization of Frenet ruled surface of Mannheim pairs $\alpha^*(s)$.

Further we can give these surface equations as in the following way;

$$\begin{cases} \varphi_1^*(s, v_1) = \alpha^*(s) + v_1 V_1^*(s) & = \alpha(s) - \lambda V_2(s) + v_1 (cos\theta\ V_1 - sin\theta\ V_3) \\ \varphi_2^*(s, v_2) = \alpha^*(s) + v_2 V_2^*(s) & = \alpha(s) - \lambda V_2(s) + v_2 (sin\theta\ V_1 + cos\theta\ V_3), \\ \varphi_3^*(s, v_3) = \alpha^*(s) + v_3 V_3^*(s) & = \alpha(s) - \lambda V_2(s) + v_3 V_2 = \alpha(s) + (v_3 - \lambda) V_2, \end{cases}$$

are the parametrization of Frenet ruled surface which are called Mannheim Tangent ruled surface, Mannheim Normal ruled surface, and Mannheim Binormal ruled surface respectively. They are called collectively Mannheim Frenet ruled surface in this study.

Theorem 2.1. *The normal vector fields $\eta_1^*, \eta_2^*, \eta_3^*$, of ruled surfaces $\varphi_1^*, \varphi_2^*, \varphi_3^*$, recpectively, along the curve $Mannheim$ partner α^*, can be expressed by the following matrix;*

$$[\eta^*] = \begin{bmatrix} \eta_1^* \\ \eta_2^* \\ \eta_3^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a^* & 0 & b^* \\ c^* & d^* & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}.$$

where

$$a^* = \frac{-v_2 k_2^*}{\sqrt{(v_2 k_2^*)^2 + (1 - v_2 k_1^*)^2}} \quad c^* = \frac{-v_3 k_2^*}{\sqrt{(v_3 k_2^*)^2 + 1}}$$
$$b^* = \frac{(1 - v_2 k_1^*)}{\sqrt{(v_2 k_2^*)^2 + (1 - v_2 k_1^*)^2}} \quad d^* = \frac{-1}{\sqrt{(v_3 k_2^*)^2 + 1}}$$

Proof. It is trivial □

Theorem 2.2. *In the Euclidean $3-$ space, the product matrix of the position of the unit normal vector fields η_1, η_2, η_3, and $\eta_1^*, \eta_2^*, \eta_3^*$ of Frenet ruled surfaces, along the Mannheim pairs α and α^* is*

$$[\eta][\eta^*]^{\mathbf{T}} = \begin{matrix} \langle \eta_1, \eta_1^* \rangle & \langle \eta_1, \eta_2^* \rangle & \langle \eta_1, \eta_3^* \rangle \\ \langle \eta_2, \eta_1^* \rangle & \langle \eta_2, \eta_2^* \rangle & \langle \eta_2, \eta_3^* \rangle \\ \langle \eta_3, \eta_1^* \rangle & \langle \eta_3, \eta_2^* \rangle & \langle \eta_3, \eta_3^* \rangle \end{matrix} \ ...(I)$$

Proof. It is easy from the matrix product;

$$[\eta][\eta^*]^{\mathbf{T}} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \begin{bmatrix} \eta_1^* & \eta_2^* & \eta_3^* \end{bmatrix}.$$

□

Theorem 2.3. *In the Euclidean $3-$ space, the product matrix of the unit normal vector fields η_1, η_2, η_3 and $\eta_1^*, \eta_2^*, \eta_3^*$ of Frenet ruled surfaces, along the Mannheim pairs α and α^*,can be given by the following matrix*

$$[\eta][\eta^*]^{\mathbf{T}} = \begin{bmatrix} 0 & a^* \sin\theta & c^* \sin\theta - d^* \cos\theta \\ 0 & a^* (a \cos\theta - b \sin\theta) & c^* (a \cos\theta - b \sin\theta) + d^* (a \sin\theta + b \cos\theta) \\ -d & a^* c \cos\theta + d b^* & c^* c \cos\theta + d^* c \sin\theta \end{bmatrix} ...(II)$$

Proof. Let $[\eta] = [A][V]$ and $[\eta^*] = [A^*][V^*]$ hence

$$[\eta][\eta^*]^{\mathbf{T}} = [A][V]([A^*][V^*])^{\mathbf{T}} = [A]\left([V][V^*]^{\mathbf{T}}\right)[A^*]^{\mathbf{T}}.$$

Where the matrix product of Frenet vector fields of the Mannheim partner α^*, *and* Mannheim curve α has the following matrix form;

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}\begin{bmatrix} V_1^* & V_2^* & V_3^* \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}$$

Hence

$$\begin{aligned} [\eta][\eta^*]^T &= [A]\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}[A^*]^T \\ &= \begin{bmatrix} 0 & a^*\sin\theta & c^*\sin\theta - d^*\cos\theta \\ 0 & a^*(a\cos\theta - b\sin\theta) & c^*(a\cos\theta - b\sin\theta) + d^*(a\sin\theta + b\cos\theta) \\ -d & a^*c\cos\theta + db^* & c^*c\cos\theta + d^*c\sin\theta \end{bmatrix} \end{aligned}$$

this product give us the result. ☐

In the Euclidean $3-space$, the position of six surface, basicly, can be examined by the position of their unit normal vector fields. We can examine the nine positions of six surfaces, basicly, according to the position of their unit normal vector fields in a matrix. Since the equality of the last two matrice (I) and (II), we have nine interesting results according to the normal vector fields with the following results.

There are two pairs of normal vector fields perpendicular to each other of Frenet ruled surface along the Mannheim pairs $\{\alpha^*, \alpha\}$ as in the following corollary;

Corollary 2.1. *Tangent ruled surface and Mannheim Tangent ruled surface curve α have perpendicular normal vector fields. Normal ruled surface and Mannheim Tangent ruled surface of Mannheim curve α have perpendicular normal vector fields.*

Proof. It is trivial since $\langle \eta_1, \eta_1^* \rangle = 0$ and since $\langle \eta_2, \eta_1^* \rangle = 0$. ☐

Corollary 2.2. *Tangent ruled surface and Mannheim normal ruled surface of Mannheim curve α have not perpendicular normal vector fields.*

Proof. Since $\langle \eta_1, \eta_2^* \rangle = a^*\sin\theta$ and $v_2 k_2^* \sin\theta \neq 0$ it is trivial. ☐

Corollary 2.3. *Tangent ruled surface and Mannheim binormal ruled surface of Mannheim curve α have not perpendicular normal vector fields, along the curve $\varphi_3^*(s) = \alpha(s) + \lambda\left(\frac{k_2}{k_1\tan\theta} - 1\right)V_2$.*

Proof. Since $\langle \eta_1, \eta_3^* \rangle = c^*\sin\theta - d^*\cos\theta$ and under the condition $c^*\sin\theta - d^*\cos\theta = 0$

$$\begin{aligned} \frac{-v_3 k_2^* \sin\theta}{\sqrt{(v_3 k_2^*)^2 + 1}} + \frac{\cos\theta}{\sqrt{(v_3 k_2^*)^2 + 1}} &= 0 \\ -v_3 k_2^* \sin\theta + \cos\theta &= 0 \end{aligned}$$

and

$$v_3 = \frac{\lambda k_2}{k_1 \tan\theta}$$

it is trivial. □

Corollary 2.4. *Normal ruled surface and Mannheim normal ruled surface of Mannheim curve α α have perpendicular normal vector fields along the curve* $\varphi_2(s) = \alpha(s) + \frac{\tan\theta}{k_1\tan\theta - k_2} V_2(s)$, $\tan\theta \neq \frac{k_2}{k_1}$.

Proof. Since $\langle \eta_2, \eta_2^* \rangle = a^*(a\cos\theta - b\sin\theta)$ and under the orthogonality condition $-v_2 k_2^*(a\cos\theta - b\sin\theta) = 0$, and $v_2 k_2^* \neq 0$. Hence

$$\begin{aligned} a\cos\theta &= b\sin\theta \\ \tan\theta &= \frac{-u_2 k_2}{(1 - u_2 k_1)} \end{aligned}$$

or

$$u_2 = \frac{\tan\theta}{k_1\tan\theta - k_2},$$

this completes the proof. □

Corollary 2.5. *Normal ruled surface and Mannheim binormal ruled surface of Mannheim curve α α have perpendicular normal vector fields along the curve* $\varphi_3^*(s) = \alpha(s) + \left(\frac{k_2(-u_2k_2\tan\theta - u_2k_1 + 1)}{(k_1^2+k_2^2)(u_2k_1\tan\theta - \tan\theta - u_2k_2)} + \frac{k_1}{(k_1^2+k_2^2)} \right) V_2$ *where* $\tan\theta \neq \frac{u_2k_2}{(u_2k_1 - 1)}$.

Proof. Since $\langle \eta_2, \eta_3^* \rangle = c^*(a\cos\theta - b\sin\theta) + d^*(a\cos\theta + b\sin\theta)$ and under the orthogonality condition

$$\frac{-v_3k_2^*}{\sqrt{(v_3k_2^*)^2 + 1}}(a\cos\theta - b\sin\theta) + \frac{-1}{\sqrt{(v_3k_2^*)^2+1}}(a\sin\theta + b\cos\theta) = 0$$

$$\begin{aligned} -v_3k_2^*(a\cos\theta - b\sin\theta) &= (a\sin\theta + b\cos\theta) \\ v_3 &= \frac{k_2(-u_2k_2\tan\theta - u_2k_1 + 1)}{(k_1^2+k_2^2)(u_2k_1\tan\theta - \tan\theta - u_2k_2)} \\ \tan\theta &\neq \frac{u_2k_2}{(u_2k_1 - 1)} \end{aligned}$$

we have the proof. □

Corollary 2.6. *Binormal ruled surface and Mannheim tangent ruled surface of Mannheim curve α have not perpendicular normal vector fields.*

Proof. Since $\langle \eta_3, \eta_1^* \rangle = -d$ and $\frac{-1}{\sqrt{(u_3k_2)^2+1}} \neq 0$ it is trivial. □

Corollary 2.7. *Binormal ruled surface and Mannheim normal ruled surface of Mannheim curve α have perpendicular normal vector fields along* $\varphi_2^*(s) = \alpha(s) + \frac{\cos\theta\sin\theta}{-u_3(k_1^2+k_2^2)\cos^2\theta + \dot\theta} V_1 + \frac{k_1}{(k_1^2+k_2^2)} V_2 + \frac{\cos^2\theta}{-u_3(k_1^2+k_2^2)\cos^2\theta + \dot\theta} V_3$, *except* $u_3 = \frac{(k_1^2+k_2^2)\cos^2\theta}{\dot\theta}$.

Proof. Since$\langle \eta_3, \eta_2^* \rangle = a^* c \cos\theta + db^*$ and under the orthogonality condition $\langle \eta_3, \eta_2^* \rangle = 0$ we have

$$\begin{aligned} -v_2 k_2^* c \cos\theta + d\left(1 - v_2 k_1^*\right) &= 0 \\ -v_2 k_2^* c \cos\theta - d v_2 k_1^* &= -d \\ v_2 &= \frac{\cos\theta}{-u_3\left(k_1^2 + k_2^2\right)\cos^2\theta + \dot{\theta}} \end{aligned}$$

where $k_1^* = -\frac{d\theta}{ds^*} = \frac{\dot{\theta}}{\cos\theta}$ and $k_2^* = \frac{k_1}{\lambda k_2}$. □

Corollary 2.8. *Binormal ruled surface and Mannheim binormal ruled surface Mannheim curve α, have perpendicular normal vector fields along the curve $\varphi_3^*(s) = \alpha(s) + \frac{k_2 \tan\theta + k_1}{k_1^2 + k_2^2} V_2$*

Proof. Since $\langle \eta_3, \eta_3^* \rangle = c^* c \cos\theta + d^* c \sin\theta$ and $\langle \eta_3, \eta_3^* \rangle = 0$,we have

$$\begin{aligned} \frac{-v_3 k_2^*}{\sqrt{\left(v_3 k_2^*\right)^2 + 1}} c \cos\theta &= \frac{1}{\sqrt{\left(v_3 k_2^*\right)^2 + 1}} c \sin\theta \\ -v_3 k_2^* c \cos\theta &= c \sin\theta \\ v_3 &= \frac{k_2 \tan\theta}{k_1^2 + k_2^2} \end{aligned}$$

hence we have the proof. □

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COMPARATIVE GROWTH ESTIMATES OF DIFFERENTIAL MONOMIALS DEPENDING UPON THEIR RELATIVE ORDERS, RELATIVE TYPE AND RELATIVE WEAK TYPE

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Abstract. In this paper the comparative growth properties of composition of entire and meromorphic functions on the basis of their relative orders (relative lower orders), relative types and relative weak types of differential monomials generated by entire and meromorphic functions have been investigated.

1. Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function relating to entire f is defined as $M_f(r) = \max\{|f(z)| : |z| = r\}$. If f is non-constant then it has the following property:

Property (A) ([2]) : A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large values of r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [2].

When f is meromorphic, $M_f(r)$ can not be defined as f is not analytic. In this situation one may define another function $T_f(r)$ known as Nevanlinna's Characteristic function of f, playing the same role as $M_f(r)$ in the following manner:

$$T_f(r) = N_f(r) + m_f(r) \ .$$

Given two meromorphic functions f and g the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their Nevanlinna's Characteristic functions.

When f is entire function, the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r) \ .$$

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We called the function $N_f(r,a)\left(\bar{N}_f(r,a)\right)$ as counting function of a-points (distinct a-points) of f. In many occasions $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively. We put

$$N_f(r,a)=\int\limits_0^r \frac{n_f(t,a)-n_f(0,a)}{t}dt+\bar{n}_f(0,a)\log r\ ,$$

where we denote by $n_f(r,a)\left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z|\le r$ and an ∞ -point is a pole of f . Also we denote by $n_{f|=1}(r,a)$,the number of simple zeros of $f-a$ in $|z|\le r$. Accordingly, $N_{f|=1}(r,a)$ is defined in terms of $n_{f|=1}(r,a)$ in the usual way and we set

$$\delta_1(a;f)=1-\limsup_{r\to\infty}\frac{N(r,a;f\mid=1)}{T_f(r)}\quad\{\text{cf. [17]}\}\,,$$

the deficiency of 'a' corresponding to the simple a- points of f i,e. simple zeros of $f-a$. In this connection Yang [16] proved that there exists at most a denumerable number of complex numbers $a\in\mathbb{C}\cup\{\infty\}$ for which

$$\delta_1(a;f)>0 \text{ and } \sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)\le 4.$$

On the other hand, $m\left(r,\frac{1}{f-a}\right)$ is denoted by $m_f(r,a)$ and we mean $m_f(r,\infty)$ by $m_f(r)$, which is called the proximity function of f. We also put

$$m_f(r)=\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta,\quad\text{where}$$

$$\log^+ x=\max(\log x,0)\ \text{ for all } x\geqslant 0\ .$$

Further we denote $\Theta(\infty;f)$ as

$$\Theta(\infty;f)=1-\limsup_{r\to\infty}\frac{\overline{N}_f(r)}{T_f(r)}\ .$$

However, a meromorphic function $b=b(z)$ is called small with respect to f if $T_b(r)=S_f(r)$ where $S_f(r)=o\{T_f(r)\}$ i.e., $\frac{S_f(r)}{T_f(r)}\to 0$ as $r\to\infty$. Moreover for any transcendental meromorphic function f , we call $P[f]=bf^{n_0}(f^{(1)})^{n_1}...(f^{(k)})^{n_k}$, to be a differential monomial generated by it where $\sum\limits_{i=0}^{k}n_i\ge 1$ (all $n_i\mid i=0,1,...,k$ are non-negative integers) and the meromorphic function b is small with respect to f. In this connection the numbers $\gamma_{P[f]}=\sum\limits_{i=0}^{k}n_i$ and $\Gamma_{P[f]}=\sum\limits_{i=0}^{k}(i+1)n_i$ are called the degree and weight of $P[f]$ respectively $\{$cf. [5]$\}$.

If f is a non-constant entire function then $T_f(r)$ is rigorously increasing and continuous function of r and its inverse $T_f^{-1}:(T_f(0),\infty)\to(0,\infty)$ exist where $\lim\limits_{s\to\infty}T_f^{-1}(s)=\infty$. Also the ratio $\frac{T_f(r)}{T_g(r)}$ as $r\to\infty$ is known as growth of f with respect to g in terms of the Nevanlinna's Characteristic functions of the meromorphic functions f and g. Further in case of meromorphic functions, the growth markers such as order and lower order which are traditional in complex analysis are defined in terms of their growth with respect to the $\exp z$ function in the following way:

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}\right),$$

and the growth of functions is said to be regular if their lower order coincides with their order.

In this connection the following two definitions are also well known:

Definition 1.1. The *type* σ_f and *lower type* $\overline{\sigma}_f$ of a meromorphic function f are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty\ .$$

If f is entire then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}},\ 0 < \rho_f < \infty\ .$$

Definition 1.2. [7] The *weak type* τ_f and the growth indicator τ_f of a meromorphic function f of finite positive lower order λ_f are defined by

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}},\ 0 < \lambda_f < \infty\ .$$

When f is entire then

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}},\ 0 < \lambda_f < \infty\ .$$

However, extending the thought of relative order of entire functions as initiated by Bernal {[1], [2]}, Lahiri and Banerjee [13] introduced the definition of relative order of a meromorphic function f with respect to another entire function g, symbolized by $\rho_g(f)$ to avoid comparing growth just with $\exp z$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one if $g(z) = \exp z$ {cf. [13] }.

Similarly, one can define the relative lower order of a meromorphic function f with respect to an entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

To compare the relative growth of two entire functions having same non zero finite *relative order* with respect to another entire function, Roy [14] introduced the notion of *relative type* of two entire functions in the following way:

Definition 1.3. [14] Let f and g be any two entire functions such that $0 < \rho_g(f) < \infty$. Then the *relative type* $\sigma_g(f)$ of f with respect to g is defined as :

$$\begin{aligned} & \sigma_g(f) \\ = \ & \inf\left\{k > 0 : M_f(r) < M_g\left(kr^{\rho_g(f)}\right) \text{ for all sufficiently large values of } r\right\} \\ = \ & \limsup_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} \ . \end{aligned}$$

Likewise, one can define the *relative lower type* of an entire function f with respect to an entire function g denoted by $\overline{\sigma}_g(f)$ as follows :

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}}, \ 0 < \rho_g(f) < \infty \ .$$

Analogously, to determine the relative growth of two entire functions having same non zero finite *relative lower order* with respect to another entire function, Datta and Biswas [8] introduced the definition of *relative weak type* of an entire function f with respect to another entire function g of finite positive *relative lower order* $\lambda_g(f)$ in the following way:

Definition 1.4. [8] The *relative weak type* $\tau_g(f)$ of an entire function f with respect to another entire function g having finite positive *relative lower order* $\lambda_g(f)$ is defined as:

$$\tau_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\lambda_g(f)}} \ .$$

Also one may define the growth indicator $\overline{\tau}_g(f)$ of an entire function f with respect to an entire function g in the following way :

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\lambda_g(f)}}, \ 0 < \lambda_g(f) < \infty \ .$$

In the case of meromorphic functions, it therefore seems reasonable to define suitably the *relative type* and *relative weak type* of a meromorphic function with respect to an entire function to determine the relative growth of two meromorphic functions having same non zero finite *relative order* or *relative lower order* with respect to an entire function. Datta and Biswas also [8] gave such definitions of *relative type* and *relative weak type* of a meromorphic function f with respect to an entire function g which are as follows:

Definition 1.5. [8] The *relative type* $\sigma_g(f)$ of a meromorphic function f with respect to an entire function g are defined as

$$\sigma_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \quad \text{where } 0 < \rho_g(f) < \infty.$$

Similarly, one can define the *lower relative type* $\overline{\sigma}_g(f)$ in the following way:

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \quad \text{where } 0 < \rho_g(f) < \infty.$$

Definition 1.6. [8] The *relative weak type* $\tau_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ is defined by

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}}.$$

In a like manner, one can define the growth indicator $\overline{\tau}_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ as

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}}.$$

Considering $g = \exp z$ one may easily verify that Definition 1.3 , Definition 1.4, Definition 1.5 and Definition 1.6 coincide with the classical definitions of type (lower type) and weak type of entire are meromorphic functions respectively.

For entire and meromorphic functions, the notion of their growth indicators such as *order*, *type* and *weak type* are classical in complex analysis and during the past decades, several researchers have already been continuing their studies in the area of comparative growth properties of composite entire and meromorphic functions in different directions using the same. But at that time, the concept of *relative order* and consequently *relative type* as well as *relative weak type* of entire and meromorphic functions with respect to another entire function was mostly unknown to complex analysists and they are not aware of the technical advantages of using the relative growth indicators of the functions. In this paper we wish to prove some newly developed results based on the growth properties of *relative order, relative type* and *relative weak type* of differential monomials generated by entire and meromorphic functions. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [11] and [15].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [3] *Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f\circ g}(r) \leqslant \{1+o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)) \ .$$

Lemma 2.2. [4] *Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) \geq T_f(\exp(r^{\mu})) \ .$$

Lemma 2.3. [12] *Let f be meromorphic and g be entire such that $0 < \rho_g < \infty$ and $0 < \lambda_f$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) > T_g(\exp(r^{\mu})),$$

where $0 < \mu < \rho_g$.

Lemma 2.4. [6] *Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) < T_f(\exp(r^{\mu})) \ .$$

Lemma 2.5. [6] *Let f be a meromorphic function of finite order and g be an entire function such that $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) < T_g(\exp(r^{\mu})) \ .$$

Lemma 2.6. [9] *Let f be an entire function which satisfy the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f\left(\alpha r^{\delta}\right).$$

Lemma 2.7. [10] *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f) = 4$. Also let g be a transcendental entire function of regular growth having non zero finite order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g) = 4$. Then the relative order and relative lower order of $P[f]$ with respect to $P[g]$ are same as those of f with respect to g.*

Lemma 2.8. [10] *If f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f) = 4$ and g be a transcendental entire function of regular growth having non zero finite type and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g) = 4$. Then the relative type and relative lower type of $P[f]$ with respect to $P[g]$ are $\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\rho_g(f)$ is positive finite.*

Lemma 2.9. [10] *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f) = 4$ and g be a transcendental entire function of regular growth having non zero finite type and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g) = 4$. Then $\tau_{P[g]}(P[f])$ and $\overline{\tau}_{P[g]}(P[f])$ are $\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\lambda_g(f)$ is positive finite i.e.,*

$$\tau_{P[g]}(P[f]) = \left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}.\tau_g(f) \text{ and}$$

$$\overline{\tau}_{P[g]}(P[f]) = \left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}.\bar{\tau}_g(f)\ .$$

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f) = 4$, g be entire and h a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h) = 4$, $0 < \rho_h(f) < \infty$, $\rho_h(f) = \rho_g$, $\sigma_g < \infty$ and $0 < \sigma_h(f) < \infty$. Also let h satisfy the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)} \leq \left(\frac{\delta\cdot\rho_h(f)\cdot\sigma_g}{\sigma_h(f)}\right)\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}\right)^{\frac{1}{\rho_h}}.$$

Proof. From (3.9), we get for all sufficiently large values of r that

$$\log T_h^{-1} T_{f \circ g}(r) \leqslant \delta(\rho_h(f) + \varepsilon) \log M_g(r) + O(1) \ . \tag{3.1}$$

Using Definition 1.1, we obtain from (3.1) for all sufficiently large values of r that

$$\log T_h^{-1} T_{f \circ g}(r) \leqslant \delta(\rho_h(f) + \varepsilon)(\sigma_g + \varepsilon) \cdot r^{\rho_g} + O(1) \ . \tag{3.2}$$

Now in view of condition (ii), we obtain from (3.2) for all sufficiently large values of r that

$$\log T_h^{-1} T_{f \circ g}(r) \leqslant \delta(\rho_h(f) + \varepsilon)(\sigma_g + \varepsilon) \cdot r^{\rho_h(f)} + O(1) \ . \tag{3.3}$$

Again in view of Definition 1.5, we get for a sequence of values of r tending to infinity that

$$T_{M[h]}^{-1} T_{M[f]}(r) \geq \left(\sigma_{M[h]}(M[f]) - \varepsilon\right) r^{\rho_{M[h]}(M[f])} \ .$$

Therefore in view of Lemma 2.7 and Lemma 2.8, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} & T_{M[h]}^{-1} T_{M[f]}(r) \\ \geq \ & \left(\sigma_h(f) \left(\frac{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]})\Theta(\infty; f)}{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty; h)}\right)^{\frac{1}{\rho_h}} - \varepsilon\right) r^{\rho_h(f)} \ . \end{aligned} \tag{3.4}$$

Therefore from (3.3) and (3.4), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r)} \leq \frac{\delta(\rho_h(f) + \varepsilon)(\sigma_g + \varepsilon) \cdot r^{\rho_h(f)} + O(1)}{\left(\sigma_h(f) \left(\frac{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]})\Theta(\infty; f)}{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty; h)}\right)^{\frac{1}{\rho_h}} - \varepsilon\right) r^{\rho_h(f)}} \ .$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r)} \leq \left(\frac{\delta \cdot \rho_h(f) \cdot \sigma_g}{\sigma_h(f)}\right) \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty; h)}{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]})\Theta(\infty; f)}\right)^{\frac{1}{\rho_h}} \ .$$

Hence the theorem follows. □

Using the notion of *lower type* and *relative lower type*, we may state the following theorem without its proof as it can be carried out in the line of Theorem 3.1 :

Theorem 3.2. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$, g be entire and h a transcendental entire function of regular growth having non zero finite order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$, $0 < \rho_h(f) < \infty$, $\rho_h(f) = \rho_g$, $\overline{\sigma}_g < \infty$ and $0 < \overline{\sigma}_h(f) < \infty$. Also let h satisfies the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r)} \leq \frac{\delta \cdot \rho_h(f) \cdot \overline{\sigma}_g}{\overline{\sigma}_h(f)} \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty; h)}{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]})\Theta(\infty; f)}\right)^{\frac{1}{\rho_h}} \ .$$

Similarly using the notion of *type* and *relative lower type,* one may state the following two theorems without their proofs because those can also be carried out in the line of Theorem 3.1 :

Theorem 3.3. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, *$g$ be entire and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\lambda_h(f)\le\rho_h(f)<\infty$, $\rho_h(f)=\rho_g$, $\sigma_g<\infty$ *and* $0<\overline{\sigma}_h(f)<\infty$. *Also let h satisfies the Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\le\frac{\delta\cdot\lambda_h(f)\cdot\sigma_g}{\overline{\sigma}_h(f)}\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.4. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, *$g$ be entire and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\rho_h(f)<\infty$, $\rho_h(f)=\rho_g$, $\sigma_g<\infty$ *and* $0<\overline{\sigma}_h(f)<\infty$. *Also let h satisfies the Property (A). Then for any $\delta>1$,*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\le\frac{\delta\cdot\rho_h(f)\cdot\sigma_g}{\overline{\sigma}_h(f)}\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}\right)^{\frac{1}{\rho_h}}.$$

Similarly, using the concept of *weak type* and *relative weak type*, we may state next four theorems without their proofs as those can be carried out with the help of Lemma 2.9 and in the line of Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4 respectively.

