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On polar relative normalizations of ruled surfaces

Ioanna-Iris Papadopoulou[a*] and Stylianos Stamatakis[a]

[a]*Department of Mathematics, Aristotle University of Thessaloniki*
[*]*Corresponding author E-mail: irisioanna@gmail.com*

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Abstract

This paper deals with skew ruled surfaces in the Euclidean space $\mathbb{E}^3$ which are equipped with polar normalizations, that is, relative normalizations such that the relative normal at each point of the ruled surface lies on the corresponding polar plane. We determine the invariants of a such normalized ruled surface and we study some properties of the Tchebychev vector field and the support vector field of a polar normalization. Furthermore, we study a special polar normalization, the relative image of which degenerates into a curve.

1. Introduction

In 1989 F. Manhart introduced the one-parameter family of relative normalizations ${}^{(a)}\bar{y}$ of a hypersurface with non-vanishing Gaussian curvature $\widetilde{K}$ in the Euclidean space $\mathbb{E}^{n+1}$ which are characterized by the support functions ${}^{(a)}q = |\widetilde{K}|^a$, $a \in \mathbb{R}$ and called Manhart's normalizations (see [2]).
G. Stamou and A. Magkos in [9] and G. Stamou, St. Stamatakis and I. Delivos in [10] studied ruled surfaces in the Euclidean space $\mathbb{E}^3$ which are equipped with Manhart's normalizations. Later, S. Stamatakis and I. Kaffas studied in [5] the asymptotic relative normalizations of a ruled surface Φ, that is, relative normalizations such that the relative normals at each point P of Φ lie on the corresponding asymptotic plane of Φ.
Following this idea the authors introduced in [7] three special relative normalizations:

1. the *central* normalizations, i.e, relative normalizations such that the relative normals at each point P of Φ lie on the corresponding central plane,
2. the *polar* normalizations, i.e, relative normalizations such that the relative normals at each point P of Φ lie on the corresponding polar plane and finally
3. the *right* normalizations, that is relative normalizations of Φ whose relative images $\overline{\Phi}$ are also ruled surfaces with the additional property that their generators are parallel to those of Φ. Some of these relative normalizations degenerate into a curve.

The central and the right normalizations were studied thoroughly in [7] and [8], respectively. In this paper we will study the polar normalizations.

2. Preliminaries

A brief discussion of some definitions, results and formulae of relative Differential Geometry of surfaces and Differential Geometry of ruled surfaces in the Euclidean space $\mathbb{E}^3$ appears in this section. We refer the reader to [3] and [4].
In the three-dimensional Euclidean space $\mathbb{E}^3$ let $\Phi = (U, \bar{x})$ be a ruled C^r-surface of nonvanishing Gaussian curvature, $r \geq 3$, defined by an injective C^r-immersion $\bar{x} = \bar{x}(u,v)$ on a region $U := I \times \mathbb{R}$ ($I \subset \mathbb{R}$ open interval) of $\mathbb{R}^2$. We introduce the so-called standard parameters $u \in I, v \in \mathbb{R}$ of Φ, such that

$$\bar{x}(u,v) = \bar{s}(u) + v\bar{e}(u), \tag{2.1}$$

Email addresses: irisioanna@gmail.com (I. I. Papadopoulou), stamata@math.auth.gr (S. Stamatakis)

and

$$|\bar{e}| = |\bar{e}'| = 1, \quad \langle \bar{s}', \bar{e}' \rangle = 0,$$

where the differentiation with respect to u is denoted by a prime and $\langle , \rangle$ denotes the standard scalar product in $\mathbb{E}^3$. Here $\varGamma : \bar{s} = \bar{s}(u)$ is the striction curve of $\varPhi$ and the parameter u is the arc length along the spherical curve $\bar{e} = \bar{e}(u)$.

The distribution parameter $\delta(u) := (\bar{s}', \bar{e}, \bar{e}')$, the conical curvature $\kappa(u) := (\bar{e}, \bar{e}', \bar{e}'')$ and the function $\lambda(u) := \cot\sigma$, where $\sigma(u) := \sphericalangle(\bar{e}, \bar{s}')$ is the striction of $\varPhi$ ($-\frac{\pi}{2} < \sigma \leq \frac{\pi}{2}$, $\operatorname{sign}\sigma = \operatorname{sign}\delta$), are the fundamental invariants of $\varPhi$ and determine uniquely the ruled surface $\varPhi$ up to Euclidean rigid motions. We also consider the central normal vector $\bar{n}(u) := \bar{e}'$ and the central tangent vector $\bar{z}(u) := \bar{e} \times \bar{n}$. It is known that the vectors of the moving frame $\mathscr{D} := \{\bar{e}, \bar{n}, \bar{z}\}$ of $\varPhi$ fulfil the following equations [3, p. 280]

$$\bar{e}' = \bar{n}, \quad \bar{n}' = -\bar{e} + \kappa \bar{z}, \quad \bar{z}' = -\kappa \bar{n}. \tag{2.2}$$

Then we have

$$\bar{s}' = \delta \lambda \bar{e} + \delta \bar{z}. \tag{2.3}$$

We denote partial derivatives of a function (or a vector-valued function) f in the coordinates $u^1 := u$, $u^2 := v$ by $f_{/i}, f_{/ij}$ etc. Then from (2.1) and (2.3) we take

$$\bar{x}_{/1} = \delta \lambda \bar{e} + v \bar{n} + \delta \bar{z}, \quad \bar{x}_{/2} = \bar{e}, \tag{2.4}$$

and thus the unit normal vector $\bar{\xi}(u,v)$ to $\varPhi$ is given by

$$\bar{\xi} = \frac{\delta \bar{n} - v \bar{z}}{w}, \quad \text{where} \quad w := \sqrt{\delta^2 + v^2}.$$

Let $I = g_{ij} \mathrm{d}u^i \mathrm{d}u^j$ and $II = h_{ij} \mathrm{d}u^i \mathrm{d}u^j$, $i, j = 1, 2$ be the first and the second fundamental form of $\varPhi$, respectively, where

$$g_{11} = w^2 + \delta^2 \lambda^2, \qquad g_{12} = \delta \lambda, \qquad g_{22} = 1, \tag{2.5}$$

$$h_{11} = -\frac{\kappa w^2 + \delta' v - \delta^2 \lambda}{w}, \qquad h_{12} = \frac{\delta}{w}, \qquad h_{22} = 0. \tag{2.6}$$

The Gaussian curvature $\widetilde{K}(u,v)$ and the mean curvature $\widetilde{H}(u,v)$ of $\varPhi$ are given by (see [3])

$$\widetilde{K} = -\frac{\delta^2}{w^4}, \quad \widetilde{H} = -\frac{\kappa w^2 + \delta' v + \delta^2 \lambda}{2 w^3}. \tag{2.7}$$

A C^s-relative normalization of $\varPhi$ is a C^s-mapping $\bar{y} = \bar{y}(u,v)$, $1 \leq s < r$, defined on U, such that

$$\operatorname{rank}(\{\bar{x}_{/1}, \bar{x}_{/2}, \bar{y}\}) = 3, \ \operatorname{rank}(\{\bar{x}_{/1}, \bar{x}_{/2}, \bar{y}_{/i}\}) = 2, \ i = 1,2, \ \forall (u,v) \in U. \tag{2.8}$$

The pair $(\varPhi, \bar{y})$ is called a relatively normalized ruled surface in $\mathbb{R}^3$ and the straight line issuing from a point $P \in \varPhi$ in the direction of $\bar{y}$ is called the relative normal of $(\varPhi, \bar{y})$ at P. The pair $\overline{\varPhi} = (U, \bar{y})$ is called the relative image of $(\varPhi, \bar{y})$.

The support function of the relative normalization $\bar{y}$ is defined by $q(u,v) := \langle \bar{\xi}, \bar{y} \rangle$ (see [1]). For $q = 1$, we have $\bar{y} = \bar{\xi}$, that is, the normalization is the Euclidean one.

Due to (2.8), q never vanishes on U. Conversely, when a support function q is given, the relative normalization $\bar{y}$ of the ruled surface $\varPhi$ is uniquely determined and can be expressed in terms of the moving frame $\mathscr{D}$ as follows [5, p. 179]:

$$\bar{y} = y_1 \bar{e} + y_2 \bar{n} + y_3 \bar{z}, \tag{2.9}$$

where

$$y_1 = -w \frac{\delta q_{/1} + q_{/2}(\kappa w^2 + \delta' v)}{\delta^2}, \quad y_2 = \frac{\delta^2 q - w^2 v q_{/2}}{\delta w}, \quad y_3 = -\frac{v q + w^2 q_{/2}}{w}. \tag{2.10}$$

For the coefficients $G_{ij}(u,v)$ of the relative metric $G(u,v)$ of $(\varPhi, \bar{y})$, which is indefinite, we have

$$G_{ij} = q^{-1} h_{ij}. \tag{2.11}$$

Then, because of (2.6), the coefficients of the inverse relative metric tensor are

$$G^{(11)} = 0, \quad G^{(12)} = \frac{w q}{\delta}, \quad G^{(22)} = w q \frac{\kappa w^2 + \delta' v - \delta^2 \lambda}{\delta^2}. \tag{2.12}$$

For a function (or a vector-valued function) f we denote by $\nabla_i^G f$ the covariant derivatives in the direction of u^i, both with respect to the relative metric. The coefficients $A_{ijk}(u,v)$ of the Darboux tensor are given by

$$A_{ijk} := q^{-1} \langle \bar{\xi}, \nabla_k^G \nabla_j^G \bar{x}_{/i} \rangle.$$

Then, by using the relative metric tensor G_{ij} for "raising and lowering the indices", the Pick invariant $J(u,v)$ of $(\varPhi, \bar{y})$ is defined by

$$J := \frac{1}{2} A_{ijk} A^{ijk}.$$

As we proved in [7] (see equation (2.2)) the Pick invariant is calculated by

$$J=\frac{3\left(w^2q_{/2}+vq\right)}{2\delta^2w^3q}\left\{w^2\left[\kappa qv+2\delta q_{/1}+q_{/2}\left(\kappa w^2+\delta'v-\delta^2\lambda\right)\right]-\delta^2q\left(\lambda v-\delta'\right)\right\}. \tag{2.13}$$

The relative shape operator has the coefficients $B_i^j(u,v)$ given by

$$\bar{y}_{/i}=:-B_i^j\bar{x}_{/j}. \tag{2.14}$$

Then, for the relative curvature $K(u,v)$ and the relative mean curvature $H(u,v)$ of $(\Phi,\bar{y})$ we have

$$K:=\det\left(B_i^j\right),\quad H:=\frac{B_1^1+B_2^2}{2}. \tag{2.15}$$

We conclude this section by mentioning that, among the surfaces of $\mathbb{E}^3$ with negative Gaussian curvature the ruled surfaces are characterized by the relation

$$3H-J-3S=0 \tag{2.16}$$

(see [6]), where $S(u,v)$ is the scalar curvature of the relative metric G of such a surface Φ, which is defined formally as the curvature of the pseudo-Riemannian manifold (Φ,G).

3. Polar normalizations

We concentrate now on the main topic of this paper, namely the polar normalizations of a skew ruled surface Φ, i.e., relative normalizations such that the relative normal at each point P of Φ lies on the corresponding polar plane $\{P;\bar{n},\bar{z}\}$. In [7] it was shown that the support function of $\bar{y}$ is of the form

$$q=f(V), \tag{3.1}$$

where $f(V)$ is an arbitrary C^2-function of

$$V=\arctan\frac{v}{\delta}-\int\kappa du. \tag{3.2}$$

By means of (2.9), (2.10), (3.1) and (3.2) we deduce that the arising relative normalization, i.e., the polar normalization of the given ruled surface Φ is

$$\bar{y}=\frac{\delta q-\dot{q}v}{w}\bar{n}-\frac{qv+\delta\dot{q}}{w}\bar{z}, \tag{3.3}$$

where the dot denotes the differentiation with respect to V. Then, from (2.2), (2.4), (2.14) and (3.3), we take the coefficients B_i^j of the relative shape operator of a polar normalization:

$$\begin{aligned}
B_1^1&=-\frac{(\kappa w^2+\delta'v)(q+\ddot{q})}{w^3},\\
B_1^2&=\frac{1}{w^3}\left\{-\dot{q}v^3-\delta^2\dot{q}v+\delta^3\left[q\left(\kappa\lambda+1\right)+\kappa\lambda\ddot{q}\right]+\delta v\left[q\left(\kappa\lambda v+v+\delta'\lambda\right)+\lambda\ddot{q}\left(\kappa v+\delta'\right)\right]\right\},\\
B_2^1&=\frac{\delta(q+\ddot{q})}{w^3},\\
B_2^2&=-\frac{\delta^2\lambda(q+\ddot{q})}{w^3}.
\end{aligned}$$

Hence, by using (2.15) and (2.7b), we obtain the relative curvature K and the relative mean curvature H:

$$K=-\delta\,\frac{(\delta q-\dot{q}v)\left(q+\ddot{q}\right)}{w^4},\quad H=\widetilde{H}\left(q+\ddot{q}\right). \tag{3.4}$$

From (3.4a) we deduce that the relative curvature K of a polar normalization vanishes identically iff

$$\delta q-\dot{q}v=0\quad\text{or}\quad q+\ddot{q}=0,$$

or, equivalently, iff

$$q=ce^{\frac{\delta V}{v}},c\in\mathbb{R}^*\quad\text{or}\quad q=c_1\cos V+c_2\sin V,\,c_1,c_2\in\mathbb{R},\,c_1^2+c_2^2\neq 0.$$

We reject the first support function since it leads to a non polar normalization. Thus we have the following

Theorem 3.1. *Let $\Phi\subset E^3$ be a polar normalized ruled surface. The relative curvature K of $(\Phi,\bar{y})$ vanishes identically iff the support function is of the form*

$$q=c_1\cos V+c_2\sin V,\quad c_1,c_2\in\mathbb{R},\quad c_1^2+c_2^2\neq 0.$$

By taking (2.7b) and (3.4b) into consideration we arrive at

Theorem 3.2. *Let $\Phi \subset E^3$ be a polar normalized ruled surface. $(\Phi, \bar{y})$ is relatively minimal $(H = 0)$ iff one of the following holds true*
(a) *the support function is of the form*

$$q = c_1 \cos V + c_2 \sin V, \quad c_1, c_2 \in \mathbb{R}, \quad c_1^2 + c_2^2 \neq 0,$$

(b) $(\Phi, \bar{y})$ *is a polar normalized right helicoid* $(\delta = c \in \mathbb{R}^*$ *and* $\kappa = \lambda = 0)$.

We notice that both the relative curvature K and the relative mean curvature H vanish identically iff the support function is of the form

$$q = c_1 \cos V + c_2 \sin V, \quad c_1, c_2 \in \mathbb{R}, \quad c_1^2 + c_2^2 \neq 0. \tag{3.5}$$

By using (2.7b) and (2.13) we find the Pick invariant

$$J = (qv + \delta \dot{q}) \left(\frac{J_{EUK}}{v} + \frac{3\widetilde{H}\dot{q}}{\delta q} \right), \tag{3.6}$$

where

$$J_{EUK} = 3v \frac{\kappa v^3 + \delta^2 (\kappa - \lambda) v + \delta^2 \delta'}{2\delta^2 w^3} \tag{3.7}$$

is the Pick invariant of the Euclidean normalization. The Pick invariant vanishes identically iff

$$qv + \delta \dot{q} = 0 \quad \text{or} \quad \frac{J_{EUK}}{v} + \frac{3\widetilde{H}\dot{q}}{\delta q} = 0,$$

or, equivalently, iff

- the support function is of the form

  $$q = c_1 e^{\frac{-Vv}{\delta}}, c_1 \in \mathbb{R}^*, \text{or}$$

- Φ is not a right helicoid and the support function is of the form

  $$q = c_2 e^{\frac{v\left[\kappa v^3 + \delta^2 (\kappa - \lambda) v + \delta^2 \delta'\right]}{\delta\left[\kappa v^2 + \delta' v + \delta^2 (\kappa + \lambda)\right]}}, \ c_2 \in \mathbb{R}^*, \text{or}$$

- Φ is a right helicoid.

We reject the two support functions since they are not polar. Hence, we deduce

Theorem 3.3. *Let $\Phi \subset E^3$ be a polar normalized ruled surface. The Pick invariant J of $(\Phi, \bar{y})$ vanishes identically iff Φ is a right helicoid.*

From (2.16), (3.4b), (3.6) and (3.7) we evaluate the scalar curvature of the relative metric

$$\begin{aligned} S = \frac{1}{2\delta^2 w^3 q} \Big\{ & -q^2 \left\{ \kappa w^4 + \delta^2 \left[\left(-v^2 + \delta^2 \right) \lambda + 2\delta' v \right] \right\} + \delta^2 \left(\kappa w^2 + \delta^2 \lambda + \delta' v \right) \dot{q}^2 \\ & + \delta q \Big\{ \left[2\delta^2 \lambda v + \left(v^2 - \delta^2 \right) \delta' \right] \dot{q} - \delta \left(\kappa w^2 + \delta' v + \delta^2 \lambda \right) \ddot{q} \Big\} \Big\}. \end{aligned}$$

4. The Tchebychev vector field and the support vector field of a polar normalization

In [5] it was shown that the coordinate functions of the Tchebychev vector $\overline{T}(u, v)$ of $(\Phi, \bar{y})$, which is defined by

$$\overline{T} := T^m \bar{x}_{/m}, \quad \text{where} \quad T^m := \frac{1}{2} A_i^{im},$$

are given by

$$T^1 = \frac{w^2 q_{/2} + vq}{\delta w}, \ T^2 = \frac{2\delta w^2 q_{/1} + \delta' q (\delta^2 - v^2)}{2\delta^2 w} + \frac{T^1 (\kappa w^2 + \delta' v - \delta^2 \lambda)}{\delta}.$$

Hence, by using (3.1) and (3.2), we deduce that the coordinate functions of the Tchebychev vector of a polar normalization are

$$T^1 = \frac{qv + \delta \dot{q}}{\delta w}, \quad T^2 = \frac{q \left(2\kappa v w^2 - 2\delta^2 \lambda v + \delta' w^2 \right) - 2\delta^3 \lambda \dot{q}}{2\delta^2 w}. \tag{4.1}$$

The divergence $\operatorname{div}^I \overline{T}$ of $\overline{T}$ with respect to the first fundamental form I of Φ, which initially reads (see [5])

$$\operatorname{div}^I \overline{T} = \frac{\left(w T^i \right)_{/i}}{w}, \tag{4.2}$$

becomes, on account of (4.1),

$$\begin{aligned}
\operatorname{div}^I \overline{T} = \frac{1}{2\delta^2 w^3} \Big\{ & 2w^2 q \Big[\left(3v^2+\delta^2\right)\kappa - \delta^2\lambda \Big] \\
& + \delta \Big\{ \Big[-\delta' v^2 + \delta^2 \left(-2\lambda v + \delta'\right) \Big] \dot{q} - 2\delta \left(\kappa w^2 + \delta^2\lambda + \delta' v\right) \ddot{q} \Big\} \Big\}.
\end{aligned}$$

The rotation $\operatorname{curl}^I \overline{T}$ of $\overline{T}$ with respect to the first fundamental form I of Φ, which initially reads (see [5])

$$\operatorname{curl}^I \overline{T} = \frac{\left(g_{12}T^1 + g_{22}T^2\right)_{/1} - \left(g_{11}T^1 + g_{12}T^2\right)_{/2}}{w}, \tag{4.3}$$

becomes, by taking (2.5) and (4.1) into consideration,

$$\begin{aligned}
\operatorname{curl}^I \overline{T} = -\frac{1}{2\delta^3 w^2} \Big\{ & 2\delta' q v^2 \left(2\kappa v + \delta'\right) + \delta^2 q \Big[4\left(\kappa\lambda+1\right) v^2 + \delta' \left(2\kappa+\lambda\right) v + \delta'^2 \Big] \\
& + \delta^3 \Big\{ \dot{q} \Big[4v + \left(\kappa+\lambda\right)\left(2\kappa v + \delta'\right) \Big] - q\left(2\kappa' v + \delta''\right) \Big\} \\
& + \delta v \Big[2\kappa^2 \dot{q} v^2 + 3\kappa\delta' \dot{q} v + \delta'^2 \dot{q} - q v \left(2\kappa' v + \delta''\right) \Big] + 2\delta^4 \Big[q\left(\kappa\lambda+1\right) + \ddot{q} \Big] \Big\}.
\end{aligned}$$

Analogously we calculate the divergence and the rotation of $\overline{T}$ with respect to the relative metric of Φ:

$$\begin{aligned}
\operatorname{div}^G \overline{T} = & \frac{1}{\delta^2 w^3 q} \Big\{ q^2 \big\{ \kappa w^4 + \delta^2 \Big[\left(v^2-\delta^2\right)\lambda - 2\delta' v \Big] \big\} + \delta^2 \dot{q}^2 \left(\kappa w^2 + \delta' v + \delta^2 \lambda\right) \\
& + \delta q \big\{ \dot{q} \Big[2\delta^2 \lambda v + \delta' \left(v^2 - \delta^2\right) \Big] - \delta \ddot{q} \left(\kappa w^2 + \delta' v + \delta^2\lambda\right) \big\} \Big\}, \\
\operatorname{curl}^G \overline{T} = & \; 0.
\end{aligned}$$

Last relation agrees with

$$\overline{T} = \nabla^G \left(\ln \frac{q}{q_{AFF}}, \overline{x} \right)$$

(see [6]), where $q_{AFF} = |\widetilde{K}|^{1/4}$ and ∇^G denotes the first Beltrami differential operator with respect to G for which holds $\nabla^G(f,g) = G^{(ij)} f_{/i} g_{/j}$. So, we have

Theorem 4.1. *Let $\Phi \subset E^3$ be a polar normalized ruled surface. The rotation of the Tchebychev vector field with respect to the relative metric of Φ vanishes identically and its potential is given by*

$$\tau(u,v) = \ln \frac{wq}{|\delta|^{1/2}} + c, \quad c \in \mathbb{R}.$$

Now let

$$\overline{Q} := \frac{1}{4} \bigtriangledown^G \left(\frac{1}{q}, \overline{x} \right) \tag{4.4}$$

be the support vector $\overline{Q}(u,v)$ of $(\Phi, \overline{y})$, which is introduced in [5]. By taking (2.12), (3.1) and (3.2) into consideration we find that the coordinate functions of the support vector field of a polar normalization are

$$Q^1 = -\frac{\dot{q}}{4wq}, \quad Q^2 = \frac{\delta\lambda\dot{q}}{4wq}. \tag{4.5}$$

By means of (2.7b), (4.2) and (4.5), we find the divergence $\operatorname{div}^I \overline{Q}$ of $\overline{Q}$ with respect to the first fundamental form I of Φ

$$\operatorname{div}^I \overline{Q} = \widetilde{H} \frac{\dot{q}^2 - q\ddot{q}}{2q^2}.$$

Hence, we derive

Theorem 4.2. *Let $\Phi \subset E^3$ be a polar normalized ruled surface. The support vector field is incompressible with respect to the first fundamental form of Φ ($\operatorname{div}^I \overline{Q} = 0$) iff*
(a) *the support function is of the form*

$$q = c_2 e^{c_1 V}, \quad c_1 \in \mathbb{R}, \quad c_2 \in \mathbb{R}^*, \text{ or}$$

(b) *Φ is a right helicoid.*

By taking (2.5), (4.3) and (4.5) into account we deduce that the rotation $\operatorname{curl}^I \overline{Q}$ of $\overline{Q}$ with respect to the first fundamental form I of Φ is

$$\operatorname{curl}^I \overline{Q} = \frac{-\delta \dot{q}^2 + q(\dot{q}v + \delta \ddot{q})}{4w^2 q^2}.$$

By taking (2.11), (4.2) and (4.5) into consideration we find the divergence $\operatorname{div}^G \overline{Q}$ of $\overline{Q}$ with respect to the relative metric of Φ

$$\operatorname{div}^G \overline{Q} = \frac{1}{4\delta w^3 q^2} \Big\{ \dot{q} \{ q \left[-\delta' v^2 + \delta^2 (-2\lambda v + \delta') \right] - 2\delta \dot{q} \left(\kappa w^2 + \delta' v + \delta^2 \lambda \right) \} \\ + \delta q \ddot{q} \left(\kappa w^2 + \delta^2 \lambda + \delta' v \right) \Big\}.$$

By using (2.6), (2.11), (3.1), (3.2), (4.3) and (4.5) we have the rotation $\operatorname{curl}^G \overline{Q}$ of $\overline{Q}$ with respect to the relative metric of Φ

$$\operatorname{curl}^G \overline{Q} = 0,$$

which agrees with the relation (4.4). Thus, we have

Theorem 4.3. *Let $\Phi \subset E^3$ be a polar normalized ruled surface. The rotation of the support vector field with respect to the relative metric of Φ vanishes identically and its potential is given by*

$$\tau(u,v) = \frac{1}{4q} + c, \quad c \in \mathbb{R}.$$

5. A special polar normalization

In this section we will study the support function of the form (3.5), which arises when the relative curvature K or the relative mean curvature H vanishes identically (see Sec. 3). By using (3.3) the corresponding relative normalization takes the form

$$\bar{y} = \left[c_1 \cos\left(\int \kappa du \right) - c_2 \sin\left(\int \kappa du \right) \right] \bar{n} - \left[c_2 \cos\left(\int \kappa du \right) + c_1 \sin\left(\int \kappa du \right) \right] \bar{z},$$

i.e., the relative normalization degenerates into a curve Γ^* with curvature

$$\kappa^* = \frac{1}{|c_1 \cos(\int \kappa du) - c_2 \sin(\int \kappa du)|}$$

and torsion

$$\sigma^* = \frac{-\kappa}{c_1 \cos(\int \kappa du) - c_2 \sin(\int \kappa du)}.$$

Since

$$\frac{\kappa^*}{\sigma^*} = \pm \frac{1}{\kappa},$$

we deduce that $\bar{y}$ *is a curve of constant slope iff Φ is a ruled surface of constant slope*.
By means of (3.6) and (3.7) we find the Pick invariant of this normalization:

$$J = \frac{3\left[c_2 \cos(\int \kappa du) + c_1 \sin(\int \kappa du)\right]}{2\delta^2 w (c_1 \cos V + c_2 \sin V)} \Big\{ \cos\left(\int \kappa du \right) \left[\kappa \left(c_2 v^2 + 2c_1 \delta v - c_2 \delta^2 \right) \right. \\ \left. + \delta \left(-c_2 \delta \lambda + c_1 \delta' \right) \right] + \sin\left(\int \kappa du \right) \left[\kappa \left(c_1 v^2 - 2c_2 \delta v - c_1 \delta^2 \right) - \delta \left(c_1 \delta \lambda + c_2 \delta' \right) \right] \Big\}.$$

Then by using (2.4) and (4.1) we deduce the Tchebychev vector

$$\overline{T} = \frac{w}{2\delta^2} (c_1 \cos V + c_2 \sin V)(2\kappa v + \delta') \bar{e} + \frac{v}{\delta} \left[c_2 \cos\left(\int \kappa du \right) + c_1 \sin\left(\int \kappa du \right) \right] \bar{n} \\ + \left[c_2 \cos\left(\int \kappa du \right) + c_1 \sin\left(\int \kappa du \right) \right] \bar{z}.$$

Finally, by taking (2.4) and (4.5) into consideration we derive the support vector

$$\overline{Q} = \frac{c_1 \sin V - c_2 \cos V}{4w(c_1 \cos V + c_2 \sin V)} (v\bar{n} + \delta \bar{z}).$$

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Product associativity in scator algebras and the quantum wave function collapse

M. Fernández-Guasti[a*+]

[a]*Lab. de Óptica Cuántica, Departamento de Física, Universidad Autónoma Metropolitana - Iztapalapa, 09340 Ciudad de México, Ap. postal. 55-534, Mexico*
[*]*Corresponding author E-mail: mfg@xanum.uam.mx*
[+]*Orcid: 0000-0002-1839-6002*

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Abstract

The scator product in $1+n$ dimensions for $n>1$, is associative if all possible product pairs have a non vanishing additive scalar component. The product is in general, not associative in the additive representation whenever the additive scalar component of a product pair is zero. A particular case of this statement is non associativity due to zero products of non zero factors. These features of scator algebra could be used to model the quantum wave function evolution and collapse in a unified description.

1. Introduction

Scator algebras are $1+n$ dimensional algebras endowed with a main involution and an order parameter. The addition operation satisfies commutative group conditions in $\mathbb{R}^{1+n}$. The product operation, defined in a subset of $\mathbb{R}^{1+n}$, is always commutative and the conditions when it fulfills associativity are the matter of this communication. The scator product is not bilinear, thus, it cannot be represented as a matrix - matrix product. For this same reason, it is in general not distributive over addition [6]. However, the scator set in $1+n$ dimensions, can be mapped into a higher dimensional space in order to recover distributivity. This procedure has been expounded for 1+2 dimensional real scators by extending this set to 1+3 dimensional space [14]. The elements of scator algebra, decorated with an overhead oval, can be written as an ordered sequence of $1+n$ real numbers, $\overset{o}{\varphi}=\left(f_0;f_1,\ldots,f_j,\ldots,f_n\right)$. The component with subindex zero, separated by a semicolon, has a distinct nature from the rest. In the *additive representation*, scator elements are described by a sum of components

$$\overset{o}{\varphi}=\left(f_0;f_1,\ldots,f_j,\ldots,f_n\right)=f_0+\sum_{j=1}^{n}f_j\mathbf{e}_j, \tag{1.1}$$

where $f_0,f_j\in\mathbb{R}$ for j from 1 to n in $\mathbb{N}$ and $\mathbf{e}_j\notin\mathbb{R}$. The component with subindex zero is labeled the *additive scalar component* whereas the subindices 1 to n stand for the *additive director components*. This representation is similar to the rectangular form of complex numbers extended to higher dimensions. In some subsets of $\mathbb{R}^{1+n}$, there exists a multiplicative representation of scators that is akin to the polar form of complex or hyperbolic numbers extended to higher dimensional spaces [11]. This communication is restricted to associativity in the additive representation.

Product associativity in scator algebra has been loosely addressed in several previous communications [6, 9, 10, 11]. It has been correctly stated that the scator product in the additive representation is in general not associative if zero product pairs are involved. *Zero products* are products of non zero scator factors that yield a zero scator, i.e. a scator with all additive components equal to zero. However, it has been stated that the product is associative if zero products are avoided. This statement is not correct. When these assertions were made, it was not foreseen that there exist non associative products in the additive representation even when zero products are not involved. The decisive parameter in the additive representation is, as we shall presently see, whether the additive scalar component of any product pair is zero.

In diverse graded algebras, notably algebra of physical space (APS) [3], space-time algebra (STA) [13] and commutative quaternions [4], the scalar component has been used to represent time [5]. We propose that the collapse of the wave function in quantum mechanics may be

Email addresses: **mfg@xanum.uam.mx (M. Fernández-Guasti)**

described via the non associativity of the scator product. The reduction occurs when the time variable, represented by the additive scalar component, is zero. If each director component constitutes a quantum state, when the scalar component of the product becomes null, all but one (single state) of the director components of the product become zero.
Sufficient conditions for the scator product associativity are addressed in section 2, while section 3, evaluates the necessary conditions for the lack of associativity. The possibility of scator algebra to describe the wavefunction evolution and collapse is outlined in section 4. The appendix deals with associativity exceptions. The remaining paragraphs in this introduction establish the necessary algebra prerequisites.

Product operation

We denote the *scator set* by

$$\mathbb{S}^{1+n}=\left\{\overset{o}{\varphi}\in\mathbb{R}^{1+n}: f_0\neq 0,\ \text{if there exists } f_j f_l\neq 0 \text{ for } j\neq l\in n\right\},$$

the subset of $\mathbb{R}^{1+n}$ where the additive scalar is different from zero if two or more director components are not zero. This set is the union of two disjoint subsets

$$\mathbb{S}^{1+n}_{\neq 0}=\left\{\overset{o}{\varphi}\in\mathbb{S}^{1+n}: f_0\neq 0\right\},\quad \mathbb{S}^{1+n}_{0}=\left\{\overset{o}{\varphi}\in\mathbb{S}^{1+n}: f_0= 0\right\}.$$

The elements of the set $\mathbb{S}^{1+n}_0$ have at most only one non vanishing director component, that is elements of the form $(0;0,\ldots,f_l,\ldots,0)=f_l\mathbf{e}_l$,

$$\mathbb{S}^{1+n}_{l}=\left\{\overset{o}{\varphi}\in\mathbb{S}^{1+n}: f_0= 0, f_j=0 \text{ for all } j\neq l \text{ from 1 to } n\right\}.$$

The set $\mathbb{S}^{1+n}_0$ can be written as the union of all subsets with at most one non vanishing director component $\mathbb{S}^{1+n}_0=\bigcup_{l=1}^{n}\mathbb{S}^{1+n}_l$. The sets $\mathbb{S}^{1+n}_l$ are not disjoint since the element with zero components everywhere is common to all of them, $\cap_{l=1}^{n}\mathbb{S}^{1+n}_l=(0;0,\ldots,0,\ldots,0)=\overset{o}{0}$.
In the additive representation, the *scator product* is defined in the $\mathbb{S}^{1+n}$ set:
The *product* of two scators $\overset{o}{\alpha}=a_0+\sum_{j=1}^{n}a_j\mathbf{e}_j$ and $\overset{o}{\beta}=b_0+\sum_{j=1}^{n}b_j\mathbf{e}_j$ in $\mathbb{S}^{1+n}_{\neq 0}$ is defined by

$$\overset{o}{\gamma}=\overset{o}{\alpha}\overset{o}{\beta}=\underbrace{a_0b_0\prod_{k=1}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)}_{g_0}+\underbrace{a_0b_0\sum_{j=1}^{n}\prod_{k\neq j}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)\left(\frac{a_j}{a_0}+\frac{b_j}{b_0}\right)\mathbf{e}_j}_{g_j};\tag{1.2a}$$

If $\overset{o}{\alpha}\in\mathbb{S}^{1+n}_l$ and $\overset{o}{\beta}\in\mathbb{S}^{1+n}_{\neq 0}$ the product $\overset{o}{\alpha}\overset{o}{\beta}$ is

$$(a_l\mathbf{e}_l)\,\overset{o}{\beta}=\mp a_lb_l+a_lb_0\mathbf{e}_l\mp\sum_{j\neq l}^{n}\left(a_l\frac{b_lb_j}{b_0}\right)\mathbf{e}_j;\tag{1.2b}$$

If $\overset{o}{\alpha}\in\mathbb{S}^{1+n}_l$ and $\overset{o}{\beta}\in\mathbb{S}^{1+n}_m$,

$$(a_l\mathbf{e}_l)\,(b_m\mathbf{e}_m)=\mp a_lb_m\delta_{lm},\tag{1.2c}$$

where δ_{lm} is a Kroneker delta.
The scator product defined in the $\mathbb{S}^{1+n}$ set is closed, commutative and there is a multiplicative identity [6]. Existence of an inverse depends on the signature of the product. Wherever applicable, here and in the rest of this manuscript, the upper $\mp$ (negative) sign corresponds to the imaginary scators product while the lower (positive) sign corresponds to the real or hyperbolic scators product. From (1.2c), $\check{\mathbf{e}}_j\check{\mathbf{e}}_j=-1$ for imaginary scators, we usually label the unit imaginary director components with a check above. When the product is defined with the (upper) negative sign, the $1+1$ dimensional scator sets with scalar component and only one director component, become isomorphic to complex algebra. In contrast, $\hat{\mathbf{e}}_j\hat{\mathbf{e}}_j=1$ for real scators, their director unit components usually labeled with a hat above. Real scators in 1+1 dimensions are isomorphic to hyperbolic numbers. The check/hat decoration is omitted here in order to cope with both scator sets simultaneously.

