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Quantum contextuality in classical information retrieval

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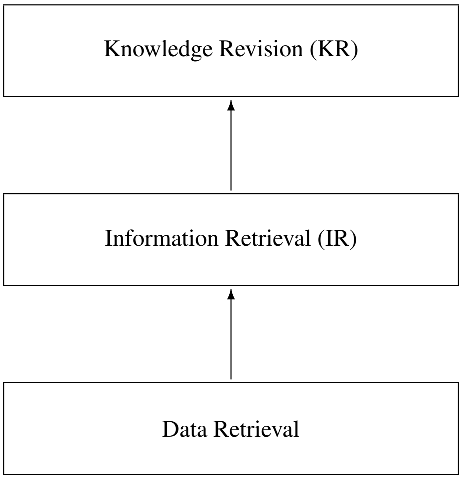
Abstract

Document ranking based on probabilistic evaluations of relevance is known to exhibit non-classical correlations, which may be explained by admitting a complex structure of the event space, namely, by assuming the events to emerge from multiple sample spaces. The structure of event space formed by overlapping sample spaces is known in quantum mechanics, they may exhibit some counter-intuitive features, called quantum contextuality. In this Note I observe that from the structural point of view quantum contextuality looks similar to personalization of information retrieval scenarios. Along these lines, Knowledge Revision is treated as operationalistic measurement and a way to quantify the rate of personalization of Information Retrieval scenarios is suggested.

1. The evolution of information needs

The notion of information needs was clearly formulated by Tailor [12]. Along with the development of IR systems the very structure of information needs, of queries was subject to evolution. Briefly, its mainstream can be described as a transition (read upwards)



(1.1)

each stage using the previous one as a background. Information Retrieval uses Data Retrieval environment yet modifying the structure of queries, as formulated by Lancaster "An information retrieval system does not inform (i.e. change the knowledge of) the user on the subject of his inquiry. It merely informs on the existence (or non-existence) and whereabouts of documents relating to his request" [5]. Then the next stage is the increasing personalization of search. The user interacts with an IR environment having a goal to update the state of his knowledge (belief) rather than to retrieve a particular document. This way Information Retrieval serves for Knowledge Revision (KR).
How quantum mechanics comes? The chain (1.1) can be compared with the transition from classical mechanics, dealing with the absolute character of the values measured, to quantum mechanics, where the result of a measurement is a result of an act of will of an observer rather

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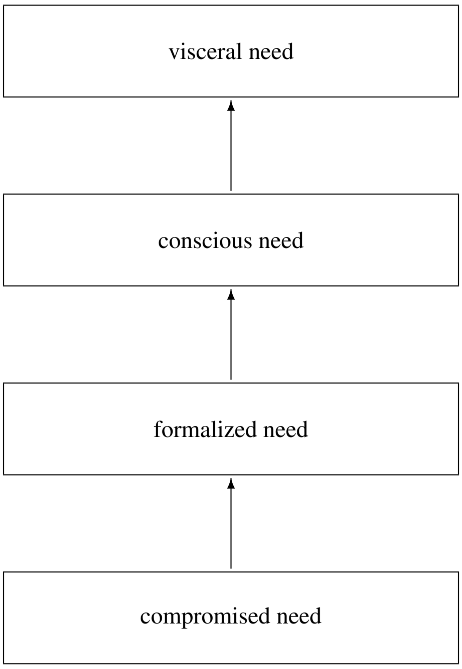
than retrieving a pre-existing value. In both extreme cases, the retrieval act is nothing but a measurement. Similar to the evolution of the notion of measurement, the retrieval metaphors evolve.
We shall deal and with the general notion of Information Needs (IN), ranging them in four levels [12]



(1.2)

with the following meaning

- The visceral need is the actual, but unexpressed, need for information.
- The conscious need is a within-brain description of the need.
- The formalized need is a formal statement of the question.
- The compromised need is the question as presented to the information system.

The chain (1.1) reflects the upwards transition in the above list, and the personalization tightly approaches to the visceral IN. In this Note I deal with the quantification of personalization – the crucial part of Knowledge Revision – using quantum metaphor. The technical basis for this quantitative approach is formed of the following research lines:

- Simulation of quantum contextuality effects by finite automata and the evaluation of the amount of memory required for this simulation [4]. Our basic idea is to revert this argumentation and to evaluate the features of a quantum system, which can be in certain sense simulated by giver IR environment.
- The evaluations of violations of classical probabilistic laws by index term probabilities, carried out by Melucci [7] and the quantitative evaluation of the amount of contextuality by Svozil [11]

2. On the nature of non-classical correlations

In general, non-classical correlations appear when Kolmogorovian probability model is no longer applicable. The basic point of Kolmogorovian model is the existence of a (single) sample space Ω. The events are subsets of Ω, while the points of the sample space are elementary and independent.
In order to test this or that model, we employ Accardi's statistical invariants [2], they allow to test the applicability of Kolmogorovian model. Given:

- a family of discrete maximal observables $\{A_\alpha : \alpha = 1,\dots T\}$ (T being finite), each observable A_α takes the finite number of values $a^{(\alpha)}_{j\alpha}$ labelled by $j_\alpha = 1,\dots,n$
- the experimentally measurable conditional probabilities $p_{j_\alpha,j_\beta}(\beta \mid \alpha)$

$$p_{j_\alpha,j_\beta}(\beta \mid \alpha) \;=\; P\left(A_\beta = a^{(\beta)}_{j\beta} \middle| A_\alpha = a^{(\alpha)}_{j\alpha}\right) \tag{2.1}$$

The problem is: does there exist a probability space $(\Omega;\mathscr{F};P)$ and T measurable partitions $A^{(\alpha)}_j$ of cardinality n (the number of distinct values of each observable is assumed to be the same)

$$A^{(\alpha)}_j, \alpha = 1,\dots T,\ j = 1,\dots n$$

such that for any $\alpha,\beta = 1,\dots T$ one has

$$P\left(A^{(\beta)} = a^{(\beta)}_j \mid A^{(\alpha)} = a^{(\alpha)}_i\right) \;=\; \frac{P\left(A^{(\beta)}_j \cup A^{(\alpha)}_i\right)}{P\left(A^{(\beta)}_j\right)} \tag{2.2}$$

In order to get the answer, a linear programming problem is to be solved [1], that is, the problem of the existence of a single sample space is finitely decidable.

In the sequel we shall need the special case of three observables A,B,C, each taking only two values a_1,a_2 for A, b_1,b_2 for B and c_1,c_2 for the observable C. The transition probability matrices for each pair of observables, being bistochastic, each has only one numeric parameter, denote the appropriate matrices as

$$\begin{array}{l} P(A \mid B) = P = \left(\begin{array}{cc} p & 1-p \\ 1-p & p \end{array} \right) \\ P(B \mid C) = Q = \left(\begin{array}{cc} q & 1-q \\ 1-q & q \end{array} \right) \\ P(C \mid A) = R = \left(\begin{array}{cc} r & 1-r \\ 1-r & r \end{array} \right) \end{array} \tag{2.3}$$

then these transition probabilities can be described by a Kolmogorovian model (that is, they are produced by a single sample space) if and only if

$$|p+q-1| \leq r \leq 1-|p-q| \tag{2.4}$$

3. Melucci operationalistic metaphor

There is a straightforward analogy between IR and the process of measurement, called Melucci metaphor. There is a search machine, which we may treat to be prepared in certain state, and there is an observer, which performs a measurement. It is typical that the preparation of query system does not assume a query asked by the user, this causes a mismatch, which is to be handled [7].
The situation when a mismatch between the preparation and measurement occurs is a source of paradoxes and counter-intuitive observable consequences of quantum mechanics. It results in the possible randomness of single accounts, though previous stages were deterministically prepared. To deal with it, context translation is introduced as handling the mismatch between state preparation and measurement. In quantum mechanics this metaphorically looks as follows [10]. Suppose an electron is prepared, using Stern-Gerlach device, in pure spin stat along z axis, always showing spin up. Then we decide to ask the so-prepared electron a complementary question: "what is direction of spin along the x axis?" Quantum mechanics tells us that the electron is completely incapable to store more than one bit of information (assuming this is not so leads to direct experimental contradictions). That is why the electron gives a random reply on this query. This is what makes it different from deterministic query agents, who are not able to handle improper input, on which they offer no answer.
Modern IR environments are no longer so rigid, they easily handle any kind of input: if you ask them, almost always you get an answer, but sometimes the relevance of this answer for you personally may be of zero value. To overcome this, search engines are configured to track user's requests, or, in other words, to keep the context associated with particular user and his present role. Altogether, each such particular action I call knowledge revision scenario. In practice this is done by seeding pebbles along the way the user goes through the jungles of World Wide Web, say, by storing browser's cookies. These pebbles are, after all, just sequences of bits. Now suppose our task is to judge to what extent the act of measurement is personalized, let us view it from a perspective of quantum measurement. To do it, recall a series of recent works summarized in [4].

