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Boolean hypercubes and the structure of vector spaces

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Abstract

The present study pretends to describe an alternative way to look at Vector Spaces as a scaffold to produce a meaningful new theoretical structure to be used in both classical and quantum QSPR. To reach this goal it starts from the fact that N-Dimensional Boolean Hypercubes contain as vertices the whole information maximally expressible by means of strings of N bits. One can use this essential property to construct the structure of N-Dimensional Vector Spaces, considering vector classes within a kind of Space Wireframe related to a Boolean Hypercube. This way of deconstruct-reconstruct Vector Spaces starts with some newly coined nomenclature, because, through the present paper, any vector set is named as a Vector Polyhedron, or a polyhedron for short if the context allows it. Also, definition of an Inward Vector Product allows to easily build up polyhedral vector structures, made of inward powers of a unique vector, which in turn one might use as Vector Space basis sets. Moreover, one can construct statistical-like vectors of a given Vector Polyhedron as an extended polyhedral sequence of vector inward powers. Furthermore, the Complete Sum of a vector is defined simply as the sum of all its elements. Once defined, one can use it to compute, by means of inward products, generalized scalar products, generalized vector norms and statistical-like indices attached to a Vector Polyhedron.

1. Introduction

It will be useless to provide an exhaustive list of literature devoted to vector spaces and their applications. Therefore, just a simplified list containing references which have been useful to the present author will be given. Obviously enough the following quotations, see for instance [1]-[8], contain reference descriptions of vector spaces for learning purposes and practical application within linear algebra algorithms as well. In chemistry one of the main uses vector spaces are employed corresponds to the description of molecules described as vectors, both in discrete and infinite dimensional spaces. In fact, the fields of classical and quantum QSPR studied within the so-called molecular spaces are effectively connected with finite dimensional vector spaces, see as a recent example, references [9]-[28].

Considering the previous reflections, it seems interesting for QSPR purposes and beyond, to study vector spaces from a new point of view, which as far as the author knows has not been described in the current literature. Therefore, one might simply describe the aim of the present study as an essay to deconstruct-reconstruct vector spaces, within the framework of Boolean Hypercubes of the same dimension.

The provided references correspond to many topics, which the author has developed mainly in the 21st century if they were reflecting in some way or another the picture of vector spaces as are discussed here. See for example references [10]-[22].

Along this paper will be used the well-known Dirac notation to represent vectors. According to this choice, N-Dimensional column vectors will be written as: $|\mathbf{a}\rangle = (a_1, a_2, \ldots, a_N)^T = \{a_I \,|I = 1, N\}$ and the associated dual row vectors as: $\langle \mathbf{a}| = (a_1, a_2, \ldots, a_N)$.

An N-Dimensional vector space defined over some scalar numerical set S will be represented by: $V_N(S)$.

Symbolized by $\mathbf{H}_N$ an *N-Dimensional Boolean Hypercube* is described by a set of 2^N vertices, which might be also noted by: $\{|\mathbf{h}_I\rangle \,|I = 0, 2^N - 1\}$. Each vertex is made by all the possible combinations of sets of N binary digits. That is, sets in the form of vectors of N elements just made by two number-symbols as: $B = \{0, 1\}$. $\mathbf{H}_N$might be seen as the unique set of distinct vector elements, which can be contained within a vector space defined as:$V_N(B)$.

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1.1. Vector spaces and vector semispaces

By a *vector semispace*[1] it is here understood a vector space where within it are not included vectors with negative elements at all, see references [10]-[22]. for an exposition of definitions, properties and applications of semispaces. One must only note the addition in a semispace forms a semigroup, and from this property, the name semispace appears coherently defined. Scalars intervening in the product of a vector by a scalar in semispaces are also considered positive definite.
In principle, it will be latter studied the fact that, to construct any vector space, one only needs vectors strictly bearing *non-null elements*. Such kind of vectors will be named here: *whole vectors*.
The set of the positive definite whole vectors in a vector space constitutes a vector semispace.
Broadly speaking, one can consider vector semispaces, as defined over the positive real, rational or natural numbers. One can consider *natural* vector spaces $V_N(\mathbb{N})$, defined over the set of natural numbers, as recently discussed, see references [23]-[24]., directly constructed in the guise of semispaces.

1.2. Vector polyhedra

In the present work, by a *Vector Polyhedron* (or shortly: Polyhedron) P it will be understood a set of M vectors:

$$P = \left\{ |\mathbf{p}_I\rangle = \begin{pmatrix} p_{1I} & p_{2I} & \dots & p_{NI} \end{pmatrix}^T | I = 1, M \right\},$$

belonging to a vector space, that is: $P \subset V_N(S)$. The vector elements of a Polyhedron might be named as *vertices*.
Whenever the value $M > N$, the Polyhedron will be made of linearly dependent vertices. Only when $M \leq N$ there is a possibility that the Polyhedron might be strictly made by a set of linearly independent vertices.
The Gram matrix $\mathbf{G}$ of a Polyhedron P contains the scalar products of the Polyhedron vertices pairs. That is, G is a symmetrical $(M \times M)$ matrix defined as:

$$\mathbf{G} = \left\{ G_{IJ} = \langle \mathbf{p}_I | \mathbf{p}_J \rangle = \sum_{K=1}^{N} p_{KI} p_{KJ} | I, J = 1, M \right\}.$$

The notation $\mathbf{G}[P]$ makes explicit, if necessary, the fact that the Gram matrix is computed over the vertices of the Polyhedron P.
Any vector Polyhedron can be considered as constructed by a linearly independent set of vertices whenever: $Det|\mathbf{G}| \neq 0$.
Vector polyhedra concept has been recently employed in various publications dealing with several topics, see for instance references [22]-[26].

2. Boolean hypercubes as vector space wireframes

2.1. Definition and characteristics of a boolean hypercube

In the present discussion, each vertex contained in a Boolean Hypercube $\mathbf{H}_N$ will be considered as made by all the possible combinations of the digits $B = \{0, 1\}$ taken in batches of N terms, which in turn can be depicted as vectors, considered as Boolean Hypercube vertices.
One can alternatively accept the two digits forming the set B as binary, integer, rational or real numbers, depending on the mathematical context where the Boolean Hypercube is used. For more information about the use which has been made of Hypercubes to study different theoretical and practical problems, see references [27]-[32].
For instance, taken B as the elementary binary digit set or bits, then the $\mathbf{H}_N$ vertices become a collection of binary strings, which one can translate into a natural numerical form in the customary manner.
Within this strict Boolean interpretation, each one of the 2^N vertices of $\mathbf{H}_N$represents a decimal natural number, which, if necessary, one can represent by the transformations:

$$\forall I = 0, 2^N - 1 \Rightarrow I = \nu[|\mathbf{h}_I\rangle] \leftrightarrow |\mathbf{h}_I\rangle = \beta[I].$$

As previously stated any $\mathbf{H}_N$ constructed in this fashion is equivalent to a vector space of the same dimension: $V_N(B)$, associated to the basic binary number set: B.

2.2. Characteristic vertices of a boolean hypercube

Two vertices are present in any Boolean Polyhedron: $\mathbf{H}_N$, representing two extreme bit constructs. Namely,
a) The *zero* vertex: $|\mathbf{0}\rangle = (0, 0, ..., 0)^T$ containing an N times repeated string of the digit $\{0\}$, whose decimal representation one might accept as the number zero, that is: $0 = \nu[|\mathbf{0}\rangle]$.
b) The *Mersenne or unity* vertex: $|\mathbf{1}\rangle = (1, 1, ..., 1)^T$, possess their components made by the digit $\{1\}$ repeated N times, containing the decimal representation of the Mersenne number $\mu(N)$, defined as: $\mu(N) = 2^N - 1 = \nu[|\mathbf{1}\rangle]$. The Mersenne number is the largest decimal representation, which can be associated to any bit string contained within the N-Dimensional Boolean Hypercube: $\mathbf{H}_N$.
Mersenne numbers have relative importance as some are prime numbers, like 7. The largest prime number computed up to date[2], is also a Mersenne number: $2^{77,232,917} - 1$

[1] In some literature, vector semispaces are also known as orthants.
[2] See, for example: https://www.mersenne.org/primes/press/M77232917.html

2.3. Canonical vertex set

A set of vertices contained into every $\mathbf{H}_N$ is also worth of mention: the *canonical vertex set*, E, which one might define using the representation:

$$E = \{|\mathbf{e}_I\rangle \,|I = 0, N-1\} \equiv \left\{|\mathbf{h}_J\rangle \,\middle|\, J = 2^I \;|I = 0, N-1\right\},$$

where each element of E could be associated to the decimal powers of two using the transformation:

$$\forall I = 0, N-1 : \nu\left[|\mathbf{e}_I\rangle\right] = 2^I.$$

From the Boolean point of view, each element of the canonical vertex set E corresponds to a string of $N-1$ digits $\{0\}$ and just one digit $\{1\}$, which one can consider positioned into every one of the elements of the associated N-Dimensional vector. The vector set E, viewed as an integer, rational or real construct, corresponds to the so-called canonical basis set, which one can suppose as belonging to any vector space of the appropriate dimension. Also, any vertex of $\mathbf{H}_N$ corresponds to a linear combination of the canonical vertex set, whose coordinates are coincident with the elements of the considered vertex in question.
In terms of the nomenclature used in many previous papers and adopted here, one might consider the canonical basis set as a vector Polyhedron. Because such a general name, as commented in the introduction, has been chosen to be applied to any kind of vector set, belonging to any vector space.
Such a nomenclature, as used here, seems preferable to the alternative possible name of vector Polytope. As one might accept, a Polytope being a specific construction case of a regular Polyhedron. Accordingly, one might consider a Hypercube $\mathbf{H}_N$as an element among vector Polyhedra but being regular as an N-Dimensional Polytope too.
Finally, one can add a comment about the fact that, while the Mersenne vertex $|\mathbf{1}\rangle \equiv \beta\left[2^N - 1\right]$ might be considered as the last vertex of some N-Dimensional Boolean Hypercube $\mathbf{H}_N$, the canonical vertex $|\mathbf{e}_N\rangle \equiv \beta\left[2^N\right]$, which corresponds to the next natural number representation in the sequence of binary numbers, might be taken as the first vector of $\mathbf{H}_{N+1}$, which is not contained in $\mathbf{H}_N$.
Concatenation $\cup$ is an operation involving Hypercubes, which can be defined to construct Hypercubes of higher dimension from lesser dimensional ones, see for example [30, 31]. That is, for instance, while it can be written: $\{0\} \cup \mathbf{H}_N \subset \mathbf{H}_{N+1}$, also one can consider that: $|\mathbf{e}_N\rangle \in \mathbf{H}_{N+1} \wedge |\mathbf{e}_N\rangle \notin \{0\} \cup \mathbf{H}_N$.

2.4. Vertex classes

One can classify the set of $\mathbf{H}_N$ vertices into $N+1$ classes, according to the number of digits $\{0\}$ or $\{1\}$, used to construct the corresponding class. Usually, there are two complementary sets of vertex classes.
Using the digit $\{1\}$ as classifier, one can start with the class holding zero digits $\{1\}$, formed by the zero vector: $|\mathbf{0}\rangle$ only, and end with the class holding N digits $\{1\}$, containing the Mersenne vertex $|\mathbf{1}\rangle$ only. The class of N vertices containing one digit $\{1\}$ and $N-1$digits $\{0\}$ is coincident with the canonical vertex set E.
One might accept that every $\mathbf{H}_N$ class is forming a vector Polyhedron by itself. The classification based on the number of digits $\{1\}$ in every $\mathbf{H}_N$ vertex within the class, might be described with the symbols: $\mathbf{C}_{[N_1;N_0]}$, with $[N_1;N_0]$ meaning the number of digits $\{1\}$ and $\{0\}$ respectively, contained in every vector belonging to the class.
Obviously enough: $N_1 + N_0 = N$. The number of vertices within each class corresponds to the combinatorial numbers: $\begin{pmatrix} N \\ N_1 \end{pmatrix} = \frac{N!}{N_1!(N-N_1)!}$, which correspond to the N-th row of the Pascal triangle. Following this thought, it must be also obtained: $\sum_{K=0}^{N} \begin{pmatrix} N \\ K \end{pmatrix} = 2^N$.
Every vertex class appears to be disjoint with respect of the remnant classes. No vertex class appears closed with respect to the vector sum though.
Complementary classes correspond to classes, which one can describe with the two symbols: $\mathbf{C}_{[K;L]}$ and $\mathbf{C}_{[L;K]}$. Complementary vertices belonging to complementary classes are pairs of vertices where the positions of the corresponding binary digits are interchanged: summed up, such kind of vertices produce the Mersenne vertex.

2.5. The complementary canonical vertex set as a basis set

The classes $\mathbf{C}_{[K;N-K]}$ and $\mathbf{C}_{[N-K;K]}$ are complementary for every value of K, with allowed values in the interval: $\{0, N\}$ and K taking here the role of the number of digits $\{1\}$ present within the vertices of the class.
The classes $\mathbf{C}_{[1;N-1]} \equiv E$ and $\mathbf{C}_{[N-1;1]} \equiv J$ are thus complementary. In fact, one might gather the canonical vertex set E as the columns of the unit matrix of the appropriate dimension, using a logical Kronecker delta[3]:

$$I_N = \{I_{N;IJ} = \delta(I = J)\,|I, J = 0, N-1\} = (|\mathbf{e}_0\rangle; |\mathbf{e}_1\rangle; \ldots |\mathbf{e}_{N-1}\rangle).$$

The complementary class J of the canonical vertex set E might be written by defining first the *unity matrix*: $\mathbf{1}_N$, as a matrix with all the columns or rows made by the Mersenne vertex repeated N times:

$$\mathbf{1}_N = \{1_{N;IJ} = 1\,|I, J = 0, N-1\} = (|\mathbf{1}\rangle; |\mathbf{1}\rangle; \ldots |\mathbf{1}\rangle)$$

and then, the complementary matrix to the unit one, can be easily written, using again a logical Kronecker delta, as:

$$\mathbf{J}_N = \mathbf{1}_N - \mathbf{I}_N = \{J_{N;IJ} = \delta(I \neq J)\,|I, J = 0, N-1\} = (|\mathbf{j}_0\rangle; |\mathbf{j}_1\rangle; \ldots |\mathbf{j}_{N-1}\rangle).$$

[3] A logical Kronecker delta is defined as a symbol written this way: $\delta(L)$, where L might be any logical expression. Then, this symbol can be used as: $\delta(L \equiv .T.) = 1$ or $\delta(L \equiv .F.) = 0$. Thus, the classical Kronecker delta might be written as: $\delta(I = J)$.

Finally, one can easily describe the $\mathbf{C}_{[N-1;1]}$ class vertex composition as:

$$\mathbf{C}_{[N-1;1]} \equiv J = \{|\mathbf{j}_I\rangle \,|I = 0, N-1\}\,.$$

Therefore, one can name the Polyhedron J as the *complementary* canonical basis set. This might be so because their elements are linearly independent. One can easily show the linear independence of the set of columns or rows of the vertex class J, considering that one might write the unity matrix spectrum as:

$$Sp\,(\mathbf{1}_N) = \{\lambda_1 = N; \lambda_K = 0\,|K = 2, N\}\,,$$

then the spectrum of the matrix $\mathbf{J}_N$, is easily computed as:

$$Sp\,(\mathbf{J}_N) = \{\mu_1 = N-1; \mu_K = -1\,|K = 2, N\}\,,$$

therefore:

$$Det\,|\mathbf{J}_N| = (-1)^{N-1}\,(N-1)\,.$$

Showing the fact that the vertices in the class J are linearly independent, therefore can be considered as a basis set complementary to the canonical basis set.

2.6. Model or seed vertex of a class $\mathbf{C}_{[N_1;N_0]}$ in $\mathbf{H}_N$

Every class holding N_1 digits $\{1\}$ and thus the corresponding N_0 digits $\{0\}$ possess a vertex, which might be called as the *model (or seed) vertex*: $\left|\mathbf{c}_{[N_1;N_0]}\right\rangle$,which can be defined as a vector with N_1 ones in the first places counted from the top and the rest filled by N_0 zeroes.
One can express this ordering of the binary digits leading to the model vertex of the class $\mathbf{C}_{[N_1;N_0]}$ by:

$$\left|\mathbf{c}_{[N_1;N_0]}\right\rangle = \begin{pmatrix} |\mathbf{1}_{N_1}\rangle \\ |\mathbf{0}_{N_0}\rangle \end{pmatrix}.$$

2.6.1. Mersenne basis set

The set of model vertices in each Boolean Hypercube $\mathbf{H}_N$, that is the sequence:

$$I = 1,\, N : \left|\mathbf{c}_{[I;N-I]}\right\rangle = \begin{pmatrix} |\mathbf{1}_I\rangle \\ |\mathbf{0}_{N-I}\rangle \end{pmatrix},$$

where the elements of the sequence contain in a first place the sequence of Mersenne vertices of dimension I: $I = 1, N : |\mathbf{1}_I\rangle$. Conveniently arranged as columns of a square, upper triangular, $(N \times N)$ matrix as follows:

$$\mathbf{M}_N = \{M_{IJ} = \delta\,(I \leq J)\,|I, J = 1, N\}\,,$$

yield a non-singular matrix. This is so as $Det\,|\mathbf{M}_N| = 1$, resulting from the fact that the diagonal of the matrix $\mathbf{M}_N$ coincides with the unit matrix, that is, one can write: $Diag\,(\mathbf{M}_N) = \mathbf{I}_N$. Thus the matrix $\mathbf{M}_N$ will be named here Mersenne matrix for obvious reasons.
Because of the non-singularity of the Mersenne matrix, its columns correspond to a new basis set polyhedron: M, contained in the Boolean Hypercube $\mathbf{H}_N$ and constructed by the Mersenne vertices of the increasing dimensions from 1 to N, which can be associated to the non-zero model vertices, contained into any Boolean Hypercube $\mathbf{H}_N$.
Contrarily to the previously discussed basis sets, which were found forming a whole Boolean Hypercube class, the Mersenne basis set is made of the whole collection of model vertices of the available non-zero classes.

2.6.2. The structure of class elements

The rest of the class vertex members can be obtained via the set of corresponding combinations of the digits $\{1\}$ and $\{0\}$ present in the model vertex. Every combination of the model vertex elements produces a vertex contained into the well-defined class $\mathbf{C}_{[N_1;N_0]}$. One might refer to the elements of a given class as the collection of *shadow vertices or shadow vectors* of the model vertex.
However, independently of the class construction, one can see, from considering the corresponding model vertex $\left|\mathbf{c}_{[N_1;N_0]}\right\rangle$, that one can also obtain all the shadow vectors within a given class, as generated by some transformation of the type:

$$\forall K = 1, \begin{pmatrix} N \\ N_1 \end{pmatrix} : T_K\left(\left|\mathbf{c}_{[N_1;N_0]}\right\rangle\right) = \left|\mathbf{c}_{[N_1;N_0];K}\right\rangle \in \mathbf{C}_{[N_1;N_0]},$$

producing all the necessary combinations of the digits $\{1\}$ and $\{0\}$, and thus forming the vertices of $\mathbf{H}_N$ belonging to the class. Of course, the transformation T_1 corresponds to the unit operator.
One can further use this previously defined transformation of the model vector: $\left|\mathbf{c}_{[N_1;N_0]}\right\rangle$, as an algorithm providing the ways on how a Mersenne vector $|\mathbf{1}_{N_1}\rangle$, belonging in fact to a lesser dimensional Boolean Hypercube $\mathbf{H}_{N_1}$, (that is: $N_1 < N$) could be transformed into the shadow vertices belonging to a given class in $\mathbf{H}_N$. This is due that one can write: $\left|\mathbf{c}_{[N_1;N_0]}\right\rangle = |\mathbf{1}_{N_1}\rangle \cup |\mathbf{0}_{N_0}\rangle$.

2.7. Whole and hollow vertices and vectors

Considering the previous definitions and characteristics of the Boolean Hypercube vertices, one can propose the following definitions.
In fact, from analyzing at the light of what has been said in the previous paragraphs, from observing all the vertices contained within a given N-Dimensional Boolean Hypercube, one can realize the unique vertex possessing the characteristic of being a truly N-Dimensional vector is the Mersenne or unity vertex $|\mathbf{1}_N\rangle$.
Shadow vertices in each class $\mathbf{C}_{[N_1;N_0]}$ are obviously related to the model vertex of this class, via the transformation discussed in the previous paragraph. Thus, all of them might be associated to the elements of a Mersenne vertex $|\mathbf{1}_{N_1}\rangle$of a lesser dimensional Boolean Hypercube $\mathbf{H}_{N_1}$.
The Mersenne vertices of Boolean Hypercubes might be associated to the kind of vectors in vector spaces, which one has previously named as *whole vectors*. Remembering that one can use such adjective to giving name to vectors of a Polyhedron collection in a specific N-Dimensional vector space, possessing no null elements at all. The whole vectors correspond to the true N-Dimensional elements of such a space.
One can call vectors associated to any other class, bearing any number of null elements as *hollow vectors*.
Therefore, Boolean Hypercube classes can be associated to the vectors which can be termed as hollow, whenever bearing one or more digits $\{0\}$. Hollow vectors, depending of the number of null elements, N_0, can be associated to the corresponding Boolean Hypercube $\mathbf{H}_N$ shadow vector elements attached to the class: $\mathbf{C}_{[N_1;N_0]}$.

3. Perfect vectors and generation of vector semispace elements

Realizing the nature of the whole vectors which can be found in a semispace, one can also consider the vectors, which are ordered canonically reversed[4], from maximal to minimal elements. That is, if one defines a whole vector of a semispacelike:

$$|\mathbf{a}\rangle = \{a_I \,|I = 1,N\} \wedge \forall I : a_I \neq 0.$$

Then, such a whole vector when chosen with canonically reversed ordered elements, might be also named a *perfect vector*, considering that:

$$a_1 > a_2 > ... > a_N.$$

Therefore, initially one can first suppose a *perfect semispace* as a subset of a semispace constructed from perfect vectors, that is: canonically reverse ordered positive definite whole vectors.
Then, one might construct the entire semispace via the $N!$ permutations of each positive definite perfect vector.
That is, suppose such a perfect vector known, then one can define the permutation operators' collection in the form: $\{P_K \,|K = 1,N!\}$, such that:

$$\forall K = 1,N! : |\mathbf{a}_K\rangle = P_K\left(|\mathbf{a}\rangle\right),$$

being every element of the vector permutation Polyhedron:

$$A = \{|\mathbf{a}_K\rangle \,|K = 1,N!\}$$

a whole vector deduced from a perfect vector $|\mathbf{a}\rangle$.
That is: the elements of every permutation vector $|\mathbf{a}_K\rangle$ might be considered reordered according to a permutation of the elements with the canonically ordered subindices of the perfect vector acting as a seed.
Hollow vectors do not need to be considered, as they can be formed as elements in lesser dimensional semispaces, and constructed by a lesser dimensional perfect vector, which one might transform into a hollow vector via the definition of a corresponding model vertex with the necessary number of digits $\{0\}$ followed by the described combination procedure.
Moreover, from a perfect vector defined in an N-Dimensional vector space, one can also deduce the N vertices of a circular Polyhedron, constructed by choosing the N circular permutations of the original perfect vector acting as a seed.

3.1. Signature hypercubes and the construction of vector spaces from a unique whole vector

Consider a Boolean Hypercube: $\mathbf{H}_N = \left\{|\mathbf{h}_I\rangle \,\middle|\, I = 0,2^N - 1\right\}$. One can readily compute the centroid $|\mathbf{h}_C\rangle$ of such a Boolean Hypercube, employing the arithmetic mean of its vertices.
Such vector mean might be considered as yielding a homothetic vector of the Mersenne vertex:

$$|\mathbf{h}_C\rangle = 2^{-N}\sum_{I=0}^{2^N-1}|\mathbf{h}_I\rangle = 2^{-1}\,|\mathbf{1}\rangle,$$

that is: obtaining in this way the Hypercube centroid as corresponding to a homothetic Mersenne vertex scaled by a factor $\frac{1}{2}$.
One can origin shift the Boolean Hypercube $\mathbf{H}_N$ using the centroid $|\mathbf{h}_C\rangle$, producing in such a manner a new Polyhedron construct, which might be called Signature Hypercube: $\mathbf{S}_N = \left\{|\mathbf{s}_I\rangle \,\middle|\, I = 0,2^N - 1\right\}$, and one might symbolically express it via the algorithm:

$$\mathbf{S}_N = 2\left(\mathbf{H}_N - \frac{1}{2}\,|\mathbf{1}\rangle\right) \Rightarrow \forall I = 0,2^N - 1 : |\mathbf{s}_I\rangle = 2\left(|\mathbf{h}_I\rangle - \frac{1}{2}\,|\mathbf{1}\rangle\right) \equiv 2\,|\mathbf{h}_I\rangle - |\mathbf{1}\rangle.$$

Therefore, the new Signature Hypercube $\mathbf{S}_N$ possess its vertices made by the digits: $\{-1;+1\}$. The vertices of $\mathbf{S}_N$ correspond to any combination of signs, which might be called a signature, which can be associated in turn to any N-dimensional positive definite whole vector.
The signature vectors have been already described in another context, see for example reference [33].

[4] Perfect vectors, of course, could also be chosen canonically ordered from minimal to maximal elements. But here have been chosen in reverse order to keep some resemblance to the ordered definition of model vertices in Boolean Hypercubes classes.