2. Associativity in the additive representation

In the proofs that follow, products with three scator factors are evaluated. Products with a larger number of factors can be obtained by induction.

2.1. Conditions for a product with non vanishing additive scalar component

From the product definition (1.2a), the product of two scators with non zero additive scalar components ($a_0b_0\neq 0$), has a non vanishing scalar component, if and only if

$$\frac{a_kb_k}{a_0b_0}\neq\pm 1,\tag{2.1}$$

for all k from 1 to n. From (1.2b), the product of non zero factors, one with zero scalar component $\overset{o}{\alpha}=a_l\mathbf{e}_l$, times $\overset{o}{\beta}\in\mathbb{S}^{1+n}_{\neq 0}$, has a non zero additive scalar component, if and only if $b_l\neq 0$. If both scator factors have null scalar, from (1.2c), their product has non zero scalar only if both scators have the same non zero director component, $l=m$. In the derivations that follow, it is assumed that none of the initial scator factors is zero.

2.2. Associative conditions

Theorem 2.1. *The scator product in $\mathbb{S}^{1+n}$ is associative in the additive representation if all possible product pairs have a non vanishing additive scalar component,* $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}=\overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right)=\overset{o}{\beta}\left(\overset{o}{\alpha}\overset{o}{\varphi}\right)$ *if* $\overset{o}{\alpha}\overset{o}{\beta},\overset{o}{\beta}\overset{o}{\varphi},\overset{o}{\alpha}\overset{o}{\varphi}\in\mathbb{S}^{1+n}_{\neq 0}$.

Proof. Case i) *Scator factors with non zero additive scalar,* $\overset{o}{\alpha},\overset{o}{\beta},\overset{o}{\varphi}\in\mathbb{S}^{1+n}_{\neq 0}$.

The product of scators $\overset{o}{\gamma}=\overset{o}{\alpha}\overset{o}{\beta}$ with $\overset{o}{\alpha},\overset{o}{\beta}\in\mathbb{S}^{1+n}_{\neq 0}$ is given by (1.2a). Since all product pairs must have non zero scalar, $\overset{o}{\alpha}\overset{o}{\beta}\in\mathbb{S}^{1+n}_{\neq 0}$; then $\frac{a_kb_k}{a_0b_0}\neq\pm 1$ for all k from 1 to n. The subsequent product with $\overset{o}{\varphi}\in\mathbb{S}^{1+n}_{\neq 0}$ is again given by (1.2a),

$$
\begin{aligned}
\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}=\,&a_0b_0\prod_{k=1}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)f_0\prod_{k=1}^{n}\left(1\mp\frac{\left(\frac{a_k}{a_0}+\frac{b_k}{b_0}\right)}{\left(1\mp\frac{a_kb_k}{a_0b_0}\right)}\frac{f_k}{f_0}\right)\\
&+a_0b_0\prod_{k=1}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)f_0\sum_{j=1}^{n}\prod_{k\neq j}^{n}\left(1\mp\frac{\left(\frac{a_k}{a_0}+\frac{b_k}{b_0}\right)}{\left(1\mp\frac{a_kb_k}{a_0b_0}\right)}\frac{f_k}{f_0}\right)\left(\frac{\left(\frac{a_j}{a_0}+\frac{b_j}{b_0}\right)}{\left(1\mp\frac{a_jb_j}{a_0b_0}\right)}+\frac{f_j}{f_0}\right)\mathbf{e}_j.
\end{aligned}
$$

This expression can be written in a symmetrical form in the coefficients of the three scators,

$$
\begin{aligned}
\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}=\,&a_0b_0f_0\prod_{k=1}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\mp\frac{a_kf_k}{a_0f_0}\mp\frac{b_kf_k}{b_0f_0}\right)\\
&+a_0b_0f_0\sum_{j=1}^{n}\prod_{k\neq j}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\mp\frac{a_kf_k}{a_0f_0}\mp\frac{b_kf_k}{b_0f_0}\right)\left(\frac{a_j}{a_0}+\frac{b_j}{b_0}+\frac{f_j}{f_0}\mp\frac{a_jb_jf_j}{a_0b_0f_0}\right)\mathbf{e}_j. \qquad (2.2)
\end{aligned}
$$

The first factor in parenthesis can be grouped back as $\left(1\mp\frac{b_kf_k}{b_0f_0}\mp\left(\frac{b_k}{b_0}+\frac{f_k}{f_0}\right)\frac{a_k}{a_0}\right)$, whereas the last factors are grouped as $\left(\frac{b_j}{b_0}+\frac{f_j}{f_0}+\left(1\mp\frac{b_jf_j}{b_0f_0}\right)\frac{a_j}{a_0}\right)$. But $\overset{o}{\beta}\overset{o}{\varphi}\in\mathbb{S}^{1+n}_{\neq 0}$, thus, the terms $1\mp\frac{b_kf_k}{b_0f_0}\neq 0$ for all k from 1 to n, can be factored to obtain

$$
\begin{aligned}
\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}=\,&a_0b_0f_0\prod_{k=1}^{n}\left(1\mp\frac{b_kf_k}{b_0f_0}\right)\left(1\mp\frac{\left(\frac{b_k}{b_0}+\frac{f_k}{f_0}\right)}{\left(1\mp\frac{b_kf_k}{b_0f_0}\right)}\frac{a_k}{a_0}\right)\\
&+b_0f_0\prod_{k=1}^{n}\left(1\mp\frac{b_kf_k}{b_0f_0}\right)a_0\sum_{j=1}^{n}\prod_{k\neq j}^{n}\left(1\mp\frac{\left(\frac{b_k}{b_0}+\frac{f_k}{f_0}\right)}{\left(1\mp\frac{b_kf_k}{b_0f_0}\right)}\frac{a_k}{a_0}\right)\left(\frac{\frac{b_j}{b_0}+\frac{f_j}{f_0}}{\left(1\mp\frac{b_jf_j}{b_0f_0}\right)}+\frac{a_j}{a_0}\right)\mathbf{e}_j=\overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right).
\end{aligned}
$$

Since the product is always commutative, the factorization, provided that $\frac{a_kf_k}{a_0f_0}\neq\pm 1$, can also be performed to construct $\overset{o}{\beta}\left(\overset{o}{\alpha}\overset{o}{\varphi}\right)$. Therefore, the product is associative for scator factors with non zero scalar components $a_0b_0f_0\neq 0$, if all three possible product pairs have non zero scalar component $\frac{a_jb_j}{a_0b_0}\neq\pm 1$, $\frac{b_jf_j}{b_0f_0}\neq\pm 1$, $\frac{a_jf_j}{a_0f_0}\neq\pm 1$, for all j from 1 to n.

Case ii) *Two scator factors have a non zero additive scalar,* $\overset{o}{\alpha},\overset{o}{\beta}\in\mathbb{S}^{1+n}_{\neq 0}$ *and one factor has zero additive scalar component,* $\overset{o}{\varphi}\in\mathbb{S}^{1+n}_{l}$.

Product $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}$. The product of scators $\overset{o}{\alpha},\overset{o}{\beta}$ is given by (1.2a), where $\overset{o}{\alpha}\overset{o}{\beta}\in\mathbb{S}^{1+n}_{\neq 0}$ since their product must have non zero scalar. The product with a scator having zero scalar component $\overset{o}{\varphi}=f_l\mathbf{e}_l$, from (1.2b) is then

$$
\begin{aligned}
\left(\overset{o}{\alpha}\overset{o}{\beta}\right)f_l\mathbf{e}_l=\,&\mp f_la_0b_0\prod_{k\neq l}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)\left(\frac{a_l}{a_0}+\frac{b_l}{b_0}\right)+f_la_0b_0\prod_{k=1}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)\mathbf{e}_l\\
&\mp\sum_{j\neq l}^{n}\left(f_l\frac{a_0b_0\prod_{k\neq l}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)\left(\frac{a_l}{a_0}+\frac{b_l}{b_0}\right)\left(\frac{a_j}{a_0}+\frac{b_j}{b_0}\right)}{\left(1\mp\frac{a_jb_j}{a_0b_0}\right)}\right)\mathbf{e}_j. \qquad (2.3)
\end{aligned}
$$

Product $\overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right)$. The product $\overset{o}{\beta}\overset{o}{\varphi}$ is obtained from (1.2b), where f_lb_l should not be zero in order to have $\overset{o}{\beta}\overset{o}{\varphi}\in\mathbb{S}^{1+n}_{\neq 0}$. The product with $\overset{o}{\alpha}$ is given by (1.2a) since both factors have non zero scalar. The resulting expression can be simplified to

$$
\begin{aligned}
\overset{o}{\alpha}\left(\overset{o}{\beta}f_l\mathbf{e}_l\right)=\,&\mp f_lb_la_0\prod_{k\neq l}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)\left(1+\frac{a_lb_0}{b_la_0}\right)\mp f_lb_la_0\left[\prod_{k\neq l}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)\left(\frac{a_l}{a_0}\mp\frac{b_0}{b_l}\right)\right]\mathbf{e}_l\\
&\mp f_lb_la_0\sum_{j\neq l}^{n}\left[\prod_{k\neq j,l}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)\left(1+\frac{a_lb_0}{b_la_0}\right)\left(\frac{a_j}{a_0}+\frac{b_j}{b_0}\right)\right]\mathbf{e}_j. \qquad (2.4)
\end{aligned}
$$

Comparison component by component of (2.3) and (2.4) shows that both expressions are identical. Evaluation of $\overset{o}{\beta}\left(\overset{o}{\alpha}\overset{o}{\varphi}\right)$ by a similar procedure shows that the three product associations give the same result.

Case iii) *One scator factor has non zero additive scalar,* $\overset{o}{\varphi}\in\mathbb{S}^{1+n}_{\neq 0}$*, and two factors have zero scalar component,* $\overset{o}{\alpha},\overset{o}{\beta}\in\mathbb{S}^{1+n}_{0}$.

The scalar component of the product $\overset{o}{\alpha}\overset{o}{\beta}=(a_l\mathbf{e}_l)(b_k\mathbf{e}_k)=\mp a_l b_k\delta_{lk}$ is not zero only if $k=l$, thus $\overset{o}{\beta}=b_l\mathbf{e}_l$.

Product $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}$. The product of $\overset{o}{\alpha}\overset{o}{\beta}=\mp a_l b_l$ times a scator $\overset{o}{\varphi}\in\mathbb{S}^{1+n}_{\neq 0}$ is $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}=\mp a_l b_l\left(f_0+\sum_{j=1}^n f_j\mathbf{e}_j\right)$.

Product $\overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right)$. The product $\overset{o}{\beta}\overset{o}{\varphi}$, is given by (1.2b), where the scalar component of this product is not zero if $f_l\neq 0$. Multiplication by $\overset{o}{\alpha}=a_l\mathbf{e}_l$, again from (1.2b), gives

$$\overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right)=a_l\mathbf{e}_l\left(b_l\mathbf{e}_l\overset{o}{\varphi}\right)=a_l b_l\left(\mp f_0+(\mp f_l)\,\mathbf{e}_l\mp\sum_{j\neq l}^{n}\left(\frac{f_0}{\mp f_l}\right)\left(\mp\frac{f_l f_j}{f_0}\right)\mathbf{e}_j\right).$$

Thus, $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}=\overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right)$. Evaluation of $\overset{o}{\beta}\left(\overset{o}{\alpha}\overset{o}{\varphi}\right)$ shows that all three associations are equal.

Case iv) *All scator factors have zero scalar,* $\overset{o}{\alpha},\overset{o}{\beta},\overset{o}{\varphi}\in\mathbb{S}^{1+n}_{0}$.

The scalar components of the products $\overset{o}{\alpha}\overset{o}{\beta}=(a_l\mathbf{e}_l)(b_k\mathbf{e}_k)=\mp a_l b_k\delta_{lk}$, $\overset{o}{\beta}\overset{o}{\varphi}=(b_k\mathbf{e}_k)\left(f_j\mathbf{e}_j\right)=\mp b_k f_j\delta_{kj}$, $\overset{o}{\alpha}\overset{o}{\varphi}=(a_l\mathbf{e}_l)\left(f_j\mathbf{e}_j\right)=\mp a_l f_j\delta_{lj}$, are not zero only if $k=l=j$, thus $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}=\overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right)=\overset{o}{\beta}\left(\overset{o}{\alpha}\overset{o}{\varphi}\right)=\mp a_j b_j f_j\mathbf{e}_j$. □

Notice that associativity holds even if the factors have zero additive scalar provided that all product pairs have non vanishing scalar. Theorem 2.1 establishes sufficient but not necessary conditions for the product associativity in the additive representation. Particular cases where the product is associative, although there are product pairs with zero scalar component, are examined in the appendix.

3. Lack of associativity in the additive representation

3.1. Conditions for the product to yield a zero additive scalar component

Factors with non vanishing scalar. Consider the product of two factors $\overset{o}{\alpha},\overset{o}{\beta}\in\mathbb{S}^{1+n}_{\neq 0}$, from (1.2a),

$$\overset{o}{\alpha}\overset{o}{\beta}=a_0b_0\prod_{k\neq l}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)\left(1\mp\frac{a_lb_l}{a_0b_0}\right)+a_0b_0\prod_{k\neq l}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)\left(\frac{a_l}{a_0}+\frac{b_l}{b_0}\right)\mathbf{e}_l$$
$$+a_0b_0\sum_{j\neq l}^{n}\left[\prod_{k\neq j,l}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)\left(1\mp\frac{a_lb_l}{a_0b_0}\right)\left(\frac{a_j}{a_0}+\frac{b_j}{b_0}\right)\right]\mathbf{e}_j,$$

where the l^{th} director component has been written out explicitly. Let this product have zero scalar component due to the l^{th} factor,

$$\left(1\mp\frac{a_lb_l}{a_0b_0}\right)=0,\Longleftrightarrow\frac{b_l}{b_0}=\pm\frac{a_0}{a_l}. \tag{3.1}$$

A necessary condition for this expression to hold is that a_l and b_l should not be zero. Then, the product $\overset{o}{\alpha}\overset{o}{\beta}\in\mathbb{S}^{1+n}_{l}$ has only one non vanishing director component given by

$$\overset{o}{\alpha}\overset{o}{\beta}=a_0b_0\prod_{k\neq l}^{n}\left(1\mp\frac{a_kb_k}{a_0b_0}\right)\left(\frac{a_l}{a_0}\pm\frac{a_0}{a_l}\right)\mathbf{e}_l. \tag{3.2}$$

One factor has vanishing scalar. If $\overset{o}{\alpha}\in\mathbb{S}^{1+n}_{l}$ and $\overset{o}{\beta}\in\mathbb{S}^{1+n}_{\neq 0}$, from (1.2b), this product has zero scalar if $a_lb_l=0$. Since the factors are not zero, $a_l\neq 0$, then the product has zero scalar component if $b_l=0$ and is given by $\overset{o}{\alpha}\overset{o}{\beta}=(a_l\mathbf{e}_l)\overset{o}{\beta}=a_lb_0\mathbf{e}_l$. Notice that this product is never zero for non zero scator factors, a result that is of paramount importance when evaluating differential quotients [8].

Both factors have vanishing scalar From (1.2c), $(a_l\mathbf{e}_l)\left(b_j\mathbf{e}_j\right)=\mp a_lb_j\delta_{lj}$. Only if $j\neq l$, the additive scalar is zero.

Zero products For scator factors with non vanishing scalar, if two or more factors are zero in expression (1.2a),

$$\left(1\mp\frac{a_lb_l}{a_0b_0}\right)=0,\ \left(1\mp\frac{a_mb_m}{a_0b_0}\right)=0\text{ for }l\neq m, \tag{3.3}$$

the additive scalar and all director components are zero, $\overset{o}{\alpha}\overset{o}{\beta}=\overset{o}{0}=0$. For scator factors with vanishing scalar, the scator product is zero only if $j\neq l$, $\overset{o}{\alpha}\overset{o}{\beta}=(a_l\mathbf{e}_l)\left(b_j\mathbf{e}_j\right)=0$. The zero product condition, where non associativity has been asserted before [6, 11, 12], will now be encompassed in the condition where a scator product pair with zero additive scalar component is involved. Besides the zero products, the product can also be zero if any of the factors is zero. This trivial case is dismissed since only non zero factors are considered as mentioned before.

3.2. Non associative conditions

Theorem 3.1. *The scator product in $\mathbb{S}^{1+n}$ is in general not associative for $n>1$ in the additive representation, if one or more of the possible product pairs has a vanishing additive scalar component,* $(\overset{o}{\alpha}\overset{o}{\beta})\overset{o}{\varphi} \neq \overset{o}{\alpha}(\overset{o}{\beta}\overset{o}{\varphi}) \neq \overset{o}{\beta}(\overset{o}{\alpha}\overset{o}{\varphi})$ *if* $\overset{o}{\alpha}\overset{o}{\beta}$ *or* $\overset{o}{\beta}\overset{o}{\varphi}$ *or* $\overset{o}{\alpha}\overset{o}{\varphi} \in \mathbb{S}_0^{1+n}$.

Proof. Case 1. *Product of three factors with non zero scalar component,* $\overset{o}{\alpha},\overset{o}{\beta},\overset{o}{\varphi} \in \mathbb{S}_{\neq 0}^{1+n}$.

Subcase 1.1 Only one product pair has a vanishing scalar component, $\overset{o}{\alpha}\overset{o}{\beta} \in \mathbb{S}_0^{1+n}$, $\overset{o}{\beta}\overset{o}{\varphi},\overset{o}{\alpha}\overset{o}{\varphi} \in \mathbb{S}_{\neq 0}^{1+n}$.

Product $(\overset{o}{\alpha}\overset{o}{\beta})\overset{o}{\varphi}$. The $\overset{o}{\gamma}=\overset{o}{\alpha}\overset{o}{\beta} \in \mathbb{S}_l^{1+n}$ product is given by (3.2) and its product with a scator $\overset{o}{\varphi} \in \mathbb{S}_{\neq 0}^{1+n}$ is then given by (1.2b),

$$(\overset{o}{\alpha}\overset{o}{\beta})\overset{o}{\varphi} = a_0 b_0 \prod_{k\neq l}^{n}\left(1 \mp \frac{a_k b_k}{a_0 b_0}\right)\left(\frac{a_l}{a_0} \pm \frac{a_0}{a_l}\right)\left(\mp f_l + f_0 \mathbf{e}_l \mp \sum_{j\neq l}^{n}\left(\frac{f_l f_j}{f_0}\right)\mathbf{e}_j\right). \tag{3.4}$$

This result is not symmetric in the three scators variables, thus already indicative of the lack of associativity, but let us not anticipate.

Product $\overset{o}{\alpha}(\overset{o}{\beta}\overset{o}{\varphi})$. Evaluate $\overset{o}{\beta}\overset{o}{\varphi} \in \mathbb{S}_{\neq 0}^{1+n}$ from (1.2a). The product with $\overset{o}{\alpha}$ using (1.2a) again, is then

$$\begin{aligned}\overset{o}{\alpha}(\overset{o}{\beta}\overset{o}{\varphi}) = {} & a_0 b_0 f_0 \prod_{k=1}^{n}\left(1 \mp \frac{b_k f_k}{b_0 f_0}\right)\left(1 \mp \frac{\left(\frac{b_k}{b_0}+\frac{f_k}{f_0}\right)}{\left(1 \mp \frac{b_k f_k}{b_0 f_0}\right)}\frac{a_k}{a_0}\right) \\ & + b_0 f_0 \prod_{k=1}^{n}\left(1 \mp \frac{b_k f_k}{b_0 f_0}\right) a_0 \sum_{j=1}^{n}\prod_{k\neq j}^{n}\left(1 \mp \frac{\left(\frac{b_k}{b_0}+\frac{f_k}{f_0}\right)}{\left(1 \mp \frac{b_k f_k}{b_0 f_0}\right)}\frac{a_k}{a_0}\right)\left(\frac{\frac{b_j}{b_0}+\frac{f_j}{f_0}}{\left(1 \mp \frac{b_j f_j}{b_0 f_0}\right)}+\frac{a_j}{a_0}\right)\mathbf{e}_j.\end{aligned} \tag{3.5}$$

The product $(\overset{o}{\alpha}\overset{o}{\beta})\overset{o}{\varphi}$ given by (3.4) and the product $\overset{o}{\alpha}(\overset{o}{\beta}\overset{o}{\varphi})$ given by (3.5) are clearly not equal for arbitrary scator coefficients. In appendix A.1, particular scator coefficients when associativity holds are evaluated in this and subsequent cases. If, in addition to $\frac{a_l b_l}{a_0 b_0}=\pm 1$, another factor $\frac{a_m b_m}{a_0 b_0}=\pm 1$, $m\neq l$, then $\overset{o}{\alpha}\overset{o}{\beta}=\overset{o}{0}$ and $(\overset{o}{\alpha}\overset{o}{\beta})\overset{o}{\varphi}$ is also zero. However, this condition does not make the product $\overset{o}{\alpha}(\overset{o}{\beta}\overset{o}{\varphi})$ given by (3.5) equal to zero. Non associativity due to zero products is a particular case of non associativity due to products with zero scalar component. Since $\overset{o}{0} \in \mathbb{S}_0^{1+n}$, they need not be treated separately.

Subcase 1.2 Two product pairs have null scalar component, $\overset{o}{\alpha}\overset{o}{\beta},\overset{o}{\beta}\overset{o}{\varphi} \in \mathbb{S}_0^{1+n}$, $\overset{o}{\alpha}\overset{o}{\varphi} \in \mathbb{S}_{\neq 0}^{1+n}$.

Product $\overset{o}{\alpha}(\overset{o}{\beta}\overset{o}{\varphi})$. Let the zero scalar in the $\overset{o}{\beta}\overset{o}{\varphi}$ product originate due to the p^{th} factor, $\left(1 \mp \frac{b_p f_p}{b_0 f_0}\right)=0$. The subsequent product with an arbitrary scator $\overset{o}{\alpha}$ with non vanishing scalar is then given by (1.2b),

$$\overset{o}{\alpha}(\overset{o}{\beta}\overset{o}{\varphi}) = b_0 f_0 \prod_{k\neq p}^{n}\left(1 \mp \frac{b_k f_k}{b_0 f_0}\right)\left(\frac{b_p}{b_0}+\frac{f_p}{f_0}\right)\left(\mp a_p + a_0 \mathbf{e}_p \mp \sum_{j\neq p}^{n}\left(\frac{a_p a_j}{a_0}\right)\mathbf{e}_j\right). \tag{3.6}$$

Comparison of Eqs. (3.4) and (3.6) shows that these two expressions are different either for $l=p$ or $l\neq p$, for otherwise arbitrary scator coefficients.

Subcase 1.3 All three product pairs have null scalar component, $\overset{o}{\alpha}\overset{o}{\beta},\overset{o}{\beta}\overset{o}{\varphi},\overset{o}{\alpha}\overset{o}{\varphi} \in \mathbb{S}_0^{1+n}$. The derivation involves, in addition to the previous case, a zero scalar due to the q^{th} factor, $\left(1 \mp \frac{a_q f_q}{a_0 f_0}\right)=0$ of the $\overset{o}{\alpha}\overset{o}{\varphi}$ product. The procedure is analogous to the previous subcase.

Case 2. *Product of two scator factors with non zero additive scalar,* $\overset{o}{\alpha},\overset{o}{\beta} \in \mathbb{S}_{\neq 0}^{1+n}$, *and one factor with zero scalar,* $\overset{o}{\varphi} \in \mathbb{S}_m^{1+n}$.

Subcase 2.1 Only one product pair has a vanishing scalar $\overset{o}{\alpha}\overset{o}{\beta} \in \mathbb{S}_l^{1+n}$, $\overset{o}{\beta}\overset{o}{\varphi},\overset{o}{\alpha}\overset{o}{\varphi} \in \mathbb{S}_{\neq 0}^{1+n}$.

Product $(\overset{o}{\alpha}\overset{o}{\beta})\overset{o}{\varphi}$. If $\overset{o}{\alpha},\overset{o}{\beta} \in \mathbb{S}_{\neq 0}^{1+n}$ and $\overset{o}{\alpha}\overset{o}{\beta} \in \mathbb{S}_l^{1+n}$, the product is given by (3.2). The product of this result times $\overset{o}{\varphi}=f_m\mathbf{e}_m \in \mathbb{S}_m^{1+n}$ is from (1.2c),

$$(\overset{o}{\alpha}\overset{o}{\beta})\overset{o}{\varphi} = \mp a_0 b_0 f_m \prod_{k\neq l}^{n}\left(1 \mp \frac{a_k b_k}{a_0 b_0}\right)\left(\frac{a_l}{a_0} \pm \frac{a_0}{a_l}\right)\delta_{lm}. \tag{3.7}$$

Product $\overset{o}{\alpha}(\overset{o}{\beta}\overset{o}{\varphi})$. The product $\overset{o}{\beta}\overset{o}{\varphi}$, since $b_m\neq 0$ is obtained from (1.2b). The product with $\overset{o}{\alpha}$, if $l\neq m$, is

$$\overset{o}{\alpha}(\overset{o}{\beta}\overset{o}{\varphi}) = \mp a_0 b_m f_m \prod_{k\neq l,m}^{n}\left(1 \mp \frac{a_k b_k}{a_0 b_0}\right)\left(1 + \frac{a_m b_0}{a_0 b_m}\right)\left(\frac{a_l}{a_0} \pm \frac{a_0}{a_l}\right)\mathbf{e}_l. \tag{3.8}$$

This single director component scator differs from (3.7), since the latter has no director component. If $l=m$,

$$\overset{o}{\alpha}(\overset{o}{\beta}\overset{o}{\varphi}) = \mp a_0 b_l f_l \left(1 \pm \frac{a_l^2}{a_0^2}\right)\prod_{k\neq l}^{n}\left(1 \mp \frac{a_k b_k}{a_0 b_0}\right) \mp a_0 b_l f_l \left(1 \pm \frac{a_l^2}{a_0^2}\right)\sum_{j\neq l}^{n}\left[\prod_{k\neq j,l}^{n}\left(1 \mp \frac{a_k b_k}{a_0 b_0}\right)\left(\frac{a_j}{a_0}+\frac{b_j}{b_0}\right)\right]\mathbf{e}_j. \tag{3.9}$$

This expression also differs from (3.7). Subcases 2.2 where two product pairs have a vanishing scalar $\overset{o}{\alpha}\overset{o}{\beta},\overset{o}{\beta}\overset{o}{\varphi} \in \mathbb{S}_0^{1+n}$, $\overset{o}{\alpha}\overset{o}{\varphi} \in \mathbb{S}_{\neq 0}^{1+n}$ and 2.3 where all three product pairs have null scalar component, $\overset{o}{\alpha}\overset{o}{\beta},\overset{o}{\beta}\overset{o}{\varphi},\overset{o}{\alpha}\overset{o}{\varphi} \in \mathbb{S}_0^{1+n}$ are similarly tackled.

Case 3. *Product of one scator factor with non zero scalar,* $\overset{o}{\varphi} \in \mathbb{S}^{1+n}_{\neq 0}$*, and two factors with zero additive scalar,* $\overset{o}{\alpha}, \overset{o}{\beta} \in \mathbb{S}^{1+n}_{0}$.
Subcase 3.1 $l \neq m$
Product $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}$. The product of two scators $\overset{o}{\alpha}, \overset{o}{\beta} \in \mathbb{S}^{1+n}_{0}$ with different non zero director component, say $\overset{o}{\alpha} = a_m\mathbf{e}_m$, $\overset{o}{\beta} = b_l\mathbf{e}_l$, from (1.2c), is zero. Their product with an arbitrary scator $\overset{o}{\varphi}$ is zero, $\left(a_m\mathbf{e}_m b_l\mathbf{e}_l\right)\overset{o}{\varphi} = 0$.
Product $\overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right)$. The product of $\overset{o}{\varphi} \in \mathbb{S}^{1+n}_{\neq 0}$ times a single director component scator $\overset{o}{\beta} = b_l\mathbf{e}_l$, obtained from (1.2b), is

$$b_l\mathbf{e}_l\overset{o}{\varphi} = b_l\left(\mp f_l + f_0\mathbf{e}_l \mp \left(\frac{f_l f_m}{f_0}\right)\mathbf{e}_m \mp \sum_{j\neq l,m}^{n}\left(\frac{f_l f_j}{f_0}\right)\mathbf{e}_j\right),$$

where the l^{th} and m^{th} terms are shown explicitly. Multiplication by $\overset{o}{\alpha} = a_m\mathbf{e}_m$, recalling that $m \neq l$, again from (1.2b) is,

$$\overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right) = a_m\mathbf{e}_m\left(b_l\mathbf{e}_l\overset{o}{\varphi}\right) = a_m b_l\left(\left(\frac{f_l f_m}{f_0}\right) \mp f_l\mathbf{e}_m \mp f_m\mathbf{e}_l + \sum_{j\neq l,m}^{n}\left(\frac{f_l f_m f_j}{f_0^2}\right)\mathbf{e}_j\right). \tag{3.10}$$

This expression is not zero and thus differs from the product $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}$.
Subcase 3.2 If $l = m$, the product $\left(\overset{o}{\alpha}\overset{o}{\beta}\right) = \left(a_m\mathbf{e}_m b_l\mathbf{e}_l\right) = a_l b_l$ has non zero scalar component. The product of $\overset{o}{\alpha}\overset{o}{\beta}$ with $\overset{o}{\varphi} \in \mathbb{S}^{1+n}_{\neq 0}$ gives a scator with non zero scalar component. Thus, a product pair with zero scalar component cannot be achieved in this subcase.
Case 4. *Product of three scator factors with zero additive scalar* $\overset{o}{\alpha}, \overset{o}{\beta}, \overset{o}{\varphi} \in \mathbb{S}^{1+n}_{0}$.
If the scator $\overset{o}{\varphi}$ also has vanishing scalar component, the product definition (1.2c) has to be used throughout. For $l \neq m$, $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi} = \left(a_l\mathbf{e}_l b_m\mathbf{e}_m\right)\left(f_m\mathbf{e}_m\right) = 0$ and $\overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right) = \left(a_l\mathbf{e}_l\right)\left(b_m\mathbf{e}_m f_m\mathbf{e}_m\right) = a_l b_m f_m\mathbf{e}_m$. Therefore, this product is not associative if two director components have different subindices. □

In accordance with Theorems 2.1 and 3.1, non vanishing additive scalar components in all possible product pairs is a sufficient condition for scator associativity while at least one product pair with vanishing scalar component is a necessary condition for non associativity.

4. Wavefunction collapse

The quantum mechanical wave function, according with the prevailing Copenhagen view, contains probabilistic information regarding the state of a system. In a simplified scheme, when an observation is made, only one state is detected from the superposition of all possible states. This process, whereby the probability of all but one state become zero and the remaining state becomes a certainty, is the collapse or reduction of the wave function. Continuous spectrum operators such as position or momentum do not collapse to a single state, but to a combination of eigenstates within a spread of eigenvalues given by the nature of the measurement. The wave function reduction, whether to a single state or a range of closely packed states determines the state of the system up to the precision imposed by the uncertainty principle. Let us recall the formal description of the two processes involved in the evolution of a quantum system as a preliminary to the scator description. In the Schrödinger representation, for a time independent Hamiltonian $\mathscr{H}$, the time evolution of the wave function $|\psi\rangle$ is given by a unitary time evolution propagator $\mathscr{U}$, so that $|\psi(t)\rangle = \mathscr{U}\,|\psi(0)\rangle$. The wave function is written as a superposition of energy eigenfunctions $|\psi\rangle = \sum_{j=1}^{n} c_j\left|\psi_j\right\rangle$, such that $\mathscr{H}\left|\psi_j\right\rangle = \mathscr{E}_j\left|\psi_j\right\rangle$. The set of eigenfunctions and their corresponding propagators in matrix form are $\psi(t) = \mathbf{U}\psi(0)$. If the systems evolves from 0 to t_1 and then from t_1 to t_2 and so on, the evolution may be written as a product of matrices $\psi(t) = \mathbf{U}_f\cdots\mathbf{U}_2\mathbf{U}_1\,\psi(0)$, provided that there are no state reductions in the process. This picture is broken down when the operator $\mathbf{R}$, reduces the wavefunction to a single j state, $\mathbf{R}\psi(t) = \left|\psi_j\right\rangle$. These two distinct procedures $\mathbf{U}$ and $\mathbf{R}$, required in order to describe a quantum system, are referred to as the quantum measurement problem [1]. Different approaches have been proposed to unify these two mechanisms under the generic name of dynamical reduction models [2]. The aim of the following scator description is to contribute towards the formalization of a reduction model.
Allow for the quantum wave function to be described by a scator function $\overset{o}{\psi}(t)$ instead of a column vector. Each eigenfunction $\left|\psi_j\right\rangle$ is associated with a scator's director component $\left|\psi_j\right\rangle \to \mathbf{e}_j$. The wavefunction eigenstates are described by the director components of a scator function $|\psi\rangle + c_0 = \overset{o}{\psi} = c_0 + \sum_{j=1}^{n} c_j\mathbf{e}_j$. This is certainly a huge formal leap, and it is not certain that it is possible, but let us continue with the sketch of the procedure in the scator formalism. Notice, that we must add a scalar function c_0 for $n \geq 2$, otherwise, the scator is not in $\mathbb{S}^{1+n}$. Since all the possible wave eigenfunctions have been ascribed to the director components of the scator, we have freedom to establish the value of the scalar function. This function can be associated with the time variable. The time evolution of the system is modeled by the product of unitary scators $\overset{o}{\Upsilon}_k$ times the system's wavefunction

$$\overset{o}{\psi}(t) = \overset{o}{\Upsilon}_f\cdots\overset{o}{\Upsilon}_2\overset{o}{\Upsilon}_1\overset{o}{\psi}(0),$$

where $\overset{o}{\Upsilon}_k \in \mathbb{S}^{1+n}$ for all k from the initial to the final time interval f and $\overset{o}{\psi}(0) \in \mathbb{S}^{1+n}$. Unitary scators are obtained from the main involution of scator algebra; The conjugate of the scator $\overset{o}{\varphi} = f_0 + \sum_{j=1}^{n} f_j\mathbf{e}_j$, is given by $\overset{o}{\varphi}^* = f_0 - \sum_{j=1}^{n} f_j\mathbf{e}_j$, that is, all its director components reverse sign while the scalar component remains unaltered. The product $\overset{o}{\varphi}\overset{o}{\varphi}^* = \left\|\overset{o}{\varphi}\right\|^2$ is a real number that establishes a magnitude or order parameter. Scators of the form $\overset{o}{\varphi}/\left\|\overset{o}{\varphi}\right\|$ constitute the set of unit magnitude scators [6]. Let the scalar component represent time evolution and wavefunction reduction, the latter taking place when the scalar component becomes zero, as the reader may have already guessed. When the scator product yields a zero scalar additive component, all director components must be zero with one possible exception as was shown in subsection 3.1. Since each director component has been chosen to represent an eigenstate, the product then models a wave function collapse, for in this circumstance only one eigenfunction subsists. Depending on which is the l^{th} factor that fulfills condition (3.1), it is the l^{th} eigenfunction from the 1 to n eigenfunctions that survives. Therefore, an **R** measurement process is attained.