Theorem 3.5. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, *$g$ be entire and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\lambda_h(f)\le\rho_h(f)<\infty$, $\lambda_h(f)=\lambda_g$, $\overline{\tau}_g<\infty$ *and* $0<\overline{\tau}_h(f)<\infty$. *Also let h satisfies the Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\le\frac{\delta\cdot\rho_h(f)\cdot\overline{\tau}_g}{\overline{\tau}_h(f)}\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.6. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, *$g$ be entire and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\lambda_h(f)\le\rho_h(f)<\infty$, $\lambda_h(f)=\lambda_g$, $\tau_g<\infty$ *and* $0<\tau_h(f)<\infty$. *Also let h satisfies the Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\le\frac{\delta\cdot\rho_h(f)\cdot\tau_g}{\tau_h(f)}\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.7. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, *$g$ be entire and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=$

4, $0<\lambda_h(f)<\infty$, $\lambda_h(f)=\lambda_g$, $\overline{\tau}_g<\infty$ and $0<\tau_h(f)<\infty$. Also let h satisfies the Property (A). Then for any $\delta>1$,

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\leq\frac{\delta\cdot\lambda_h(f)\cdot\overline{\tau}_g}{\tau_h(f)}\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.8. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, g be entire and h a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\lambda_h(f)\leq\rho_h(f)<\infty$, $\lambda_h(f)=\lambda_g$, $\overline{\tau}_g<\infty$ and $0<\tau_h(f)<\infty$. Also let h satisfies the Property (A). Then for any $\delta>1$,*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\leq\frac{\delta\cdot\rho_h(f)\cdot\overline{\tau}_g}{\tau_h(f)}\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.9. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, g be entire and h a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\lambda_h(f)\leq\rho_h(f)<\rho_g\leq\infty$ and $\sigma_h(f)<\infty$. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\geq\frac{\lambda_h(f)}{\sigma_h(f)}\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}\right)^{\frac{1}{\rho_h}}.$$

Proof. Since $\rho_h(f)<\rho_g$ and $T_h^{-1}(r)$ is a increasing function of r, we get from Lemma 2.2 for a sequence of values of r tending to infinity that

$$\begin{aligned}\log T_h^{-1}T_{f\circ g}(r)&\geq\log T_h^{-1}T_f(\exp(r^{\mu}))\\ i.e.,\ \log T_h^{-1}T_{f\circ g}(r&\geq(\lambda_h(f)-\varepsilon)\cdot r^{\mu}\\ i.e.,\ \log T_h^{-1}T_{f\circ g}(r)&\geq(\lambda_h(f)-\varepsilon)\cdot r^{\rho_h(f)}\,.\end{aligned}\tag{3.5}$$

Again in view of Definition 1.5, we get for all sufficiently large values of r that

$$T_{M[h]}^{-1}T_{M[f]}(r)\leq\left(\sigma_{M[h]}(M[f])+\varepsilon\right)r^{\rho_{M[h]}(M[f])}\,.$$

Therefore in view of Lemma 2.7 and Lemma 2.8, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned}&T_{M[h]}^{-1}T_{M[f]}(r)\\ \leq&\left(\sigma_h(f)\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}+\varepsilon\right)r^{\rho_h(f)}\,.\end{aligned}\tag{3.6}$$

Now from (3.5) and (3.6), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\geq\frac{(\lambda_h(f)-\varepsilon)\,r^{\rho_h(f)}}{\left(\sigma_h(f)\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}+\varepsilon\right)r^{\rho_h(f)}}\,.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}\,(r)}\geq\frac{\lambda_h\,(f)}{\sigma_h\,(f)}\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}\right)^{\frac{1}{\rho_h}}.$$

Thus the theorem follows. □

In the line of Theorem 3.9, the following theorem can be proved and therefore its proof is omitted:

Theorem 3.10. *Let f be a meromorphic function, g a transcendental entire function of finite order or of non-zero lower order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ and h a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\lambda_h\,(f)$, $0<\rho_h\,(g)<\rho_g\leq\infty$ and $\sigma_h\,(g)<\infty$. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}\,(r)}\geq\frac{\lambda_h\,(f)}{\sigma_h\,(g)}\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_h}}.$$

The following two theorems can also be proved in the line of Theorem 3.9 and Theorem 3.10 respectively and with help of Lemma 2.3. Hence their proofs are omitted.

Theorem 3.11. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, g be entire and h a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\lambda_h\,(g)$, $0<\lambda_f$, $0<\rho_h\,(f)<\rho_g<\infty$ and $\sigma_h\,(f)<\infty$. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}\,(r)}\geq\frac{\lambda_h\,(g)}{\sigma_h\,(f)}\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.12. *Let f be a meromorphic function, g a transcendental entire function of finite order or of non-zero lower order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ and h a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\lambda_h\,(g)$, $0<\lambda_f$, $0<\rho_h\,(g)<\rho_g<\infty$ and $\sigma_h\,(g)<\infty$. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}\,(r)}\geq\frac{\lambda_h\,(g)}{\sigma_h\,(g)}\left(\frac{\Gamma_{P[h]}-(\Gamma_{P[h]}-\gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_h}}.$$

Now we state the following four theorems without their proofs as those can be carried out with the help of Lemma 2.9 and in the line of Theorem 3.9, Theorem 3.10, Theorem 3.11 and Theorem 3.12 and with the help of Definition 1.6:

Theorem 3.13. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, g be entire and h a* *transcendental entire function of regular growth having non zero finite order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;h) = 4$, $0 < \lambda_h(f) < \rho_g \leq \infty$ *and* $\overline{\tau}_h(f) < \infty$. *Then*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r)} \geq \frac{\lambda_h(f)}{\overline{\tau}_h(f)} \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]})\Theta(\infty;f)} \right)^{\frac{1}{\rho_h}} .$$

Theorem 3.14. *Let f be a meromorphic function, g a transcendental entire function of finite order or of non-zero lower order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;g) = 4$ *and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;h) = 4$, $0 < \lambda_h(f)$, $0 < \lambda_h(g) < \rho_g \leq \infty$ *and* $\overline{\tau}_h(g) < \infty$. *Then*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[g]}(r)} \geq \frac{\lambda_h(f)}{\overline{\tau}_h(g)} \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[g]} - (\Gamma_{P[g]} - \gamma_{P[g]})\Theta(\infty;g)} \right)^{\frac{1}{\rho_h}} .$$

Theorem 3.15. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;f) = 4$, *$g$ be entire and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;h) = 4$, $0 < \lambda_h(g) < \rho_g < \infty$, $0 < \lambda_f$ *and* $\overline{\tau}_h(f) < \infty$. *Then*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r)} \geq \frac{\lambda_h(g)}{\overline{\tau}_h(f)} \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]})\Theta(\infty;f)} \right)^{\frac{1}{\rho_h}} .$$

Theorem 3.16. *Let f be a meromorphic function, g a transcendental entire function of finite order or of non-zero lower order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;g) = 4$ *and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;h) = 4$, $0 < \lambda_h(g) < \rho_g < \infty$, $0 < \lambda_f$ *and* $\overline{\tau}_h(g) < \infty$. *Then*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[g]}(r)} \geq \frac{\lambda_h(g)}{\overline{\tau}_h(g)} \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[g]} - (\Gamma_{P[g]} - \gamma_{P[g]})\Theta(\infty;g)} \right)^{\frac{1}{\rho_h}} .$$

Theorem 3.17. *Let f be a transcendental meromorphic function of non zero finite order and lower order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;f) = 4$, *$g$ be entire and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;h) = 4$, $0 < \lambda_g < \rho_h(f) < \infty$ *and* $\overline{\sigma}_h(f) > 0$. *Then*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r)} \leq \frac{\rho_h(f)}{\overline{\sigma}_h(f)} \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty;h)}{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]})\Theta(\infty;f)} \right)^{\frac{1}{\rho_h}} .$$

Proof. As $\lambda_g < \rho_h(f)$ and $T_h^{-1}(r)$ is a increasing function of r, it follows from Lemma 2.4 for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f \circ g}(r) &< \log T_h^{-1} T_f(\exp(r^{\mu})) \\ i.e.,\ \log T_h^{-1} T_{f \circ g}(r &< (\rho_h(f) + \varepsilon) \cdot r^{\mu} \\ i.e.,\ \log T_h^{-1} T_{f \circ g}(r) &< (\rho_h(f) + \varepsilon) \cdot r^{\rho_h(f)} . \end{aligned} \tag{3.7}$$

Further in view of Definition 1.5, we obtain for all sufficiently large values of r that

$$T_{M[h]}^{-1}T_{M[f]}(r) \geq \left(\overline{\sigma}_{M[h]}(M[f]) - \varepsilon\right) r^{\rho_{M[h]}(M[f])} .$$

Therefore in view of Lemma 2.7 and Lemma 2.8, we get from above that

(3.8)
$$T_{M[h]}^{-1}T_{M[f]}(r) \geq \left(\overline{\sigma}_h(f)\left(\frac{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]})\Theta(\infty; f)}{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty; h)}\right)^{\frac{1}{\rho_h}} - \varepsilon\right) r^{\rho_h(f)} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, therefore from (3.7) and (3.8) we have for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)} \leq \frac{(\rho_h(f) + \varepsilon)\cdot r^{\rho_h(f)}}{\left(\overline{\sigma}_h(f)\left(\frac{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]})\Theta(\infty; f)}{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty; h)}\right)^{\frac{1}{\rho_h}} - \varepsilon\right) r^{\rho_h(f)}}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)} \leq \frac{\rho_h(f)}{\overline{\sigma}_h(f)}\left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty; h)}{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]})\Theta(\infty; f)}\right)^{\frac{1}{\rho_h}} .$$

Hence the theorem is established. □

In the line of Theorem 3.17, the following theorem can be proved and therefore its proof is omitted:

Theorem 3.18. *Let f be a meromorphic function with non zero finite order and lower order, g a transcendental entire function of finite order or of non-zero lower order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a; g) = 4$ and h a transcendental entire function of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a; h) = 4$, $\rho_h(f) < \infty$, $0 < \lambda_g < \rho_h(g) < \infty$ and $\overline{\sigma}_h(g) > 0$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}(r)} \leq \frac{\rho_h(f)}{\overline{\sigma}_h(g)}\left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty; h)}{\Gamma_{P[g]} - (\Gamma_{P[g]} - \gamma_{P[g]})\Theta(\infty; g)}\right)^{\frac{1}{\rho_h}} .$$

Moreover, the following two theorems can also be deduced in the line of Theorem 3.9 and Theorem 3.10 respectively and with help of Lemma 2.5 and therefore their proofs are omitted.

Theorem 3.19. *Let f be a transcendental meromorphic function of finite order or of non zero lower order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a; f) = 4$, g be entire and h a transcendental entire function of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a; h) = 4$, $\rho_h(g) < \infty$, $0 < \lambda_g < \rho_h(f) < \infty$ and $\overline{\sigma}_h(f) > 0$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)} \leq \frac{\rho_h(g)}{\overline{\sigma}_h(f)}\left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]})\Theta(\infty; h)}{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]})\Theta(\infty; f)}\right)^{\frac{1}{\rho_h}} .$$

Theorem 3.20. *Let f be a meromorphic function with finite order, g a transcendental entire function of finite order or of non-zero lower order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a; g) =$*

4 and h a transcendental entire function of regular growth having non zero finite order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;h) = 4$, $0 < \lambda_g < \rho_h(g) < \infty$ *and* $\overline{\sigma}_h(g) > 0$. *Then*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[g]}(r)} \leq \frac{\rho_h(g)}{\overline{\sigma}_h(g)} \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]}) \Theta(\infty; h)}{\Gamma_{P[g]} - (\Gamma_{P[g]} - \gamma_{P[g]}) \Theta(\infty; g)} \right)^{\frac{1}{\rho_h}} .$$

Finally we state the following four theorems without their proofs as those can be carried out in view of Lemma 2.9 and in the line of Theorem 3.17, Theorem 3.18, Theorem 3.19 and Theorem 3.20 using the concept of *relative weak type:*

Theorem 3.21. *Let f be a transcendental meromorphic function of non zero finite order and lower order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;f) = 4$, *g be entire and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;h) = 4$ $0 < \lambda_g < \lambda_h(f) \leq \rho_h(f) < \infty$ *and* $\tau_h(f) > 0$. *Then*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r)} \leq \frac{\rho_h(f)}{\tau_h(f)} \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]}) \Theta(\infty; h)}{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]}) \Theta(\infty; f)} \right)^{\frac{1}{\rho_h}} .$$

Theorem 3.22. *Let f be a meromorphic function with non zero finite order and lower order, g a transcendental entire function of finite order or of non-zero lower order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;g) = 4$ *and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;h) = 4$, $\rho_h(f) < \infty$, $0 < \lambda_g < \lambda_h(g) < \infty$ *and* $\tau_h(g) > 0$. *Then*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[g]}(r)} \leq \frac{\rho_h(f)}{\tau_h(g)} \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]}) \Theta(\infty; h)}{\Gamma_{P[g]} - (\Gamma_{P[g]} - \gamma_{P[g]}) \Theta(\infty; g)} \right)^{\frac{1}{\rho_h}} .$$

Theorem 3.23. *Let f be a transcendental meromorphic function of finite order or of non zero lower order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;f) = 4$, *g be entire and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;h) = 4$, $\rho_h(g) < \infty$, $0 < \lambda_g < \lambda_h(f) < \infty$ *and* $\tau_h(f) > 0$. *Then*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r)} \leq \frac{\rho_h(g)}{\tau_h(f)} \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]}) \Theta(\infty; h)}{\Gamma_{P[f]} - (\Gamma_{P[f]} - \gamma_{P[f]}) \Theta(\infty; f)} \right)^{\frac{1}{\rho_h}} .$$

Theorem 3.24. *Let f be a meromorphic function with finite order, g a transcendental entire function of finite order or of non-zero lower order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;g) = 4$ *and h a transcendental entire function of regular growth having non zero finite order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;h) = 4$, $0 < \lambda_g < \lambda_h(f) \leq \rho_h(g) < \infty$ *and* $\tau_h(g) > 0$. *Then*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[g]}(r)} \leq \frac{\rho_h(g)}{\tau_h(g)} \left(\frac{\Gamma_{P[h]} - (\Gamma_{P[h]} - \gamma_{P[h]}) \Theta(\infty; h)}{\Gamma_{P[g]} - (\Gamma_{P[g]} - \gamma_{P[g]}) \Theta(\infty; g)} \right)^{\frac{1}{\rho_h}} .$$

Theorem 3.25. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, g be entire and h a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\lambda_h(f)\leq\rho_h(f)<\infty$ and $\sigma_g<\infty$. Also h satisfy the Property (A). Then for any $\delta>1$,*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[f]}(\exp r^{\rho_g})}\leq\frac{\delta\cdot\sigma_g\cdot\rho_h(f)}{\lambda_h(f)}\ .$$

Proof. Let us suppose that $\alpha>2$.
Since $T_h^{-1}(r)$ is an increasing function r, it follows from Lemma 2.1, Lemma 2.6 and the inequality $T_g(r)\leq\log M_g(r)$ $\{cf.\ [11]\}$ for all sufficiently large values of r that

$$\begin{aligned}
T_h^{-1}T_{f\circ g}(r) &\leqslant T_h^{-1}\left[\{1+o(1)\}T_f(M_g(r))\right]\\
i.e.,\ T_h^{-1}T_{f\circ g}(r) &\leqslant \alpha\left[T_h^{-1}T_f(M_g(r))\right]^{\delta}\\
i.e.,\ \log T_h^{-1}T_{f\circ g}(r) &\leqslant \delta\log T_h^{-1}T_f(M_g(r))+O(1) \qquad (3.9)
\end{aligned}$$

$$\begin{aligned}
&i.e.,\ \frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[f]}(\exp r^{\rho_g})}\\
&\leq\frac{\delta\log T_h^{-1}T_f(M_g(r))+O(1)}{\log T_{M[h]}^{-1}T_{M[f]}(\exp r^{\rho_g})}=\frac{\delta\log T_h^{-1}T_f(M_g(r))+O(1)}{\log M_g(r)}\cdot\\
&\qquad\frac{\log M_g(r)}{r^{\rho_g}}\cdot\frac{\log\exp r^{\rho_g}}{\log T_{M[h]}^{-1}T_{M[f]}(\exp r^{\rho_g})} \qquad (3.10)
\end{aligned}$$

$$\begin{aligned}
&i.e.,\ \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[f]}(\exp r^{\rho_g})}\\
&\leq\limsup_{r\to\infty}\frac{\delta\log T_h^{-1}T_f(M_g(r))+O(1)}{\log M_g(r)}\cdot\limsup_{r\to\infty}\frac{\log M_g(r)}{r^{\rho_g}}\cdot\\
&\qquad\limsup_{r\to\infty}\frac{\log\exp r^{\rho_g}}{\log T_{M[h]}^{-1}T_{M[f]}(\exp r^{\rho_g})}
\end{aligned}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[f]}(\exp r^{\rho_g})}\leq\delta\cdot\rho_h(f)\cdot\sigma_g\cdot\frac{1}{\lambda_{M[h]}(M[f])}\ .$$

Therefore in view of Lemma 2.7, we obtain from above that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[f]}(\exp r^{\rho_g})}\leq\frac{\delta\cdot\sigma_g\cdot\rho_h(f)}{\lambda_h(f)}\ .$$

Thus the theorem is established. □

In the line of Theorem 3.25 the following theorem can be proved :

Theorem 3.26. *Let f be a meromorphic function, g a transcendental entire function of finite order or of non-zero lower order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ and h a transcendental entire function of regular growth having non zero finite order with*

$\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $\lambda_h(g)>0$, $\rho_h(f)<\infty$, $\sigma_g<\infty$ *and also* h *satisfy the Property (A). Then for any* $\delta>1$,

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[g]}(\exp r^{\rho_g})}\leq\frac{\delta\cdot\sigma_g\cdot\rho_h(f)}{\lambda_h(g)}\ .$$

Using the notion of l*ower type*, we may state the following two theorems without their proofs because those can be carried out in the line of Theorem 3.25 and Theorem 3.26 respectively.

Theorem 3.27. *Let* f *be a transcendental meromorphic function of finite order or of non-zero lower order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, g *be entire and* h *a transcendental entire function of regular growth having non zero finite order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\lambda_h(f)\leq\rho_h(f)<\infty$, $\overline{\sigma}_g<\infty$ *and also* h *satisfy the Property (A). Then for any* $\delta>1$,

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[f]}(\exp r^{\rho_g})}\leq\frac{\delta\cdot\overline{\sigma}_g\cdot\rho_h(f)}{\lambda_h(f)}\ .$$

Theorem 3.28. *Let* f *be a meromorphic function,* g *a transcendental entire function of finite order or of non-zero lower order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ *and* h *a transcendental entire function of regular growth having non zero finite order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $\lambda_h(g)>0$, $\rho_h(f)<\infty$, $\overline{\sigma}_g<\infty$ *and also* h *satisfy the Property (A). Then for any* $\delta>1$,

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[g]}(\exp r^{\rho_g})}\leq\frac{\delta\cdot\overline{\sigma}_g\cdot\rho_h(f)}{\lambda_h(g)}\ .$$

Using the concept of the growth indicators τ_g and $\overline{\tau}_g$ of an entire function g, we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 3.25, Theorem 3.26, Theorem 3.27 and Theorem 3.28 respectively.

Theorem 3.29. *Let* f *be a transcendental meromorphic function of finite order or of non-zero lower order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, g *be entire and* h *a transcendental entire function of regular growth having non zero finite order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $0<\lambda_h(f)\leq\rho_h(f)<\infty$, $\overline{\tau}_g<\infty$ *and also* h *satisfy the Property (A). Then for any* $\delta>1$,

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[f]}(\exp r^{\lambda_g})}\leq\frac{\delta\cdot\overline{\tau}_g\cdot\rho_h(f)}{\lambda_h(f)}\ .$$

Theorem 3.30. *Let* f *be a meromorphic function,* g *a transcendental entire function of finite order or of non-zero lower order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ *and* h *a transcendental entire function of regular growth having non zero finite order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$, $\lambda_h(g)>0$, $\rho_h(f)<\infty$, $\overline{\tau}_g<\infty$ *and also* h *satisfy the*

Property (A). Then for any $\delta > 1$,

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[g]}(\exp r^{\lambda_g})} \le \frac{\delta \cdot \overline{\tau}_g \cdot \rho_h(f)}{\lambda_h(g)} .$$

Theorem 3.31. *Let* f *be a transcendental meromorphic function of finite order or of non-zero lower order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$, g *be entire and* h *a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$, $0 < \lambda_h(f) \le \rho_h(f) < \infty$, $\tau_g < \infty$ *and also* h *satisfy the Property (A). Then for any* $\delta > 1$,

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}(\exp r^{\lambda_g})} \le \frac{\delta \cdot \tau_g \cdot \rho_h(f)}{\lambda_h(f)} .$$

Theorem 3.32. *Let* f *be a meromorphic function,* g *a transcendental entire function of finite order or of non-zero lower order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$ *and* h *a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$, $\lambda_h(g) > 0$, $\rho_h(f) < \infty$, $\tau_g < \infty$ *and also* h *satisfy the Property (A). Then for any* $\delta > 1$,

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[g]}(\exp r^{\lambda_g})} \le \frac{\delta \cdot \tau_g \cdot \rho_h(f)}{\lambda_h(g)} .$$

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TRIVARIATE FIBONACCI AND LUCAS POLYNOMIALS

E. GOKCEN KOCER AND HATICE GEDIKCE

Abstract. In this article, we study the Trivariate Fibonacci and Lucas polynomials. The classical Tribonacci numbers and Tribonacci polynomials are the special cases of the trivariate Fibonacci polynomials. Also, we obtain some properties of the trivariate Fibonacci and Lucas polynomials. Using these properties, we give some results for the Tribonacci numbers and Tribonacci polynomials.

1. Introduction

In [4], the Tribonacci sequence originally was studied in 1963 by M. Feinberg. For any integer $n > 2$, the Tribonacci numbers T_n were defined by the recurrence relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}; \quad T_0 = 0,\ T_1 = 1,\ T_2 = 1.$$

In [2], the author derived the different recurrence relations on the Tribonacci numbers and their sums and got some identities of the Tribonacci numbers and their sums by using the companion matrices and generating matrices. In [5], the authors defined the generalized Tribonacci numbers and derived an explicit formula for the generalized Tribonacci numbers with negative subscripts. In [6], Lin obtained the Binet's formula and De Moivre types identities for the Tribonacci Numbers. In [7], the author got a formula for Tribonacci numbers by using an analytic method. In [8], the author obtained some identities for the Tribonacci numbers. Also, Pethe defined the complex Tribonacci numbers at Gaussian integers. In [10], Spickerman got the Binet's formula and generating function for the Tribonacci sequence and obtained an application for the Tribonacci numbers.

In [1], the authors got the Tribonacci Numbers from Tribonacci triangles and discussed the properties of functions related to Tribonacci Numbers. Also, Alladi

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and Hoggatt defined the Tribonacci triangle as follows

$n\diagdown i$	0	1	2	3	4	5	$\ldots$
0	1						
1	1	1					
2	1	3	1				
3	1	5	5	1			
4	1	7	13	7	1		
5	1	9	25	25	9	1	
$\vdots$							

Table1: Tribonacci Triangle

It is interesting to note that, the sum of the elements on the rising diagonal lines in the Tribonacci triangle is $1, 1, 2, 4, 7, 13, 24, \ldots$ which are the Tribonacci numbers.

In 1973, the Tribonacci polynomials was defined by Hoggatt and Bicknell [3]. For any integer $n > 2$, the recurrence relation of the Tribonacci polynomials is as follows

$$t_n(x) = x^2 t_{n-1}(x) + x t_{n-2}(x) + t_{n-3}(x)$$

where $t_0(x) = 0,\ t_1(x) = 1,\ t_2(x) = x^2$.

Some of Tribonacci polynomials are $0,\ 1,\ x^2,\ x^4 + x,\ x^6 + 2x^3 + 1,\ x^8 + 3x^5 + 3x^2,\ x^{10} + 4x^7 + 6x^4 + 2x,\ \ldots$. It's clear that $t_n(1) = T_n$, where T_n is $n-th$ Tribonacci number.

In [3], the authors gave the generating matrices for the Tribonacci, quadranacci and $r-$ bonacci polynomials. Also, they obtained the interesting determinantal properties for these polynomials. In [11], the authors defined the bivariate and trivariate Fibonacci polynomials and obtained the some properties of these polynomials.

There are different studies associated with the Tribonacci numbers and polynomials. One of them is incomplete Tribonacci numbers and polynomials in [9]. Ramirez and Sirvent defined the Tribonacci polynomial triangle as follows

$n\diagdown i$	0	1	2	3	4	5	$\ldots$
0	1						
1	x^2	x					
2	x^4	$2x^3 + 1$	x^2				
3	x^6	$3x^5 + 2x^2$	$3x^4 + 2x$	x^3			
4	x^8	$4x^7 + 3x^4$	$6x^6 + 6x^3 + 1$	$4x^5 + 3x^2$	x^4		
5	x^{10}	$5x^9 + 4x^6$	$10x^8 + 12x^5 + 3x^2$	$10x^7 + 12x^4 + 3x$	$5x^6 + 4x^3$	x^5	
$\vdots$							

Table 2: Tribonacci Polynomial Triangle

In this study, based on the definition of Tan and Zhang [11], we make a new genaralization of the Tribonacci polynomials.