4. Quantifying the personality in Knowledge Revision scenarios

In brief, quantum contextuality manifests itself as follows: when measuring quantum systems, the result may depend on which other compatible observables are measured simultaneously. Furthermore, these other observables may be just intended to be measured rather than really measured. This cloud of potentially co-measurable values is referred to as **context**. When simulating a quantum system by agents with internal memory (recall that, as told above, quantum system are so smart that they behave in this way without having internal memory), the agent will attain different internal states in course of carrying out a sequence of elementary queries. The minimal amount of memory needed to simulate particular manifestations of quantum contextuality is called memory cost of this quantum effect. The paper [4] explores the memory cost of simulating quantum contextuality effect observed on singlet states of positronium. It gives a clue to draw a correspondence:

$$\text{quantum contextuality} \longrightarrow \text{its memory cost}$$

In general, the memory cost increases as more and more contextuality constraints are considered. The complexity of contextuality constrains depends, in its turn, on the dimension of the state space of the system in question.
I suggest the following technical idea. The argumentation of the authors of [4] is reverted. We start with an IR environment and ask how complex quantum contextual features it may exhibit? Furthermore, we may reduce the answer to just a number (or a string of numbers), namely, the dimensionality (or a tensor product structure – TPS [13]) of a quantum system demonstrating similar context dependence.

$$\text{KR scenario} \longrightarrow \text{quantum system}$$

How to do this? What is to be simulated? Here, I dwell only on the logical and certain probabilistic aspects of simulation. To do it, the proper tools to deal with the structure of the collection of properties of a system are introduced.

Overlapping contexts. It was observed by different authors that complex IR systems are not well described by probabilistic models based on a single sample space. In [9] it was explicitly shown that Bayesian reasoning in its direct for fails and, in order to get adequate evaluations, when writing conditional probabilities $P(A \mid B)$ one should take care about specifying the context – a particular sample space, in which these conditional probabilities are calculated. In the meantime, the small sample spaces are not separated - thy overlap, there are events belonging to different contexts. It occurs that the classical contingency table

	RETRIEVED	NOT RETRIEVED	
RELEVANT	$A \cap B$	$A \cap \bar{B}$	A
NON-RELEVANT	$\bar{A} \cap B$	$\bar{A} \cap \bar{B}$	$\bar{A}$
	B	$\bar{B}$	

ceases to be adequate. The reason is that even within a single scenario both A and B may belong to different contexts, in particular, $\bar{A}$ is no longer uniquely defined by A (the same to B and $\bar{B}$). How to capture this structure? A tool of combinatorial nature is needed to describe overlapping contexts. First note that a single sample space is structureless, all its elements are equally (un)related with each other. In case of overlapping contexts this is no longer the case. A graphical (and combinatorial) way to capture such relations was suggested by R.Greechie (see [3] for an overview). The idea is to

(i) consider all the elements of all sample spaces together
(ii) label each element with a tag pointing to appropriate context

The Kolmogorovian probabilities (and hence Bayesian inference) come from the fact that the logic of statements about the appropriate sample space is Boolean. In case of pasted contexts this is no longer so, the structure of all the statements about the IR environment is no longer Boolean.

How contextuality effects come? Mainly, in the form of Kochen-Specker reasoning stating that particular hypothetical probability assignments do not exist such as a total probability distribution on the whole diagram viewed as a single sample space. The consequence of such results is signaling that the evaluation of conditional probabilities based on standard Bayes model will be no loner adequate. For examples of such violations in quantum mechanics see [4], in IR this also takes place, see, for instance [7]. Quantitatively it looks as follows.

'How much contextuality'? So far, only qualitative ideas were provided. The next step is to try to evaluate them, putting the question 'How much contextuality'? A possible transparent answer was recently proposed in [11]. We take a representative sampling of observables, and simply check the ratio of the triples, for which Accardi inequalities (2.4) are violated.
Using the ideas of [11], the rate of personalization can be evaluated in a similar way. First, by random sampling, triples of properties, that is, yes-no queries are picked. Then, for each triple, the transition probability matrices (2.3) are calculated. For each particular sample triple the inequalities (2.4) are checked. Then the ratio of samples is calculated:

$$\mathbf{Pers} = \frac{\text{number of triples violating (2.4)}}{\text{total number of sampled triples}} \tag{4.1}$$

Conclusions

Vector models of IR become more and more popular, first of all because they make it possible to carry out multi-document actions. In this paper I dwell on a QIA framework [8]. The basic ingredient of QIA framework is a Hilbert space $\mathscr{H}$ called the information need space. In its simplest form, IN space is linear space of elementary (atomic) topics. In my approach, I suggest to start introducing the IN space to satisfy the necessary amount of capturing contextuality. The ideology of IN space is the closest to that of quantum mechanics. In QM, the state space of a system is a space of some internal (in the deepest possible sense) features of a system, while the observables are expressed in terms of operators and other derived structures on the state space. Similar things happen in QIA approach. The space of information needs exists per se, we may treat it as spanned on elementary entities, but this will be nothing, but a representation of this space. The source of emergence of this space lies in the multicontextual structure described in the previous section. Furthermore, as pointed in [6], [7], the correlations, which occur in IR environment may even be stronger than quantum ones. In this case a straightforward Hilbert space model may fail to work properly, and we may call 'foil quantum theories[1]" to grip these situations.

So far, I was interested in information retrieval situations, when the result of a particular action may depend on other actions, which the IR agent could in principle do alongside with the actions actually performed. This phenomenon is called contextuality, we encounter it in IR, we have to take it into account, to work with it. A similar kind of dependence takes place in quantum mechanics.

Quantum Mechanics	$\leftrightarrow$	Information Retrieval
contextuality		personalization

The difference is that in QM contextuality appears by itself, not being originated by some 'internal mechanisms'. The situations where contextuality occurs depend on the state space of the system the structure of observables involved. In the realm of QM we can quantitatively evaluate the rate of contextuality [11]. The origin of contextuality effects in IR stems from personalization of query scenarios. The personalization, in turn, can be quantified by memory resources required to keep tracking the information needs of a particular user (note that 'user' in this context might not be a single person, nor even a 'person' at all). The idea of this Note was to demonstrate that using quantum mechanics formalism, we can quantify the rate of personalization in particular IR environments. To do this, I suggest to reverse the procedure of estimation of memory cost of quantum contextuality based on simulating quantum systems by finite automata. Instead, a KR scenario (which as a matter of fact is a sequence of queries upon a finite automaton) is suggested to be simulated by appropriate scenario of quantum measurement, demonstrating the same contextuality features. As a result, a Hilbert space of appropriate quantum system emerges together with a collection of observables. This Hilbert space is suggested to play the role of information need space, which is developed within QIA (quantum information access) framework for Information Retrieval. Technically, the IN space is built starting from Greechie-like diagrams (pasted overlapping contexts, see Section 4 above, capturing the particular IR environment. QIA framework provides more flexible machinery to deal with information needs than any classical probabilistic approach by that simple reason that it incorporates the latter. But

[1]Mathematically rigorous constructions, describing ways the world could have been were it not quantum mechanical

we should be aware that it is not ultimately general. In quantum realm, we have non-classical correlations and the present state of our knowledge shows that quantum mechanics is enough to explain all them. However, IR may in principle provide stronger-than-quantum correlations. For them, 'foils of quantum theory' - the operational theories, which do not compete with quantum mechanics, but generalize it to the extent not demanded in modern physics [6, 14], these theories may be of help in Information Processing.

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Stability conditions for non-autonomous linear differential equations in a Hilbert space via commutators

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Abstract

In a Hilbert space $\mathscr{H}$ we consider the equation $dx(t)/dt = (A+B(t))x(t)$ $(t \geq 0)$, where A is a constant bounded operator, and $B(t)$ is a piece-wise continuous function defined on $[0,\infty)$ whose values are bounded operators in $\mathscr{H}$. Conditions for the exponential stability are derived in terms of the commutator $AB(t) - B(t)A$. Applications to integro-differential equations are also discussed. Our results are new even in the finite dimensional case.