Then, to every perfect vector in a semispace, as previously commented, one can attach a set of permutations of its elements, yielding $N!$ whole vectors. Afterwards, from every permuted whole vector constructed in this way, one can build an additional set of 2^N vectors bearing the corresponding signatures, attaching the signs of every vertex in $\mathbf{S}_N$ to the positive definite whole vector elements.
Consequently, to every perfect vector of a semispace, one can easily construct as many as $M(N) = 2^N(N!)$ different vectors.
For example, considering a $N = 10$ dimensional vector semispace, for each perfect vector belonging to it, one can build a set of 3715891200 vectors belonging to the 10-Dimensional vector space. That is, some Polyhedron made of a large amount of vector elements, might be generated from only one perfect vector as a seed.

3.2. Vector classes

The class patterns of the Boolean Hypercube vertices might be used to describe both hollow and whole vectors. First, hollow vectors will be analyzed and then whole vectors will be classified. In fact, the relevant result of this paragraph can be resumed by the statement that, in any vector space the structure which matters is just the set of whole vectors.

3.2.1. Hollow vector classes

To start with, one can consider the hollow vectors first. Any hollow vector can have from one to all the elements null. This last occurrence will define the zero vector of the space, considered coincident with the zero vertex $|\mathbf{0}\rangle$ of the corresponding Boolean Hypercube.
Therefore, it can be accepted that hollow vectors can be also gathered into classes, which can be noted in turn as: $\left\{\mathbf{L}_{[K]}\,|K=0,N\right\}$. The subindex between brackets meaning the number of zero elements present anywhere in the vector.
Of course, the class $\mathbf{L}_{[0]}$ contains all the whole vectors and has been considered as a hollow class just for the sake of completeness.
Any vector $|\mathbf{a}\rangle$ belonging to a hollow class $\mathbf{L}_{[K]}$ can be referred to a vector seed, belonging in turn to some Boolean Hypercube class of the type: $\mathbf{C}_{[N-K,K]}$. The difference in both type of classes is such that the unity vector $|\mathbf{1}_{N-K}\rangle$ has been substituted say, by a whole perfect vector of dimension $N-K$: $|\mathbf{a}_{N-K}\rangle$.
Within the vector $|\mathbf{a}_{N-K}\rangle$ all the non-zero vector elements are gathered in the same order as in the related hollow vector of $\mathbf{C}_{[N-K,K]}$. The decomposition of the original vector into the whole vector above and a hollow vector, that is, like:

$$|\mathbf{a}\rangle \Rightarrow \begin{pmatrix} |\mathbf{a}_{N-K}\rangle \\ |\mathbf{0}_K\rangle \end{pmatrix},$$

where the zeroes are gathered at the bottom of the vector, makes it possible to consider the resultant reordered vector as the seed of the hollow class $\mathbf{L}_{[K]}$.
The remaining elements of the class could be deduced from the elements of the Boolean Hypercube class $\mathbf{C}_{[N-K,K]}$, substituting the unit elements of such class by the elements of the condensed $(N-K)$-Dimensional whole vector $|\mathbf{a}_{N-K}\rangle$.
Therefore, hollow vectors might be considered as whole vector elements of a $(N-K)$-Dimensional subspace. That is: in fact, hollow vectors might be considered as whole vectors contained into a lower dimensional space.
Consequently, the analysis of the elements of any vector space essentially might refer to the study of its whole vectors.

3.2.2. Whole vectors

Whole vectors might be classified into a set of classes, which can be referred to the Boolean Hypercube classes, in a similar way as the hollow vectors have previously been studied.
Whole vectors might have elements only repeated once, thus producing upon canonical or reverse canonical ordering a perfect vector, while the next alternative possibilities might consist into an element repeated twice, thrice… and so on. The last possible occurrence is having an element repeated N times, producing an homothecy of the Mersenne vertex, or a whole vector like: $|\mathbf{a}\rangle = \alpha\,|\mathbf{1}\rangle$.
Therefore, in this manner one can describe a set of whole vectors classes too, which can be described as: $\left\{\mathbf{W}_{[K]}\,|K=1,N\right\}$. Any vector in a class $\mathbf{W}_{[K]}$ can be associated to the Boolean Hypercube class: $\mathbf{C}_{[K,N-K]}$. Constructed with the Mersenne part of the model class vector, substituted by the repeated element α multiplying the same Mersenne vector: $|\mathbf{1}_K\rangle$ and the bottom part formed by a whole vector $|\mathbf{a}_{N-K}\rangle$ bearing the non-repeated elements disposed in the same order as in the original vector. Or what is the same:

$$|\mathbf{a}\rangle \Rightarrow \begin{pmatrix} \alpha\,|\mathbf{1}_K\rangle \\ |\mathbf{a}_{N-K}\rangle \end{pmatrix},$$

an arrangement which, can be considered as the model vector of the whole class $\mathbf{W}_{[K]}$.
The original vector might be reordered as any of the other members of the class, just considering the shadow vertex of $\mathbf{C}_{[K,N-K]}$, where the original digits $\{1\}$ might be substituted by the repeated elements and the original Boolean Hypercube digits $\{0\}$, exchanged by the original non-repeated elements of the vector.
As any homothetic vector of the Mersenne vertex corresponds to an element of a monodimensional space, the whole vectors belonging to the class $\mathbf{W}_{[K]}$, might be considered as belonging to a $(N-K+1)$-Dimensional subspace.
Therefore, only whole vectors belonging to the class $\mathbf{W}_{[1]}$ could be considered as true vectors belonging to the vector space$V_N(S)$.

4. Inward vector products and inward vector powers

Before proceeding with more consequences of the above definitions, a fundamental simple operation: the *inward product of two vectors*. must be described.

4.1. Inward product of two vectors

In a vector space $V_N(S)$ a binary operation noted as a product, the *inward product*[5], can be defined between every pair of vectors. Inward products have been described and employed in many places by the author, see for example references [15], [34]-[36].
To visualize such a product of two vectors, one has just to transform the vectors into a pair of diagonal matrices, then execute their product and retransform the resultant diagonal matrix back into a vector.
Suppose two vectors in $V_N(S)$:

$$\{|\mathbf{a}\rangle, |\mathbf{b}\rangle\} \subset V_N(S) \to |\mathbf{a}\rangle = \{a_I \,|I = 1,N\} \wedge |\mathbf{b}\rangle = \{b_I \,|I = 1,N\}.$$

The following algorithm easily defines their inward product:

$$|\mathbf{a}\rangle * |\mathbf{b}\rangle = |\mathbf{p}\rangle = \{p_I \,|I = 1,N\} \in V_N(S) \to \forall I = 1,N : p_I = a_I b_I.$$

The inward product is commutative, associative and distributive with respect to the vector summation.
The neutral element for the inward product is made precisely of the Mersenne or unity vector $|\mathbf{1}\rangle$ of the appropriate dimension, because it is trivial to demonstrate that:

$$\forall |\mathbf{a}\rangle \in V_N(S) \to |\mathbf{a}\rangle * |\mathbf{1}\rangle = |\mathbf{1}\rangle * |\mathbf{a}\rangle = |\mathbf{a}\rangle.$$

As an obvious application example, the generation of the set of signed vectors from a whole vector, as previously discussed, using the vertices of a Signature Hypercube, can be performed via the inward product of the signature vertices by the whole vector of the appropriate dimension. Suppose known the appropriate Signature Hypercube: $\mathbf{S}_N = \left\{|\mathbf{s}_I\rangle \,\middle|\, I = 0, 2^N - 1\right\}$ and some whole vector $|\mathbf{a}\rangle \in V_N(S)$, the set of all signed vectors, which can be associated to any positive definite whole vector can be simply obtained using:

$$|\mathbf{a}\rangle \to I = 0, 2^N - 1 : |\mathbf{a}_I\rangle = |\mathbf{s}_I\rangle * |\mathbf{a}\rangle.$$

Obviously enough, the inward vector product can be applied to more than a pair of vectors. Suppose a Polyhedron of some vector space: $P = \{|\mathbf{a}_I\rangle \,|I = 1,M\} \subset V_N$, then it can be written a vector resultant of the inward product of the whole Polyhedron vertices:

$$|\mathbf{p}\rangle = \underset{K=1}{\overset{M}{*}} |\mathbf{a}_K\rangle = |\mathbf{a}_1\rangle * |\mathbf{a}_2\rangle * ... * |\mathbf{a}_M\rangle \in V_N \to \forall I = 1,N : p_I = \prod_{K=1}^{M} a_{IK}.$$

4.2. Inward inverse vectors

Only whole vectors can be supposedly attached to some true inward inverse vector. That is, whenever $|\mathbf{a}\rangle \in V_N(S)$ is a whole vector, then the inward inverse of this vector can be easily defined as the whole vector:

$$|\mathbf{a}\rangle^{[-1]} \in V_N(S) \to |\mathbf{a}\rangle^{[-1]} = \left\{a_I^{-1} \,|I = 1,N\right\};$$

therefore, it is easy to write:

$$|\mathbf{a}\rangle * |\mathbf{a}\rangle^{[-1]} = |\mathbf{a}\rangle^{[-1]} * |\mathbf{a}\rangle = |\mathbf{1}\rangle.$$

Hollow vectors can be transformed into the model or seed vectors, while their shadow Boolean vector is kept as a class structure token. Once the separated non-zero part of the seed vector is made, then the inward inverse of this whole vector can be computed. This can be followed by a pseudoinverse vector, which can be constructed using the corresponding shadow vector and locating the inverse non-zero elements accordingly into the final vector form.

4.3. Inward power of a vector

One of the interesting applications of inward products corresponds to the possibility to compute successive inward powers of a given vector. Suppose known: $\forall |\mathbf{a}\rangle \in V_N(S)$, its inward p-th power: $|\mathbf{a}\rangle^{[p]}$ can be constructed via the following algorithm:

$$\left|\mathbf{a}^{[p]}\right\rangle = \underset{k=1}{\overset{p}{*}} |\mathbf{a}\rangle = |\mathbf{a}\rangle * |\mathbf{a}\rangle ... * |\mathbf{a}\rangle = \left\{a_I^p \,|I = 1,N\right\}.$$

An inward power Polyhedron can be associated to the collection of a given vector and their inward powers up to some fixed limit. For instance:

$$P_P(|\mathbf{a}\rangle) = \left\{\left|\mathbf{a}^{[p]}\right\rangle |p = 1,P\right\} = \left\{|\mathbf{a}\rangle, |\mathbf{a}\rangle^{[2]}, ...|\mathbf{a}\rangle^{[P]}\right\}.$$

[5] Such product is also named as Hadamard or Schur or Diagonal product.

4.4. Inward function of a vector

Of course, the inward power allows the construction of inward smooth vector functions, via Taylor series around some point $|\mathbf{x}_0\rangle$, for instance:

$$|\varphi[\mathbf{x}]\rangle = \{\varphi(x_I)|I=1,N\} \Rightarrow |\varphi[\mathbf{x}]\rangle = \sum_{k=0}^{\infty} \frac{1}{k!}(|\mathbf{x}\rangle - |\mathbf{x}_0\rangle)^{[k]} \left| \frac{\partial^k \varphi[\mathbf{x}_0]}{\partial \mathbf{x}^{[k]}} \right\rangle,$$

with the auxiliary definition

$$\left| \frac{\partial^k \varphi[\mathbf{x}_0]}{\partial \mathbf{x}^{[k]}} \right\rangle = \left\{ \left. \frac{d^k \varphi(x_I)}{dx_I^k} \right|_{x_I = x_{I,0}} |I=1,N \right\}.$$

The notation used here into the vector associated to an inward function of a vector: $|\varphi[\mathbf{x}]\rangle$ has been used to differentiate it from a multivariate scalar function of a vector, which can be written as: $\varphi(|\mathbf{x}\rangle)$.

5. The making of a generalized Minkowski vector space

5.1. Complexity hypercubes

One might start remembering again the Signature Hypercube: $\mathbf{S}_N$, which has been deduced from a Boolean Hypercube $\mathbf{H}_N$ of the appropriate dimension. As the vertices of $\mathbf{S}_N$ possess elements defined into the two-digit set: $\{-1;+1\}$, then one can easily imagine another Hypercube of the same dimension derived from: $\mathbf{S}_N$. This can be done just describing the operation which one might call inward square root of a vector:

$$\forall |\mathbf{a}\rangle = \{a_I|I=1,N\} \Rightarrow \left|\mathbf{a}^{\left[\frac{1}{2}\right]}\right\rangle = \left\{(a_I)^{\frac{1}{2}} \equiv \sqrt{a_I}|I=1,N\right\},$$

and apply it to each Signature Hypercube vertex, that is:

$$\mathbf{M}_N = \mathbf{S}_N^{\left[\frac{1}{2}\right]} \Rightarrow \forall I = 0, 2^N - 1 : |\mathbf{m}_I\rangle = \left|\mathbf{s}_I^{\left[\frac{1}{2}\right]}\right\rangle \equiv (2|\mathbf{h}_I\rangle - |\mathbf{1}\rangle)^{\left[\frac{1}{2}\right]}.$$

It is easily seen that the elements of the resultant Complexity Hypercube will be now described by the two digits: $\left\{\sqrt{-1};+1\right\} \to \{+i;+1\}$, where $i = \sqrt{-1}$ corresponds to the imaginary unit.
The Complexity Hypercube acts as another 2^N element source of an alternative kind of powerful binary attribute, which like signs, can be associated to whole N-Dimensional real vectors.
Again, two vectors can be chosen among all the vertices of a complexity Hypercube, the usual Mersenne vertex $|\mathbf{1}\rangle$ and the imaginary Mersenne vertex: $|\mathbf{i}\rangle = i|\mathbf{1}\rangle$, which appears as a pure imaginary homothecy of the Mersenne vertex.
In fact, one can generate from the elements of $\mathbf{M}_N$ the Signature Hypercube vertices, using the definition of the inward p-th power of a vector and simply considering that the following relationship holds:

$$\forall I = 0, 2^N - 1 : \left|\mathbf{m}_I^{[2]}\right\rangle = |\mathbf{s}_I\rangle,$$

just bearing in mind the trivial equality: $i^2 = -1$. Then, in this sense, the number of vectors which can be deduced from a perfect vector might be supposedly correspond to: $N!2^{2N}$

5.2. Complex vectors via complexity hypercubes

Given a perfect N-Dimensional real vector $|\mathbf{a}\rangle$, one can therefore construct a complex vector.
Thus, using the following algorithm, accordingly to the previous inward product definition:

$$\forall I = 0, 2^N - 1 : |\mathbf{c}_I\rangle = |\mathbf{m}_I\rangle * |\mathbf{a}\rangle,$$

it is obtained the vector set which might be easily considered as a generalized complex number set.
This is so, because it might be stated that a typical complex number, written as: $c = a + ib$, can be expressed as the sum of two one-Dimensional elements, forming the real and imaginary part.
Thinking of a two-dimensional case here, according to what has been previously said about Complexity Hypercubes, one can write four possible complexity attribute vertices as follows:

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} i \\ i \end{pmatrix} \right\}.$$

Every one of these vertices produces, upon inward product with any given whole real vector, four kinds of vectors: a) the same two-dimensional real Mersenne vector, b) an imaginary-real one, c) a real-imaginary vector and d) the purely imaginary Mersenne vector.
One can use the four vector complex characteristics summed upon a set in diverse ways, providing as a result vectors, which might be completely real, mixed real and complex, mixed imaginary and real, mixed imaginary and complex, completely complex and completely imaginary.
For instance, an N-Dimensional version of a complex number can be constructed in the following way, using two whole real vectors

$$\forall \{|\mathbf{a}\rangle, |\mathbf{b}\rangle\} \in V_N() : |\mathbf{1}\rangle * |\mathbf{a}\rangle + |\mathbf{i}\rangle * |\mathbf{b}\rangle = |\mathbf{a}\rangle + i|\mathbf{b}\rangle = \{a_I + ib_I|I=1,N\},$$

and the above expression can be written in this way above, because: $|\mathbf{i}\rangle = i\,|\mathbf{1}\rangle$.
The usual operations, like conjugation, described in complex algebra can be employed within this Complexity Hypercubes framework as in the scalar complex numbers case.
Vector spaces constructed using Complexity Hypercubes and whole vectors combine the character of the well-known Minkowski spaces, providing a generalization of them.
Of course, in the same way as there have been described vector classes following the number of null or equal elements entering the vector, the same organizing process could be associated to the vectors obtained from the vertices of a Complexity Hypercube.
Therefore, Complexity Hypercubes can be supposed to be in the definition of whole complex vectors.

5.3. Geometrical considerations on the unity vectors

Among the set of complex vectors constructed via Complexity Hypercubes one must consider the two vectors already described: $\{|\mathbf{1}\rangle, |\mathbf{i}\rangle \equiv i\,|\mathbf{1}\rangle\}$, as defining the unity vectors in real and pure imaginary subspaces.
Using Minkowski ideas, that is: not conjugating the imaginary unit when performing products, one can obtain the relative directions of both vectors. There are needed the scalar products:

$$\langle \mathbf{1}|\mathbf{1}\rangle = N \wedge \langle \mathbf{i}|\mathbf{i}\rangle = -N \wedge \langle \mathbf{1}|\mathbf{i}\rangle = \langle \mathbf{i}|\mathbf{1}\rangle = iN,$$

they permit to compute the cosine of the subtended angle, like:

$$\cos(\theta) = \frac{\langle \mathbf{1} \mid \mathbf{i}\rangle}{\sqrt{\langle \mathbf{1} \mid \mathbf{1}\rangle \langle \mathbf{i} \mid \mathbf{i}\rangle}} = \frac{iN}{\sqrt{N(-N)}} = \frac{iN}{iN} = 1$$

which constitutes an obvious result indicating that the unity vectors are collinear.
If conjugation is used as in the usual way scalar products in (pre-)Hilbert spaces are defined, then the scalar products become:

$$\langle \mathbf{1} \mid \mathbf{1}\rangle = N \wedge \langle \mathbf{i} \mid \mathbf{i}\rangle = N \wedge \langle \mathbf{1} \mid \mathbf{i}\rangle = iN \wedge \langle \mathbf{i} \mid \mathbf{1}\rangle = \langle \mathbf{1} \mid \mathbf{i}\rangle^* = -iN$$

providing a cosine like:

$$\cos(\theta) = \frac{|\langle \mathbf{1} \mid \mathbf{i}\rangle|}{\sqrt{\langle \mathbf{1} \mid \mathbf{1}\rangle \langle \mathbf{i} \mid \mathbf{i}\rangle}} = \frac{N}{\sqrt{NN}} = 1,$$

a similar result as in the Minkowski case but considering that the module of the numerator has been employed to deal with real cosine values, as the result of the scalar products manipulation.
Squared Euclidian distance might be useful to complete the geometrical background of the meaning of both Mersenne real and imaginary vectors.
Owing to the definition of Euclidian distances as positive definite scalars, one needs to employ the usual scalar product conjugation to compute the distance in this case, producing:

$$D^2 = \langle \mathbf{1} \mid \mathbf{1}\rangle + \langle \mathbf{i} \mid \mathbf{i}\rangle - (\langle \mathbf{1} \mid \mathbf{i}\rangle + \langle \mathbf{i} \mid \mathbf{1}\rangle) = 2N - (iN + (-i)N) = 2N.$$

Corresponding to a result reinforcing the collinearity of both unity vectors.

6. Complete sum of a vector

Suppose now an N-Dimensional vector space: $V_N(S)$, defined over some numerical set S. Then, the vectors of the space can be associated to the definitions:

$$|\mathbf{a}\rangle \in V_N(S) \rightarrow |\mathbf{a}\rangle = \{a_I \,|I = 1, N\} \rightarrow \forall I = 1, N : a_I \in S.$$

The complete sum of a vector defined within$V_N(S)$ might be defined using the symbol $\langle\ \rangle$ applied over a given vector or to a complicated set of vector operations yielding a vector. That is, the complete sum is constructed in such a way that:

$$\langle |\mathbf{a}\rangle \rangle = \sum_{I=1}^{N} a_I \rightarrow \langle |\mathbf{a}\rangle \rangle \in S.$$

The complete sum of a vector could be associated to some linear operator kind, as it is easy to prove that:

$$\forall \{|\mathbf{a}_J\rangle\} \subset V_N \wedge \{\alpha_J\} \subset S : |\mathbf{s}\rangle = \sum_J \alpha_J\,|\mathbf{a}_J\rangle \Rightarrow \langle |\mathbf{s}\rangle \rangle = \sum_J \alpha_J \langle |\mathbf{a}_J\rangle \rangle.$$

In a vector semispace, the resulting vector permutation Polyhedron, associated to any perfect vector, possess the obvious property of having the complete sum of every permutated whole vector equal to the complete sum of the perfect vector seed.
The complete sum of the vertex elements of any hollow class $\mathbf{C}_{[N_1;N_0]}$ of a Boolean Hypercube$\mathbf{H}_N$ corresponds to N_1, that is the number of digits $\{1\}$ contained in every vertex of the class.

7. Generalized scalar products and norms

Using the inward product of an indefinite number of vectors and applying the complete sum to the resultant vector, the scalar result of these two operations can be considered as a generalized scalar product.
Suppose a given Polyhedron defined over a vector space, defined in turn over some number set: $P = \{|\mathbf{a}_K\rangle\,|K = 1,M\} \subset V_N(S)$, the M-th order scalar product corresponds to an element of the reference set S, obtained by the complete sum of the inward product of the vectors of the Polyhedron. It might be defined accordingly as:

$$\langle P\rangle = \left\langle \underset{K=1}{\overset{M}{*}} |\mathbf{a}_K\rangle \right\rangle = \sum_{I=1}^{N}\left(\prod_{K=1}^{M} a_{IK}\right) \in S.$$

Such a definition permits to construct the p-th norm of a vector: $|\mathbf{a}\rangle \in V_N(S)$, using the algorithm:

$$\left\langle \left|\mathbf{a}^{[p]}\right\rangle \right\rangle = \sum_{I=1}^{N}\left(a_I^p\right);$$

that is, as the complete sum of the p-th inward power of a given vector.

8. Inward power polyhedra generation

Suppose a M-vertex Polyhedron: $P = \{|\mathbf{a}_I\rangle\,|I = 1,M\}$, constructed in a N-Dimensional Vector Space $V_N(\mathbb{R})$, defined over the real field. One can write the Polyhedron as a row vector whose elements are the vertices of the Polyhedron and this circumstance might be expressed as:

$$\langle P| = \langle|\mathbf{a}_1\rangle, |\mathbf{a}_2\rangle, \ldots |\mathbf{a}_M\rangle| = (|\mathbf{a}_1\rangle, |\mathbf{a}_2\rangle, \ldots |\mathbf{a}_M\rangle).$$

The centroid of the Polyhedron might be simply defined as:

$$|\mathbf{c}\rangle = M^{-1}\sum_I |\mathbf{a}_I\rangle.$$

With this information, a new translated Polyhedron: $T = \{|\mathbf{t}_I\rangle\,|I = 1,M\}$ can be defined as:

$$\forall I = 1,M : |\mathbf{t}_I\rangle = |\mathbf{a}_I\rangle - |\mathbf{c}\rangle;$$

so, the centroid of the translated Polyhedron becomes null:

$$|\mathbf{0}\rangle = M^{-1}\sum_I |\mathbf{t}_I\rangle$$

Inward powers of the original polyhedron vertices can be also easily defined and written as:

$$\left\langle P^{[p]}\right| = \left\langle |\mathbf{a}_1\rangle^{[p]}, |\mathbf{a}_2\rangle^{[p]}, \ldots |\mathbf{a}_M\rangle^{[p]}\right|,$$

with the implied inward powers constructed with the already defined algorithm:

$$\forall I = 1,M : |\mathbf{a}_I\rangle^{[p]} = \underset{k=1}{\overset{p}{*}} |\mathbf{a}_I\rangle = |\mathbf{a}_I\rangle * |\mathbf{a}_I\rangle * \ldots |\mathbf{a}_I\rangle = \left\{a_{JI}^p\,|J = 1,N\right\}.$$

Thus, the centroid of a general inward power Polyhedron might be written as:

$$\left|\mathbf{c}^{(p)}\right\rangle = M^{-1}\sum_I |\mathbf{a}_I\rangle^{[p]}$$

and the translated Polyhedron vertices might be described as:

$$\forall I = 1,M : \left|\mathbf{t}_I^{(p)}\right\rangle = |\mathbf{a}_I\rangle^{[p]} - \left|\mathbf{c}^{(p)}\right\rangle,$$

providing the vertices of the translated Polyhedron:

$$\left\langle T^{[p]}\right| = \left\langle \left|\mathbf{t}_1^{(p)}\right\rangle, \left|\mathbf{t}_2^{(p)}\right\rangle, \ldots \left|\mathbf{t}_M^{(p)}\right\rangle\right|.$$

Thus, in the same way as in the original Polyhedron, within the inward power translated Polyhedron it will also hold the property, consisting into that the centroids of any translated inward power Polyhedra are null:

$$|\mathbf{0}\rangle = M^{-1}\sum_I \left|\mathbf{t}_I^{(p)}\right\rangle.$$

Therefore, knowing some inward power Polyhedra sequence:

$$p = 1,2,\ldots : \left\langle P^{[p]}\right|,$$

there is a trivial matter to compute the corresponding centroid sequence:

$$p = 1,2,\ldots : \left|\mathbf{c}^{(p)}\right\rangle,$$

which can be used to build up a set of translated Polyhedra:

$$p = 1,2,\ldots : \left\langle T^{[p]}\right|,$$

whose centroids could be homogenously associated to the null vector $|\mathbf{0}\rangle$.
In this way, every member of the translated Polyhedra, contained into the inward power sequence, possess a common centroid coincident with the null vector.