2. Trivariate Fibonacci and Lucas Polynomials

Definition 2.1. Let $n > 2$ be integer. The recurrence relation of the trivariate Fibonacci and Lucas polynomials are as follows

$$H_n(x, y, z) = x H_{n-1}(x, y, z) + y H_{n-2}(x, y, z) + z H_{n-3}(x, y, z) \tag{2.1}$$

with the initial conditions

$$H_0(x,y,z)=0,\quad H_1(x,y,z)=1,\quad H_2(x,y,z)=x$$

and

$$K_n(x,y,z)=xK_{n-1}(x,y,z)+yK_{n-2}(x,y,z)+zK_{n-3}(x,y,z) \tag{2.2}$$

with the initial conditions

$$K_0(x,y,z)=3,\quad K_1(x,y,z)=x,\quad K_2(x,y,z)=x^2+2y$$

respectively.

It is not difficult to see that $H_n(1,1,1)=T_n$, where T_n is $n-th$ Tribonacci number and $H_n\left(x^2,x,1\right)=t_n(x)$, where $t_n(x)$ is $n-th$ Tribonacci polynomial, are special cases of the trivariate Fibonacci polynomials.

The characteristic equation of the recurrences in (2.1) and (2.2) is as

$$\lambda^3-x\lambda^2-y\lambda-z=0. \tag{2.3}$$

The Binet's formula for the trivariate Fibonacci and Lucas polynomials are as follows

$$H_n(x,y,z)=\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}$$

and

$$K_n(x,y,z)=\alpha^n+\beta^n+\gamma^n$$

where α, β and γ are roots of the characteristic equation (2.3), respectively.

Now, we show that some of trivariate Fibonacci and Lucas polynomials in Table 3.

n	$H_n(x,y,z)$	$K_n(x,y,z)$
0	0	3
1	1	x
2	x	x^2+2y
3	x^2+y	$x^3+3xy+3z$
4	$x^3+2xy+z$	$x^4+4x^2y+4xz+2y^2$
5	$x^4+3x^2y+2xz+y^2$	$x^5+5x^3y+5xy^2+5x^2z+5yz$
6	$x^5+4x^3y+3xy^2+3x^2z+2yz$	$x^6+6x^4y+9x^2y^2+6x^3z+12xyz+2y^3+3z^2$
$\vdots$	$\vdots$	$\vdots$

Table 3:Trivariate Fibonacci and Lucas Polynomials

The generating functions of the trivariate Fibonacci and Lucas poynomials are as follows

$$h(t)=\sum_{n=0}^{\infty}H_n(x,y,z)\,t^n=\frac{t}{1-xt-yt^2-zt^3} \tag{2.4}$$

and

$$k(t)=\sum_{n=0}^{\infty}K_n(x,y,z)\,t^n=\frac{3-2xt-yt^2}{1-xt-yt^2-zt^3}. \tag{2.5}$$

Taking $x=y=z=1$ in (2.4), we obtain the generating function of the Tribonacci numbers. Writing x^2 instead of x, x instead of y and taking $z=1$ in (2.4), we have the generating function of the Tribonacci polynomials.

Theorem 2.1. *Let $H_n(x,y,z)$ and $K_n(x,y,z)$ be $n-th$ trivariate Fibonacci and Lucas polynomials, respectively. Then, we get*

$$K_n(x,y,z) = xH_n(x,y,z) + 2yH_{n-1}(x,y,z) + 3zH_{n-2}(x,y,z). \tag{2.6}$$

Proof. Using the generating function of the trivariate Lucas polynomials, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} K_n(x,y,z)\,t^n &= \frac{3-2xt-yt^2}{1-xt-yt^2-zt^3} \\
&= 3\frac{1}{1-xt-yt^2-zt^3} - 2x\frac{t}{1-xt-yt^2-zt^3} - y\frac{t^2}{1-xt-yt^2-zt^3} \\
&= 3\sum_{n=0}^{\infty} H_{n+1}(x,y,z)\,t^n - 2x\sum_{n=0}^{\infty} H_n(x,y,z)\,t^n - y\sum_{n=0}^{\infty} H_{n-1}(x,y,z)\,t^n \\
&= \sum_{n=0}^{\infty} \left(3H_{n+1}(x,y,z) - 2xH_n(x,y,z) - yH_{n-1}(x,y,z)\right) t^n.
\end{aligned}$$

From the recurrence relation in (2.1), we can write

$$\sum_{n=0}^{\infty} K_n(x,y,z)\,t^n = \sum_{n=0}^{\infty} \left(xH_n(x,y,z) + 2yH_{n-1}(x,y,z) + 3zH_{n-2}(x,y,z)\right) t^n.$$

Comparing of the coefficients of t^n, we have the desired result. □

Theorem 2.2. *The sum of the trivariate Fibonacci and Lucas polynomials are as follows*

$$\sum_{s=0}^{n} H_s(x,y,z) = \frac{H_{n+2}(x,y,z) + (1-x)H_{n+1}(x,y,z) + zH_n(x,y,z) - 1}{x+y+z-1} \tag{2.7}$$

and

$$\sum_{s=0}^{n} K_s(x,y,z) = \frac{K_{n+2}(x,y,z) + (x-1)K_{n+1}(x,y,z) + zK_n(x,y,z) - (3-2x-y)}{x+y+z-1} \tag{2.8}$$

for $x+y+z \neq 1$, respectively.

Proof. Using the Binet's formulas, it can be proved. □

Taking $x=y=z=1$ in (2.7), we have the sum of the Tribonacci numbers as

$$\sum_{s=0}^{n} T_s = \frac{T_{n+2} + T_n - 1}{2}.$$

Similarly, we obtain the sum of the Tribonacci polynomials as

$$\sum_{s=0}^{n} t_s(x) = \frac{t_{n+2}(x) + \left(1-x^2\right)t_{n+1}(x) + t_n(x) - 1}{x^2+x}.$$

Similar to Table 1 and Table 2, we can give the trivariate Fibonacci polynomial triangle as follows

$n \diagdown i$	0	1	2	3	4	$\ldots$
0	1					
1	x	y				
2	x^2	$2xy+z$	y^2			
3	x^3	$3x^2y+2xz$	$3xy^2+2yz$	y^3		
4	x^4	$4x^3y+3x^2z$	$6x^2y^2+6xyz+z^2$	$4xy^3+3y^2z$	y^4	
$\vdots$						

Table 4: Trivariate Fibonacci Polynomial Triangle

$G(n,i,x,y,z)$ is the element in the $n-th$ row and $i-th$ column of the trivariate Fibonacci polynomial triangle. Then, we get

$$G(n,i,x,y,z) = \sum_{j=0}^{i} \binom{i}{j} \binom{n-j}{i} x^{n-i-j} y^{i-j} z^{j} \tag{2.9}$$

and

$$G(n+1,i,x,y,z) = xG(n,i,x,y,z) + yG(n,i-1,x,y,z) + zG(n-1,i-1,x,y,z)$$

where

$$G(n,0,x,y,z) = x^n, \quad G(n,n,x,y,z) = y^n.$$

The sum of elements on the rising diagonal lines in the trivariate Fibonacci polynomial triangle is the trivariate Fibonacci polynomial $H_n(x,y,z)$. Thus, we have

$$H_n(x,y,z) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} G(n-i-1,i,x,y,z).$$

Consequently, we obtain an explicit formula for the trivariate Fibonacci polynomial $H_n(x,y,z)$ as

$$H_n(x,y,z) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n-i-j-1}{i} x^{n-2i-j-1} y^{i-j} z^{j}. \tag{2.10}$$

Taking $x=y=z=1$ in (2.10), we obtain the explicit formula for the Tribonacci numbers as

$$H_n(1,1,1) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n-i-j-1}{i}$$

Also, we have

$$H_n\left(x^2,x,1\right) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n-i-j-1}{i} x^{2n-3(i+j)-2}$$

which is the explicit formula for the Tribonacci polynomials in [9].

Similarly, we have an explicit formula for the trivariate Lucas polynomials as follows

$$K_n(x,y,z) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{i} \frac{n}{n-i-j} \binom{i}{j} \binom{n-i-j}{i} x^{n-2i-j} y^{i-j} z^{j}. \tag{2.11}$$

Theorem 2.3. *Let $H_n(x,y,z)$ and $K_n(x,y,z)$ be $n-th$ trivariate Fibonacci and Lucas polynomials, respectively. Then, we get*

$$x\frac{\partial K_n(x,y,z)}{\partial x}+y\frac{\partial K_n(x,y,z)}{\partial y}+z\frac{\partial K_n(x,y,z)}{\partial z}=nH_{n+1}(x,y,z).$$

Proof. Using partial derivations of the explicit formula of the trivariate Lucas polynomial $K_n(x,y,z)$, we have

$$\begin{aligned}
\frac{\partial K_n(x,y,z)}{\partial x} &= \frac{\partial}{\partial x}\left(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{i}\frac{n}{n-i-j}\binom{i}{j}\binom{n-i-j}{i}x^{n-2i-j}y^{i-j}z^{j}\right)\\
&= \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{i}\frac{n}{n-i-j}(n-2i-j)\binom{i}{j}\binom{n-i-j}{i}x^{n-2i-j-1}y^{i-j}z^{j}\\
&= n\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{n-i-j-1}{i}x^{n-2i-j-1}y^{i-j}z^{j}\\
&= nH_n(x,y,z).
\end{aligned}$$

Similarly, we obtain

$$\frac{\partial K_n(x,y,z)}{\partial y}=nH_{n-1}(x,y,z)$$

and

$$\frac{\partial K_n(x,y,z)}{\partial z}=nH_{n-2}(x,y,z).$$

Using the recurrence relation (2.1), we have

$$x\frac{\partial K_n(x,y,z)}{\partial x}+y\frac{\partial K_n(x,y,z)}{\partial y}+z\frac{\partial K_n(x,y,z)}{\partial z}=nH_{n+1}(x,y,z).$$

□

The generating matrix of the Tribonacci polynomials was introduced in [3, 4]. Similarly, the trivariate Fibonacci polynomials are generated by the matrix Q,where

$$Q=\begin{pmatrix} x & 1 & 0\\ y & 0 & 1\\ z & 0 & 0\end{pmatrix}$$

with the help of mathematical induction on n, we get

$$Q^n=\begin{pmatrix} H_{n+1} & H_n & H_{n-1}\\ yH_n+zH_{n-1} & yH_{n-1}+zH_{n-2} & yH_{n-2}+zH_{n-3}\\ zH_n & zH_{n-1} & zH_{n-2}\end{pmatrix},$$

where H_n is $n-th$ trivariate Fibonacci polynomial, namely $H_n(x,y,z)=H_n$.

Theorem 2.4. *Let m and n be positive integers. Then, we get*

$$\begin{aligned}
H_{m+n}(x,y,z) &= H_{m+1}(x,y,z)H_n(x,y,z)+H_m(x,y,z)H_{n+1}(x,y,z)\\
&\quad +zH_{m-1}(x,y,z)H_{n-1}(x,y,z)\\
&\quad -xH_m(x,y,z)H_n(x,y,z). \qquad (2.12)
\end{aligned}$$

Proof. It can be proved by using the identity $Q^{n+m}=Q^nQ^m$ and matrix equality. □

The identity in (2.12) is similar to Honsberger formula for the Fibonacci like sequences. From the special cases of (2.12), we obtain some identities for the trivariate Fibonacci polynomials. Therefore, taking $m = n$ in (2.12), we have

$$H_{2n}(x,y,z) = zH_{n-1}^2(x,y,z) - xH_n^2(x,y,z) + 2H_{n+1}(x,y,z)\,H_n(x,y,z)$$

Writing $n+1$ instead of m in (2.12), and using the recurennce relation in (2.1), we obtain

$$H_{2n+1}(x,y,z) = H_{n+1}^2(x,y,z) + yH_n^2(x,y,z) + 2zH_n(x,y,z)\,H_{n-1}(x,y,z)\,.$$

Theorem 2.5. *Let $H_n(x,y,z)$ be $n-th$ trivariate Fibonacci polynomial. Then, we get*

$$\left| \begin{array}{ccc} H_{n+2}(x,y,z) & H_{n+1}(x,y,z) & H_n(x,y,z) \\ H_{n+1}(x,y,z) & H_n(x,y,z) & H_{n-1}(x,y,z) \\ H_n(x,y,z) & H_{n-1}(x,y,z) & H_{n-2}(x,y,z) \end{array} \right| = -z^{n-1}. \tag{2.13}$$

Proof. It's note that $\det(Q) = z$, $\det(Q^n) = z^n$. Using the determinants of the matrices Q and Q^n , we obtain

$$\left| \begin{array}{ccc} H_{n+1} & H_n & H_{n-1} \\ yH_n + zH_{n-1} & yH_{n-1} + zH_{n-2} & yH_{n-2} + zH_{n-3} \\ zH_n & zH_{n-1} & zH_{n-2} \end{array} \right| = z^n.$$

Multiplying the first row of Q^n by x and then adding to second row, then, exchanging rows 1 and 2, we have

$$\left| \begin{array}{ccc} H_{n+2}(x,y,z) & H_{n+1}(x,y,z) & H_n(x,y,z) \\ H_{n+1}(x,y,z) & H_n(x,y,z) & H_{n-1}(x,y,z) \\ zH_n(x,y,z) & zH_{n-1}(x,y,z) & zH_{n-2}(x,y,z) \end{array} \right| = -z^n$$

From the properties of determinant, we obtain

$$\left| \begin{array}{ccc} H_{n+2}(x,y,z) & H_{n+1}(x,y,z) & H_n(x,y,z) \\ H_{n+1}(x,y,z) & H_n(x,y,z) & H_{n-1}(x,y,z) \\ H_n(x,y,z) & H_{n-1}(x,y,z) & H_{n-2}(x,y,z) \end{array} \right| = -z^{n-1}.$$

□

In this way, we obtain the interesting determinantal property for the trivariate Fibonacci polynomials. The result of the determinant in (2.13) is similar to the Cassini like identity for the trivariate Fibonacci polynomials. Taking $x = y = z = 1$ in (2.13), we obtain the determinantal property for the Tribonacci numbers. Writing x^2 instead of x, x instead of y and taking $z = 1$, we have the determinantal property for the Tribonacci polynomials in [3].

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ON SHERMAN'S TYPE INEQUALITIES FOR n-CONVEX FUNCTION WITH APPLICATIONS

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Abstract. New generalizations of Sherman's inequality for convex functions of higher order are obtained by using Hermite's interpolating polynomials and Green's function. The Ostrowski and Grüss type bounds for the identity related to generalized Sherman's inequality are established. Some applications are discussed.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $\mathbf{x} = (x_1, ..., x_m)$, $\mathbf{y} = (y_1, ..., y_m) \in I^m$, where $m \geq 2$. Let $x_{[i]}$ and $y_{[i]}$ denote the elements of $\mathbf{x}$ and $\mathbf{y}$ sorted in decreasing order. We say that $\mathbf{x}$ *majorizes* $\mathbf{y}$ or $\mathbf{y}$ is *majorized* by $\mathbf{x}$ and write $\mathbf{y} \prec \mathbf{x}$ if

$$\sum_{i=1}^{k} y_{[i]} \leq \sum_{i=1}^{k} x_{[i]}, \quad k = 1, ..., m-1, \tag{1.1}$$

and the equation holds for $k = m$.

In majorization theory, the next result, well known as *Majorization theorem,* plays a very important role (see [15]).

Theorem 1.1. *Let $\phi : I \to \mathbb{R}$ be a convex continuous function on an interval I and $\mathbf{x} = (x_1, ..., x_m)$, $\mathbf{y} = (y_1, ..., y_m) \in I^m$. If $\mathbf{y} \prec \mathbf{x}$, then*

$$\sum_{i=1}^{m} \phi(y_i) \leq \sum_{i=1}^{m} \phi(x_i).$$

Recently some generalizations of majorization theorem with applications are obtained (see [1]-[5], [12]).

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S. Sherman [16], considering a weighted relation of majorization

$$\sum_{i=1}^{k} v_i y_i \leq \sum_{j=1}^{l} u_j x_j,$$

for nonnegative weights u_j and v_i, proved the general result which include the row stochastic $k \times l$ matrix, i.e. matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{kl}(\mathbb{R})$ such that

$$\begin{aligned} a_{ij} &\geq 0 \quad \text{for all } i = 1, ..., k,\ j = 1, ..., l, \\ \sum_{j=1}^{l} a_{ij} &= 1 \quad \text{for all } i = 1, ..., k, \end{aligned}$$

and holds under relations

$$\begin{aligned} y_i &= \sum_{j=1}^{l} x_j a_{ij}, \quad \text{for} \quad i = 1, ..., k, \\ u_j &= \sum_{i=1}^{k} v_i a_{ij}, \quad \text{for} \quad j = 1, ..., l. \end{aligned} \tag{1.2}$$

His result can be formulated as the following theorem.

Theorem 1.2. *Let $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^k$, $\mathbf{u} \in [0, \infty)^l$ and $\mathbf{v} \in [0, \infty)^k$ be such that (1.2) holds for some row stochastic matrix $\mathbf{A} \in \mathcal{M}_{kl}(\mathbb{R})$. Then for every convex function $\phi : [\alpha, \beta] \to \mathbb{R}$ we have*

$$\sum_{q=1}^{k} v_q \phi(y_q) \leq \sum_{p=1}^{l} u_p \phi(x_p). \tag{1.3}$$

From Sherman's theorem we can easily get Majorization theorem by setting $k = l$ and $\mathbf{v} = (1, ..., 1)$. Specially, when $k = l$ and all weights $v_i = u_j$ are equal, the condition (1.2), i.e. $\mathbf{u} = \mathbf{vA}$, assures the stochasticity on columns, so in that case we deal with *doubly stochastic matrices.* It is well known that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^l$ is valid

$$\mathbf{y} \prec \mathbf{x} \quad \text{if and only if} \quad \mathbf{y} = \mathbf{xA}$$

for some doubly stochastic matrix $\mathbf{A} \in \mathcal{M}_{ll}(\mathbb{R})$.

The aim of this paper is to establish generalizations of Sherman's result which hold for real, not necessary nonnegative vectors $\mathbf{u}$, $\mathbf{v}$ and matrix $\mathbf{A}$ and for convex functions of higher order. Recently some related results are obtained (see [6], [10]).

The class of convex functions of higher order, i.e. the notion of n-convexity was defined in terms of divided differences by T. Popoviciu. A function $\phi : [\alpha, \beta] \to \mathbb{R}$ is n-convex, $n \geq 0$, if its nth order divided differences $[x_0, ..., x_n; \phi]$ are nonnegative for all choices of $(n + 1)$ distinct points $x_i \in [\alpha, \beta]$, $i = 0, ..., n$. Thus, a 0-convex function is nonnegative, 1-convex function is nondecreasing and 2-convex function is convex in the usual sense. If $\phi^{(n)}$ exists, then ϕ is n-convex iff $\phi^{(n)} \geq 0$ (see [15]).

At the end we point definition and some basic facts about exponential convexity. For more details see [6], [11]. Here I denotes an open interval in $\mathbb{R}$.

Definition 1.1. [14] For a fixed $n \in \mathbb{N}$, a function $\phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} p_i p_j \phi\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices $p_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, ..., n$. A function $\phi : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.1. Let $\phi : I \to \mathbb{R}$ be a given function.

- ϕ is exponentially convex in the Jensen sense on I, if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.
- A positive function ϕ is log-convex, i.e. $\log \phi$ is convex, in the Jensen sense on I iff it is 2-exponentially convex in the Jensen sense on I.
- A positive function ϕ is log-convex on I if it is continuous and log-convex in the Jensen sense on I
- A positive exponentially convex function ϕ on I is also log-convex on I.

2. Preliminaries

We use notations and terminology from [7].

Let $-\infty < \alpha < \beta < \infty$ and let $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$ be r $(r \geq 2)$ distinct points. For $\phi \in C^n([\alpha, \beta])$ $(n \geq r)$ a unique polynomial $\rho_H(s)$ of degree $(n-1)$ exists, such that **Hermite conditions** hold

$$\rho_H^{(i)}(a_j) = \phi^{(i)}(a_j);\ 0 \leq i \leq k_j,\ 1 \leq j \leq r, \tag{H}$$

where $\sum\limits_{j=1}^{r} k_j + r = n$.

Specially, for $r = 2, 1 \leq m \leq n-1$, $k_1 = m-1$ and $k_2 = n-m-1$ we have **type** $(m, n-m)$ **conditions:**

$$\rho_{(m,n)}^{(i)}(\alpha) = \phi^{(i)}(\alpha),\ 0 \leq i \leq m-1,$$

$$\rho_{(m,n)}^{(i)}(\beta) = \phi^{(i)}(\beta),\ 0 \leq i \leq n-m-1.$$

For $n = 2m$, $r = 2$ and $k_1 = k_2 = m-1$ we have **two-point Taylor conditions:**

$$\rho_{2T}^{(i)}(\alpha) = \phi^{(i)}(\alpha),\ \rho_{2T}^{(i)}(\beta) = \phi^{(i)}(\beta),\ 0 \leq i \leq m-1.$$

Theorem 2.1. *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$ be r $(r \geq 2)$ distinct points and $\phi \in C^n([\alpha, \beta])$. Then*

$$\phi(t) = \rho_H(t) + R_{H,n}(\phi, t), \tag{2.1}$$

where $\rho_H(t)$ is the Hermite inrepolating polynomial, i.e.

$$\rho_H(t) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) \phi^{(i)}(a_j),$$

H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!} \frac{\omega(t)}{(t-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k!} \frac{d^k}{dt^k} \left(\frac{(t-a_j)^{k_j+1}}{\omega(t)} \right) \Bigg|_{t=a_j} (t-a_j)^k, \tag{2.2}$$

with

$$\omega(t) = \prod_{j=1}^{r} (t-a_j)^{k_j+1}, \tag{2.3}$$

and the remainder is given by

$$R_{H,n}(\phi,t)=\int_{\alpha}^{\beta}G_{H,n}(t,s)\phi^{(n)}(s)ds,$$

where $G_{H,n}(t,s)$ *is defined by*

$$G_{H,n}(t,s)=\begin{cases}\sum\limits_{j=1}^{l}\sum\limits_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t);\ s\le t,\\ -\sum\limits_{j=l+1}^{r}\sum\limits_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t);\ s\ge t,\end{cases}\tag{2.4}$$

for all $a_l\le s\le a_{l+1}$*;* $l=0,\dots,r$ *with* $a_0=\alpha$ *and* $a_{r+1}=\beta$*.*

Remark 2.1. For type $(m,n-m)$ conditions, from Theorem 2.1 we have

$$\phi(t)=\rho_{(m,n)}(t)+R_{(m,n)}(\phi,t)$$

where $\rho_{(m,n)}(t)$ is $(m,n-m)$ interpolating polynomial, i.e.

$$\rho_{(m,n)}(t)=\sum_{i=0}^{m-1}\tau_i(t)\phi^{(i)}(\alpha)+\sum_{i=0}^{n-m-1}\eta_i(t)\phi^{(i)}(\beta),$$

with

$$\tau_i(t)=\frac{1}{i!}(t-\alpha)^i\Big(\frac{t-\beta}{\alpha-\beta}\Big)^{n-m}\sum_{p=0}^{m-1-i}\binom{n-m+p-1}{p}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^p,\tag{2.5}$$

$$\eta_i(t)=\frac{1}{i!}(t-\beta)^i\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^{m}\sum_{p=0}^{n-m-1-i}\binom{m+p-1}{p}\Big(\frac{t-\beta}{\alpha-\beta}\Big)^p,\tag{2.6}$$

and the remainder is given by

$$R_{(m,n)}(\phi,t)=\int_{\alpha}^{\beta}G_{(m,n)}(t,s)\phi^{(n)}(s)ds$$

with

$$G_{(m,n)}(t,s)=\begin{cases}\sum\limits_{j=0}^{m-1}\Big[\sum\limits_{p=0}^{m-1-j}\binom{n-m+p-1}{p}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^p\Big]\times\\ \frac{(t-\alpha)^j(\alpha-s)^{n-j-1}}{j!(n-j-1)!}\Big(\frac{\beta-t}{\beta-\alpha}\Big)^{n-m}, & \alpha\le s\le t\le\beta\\ -\sum\limits_{i=0}^{n-m-1}\Big[\sum\limits_{q=0}^{n-m-i-1}\binom{m+q-1}{q}\Big(\frac{\beta-t}{\beta-\alpha}\Big)^q\Big]\times\\ \frac{(t-\beta)^i(\beta-s)^{n-i-1}}{i!(n-i-1)!}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^{m}, & \alpha\le t\le s\le\beta.\end{cases}\tag{2.7}$$

For Type Two-point Taylor conditions, from Theorem 2.1 we have

$$\phi(t)=\rho_{2T}(t)+R_{2T}(\phi,t)$$

where $\rho_{2T}(t)$ is the two-point Taylor interpolating polynomial i.e,

$$\rho_{2T}(t)=\sum_{i=0}^{m-1}\sum_{p=0}^{m-1-i}\binom{m+p-1}{p}\Big[\frac{(t-\alpha)^i}{i!}\Big(\frac{t-\beta}{\alpha-\beta}\Big)^m\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^p\phi^{(i)}(\alpha)$$
$$+\frac{(t-\beta)^i}{i!}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^m\Big(\frac{t-\beta}{\alpha-\beta}\Big)^p\phi^{(i)}(\beta)\Big]$$

and the remainder is given by

$$R_{2T}(\phi,t)=\int_{\alpha}^{\beta}G_{2T}(t,s)\phi^{(n)}(s)ds$$

with

$$G_{2T}(t,s)=\begin{cases}\frac{(-1)^m}{(2m-1)!}p^m(t,s)\sum\limits_{j=0}^{m-1}\binom{m-1+j}{j}(t-s)^{m-1-j}q^j(t,s), & s\le t;\\ \frac{(-1)^m}{(2m-1)!}q^m(t,s)\sum\limits_{j=0}^{m-1}\binom{m-1+j}{j}(s-t)^{m-1-j}p^j(t,s), & s\ge t;\end{cases}\tag{2.8}$$

where $p(t,s)=\frac{(s-\alpha)(\beta-t)}{\beta-\alpha}$, $q(t,s)=p(s,t)$, $\forall\, t,s\in[\alpha,\beta]$.

The following lemma describes the positivity of $G_{H,n}(t,s)$ (see [8], [13]).

Lemma 2.1. *The function $G_{H,n}(t,s)$, defined by (2.4), has the following properties:*

(i) $\frac{G_{H,n}(t,s)}{\omega(t)}>0$, $a_1\le t\le a_r$, $a_1<s<a_r$;

(ii) $G_{H,n}(t,s)\le\frac{1}{(n-1)!(\beta-\alpha)}|\omega(t)|$;

(iii) $\int_{\alpha}^{\beta}G_{H,n}(t,s)ds=\frac{\omega(t)}{n!}$.