1. Introduction

Let $\mathscr{H}$ be a Hilbert space with a scalar product $\langle .,. \rangle$, the norm $\|.\| = \sqrt{\langle .,. \rangle}$ and unit operator I. In addition, $\mathscr{B}(\mathscr{H})$ denotes the algebra of bounded linear operators in $\mathscr{H}$. For an $A \in \mathscr{B}(\mathscr{H})$, A^* is the adjoint operator, $\sigma(A)$ is the spectrum of A, $\Re A := (A+A^*)/2$, $\Im A := (A-A^*)/2i$, $\|A\|$ denotes the operator norm of A.
We consider the equation

$$\frac{du(t)}{dt} = (A+B(t))u(t) \quad (t \geq 0), \tag{1.1}$$

where A is a constant bounded operator and $B(t) : [0,\infty) \to \mathscr{B}(\mathscr{H})$ is a strongly piece-wise continuous function. A solution of (1.1) is a function $u(t)$, defined on $[0,\infty)$ with values in $\mathscr{H}$, absolutely continuous in t and satisfying the given initial condition and (1.1) almost everywhere on $[0,\infty)$. The existence of solutions follows from the a priory estimates proved below. We will say that equation (1.1) is exponentially stable, if there are positive constants M and ε, such that any solution $u(t)$ of (1.1) satisfies $\|u(t)\| \leq Me^{-\varepsilon t}\|u(0)\|$ $(t \geq 0)$.
Equation (1.1) can be considered as the equation

$$\frac{dx(t)}{dt} = C(t)x(t), \tag{1.2}$$

with a variable linear operator $C(t)$. This identification which is a common device in the theory of concrete differential or integro-differential equations when passing from a given equation to an abstract evolution equation turns out to be useful also here. Observe that $C(t)$ in the considered case has a special form: it is the sum of operators A and $B(t)$. This fact allows us to use the information about the coefficients more completely than the theory of differential equations (1.2) containing an arbitrary operator $C(t)$.
The basic method for the stability analysis of (1.2) is the direct Lyapunov method, cf. [2]. By that method many very strong results are obtained, but finding Lyapunov's functions is often connected with serious mathematical difficulties.
For a selfadjoint operator S put $\Lambda(S) = \sup\ \sigma(S)$ and $\lambda(S) = \inf\ \sigma(S)$. So $\Lambda(\Re C(s)) = \sup\ \sigma(\Re C(s))$ and $\lambda(\Re C(s)) = \inf\ \sigma(\Re C(s))$. The important tool of the stability analysis is the Wintner inequalities [7, Theorem III.4.7]:

$$\exp[\int_s^t \lambda(\Re C(s_1))ds_1] \leq \frac{\|u(t)\|}{\|u(s)\|} \leq \exp[\int_s^t \Lambda(\Re C(s_1))ds_1] \quad (t \geq s \geq 0), \tag{1.3}$$

for any solution $x(t)$ of equation (1.2). If $C(t)$ is not dissipative, i.e. if $C(t)+C^*(t)$ is not negative definite for sufficiently large t, then the just mentioned inequalities do not give us stability conditions even in the case of a constant operator. In addition, in [14] the stability test

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for (1.2) has been derived for equations whose operator coefficients have "small" derivatives. The approach in [14] is the extension of the freezing method for ordinary differential equations. In this paper, we suggest a stability test via the commutator $K(t) = AB(t) - B(t)A$, which in the appropriate situations improves the published results. To the best of our knowledge, our results are new even in finite dimensional case, cf. [20].
As an illustrative example we consider a class of the so called Barbashin integro-differential equations, which play an essential role in numerous applications, in particular, in kinetic theory [5], transport theory [18], continuous mechanics [1], radiation theory [4], the dynamics of populations [21], etc.

2. The main result

Assume that

$$\alpha(A) := \sup \Re\sigma(A) < 0 \tag{2.1}$$

and put

$$W := 2\int_0^\infty e^{A^*t}e^{At}dt,\ \zeta(A) := 2\int_0^\infty \|e^{At}\| \int_0^t \|e^{As}\|\|e^{A(t-s)}\|ds\,dt$$

and

$$\psi(W,B(t)) := \begin{cases} \Lambda(\Re B(t))\|W\| & \text{if } \Lambda(\Re B(t)) > 0, \\ \Lambda(\Re B(t))\lambda(W) & \text{if } \Lambda(\Re B(t)) \le 0. \end{cases}$$

Below we suggest estimates for $\|W\|$ and $\lambda(W)$. Furthermore, let $[A_1,A_2] = A_1A_2 - A_2A_1$ (the commutator of $A_1, A_2 \in \mathscr{B}(\mathscr{H})$). So $K(t) = [A,B(t)]$.
Now we are in a position to formulate our main result.

Theorem 2.1. *Let the conditions (2.1) and*

$$\sup_{t\ge 0}\left(\psi(W,B(t)) + \|K(t)\|\zeta(A)\right) < 1 \tag{2.2}$$

hold. Then equation (1.1) is exponentially stable.

This theorem is proved in the next section. If

$$\|e^{As}\| \le ce^{-\nu s}\ \ (s \ge 0; c, \nu = const > 0), \tag{2.3}$$

then

$$\langle Wv, v\rangle = 2\int_0^\infty \|e^{At}v\|^2 dt \le 2c^2\int_0^\infty e^{-2\nu t}dt\|v\|^2\ \ (v \in \mathscr{H}).$$

Consequently,

$$\|W\| \le \frac{c^2}{\nu} \text{ and } \zeta(A) \le 2c^3\int_0^\infty e^{-\nu t}\int_0^t e^{-\nu s}e^{-\nu(t-s)}ds\,dt = 2c^3\int_0^\infty e^{-2\nu t}t dt = \frac{c^3}{2\nu^2}. \tag{2.4}$$

Now let us estimate $\lambda(W)$. Due to the Wintner inequalities (1.3),

$$\|e^{At}v\| \ge e^{\lambda(\Re A)t}\|v\|\ \ (v \in \mathscr{H}).$$

So in view of (2.1), $\lambda(\Re A)$ is negative. Consequently,

$$\langle Wv, v\rangle = 2\int_0^\infty \|e^{At}v\|^2 dt \ge 2\int_0^\infty e^{2\lambda(\Re A)t}\|v\|^2 dt \ge \|v\|^2/|\lambda(\Re A)|\ \ (v \in \mathscr{H}).$$

Thus

$$\lambda(W) \ge 1/|\lambda(\Re A)|. \tag{2.5}$$

If A is a normal operator: $AA^* = A^*A$, then $\|e^{At}\| = e^{\alpha(A)t}\ \ (t \ge 0)$, and according to (2.4),

$$\|W\| \le \frac{1}{|\alpha(A)|}, \zeta(A) = \frac{1}{2|\alpha(A)|^2} \text{ and, in addition, } \lambda(\Re A) = \beta(A),$$

where $\beta(A) := \inf\ \Re\sigma(A)$. Consequently, $\psi(W,B(t)) = \psi_0(A,B(t))$, where

$$\psi_0(A,B(t)) = \begin{cases} \frac{\Lambda(\Re B(t))}{|\alpha(A)|} & \text{if } \Lambda(\Re B(t)) > 0, \\ \frac{\Lambda(\Re B(t))}{|\beta(A)|} & \text{if } \Lambda(\Re B(t)) \le 0. \end{cases}$$

So we arrive at

Corollary 2.2. *Let A be a normal operator, and the conditions (2.1) and*

$$\sup_{t\ge 0}\left(\psi_0(A,B(t)) + \frac{\|K(t)\|}{2|\alpha(A)|^2}\right) < 1 \tag{2.6}$$

hold. Then equation (1.1) is exponentially stable.

Theorem 2.1 *is sharp* in the following sense: if $B(t)=0$, then $\psi(A,B(t))=\|K(t)\|=0$, and (2.2) obviously holds. But condition (2.1) is necessary in this case.
Traditionally (1.1) is considered as a perturbation of the equation $du/dt = Au$ with stable A. Besides, it is supposed that

$$\int_0^\infty \|e^{sA}\|ds \sup_t \|B(t)\| < 1, \tag{2.7}$$

e.g. [2, 14] and references therein. We do not assume this condition. For example, if A and $B(t)$ commute, then takes the form

$$\sup_{t\geq 0} \psi_0(A,B(t)) < 1$$

which is sharper than (2.7).
Moreover, in the contrary to the Wintner inequalities, *we do not require the dissipativity* of $A+B(t)$.

3. Proof of theorem 2.1

Lemma 3.1. *Let A,B be constant bounded operators and $K=[A,B]$. Then*

$$[e^{At},B] = \int_0^t e^{As}Ke^{A(t-s)}ds \ \ (t\geq 0). \tag{3.1}$$

Proof: For the proof see [15].
Under condition (2.1), the Lyapunov equation

$$WA + A^*W = -2I \tag{3.2}$$

has a unique solution $W\in\mathscr{B}(\mathscr{H})$ and it can be represented as in Section 2, cf. [7, Theorem I.5.1] (see also equation (4.12) from Chapter I of [7]). For two selfadjoint operators S and S_1 the inequality $S<S_1$ $(S\leq S_1)$ means $(Sh,h)<(S_1h,h)$ $((Sh,h)\leq(S_1h,h))$ $(h\in\mathscr{H})$. In particular, the inequality $S<0$ $(S>0)$ means that S is strongly negative (strongly positive) definite.