8.1. Comparing vertex differences in polyhedra

Differences between Polyhedron vertices might be simply obtained with usual tools.
For example, to represent with a vector the differences between first and second order translated Polyhedra can be done in the following way. Suppose known a translated vector: $|\mathbf{t}\rangle = |\mathbf{a}\rangle - |\mathbf{c}\rangle$ and the corresponding translated second inward power of it: $\left|\mathbf{t}^{(2)}\right\rangle = |\mathbf{a}\rangle^{[2]} - \left|\mathbf{c}^{(2)}\right\rangle$. It has been dropped the vertex subindex in order the symbol can be thought as representative of any Polyhedron vertex.
Scalar products are basic to obtain by further manipulation the possible differences. Now, considering that:$|\mathbf{a}\rangle^{[2]} = |\mathbf{a}\rangle * |\mathbf{a}\rangle$ and that the scalar product can be easily expressed as the complete sum of an inward product of two vectors:

$$\langle \mathbf{t}|\mathbf{a}\rangle = \langle |\mathbf{t}\rangle * |\mathbf{a}\rangle\rangle = \sum_J t_J a_J;$$

so, it can be written:

$$\begin{aligned}
\left\langle \mathbf{t} \left| \mathbf{t}^{(2)} \right.\right\rangle &= \langle |\mathbf{a}\rangle * (|\mathbf{a}\rangle * |\mathbf{a}\rangle)\rangle - \left(\left\langle \mathbf{a} \left| \mathbf{c}^{(2)} \right.\right\rangle + \langle |\mathbf{c}\rangle * (|\mathbf{a}\rangle * |\mathbf{a}\rangle)\rangle\right) + \left\langle \mathbf{c} \left| \mathbf{c}^{(2)} \right.\right\rangle \\
&= \left\langle |\mathbf{a}\rangle^{[3]} \right\rangle - M^{-1}\sum_I \langle |\mathbf{a}\rangle * (|\mathbf{a}_I\rangle * |\mathbf{a}_I\rangle)\rangle - M^{-1}\sum_I \langle |\mathbf{a}_I\rangle * (|\mathbf{a}\rangle * |\mathbf{a}\rangle)\rangle + M^{-2}\sum_I\sum_J \langle |\mathbf{a}_I\rangle * (|\mathbf{a}_J\rangle * |\mathbf{a}_J\rangle)\rangle \\
&= \left\langle |\mathbf{a}\rangle^{[3]} \right\rangle - M^{-1}\sum_I \left\langle |\mathbf{a}\rangle * |\mathbf{a}_I\rangle^{[2]} \right\rangle - M^{-1}\sum_I \left\langle |\mathbf{a}_I\rangle * |\mathbf{a}\rangle^{[2]} \right\rangle + M^{-2}\sum_I\sum_J \left\langle |\mathbf{a}_I\rangle * |\mathbf{a}_J\rangle^{[2]} \right\rangle
\end{aligned}$$

This construction above might be considered most interesting, because the centroids of the sequence of inward powers $\left\langle P^{[p]} \right|$ are alternatively attached to a sequence of centroids $\left|\mathbf{c}^{(p)}\right\rangle$,which is far from being homogeneous.
Every term in the above equation corresponds to a scalar product of three vectors, which in general can be written as:

$$\forall |\mathbf{a}\rangle, |\mathbf{b}\rangle, |\mathbf{c}\rangle \in V_N : \langle |\mathbf{a}\rangle * |\mathbf{b}\rangle * |\mathbf{c}\rangle\rangle = \sum_{I=1}^{N} a_I b_I c_I.$$

Therefore, using the previous definitions of the preceding section, for instance:

$$\left\langle |\mathbf{a}\rangle^{[3]} \right\rangle = \sum_{I=1}^{N} a_I^3$$

might represent a third order Minkowski pseudonorm of the vector $|\mathbf{a}\rangle$.

9. Statistical-like vectors and power series of a polyhedron

The centroid inhomogeneity of powers of a Polyhedron might be instead associated to the definition of a sequence of statistical-like vectors, which can be seen, in turn, as an N-Dimensional generalization of the well-known statistical moments, which could be connected to any set of scalar values of an aleatory variable.
The first element of this Polynomial inward power sequence corresponds to the null vector, which can be associated on the other hand to the mean value vector of the translated original Polyhedron vertices. While the second inward power term is easily associated to a vector, which corresponds to the generalization of the variance. This variance-like vector might be written as the centroid difference and further simplified:

$$\left|\mathbf{m}^{(2)}\right\rangle = M^{-1}\sum_I (|\mathbf{a}_I\rangle - |\mathbf{c}\rangle)^{[2]} = \left|\mathbf{c}^{(2)}\right\rangle - |\mathbf{c}\rangle^{[2]}.$$

Higher order statistical-like moments can be written in the same way:

$$p = 1,2,\ldots : \left|\mathbf{m}^{(p)}\right\rangle = M^{-1}\sum_I (|\mathbf{a}_I\rangle - |\mathbf{c}\rangle)^{[p]} = M^{-1}\sum_{k=0}^{p} (-1)^k \binom{p}{k} \sum_I \left(|\mathbf{a}_I\rangle^{[p-k]} * |\mathbf{c}\rangle^{[k]}\right),$$

which can be simplified using a bit more involved expression:

$$p = 1,2,\ldots : \left|\mathbf{m}^{(p)}\right\rangle = \left|\mathbf{c}^{(p)}\right\rangle + \delta\,(p>2)\,M^{-1}\sum_{k=1}^{p-2} (-1)^k \binom{p}{k} \sum_I \left(|\mathbf{a}_I\rangle^{[p-k]} * |\mathbf{c}\rangle^{[k]}\right) + (-1)^{p-1}\,(p-1)\,|\mathbf{c}\rangle^{[p]}.$$

Note that, whenever the inward powers defining the statistical-like vector moments are obtained with the inward absolute power of the differences involved:

$$p = 1,2,\ldots : \left|\mathbf{d}^{(p)}\right\rangle = M^{-1}\sum_I ||\mathbf{a}_I\rangle - |\mathbf{c}\rangle|^{[p]},$$

using the inward function concept of a vector as follows:

$$\forall |\mathbf{a}\rangle = \{a_I \,|I = 1,N\} \to f\,[|\mathbf{a}\rangle] = \{f\,(a_I)\,|I = 1,N\},$$

then, the sequence of vectors: $\left\{\left|\mathbf{d}^{(p)}\right\rangle |\, p = 1,2,\ldots\right\}$ belongs to a vector semispace. The complete sum of each vector in the above sequence of statistical vector moments, describes a positive definite scalar or a p-th order Minkowski norm:

$$p = 1,2,\ldots : \left\langle \left|\mathbf{d}^{(p)}\right\rangle \right\rangle = m_p.$$

The scalar set generated in this way also can be supposedly imagined that corresponds to the collective distances of increasing order, involving all the original Polyhedron vertices. One might note that the even inward powers of both sequences are equivalent in both cases, that is:

$$p = 1,2,\ldots : \left|\mathbf{d}^{(p)}\right\rangle = \left|\mathbf{m}^{(p)}\right\rangle.$$

Such kind of vector momenta sequences cannot be defined within the translated inward power Polyhedra. They shall be considered as another inward power sequence associated just to the first order translated Polyhedron: $T = \{|\mathbf{t}_I\rangle\,|I = 1,M\}$.

$$p = 1,2,\ldots : \left\langle S^{[p]}\right| = \left\langle |\mathbf{t}_1\rangle^{[p]}, |\mathbf{t}_2\rangle^{[p]}, \ldots |\mathbf{t}_M\rangle^{[p]}\right|,$$

corresponding to an inhomogeneous centroid sequence of inward power Polyhedra. Such centroids coincide with the set of statistical-like momenta.
Computed in any way, the sequence of complete sums of the statistical-like vectors of a given Polyhedron generates a vector which might be considered as a vector defining the Polyhedron in a condensed form. In this way two or more polyhedra can be compared.

9.1. Polyhedron power series

The homogeneous centroid inward power Polyhedra, might be employed to describe a Polyhedron series which can be expressed in the simple form:

$$\langle Z| = \sum_p \theta_p \left\langle T^{[p]}\right| \rightarrow \forall I = 1,M : |\mathbf{z}_I\rangle = \sum_p \theta_p |\mathbf{t}_I\rangle^{[p]}.$$

The vertices of the Polyhedron described by the above series might be used as operators acting on the original Polyhedron vertices in such a manner that one can write the scalar products between the series Polyhedron vertices and the original Polyhedron ones:

$$\forall I = 1,M : \langle \mathbf{z}_I|\mathbf{a}_I\rangle = \pi_I \equiv \sum_p \theta_p \left({}^{[p]}\langle \mathbf{t}_I|\mathbf{a}_I\rangle\right).$$

The scalar products appearing in the right-hand side of the above expression can be taken, as it has been already commented, as generalized scalar products. To make clearer this last statement, one can write the simple scalar product by means of the complete sum of the inward product of two vectors, as discussed before. Then, according to the already described inward powers, a generalized scalar product involving several vectors can be defined just in the following way:

$${}^{[p]}\langle \mathbf{t} \mid \mathbf{a}\rangle = \left\langle |\mathbf{t}\rangle^{[p]} * |\mathbf{a}\rangle\right\rangle = \left\langle \left(\overset{p}{\underset{k=1}{*}} |\mathbf{t}\rangle\right) * |\mathbf{a}\rangle\right\rangle = \sum_J t_J^p a_J.$$

9.2. Expression of an expectation value

Therefore, the equality:

$$\forall I = 1,M : \pi_I \equiv \sum_p \theta_p \left({}^{[p]}\langle \mathbf{t}_I|\mathbf{a}_I\rangle\right) = \sum_p \theta_p Z_{Ip}$$

might be transformed easily into a linear matrix equation:

$$|\pi\rangle = \mathbf{Z}\,|\theta\rangle \tag{9.1}$$

with obvious definitions:

$$|\pi\rangle = \{\pi_I\}; |\theta\rangle = \{\theta_p\}; \mathbf{Z} = \{Z_{Ip}\}.$$

If the $|\pi\rangle$ vector components are well-defined and known, then the components of the coefficient vector $|\theta\rangle$ might be computed if there are some sound solutions of the associated equation (9.1).

9.3. Exponential approximation

Another possibility might be found using directly an inward exponential function with the translated original Polynomial vertices as variable:

$$\forall I = 1,M : |\mathbf{z}_I\rangle = \exp(-\alpha\,|\mathbf{t}_I\rangle) = \sum_{p=0}^{\infty} \frac{(-\alpha)^p}{p!} |\mathbf{t}_I\rangle^{[p]} = |\mathbf{1}\rangle - \alpha\,|\mathbf{t}_I\rangle + \frac{\alpha^2}{2}|\mathbf{t}_I\rangle^{[2]} + O(3) \tag{9.2}$$

the exponential parameter value might be obtained as follows. Using the exponential vectors defined in equation (9.2). operator-like, then:

$$\forall I = 1,M : \langle \mathbf{z}_I \mid \mathbf{a}_I\rangle = \pi_I \approx \langle \mathbf{1} \mid \mathbf{a}_I\rangle - \alpha\,\langle \mathbf{t}_I \mid \mathbf{a}_I\rangle + \frac{\alpha^2}{2}\left({}^{[2]}\langle \mathbf{t}_I \mid \mathbf{a}_I\rangle\right) + O(3) \tag{9.3}$$

therefore, if it exists, the exponential parameter α might be computed by solving:

$$\forall I = 1,M : \langle \mathbf{1}|\mathbf{a}_I\rangle - \pi_I - \alpha\,\langle \mathbf{t}_I|\mathbf{a}_I\rangle + \frac{\alpha^2}{2}\left({}^{[2]}\langle \mathbf{t}_I|\mathbf{a}_I\rangle\right) = 0$$

which can be written in general, dropping the vertex indices, like:

$$\tau_0 - \pi - \tau_1\alpha + \frac{\tau_2}{2}\alpha^2 = 0 \rightarrow \alpha = \frac{1}{\tau_2}\left(\tau_1 \pm \sqrt{(\tau_1)^2 + 2\tau_2(\pi - \tau_0)}\right).$$

Of course, taking more terms in equation (9.3) the problem corresponds to obtain the roots of a higher order polynomial in α. It must be noted too that the successive inward powers of the translated vertices correspond to the original Polynomial non-averaged moments.

10. Linear independence of inward powers of a whole vector

Another question might be proposed as follows. Are two inward powers of the same vector linearly independent? For instance: $|\mathbf{a}\rangle$ and $|\mathbf{a}\rangle^{[2]}$ are in general linearly independent? To test this issue it is easy, because if they are linearly independent the equation:

$$\alpha\,|\mathbf{a}\rangle+\beta\,|\mathbf{a}\rangle^{[2]}=|\mathbf{0}\rangle$$

implies that $\alpha=\beta=0$. Therefore, to test the truth of this double equality, the previous equation can be rewritten as:

$$\alpha\,|\mathbf{a}\rangle+\beta\,(|\mathbf{a}\rangle*|\mathbf{a}\rangle)=|\mathbf{0}\rangle$$

and furthermore, the equality can be rearranged owing to the nature of the inward product and the existence of the unity vector: $|\mathbf{1}\rangle=\{1_I=1\,|I=1,N\}$, which acts as neutral element of the inward product, besides taking: $|\mathbf{a}\rangle\neq|\mathbf{0}\rangle$, or assuming it is a perfect or whole vector, then one can write:

$$|\mathbf{a}\rangle*(\alpha\,|\mathbf{1}\rangle+\beta\,|\mathbf{a}\rangle)=|\mathbf{0}\rangle\rightarrow\alpha\,|\mathbf{1}\rangle+\beta\,|\mathbf{a}\rangle=|\mathbf{0}\rangle\rightarrow\alpha\,|\mathbf{1}\rangle=-\beta\,|\mathbf{a}\rangle\,,$$

thus, whenever $|\mathbf{a}\rangle\neq\lambda\,|\mathbf{1}\rangle$, one arrives to the fact that necessarily: $\alpha=\beta=0$, and then this implies the linear independence of the two inward powers.
This result can be easily generalized writing:

$$p<q:\alpha|\mathbf{a}\rangle^{[p]}+\beta|\mathbf{a}\rangle^{[q]}=|\mathbf{0}\rangle\rightarrow|\mathbf{a}\rangle^{[p]}*\left(\alpha\,|\mathbf{1}\rangle+\beta\,|\mathbf{a}\rangle^{[q-p]}\right)=|\mathbf{0}\rangle\,.$$

Thus, a similar argument as before can be employed, as if:

$$|\mathbf{a}\rangle\neq\lambda\,|\mathbf{1}\rangle\rightarrow|\mathbf{a}\rangle^{[q-p]}\neq\lambda\,|\mathbf{1}\rangle\,.$$

If any pair of powers of a whole vector are linearly independent, any Polyhedron made of powers of a unique vector can furnish linearly independent generators, basis sets of any given vector space.
However, there appears just here a paradox, because if the successive inward powers, which can be written as: $\left\{|\mathbf{a}\rangle^{[p]}\,|p=1,2,...n,...\right\}$, of a non-unity vector are linearly independent, then in case the vectors belong to some vector space V_N, just a limited number N of them can be taken together as a linearly independent set generating the space, a basis set, and it can be written, say: $\left\{|\mathbf{a}\rangle^{[p]}\,|p=1,2,...N\right\}$ as a linearly independent set. Even one can think about using a first term in the series:$|\mathbf{a}\rangle^{[0]}=|\mathbf{1}\rangle$ in case of need or fancy, but then of course, the linearly independent set will be written as:

$$\left\{|\mathbf{a}\rangle^{[p]}\,|p=0,2,...,N-1\right\}.$$

This situation might be easily expressed in general, for instance, using:

$$\forall q>N,\exists\left\{\alpha_p^{(q)}\,|p=1,N\right\}:\sum_{p=1}^{N}\alpha_p^{(q)}|\mathbf{a}\rangle^{[p]}=|\mathbf{a}\rangle^{[q]}$$

If the seed vector of any Power Polyhedron, like the one previously defined, is a whole or perfect vector, then inverse powers are also allowed to construct such classes of Polyhedra.
When dealing with vector spaces V_N, of dimension N, then just a limited number N of linearly independent vectors form a basis set. This means that any power of a given vector can be constructed with the N chosen power vectors of the basis set.
This means that, for instance, in 2-dimensional spaces the whole vectors: $|\mathbf{a}\rangle$ and $|\mathbf{a}\rangle^{[2]}$ can generate any inward power of $|\mathbf{a}\rangle$, including the Mersenne vertex. This can be set up like the set of equivalent equalities:

$$|\mathbf{1}\rangle=\alpha\,|\mathbf{a}\rangle+\beta|\mathbf{a}\rangle^{[2]}\rightarrow|\mathbf{1}\rangle=|\mathbf{a}\rangle*(\alpha\,|\mathbf{1}\rangle+\beta\,|\mathbf{a}\rangle)\rightarrow|\mathbf{a}\rangle^{[-1]}=\alpha\,|\mathbf{1}\rangle+\beta\,|\mathbf{a}\rangle\,,$$

meaning that:

$$\left.\begin{array}{l}a_1^{-1}=\alpha+\beta a_1\\ a_2^{-1}=\alpha+\beta a_2\end{array}\right\}\Rightarrow\left.\begin{array}{c}\frac{a_1^{-1}-a_2^{-1}}{a_1-a_2}=\beta\\ \frac{1}{2}\left[\left(a_1^{-1}+a_2^{-1}\right)-\beta\,(a_1+a_2)\right]=\alpha\end{array}\right\}$$

Moreover, such a composition might be done with any pair of powers of the vector $|\mathbf{a}\rangle$. Of course, any vector of the 2-dimensional space might be generated with any pair of powers of the vector.
The following statements might be considered. Any perfect or whole vector $|\mathbf{a}\rangle$ of any N-dimensional space V_N can be used to generate a basis set, based uniquely on the vector $|\mathbf{a}\rangle$. A basis set of V_N can be constructed using N arbitrary but different inward powers of the perfect or whole vector $|\mathbf{a}\rangle$. That is the same to say that, considering some vector space V_N, then any Polyhedron made by N distinct inward powers of an arbitrary whole vector, corresponds to a basis set of V_N.

11. Conclusions

Boolean Hypercubes might schematically contain the structure of any vector space. The concept of model or seed vertices of the Hypercube permit to construct vertex classes, which might have a similar reference in vector spaces. Perfect vectors can be considered the true elements of any vector space from where $N!$ whole (that is: not bearing null elements) vectors can be deduced. While hollow (that is: bearing null elements) vectors might be considered vectors of spaces of lesser dimensions. Signature Hypercubes can be employed via inward products as possible sign generators, which can be attached in turn to any whole vector. Such construct can be employed in many instances, like the construction of Minkowski spaces. The Polyhedron obtained by the inward powers of a whole vector, can be used to construct a basis set of the attached vector space. Inward products involving vectors and the complete sum of a vector might be employed in vector spaces to describe generalized scalar products, hence also vector norms. Statistical-like vectors of Polyhedra contained in a vector space can be easily defined. They can lead, via their complete sum, to a set of scalars, which ordered as a vector of arbitrary dimensions can resume in a condensed manner the attached Polyhedron. From these considerations, a new look into vector spaces emerges. Application of the developed theoretical frame might have powerful applications into quantum QSPR and beyond, that is: in the characterization and comparison of molecular sets represented, as it is customary, in the form of finite dimensional vectors.

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A new class activation functions with application in the theory of impulse techniques

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Abstract

In this note we define the new activation functions, based on the well-known hyperbolic tangent and half–hyperbolic tangent activation functions. We consider the Hausdorff distance between the "double step" function $\sigma^*(t)$ (resp. function $\sigma^{**}(t)$) and the new classes of activation functions. The results have independent significance in the study of issues related to neural networks and impulse techniques. Numerical examples, illustrating our results are presented using programming environment Mathematica.

1. Introduction

The typical examples of impulse functions from antenna feeder technique has the following shape (see, Figures 1.1-1.2):

$$\sigma^*(t)=\begin{cases}1, t\in[1,+\infty)\\ 0, t\in[-1,1)\\ -1, t\in(-\infty,-1)\end{cases} \tag{1.1}$$

$$\sigma^{**}(t)=\begin{cases}1, t\in[-\infty,-1)\cup(1,+\infty)\\ 0, t\in[-1,1].\end{cases} \tag{1.2}$$

In [3] the following basic problems are considered – approximation of functions and point sets by algebraic and trigonometric polynomials in Hausdorff metric [2] as well as their applications in the field of antenna-feeder technique, analysis and synthesis of antenna patterns and filters, noise minimization by suitable approximation of impulse functions.
The polynomial Hausdorff approximations of the signals of type (1.1)-(1.2) are visualized on Figure 1.3-1.4.
These polynomials play an important role in approximation of antenna factor for scanning of directed chart.
Evidently the task of great difficulty is to determine the coefficients of the polynomials and values of the best Hausdorff approximation.
For other results, see [4]–[14], [16].

2. Main results

Definition 2.1. *[1], [2] The Hausdorff distance (the H–distance) [1] $\rho(f,g)$ between two interval functions f,g on $\Omega\subseteq\mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega\times\mathbb{R}$. More precisely,*

$$\rho(f,g)=\max\{\sup_{A\in F(f)}\inf_{B\in F(g)}\|A-B\|,\ \sup_{B\in F(g)}\inf_{A\in F(f)}\|A-B\|\}, \tag{2.1}$$

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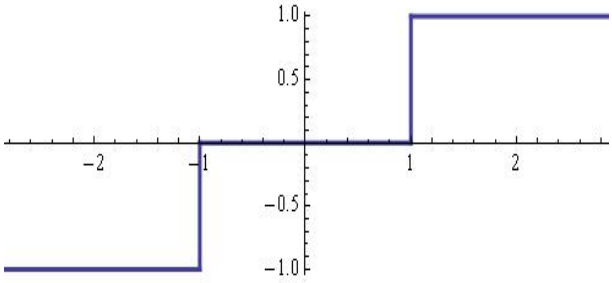

Figure 1.1: The signal of "double step" $\sigma^*(t)$ – type.

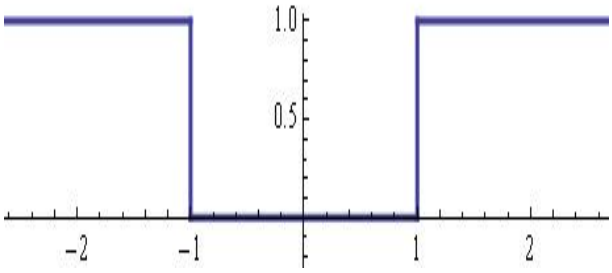

Figure 1.2: The signal of $\sigma^{**}(t)$ – type.

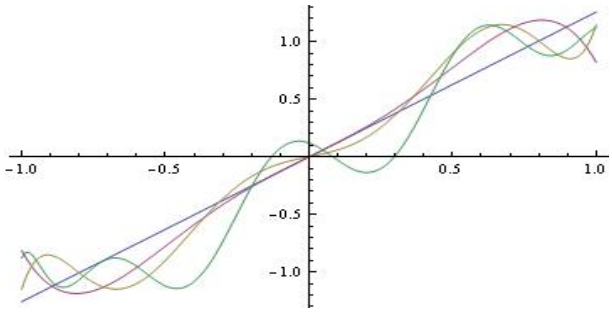

Figure 1.3: The polynomial approximation of the $\sigma^*(t)$ – type function.

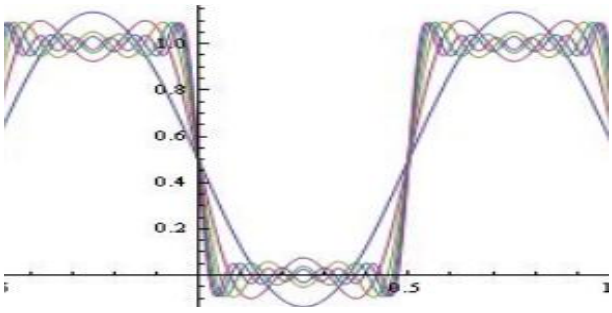

Figure 1.4: The polynomial approximation of the $\sigma^{**}(t)$ – type function.

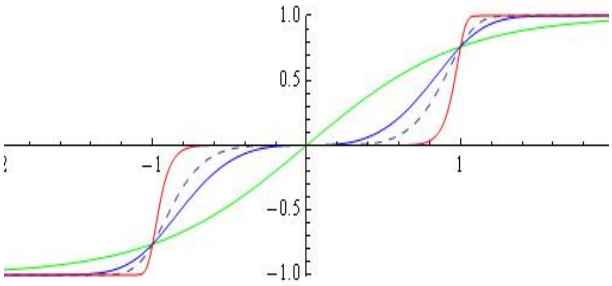

Figure 2.1: The family of activation functions $\varphi^*(t)$ for $m = 1$ (green); $m = 3$ (blue); $m = 5$ (dashed); $m = 15$ (red); ($\beta = 1$ is fixed).