If the products have a non vanishing additive scalar, all director components prevail and since each represents a wave eigenfunction, a superposition of all of them is obtained. The products are then associative (Theorem 2.1) and can be performed with any precedence or grouped together, i.e. $\overset{o}{\Upsilon}_3\left(\overset{o}{\Upsilon}_2\left(\overset{o}{\Upsilon}_1\overset{o}{\psi}(0)\right)\right) = \left(\overset{o}{\Upsilon}_3\overset{o}{\Upsilon}_2\overset{o}{\Upsilon}_1\right)\overset{o}{\psi}(0) = \overset{o}{\Upsilon}_i\overset{o}{\psi}(0)$. The way these products are associated, represent an evolution that can be traced in several steps, grouped with any precedence or the evolution performed in one single step. Whether these products commute, depend on the operators involved in the scator coefficients. This is analogous to the usual quantum description with complex algebra. Complex algebra is commutative, but the complex numbers involve operators in quantum mechanics that may or may not commute. Physically, time evolution **U** processes are then described when the scator products have non vanishing additive scalar.

The scator product provides a unified description of the **U** and **R** processes. For example, $\left(\overset{o}{\Upsilon}_4\overset{o}{\Upsilon}_i\right) \in \mathbb{S}_0^{1+n}$ represents a collapse and the product $\overset{o}{\Upsilon}_5\left(\overset{o}{\Upsilon}_4\overset{o}{\Upsilon}_i\right) \in \mathbb{S}_{\neq 0}^{1+n}$ the subsequent **U** evolution. When the product is not associative, it reflects the fact that the subsequent evolution of the system is altered by the measurement or collapse of the wave function. After a collapse, the time evolution of the system is again described by another wave function, where the initial conditions are given by the state of the collapsed system. Notice that the product of the collapsed system, say in the $\left(\overset{o}{\Upsilon}_4\overset{o}{\Upsilon}_i\right) = a_l\mathbf{e}_l$ state, regains a finite probability for all possible states when its product is taken with a unitary evolution scator $\overset{o}{\Upsilon}_5 = \overset{o}{\beta}$, with arbitrary non vanishing director components (equal to the number of possible states) given by the product definition (1.2b). Therefore, one and the same mathematical procedure, the scator product, is used to describe the complete time evolution including the possible wave function reductions of the physical system. This scheme is in the vein of the Penrose proposal [15, Sec.22.1, p.529] whereby the same mathematical object, the scator product in this approach, describes the **U** and **R** processes. It could be argued that the two distinct procedures have only been deferred to the two definitions of the scator product (1.2a) and (1.2b), depending on whether the scalar component of $\overset{o}{\alpha}$ vanishes or not. However, it is possible to obtain the product (1.2b) from (1.2a) through a limit. In order to remain within the $\mathbb{S}^{1+n}$ set, the limit of all but one of the director components should be taken first and thereafter the scalar component limit should be taken [7]. The $\overset{o}{\alpha}$ director component limits from (1.2a) is

$$\lim_{a_{k\neq l}\to 0}\left(\overset{o}{\alpha}\overset{o}{\beta}\right) = \left(a_0b_0 \mp a_lb_l\right) + \left(a_lb_0 + a_0b_l\right)\mathbf{e}_l + \sum_{j\neq l}^{n}\left(a_0b_0 \mp a_lb_l\right)\left(\frac{b_j}{b_0}\right)\mathbf{e}_j.$$

Thereafter, the scalar component limit is

$$\lim_{a_0\to 0}\left(\lim_{a_{k\neq l}\to 0}\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\right) = \mp a_lb_l + a_lb_0\mathbf{e}_l \mp \sum_{j\neq l}^{n} a_lb_l\left(\frac{b_j}{b_0}\right)\mathbf{e}_j.$$

But this result is identical to (1.2b), thus this latter definition is a removable singularity of the product function (1.2a) with arbitrary but non vanishing scalar components. Another asset of this formalism is that no 'tails' are present when the collapse takes place. The tail problem arises because finite, albeit small, amplitude probabilities for states other than the reduced state are present in some proposals, such as the GRW scheme [2]. In the present scheme, all states different from the collapsed state are strictly zero. There are other issues that require careful assessment to confirm whether this proposal is plausible. Two ingredients are essential to achieve a unified dynamical scheme, nonlinearity and an stochastic process. Here, the nonlinearity is ultimately due to the scator product definition. However, an stochastic process still needs to be incorporated. Nonetheless, according to this initial assessment, scator algebra seems to be a promising route for a unified description of quantum dynamics.

5. Conclusions

Product associativity in either real or imaginary scator algebras depends on whether the additive scalar component of any product pair is zero. The scator product is associative in $\mathbb{S}^{1+n}$ if all possible product pairs have a non vanishing additive scalar component. Recall that the scator set $\mathbb{S}^{1+n}$ where the product is defined in the additive representation, is the subset of $\mathbb{R}^{1+n}$ where the additive scalar is different from zero if two or more director components are not zero. The scator product is in general not associative in the additive representation in $\mathbb{S}^{1+n}$ for dimensions with $n > 1$, if one or more of the possible product pairs has a vanishing additive scalar component. At least one product pair with zero scalar component is a necessary condition for non associativity. These assertions are embodied in theorems 2.1 and 3.1.

The additive scalar component of any product pair is the decisive parameter in order to establish whether the scator product is associative or not. The stronger condition stated in previous communications for lack of associativity, if all components of any product pair are zero (i.e. a zero scator), has been included in the zero additive scalar component necessary condition (i.e. a scator with possibly one non vanishing director component).

The main involution is scator algebras is conjugation [6]. The *conjugate* of a scator $\overset{o}{\varphi} = f_0 + \sum_{j=1}^{n} f_j\mathbf{e}_j$, is defined by the negative of its director components while the scalar component remains unchanged, $\overset{o}{\varphi}^{*} \equiv f_0 - \sum_{j=1}^{n} f_j\mathbf{e}_j$. The *magnitude* of a scator is defined by the positive square root of the scator times its conjugate $\left\|\overset{o}{\varphi}\right\| = \sqrt{\overset{o}{\varphi}\overset{o}{\varphi}^{*}}$. From the product definition, clearly $\left\|\overset{o}{\varphi}\right\|^2 \in \mathbb{R}$.

Corollary 5.1. *In $\mathbb{S}^{1+n}$, the magnitude of two or more scator products is equal to the product of their magnitudes if the products are associative.*

Proof. The square magnitude of the product of two scators $\overset{o}{\alpha}, \overset{o}{\beta}$, according to the magnitude definition is $\left\|\overset{o}{\alpha}\overset{o}{\beta}\right\|^2 = \left(\overset{o}{\alpha}\overset{o}{\beta}\right)\left(\overset{o}{\alpha}^{*}\overset{o}{\beta}^{*}\right)$. Since the scator product is commutative $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\left(\overset{o}{\alpha}^{*}\overset{o}{\beta}^{*}\right) = \left(\overset{o}{\alpha}\overset{o}{\beta}\right)\left(\overset{o}{\beta}^{*}\overset{o}{\alpha}^{*}\right)$. If the product is associative $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\left(\overset{o}{\beta}^{*}\overset{o}{\alpha}^{*}\right) = \overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\beta}^{*}\right)\overset{o}{\alpha}^{*} = \left(\overset{o}{\alpha}\overset{o}{\alpha}^{*}\right)\left(\overset{o}{\beta}\overset{o}{\beta}^{*}\right)$, thus $\left\|\overset{o}{\alpha}\overset{o}{\beta}\right\| = \left\|\overset{o}{\alpha}\right\|\left\|\overset{o}{\beta}\right\|$. $\square$

The magnitude of the scator products is equal to the product of the scator magnitudes if all possible product pairs have a non vanishing additive scalar component. Therefore, the $\overset{o}{\alpha}\overset{o}{\beta} \neq 0$ condition stated in Lemma 4.1 [12], is now corrected and embraced in the condition stated in this corollary. Notice that $\overset{o}{\varphi} + \overset{o}{\varphi}^* = 2f_0$, thus a scator product with non zero scalar component can be stated in terms of the involution as $\overset{o}{\alpha}\overset{o}{\beta} + \overset{o}{\alpha}^*\overset{o}{\beta}^* \neq 0$.

A. Appendix: Associativity exceptions

There are three generic conditions where the scalar component of a product pair is zero but associativity holds: i) The $\mathbb{S}^{1+1_j}$ subsets; ii) Two scator factors are linearly dependent on all director coefficients except for the two that produce a zero scalar ; iii) All product pairs give zero products.
The foremost exception are scator sets with only one director component

$$\mathbb{S}^{1+1_j} = \left\{ \overset{o}{\varphi} = f_0 + \sum_{k=1}^{n} f_k \mathbf{e}_k, \in \mathbb{S}^{1+n} : f_k = 0, \text{ for any } k \neq j \right\}.$$

Numbers in the $\mathbb{S}^{1+1_j}$ subset are of the form $\overset{o}{\varphi} = f_0 + f_j\mathbf{e}_j$, these subsets are isomorphic to the complex field for each j, [10, Proposition 4.12]. The scator product definition (1.2a)-(1.2c) can then be grouped in the usual single product definition for complex numbers (or hyperbolic numbers). The scator product is thus associative in $\mathbb{S}^{1+1_j}$ even if a product pair has zero scalar component.
In the proof of Theorem 3.1, non associativity was analyzed in various cases where different expressions were obtained depending on the precedence of evaluation. These expressions are forced here to be equal by imposing the necessary conditions on the scator coefficients.

A.1. Linear dependence except for l^{th} component (Case 1)

The product in subcase 1.1 of Theorem 3.1 is not associative unless $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi}$ given by (3.4) and $\overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right)$ given by (3.5) are equal. Equating the scalar terms imposes

$$\prod_{k\neq l}^{n}\left(1 \mp \frac{a_k b_k}{a_0 b_0}\right) = \prod_{k\neq l}^{n}\left(1 \mp \frac{a_k b_k}{a_0 b_0} \mp \frac{b_k f_k}{b_0 f_0} \mp \frac{a_k f_k}{a_0 f_0}\right). \tag{A.1}$$

The $\mathbf{e}_j$ components in (3.4) and (3.5) are equal only if, in addition to the above condition,

$$\frac{f_j}{f_0} = \frac{\left(\frac{a_j}{a_0} + \frac{b_j}{b_0} + \frac{f_j}{f_0} \mp \frac{a_j}{a_0}\frac{b_j f_j}{b_0 f_0}\right)}{\left(1 \mp \frac{b_j f_j}{b_0 f_0} \mp \frac{a_j b_j}{a_0 b_0} \mp \frac{a_j f_j}{a_0 f_0}\right)},$$

for all $j \neq l$ from 1 to n. This equality holds provided that i) $\frac{a_j}{a_0} + \frac{b_j}{b_0} \neq 0$ and $f_j^2 = \mp f_0^2$, but this condition cannot be fulfilled for imaginary scators except if $f_j = f_0 = 0$, but then $\overset{o}{\varphi} \in \mathbb{S}_0^{1+n}$, contrary to the initial premise. For real scators $f_j^2 = f_0^2$, implies that the scator lies in the null geodesic surface [12]. ii) If $\frac{a_j}{a_0} + \frac{b_j}{b_0} = 0$, then $b_j = -b_0\frac{a_j}{a_0}$. The product of scators

$$\overset{o}{\alpha} = a_0\left(1 + \sum_{j=1}^{n}\frac{a_j}{a_0}\mathbf{e}_j\right), \quad \overset{o}{\beta} = b_0\left(1 - \sum_{j\neq l}^{n}\frac{a_j}{a_0}\mathbf{e}_j \pm \frac{a_0}{a_l}\mathbf{e}_l\right), \quad \overset{o}{\varphi} = f_0 + \sum_{j=1}^{n} f_j\mathbf{e}_j,$$

is then associative although $\overset{o}{\alpha}\overset{o}{\beta} \in \mathbb{S}_0^{1+n}$. The scators $\overset{o}{\alpha}$ and $\overset{o}{\beta}$ are linearly dependent on all $\mathbf{e}_j$ coefficients except for the $\mathbf{e}_l$ coefficient, where $a_l b_l = \pm a_0 b_0$ produces a product $\overset{o}{\alpha}\overset{o}{\beta}$ with zero scalar component. Associativity in subcase 1.2 of Theorem 3.1 is obtained by imposing the equality of equations (3.4) and (3.6). This equality again requires linear dependence on all but one of the director components.

A.2. Zero product pairs (Cases 2, 3 and 4)

Subcase 2.1 If $l \neq m$, equations (3.7) and (3.8), require the scalar component to be equal to the director component $\mathbf{e}_l$, a condition that can only be satisfied if both coefficients are zero. A factor in the product $\prod_{k\neq l,m}^{n}\left(1 \mp \frac{a_k b_k}{a_0 b_0}\right)$ of equation (3.8) then has to be zero. Hence $\left(\overset{o}{\alpha}\overset{o}{\beta}\right)\overset{o}{\varphi} = \overset{o}{\alpha}\left(\overset{o}{\beta}\overset{o}{\varphi}\right) = \overset{o}{\beta}\left(\overset{o}{\alpha}\overset{o}{\varphi}\right) = 0$. Subcase 2.1 with $l = m$, as well as subcase 2.2 and cases 3 and 4 are similarly obtained and again associativity holds when all product pairs are zero.

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Study of Kenmotsu manifolds with semi-symmetric metric connection

Sudhakar Kr Chaubey[a*] and Sunil Kr Yadav[b]

[a]*Section of Mathematics, Department of Information Technology, Shinas College of Technology, Shinas, P.O. Box 77 Postal Code 324, Oman*
[b]*Department of Mathematics, Poornima college of Engineering, Jaipur-302022, Rajasthan, India*
[*]*Corresponding author E-mail: sk22_math@yahoo.co.in; sudhakar.chaubey@shct.edu.om*

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Abstract

The present paper deals with the study of Kenmotsu manifolds equipped with a semi-symmetric metric connection. The properties of $\eta-$parallel Ricci tensor, globally symmetric and $\phi-$symmetric Kenmotsu manifolds with the semi-symmetric metric connection are evaluated. In the end, we construct an example of a $3-$dimensional Kenmotsu manifold admitting semi-symmetric metric connection and verify our some results.

1. Introduction

The notion of curvatures play a central role in the differential geometry and physics. For instant, the magnitude of a force required to move an object at constant speed along a curve path is, according to Newton's laws, a constant multiple of the curvature of the trajectory. The motion of a body in a gravitational field is determined, according to Einstein, by the curvature of spacetime [1]. The space of constant curvature also plays a key role in differential geometry and mathematical physics (specially in the theory of relativity and cosmology). In 1926, Cartan ([2], [3]) introduced the notion of a locally symmetric Riemannian manifold which is the natural generalization of manifolds of constant curvature. The condition of local symmetry is equivalent to the fact that at every point $x \in M$, the local geodesic symmetry $F(x)$ is an isometry [4]. The idea of locally $\phi-$symmetric Sasakian manifold was introduced by Takahashi [35] in 1977. Since then, the properties of such manifolds have been studied by several geometers on different spaces.

The study of odd dimensional manifolds with contact and almost contact structures were initiated by Boothby and Wong [5] in 1958 rather from topological point of view. Sasaki and Hatakeyama [6] re-investigated them using tensor calculus in 1961. In 1972, K. Kenmotsu studied a class of almost contact metric manifolds and call them Kenmotsu manifold [7]. He proved that if a Kenmotsu manifold satisfies the condition $R(X,Y).R = 0$, then the manifold is of negative curvature -1, where R is the Riemannian curvature tensor of type $(1,3)$ and $R(X,Y)$ denotes the derivation of the tensor algebra at each point of the tangent space $T(M)$. The properties of Kenmotsu manifolds have been noticed in ([9] - [17]) and by others.

The notion of a semi-symmetric linear connection on a differentiable manifold has been introduced by Friedmann and Schouten [20] in 1924. Hayden [21] in 1932, introduced and studied the idea of semi-symmetric linear connection with torsion on a Riemannian manifold. After a long interval, Yano [22] started the systematic study of a semi-symmetric metric connection on a Riemannian manifold in 1970. Since then the properties of semi-symmetric metric connection on different spaces have studied in ([27]-[32]) and the references therein.

Motivated from the above studied, authors start the study of the properties of Kenmotsu manifold equipped with a semi-symmetric metric connection. We organize the present paper as follows: Section 2 contains the basic known results of Kenmotsu manifolds and $\eta-$parallel Ricci tensor. The brief results of the semi-symmetric metric connection on a Kenmotsu manifold are given in section 3. Section 4 deals the study of $\eta-$parallel Ricci tensor with respect to the semi-symmetric metric connection on the Kenmotsu manifold and find some geometrical results. The properties of concircular and projective curvature tensors endowed with a semi-symmetric metric connection are investigated in section 5. In last section, we construct an example of Kenmotsu manifold equipped with semi-symmetric connection and verify our results.

Email addresses: **sk22_math@yahoo.co.in (S. K. Chaubey), prof_sky16@yahoo.com (S. K. Yadav)**

2. Preliminaries

An odd dimensional differentiable manifold M $(dimM = n = 2m+1)$ of class C^{∞} is said to have a $(\phi, \xi, \eta)-$structure or almost contact structure if it admits a tensor field ϕ of endomorphisms of the tangent spaces, a vector field ξ, and a $1-$form η satisfying

$$\eta(\xi) = 1 \;\; and \;\; \phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

where I denotes the identity transformation [25]. From (2.1), it can be easily see that $\phi\xi = 0$, $\eta \circ \phi = 0$ and rank $\phi = n-1$. A Riemannian metric g of type $(0,2)$ is said to be compatible with the almost contact structure (ϕ, ξ, η) if the relation

$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y) \tag{2.2}$$

holds for arbitrary vector fields X and Y. An almost contact structure (ϕ, ξ, η) equipped with a compatible Riemannian metric g is known as almost contact metric structure (ϕ, ξ, η, g) and the manifold M endowed with the almost contact structure is called an almost contact metric manifold. If moreover,

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.3}$$

holds for all X on $M(\phi, \xi, \eta, g)$, then the manifold is said to be Kenmotsu manifold [7]. Here ∇ denotes the Levi-Civita connection of the metric g. For proving our main results in next sections, we are going to recall some basic known results of Kenmotsu manifold as:

$$(\nabla_X \phi)(Y) = -\eta(Y)\phi X - g(X, \phi Y)\xi, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.5}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.6}$$

$$\eta(R(X,Y)Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z), \tag{2.7}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.9}$$

$$S(X,\xi) = -(n-1)\eta(X). \tag{2.10}$$

A Kenmotsu manifold M is said to be $\eta-$Einstein if its Ricci tensor S takes the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.11}$$

for arbitrary vector fields X and Y, where a and b are smooth functions on (M,g) [7]. If $b = 0$, then $\eta-$Einstein manifold becomes Einstein manifold. It is well known that in a Kenmotsu manifold $a + b = -(n-1)$ (see [7], p. 97).
The notion of $\eta-$parallelism on a Sasakian manifold was introduced by M. Kon [26]. A Ricci tensor S of an $n-$dimensional Kenmotsu manifold M is said to be $\eta-$parallel if it satisfies the tensorial relation

$$(\nabla_X S)(\phi Y, \phi Z) = 0 \tag{2.12}$$

for all X, Y, $Z \in \chi(M)$.

3. Semi-symmetric metric connection

Let M be an $n-$dimensional Kenmotsu manifold and ∇ denotes the Levi-Civita connection on it. A linear connection $\tilde{\nabla}$ on M is said to be a semi-symmetric if the torsion tensor $\tilde{T}$ of type $(1,2)$ defined as

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$$

satisfies

$$\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y \tag{3.1}$$

for all vector fields X and Y on M. If moreover, the semi-symmetric connection $\tilde{\nabla}$ holds the relation

$$\tilde{\nabla} g = 0 \tag{3.2}$$

is called semi-symmetric metric connection. A semi-symmetric connection $\tilde{\nabla}$ is said to be non-metric if $\tilde{\nabla} g \neq 0$. A relation between a semi-symmetric metric and Levi-Civita connections is given by the relation

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{3.3}$$

for all vector fields X, $Y \in \chi(M)$ ([22], p. 15). With the help of equations (2.1), (2.2), (2.3), (2.4) and (3.3), we can easily observe that

$$(\tilde{\nabla}_X \eta)(Y) = (\nabla_X \eta)(Y) - \eta(X)\eta(Y) + g(X,Y) \tag{3.4}$$

and

$$(\tilde{\nabla}_X \phi)(Y) = -g(X,\phi Y)\xi - 2\eta(Y)\phi X. \tag{3.5}$$

If R and $\tilde{R}$ denote the curvature tensors with respect to the Levi-Civita and semi-symmetric metric connections of the manifold M respectively, then it is related by the relation

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \theta(X,Z)Y - \theta(Y,Z)X + g(X,Z)LY - g(Y,Z)LX \tag{3.6}$$

for all X, Y, $Z \in \chi(M)$, where

$$\theta(X,Y) = g(LX,Y) = (\nabla_X \eta)(Y) - \eta(X)\eta(Y) + \frac{1}{2}g(X,Y) \tag{3.7}$$

is a symmetric tensor of type (0, 2) on M. In consequence of (2.1), (2.5) and (3.7), equation (3.6) assumes the form

$$\tilde{R}(X,Y)Z = R(X,Y)Z - 3g(Y,Z)X + 3g(X,Z)Y + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y + 2\eta(X)g(Y,Z)\xi - 2\eta(Y)g(X,Z)\xi. \tag{3.8}$$

The contraction of equation (3.8) along the vector field X gives

$$\tilde{S}(Y,Z) = S(Y,Z) - (3n-5)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z), \tag{3.9}$$

which is equivalent to

$$\tilde{Q}Y = QY - (3n-5)Y + 2(n-2)\eta(Y)\xi, \tag{3.10}$$

where $\tilde{Q}$ and Q denote the Ricci operators corresponding to the connections $\tilde{\nabla}$ and ∇ respectively and defined as $\tilde{S}(Y,Z) = g(\tilde{Q}Y,Z)$ and $S(Y,Z) = g(QY,Z)$. Let $\{e_i, i = 1,2,3,...,n\}$ be an orthonormal basis of the tangent space at each point of the manifold M. Setting $Y = Z = e_i$ in (3.9) and taking summation over i, $1 \leq i \leq n$, we get

$$\tilde{r} = r - n(3n-7) - 4, \tag{3.11}$$

where

$$\tilde{r} = \sum_{i=1}^{n} \tilde{S}(e_i,e_i) \quad \textit{and} \quad r = \sum_{i=1}^{n} S(e_i,e_i)$$

represent the scalar curvatures with respect to the connections $\tilde{\nabla}$ and ∇ respectively. In view of equations (2.1), (2.10) and (3.9), we can find that

$$\tilde{S}(Y,\xi) = -2(n-1)\eta(Y). \tag{3.12}$$

With the help of (2.1), (2.8), (2.9) and (3.8), we can easily calculate the following:

$$\tilde{R}(\xi,Y)Z = 2\{\eta(Z)Y - g(Y,Z)\xi\} \tag{3.13}$$

and

$$\tilde{R}(X,Y)\xi = 2\{\eta(X)Y - \eta(Y)X\}. \tag{3.14}$$

The equation (3.14) shows that the manifold M equipped with $\tilde{\nabla}$ is regular.

4. $\eta-$parallel Ricci tensor with respect to semi-symmetric metric connection

In this section, we study the geometrical properties of $\eta-$parallel Ricci tensor with respect to the semi-symmetric metric connection $\tilde{\nabla}$. In [8], authors studied the properties of $\eta-$parallel Ricci tensor and proved several results. Analogous to the definition of $\eta-$parallelism given by M. Kon [26] on Sasakian manifolds, we define

Definition 4.1. *A Ricci tensor $\tilde{S}$ of an $n-$dimensional Kenmotsu manifold M endowed with a semi-symmetric metric connection $\tilde{\nabla}$ is said to be $\eta-$parallel for $\tilde{\nabla}$ if it satisfies the relation $(\tilde{\nabla}_X \tilde{S})(\phi Y, \phi Z) = 0$, for arbitrary vector fields X, Y and Z.*

From (3.3), it is obvious that

$$\tilde{\nabla}_X(\tilde{Q}Y) = \nabla_X(\tilde{Q}Y) + g(\tilde{Q}Y,\xi)X - g(X,\tilde{Q}Y)\xi.$$

With the help of (2.1), (2.5), (3.3) and (3.10), we can find

$$\tilde{\nabla}_X(\tilde{Q}Y) = (\tilde{\nabla}_X\tilde{Q})(Y) + Q(\nabla_X Y) + 2(n-1)g(X,Y)\xi + \eta(Y)QX + 2(n-2)\{\eta(\nabla_X Y) - \eta(X)\eta(Y)\}\xi - (3n-5)\{\nabla_X Y + \eta(Y)X\},$$

and

$$\nabla_X(\tilde{Q}Y) = (\nabla_X Q)(Y) + Q(\nabla_X Y) - (3n-5)\nabla_X Y + 2(n-2)[\{\eta(\nabla_X Y) - \eta(X)\eta(Y) + g(X,Y)\}\xi + \eta(Y)\{X - \eta(X)\xi\}].$$

From the above results, we obtain

$$(\tilde{\nabla}_X\tilde{Q})(Y) = (\nabla_X Q)(Y) - \eta(Y)QX - g(X,QY)\xi - 8(n-2)\eta(Y)\eta(X)\xi + (3n-7)\{\eta(Y)X + g(X,Y)\xi\}.$$

In view of (2.1), (2.10) and $(\tilde{\nabla}_X\tilde{S})(Y,Z) = g((\tilde{\nabla}_X\tilde{Q})(Y),Z)$, above relation assumes the form

$$(\tilde{\nabla}_X\tilde{S})(Y,Z) = (\nabla_X S)(Y,Z) - \eta(Y)S(X,Z) - \eta(Z)S(X,Y) + (3n-7)\{\eta(Z)g(X,Y) + \eta(Y)g(X,Z)\} - 8(n-2)\eta(X)\eta(Y)\eta(Z).$$

Replacing the vector fields Y by ϕY and Z by ϕZ in (**??**) and then using (2.1), we obtain

$$(\tilde{\nabla}_X\tilde{S})(\phi Y,\phi Z) = (\nabla_X S)(\phi Y,\phi Z). \tag{4.1}$$

In view of (2.12), (4.1) and Definition 4.1, we can state the following:

Theorem 4.2. *Let M be an $n-$dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$. Then the Ricci tensor $\tilde{S}$ on M is $\eta-$parallel with respect to the connection $\tilde{\nabla}$ if and only if the manifold has $\eta-$parallel Ricci tensor S for the Levi-Civita connection ∇.*

The straight forward calculations from the equations (2.1), (2.2), (2.6) and (3.9) give

$$\tilde{S}(\phi Y,\phi Z) = \tilde{S}(Y,Z) + 2(n-1)\eta(Y)\eta(Z) \tag{4.2}$$

for all X, $Y \in \chi(M)$. Differentiating (4.2) covariantly along the vector field X, we have

$$(\tilde{\nabla}_X\tilde{S})(\phi Y,\phi Z) = \tilde{\nabla}_X\tilde{S}(\phi Y,\phi Z) - \tilde{S}(\tilde{\nabla}_X(\phi Y),\phi Z) - \tilde{S}(\phi Y,\tilde{\nabla}_X(\phi Z)).$$

With the help of equations (2.1), (3.3), (3.4), (3.5), (3.12) and (4.2), last equation assumes the form

$$(\tilde{\nabla}_X\tilde{S})(\phi Y,\phi Z) = (\tilde{\nabla}_X\tilde{S})(Y,Z) + 2\{\eta(Y)\tilde{S}(X,Z) + \eta(Z)\tilde{S}(X,Y)\} + 4(n-1)\{\eta(Y)g(X,Z) + \eta(Z)g(X,Y)\}. \tag{4.3}$$

Let us suppose that the manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ has $\eta-$parallel Ricci tensor $\tilde{S}$ for the connection $\tilde{\nabla}$, *i.e.* $(\tilde{\nabla}_X\tilde{S})(\phi Y,\phi Z) = 0$, then from (4.3), we obtain

$$(\tilde{\nabla}_X\tilde{S})(Y,Z) = -2\{\eta(Y)\tilde{S}(X,Z) + \eta(Z)\tilde{S}(X,Y)\} - 4(n-1)\{\eta(Y)g(X,Z) + \eta(Z)g(X,Y)\}. \tag{4.4}$$

Thus, we can state the following:

Theorem 4.3. *An $n-$dimensional Kenmotsu manifold M endowed with a semi-symmetric metric connection $\tilde{\nabla}$ has $\eta-$parallel Ricci tensor for the connection $\tilde{\nabla}$ if and only if the relation (4.4) holds on M.*

Let $\{e_i, i = 1,2,3,...,n\}$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $Y = Z = e_i$ in (4.4) and taking summation over i, $1 \le i \le n$, we get

$$d\tilde{r}(X) = 0$$

for all $X \in \chi(M)$. This shows that the scalar curvature $\tilde{r}$ is constant with respect to the connection $\tilde{\nabla}$. Hence we state:

Corollary 4.4. *If the Ricci tensor $\tilde{S}$ of an $n-$dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is $\eta-$parallel, then the scalar curvature for the connection $\tilde{\nabla}$ is constant.*

From (3.11) and Corollary 4.4, it is obvious that

$$d\tilde{r}(X) = dr(X) = 0,$$

$\forall X \in \chi(M)$, which implies that the scalar curvature with respect to the Levi-Civita connection is constant. Thus we have,

Corollary 4.5. *Let an $n-$dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ has $\eta-$parallel Ricci tensor $\tilde{S}$. Then the scalar curvature of the manifold is constant.*

Moreover, since $\tilde{S}(Y,Z) = g(\tilde{Q}Y,Z)$, we can find from (3.3) that

$$\tilde{\nabla}_X|\tilde{Q}|^2 = 2\sum_{i=1}^{n} g((\tilde{\nabla}_X\tilde{Q})e_i,\tilde{Q}e_i) = 0,$$

which implies that $|\tilde{Q}|^2$ =constant. Thus we have

Corollary 4.6. *If an $n-$dimensional Kenmotsu manifold endowed with a semi symmetric metric connection $\tilde{\nabla}$ has $\eta-$parallel Ricci tensor, then the length of Ricci operator with $\tilde{\nabla}$ is constant on M.*

Let us suppose that the Ricci tensor $\tilde{S}$ on an $n-$dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is Codazzi type, *i.e.*, $(\tilde{\nabla}_X\tilde{S})(Y,Z) = (\tilde{\nabla}_Y\tilde{S})(X,Z)$. Thus in view of this definition and equation (4.4), we obtain

$$\eta(Y)\tilde{S}(X,Z) - \eta(X)\tilde{S}(Y,Z) = 2(n-1)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}.$$

Setting $Y = \xi$ in last expression and using (2.1) and (3.12), we find

$$\tilde{S}(X,Z) = -2(n-1)g(X,Z),$$

which is equivalent to

$$S(X,Z) = (n-3)g(X,Z) - 2(n-2)\eta(X)\eta(Z). \tag{4.5}$$

This shows that the manifold M is an $\eta-$Einstein manifold with the scalars $a = n-3$ and $b = -2(n-2)$. It is obvious that $a+b = -(n-1)$ [For instant, see [7], p-97]. Conversely, if we suppose that the manifold M satisfies (4.5), then we can easily find that the Ricci tensor with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is of Codazzi type. Thus we can state:

Corollary 4.7. *Let an $n(>3)-$dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ has $\eta-$parallel Ricci tensor for $\tilde{\nabla}$. Then the Ricci tensor $\tilde{S}$ on M to be of Codazzi type if and only if the manifold M is Einstein for $\tilde{\nabla}$ or an $\eta-$Einstein for Levi-Civita connection.*

Again, we consider that the Ricci tensor $\tilde{S}$ with respect to a semi-symmetric metric connection $\tilde{\nabla}$ is cyclic parallel, *i.e.*,

$$(\tilde{\nabla}_X\tilde{S})(Y,Z) + (\tilde{\nabla}_Y\tilde{S})(Z,X) + (\tilde{\nabla}_Z\tilde{S})(X,Y) = 0.$$

In view of (4.4), above relation converts into the form

$$\eta(X)\tilde{S}(Y,Z) + \eta(Y)\tilde{S}(Z,X) + \eta(Z)\tilde{S}(X,Y) + 2(n-1)\{\eta(X)g(Y,Z) + \eta(Y)g(X,Z) + \eta(Z)g(X,Y)\} = 0.$$

Putting $Z = \xi$ in last expression and using (2.1) and (3.12), we get (4.5). Hence we can state:

Corollary 4.8. *Let M be an $n(>3)-$dimensional Kenmotsu manifold endowed with a semi-symmetric metric connection $\tilde{\nabla}$, has $\eta-$parallel Ricci tensor $\tilde{S}$, then the Ricci tensor with respect to the connection $\tilde{\nabla}$ to be cyclic parallel if and only if the manifold is $\eta-$Einstein.*

5. Concircular curvature tensor with semi-symmetric metric connection $\tilde{\nabla}$

It is well known that a geodesic circle (a curve whose first curvature is constant and second curvature is identically zero) does not transform into a geodesic circle by the conformal transformation

$$\overline{g_{ij}} = \psi^2 g_{ij}, \tag{5.1}$$

where g_{ij} denotes the fundamental tensor. Yano [18] proved that a conformal transformation, defined in (5.1), satisfying the partial differential equation

$$\psi_{i;j} = \phi g_{ij} \tag{5.2}$$

alters a geodesic circle into a geodesic circle. Such a transformation is known as concircular transformation and the geometry deals with such transformation is called the concircular geometry [18]. A tensor field C of type $(1,3)$ on a Riemannian manifold, which remains invariant under the concircular transformation, defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{5.3}$$

where R is the curvature tensor and r denotes the scalar curvature, is known as concircular curvature tensor [19]. Analogous to the definition of (5.3), we can define

Definition 5.1. *Let M be an $n-$dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$. A concircular curvature tensor $\tilde{C}$ with respect to the connection $\tilde{\nabla}$ on M is a tensor field of type $(1,3)$ and satisfies the relation*

$$\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{5.4}$$

for all vector fields X, Y, $Z \in \chi(M)$. Here $\tilde{R}$ and $\tilde{r}$ are the curvature tensor and scalar curvature of the manifold M corresponding to the connection $\tilde{\nabla}$ respectively.

Definition 5.2. *Let M be an $n-$dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$. A projective curvature tensor $\tilde{P}$ with respect to the connection $\tilde{\nabla}$ on M is a tensor field of type $(1,3)$ and satisfies the relation*

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-1)}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\}, \tag{5.5}$$

for all vector fields X, Y, $Z \in \chi(M)$. Here $\tilde{R}$ and $\tilde{S}$ denote the curvature and Ricci tensors of the manifold M corresponding to the connection $\tilde{\nabla}$ respectively.