Green's function of Lagrange type is defined on $[\alpha,\beta]\times[\alpha,\beta]$ by

$$G(t,s)=\begin{cases}\frac{(t-\beta)(s-\alpha)}{\beta-\alpha}\;, & \alpha\le s\le t\\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t\le s\le\beta\end{cases}.\tag{2.9}$$

It is convex and continuous in both variables (see [17]).

3. Main results

The next identity related to generalized Sherman's inequality holds.

Theorem 3.1. *Let $n\ge 4$ and $\phi\in C^n([\alpha,\beta])$, $\alpha\le a_1<a_2\cdots<a_r\le\beta$ $(r\ge 2)$ be the given points and $k_1,...,k_r\in\mathbb{N}$ with $\sum\limits_{j=1}^{r}k_j+r=n$. Let $\mathbf{x}\in[\alpha,\beta]^l$, $\mathbf{y}\in[\alpha,\beta]^k$, $\mathbf{u}\in\mathbb{R}^l$ and $\mathbf{v}\in\mathbb{R}^k$ be such that (1.2) holds for some matrix $\mathbf{A}\in\mathcal{M}_{kl}(\mathbb{R})$ whose entries satisfy the condition $\sum\limits_{j=1}^{l}a_{ij}=1$, $i=1,...,k$. Then*

$$\begin{aligned}&\sum_{p=1}^{l}u_p\phi(x_p)-\sum_{q=1}^{k}v_q\phi(y_q)\\ &=\int_{\alpha}^{\beta}\left[\sum_{p=1}^{l}u_pG(x_p,t)-\sum_{q=1}^{k}v_qG(y_q,t)\right]\sum_{j=1}^{r}\sum_{i=0}^{k_j}H_{ij}(t)\phi^{(i+2)}(a_j)dt\\ &+\int_{\alpha}^{\beta}\int_{\alpha}^{\beta}\left[\sum_{p=1}^{l}u_pG(x_p,t)-\sum_{q=1}^{k}v_qG(y_q,t)\right]G_{H,n-2}(t,s)\phi^{(n)}(s)dsdt,\end{aligned}\tag{3.1}$$

where G, H_{ij} and $G_{H,n-2}$ are defined as in (2.9), (2.2) and (2.4), respectively.

Proof. For any function $\phi\in C^2([\alpha,\beta])$, we can show integration by parts that the following identity holds

$$\phi(x)=\frac{\beta-x}{\beta-\alpha}\phi(\alpha)+\frac{x-\alpha}{\beta-\alpha}\phi(\beta)+\int_{\alpha}^{\beta}G(x,t)\phi''(t)dt,\tag{3.2}$$

where G is defined by (2.9).

By an easy calculation, applying (3.2) in $\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{k} v_q\phi(y_q)$ and using (1.2), we get

$$\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{k} v_q\phi(y_q)$$
$$= \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t)\right] \phi''(t)dt. \tag{3.3}$$

By Theorem 2.1, the function $\phi''(t)$ can be expressed as

$$\phi''(t) = \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i+2)}(a_j) + \int_{\alpha}^{\beta} G_{H,n-2}(t,s)\phi^{(n)}(s)ds. \tag{3.4}$$

Now, combining (3.3) and (3.4), we get (3.1). □

Using the previous identity we get the following generalization of Sherman's theorem which hold for real, not necessary nonnegative vectors **u**, **v** and matrix **A**.

Theorem 3.2. *Let $n \geq 4$ and $\phi \in C^n([\alpha,\beta])$ be n-convex on $[\alpha,\beta]$, $\alpha = a_1 < a_2 \cdots < a_r = \beta$ $(r \geq 2)$ be the given points and $k_1, ..., k_r \in \mathbb{N}$ with $\sum_{j=1}^{r} k_j + r = n$. Let $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^k$, $\mathbf{u} \in \mathbb{R}^l$ and $\mathbf{v} \in \mathbb{R}^k$ be such that (1.2) holds for some matrix $\mathbf{A} \in \mathcal{M}_{kl}(\mathbb{R})$ whose entries satisfy the condition $\sum_{j=1}^{l} a_{ij} = 1$, $i = 1, ..., k$ and*

$$\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t) \geq 0, \quad t \in [\alpha,\beta]. \tag{3.5}$$

(i) *If k_j is odd for each $j = 2, .., r$, then*

$$\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{k} v_q\phi(y_q)$$
$$\geq \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t)\right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i+2)}(a_j)dt. \tag{3.6}$$

(ii) *If k_j is odd for each $j = 2, .., r-1$ and k_r is even, then the reverse inequality in (3.6) holds.*

Proof. (i) Since $\phi \in C^n([\alpha,\beta])$ is n-convex, then $\phi^{(n)} \geq 0$.
Clearly, $(t-a_1)^{k_1+1} \geq 0$ for any $t \in [\alpha,\beta]$ and if k_j is odd for each $j = 2, .., r$, then the function ω, defined by (2.3), satisfied $\omega(t) \geq 0$ for any $t \in [\alpha,\beta]$. Therefore, by Lemma 2.1 (i) it follows that $G_{H,n-2}(t,s) \geq 0$. Hence, we can apply Theorem 3.1 to obtain (3.6).
(ii) This part we can prove similarly. □

Under Sherman's assumptions of non-negativity of vectors **u**, **v** and matrix **A** the following generalizations hold.

Theorem 3.3. *Let $n \geq 4$ and $\phi \in C^n([\alpha, \beta])$ be n-convex on $[\alpha, \beta]$, $\alpha = a_1 < a_2 \cdots < a_r = \beta$ $(r \geq 2)$ be the given points and $k_1, ..., k_r \in \mathbb{N}$ with $\sum_{j=1}^{r} k_j + r = n$. Let $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^k$, $\mathbf{u} \in [0, \infty)^l$ and $\mathbf{v} \in [0, \infty)^k$ be such that (1.2) holds for some row stochastic matrix $\mathbf{A} \in \mathcal{M}_{kl}(\mathbb{R})$.*

(i) *If k_j is odd for each $j = 2, .., r$, then (3.6) holds.*

(ii) *If k_j is odd for each $j = 2, .., r-1$ and k_r is even, then the reverse inequality in (3.6) holds.*

(iii) *If (3.6) holds and the function*

$$\bar{F}(\cdot) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \int_{\alpha}^{\beta} G(\cdot, t) H_{ij}(t) \phi^{(i+2)}(a_j) dt \tag{3.7}$$

is convex on $[\alpha, \beta]$, then (1.3) holds.

Proof. (i) Since the function $G(., t)$, $t \in [\alpha, \beta]$, is convex, then by Sherman's theorem we have

$$\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \geq 0, \quad t \in [\alpha, \beta].$$

Applying Theorem 3.2 and Lemma 2.1 (i) we get (3.6).
(ii) Similarly we can prove this part.
(iii) If (3.6) holds, the right hand side of (3.6) can be rewriting in the form

$$\sum_{p=1}^{l} u_p \bar{F}(x_p) - \sum_{q=1}^{k} v_q \bar{F}(y_q),$$

where $\bar{F}$ is defined by (3.7). If $\bar{F}$ is convex, then by Sherman's theorem we have

$$\sum_{p=1}^{l} u_p \bar{F}(x_p) - \sum_{q=1}^{k} v_q \bar{F}(y_q) \geq 0,$$

i.e. the right hand side of (3.6) is nonnegative, so (1.3) immediately follows. □

As a direct consequence of the previous result, considering particular case of Hermite interpolating polynomial with type $(m, n-m)$ conditions, we get the following corollary.

Corollary 3.1. *Let $n \geq 4$, $1 \leq m \leq n-1$ and $\phi \in C^n([\alpha, \beta])$ be n-convex. Let $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^k$, $\mathbf{u} \in [0, \infty)^l$ and $\mathbf{v} \in [0, \infty)^k$ be such that (1.2) holds for some row stochastic matrix $\mathbf{A} \in \mathcal{M}_{kl}(\mathbb{R})$.*

(i) *If $n - m$ is even, then*

$$\begin{aligned} &\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{k} v_q \phi(y_q) \\ &\geq \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right] \left(\sum_{i=0}^{m-1} \tau_i(t) \phi^{(i+2)}(\alpha) + \sum_{i=0}^{n-m-1} \eta_i(t) \phi^{(i+2)}(\beta) \right) dt, \end{aligned} \tag{3.8}$$

where G, τ_i and η_i are defined as in (2.9), (2.5) and (2.6), respectively.

(ii) *If $n-m$ is odd, then the reverse inequality in (3.8) holds.*
(iii) *If (3.8) holds and the function*

$$\tilde{F}(\cdot) = \int_\alpha^\beta G(\cdot,t)\left(\sum_{i=0}^{m-1}\tau_i(t)\phi^{(i)}(\alpha) + \sum_{i=0}^{n-m-1}\eta_i(t)\phi^{(i)}(\beta)\right)dt \tag{3.9}$$

is convex on $[\alpha,\beta]$, then (1.3) holds.

Considering particular case of Hermite interpolating polynomial with two-point Taylor conditions we get the next generalizations.

Corollary 3.2. *Let $m \geq 2$ and $\phi \in C^{2m}([\alpha,\beta])$ be $2m$-convex. Let $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^k$, $\mathbf{u} \in [0,\infty)^l$ and $\mathbf{v} \in [0,\infty)^k$ be such that (1.2) holds for some row stochastic matrix $\mathbf{A} \in \mathcal{M}_{kl}(\mathbb{R})$.*

(i) *If m is even, then*

$$\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{k} v_q\phi(y_q) \geq \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t)\right] F(t)dt, \tag{3.10}$$

where

$$F(t) = \sum_{i=0}^{m-1}\sum_{p=0}^{m-1-i}\binom{m+p-1}{p}\left[\frac{(t-\alpha)^i}{i!}\left(\frac{t-\beta}{\alpha-\beta}\right)^m\left(\frac{t-\alpha}{\beta-\alpha}\right)^p\phi^{(i+2)}(\alpha)\right.$$
$$\left.+\frac{(t-\beta)^i}{i!}\left(\frac{t-\alpha}{\beta-\alpha}\right)^m\left(\frac{t-\beta}{\alpha-\beta}\right)^p\phi^{(i+2)}(\beta)\right].$$

(ii) *If m is odd, then the reverse inequality in (3.10) holds.*
(iii) *If (3.10) holds and the function*

$$\hat{F}(\cdot) = \int_\alpha^\beta G(\cdot,t)F(t)dt$$

is convex on $[\alpha,\beta]$, then (1.3) holds.

4. Grüss and Ostrowski type inequalities related to generalized Sherman's inequality

P. Cerone and S. S. Dragomir [9], considering the Čebyšev functional

$$T(f,g) := \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)g(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)dt$$

for Lebesgue integrable functions $f,g : [\alpha,\beta] \to \mathbb{R}$, proved the following two results which contain the Grüss and Ostrowski type inequalities.

Theorem 4.1. *Let $f : [\alpha,\beta] \to \mathbb{R}$ be Lebesgue integrable and $g : [\alpha,\beta] \to \mathbb{R}$ be absolutely continuous with $(\cdot-\alpha)(\beta-\cdot)(g')^2 \in L[\alpha,\beta]$. Then*

$$|T(f,g)| \leq \frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (x-\alpha)(\beta-x)[g'(x)]^2dx\right)^{\frac{1}{2}}. \tag{4.1}$$

The constant $\frac{1}{\sqrt{2}}$ in (4.1) is the best possible.

Theorem 4.2. *Let $g : [\alpha, \beta] \to \mathbb{R}$ be monotonic nondecreasing and $f : [\alpha, \beta] \to \mathbb{R}$ be absolutely continuous with $f' \in L_\infty[\alpha, \beta]$. Then*

$$|T(f, g)| \leq \frac{1}{2(\beta-\alpha)} \|f'\|_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)dg(x). \tag{4.2}$$

The constant $\frac{1}{2}$ in (4.2) is the best possible.

To avoid many notations, under assumptions of Theorem 3.1, we define the function $\mathcal{B} : [\alpha, \beta] \to \mathbb{R}$ by

$$\mathcal{B}(s) = \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{m} v_q G(y_q, t)\right] G_{H,n-2}(t, s)dt. \tag{4.3}$$

Then $T(\mathcal{B}, \mathcal{B})$ denotes the Čebyšev functional

$$T(\mathcal{B}, \mathcal{B}) = \frac{1}{\beta-\alpha} \int_\alpha^\beta \mathcal{B}^2(s)ds - \left(\frac{1}{\beta-\alpha} \int_\alpha^\beta \mathcal{B}(s)ds\right)^2.$$

Theorem 4.3. *Suppose that all the assumptions of Theorem 3.1 hold. Additionally, let $\phi^{(n)}$ be absolutely continuous on $[\alpha, \beta]$ with $(\cdot - \alpha)(\beta - \cdot)(\phi^{(n+1)})^2 \in L[\alpha, \beta]$ and $\mathcal{B}$ be defined as in (4.3). Then the following representation holds*

$$\begin{aligned}
&\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{k} v_q \phi(y_q) \\
&= \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t)\right] \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) \phi^{(i+2)}(a_j)dt \\
&+ \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{\beta-\alpha} \int_\alpha^\beta \mathcal{B}(s)ds + R(\phi; \alpha, \beta)
\end{aligned} \tag{4.4}$$

and the remainder $R(\phi; \alpha, \beta)$ satisfies the estimation

$$|R(\phi; \alpha, \beta)| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}} [T(\mathcal{B}, \mathcal{B})]^{\frac{1}{2}} \left|\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right|^{\frac{1}{2}} \tag{4.5}$$

Proof. Applying Theorem 4.1 for $f \to \mathcal{B}$ and $g \to \phi^{(n)}$, we get

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha} \int_\alpha^\beta \mathcal{B}(s)\phi^{(n)}(s)ds - \frac{1}{\beta-\alpha} \int_\alpha^\beta \mathcal{B}(s)ds \cdot \frac{1}{\beta-\alpha} \int_\alpha^\beta \phi^{(n)}(s)ds\right| \\
&\leq \frac{1}{\sqrt{2}} [T(\mathcal{B}, \mathcal{B})]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}} \left(\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, we have

$$\int_\alpha^\beta \mathcal{B}(s)\phi^{(n)}(s)ds = \frac{\left(\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)\right)}{\beta-\alpha} \int_\alpha^\beta \mathcal{B}(s)ds + R(\phi; \alpha, \beta),$$

where the remainder $R(\phi; \alpha, \beta)$ satisfies the estimation (4.5).
Now from the identity (3.1) we obtain (4.4). □

Theorem 4.4. *Suppose that all the assumptions of Theorem 3.1 hold. Additionally, let $\phi^{(n+1)} \geq 0$ on $[\alpha, \beta]$ and $\mathcal{B}$ be defined as in (4.3). Then the representation (4.4) holds and $R(\phi; \alpha, \beta)$ satisfies the estimation*

$$|R(\phi;\alpha,\beta)| \leq \|\mathcal{B}'\|_{\infty} \left\{ \frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha} \right\}. \tag{4.6}$$

Proof. Applying Theorem 4.2 for $f \to \mathcal{B}$ and $g \to \phi^{(n)}$, we get

$$\begin{aligned} &\left| \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathcal{B}(s)\phi^{(n)}(s)ds - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \mathcal{B}(s)ds \cdot \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \phi^{(n)}(s)ds \right| \\ &\leq \frac{1}{2(\beta-\alpha)} \|\mathcal{B}'\|_{\infty} \int_{\alpha}^{\beta} (s-\alpha)(\beta-s)\phi^{(n+1)}(s)ds. \end{aligned} \tag{4.7}$$

Since

$$\begin{aligned} &\int_{\alpha}^{\beta} (s-\alpha)(\beta-s)\phi^{(n+1)}(s)ds = \int_{\alpha}^{\beta} \left[2s-(\alpha+\beta)\right]\phi^{(n)}(s)ds \\ &= (\beta-\alpha)\left[\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right] - 2\left[\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right], \end{aligned}$$

using identity (3.1) and the inequality (4.7) we deduce (4.6). □

Theorem 3.2 gives the lower bound for the expression

$$\begin{aligned} &\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{k} v_q\phi(y_q) \\ &- \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t) \right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i+2)}(a_j)dt. \end{aligned}$$

The upper bound is presented in the next theorem.

Theorem 4.5. *Suppose that all the assumptions of Theorem 3.1 hold. Additionally, let $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$, $\left|\phi^{(n)}\right|^p \in L_p[\alpha, \beta]$ and $\mathcal{B}$ be defined as in (4.3). Then*

$$\begin{aligned} &\left| \sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{k} v_q\phi(y_q) \right. \\ &\left. - \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t) \right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i+2)}(a_j)dt \right| \\ &\leq \left\|\phi^{(n)}\right\|_p \|\mathcal{B}\|_q . \end{aligned} \tag{4.8}$$

The constant $\|\mathcal{B}\|_q$ is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Applying Hölder's inequality to the identity (3.1) we obtain

$$\begin{aligned}(4.9)\qquad &\left|\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{k} v_q\phi(y_q)\right.\\ &\left.-\int_{\alpha}^{\beta}\left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t)\right]\sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i+2)}(a_j)dt\right|\\ &= \left|\int_{\alpha}^{\beta}\mathcal{B}(s)\phi^{(n)}(s)ds\right| \le \left\|\phi^{(n)}\right\|_p \|\mathcal{B}\|_q .\end{aligned}$$

For the proof of the sharpness of the constant $\|\mathcal{B}\|_q$ let us find a function ϕ for which the equality in (4.9) holds.
For $1 < p < \infty$ take ϕ to be such that

$$\phi^{(n)}(s) = \operatorname{sgn}\mathcal{B}(s)\,|\mathcal{B}(s)| .$$

For $p = \infty$ take $\phi^{(n)}(s) = \operatorname{sgn}\mathcal{B}(s)$.
For $p = 1$ we prove that

$$(4.10)\qquad \left|\int_{\alpha}^{\beta}\mathcal{B}(s)\phi^{(n)}(s)ds\right| \le \max_{s\in[\alpha,\beta]}|\mathcal{B}(s)|\left(\int_{\alpha}^{\beta}\left|\phi^{(n)}(s)\right|ds\right)$$

is the best possible inequality.
Assume that $|\mathcal{B}(s)|$ attains its maximum at $s_0 \in [\alpha,\beta]$. First we assume that $\mathcal{B}(s_0) > 0$. For ε small enough we define $\phi_\varepsilon(s)$ by

$$\phi_\varepsilon(s) = \begin{cases} 0, & \alpha \le s \le s_0,\\ \frac{1}{\varepsilon\, n!}(s-s_0)^n, & s_0 \le s \le s_0+\varepsilon,\\ \frac{1}{n!}(s-s_0)^{n-1}, & s_0+\varepsilon \le s \le \beta.\end{cases}$$

Then for ε small enough we have

$$\left|\int_{\alpha}^{\beta}\mathcal{B}(s)\phi^{(n)}(s)ds\right| = \left|\int_{s_0}^{s_0+\varepsilon}\mathcal{B}(s)\frac{1}{\varepsilon}ds\right| = \frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon}\mathcal{B}(s)ds.$$

Now from (4.10) we have

$$\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon}\mathcal{B}(s)ds \le \mathcal{B}(s_0)\int_{s_0}^{s_0+\varepsilon}\frac{1}{\varepsilon}ds = \mathcal{B}(s_0).$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon}\mathcal{B}(s)ds = \mathcal{B}(s_0)$$

then the statement follows.
In case $\mathcal{B}(s_0) < 0$, we define $\phi_\varepsilon(s)$ by

$$\phi_\varepsilon(s) = \begin{cases} \frac{1}{n!}(s-s_0-\varepsilon)^{n-1}, & \alpha \le s \le s_0,\\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & s_0 \le s \le s_0+\varepsilon,\\ 0, & s_0+\varepsilon \le s \le \beta,\end{cases}$$

and the rest of the proof is the same as above. $\square$

5. Some applications

Motivated by the inequality (3.6), under the assumptions of Theorems 3.2, we define the linear functional $\Lambda : C^n([\alpha,\beta]) \to \mathbb{R}$ by

$$\Lambda(\phi) = \sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{k} v_q\phi(y_q)$$

$$- \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t)\right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i+2)}(a_j)dt. \tag{5.1}$$

Remark 5.1. Note that if $\phi \in C^n([\alpha,\beta])$ is n-convex, then by Theorem 3.2 we have

$$\Lambda(\phi) \geq 0.$$

Using the linearity and positivity of defined functional we derive mean-value theorems of the Lagrange and Cauchy type.

Theorem 5.1. *Let $\phi \in C^n([\alpha,\beta])$ and $\Lambda : C^n([\alpha,\beta]) \to \mathbb{R}$ be the linear functional defined by (5.1). Then there exist $\xi \in [\alpha,\beta]$ such that*

$$\Lambda(\phi) = \phi^{(n)}(\xi)\Lambda(\varphi),$$

where $\varphi(x) = \frac{x^n}{n!}$.

Proof. Similar to the proof of Theorem 4.1 in [11]. □

Theorem 5.2. *Let $\phi, \psi \in C^n([\alpha,\beta])$ and $\Lambda : C^n([\alpha,\beta]) \to \mathbb{R}$ be the linear functional defined by (5.1). Then there exists $\xi \in [\alpha,\beta]$ such that*

$$\frac{\Lambda(\phi)}{\Lambda(\psi)} = \frac{\phi^{(n)}(\xi)}{\psi^{(n)}(\xi)}, \tag{5.2}$$

provided that the denominators are non-zero

Proof. Similar to the proof of Corollary 4.2 in [11]. □

Remark 5.2. If $\frac{\phi^{(n)}}{\psi^{(n)}}$ is an invertible function, then we get

$$\xi = \left(\frac{\phi^{(n)}}{\psi^{(n)}}\right)^{-1}\left(\frac{\Lambda(\phi)}{\Lambda(\psi)}\right)$$

which is exactly mean of Chauchy type of the segment $[\alpha,\beta]$.

Applying Exponential convexity method [11], we may interpret our results in the form of exponentially convex functions or in the special case log convex functions. In order to obtain such results, we define the families of functions as follows.

For every choice of $l+1$ mutually different points $x_0, x_1, ..., x_l \in [\alpha,\beta]$ we define

- $\mathcal{F}_1 = \{\phi_t : [\alpha,\beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is n-exponentially convex in the Jensen sense on $I\}$
- $\mathcal{F}_2 = \{\phi_t : [\alpha,\beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is exponentially convex in the Jensen sense on $I\}$
- $\mathcal{F}_3 = \{\phi_t : [\alpha,\beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is 2-exponentially convex in the Jensen sense on $I\}$

Theorem 5.3. *Let Λ be the linear functional defined as in (5.1) associated with family $\mathcal{F}_1$. Then the following statements hold:*

(i) *The function* $t \mapsto \Lambda(\phi_t)$ *is* n*-exponentially convex in the Jensen sense on* I.

(ii) *If the function* $t \mapsto \Lambda(\phi_t)$ *is continuous on* I, *then it is* n*-exponentially convex on* I.

Proof. (i) We define the function $h : [\alpha, \beta] \to \mathbb{R}$ by

$$h(x) = \sum_{j,k=1}^{n} p_j p_k \phi_{s_{jk}}(x),$$

where $p_j, s_j \in \mathbb{R}$, $j = 1, ..., n$, $s_{jk} = \frac{s_j+s_k}{2}$, $1 \le j, k \le n$, and $\phi_{s_{jk}} \in \mathcal{F}_1$.
Since $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is n-exponentially convex in the Jensen sense on I, then

$$[x_0, x_1, ..., x_l; h] = \sum_{j,k=1}^{n} p_j p_k \left[x_0, x_1, ..., x_l; \phi_{s_{jk}}\right] \ge 0,$$

i.e. h is l-convex. Therefore, we have

$$\Lambda(h) = \sum_{j,k=1}^{n} p_j p_k \Lambda\left(\phi_{s_{jk}}\right) \ge 0.$$

Hence, the function $t \mapsto \Lambda(\phi_t)$ is n-exponentially convex in the Jensen sense on I.
(ii) Follows from (i) and Definition 1.1. □

The following corollary is an easy consequence of the previous theorem.

Corollary 5.1. *Let* Λ *be the linear functional defined as in (5.1) associated with family* $\mathcal{F}_2$. *Then the following statements hold:*

(i) *The function* $t \mapsto \Lambda(\phi_t)$ *is exponentially convex in the Jensen sense on* I.

(ii) *If the function* $t \mapsto \Lambda(\phi_t)$ *is continuous on* I, *then it is exponentially convex on* I.

Corollary 5.2. *Let* Λ *be the linear functional defined as in (5.1) associated with family* $\mathcal{F}_3$. *Then the following statements hold:*

(i) *If the function* $t \mapsto \Lambda(\phi_t)$ *is continuous on* I, *then it is* 2*-exponentially convex on* I. *If* $t \mapsto \Lambda(\phi_t)$ *is additionally positive, then it is also log-convex on* I. *Furthermore, for every choice* $r, s, t \in I$, *such that* $r < s < t$, *it holds*

$$[\Lambda(\phi_s)]^{t-r} \le [\Lambda(\phi_r)]^{t-s} [\Lambda(\phi_r)]^{s-r}.$$

(ii) *If the function* $t \mapsto \Lambda(\phi_t)$ *is positive and differentiable on* I, *then for all* $r, s, u, v \in I$ *such that* $r \le u$, $s \le v$, *we have*

$$\mu_{r,s}(\Lambda, \mathcal{F}_3) \le \mu_{u,v}(\Lambda, \mathcal{F}_3), \tag{5.3}$$

where

$$\mu_{r,s}(\Lambda, \mathcal{F}_3) = \begin{cases} \left(\frac{\Lambda(\phi_r)}{\Lambda(\phi_s)}\right)^{\frac{1}{r-s}}, & r \ne s, \\ \exp\left(\frac{\frac{d}{dr}(\Lambda(\phi_r))}{\Lambda(\phi_r)}\right), & r = s. \end{cases} \tag{5.4}$$

Proof. (i) The first part of statement is an easy consequence of Theorem 5.3 and the second one of Remark 1.1.
Since $t \mapsto \Lambda(\phi_t)$ is log-convex on I, i.e. $t \mapsto \log \Lambda(\phi_t)$ is convex on I, then by definition we have

$$(r-t)\log\Lambda(\phi_t) + (t-s)\log\Lambda(f_r) + (s-r)\log\Lambda(f_r) \ge 0$$

for every choice $r, s, t \in I$, such that $r < s < t$. Therefore, we have

$$[\Lambda(\phi_s)]^{t-r} \leq [\Lambda(\phi_r)]^{t-s} [\Lambda(\phi_r)]^{s-r}.$$

(ii) Since $t \mapsto \log \Lambda(\phi_t)$ is convex on I, by definition we have

$$\frac{\log \Lambda(\phi_r) - \log \Lambda(\phi_s)}{r-s} \leq \frac{\log \Lambda(\phi_u) - \log \Lambda(\phi_v)}{u-v} \tag{5.5}$$

for $r \leq u$, $s \leq v$, $r \neq u$, $s \neq v$. Therefore, we have

$$\mu_{r,s}(\Lambda, \mathcal{F}_3) \leq \mu_{u,v}(\Lambda, \mathcal{F}_3).$$

Case $r = s$, $u = v$ follows from (5.5) as limiting case. □

Using obtained mean-valued theorems and results regarding the exponential convexity, we may deduce some new classes of two-parameter Cauchy-type means.