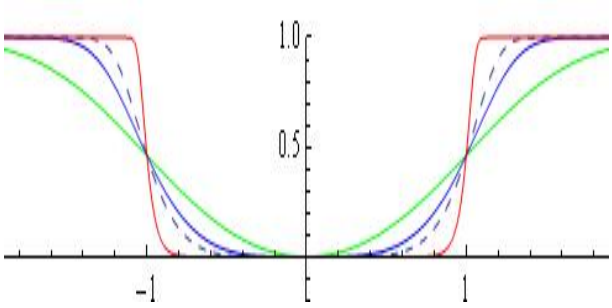

Figure 2.2: The family of activation functions $\varphi^{**}(t)$ for $m = 2$ (green); $m = 4$ (blue); $m = 6$ (dashed); $m = 20$ (red); ($\beta = 1$ is fixed).

wherein $\|.\|$ *is any norm in* $\mathbb{R}^2$*, e. g. the maximum norm* $\|(t,x)\| = \max\{|t|,|x|\}$*; hence the distance between the points* $A = (t_A, x_A)$*,* $B = (t_B, x_B)$ *in* $\mathbb{R}^2$ *is* $\|A-B\| = max(|t_A - t_B|, |x_A - x_B|)$.

Definition 2.2. *Define the following "new activation function" as*

$$\varphi^*(t) = 1 - \frac{2e^{-\beta t^m}}{e^{\beta t^m} + e^{-\beta t^m}}, \tag{2.2}$$

where m is an odd parameter.

Definition 2.3. *Define the following "new half–activation function as:*

$$\varphi^{**}(t) = \frac{1 - e^{-\beta t^m}}{1 + e^{-\beta t^m}}, \tag{2.3}$$

where m is an even parameter.

The Hausdorff distance $d = d(\varphi^*(t), \sigma^*(t)$ between the activation function $\varphi^*(t)$ and the function $\sigma^*(t)$ can be calculating by solving nonlinear equation:

$$\varphi^*(1-d) = d. \tag{2.4}$$

Analogously, for the Hausdorff distance $d = d(\varphi^{**}(t), \sigma^{**}(t)$ between the activation function $\varphi^{**}(t)$ and the function $\sigma^{**}(t)$ we have

$$\varphi^{**}(1-d) = d. \tag{2.5}$$

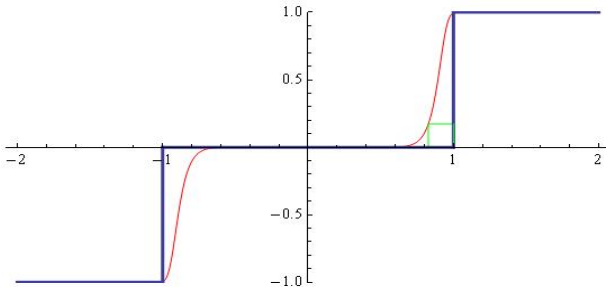

Figure 2.3: The activation functions $\varphi^*(t)$ for $\beta = 3$; $m = 15$; Hausdorff distance $d = 0.172738$.

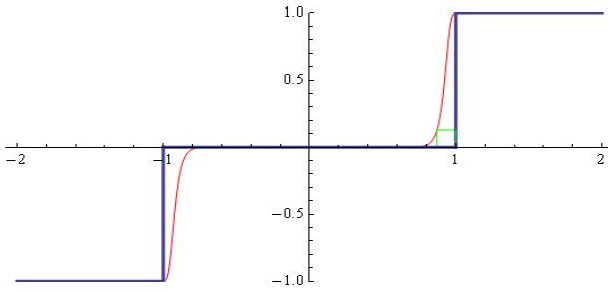

Figure 2.4: The activation functions $\varphi^*(t)$ for $\beta = 4$; $m = 25$; Hausdorff distance $d = 0.128338$.

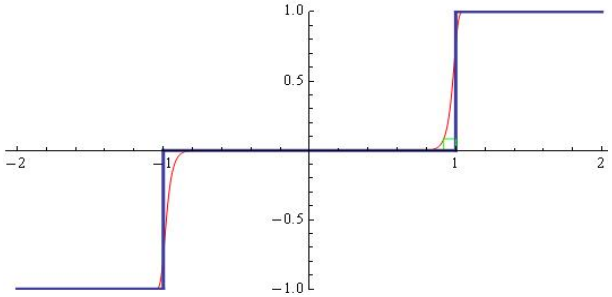

Figure 2.5: The activation functions $\varphi^*(t)$ for $\beta = 1$; $m = 29$; Hausdorff distance $d = 0.0824019$.

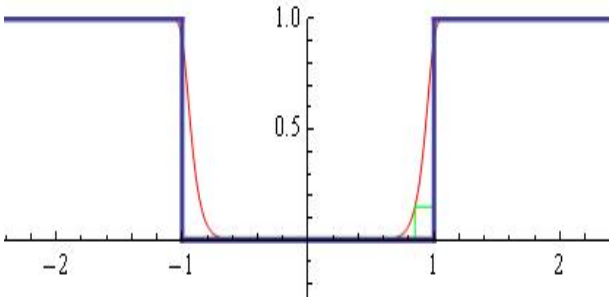

Figure 2.6: The activation functions $\varphi^{**}(t)$ for $\beta = 3$; $m = 14$; Hausdorff distance $d = 0.15085$.

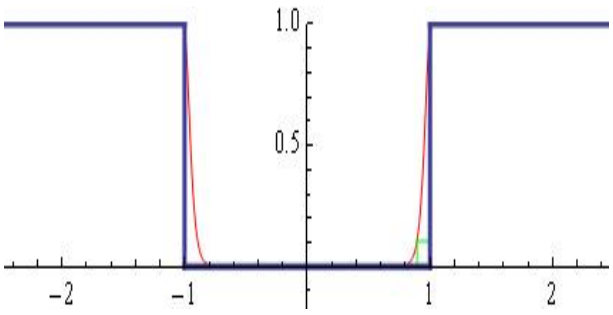

Figure 2.7: The activation functions $\varphi^{**}(t)$ for $\beta = 4$; $m = 26$; Hausdorff distance $d = 0.106529$.

```
φ1[θ_] := (1 - Exp[-β * (Pi * Cos[θ] + a)^m]) / (1 + Exp[-β * (Pi * Cos[θ] + a)^m])
Manipulate[PolarPlot[(1 - Exp[-β * (Pi * Cos[θ] + a)^m]) / (1 + Exp[-β * (Pi * Cos[θ] + a)^m]) ,
  {θ, -2 Pi, 2 Pi}], {β, 0.05, 10., Appearance → "Open"}, {m, 1, 30, Appearance → "Open"},
 {a, 0.1, 2 * Pi, Appearance → "Open"}]
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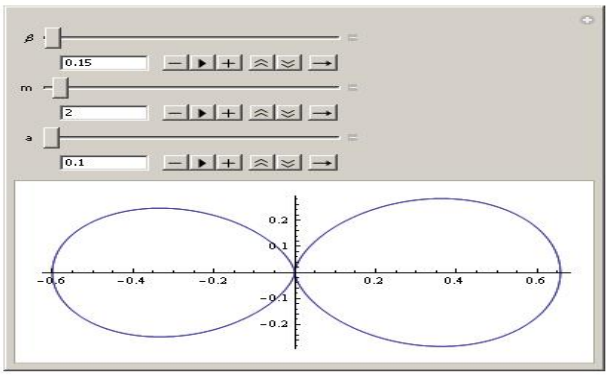

Figure 2.8: Typical emitting chart $(\varphi^{**}(\theta))$ for $\beta = 0.15$; $a = 0.1$; $m = 2$.