Lemma 3.2. *If condition* (2.1) *holds, then*

$$\Re(WB(t)) = \frac{1}{2}(WB(t)+(WB(t))^*) \leq (\psi(W,B(t))+\|K(t)\|\zeta(A))I.$$

Proof. Making use of (2.1) we can write

$$\Re(WB(t)) = \frac{1}{2}(WB(t)+B^*(t)W) = \int_0^\infty (e^{A^*t_1}e^{At_1}B(t)+B^*(t)e^{A^*t_1}e^{At_1})dt_1.$$

But

$$e^{At_1}B(t) = B(t)e^{At_1}+[e^{At_1},B(t)], B^*(t)e^{A^*t_1} = e^{A^*t_1}B^*(t)+[B^*(t),e^{A^*t_1}].$$

So $\Re(WB(t)) = J_1+J_2$, where

$$J_1 = \int_0^\infty e^{A^*t_1}(B(t)+B^*(t))e^{At_1}dt \text{ and } J_2 = \int_0^\infty (e^{A^*t_1}[e^{At_1},B(t)]+(e^{A^*t_1}[e^{At_1},B(t)])^*)dt_1.$$

We have

$$J_1 \leq 2\Lambda(\Re B(t))\int_0^\infty e^{A^*t_1}e^{At_1}dt_1 = \Lambda(\Re B(t))W.$$

If $\Lambda(\Re B(t))>0$, then $J_1\leq\Lambda(\Re B(t))\|W\|I$. If $\Lambda(\Re B(t))<0$, then $J_1\leq\Lambda(\Re B(t))\lambda(W)I$. So $J_1\leq\psi(W,B(t))I$.
In addition, by Lemma 3.1

$$\|J_2\| \leq 2\int_0^\infty \|e^{At_1}\|\|[e^{At_1},B(t)]\|dt_1 \leq 2\int_0^\infty \|e^{At_1}\|\|K(t)\|\int_0^{t_1}\|e^{As}\|\|e^{A(t_1-s)}\|ds\,dt_1$$
$$= \|K(t)\|\zeta(A).$$

This proves the lemma. □

Proof of Theorem 2.1: Due to the Lyapunov equation and Lemma 3.2 we have,

$$\Re W(A+B(t)) \leq -(1-\psi(W,B(t))-\|K(t)\|\zeta(A))I.$$

So (2.2) implies

$$\Re W(A+B(t)) < \sup_t(-1+\psi(W,B(t))+\|K(t)\|\zeta(A))I < 0. \tag{3.3}$$

Applying the right-hand Wintner inequality (1.3) with the scalar product $(.,.)_W$ defined by $(h,g)_W = \langle Wh,g\rangle$ $(h,g\in\mathscr{H})$, we can assert that equation (1.1) is exponentially stable, as claimed. □

4. Equations with finite dimensional operators

In this section $\mathscr{H} = \mathbb{C}^n$-the n-dimension complex Euclidean space, A and $B(t)$ are $n \times n$ matrices. Put

$$g(A) = [N_2^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2\,]^{1/2},$$

where $\lambda_k(A)$ $(k = 1, ..., n)$ are the eigenvalues of A, counted with their multiplicities; $N_2(A) = (\text{trace } AA^*)^{1/2}$ is the Frobenius (Hilbert-Schmidt) norm of A. The following relations are checked in [12, Section 2.1]: $g^2(A) \le N_2^2(A) - |\text{trace } A^2|$,

$$g(e^{i\tau}A + zI) = g(A)\ (\tau \in \mathbb{R}, z \in \mathbb{C},) \text{ and } g^2(A) \le \frac{N_2^2(A - A^*)}{2}.$$

If A is a normal matrix, then $g(A) = 0$.
It is shown in [12, Example 2.7.3], that

$$\|e^{At}\| \le e^{\alpha(A)t} \sum_{k=0}^{n-1} \frac{t^k g^k(A)}{(k!)^{3/2}} \quad (t \ge 0).$$

So

$$\|W\| \le 2\int_0^\infty \|e^{At}\|^2 dt \le 2\int_0^\infty e^{2\alpha(A)t} \left(\sum_{k=0}^{n-1} \frac{t^k g^k(A)}{(k!)^{3/2}}\right)^2 dt = \chi_n(A),$$

where

$$\chi_n(A) = \sum_{j,k=0}^{n-1} \frac{g^{j+k}(A)(k+j)!}{2^{j+k}|\alpha(A)|^{j+k+1}(j!\,k!)^{3/2}}.$$

Put

$$p_n(A,t) = \sum_{k=0}^{n-1} \frac{t^k g^k(A)}{(k!)^{3/2}} \quad (t \ge 0).$$

Then $\|e^{At}\| \le e^{\alpha(A)t} p_n(A,t)$ and $\zeta(A) \le \zeta_n(A)$, where

$$\zeta_n(A) := 2\int_0^\infty e^{2\alpha(A)t} p_n(A,t) \int_0^t p_n(A, t-s) p(A,s) ds\, dt.$$

Moreover, according to (2.5), $\psi(W, B(t)) \le \hat{\psi}_n(A, B(t))$, where

$$\hat{\psi}_n(A, B(t)) := \begin{cases} \chi_n(A)\Lambda(\Re B(t)) & \text{if } \Lambda(\Re B(t)) > 0, \\ \frac{\Lambda(\Re B(t))}{|\lambda(\Re A)|} & \text{if } \Lambda(\Re B(t)) \le 0. \end{cases}$$

Now Theorem 2.1 and (2.5) imply

Corollary 4.1. *Let $\mathscr{H} = \mathbb{C}^n$, A be a Hurwitzian matrix (i.e. condition (2.1) holds), and*

$$\sup_{t \ge 0} (\hat{\psi}_n(A, B(t)) + \|K(t)\|\zeta_n(A)) < 1.$$

Then (1.1) is exponentially stable.

5. Equations with infinite dimensional operators

In this section we consider equation (1.1) in the infinite dimensional space assuming that

$$\Im A \text{ is a Hilbert-Schmidt operator.} \tag{5.1}$$

i.e. $N_2(\Im A) = (\text{trace } (\Im A)^2)^{1/2} < \infty$. Put

$$\hat{u}(A) = [2N_2^2(\Im A) - 2\sum_{k=1}^{\infty} |\Im \hat{\lambda}_k(A)|^2\,]^{1/2},$$

where $\hat{\lambda}_k(A), k = 1, 2, ...,$ are nonreal eigenvalues of A, enumerated with their multiplicities in the decreasing order of the absolute values of their imaginary parts. Recall the classical Weyl inequality

$$N_2^2(\Im A) \ge \sum_{k=1}^{\infty} |\Im \hat{\lambda}_k(A)|^2,$$

cf. [12, p. 98]. So $\hat{u}(A) \le \sqrt{2}N_2(\Im A)$. If A is a normal operator, then $\hat{u}(A) = 0$, cf. [12, Section 7.7]. As is shown in [12, Example 7.10.3],

$$\|e^{At}\| \le e^{\alpha(A)t} \sum_{k=0}^{\infty} \frac{t^k \hat{u}^k(A)}{(k!)^{3/2}} \quad (t \ge 0),$$

So

$$\|W\| \le 2\int_0^\infty \|e^{At}\|^2 dt \le 2\int_0^\infty e^{\alpha(A)t} \left(\sum_{k=0}^{\infty} \frac{t^k \hat{u}^k(A)}{(k!)^{3/2}}\right)^2 dt = \tilde{\chi}(A),$$

where

$$\tilde{\chi}(A) = \sum_{j,k=0}^{\infty} \frac{\hat{u}^{j+k}(A)(k+j)!}{2^{j+k}|\alpha(A)|^{j+k+1}(j!\,k!)^{3/2}}.$$

Put

$$\tilde{p}(A,t) = \sum_{k=0}^{\infty} \frac{t^k \hat{u}^k(A)}{(k!)^{3/2}} \ \ (t \geq 0).$$

Then $\|e^{At}\| \leq e^{\alpha(A)t}\hat{p}(A,t)$ and

$$\zeta(A) \leq \tilde{\zeta}(A) := 2\int_0^{\infty} e^{2\alpha(A)t}\tilde{p}(t,A)\int_0^t \tilde{p}(t-s,A)\tilde{p}(s,A)ds\,dt.$$

Moreover, $\psi(W,B(t)) \leq \tilde{\psi}(A,B(t))$, where

$$\tilde{\psi}(A,B(t)) := \begin{cases} \tilde{\chi}(A)\Lambda(\Re B(t)) & \text{if } \Lambda(\Re B(t)) > 0, \\ \frac{\Lambda(\Re B(t))}{|\lambda(\Re A)|} & \text{if } \Lambda(\Re B(t)) \leq 0. \end{cases}$$

Now Theorem 2.1 and (2.5) imply

Corollary 5.1. *If the conditions* (2.1)*,* (5.1) *and*

$$\sup_{t\geq 0}\left(\tilde{\psi}(A,B(t)) + \|K(t)\|\tilde{\zeta}(A)\right) < 1,$$

hold, then (1.1) *is exponentially stable.*

6. Example

Put $\Omega = [0,1]\times[0,1]$. In this section $\mathscr{H} = L^2(\Omega)$ is the Hilbert spaces of complex square integrable functions defined on Ω with the traditional scalar product and norm.
Consider the equation

$$\frac{\partial u(t,x,y)}{\partial t} = c(x)u(t,x,y) + \int_0^1 k_1(x,s)u(t,s,y)ds + \int_0^1 k_2(t,y,s)u(t,x,s)ds \tag{6.1}$$

$$(0 \leq x,y \leq 1; t \geq 0),$$

where $c(\cdot):[0,1]\to\mathbb{R}$ is piece-wise continuous, $k_1(\cdot,\cdot):[0,1]^2\to\mathbb{C}$, $k_2(\cdot,\cdot,\cdot):[0,\infty)\times[0,1]^2\to\mathbb{C}$, are given functions satisfying the conditions pointed below. Equation of the type (6.1) is the Barbashin type integro-differential equation or simply the Barbashin equation, [2]. The stability of (6.1) can also be investigated by perturbations of the simple equation

$$\frac{\partial u(t,x,y)}{\partial t} = c(x)u(t,x,y),$$

cf. [2, Section 2.5], but this approach gives rather rough results if the norm of k_1 and k_2 are large enough.
Define the operators A and $B(t)$ by