For example, consider the family of functions

$$\Omega = \{\varphi_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

defined by

$$\varphi_t(x) = \frac{e^{-x\sqrt{t}}}{\left(-\sqrt{t}\right)^n}.$$

Since $\frac{d^n \varphi_t}{dx^n}(x) = e^{-x\sqrt{t}} > 0$, the function φ_t is n-convex function for every $t > 0$. Moreover, the function $t \mapsto \frac{d^n \varphi_t}{dx^n}(x)$ is exponentially convex. Therefore, using the same arguments as in proof of Theorem 5.3, we conclude that the function $t \mapsto [x_0, x_1, ..., x_l; \varphi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Then from Corollary 5.1 it follows that $t \mapsto \Lambda(\varphi_t)$ is exponentially convex in the Jensen sense. It is easy to verify that the function $t \mapsto \Lambda(\varphi_t)$ is continuous, so it is exponentially convex.

For this family of functions, with assumption that $[\alpha, \beta] \subset (0,\infty)$ and $t \mapsto \Lambda(\varphi_t)$ is positive, (5.4) becomes

$$\mu_{\eta,\zeta} = \left(\frac{\zeta^n}{\eta^n} \cdot \frac{\sum_{p=1}^{l} u_p e^{-x_p\sqrt{\eta}} - \sum_{q=1}^{k} v_q e^{-y_q\sqrt{\eta}} - A_1}{\sum_{p=1}^{l} u_p e^{-x_p\sqrt{\zeta}} - \sum_{q=1}^{k} v_q e^{-y_q\sqrt{\zeta}} - B_1} \right)^{\frac{1}{\eta-\zeta}}, \quad \eta \neq \zeta,$$

$$\mu_{\eta,\eta} = \exp\left(\frac{\sum_{q=1}^{k} v_q y_q e^{-y_q\sqrt{\eta}} - \sum_{p=1}^{l} u_p x_p e^{-x_p\sqrt{\eta}} + A_2}{2\sqrt{\eta}\left(\sum_{p=1}^{l} u_p e^{-x_p\sqrt{\eta}} - \sum_{q=1}^{k} v_q e^{-y_q\sqrt{\eta}} - A_1 \right)} - \frac{n}{\eta} \right),$$

where

$$A_1 = \int_\alpha^\beta \left(\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right) \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t)(-1)^{i+2} \eta^{1+\frac{i}{2}} e^{-a_j \sqrt{\eta}} dt,$$

$$A_2 = \int_\alpha^\beta \left(\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right) \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) \frac{d^{i+2}}{dx^{i+2}} (x e^{-x\sqrt{\eta}})|_{x=a_j} dt,$$

$$B_1 = \int_\alpha^\beta \left(\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right) \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t)(-1)^{i+2} \zeta^{1+\frac{i}{2}} e^{-a_j \sqrt{\zeta}} dt.$$

Using Theorem 5.2 it follows that

$$\mu_{\eta,\zeta}(\Lambda, \Omega) = -\left(\sqrt{\eta} + \sqrt{\zeta}\right) \log \mu_{\eta,\zeta}(\Lambda, \Omega)$$

satisfies

$$\alpha \leq \mu_{\eta,\zeta}(\Lambda, \Omega) \leq \beta,$$

i.e. $\mu_{\eta,\zeta}(\Lambda, \Omega)$ is mean. By Corollary 5.2, using (5.3), it follows that this mean is monotonic.

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SCREEN SEMI-INVARYANT HALF-LIGHTLIKE SUBMANIFOLDS OF A SEMI-RIEMANNIAN PRODUCT MANIFOLD WITH QUARTER-SYMMETRIC CONNECTION

OGUZHAN BAHADIR

Abstract. In this paper, we study half-lightlike submanifolds of a semi-Riemannian product manifold. We introduce a classes half-lightlike submanifolds of called screen semi-invariant half-lightlike submanifolds. We defined some special distribution of screen semi-invariant half-lightlike submanifold. We give some equivalent conditions for integrability of distributions with respect to the Levi-Civita connection of semi-Riemannian manifolds and quarter-symmetric non-metric connection of semi-Riemannian manifolds and some results.

1. Introduction

The theory of degenerate submanifolds of semi-Riemannian manifolds is one of a important topics of diferential geometry. The geometry of lightlike submanifolds a semi-Riemannian manifold was presented in [7] (see also [8]) by K.L. Duggal and A. Bejancu. Differential Geometry of Lightlike Submanifolds was presented in [17] by K. L. Duggal and B. Sahin. In [12],[13], [14], [15], K. L. Duggal and B. Sahin introduced and studied geometry of classes of lightlike submanifolds in indefinite Kaehler and indefinite Sasakian manifolds which is an umbrella of CR-lightlike, SCR-lightlike, Screen real GCR-lightlie submanifolds. In [16], M. Atceken and E. Kilic introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. In [18], E. Kilic and B. Sahin introduced radical anti-invariant lightlike submanifolds of a semi-Riemannian product and gave some examples and results for lightlike submanifolds. In [19] E. Kilic and O. Bahadir studied lightlike hypersurfaces of a semi-Riemannian product manifold with respect to quarter symmetric non-metric connection. In [20] O. Bahadir give some equivalent conditions for integrability of distributions with respect to Levi Civita connection of semi-Riemannian manifolds and some results.

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In this paper, we study half-lightlike submanifolds of a semi-Riemannian product manifold. In Section 2, we give some basic concepts. In Section 3, we introduce screen semi-invariant half-lightlike submanifolds. We defined some special distribution of screen semi-invariant half-lightlike submanifold. In Section 4, we consider half-lightlike submanifolds of a semi-Riemannian product manifold with quarter symmetric non-metric connection determined by the product structure. We compute some results with respect to the quarter-symmetric non-metric connection.

2. Half-lightlike submanifolds

Let $(\widetilde{M}, \widetilde{g})$ be an $(m+2)$-dimensional $(m > 1)$ semi-Riemannian manifold of index $q \geq 1$ and M a submanifold of codimension 2 of $\widetilde{M}$. If $\widetilde{g}$ is degenerate on the tangent bundle TM on M, then M is called a lightlike submanifold of $\widetilde{M}$ [17]. Denote by g the induced degenerate metric tensor of $\widetilde{g}$ on M. Then there exists locally (or globally) a vector field $\xi \in \Gamma(TM)$, $\xi \neq 0$, such that $g(\xi, X) = 0$ for any $X \in \Gamma(TM)$. For any tangent space T_xM, $(x \in M)$, we consider

$$T_xM^{\perp} = \{u \in T_x\widetilde{M} : \widetilde{g}(u, v) = 0, \forall v \in T_xM\}, \tag{2.1}$$

a degenerate 2-dimensional orthogonal (but not complementary) subspace of $T_x\widetilde{M}$. The radical subspace $Rad\ T_xM = T_xM \cap T_xM^{\perp}$ depends on the point $x \in M$. If the mapping

$$Rad\ TM : x \in M \longrightarrow Rad\ T_xM \tag{2.2}$$

defines a radical distribution on M of rank $r > 0$, then the submanifold M is called $r-$lightlike submanifold. If $r = 1$, then M is called half-lightlike submanifold of $\widetilde{M}$ [17]. Then there exist $\xi, u \in T_xM^{\perp}$ such that

$$\widetilde{g}(\xi, v) = 0, \quad \widetilde{g}(u, u) \neq 0, \forall v \in T_xM^{\perp}. \tag{2.3}$$

Furthermore, $\xi \in Rad\ T_xM$, and

$$\widetilde{g}(\xi, X) = \widetilde{g}(\xi, v) = 0, \forall X \in \Gamma(TM),\ v \in \Gamma(TM^{\perp}). \tag{2.4}$$

Thus, $Rad\,TM$ is locally (or globally) spanned by ξ. By denote the complementary vector bundle $S(TM)$ of $Rad\,TM$ in TM which is called screen bundle of M. Thus we have the following decomposition

$$TM = Rad\ TM \perp S(TM), \tag{2.5}$$

where $\perp$ denotes the orthogonal-direct sum. In this paper, we assume that M is half-lightlike. Then there exists complementary non-degenerate distribution $S(TM^{\perp})$ of $Rad\ TM$ in $TM^{\perp}$ such that

$$TM^{\perp} = Rad\ TM \perp S(TM^{\perp}). \tag{2.6}$$

Choose $u \in S(TM^{\perp}$ as a unit vector field with $\widetilde{g}(u, u) = \epsilon = \pm 1$. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\widetilde{M}$. We note that ξ and u belong to $S(TM)^{\perp}$. Thus we have

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section ξ of $Rad\ TM$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely determined null vector field $N \in \Gamma(ltr(TM))$ satisfying

$$\widetilde{g}(\xi, N) = 1,\ \widetilde{g}(N, N) = \widetilde{g}(N, X) = \widetilde{g}(N, u) = 0, \forall X \in \Gamma(TM), \tag{2.7}$$

where N, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \perp ltr(TM)$ are called the lightlike transversal vector field, lightlike transversal vector bundle and transversal vector bundle of M with respect to $S(TM)$, respectively. Then we have the following decomposition:

$$T\widetilde{M} = TM \oplus tr(TM) = S(TM) \perp \{Rad\ TM \oplus ltr(TM)\} \perp S(TM^{\perp}). \tag{2.8}$$

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ and P the projection of TM on $S(TM)$ with respect to the decomposition (2.5). Thus, for any $X \in \Gamma(TM)$, we can write $X = PX + \eta(X)\xi$, where η is a local differential 1-form on M given by $\eta(X) = \widetilde{g}(X, N)$. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + D_1(X, Y)N + D_2(X, Y)u, \tag{2.9}$$
$$\widetilde{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{2.10}$$
$$\widetilde{\nabla}_X N = -A_N X + p_1(X)N + p_2(X)u, \tag{2.11}$$
$$\widetilde{\nabla}_X u = -A_u X + \varepsilon_1(X)N + \varepsilon_2(X)u, \tag{2.12}$$
$$\nabla_X PY = \nabla^*_X PY + E(X, PY)\xi, \tag{2.13}$$
$$\nabla_X \xi = -A^*_\xi X - p_1(X)\xi, \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$, $u \in s(TM^{\perp})$, $U \in \Gamma(tr(TM))$, where ∇, ∇^* and ∇^t are induced linear connections on M, $S(TM)$ and $tr(TM)$, respectively, D_1 and D_2 are called the lightlike second fundamental and screen second fundemental form of M respectively, E is called the local second fundamental form on $S(TM)$. A_U, A_N, A^*_ξ and A_u are linear operators on TM and τ, ρ and ϕ are $1-$forms on TM. We note that, the induced connection ∇ is torsion-free but it is not metric connection on M and satisfies

$$(\nabla_X g)(Y, Z) = D_1(X, Y)\eta(Z) + D_1(X, Z)\eta(Y), \tag{2.15}$$

for any $X, Y, Z \in \Gamma(TM)$. However the connection ∇^* on $S(TM)$ is metric. From the above statements, we have

$$D_1(X, PY) = g(A^*_\xi X, PY), \quad g(A^*_\xi X, N) = 0, \quad D_1(X, \xi) = 0, \quad \widetilde{g}(A_N X, N) = 0, \tag{2.16}$$

$$E(X, PY) = g(A_N X, PY), \quad \epsilon D_2(X, Y) = g(A_u X, Y) - \varepsilon_1(X)\eta(Y), \tag{2.17}$$

$$\epsilon\rho(X) = \widetilde{g}(A_u X, N),\ p_1(X) = -\eta(\nabla_X \xi), \quad p_2(X) = \epsilon\eta(A_u X), \quad \varepsilon_1(X) = -\epsilon D_2(X, \xi), \tag{2.18}$$

for any $X, Y \in \Gamma(TM)$. From (2.17) and (2.18), A^*_ξ and A_N are $\Gamma(S(TM))-$valued shape operators related to D_1 and E, respectively and $A^*_\xi \xi = 0$.

Using torsion free linear connection $\nabla$and (2.13) we have

$$\begin{aligned}[X,Y] &= \{\nabla^*_X PY - \nabla^*_Y PX + \eta(X)A^*_\xi Y - \eta(Y)A^*_\xi X\} \\ &\quad +\{E(X,PY) - E(Y,PX) + X(\eta(Y)) \\ &\quad -Y(\eta(X)) + \eta(X)p_1(Y) - \eta(Y)p_1(X)\}\xi.\end{aligned}$$

The last equation and (2.17)

$$\begin{aligned} g(\nabla^*_X PY, PZ) - g(\nabla^*_X PZ, PY) - g([X,Y], PZ) \\ = \eta(Y)D_1(X,PZ) - \eta(X)D_1(Y,PZ), \\ 2d\eta(X,Y) = E(Y,PX) - E(X,PY) \\ +p_1(X)\eta(Y) - p_1(Y)\eta(X). \end{aligned} \tag{2.19}$$

From the second equation (2.19) we have

$$\eta([PX,PY]) = E(PX,PY) - E(PY,PX). \tag{2.20}$$

From (2.18) and (2.20), we have the following theorem.

Theorem 2.1. *Let M be a half-lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$. Then the following assertions are equivalent:.*
(1) *The screen distribution $S(TM)$ is integrable.*
(2) *The second fundamental form of $S(TM)$ is symmetric on $\Gamma(s(TM)$.*
(3) *The shape operator A_N of the immersion of M in $\widetilde{M}$ is symmetric with respect to g on $\Gamma(s(TM)$.*

Next by using (2.14), (2.15), (2.17) and (2.18) we obtain

Theorem 2.2. *Let M be a half-lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$. Then the following assertions are equivalent:*
(1)*The induced connection ∇ on M is a metric connection.*
(2) *D_1 vanishes identically on M.*
(3) *A^*_ξ vanishes identically on M.*
(4) *ξ is a Killing vector field.*
(5) *$TM^\perp$ is a parallel distribution with respect to ∇.*

Theorem 2.3. *Let (M,g) be a proper totally umbilical half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}(c),\widetilde{g})$ of constant sectional curvature c. Then the following assertions are equivalent:*
(i) *The screen distribution $s(TM)$ is integrable.*
(ii) *Each $1-$ form p_1 is closed on $s(TM)$, i.e., $dp_1 = 0$*
(iii) *Each $1-$ form p_2 induced by $s(TM)$ satisfies*

$$2dp_2(X,Y) = p_1(X)p_2(Y) - p_2(X)p_1(Y), \quad \forall X,Y \in \Gamma(TM).$$

For basic information on the geometry of lightlike submanifolds, we refer to [7], [17].

Let $(\widetilde{M}$ be an $n-$ dimensional diferentiable manifold with a tensor field F of type $(1,1)$ on $\widetilde{M}$ such that $F^2 = I$. Then M is called an almost product manifold with almost product structure F. If we put $\pi = \frac{1}{2}(I+F)$, $\sigma = \frac{1}{2}(I-F)$ then we have

$$\pi + \sigma = I,\ \pi^2 = \pi,\ \sigma^2 = \sigma,\ \pi\sigma = \sigma\pi = 0,\ F = \pi - \sigma.$$

Thus π and σ define two complementary distributions and the eigenvalue of F are ∓ 1. If an almost product manifold $\widetilde{M}$ admits a semi-Riemannian metric $\widetilde{g}$ such that

$$\widetilde{g}(FX, FY) = \widetilde{g}(X, Y),\ \widetilde{g}(FX, Y) = \widetilde{g}(X, FY), \forall X, Y \in \Gamma(\widetilde{M}),$$

then $(\widetilde{M}, \widetilde{g})$ is called semi-Riemannian almost product manifold. If, for any X, Y vector fields on $\widetilde{M}$, $(\widetilde{\nabla}_X F)Y = 0$, that is

$$\widetilde{\nabla}_X FY = F\widetilde{\nabla}_X Y,$$

then M is called an semi-Riemannian product manifold, where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$.

3. Screen Semi-Invariant Lightlike Submanifolds

Let (M, g) be a half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$ For any $X \in \Gamma(TM)$ we can write

$$FX = fX + wX, \tag{3.1}$$

where f and w are the projections on of $\Gamma(T\widetilde{M})$ onto TM and $trTM$, respectively, that is, fX and wX are tangent and transversal components of FX. From (2.8) and (3.1), we can write

$$FX = fX + w_1(X)N + w_2(X)u, \tag{3.2}$$

where $w_1(X) = \widetilde{g}(FX, \xi)$, $w_2(X) = \epsilon\widetilde{g}(FX, u)$.

Definition 3.1. Let (M, g) be a half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. If $FRad\ TM \subset S(TM)$, $Fltr(TM) \subset S(TM)$ and $F(S(TM^{\perp})) \subset S(TM)$ then we say that M is a screen semi-invaryant (SSI) half-lightlike submanifold.

If $FS(TM) = S(TM)$, then we say that M is a screen invaryant half-lightlike submanifold.

Now, let M be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. If we set $L_1 = FRad\ TM$, $L_2 = Fltr(TM)$ and $L_3 = F(S(TM^{\perp}))$, then we can write

$$S(TM) = L_0 \perp \{L_1 \oplus L_2\} \perp L_3, \tag{3.3}$$

where L_0 is a $(m-4)-$dimensional distribution. Hence we have the following decompositions:

$$TM = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp Rad\ TM, \tag{3.4}$$

$$T\widetilde{M} = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp S(TM^{\perp}) \perp \{Rad\ TM \oplus ltr(TM)\}. \tag{3.5}$$

Let (M, g) be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. If we set

$$L = L_0 \perp L_1 \perp Rad\ TM \qquad L^{\perp} = L_2 \perp L_3,$$

then we can write

$$TM = L \oplus L^{\perp}.$$

We note that the distribution L is a invariant distribution and the distribution $L^{\perp}$ is anti-invariant distribution with respect to F on M.

4. Quarter-symmetric Non-metric Connections

Let (M, g, F) be a semi-Riemannian product manifold and $\widetilde{\nabla}$ be the Levi-Civita connection on M. If we set

$$\widetilde{D}_X Y = \widetilde{\nabla}_X Y + \pi(Y)FX \tag{4.1}$$

for any $X, Y \in \Gamma(T\widetilde{M})$, then $\widetilde{D}$ is a linear connection on $\widetilde{M}$, where u is a 1-form on $\widetilde{M}$ with U as associated vector field, that is

$$\pi(X) = \widetilde{g}(X, U).$$

The torsion tensor of $\widetilde{D}$ on $\widetilde{M}$ denoted by $\widetilde{T}$. Then we obtain

$$\widetilde{T}(X, Y) = \pi(Y)FX - \pi(X)FY, \tag{4.2}$$

and

$$(\widetilde{D}_X \widetilde{g})(Y, Z) = -\pi(Y)\widetilde{g}(FX, Z) - \pi(Z)\widetilde{g}(FX, Y), \tag{4.3}$$

for any $X, Y \in \Gamma(T\widetilde{M})$. Thus $\widetilde{D}$ is a quarter-symmetric non-metric connection on $\widetilde{M}$. From (4.1) we have

$$(\widetilde{D}_X F)Y = \pi(FY)FX - \pi(Y)X. \tag{4.4}$$

Replacing X by FX and Y by FY in (4.4) we obtain

$$(\widetilde{D}_{FX} F)FY = \pi(Y)X - \pi(FY)FX. \tag{4.5}$$

Thus we have

$$(\widetilde{D}_X F)Y + (\overline{D}_{FX} F)FY = 0. \tag{4.6}$$

If we set

$$'F(X, Y) = \widetilde{g}(FX, Y) \tag{4.7}$$

for any $X, Y \in \Gamma(T\overline{M})$, from (4.1) we get

$$(\widetilde{D}_X\, 'F)(Y, Z) = (\widetilde{\nabla}_X\, 'F)(Y, Z) - \pi(Y)\widetilde{g}(X, Z) - \pi(Z)\widetilde{g}(X, Y). \tag{4.8}$$

From (4.1) the curvature tensor $\widetilde{R}^D$ of the quarter-symmetric non-metric connection $\widetilde{D}$ is given by

$$\widetilde{R}^D(X, Y)Z = \widetilde{R}(X, Y)Z + \widetilde{\lambda}(X, Z)FY - \widetilde{\lambda}(Y, Z)FX, \tag{4.9}$$

for any $X, Y, Z \in \Gamma(T\widetilde{M})$, where $\widetilde{\lambda}$ is a $(0, 2)$-tensor given by $\widetilde{\lambda}(X, Z) = (\widetilde{\nabla}_X \pi)(Z) - \pi(Z)\pi(FX)$. If we set $\widetilde{R}^D(X, Y, Z, W) = \widetilde{g}(\overline{R}^D(X, Y)Z, W)$, then, from (4.9), we obtain

$$\widetilde{R}^D(X, Y, Z, W) = -\widetilde{R}^D(Y, X, Z, W).$$

We note that the Riemannian curvature tensor $\widetilde{R}^D$ of $\widetilde{D}$ does not satisfy the other curvature-like properties. But, from (4.9), we have

$$\begin{aligned}
\widetilde{R}^D(X, Y)Z + \widetilde{R}^D(Y, Z)X + \widetilde{R}^D(Z, X)Y &= (\widetilde{\lambda}(Z, Y) - \widetilde{\lambda}(Y, Z))FX \\
&+ (\widetilde{\lambda}(X, Z) - \widetilde{\lambda}(Z, X))FY \\
&+ (\widetilde{\lambda}(Y, X) - \widetilde{\lambda}(X, Y))FZ.
\end{aligned}$$

Thus we have the following proposition.

Proposition 4.1. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\widetilde{M}$. Then the first Bianchi identity of the quarter-symmetric non-metric connection $\widetilde{D}$ on M is provided if and only if $\widetilde{\lambda}$ is symmetric.*

Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M},\widetilde{g}\right)$ with quarter-symmetric non-metric connection $\widetilde{D}$. Then the Gauss and Weingarten formulas with respect to $\widetilde{D}$ are given by, respectively,

$$\widetilde{D}_X Y = D_X Y + \widetilde{D}_1(X,Y)N + \widetilde{D}_2(X,Y)u, \tag{4.10}$$

$$\widetilde{D}_X N = -\widetilde{A}_N X + \widetilde{p}_1(X)N + \widetilde{p}_2(X)u, \tag{4.11}$$

$$\widetilde{D}_X u = -\widetilde{A}_u X + \widetilde{\varepsilon}_1(X)N + \widetilde{\varepsilon}_2(X)u. \tag{4.12}$$

for any $X,Y \in \Gamma(TM)$, where $D_X Y,\ \widetilde{A}_N X,\ \widetilde{A}_u X \in \Gamma(TM)$, $\widetilde{D}_1(X,Y) = \widetilde{g}(\widetilde{D}_X Y,\xi)$, $\widetilde{D}_2(X,Y) = \epsilon\widetilde{g}(\widetilde{D}_X Y,u)$, $\widetilde{p}_1(X) = \widetilde{g}(\widetilde{D}_X N,\xi)$, $\widetilde{p}_2(X) = \epsilon\widetilde{g}(\widetilde{D}_X N,u)$, $\widetilde{\varepsilon}_1(X) = \widetilde{g}(\widetilde{D}_X u,\xi)$, $\widetilde{\varepsilon}_2(X) = \epsilon\widetilde{g}(\widetilde{D}_X u,u)$. Here, $\widetilde{D}_1$ and $\widetilde{D}_2$ the lightlike second fundamental form and the screen second fundamental form of M with respect to $\widetilde{D}$ respectively. Both $\widetilde{A}_N$ and $\widetilde{A}_u$ are linear operators on $\Gamma(TM)$. From (2.9), (2.11), (2.12), (4.1), (4.10), (4.11) and (4.12) we obtain

$$D_X Y = \nabla_X Y + \pi(Y)fX, \tag{4.13}$$

$$\widetilde{D}_1(X,Y) = D_1(X,Y) + \pi(Y)w_1(X), \tag{4.14}$$

$$\widetilde{D}_2(X,Y) = D_2(X,Y) + \pi(Y)w_2(X), \tag{4.15}$$

$$\widetilde{A}_N X = A_N X - \pi(N)fX, \tag{4.16}$$

$$\widetilde{p}_1(X) = p_1(X) + \pi(N)w_1(X), \tag{4.17}$$

$$\widetilde{p}_2(X) = p_2(X) + \pi(N)w_2(X), \tag{4.18}$$

$$\widetilde{A}_u X = A_u X - \pi(u)fX, \tag{4.19}$$

$$\widetilde{\varepsilon}_1(X) = \varepsilon_1(X) + \pi(u)w_1(X), \tag{4.20}$$

$$\widetilde{\varepsilon}_2(X) = \varepsilon_2(X) + \pi(u)w_2(X). \tag{4.21}$$

for any $X,Y \in \Gamma(TM)$. From (2.15), (4.1) we get

$$\begin{aligned}(D_x g)(Y,Z) &= D_1(X,Y)\eta(Z) + D_1(X,Z)\eta(Y)\\ &\quad -\pi(Y)g(fX,Z) - \pi(Z)g(fX,Y),\end{aligned} \tag{4.22}$$

On the other hand, the torsion tensor of the induced connection D is

$$T^D(X,Y) = \pi(Y)fX - \pi(X)fY. \tag{4.23}$$

From last two equations we have the following proposition.