```
φ1[θ_] := (1 - Exp[-β * (Pi * Cos[θ] + a)^m]) / (1 + Exp[-β * (Pi * Cos[θ] + a)^m])
Manipulate[PolarPlot[(1 - Exp[-β * (Pi * Cos[θ] + a)^m]) / (1 + Exp[-β * (Pi * Cos[θ] + a)^m]) ,
  {θ, -2 Pi, 2 Pi}], {β, 0.05, 10., Appearance → "Open"}, {m, 1, 30, Appearance → "Open"},
 {a, 0.1, 2 * Pi, Appearance → "Open"}]
```

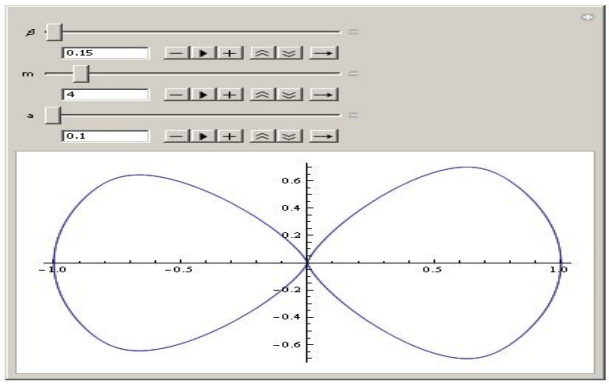

Figure 2.9: Typical emitting chart $(\varphi^{**}(\theta))$ for $\beta = 0.15$; $a = 0.1$; $m = 4$.

Some families of activation functions $\varphi^{*}(t)$ and $\varphi^{**}(t)$ are visualized on Figures 2.1-2.7.
After the substitution $t = kl\cos\theta + a$, where
– $k = \frac{2\pi}{\lambda}$, λ is the wave length;
– a is the phase difference;
– θ is the azimuthal angle;
– l is the distance between the emitters ($l = \frac{\lambda}{2}$ is fixed),
the activation function $\varphi^{**}(t)$ (or emitting chart of antenna factor can be written in the form

$$\varphi^{**}(\theta) = \frac{1 - e^{-\beta(\pi\cos\theta + a)^m}}{1 + e^{-\beta(\pi\theta + a)^m}}. \tag{2.6}$$

Typical emitting chart are visualized on Figures 2.8-2.9.
If $l = \lambda$ we have the chart - Figure 2.10.
Of course, the question of the practical realization of the activation functions which are generated as emitting charts remains open.
The mathematical apparatus proposed in the article can be successfully used for imitation and simulation of such charts.
I will explicitly say that the results have independent significance in the study of issues related to impulse technics [3], [15] and neural networks (see, [17]–[21]).
Remarks.

For the special case $m = 1$ we have the following
Theorem [16]. For the Hausdorff distance d between the *sgn* function and the half–hyperbolic function the following inequalities hold for $\beta \geq 5$:

$$d_l = \frac{1}{0.5(2+\beta)} < d < \frac{\ln(0.5(2+\beta))}{0.5(2+\beta)} = d_r.$$

Following the ideas given in [16], the reader may formulate the corresponding approximation problem for each number m.

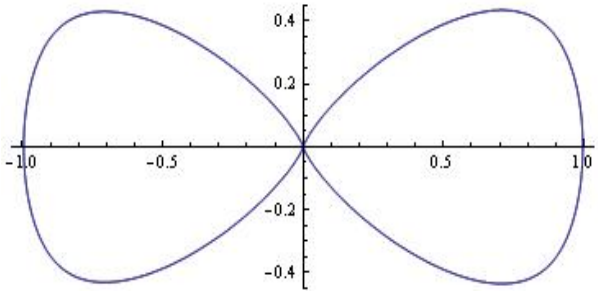

Figure 2.10: The emitting chart for $\beta = 0.0001$; $a = 0.01$; $m = 6$; $l = \lambda$.

3. Acknowledgements

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Availability analysis of a consecutive three stages deteriorating standby system considering maintenance and replacement

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Abstract

This paper deals with the modelling and evaluation of availability of a system subject to three consecutive stages of deterioration: minor, medium and major deteriorations under minor and major maintenance, and replacement at deterioration and failure respectively. The system has three possible modes: working with full capacity, deterioration and failure mode. In this paper, probabilistic models have been developed to evaluate the relationship between availability and the performance of a standby deteriorating system. Various graphs have been plotted to discover the impact of the deterioration and failure on steady-state availability. The system is analysed using first order linear differential equations.

1. Introduction

The process industry comprises of large complex engineering systems, subsystems arranged in standby, series, parallel or a combination of them. For efficient and economical operation of a process plant, each system or the subsystem should work failure free under the existing operative plant conditions. However, once the system starts, potential fault conditions gradually accumulate leading to degraded performance. The study of a cold standby repairable deteriorating system reliability model is one important model application of reliability theory. In practical engineering applications, most repairable systems are deteriorative that system failure often cannot be as good as new, it is more reasonable for these deteriorating repairable systems to assume that the successive working times of the system after repair will become shorter and shorter while the consecutive repair times of the system after failure will become longer and longer. Most of these systems are subjected to random deterioration which can result in unexpected failures and disastrous effect on the system availability and the prospect of the economy. Therefore it is important to find a way to slow down the deterioration rate, and to prolong the equipment's life span. Maintenance policies are vital in the analysis of deterioration and deteriorating systems as they help in improving reliability and availability of the systems. Maintenance models can assume minor maintenance, major maintenance before system failure, perfect repair (as good as new), minimal repair (as bad as old), imperfect repair and replacement at system failure. For repairable system, availability is a very meaningful measure, and achieving a high or required level of availability is an essential requisite. Improving the availability of system, the production and associated profit will also increase. Increase in production lead to the increase of profit. This can be achieve be maintaining reliability and availability at highest order. To achieve high production and profit, the system should remain operative (availability) for maximum possible duration.

The concept of deterioration and its effect on the reliability and availability model of a repairable system is vital in the study of system performance. For this reason, several models on deteriorating systems under different conditions have been studied by several researchers such as [1], [2], [3], [5], [7], [9], [13], [14]. Analysis of reliability and availability model for deteriorating system have been studied under different conditions such as [4] who investigated reliability analysis of a deteriorating system with delayed vacation of repairman, [8]. A Reliability-based Opportunistic Predictive Maintenance Model for k-out-of-n Deteriorating Systems, [12] proposed the Bayesian reliability estimation for deteriorating systems with limited samples using the maximum entropy approach, [15]. Modelling the reliability and availability characteristics of a system with three stages of deterioration, [16] deal with the study of deteriorating cold standby repairable system with priority in use.

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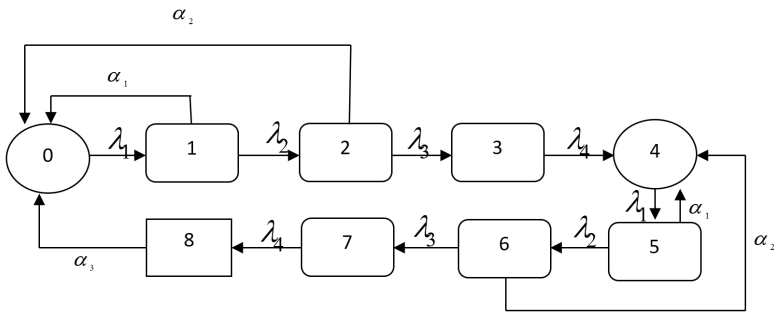

Figure 2.1: Transition diagram of the system with maintenance and replacement

State	Description
S_0	Initial state, one is working, the other unit is on standby. The system is working.
S_1	The working unit is in minor deterioration mode and is under online minor maintenance, the other unit is on standby. The system is working.
S_2	The working unit is in medium deterioration mode and is under online major maintenance, the other unit is on standby. The system is working.
S_3	The working unit is in major deterioration mode. The system is working.
S_4	The working unit has failed; the standby unit is switch to operation. The system is working.
S_5	One unit has failed; the working unit is in minor deterioration mode and is under online minor maintenance. The system is working.
S_6	One unit has failed: the working unit is in medium deterioration mode and is under online major maintenance. The system is working.
S_7	One unit has failed: the working unit is in major deterioration mode. The system is working.
S_8	Both units have failed. The system is inoperative and replaced with new one.

This paper considers a two unit cold standby system with three consecutive stages of deterioration before failure and derived its corresponding mathematical models. Furthermore, we study availability of the system using Kolmogorov's forward equation method. The focus of our analysis is primarily to capture the effect of minor, medium and major deterioration, failure and replacement rates on the availability.

The organization of the paper is as follows. Section 2 contains a description of the system under study. Section 3 presents formulations of the models. The results of our numerical simulations are presented in section 4. Finally, we make some concluding remarks in Section 5.

2. Description and States of the System

In this paper, two unit cold standby system is considered. It is assumed that the system most pass through consecutive stages of deterioration which are minor, medium and major deterioration before failure. It is also assumed that switching is perfect, e.g. never fails and never does any deterioration. Primary units are considered to be repairable. At early state of the system life, the operating unit is exposed to minor deterioration with rate λ_1 and this deterioration is rectified through minor maintenance α_1 which revert the unit to its earliest position before deterioration. If not maintained, the unit is allowed to continue operating under the condition of minor deterioration which later grows to medium deterioration with rate λ_1 . At this stage, the strength of the unit still strong that it can reverted to early state with major maintenance with rate α_2. However, the system can move to major deterioration stage with rate λ_3 where the strength of the unit has decreases to the extent that it cannot be reverted to its early state, neither that of minor nor medium deterioration stages. Here the unit is allowed to continue operation until it fails. Only one primary unit can be served at a time. Each of the primary units fails independently of the state of the others with parameter λ_4. Whenever one of these units fails; the standby unit is switched to operation and the three stages of deterioration which continue until the unit fails and the system is immediately replaced by with a new one with rate α_3.

3. Formulation of the Models

In order to analyse the system availability of the system, we define to be the probability that the system at is in state . Also let be the row vector of these probabilities at time . The initial condition for this problem is:

$$P(0) = [p_0(0), p_1(0), p_2(0), \ldots, p_8(0)] = [1,0,0,0,0,0,0,0,0]$$

We obtain the following differential equations:

$$
\begin{aligned}
p_0'(t) &= -\lambda_1 p_0(t) + \alpha_1 p_1(t) + \alpha_2 p_2(t) + \alpha_3 p_8(t) \\
p_1'(t) &= -(\alpha_1 + \lambda_2) p_1(t) + \lambda_1 p_0(t) \\
p_2'(t) &= -(\alpha_2 + \lambda_3) p_2(t) + \lambda_2 p_1(t) \\
p_3'(t) &= -\lambda_4 p_3(t) + \lambda_3 p_2(t) \\
p_4'(t) &= -\lambda_1 p_4(t) + \lambda_4 p_3(t) + \alpha_1 p_5(t) + \alpha_2 p_6(t) \\
p_5'(t) &= -(\alpha_1 + \lambda_2) p_5(t) + \lambda_1 p_4(t) \\
p_6'(t) &= -(\alpha_2 + \lambda_3) p_6(t) + \lambda_2 p_5(t) \\
p_7'(t) &= -\lambda_4 p_7(t) + \lambda_3 p_6(t) \\
p_8'(t) &= -\alpha_3 p_8(t) + \lambda_4 p_7(t)
\end{aligned}
\tag{3.1}
$$

This can be written in the matrix form as

$$
\dot{P} = MP \tag{3.2}
$$

where

$$
M = \begin{pmatrix}
-\lambda_1 & \alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & \alpha_3 \\
\lambda_1 & -(\alpha_1+\lambda_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & -(\alpha_2+\lambda_3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & -\lambda_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & -\lambda_1 & \alpha_1 & \alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1 & -(\alpha_1+\lambda_2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2 & -(\alpha_2+\lambda_3) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & -\lambda_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & -\alpha_3
\end{pmatrix}
$$

equation (3.2) is expressed explicitly in the form

$$
\begin{pmatrix}
p_0'(t) \\ p_1'(t) \\ p_2'(t) \\ p_3'(t) \\ p_4'(t) \\ p_5'(t) \\ p_6'(t) \\ p_7'(t) \\ p_8'(t)
\end{pmatrix}
=
\begin{pmatrix}
-\lambda_1 & \alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & \alpha_3 \\
\lambda_1 & -(\alpha_1+\lambda_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & -(\alpha_2+\lambda_3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & -\lambda_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & -\lambda_1 & \alpha_1 & \alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1 & -(\alpha_1+\lambda_2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2 & -(\alpha_2+\lambda_3) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & -\lambda_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & -\alpha_3
\end{pmatrix}
\begin{pmatrix}
p_0(t) \\ p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \\ p_5(t) \\ p_6(t) \\ p_7(t) \\ p_8(t)
\end{pmatrix}
$$

The steady-state availability (the proportion of time the system is in a functioning condition or equivalently, the sum of the probabilities of operational states) is given by

$$
A_{V1}(\infty) = p_0(\infty) + p_1(\infty) + p_2(\infty) + p_3(\infty) \tag{3.3}
$$

In the steady state, the derivatives of the state probabilities become zero and therefore equation (3.2) becomes

$$
MP = 0 \tag{3.4}
$$

this is in matrix form

$$
\begin{pmatrix}
-\lambda_1 & \alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & \alpha_3 \\
\lambda_1 & -(\alpha_1+\lambda_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & -(\alpha_2+\lambda_3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & -\lambda_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & -\lambda_1 & \alpha_1 & \alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1 & -(\alpha_1+\lambda_2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2 & -(\alpha_2+\lambda_3) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & -\lambda_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & -\alpha_3
\end{pmatrix}
\begin{pmatrix}
p_0(t) \\ p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \\ p_5(t) \\ p_6(t) \\ p_7(t) \\ p_8(t)
\end{pmatrix}
=
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
$$

Subject to following normalizing conditions:

$$
p_0(\infty) + p_1(\infty) + p_2(\infty) + p_3(\infty) + p_4(\infty) + \ldots + p_8(\infty) = 1 \tag{3.5}
$$

Following [10] and [11] we substitute equation (3.5) in the last row of equation (3.4) to compute the steady-state probabilities.

$$
\begin{pmatrix}
-\lambda_1 & \alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & \alpha_3 \\
\lambda_1 & -(\alpha_1+\lambda_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & -(\alpha_2+\lambda_3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & -\lambda_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & -\lambda_1 & \alpha_1 & \alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1 & -(\alpha_1+\lambda_2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_2 & -(\alpha_2+\lambda_3) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & -\lambda_4 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
p_0(\infty) \\ p_1(\infty) \\ p_2(\infty) \\ p_3(\infty) \\ p_4(\infty) \\ p_5(\infty) \\ p_6(\infty) \\ p_7(\infty) \\ p_8(\infty)
\end{pmatrix}
=
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1
\end{pmatrix}
$$

Solving equation (3.4), we obtain the steady-state probabilities

$$p_0(\infty), p_1(\infty), p_2(\infty), p_3(\infty), \ldots, p_8(\infty)$$

The expressions for the steady-state availability involving minor and major maintenance and replacement given in equations (3.3) above is given by

$$A_{V1}(\infty) = \frac{2\alpha_3\lambda_4(\alpha_2\lambda_2+\lambda_2\lambda_3+\alpha_1\lambda_3+\alpha_1\alpha_2)+2\alpha_3\lambda_1\lambda_4(\alpha_2+\lambda_3)+2\alpha_3\lambda_1\lambda_2\lambda_4+2\alpha_3\lambda_1\lambda_2\lambda_3}{\lambda_1\lambda_2\lambda_3\lambda_4+2\alpha_3\lambda_1\lambda_2\lambda_3+2\alpha_3\lambda_1\lambda_2\lambda_4+2\alpha_3\lambda_1\lambda_3\lambda_4+2\alpha_2\alpha_3\lambda_1\lambda_4+2\alpha_3\lambda_2\lambda_3\lambda_4+2\alpha_2\alpha_3\lambda_2\lambda_4+2\alpha_1\alpha_3\lambda_3\lambda_4+2\alpha_1\alpha_2\alpha_3\lambda_4}$$

Special case:

(i) System with replacement only at failure

$$A_{V2}(\infty) = \frac{\alpha_3\lambda_2\lambda_3\lambda_4+2\alpha_3\lambda_1\lambda_3\lambda_4+2\alpha_3\lambda_1\lambda_2\lambda_4+2\alpha_3\lambda_1\lambda_2\lambda_3}{\lambda_1\lambda_2\lambda_3\lambda_4+2\alpha_3\lambda_1\lambda_2\lambda_3+2\alpha_3\lambda_1\lambda_2\lambda_4+2\alpha_3\lambda_1\lambda_3\lambda_4+2\alpha_3\lambda_2\lambda_3\lambda_4}$$

(ii) System with minor maintenance and replacement

$$A_{V3}(\infty) = \frac{2\alpha_3\lambda_3\lambda_4(\alpha_1+\lambda_2)+2\alpha_3\lambda_1\lambda_2\lambda_4+2\alpha_3\lambda_1\lambda_2\lambda_3+2\alpha_3\lambda_1\lambda_3\lambda_4}{\lambda_1\lambda_2\lambda_3\lambda_4+2\alpha_3\lambda_1\lambda_2\lambda_3+2\alpha_3\lambda_1\lambda_2\lambda_4+2\alpha_3\lambda_1\lambda_3\lambda_4+2\alpha_3\lambda_2\lambda_3\lambda_4+2\alpha_3\lambda_2\lambda_3\lambda_4+2\alpha_1\alpha_3\lambda_3\lambda_4}$$

(iii) System with major maintenance and replacement

$$A_{V4}(\infty) = \frac{2\alpha_3\lambda_1\lambda_4(\alpha_2+\lambda_3)+2\alpha_3\lambda_2\lambda_4(\alpha_2+\lambda_3)+2\alpha_3\lambda_1\lambda_2\lambda_4+2\alpha_3\lambda_1\lambda_2\lambda_3}{\lambda_1\lambda_2\lambda_3\lambda_4+2\alpha_3\lambda_1\lambda_2\lambda_3+2\alpha_3\lambda_1\lambda_2\lambda_4+2\alpha_3\lambda_1\lambda_3\lambda_4+2\alpha_2\alpha_3\lambda_1\lambda_4+2\alpha_3\lambda_2\lambda_3\lambda_4+2\alpha_2\alpha_3\lambda_2\lambda_4}$$

4. Numerical Examples

Numerical examples are presented to demonstrate the impact of deterioration, failure and replacement rates on steady-state availability based on given values of the parameters. For the purpose of numerical example, the following sets of parameter values are used:

Case 1: We fix $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $\alpha_3 = 0.04$, $\lambda_2 = 0.5$, $\lambda_3 = 0.9$, $\lambda_4 = 0.2$ and vary λ_1 between 0 and 1 as in Figure 4.1.
Case 2: We fix $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $\alpha_3 = 0.04$, $\lambda_1 = 0.1$, $\lambda_3 = 0.09$, $\lambda_4 = 0.2$ and vary λ_2 between 0 and 1 as in Figure 4.2.
Case 3: We fix $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $\alpha_3 = 0.04$, $\lambda_1 = 0.1$, $\lambda_2 = 0.5$, $\lambda_4 = 0.2$ and vary λ_3 between 0 and 1 as in Figure 4.3.
Case 4: We fix $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $\alpha_3 = 0.04$, $\lambda_1 = 0.1$, $\lambda_2 = 0.5$, $\lambda_3 = 0.9$ and vary λ_4 between 0 and 1 as in Figure 4.4.
Case 5: We fix $\alpha_1 = 0.93$, $\alpha_2 = 0.43$, $\lambda_1 = 0.6$, $\lambda_2 = 0.5$, $\lambda_3 = 0.39$, $\lambda_4 = 0.42$ and vary α_3 between 0 and 1 as in Figure 4.5.

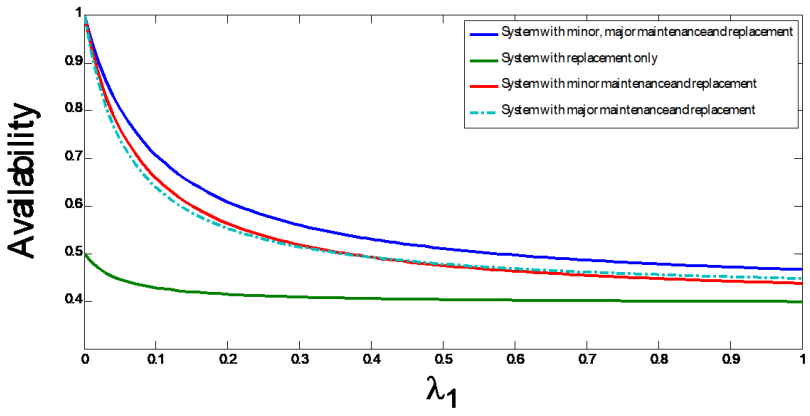

Figure 4.1: Relation between availability and minor deterioration

The results which compare the steady state availability with respect to λ_1 (minor deterioration rate), λ_2 (medium deterioration rate), λ_3 (major deterioration rate), λ_4 (failure rate) and α_3 (replacement rate) for all the four systems considered are depicted in Figures 4.1 to 4.5. Figure 4.1 show that system availability decrease as increases for any system. Furthermore, system with maintenance and replacement seems to be most effective and reliable among all the four systems. This is as a result of regular maintenance (both minor and major) and replacement at invoked whenever the system is in either minor or medium deterioration or failed states. It is shown that system with both maintenance and replacement produces more availability than the other systems. It is evident from Figures 4.2 – 4.5 that system with maintenance and replacement is more reliable and than the systems with minor maintenance and replacement, major maintenance with replacement and replacement only at failure. Thus, system with maintenance and replacement is the optimal configuration in this study. This shows that maintenance and replacement could make a significant difference while system is in minor, medium or major deterioration state.

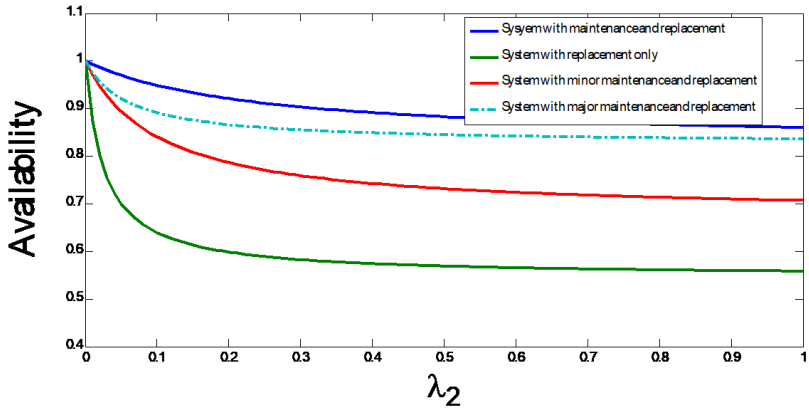

Figure 4.2: Relation between availability and medium deterioration

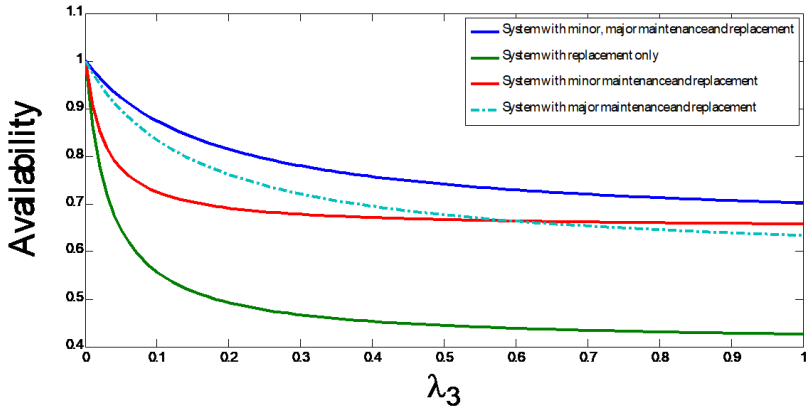

Figure 4.3: Relation between availability and major deterioration

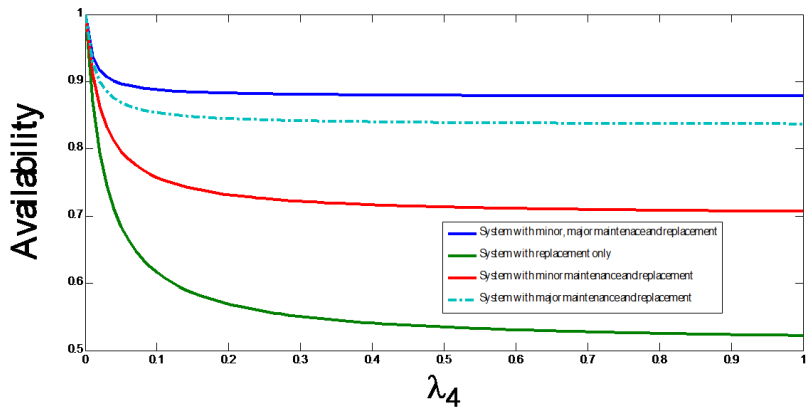

Figure 4.4: Relation between availability and system failure rate

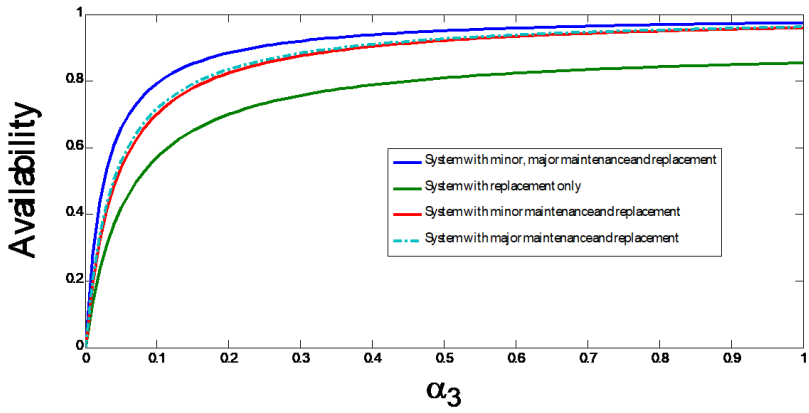

Figure 4.5: Relation between availability and replacement rate

5. Conclusion

This paper studied a two unit cold standby with three consecutive stages of deterioration before failure. Explicit expression for the steady-state availability was derived. The numerical simulations presented in Figures 4.1 – 4.5 provide a description of the effect of the deterioration rates, failure rate and replacement rate on steady-state availability. On the basis of the numerical results obtained for particular cases, it is suggested that the system availability can be improved significantly by:

(i) Adding more cold standby units.
(ii) Increasing the replacement rate.
(iii) Reducing the failure rate of the system by hot or cold duplication method.
(iv) Incorporating minor and major maintenance to system at minor and medium deterioration stages.
(v) Exchange the system at major deterioration with new one before.

The system can further be developed into system with multiple standbys in solving reliability and availability problems.

Conflict of Interests

The authors declare that there is no conflict of interests.

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Diagnosis of axial displacement in transformer windings using finite element analysis

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Abstract

Transformer is one of the main equipment in the power system and power quality of the power system mainly depends on the working of the power transformer. Leakage reactance is one of the main parameters of the transformer. The design of the transformer has a significant impact on the leakage reactance of the transformer. The mechanical strength of the transformer mainly depends on the leakage reactance of the transformer. This paper investigates the leakage reactance of the transformer in normal conditions and during the axial displacement of the high voltage and low voltage windings using finite element analysis. Normal results of finite element analysis method are also compared with the experimental short-circuit test. Results show that leakage reactance of the transformer exponentially proportional to the displacement of the windings. It is observed that with the axial displacement of the 6% of the winding, leakage reactance increased by 10%. Results also show that effect of the axial displacement in high voltage and low voltage winding is almost same.

1. Nomenclature

LV	Low Voltage
HV	High Voltage
f	Frequency
I_s	Rated secondary current
V_s	Rated secondary voltage
W_M	Magnetic field energy
H	Magnetic field strength
μ	Permeability
B	Magnetic flux density
A	Magnetic vector field potential
J	Current density
Z_b	Base impedance
L	Inductance
FEM	Finite element method

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Figure 3.1: 34.5/4 kv 1250 kVA tested Transformer

2. Introduction

Transformers are most expensive apparatus in a power network; reliability and stability of the transformer mainly depend on the leakage reactance of the transformer [1]. Efficiency and the voltage drop in a transformer on load are chiefly affected by its leakage reactance, which must be kept as low as design, and manufacturing techniques would permit. Transformers are usually expected to work for more than 30 years [2] and during this period many different mechanical defects occur in transformers and damage level of the transformer increased with the time duration if the mechanical defect is not removed in the earliest stage. Many techniques are being used for the diagnostic techniques to mechanical defects detection. The most common diagnostic technique for the detection of the mechanical fault is the calculation of the leakage reactance and short circuit impedance [3]. In past, estimation of the flux in the different parts of the transformer was the most common used method for the calculation of the leakage reactance [4]. However, nowadays finite element analysis is one of the most common used technique for the calculation of the transformer [5]-[6]. The main drawback of FEM is that the accurate boundary conditions are very difficult to define [7]-[8]. Several studies have been conducted on the deformation of the transformer windings [9][10]-, but the effects of the displacement of the transformer windings are not highlighted that much. Even with the small displacement in transformer windings can increase leakage reactance and electromagnetic forces, which may cause heavy damage to the transformer. There are many available analytical methods for the calculations of the leakage reactance, however; it is not applicable when windings are displaced whether due to the transportation of the transformer or due to the mechanical fault in the transformer. In this study, short circuit test, transient analysis, magnetostatic analysis, and post-processing were performed to study the performance characteristics of the transformer during normal and axial displacement conditions. 3-dimensional transformer model was used in the transient and magnetostatic analysis to determine the magnetic energy and flux distribution in the transformer. In post-processing leakage reactance is calculated from obtained magnetic energy.