$$(Aw)(x,y) = c(x)w(x,y) + \int_0^1 k_1(x,s)w(s,y)ds$$

and

$$(B(t)w)(x,y) = \int_0^1 k_2(t,x,s)w(x,s)ds \ \ (x,y\in[0,1];\ w\in L^2(\Omega)),$$

respectively. Under consideration we have $[A,B(t)] = 0$ for all $t \geq 0$. Moreover, assume that

$$N_2(A-A^*) = \left(\int_0^1\int_0^1 |k_1(x,s)-\bar{k}_1(s,x))|^2\,ds\,dx\right)^{1/2} < \infty$$

and k_2 provides the boundedness of $B(t)$. Various estimates for $\alpha(A)$ under considerations can be found in [13]. In particular, if $k_1(x,s)=0$ for $x\leq s$, then $\alpha(A) = \sup_x c(x)$. Furthermore, it is not hard to check that

$$\Lambda(\Re B(t)) = \frac{1}{2}\sup_{v\in L^2(0,1)}\int_0^1\int_0^1 (k_2(t,y,s)+\bar{k}_2(t,s,y))v(s)\,\bar{v}(y)\,ds\,dy$$

and

$$\lambda(\Re A) = \frac{1}{2}\inf_{v\in L^2(0,1)}\int_0^1\int_0^1 (k_1(x,s)+\bar{k}_1(s,x))v(s)\,\bar{v}(x)\,ds\,dx.$$

Now we can directly apply Corollary 5.1.
Note that the theory of of various classes of integro-differential equations is rather rich, cf. [3, 6], [8]-[11], [16, 17, 19, 22, 23] and references therein, but the stability conditions in terms of the commutators have not been derived.

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The new UP-isomorphism theorems for UP-algebras in the meaning of the congruence determined by a UP-homomorphism

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Abstract

The aim of this paper is to construct the new fundamental theorem of UP-algebras in the meaning of the congruence determined by a UP-homomorphism. We also give an application of the theorem to the first, second, and third UP-isomorphism theorems in UP-algebras.

1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [7], BCI-algebras [8], BCH-algebras [4], KU-algebras [15], SU-algebras [10], UP-algebras [6] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [8] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [7, 8] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.
The isomorphism theorems play an important role in a general logical algebra, which were studied by several researches such as: In 1998, Jun, Hong, Xin and Roh [9] proved isomorphism theorems by using Chinese Remainder Theorem in BCI-algebras. In 2001, Park, Shim and Roh [14] proved isomorphism theorems of IS-algebras. In 2004, Hao and Li [3] introduced the concept of ideals of an ideal in a BCI-algebra and some isomorphism theorems are obtained by using this concept. They obtained several isomorphism theorems of BG-algebras and related properties. In 2006, Kim [12] introduced the notion of KS-semigroups. He characterized ideals of a KS-semigroup and proved the first isomorphism theorem for KS-semigroups. In 2007, Dar and Akram [2] introduced the notion of K-homomorphism of K-algebras. In 2008, Kim and Kim [11] introduced the notion of BG-algebras which is a generalization of B-algebras. They obtained several isomorphism theorems of BG-algebras and related properties. In 2009, Paradero-Vilela and Cawi [13] characterized KS-semigroup homomorphisms and proved the isomorphism theorems for KS-semigroups. In 2011, Keawrahun and Leerawat [10] introduced the notion of SU-semigroups and proved the isomorphism theorems for SU-semigroups. In 2012, Asawasamrit [1] introduced the notion of KK-algebras and studied isomorphism theorems of KK-algebras. In 2015, Iampan [5] studied UP-isomorphism theorems of UP-algebras.
In this paper, we construct the new fundamental theorem of UP-algebras in the meaning of the congruence determined by a UP-homomorphism. We also give an application of the theorem to the first, second, and third UP-isomorphism theorems in UP-algebras.

Before we begin our study, we will introduce to the definition of a UP-algebra.

Definition 1.1. *[6] An algebra $A = (A, \cdot, 0)$ of type $(2,0)$ is called a UP-algebra, where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,*

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(UP-2) $0 \cdot x = x$,
(UP-3) $x \cdot 0 = 0$,
(UP-4) $x \cdot y = y \cdot x = 0$ *implies* $x = y$.

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Example 1.2. *[6] Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathscr{P}(X)$. Then $(\mathscr{P}(X), \cdot, \emptyset)$ and $(\mathscr{P}(X), *, X)$ are UP-algebras and we shall call it the power UP-algebra of type 1 and the power UP-algebra of type 2, respectively.*

Example 1.3. *[6] Let $A = \{0, a, b, c\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

(1.1)

Then $(A, \cdot, 0)$ is a UP-algebra.

In what follows, let A and B denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.4. [6] *In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \cdot x = 0$,
(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *implies* $x \cdot z = 0$,
(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,
(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,
(5) $x \cdot (y \cdot x) = 0$,
(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*
(7) $x \cdot (y \cdot y) = 0$.

Definition 1.5. *[6] Let A be a UP-algebra. A nonempty subset B of A is called a UP-ideal of A if it satisfies the following properties:*

(1) the constant 0 of A is in B, and
(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ implies $x \cdot z \in B$.

Definition 1.6. *[6] Let $A = (A, \cdot, 0)$ be a UP-algebra. A subset S of A is called a UP-subalgebra of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.*

Proposition 1.7. [6] *A nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Definition 1.8. *[6] Let A be a UP-algebra. An equivalence relation ρ on A is called a congruence if for any $x, y, z \in A$,*

$$x \rho y \text{ implies } x \cdot z \rho y \cdot z \text{ and } z \cdot x \rho z \cdot y.$$

Lemma 1.9. [6] *An equivalence relation ρ on A is a congruence if and only if for any $x, y, u, v \in A$, $x \rho y$ and $u \rho v$ imply $x \cdot u \rho y \cdot v$.*

Definition 1.10. *[6] Let A be a UP-algebra and B a UP-ideal of A. Define the binary relation $\sim_B$ on A as follows: for all $x, y \in A$,*

$$x \sim_B y \text{ if and only if } x \cdot y \in B \text{ and } y \cdot x \in B. \tag{1.2}$$

Proposition 1.11. [6] *Let A be a UP-algebra and B a UP-ideal of A with a binary relation $\sim_B$ defined by* (1.2). *Then $\sim_B$ is a congruence on A.*

Let A be a UP-algebra and ρ a congruence on A. If $x \in A$, then the ρ-class of x is the $(x)_\rho$ defined as follows:

$$(x)_\rho = \{y \in A \mid y \rho x\}.$$

Then the set of all ρ-classes is called the *quotient set of A by ρ*, and is denoted by A/ρ. That is,

$$A/\rho = \{(x)_\rho \mid x \in A\}.$$

Theorem 1.12. [6] *Let A be a UP-algebra and B a UP-ideal of A. Then $(A/\sim_B, *, (0)_{\sim_B})$ is a UP-algebra under the $*$ multiplication defined by $(x)_{\sim_B} * (y)_{\sim_B} = (x \cdot y)_{\sim_B}$ for all $x, y \in A$, called the quotient UP-algebra of A induced by the congruence $\sim_B$.*

Definition 1.13. *[6] Let $(A, \cdot, 0)$ and $(A', \cdot', 0')$ be UP-algebras. A mapping f from A to A' is called a UP-homomorphism if*

$$f(x \cdot y) = f(x) \cdot' f(y) \text{ for all } x, y \in A.$$

A UP-homomorphism $f : A \to A'$ is called a

(1) UP-epimorphism if f is surjective,
(2) UP-monomorphism if f is injective,
(3) UP-isomorphism if f is bijective. Moreover, we say A is UP-isomorphic to A', symbolically, $A \cong A'$, if there is a UP-isomorphism from A to A'.

Let f be a mapping from A to A', and let B be a nonempty subset of A, and B' of A'. The set $\{f(x) \mid x \in B\}$ is called the image of B under f, denoted by $f(B)$. In particular, $f(A)$ is called the image of f, denoted by $\mathrm{Im}(f)$. Dually, the set $\{x \in A \mid f(x) \in B'\}$ is said the inverse image of B' under f, symbolically, $f^{-1}(B')$. Especially, we say $f^{-1}(\{0'\})$ is the kernel of f, written by $\mathrm{Ker}(f)$. That is,

$$\mathrm{Im}(f) = \{f(x) \in A' \mid x \in A\}$$

and

$$\mathrm{Ker}(f) = \{x \in A \mid f(x) = 0'\}.$$

Theorem 1.14. [6] *Let A be a UP-algebra and B a UP-ideal of A. Then the mapping $\pi_B \colon A \to A/\sim_B$ defined by $\pi_B(x) = (x)_{\sim_B}$ for all $x \in A$ is a UP-epimorphism, called the natural projection from A to $A/\sim_B$.*

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0. \tag{1.3}$$

Proposition 1.15. [6] *Let A be a UP-algebra with a binary relation $\leq$ defined by (1.3). Then $(A, \leq)$ is a partially ordered set with 0 as the greatest element.*

We often call the partial ordering $\leq$ defined by (1.3) the *UP-ordering* on A. From now on, the symbol $\leq$ will be used to denote the UP-ordering, unless specified otherwise.