Proposition 4.2. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M},\widetilde{g}\right)$ with quarter-symmetric non-metric connection $\overline{D}$. Then the induced connection D is a quarter-symmetric non-metric connection on the half-lightlike submanifold M.*

From (4.2), (4.14) and (4.15) we have the following theorem

For any $X,Y \in \Gamma(TM)$, $\xi \in \Gamma(RadTM)$ we can write

$$D_X PY = D^*_X PY + E^*(X,PY)\xi, \tag{4.24}$$

$$D_X \xi = -\widetilde{A}^*_\xi X - \widetilde{p}_1(X)\xi, \tag{4.25}$$

where $D_X^* PY \quad \widetilde{A}_\xi^* X \in \Gamma(S(TM))$, $E^*(X, PY) = \widetilde{g}(D_X \, PY, N)$ and $\widetilde{p}_1(X) = -\widetilde{g}(D_X \xi, N)$. From (2.13), (2.14), (4.24) and (4.25), we obtain

$$D_X^* PY = \nabla_X^* PY + \pi(PY) PfX, \tag{4.26}$$

$$E^*(X, PY) = E(X, PY) + \pi(PY)\eta(fX), \tag{4.27}$$

$$\widetilde{A}_\xi^* X = A_\xi^* X - \pi(\xi) PfX, \tag{4.28}$$

$$\widetilde{u}_1(X) = u_1(X) + \pi(\xi)\eta(fX). \tag{4.29}$$

Proposition 4.3. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then D^* the induced connection is quarter-symmetric non-metric connection on $s(TM)$*

Proof. *For any $X, Y, Z \in \Gamma(s(TM))$, we know that ∇^* is metric connection. Thus from (4.26), we get*

$$(D_X^* g)(Y, Z) = -\pi(Y) g(PfX, Z) - \pi(Z) g(Y, PfX). \tag{4.30}$$

Let T^{D^*} be torsion tensor with respect to D^*. From (4.26), we obtain

$$T^{D^*}(X, Y) = \pi(Y) PfX - \pi(X) PfY. \tag{4.31}$$

Then from (4.30) and (4.31), we have proof.

We know that $\widetilde{\nabla} F = 0$. From (4.1) and (4.13) we obtain

$$(\widetilde{D}_X F) Y = \pi(FY) FX - \pi(Y) X, \tag{4.32}$$

and

$$(D_X f) Y = (\nabla_X f) Y + \pi(fY) fX - \pi(Y) f^2 X. \tag{4.33}$$

From (4.32) and (4.33) we have the following propositions.

Proposition 4.4. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. F is not parallel with respect to quarter-symmetric non-metric connection $\widetilde{D}$.*

Proposition 4.5. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. f is not parallel with respect to quarter-symmetric non-metric connection D.*

From (4.14) we have

$$\begin{aligned} \widetilde{D}_1(X, Y) - \widetilde{D}_1(Y, X) &= D_1(X, Y) - D_1(Y, X) + g(\pi(Y) FX - \pi(X) FY, \xi) \\ &= g(\widetilde{T}(X, Y), \xi). \end{aligned} \tag{4.34}$$

Similarly from (4.15) we obtain

$$\widetilde{D}_2(X, Y) - \widetilde{D}_2(Y, X) = g(\widetilde{T}(X, Y), u). \tag{4.35}$$

From the (4.34) and (4.35) we have the following theorems

Theorem 4.1. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then the lightlike second fundemental form $\widetilde{D}_1$ of quarter symmetric non-metric connection is symmetric if and only if there is no $ltrTM$ component of the torsion $\widetilde{T}$.*

Theorem 4.2. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then the screen second fundemental form $\widetilde{D}_2$ of quarter symmetric non-metric connection $\widetilde{D}$ is symmetric if and only if there is no $s(TM^{\perp})$ component of the torsion $\widetilde{T}$.*

Theorem 4.3. *Let M be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then the second fundemental form of $s(TM)$ is symmetric with respect to quarter symmetric non-metric connection if and only if there is no $RadTM$ component of the torsion tesor T^D.*

Proof. *For any $X, Y \in \Gamma(s(TM))$, since E is symmetric, from (4.27) we obtain*

$$E^*(X,Y) - E^*(Y,X) = \pi(Y)\eta(fX) - \pi(X)\eta(fY) = g(T^D(X,Y), N).$$

Thus proof is completed.

Lemma 4.1. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then we have the following equation;*

$$\widetilde{D}_i(X,Y) = D_i(X,Y),\ i \in \{1,2\},\ \forall X \in \Gamma(L_0) \text{ and } Y \in \Gamma(TM)$$

.

Proof. *For any $X \in \Gamma(L_0)$, we know that $wX = 0$. Then from (4.14) and (4.15) proof is completed.*

From the above lemma we have the following theorem.

Theorem 4.4. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then M is L_0- totally geodesic with respect to quarter symmetric non-metric connection if and only if M is L_0- totally geodesic with respect to connection ∇.*

Theorem 4.5. *Let M be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then the following equivalent;*

(i) *$L^{\perp}$ is integrable.*

(ii) *$\widetilde{A}_{FY}X = \widetilde{A}_{FX}Y$, $X, Y \in \Gamma(L^{\perp})$*

(iii) *E_1^* second fundemental form of $s(TM)$ with quarter symmetric non-metric connection is symmetric on $L^{\perp}$.*

Proof. *For any $X, Y \in \Gamma(L^{\perp})$ we obtain*

$$\begin{aligned} g([X,Y], FN) &= g(F[X,Y], N) \\ &= g(\widetilde{\nabla}_X FY - \widetilde{\nabla}_Y FX, N) \\ &= g(A_{FX}Y - A_{FY}X, N). \end{aligned}$$

and for any $Z \in \Gamma(L_0)$ we get

$$\begin{aligned} g([X,Y], Z) &= g(F[X,Y], FZ) \\ &= g(\widetilde{\nabla}_X FY - \widetilde{\nabla}_Y FX, FZ) \\ &= g(A_{FX}Y - A_{FY}X, FZ). \end{aligned}$$

From (4.16) ve (4.19) we know that

$$\widetilde{A}_{FY}X = A_{FY}X.$$

Thus we get $(i) \Leftrightarrow (ii)$.

From (4.27) we know that $E_1^*(X,Y) = E_1(X,Y)$ *and since teorem (2.1), we get* $(i) \Leftrightarrow (iii)$.

For any $X, Y, Z \in \Gamma(L^\perp)$ from (2.15) and (4.22) we obtain

$$(\nabla_X g)(Y,Z) = 0. \tag{4.36}$$

and

$$(D_X g)(Y,Z) \quad = 0.$$

Thus we have the following proposition

Proposition 4.6. *Let* M *be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold* $\left(\widetilde{M}, \widetilde{g}\right)$. *Then we have*

$$\nabla_X g = 0 \text{ and } D_X g = 0, \; for \text{ any } X, Y \in \Gamma(L^\perp).$$

Corollary 4.1. *Let* M *be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold* $\left(\widetilde{M}, \widetilde{g}\right)$. *Then the following assertions are equivalent:*

(i) $\widetilde{D}_i(X,Y) = D_i(X,Y), i = 1,2, \; X, Y \in \Gamma(L)$

(ii) $\widetilde{D}_1$ *and* $\widetilde{D}_2$ *is symmetric on* L.

(iii) *If* M *is* $L-$ *totally geodesic then* M *is* $L-$ *totally geodesic with respect to quarter symmetric non-metric connection.*

(iv) *If* M *is* $L-$ *totally umbilic then* M *is* $L-$ *totally umbilic with respect to quarter symmetric non-metric connection.*

Proof. *For any* $X, Y \in \Gamma(L)$
since $w_1(X) = 0 = w_2(X)$, *we obtain*

$$\widetilde{D}_1(X,Y) = D_1(X,Y),$$

$$\widetilde{D}_2(X,Y) = D_2(X,Y).$$

Thus proof is completed.

Theorem 4.6. *Let* M *be a mixed geodesic semi-invariant half-lightlike submanifold of a screen semi-Riemannian product manifold* $\left(\widetilde{M}, \widetilde{g}\right)$. *Then for any* $X \in \Gamma(L)$ *and* $Y \in \Gamma(L^\perp)$ *we have*

$$\widetilde{D}_i(X,Y) = 0, \; i = 1,2.$$

Proof. *For any* $X \in \Gamma(L)$ *and* $Y \in \Gamma(L^\perp)$ *we obtain*

$$\widetilde{D}_1(X,Y) = \widetilde{g}(\widetilde{D}_X Y, \xi) = \widetilde{g}(\widetilde{\nabla}_X Y, \xi) = D_1(X,Y),$$

and

$$\widetilde{D}_2(X,Y) = \widetilde{g}(\widetilde{D}_X Y, u) = \widetilde{g}(\widetilde{\nabla}_X Y, u) = D_2(X,Y).$$

thus proof is completed.

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ON THE STRICTION CURVES OF INVOLUTIVE FRENET RULED SURFACES IN $\mathbb{E}^3$

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Abstract. In this article we conceive eight ruled surfaces related to the evolute curve α and involute α^*. They are called as Frenet ruled surface and involutive Frenet ruled surfaces, cause of their generators are Frenet vector fields of evolute curve α. First we give tangent vector fields of striction curves of all Frenet ruled surfaces and the tangent vector fields of striction curves of involutive Frenet ruled surfaces are given according to Frenet apparatus of evolute curve α. Further we give only one matrix in which we can see sixteen position of these tangent vector fields, such that we can say there is six position the tangent vector fields are perpendicular.

1. General Information

Deriving curves based on the other curves is a subject in geometry. Bertrand curves, involute-evolute curves are this kind of curves. By using the analogous means we generate ruled surface based on the other ruled surface. The properties of the B-scroll are also examined in Euclidean 3-space, Lorentzian 3-space and n-space with time-like directrix curve and null rulings (see [2], [5], [6]). Differential geometric elements of the *involute* $\tilde{D}$ *scroll* are examined in [10]. Let Frenet vector fields be $V_1(s), V_2(s), V_3(s)$ of α and let first and second curvatures of the curve $\alpha(s)$ be $k_1(s)$ and $k_2(s)$, respectively. The quantities $\{V_1, V_2, V_3, k_1, k_2\}$ are Frenet-Serret elements of the curves. Frenet formulae are,

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{1.1}$$

The Darboux vector makes a path of curvature k_1 and torsion k_2, curvature is the measuring of the rotation of the Frenet frame on the binormal unit vector, and torsion is the measurement of the rotation of the Frenet frame on the tangent unit

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vector. For any unit speed curve α, according to the Frenet-Serret elements, the Darboux vector can be defined

$$D(s) = k_2(s)V_1(s) + k_1(s)V_3(s) \tag{1.2}$$

where curvature functions are defined by $k_1(s) = \|V_1(s)\|$ and $k_2(s) = -\left\langle V_2, \dot{V}_3 \right\rangle$. The Darboux vector field of α and it has the bellowing symmetrical properties, [3].

$$\tilde{D}(s) = \frac{k_2}{k_1}(s)V_1(s) + V_3(s) \tag{1.3}$$

throughout $\alpha(s)$ under the condition that $k_1(s) \neq 0$ and it is called the modified Darboux vector field of α [8].
Let unit speed regular curve $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be given. For $\forall s \in I$, then the curve α^* is called the involute of the curve α, if the tangent at the point $\alpha(s)$ to the curve α passes through the tangent at the point $\alpha^*(s)$ to the curve α^*, then we can write that

$$\alpha^*(s) = \alpha(s) + (c - s)V_1(s), c = const.$$

The distance between corresponding points of the involute curve in $\mathbb{E}^3$ is $d\big(\alpha(s), \alpha^*(s)\big) = |c - s|$, c is constant , $\forall s \in I$, ([4],[9]). The Frenet vector fields of the *involute* α^*, based on the its evolute curve α are

$$\begin{cases} V_1^* = V_2, \\ V_2^* = \frac{-k_1}{(k_1^2+k_2^2)^{\frac{1}{2}}}V_1 + \frac{k_2}{(k_1^2+k_2^2)^{\frac{1}{2}}}V_3 \\ V_3^* = \frac{k_2}{(k_1^2+k_2^2)^{\frac{1}{2}}}V_1 + \frac{k_1}{(k_1^2+k_2^2)^{\frac{1}{2}}}V_3 \end{cases} \tag{1.4}$$

and

$$\tilde{D}^* = \frac{k_2}{(k_1^2+k_2^2)^{\frac{1}{2}}}V_1 - \frac{k_1'k_2 - k_1k_2'}{(k_1^2+k_2^2)^{\frac{3}{2}}}V_2 + \frac{k_1V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}. \tag{1.5}$$

The first curvature and second curvature of *involute* α^* are, respectively [9],

$$k_1^* = \frac{\sqrt{k_1^2+k_2^2}}{(c-s)k_1}, \quad k_2^* = \frac{-k_2^2\left(\frac{k_1}{k_2}\right)'}{(c-s)k_1\,(k_1^2+k_2^2)}. \tag{1.6}$$

Since $\eta = k_1^2 + k_2^2 \neq 0$, and $\mu = \left(\frac{k_2}{k_1}\right)'$, we have

$$\eta^* = k_1^{*2} + k_2^{*2} = \left(\frac{\sqrt{k_1^2+k_2^2}}{\lambda k_1}\right)^2 + \left(\frac{k_2'k_1 - k_1'k_2}{\lambda k_1\,(k_1^2+k_2^2)}\right)^2 = \frac{\eta^3 + k_1^4\mu^2}{\lambda^2\eta^2k_1^2}, \tag{1.7}$$

$$\mu^* = \left(\frac{k_2^*}{k_1^*}\right)'\frac{ds}{ds^*} = \frac{\frac{k_2'k_1-k_1'k_2}{\lambda k_1(k_1^2+k_2^2)}}{\frac{\sqrt{k_1^2+k_2^2}}{\lambda k_1}}\frac{1}{\lambda k_1} = \frac{k_2'k_1 - k_1'k_2}{\lambda k_1\,(k_1^2+k_2^2)^{\frac{3}{2}}} = \frac{\mu k_1}{\lambda\eta^{\frac{3}{2}}}, \tag{1.8}$$

$$\left(\frac{k_1^*}{\eta^*}\right)' = \left(\frac{\frac{\sqrt{k_1^2+k_2^2}}{\lambda k_1}}{\frac{(k_1^2+k_2^2)^3+(k_2'k_1-k_1'k_2)^2}{\lambda^2k_1^2(k_1^2+k_2^2)^2}}\right)'\frac{1}{\lambda k_1} = \left(\frac{\eta^{\frac{5}{2}}\lambda k_1}{\eta^3+k_1^2\mu}\right)'\frac{1}{\lambda k_1}. \tag{1.9}$$

A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation ,

$$\varphi(s,v)=\alpha(s)+vx(s), \tag{1.10}$$

where α and x are curves in $\mathbb{E}^3$. We call φ a ruled patch. The curve α is called the directrix or base curve of the ruled surface, and x is called the director curve, [1]. The striction point on a ruled surface is the foot of the common normal between two consecutive generators. The set of striction points defines the striction curve given by [1]

$$c(s)=\alpha(s)-\frac{\langle \alpha_s, x_s\rangle}{\langle x_s, x_s\rangle}x(s). \tag{1.11}$$

2. On the striction curves of Involutive Frenet ruled surfaces in $\mathbb{E}^3$

Theorem 2.1. *The striction curves of Frenet ruled surfaces are,* [7]

$$\begin{bmatrix} c_1-\alpha \\ c_2-\alpha \\ c_3-\alpha \\ c_4-\alpha \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1}{k_2^2+k_2^2} & 0 \\ 0 & 0 & 0 \\ \frac{-k_2}{k_1\left(\frac{k_2}{k_1}\right)'} & 0 & \frac{-1}{\left(\frac{k_2}{k_1}\right)'} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{2.1}$$

Theorem 2.2. *Tangent vector fields* T_1, T_2, T_3, *and* T_4 *of striction curves along Frenet ruled surface are given by*

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ \frac{k_2^2}{\eta\|c_2'(s)\|} & \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|} & \frac{k_1k_2}{\eta\|c_2'(s)\|} \\ 1 & 0 & 0 \\ \frac{\mu-\mu'-\frac{k_2}{k_1}}{\mu\|c_4'(s)\|} & 0 & \frac{\mu'}{\mu^2\|c_4'(s)\|} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where $k_1^2+k_2^2=\eta$, $\left(\frac{k_2}{k_1}\right)'=\mu$.

Proof. It is given this matrix, so we get equalyties as follows:

$$T_1(s)=T_3(s)=\alpha'(s)=V_1$$

Since $c_2(s)=\alpha(s)+\frac{k_1}{k_1^2+k_2^2}V_2$ and

$$T_2(s)=\frac{k_2^2}{(k_1^2+k_2^2)\|c_2'(s)\|}V_1+\frac{\left(\frac{k_1}{\eta}\right)'}{(k_1^2+k_2^2)\|c_2'(s)\|}V_2+\frac{k_1k_2}{(k_1^2+k_2^2)\|c_2'(s)\|}V_3.$$

Also

$$\begin{aligned} T_4(s) &= \frac{\left(\left(\frac{k_2}{k_1}\right)'\right)^2-\left(\frac{k_2}{k_1}\right)'\left(\frac{k_2}{k_1}\right)''-\frac{k_2}{k_1}\left(\frac{k_2}{k_1}\right)'}{\left(\left(\frac{k_2}{k_1}\right)'\right)^2\|c_4'(s)\|}V_1-\frac{-1\left(\frac{k_2}{k_1}\right)''}{\left(\left(\frac{k_2}{k_1}\right)'\right)^2\|c_4'(s)\|}V_3, \\ T_4(s) &= \frac{\mu^2-\mu\mu'-\frac{k_2}{k_1}\mu}{\mu^2\|c_4'(s)\|}V_1+\frac{\mu'}{\mu^2\|c_4'(s)\|}V_3. \end{aligned}$$

□

Definition 2.1. Let $\alpha^*(s)$ be involute of $\alpha(s)$ with arc-lenght parameter s. The equations

$$\begin{cases} \varphi_1^*(s,v_1) = \alpha^*(s) + v_1 V_1^*(s) \\ \varphi_2^*(s,v_2) = \alpha^*(s) + v_2 V_2^*(s) \\ \varphi_3^*(s,v_3) = \alpha^*(s) + v_3 V_3^*(s) \\ \varphi_4^*(s,v_4) = \alpha^*(s) + v_4 \tilde{D}^*(s) \end{cases}$$

are the parametrization of Frenet ruled surface of involute curve $\alpha^*(s)$.

The above definition can be written as follows.

$$\begin{cases} \varphi_1^*(s,v_1) = \alpha(s) + (\sigma - s)V_1(s) + v_1 V_2(s), \\ \varphi_2^*(s,v_2) = \alpha(s) + (\sigma - s)V_1(s) + v_2 \left(\frac{-k_1 V_1 + k_2 V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}} \right), \\ \varphi_3^*(s,v_3) = \alpha(s) + (\sigma - s)V_1(s) + v_3 \left(\frac{k_2 V_1 + k_1 V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}} \right), \\ \varphi_4^*(s,v_4) = \alpha(s) + (\sigma - s)V_1(s) \\ \qquad + v_4 \left(\frac{k_2}{\sqrt{k_1^2+k_2^2}} V_1 - \frac{k_1' k_2 - k_1 k_2'}{(k_1^2+k_2^2)^{\frac{3}{2}}} V_2 + \frac{k_1 V_3}{\sqrt{k_1^2+k_2^2}} \right) \end{cases}$$

Theorem 2.3. *The equations of the striction curves of involutive Frenet ruled surfaces on the evolute curve α according to Frenet elements of evolute curve α,* [7]

$$\begin{bmatrix} c_1^* - \alpha \\ c_2^* - \alpha \\ c_3^* - \alpha \\ c_4^* - \alpha \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ \lambda\left(1 - \frac{k_1^2}{\eta(1+m)}\right) & 0 & \lambda \frac{k_1 k_2}{\eta(1+m)} \\ \lambda & 0 & 0 \\ \lambda - \frac{k_2}{m' \ \eta^{\frac{1}{2}}} & -\frac{m}{m'} & \frac{k_1}{m' \eta^{\frac{1}{2}}} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{2.2}$$

Theorem 2.4. *Tangent vector fields ${T_1}^*, {T_2}^*, {T_3}^*, {T_4}^*$ of striction curves of involutive Frenet ruled surface according to Frenet elements by themselves are given by*

$$\begin{bmatrix} {T_1}^* \\ {T_2}^* \\ {T_3}^* \\ {T_4}^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{-b^* k_1 + c^* k_2}{(k_1^2 + k_2^2)^{\frac{1}{2}}} & a^* & \dfrac{b^* k_2 + c^* k_1}{(k_1^2 + k_2^2)^{\frac{1}{2}}} \\ 0 & 1 & 0 \\ \dfrac{e^* k_2}{(k_1^2 + k_2^2)^{\frac{1}{2}}} & d^* & \dfrac{e^* k_1}{(k_1^2 + k_2^2)^{\frac{1}{2}}} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{2.3}$$

where

$$\begin{aligned} a^* &= \frac{{k_2^*}^2}{\eta^* \left\| {c_2^*}'(s) \right\|}, \quad b^* = \frac{\left(\frac{k_1^*}{\eta^*}\right)'}{\left\| {c_2^*}'(s) \right\|}, \quad c^* = \frac{k_1^* k_2^*}{\eta^* \left\| {c_2^*}'(s) \right\|} \\ d^* &= \frac{\mu^* - {\mu^*}' - \frac{k_2^*}{k_1^*}}{\mu^* \left\| {c_4^*}'(s) \right\|}, \quad e^* = \frac{{\mu^*}'}{{\mu^*}^2 \left\| {c_4^*}'(s) \right\|} \end{aligned}$$

and ${k_1^*}^2 + {k_2^*}^2 = \eta^*$, $\left(\frac{k_2^*}{k_1^*}\right)' = \mu^*$.

Proof. Tangent vector fields ${T_1}^*, {T_2}^*, {T_3}^*, {T_4}^*$ of striction curves of involutive Frenet ruled surface matrix form as follows;

$$\begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \\ T_4^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d^* & 0 & e^* \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}.$$

In the above matrix by using the equation (1.2), we can write

$$\begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \\ T_4^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d^* & 0 & e^* \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{-k_1}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} & 0 & \frac{k_2}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} \\ \frac{k_2}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} & 0 & \frac{k_1}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

or

$$\begin{bmatrix} {T_1}^* \\ {T_2}^* \\ {T_3}^* \\ {T_4}^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{-b^*k_1 + c^*k_2}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} & a^* & \dfrac{b^*k_2 + c^*k_1}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} \\ 0 & 1 & 0 \\ \dfrac{e^*k_2}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} & d^* & \dfrac{e^*k_1}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

□

Theorem 2.5. *The product of tangent vector fields* $T_1^*,\ T_2^*,\ T_3^*,\ T_4^*$ *and tangent vector fields* $T_1,\ T_2,\ T_3,\ T_4,$ *of striction curves belonging to Frenet ruled surfaces and involutive Frenet ruled surfaces are given by,*

$$[T]\,[T^*]^{\mathbf{T}} = \frac{1}{\eta^{\frac{1}{2}}} \begin{bmatrix} 0 & -k_1b^* + k_2c^* & 0 & k_2e^* \\ b\eta^{\frac{1}{2}} & X & b\eta^{\frac{1}{2}} & b\eta^{\frac{1}{2}}d^* + (ak_2+ck_1)\,e^* \\ 0 & -k_1b^* + k_2c^* & 0 & k_2e^* \\ 0 & Y & 0 & e^*\,(dk_2+ek_1) \end{bmatrix} \tag{2.4}$$

where $X = b\eta^{\frac{1}{2}}a^* + (-ak_1+ck_2)\,b^* + (ak_2+ck_1)\,c^*$ *and* $Y = b^*\,(-dk_1+ek_2) + c^*\,(dk_2+ek_1)$

Proof. By using matrices (2.3) and (2.4), we can write

$$\begin{aligned}
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}\begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \\ T_4^* \end{bmatrix}^{\mathbf{T}} &= \begin{bmatrix} 1 & 0 & 0 \\ a & b & c \\ 1 & 0 & 0 \\ d & 0 & e \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d* & 0 & e* \end{bmatrix}\begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}\right)^{\mathbf{T}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ a & b & c \\ 1 & 0 & 0 \\ d & 0 & e \end{bmatrix}\left(\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}\begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}^{\mathbf{T}}\right)\begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d* & 0 & e* \end{bmatrix}^{\mathbf{T}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ a & b & c \\ 1 & 0 & 0 \\ d & 0 & e \end{bmatrix}\left(\frac{1}{\eta^{\frac{1}{2}}}\begin{bmatrix} 0 & -k_1 & k_2 \\ \eta^{\frac{1}{2}} & 0 & 0 \\ 0 & k_2 & k_1 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d* & 0 & e* \end{bmatrix}^{\mathbf{T}} \\
&= \frac{1}{\eta^{\frac{1}{2}}}\begin{bmatrix} 0 & -k_1 b^* + k_2 c^* & 0 & k_2 e^* \\ b\eta^{\frac{1}{2}} & X & b\eta^{\frac{1}{2}} & b\eta^{\frac{1}{2}} d^* + (ak_2 + ck_1)\, e^* \\ 0 & -k_1 b^* + k_2 c^* & 0 & k_2 e^* \\ 0 & Y & 0 & e^*\,(dk_2 + ek_1) \end{bmatrix}.
\end{aligned}$$

□

The position of the unit tangent vector field $T_1^*, T_2^*, T_3^*, T_4^*$ of ruled surfaces $\varphi_1^*, \varphi_2^*, \varphi_3^*, \varphi_4^*$, respectively, on the curve α^*, can be expressed by the bellowing matrix;

$$[T]\,[T^*]^{\mathbf{T}} = \begin{bmatrix} \langle T_1, T_1^* \rangle & \langle T_1, T_2^* \rangle & \langle T_1, T_3^* \rangle & \langle T_1, T_4^* \rangle \\ \langle T_2, T_1^* \rangle & \langle T_2, T_2^* \rangle & \langle T_2, T_3^* \rangle & \langle T_2, T_4^* \rangle \\ \langle T_3, T_1^* \rangle & \langle T_3, T_2^* \rangle & \langle T_3, T_3^* \rangle & \langle T_3, T_4^* \rangle \\ \langle T_4, T_1^* \rangle & \langle T_4, T_2^* \rangle & \langle T_4, T_3^* \rangle & \langle T_4, T_4^* \rangle \end{bmatrix}, \tag{2.5}$$

here $[T^*]^{\mathbf{T}}$ is the tranpose matrix of $[T^*]$.

The six pairs of Frenet ruled surface and involutive Frenet ruled surface have striction curves with orthogonal tangent vector fields, these are

Tangent and involutive tangent ruled surfaces of the α,
involutive binormal and tangent ruled surface of the α,
involutive tangent and binormal ruled surface of the α,
Binormal and involutive binormal ruled surfaces of the α,
Darboux and involutive tangent ruled surfaces of an α,
Darboux and involutive binormal ruled surfaces of an α.