3. Analyzed transformer

In this study 1250 kVA, three-phase distribution transformer is used. M5 grain oriented silicon steel was used for the manufacturing of the transformer core. Main parameters and figure of the used transformer is given in Table 1 and figure 3.1 respectively.

Ratings	Capacity (MVA)	1.25
	High Voltage (kV)	34.5
	Low Voltage (kV)	0.4
	HV Current (A)	12.08
	LV Current (A)	1804.37
	Frequency (Hz)	50
Core	Material	M5
	Flux Density	1.53
Windings	Material	Aluminum
	HV Turns	2390
	LV turns	16

Table 1: Transformer Data

B-H, B-P and Magnetization Curves of the core material of the studied transformer are shown in figure-2, 3 and 4 respectively.

4. Finite element analysis

The integral and differential equation can be solved by using FEM analysis [11]. In electrical engineering, FEM is most commonly used for the magnetostatic analysis, electromagnetic analysis and to determine the thermal characters and conductivity [12]. Reactance can be calculated accurately by using numerical method like FEM [13]. Due to the improvement in the computational techniques and modelling of the complex structured transformers become easier in FEM and effects of the different parameters on the transformer can be evaluated easily and accurately. The effects of the displacement in the transformer windings can be evaluated by using FEM. Geometry details of analysed transformer are given in Figure 4.1.

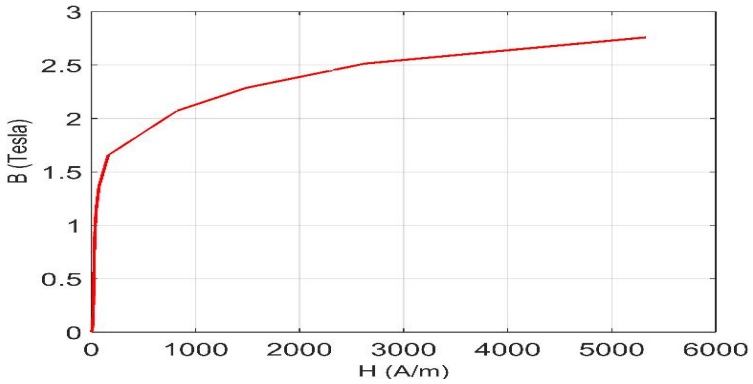

Figure 3.2: B-H curve of the Transformer

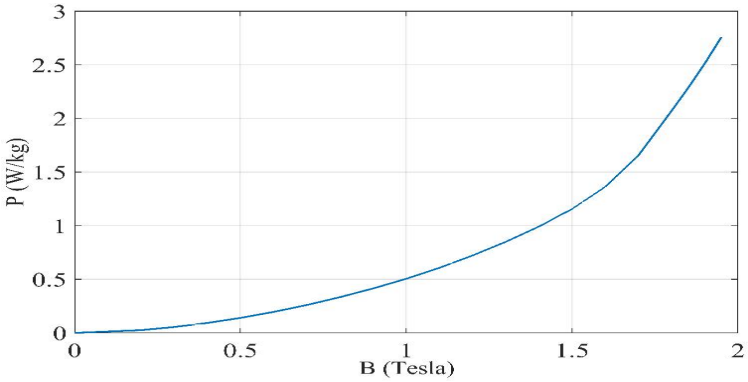

Figure 3.3: B-P curve of the Transformer

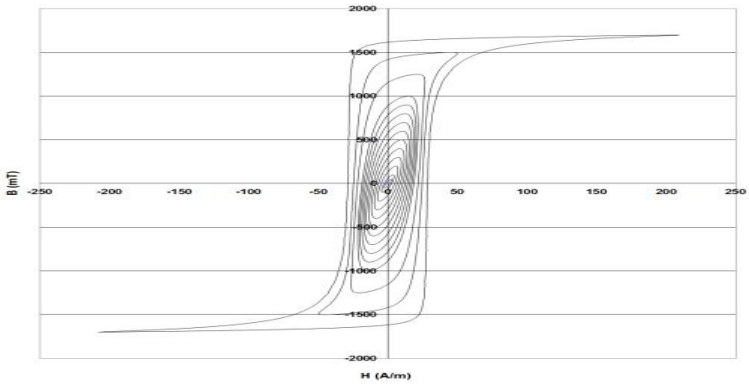

Figure 3.4: Magnetization Curves of the core material

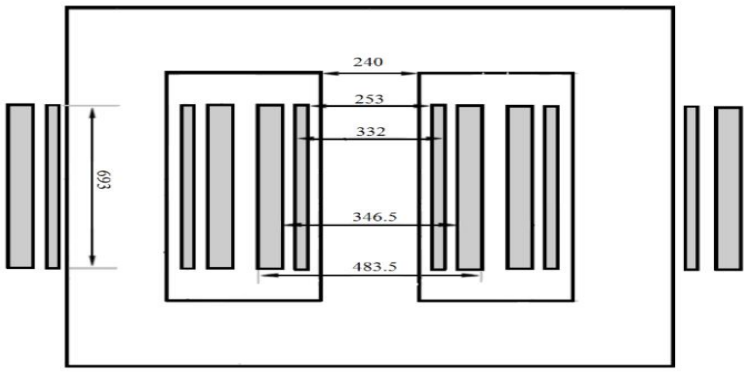

Figure 4.1: Geometry details of Analyzed Transformer

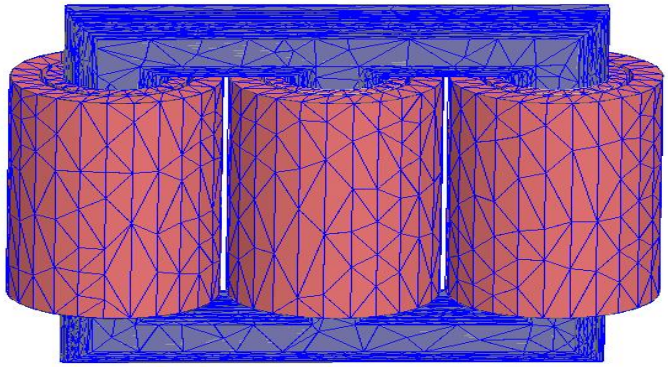

Figure 4.2: 3D Mesh operation of Studied Transformer.

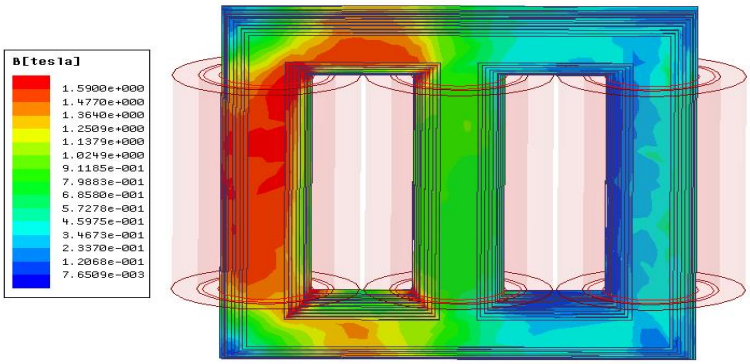

Figure 4.3: Magnetic flux distribution

3-D transformer model under mesh operation is shown in Figure 4.2, the number of the mesh generated in this model is 43146 elements. Figure 4.3 is showing magnetic flux distribution in the core of the transformer.
Magnetic energy is obtained from Ansys Maxwell FEM software. In the post-processing, obtained energy is converted into the inductance using equation (4.1) [14].

$$L = \frac{2W_m}{I^2} \tag{4.1}$$

Total magnetic field energy of the magnetic field in each part in a volume can be calculated as by using equation (4.2)

$$W_m = \int_V \frac{1}{2} BH dv \tag{4.2}$$

where $H = \frac{B}{\mu}$. Equation (4.2) can be rewritten as

$$W_m = \frac{1}{2\mu} \int_V B^2 dv \tag{4.3}$$

From equation (4.1) and (4.2)

$$L = \frac{2W_m}{I^2} = W_m = \frac{1}{I^2} \int_V BH dv \tag{4.4}$$

Another method for the calculation of the magnetic energy is by finding product of the magnetic vector potential and current density in volume

$$W_m = \frac{1}{2} \int_V AJ dv \tag{4.5}$$

From equation (4.1) and (4.5)

$$L = \frac{2W_m}{I^2} = \frac{1}{I^2} \int_V AJ dv \tag{4.6}$$

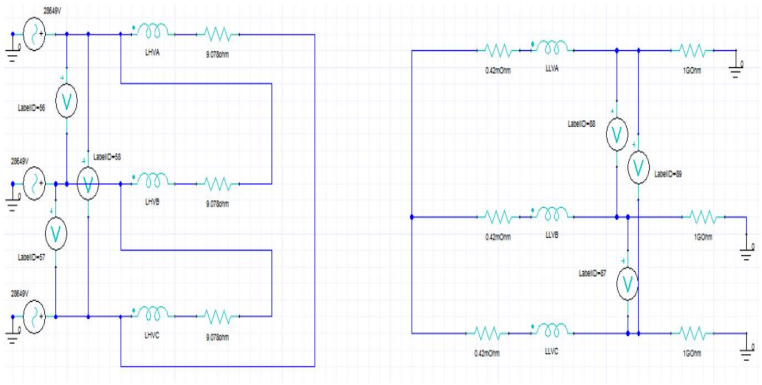

Figure 4.4: External Excitation Circuit of Three-phase Transformer.

The stored magnetic energy in the window space in the 2-D magnetic field can be calculated by using equation (4.7) and equation (4.8) [14]-[15].

$$W_m = \frac{1}{2} t \int \int BH dx dy \tag{4.7}$$

$$W_m = \frac{1}{2} t \int \int JA dx dy \tag{4.8}$$

The leakage reactance of the transformer is calculated by using equation (4.9).

$$X = \frac{2\pi f L}{Z_b} \times 100 \tag{4.9}$$

The base impedance of the transformer can be calculated by using equation.

$$Z_b = \frac{kV^2(HV \quad side)}{MVA} \tag{4.10}$$

External excitation circuit of the transformer using FEM is shown in Figure 5.1.

5. FEM and short-circuit test results

ANSYS MAXWELL® is used for the calculation of the leakage reactance. For normal conditions results of the finite element analysis are compared with the short-circuit test of the transformer. Magneto-static analysis, transient analysis, and post-processing solutions are used for the determination of the leakage reactance. Calculated experimental and finite element analysis leakage reactance is given in Table 2. For short-circuit test, voltage is applied to the HV windings and LV windings are short circuited in order to pass a rated current in HV and LV windings. The test is performed according to the IEC 60076-1. Measurement of the leakage reactance is performed quickly so that the temperature rises do not affect the measured results.

Short-circuit Test	FEM Method	Analytical Method
6.15	6.17	6.15

Table 2: Calculation of leakage reactance during normal conditions

Table 3 is showing the leakage reactance during the axial displacement of the winding. In finite element analysis, at a time one winding is displaced and second remains constant. Results show that effect of axial displacement of the LV and HV windings is almost same.
The relationship between the leakage reactance and winding displacement is shown in figure-9. Results show that the leakage reactance is directly proportional to the displacement of the transformer windings. The axial height of the windings was 693mm and with the displacement of the 20mm leakage reactance increase by 3% and with the displacement of 40*mm* leakage reactance increased by 12.5%.

6. Conclusion

This paper investigates the effect of the axial displacements of the windings on the leakage reactance of the transformer. Finite element method is used to determine the leakage reactance of the transformer and FEM results are validated with the short-circuit tests of the transformer. Results show that finite element analysis technique is the highly efficient technique to determine the leakage reactance of the transformer during the normal and displaced conditions. This method can help the transformer manufacturers in the designing stage of the transformer and during the axial displacement of the HV or LV winding in the transformer.

Displacement (cm)	Leakage Reactance	Displacement (cm)	Leakage Reactance
0.2	6.175	2.2	6.41
0.4	6.179	2.4	6.45
0.6	6.19	2.6	6.5
0.8	6.2	2.8	6.55
1.0	6.21	3.0	6.61
1.2	6.24	3.2	6.67
1.4	6.258	3.4	6.73
1.6	6.29	3.6	6.8
1.8	6.33	3.8	6.87
2.0	6.37	4.0	6.945

Table 3: Calculation of leakage reactance during normal conditions

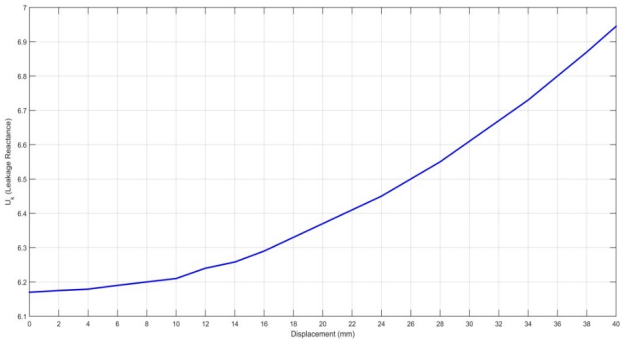

Figure 5.1: Relationship between displacement and leakage reactance

7. Acknowledgement

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Evolution equations in Fréchet spaces

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Abstract

This paper deals with the existence of mild solutions for a class of evolution equations. The technique used is a generalization of the classical Darbo fixed point theorem for Fréchet spaces associated with the concept of measure of noncompactness.

1. Introduction

There has been a significant development in functional evolution equations in recent years; see the monographs [2, 3, 14, 17, 18] and the references therein. By means of a nonlinear alternative of Leray–Schauder type for contraction operators on Fréchet spaces [13], Baghli and Benchohra [4, 5] provided sufficient conditions for the existence of mild solutions of some classes of evolution equations, while in [6, 7, 8] the authors presented some global existence and stability results for functional evolution equations and inclusions in the space of continuous and bounded functions. In [1], an iterative method is used for the existence of mild solutions of evolution equations and inclusions. Using the Tichonov's fixed point theorem, Olszowy and Wędrychowicz [16] considered a class of evolution equations on unbounded intervals. However in the previous papers some restrictions like, the compactness of the semigroup, the Lipschitz condition on the nonlinear term or the boundedness of the obtained mild solutions, are supposed.

In the present paper, we discuss the existence of mild solutions for the evolution equation

$$u'(t) = A(t)u(t) + f(t,u(t)); \ \text{ if } t \in \mathbb{R}_+ := [0,\infty), \tag{1.1}$$

with the initial condition

$$u(0) = u_0 \in E, \tag{1.2}$$

where $f : \mathbb{R}_+ \times E \to E$ is a given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space, and $\{A(t)\}_{t>0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of bounded linear operators $\{U(t,s)\}_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+}$; for $(t,s) \in \Lambda := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 \le s \le t < +\infty\}$.

This paper initiates the existence of solutions for evolution equations with an application of a generalization of the classical Darbo fixed point theorem, and the concept of measure of noncompactness in Fréchet spaces, The paper is organized as follows. In Section 2 some preliminary results are introduced. The main results is presented in Section 3, while the last section is devoted to an illustrative example.

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2. Preliminaries

Let $I := [0,T];\ T > 0$. A measurable function $u : I \to E$ is Bochner integrable if and only if $\|u\|$ is Lebesgue integrable. For properties of the Bochner integral, see for instance, Yosida [19].
By $B(E)$ we denote the Banach space of all bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup_{\|u\|=1} \|N(u)\|.$$

As usual, $L^1(I,E)$ denotes the Banach space of measurable functions $u : I \to E$ which are Bochner integrable and normed by

$$\|u\|_{L^1} = \int_0^T \|u(t)\|dt.$$

As usual, by $\mathscr{C} := C(I)$ we denote the Banach space of all continuous functions from I into E with the norm $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \sup_{t\in I} \|u(t)\|.$$

In what follows, for the family $\{A(t),\ t \geq 0\}$ of closed densely defined linear unbounded operators on the Banach space E we assume that it satisfies the following assumptions (see [3], p. 158).

- (P_1) The domain $D(A(t))$ is independent of t and is dense in E,
- (P_2) For $t \geq 0$, the resolvent $R(\lambda,A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leq 0$, and there is a constant K independent of λ and t such that
$$\|R(t,A(t))\| \leq K(1+|\lambda|)^{-1},\ \text{for } Re\lambda \leq 0,$$
- (P_3) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that
$$\|(A(t) - A(\theta))A^{-1}(\tau)\| \leq L|t-\tau|^\alpha,\ \text{for } t,\theta,\tau \in I.$$

Lemma 2.1. *([3], p. 159) Under assumptions $(P_1)-(P_3)$, the Cauchy problem*

$$u'(t) - A(t)u(t) = 0,\ t \in I,\ u(0) = y_0,$$

has a unique evolution system $U(t,s)$, $(t,s) \in \Delta := \{(t,s) \in J \times J : 0 \leq s \leq t \leq T\}$ satisfying the following properties:

1. *$U(t,t) = I$ where I is the identity operator in E,*
2. *$U(t,s)\,U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t \leq T$,*
3. *$U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(t,s) \in \Delta$ and for each $u \in E$, the mapping $(t,s) \to U(t,s)u$ is continuous.*

More details on evolution systems and their properties can be found in the books of Ahmed [3] and Pazy [17].

Let $C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into E, equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[0,n]} \|v(t)\|;\ n \in \mathbb{N},$$

and the distance

$$d(u,v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|u-v\|_n}{1+\|u-v\|_n};\ u,v \in C(\mathbb{R}_+).$$

We recall the following definition of the notion of a sequence of measures of noncompactness [10, 11].

Definition 2.2. *Let $\mathscr{M}_X$ be the family of all nonempty and bounded subsets of a Fréchet space X. A family of functions $\{\mu_n\}_{n\in\mathbb{N}}$ where $\mu_n : \mathscr{M}_X \to [0,\infty)$ is said to be a family of measures of noncompactness in the real Fréchet space X if it satisfies the following conditions for all $B, B_1, B_2 \in \mathscr{M}_X$:*

- *(a) $\{\mu_n\}_{n\in\mathbb{N}}$ is full, that is: $\mu_n(B) = 0$ for $n \in \mathbb{N}$ if and only if B is precompact,*
- *(b) $\mu_n(B_1) \leq \mu_n(B_2)$ for $B_1 \subset B_2$ and $n \in \mathbb{N}$,*
- *(c) $\mu_n(ConvB) = \mu_n(B)$ for $n \in \mathbb{N}$,*
- *(d) If $\{B_i\}_{i=1,\cdots}$ is a sequence of closed sets from $\mathscr{M}_X$ such that $B_{i+1} \subset B_i;\ i = 1,\cdots$ and if $\lim_{i\to\infty} \mu_n(B_i) = 0$, for each $n \in \mathbb{N}$, then the intersection set $B_\infty := \cap_{i=1}^{\infty} B_i$ is nonempty.*

Some Properties:

- (e) We call the family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ to be homogeneous if $\mu_n(\lambda B) = |\lambda|\mu_n(B)$; for $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$.
- (f) If the family $\{\mu_n\}_{n\in\mathbb{N}}$ satisfied the condition $\mu_n(B_1 \cup B_2) \leq \mu_n(B_1) + \mu_n(B_2)$, for $n \in \mathbb{N}$, it is called subadditive.
- (g) It is sublinear if both conditions (e) and (f) hold.
- (h) We say that the family of measures $\{\mu_n\}_{n\in\mathbb{N}}$ has the maximum property if

$$\mu_n(B_1 \cup B_2) = \max\{\mu_n(B_1), \mu_n(B_2)\},$$

(i) The family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ is said to be regular if and only if the conditions (a), (g) and (h) hold; (full sublinear and has maximum property).

Example 2.3. *For $B\in\mathscr{M}_X$, $x\in B$, $n\in\mathbb{N}$ and $\varepsilon>0$, let us denote by $\omega^n(x,\varepsilon)$ for $n\in\mathbb{N}$; the modulus of continuity of the function x on the interval $[0,n]$; that is,*

$$\omega^n(x,\varepsilon)=\sup\{|x(t)-x(s)|:t,s\in[0,n],|t-s|\leq\varepsilon\}.$$

Further, let us put

$$\omega^n(B,\varepsilon)=\sup\{\omega^n(x,\varepsilon):x\in B\},$$

$$\omega_0^n(B)=\lim_{\varepsilon\to0^+}\omega^n(B,\varepsilon),$$

$$\bar{\alpha}^n(B)=\sup_{t\in[0,n]}\alpha(B(t)):=\sup_{t\in[0,n]}\alpha(\{x(t):x\in B\}),$$

and

$$\beta_n(B)=\omega_0^n(B)+\bar{\alpha}^n(B).$$

The family of mappings $\{\beta_n\}_{n\in\mathbb{N}}$ where $\beta_n:\mathscr{M}_X\to[0,\infty)$, satisfies the conditions (a)-(d) from Definition 2.2.

Definition 2.4. *A nonempty subset $B\subset X$ is said to be bounded if*

$$\sup_{v\in X}\|v\|_n<\infty;$$

for $n\in\mathbb{N}$.

Lemma 2.5. *[9] If Y is a bounded subset of Fréchet space X, then for each $\varepsilon>0$, there is a sequence $\{y_k\}_{k=1}^{\infty}\subset Y$ such that*

$$\mu_n(Y)\leq2\mu_n(\{y_k\}_{k=1}^{\infty})+\varepsilon;\ \text{for } n\in\mathbb{N}.$$

Lemma 2.6. *[15] If $\{u_k\}_{k=1}^{\infty}\subset L^1(I)$ is uniformly integrable, then $\mu_n(\{u_k\}_{k=1}^{\infty})$ is measurable for $n\in\mathbb{N}$, and*

$$\mu_n\left(\left\{\int_0^t u_k(s)ds\right\}_{k=1}^{\infty}\right)\leq2\int_0^t\mu_n(\{u_k(s)\}_{k=1}^{\infty})ds,$$

for each $t\in[0,n]$.

Definition 2.7. *Let Ω be a nonempty subset of a Fréchet space X, and let $A:\Omega\to X$ be a continuous operator which transforms bounded subsets of onto bounded ones. One says that A satisfies the Darbo condition with constants $(k_n)_{n\in\mathbb{N}}$ with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$, if*

$$\mu_n(A(B))\leq k_n\mu_n(B)$$

for each bounded set $B\subset\Omega$ and $n\in\mathbb{N}$. If $k_n<1$; $n\in\mathbb{N}$ then A is called a contraction with respect to $\{\mu_n\}_{n\in\mathbb{N}}$.

In the sequel we will make use of the following generalization of the classical Darbo fixed point theorem for Fréchet spaces.

Theorem 2.8. *[10, 11] Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space F and let $V:\Omega\to\Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$. Then V has at least one fixed point in the set Ω.*

3. Existence of mild solutions

In this section, we present the main results for the global existence of solutions for our problem. Let us introduce the definition of the mild solution of the problem (1.1)-(1.2).

Definition 3.1. *A continuous function $u(\cdot):I\to E$ is said a mild solution of the problem (1.1)-(1.2), if u satisfies the following integral equation*

$$u(t)=U(t,0)u_0+\int_0^t U(t,s)\,f(s,u(s))ds,\quad \text{for each } t\in\mathbb{R}_+.$$

Let us introduce the following hypotheses.

(H_1) There exists a constant $M\geq1$ such that

$$\|U(t,s)\|_{B(E)}\leq M;\ \text{for every } (t,s)\in\Lambda.$$

(H_2) The function $t\mapsto f(t,u)$ is measurable on $\mathbb{R}_+$ for each $u\in E$, and the function $u\mapsto f(t,u)$ is continuous on E for a.e. $t\in\mathbb{R}_+$.

(H_3) There exists a continuous function $p:\mathbb{R}_+\to[0,\infty)$ such that

$$\|f(t,u)\|\leq p(t)(1+\|u\|);\ \text{for a.e. } t\in\mathbb{R}_+,\ \text{and each } u\in E.$$

(H_4) For each bounded and measurable set $B\subset E$ and for each $t\in\mathbb{R}_+$, we have

$$\mu(f(t,B))\leq p(t)\mu(B),$$

where μ is a measure of noncompactness on the Banach space E.

For $n \in \mathbb{N}$, let

$$p_n^* = \sup_{t\in[0,n]} p(t),$$

and define on $C(\mathbb{R}_+)$ the family of measure of noncompactness by

$$\mu_n(D) = \sup_{t\in[0,n]} e^{-4Mp_n^*\tau t}\mu(D(t)),$$

where $\tau > 1$, and $D(t) = \{v(t) \in E : v \in D\}$; $t \in [0,n]$.

Theorem 3.2. *Assume that the hypotheses $(H_1)-(H_4)$ are satisfied, and $nMp_n^* < 1$ for each $n \in \mathbb{N}$. Then the problem (1.1)-(1.2) has at least one mild solution.*

Proof. Consider the operator $N : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by:

$$(Nu)(t) = U(t,0)u_0 + \int_0^t U(t,s)\ f(s,u(s))ds. \tag{3.1}$$

Clearly, the fixed points of the operator N are solution of the problem (1.1)-(1.2).

For any $n \in \mathbb{N}$, let R_n be a positive real number with

$$R_n \geq \frac{M\|u_0\| + np_n^*M}{1 - nMp_n^*},$$

and we consider the ball

$$B_{R_n} := B(0,R_n) = \{w \in C(\mathbb{R}_+) : \|w\|_n \leq R_n\}.$$

For any $n \in \mathbb{N}$, and each $u \in B_{R_n}$ and $t \in [0,n]$ we have

$$\begin{aligned}
\|(Nu)(t)\| &\leq \|U(t,0)\|_{B(E)}\|u_0\| + \int_0^t \|U(t,s)\|_{B(E)}\|f(s,u(s))\|ds \\
&\leq M\|u_0\| + M\left(\int_0^t p(s)(1+\|u(s)\|)ds\right) \\
&\leq M\|u_0\| + M(1+\|u\|_n)\int_0^t p(s)ds \\
&\leq M\|u_0\| + nMp_n^*(1+R_n) \\
&\leq R_n.
\end{aligned}$$

Thus

$$\|N(u)\|_n \leq R_n. \tag{3.2}$$

This proves that N transforms the ball B_{R_n} into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 2.8. The proof will be given in several steps.

Step 1. *$N : B_{R_n} \to B_{R_n}$ is continuous.*
Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that $u_k \to u$ in B_{R_n}. Then, for each $t \in [0,n]$, we have

$$\begin{aligned}
\|(Nu_k)(t) - (Nu)(t)\| &\leq \int_0^t \|U(t,s)\|_{B(E)}\|f(s,u_k(s)) - f(s,u(s))\|ds \\
&\leq M\int_0^t \|f(s,u_k(s)) - f(s,u(s))\|ds.
\end{aligned}$$

Since $u_k \to u$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|N(u_k) - N(u)\|_n \to 0 \quad \text{as } k \to \infty.$$

Step 2. *$N(B_{R_n})$ is bounded.*
Since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded, then $N(B_{R_n})$ is bounded.

Step 3. *For each bounded subset D of B_{R_n}, $\mu_n(N(D)) \leq \ell_n \mu_n(D)$.*
From Lemmas 2.5 and 2.6, for any $D \subset B_{R_n}$ and any $\varepsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0,n]$, we have

$$\begin{aligned}
\mu((ND)(t)) &= \mu\left(\left\{U(t,0)u_0 + \int_0^t U(t,s)\ f(s,u(s))ds;\ u \in D\right\}\right) \\
&\leq 2\mu\left(\left\{\int_0^t U(t,s)f(s,u_k(s))ds\right\}_{k=1}^{\infty}\right) + \varepsilon \\
&\leq 4\int_0^t \mu\left(\left\|U(t,s)\right\|_{B(E)}\{f(s,u_k(s))\}_{k=1}^{\infty}\right) ds + \varepsilon \\
&\leq 4M\int_0^t \mu\left(\{f(s,u_k(s)\}_{k=1}^{\infty}\right) ds + \varepsilon \\
&\leq 4M\int_0^t p(s)\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \varepsilon \\
&\leq 4Mp_n^*\int_0^t e^{4Mp_n^*\tau s}e^{-4Mp_n^*\tau s}\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \varepsilon \\
&\leq \frac{\left(e^{4Mp_n^*\tau t}-1\right)}{\tau}\mu_n(D) + \varepsilon \\
&\leq \frac{e^{4Mp_n^*\tau t}}{\tau}\mu_n(D) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then

$$\mu((ND)(t)) \leq \frac{e^{4Mp_n^*\tau t}}{\tau}\mu_n(D).$$

Thus

$$\mu_n(N(D)) \leq \frac{1}{\tau}\mu_n(D).$$

As a consequence of steps 1 to 3 together with Theorem 2.8, we can conclude that N has at least one fixed point in B_{R_n} which is a mild solution of problem (1.1)-(1.2).

□

4. An example

As an application of our results, we consider the following functional evolution problem of the form

$$\begin{cases}
\dfrac{\partial z}{\partial t}(t,x) = a(t,x)\dfrac{\partial^2 z}{\partial x^2}(t,x) + Q(t,z(t,x)); & t \in \mathbb{R}_+,\ x \in [0,\pi], \\
z(t,0) = z(t,\pi) = 0; & t \in \mathbb{R}_+. \\
z(0,x) = \Phi(x); & x \in [0,\pi],
\end{cases} \tag{4.1}$$

where $a(t,x) : \mathbb{R}_+ \times [0,\pi] \to \mathbb{R}$ is a continuous function and is uniformly Hölder continuous in t, $Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0,\pi] \to \mathbb{R}$ are continuous functions.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)w = a(t,x)w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous}, w'' \in E,\ w(0) = w(\pi) = 0\}.$$

Then $A(t)$ generates an evolution system $U(t,s)$ (see [12]).

For $x \in [0,\pi]$, we have

$$y(t)(x) = z(t,x); \quad t \in \mathbb{R}_+,$$

$$f(t,y)(x) = Q(t,z(t,x)); \quad t \in \mathbb{R}_+,$$

and

$$u_0(x) = \Phi(x); \quad x \in [0,\pi].$$

Thus, under the above definitions of f, u_0 and $A(\cdot)$, the system (4.1) can be represented by the functional evolution problem (1.1)-(1.2). Furthermore, more appropriate conditions on Q ensure the hypotheses $(H_1)-(H_4)$. Consequently, Theorem 3.2 implies that the evolution problem (4.1) has at least one mild solution.

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Reduction of non-variational bi-Hamiltonian system of shallow-water waves propagation via symmetry approach

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Abstract

In this paper, non-variational bi-Hamiltonian system of shallow-water waves propagation is considered. Lie point generators are calculated and one dimensional optimal system of its subalgebras up to conjugacy classes are reported. Then similarity variables are computed by using these conjugacy classes which are further utilized for the reduction of considered system. Then, a transformation is used to convert the system from non-variational to variational system, thus standard Lagrangian is computed. Noether operators are calculated by using Noether approach and local conserved quantity is discussed for the new fourth order system of partial differential equations (PDEs). Further, inverse transformation is applied to get the corresponding local conserved quantity for the considered non-variational problem. Moreover, this local conservation law with the help of double reduction theorem is utilized to reduce the system.

1. Introduction

Exact solutions of a system of nonlinear PDEs have lot of importance in the solution of initial or boundary value problems. It also helps to see the exact inner picture of the phenomena. For higher order nonlinear system of PDEs, one does not has the enough methods to find the exact solutions. Symmetry approach is one of the powerful tools for analyzing such systems. Lie group theory is also one of the most interesting and active field of research for the solution of nonlinear PDEs. It is a systematic technique to construct a class of exact solutions or reduce the considered differential equation into one of its simplest form. It also provides applicable approach to find the closed form solution of differential equation. There are lot of efficient techniques available for the computation of exact solutions of differential equations. Some well known techniques are separation of variables, traveling wave solution, self-similar solutions, solution by using ansatz and exponential self-similar solutions. These approaches are the particular case of Lie group method (see [15]).
Reduction of a differential equation by using Noether approach is one of the aspect of Lie theory. Conserved quantities play a vital role in the theory of differential equations. Solution and reduction of PDEs are some of the aspects where conserved quantities have its significant importance. On the derivation of conservation laws active research efforts have been made in the last few decades. One of the elegant and systematic approach is developed by Noether [12]. Direct method [2], characteristics approach [20] and the partial Noether approach [10] are the approaches which have been used frequently in literature. Computing utilities for the calculation of conservation laws are also practised in past (see [4, 6, 7, 8, 9, 21, 22]).