Theorem 1.16. [6] *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f \colon A \to B$ be a UP-homomorphism. Then the following statements hold:*

(1) $f(0_A) = 0_B$,
(2) for any $x, y \in A$, if $x \leq y$, then $f(x) \leq f(y)$,
(3) if C is a UP-subalgebra of A, then the image $f(C)$ is a UP-subalgebra of B. In particular, $\mathrm{Im}(f)$ is a UP-subalgebra of B,
(4) if D is a UP-subalgebra of B, then the inverse image $f^{-1}(D)$ is a UP-subalgebra of A. In particular, $\mathrm{Ker}(f)$ is a UP-subalgebra of A,
(5) if C is a UP-ideal of A such that $\mathrm{Ker}(f) \subseteq C$, then the image $f(C)$ is a UP-ideal of $f(A)$,
(6) if D is a UP-ideal of B, then the inverse image $f^{-1}(D)$ is a UP-ideal of A. In particular, $\mathrm{Ker}(f)$ is a UP-ideal of A, and
(7) $\mathrm{Ker}(f) = \{0_A\}$ if and only if f is injective.

2. Main results

In this section, we introduce the congruence determined by a UP-homomorphism and prove the new fundamental theorem of UP-algebras in the meaning of the congruence determined by a UP-homomorphism. We also prove the first, second, and third UP-isomorphism theorems in UP-algebras.

Definition 2.1. *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f \colon A \to B$ a UP-homomorphism. Define the binary relation $\sim_f$ on A as follows: for all $x, y \in A$,*

$$x \sim_f y \text{ if and only if } f(x) = f(y). \tag{2.1}$$

Theorem 2.2. *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f \colon A \to B$ a UP-homomorphism with a binary relation $\sim_f$ on A defined by (2.1). Then $\sim_f$ is a congruence on A, called the congruence determined by f.*

Proof. *Reflexive:* For all $x \in A$, we have $f(x) = f(x)$. Thus $x \sim_f x$.
Symmetric: Let $x, y \in A$ be such that $x \sim_f y$. Then $f(x) = f(y)$, so $f(y) = f(x)$. Thus $y \sim_f x$.
Transitive: Let x, y, z be such that $x \sim_f y$ and $y \sim_f z$. Then $f(x) = f(y)$ and $f(y) = f(z)$, so $f(x) = f(z)$. Thus $x \sim_f z$.
Therefore, $\sim_f$ is an equivalence relation on A. Finally, let $x, y, u, v \in A$ be such that $x \sim_f u$ and $y \sim_f v$. Then $f(x) = f(u)$ and $f(y) = f(v)$. Since f is a UP-homomorphism, we get

$$f(x \cdot y) = f(x) \bullet f(y) = f(u) \bullet f(v) = f(u \cdot v).$$

Thus $x \cdot y \sim_f u \cdot v$. By Lemma 1.9, we have $\sim_f$ is a congruence on A. □

Theorem 2.3. *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f \colon A \to B$ a UP-homomorphism. Then $(A/\sim_f, *, (0_A)_{\sim_f})$ is a UP-algebra under the $*$ multiplication defined by $(x)_{\sim_f} * (y)_{\sim_f} = (x \cdot y)_{\sim_f}$ for all $x, y \in A$, called the quotient UP-algebra of A induced by the congruence $\sim_f$.*

Proof. Let $x, y, u, v \in A$ be such that $(x)_{\sim_f} = (y)_{\sim_f}$ and $(u)_{\sim_f} = (v)_{\sim_f}$. Since $\sim_f$ is an equivalence relation on A, we get $x \sim_f y$ and $u \sim_f v$. By Lemma 1.9, we have $x \cdot u \sim_f y \cdot v$. Hence, $(x)_{\sim_f} * (u)_{\sim_f} = (x \cdot u)_{\sim_f} = (y \cdot v)_{\sim_f} = (y)_{\sim_f} * (v)_{\sim_f}$, showing $*$ is well defined.
(UP-1): Let $x, y, z \in A$. By (UP-1), we have $((y)_{\sim_f} * (z)_{\sim_f}) * (((x)_{\sim_f} * (y)_{\sim_f}) * ((x)_{\sim_f} * (z)_{\sim_f})) = ((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)))_{\sim_f} = (0_A)_{\sim_f}$.
(UP-2): Let $x \in A$. By (UP-2), we have $(0_A)_{\sim_f} * (x)_{\sim_f} = (0_A \cdot x)_{\sim_f} = (x)_{\sim_f}$.
(UP-3): Let $x \in A$. By (UP-3), we have $(x)_{\sim_f} * (0_A)_{\sim_f} = (x \cdot 0_A)_{\sim_f} = (0_A)_{\sim_f}$.
(UP-4): Let $x, y \in A$ be such that $(x)_{\sim_f} * (y)_{\sim_f} = (y)_{\sim_f} * (x)_{\sim_f} = (0_A)_{\sim_f}$. Then $(x \cdot y)_{\sim_f} = (y \cdot x)_{\sim_f} = (0_A)_{\sim_f}$, it follows that $f(x) \bullet f(y) = f(x \cdot y) = f(0_A) = f(y \cdot x) = f(y) \bullet f(x)$. By Theorem 1.16 (1), we have $f(x) \bullet f(y) = f(y) \bullet f(x) = 0_B$. By (UP-4), we have $f(x) = f(y)$. Thus $x \sim_f y$, so $(x)_{\sim_f} = (y)_{\sim_f}$.
Hence, $(A/\sim_f, *, (0_A)_{\sim_f})$ is a UP-algebra. □

Theorem 2.4. *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f \colon A \to B$ a UP-homomorphism. Then the mapping $\pi_f \colon A \to A/\sim_f$ defined by $\pi_f(x) = (x)_{\sim_f}$ for all $x \in A$ is a UP-epimorphism, called the natural projection from A to $A/\sim_f$.*

Proof. Let $x, y \in A$ be such that $x = y$. Then $(x)_{\sim_f} = (y)_{\sim_f}$, so $\pi_f(x) = \pi_f(y)$. Thus π_f is well defined. Note that by the definition of π_f, we have π_f is surjective. Let $x, y \in A$. Then

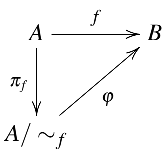
$$\pi_f(x \cdot y) = (x \cdot y)_{\sim_f} = (x)_{\sim_f} * (y)_{\sim_f} = \pi_f(x) * \pi_f(y).$$

Thus π_f is a UP-homomorphism. So we conclude that π_f is a UP-epimorphism. □

Theorem 2.5. *(Fundamental Theorem of UP-homomorphisms) Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f : A \to B$ a UP-homomorphism. Then there exists uniquely a UP-homomorphism φ from $A/\sim_f$ to B such that $f = \varphi \circ \pi_f$. Moreover,*

(1) π_f is a UP-epimorphism and φ a UP-monomorphism, and
(2) f is a UP-epimorphism if and only if φ is a UP-isomorphism.

As f makes the following diagram commute,



Proof. By Theorem 2.3, we have $(A/\sim_f, *, (0_A)_{\sim_f})$ is a UP-algebra. Define a mapping $\varphi : A/\sim_f \to B$ by

$$\varphi((x)_{\sim_f}) = f(x) \text{ for all } (x)_{\sim_f} \in A/\sim_f . \tag{2.2}$$

Indeed, let $(x)_{\sim_f}, (y)_{\sim_f} \in A/\sim_f$ be such that $(x)_{\sim_f} = (y)_{\sim_f}$. Then $x \sim_f y$, so

$$\varphi((x)_{\sim_f}) = f(x) = f(y) = \varphi((y)_{\sim_f}).$$

For any $x, y \in A$, we see that

$$\begin{aligned} \varphi((x)_{\sim_f} * (y)_{\sim_f}) &= \varphi((x \cdot y)_{\sim_f}) \\ &= f(x \cdot y) \\ &= f(x) \bullet f(y) \\ &= \varphi((x)_{\sim_f}) \bullet \varphi((y)_{\sim_f}). \end{aligned}$$

Thus φ is a UP-homomorphism. Also, since

$$(\varphi \circ \pi_f)(x) = \varphi(\pi_f(x)) = \varphi((x)_{\sim_f}) = f(x) \text{ for all } x \in A,$$

we obtain $f = \varphi \circ \pi_f$. We have shown the existence. Let φ' be a mapping from $A/\sim_f$ to B such that $f = \varphi' \circ \pi_f$. Then for any $(x)_{\sim_f} \in A/\sim_f$, we have

$$\begin{aligned} \varphi'((x)_{\sim_f}) &= \varphi'(\pi_f(x)) \\ &= (\varphi' \circ \pi_f)(x) \\ &= f(x) \\ &= (\varphi \circ \pi_f)(x) \\ &= \varphi(\pi_f(x)) \\ &= \varphi((x)_{\sim_f}). \end{aligned}$$