Theorem 2.6. *Tangent vector fields of striction curves on tangent ruled surface and involutive normal ruled surface and binormal ruled surface have orthogonal under the condition are* $\dfrac{k_2}{k_1} = \dfrac{\left(\frac{k_1^*}{\eta^*}\right)' \eta^*}{k_1^* k_2^*}$.

Proof. Since the equations (2.4) and (2.5), we have

$$\langle T_1, T_2^* \rangle = \langle T_3, T_2^* \rangle = \frac{-k_1 b^* + k_2 c^*}{\eta^{\frac{1}{2}}} = 0 \Longrightarrow \frac{k_2}{k_1} = \frac{\left(\frac{k_1^*}{\eta^*}\right)' \eta^*}{k_1^* k_2^*},$$

this completes the proof. □

Theorem 2.7. *Tangent vector fields of striction curves on tangent ruled surface and binormal ruled surface and involutive Darboux ruled surface have orthogonal under the condition are* $\dfrac{k_2^{'}k_1 - k_1^{'}k_2}{\lambda k_1\left(k_1^2+k_2^2\right)^{\frac{3}{2}}} = constant.$

Proof. From the equations (2.4) and (2.5), we have

$$\begin{aligned}
\langle T_1, T_4^* \rangle &= \langle T_3, T_4^* \rangle = \frac{1}{\eta^{\frac{1}{2}}} k_2 e^* = 0 \Longrightarrow k_2 e^* = 0, k_2 \neq 0 \\
e^* &= 0 \Longrightarrow (\mu^*)' = 0 \Longrightarrow \frac{k_2^{'}k_1 - k_1^{'}k_2}{\lambda k_1\left(k_1^2+k_2^2\right)^{\frac{3}{2}}} = const.,
\end{aligned}$$

this completes the proof. □

Theorem 2.8. ***i)*** *Tangent vector fields of striction curves on normal and involutive tangent ruled surfaces have orthogonal under the condition are* $\left(\frac{k_1}{k_1^2+k_2^2}\right)' = 0.$
ii) *Tangent vector fields of striction curves on normal and involutive binormal ruled surfaces have orthogonal under the condition are* $\left(\frac{k_1}{k_1^2+k_2^2}\right)' = 0.$

Proof. **i)** By using the equations (2.4) and (2.5), we can write

$$\langle T_2, T_1^* \rangle = b = \frac{\left(\frac{k_1}{k_1^2+k_2^2}\right)'}{\|c_2'(s)\|} = 0 \Longrightarrow \left(\frac{k_1}{k_1^2+k_2^2}\right)' = 0,$$

this completes the proof.
ii) Since $\langle T_2, T_3^* \rangle = b$, it is trivial. □

Theorem 2.9. *Tangent vector fields of striction curves along normal and involutive normal ruled surfaces are orthogonal under the condition*

$$b\eta^{\frac{1}{2}} a^* + (-ak_1 + ck_2)\, b^* + (ak_2 + ck_1)\, c^* = 0.$$

Proof. Since the equations (2.4) and (2.5), we have

$$\langle T_2, T_2^* \rangle = \frac{X}{\eta^{\frac{1}{2}}} = 0 \Longrightarrow X = b\eta^{\frac{1}{2}} a^* + (-ak_1 + ck_2)\, b^* + (ak_2 + ck_1)\, c^* = 0,$$

this completes the proof. □

Theorem 2.10. *Tangent vector fields of striction curves along normal and involutive Darboux ruled surfaces are orthogonal under the condition*

$$b\eta^{\frac{1}{2}} d^* + (ak_2 + ck_1)\, e^* = 0.$$

Proof. Since $\langle T_2, T_4^* \rangle = \dfrac{b\eta^{\frac{1}{2}} d^* + (ak_2 + ck_1)\, e^*}{\eta^{\frac{1}{2}}}$ in the equations (2.4) and (2.5) and under the orthogonality condition $b\eta^{\frac{1}{2}} d^* + (ak_2 + ck_1)\, e^* = 0.$ □

Theorem 2.11. *Tangent vector fields of striction curves along Darboux ruled surface and involutive normal ruled surface are orthogonal under the condition*

$$\frac{k_1}{k_2} = \frac{(dc^* + eb^*)}{(db^* - ec^*)}.$$

Proof. Since the equations (2.4) and (2.5), we have

$$\begin{aligned}\langle T_4, T_2^* \rangle &= \frac{Y}{\eta^{\frac{1}{2}}} = 0 \Longrightarrow Y = b^* \left(-dk_1 + ek_2\right) + c^* \left(dk_2 + ek_1\right) = 0 \\ &\Longrightarrow \frac{k_1}{k_2} = \frac{(dc^* + eb^*)}{(db^* - ec^*)},\end{aligned}$$

this completes the proof. □

Theorem 2.12. *Tangent vector fields of striction curves on involutive Darboux ruled surface and Darboux ruled surface are orthogonal under the condition* $(dk_2 + ek_1) = 0$ *or* $\left(\frac{k_2^*}{k_1^*}\right)' = const.$

Proof. By using the equations (2.4) and (2.5), we can write

$$\begin{aligned}\langle T_4, T_4^* \rangle &= \frac{e^* \left(dk_2 + ek_1\right)}{\eta^{\frac{1}{2}}} = 0 \Longrightarrow (dk_2 + ek_1) = 0 \;\; or e^* = 0 \\ e^* &= 0 \Longrightarrow \mu^* = const. \Longrightarrow \left(\frac{k_2^*}{k_1^*}\right)' = const.\end{aligned}$$

this completes the proof. □

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SPHERICAL PRODUCT SURFACES IN THE GALILEAN SPACE

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Abstract. In the present paper, we consider the spherical product surfaces in a Galilean 3-space $\mathbb{G}_3$. We derive a classification result for such surfaces of constant curvature in $\mathbb{G}_3$. Moreover, we analyze some special curves on these surfaces in $\mathbb{G}_3$.

1. Introduction

The tight embeddings of product spaces were investigated by N.H. Kuiper (see [17]) and he introduced a different tight embedding in the (n_1+n_2-1) −dimensional Euclidean space $\mathbb{R}^{n_1+n_2-1}$ as follows: Let

$$\begin{aligned} c_1 &: M^m \longrightarrow \mathbb{R}^{n_1}, \\ c_1(u_1,...,u_m) &= (f_1(u_1,...,u_m),...,f_{n_1}(u_1,...,u_m)) \end{aligned}$$

be a tight embedding of a $m-$dimensional manifold M^m satisfying Morse equality and

$$\begin{aligned} c_2 &: \mathbb{S}^{n_2-1} \longrightarrow \mathbb{R}^{n_2}, \\ c_1(v_1,...,v_{n_2-1}) &= (g_1(v_1,...,v_{n_2-1}),...,g_{n_2}(v_1,...,v_{n_2-1})) \end{aligned}$$

the standard embedding of (n_2-1) −sphere in $\mathbb{R}^{n_2}$, where $u=(u_1,...,u_m)$ and $v=(v_1,...,v_{n_2-1})$ are the local coordinate systems on M^m and $\mathbb{S}^{n_2-1}$, respectively. Then a new *tight embedding* is given by

$$\begin{gathered} \mathbf{x}=c_1\otimes c_2 : M^m\times\mathbb{S}^{n_2-1}\longrightarrow\mathbb{R}^{n_1+n_2-1}, \\ (u,v)\longmapsto(f_1(u),...,f_{n_1-1}(u),f_{n_1}(u)g_1(v),...,f_{n_1}(u)g_{n_2}(v)). \end{gathered}$$

Such embeddings are obtained from c_1 by rotating $\mathbb{R}^{n_1}$ about $\mathbb{R}^{n_1-1}$ in $\mathbb{R}^{n_1+n_2-1}$ (cf. [4]).

B. Bulca et al. [6, 7] called such embeddings *rotational embeddings* and considered the spherical product surfaces in Euclidean spaces, which are a special type

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of the rotational embeddings as taking $m = 1, n_1 = 2, 3$ and $n_2 = 2$ in above definition.

The surfaces of revolution in $\mathbb{R}^3$ can be considered as simplest models of spherical product surfaces as well as the quadrics and the superquadrics [5].

On the other hand, the Galilean geometry is one model of the real Cayley-Klein geometries which has projective signature $(0, 0, +, +)$. In particular, the Galilean plane $\mathbb{G}_2$ is one of three Cayley-Klein planes (including Euclidean and Lorentzian planes) with a parabolic measure of distance. This projective-metric plane has an absolute figure $\{f, P\}$ for an absolute (ideal) line f and an absolute point P on f.

Many kind of surfaces in the (pseudo-) Galilean 3-space $\mathbb{G}_3$ (further details of $\mathbb{G}_3$ see Section 2) have been studied in [3], [8]-[10], [15, 16], [22]-[28] such as ruled surfaces, translation surfaces, tubular surfaces, etc.

In the present paper, we consider the spherical product surfaces of two Galilean plane curves in $\mathbb{G}_3$. We obtain several classifications for the spherical product surfaces of constant curvature in $\mathbb{G}_3$. Then some special curves on such surfaces are also analyzed.

2. Preliminaries

For later use, we provide a brief review of Galilean geometry from [12, 13], [18]-[28].

The Galilean 3-space $\mathbb{G}_3$ can be defined in three-dimensional real projective space $P_3(\mathbb{R})$ and its *absolute figure* is an ordered triple $\{\omega, f, I\}$, where ω is the ideal (absolute) plane, f a line in ω and I is the fixed elliptic involution of the points of f. The homogeneous coordinates in $\mathbb{G}_3$ is introduced in such a way that the ideal plane ω is given by $x_0 = 0$, the ideal line f by $x_0 = x_1 = 0$ and the elliptic involution by

$$(0 : 0 : x_2 : x_3) \longrightarrow (0 : 0 : x_3 : -x_2).$$

By means of the affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, the *similarity group* H_8 of $\mathbb{G}_3$ has the following form

$$\begin{aligned} \bar{x} &= a + bx \\ \bar{y} &= c + dx + r(\cos\theta)\,y + r(\sin\theta)\,z \\ \bar{z} &= e + fx + r(-\sin\theta)\,y + r(\cos\theta)\,z, \end{aligned}$$

where a, b, c, d, e, f, r and θ are real numbers. In particular, for $b = r = 1$, the group becomes the *group of isometries* (*proper motions*), $B_6 \subset H_8$, of $\mathbb{G}_3$.

A plane is called *Euclidean* if it contains f, otherwise it is called *isotropic*, i.e., the planes $x = const.$ are Euclidean, in particular the plane ω. Other planes are isotropic.

We introduce the metric relations with respect to the absolute figure. The *Galilean distance* between the points $P_i = (u_i, v_i, w_i)$ $(i = 1, 2)$ is given by

$$d(P_1, P_2) = \begin{cases} |u_2 - u_1|, & \text{if } u_1 \neq 0 \text{ or } u_2 \neq 0, \\ \sqrt{(v_2 - v_1)^2 + (w_2 - w_1)^2}, & \text{if } u_1 = 0 \text{ and } u_2 = 0. \end{cases}$$

The *Galilean scalar product* between two vectors $\mathbf{X} = (x_1, x_2, x_3)$ and $\mathbf{Y} = (y_1, y_2, y_3)$ is given by

$$\mathbf{X} \cdot \mathbf{Y} = \begin{cases} x_1 y_1, & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0, \\ x_2 y_2 + x_3 y_3, & \text{if } x_1 = 0 \text{ and } y_1 = 0. \end{cases}$$

In this sense, the *Galilean norm* of a vector $\mathbf{X}$ is $\|\mathbf{X}\| = \sqrt{\mathbf{X}\cdot\mathbf{X}}$. A vector $\mathbf{X} = (x_1, x_2, x_3)$ is called *isotropic* if $x_1 = 0$, otherwise it is called *non-isotropic*.

The *cross product* in the sense of Galilean space is

$$\mathbf{X} \times_{\mathbb{G}} \mathbf{Y} = \left(0, -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right).$$

Let D be an open subset of $\mathbb{R}^2$ and M^2 a surface in $\mathbb{G}_3$ parametrized by

$$\mathbf{r} : D \longrightarrow \mathbb{G}_3, \ \ (u_1, u_2) \longmapsto (r_1(u_1, u_2), r_2(u_1, u_2), r_3(u_1, u_2)),$$

where r_k is a smooth real-valued function on D, $1 \le k \le 3$. Denote

$$(r_k)_{u_i} = \partial r_k / \partial u_i \text{ and } (r_k)_{u_i u_j} = \partial^2 r_k / \partial u_i \partial u_j, \ 1 \le k \le 3 \text{ and } 1 \le i, j \le 2.$$

Then such a surface is *admissible* (i.e., without Euclidean tangent planes) if and only if $(r_1)_{u_i} \neq 0$ for some $i = 1, 2$.

Let us introduce

$$g_i = (r_1)_{u_i}, \ h_{ij} = (r_2)_{u_i}(r_2)_{u_j} + (r_3)_{u_i}(r_3)_{u_j}, \ i, j = 1, 2.$$

Hence the first fundamental form of M^2 is

$$\mathbf{I} = ds_1^2 + \varepsilon ds_2^2,$$

where

$$ds_1^2 = (g_1 du_1 + g_2 du_2)^2, \ \ ds_2^2 = h_{11} du_1^2 + 2h_{12} du_1 du_2 + h_{22} du_2^2$$

and

$$\varepsilon = \begin{cases} 0 & \text{if the direction } du_1 : du_2 \text{ is non-isotropic,} \\ 1 & \text{if the direction } du_1 : du_2 \text{ is isotropic.} \end{cases}$$

Define the function w as

$$w = \sqrt{\left((r_1)_{u_2}(r_3)_{u_1} - (r_1)_{u_1}(r_3)_{u_2}\right)^2 + \left((r_1)_{u_1}(r_2)_{u_2} - (r_1)_{u_2}(r_2)_{u_1}\right)^2}.$$

Thus a side tangential vector $\mathbf{S}$ in the tangent plane of M^2 is defined by

$$\mathbf{S} = \frac{1}{w}\left(0, (r_1)_{u_2}(r_2)_{u_1} - (r_1)_{u_1}(r_2)_{u_2}, (r_1)_{u_2}(r_3)_{u_1} - (r_1)_{u_1}(r_3)_{u_2}\right). \tag{2.1}$$

The unit normal vector field $\mathbf{U}$ of M^2 is an isotropic vector field given by

$$\mathbf{U} = \frac{1}{w}\left(0, (r_1)_{u_2}(r_3)_{u_1} - (r_1)_{u_1}(r_3)_{u_2}, (r_1)_{u_1}(r_2)_{u_2} - (r_1)_{u_2}(r_2)_{u_1}\right). \tag{2.2}$$

In the sequel, the second fundamental form $\mathbf{II}$ of M^2 is

$$\mathbf{II} = L_{11} du_1^2 + 2L_{12} du_1 du_2 + L_{22} du_2^2,$$

where

$$\begin{aligned} L_{ij} &= \frac{1}{g_1}\left(g_1\left(0, (r_2)_{u_i u_j}, (r_3)_{u_i u_j}\right) - (g_i)_{u_j}\left(0, (r_2)_{u_1}, (r_3)_{u_1}\right)\right) \cdot \mathbf{U} \\ &= \frac{1}{g_2}\left(g_2\left(0, (r_2)_{u_i u_j}, (r_3)_{u_i u_j}\right) - (g_i)_{u_j}\left(0, (r_2)_{u_2}, (r_3)_{u_2}\right)\right) \cdot \mathbf{U}. \end{aligned}$$

A surface is called *totally geodesic* if its second fundamental form is identically zero.

The *third fundamental form* of M^2 is

$$\mathbf{III} = P_{11} du_1^2 + 2P_{12} du_1 du_2 + P_{22} du_2^2,$$

where

$$P_{11} = \mathbf{U}_{u_1} \cdot \mathbf{U}_{u_1}, \ P_{12} = \mathbf{U}_{u_1} \cdot \mathbf{U}_{u_2}, \ P_{22} = \mathbf{U}_{u_2} \cdot \mathbf{U}_{u_2}. \tag{2.3}$$

The *Gaussian curvature* K and the *mean curvature* H of M^2 are of the form

$$K = \frac{L_{11}L_{22} - L_{12}^2}{w^2} \text{ and } H = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{2w^2}. \tag{2.4}$$

A surface in $\mathbb{G}_3$ is said to be *minimal* (resp. *flat*) if its mean curvature (resp. Gaussian curvature) vanishes.

3. Spherical product surfaces of constant curvature in $\mathbb{G}_3$

Let $c_i : I_i \subset \mathbb{R} \longrightarrow \mathbb{G}_2$, $i = 1, 2$, be two Galilean plane curves given by

$$c_1(u) = (p_1(u), p_2(u)) \text{ and } c_2(v) = (q_1(v), q_2(v)),$$

where p_i and q_i $(i = 1, 2)$ are respectively smooth real-valued non-constant functions on the intervals I_1 and I_2. Thus the *spherical product surface* M^2 of the two plane curves in $\mathbb{G}_3$ is defined by

$$\mathbf{r} := c_1 \otimes c_2 : I_1 \times I_2 \longrightarrow \mathbb{G}_3, \quad (u, v) \longmapsto (p_1(u), p_2(u)\, q_1(v), p_2(u)\, q_2(v)). \tag{3.1}$$

We call the curves c_1 and c_2 *generating curves.* Denote $p_i' = \frac{dp_i}{du}, q_i' = \frac{dq_i}{dv}$, etc. Since p_i and q_i are non-constant, M^2 is always admissible.

It follows from (2.1), (2.2) and (3.1) that the side tangent vector field $\mathbf{S}$ is

$$\mathbf{S} = \frac{1}{\sqrt{(q_1')^2 + (q_2')^2}} (0, -q_1', -q_2') \tag{3.2}$$

and the unit normal vector field $\mathbf{U}$ becomes

$$\mathbf{U} = \frac{1}{\sqrt{(q_1')^2 + (q_2')^2}} (0, -q_2', q_1'). \tag{3.3}$$

Remark 3.1. The equality (3.3) immediately implies from (2.3) that a spherical product surface in $\mathbb{G}_3$ has degenerate third fundamental form, i.e., $P_{11}P_{22} - P_{12}^2 = 0$.

For the coefficients of the first fundamental form, we have $g_1 = p_1'$ and $g_2 = 0$. Also the coefficients of the second fundamental form are

$$L_{11} = -\frac{(p_1')(q_1)^2}{\sqrt{(q_1')^2 + (q_2')^2}} \alpha' \beta', \; L_{12} = 0, \; L_{22} = \frac{p_2(q_1')^2}{\sqrt{(q_1')^2 + (q_2')^2}} \gamma', \tag{3.4}$$

where

$$\alpha = \frac{p_2'}{p_1'}, \; \beta = \frac{q_2}{q_1}, \; \gamma = \frac{q_2'}{q_1'}. \tag{3.5}$$

Remark 3.2. It is easy to see that when c_2 is a line passing through the origin, then $\beta = const.$ and hence the spherical product surface is totally geodesic.

Therefore, the next results classify the spherical product surfaces in $\mathbb{G}_3$ with constant mean curvature and null Gaussian curvature.

Theorem 3.1. *There does not exist a spherical product surface in $\mathbb{G}_3$ with constant mean curvature except isotropic planes.*

Proof. Let M^2 be a spherical product surface given by (3.1) in $\mathbb{G}_3$ with constant mean curvature H_0. From (2.4), we have

$$2H_0 = \frac{(q_1')^2}{p_2\left((q_1')^2 + (q_2')^2\right)^{\frac{3}{2}}}\gamma'. \tag{3.6}$$

Then differentiating of (3.6) with respect to u yields that

$$0 = \frac{p_2'(q_1')^2}{-(p_2)^2\left((q_1')^2 + (q_2')^2\right)^{\frac{3}{2}}}\gamma'. \tag{3.7}$$

Since the functions p_i and q_i are non-constant functions, it follows from (3.7) that $\gamma' = 0$ and thus $H_0 = 0$. Considering $\gamma = const.$ in (3.5), then it turns to

$$q_2 = \lambda_1 q_1 + \lambda_2,\ \lambda_1 \neq 0, \tag{3.8}$$

which implies that c_2 is a line. Moreover, from (3.3), we have the constant unit normal vector field $\mathbf{U}$ as

$$\mathbf{U} = \frac{1}{\sqrt{1 + (\lambda_1)^2}}(0, -\lambda_1, 1),\ \lambda_1 \neq 0. \tag{3.9}$$

This means that the spherical product surface is an open part of an isotropic plane, which proves the theorem. □

Theorem 3.2. *A spherical product surface of the curves c_1 and c_2 in $\mathbb{G}_3$ is flat if and only if either it is an isotropic plane or the generating curve c_1 is a line.*

Proof. Assume that M^2 is a flat spherical product surface of the curves c_1 and c_2 in $\mathbb{G}_3$. For the Gaussian curvature K, by using (2.4), we get

$$0 = K = \frac{(q_1)^2 (q_1')^2}{p_1' p_2\left((q_1')^2 + (q_2')^2\right)^2}\alpha'\beta'\gamma'.$$

Thus three cases occur:

Case (A) $\alpha = const.$ Then, we deduce

$$p_1 = \lambda_3 p_2 + \lambda_4,\ \lambda_3 \neq 0,$$

which implies that c_1 is a line.

Case (B) $\beta = const.$ Hence $\frac{q_2}{q_1} = const.$ for all $v \in I_2$ and the generating curve c_2 is a line passing through the origin. This gives that M^2 is a totally geodesic surface and an open part of an isotropic plane.

Case (C) $\gamma = const.$ This case was already analyzed via (3.8) and in this case M^2 is an open part of an isotropic plane.

Therefore the proof is completed. □

By using Theorem 3.1 and Theorem 3.2, we have the following classification result.

Corollary 3.1. ***(Classification)*** *For a spherical product surface M^2 of the curves c_1 and c_2 in $\mathbb{G}_3$, the following statements hold:*

(A) If c_1 is a line, then M^2 is flat but not minimal,

(B) If c_2 is a line passing through the origin, then M^2 is a totally geodesic surface and an open part of an isotropic plane,

(C) If c_2 is a line of the form $y = mx + n$, $m, n \neq 0$, then M^2 is an open part of an isotropic plane,

(D) There does not exist a spherical product surface with constant mean curvature except isotropic planes.

Example 3.1. Let us consider the spherical product surface of the Euclidean ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and the line $y = 0.5x + 2.5$. Thus we parametrize the surface being flat but not minimal as follows

$$\mathbf{r}(u, v) = (u - 3, (0.5u + 1)(2 \sin v), (0.5u + 1)(3 \cos v)), \ 0 \leq u, v \leq 2\pi.$$

We plot it as in Fig. 1.

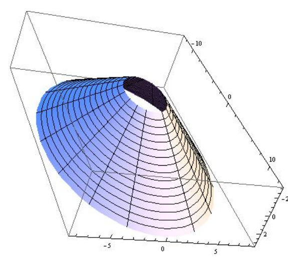

FIGURE 1. The flat spherical product surface of an Euclidean ellipse and a line, $K = 0$.

4. Curves on spherical product surfaces in $\mathbb{G}_3$

There exist a frame field, also called the *Darboux frame field,* for the curves lying on surfaces apart from the Frenet frame field. For details, see [11, 14]. Let γ be a curve lying on the surface M^2 with unit normal vector field $\mathbf{U}$. By taking $\mathbf{T} = \gamma_* \left(\frac{d}{dt}\right)$ one can get a new frame field $\{\mathbf{T}, \mathbf{T} \times \mathbf{U}, \mathbf{U}\}$ which is the Darboux frame field of γ with respect to M^2.

On the other hand, the second derivative $\ddot{\gamma}$ of the curve γ on M^2 has a component perpendicular to M^2 and a component tangent to M^2, i.e.,

$$\ddot{\gamma} = \tan(\ddot{\gamma}) + \operatorname{nor}(\ddot{\gamma}), \tag{4.1}$$

where the dot " $\cdot$ " denotes the derivative with respect to the parameter of the curve. The norms $\|\tan(\ddot{\gamma})\|$ and $\|\operatorname{nor}(\ddot{\gamma})\|$ are called the *geodesic curvature* and the *normal curvature* of γ on M^2, respectively. The curve γ is called *geodesic* (resp. *asymptotic line*) if and only if its geodesic curvature κ_g (resp. normal curvature κ_n) vanishes.

Let us consider the spherical product surface $\mathbf{r} = c_1 \otimes c_2$ in $\mathbb{G}_3$ given by (3.1). As in the previous section, put

$$c_1(u) = (p_1(u), p_2(u)) \text{ and } c_2(v) = (q_1(v), q_2(v)).$$

The geodesic curvatures of the $u-$parameter curves and $v-$parameter curves on $\mathbf{r} = c_1 \otimes c_2$ are respectively given by (see [10])

$$\kappa_g^u = \mathbf{S} \cdot \mathbf{r}_{uu} = \begin{cases} 0, & \text{if } p_1 \text{ is non-linear} \\ \frac{-p_2''\left(q_1 q_1' + q_2 q_2'\right)}{\sqrt{\left(q_1'\right)^2 + \left(q_2'\right)^2}}, & \text{if } p_1 \text{ is linear} \end{cases} \tag{4.1}$$

and

$$\kappa_g^v = \mathbf{S} \cdot \mathbf{r}_{vv} = \frac{-p_2 \left(q_1' q_1'' + q_2' q_2'' \right)}{\sqrt{\left(q_1'\right)^2 + \left(q_2'\right)^2}}. \tag{4.2}$$

By considering (4.1) and (4.2), we derive the following result.

Theorem 4.1. *Let M^2 be a spherical product surface of the curves $c_1(u) = (p_1(u), p_2(u))$ and $c_2(v) = (q_1(v), q_2(v))$ in $\mathbb{G}_3$. Then we have*

(A) If p_1 is a non-linear function, then the $u-$parameter curves are geodesic lines. Otherwise (when p_1 is a linear function) the $u-$ parameter curves are geodesic lines if and only if either

(A.1) p_2 is a linear function, or

(A.2) c_2 is an Euclidean circle.