Let M be an $n(>3)-$dimensional Kenmotsu manifold admitting a semi-symmetric metric connection $\tilde{\nabla}$ has $\eta-$parallel Ricci tensor $\tilde{S}$ for $\tilde{\nabla}$. If additionally, Ricci tensor $\tilde{S}$ is either Codazzi type or cyclic parallel, then equation (4.5) holds on M. Let $\{e_i, i=1,2,3,...,n\}$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $Y=Z=e_i$ in (4.5) and taking summation over i, $1\leq i\leq n$, we get $\tilde{r}=-2n(n-1)$. By considering this fact and equations (5.4) and (5.5), we find that

$$\tilde{C}(X,Y)Z=\tilde{P}(X,Y)Z. \tag{5.6}$$

Conversely, if the relation (5.6) holds, then from (5.4) and (5.5), we can easily obtain

$$\tilde{S}(X,Z)Y-\tilde{S}(Y,Z)X=\frac{\tilde{r}}{n}\{g(Y,Z)X-g(X,Z)Y\},$$

which is equivalent to

$$\tilde{S}(X,Z)\eta(Y)-\tilde{S}(Y,Z)\eta(X)=\frac{\tilde{r}}{n}\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}.$$

Putting $Y=\xi$ in last equation and using (2.1) and (3.12), we get

$$\tilde{S}(X,Z)+2(n-1)\eta(Z)\eta(X)=\frac{\tilde{r}}{n}\{\eta(Z)\eta(X)-g(X,Z)\}.$$

Let $\{e_i, i=1,2,3,...,n\}$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $X=Z=e_i$ in the last equation and taking summation over i, $1\leq i\leq n$, we immediately get $\tilde{r}=-2n(n-1)$ and hence from (5.4) and (5.5), we obtain (4.5). From the above discussion, we can state the following corollary:

Corollary 5.3. *Let M be an $n(>3)-$dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$ has $\eta-$parallel Ricci tensor for $\tilde{\nabla}$. Then the concircular and projective curvature tensors for $\tilde{\nabla}$ coincide if and only if the Ricci tensor $\tilde{S}$ is either Codazzi type or cyclic parallel.*

From the Corollaries (4.7), (4.8) and (5.3), we observe the following:

Corollary 5.4. *If the Ricci tensor $\tilde{S}$ of an $n(>3)-$dimensional Kenmotsu manifold M endowed with a semi-symmetric metric connection $\tilde{\nabla}$ is $\eta-$parallel . Then the following results on M are equivalent*
(i) Ricci tensor $\tilde{S}$ is of Codazzi type,
(ii) Ricci tensor $\tilde{S}$ is cyclic parallel,
(iii) $\tilde{S}=-2(n-1)g(Y,Z)$,
(iv) $\tilde{C}(X,Y)Z=\tilde{P}(X,Y)Z$,
(v) $\tilde{r}=-2n(n-1)$.

Let us suppose that the manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is either concircularly or projectively flat with respect to the connection $\tilde{\nabla}$, then in consequence of (4.5), (5.4), (5.5) and (5.6), we find that

$$\tilde{R}(X,Y)Z=-2\{g(Y,Z)X-g(X,Z)Y\}, \tag{5.7}$$

which shows that the manifold M equipped with $\tilde{\nabla}$ is of constant curvature. Therefore we have:

Remark 5.5. *The idea of constant curvature plays a central role in the theory of relativity and cosmology. The simplest cosmological model can be constructed by assuming that the universe is isotropic and homogeneous. This is known as cosmological principle. When we translated this principle to Riemannian geometry, professes that the three dimensional position space is a space of maximal symmetry [24], i.e., a space of constant curvature whose curvature depends upon time. The cosmological solutions of Einstein equations which contain a three dimensional space like surfaces of a constant curvature are the Robertson-Walker metric, while four dimensional space of constant curvature is the de Sitler model of the universe ([23], [24]).*

In consequence of (3.8) and (5.7), we immediately get

$$R(X,Y)Z=\{g(Y,Z)X-g(X,Z)Y\}+2\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X-\eta(X)g(Y,Z)\xi+\eta(Y)g(X,Z)\xi\}, \tag{5.8}$$

which shows that it is a certain class of generalized Sasakian space form (for instance, see [33], [34]). From (5.8), it is obvious that $f_1=1$, $f_2=0$ and $f_3=2$. Also, if we suppose that the manifold M equipped with $\tilde{\nabla}$ satisfies (5.8), then equations (3.8), (5.4), (5.5) and (5.8) give $\tilde{C}=\tilde{P}=0$. Kim [34] proved that a generalized Sasakian-space form is conformally flat if and only if $f_2=0$. Thus from (5.8) and result of Kim, we have the following theorem:

Theorem 5.6. *Let M be an $n(>3)-$dimensional Kenmotsu manifold endowed with a semi-symmetric metric connection $\tilde{\nabla}$. Then the manifold is conformally flat if and only if it is either projectively or concircularly flat for $\tilde{\nabla}$.*

Taking covariant derivative of (5.4) with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along the vector field W, we get

$$(\tilde{\nabla}_W\tilde{C})(X,Y)Z=(\tilde{\nabla}_W\tilde{R})(X,Y)Z-\frac{d\tilde{r}(W)}{n(n-1)}\{g(Y,Z)X-g(X,Z)Y\}. \tag{5.9}$$

Let us suppose that the Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ has $\eta-$parallel Ricci tensor, *i.e.*, $(\tilde{\nabla}_X\tilde{S})(\phi Y,\phi Z)=0$. Thus from the Corollary 4.4, equation (5.9) assumes the form

$$(\tilde{\nabla}_W\tilde{C})(X,Y)Z=(\tilde{\nabla}_W\tilde{R})(X,Y)Z. \tag{5.10}$$

Before going to discuss our results, we define the following definitions as:

Definition 5.7. *An $n-$dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is said to be a global symmetric Kenmotsu manifold with respect to the connection $\tilde{\nabla}$ if its non vanishing curvature tensor $\tilde{R}$ satisfies*

$$(\tilde{\nabla}_W \tilde{R})(X,Y)Z = 0,$$

for arbitrary vector fields X, Y, Z and W.

Definition 5.8. *An $n-$dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is said to be a global concircularly symmetric Kenmotsu manifold with respect to the connection $\tilde{\nabla}$ if its concircular curvature tensor $\tilde{C}$ satisfies*

$$(\tilde{\nabla}_W \tilde{C})(X,Y)Z = 0,$$

for arbitrary vector fields X, Y, Z and W.

Definition 5.9. *An $n-$dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is said to be a globally $\phi-$symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if its non vanishing curvature tensor $\tilde{R}$ satisfies*

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = 0,$$

for arbitrary vector fields X, Y, Z and W.

Definition 5.10. *An $n-$dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is said to be a globally $\phi-$concircularly symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if its concircular curvature tensor $\tilde{C}$ satisfies*

$$\phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = 0,$$

for all X, Y, Z, $W \in \chi(M)$.

In consequence of equation (5.10) and Definitions 5.7 and 5.8, we can observe the following:

Theorem 5.11. *Let M be an $n-$dimensional Kenmotsu manifold with a semi-symmetric metric connection $\tilde{\nabla}$ and the Ricci tensor $\tilde{S}$ of M is $\eta-$parallel. Then M is globally symmetric if and only if it is globally concircularly symmetric with respect to the connection $\tilde{\nabla}$.*

Operating ϕ^2 on both sides of (5.10), we have

$$\phi^2(\tilde{\nabla}_W \tilde{C})(X,Y)Z = \phi^2(\tilde{\nabla}_W \tilde{R})(X,Y)Z.$$

Thus, with the help of above equation and Definitions 5.3 and 5.4, we can state:

Theorem 5.12. *If an $n-$dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ has $\eta-$parallel Ricci tensor $\tilde{S}$, then the manifold M to be globally $\phi-$symmetric if and only if it is globally $\phi-$concircularly symmetric.*

It is observed that a globally $\phi-$concircularly symmetric Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is an $\eta-$Einstein manifold. Thus, by considering this fact and Theorem 5.12, we have

Corollary 5.13. *If an $n-$dimensional Kenmotsu manifold M endowed with a semi-symmetric metric connection $\tilde{\nabla}$ has $\eta-$parallel Ricci tensor $\tilde{S}$, then it is an $\eta-$Einstein manifold.*

6. Example

In this section, we construct an example of the Kenmotsu manifold admitting a semi-symmetric metric connection and after that we validate our results.

Example 6.1. *Let*

$$M^3 = \{(x,y,z) \in \mathbb{R}^3 : x,y,z(\neq 0) \in \mathbb{R}\},$$

be a three dimensional smooth manifold, where (x,y,z) denotes the standard coordinate of a point in $\mathbb{R}^3$. Let us suppose that

$$e_1 = e^z\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad e_2 = e^z\frac{\partial}{\partial y}, \quad e_3 = -\frac{\partial}{\partial z}$$

be a set of linearly independent vector field at each point of the manifold M^3 and therefore it form a basis for the tangent space $\chi(M^3)$. We also define the Riemannian metric g of the manifold M^3 as $g(e_i,e_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta and $i,j = 1,2,3$. Let us consider the $1-$form η defined by $\eta(Z) = g(Z,e_3)$ for any $Z \in \chi(M^3)$ and a tensor field ϕ of type $(1,1)$ defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

By the linearity properties of ϕ and g, we can easily verify the following relations

$$\phi^2 X = -X + \eta(X)e_3, \quad \eta(e_3) = 1, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for arbitrary vector fields $X,Y \in \chi(M^3)$. This shows that $\xi = e_3$ and the structure (ϕ,ξ,η,g) defines an almost contact metric structure on M^3. If ∇ represents the Levi-Civita connection with respect to the Riemannian metric g, then with help of above, we can easily calculate that

$$[e_1,e_2] = 0, \quad [e_1,e_3] = e_1, \quad [e_2,e_3] = e_2.$$

We recall the Koszul's formula as

$$2g(\nabla_X Y,Z) = Xg(Y,Z)+Yg(X,Z)-Zg(X,Y)-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y])$$

for arbitrary vector fields $X,Y,Z\in\chi(M^3)$*. It is obvious from Koszul's formula that*

$$\nabla_{e_1}e_1=-e_3,\quad \nabla_{e_1}e_2=0,\quad \nabla_{e_1}e_3=e_1,\quad \nabla_{e_2}e_1=0,\quad \nabla_{e_2}e_2=-e_3,\quad \nabla_{e_2}e_3=e_2,\quad \nabla_{e_3}e_1=0,\quad \nabla_{e_3}e_2=0,\quad \nabla_{e_3}e_3=0.$$

From the above calculations, we can observe that $\nabla_X\xi = X-\eta(X)\xi$*, for* $\xi=e_3$*. Thus the manifold* (M^3,g) *is a Kenmotsu manifold of dimension* 3 *and the structure* (ϕ,η,ξ,g) *denotes the Kenmotsu structure on the manifold* M^3*.*
It is obvious from the above results that

$$\begin{aligned}
&R(e_1,e_2)e_3=0,\quad R(e_1,e_3)e_3=-e_1,\quad R(e_3,e_2)e_2=-e_3,\quad R(e_3,e_1)e_1=-e_3,\quad R(e_2,e_1)e_1=-e_2,\\
&R(e_2,e_3)e_3=-e_2,\quad R(e_1,e_2)e_2=-e_1,\quad R(e_3,e_1)e_2=0,\quad S(e_1,e_1)=-2,\quad S(e_2,e_2)=-2,\\
&S(e_3,e_3)=-2,\quad S(\phi e_1,\phi e_1)=-2,\quad S(\phi e_2,\phi e_2)=-2,\quad S(\phi e_3,\phi e_3)=0,\quad S(\phi e_i,\phi e_j)=0,\ for\ all\ \ i,j=1,2,3\ (i\neq j).
\end{aligned}$$

From the above relations, we can easily calculate that $(\nabla_X S)(\phi e_i,\phi e_j)=0$ *for all* $X\in\chi(M^3)$ *and* $i,j=1,2,3$*. Hence the manifold* M^3 *is* $\eta-$*parallel.*
In consequence of (3.3) and above results, we can find that

$$\tilde\nabla_{e_1}e_1=-2e_3,\quad \tilde\nabla_{e_1}e_2=0,\quad \tilde\nabla_{e_1}e_3=2e_1,\quad \tilde\nabla_{e_2}e_1=0,\quad \tilde\nabla_{e_2}e_2=-2e_3,\tilde\nabla_{e_2}e_3=2e_2,\quad \tilde\nabla_{e_3}e_1=0,\quad \tilde\nabla_{e_3}e_2=0,\quad \tilde\nabla_{e_3}e_3=0$$

and also the components of torsion tensor $\tilde T$ *are*

$$\tilde T(e_i,e_i)=\tilde\nabla_{e_i}e_i-\tilde\nabla_{e_i}e_i-[e_i,e_i]=0,\ for\ \ i=1,2,3\quad and\quad \tilde T(e_1,e_2)=0,\quad \tilde T(e_1,e_3)=e_1,\quad \tilde T(e_2,e_3)=e_2.$$

These equations show that $\tilde T\neq 0$ *and therefore by equation (3.1), we can say that the linear connection defined in (3.3) is a semi-symmetric connection on* (M^3,g)*. By the straight forward calculation, we can also find*

$$(\tilde\nabla_{e_1}g)(e_i,e_j)=0,\qquad (\tilde\nabla_{e_2}g)(e_i,e_j)=0,\qquad (\tilde\nabla_{e_3}g)(e_i,e_j)=0$$

for all $i,j=1,2,3$*. This demonstrates that the equation (3.2) holds on* M^3 *and hence the linear connection defined in (3.3) is a semi-symmetric metric connection on* M^3*. Thus we can say that the manifold* (M^3,g) *be a three dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection* $\tilde\nabla$ *defined in (3.3).*
With the help of above discussions, we can calculate the curvature and Ricci tensors with respect to the semi-symmetric metric connection $\tilde\nabla$ *as*

$$\begin{aligned}
&\tilde R(e_1,e_2)e_3=0,\ \tilde R(e_1,e_3)e_3=-2e_1,\ \tilde R(e_3,e_2)e_2=-2e_3,\ \tilde R(e_3,e_1)e_1=-2e_3,\ \tilde R(e_2,e_1)e_1=-4e_2,\ \tilde R(e_2,e_3)e_3=-2e_2,\\
&\tilde R(e_1,e_2)e_2=-4e_1,\ \tilde R(e_3,e_1)e_2=0,\ \tilde S(e_1,e_1)=-6,\ \tilde S(e_2,e_2)=-6,\ \tilde S(e_3,e_3)=-4,\ \tilde r=-16
\end{aligned}$$

and other components can be calculated by symmetric and skew-symmetric properties. We can easily observe that the equations [(3.8) - (3.14)] are verified. Also,

$$\tilde S(\phi e_1,\phi e_1)=-6,\ \tilde S(\phi e_2,\phi e_2)=-6,\ \tilde S(\phi e_3,\phi e_3)=0,\qquad \tilde S(\phi e_i,\phi e_j)=0,\ for\ \ all\ \ i,j=1,2,3(i\neq j).$$

It is clear from the above discussions that $(\tilde\nabla_X\tilde S)(\phi e_i,\phi e_j)=0$ *for all* $X\in\chi(M^3)$ *and* $i,j=1,2,3$*. Hence the manifold* M^3 *equipped with a semi-symmetric metric connection* $\tilde\nabla$ *has* $\eta-$*parallel Ricci tensor for the connection* $\tilde\nabla$*. From the above discussions, we come to the conclusion:*
"If the manifold M^3 *has* $\eta-$*parallel Ricci tensor with respect to Levi-Civita connection* ∇*, then it contains also* $\eta-$*parallel Ricci tensor with respect to the semi-symmetric metric connection* $\tilde\nabla$*".*
Hence the statement of the Theorem 4.2.
It is obvious from the above relations that $\tilde r=-16\,(constant)$ *and hence the Corollary 4.4 is satisfied on* M^3*.*

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Proper UP-filters of UP-algebra

Daniel A. Romano[a*]

[a]*International Mathematical Virtual Institute,6, Kordunaşka Street, 78000 Banja Luka, Bosnia and Herzegovina*
[*]*Corresponding author E-mail: bato49@hotmail.com*

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Abstract

In this paper we introduce and discus a concept of proper UP-filters in UP-algebras. We present a connection between proper UP-filters and UP-ideals. We expect this concept of filters to play a significant role in research and understanding of algebras.

1. Introduction

The basic concepts of UP-algebra are taken from the text [3]. The author in his article has introduced and analyzed the concepts of UP-algebra, UP-subalgebra and UP-ideal. In the article [6] the authors introduced the concept of UP-filters in UP algebra. This latter concept has a non-standard attitude towards the concept of UP-ideals. That made us confused - we expected the filter to have a standard attitude towards the ideal. We were interested in why the authors of the concept of UP-filters opted for such definition of the UP-filter. Our first reaction to such a UP-filter determination was - the offered definition is not correct. Then we thought that the text of the UP-filter definition in the article [6] was incorrectly written. Viewing the available literature about the concept of filters in BCC-algebra (See [1, 2]) and KU-algebra ([4, 5]), algebras which close to UP-algebra, did not yield the expected results. It was possible to find the term 'deductive system' some authors called the filter. But this concept had a non-standard relationship to the concept of the ideal. That was the motive for our research of UP-algebra. In order to obtain a satisfactory definition of the proper UP-filter, we have permuted the positions of the logical atoms in the definition of the UP-ideals. In this text one obtained intriguing reflection of these variation is exposed.

Since in each of the previously known algebras the ideals play an important role, this is the case in this UP algebra, too. Filters in algebras, as substructures these algebras associated with ideals, could also play a significant role in our understanding of algebras.

2. Preliminaries

First, let us recall the definition of UP-algebra.

Definition 2.1 ([3], Definition 1.3)**.** *An algebra* $A=(A,\cdot,0)$ *of type* $(2,0)$ *is called a* UP- algebra *if it satisfies the following axioms:*
(UP - 1)*:* $(\forall x,y,z\in A)((y\cdot z)\cdot((x\cdot y)\cdot(x\cdot z))=0)$*,*
(UP - 2)*:* $(\forall x\in A)(0\cdot x=x)$*,*
(UP - 3)*:* $(\forall x\in A)(x\cdot 0=0)$*,*
(UP - 4)*:* $(\forall x,y\in A)((x\cdot y=0\wedge y\cdot x=0)\Longrightarrow x=y)$*.*

Second, in the following we give definition of the concept of UP-ideals of UP-algebra.

Definition 2.2 ([3], Definition 2.1)**.** *Let A be a UP-algebra. A subset J of A is called a* UP-ideal *of A if it satisfies the following properties:*
(1) $0\in J$*, and*
(2) $(\forall x,y,z\in A)(x\cdot(y\cdot z)\in J\wedge y\in J\Longrightarrow x\cdot z\in J)$*.*

For this article, the recognizable feature of the UP-ideal is given in statement (1) of Proposition 2.7 in the article [3]:

Let A be a UP-algebra and B a UP-ideal of A. Then

$$(\forall x,y\in A)((x\in B\wedge x\leq y)\Longrightarrow y\in B).$$

Email addresses: **bato49@hotmail.com (D. A. Romano)**

The concept of UP-filters is introduced by the following definition.

Definition 2.3 ([6], Definition 1.11)**.** *Let A be a UP-algebra. A subset F of A is called a* UP-filter *of A, if it satisfies the following properties:*
(i) $0 \in F$,
(ii) $(\forall x, y \in A)((x \in F \wedge x \cdot y \in F) \Longrightarrow y \in F)$.

3. The main results

Our intention in this short notice is to construct a substructure G in UP-algebras that will have the following property

$$(\forall x, y \in A)((y \in G \wedge x \leq y) \Longrightarrow x \in G)$$

and has a standard attitude toward the UP-ideal. We can transform this formula into the following formula

$$(\forall x, y \in A)((y \in G \wedge \neg(x \cdot y \in G)) \Longrightarrow x \in G).$$

The previous formula was the basis for us concept of UP-filters. The concept of proper UP-filter in a UP-algebra we introduce by the following definition.

Definition 3.1. *Let A be a UP-algebra. A subset G of A is called a* proper UP-filter *of A if it satisfies the following properties:*
(3) $\neg(0 \in G)$, *and*
(4) $(\forall x, y, z \in A)((\neg(x \cdot (y \cdot z) \in G) \wedge x \cdot z \in G) \Longrightarrow y \in G)$.

Subsets $\emptyset$ and $A_0 = \{x \in A : x \neq 0\}$ are trivial proper UP-filters and UP-algebras A. So, the family of all proper UP-filters of a UP-algebra is not empty.

Example 3.2. *Let A is as in Example 2.2 in* [3]. *Then the sets* $\{2,4\}$ *and* $\{3,4\}$ *are proper UP-filters of A.*

Example 3.3. *Let* $f : A \longrightarrow B$ *be a UP-homomorphism of UP-algebras. Then the set* $Coker(f) = \{x \in A : f(x) \neq 0\}$ *is a UP-filter of A. In addition, if H is a proper UP-filter of B, then* $f^{-1}(H)$ *is a proper UP-filter of A. Specifically, if* H/J *is a proper filter in* A/J*, where J is a UP-ideal in A, then* $\pi^{-1}(H/J)$ *is a proper UP-filter of A, where* $\pi : A \longrightarrow A/J$ *is the canonical UP-epimorphism.*

This determined substructure of a UP-algebra A has the following property.

Theorem 3.4. *Let A be a UP-algebra and G a proper UP-filter of A. Then*
(5) $(\forall x, y \in A)((\neg(x \cdot y \in G) \wedge y \in G) \Longrightarrow x \in G)$.
(6) $(\forall x, y \in A)(x \cdot y \in G \Longrightarrow y \in G)$.

Proof. The first statement follows directly from definition when we put $x = 0$, $y = x$ and $z = y$.
If we put y = z in formula (4), we get $\neg(x \cdot (y \cdot y) = x \cdot 0 = 0 \in G)$ and $x \cdot y \in G$. Thus $y \in G$. Therefore, (6) is proved. □

Corollary 3.5. *Let A be a UP-algebra and G a proper UP-filter of A. Then*
(7) $(\forall x, y \in A)((x \leqslant y \wedge y \in G) \Longrightarrow x \in G)$.

Proof. Let $x, y \in A$ be arbitrary elements such that $x \leqslant y$ and $y \in G$. Thus $\neg(x \cdot y = 0 \in G)$ and $y \in G$. Then $x \in G$ by (5). □

Remark 3.6. *The usual term used for property* (7) *of an algebra subset is a 'deductive system'. So, the concept of 'proper filters', introduced by definition 3.1, is a deductive system in the UP-algebra A. The important difference between the concept of 'deductive systems' and our concept of 'proper UP-filters' is in the requirement* (3).

Theorem 3.7. *A subset G of a UP-algebra A is a proper UP-filter of A if and only if the set* $A \backslash G$ *is a UP-ideal of A.*

Proof. Suppose that G is a proper UP-filter in UP-algebra A. It is clear $0 \in A \backslash G$. Let $x, y, z \in A$ be arbitrary elements such that $\neg(x \cdot (y \cdot z) \in G)$ and $\neg(y \in G)$. Thus $\neg(x \cdot z \in G)$. Indeed, if it were not, from $\neg(x \cdot (y \cdot z) \in G)$ and $x \cdot z \in G$ would follow $y \in G$ what is in a contradiction with $\neg(y \in G)$. So it have to be $\neg(x \cdot z \in G)$. Therefore, the set $A \backslash G$ is a UP-ideal of A.
In opposite, let J be a UP-ideal of UP-algebra A. It is obvious that $\neg(0 \in A \backslash J)$ is valid. Let $x, y, z \in A$ be arbitrary elements such that $\neg(x \cdot (y \cdot z) \in A \backslash J)$ and $x \cdot z \in A \backslash J$. Thus $y \in A \backslash J$. Indeed, if it were $y \in J$, then $x \cdot z \in J$ would follow from $x \cdot (y \cdot z) \in J$ and $y \in J$, contrary to the hypothesis $\neg(x \cdot z \in J)$. Therefore, the set $A \backslash J$ is a proper UP-filter of A. □

Theorem 3.8. *The family* $\mathfrak{G}_A$ *of all proper UP-filters in UP-algebra A forms a completely lattice.*

Proof. Let A be a UP-algebra and $\{G_{i \in I}\}$ a family of proper UP-filters of A.
(a) Let $x, y, z \in A$ be elements such that $\neg(x \cdot (y \cdot z) \in \bigcup_{i \in I} G_i)$ and $x \cdot z \in \bigcup_{i \in I} G_i$. Then there exists an index $i \in I$ such that $\neg(x \cdot (y \cdot z) \in G_i)$ and $x \cdot z \in G_i$. Thus $y \in G_i$ by (4). Therefore, $y \in \bigcup_{i \in I} G_i$.
(b) Let $\mathfrak{X}$ be a family of all proper UP-filter contained in $\bigcap_{i \in I} G_i$. Thus, by part (a) of this proof, the union $\bigcup \mathfrak{X}$ is a proper UP-filter of A.
(c) If we define $\sqcap_{i \in I} G_i = \bigcup \mathfrak{X}$ and $\sqcup_{i \in I} G_i = \bigcup_{i \in I} G_i$, then $(\mathfrak{G}, \sqcap, \sqcup)$ is a completely lattice. □

4. Final observation

In the present paper, we have introduced a new algebraic substructure in UP-algebra, called a proper UP-filter. We present some connections between proper UP-filters and UP-ideals. This concept of UP-filters has almost a standard connection with the UP-ideal. The author believes that this new structure enriches the family of substructures in UP-algebras. Of course, while the academic community of researchers algebras accept this concept of filters in algebras, it can be expected that further research involves relations of the concept of proper UP-filters and some other concepts, for example as concepts of orders, homomorphisms and congruences in algebras.

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Asymptotically $\mathscr{I}$-Cesàro Equivalence of Sequences of Sets

Uğur Ulusu[a] and Erdinç Dündar[a*]

[a]*Department of Mathematics, Faculty of Science and Literature, Afyon Kocatepe University, 03200, Afyonkarahisar, Turkey*

[*]*Corresponding author E-mail: edundar@aku.edu.tr*

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Abstract

In this paper, we defined concepts of asymptotically $\mathscr{I}$-Cesàro equivalence and investigate the relationships between the concepts of asymptotically strongly $\mathscr{I}$-Cesàro equivalence, asymptotically strongly $\mathscr{I}$-lacunary equivalence, asymptotically p-strongly $\mathscr{I}$-Cesàro equivalence and asymptotically $\mathscr{I}$-statistical equivalence of sequences of sets.

1. Introduction

The concept of convergence of sequences of real numbers $\mathbb{R}$ has been transferred to statistical convergence by Fast [5] and independently by Schoenberg [16]. $\mathscr{I}$-convergence was first studied by Kostyrko et al. [9] in order to generalize of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of the set of natural numbers $\mathbb{N}$. Das et al. [4] introduced new notions, namely $\mathscr{I}$-statistical convergence and $\mathscr{I}$-lacunary statistical convergence by using ideal.

There are different convergence notions for sequence of sets. One of them handled in this paper is the concept of Wijsman convergence (see, [1], [3], [11], [21], [22]). The concepts of statistical convergence and lacunary statistical convergence of sequences of sets were studied in [11, 18] in Wijsman sense. Also, new convergence notions, for sequences of sets, which is called Wijsman $\mathscr{I}$-convergence, Wijsman $\mathscr{I}$-statistical convergence and Wijsman $\mathscr{I}$-Cesàro summability by using ideal were introduced in [7], [8], [20].

Marouf [10] peresented definitions for asymptotically equivalent and asymptotic regular matrices. This concepts was investigated in [12, 13, 14]. The concept of asymptotically equivalence of sequences of real numbers which is defined by Marouf [10] has been extended by Ulusu and Nuray [19] to concepts of Wijsman asymptotically equivalence of set sequences. Moreover, natural inclusion theorems are presented. Kişi et al. [8] introduced the concepts of Wijsman $\mathscr{I}$-asymptotically equivalence of sequences of sets.

2. Definitions and notations

Now, we recall the basic definitions and concepts (See [1, 2, 6, 7, 8, 9, 10, 11, 15, 19, 20]).

Let (Y,ρ) be a metric space. For any point $y \in Y$ and any non-empty subset U of Y, we define the distance from y to U by $d(y,U) = \inf_{u \in U} \rho(y,u)$.

Let (Y,ρ) be a metric space and U, U_i be any non-empty closed subsets of Y. The sequence $\{U_i\}$ is Wijsman convergent to U if for each $y \in Y$,

$$\lim_{i \to \infty} d(y,U_i) = d(y,U).$$

Let (Y,ρ) be a metric space and U, U_i be any non-empty closed subsets of Y. The sequence $\{U_i\}$ is Wijsman statistical convergent to U if $\{d(y,U_i)\}$ is statistically convergent to $d(y,U)$; i.e., for every $\varepsilon > 0$ and for each $y \in Y$,

$$\lim_{n \to \infty} \frac{1}{n} \Big| \big\{ i \leq n : |d(y,U_i) - d(y,U)| \geq \varepsilon \big\} \Big| = 0.$$

Email addresses: ulusu@aku.edu.tr (U. Ulusu), edundar@aku.edu.tr (E. Dündar)

A family of sets $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathscr{I}$, (ii) For each $U, V \in \mathscr{I}$ we have $U \cup V \in \mathscr{I}$, (iii) For each $U \in \mathscr{I}$ and each $V \subseteq U$ we have $V \in \mathscr{I}$.
An ideal is called non-trivial ideal if $\mathbb{N} \notin \mathscr{I}$ and non-trivial ideal is called admissible ideal if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$.
A family of sets $\mathscr{F} \subseteq 2^{\mathbb{N}}$ is a filter if and only if (i) $\emptyset \notin \mathscr{F}$, (ii) For each $U, V \in \mathscr{F}$ we have $U \cap V \in \mathscr{F}$, (iii) For each $U \in \mathscr{F}$ and each $V \supseteq U$ we have $V \in \mathscr{F}$.

Proposition 2.1. *([9]) $\mathscr{I}$ is a non-trivial ideal in $\mathbb{N}$ if and only if*

$$\mathscr{F}(\mathscr{I}) = \{E \subset \mathbb{N} : (\exists U \in \mathscr{I})(E = \mathbb{N} \backslash U)\}$$

is a filter in $\mathbb{N}$.

Throughout the paper, we let (Y, ρ) be a separable metric space, $\mathscr{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and U, U_i be any non-empty closed subsets of Y.
The sequence $\{U_i\}$ is Wijsman $\mathscr{I}$-convergent to U, if for every $\varepsilon > 0$ and for each $y \in Y$, $U(y, \varepsilon) = \left\{i \in \mathbb{N} : |d(y, U_i) - d(y, U)| \geq \varepsilon\right\}$ belongs to $\mathscr{I}$.
The sequence $\{U_i\}$ is Wijsman $\mathscr{I}$-statistical convergent to U, if for every $\varepsilon > 0$, $\delta > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{i \leq n : |d(y, U_i) - d(y, U)| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathscr{I}$$

and we write $U_i \stackrel{S(\mathscr{I}_W)}{\longrightarrow} U$.
The sequence $\{U_i\}$ is Wijsman $\mathscr{I}$-Cesàro summable to U, if for every $\varepsilon > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \left|\frac{1}{n}\sum_{i=1}^{n} d(y, U_i) - d(y, U)\right| \geq \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \stackrel{C_1(\mathscr{I}_W)}{\longrightarrow} U$.
The sequence $\{U_i\}$ is Wijsman strongly $\mathscr{I}$-Cesàro summable to U, if for every $\varepsilon > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y, U_i) - d(y, U)| \geq \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \stackrel{C_1[\mathscr{I}_W]}{\longrightarrow} U$.

The sequence $\{U_i\}$ is Wijsman p-strongly $\mathscr{I}$-Cesàro summable to U, if for every $\varepsilon > 0$, for each p positive real number and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y, U_i) - d(y, U)|^p \geq \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \stackrel{C_p[\mathscr{I}_W]}{\longrightarrow} U$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. In this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Let θ be a lacunary sequence. The sequence $\{U_i\}$ is Wijsman strongly $\mathscr{I}$-lacunary summable to U, if for every $\varepsilon > 0$ and for each $y \in Y$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{i \in I_r} |d(y, U_i) - d(y, U)| \geq \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \stackrel{N_\theta[\mathscr{I}_W]}{\longrightarrow} U$.

Two nonnegative sequences $a = (a_i)$ and $b = (b_i)$ are said to be asymptotically equivalent if

$$\lim_i \frac{a_i}{b_i} = 1$$

and denoted by $a \sim b$.

We define $d(y; U_i, V_i)$ as follows:

$$d(y; U_i, V_i) = \begin{cases} \dfrac{d(y, U_i)}{d(y, V_i)} & , \quad y \notin U_i \cup V_i \\ \mathscr{L} & , \quad y \in U_i \cup V_i. \end{cases}$$

The sequences $\{U_i\}$ and $\{V_i\}$ are Wijsman asymptotically equivalent of multiple $\mathscr{L}$, if for each $y \in Y$,

$$\lim_{i \to \infty} d(y; U_i, V_i) = \mathscr{L}.$$

The sequences $\{U_i\}$ and $\{V_i\}$ are Wijsman asymptotically statistical equivalent of multiple $\mathscr{L}$, if for every $\varepsilon > 0$ and for each $y \in Y$,

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{i \le n : |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\}\right| = 0.$$

The sequences $\{U_i\}$ and $\{V_i\}$ are Wijsman asymptotically $\mathscr{I}$-equivalent of multiple $\mathscr{L}$, if for every $\varepsilon > 0$ and each $y \in Y$

$$\left\{i \in \mathbb{N} : |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \overset{\mathscr{I}_W^L}{\sim} V_i$ and simply Wijsman asymptotically $\mathscr{I}$-equivalent if $\mathscr{L} = 1$.

The sequences $\{U_i\}$ and $\{V_i\}$ are Wijsman asymptotically $\mathscr{I}$-statistical equivalent of multiple $\mathscr{L}$, if for every $\varepsilon > 0$, $\delta > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{i \le n : |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\}\right| \ge \delta\right\} \in \mathscr{I}$$

and we write $U_i \overset{S(\mathscr{I}_W^L)}{\sim} V_i$ and simply Wijsman asymptotically $\mathscr{I}$-statistical equivalent if $\mathscr{L} = 1$.

Let θ be a lacunary sequence. The sequences $\{U_i\}$ and $\{V_i\}$ are said to be Wijsman asymptotically strongly $\mathscr{I}$-lacunary equivalent of multiple $\mathscr{L}$, if for every $\varepsilon > 0$ and for each $y \in Y$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{i\in I_r} |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \overset{N_\theta[\mathscr{I}_W^L]}{\sim} V_i$ and simply Wijsman asymptotically strongly $\mathscr{I}$-lacunary equivalent if $\mathscr{L} = 1$.

3. Main results

In this section, we defined notions of asymptotically $\mathscr{I}$-Cesàro equivalence of sequences of sets. Also, we investigate the relationships between the concepts of asymptotically strongly $\mathscr{I}$-Cesàro equivalence, asymptotically strongly $\mathscr{I}$-lacunary equivalence, asymptotically p-strongly $\mathscr{I}$-Cesàro equivalence and asymptotically $\mathscr{I}$-statistical equivalence of sequences of sets.