A first systematic method for finding the conservation laws [12] of the variational problems is given by Emmy Noether. She found a link between the symmetries and conservation laws [12]. She concluded that each symmetry corresponds to a conserved quantity. This approach is valid for the variational problems and thus depends on the existence of standard Lagrangian. There is a big class of PDEs and system of PDEs which do not posses standard Lagrangian, thus Noether approach is not applicable. In this article, we will apply Noether approach to the non-variational system of PDEs. For this we will use the following steps:

(i): Convert the non-variational system of PDEs into variational problem by using suitable transformation.

(ii): Standard Lagrangian is calculated and Noether approach is applied in new coordinates.

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(iii): Noether operators and corresponding conservation laws are computed.

(iv): Use an inverse transformation to convert the conserved quantities into the coordinates of original problem.
The bi-Hamiltonian Boussinesq system [1]:

$$u_t - \Omega v_x = 0, \quad v_t - \Omega\left(u_{xxx} + 8uu_x\right) = 0, \tag{1.1}$$

describes shallow-water wave propagation in both directions, where Ω is a real constant.
In the past, lot of work has been done on the different forms of shallow-water waves propagation by means of Lie symmetry analysis. In [14], M. Pandey discussed the one dimensional shallow water equation by using Lie approach. Exact solutions of one dimensional axis symmetric flow of shallow water equations involving bores was computed with the help of point symmetries in [17]. Plebanski's second heavenly equation and $2+1$-dimensional hamiltonian integrable system and is discussed in [11, 23] by means of symmetries. The group analysis and an infinite aggregate of non-degenerate solutions of the equations of one-dimensional model of shallow water with a straight bottom is discussed in [5].
The pattern of this paper is as follow. In Section 2, basic operators are discussed. The Lie point generators and reduction by using similarity variables for the system (1.1) are discussed in Section 3. Local conserved quantity and double reduction for the considered system is computed in Section 4. At last conclusion is provided.

2. Basic operators

Let (t,x) be independent variables and (u,v) be dependent variables. Consider the third order system of PDEs with two independent and dependent variables i.e.

$$E_1(t,x,u,v,u_t,v_t,...) = 0, \quad E_2(t,x,u,v,u_t,v_t,...) = 0. \tag{2.1}$$

(1): A vector field:

$$Y = \tau(t,x,u,v)\frac{\partial}{\partial t} + \xi(t,x,u,v)\frac{\partial}{\partial x} + \phi(t,x,u,v)\frac{\partial}{\partial u} + \eta(t,x,u,v)\frac{\partial}{\partial v}$$

is a Lie point symmetry generator of (2.1) if it satisfies the invariance condition i.e.

$$Y^{[3]}E_1 = 0, \quad Y^{[3]}E_2 = 0, \quad \text{whenever} \quad E_1 = 0, \quad E_2 = 0, \tag{2.2}$$

where $Y^{[3]}$ is the third prolongation [13].

(2): If $u \to U_x$ and $v \to V_x$ in (2.1) then the given third order system becomes fourth order system in U, V variables, i.e.

$$G_1(t,x,U,V,U_t,V_t,...) = 0, \quad G_2(t,x,U,V,U_t,V_t,...) = 0. \tag{2.3}$$

(3): The Euler operator is:

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} - D_t\frac{\partial}{\partial U_t} - D_x\frac{\partial}{\partial U_x} + D_t^2\frac{\partial}{\partial U_{tt}} + D_x^2\frac{\partial}{\partial U_{xx}} + ..., \tag{2.4}$$

$$\frac{\delta}{\delta V} = \frac{\partial}{\partial V} - D_t\frac{\partial}{\partial V_t} - D_x\frac{\partial}{\partial V_x} + D_t^2\frac{\partial}{\partial V_{tt}} + D_x^2\frac{\partial}{\partial V_{xx}} + ..., \tag{2.5}$$

where

$$D_t = \frac{\partial}{\partial t} + U_t\frac{\partial}{\partial U} + V_t\frac{\partial}{\partial V} + U_{tt}\frac{\partial}{\partial U_t} + V_{tt}\frac{\partial}{\partial V_t} + U_{tx}\frac{\partial}{\partial U_x} + ... \tag{2.6}$$

and

$$D_x = \frac{\partial}{\partial x} + U_x\frac{\partial}{\partial U} + V_x\frac{\partial}{\partial V} + U_{xx}\frac{\partial}{\partial U_x} + V_{xx}\frac{\partial}{\partial V_x} + U_{tx}\frac{\partial}{\partial U_t} + ... \tag{2.7}$$

are known as the total derivative operators.

(4): The generalized operator is defined by

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial U} + \eta\frac{\partial}{\partial V} + \phi^x\frac{\partial}{\partial U_x} + \eta^x\frac{\partial}{\partial V_x} + \phi^t\frac{\partial}{\partial U_t} + ...\tag{2.8}$$

(5): A standard Lagrangian $L = L(t,x,U,V,U_t,V_t...) \in A$ (space of differential functions) and satisfies:

$$\frac{\delta L}{\delta U} = 0 \text{ and } \frac{\delta L}{\delta V} = 0.$$

(6): The generalized operator is known as Noether operator associated with a standard Lagrangian L if it satisfies:

$$X^{[2]}L + L(\tau_t + \tau_U U_t + \tau_V V_t + \xi_x + \xi_U U_x + \xi_V V_x) = B_t^1 + B_U^1 U_t + B_V^1 V_t + B_x^2 + B_U^2 U_x + B_V^2 V_x. \tag{2.9}$$

In Eq.(2.9), B^i's are known as the gauge terms, while $X^{[2]}$ is the second prolongation of the generator X.

(7): The equation:

$$(T^1)_t + (T^2)_x = 0 \tag{2.10}$$

evaluated on the solution space given by (2.1) is known as the conservation laws for Eq. (2.1). The vector $T = (T^1, T^2)$ is a conserved vector where T^1, T^2 are its components.

The conserved vectors of the system (2.1) associated with a Noether operator X can be determined from the formula:

$$T^i = B^i - N^i(L) \quad i = 1,2, \tag{2.11}$$

where

$$\begin{aligned} N^1 &= \tau + (\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_t} + (\eta - \tau V_t - \xi V_x)\frac{\delta}{\delta V_t} - D_t(\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_{tt}} \\ &\quad -D_t(\eta - \tau V_t - \xi V_x)\frac{\delta}{\delta V_{tt}} - D_x(\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_{tx}} + ..., \end{aligned} \tag{2.12}$$

$$\begin{aligned} N^2 &= \xi + (\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_x} + (\eta - \tau V_t - \xi V_x)\frac{\delta}{\delta V_x} - D_x(\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_{xx}} \\ &\quad -D_x(\eta - \tau V_t - \xi V_x)\frac{\delta}{\delta V_{xx}} - D_t(\phi - \tau U_t - \xi U_x)\frac{\delta}{\delta U_{tx}} + ...\cdot \end{aligned} \tag{2.13}$$

Following statements are taken from [3, 18, 19].

(8): For nonlocal variable ω such that $T^1 = \omega_x$ and $T^2 = -\omega_t$. While using similarity variables one can have $T^r = \omega_s$ and $T^s = -\omega_r$, where

$$T^r = \frac{T^1 D_t(r) + T^2 D_x(r)}{D_t(r)D_x(s) - D_x(r)D_t(s)}, \quad T^s = \frac{T^1 D_t(s) + T^2 D_x(s)}{D_t(r)D_x(s) - D_x(r)D_t(s)}, \tag{2.14}$$

so that $D_r T^r + D_s T^s = 0$.

Theorem 2.1. *If system (2.1) admits a nontrivial conserved form that has at least one associated symmetry in every reduction can be reduced to a non linear system of 2^{nd} order ordinary differential equations namely $T^r = K$, while T^r is defined in (2.14).*

3. Lie Point Symmetries

In this section, we will apply standard method for finding infinitesimal generators for (1.1). Using (1.1) in (2.2) one can have an over-determined system of linear PDEs that corresponds the following Lie point symmetries [13]:

$$Y_1 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - \frac{3v}{2}\frac{\partial}{\partial v}, \quad Y_2 = \frac{\partial}{\partial t}, \quad Y_3 = \frac{\partial}{\partial x}, \quad Y_4 = \frac{\partial}{\partial v}. \tag{3.1}$$

After rescalling

$$\mathbf{Y_1} = \frac{2}{3}Y_1, \ \mathbf{Y_2} = Y_2, \ \mathbf{Y_3} = Y_3 \ \textit{and} \ \mathbf{Y_4} = Y_4$$

.

By the definition of a subalgebra, one can easily conclude that there are infinite number of a one-dimensional subalgebras of

$$L = \{\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y_3}, \mathbf{Y_4}\}.$$

The greatest or best representative of these algebras is called optimal system of one-dimensional subalgebra. In view of all these we can write an arbitrary element from L i.e.

$$\mathbf{Y} = a\mathbf{Y_1} + b\mathbf{Y_2} + c\mathbf{Y_3} + d\mathbf{Y_4}, \tag{3.2}$$

where a, b, c and d are real constants. Before going to write down optimal system for the one-dimensional algebra it should be noted that $\pounds_i$ and $\pounds_j$ are said to be equivalent conjugacy classes if

$$\pounds_i = Ad\ Y_i\Big(\pounds_j\Big),$$

where $Y_i \in L_1$ and

$$Ad\left[\exp\Big(tY_i\Big)\right]Y_j = Y_j - t[Y_i,\ Y_j] + \frac{t^2}{2}\Big[Y_i, [Y_i,\ Y_j]\Big] - \cdots.$$

Now our next task is to simplify (3.2) by applying a carefully chosen adjoint transformation [16] that gives:

$$\pounds_1 =< \mathbf{Y_1} >, \quad \pounds_2 =< \mathbf{Y_2} >, \quad \pounds_3 =< \mathbf{Y_3} >, \quad \pounds_4 =< \mathbf{Y_4} >,$$

$$\pounds_5 =< \mathbf{Y_4} + \varepsilon\mathbf{Y_3} >, \quad \pounds_6 =< \mathbf{Y_2} + \varepsilon\mathbf{Y_3} >, \pounds_7 =< \mathbf{Y_4} + \varepsilon\mathbf{Y_2} + \alpha\mathbf{Y_3} >,$$

where $\varepsilon = \pm 1$ and $\alpha \neq 0$.

3.0.1. Reduction by calculating similarity variables

$< \mathbf{Y_1} >$
This operator has the following characteristic system

$$\frac{dt}{t} = \frac{dx}{(\frac{x}{2})} = \frac{du}{-u} = \frac{dv}{(\frac{-3v}{2})}. \tag{3.3}$$

(3.3) can be separated into the following linear equations

$$(i): \ \frac{dt}{t} = \frac{dx}{(\frac{x}{2})}, \quad (ii): \ \frac{dt}{t} = \frac{du}{-u}, \quad (iii): \ \frac{dx}{(\frac{x}{2})} = \frac{dv}{(\frac{-3v}{2})}. \tag{3.4}$$

Integrating (3.4 (i)), (3.4 (ii)) and (3.4 (iii)) one by one we get the following results respectively:

$$\xi = \frac{t}{x^2}, \quad u = \frac{P(\xi)}{t}, \quad v = \frac{Q(\xi)}{x^3}.$$

Substituting the obtained invariants in (1.1) we obtain the following system of ordinary differential equation:

$$\frac{dP}{d\xi} + 2\Omega\xi^2\frac{dQ}{d\xi} - \frac{P}{\xi} + 3\Omega\xi Q = 0,$$

$$8\Omega\xi^3\frac{d^3P}{d\xi^3} + 36\Omega\xi^2\frac{d^2P}{d\xi^2} + 8(3\xi + 2P)\Omega\frac{dP}{d\xi} - \xi\frac{dQ}{d\xi} = 0. \tag{3.5}$$

Similarly, working on the same line we discuss each class one by one.
$< \mathbf{Y_2} >$

$$\xi = x, \quad u = P(\xi), \quad v = Q(\xi),$$

system after using similarity variables we have:

$$\frac{dQ}{d\xi} = 0, \quad \frac{d^3P}{d\xi^3} + 8P\frac{dP}{d\xi} = 0. \tag{3.6}$$

$< \mathbf{Y_3} >$
For this case, one can have constant solution.
$< \mathbf{Y_4} >$
It does not give any new solution.
$< \mathbf{Y_4} + \varepsilon\mathbf{Y_3} >$

$$\xi = t, \quad u = P(\xi), \quad v = \frac{x}{\varepsilon} + Q(\xi).$$

System (1.1) after using similarity variables yields the following solution:

$$v = \frac{x}{\varepsilon} + c_1, \quad u = \frac{\Omega t}{\varepsilon} + c_2. \tag{3.7}$$

$< \mathbf{Y_2} + \varepsilon\mathbf{Y_3} >$
Similarity variables:

$$\xi = \varepsilon t - x, \quad u = P(\xi), \text{ and } v = Q(\xi)$$

converts the system (1.1) into:

$$\frac{dP}{d\xi} + \Omega\frac{dQ}{d\xi} = 0, \quad \Omega^2\frac{d^3P}{d\xi^3} + (\varepsilon + 8\Omega^2P)\frac{dP}{d\xi} = 0. \tag{3.8}$$

$< \mathbf{Y_4} + \varepsilon\mathbf{Y_2} + \alpha\mathbf{Y_3} >$

$$\xi = \varepsilon\alpha t - \varepsilon x, \quad u = P(\xi), \quad v = \frac{t}{\varepsilon} + Q(\xi).$$

System after using similarity variables we have:

$$\alpha\frac{dP}{d\xi} + \Omega\varepsilon\frac{dQ}{d\xi} = 0, \quad \Omega\varepsilon^3\frac{d^3P}{d\xi^3} - 8\Omega P\frac{dP}{d\xi} - \alpha\frac{dQ}{d\xi} - \frac{1}{\varepsilon} = 0. \tag{3.9}$$

4. Noether operators and conservation laws

In this section, we will calculate the conserved quantities of the system of shallow-water waves propagation.

4.1. System of shallow-water waves propagation

It should be noted that system (1.1) belongs to the class of non-variational problems and hence does not possess a standard Lagrangian. In order to make system (1.1) variational let us take $u = U_x$ and $v = V_x$. The system (1.1) becomes

$$U_{tx} - \Omega V_{xx} = 0, \quad V_{tx} - \Omega\left(U_{xxxx} + 8U_xU_{xx}\right) = 0. \tag{4.1}$$

The standard Lagrangian for system (4.1) is:

$$L = \frac{\Omega}{2}U_{xx}^2 + \frac{\Omega}{2}V_x^2 + \frac{4}{3}\Omega U_x^3 - \frac{1}{2}U_tV_x - \frac{1}{2}U_xV_t. \tag{4.2}$$

Using Eq.(4.2) in Eq.(2.9) and separating with respect to derivatives of U and V one gets:

$$(i): \xi_x = 0, \ (ii): \xi_t = 0, \ (iii): \xi_U = 0, \ (iv): \xi_V = 0, \ (v): \eta_U = 0, \tag{4.3}$$

$$(i): \ \tau_V = 0, \ (ii): \ \tau_U = 0, \ (iii): \tau_x = 0, \ (iv): \ \tau_t = 0, \ (v): \ \eta_V = 0, \tag{4.4}$$

$$(i): \ \phi_U = 0, \ (ii): \ \phi_V = 0, \ (iii): \ \phi_x = 0, \ (iii): \ \phi_t = 0, \tag{4.5}$$

$$(i): B_U^1 = -\frac{1}{2}\eta_x, \ (ii): B_V^1 = 0, \ (iii): B_U^2 = -\frac{1}{2}\eta_t, \ (iv): B_V^2 = \Omega\eta_x \tag{4.6}$$

and

$$B_t^1 + B_x^2 = 0. \tag{4.7}$$

The solution of Eqs. (4.3)-(4.7) is

$$\tau = c_1, \quad \xi = c_2, \quad \phi = a, \quad \eta = b + c_3x, \tag{4.8}$$

$$B^1 = -\frac{1}{2}c_3U + \lambda(t,x), \quad B^2 = -\frac{1}{2}b_tU - \frac{1}{2}a_tV + c_3\Omega V + \delta(t,x). \tag{4.9}$$

where

$$\lambda_t + \delta_x = 0, \tag{4.10}$$

while a and b are any arbitrary functions of t. Without loss of generality one can take $\lambda = 0 = \delta$. Thus the Noether operators for the system (4.1) will be:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial V}, \quad X_{(a,b)} = a\frac{\partial}{\partial U} + b\frac{\partial}{\partial V}. \tag{4.11}$$

Using (2.11) and after applying the inverse transformation, i.e. $U \to \int u dx, V \to \int v dx$ we can get the corresponding conserved vectors for the system (1.1). One can easily verify that only X_2 leads to local conserved quantity:

$$T_2^1 = -uv, \quad T_2^2 = \frac{\Omega}{2}v^2 - \frac{\Omega}{2}u_x^2 + \frac{8\Omega}{3}u^3 + \Omega uu_{xx}. \tag{4.12}$$

4.2. Double reduction by Noether approach

In this section, we will apply double reduction theory [3, 18, 19] for the reduction of the system (1.1). One can see that $\mathbf{Y_1}$ and $\mathbf{Y_2}$ are the associated Lie point symmetries thus using a linear transformation:

$$Y = \frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x},$$

the corresponding similarity variables will be:

$$s = t, \ \ r = \beta t - x, \ \ u = u(r), \ \ v = v(r). \tag{4.13}$$

Using nonlocal conservation law (4.12) in (2.14) we have

$$T^r = -\beta uv - \frac{\Omega v^2}{2} + \frac{\Omega u_r^2}{2} - \frac{8\Omega u^3}{3} - \Omega uu_{rr}. \tag{4.14}$$

Using Theorem (2.1) we get:

$$-\Omega u\frac{d^2u}{dr^2} + \frac{\Omega}{2}\left(\frac{du}{dr}\right)^2 - \frac{8\Omega u^3}{3} - \beta uv - \frac{\Omega v^2}{2} = K, \tag{4.15}$$

where K is a real constant.
So system (1.1) after reduction converts to:

$$\frac{dv}{dr} = -\beta\Omega\frac{du}{dr}, \quad -\Omega u\frac{d^2u}{dr^2} + \frac{\Omega}{2}\left(\frac{du}{dr}\right)^2 - \frac{8\Omega u^3}{3} - \beta uv - \frac{\Omega v^2}{2} = K. \tag{4.16}$$

4.3. Conclusion

In this paper, we presented a complete classification of one dimensional subalgebra of the symmetry algebra of the system of shallow-water waves propagation up to conjugacy classes. These conjugacy classes were further used to reduce the considered system (1.1) into system of ordinary differential equation.
Further, Noether approach was applied to the non-variational third order systems of PDEs (1.1). In order to convert the considered system into variational problem the transformation $u = U_x$ and $v = V_x$ was applied. Moreover, local conserved quantity for new fourth order systems in U , V variables was computed by using Noether approach. Then an inverse transformation $U \to \int udx$, $V \to \int vdx$ was used to obtain the local conserved quantity for the main problem. The obtained local conservation law with corresponding Lie point symmetries was used to get double reduction for the system (1.1).

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Existence of solutions for nonlocal boundary value problem for Caputo nonlinear fractional differential inclusion

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Abstract

This paper deals with the existence of solutions for nonlinear fractional differential inclusions supplemented with three-point boundary conditions. First, we investigate it for L^1-Caratheodory convex-compact valued multifunction. Then, we investigate it for nonconvex-compact valued multifunction via some conditions. Two illustrative examples are presented at the end of the paper to illustrate the validity of our results.

1. Introduction

The concept of fractional calculus has played an important role in improving the work based on integer-order (classical) calculus in several diverse disciplines of science and engineering. In fact, quantum calculus has a rich history and the details of this basic notions, results and methods can be found in the text [2, 26, 37] and papers [10, 22]. The nonlocal nature of a fractional order differential operator, which take into account hereditary properties of various material and processes, has helped to improve the mathematical modeling of many natural phenomena and physical processes, see for example [4, 5, 21]. The increasing interest of fractional differential equations and inclusions are motivated by their applications in various fields of science such as physics chemistry, biology, economics, fluid mechanics, control theory, etc, we refer the reader to [9, 17, 30] and the references therein. Realistic problems arising from economics, optimal control, stochastic analysis can be modelled as differential inclusion. So much attention has been paid by many authors to study this kind of problems, see [4, 5, 36]. On the other hand boundary value problems with local and nonlocal boundary conditions constitute a very interesting and important class of problems. They include two, three and multipoint boundary value problems. The existence and multiplicity of positive solutions for such problems have received a great deal of attentinos. To identify a few, we refer the reader to [8, 11, 13, 18, 19, 20, 24, 25, 27, 28, 29, 31, 32, 33, 34].

In this paper, we are interested in the existence of solutions for the Caputo fractional differential inclusion

$$ {}^{c}D^{\alpha}u(t) \in F\left(t,u(t),u'(t)\right), \quad t \in J=[0,1], \tag{1.1} $$

subject to three-point boundary conditions

$$ \begin{cases} \beta u(0)+\gamma u(1)=u(\eta), \\ u(0)=\int_0^{\eta} u(s)\,ds, \\ \beta\, {}^{c}D^{p}u(0)+\gamma\, {}^{c}D^{p}u(1)={}^{c}D^{p}u(\eta), \end{cases} \tag{1.2} $$

where $2<\alpha\leq 3$, $1<p\leq 2$, $0<\eta<1$, $\beta,\gamma\in\mathbb{R}^{+}$, $f\in C\left([0,1]\times\mathbb{R}^{2},\mathbb{R}\right)$ and ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α.

The current paper is organized as follows. In section 2, we introduce some definitions and preliminary results that will be used in the remainder of the paper. In section 3, we present existence results for the problem (1.1) – (1.2) when the right-hand side is convex-compact as well as nonconvex-compact values. In the first result we use the fixed-point theorem (Lemma 2.12) for multivalued maps (see [3]) while in the second result we prove the existence by applying a fixed-point theorem for contraction multivalued maps due to Covitz and Nadler and we give two examples to illustrate our results.

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2. Preliminaries

In this section, we introduce some necessary definitions and lemmas of fractional calculus to facilitate the analysis of the problem (1.1) – (1.2). These details can be found in the recent literature; see [1, 12, 16] and the references therein.

Definition 2.1. *Let $\alpha > 0$, $n-1 < \alpha < n$, $n = [\alpha]+1$ and $u \in C([0,\infty),\mathbb{R})$. The Caputo derivative of fractional order α for the function u is defined by*

$$^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)\,ds.$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,\infty) \to \mathbb{R}$ is given by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Euler gamma function, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.3. *[22] Let $\alpha, \beta \geq 0$ and $u \in L^p(0,1)$, $0 \leq p \leq +\infty$. Then the next formulas hold.*
(i) $\left(I^{\beta}I^{\alpha}u\right)(t) = I^{\alpha+\beta}u(t)$,
(ii) $\left(D^{\beta}I^{\alpha}u\right)(t) = I^{\alpha-\beta}u(t)$,
(iii) $\left(D^{\alpha}I^{\alpha}u\right)(t) = u(t)$.

Lemma 2.4. *[26] Let $\alpha > 0$, $n-1 < \alpha < n$ and the function $g : [0,T] \to \mathbb{R}$ be continuous for each $T > 0$. Then, the general solution of the fractional differential equation $^{c}D^{\alpha}g(t) = 0$ is given by*

$$g(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha]+1$.

Lemma 2.5. *[4] Assume that $u \in C[0,1] \cap L^1(0,1)$ with a Caputo fractional derivative of order $\alpha > 0$ that belongs to $u \in C^n[0,1]$, then*

$$I^{\alpha}\,{}^{c}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha]+1$.

We will present notations, definitions and preliminary facts from multivalued analysis which are used throughout this paper. Here $(C[0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|u\| = sup\{|u(t)| : for all\, t \in [0,1]\}$, $L^1([0,1],\mathbb{R})$, the Banach space of measurable functions $u : [0,1] \to \mathbb{R}$ which are Lebesgue integrable, normed by $\|u\|_{L^1} = \int_0^1 |u(t)|\,dt$, and $AC^i([0,1],\mathbb{R})$ the space of $i-times$ differentiable functions $u : [0,1] \to \mathbb{R}$ whose $i-th$ derivative $u^{(i)}$ is absolutely continuous.
Let (X,d) be a metric space induced from the normed space $(X,\|\cdot\|)$. We denote

$$P_0(X) = \{A \in P(X) : A \neq \phi\},$$

$$P_b(X) = \{A \in P_0(X) : A\, is\, bounded\},$$

$$P_{cl}(X) = \{A \in P_0(X) : A\, is\, closed\},$$

$$P_{cp,cv}(X) = \{A \in P_0(X) : A\, is\, compact\ and\ convex\},$$

$$P_{b,cl}(X) = \{A \in P_0(X) : A\, is\, closed\ and\ bounded\}.$$

Definition 2.6. *A multivalued maps $G : X \to P(X)$.*
(1) is convex (closed) valued if $G(X)$ is convex (closed) for all $u \in X$,
(2) is bounded on bounded sets if $G(B) = \bigcup_{u\in B} G(u)$ is bounded in X for all $B \in P_b(X)$ i.e. $\sup_{u\in B}\{sup\{|v|, v \in G(u)\}\} < \infty$,
(3) is called upper semicontinuous (u.s.c) on X if for each $u_0 \in X$, the set $G(u_0)$ is a nonempty closed subset of X and if for each open set N of X containing $G(u_0)$ there exists an open neighborhood N_0 of u_0 such that $G(N_0) \subseteq N$,
(4) is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$,
(5)has a fixed point if there is $u \in X$ such that $u \in G(X)$. The fixed point set of the multi-valued operator G will be denote by Fix G.

Remark 2.7. *It is well known that, if the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c if and only if G has closed graph i.e., $u_n \to u$, $v_n \to v$, $v_n \in G(u_n)$ imply $v \in G(u)$.*

Definition 2.8. *A multi-valued maps $G:[0,1]\to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y\in\mathbb{R}$ the function*

$$t\longmapsto d(y,G(t))=inf\{\|y-z\|:z\in G(t)\},$$

is measurable.

Definition 2.9. *A multi-valued maps $F:[0,1]\times\mathbb{R}\times\mathbb{R}\to 2^{\mathbb{R}}$ is said to be Caratheodory if,*
(i) $t\longmapsto F(t,u,v)$ for all $u,v\in\mathbb{R}$,
(ii) $t\longmapsto F(t,u,v)$ is upper semi-continuous for almost all $t\in[0,1]$. Further a Caratheodory function is called L^1- Caratheodory,
(iii) for each $\rho>0$, there exists $\phi_\rho\in L^1([0,1],\mathbb{R}^+)$ such that $\|F(t,u,v)\|=sup\{|v|,v\in F(t,u,v)\}\leq\phi_\rho(t)$, for all $|u|,|v|<\rho$.

Definition 2.10. *Let Y be a nonempty closed subset of a Banach spase E and $G:Y\to P_{cl}(E)$ be a multivalued operator with nonempty closed values. G is said to be lower semicontinuous (l.s.c) if the set $\{x\in X:G(x)\cap U\neq\phi\}$ is open for any open set U in E.*

For $y\in(C[0,1],\mathbb{R})$, define the set of selection of F by

$$S_{F,u}=\left\{v\in AC([0,1],\mathbb{R}),v\in F\left(t,u(t),u'(t)\right),\ \text{for almost all } t\in[0,1]\right\}.$$

For $P(X)=2^X$, consider the Pompeiu-Hausdorff metric (see[**?**])
$H_d:2^{\mathbb{X}}\times 2^{\mathbb{X}}\to[0,\infty)$ given by

$$H_d(A,B)=max\left\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)\right\},$$

where $d(a,B)=\inf_{b\in B}d(a,b)$ and $d(b,A)=\inf_{a\in A}d(a,b)$. Then $\left(P_{b,cl}(X),H_d\right)$ is a metric space and $(P_{cl}(X),H_d)$ is a generalized metric space see [7].

Let Y be a nonempty closed subset of a Banach spase E and $G:Y\to P_{cl}(E)$ be a multivalued operator with nonempty closed values.
G is said to be lower semicontinuous (l.s.c) if the set $\{x\in X:G(x)\cap U\neq\phi\}$ is open for any open set U in E.
G has a fixed point if there is $x\in Y$ such that $x\in G(x)$. For more details on the multi-valued maps, see the book of Aubin and Celina [14], Demling [15], Gorniewicz [16] and Hu and Papageorgiou [35].

Lemma 2.11. *[1] Let X be a Banach space. $F:[0,1]\times\mathbb{X}\to P_{cp,cv}(X)$ an L^1-Caratheodory multifunction and Θ a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator $(\Theta\circ S_F)(u)=\Theta\left(S_{F,u}\right)$ is a closed graph operator in $C([0,1],X)\times C([0,1],X)$.*