(B) The $v-$ parameter curves are geodesic lines if and only if c_2 is curve satisfying the equation

$$q_1 = \pm \int \sqrt{\lambda_2 - \left(q_2'\right)^2} dv.$$

Proof. From (4.1), the statement (A) of the theorem is clear. Now let assume that p_1 is a linear function. Then, by (4.1), we deduce that the $u-$parameter curves are geodesic lines (i.e. κ_g^u vanishes) if and only if either p_2 is a linear function (this implies the statement (A.1) of the theorem) or

$$q_1 q_1' + q_2 q_2' = 0. \tag{4.3}$$

From (4.3), we conclude $q_1^2 + q_2^2 = \lambda_1$ for some constant $\lambda_1 > 0$. It means that c_2 is an Euclidean circle with radius $\sqrt{\lambda_1}$ and centered at origin. This proves the statement (A.2) of the theorem.

If κ_g^v is equivalently zero, then we have from (4.2) that $q_1' q_1'' + q_2' q_2'' = 0$, i.e.,

$$q_1 = \pm \int \sqrt{\lambda_2 - \left(q_2'\right)^2} dv,$$

which completes the proof. □

The normal curvatures of the parameter curves on $\mathbf{r} = c_1 \otimes c_2$ (see [10]) are respectively given by

$$\kappa_n^u = \mathbf{U} \cdot \mathbf{r}_{uu} = \begin{cases} 0, & \text{if } p_1 \text{ is non-linear} \\ \frac{-p_2'' \left(q_1 q_2' - q_1' q_2 \right)}{\sqrt{\left(q_1'\right)^2 + \left(q_2'\right)^2}}, & \text{if } p_1 \text{ is linear} \end{cases} \tag{4.4}$$

and

$$\kappa_n^v = \mathbf{U} \cdot \mathbf{r}_{vv} = \frac{p_2 \left(q_1' q_2'' - q_1'' q_2' \right)}{\sqrt{\left(q_1'\right)^2 + \left(q_2'\right)^2}}. \tag{4.5}$$

Theorem 4.2. *Let M^2 be a spherical product surface of the curves $c_1(u) = (p_1(u), p_2(u))$ and $c_2(v) = (q_1(v), q_2(v))$ in $\mathbb{G}_3$. Then we have the following:*

(A) If p_1 is a non-linear function, then the $u-$parameter curves are asymptotic lines. Otherwise (when p_1 is a linear function) the $u-$ parameter curves are asymptotic lines if and only if either

(A.1) p_2 is a linear function, or

(A.2) M^2 is a totally geodesic surface.

(B) The $v-$ parameter curves are asymptotic lines if and only if M^2 is an open part of an isotropic plane.

Proof. From (4.4), the statement (A) of the theorem is obvious. If p_1 is a linear function, then by (4.4) we derive that the $u-$parameter curves are asymptotic lines if and only if either p_2 is a linear function (it gives the proof of the statement (A.1) of the theorem), or

$$q_1 q_2' - q_1' q_2 = 0. \tag{4.6}$$

It follows from (4.6) that $q_2 = \lambda_1 q_1$ for nonzero constant λ_1. Considering Remark 3.2 implies that M^2 is totally geodesic surface, which proves the statement (A.2).

Also, in case when $v-$parameter curves are asymptotic lines, from (4.5), the following satisfies

$$q_2 = \lambda_2 q_1 + \lambda_3, \ \lambda_2 \neq 0. \tag{4.7}$$

From (3.3), the equality (4.7) implies the statement (B) of the theorem.

Thus the proof is completed. □

A curve γ on a regular surface M^2 is called a *principal curve* if and only if the its velocity vector field always points in a principal direction. Moreover, a surface M^2 is called a *principal surface* if and only if its parameter curves are principal curves (cf. [14]).

A principal curve γ on a surface in $\mathbb{G}_3$ is determined by the following formula

$$\det\left(\dot{\gamma}, \mathbf{U}, \dot{\mathbf{U}}\right) = 0, \tag{4.8}$$

where $\mathbf{U}$ is the unit normal vector field of the surface (see [10]). Considering (3.1), (3.3) and (4.8), we immediately derive

$$\det\left(\mathbf{r}_u, \mathbf{U}, \mathbf{U}_u\right) = 0 \text{ and } \det\left(\mathbf{r}_v, \mathbf{U}, \mathbf{U}_v\right) = 0,$$

which yields the following.

Corollary 4.1. *The spherical product surfaces in $\mathbb{G}_3$ are principal ones.*

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THE CHARACTERIZATIONS OF SPACELIKE CURVES IN R_1^4

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Abstract. In this paper, we study the geometry of position vectors of a spacelike curve in the Minkowski 4-space. We give some characterizations for spacelike curves to lie on some subspaces of R_1^4.

1. Introduction

The Frenet frames for spacelike, timelike and null curves have been studied and developed by several authors [5], [3], [11], [1] and [2]. A. Fernandez, A. Gimenez and P. Lucas introduced a Frenet frame with curvature functions for a null curve in a Lorentzian manifold and studied null helices in Lorentzian space forms [2]. C. Coken and U. Ciftci studied null curves in the 4-dimensional Minkowski space R_1^4 , and give some results for psoudospherical null curves and Bertrand null curves.

K. Ilarslan and O. Boyacioglu studied position vectors of a timelike and a null helice in R_1^3 [5]. K. Ilarslan and E. Nesovic gave some characterizations for null curves in R_1^4 and they obtained some relations between null normal curves and null osculating curves as well as between null rectifying curves and null osculating curves [6].

K. Ilarslan studied spacelike normal curves in Minkowski space E_1^3 and gave some characterizations of spacelike normal curves with spacelike, timelike and null principal normal [6]. K. Ilarsalan, E. Nesovic and M. Petrovic-Torgasev characterized non-null and null rectifying curves, lying fully in the Minkowski 3-space [7].

A. T. Ali and M. Onder characterize rectifying spacelike curves in terms of their curvature functions in Minkowski spacetime [3]. M. Onder, H. Kocayigit and M. Kazaz gave some characterizations for spacelike helices in Minkowski spacetime and found the differential equations characterizing the spacelike helices in Minkowski 4-space [11].

M. A. Akgun and A. I. Sivridag studied null Cartan curves in Minkowski 4-space and give some theorems for null Cartan curves to lie on some subspaces of R_1^4 [13].

M. A. Akgun and A. I. Sivridag studied spacelike and timelike curves to lie on some subspaces of R_1^4 and give some theorems in [14] and [15].

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This paper organized following: In section 2 we give some basic knowledge related with curves in Minkowski space-time. Section 3 is the original part of this paper. In this section we investigate the conditions for spacelike curves to lie on some subspaces of R_1^4 and we give some characterizations and theorems for these curves.

2. Preliminaries

Let R_1^4 denote Minkowski space together with a flat Lorentz metric $\langle , \rangle$ of signature $(-,+,+,+)$. A vector X is said to be timelike if $\langle X, X\rangle < 0$, spacelike if $\langle X, X\rangle > 0$ or $X = 0$ and null(lightlike) if $\langle X, X\rangle = 0$ and $X \neq 0$. The norm of a vector $X \in R_1^4$ is denoted by $\|X\|$ and defined by $\|X\| = \sqrt{|\langle X, X\rangle|}$.

A curve α in R_1^4 is called a null curve if $\langle \alpha'(s), \alpha'(s)\rangle = 0$ and $\alpha'(s) \neq 0$, timelike curve if $\langle \alpha'(s), \alpha'(s)\rangle < 0$ and spacelike curve if $\langle \alpha'(s), \alpha'(s)\rangle > 0$ for all $s \in R$.

Let α be a spacelike curve in R_1^4 with the Frenet frame $\{T, N, B_1, B_2\}$ and let N be null vector and B_1 be null vector. In this case there exists only one Frenet frame $\{T, N, B_1, B_2\}$ for which $\alpha(s)$ is a spacelike curve with Frenet equations

$$\begin{aligned} \nabla_T T &= k_1 N \\ \nabla_T N &= k_2 B_2 \\ \nabla_T B_1 &= -k_1 T + k_3 B_2 \\ \nabla_T B_2 &= -k_3 N - k_2 B_1 \end{aligned}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying the equations

$$\langle B_1, B_1\rangle = \langle N, N\rangle = 0, \quad \langle T, T\rangle = \langle B_2, B_2\rangle = 1, \quad \langle N, B_1\rangle = 1$$

[12]

3. The Characterizations of Spacelike Curves in R_1^4

In this section we will investigate some characterizations of spacelike curves to lie on some subspaces of R_1^4.

Let α be a spacelike curve in R_1^4 with the Frenet frame $\{T, N, B_1, B_2\}$. Then, the subspaces of R_1^4 spanned by $\{T, N\}$, $\{T, B_1\}$, $\{T, B_2\}$, $\{N, B_1\}$, $\{N, B_2\}$, $\{B_1, B_2\}$, $\{T, N, B_1\}$, $\{T, N, B_2\}$, $\{T, B_1, B_2\}$ and $\{N, B_1, B_2\}$.

Case 1) First we will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T, N\}$. In this case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)N \tag{3.1}$$

for some differentiable functions λ and μ of s, which is the arc-length parameter of $\alpha(s)$. Differentiating (3.1) with respect to s and by using the Frenet equations we find that

$$\alpha'(s) = \lambda'(s)T + (\lambda(s)k_1(s) + \mu'(s))N + \mu(s)k_2(s)B_2$$

where $\alpha' = T$. Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} \lambda'(s) = 1 \\ \lambda(s)k_1(s) + \mu'(s) = 0 \\ \mu(s)k_2(s) = 0 \end{cases}$$

If $\mu(s) = 0$ we find $k_1(s) = 0$ and $\lambda(s) = s + c$. So we have

$$\alpha(s) = (s + c)T$$

If $k_2(s) = 0$,then we find $\mu(s) = -\int (s+c)k_1(s)ds$. So we have

$$\alpha(s) = (s+c)T - (\int (s+c)k_1(s)ds)N.$$

Thus we have the following theorem.

Theorem 3.1. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{T, N\}$ if and only if it is in the form*

$$\alpha(s) = (s+c)T$$

where $k_1(s) = 0$ or

$$\alpha(s) = (s+c)T - (\int (s+c)k_1(s)ds)N$$

where $k_2(s) = 0$

Case 2) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T, B_1\}$. In this case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)B_1 \tag{3.2}$$

for some differentiable functions λ and μ. Differentiating (3.2) with respect to s and by using the Frenet equations we find that

$$\alpha'(s) = (\lambda'(s) - \mu(s)k_1(s))T + \lambda(s)k_1(s)N + \mu'(s)B_1 + \mu(s)k_3(s)B_2.$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} \lambda'(s) - \mu(s)k_1(s) = 1 \\ \lambda(s)k_1(s) = 0 \\ \mu(s)k_3(s) = 0 \\ \mu'(s) = 0 \end{cases}$$

From the equation $\lambda(s)k_1(s) = 0$, if $\lambda(s) = 0$ then we can write $\mu(s) = -\frac{1}{k_1(s)} = cons.$ and $k_3(s) = 0$. So we have

$$\alpha(s) = -\frac{1}{k_1(s)}B_1.$$

If $k_1(s) = 0$ and $\mu(s) = 0$ we find $\lambda(s) = s + c$. So we have

$$\alpha(s) = (s+c)T.$$

If $k_1(s) = k_3(s) = 0$ then we find $\mu(s) = c_2$ and $\lambda(s) = s + c_1$. So we have

$$\alpha(s) = (s+c_1)T + c_2B_1.$$

Thus we have the following theorem.

Theorem 3.2. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{T, B_1\}$ if and only if it is in the form*

$$\alpha(s) = -\frac{1}{k_1(s)}B_1$$

where $k_3(s) = 0$ or

$$\alpha(s) = (s+c)T$$

where $k_1(s) = 0$ or

$$\alpha(s) = (s+c_1)T + c_2B_1$$

where $k_1(s) = k_3(s) = 0$ *and* c, c_1 *and* c_2 *are constants.*

Case 3) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T, B_2\}$. In this case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)B_2, \tag{3.3}$$

for some differentiable functions λ and μ of the parameter s. Differentiating (3.3) with respect to s and by using the Frenet equations we find that

$$\alpha'(s) = \lambda'(s)T + (\lambda(s)k_1(s) - \mu(s)k_3(s))N - \mu(s)k_2(s)B_1 + \mu'(s)B_2.$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} \lambda'(s) = 1 \\ \lambda(s)k_1(s) - \mu(s)k_3(s) = 0 \\ \mu(s)k_2(s) = 0 \\ \mu'(s) = 0 \end{cases} \tag{3.4}$$

From (3.4) if $\mu(s) = 0$ then we find $\lambda(s) = s + c$ and $k_1(s) = 0$. So we have

$$\alpha(s) = (s + c)T.$$

If $k_2(s) = 0$ then we find $\lambda(s) = s + c_1$ and $\mu(s) = c_2$. So we have

$$\alpha(s) = (s + c_1)T + c_2B_2.$$

Thus we have the following theorem.

Theorem 3.3. *A spacelike curve* α *in* R^4_1 *lies on the subspace spanned by* $\{T, B_2\}$ *if and only if it is in the form*

$$\alpha(s) = (s + c)T.$$

where $k_1(s) = 0$ *or*

$$\alpha(s) = (s + c_1)T + c_2B_2.$$

where $k_2(s) = 0$ *and the curvature functions satisfy the equation* $\frac{k_1(s)}{k_3(s)} = \frac{c_2}{s+c_1}$.

Case 4) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{N, B_1\}$. In this case we can write

$$\alpha(s) = \lambda(s)N + \mu(s)B_1 \tag{3.5}$$

for some differentiable functions λ and μ of the parameter s. Differentiating (3.5) with respect to s and by using the Frenet equations we find that

$$\alpha'(s) = -\mu(s)k_1(s)T + \lambda'(s)N + \mu'(s)B_1 + (\lambda(s)k_2(s) + \mu(s)k_3(s))B_2.$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} -\mu(s)k_1(s) = 1 \\ \lambda'(s) = 0 \\ \mu'(s) = 0 \\ \lambda(s)k_2(s) + \mu(s)k_3(s) = 0 \end{cases} \tag{3.6}$$

From (3.6) we can write $\lambda(s) = c_1$ and $\mu(s) = -\frac{1}{k_1(s)} = c_2$. . So we have

$$\alpha(s) = c_1N - \frac{1}{k_1(s)}B_1.$$

Thus we have the following theorem.

Theorem 3.4. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{N, B_1\}$ if and only if it is in the form*

$$\alpha(s) = c_1 N - \frac{1}{k_1(s)} B_1$$

where c_1, c_2 are constants and the curvature functions satisfy the equation $c_1 k_2(s) + c_2 k_3(s) = 0$.

Case 5) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{N, B_2\}$. In this case we can write

$$\alpha(s) = \lambda(s) N + \mu(s) B_2 \tag{3.7}$$

for some differentiable functions λ and μ of the parameter s. Differentiating (3.7) with respect to s and by using the Frenet equations we find that

$$\alpha'(s) = (\lambda'(s) - \mu(s)k_3(s))N - \mu(s)k_2(s)B_1 + (\lambda(s)k_2(s) + \mu'(s))B_2. \tag{3.8}$$

Since $\alpha(s)$ is a spacelike curve from (3.8) there is a contradiction. Thus we have the following theorem.

Theorem 3.5. *A spacelike curve α in R_1^4 does not lie on the subspace spanned by $\{N, B_2\}$.*

Case 6) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{B_1, B_2\}$. In this case we can write

$$\alpha(s) = \lambda(s) B_1 + \mu(s) B_2 \tag{3.9}$$

for some differentiable functions λ and μ of the parameter s. Differentiating (3.9) with respect to s and by using the Frenet equations we find that

$$\alpha'(s) = -\lambda(s)k_1(s)T - \mu(s)k_3(s)N + (\lambda'(s) - \mu(s)k_2(s))B_1 + (\lambda(s)k_3(s) + \mu'(s))B_2.$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} -\lambda(s)k_1(s) = 1 \\ \mu(s)k_3(s) = 0 \\ \lambda'(s) - \mu(s)k_2(s) = 0 \\ \lambda(s)k_3(s) + \mu'(s) = 0 \end{cases} \tag{3.10}$$

From (3.10) if $\mu(s) = 0$ then we can write $\lambda(s) = -\frac{1}{k_1(s)}$. So we have

$$\alpha(s) = -\frac{1}{k_1(s)} B_1.$$

If $k_3(s) = 0$ we find $\mu(s) = \frac{k_1'(s)}{k_1^2(s)k_2(s)} = cons$. So we have

$$\alpha(s) = (-\frac{1}{k_1(s)})B_1 + (\frac{k_1'(s)}{k_1^2(s)k_2(s)})B_2.$$

Theorem 3.6. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{B_1, B_2\}$ if and only if it is in the form of*

$$\alpha(s) = -\frac{1}{k_1(s)} B_1$$

or

$$\alpha(s) = (-\frac{1}{k_1(s)})B_1 + (\frac{k_1'(s)}{k_1^2(s)k_2(s)})B_2$$

where $k_3(s) = 0$.

Case 7) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T, N, B_1\}$. In this case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)N + \gamma(s)B_1 \tag{3.11}$$

for some differentiable functions λ, μ and γ of the parameter s. Differentiating (3.11) with respect to s and by using the Frenet equations we find that

$$\begin{aligned}\alpha'(s) = \quad & (\lambda'(s) - \gamma(s)k_1(s))T + (\lambda(s)k_1(s) + \mu'(s))N + \gamma'(s)B_1 \\ & +(\mu(s)k_2(s) + \gamma(s)k_3(s))B_2.\end{aligned}$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\left\{\begin{array}{c} \lambda'(s) - \gamma(s)k_1(s) = 1 \\ \lambda(s)k_1(s) + \mu'(s) = 0 \\ \gamma'(s) = 0 \\ \mu(s)k_2(s) + \gamma(s)k_3(s) = 0 \end{array}\right. \tag{3.12}$$

From (3.12) we can write $\gamma(s) = c_1$. If we use the equation $\mu(s)k_2(s)+\gamma(s)k_3(s) = 0$ we find $\mu(s) = -c_1\frac{k_3(s)}{k_2(s)}$. From the equation $\lambda(s)k_1(s) + \mu'(s) = 0$ we obtain $\lambda(s) = c_1\frac{k_3'(s)k_2(s)-k_3(s)k_2'(s)}{k_2^2(s)k_1(s)}$. So we have

$$\alpha(s) = (c_1\frac{k_3'(s)k_2(s) - k_3(s)k_2'(s)}{k_2^2(s)k_1(s)})T - (c_1\frac{k_3(s)}{k_2(s)})N + c_1B_1.$$

Thus we have the following theorem.

Theorem 3.7. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{T, N, B_1\}$ if and only if it is in the form*

$$\alpha(s) = (c_1\frac{k_3'(s)k_2(s) - k_3(s)k_2'(s)}{k_2^2(s)k_1(s)})T - (c_1\frac{k_3(s)}{k_2(s)})N + c_1B_1.$$

where c_1 is a constant.

Case 8) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T, N, B_2\}$. In this case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)N + \gamma(s)B_2 \tag{3.13}$$

for some differentiable functions λ, μ and γ of the parameter s. Differentiating (3.13) with respect to s and by using the Frenet equations we find that

$$\begin{aligned}\alpha'(s) = \quad & \lambda'(s)T + (\lambda(s)k_1(s) + \mu'(s) - \gamma(s)k_3(s))N + \gamma(s)k_2(s)B_1 \\ & +(\gamma'(s) + \mu(s)k_2(s))B_2.\end{aligned}$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations:

$$\left\{\begin{array}{c} \lambda'(s) = 1 \\ \lambda(s)k_1(s) + \mu'(s) - \gamma(s)k_3(s) = 0 \\ \gamma(s)k_2 = 0 \\ \gamma'(s) + \mu(s)k_2(s) = 0 \end{array}\right. \tag{3.14}$$

From (3.14) we find $\lambda(s) = s + c$. If $\gamma(s) = 0$ we can write the equations

$$\begin{array}{c} \mu(s)k_2(s) = 0 \\ \lambda(s)k_1(s) + \mu'(s) = 0 \end{array} \tag{3.15}$$

From (3.15) if $\mu(s) = 0$ then we can write $\lambda(s) = s + c_1$ and $k_1(s) = 0$. So we have

$$\alpha(s) = (s + c_1)T.$$

If $k_2(s) = 0$ then we can write

$$\mu(s) = -\int (s + c_1)k_1(s)ds + c_2.$$

So we have

$$\alpha(s) = (s + c_1)T + (-\int (s + c_1)k_1(s)ds + c_2)N.$$

From (3.14) if $k_2(s) = 0$ then we can write $\gamma(s) = c_2$ and $\mu(s) = c_2 \int k_3(s)ds - \int k_1(s)(s + c_1)ds + c$. So we have

$$\alpha(s) = (s + c_1)T + (c_2 \int k_3(s)ds - \int k_1(s)(s + c_1)ds + c)N + c_2 B_2.$$

Thus we have the following theorem.

Theorem 3.8. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{T, N, B_2\}$ if and only if it is in the form*

$$\alpha(s) = (s + c_1)T$$

where $k_1(s) = 0$ or

$$\alpha(s) = (s + c_1)T + (-\int (s + c_1)k_1(s)ds + c_2)N$$

where $k_2(s) = 0$ or

$$\alpha(s) = (s + c_1)T + (c_2 \int k_3(s)ds - \int k_1(s)(s + c_1)ds + c)N + c_2 B_2$$

where $k_2(s) = 0$.

Case 9) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T, B_1, B_2\}$. In this case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)B_1 + \gamma(s)B_2 \tag{3.16}$$

for some differentiable functions λ, μ and γ of the parameter s. Differentiating (3.16) with respect to s and by using the Frenet equations we find that

$$\begin{aligned} \alpha'(s) &= (\lambda'(s) - \mu(s)k_1(s))T + (\lambda(s)k_1(s) - \gamma(s)k_3(s))N + (\mu'(s) - \gamma(s)k_2(s))B_1 \\ &+ (\mu(s)k_3(s) + \gamma'(s))B_2. \end{aligned}$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} \lambda'(s) - \mu(s)k_1(s) = 1 \\ \lambda(s)k_1(s) - \gamma(s)k_3(s) = 0 \\ \mu'(s) - \gamma(s)k_2(s) = 0 \\ \mu(s)k_3(s) + \gamma'(s) = 0 \end{cases} \tag{3.17}$$

From the equation $\lambda'(s) - \mu(s)k_1(s) = 1$ we can write $\frac{d\lambda(s)}{ds} + \frac{k_1(s)}{k_3(s)}\gamma'(s) = 1$. From the last equation we have

$$\frac{d}{ds}(\frac{k_3(s)}{k_1(s)}\gamma(s)) + \frac{k_1(s)}{k_3(s)}\frac{d\gamma(s)}{ds} = 1 \tag{3.18}$$

By using exchange variable $t = \int_0^s \frac{k_3(s)}{k_1(s)} ds$ in (3.18) we have

$$2\frac{d\gamma(s)}{ds} = 1 \tag{3.19}$$

The solution of (3.19) is $\gamma(s) = \frac{t}{2} + c$. Replacing variable $t = \int \frac{k_3(s)}{k_1(s)} ds$ in the last equation we find

$$\gamma(s) = \frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)} ds + c. \tag{3.20}$$

If we use (3.20) in (3.17) we find $\mu(s) = -\frac{1}{2k_1(s)}$ and

$$\lambda(s) = \frac{k_3(s)}{k_1(s)}(\frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)} ds + c)$$

So we have

$$\alpha(s) = (\frac{k_3(s)}{k_1(s)}(\frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)} ds + c))T - (\frac{1}{2k_1(s)})B_1 + (\frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)} ds + c)B_2.$$

Thus we have the following theorem.

Theorem 3.9. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{T, B_1, B_2\}$ if and only if it is in the form*

$$\alpha(s) = (\frac{k_3(s)}{k_1(s)}(\frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)} ds + c))T - (\frac{1}{2k_1(s)})B_1 + (\frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)} ds + c)B_2.$$

Case 10) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{N, B_1, B_2\}$. In this case we can write

$$\alpha(s) = \lambda(s)N + \mu(s)B_1 + \gamma(s)B_2 \tag{3.21}$$

for some differentiable functions λ, μ and γ of the parameter s. Differentiating (3.21) with respect to s and by using the Frenet equations we find that

$$\begin{aligned} \alpha'(s) &= -\mu(s)k_1(s)T + (\lambda'(s) - \gamma(s)k_3(s))N + (\mu'(s) - \gamma(s)k_2(s))B_1 \\ &+ (\lambda(s)k_2(s) + \mu(s)k_3(s) + \gamma'(s))B_2. \end{aligned}$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} -\mu(s)k_1(s) = 1 \\ \lambda'(s) - \gamma(s)k_3(s) = 0 \\ \mu'(s) - \gamma(s)k_2(s) = 0 \\ \lambda(s)k_2(s) + \mu(s)k_3(s) + \gamma'(s) = 0 \end{cases} \tag{3.22}$$

From (3.22) we can write $\mu(s) = -\frac{1}{k_1(s)}$. From the equation $\gamma(s) = \frac{k_1'(s)}{k_1^2(s)k_2(s)}$ and from the equation $\lambda(s)k_2(s) + \mu(s)k_3(s) + \gamma'(s) = 0$ we obtain

$$\lambda(s) = \frac{k_3(s)}{k_1(s)k_2(s)} - \frac{k_1''(s)k_1(s)k_2(s) - k_1'(s)(2k_1'(s)k_2(s) + k_1(s)k_2'(s))}{(k_1(s)k_2(s))^3}$$

So we have

$$\begin{aligned} \alpha(s) &= (\frac{k_3(s)}{k_1(s)k_2(s)} - \frac{k_1''(s)k_1(s)k_2(s) - k_1'(s)(2k_1'(s)k_2(s) + k_1(s)k_2'(s))}{(k_1(s)k_2(s))^3})N \\ &- (\frac{1}{k_1(s)})B_1 + (\frac{k_1'(s)}{k_1^2(s)k_2(s)})B_2. \end{aligned}$$

Thus we have the following theorem.

Theorem 3.10. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{N, B_1, B_2\}$ if and only if it is in the form*

$$\begin{aligned}\alpha(s) &= \left(\frac{k_3(s)}{k_1(s)k_2(s)} - \frac{k_1''(s)k_1(s)k_2(s) - k_1'(s)(2k_1'(s)k_2(s) + k_1(s)k_2'(s))}{(k_1(s)k_2(s))^3}\right)N \\ &- \left(\frac{1}{k_1(s)}\right)B_1 + \left(\frac{k_1'(s)}{k_1^2(s)k_2(s)}\right)B_2.\end{aligned}$$

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