Definition 3.1. *The sequences $\{U_i\}$ and $\{V_i\}$ are asymptotically $\mathscr{I}$-Cesàro equivalence of multiple $\mathscr{L}$, if for every $\varepsilon > 0$ and for each $y \in Y$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \overset{C_1^L(\mathscr{I}_W)}{\sim} V_i$ and simply asymptotically $\mathscr{I}$-Cesàro equivalent if $\mathscr{L} = 1$.

Definition 3.2. *The sequences $\{U_i\}$ and $\{V_i\}$ are asymptotically strongly $\mathscr{I}$-Cesàro equivalence of multiple $\mathscr{L}$, if for every $\varepsilon > 0$ and for each $y \in Y$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \overset{C_1^L[\mathscr{I}_W]}{\sim} V_i$ and simply asymptotically strongly $\mathscr{I}$-Cesàro equivalent if $\mathscr{L} = 1$.

Theorem 3.3. *Let θ be a lacunary sequence. If $\liminf_r q_r > 1$ then,*

$$U_i \overset{C_1^L[\mathscr{I}_W]}{\sim} V_i \Rightarrow U_i \overset{N_\theta^L[\mathscr{I}_W]}{\sim} V_i.$$

Proof. If $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r \ge 1+\delta$ for all $r \ge 1$. Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r}{h_r} \le \frac{1+\delta}{\delta} \quad \text{and} \quad \frac{k_{r-1}}{h_r} \le \frac{1}{\delta}.$$

Let $\varepsilon > 0$ and for each $y \in Y$, we define the set

$$S = \left\{k_r \in \mathbb{N} : \frac{1}{k_r}\sum_{i=1}^{k_r} |d(y;U_i,V_i) - \mathscr{L}| \ < \varepsilon\right\}.$$

We can easily say that $S \in \mathscr{F}(\mathscr{I})$, which is a filter of the ideal $\mathscr{I}$, so we have

$$\begin{aligned}
\tfrac{1}{h_r}\sum_{i\in I_r} |d(y;U_i,V_i) - \mathscr{L}| &= \tfrac{1}{h_r}\sum_{i=1}^{k_r} |d(y;U_i,V_i) - \mathscr{L}| - \tfrac{1}{h_r}\sum_{i=1}^{k_{r-1}} |d(y;U_i,V_i) - \mathscr{L}| \\
&= \tfrac{k_r}{h_r}\cdot\tfrac{1}{k_r}\sum_{i=1}^{k_r} |d(y;U_i,V_i) - \mathscr{L}| \\
&\quad -\tfrac{k_{r-1}}{h_r}\cdot\tfrac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}} |d(y;U_i,V_i) - \mathscr{L}| \\
&\le \left(\frac{1+\delta}{\delta}\right)\varepsilon - \frac{1}{\delta}\varepsilon'
\end{aligned}$$

for each $y \in Y$ and for each $k_r \in S$. Choose $\eta = \left(\frac{1+\delta}{\delta}\right)\varepsilon + \frac{1}{\delta}\varepsilon'$. Therefore, for each $y \in Y$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r}\sum_{i \in I_r} |d(y;U_i,V_i) - \mathscr{L}| \ < \eta \right\} \in \mathscr{F}(\mathscr{I}).$$

Therefore, $U_i \overset{N_\theta^L[\mathscr{I}_W]}{\sim} V_i$. □

Theorem 3.4. *Let θ be a lacunary sequence. If $\limsup_r q_r < \infty$ then,*

$$U_i \overset{N_\theta^L[\mathscr{I}_W]}{\sim} V_i \Rightarrow U_i \overset{C_1^L[\mathscr{I}_W]}{\sim} V_i.$$

Proof. If $\limsup_r q_r < \infty$, then there exists $K > 0$ such that $q_r < K$ for all $r \geq 1$. Let $U_i \overset{N_\theta^L[\mathscr{I}_W]}{\sim} V_i$ and for each $y \in Y$, we define the sets T and R

$$T = \left\{ r \in \mathbb{N} : \frac{1}{h_r}\sum_{i \in I_r} |d(y;U_i,V_i) - \mathscr{L}| \ < \varepsilon_1 \right\}$$

and

$$R = \left\{ n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}| \ < \varepsilon_2 \right\}.$$

Let

$$a_j = \frac{1}{h_j}\sum_{i \in I_j} |d(y;U_i,V_i) - \mathscr{L}| \ < \varepsilon_1$$

for each $y \in Y$ and for all $j \in T$. It is obvious that $T \in \mathscr{F}(\mathscr{I})$. Choose n is any integer with $k_{r-1} < n < k_r$, where $r \in T$. Then, for each $y \in Y$ we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}| &\leq \frac{1}{k_{r-1}}\sum_{i=1}^{k_r} |d(y;U_i,V_i) - \mathscr{L}| \\
&= \frac{1}{k_{r-1}}\Bigg(\sum_{i \in I_1} |d(y;U_i,V_i) - \mathscr{L}| + \sum_{i \in I_2} |d(y;U_i,V_i) - \mathscr{L}| \\
&\quad + \cdots + \sum_{i \in I_r} |d(y;U_i,V_i) - \mathscr{L}| \Bigg) \\
&= \frac{k_1}{k_{r-1}}\left(\frac{1}{h_1}\sum_{i \in I_1} |d(y;U_i,V_i) - \mathscr{L}| \right) \\
&\quad + \frac{k_2-k_1}{k_{r-1}}\left(\frac{1}{h_2}\sum_{i \in I_2} |d(y;U_i,V_i) - \mathscr{L}| \right) \\
&\quad + \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}\left(\frac{1}{h_r}\sum_{i \in I_r} |d(y;U_i,V_i) - \mathscr{L}| \right) \\
&= \frac{k_1}{k_{r-1}}a_1 + \frac{k_2-k_1}{k_{r-1}}a_2 + \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}a_r \\
&\leq \left(\sup_{j \in T} a_j \right)\frac{k_r}{k_{r-1}} < \varepsilon_1 \cdot K.
\end{aligned}$$

Choose $\varepsilon_2 = \frac{\varepsilon_1}{K}$ and in view of the fact that

$$\bigcup \{ n : k_{r-1} < n < k_r, \, r \in T \} \subset R,$$

where $T \in \mathscr{F}(\mathscr{I})$, it follows from our assumption on θ that the set R also belongs to $\mathscr{F}(\mathscr{I})$ and therefore, $U_i \overset{C_1^L[\mathscr{I}_W]}{\sim} V_i$. □

We have the following Theorem by Theorem 3.3 and Theorem 3.4.

Theorem 3.5. *Let θ be a lacunary sequence. If $1 < \liminf_r q_r < \limsup_r q_r < \infty$ then,*

$$U_i \overset{C_1^L[\mathscr{I}_W]}{\sim} V_i \Leftrightarrow U_i \overset{N_\theta^L[\mathscr{I}_W]}{\sim} V_i.$$

Definition 3.6. *The sequences $\{U_i\}$ and $\{V_i\}$ are asymptotically p-strongly $\mathscr{I}$-Cesàro equivalence of multiple $\mathscr{L}$ if for every $\varepsilon > 0$, for each p positive real number and for each $y \in Y$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \geq \varepsilon \right\} \in \mathscr{I}$$

and we write $U_i \overset{C_p^L[\mathscr{I}_W]}{\sim} V_i$ and simply asymptotically p-strongly $\mathscr{I}$-Cesàro equivalent if $\mathscr{L} = 1$.

Theorem 3.7. *If the sequences $\{U_i\}$ and $\{V_i\}$ are asymptotically p-strongly $\mathscr{I}$-Cesàro equivalence of multiple $\mathscr{L}$ then, $\{U_i\}$ and $\{V_i\}$ are asymptotically $\mathscr{I}$-statistical equivalence of multiple $\mathscr{L}$.*

Proof. Let $U_i \overset{C_p^L[\mathscr{I}_W]}{\sim} V_i$ and $\varepsilon > 0$ given. Then, for each $y \in Y$ we have

$$\begin{aligned}\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p &\geq \sum_{\substack{i=1 \\ |d(y;U_i,V_i)-\mathscr{L}|\geq\varepsilon}}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \\ &\geq \varepsilon^p \cdot |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}|\end{aligned}$$

and so

$$\frac{1}{\varepsilon^p \cdot n} \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \geq \frac{1}{n} |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}| .$$

Hence, for each $y \in Y$ and for a given $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}| \geq \delta \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \geq \varepsilon^p \cdot \delta \right\} \in \mathscr{I}.$$

Therefore, $U_i \overset{S(\mathscr{I}_W)}{\sim} V_i$. □

Theorem 3.8. *Let $d(y,U_i) = \mathscr{O}\big(d(y,V_i)\big)$. If $\{U_i\}$ and $\{V_i\}$ are asymptotically $\mathscr{I}$-statistical equivalence of multiple $\mathscr{L}$ then, $\{U_i\}$ and $\{V_i\}$ are asymptotically p-strongly $\mathscr{I}$-Cesàro equivalence of multiple $\mathscr{L}$.*

Proof. Suppose that $d(y,U_i) = \mathscr{O}\big(d(y,V_i)\big)$ and $U_i \overset{S(\mathscr{I}_W)}{\sim} V_i$. Then, there is a $K > 0$ such that $|d(y;U_i,V_i) - \mathscr{L}| \leq K$, for all i and for each $y \in Y$. Given $\varepsilon > 0$ and for each $y \in Y$, we have

$$\begin{aligned}\frac{1}{n} \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p &= \frac{1}{n} \sum_{\substack{i=1 \\ |d(y;U_i,V_i)-\mathscr{L}|\geq\varepsilon}}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p + \frac{1}{n} \sum_{\substack{i=1 \\ |d(y;U_i,V_i)-\mathscr{L}|<\varepsilon}}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \\ &\leq \frac{1}{n} K^p |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}| + \frac{1}{n} \varepsilon^p |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| < \varepsilon\}| \\ &\leq \frac{K^p}{n} |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}| + \varepsilon^p .\end{aligned}$$

Then, for any $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \geq \delta \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}| \geq \frac{\delta^p}{K^p} \right\} \in \mathscr{I}.$$

Therefore, $U_i \overset{C_p^L[\mathscr{I}_W]}{\sim} V_i$. □

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Fractional Ulam-stability of fractional impulsive differential equation involving Hilfer-Katugampola fractional differential operator

S. Harikrishnan[a*], Rabha W. Ibrahim[b] and K. Kanagarajan[a]

[a]*Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore-641020, India*
[b]*Faculty of Computer Science and Information Technology, University Malaya, 50603, Malaysia*
[*]*Corresponding author E-mail: rabhaibrahim@yahoo.com*

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Abstract

In this note, we set up existence, uniqueness as well as the stability of a special class of fractional differential equation (FDE) with Hilfer-Katugampola fractional differential operator (HKFDO). The outcomes are given by employing the Schaefer's fixed point theorem and Banach contraction principle. Moreover, we modify the fractional Ulam stability (FUS) concept utilizing HKFDO.

1. Introduction

The idea of impulsive differential equations has had attention many investigators. Its developments over more than twenty years in almost all science. It performed as an essential role in present day in current applied mathematical model of real techniques bobbing up in phenomena studied in physics, chemical generation, population studies and political economy; one can follow the monograph of Lakshmikantham et al. [12]. The analysis of impulsive differential equation involving classical derivatives one can refer to [2, 13, 14, 16]. Nowadays the investigation of FDE involving Hilfer fractional operator introduced by Hilfer [4] is increasing rapidly one can refer to [3, 9, 10]. Later on the generalized fractional derivative introduced by U.N. Katugampola [11] is unified with Hilfer fractional derivative by Oliveira and E. Capelas de Oliveira in [15] is named as Hilfer-Katugampola fractional derivative.

The fractional Ulam-Hyers stability (FUHRS) of FDE has been studied in [5, 17] utilizing the classical fractional calculus. While, this form of stability has been formalized in a complex domain for the Cauchy problem in [6]-[8]. Here, we shall introduce a generalization for FUHRS involving a multi- power of fractional calculus.

Consider the impulsive differential equation involving Hilfer-Katugampola fractional derivative of the form

$$\begin{cases} {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) = f(t,\mathfrak{v}(t)), \quad t \in I' := I \setminus \{t_1,\dots,t_m\}, I := [0,b] \\ \Delta^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{v}(t)|_{t=t_k} = \chi_k \mathfrak{v}(t_k^-), \\ {}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{v}(0) = \mathfrak{v}_0, \quad \gamma = \alpha+\beta-\alpha\beta, \end{cases} \tag{1.1}$$

where ${}^{\rho}\mathfrak{D}^{\alpha,\beta}$ is Hilfer-Katugampola fractional differential operator of order $\alpha(0<\alpha<1)$, $\beta(0\le\beta\le1)$, ${}^{\rho}\mathfrak{J}^{1-\gamma}$ is a generalization fractional integral operator of order $1-\gamma$, $\rho>0$, $f: I\times R\to R$ is a given continuous function, $\chi_k: R\to R$, and $0=t_0<t_1<\dots<t_m<t_{m+1}=b$, $\Delta^{\rho}I^{1-\gamma}\mathfrak{v}|_{t=t_k} = {}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{v}(t_k^+) - {}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{v}(t_k^-)$, and ${}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{v}(t_k^+) = \lim_{h\to0+}\mathfrak{v}(t_k+h)$, ${}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{v}(t_k^-) = \lim_{h\to0-}\mathfrak{v}(t_k+h)$ are the right and left limits of $\mathfrak{v}(t)$ at $t=t_k$ respectively.

The paper constructed as follows: In Section 2, we present the main definitions and preliminaries. In Section 3, we deal with the finding results. In Section 4, we introduce a generalization of a special class of FUHRS.

Email address: **hkkhari1@gmail.com (S. Harikrishnan), rabhaibrahim@yahoo.com (R. W. Ibrahim), kanagarajank@gmail.com (K. Kanagarajan)**

2. Preliminaries

Here, we recall some of well known concepts (see [1, 13, 15]). Consider the space

$$PC(I,R)=\left\{\mathfrak{v}:I\to R:\mathfrak{v}(t)\in C(t_k,t_{k+1}],k=0,...,m;\text{ there exists }\mathfrak{v}(t_k^+)\text{and}\mathfrak{v}(t_k^-)\right\}.$$

Now we consider the weighted space $PC_\gamma(I,R)$.

$$PC_{\gamma,\rho}(I,R)=\left\{\mathfrak{v}:\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^\gamma\mathfrak{v}|_{[t_k,t_{k+1}]}\in C[t_k,t_{k+1}],k=0,...,m\text{ where }0\le\gamma<1\right\}.$$

Obviously, it is a Banach space with norm

$$\|\mathfrak{v}\|_{PC_{\gamma,\rho}}=\sup_{(t_k,t_{k+1}]}\left\{\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^\gamma\mathfrak{v}(t)\right\},k=0,...,m.$$

The following spaces are used to solve the problem:

$$PC_{1-\gamma,\rho}^{\alpha,\beta}(I,R)=\left\{f\in PC_{1-\gamma,\rho}(I,R),{}^\rho\mathfrak{D}^{\alpha,\beta}f\in PC_{\mu,\rho}(I,R)\right\}$$

and

$$PC_{1-\gamma,\rho}^{\gamma}(I,R)=\left\{f\in PC_{1-\gamma,\rho}(I,R),{}^\rho\mathfrak{D}^{\gamma}f\in PC_{1-\gamma,\rho}(I,R)\right\}.$$

It is obvious that

$$PC_{1-\gamma,\rho}^{\gamma}(I,R)\subset PC_{1-\gamma,\rho}^{\alpha,\beta}(I,R).$$

Definition 2.1. *The generalized left-sided fractional integral ${}^\rho I_{a^+}^\alpha f$ of order $\alpha\in C(\Re(\alpha))$ is defined by*

$$\left({}^\rho\mathfrak{J}_{a^+}^\alpha\right)f(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(t^\rho-s^\rho)^{\alpha-1}s^{\rho-1}f(s)ds,\ t>a. \tag{2.1}$$

The generalized fractional differential operator, corresponding to the generalized fractional integral operator (2.1), is defined for $0\le a<t$, by

$$\left({}^\rho\mathfrak{D}_{a^+}^\alpha f\right)(t)=\frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int_a^t(t^\rho-s^\rho)^{n-\alpha+1}s^{\rho-1}f(s)ds, \tag{2.2}$$

if the integral exists.

Definition 2.2. *The Hilfer-Katugampola fractional derivative with respect to t, with the fractional power $\rho>0$, is defined by*

$$\begin{aligned}\left({}^\rho\mathfrak{D}_{a^\pm}^{\alpha,\beta}f\right)(t)&=\left(\pm{}^\rho\mathfrak{J}_{a^\pm}^\alpha\left(t^{\rho-1}\frac{d}{dt}\right){}^\rho\mathfrak{J}_{a^\pm}^{(1-\beta)(1-\alpha)}\right)(t)\\&=\left(\pm{}^\rho\mathfrak{J}_{a^\pm}^\alpha\delta_\rho{}^\rho\mathfrak{J}_{a^\pm}^{(1-\beta)(1-\alpha)}\right)(t).\end{aligned} \tag{2.3}$$

- *The operator ${}^\rho\mathfrak{D}_{a^+}^{\alpha,\beta}$ can be written as*

$${}^\rho\mathfrak{D}_{a^+}^{\alpha,\beta}={}^\rho\mathfrak{J}_{a^+}^{\beta(1-\alpha)}\delta_\rho{}^\rho\mathfrak{J}_{a^+}^{1-\gamma}={}^\rho\mathfrak{J}_{a^+}^{\beta(1-\alpha)}{}^\rho\mathfrak{D}_{a^+}^{\gamma},\ \ \gamma=\alpha+\beta-\alpha\beta.$$

- *The fractional operator ${}^\rho\mathfrak{D}_{a^+}^{\alpha,\beta}$ is considered as interpolation, with the convenient parameters, of the following fractional derivatives, Hilfer fractional differential operator when $(\rho\to1)$, Hilfer-Hadamard fractional derivative when $(\rho\to0)$, generalized fractional derivative when $(\beta=0)$, Caputo-type fractional derivative when $(\beta=1)$, Riemann-Liouville fractional differential operator when $(\beta=0,\rho\to1)$, Hadamard fractional operator when $(\beta=0,\rho\to0)$, Caputo fractional operator when $(\beta=1,\rho\to1)$. Caputo-Hadamard fractional operator when $(\beta=1,\rho\to0)$, Liouville fractional operator when $(\beta=0,\rho\to1,a=0)$, Hadamard fractional operator when $(\beta=0,\rho\to1,a=-\infty)$, We consider the following parameters α,β,γ satisfying*

$$\gamma=\alpha+\beta-\alpha\beta,\ 0\le\gamma<1,\alpha>0,\ \beta<1.$$

For $\alpha>0$, $\beta>0$ and $0\le\gamma<1$. The properties are given as follows,

1. If $f\in C_\gamma(I,R)$, then we have the following semigroup property

$$({}^\rho\mathfrak{J}^{\alpha}{}^\rho\mathfrak{J}^{\beta}f)(t)=({}^\rho\mathfrak{J}^{\alpha+\beta})(t).$$

2. If $f\in C_\gamma(I,R)$, then

$$({}^\rho\mathfrak{D}^{\alpha}{}^\rho\mathfrak{J}^{\alpha}f)(t)=f(t).$$

3. If $t > 0$ then

$$^{\rho}\mathfrak{I}^{\alpha}\left(\frac{t^{\rho}}{\rho}\right)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha+\beta-1},$$

and

$$^{\rho}\mathfrak{D}^{\alpha}\left(\frac{t^{\rho}}{\rho}\right)^{\beta-1}(t) = 0.$$

4. If $f \in PC_{\gamma}$ and $^{\rho}\mathfrak{I}^{1-\alpha}f \in PC_{\gamma}^{1}(I,R)$, then

$$\left(^{\rho}\mathfrak{I}^{\alpha\rho}\mathfrak{D}^{\alpha}\right)(t) = f(t) - \frac{\left(^{\rho}\mathfrak{I}^{1-\alpha}f\right)(0)}{\Gamma(\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1},$$

5. If $\alpha > \gamma$, then $^{\rho}\mathfrak{I}^{\alpha}f$ is continuous on $[0,b]$

$$^{\rho}\mathfrak{I}^{\alpha}f(0) = \lim_{t\to 0}{}^{\rho}\mathfrak{I}^{\alpha}f(t) = 0.$$

6. If $f \in PC_{\gamma,\rho}^{\gamma}(I,R)$, then

$$^{\rho}\mathfrak{I}^{\gamma\rho}\mathfrak{D}^{\gamma}f(t) = {}^{\rho}\mathfrak{I}^{\alpha\rho}\mathfrak{D}^{\alpha,\beta}f(t) \tag{2.4}$$

and

$$^{\rho}\mathfrak{D}^{\gamma\rho}\mathfrak{I}^{\alpha}f(t) = {}^{\rho}\mathfrak{D}^{\beta(1-\alpha)}f(t). \tag{2.5}$$

7. Let $f \in L^{1}(0,b)$. If $^{\rho}\mathfrak{D}^{\beta(1-\alpha)}f$ occurs on $L^{1}(0,b)$, then

$$^{\rho}\mathfrak{D}^{\alpha,\beta\rho}\mathfrak{I}^{\alpha}f(t) = {}^{\rho}\mathfrak{I}^{\beta(1-\alpha)\rho}\mathfrak{D}^{\beta(1-\alpha)}f(t).$$

8. If $f \in PC_{\gamma,\rho}(I,R)$ and $^{\rho}\mathfrak{I}^{1-\beta(1-\alpha)} \in PC_{1-\gamma}^{1}(I,R)$, then $^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{I}^{\alpha}$ exists on $[0,b]$ and

$$^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{I}^{\alpha}f(t) = f(t).$$

Lemma 2.3. *Let $\mathfrak{v} \in PC_{1-\gamma}(I,R)$ satisfies the following inequality*

$$|\mathfrak{v}(t)| \le c_1 + c_2\int_0^t \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}|\mathfrak{v}(t)|\,ds + \sum_{0<t_k<t}\chi_k|\mathfrak{v}(t_k)|,$$

where c_1 is a non-negative, continuous and non-decreasing function on I and c_2, χ_i are constants. Then

$$|\mathfrak{v}(t)| \le c_1\left(1+\chi E_{\alpha}(c_2\Gamma(\alpha)t^{\alpha})^{k}E_{\alpha}(c_2\Gamma(\alpha)t^{\alpha}\right)\ for\ t \in (t_k.t_{k+1}],$$

where $\chi = \sup\{\chi_k : k = 1,2,3,...,\}$.

Theorem 2.4. *(Schaefer's fixed point theorem) Let $\mathfrak{P} : K \to K$ be completely continuous operator. If set $E[\mathfrak{P}] = \{\mathfrak{v} \in K : \mathfrak{v} = \delta(\mathfrak{P}\mathfrak{v}),\ for\ \ some\ \delta \in [0,b]\}$ is bounded, Then $\mathfrak{P}$ has fixed point.*

Lemma 2.5. *A function $\mathfrak{v}$ is the solution of fractional impulsive differential equation*

$$\begin{cases} ^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) = f(t,\mathfrak{v}(t)), t \in I' \\ \Delta^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{v}(t)_{t=t_k} = \chi_k\mathfrak{v}(t_k^{-}), \\ ^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{v}(0) = a, \end{cases}$$

if and only if $\mathfrak{v}$ achieves the integral equation

$$\mathfrak{v}(t) = \left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left[a + \sum_{0<t_k<t}\chi_k\mathfrak{v}(t_k) + \sum_{0<t_k<t}{}^{\rho}\mathfrak{I}_{t_{k-1}}^{1-\beta(1-\alpha)}f(t_k,\mathfrak{v}(t_k))\right] + {}^{\rho}\mathfrak{I}_{t_k}^{\alpha}f(t,\mathfrak{v}(t)). \tag{2.6}$$

3. Findings

We make the following hypotheses to prove our main results.

(H1) Let $f: I\times R \longrightarrow R$ be a continuous function and a positive constant $L>0$ accomplishing $|f(t,\mathfrak{v})-f(t,\overline{\mathfrak{v}})|\leq L|\mathfrak{v}-\overline{\mathfrak{v}}|$, for all $\mathfrak{v},\overline{\mathfrak{v}}\in R$.
(H2) Let $f: I\times R\rightarrow R$ be a completely continuous function and a function $\mu\in L^1$ fulfilling $|f(t,\mathfrak{v})|\leq|\mu(t)|$, for all $t\in I$, $\mathfrak{v}\in R$.
(H3) Let the functions $\chi_k: R\rightarrow R$ be continuous and a constant $L_k^*>0$ achieving

$$\left|\chi_k(\mathfrak{v}(t_k^-))-\chi_k(\overline{\mathfrak{v}}(t_k^-))\right|\leq L_k^*|\mathfrak{v}(t_k)-\overline{\mathfrak{v}}(t_k)|, \text{ for all } \mathfrak{v},\overline{\mathfrak{v}}\in R,\ k=1,2,...,m.$$

(H4) Let the functions $\chi_k: R\rightarrow R$ be continuous and a constant $\mu\in L^1$ satisfying

$$\left|\chi_k(\mathfrak{v}(t_k^-))\right|\leq|\mu^*(t)|, \text{ for all } \mathfrak{v}\in R,\ k=1,2,...,m.$$

(H5) There is an increasing function $\varphi\in PC_{1-\gamma,\rho}(I,R)$ and there occurs $\lambda_\varphi>0$ such that for any $t\in I$

$$^{\rho}\mathfrak{I}^{\alpha}\varphi(t)\leq\lambda_\varphi\varphi(t).$$

Theorem 3.1. *Assume that [H1] - [H4] are satisfied. Then, Eq.(1.1) has at least one solution.*

Proof. The proof will be given in several steps.
Consider the operator $\mathfrak{P}: PC_{1-\gamma,\rho}(I,R)\rightarrow PC_{1-\gamma,\rho}(I,R)$. The equivalent integral Eq. (2.6) which can be written in the operator form

$$\mathfrak{v}(t)=\mathfrak{P}\mathfrak{v}(t)$$

where

$$\begin{aligned}\mathfrak{P}\mathfrak{v}(t)=&\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left[a+\sum_{0<t_k<t}\chi_k\mathfrak{v}(t_k)+\sum_{0<t_k<t}{}^{\rho}\mathfrak{I}_{t_{k-1}}^{1-\beta(1-\alpha)}f(t_k,\mathfrak{v}(t_k))\right]\\&+{}^{\rho}\mathfrak{I}_{t_k}^{\alpha}f(t,\mathfrak{v}(t)).\end{aligned}\tag{3.1}$$

We shall show that the operator $\mathfrak{P}$ is continuous and completely continuous.

Claim 1: $\mathfrak{P}$ is continuous.
Let $\mathfrak{v}_n$ be a sequence such that $\mathfrak{v}_n\rightarrow\mathfrak{v}$ in $PC_{1-\gamma,\rho}(I,R)$. Then for each $t\in I$,

$$\begin{aligned}\left|((\mathfrak{P}\mathfrak{v}_n)(t)-(\mathfrak{P}\mathfrak{v})(t))\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}\right|\leq&\frac{1}{\Gamma(\gamma)}\left[\sum_{0<t_k<t}|\chi_k(\mathfrak{v}_n(t_k))-\chi_k(\mathfrak{v}(t_k))|+\sum_{0<t_k<t}\mathfrak{I}_{t_{k-1}}^{1-\beta(1-\alpha)}|f(t_k,\mathfrak{v}_n(t_k))-f(t_k,\mathfrak{v}(t_k))|\right]\\&+\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}\mathfrak{I}_{t_k}^{\alpha}|f(t,\mathfrak{v}_n(t))-f(t,\mathfrak{v}(t))|,\end{aligned}$$

since f is continuous, then we have

$$\|(\mathfrak{P}\mathfrak{v}_n)(t)-(\mathfrak{P}\mathfrak{v})(t)\|_{PC_{1-\gamma,\rho}}\rightarrow0 \text{ as } n\rightarrow\infty.$$

Claim 2: We show that $\mathfrak{P}$ is the mapping of two bounded set.
For $r>0$, there exists a positive constant l such that
$B_r=\left\{\mathfrak{v}\in PC_{1-\gamma,\rho}(I,R):\|\mathfrak{v}\|_{PC_{1-\gamma}}\leq r\right\}$, we have $\|(N\mathfrak{v})\|_{PC_{1-\gamma,\rho}}\leq l$.

$$\begin{aligned}\left|(\mathfrak{P}\mathfrak{v})(t)\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}\right|&\leq\frac{1}{\Gamma(\gamma)}\left[a+\sum_{0<t_k<t}|\chi_k(\mathfrak{v}(t_k))|+\sum_{0<t_k<t}\mathfrak{I}_{t_{k-1}}^{1-\beta(1-\alpha)}|f(t_k,\mathfrak{v}(t_k))|\right]+\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}\mathfrak{I}_{t_k}^{\alpha}|f(t,\mathfrak{v}(t))|\\&\leq\frac{1}{\Gamma(\gamma)}\left[a+m\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{\gamma-1}\|\mu^*(t)\|_{PC_{1-\gamma,\rho}}+\|\mu(t)\|_{PC_{1-\gamma,\rho}}\left(\frac{t_k^\rho-t_{k-1}^\rho}{\rho}\right)^{\alpha}\frac{mB(\gamma,1-\beta(1-\alpha))}{\Gamma(1-\beta(1-\alpha))}\right]\\&\quad+\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{\alpha+\gamma-1}\|\mu(t)\|_{PC_{1-\gamma,\rho}}\\&\leq\frac{1}{\Gamma(\gamma)}\left[a+m\left(\frac{b^\rho}{\rho}\right)^{\gamma-1}\|\mu^*(t)\|_{PC_{1-\gamma,\rho}}+\frac{mB(\gamma,1-\beta(1-\alpha))}{\Gamma(1-\beta(1-\alpha))}\left(\frac{b^\rho}{\rho}\right)^{\alpha}\|\mu(t)\|_{PC_{1-\gamma,\rho}}\right]\\&\quad+\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{b^\rho}{\rho}\right)^{\alpha}\|\mu(t)\|_{PC_{1-\gamma,\rho}}\\&=l.\end{aligned}$$

Claim 3: We show that $\mathfrak{P}$ maps bounded sets into equicontinuous set.
Let $t_1, t_2 \in I, t_1 < t_2, B_r$ be a bounded set of $PC_{1-\gamma,\rho}(I,R)$ as in Claim 2, and $\mathfrak{v} \in B_r$. Then,

$$\begin{aligned}
&\left|(\mathfrak{P}\mathfrak{v})(t_1)\left(\frac{t_1^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} - (\mathfrak{P}\mathfrak{v})(t_2)\left(\frac{t_2^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\right| \\
&\le \frac{1}{\Gamma(\gamma)}\left[a + \sum_{0<t_k<t_1}\chi_k \mathfrak{v}(t_k) + \sum_{0<t_k<t_1} {}^\rho\mathfrak{I}_{t_{k-1}}^{1-\beta(1-\alpha)} f(t_k, \mathfrak{v}(t_k))\right] + \left(\frac{t_1^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} {}^\rho\mathfrak{I}_{t_k}^\alpha f(t_1, \mathfrak{v}(t_1)) \\
&\quad - \frac{1}{\Gamma(\gamma)}\left[a + \sum_{0<t_k<t_2}\chi_k \mathfrak{v}(t_k) + \sum_{0<t_k<t_2} {}^\rho\mathfrak{I}_{t_{k-1}}^{1-\beta(1-\alpha)} f(t_k, \mathfrak{v}(t_k))\right] - \left(\frac{t_2^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} {}^\rho\mathfrak{I}_{t_k}^\alpha f(t_2, \mathfrak{v}(t_2)) \\
&\le \frac{1}{\Gamma(\gamma)}\left[\sum_{0<t_k<t_1-t_2}\chi_k \mathfrak{v}(t_k) + \sum_{0<t_k<t_1-t_2} {}^\rho I_{t_{k-1}}^{1-\beta(1-\alpha)} f(t_k, \mathfrak{v}(t_k))\right] \\
&\quad + \|f\|_{PC_{1-\gamma,\rho}} \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left|\left(\frac{t_1^\rho - t_k^\rho}{\rho}\right)^\alpha - \left(\frac{t_2^\rho - t_k^\rho}{\rho}\right)^\alpha\right|
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. From Claim 1 to 3, together with Arzela-Ascoli theorem, we conclude that $\mathfrak{P}: PC_{1-\gamma,\rho}(I,R) \to PC_{1-\gamma,\rho}(I,R)$ is continuous and completely continuous.

Claim 4: A priori bounds.
Now we prove that

$$\omega = \left\{\mathfrak{v} \in PC_{1-\gamma,\rho}(I,R) : \mathfrak{v} = \delta N(\mathfrak{v}), 0 < \delta < 1\right\}$$

is bounded set.
Let $\mathfrak{v} \in \omega$, $\mathfrak{v} = \delta\mathfrak{P}(\mathfrak{v})$ for some $0 < \delta < 1$. Thus for each $t \in I$. We have

$$\mathfrak{v}(t) = \delta\left[\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1} \frac{1}{\Gamma(\gamma)}\left[a + \sum_{0<t_k<t}\chi_k \mathfrak{v}(t_k) + \sum_{0<t_k<t} {}^\rho\mathfrak{I}_{t_{k-1}}^{1-\beta(1-\alpha)} f(t_k, \mathfrak{v}(t_k))\right] + {}^\rho\mathfrak{I}_{t_k}^\alpha f(t, \mathfrak{v}(t))\right].$$

We show this Claim by letting the estimation in Claim 2. Finally, by Theorem 2.4, we deduce that $\mathfrak{P}$ has a fixed point and it is the solution of problem (1.1). □

Theorem 3.2. *Assume that the hypothesis (H1) and (H3) are fulfilled. If*

$$\left[\frac{1}{\Gamma(\gamma)}\left(mL^*\left(\frac{b^\rho}{\rho}\right)^{\gamma-1} + \frac{mLB(\gamma,1-\beta(1-\alpha))}{\Gamma(1-\beta(1-\alpha))}\left(\frac{b^\rho}{\rho}\right)^\alpha\right) + \frac{LB(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{b^\rho}{\rho}\right)^\alpha\right] < 1$$

then, Eq. (1.1) has a unique solution.