Lemma 2.12. *[3] Let E be a Banach space. C a closed convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\overline{U}\to P_{cp,cv}(C)$ is an upper semi-continuous compact map, where $P_{cp,cv}(C)$ denotes the family of nonempty, compact convex subset of C. Then either F has a fixed point in $\overline{U}$ or there exist $u\in\partial U$ and $\lambda\in(0,1)$ such that $u\in\lambda F(U)$.*

Lemma 2.13. *[12] A multifunction $F:X\to C(X)$ is called a contraction whenever there exists $\gamma\in(0,1)$ such that $H_d(N(u),N(v))\leq\gamma d(u,v)$ for all $u,v\in X$*

Lemma 2.14. *(Covitz-Nadler) Let (X,d) be a complete metric space. If $N:X\to P_{cl}(X)$ is a contraction, then $FixN\neq\emptyset$.*

3. Existence results

Let $X=\{u:\ u,u'\in C([0,1],\mathbb{R})\}$ endowed with the norm defined by $\|u\|=\sup_{t\in[0,1]}|u(t)|+\sup_{t\in[0,1]}|u'(t)|$ such that $\|u\|<\infty$. Then $(X,\|.\|)$ is a Banach space.

Lemma 3.1. *Let $y\in C([0,1],\mathbb{R})$. Then the integral solution of the linear fractional differential equation*

$$^cD^\alpha u(t)=y(t)\quad t\in[0,1],\ \alpha\in(2,3], \tag{3.1}$$

subject to three-point boundary conditions

$$\beta u(0)+\gamma u(1)=u(\eta),\quad \beta\geq 0,\ \gamma\geq 0, \tag{3.2}$$

$$u(0)=\int_0^\eta u(s)\,ds,\quad \eta\in(0,1), \tag{3.3}$$

$$\beta^cD^pu(0)+\gamma^cD^pu(1)=^cD^pu(\eta),\quad p\in(1,2], \tag{3.4}$$

is given by

$$\begin{aligned}u(t) &= \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds-\frac{\Lambda_1(t)}{Q_1(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds\\ &-\frac{\Lambda_2(t)M_1}{6(1-\eta)Q_1}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right]\\ &+\frac{\Lambda_1(t)}{Q_1(\beta+\gamma-1)}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right],\end{aligned} \tag{3.5}$$

where

$$\Lambda_1(t)=(\beta+\gamma-1)\left(\eta^2+2(1-\eta)t\right),\ M_1=\frac{\Gamma(3-p)}{\gamma-\eta^{2-p}}$$

$$\Lambda_2(t)=\left(\eta^3(\beta+\gamma-1)+3\left(\gamma-\eta^2\right)(1-\eta)\right)\left(\eta^2+2(1-\eta)t\right)-Q_1\left(\eta^3+3(1-\eta)t^2\right),$$

and

$$Q_1=2(1-\eta)(\gamma-\eta)+\eta^2(\beta+\gamma-1)\neq 0.$$

Proof. In view of Lemma 2.3 and Lemma 2.5, the solution of equation (3.1) can be written as

$$u(t)=I^{\alpha}y(t)+c_0+c_1t+c_2t^2=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+c_0+c_1t+c_2t^2, \tag{3.6}$$

where $c_0,c_1,c_2\in\mathbb{R}$ are arbitrary constants.
Differentiating both sides of (3.6) and applying Definition 2.1, Lemma 2.3 and Lemma 2.5, we obtain

$${}^cD^pu(t)=I^{\alpha-p}y(t)+c_2\frac{2t^{2-p}}{\Gamma(3-p)}=\int_0^t\frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds+\frac{2t^{2-p}}{\Gamma(3-p)}c_2, \tag{3.7}$$

where $\alpha\in(2,3]$ and $p\in(1,2]$.
Integrating both sides of (3.6), we obtain

$$\int_0^{\eta}u(t)\,dt=\int_0^{\eta}\left(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right)dt+c_0\eta+\frac{1}{2}c_1\eta^2+\frac{1}{3}c_2\eta^3. \tag{3.8}$$

By using the boundary condition (3.2) in (3.6), we obtain

$$c_0(\beta+\gamma-1)+c_1(\gamma-\eta)+c_2\left(\gamma-\eta^2\right)=\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds \tag{3.9}$$

By using the boundary condition (3.3) in (3.6) and (3.8), we obtain

$$(1-\eta)c_0-\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\frac{1}{2}c_1\eta^2-\frac{1}{3}\eta^3=0. \tag{3.10}$$

By using the boundary condition (3.4) in (3.7), we obtain

$$c_2=\frac{M_1}{2}\left(\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right). \tag{3.11}$$

Solving the above system of the equations (3.9), (3.10) and (3.11) for c_0, c_1, c_2, we get

$$c_2=\frac{M_1}{2}\left(I^{\alpha-p}y(\eta)-\gamma I^{\alpha-p}y(1)\right)=\frac{M_1}{2}\left(\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right),$$

$$\begin{aligned}c_0 &= -\frac{2\eta^2(\beta+\gamma-1)}{2(1-\eta)Q_1}\int_0^{\eta}\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds+\frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds\\ &\quad-\frac{\left(\eta^2\left[\eta^3(\beta+\gamma-1)+3\left(\gamma-\eta^2\right)(1-\eta)\right]-\eta^3Q_1\right)M_1}{2(1-\eta)Q_1}\left[I^{\alpha-p}y(\eta)-\gamma I^{\alpha-p}y(1)\right]\\ &\quad+\frac{\eta^2}{Q_1}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right],\end{aligned}$$

and

$$\begin{aligned}c_1 &= \frac{-2(\beta+\gamma-1)}{Q_1}\int_0^{\eta}\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds-\frac{\left(\eta^3(\beta+\gamma-1)+3\left(\gamma-\eta^2\right)(1-\eta)\right)M_1}{3Q_1}\left[I^{\alpha-p}y(\eta)-\gamma I^{\alpha-p}y(1)\right]\\ &\quad+\frac{2(1-\eta)}{Q_1}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right],\end{aligned}$$

where

$$I^{\alpha-p}y(\eta)-\gamma I^{\alpha-p}y(1)=\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^{1}\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds.$$

Substituting the values of constants c_0, c_1 and c_2 in (3.6), we get (3.5). The proof is complete. □

Throughout the paper, we let

$$M=\frac{\Gamma(3-p)}{\left|\gamma-\eta^{2-p}\right|}\neq 0,\ |\beta+\gamma-1|\neq 0,\ \left|\gamma-\eta^2\right|\neq 0,\ Q=\left|2(1-\eta)(\gamma-\eta)+\eta^2|\beta+\gamma-1|\right|\neq 0,$$

$$A(t)=|\beta+\gamma-1|\left(\eta^2+2(1-\eta)t\right),$$

and

$$B(t)=\left(\eta^3|\beta+\gamma-1|+3\left|\gamma-\eta^2\right|(1-\eta)\right)\left(\eta^2+2(1-\eta)t\right)-Q\left(\eta^3+3(1-\eta)t^2\right).$$

The following inequalities hold:

$$|A(t)|\leq|\beta+\gamma-1|\left(\eta^2+2(1-\eta)\right)=A_1,$$

$$|B(t)|\leq\left|\left(\eta^3|\beta+\gamma-1|+3\left|\gamma-\eta^2\right|(1-\eta)\right)\left(\eta^2+2(1-\eta)\right)-Q\left(\eta^3+3(1-\eta)\right)\right|=B_1,$$

$$\left|A'(t)\right|\leq 2|\beta+\gamma-1|(1-\eta)=A_1',$$

and

$$\left|B'(t)\right|\leq 2(1-\eta)\left|\left(\eta^3|\beta+\gamma-1|+3\left|\gamma-\eta^2\right|(1-\eta)\right)-3Q\right|=B_1'.$$

To simplify the proofs in the forthcoming theorems, we establish the bounds for the integrals arising in the sequel.

Lemma 3.2. *For $y\in C([0,1],\mathbb{R})$, we have*

$$\left|\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds\right|\leq\frac{\eta^{\alpha+1}}{\Gamma(\alpha+2)}\|y\|.$$

Proof. Obviously,

$$\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau=\left[-\frac{(s-\tau)^{\alpha}}{\Gamma(\alpha)}\right]_0^{s}=\frac{s^{\alpha}}{\alpha\Gamma(\alpha)}=\frac{s^{\alpha}}{\Gamma(\alpha+1)}.$$

Hence

$$\left|\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds\right|\leq\|y\|\int_0^{\eta}\frac{s^{\alpha}}{\Gamma(\alpha+1)}ds=\frac{\eta^{\alpha+1}}{\Gamma(\alpha+2)}\|y\|.$$

□

For the sake of brevity, we set

$$\triangle_1=\frac{\eta^{\alpha+1}}{(1-\eta)\Gamma(\alpha+2)}+\frac{A_1\eta^{\alpha+1}}{Q(1-\eta)\Gamma(\alpha+2)}+\frac{MB_1\left(\eta^{\alpha-p}+\gamma\right)}{(1-\eta)Q\Gamma(\alpha-p+1)}+\frac{A_1\left(\eta^{\alpha}+\gamma\right)}{Q|\beta+\gamma-1|\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)},\tag{3.12}$$

and

$$\triangle_2=\frac{A_1'\eta^{\alpha+1}}{Q(1-\eta)\Gamma(\alpha+2)}+\frac{MB_1'\left(\eta^{\alpha-p}+\gamma\right)}{(1-\eta)Q\Gamma(\alpha-p+1)}+\frac{A_1'\left(\eta^{\alpha}+\gamma\right)}{Q|\beta+\gamma-1|\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}.\tag{3.13}$$

An element $u\in AC^2([0,1,\mathbb{R}])$ is called a solution of the problem (1.1) whenever it satisfies the integral boundary conditions and there exists a function $y\in S_{F,u}$ such that

$$\begin{aligned}u(t) &= \int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds-\frac{B(t)M}{6(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^{1}\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right]\\ &\quad+\frac{A(t)}{Q|\beta+\gamma-1|}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds.\right]-\frac{A(t)}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds.\end{aligned}\tag{3.14}$$

for all $t\in J$.

For investigation of the problem (1.1) – (1.2) we provide two different methods.

Theorem 3.3. . *Suppose that $F : J\times\mathbb{R}\times\mathbb{R}\to P_{cp,cv}(\mathbb{R})$ is L^1-Caratheodory multifunction and there exist a bounded continuous nondecreasing map*
$\psi:[0,\infty)\to(0,\infty)$ and a continuous function $p:J\to(0,\infty)$ such that
$\|F(t,u(t)),u'(t)\| = sup\{|v| : v\in F(t,u(t),u'(t))\le p(t)\psi(\|u\|)\}$, for all $t\in J$ and $u\in X$. Then the inclusion problem (1.1) − (1.2) has at least one solution.

Proof. Define the operator

$$N(u)=\left\{\begin{array}{c} h\in X,: \text{there exists } y\in S_{F,u} \text{ such that } h(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds \\ -\frac{B(t)M}{6(1-\eta)Q}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right] \\ +\frac{A(t)}{Q|\beta+\gamma-1|}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right] \\ +\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds-\frac{A(t)}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds, t\in J \end{array}\right\}$$

We show that the operator N has a fixed point. First, we show that N maps bounded sets of X into bounded sets. Suppose that $r>0$ and $B_r=\{u\in X : \|u\|\le r\}$. Let $u\in B_r$ and $h\in N(u)$. Choose $v\in S_{F,u}$ such that $h(t)$ defined above for almost all $t\in J$. Thus

$$\begin{aligned}|h(t)| &\le \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y(\tau)|\,d\tau\right)ds+\frac{|A(t)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y(\tau)|\,d\tau\right)ds\\ &\quad+\frac{|B(t)|M}{12(1-\eta)Q}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds\right]\\ &\quad+\frac{(\gamma-\eta)|A(t)|}{2Q(\beta+\gamma-1)}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds\right]\\ &\le \Delta_1\|p\|_\infty\psi(\|u\|),\end{aligned}$$

and

$$\begin{aligned}|h'(t)| &\le \int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|y(s)|\,ds+\frac{|A'(t)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y(\tau)|\,d\tau\right)ds\\ &\quad+\frac{|B'(t)|M}{12(1-\eta)Q}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds\right]\\ &\quad+\frac{(\gamma-\eta)|A'(t)|}{2Q(\beta+\gamma-1)}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds\right]\\ &\le \Delta_2\|p\|_\infty\psi(\|u\|),\end{aligned}$$

for all $t\in J$, where $\|p\|_\infty=\underset{t\in J}{sup}|p(t)|$.
Hence,

$$\|h\|=\max_{t\in J}|h(t)|+\max_{t\in J}|h'(t)|\le(\Delta_1+\Delta_2)\|p\|_\infty\psi(\|u\|)$$

Now, we show that N maps bounded sets into equicontinuous subsets of X. Let $u\in B_r$ and $t_1,t_2\in J$ with $t_1<t_2$. Then we have

$$\begin{aligned}|h(t_2)-h(t_1)| &\le \int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds+\int_0^{t_1}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds\\ &\quad+\frac{|B(t_2)-B(t_1)|M}{12(1-\eta)Q}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds\right]\\ &\quad+\frac{(\gamma-\eta)|A(t_2)-A(t_1)|}{2Q(\beta+\gamma-1)}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds\right]\\ &\quad+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y(\tau)|\,d\tau\right)ds+\frac{|A(t_2)-A(t_1)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y(\tau)|\,d\tau\right)ds\end{aligned}$$

$$
\begin{aligned}
\leq\ & \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\|p\|_\infty \psi(\|u\|)\right] ds + \int_{t_1}^{t_2} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\|p\|_\infty \psi(\|u\|)\right] ds \\
& + \frac{|B(t_2)-B(t_1)|M}{12(1-\eta)Q} \left[\int_0^{\eta} \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} \left[\|p\|_\infty \psi(\|u\|)\right] ds + \gamma \int_0^{1} \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} \left[\|p\|_\infty \psi(\|u\|)\right] ds \right] \\
& + \frac{(\gamma-\eta)|A(t_2)-A(t_1)|}{2Q(\beta+\gamma-1)} \left[\int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\|p\|_\infty \psi(\|u\|)\right] ds + \gamma \int_0^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\|p\|_\infty \psi(\|u\|)\right] ds \right] \\
& + \frac{1}{1-\eta} \int_0^{\eta} \left(\int_0^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left[\|p\|_\infty \psi(\|u\|)\right] d\tau \right) ds + \frac{|A(t_2)-A(t_1)|}{Q(1-\eta)} \int_0^{\eta} \left(\int_0^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \|p\|_\infty \psi(\|u\|) d\tau \right) ds,
\end{aligned}
$$

It is seen that $|(h)(t_2)-(h)(t_1)| \to 0$, as $t_2 \to t_1$. Also, we have

$$
\begin{aligned}
\left|h'(t_2)-h'(t_1)\right| \leq\ & \int_0^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} |y(s)| ds + \int_0^{t_1} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} |y(s)| ds \\
& + \frac{|B'(t_2)-B'(t_1)|M}{12(1-\eta)Q} \left[\int_0^{\eta} \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} |y(s)| ds + \gamma \int_0^{1} \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} |y(s)| ds \right] \\
& + \frac{(\gamma-\eta)|A'(t_2)-A'(t_1)|}{2Q(\beta+\gamma-1)} \left[\int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} |y(s)| ds + \gamma \int_0^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |y(s)| ds \right] \\
& + \frac{|A'(t_2)-A'(t_1)|}{Q(1-\eta)} \int_0^{\eta} \left(\int_0^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |y(\tau)| d\tau \right) ds \\
\leq\ & \int_0^{t_1} \frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha-1)} \left[\|p\|_\infty \psi(\|u\|)\right] ds + \int_{t_1}^{t_2} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left[\|p\|_\infty \psi(\|u\|)\right] ds \\
& + \frac{|B'(t_2)-B'(t_1)|M}{12(1-\eta)Q} \left[\int_0^{\eta} \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} \left[\|p\|_\infty \psi(\|u\|)\right] ds + \gamma \int_0^{1} \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} \left[\|p\|_\infty \psi(\|u\|)\right] ds \right] \\
& + \frac{(\gamma-\eta)|A'(t_2)-A'(t_1)|}{2Q(\beta+\gamma-1)} \left[\int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\|p\|_\infty \psi(\|u\|)\right] ds + \gamma \int_0^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\|p\|_\infty \psi(\|u\|)\right] ds \right] \\
& + \frac{|A'(t_2)-A'(t_1)|}{Q(1-\eta)} \int_0^{\eta} \left(\int_0^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \|p\|_\infty \psi(\|u\|) d\tau \right) ds.
\end{aligned}
$$

Again, we see that $|(h)(t_2)-(h)(t_1)| \to 0$, as $t_2 \to t_1$. Also, we have $\|(h)(t_2)-(h)(t_1)\| \to 0$, as $t_2 \to t_1$. Thus N is equicontinuous and so N is relatively compact on B_r. Consequently the Ascoli-Arzela theorem implies that N is compact on B_r.

Now, we show that N has a closed graph. Let $u_n \to u_0$, $h_n \in N(u_n)$ for all n and $h \to h_0$. We prove that $h_0 \in N(u_0)$. For each n choose $y_n \in S_{F,u_n}$ such that, for all $t \in J$,

$$
\begin{aligned}
h_n(t) =\ & \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y_n(s) ds + \frac{1}{1-\eta} \int_0^{\eta} \left(\int_0^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_n(\tau) d\tau \right) ds - \frac{A(t)}{Q(1-\eta)} \int_0^{\eta} \left(\int_0^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_n(\tau) d\tau \right) ds \\
& - \frac{B(t)M}{12(1-\eta)Q} \left[\int_0^{\eta} \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_n(s) ds - \gamma \int_0^{1} \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_n(s) ds \right] \\
& + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)} \left[\int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y_n(s) ds - \gamma \int_0^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y_n(s) ds \right].
\end{aligned}
$$

Consider the continuous linear operator $\theta : L^1(J,\mathbb{R}) \to X$ defined by

$$\begin{aligned}\theta(y)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\right].\end{aligned}$$

By using Lemma 2.13, $\theta \circ S_F$ is closed graph operator. Since $u_n \to u$ and $h_n \in \theta\left(S_{F,u_n}\right)$ for all $n \in \mathbb{N}$, there exist $y_0 \in S_{F,,u_0}$ such that

$$\begin{aligned}h_0(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y_0(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_0(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_0(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_0(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_0(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y_0(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y_0(s)\,ds\right].\end{aligned}$$

Thus N has a closed graph.
Now we show that $N(u)$ is convex for all $u \in X$. Let $h_1, h_2 \in N(u)$ and $w \in [0,1]$. Choose $y_1, y_2 \in S_{F,u}$. Then

$$\begin{aligned}wh_1(t) - (1-w)h_2(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [wy_1(s) - (1-w)y_2(s)]\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} [y_1(\tau) - (1-w)y_2(\tau)]\,d\tau\right) ds \\ &\quad - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} [y_1(\tau) - (1-w)y_2(\tau)]\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} [wy_1(s) - (1-w)y_2(s)]\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} [wy_1(s) - (1-w)y_2(s)]\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} [wy_1(s) - (1-w)y_2(s)]\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} [wy_1(s) - (1-w)y_2(s)]\,ds\right]\end{aligned}$$

for all $t \in J$. Since F has convex values, $S_{F,u}$ is convex and so $wh_1(t) - (1-w)h_2(t) \in N(u)$.
If there exists $\lambda \in (0,1)$ such that $u \in \lambda N(u)$ then there exists $y \in S_{F,u}$ such that

$$\begin{aligned}u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\right],\end{aligned}$$

for almost all $t \in J$. Choose $L > 0$ such that $\frac{L}{(\Delta_1+\Delta_2)\|p\|_\infty \psi(\|u\|)} > 1$ for all $u \in X$. Thus $\|u\| < L$. Now, put $U = \{u \in X : \|u\| < L+1\}$. Note there are no $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in \lambda N(u)$ and the operator $N : \overline{U} \to P_{cp,cv}\left(\overline{U}\right)$ is upper semi-continuous because it is completely continuous. Now, by using Lemma 2.12, N has fixed point in $\overline{U}$ which is solution of the inclusion problem (1.1). This complete the proof. □

We provide another result about the existence of solutions for the problem (1.1) – (1.2) by changing the assumptions of convex values for multifunction.

Theorem 3.4. *Let $m \in C\left(J,\mathbb{R}^+\right)$ be such that $\|m\|_\infty(\Delta_1+\Delta_2) < 1$.*
Suppose that $F : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is an integrable bounded multifunction such that the map $t \longmapsto F(t,u,v,w)$ is measurable and $H_d\left(F\left(t,u_1,u_2,u_3\right),F\left(t,v_1,v_2,v_3\right)\right) \leq m(t)\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\right)$ for almost all $t \in J$ and $u,v,w,u_1,u_2,u_3,v_1,v_2,v_3 \in \mathbb{R}$. Then the problem (1.1) – (1.2) has a solution.

Proof. Note that, the multivalued map $t \longmapsto F(t,u(t),v(t),w(t))$ is measurable and closed for all $u \in X$. Hence, it has a measurable selection and so the set $S_{F,u}$ is nonempty. Now, consider the operator $N : X \to 2^X$ defined by

$$N(u) = \left\{h \in X : \text{there exists } v \in S_{F,u} \text{ such that } h(t) = u(t),\ t \in J\right\},$$

where $u(t)$ defined by (3.5), for all $t \in J$.
First, we show that $N(u)$ is a closed subset of X for all $u \in X$. Let $u \in X$ and $\{u_n\}_{n\geq 1}$ be a sequence in $N(u)$ with $u_n \to u$ for each n, such that

$$\begin{aligned} u_n(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y_n(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_n(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_n(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_n(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_n(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y_n(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y_n(s)\,ds\right], \end{aligned}$$

for almost all $t \in J$. Since F has compact values, $\{y_n\}_{n\geq 1}$ has a subsequence which converges to some $y \in L^1(J,\mathbb{R})$. It is easy to check that $y \in S_{F,u}$ and $u_n(t) \to u(t)$ for all $t \in J$. This implies that $u \in N(u)$. Thus the multifunction N has closed values.
Now, we show that N is a contractive multifunction with constant
$l = \|m\|_\infty(\Delta_1 + \Delta_2) < 1$.
Let $u, v \in X$ and $h_1 \in N(v)$. Choose $y_1 \in S_{F,,v}$ such that

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y_1(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_1(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_1(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_1(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_1(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y_1(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y_1(s)\,ds\right], \end{aligned}$$

for almost all $t \in J$. Since
$H_d(F(t,u(t),u'(t)),F(t,v(t),v'(t))) \leq m(t)(|u(t)-v(t)|+|u'(t)-v'(t)|)$ for almost all $t \in J$ there exists $w \in F(t,u(t),u'(t))$ such that

$$|y_1(t) - w| \leq m(t)\left(|u(t)-v(t)| + \left|u'(t)-v'(t)\right|\right),$$

for almost all $t \in J$.
Define the multifunction $U : J \to 2^{\mathbb{R}}$ by

$$U(t) = \left\{w \in \mathbb{R} : |y_1(t) - w| \leq m(t)\left(|u(t)-v(t)| + \left|u'(t)-v'(t)\right|\right) \text{ for almost all } t \in J\right\}.$$

It is easy to chek that the multifunction $U(\cdot) \cap F(\cdot),u(\cdot),u'(\cdot)$ is measurable. Thus, we can choose $y_2 \in S_{F,u}$ such that

$$|y_1(t) - y_2(t)| \leq m(t)\left(|u(t)-v(t)| + \left|u'(t)-v'(t)\right|\right),$$

for almost all $t \in J$. Now, consider $h_2 \in N(u)$ which is defined by

$$\begin{aligned} h_2(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y_2(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_2(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_2(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_2(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_2(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y_2(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y_2(s)\,ds\right]. \end{aligned}$$

Hence, we get

$$\begin{aligned}
|h_1(t)-h_2(t)| &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|y_1(s)-y_2(s)|\,ds+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y_1(\tau)-y_2(\tau)|\,d\tau\right)ds\\
&+\frac{|A(t)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y_1(\tau)-y_2(\tau)|\,d\tau\right)ds\\
&+\frac{|B(t)|M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y_1(s)-y_2(s)|\,ds+\gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y_1(s)-y_2(s)|\,ds\right]\\
&+\frac{(\gamma-\eta)|A(t)|}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|y_1(s)-y_2(s)|\,ds+\gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|y_1(s)-y_2(s)|\,ds\right]\\
&\leq \Delta_1\|m\|_\infty\|u-v\|,
\end{aligned}$$

and so $\|h_1-h_2\|\leq(\triangle_1+\triangle_2)\|m\|_\infty\|u-v\|=l\|u-v\|$. This implies that the multifunction N is a contraction which closed values. Thus, by using the result of Covitz and Nadler, N has a fixed point which is solution for the inclusion problem.

□

We construct two examples to illustrate the applicability of the results presented.

Example 3.5. *Consider the problem*

$$^cD^3u(t)\in F(t,u,v),\quad t\in[0,1], \tag{3.15}$$

subject to the three-point boundary conditions

$$\begin{cases} \frac{1}{100}u(0)+\frac{1}{10}u(1)=u\left(\frac{1}{2}\right),\\ u(0)=\int_0^{0,5}u(s)\,ds,\\ \frac{1}{100}{}^cD^{\frac{3}{2}}u(0)+\frac{1}{10}{}^cD^{\frac{3}{2}}u(1)={}^cD^{\frac{3}{2}}u\left(\frac{1}{2}\right)\end{cases}, \tag{3.16}$$

where $\eta=0,5,\beta=0,01,\gamma=0,1,p=1,5$ *and* $F(t,u,v):[0,1]\times\mathbb{R}^2\to 2^{\mathbb{R}}$ *multivalued map given by*

$$u\longmapsto F(t,u,v)=\left(\left(\frac{3+t^2}{4}\right)\left(\frac{|u|}{1+u}+sin(v)\right),\frac{|u|^3}{2\left(1+|u|^3\right)}+5t^3+4\right),\ u,v\in\mathbb{R}$$

verifying (H_1).
Obviously, for $f\in F$, *we have*

$$|f|=max\left(\left(\frac{3+t^2}{4}\right)\left(\frac{|u|}{1+|u|}+sin(v)\right),\frac{|u|^3}{2\left(1+|u|^3\right)}+5t^3+4\right)\leq\frac{19}{2},\ u,v\in\mathbb{R}.$$

Thus

$$\|F(t,u)\|=sup\{|f|:f\in F(t,u,v)\}\leq\frac{19}{2},\ u,v\in\mathbb{R},$$

where $p(t)=1$ *and* $\psi(t)=\frac{19}{2}$, *then one can check that the assumptions of Theorem 3.3 hold. and so the problem* (3.15) – (3.16) *has at least one solution.*

Example 3.6. *Consider the problem* (3.15) – (3.16), *where* $\eta=0,5,\beta=0,01,\gamma=0,1,p=1,5$ *and* $F(t,u,v):[0,1]\times\mathbb{R}^2\to 2^{\mathbb{R}}$ *multivalued map given by*

$$u\longmapsto F(t,u,v)=\left(0,\frac{t|u|}{2(1+|u|)}+\frac{|v|^3}{2\left(1+|v|^3\right)}\right),\ u,v\in\mathbb{R}.$$

Obviously,

$$H_d(F(t,u_1,u_2),F(t,v_1,v_2))\leq\left(\frac{t}{2}+\frac{1}{2}\right)\sum_{i=1}^2|u_i-v_i|,\ u,v\in\mathbb{R},\ t\in[0,1].$$

If $m(t)=\frac{t}{2}+\frac{1}{2}$ *for all* $t\in[0,1]$ $H_d(F(t,u_1,u_2),F(t,v_1,v_2))\leq m(t)\sum_{i=1}^2|u_i-v_i|$.

On the other hand, it can be easily found that $M=1,4597546147$, $Q=\frac{9}{400}$, $\Delta_1\cong 0,4141664514$ *and* $\Delta_2\cong 0?9758011659$.
Finally, since $\|m\|_\infty(\triangle_1+\triangle_2)\cong 0,143492<1$, *thus all assumptions of Theorem 3.4 are satisfied. Hence, The inclusion problem has at least one solution.*

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