Hence, $\varphi = \varphi'$, showing the uniqueness.
(1) By Theorem 2.4, we have π_f is a UP-epimorphism. Also, let $(x)_{\sim_f}, (y)_{\sim_f} \in A/\sim_f$ be such that $\varphi((x)_{\sim_f}) = \varphi((y)_{\sim_f})$. Then $f(x) = f(y)$, so $x \sim_f y$. Thus $(x)_{\sim_f} = (y)_{\sim_f}$. Therefore, φ a UP-monomorphism.
(2) Assume that f is a UP-epimorphism. By (1), it suffices to prove φ is surjective. Let $y \in B$. Then there exists $x \in A$ such that $f(x) = y$. Thus $y = f(x) = \varphi((x)_{\sim_f})$, so φ is surjective. Hence, φ is a UP-isomorphism.
Conversely, assume that φ is a UP-isomorphism. Then φ is surjective. Let $y \in B$. Then there exists $(x)_{\sim_f} \in A/\sim_f$ such that $\varphi((x)_{\sim_f}) = y$. Thus $f(x) = \varphi((x)_{\sim_f}) = y$, so f is surjective. Hence, f is a UP-epimorphism. □

Theorem 2.6. *(First UP-isomorphism Theorem) Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f : A \to B$ a UP-homomorphism. Then*

$$A/\sim_f \cong \mathrm{Im}(f).$$

Proof. By Theorem 1.16 (3), we have $\mathrm{Im}(f)$ is a UP-subalgebra of B. Thus $f : A \to \mathrm{Im}(f)$ is a UP-epimorphism. Applying Theorem 2.5 (2), we obtain $A/\sim_f \cong \mathrm{Im}(f)$. □

Lemma 2.7. *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, $f : A \to B$ a UP-homomorphism, and H a UP-subalgebra of A. Denote $H_{\sim_f} = \bigcup_{h \in H} (h)_{\sim_f}$. Then $H_{\sim_f}$ is a UP-subalgebra of A.*

Proof. Clearly, $\emptyset \neq H_{\sim_f} \subseteq A$. Let $a,b \in H_{\sim_f}$. Then $a \in (x)_{\sim_f}$ and $b \in (y)_{\sim_f}$ for some $x,y \in H$, so $(a)_{\sim_f} = (x)_{\sim_f}$ and $(b)_{\sim_f} = (y)_{\sim_f}$. Theorem 2.3 gives $(A/\sim_f, *, (0_A)_{\sim_f})$ is a UP-algebra, so

$$(a \cdot b)_{\sim_f} = (a)_{\sim_f} * (b)_{\sim_f} = (x)_{\sim_f} * (y)_{\sim_f} = (x \cdot y)_{\sim_f}.$$

Thus $a \cdot b \in (x \cdot y)_{\sim_f}$. Since $x,y \in H$, it follows from Proposition 1.7 that $x \cdot y \in H$. Thus $a \cdot b \in (x \cdot y)_{\sim_f} \subseteq H_{\sim_f}$. Hence, $H_{\sim_f}$ is a UP-subalgebra of A. □

Theorem 2.8. *(Second UP-isomorphism Theorem) Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, $f\colon A \to B$ a UP-homomorphism, and H a UP-subalgebra of A. Denote $H_{\sim_f}/\sim_f = \{(x)_{\sim_f} \mid x \in H_{\sim_f}\}$. Then*

$$H/\sim_{\pi_f|_H} \cong H_{\sim_f}/\sim_f .$$

Proof. By Lemma 2.7, we have $H_{\sim_f}$ is a UP-subalgebra of A. Then it is easy to check that $H_{\sim_f}/\sim_f$ is a UP-subalgebra of $A/\sim_f$, thus $(H_{\sim_f}/\sim_f, *, (0_A)_{\sim_f})$ itself is a UP-algebra. Also, it is obvious that $H \subseteq H_{\sim_f}$, then

$$(\pi_f|_H =) g\colon H \to H_{\sim_f}/\sim_f, x \mapsto (x)_{\sim_f}, \tag{2.3}$$

is a mapping. Indeed, g is the restriction of π_f to H. Thus g is a UP-epimorphism. Indeed, $H_{\sim_f}/\sim_f = H/\sim_f$. Theorem 2.6 gives $H/\sim_{\pi_f|_H} \cong H_{\sim_f}/\sim_f$. □

Theorem 2.9. *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, $f\colon A \to B$ and $g\colon A \to B$ UP-homomorphisms with $\sim_f \subseteq \sim_g$. Define the binary relation $\sim_g/\sim_f$ on $A/\sim_f$ as follows: for all $x,y \in A$,*

$$(x)_{\sim_f} \sim_g/\sim_f (y)_{\sim_f} \text{ if and only if } x \sim_g y. \tag{2.4}$$

Then $\sim_g/\sim_f$ is a congruence on $A/\sim_f$.

Proof. By Theorem 2.3, we have $(A/\sim_f, *, (0_A)_{\sim_f})$ is a UP-algebra.
Reflexive: For all $x \in A$, we have $x \sim_g x$. Thus $(x)_{\sim_f} \sim_g/\sim_f (x)_{\sim_f}$.
Symmetric: Let $x,y \in A$ be such that $(x)_{\sim_f} \sim_g/\sim_f (y)_{\sim_f}$. Then $x \sim_g y$, so $y \sim_g x$. Thus $(y)_{\sim_f} \sim_g/\sim_f (x)_{\sim_f}$.
Transitive: Let x,y,z be such that $(x)_{\sim_f} \sim_g/\sim_f (y)_{\sim_f}$ and $(y)_{\sim_f} \sim_g/\sim_f (z)_{\sim_f}$. Then $x \sim_g y$ and $y \sim_g z$, so $x \sim_g z$. Thus $(x)_{\sim_f} \sim_g/\sim_f (z)_{\sim_f}$.
Therefore, $\sim_g/\sim_f$ is an equivalence relation on $A/\sim_f$. Finally, let $x,y,u,v \in A$ be such that $(x)_{\sim_f} \sim_g/\sim_f (u)_{\sim_f}$ and $(y)_{\sim_f} \sim_g/\sim_f (v)_{\sim_f}$. Then $x \sim_g u$ and $y \sim_g v$. The binary relation $\sim_g$ is a congruence on A by Theorem 2.2, that is $x \cdot y \sim_g u \cdot v$. Thus $(x \cdot y)_{\sim_f} \sim_g/\sim_f (u \cdot v)_{\sim_f}$, so $(x)_{\sim_f} * (y)_{\sim_f} \sim_g/\sim_f (u)_{\sim_f} * (v)_{\sim_f}$. Hence, $\sim_g/\sim_f$ is a congruence on $A/\sim_f$. □

Theorem 2.10. *(Third UP-isomorphism Theorem) Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, $f\colon A \to B$ and $g\colon A \to B$ UP-homomorphisms with $\sim_f \subseteq \sim_g$. Then*

$$(A/\sim_f)/(\sim_g/\sim_f) \cong A/\sim_g .$$

Proof. By Theorem 2.3, we obtain $(A/\sim_f, *, (0_A)_{\sim_f})$ and $(A/\sim_g, *', (0_A)_{\sim_g})$ are UP-algebras. By Theorem 2.4, we obtain

$$\pi_f\colon A \to A/\sim_f, x \mapsto (x)_{\sim_f}$$

and

$$\pi_g\colon A \to A/\sim_g, x \mapsto (x)_{\sim_g}$$

are UP-epimorphisms. Applying Theorem 2.5 (2), there exists a UP-isomorphism

$$g/f\colon A/\sim_f \to A/\sim_g, (x)_{\sim_f} \mapsto (x)_{\sim_g}. \tag{2.5}$$

Indeed, $A/\sim_f \cong A/\sim_g$. By Theorem 2.9 and 2.3, we have $(A/\sim_f)/\sim_{g/f}$ is a UP-algebra. By Theorem 2.4, we obtain

$$\pi_{g/f}\colon A/\sim_f \to (A/\sim_f)/\sim_{g/f}, (x)_{\sim_f} \mapsto ((x)_{\sim_f})_{\sim_{g/f}}$$

is a UP-epimorphism. Applying Theorem 2.5 (2), there exists a UP-isomorphism

$$\varphi\colon (A/\sim_f)/\sim_{g/f} \to A/\sim_g, ((x)_{\sim_f})_{\sim_{g/f}} \mapsto (x)_{\sim_g}. \tag{2.6}$$

That is,

$$(A/\sim_f)/\sim_{g/f} \cong A/\sim_g .$$

We shall show that $\sim_{g/f} = \sim_g/\sim_f$. For any $(x)_{\sim_f}, (y)_{\sim_f} \in A/\sim_f$,

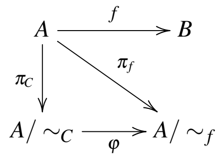
$$\begin{aligned} (x)_{\sim_f} \sim_{g/f} (y)_{\sim_f} &\Leftrightarrow (g/f)((x)_{\sim_f}) = (g/f)((y)_{\sim_f}) \\ &\Leftrightarrow (x)_{\sim_g} = (y)_{\sim_g} \\ &\Leftrightarrow x \sim_g y \\ &\Leftrightarrow (x)_{\sim_f} \sim_g/\sim_f (y)_{\sim_f} \end{aligned}$$

by (2.1) and (2.4). Thus $\sim_{g/f} = \sim_g/\sim_f$. Hence, $(A/\sim_f)/(\sim_g/\sim_f) \cong A/\sim_g$. □

Corollary 2.11. *Let* $(A, \cdot, 0_A)$ *and* $(B, \bullet, 0_B)$ *be UP-algebras,* $f : A \to B$ *a UP-homomorphism, and* C *a UP-ideal of* A*. Then*

$$A/\sim_C \cong A/\sim_f .$$

As π_f *makes the following diagram commute,*



Proof. It is straightforward by Theorem 1.12, 1.14, 2.4, and 2.5 (2). □

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A horizontal endomorphism of the canonical superspray

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Abstract

Giving up the homogeneity condition of a Lagrange superfunction, we prove that there is a unique horizontal endomorphism h (nonlinear connection) on a supermanifold $\mathscr{M}$, such that h is conservative and its torsion vanishes. There are several examples for nonhomogeneous Lagrangians such that this result is not true.