Proof. Let $\mathfrak{v}, \overline{\mathfrak{v}} \in PC_{1-\gamma,\rho}(I,R)$ and $t \in I$, then we have

$$\begin{aligned}
&\left|(\mathfrak{P}\mathfrak{v}(t) - \mathfrak{P}\overline{\mathfrak{v}}(t))\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\right| \\
&\le \frac{1}{\Gamma(\gamma)}\left[\sum_{0<t_k<t}|\chi_k \mathfrak{v}(t_k) - \chi_k \overline{\mathfrak{v}}(t_k)| + \sum_{0<t_k<t} {}^\rho\mathfrak{I}_{t_{k-1}}^{1-\beta(1-\alpha)} |f(t_k, \mathfrak{v}(t_k)) - f(t_k, \overline{\mathfrak{v}}(t_k))|\right] \\
&\quad + \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} {}^\rho\mathfrak{I}_{t_k}^\alpha |f(t, \mathfrak{v}(t)) - f(t, \overline{\mathfrak{v}}(t))| \\
&\le \frac{1}{\Gamma(\gamma)}\left[mL^*\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\|\mathfrak{v} - \overline{\mathfrak{v}}\|_{PC_{1-\gamma,\rho}} + \frac{mLB(\gamma,1-\beta(1-\alpha))}{\Gamma(1-\beta(1-\alpha))}\left(\frac{t_k^\rho - t_{k-1}^\rho}{\rho}\right)^\alpha \|\mathfrak{v} - \overline{\mathfrak{v}}\|_{PC_{1-\gamma,\rho}}\right] \\
&\quad + \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} \frac{LB(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\alpha+\gamma-1} \|\mathfrak{v} - y\|_{PC_{1-\gamma,\rho}} \\
&\le \frac{1}{\Gamma(\gamma)}\left[mL^*\left(\frac{b^\rho}{\rho}\right)^{\gamma-1}\|\mathfrak{v} - \overline{\mathfrak{v}}\|_{PC_{1-\gamma,\rho}} + \frac{mLB(\gamma,1-\beta(1-\alpha))}{\Gamma(1-\beta(1-\alpha))}\left(\frac{b^\rho}{\rho}\right)^\alpha \|\mathfrak{v} - \overline{\mathfrak{v}}\|_{PC_{1-\gamma,\rho}}\right] \\
&\quad + \frac{LB(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{b^\rho}{\rho}\right)^\alpha \|\mathfrak{v} - \overline{\mathfrak{v}}\|_{PC_{1-\gamma,\rho}} \\
&\le \left[\frac{1}{\Gamma(\gamma)}\left(mL^*\left(\frac{b^\rho}{\rho}\right)^{\gamma-1} + \frac{mLB(\gamma,1-\beta(1-\alpha))}{\Gamma(1-\beta(1-\alpha))}\left(\frac{b^\rho}{\rho}\right)^\alpha\right) + \frac{LB(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{b^\rho}{\rho}\right)^\alpha\right]\|\mathfrak{v} - \overline{\mathfrak{v}}\|_{PC_{1-\gamma,\rho}} \\
&= \|\mathfrak{v} - \overline{\mathfrak{v}}\|_{PC_{1-\gamma,\rho}}.
\end{aligned}$$

This yields that $\mathfrak{P}$ admits a unique fixed point, which is a solution of Eq. (1.1). □

4. FUS Analysis

In this section, we exam the FUS for our FDE (1.1). Let $\varphi : I \to R^{+}$ be a continuous function and $\varepsilon > 0$. We need the following inequalities:

$$\begin{cases} \left|{}^{\rho}\mathfrak{D}^{\alpha}\mathfrak{u}(t) - f(t,\mathfrak{u}(t))\right| & \leq & \varepsilon, \\ \left|\Delta^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{u}(t)_{t=t_k} - \chi_k(\mathfrak{u}(t_k^-))\right| & \leq & \varepsilon, \end{cases} \tag{4.1}$$

$$\begin{cases} \left|{}^{\rho}\mathfrak{D}^{\alpha}\mathfrak{u}(t) - f(t,\mathfrak{u}(t))\right| & \leq & \varepsilon\varphi(t), \\ \left|\Delta^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{u}(t)_{t=t_k} - \chi_k(\mathfrak{u}(t_k^-))\right| & \leq & \varepsilon\varphi(t), \end{cases} \tag{4.2}$$

$$\begin{cases} \left|{}^{\rho}\mathfrak{D}^{\alpha}\mathfrak{u}(t) - f(t,\mathfrak{u}(t))\right| & \leq & \varphi(t), \\ \left|\Delta^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{u}(t)_{t=t_k} - \chi_k(\mathfrak{u}(t_k^-))\right| & \leq & \varphi(t), \end{cases} \tag{4.3}$$

Definition 4.1. *The Eq. (1.1) is FUS if there finds a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $\mathfrak{u} \in PC_{1-\gamma,\rho}(I,R)$ of the inequality (4.1) there exists a solution $\mathfrak{v} \in PC_{1-\gamma,\rho}(I,R)$ of Eq. (1.1) with*

$$|\mathfrak{u}(t) - \mathfrak{v}(t)| \leq C_f\ \varepsilon, \quad t \in I.$$

Definition 4.2. *The Eq. (1.1) is FUS if there occurs a function $\varphi \in PC_{1-\gamma,\rho}(I,R)$, $\varphi_f(0) = 0$ satisfying that for each solution $\mathfrak{u} \in PC_{1-\gamma,\rho}(I,R)$ of the inequality (4.1) there occurs a solution $\mathfrak{v} \in PC_{1-\gamma,\rho}(I,R)$ of Eq. (1.1) with*

$$|\mathfrak{u}(t) - \mathfrak{v}(t)| \leq \varphi_f\ \varepsilon, \quad t \in I.$$

Definition 4.3. *The Eq. (1.1) is FUHR stable with respect to $\varphi \in PC_{1-\gamma,\rho}(I,R)$ if there occurs a real number $C_{f,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $\mathfrak{u} \in PC_{1-\gamma,\rho}(I,R)$ of the inequality (4.2) there exists a solution $\mathfrak{v} \in PC_{1-\gamma,\rho}(I,R)$ of Eq. (1.1) with*

$$|\mathfrak{u}(t) - \mathfrak{v}(t)| \leq C_{f,\varphi}\ \varepsilon\varphi(t), \quad t \in I.$$

Definition 4.4. *The Eq. (1.1) is FUHRS with respect to $\varphi \in PC_{1-\gamma,\rho}(I,R)$ if there finds a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{u} \in PC_{1-\gamma,\rho}(I,R)$ of the inequality (4.3) there occurs a solution $\mathfrak{v} \in C_{1-\gamma,\rho}(I,R)$ of Eq. (1.1) with*

$$|\mathfrak{u}(t) - \mathfrak{v}(t)| \leq C_{f,\varphi}\varphi(t), \quad t \in I.$$

Remark 4.5. *A function $\mathfrak{u} \in PC_{1-\gamma,\rho}(I,R)$ is a solution of the inequality (4.1) if and only if there finds a function $g \in PC_{1-\gamma,\rho}(I,R)$ such that*

(i) $|g(t)| \leq \varepsilon$, $|g_k| < \varepsilon, t \in I$.
(ii) ${}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{u}(t) = f(t,\mathfrak{u}(t)) + g(t)$, $t \in I'$.
(iii) $\Delta^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{u}(t)_{t=t_k} = \chi_k\mathfrak{u}(t_k^-) + g_k$.

Remark 4.6. *If $\mathfrak{u}$ is a solution of the inequality (4.1), then $\mathfrak{u}$ is a solution of the following integral inequality*

$$\left|\mathfrak{u}(t) - \left(\frac{t^{\rho} - t_k^{\rho}}{\rho}\right)^{\gamma-1} \frac{1}{\Gamma(\gamma)}\left[a + \sum_{0<t_k<t}\chi_k\mathfrak{u}(t_k) + \sum_{0<t_k<t}{}^{\rho}I_{a_+}^{1-\beta(1-\alpha)}f(t_k,\mathfrak{u}(t_k))\right] - {}^{\rho}I_{a_+}^{\alpha}f(t,\mathfrak{u}(t))\right|$$
$$\leq \varepsilon\left[\left(\frac{b}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left(m + \frac{m}{\Gamma(2-\beta(1-\alpha))}\left(\frac{b}{\rho}\right)^{1-\beta(1-\alpha)}\right) + \frac{1}{\Gamma(\alpha+1)}\left(\frac{b}{\rho}\right)^{\alpha-1}\right].$$

Moreover, by Remark 4.5, one can realize that

$${}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{u}(t) = f(t,\mathfrak{u}(t)) + g(t),\ t \in I$$
$$\Delta^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{u}(t)_{t=t_k} = \chi_k\mathfrak{u}(t_k^-) + g_k.$$

Then

$$\mathfrak{u}(t) = \left(\frac{t^{\rho} - t_k^{\rho}}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left[a + \sum_{0<t_k<t}\chi_k\mathfrak{u}(t_k) + \sum_{0<t_k<t}{}^{\rho}\mathfrak{I}_{a_+}^{1-\beta(1-\alpha)}f(t_k,\mathfrak{u}(t_k))\right] + {}^{\rho}\mathfrak{I}_{a_+}^{\alpha}f(t,\mathfrak{u}(t)).$$

From this it follows that

$$\left|\mathfrak{u}(t) - \left(\frac{t^{\rho} - t_k^{\rho}}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left[a + \sum_{0<t_k<t}\chi_k\mathfrak{u}(t_k) + \sum_{0<t_k<t}{}^{\rho}\mathfrak{I}_{a_+}^{1-\beta(1-\alpha)}f(t_k,\mathfrak{u}(t_k))\right] - {}^{\rho}\mathfrak{I}_{a_+}^{\alpha}f(t,\mathfrak{u}(t))\right|$$
$$\leq \left(\frac{t^{\rho} - t_k^{\rho}}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left[\sum_{0<t_k<t}|g_k| + \sum_{0<t_k<t}{}^{\rho}\mathfrak{I}_{a_+}^{1-\beta(1-\alpha)}|g(t_k)|\right] + {}^{\rho}\mathfrak{I}_{a_+}^{\alpha}|g(t)|$$
$$\leq \varepsilon\left[\left(\frac{b}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left(m + \frac{m}{\Gamma(2-\beta(1-\alpha))}\left(\frac{b}{\rho}\right)^{1-\beta(1-\alpha)}\right) + \frac{1}{\Gamma(\alpha+1)}\left(\frac{b}{\rho}\right)^{\alpha}\right].$$

We have similar remarks for the inequality (4.2) and (4.3).
Now, we give the main results, FUHRS results.

Theorem 4.7. *The hypothesis [H1], [H3] and [H5] holds. Then Eq.(1.1) is FUHRS.*

Proof. Let $\mathfrak{u}$ be solution of 4.3 and by Theorem 3.2 there $\mathfrak{v}$ is unique solution of the problem

$$\begin{aligned} {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) &= f(t,\mathfrak{v}(t),\mathfrak{v}(\lambda t)), \quad t\in I,\\ \Delta^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{u}(t)_{t=t_k} &= \chi_k\mathfrak{u}(t_k^-)+g_k,\\ {}^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{v}(0) &= {}^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{u}(0). \end{aligned}$$

Then, we have

$$\mathfrak{v}(t)=\left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left[a+\sum_{0<t_k<t}\chi_k\mathfrak{v}(t_k)+\sum_{0<t_k<t}{}^{\rho}\mathfrak{I}_{a_+}^{1-\beta(1-\alpha)}f(t_k,\mathfrak{v}(t_k))\right]+{}^{\rho}\mathfrak{I}_{a_+}^{\alpha}f(t,\mathfrak{v}(t)).$$

By differentiating inequality (4.3), we have

$$\begin{aligned} &\left|\mathfrak{u}(t)-\left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left[a+\sum_{0<t_k<t}\chi_k\mathfrak{u}(t_k)+\sum_{0<t_k<t}{}^{\rho}\mathfrak{I}_{a_+}^{1-\beta(1-\alpha)}f(t_k,\mathfrak{u}(t_k))\right]-{}^{\rho}\mathfrak{I}_{a_+}^{\alpha}f(t,\mathfrak{u}(t))\right|\\ &\le\left(\left(\frac{b^{\rho}}{\rho}\right)^{\gamma-1}\frac{m}{\Gamma(\gamma)}(1+\lambda_{\varphi})+\lambda_{\varphi}\right)\varphi(t). \end{aligned}$$

Hence, it follows

$$\begin{aligned} |\mathfrak{u}(t)-\mathfrak{v}(t)| &\le\left|\mathfrak{u}(t)-\left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left[a+\sum_{0<t_k<t}\chi_k\mathfrak{v}(t_k)+\sum_{0<t_k<t}{}^{\rho}\mathfrak{I}_{a_+}^{1-\beta(1-\alpha)}f(t_k,\mathfrak{v}(t_k))\right]-{}^{\rho}\mathfrak{I}_{a_+}^{\alpha}f(t,\mathfrak{v}(t))\right|\\ &\le\left|\mathfrak{u}(t)-\left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left[a+\sum_{0<t_k<t}\chi_k\mathfrak{u}(t_k)+\sum_{0<t_k<t}{}^{\rho}\mathfrak{I}_{a_+}^{1-\beta(1-\alpha)}f(t_k,\mathfrak{u}(t_k))\right]-{}^{\rho}\mathfrak{I}_{a_+}^{\alpha}f(t,\mathfrak{u}(t))\right|\\ &\quad+\left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{\gamma-1}\frac{1}{\Gamma(\gamma)}\left(|\chi_k\mathfrak{u}(t_k)-\chi_k\mathfrak{v}(t_k)|+{}^{\rho}\mathfrak{I}_{a_+}^{1-\beta(1-\alpha)}|f(t_k,\mathfrak{u}(t_k))-f(t_k,\mathfrak{v}(t_k))|\right)\\ &\quad+{}^{\rho}\mathfrak{I}_{a_+}^{\alpha}|f(t,\mathfrak{u}(t))-f(t,\mathfrak{v}(t))|\\ &\le\left(\left(\frac{b^{\rho}}{\rho}\right)^{\gamma-1}\frac{m}{\Gamma(\gamma)}(1+\lambda_{\varphi})+\lambda_{\varphi}\right)\varphi(t)+\left(\frac{b^{\rho}}{\rho}\right)^{\gamma-1}\frac{L_k^*}{\Gamma(\gamma)}|\mathfrak{u}(t_k)-\mathfrak{v}(t_k)|\\ &\quad+\left[\frac{m}{\Gamma(\gamma)}\left(\frac{b^{\rho}}{\rho}\right)^{\alpha}\frac{mL}{\Gamma(2-\beta(1-\alpha))}+\frac{L}{\Gamma(\alpha)}\left(\frac{b^{\rho}}{\rho}\right)^{\alpha}\right]|\mathfrak{u}(t)-\mathfrak{v}(t)|. \end{aligned}$$

By the properties, there occurs a constant $M^*>0$ independent of $\lambda_{\varphi}\varphi(t)$ such that

$$|\mathfrak{u}(t)-\mathfrak{v}(t)|\le M^*\lambda_{\varphi}\varphi(t):=C_{f,\varphi}\varphi(t).$$

Thus, Eq.(1.1) is FUHRS. □

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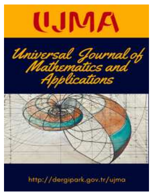

Variational iteration method combined with new transform to solve fractional partial differential equations

Mountassir Hamdi Cherif[a*] and Djelloul Ziane[b]

[a]*Department of Mathematics and Informatics, Electrical and Energy Engineering Graduate School (ESGEE), Oran, Algeria*
[a,b]*Laboratory of mathematics and its applications (LAMAP), University of Oran1 Ahmed Ben Bella, Oran, 31000, Algeria*
[*]*Corresponding author E-mail: mountassir27@yahoo.fr*

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Abstract

The aim of this paper is to combined the variational iteration method with Aboodh transform method to solve linear and nonlinear fractional partial differential equations. Some illustrative examples are given as the linear and nonlinear fractional Klein-Gordon equations and the time fractional diffusion equation. The results reveal that this method is very effective, simple and can be applied to other physical differential equations with fractional order. The fractional derivative is taken in the Caputo sense.

1. Introduction

Fractional calculus has successfully been used to study the mathematical and physical problems arising in science and engineering. Fractional differential equations are applied to describe the dynamical systems in physics and engineering. It is one of the hot topics for finding the solutions for the fractional differential equations for scientists and engineers. Due to the importance of knowledge of the solutions of these type of equations, we find that many researchers have done and are still doing great efforts to find methods to solve this type of equations. These efforts resulted in the consolidation of this research field in many methods, among them we find the homotopy analysis method ([28], [29]), Adomian decomposition method ([7], [8]), variational iteration method (VIM) ([12], [14]) and homotopy perturbation method ([13], [15]), which have become known in a large number of researchers in this area. Recently, a new option has appeared, including the composition of some transform methods with the previously mentioned methods to facilitate and improve the resolution speed of this type of equations. For example, we only mention some of these transform methods, such as Laplace transform method [11], sumudu transform method [2] or Aboodh transform method [20]. Among wich are the Laplace homotopy analysis method [25], Adomian decomposition method coupled with Laplace transform method [27], variational iteration method coupled with Laplace transform method [4], homotopy perturbation transform method [30], homotopy analysis Sumudu transform method [31], modified fractional homotopy analysis transform method [21], Sumudu decomposition method for nonlinear equations [5], variational iteration Sumudu transform method [3], homotopy perturbation Sumudu transform method [16], Aboodh decomposition method [26], fractional Aboodh decomposition method [22], Aboodh transform homotopy perturbation method [19].
The objective of this study is to combine two powerful methods, the first method is " variational iteration method", the second is called "the Aboodh transform method", for solving linear and nonlinear fractional partial differential equations, thus, we get the modified method "fractional variational iteration Aboodh transform method" (FVIATM). Several examples are given to re-confirm the effeciency of the suggested algorithm, the fractional derivative is described in this study in the sense of Caputo.

2. Preliminaries

In this section, we give some basic notions about fractional calculus, Aboodh transform and Aboudh transform of fractional derivatives which are used further in this paper.

Email addresses: **mountassir27@yahoo.fr (M. H. Cherif), djeloulz@yahoo.com (D. Ziane)**

2.1. Fractional calculus

We give some basic definitions and properties of the fractional calculus theory as the Riemann-Liouville fractional integrals and Caputo fractional derivative (see [10], [17]).

Definition 2.1. *Let $\Omega=[a,b]$ $(-\infty<a<b<+\infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann–Liouville fractional integral $I_{0+}^{\alpha}f$ of order $\alpha\in\mathbb{R}$ $(\alpha>0)$ is defined by*

$$(I_{0+}^{\alpha}f)(\tau)=\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\frac{f(\varsigma)d\varsigma}{(\tau-\varsigma)^{1-\alpha}},\quad \tau>0,\ \alpha>0$$
$$(I_{0+}^{0}f)(\tau)=f(\tau)$$

Here $\Gamma(\cdot)$ is the gamma function.

Theorem 2.2. *Let $\alpha\geq 0$ and let $n=[\alpha]+1$. If $f(\tau)\in AC^{n}[a,b]$, then the Caputo fractional derivative $({}^{c}D_{0+}^{\alpha}f)(\tau)$ exist almost evrywhere on $[a,b]$. If $\alpha\notin\mathbb{N}$, $({}^{c}D_{0+}^{\alpha}f)(\tau)$ is represented by*

$$({}^{c}D_{0+}^{\alpha}f)(\tau)=\frac{1}{\Gamma(n-\alpha)}\int_0^{\tau}\frac{f^{(n)}(\varsigma)d\varsigma}{(\tau-\varsigma)^{\alpha-n+1}},\tag{2.1}$$

where $D=\frac{d}{dr}$and $n=[\alpha]+1$.

Remark 2.3. *In this paper, we consider the time-fractional derivative in the Caputo's sense. When $\alpha\in\mathbb{R}^{+}$, the time-fractional derivative is defined as*

$$\begin{aligned}({}^{c}D_{\tau}^{\alpha}u)(r,\tau)&=\frac{\partial^{\alpha}u(r,\tau)}{\partial\tau^{\alpha}}\\&=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^{\tau}(\tau-\varsigma)^{m-\alpha-1}\frac{\partial^{m}u(r,\varsigma)d\varsigma}{\partial\varsigma^{m}}, & m-1<\alpha<m,\\ \frac{\partial^{m}u(r,\tau)}{\partial\tau^{m}}, & \alpha=m,\end{cases}\end{aligned}$$

where $m\in\mathbb{N}^{*}$.

2.2. Definitions and properties of the Aboodh transform

The Aboodh transform was defined by K. S. Aboodh [20] in 2013. In this section, we give some basic definitions and properties of this transform (see [1], [18], [20]).

2.2.1. Definitions

The Aboodh transform is defined for functions of exponential order. We consider functions belonging to a class B, where B defined by

$$B=\left\{u(\tau):\ |u(\tau)|<Me^{k_j|\tau|},\ if\ \tau\in(-1)^{j}\times[0,\ \infty,j=1,2;M,\ k_1,k_2>0\right\}.$$

Definition 2.4. *The Aboodh integral transform of the function u in B is defined by the integral equation*

$$A[u(\tau)]=U(v)=\frac{1}{v}\int_0^{\infty}u(\tau)e^{-v\tau}d\tau;\ \ \tau\geq o,\ \ v\in(k_1,k_2).\tag{2.2}$$

The variable v in this transform is used to factor the variable τ in the argument of the function u.

Proposition 2.5. *The Aboodh transform of the time-fractional derivative in the Caputo's sense is defined as*

$$A\left[({}^{c}D_{0+}^{\alpha}u)(\tau);v\right]=v^{\alpha}A[u(\tau)]-\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{v^{2-\alpha+k}}\ ,\ \ n-1<\alpha\leq n,\ \ n=1,2,\ldots\tag{2.3}$$

And the Aboodh transform of the function $u(r,\tau)$ with Caputo fractional derivative of order α is given by

$$A\left[({}^{c}D_{0+}^{\alpha}u)(r,\tau);v\right]=v^{\alpha}A[u(r,\tau)]-\sum_{k=0}^{n-1}\frac{u^{(k)}(r,0)}{v^{2-\alpha+k}}\ ,\ \ n-1<\alpha\leq n,\ \ n=1,2,\ldots\tag{2.4}$$

2.2.2. Somme properties of the Aboodh transform

1. The Aboodh transform of the nth derivative of $u(\tau)$ is given by

$$A[u^{(n)}(\tau)] = U_n(v) = v^n A[u(\tau)] - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{v^{2-n+k}} \tag{2.5}$$

2. Some elementary functions and their transformations

$u(\tau)$	$A[u(\tau)]$
1	$\frac{1}{v^2}$
τ	$\frac{1}{v^3}$
τ^n	$\frac{n!}{v^{n+2}}$, $n = 0,1,2,\dots$
τ^α	$\frac{\Gamma(\alpha+1)}{v^{\alpha+2}}$, $\alpha \geq 0.$

3. Analysis of fractional variational iteration Aboodh transform method (FVIATM)

To illustrate the basic idea of this method, we consider a general nonlinear partial differential equation of fractional order

$${}^cD_\tau^\alpha U(r,\tau) + RU(r,\tau) + NU(r,\tau) = g(r,\tau), \tag{3.1}$$

where $m-1 < \alpha \leq m$, $m = 1,2,\dots$ and the initial conditions

$$\left[\frac{\partial^{m-1}U(r,\tau)}{\partial\tau^{m-1}}\right]_{\tau=0} = h_{m-1}(r), \tag{3.2}$$

where ${}^cD_\tau^\alpha = \frac{\partial^\alpha}{\partial\tau^\alpha}$ is the Caputo fractional derevative, R is the linear differential operator, N represents the general nonlinear differential operator, and $g(r,\tau)$ is the source term.
Applying Aboodh transform on both sides of (3.1), we obtain

$$A\left[{}^cD_\tau^\alpha U(r,\tau)\right] + A\left[RU(r,\tau)\right] + A\left[NU(r,\tau)\right] = A\left[g(r,\tau)\right]. \tag{3.3}$$

Using the differentiation property of Aboodh transform, we have

$$A\left[U(r,\tau)\right] = \frac{1}{v^\alpha}\sum_{k=0}^{n-1}\frac{U^{(k)}(r,0)}{v^{2-\alpha+k}} + \frac{1}{v^\alpha}A\left[g(r,\tau)\right] - \frac{1}{v^\alpha}A\left[RU(r,\tau) + NU(r,\tau)\right]. \tag{3.4}$$

Operating with the inverse Aboodh transform on both sides of (3.4), we obtain

$$U(r,\tau) = H(r,\tau) - A^{-1}\left(\frac{1}{v^\alpha}A\left[RU(r,\tau) + NU(r,\tau)\right]\right), \tag{3.5}$$

where $H(r,\tau)$, represents the term arising from the source term and the prescribed initial conditions.
Applying $\frac{\partial}{\partial\tau}$ on both sides of (3.5), we have

$$\frac{\partial U(r,\tau)}{\partial\tau} + \frac{\partial}{\partial\tau}A^{-1}\left(\frac{1}{v^\alpha}A\left[RU(r,\tau) + NU(r,\tau)\right]\right) - \frac{\partial H(r,\tau)}{\partial\tau} = 0. \tag{3.6}$$

According to the variational iteration method ([12], [14]), we can construct a correct functional as follows

$$U_{n+1}(r,\tau) = U_n(r,\tau) - \int_0^\tau \left[\frac{\partial U_n(r,\varsigma)}{\partial\varsigma} + \frac{\partial}{\partial\varsigma}A^{-1}\left(\frac{1}{v^\alpha}A\left[RU_n(r,\varsigma) + NU_n(r,\varsigma)\right]\right) - \frac{\partial H(r,\varsigma)}{\partial\varsigma}\right] d\varsigma. \tag{3.7}$$

Recall that $U(r,\tau) = \lim_{n\to\infty} U_n(r,\tau)$.
That may give the exact solution if a closed form one exists, or we can use the $(n+1)th$ approximation for numerical purposes. The convergence of the variational iteration method is introduit by Tatari et all. in [24]. Though the variational iteration method leads to fast convergent solutions, unnecessary calculation arises in the solution procedure.

4. Applications

To illustrate the efficiency of the fractional variational iteration Aboodh transform method, we apply this method to solve some linear and nonlinear time-fractional partial differential equations with Caputo fractional derivative.

Example 4.1. *Consider the following time fractional diffusion equation*

$$\begin{aligned} {}^cD_\tau^\alpha U(r,\tau) &= \tfrac{r^2}{2}U_{rr}(r,\tau), \quad 0<\alpha\le 1, \\ U(r,0) &= r^2. \end{aligned} \tag{4.1}$$

and which subject to the boundary conditions $U(0,\tau)=0$ *and* $U(1,\tau)=f(\tau)$.
Applying Aboodh transform on both sides of (4.1) and using its differentiation property, we obtain

$$A\left[U(r,\tau)\right] = \frac{1}{v^2}r^2 + \frac{1}{v^\alpha}A\left[\frac{r^2}{2}U_{rr}(r,\tau)\right]. \tag{4.2}$$

Taking the inverse Aboodh transform of (4.2), we have

$$U(r,\tau) = r^2 + A^{-1}\left(\frac{1}{v^\alpha}A\left[\frac{r^2}{2}U_{rr}(r,\tau)\right]\right). \tag{4.3}$$

Applying $\frac{\partial}{\partial\tau}$ *on both sides of (4.3), we get*

$$\frac{\partial U(r,\tau)}{\partial\tau} = \frac{\partial}{\partial\tau}A^{-1}\left(\frac{1}{v^\alpha}A\left[\frac{r^2}{2}U_{rr}(r,\tau)\right]\right). \tag{4.4}$$

According to the variational iteration method, we can construct a correct functional as follows

$$U_{n+1}(r,\tau) = U_n(r,\tau) - \int_0^\tau \left[\frac{\partial U_n(r,\varsigma)}{\partial\varsigma} - \frac{\partial}{\partial\varsigma}A^{-1}\left(\frac{1}{v^\alpha}A\left[\frac{r^2}{2}(U_n)_{rr}(r,\tau)\right]\right)\right]d\varsigma. \tag{4.5}$$

By using the iteration formula (4.5), the first terms are given by

$$\begin{aligned} U_0(r,\tau) &= r^2, \\ U_1(r,\tau) &= r^2 + r^2\tfrac{\tau^\alpha}{\Gamma(\alpha+1)}, \\ U_2(r,\tau) &= r^2 + r^2\tfrac{\tau^\alpha}{\Gamma(\alpha+1)} + r^2\tfrac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}, \\ U_3(r,\tau) &= r^2 + r^2\tfrac{\tau^\alpha}{\Gamma(\alpha+1)} + r^2\tfrac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + r^2\tfrac{\tau^{3\alpha}}{\Gamma(3\alpha+1)} \\ &\vdots \\ U_n(r,\tau) &= \textstyle\sum_{k=0}^n \tfrac{r^2\tau^{k\alpha}}{\Gamma(k\alpha+1)}. \end{aligned} \tag{4.6}$$

Recall that the solution is given by

$$U(r,\tau) = \lim_{n\to\infty} U_n(r,\tau) = \lim_{n\to\infty}\sum_{k=0}^n \frac{r^2\tau^{k\alpha}}{\Gamma(k\alpha+1)} = r^2E_\alpha(\tau^\alpha), \tag{4.7}$$

which is the exact solution of time fractional diffusion equation (4.1) obtained by fractional variational iteration method in [9], but with less calculations. In the case $\alpha=1$, *it is given by* $U(r,\tau)=r^2e^\tau$.

Example 4.2. *Consider the linear fractional Klein-Gordon equation*

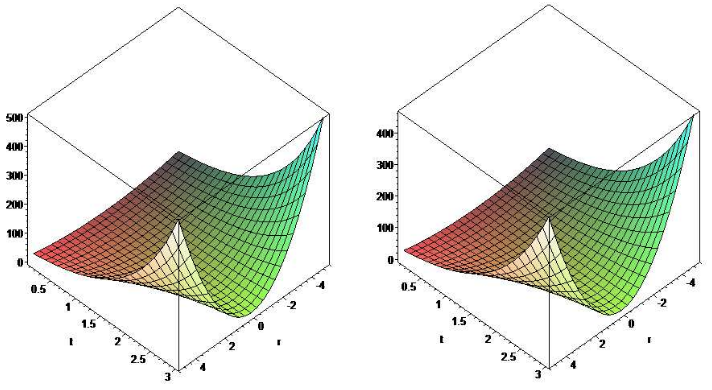
$${}^cD_\tau^\alpha U(r,\tau) = U_{rr}(r,\tau) - U(r,\tau), \quad 1<\alpha\le 2, \tag{4.8}$$

with the initial conditions

$$U(r,0)=0, \quad U_\tau(r,0)=r. \tag{4.9}$$

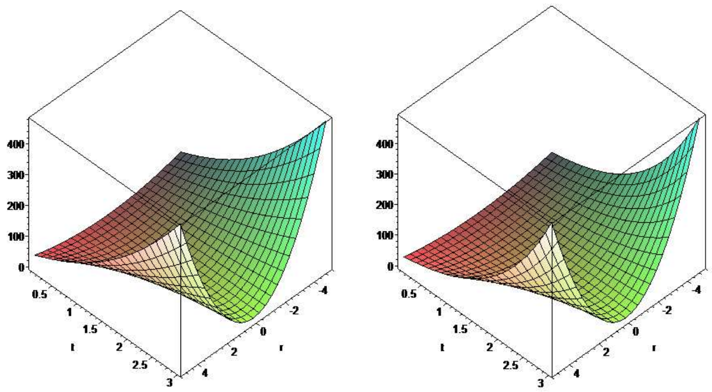
By applying the Aboodh transform on both sides of (4.8), we get

$$A\left[U(r,\tau)\right] = \frac{1}{v^3}r + \frac{1}{v^\alpha}A\left[U_{rr}(r,\tau) - U(r,\tau)\right]. \tag{4.10}$$



(a) (b)

Figure 4.1: (a) The exact solution, (b) The approximate solution when $\alpha = 1$ of (4.1).



(c) (d)

Figure 4.2: (c) and (d) The approximate solutions of (4.1) when $\alpha = 0.5$ and $\alpha = 0.9$ respectively.

Taking the inverse Aboodh transform of (4.10), we have

$$U(r,\tau) = r\tau + A^{-1}\left(\frac{1}{v^{\alpha}}A\left[U_{rr}(r,\tau) - U(r,\tau)\right]\right). \tag{4.11}$$

Applying $\frac{\partial}{\partial\tau}$ on both sides of (4.11), we get

$$\frac{\partial U(r,\tau)}{\partial\tau} = r + \frac{\partial}{\partial\tau}A^{-1}\left(\frac{1}{v^{\alpha}}A\left[U_{rr}(r,\tau) - U(r,\tau)\right]\right). \tag{4.12}$$

According to the variational iteration method, we can construct a correct functional as follows

$$U_{n+1}(r,\tau) = U_n(r,\tau) - \int_0^{\tau}\left[\frac{\partial U_n(r,\varsigma)}{\partial\varsigma} - r - \frac{\partial}{\partial\varsigma}A^{-1}\left(\frac{1}{v^{\alpha}}A\left[U_{nrr}(r,\tau) - U_n(r,\tau)\right]\right)\right]d\varsigma. \tag{4.13}$$

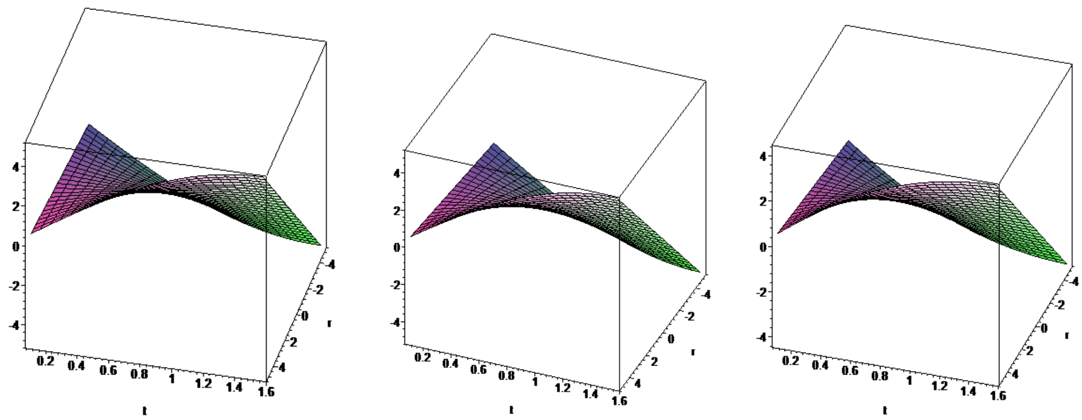
Consequently, the first terms are obtained by

$$\begin{gathered} U_0(r,\tau) = r\tau, \\ U_1(r,\tau) = r\tau - r\tfrac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}, \\ U_2(r,\tau) = r\tau - r\tfrac{\tau^{\alpha+1}}{\Gamma(\alpha+2)} + r\tfrac{\tau^{2\alpha+1}}{\Gamma(2\alpha+2)}, \\ U_3(r,\tau) = r\tau - r\tfrac{\tau^{\alpha+1}}{\Gamma(\alpha+2)} + r\tfrac{\tau^{2\alpha+1}}{\Gamma(2\alpha+2)} - r\tfrac{\tau^{3\alpha+1}}{\Gamma(3\alpha+2)}, \\ \vdots \\ U_n(r,\tau) = r\sum_{k=0}^{n}(-1)^k\tfrac{\tau^{k\alpha+1}}{\Gamma(k\alpha+2)} \end{gathered} \tag{4.14}$$

The approximate solution in a series form of (4.8)-(4.9) when $\alpha \longrightarrow 2$, is given by

$$U(r,\tau) = \lim_{n\to\infty} U_n(r,\tau) = r\sin\tau, \tag{4.15}$$

which is the exact solution of linear Klein-Gordon equation presented in [23].



(a') (b') (c')

Figure 4.3: (a') The exact solution and (b') The approximate solution in the case $\alpha = 2$, (c') The approximate solution when $\alpha = 1.5$ of (4.8)-(4.9).

Example 4.3. *We consider the nonlinear fractional Klein-Gordon equation of the form*

$$\begin{aligned} {}^{c}D_{\tau}^{\alpha}U(r,\tau) &= U_{rr}(r,\tau) - U^{2}(r,\tau) + r^{2}\tau^{2}, \ \ 1<\alpha\leq 2, \\ U(r,0) &= 0, \ \ U_{\tau}(r,0) = r. \end{aligned} \tag{4.16}$$

Applying Aboodh transform on both sides of (4.16), we have

$$A\left[U(r,\tau)\right] = \frac{1}{v^{3}}r + 2r^{2}\frac{1}{v^{\alpha+4}} + \frac{1}{v^{\alpha}}A\left[U_{rr}(r,\tau) - U^{2}(r,\tau)\right]. \tag{4.17}$$

By inverse Aboodh transform and derivative, we get

$$\frac{\partial U(r,\tau)}{\partial \tau} = r + 2(\alpha+2)r^{2}\frac{\tau^{\alpha+1}}{\Gamma(\alpha+3)} + \frac{\partial}{\partial\tau}A^{-1}\left(\frac{1}{v^{\alpha}}A\left[U_{rr}(r,\tau) - U^{2}(r,\tau)\right]\right). \tag{4.18}$$

Now, applying the variational iteration method, we obtain

$$U_{n+1}(r,\tau) = U_{n}(r,\tau) - \int_{0}^{\tau}\left[\frac{\partial U_{n}(r,\varsigma)}{\partial\varsigma} - r - 2(\alpha+2)r^{2}\frac{\tau^{\alpha+1}}{\Gamma(\alpha+3)} - \frac{\partial}{\partial\varsigma}A^{-1}\left(\frac{1}{v^{\alpha}}A\left[U_{nrr}(r,\tau) - U_{n}^{2}(r,\tau)\right]\right)\right]d\varsigma. \tag{4.19}$$

The first terms of approximate solution are obtained successively

$$\begin{aligned} U_{0}(r,\tau) &= r\tau + \tfrac{2r^{2}}{\Gamma(\alpha+3)}\tau^{\alpha+2}, \\ U_{1}(r,\tau) &= r\tau + \tfrac{4}{\Gamma(\alpha+3)\Gamma(2\alpha+3)}\tau^{2\alpha+2} \\ -\tfrac{4r^{3}\Gamma(\alpha+4)}{\Gamma(\alpha+3)\Gamma(2\alpha+4)}\tau^{2\alpha+3} &- \tfrac{4r^{4}\Gamma(2\alpha+5)}{\Gamma^{2}(\alpha+3)\Gamma(3\alpha+5)}\tau^{3\alpha+4}, \\ &\vdots \end{aligned} \tag{4.20}$$

and so on. Therefore the solution of (4.16) in series form when $\alpha = 2$, is given by

$$U(r,\tau) = \lim_{n\to\infty}U_{n}(r,\tau) = r\tau. \tag{4.21}$$

Example 4.4. *We consider the following nonlinear time-fractional partial differential equation*

$${}^{c}D_{\tau}^{\alpha}U - \frac{3}{8}\left[(U_{rr})^{2}\right]_{r} = \frac{3}{2}\tau, \quad 2<\alpha\leq 3, \tag{4.22}$$

with the initial conditions

$$U(r,0) = \frac{1}{2}r^{2}, \ \ U_{\tau}(r,0) = \frac{1}{3}r^{3}, \ \ U_{\tau\tau}(r,0) = 0. \tag{4.23}$$

According to the formula (3.7), we can construct the following iteration formula

$$U_{n+1}(r,\tau) = -\frac{1}{2}\tau^{2} + \frac{1}{3}r^{3}\tau + \frac{3}{2}\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)} - A^{-1}\left(\frac{1}{v^{\alpha}}A\left[-\frac{3}{8}\left[(U_{nrr})^{2}\right]_{r}\right]\right). \tag{4.24}$$

Using the iteration formula (4.24), we obtain

$$\begin{aligned} U_0(r,\tau) &= -\tfrac{1}{2}r^2+\tfrac{1}{3}r^3\tau, \\ U_1(r,\tau) &= -\tfrac{1}{2}r^2+\tfrac{1}{3}r^3\tau+6r\tfrac{\tau^{\alpha+2}}{\Gamma(\alpha+3)}, \\ U_2(r,\tau) &= -\tfrac{1}{2}r^2+\tfrac{1}{3}r^3\tau+6r\tfrac{\tau^{\alpha+2}}{\Gamma(\alpha+3)}, \\ U_3(r,\tau) &= -\tfrac{1}{2}r^2+\tfrac{1}{3}r^3\tau+6r\tfrac{\tau^{\alpha+2}}{\Gamma(\alpha+3)}, \\ &\vdots \end{aligned} \tag{4.25}$$

The approximate solution in a series form, is given by

$$U(r,\tau) = -\frac{1}{2}r^2+\frac{1}{3}r^3\tau+6r\frac{\tau^{\alpha+2}}{\Gamma(\alpha+3)}. \tag{4.26}$$

As $\alpha \longrightarrow 3$, *we get*

$$U(r,\tau) = -\frac{1}{2}r^2+\frac{1}{3}r^3\tau+\frac{1}{20}r\tau^5.$$

which is an exact solution of the nonlinear partial differential equation of order three (4.22)-(4.23) obtained by the modified homotopy analysis method in [6].

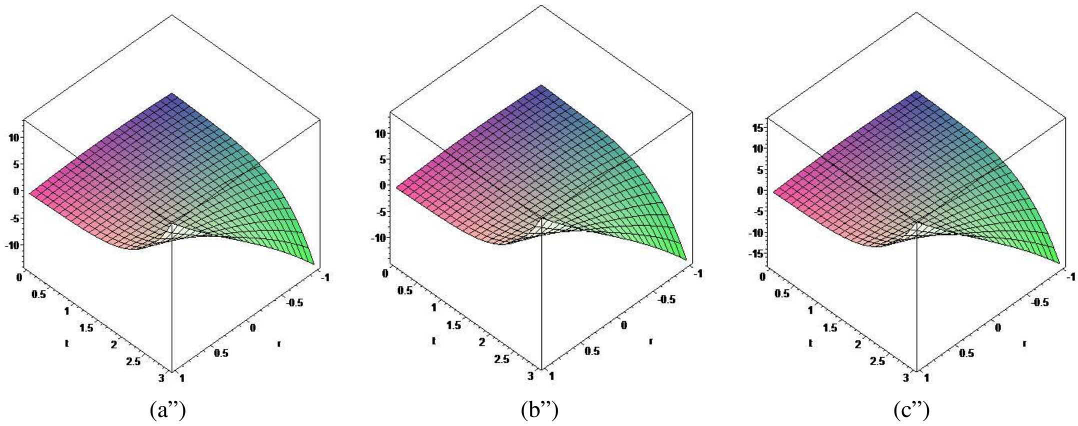


Figure 4.4: (a") The exact solution, (b") and (c") The approximate solutions in the case $\alpha = 2.9$ and $\alpha = 1.5$ respectively of (4.22)-(4.23).

5. Conclusion

In this work, a variational iteration method (VIM) and new transform method called "Aboodh transform" are successfully combined to form a powerful analytical method for solving fractional partial differential equations. The new analytical method gives a series solution which converges rapidly to the exact solution. The simplicity and high precision of the new analytical method are clearly illustrated, for example, by the resolution of some equations such as the time fractional diffusion equation, the linear and nonlinear fractional Klein-Gordon equation of order 2 and an example of nonlinear time fractional partial differential equation of order three.

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Scalar characterization in Banach-Jordan algebras

Abdelaziz Maouche[a*]

[a]*Department of Mathematics and Statistics Faculty of Science, Sultan Qaboos University, Oman*
[*]*Corresponding author E-mail: maouche@squ.edu.om*

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Abstract

Using a Diagonalization Theorem obtained when the spectrum is Lipschitzian, we extend a result of G. Braatvedt on scalar characterization in Banach algebras to Banach-Jordan algebras. We also establish that any element of a semisimple Banach-Jordan algebra with the property that all elements in some neighbourhood of the identity are spectrally invariant under multiplication by the quadratic U operator, has analogs with the identity.

1. Preliminaries

A unital Banach-Jordan algebra is a vector space with a binary product

$$(x,y) \mapsto x \cdot y$$

satisfying the identities:

$$x \cdot y = y \cdot x,\ (x \cdot y)^2 = x^2 \cdot (y \cdot x^2) \quad \forall x, y \in A,$$

and endowed with a complete norm $\| \cdot \|$ such that, for all $x, y \in A$,

$$\| x \cdot y \| \leq \| x \| \| y \| .$$

N. Jacobson introduced the notion of invertibility in Jordan algebras, which generalizes the notion of invertibility in associative algebras. Given x in A we say that x is invertible in A if there exists y in A such that $x \cdot y = 1$ and $x^2 \cdot y = x$. This element y is unique and is usually denoted by x^{-1}. It turns out that this notion of inverse is intimalely related to the quadratic map $\mathrm{U} : A \mapsto \mathscr{BL}(A)$ defined by

$$\mathrm{U}_x y = 2x \cdot (x \cdot y) - x^2 \cdot y$$

for any $x, y \in A$. Keeping in mind that the mapping $x \mapsto U_x$ from A into $\mathscr{BL}(A)$ is continuous, the invertible elements $\Omega = \{a \in A : \mathrm{U}_a \text{ is invertible}\}$ form an open subset of A; in particular, Ω is locally connected as shown by O. Loos in [7], so its connected components are open. Also, the space $\mathbb{C}[x]$ spanned by all powers of x is a commutative associative subalgebra with respect to the linear Jordan product. By continuity, the same holds for its closure $\mathscr{C}$ in A. We refer the reader to Chapter 4 of [6] for more details on spectral theory in Banach-Jordan algebras. For general theory of Jordan algebras see [8] and [10].

Theorem 1.1. *An element x of A is invertible if and only if U_x is invertible in $\mathscr{L}(A)$, the algebra of linear operators on A, in which case $\mathrm{U}_{x^{-1}} = \mathrm{U}_x^{-1}$. If $x, y \in A$, then they are both invertible if and only if $\mathrm{U}_x(y)$ is invertible in A. In particular, x is invertible if and only if x^n is invertible for every integer $n \geq 1$.*

This theorem implies that the set of invertible elements $\Omega(A)$ is invariant when taking powers, but unfortunately, it is not stable for the product. For $x \in A$ we denote respectively by $\mathrm{Sp}(x) = \{\lambda 1 - x \notin \Omega(A)\}$ and $\rho_A(x) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(x)\}$ the spectrum and spectral radius of x.
In what follows, an important tool will be the theory of subharmonic functions, based essentially on the celebrated result of Aupetit and Zraibi, which allow us to use analytic tools in Banach-Jordan algebras.

***Email addresses*: maouche@squ.edu.om (A. Maouche)**

Theorem 1.2 (Aupetit-Zraibi)**.** *Let $f : D \to A$ be a holomorphic function from a domain D of $\mathbb{C}$ into a Banach-Jordan algebra. Then the mapping $\lambda \to \mathrm{Sp}(f(\lambda))$ is an analytic multifunction. Consequently, $\lambda \longmapsto \rho(f(\lambda))$ and $\lambda \longmapsto \log\rho(f(\lambda))$ are subharmonic on D.*

We will require the following fundamental result from the theory of subharmonic functions [1].

Theorem 1.3 (Maximum Principle for Subharmonic Functions)**.** *Let f be a subharmonic function on a domain D of $\mathbb{C}$. If there exists $\lambda_0 \in D$ such that $f(\lambda) \leq f(\lambda_0)$ for all $\lambda \in D$, then $f(\lambda) = f(\lambda_0)$ for all λ in D.*

Another important ingredient is Aupetit's characterization of the McCrimmon radical $\mathrm{Rad}(A)$ of A (see [3]) and its corollaries.

Theorem 1.4 (Aupetit)**.** *Let a be an element of a Banach-Jordan algebra A. Then a is in the McCrimmon radical of A if and only if $\sup\{\rho(x+ta) : t \in \mathbb{C}\} < \infty$ for every x in A.*

Corollary 1.5. *An element a of a Banach-Jordan algebra A is in the McCrimmon radical of A if and only if $\sup\rho(U_x a) = 0$ for every x in A.*

Corollary 1.6. *An element a of a Banach- Jordan algebra A is in the McCrimmon radical of A if and only if there exists $C \geq 0$ such that $\rho(x) \leq C \parallel x-a \parallel$ for every x in a neighborhood of a.*

2. Some results under the condition of a Lipschitzian spectrum

The next lemma is a spectral characterization of the Jacobson radical in terms of the Lipshitzian behaviour of the spectrum. It was obtained by Aupetit in [3] for Banach algebras and we extend it here to Banach-Jordan algebras.

Lemma 2.1. *Let $q \in A$ be a quasi-nilpotent element. Suppose that there exists $r, C > 0$ such that $\rho(x) \leq C \parallel x-q \parallel$, for $\parallel x-q \parallel < r$, then $q \in \mathrm{Rad}(A)$.*

Proof. Let $y \in A$ be arbitrary. For $|\lambda| > \frac{\|y\|}{r}$, we have $\rho(q+\frac{y}{\lambda}) \leq C\frac{\|y\|}{|\lambda|}$, consequently $\rho(y+\lambda q) \leq C \parallel y \parallel$. Hence the upper semi-continuous function $\lambda \mapsto \rho(y+\lambda q)$ is bounded on the complex plane. Being subharmonic, it is constant by Liouville's Theorem for subharmonic functions. Thus $\rho(y+q) = \rho(y)$, for every $y \in A$ and by Aupetit's characterization of the radical [3], we obtain $q \in \mathrm{Rad}(A)$. □

We recall that the spectrum is said to be Lipschitzian at an element a of a Banach-Jordan algebra if there exists two positive constants r and C such that $\Delta(\mathrm{Sp}(x), \mathrm{Sp}(a)) \leq C \parallel x-a \parallel$ for all x satisfying $\parallel x-a \parallel < r$, where Δ represents the Hausdorff distance on compact sets of the complex plane defined by

$$\Delta(\sigma_1,\sigma_2) = \max\{\sup_{\lambda\in\sigma_2}\{\mathrm{dist}(\lambda,\sigma_1)\}, \sup_{\lambda\in\sigma_1}\{\mathrm{dist}(\lambda,\sigma_2)\}\}$$

where $\mathrm{dist}(\lambda,\sigma) = \inf\{|\lambda-\mu| : \mu \in \sigma\}$ is the distance of the point λ to the compact set σ (see [1]). Using the previous lemma, we obtained the following theorem in [9].

Theorem 2.2. *Let A be a semisimple complex Banach-Jordan algebra and let $a \in A$ have finite spectrum, $\mathrm{Sp}(a) = \{\alpha_1, \cdots, \alpha_n\}$. Suppose that the spectral mapping $x \mapsto \mathrm{Sp}_A(x)$ is Lipschitzian at a. Then there exist n nonzero orthogonal projections $p_1, \cdots, p_n$ whose sum is 1 and such that $a = \alpha_1 p_1 + \cdots + \alpha_n p_n$.*

The next theorem obtained in [5] for Banach algebras is in fact a particular case to our theorem quoted above. We extend here this theorem to Jordan algebras along with a new proof.

Theorem 2.3. *Let A be a semisimple complex Banach-Jordan algebra and let $a \in A$. If the spectrum is Lipschitzian at a and $\mathrm{Sp}(a) = \{\alpha\}$, then $a = \alpha 1$.*

Proof. Since the spectrum is Lipschitzian at a, it follows that there exists two positive constants r and C such that

$$\Delta(\mathrm{Sp}(x), \mathrm{Sp}(a)) \leq C \parallel x-a \parallel$$

for all x satisfying $\parallel x-a \parallel < r$. Clearly,

$$\Delta(\mathrm{Sp}(x), \mathrm{Sp}(a)) = \Delta(\mathrm{Sp}(x-\alpha), \mathrm{Sp}(a-\alpha)).$$

Since $\mathrm{Sp}(a-\alpha) = \{0\}$, taking $x - \alpha 1 = y$, we get from our assumption

$$\begin{aligned}\rho(y) &= \Delta(\mathrm{Sp}(y), \{0\}) \\ &= \Delta(\mathrm{Sp}(y), \mathrm{Sp}(a-\alpha 1)) \\ &\leq C \parallel x-a \parallel \\ &= C \parallel y-(a-\alpha 1) \parallel\end{aligned}$$

for all $\parallel y-(a-\alpha 1) \parallel < r$. It follows from Corollary 2 of Aupetit's characterization of the radical, that $a-\alpha 1 \in \mathrm{Rad}(A) = \{0\}$. Thus $a = \alpha 1$. □

3. Scalar characterization in a Banach-Jordan algebra

Another result obtained in [5] is the following multiplicative scalar characterization of elements in a Banach algebra.

Theorem 3.1. *Let A be a semisimple complex Banach algebra and let $a \in A$. Then $a = 1$ if and only if $\operatorname{Sp}(ax) = \operatorname{Sp}(x)$ for all x in a neighborhood of 1.*

We extend this result to Banach-Jordan algebras with a slightly different conclusion. It is clear that if $a = 1$ then $\operatorname{Sp}(\mathrm{U}_a x) = \operatorname{Sp}(x)$ for all x in A. But the converse is not exactly as for Banach algebras. Indeed, instead of using the linear Jordan product as an analogue of multiplication in Banach algebras, we consider multiplication by the quadratic U operator in the situation of Banach-Jordan algebras. Precisely, we prove that if an element a in a semisimple Banach-Jordan algebra has the property that multiplication by U_a leaves all elements in some neighbourhood of the identity spectrally invariant, then clearly that element squares to the identity.

Theorem 3.2. *Let A be a complex semisimple Banach-Jordan algebra and a nonzero element a of A. If $\operatorname{Sp}(\mathrm{U}_a x) = \operatorname{Sp}(x)$ for all x in a neighborhood of 1 then $a^2 = 1$. In particular, a is invertible and $a^{-1} = a$.*

Proof. Note that if $a = 1$ then $\operatorname{Sp}(\mathrm{U}_a x) = \operatorname{Sp}(x)$. We are interested by the converse. Suppose that $\operatorname{Sp}(\mathrm{U}_a x) = \operatorname{Sp}(x)$ for all x in a neighborhood $\mathscr{V}(1)$ of 1. Let $x = 1$, then $\operatorname{Sp}(\mathrm{U}_a 1) = \operatorname{Sp}(a^2) = \operatorname{Sp}(1) = \{1\}$. So $\operatorname{Sp}(a) \subseteq \{-1, 1\}$, hence a is invertible. Let $y \in A$ arbitrary. Take λ sufficiently small, say $\lambda \in B(0, \varepsilon)$, such that

$$\operatorname{Sp}(\lambda y + a^2) = \operatorname{Sp}(\mathrm{U}_a(\lambda \mathrm{U}_{a^{-1}} y + 1)) = \operatorname{Sp}(\lambda \mathrm{U}_{a^{-1}} y + 1) = \operatorname{Sp}(\lambda \mathrm{U}_{a^{-1}} y) + 1.$$

So,

$$\operatorname{Sp}(\lambda y + a^2 - 1) = \operatorname{Sp}(\lambda \mathrm{U}_{a^{-1}} y)$$

and

$$\rho(y + \frac{1}{\lambda}(a^2 - 1)) = \rho(\mathrm{U}_{a^{-1}} y)$$

for all $0 \neq \lambda \in B(0, \varepsilon)$. Furthermore,

$$\begin{aligned} \rho(y + \frac{1}{\lambda}(a^2 - 1)) &\leq \| y + \frac{1}{\lambda}(a^2 - 1)) \| \\ &\leq \| y \| + |\frac{1}{\lambda}| \, \| a^2 - 1 \| \\ &\leq \| y \| + \frac{1}{\varepsilon} \, \| a^2 - 1 \| \end{aligned}$$

for all $\lambda \in \mathbb{C} \setminus B(0, \varepsilon)$.
Hence, there exists $M > 0$ such that

$$\rho(y + \frac{1}{\lambda}(a^2 - 1)) \leq M$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$. Furthermore,

$$\limsup_{\lambda \to 0} \rho\left(y + \frac{1}{\lambda}(a^2 - 1)\right) \leq M.$$

Hence, taking $\mu = \frac{1}{\lambda}$ it follows that the subharmonic function

$$\phi : \mu \mapsto \rho\left(y + \mu(a^2 - 1)\right)$$

is bounded on $\mathbb{C}$, and

$$\limsup_{\mu \to \infty} \phi(\mu) \leq M.$$

By Liouville's theorem for subharmonic functions, ϕ is constant. Hence,

$$\rho\left(y + \mu(a^2 - 1)\right) = \rho(y)$$

for all $\mu \in \mathbb{C}$. By Aupetit's characterization of the radical, we obtain

$$a^2 - 1 \in \operatorname{Rad}(A) = \{0\},$$

that is $a^2 = 1$.

□

Our last result concerns bounded elements in a finite dimensional Banach-Jordan algebra. Exactly as for Banach algebras, we get the following theorem which extends another result of G. Braatvedt from associative Banach algebras to Non-associative Banach algebras. The proof follows the same arguments as the associative one. Recall that an element a of a Banach-Jordan algebra is said to be power bounded if there exists a positive constant M such that $\| a^n \| \leq M$ for all $n \in \mathbb{N}$ (more details on powers of elements in Banach-Jordan algebras can be found in [5]).

Theorem 3.3. *Let A be a finite-dimensional Banach-Jordan algebra and $a \in A$. If $\mathrm{Sp}(a) = \{1\}$ and a is power bounded, then $a = 1$.*

Proof. Note that since $\mathrm{Sp}(a) = \{1\}$, then $\mathrm{Sp}(a-1) = \{0\}$. Hence, $(a-1)$ is quasi-nilpotent and therefore nilpotent since A is finite-dimensional. Thus $(a-1)^N = 0$ for some $N \in \mathbb{N}$. Hence for all $n \geq N$, we get

$$a^n = ((a-1)+1)^n = \sum_{k=0}^{n} \binom{k}{n}(a-1)^k = \sum_{k=0}^{N-1} \binom{k}{n})(a-1)^k.$$

Since a is power bounded, for some $M > 0$ and all $n \in \mathbb{N}$ we have $\| a^n \| \leq M$. Hence for all $n \in \mathbb{N}$,

$$\| \sum_{k=0}^{N-1} \binom{k}{n}(a-1)^k \| \leq M \Longrightarrow \| \sum_{k=0}^{N-2} \binom{k}{n}(a-1)^k + \binom{n}{N-1}(a-1)^{N-1} \| \leq M$$

Dividing both sides by $\binom{n}{N-1}$ gives

$$\| \sum_{k=0}^{N-2} \frac{(N-1)!}{(n-k)(n-(k+1))\cdots(n-(N-2))k!}(a-1)^k + (a-1)^{N-1} \| \leq \frac{M}{\binom{n}{N-1}}$$

(because $k \leq N-2 < N-1$). Now considering the limit as $n \to \infty$ gives $0 \leq \| (a-1)^{N-1} \| \leq 0$, and so $(a-1)^{N-1} = 0$. It follows by induction that $(a-1) = 0$ that is $a = 1$.

□

Remark 3.4. *The previous theorem is valid for Banach-Jordan algebras because the proof takes place in a subalgebra generated by* 1 *and* a, *that is a full subalgebra of a Banach-Jordan algebra. In that case everything works as in classical Banach algebras as described and well explained in Chapter 4 of [6].*

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Solvability for a nonlinear third-order three-point boundary value problem

Habib Djourdem[a*] and Slimane Benaicha[a]

[a]*Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran1, Ahmed Benbella, Algeria*
[*]*Corresponding author E-mail: djourdem.habib7@gmail.com*

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Abstract

In this article, the existence of positive solutions for a nonlinear third-order three-point boundary value problem with integral condition is investigated. By using Leray-Schauder fixed point theorem, sufficient conditions for the existence of at least one positive solution are obtained. Illustrative examples are also presented to show the applicability of our results.

1. Introduction

This paper is devoted to the existence of positive solutions for the following third-order nonlocal integral boundary value problem (BVP):

$$u'''(t)+a(t)f(t,u(t))=0, \quad 0<t<T, \tag{1.1}$$

$$u(0)=u''(0)=0,\ u(T)=\alpha\int_0^\eta u(s)\,ds, \tag{1.2}$$

where $0<\eta<T$, $0<\alpha<\frac{2T}{\eta^2}$ and

(H_1) $f([0,T]\times[0,+\infty),[0,+\infty))$;

(H_2) $a\in C([0,T],[0,+\infty))$ and there exists $t_0\in[\eta,T]$ such that $a(t_0)>0$.

Set

$$f_0=\lim_{u\to 0^+}\frac{f(t,u)}{u}, \qquad f_\infty=\lim_{u\to\infty}\frac{f(t,u)}{u},$$

then $f_0=0$ and $f_\infty=\infty$ correspond to the superlinear case, $f_0=\infty$ and $f_\infty=0$ correspond to the sublinear case.
Third-order boundary-value problems for differential equations arise in variety of different areas of applied mathematics and physics. They have been many scholars' research object. For example, heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics can produce boundary-value problems with integral boundary conditions; see [3, 9, 11]. They include two, three, multipoint, and nonlocal boundary-value problems as special cases. By using the Krasnoselskii's fixed point theorem, Liu and Ma [19] studied the problem

$$u'''(t)+f(u(t))=0, \quad 0<t<1, \tag{1.3}$$

Email addresses: **djourdem.habib7@gmail.com (H. Djourdem), slimanebenaicha@yahoo.fr (S. Benaicha)**

subject to integral boundary condition of the form

$$u'(0)=0,\, u'(1)=0,\, u(0)=\int_0^1 k(s)\,u(s)\,ds. \tag{1.4}$$

Benaicha and Haddouchi [17] considered the fourth-order two-point boundary value problem

$$u''''(t)+f(u(t))=0, \quad t\in(0,1), \tag{1.5}$$

$$u'(0)=u'(1)=u''(0)=0,\, u(0)=\int_0^1 a(s)\,u(s)\,ds. \tag{1.6}$$

We quote also the reasearchs of [2, 4, 5, 6, 7, 8, 12, 13, 14, 15, 16, 18, 20] which concern the differential equations under various boundary conditions and by different approaches.
Motivated by the works mentioned above, we obtain the existence results for the problem (1.1)-(1.2) by using the Leray-Shauder fixed point theorem if $f_0=0$ (condition $f_\infty=\infty$ being unnecessary) , as well as, for $f_\infty=0$ (condition $f_0=\infty$ being unnecessary). In this way we remove the half of the assumptions to prove the existence of a solution when using Krasnoselskii's fixed point theorem.(See [10, 17, 19]).
Moreover, we establish our results for t in $[0,T]$.
Our main tool is the following Leray-Schauder fixed point theorem.

Theorem 1.1. *[1] Let Ω be the convex subset of Banach space E, $0\in\Omega$, $\Phi:\Omega\to\Omega$ be completely continuous operator. Then, either*
(i) Φ has at least one fixed point in Ω;
or
(ii) the set $\{x\in\Omega \mid u=\lambda\Phi u,\, 0<\lambda<1\}$ is unbounded.

2. Background

To prove the main existence results we will employ several straightforward lemmas.

Lemma 2.1. *Let $2T\neq\alpha\eta^2$. Then for $y\in C([0,T],[0,\infty))$, the problem*

$$u'''(t)+y(t)=0, \tag{2.1}$$

$$u(0)=u''(0)=0,\; u(T)=\alpha\int_0^\eta u(s)\,ds, \quad \eta\in(0,T), \quad \alpha>0, \tag{2.2}$$

has a unique solution given by

$$\begin{aligned} u(t) &= \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 y(s)\,ds-\frac{\alpha t}{3\left(2T-\alpha\eta^2\right)}\int_0^\eta (\eta-s)^3 y(s)\,ds \\ &\quad -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds. \end{aligned}$$

Proof. From equation (2.1) we have $u'''(t)=-y(t)$. Then, integrating from 0 to t we obtain

$$u''(t)=-\int_0^t y(s)\,ds.$$

For $t\in[0,T]$ we have, by integrating in t and using integration by parts,

$$\begin{aligned} u'(t) &= u'(0)-\int_0^t\left(\int_0^x y(s)\,ds\right)dx \\ &= u'(0)-\int_0^t (t-s)\,y(s)\,ds \end{aligned}$$

$$\begin{aligned} u(t) &= u'(0)\,t-\int_0^t\left(\int_0^x (x-s)\,y(s)\,ds\right)dx \\ &= u'(0)\,t-\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds. \end{aligned} \tag{2.3}$$

Thus, for $t=T$ we find

$$u(T)=u'(0)\,T-\frac{1}{2}\int_0^T (T-s)^2 y(s)\,ds. \tag{2.4}$$

Integrating again from 0 to η the expression (2.3), where $\eta\in(0,T)$, we obtain

$$\begin{aligned} \int_0^\eta u(s)\,ds &= \frac{1}{2}u'(0)\,\eta^2-\frac{1}{2}\int_0^\eta\left(\int_0^x (x-s)^2 y(s)\,ds\right)dx \\ &= \frac{1}{2}u'(0)\,\eta^2-\frac{1}{6}\int_0^\eta (\eta-s)^3 y(s)\,ds. \end{aligned} \tag{2.5}$$

From (2.2) and (2.4) we have

$$\int_0^{\eta} u(s)\,ds = \frac{1}{\alpha}u(T) = u'(0)\frac{T}{\alpha} - \frac{1}{2\alpha}\int_0^T (T-s)^2 y(s)\,ds.$$

Then, using (2.5) we see that

$$u'(0)\frac{T}{\alpha} - \frac{1}{2\alpha}\int_0^T (T-s)^2 y(s)\,ds = \frac{1}{2}u'(0)\eta^2 - \frac{1}{6}\int_0^{\eta} (\eta-s)^3 y(s)\,ds.$$

Thus,

$$u'(0)\left(\frac{2T-\alpha\eta^2}{2\alpha}\right) = \frac{1}{2\alpha}\int_0^T (T-s)^2 y(s)\,ds - \frac{1}{6}\int_0^{\eta} (\eta-s)^3 y(s)\,ds$$

or

$$u'(0) = \frac{1}{(2T-\alpha\eta^2)}\int_0^t (T-s)^2 y(s)\,ds - \frac{\alpha}{3(2T-\alpha\eta^2)}\int_0^{\eta} (\eta-s)^3 y(s)\,ds.$$

Therefore, the BVP (2.1)–(2.2) has a unique solution

$$\begin{aligned} u(t) = {} & \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 y(s)\,ds - \frac{\alpha t}{3(2T-\alpha\eta^2)}\int_0^{\eta} (\eta-s)^3 y(s)\,ds \\ & -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds. \end{aligned}$$

□

The existence of positive solutions of the problem (2.1)–(2.2) is given in the next result.

Lemma 2.2. . *Let $0<\alpha<\frac{2T}{\eta^2}$. If $y \in C([0,T],[0,+\infty))$, then the unique solution of the problem (2.1)–(2.2) satisfies $u(t) \geq 0$ for $t \in [0,T]$.*

Proof. From $u'''(t) = -y(t)$, $t \in [0,T]$, we get that $u''(t)$ is decreasing on $[0,T]$. Then, the condition $u''(0) = 0$ ensures that have $u''(t) \leq 0$, $t \in [0,T]$, which implies $u(t)$ is concave. Observe also that if $u(T) \geq 0$, the concavity of u and the fact that $u(0) = 0$ imply that $u(t) \geq 0$ for $t \in [0,T]$.
Since the graph of u is concave down $(0,T)$, we get

$$\int_0^{\eta} u(s)\,ds \geq \frac{1}{2}\eta u(\eta) \tag{2.6}$$

where $\frac{1}{2}\eta u(\eta)$ is the area of triangle under the curve $u(t)$ from $t=0$ to $t=\eta$ for $\eta \in (0,T)$.
If we assume that $u(T) < 0$, then from 2.2 we have

$$\int_0^{\eta} u(s)\,ds < 0. \tag{2.7}$$

By concavity of u and $\int_0^{\eta} u(s)\,ds < 0$, it implies that $u(\eta) < 0$.
Hence

$$u(T) = \alpha\int_0^{\eta} u(s)\,ds \geq \frac{2T}{\eta^2} \times \frac{1}{2}\eta u(\eta) = \frac{T}{\eta}u(\eta),$$

which contradicts the concavity of u. □

Lemma 2.3. *Let $\alpha > \frac{2T}{\eta^2}$. If $y \in C([0,T],[0,+\infty))$, then the problem (2.1)-(2.2) has no positive solution.*

Proof. Suppose that the problem (2.1)-(2.2) has a positive solution u.
If $u(T) > 0$, then $\int_0^{\eta} u(s)\,ds > 0$. It implies that $u(\eta) > 0$ and

$$\frac{u(T)}{T} = \frac{\alpha}{T}\int_0^{\eta} u(s)\,ds > \frac{2}{\eta^2}\left(\frac{1}{2}\eta u(\eta)\right) = \frac{u(\eta)}{\eta}$$

This contradicts the concavity of u.
If $u(T) = 0$, then $\int_0^{\eta} u(s)\,ds = 0$, this is $u(t) \equiv 0$ for all $t \in [0,\eta]$. If there exists $t_0 \in (\eta,T)$ such that $u(t_0) > 0$, then $u(0) = u(\eta) < u(t_0)$, which contradicts the concavity of u. Therefore, no positive solutions exist. □

Lemma 2.4. . *Let $0<\alpha<\frac{2T}{\eta^2}$. If $y \in C([0,T],[0,+\infty))$, then the unique solution of the problem (2.1)–(2.2) satisfies*

$$\min_{t\in[\eta,T]} u(t) \geq \gamma\|u\|, \ \|u\| = \max_{t\in[0,T]} |u(t)|, \tag{2.8}$$

where

$$\gamma := min\left\{\frac{\eta}{T}, \frac{\alpha\eta^2}{2T}, \frac{\alpha\eta(T-\eta)}{2T-\alpha\eta^2}\right\}. \tag{2.9}$$

Proof. Set $u(\tau) = \|u\|$. We consider three cases.
Case 1. If $\eta \leq \tau \leq T$ and $\min_{t\in[\eta,T]} u(t) = u(\eta)$, then the concavity of u implies that

$$\frac{u(\eta)}{\eta} \geq \frac{u(\tau)}{\tau} \geq \frac{u(\tau)}{T}$$

Thus,

$$\min_{t\in[\eta,T]} u(t) \geq \frac{\eta}{T}\|u\|.$$

Case 2. If $\eta \leq \tau \leq T$ and $\min_{t\in[\eta,T]} u(t) = u(T)$, then (2.2)-(2.6) and the concavity of u implies

$$u(T) = \alpha \int_0^\eta u(s)\,ds \geq \alpha \frac{\eta^2}{2}\left[\frac{u(\eta)}{\eta}\right] \geq \alpha \frac{\eta^2}{2}\left[\frac{u(\tau)}{\tau}\right]) \geq \frac{\alpha\eta^2}{2T}u(\tau).$$

Therefore,

$$\min_{t\in[\eta,T]} u(t) \geq \frac{\alpha\eta^2}{2T}\|u\|.$$

Case 3. If $\tau \leq \eta \leq T$, then $\min_{t\in[\eta,T]} u(t) = u(T)$. Using the concavity of u and (2.2)-(2.6), we have

$$\begin{aligned}
\frac{u(\tau)-u(T)}{\tau - T} &\geq \frac{u(T)-u(\eta)}{T-\eta} \\
u(\tau) &\leq u(T) + \frac{u(T)-u(\eta)}{T-\eta}(\tau - T) \\
u(\tau) &\leq u(T) + \frac{u(T)-u(\eta)}{T-\eta}(0 - T) \\
&\leq u(T)\left[1 - T\frac{1-\frac{2}{\alpha\eta}}{T-\eta}\right] \\
&= u(T)\left[\frac{2T-\alpha\eta^2}{\alpha\eta(T-\eta)}\right].
\end{aligned} \tag{2.10}$$

This implies that

$$\min_{t\in[\eta,T]} u(t) \geq \frac{\alpha\eta(T-\eta)}{2T-\alpha\eta^2}\|u\|.$$

This completes the proof. □

3. Main results

In this section, we establish the existence of positive solution for the (BVP) (1.1)-(1.2).

Let

$$E = C[0,T],\ \beta = \int_0^T (T-s)^2 a(s)\,ds$$

Theorem 3.1. *Assume (H1) and (H2) hold and $0 < \alpha < \frac{2T}{\eta^2}$. If $f_0 = 0$, then the problem (1.1)-(1.2) has at least one positive solution.*

Proof. From Lemma 2.1, u is a solution to the boundary value problem (1.1)-(1.2) if and only if u is a fixed point of operator A, where A is defined by

$$\begin{aligned}
Au(t) = &\frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(s,u(s))\,ds \\
&- \frac{\alpha t}{3(2T-\alpha\eta^2)}\int_0^\eta (\eta-s)^3 a(s) f(s,u(s))\,ds - \frac{1}{2}\int_0^t (t-s)^2 a(s) f(s,u(s))\,ds.
\end{aligned} \tag{3.1}$$

Denote that

$$\Omega = \left\{u \mid u \in C([0,T],\mathbb{R}),\, u \geq 0,\ \min_{t\in[\eta,T]} u(t) \geq \gamma\|u\|\right\},$$

where γ is defined in (2.9). Then Ω is the convex subset of E.
We choose $\varepsilon > 0$ and $\varepsilon \leq \frac{2T-\alpha\eta^2}{T\beta}$. By $f_0 = 0$, it there exists constant $M > 0$, such that $f(u) < \varepsilon u$ for $0 < u < M$. For $u \in \Omega$, from Lemma 2.2 and Lemma 2.4, we have $Au(t) \geq 0$ and $\min_{t\in[\eta,T]} Au(t) \geq \gamma\|Au\|$.

On the other hand,

$$
\begin{aligned}
Au(t) &\leq \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(u(s))\,ds \\
&\leq \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\varepsilon u(s)\,ds \\
&\leq \|u\|\frac{T\varepsilon}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\,ds \\
&\leq \|u\| \leq M.
\end{aligned}
$$

Thus $\|Au\| \leq \|u\|$. $u \in K \cap \partial\Omega_1$. Hence $A\Omega \subset \Omega$. It easy to check that $A:\Omega\to\Omega$ is completely continuous. For $u\in\Omega$ and $0<\lambda<1$, we have $u(t)=\lambda Au(t)<Au(t)\leq M$, which implies $\|u\|\leq M$. So $\{u\in\Omega \mid u=\lambda Au,\ 0<\lambda<1\}$ is bounded. By Theorem 1.1 the operator A has at least one fixed point in Ω. Thus the problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

Theorem 3.2. *Assume (H1) and (H2) hold, and $0<\alpha<\frac{2T}{\eta^2}$. If $f_\infty=0$, then the problem (1.1)-(1.2) has at least one positive solution.*

Proof. Choose $\varepsilon<\frac{2T-\alpha\eta^2}{2T\beta}$. By $f_\infty=0$, we know there exists Constant N, such that $f(u)<\varepsilon u$ for $u>N$.
Select

$$M \geq N+1+\frac{2T\beta}{2T-\alpha\eta^2}\max_{0\leq u\leq N} f(u)$$

Let

$$\Omega=\left\{u \mid u\in C[0,T],\ u\geq 0,\ \|u\|\leq M,\ \min_{t\in[\eta,T]} u(t)\geq\gamma\|u\|\right\},$$

then Ω is the convex subset of E. For $u\in\Omega$, by Lemma 2.2 and Lemma 2.4 we know $Au(t)\geq 0$ and $\min_{t\in[\eta,T]} Au(t() \geq \gamma\|Au\|)$.
On the other hand,

$$
\begin{aligned}
Au(t) &\leq \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(u(s))\,ds \\
&\leq \frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\varepsilon u(s)\,ds \\
&= \frac{T}{2T-\alpha\eta^2}\int_{I_1=\{s\in[0,T],\,u(s)>N\}} (T-s)^2 a(s) f(u(s))\,ds+\frac{T}{2T-\alpha\eta^2}\int_{I_2=\{s\in[0,T],\,u(s)\leq N\}} (T-s)^2 a(s) f(u(s))\,ds \\
&\leq \frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\varepsilon u(s)\,ds+\frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\,ds.\max_{0\leq u\leq N} f(u) \\
&\leq \frac{T\varepsilon}{2T-\alpha\eta^2}\|u\|\int_0^T (T-s)^2 a(s)\,ds+\frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\,ds.\max_{0\leq u\leq N} f(u) \\
&\leq \frac{T\varepsilon}{2T-\alpha\eta^2}M\int (T-s)^2 a(s)\,ds+\frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\,ds.\max_{0\leq u\leq N} f(u) \\
&\leq \frac{T\varepsilon}{2T-\alpha\eta^2}M\beta+\frac{T}{2T-\alpha\eta^2}\beta\max_{0\leq u\leq N} f(u) \\
&\leq \frac{1}{2}M+\frac{1}{2}M=M.
\end{aligned}
$$

Thus $\|Au\|\leq M$. Hence, $A\Omega\subset\Omega$. IT easy to check that $A:\Omega\to\Omega$ is completely continuous.
For $u\in\Omega$ and $u=\lambda Au$, $0<\lambda<1$, we have $u(t)=\lambda Au(t)<Au(t)\leq M$, which implies $\|u\|\leq M$. So, $\{u\in\Omega: u=\lambda Au,\ 0<\lambda<1\}$ is bounded. By Theorem 1.1, we know the operator A has at least one fixed point in Ω. Thus the problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

Theorem 3.3. *Assume (H1) and (H2) hold, and $0<\alpha<\frac{2T}{\eta^2}$. If there exists constant $\rho_1>0$, such that $f(u)\leq\frac{(2T-\alpha\eta^2)\rho_1}{T\beta}T\beta$ for $0<u<\rho_1$, then the problem (1.1)-(1.2) has at least one positive solution.*

Proof. Let $\Omega=\left\{u \mid u\in C[0,1],\ u\geq 0,\ \|u\|\leq\rho_1,\ \min_{t\in[\eta,T]} u(t)\geq\gamma\|u\|\right\}$, then Ω is the convex subset of E.
For $u\in\Omega$, by Lemma 2.2 and Lemma 2.4, we have

$$Au(t)\geq 0 \text{ and } \min_{t\in[\eta,T]} Au(t)\geq\gamma\|Au\|. \tag{3.2}$$

On the other hand

$$
\begin{aligned}
Au(t) &\leq \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(u(s)) \\
&\leq \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2\frac{(2T-\alpha\eta^2)\rho_1}{T\beta}\,ds=\rho_1.
\end{aligned}
$$

Then $\|Au\|\leq\rho_1$. Hence, $A\Omega\subset\Omega$. It easy to check yhat $A:\Omega\to\Omega$ is completely continuous.
For $u\in\Omega$ and $u=\lambda Au$, $0<\lambda<1$, we have $u(t)=\lambda Au(t)<Au(t)\leq\rho_1$, which implies $\|u\|\leq d$. So $\{u\in\Omega: u=\lambda Au,\ 0<\lambda<1\}$ is bounded. By Theorem 1.1, we know the operator A has at least one fixed point in Ω. Thus the problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

Theorem 3.4. *Assume (H1) and (H2) hold, and $0<\alpha<\frac{2T}{\eta^2}$. If there exists constant $\rho_2>0$, such that $f(u)\leq\frac{(2T-\alpha\eta^2)\rho_2}{T\beta}$ for $0<u<\rho_1$, then the problem (1.1)-(1.2) has at least one positive solution.*

Proof. Choose

$$d>1+\rho_2+\frac{T\beta}{2T-\alpha\eta^2}\cdot\max_{0\leq u\leq\rho_2}f(u).$$

Let

$$\Omega=\left\{u\mid u\in C[0,T],\,u\geq 0,\,\|u\|\leq d,\,\min_{t\in[\eta,T]}u(t)\geq\gamma\|u\|\right\},$$

then Ω is the convex subset of E.
For $u\in\Omega$, by Lemma 2.2 and Lemma 2.4, we know $Au(t)\geq 0$ and $\min_{t\in[\eta,T]}Au(t)\geq\gamma\|Au\|$.
On the other hand,

$$\begin{aligned}
Au(t)&\leq\frac{t}{2T-\alpha\eta^2}\int_0^T(T-s)^2a(s)f(u(s))ds\\
&\leq\frac{T}{2T-\alpha\eta^2}\int_0^T(T-s)^2a(s)f(u(s))ds\\
&=\frac{T}{2T-\alpha\eta^2}\int_{I_1=\{s\in[0,T],\,u(s)>\rho_2\}}(T-s)^2a(s)f(u(s))ds+\frac{T}{2T-\alpha\eta^2}\int_{I_2=\{s\in[0,T],\,u(s)\leq\rho_2\}}(T-s)^2a(s)f(u(s))ds\\
&\leq\frac{T}{2T-\alpha\eta^2}\int_0^T(T-s)^2a(s)\frac{(2T-\alpha\eta^2)\rho_2}{T\beta}ds+\frac{T}{2T-\alpha\eta^2}\int_0^T(T-s)^2a(s)\cdot\max_{0\leq u\leq\rho_2}f(u)ds\\
&\leq\rho_2+\frac{T\beta}{2T-\alpha\eta^2}\cdot\max_{0\leq u\leq\rho_2}f(u)<d.
\end{aligned}$$

Thus $\|Au\|\leq d$. Hence $A\Omega\subset\Omega$. It easy to check that the operator A is completely continuous. For $u\in\Omega$ and $u=\lambda Au$, $0<\lambda<1$, we have $u(t)=\lambda Au(t)<Au(t)\leq d$, which implies $\|u\|\leq d$. So $\{u\in\Omega:\,u=\lambda Au,\,0<\lambda<1\}$ is bounded. By Theorem 1.1, we know the operator A has at least one fixed point in Ω. Thus the problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

4. Examples

Example 4.1. *Consider the boundary value problem*

$$u'''(t)+\frac{t^2u}{t+e^u}=0,\quad 0<t<\frac{5}{4},\tag{4.1}$$

$$u(0)=0,\,u''(0)=0,\,u\left(\frac{5}{4}\right)=35\int_0^{\frac{1}{4}}u(s)ds,\tag{4.2}$$

where $\alpha=35$, $\eta=\frac{1}{4}$, $T=\frac{5}{4}$, $0<\alpha=35<40=\frac{2T}{\eta^2}$, $f(t,u)=\frac{u}{t+e^u}\in C([0,T]\times[0,\infty),[0,\infty))$ and $a(t)=t^2>0$ for $t\in\left[\frac{1}{4},\frac{5}{4}\right]$. Since $f_\infty=0$ and from Theorem 3.2, we can get that the (4.1)- (4.2) has at least one positive solution. Consequently, we cannot apply the Krasnoselskii's fixed point theorem like in [10, 17, 19]

Example 4.2. *Consider the boundary value problem*

$$u'''(t)+e^t\left(u-\frac{u}{\sqrt{1+u}}\right)=0,\quad 0<t<\frac{3}{4},\tag{4.3}$$

$$u(0)=0,\,u''(0)=0,\,u\left(\frac{3}{4}\right)=15\int_0^{0,2}u(s)ds,\tag{4.4}$$

where $\alpha=15$, $\eta=0,2=\frac{1}{5}$, $T=\frac{3}{4}$, $0<\alpha=15<37,5=\frac{2T}{\eta^2}$, $f(t,u)=u-\frac{u}{\sqrt{1+u}}\in C([0,T]\times[0,\infty),[0,\infty))$ and $a(t)=e^t>0$ for $t\in\left[\frac{1}{5},\frac{3}{4}\right]$. Obviously $f_0=0$. From Theorem 3.1, the (4.3)-(4.4) has at least one positive solution. On the other hand, we have $f_0=1$, then the function f is not superlinear. Consequently, we cannot apply the Krasnoselskii's fixed point theorem like in [10, 17, 19]

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An approach to neutrosophic ideals

Vildan Çetkin[a*] and Halis Aygün[a]

[a]*Department of Mathematics, Kocaeli University, Umuttepe Campus, 41380, Kocaeli, TURKEY*
[*]*Corresponding author E-mail: vildan.cetkin@kocaeli.edu.tr*

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Abstract

In this paper, we aim to introduce an approach to single-valued neutorosophic ideals over a given classical ring and over a given neutrosophic subring, respectively, as a continuation of our researches on algebraic structures over single-valued neutrosophic sets. We first propose the two types of neutrosophic ideals and then present their elementary properties.

1. Introduction

In many practical situations and in many complex systems like biological, behavioral and chemical etc., different types of uncertainties are encountered. Since the classical set is invalid to handle the described uncertainties, Zadeh [17] first gave the definition of a fuzzy set. According to this definition, a fuzzy set is a function described by a membership value takes degrees in the real unit interval. But, later it has been seen that this definition is inadequate by consideration not only the degree of membership but also the degree of nonmembership. So, Atanassov [2] described a set which is called an intuitionistic fuzzy set to handle mentioned ambiguity. Since this set have some problems in applications,Smarandache [15] introduced neutrosophy to deal with the problems involves indeterminate and inconsistent information. ”It is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra”[15]. Neutrosophic set is a generalization of the fuzzy set and intuitionistic fuzzy set, where the truth-membership, indeterminacy-membership, and falsity-membership are represented independently. Wang et al.[16] specified the definition of a neutrosophic set, named as a single valued neutrosophic set to make more applicable the theory to real life problems. The single valued neutrosophic set is a generalization of a classical set, fuzzy set, intuitionistic fuzzy set and paraconsistent set etc. Vasantha Kandasamy and Florentin Smarandache [9] studied the concept of neutrosophic algebraic structures.
In addition, single valued neutrosophic set is applied to algebraic and topological directions (see [1, 3, 4, 11, 13, 14]). Liu [10] defined the concept of a fuzzy ring and fuzzy ideal. Later, Martinez [12] and Dixit et al.[6] studied on fuzzy ring and obtain certain ring theoretical analogous. Hur et al.[7] proposed the notion of an intuitionistic fuzzy subring. Vasantha Kandasamy and Florentin Smarandache [8] studied the neutrosophic rings. In this work, in a different direction from [8], we give an approach to a single valued neutrosophic ideal of a classical ring as a continuation of neutrosophic algebraic structures discussed in [4, 5]. We define neutrosophic ideal and study some properties of this structure. Moreover, we examine homomorphic image and preimage of a neutrosophic ideal. By this way, we obtain the generalized form of the fuzzy ideal and intuitionistic fuzzy ideal of a classical ring.

2. Preliminaries

In this chapter, we recall the concepts of a neutrosophic set and a single valued neutrosophic set. Throughout this section, X denotes the universal set which is nonempty.

Definition 2.1. *[15] A neutrosophic set N on X is defined by : $N = \{< x, t_N(x), i_N(x), f_N(x) >, x \in X\}$ where $t_N, i_N, f_N : X \to]^-0, 1^+[$ are functions satisfy the inequality $^-0 \le t_N(x) + i_N(x) + f_N(x) \le 3^+$.*

From philosophical point of view, the neutrosophic set takes the value from real standard or non standard subsets of $]^-0, 1^+[$. But it is hard to consider the degree which belongs to a real standard or a non-standard subset of $]^-0, 1^+[$, in real world applications, especially in medical,

Email address: **vildan.cetkin@kocaeli.edu.tr (V. Çetkin), halis@kocaeli.edu.tr (H. Aygün)**

engineering and statistical problems etc. Hence throughout this work, we deal with the following specified definition of a neutrosophic set which is called a single valued neutrosophic set.

Definition 2.2. *[16] A single valued neutrosophic set (SVNS) N on X is characterized by the truth-membership function t_N, the indeterminacy-membership function i_N and the falsity-membership function f_N. For each point x in X, the values $t_N(x), i_N(x), f_N(x)$ take place in the real unit interval $[0,1]$.*
A neutrosophic set N can be written as
$$N = \sum_{i=1}^{n} < t_N(x_i), i_N(x_i), f_N(x_i) > /x_i,\ x_i \in X.$$

Since the membership functions t_N, i_N, f_N are defined from the universal set X into the unit interval $[0,1]$ as $t_N, i_N, f_N : X \to [0,1]$, a (single valued) neutrosophic set N will be denoted by a mapping described by $N : X \to [0,1] \times [0,1] \times [0,1]$ and where, $N(x) = (t_N(x), i_N(x), f_N(x))$, for simplicity. The family of all single-valued neutrosophic sets on X is denoted by $SNS(X)$.

Definition 2.3. *[13, 16] Let $N, M \in SNS(X)$. Then*
(1) N is contained in M, denoted as $N \subseteq M$, if and only if $N(x) \leq M(x)$. This means that $t_N(x) \leq t_M(x), i_N(x) \leq i_M(x)$ and $f_N(x) \geq f_M(x)$. Two sets N, M are called equal, i.e., $N = M$ iff $N \subseteq M$ and $M \subseteq N$.
(2) the union $K = N \cup M$ is defined as $K(x) = N(x) \vee M(x)$ where $N(x) \vee M(x) = (t_N(x) \vee t_M(x), i_N(x) \vee i_M(x), f_N(x) \wedge f_M(x))$, for each $x \in X$. This means that $t_K(x) = \max\{t_N(x), t_M(x)\}, i_K(x) = \max\{i_N(x), i_M(x)\}$ and $f_K(x) = \min\{f_N(x), f_M(x)\}$.
(3) the intersection $K = N \cap M$ is defined as $K(x) = N(x) \wedge M(x)$ where $N(x) \wedge M(x) = (t_N(x) \wedge t_M(x), i_N(x) \wedge i_M(x), f_N(x) \vee f_M(x))$, for each $x \in X$. This means that $t_K(x) = \min\{t_N(x), t_M(x)\}, i_K(x) = \min\{i_N(x), i_M(x)\}$ and $f_K(x) = \max\{f_N(x), f_M(x)\}$.
(4) the complement of N is denoted by N^c and it is defined as $N^c(x) = (f_N(x), 1 - i_N(x), t_N(x))$, for each $x \in X$. Here $(N^c)^c = N$.

The details of the set theoretical operations can be found in [13, 16].

Definition 2.4. *Let $g : X_1 \to X_2$ be a function and N, M be the neutrosophic sets of X_1 and X_2, respectively. Then the image of N is a neutrosophic set of X_2 and it is defined as follows:*
$g(N)(y) = (t_{g(N)}(y), i_{g(N)}(y), f_{g(N)}(y)) = (g(t_N)(y), g(i_N)(y), g(f_N)(y)), \forall y \in X_2$ *where*
$$g(t_N)(y) = \begin{cases} \bigvee t_N(x), & \text{if } x \in g^{-1}(y); \\ 0, & \text{otherwise} \end{cases}, \quad g(i_N)(y) = \begin{cases} \bigvee i_N(x), & \text{if } x \in g^{-1}(y); \\ 0, & \text{otherwise} \end{cases},$$
$$g(f_N)(y) = \begin{cases} \bigwedge f_N(x), & \text{if } x \in g^{-1}(y); \\ 1, & \text{otherwise.} \end{cases}$$
And the preimage of M is a neutrosophic set of X_1 and it is defined as follows:
$g^{-1}(M)(x) = (t_{g^{-1}(M)}(x), i_{g^{-1}(M)}(x), f_{g^{-1}(M)}(x)) = (t_M(g(x)), i_M(g(x)), f_M(g(x))) = M(g(x)), \forall x \in X_1.$

Definition 2.5. *[4] Let $N \in SNS(X)$ and $\beta \in [0,1]$. Define the β-level sets of N as follows:*
$(t_N)_\beta = \{x \in X \mid t_N(x) \geq \beta\}, (i_N)_\beta = \{x \in X \mid i_N(x) \geq \beta\}$, *and* $(f_N)^\beta = \{x \in X \mid f_N(x) \leq \beta\}$.

Following properties are easily proved by using the definitions.
(1) If $N \subseteq M$ and $\beta \in [0,1]$, then $(t_N)_\beta \subseteq (t_M)_\beta, (i_N)_\beta \subseteq (i_M)_\beta$, and $(f_N)^\beta \supseteq (f_M)^\beta$.
(2) $\beta \leq \gamma$ implies $(t_N)_\beta \supseteq (t_N)_\gamma, (i_N)_\beta \supseteq (i_N)_\gamma$, and $(f_N)^\beta \subseteq (f_N)^\gamma$.

Definition 2.6. *[5] Let $R = (R, +, \cdot)$ be a classical ring and N be a neutrosophic set on R. Then N is called a neutrosophic subring of R if the following properties are satisfied: for each $r, s \in R$,*
(R1) $N(r+s) \geq N(r) \wedge N(s)$.
(R2) $N(-r) \geq N(r)$.
(R3) $N(r \cdot s) \geq N(r) \wedge N(s)$.

From now on, R denotes a classical ring, unless otherwise specified.

Example 2.7. *[5] Let us take into consideration the classical ring $R = \mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ with the operations $\oplus$ and $\odot$ defined as $\overline{x} \oplus \overline{y} = \overline{x+y}$ and $\overline{x} \odot \overline{y} = \overline{x \cdot y}$ for all $\overline{x}, \overline{y} \in \mathbb{Z}_4$, respectively. Define the neutosophic set N on R as follows:*
$N = \{< 0.8, 0.4, 0.1 > /\overline{0} + < 0.5, 0.3, 0.5 > /\overline{1} + < 0.7, 0.4, 0.3 > /\overline{2} + < 0.5, 0.3, 0.5 > /\overline{3}\}$.
It is clear that the neutrosophic set N is a neutrosophic subgring of R.

Theorem 2.8. *[5] Let R be a classical ring and $N \in SNS(R)$. Then $N \in NSR(R)$ if and only if the following properties are satisfied for all $r, s \in R$;*
(1) $N(r-s) \geq N(r) \wedge N(s)$.
(2) $N(r \cdot s) \geq N(r) \wedge N(s)$.

3. Neutrosophic ideals

In this section, we propose two definitions as neutrosophic ideal of a neutrosophic subring and a neutrosophic ideal of a classical ring. We investigate some properties and characterizations of a neutrosophic ideal of a given classical ring.

Definition 3.1. *Let R be a classical ring and I be a neutrosophic set on R. Then I is called a neutrosophic left ideal over R if the followings are satisfied for each $r, s \in R$,*
(LI1) $I(r-s) \geq I(r) \wedge I(s)$.
(LI2) $I(r \cdot s) \geq I(s)$.

Definition 3.2. *Let R be a classical ring and I be a neutrosophic set on R. Then I is called a neutrosophic right ideal over R if the followings are satisfied for each $r,s \in R$,*
(RI1) $I(r-s) \geq I(r) \wedge I(s)$.
(RI2) $I(r \cdot s) \geq I(r)$.

Definition 3.3. *Let R be a classical ring and I be a neutrosophic set on R. Then I is called a neutrosophic ideal over R if the followings are satisfied for each $r,s \in R$,*
(I1) $I(r-s) \geq I(r) \wedge I(s)$.
(I2) $I(r \cdot s) \geq \max\{I(r), I(s)\}$.

Remark 3.4. *Each neutrosophic ideal over a classical ring R is a neutrosophic subring of R, but the converse is not true in general. For instance, let R be a ring and let $C = \{c \in R \mid cr = rc$ for all $r \in R\}$ denote the center of R. Define a neutrosophic set N on R as follows:*

$$N(s) = \begin{cases} (1,1,0), & if\ s \in C \\ (0,0,1), & otherwise \end{cases}$$

It is clear that N is a neutrosophic subring of R, but may not be an ideal.

Theorem 3.5. *Let I and J be two neutrosophic left (respectively, right) ideals of a classical ring R. Then the intersection $I \cap J$ is a neutrosophic left (respectively, right) ideal of R.*

Proof. Let $r,s \in R$ be arbitrary and I,J be the left ideals of R. Let us show that
$(I \cap J)(r-s) \geq (I \cap J)(r) \wedge (I \cap J)(s)$, and $(I \cap J)(r \cdot s) \geq (I \cap J)(s)$. First consider the truth-membership degree of the intersection for the first condition,

$$\begin{aligned} t_{I \cap J}(r-s) &= t_I(r-s) \wedge t_J(r-s) \\ &\geq (t_I(r) \wedge t_I(s)) \wedge (t_J(r) \wedge t_J(s)) \\ &= (t_I(r) \wedge t_J(r)) \wedge (t_I(s) \wedge t_J(s)) = t_{I \cap J}(r) \wedge t_{I \cap J}(s). \end{aligned}$$

The other inequalities $i_{I \cap J}(r-s) \geq i_{I \cap J}(r) \wedge i_{I \cap J}(s)$ and $f_{I \cap J}(r-s) \leq f_{I \cap J}(r) \vee f_{I \cap J}(s)$ are similarly proved for each $r,s \in R$. For the second condition, let us consider the falsity degree of the intersection,
$f_{I \cap J}(r \cdot s) = f_I(r \cdot s) \vee f_J(r \cdot s) \leq f_I(s) \vee f_J(s) = f_{I \cap J}(s)$.
The other inequalities $t_{I \cap J}(r \cdot s) \geq t_{I \cap J}(s)$ and $i_{I \cap J}(r \cdot s) \geq i_{I \cap J}(s)$ are similarly proved for each $r,s \in R$.
Consequently, $I \cap J$ is a neutrosophic ideal of R, as desired. □

Theorem 3.6. *Let R be a classical ring and I be a neutrosophic set on R. Then I is a neutrosophic (respectively, left, right) ideal over R if and only if for arbitrary $\beta \in [0,1]$, if β-level sets of I are nonempty, then $(t_I)_\beta, (i_I)_\beta$ and $(f_I)^\beta$ are all classical (respectively, left, right) ideals of R.*

Proof. Let I be a neutrosophic left ideal of R, $\beta \in [0,1]$ and $r,s \in (t_I)_\beta$(similarly $r,s \in (i_I)_\beta, (f_I)^\beta$). By the assumption,
$t_I(r-s) \geq t_I(r) \wedge t_I(s) \geq \beta \wedge \beta = \beta$ (and similarly, $i_I(r-s) \geq \beta$ and $f_I(r-s) \leq \beta$). Hence $r-s \in (t_I)_\beta$, (and similarly $r-s \in (i_I)_\beta, (f_I)^\beta$) for each $\beta \in [0,1]$. In a similar way, we obtain $r \cdot s \in (t_I)_\beta$ (respectively, $r \cdot s \in (i_I)_\beta$ and $r \cdot s \in (f_I)^\beta$), for each $r \in R$ and $s \in (t_I)_\beta$ (respectively, $s \in (i_I)_\beta$ and $s \in (f_I)^\beta$). These mean that $(t_I)_\beta$ (and similarly $(i_I)_\beta, (f_I)^\beta$) is a classical ideal of R for each $\beta \in [0,1]$.
Conversely, suppose $(t_I)_\beta, (i_I)_\beta$ and $(f_I)^\beta$ are classical ideals of R. Let $r,s \in R$ and $\beta = t_I(r) \wedge t_I(s)$, then $r,s \in (t_I)_\beta$. Since $(t_I)_\beta$ is a left ideal of R, then $r-s \in (t_I)_\beta$. This means that $t_I(r-s) \geq \beta = t_I(r) \wedge t_I(s)$.
Now let $r \in (t_I)_\beta$ and $s \in R$ such that $\beta = t_I(s)$. This shows that $t_I(r \cdot s) \geq \beta = t_I(s)$.
In similar computations, we obtain the desired inequalities as follows.
$i_I(r-s) \geq i_I(r) \wedge i_I(s),\ i_(r \cdot s) \geq i_I(s)$ and $f_I(r-s) \leq f_I(r) \vee f_I(s),\ f_I(r \cdot s) \leq f_I(s)$.
This completes the proof. □

Theorem 3.7. *Let I be a neutrosophic (left, right) ideal of R and $X_I = \{r \in R \mid I(r) = I(0)\}$, where 0 is the unit of the sum operation of R. Then the classical subset X_I of R is an (left, right) ideal of R.*

Proof. Let I be a neutrosophic ideal of R and take $r,s \in X_I$. First we need to show that the set X_I is a subgroup of R under sum operation. By the assumption, $I(r) = I(0) = I(s)$ and by the condition (I1), the following inequality is true
$I(r-s) \geq I(r) \wedge I(s) = I(0) \wedge I(0) = I(0)$.
Since, the inequality $I(0) \geq I(r-s)$ is always satisfied, we obtain that $I(r-s) = I(0)$. So, $r-s \in X_I$.
Now take $r \in X_I$ and $s \in R$. Second we need to show $r \cdot s \in X_I$, i.e., $I(r \cdot s) = I(0)$.
Since $I(r) = I(0)$ and by the condition (I2),
$I(r \cdot s) \geq \max\{I(r), I(s)\} = \max\{I(0), I(s)\} = I(0)$.
Since always $I(0) \geq I(r \cdot s)$, then $I(r \cdot s) = I(0)$. Hence, $r \cdot s \in X_I$. Similarly, $s \cdot r \in X_I$.
In conclude, X_I is an ideal of R. □

Let N and M be two neutrosophic sets on R, then $N \Diamond M$ is a neutrosophic set on R and it is defined by
$(N \Diamond M)(z) = (\sup\limits_{z=x \cdot y} \min\{t_N(x), t_M(y)\}, \sup\limits_{z=x \cdot y} \min\{i_N(x), i_M(y)\}, \inf\limits_{z=x \cdot y} \max\{f_N(x), f_M(y)\})$,
otherwise, $(N \Diamond M)(z) = (0,0,1)$, where $x,y,z \in R$.

Theorem 3.8. *Let R be a ring and I be a neutosophic left (right) ideal over R iff the followings are satisfied:*
(1) $I(r-s) \geq I(r) \wedge I(s)$*, for each* $r,s \in R$.
(2) $\chi_R \Diamond I \leq I$ *(respectively,* $I \Diamond \chi_R \leq I$*), where if* $r \in R$*, then* $\chi_R(r) = (1,1,0)$.

Proof. Suppose I is a neutrosophic left ideal over R and take $z \in R$, then

$$\begin{aligned}(\chi_R \Diamond I)(z) &= (\sup_{z=r\cdot s} \min\{t_{\chi_R}(r), t_I(s)\}, \sup_{z=r\cdot s} \min\{i_{\chi_R}(r), i_I(s)\}, \inf_{z=r\cdot s} \max\{f_{\chi_R}(r), f_I(s)\}) \\ &= (\sup_{z=r\cdot s} t_I(s), \sup_{z=r\cdot s} i_I(s), \inf_{z=r\cdot s} f_I(s)) \\ &\leq I(r\cdot s) = I(z).\end{aligned}$$

Hence, $\chi_R \Diamond I \leq I$.
Conversely, let I be a neutrosophic set on R which satisfies the corresponding two conditions.
(1) $I(r-s) \geq I(r) \wedge I(s)$ (2) $\chi_R \Diamond I \leq I$.
Take arbitrary $r, s \in R$, then

$$\begin{aligned}I(r\cdot s) &\geq (\chi_R \Diamond I)(r\cdot s) \\ &= (\sup_{r\cdot s=p\cdot q} \min\{t_{\chi_R}(p), t_I(q)\}, \sup_{r\cdot s=p\cdot q} \min\{i_{\chi_R}(p), i_I(q)\}, \inf_{r\cdot s=p\cdot q} \max\{f_{\chi_R}(p), f_I(q)\}) \\ &\geq (\min\{t_{\chi_R}(r), t_I(s)\}, \min\{i_{\chi_R}(r), i_I(s)\}, \max\{f_{\chi_R}(r), f_I(s)\}) \\ &= (t_I(s), i_I(s), f_I(s)) = I(s).\end{aligned}$$

This implies the neutrosophic set I is a neutrosophic left ideal over R.
The other situations are proved similarly. □

Theorem 3.9. *Let R_1, R_2 be the classical rings and $g : R_1 \to R_2$ be a homomorphism of rings. If J is a left (respectively, right) ideal of R_2, then the preimage $g^{-1}(J)$ is a left (respectively, right) ideal of R_1.*

Proof. Suppose that J is a neutrosophic left ideal of R_2 and $r_1, r_2 \in R_1$. Since g is a homomorphism of rings, the following inequality is obtained.

$$\begin{aligned}g^{-1}(J)(r_1 - r_2) &= (t_J(g(r_1 - r_2)), i_J(g(r_1 - r_2)), f_J(g(r_1 - r_2))) \\ &= (t_J(g(r_1) - g(r_2)), i_J(g(r_1) - g(r_2)), f_J(g(r_1) - g(r_2))) \\ &\geq (t_J(g(r_1)) \wedge t_J(g(r_2)), i_J(g(r_1)) \wedge i_J(g(r_2)), f_J(g(r_1)) \vee f_J(g(r_2))) \\ &= (t_J(g(r_1)), i_J(g(r_1)), f_J(g(r_1))) \wedge (t_J(g(r_2)), i_J(g(r_2)), f_J(g(r_2))) \\ &= g^{-1}(J)(r_1) \wedge g^{-1}(J)(r_2).\end{aligned}$$

In similar computations, it is clear that $g^{-1}(J)(r\cdot s) \geq g^{-1}(J)(s)$, for each $r, s \in R$.
Therefore, $g^{-1}(J)$ is a neutorosophic left ideal of R_1. □

Theorem 3.10. *Let R_1, R_2 be the classical rings and $g : R_1 \to R_2$ be a homomorphism of rings. If I is a neutrosophic left (respectively, right) ideal of R_1, then $g(I)$, the image of I, is a neutrosophic left (respectively, right) ideal of R_2.*

Proof. The proof is obtained by using the definitions of a left (respectively, right) ideal of a classical ring, and the image of a neutrosophic set. □

In the following, we introduce the neutrosophic ideal of a neutrosophic subring.

Definition 3.11. *Let N be a neutrosophic subring of a classical ring R. A non-null neutrosophic set M is called a neutrosophic ideal of N, if the following conditions are valid for each $r, s \in R$,*
(1) $M(r-s) \geq M(r) \wedge M(s)$.
(2) $M(r\cdot s) \geq M(r) \wedge M(s)$.
(3) $M(r) \leq N(r)$.

Theorem 3.12. *Let M_1 and M_2 be the neutrosophic ideals of the neutrosophic subrings of N_1 and N_2, respectively. Then the intersection $M_1 \cap M_2$ is a neutrosophic ideal of $N_1 \cap N_2$.*

Proof. Similar to the proof of Theorem 3.5. □

4. Conclusion

Just as normal subgroups played a crucial role in the theory of groups, so ideals play an analogous role in the study of rings. A single valued neutrosophic set is a kind of neutrosophic set which is suitable to use in real world applications. Therefore, the study of single valued neutrosophic sets and their properties have a considerable significance in the sense of applications as well as in understanding the fundamentals of uncertainty. So, we decided to propose the definitions of a neutrosophic ideals of a classical ring and of a neutrosophic subring, in the sense of [4, 5], and observe their fundamental properties. For further research one can handle cyclic (respectively, symmetric, abelian) neutrosophic group structure, and some of other algebraic structures.

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