1. Introduction

The fundamental relation between the horizontal endomorphisms and semisprays was discovered, independently, by M. Crampin [3] and J. Grifone [6, 7]. The conditions for a system of second order differential equations to be derivable from a Lagrangian are related to the differential geometry of the tangent bundle of configuration space. These conditions are simply expressed in terms of the horizontal distribution which is associated with any vector field representing a system of second-order differential equations.

In supergeometry, relationship between nonlinear connections and supersprays structures to be discussed. Also it was shown that there exists a homogeneous superspray, so called the Euler-Lagrange supervector field, which is induced by a Finsler metric [8, 13]. This superspray can help us to introduce a horizontal endomorphism which will be used to obtain the main result. So we will show that on a Finsler supermanifold $(\mathscr{M},F)$, there is a unique horizontal endomorphism h which is conservative (see theorem 3.6) i.e. $d_hL=0$. The property $d_hL=0$ tells us that the Lagrangian L is constant along the horizontal curves of the nonlinear connection and hence it is constant along the geodesics of the superspace. This result is not true for an arbitrary Lagrangian L. We will find non homogeneous Lagrangian superfunctions for which $d_hL\neq 0$.

The paper is organized as follows: Section 2 deals with the vertical and complete lift of supervector fields to the tangent superbundle. It contains a brief review of the notion of superspray and the relationship between supersprays and nonlinear connections. We also introduce the notion of Euler-Lagrange supervector field which is an important tool to construct the horizontal endomorphism. In section three, we introduce a horizontal endomorphism h on a supermanifold $\mathscr{M}$, such that h is conservative and its torsion vanishes. We consider an example for a nonhomogeneous Lagrangian such that this result is not true.

2. Preliminary

The basic structure for building up supermanifolds is the Grassmann algebra. With $B_L=(B_L)_0+(B_L)_1$ we shall denote a real Grassmann algebra with L generators. If $L=\infty, B_L$ is given a suitable Banach norm, making B_∞ a Banach-Grassmann algebra as defined in [9]. Here B_L is a graded commutative algebra, namely ,

$$ab\in(B_L)_{|a|+|b|},\quad ab=(-1)^{|a||b|}ba,$$

where the element $a,b\in B_L$ are the homogeneous. A $(m,n)-$dimensional supermanifold $\mathscr{M}$ is defined on $B_L^{m,n}$(see details in [4]). Throughout this paper, $\mathscr{M}$ will denote an (m,n)-dimensional supermanifold.

The concept of nonlinear connection (N-connection) was introduced in component form in a number of works by Cartan [2], Kawaguchi [10, 11] and Ehresmann [5]. But the first global definition is due to Barthel [1]. The geometry of N-connection in superspaces are considered in detail in [16], [14].

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Let us consider a vector superbundle $\mathscr{E}=(E,\pi_E,\mathscr{M})$ whose type fiber is $\mathscr{F}$ and $\pi^T: T\mathscr{E}\to T\mathscr{M}$ is the superdifferential of the map π_E. The kernel of this vector superbundle morphism being a subbundle of (TE,τ_E,E) is called the vertical subbundle over $\mathscr{E}$ and is denoted by $V\mathscr{E}=(VE,\tau_V,E)$. Its total space is $V\mathscr{E}=\bigcup_{u\in\mathscr{E}} \quad V_u, \quad$ where $V_u=ker\pi^T, \quad u\in\mathscr{E}$.
A nonlinear connection, N-connection [15, 16], in vector superbundle $\mathscr{E}$ is a splitting on the left of the exact sequence

$$0\longmapsto V\mathscr{E}\overset{i}{\longmapsto} T\mathscr{E}\longmapsto T\mathscr{E}/V\mathscr{E}\longmapsto 0, \tag{2.1}$$

i.e. a morphism of vector superbundles $N: T\mathscr{E}\to V\mathscr{E}$ such that $N\circ i$ is the identity on $V\mathscr{E}$.
The kernel of the morphism N is called the horizontal subbundle and is denoted by (HE,τ_H,E). From the exact sequence (2.1) it follows that N-connection structure can be equivalently defined as a distribution $T_uE=H_uE\oplus V_uE, \quad u\in E$ on E defining a global decomposition, as a Whitney sum,

$$T\mathscr{E}=H\mathscr{E}\oplus V\mathscr{E}. \tag{2.2}$$

Locally a nonlinear connection in $\mathscr{E}$ is given by its coefficients

$$N_i^j(x,y,\eta,\theta),N_i^\beta(x,y,\eta,\theta),N_\alpha^j(x,y,\eta,\theta),N_\alpha^\beta(x,y,\eta,\theta).$$

In the tangent superbundle a local basis adapted to the given nonlinear connection N is introduced by

$$(\frac{\delta}{\delta x_i},\frac{\delta}{\delta\eta_\alpha},\frac{\partial}{\partial y_i},\frac{\partial}{\partial\theta_\alpha}),$$

where

$$\frac{\delta}{\delta x_i}:=\frac{\partial}{\partial x_i}-N_i^j\frac{\partial}{\partial y_j}-N_i^\alpha\frac{\partial}{\partial\theta_\alpha} \tag{2.3}$$

and

$$\frac{\delta}{\delta\eta_\alpha}:=\frac{\partial}{\partial\eta_\alpha}-N_\alpha^i\frac{\partial}{\partial y_i}-N_\alpha^\beta\frac{\partial}{\partial\theta_\beta}. \tag{2.4}$$

Let $X=X^i\frac{\partial}{\partial x_i}+X^\alpha\frac{\partial}{\partial\eta_\alpha}$ be a supervector field in a coordinate neighborhood $\mathscr{U}$ of $\mathscr{M}$, then the vertical lift X^v and the complete lift X^c of X have the form

$$X^v=X^i\frac{\partial}{\partial y_i}+X^\alpha\frac{\partial}{\partial\theta_\alpha},$$

and

$$\begin{aligned} X^c \;=\; & \sum_{i=1}^m\left(X^i\frac{\partial}{\partial x_i}+\left(\sum_{j=1}^m y_j\frac{\partial X^i}{\partial x_j}+\sum_{\gamma=1}^n\theta_\gamma\frac{\partial X^i}{\partial\eta_\gamma}\right)\frac{\partial}{\partial y_i}\right)\\ +\; & \sum_{\alpha=1}^n\left(X^\alpha\frac{\partial}{\partial\eta_\alpha}+\left(\sum_{j=1}^m y_j\frac{\partial X^\alpha}{\partial x_j}+\sum_{\gamma=1}^n\theta_\gamma\frac{\partial X^\alpha}{\partial\eta_\gamma}\right)\frac{\partial}{\partial\theta_\gamma}\right). \end{aligned}$$

Definition 2.1. *A vertical endomorphism on the tangent superbundle $T\mathscr{M}$ is a (super) tensor field*

$$J:\mathscr{X}(T\mathscr{M})\mapsto\mathscr{X}(T\mathscr{M})$$

satisfies in $ImJ=KerJ, \quad J^2=0$.

If J is a vertical endomorphism, the vertical differentiation d_J is the mapping $d_J=[i_J,d]=i_Jod-doi_J$. In particular, for any superfunction f on $\mathscr{M}$, we have $d_Jf=i_Jdf$.
Let $(x_i;\eta_\alpha)$ be local coordinates on $\mathscr{M}$ and $(x_i,y_i;\eta_\alpha,\theta_\alpha)$ the corresponding local coordinates on $T\mathscr{M}$. The Liouville supervector field C on $\mathscr{X}(T\mathscr{M})$ defined by

$$C=y_i\frac{\partial}{\partial y_i}+\theta_\alpha\frac{\partial}{\partial\theta_\alpha}. \tag{2.5}$$

Definition 2.2. *A morphism $h:\mathscr{X}(T\mathscr{M})\mapsto\mathscr{X}(T\mathscr{M})$ is said to be a horizontal endomorphism on $\mathscr{M}$ if it satisfies the following conditions:*
(i) $h^2=h$
(ii)$Kerh=\mathscr{X}^v(T\mathscr{M})$.

Assume h is a horizontal endomorphism. The supervector 1-form, or simply the vector 1-form, $[h,C]$ is said to be the tension of h. The vector 2-form $[J,h]$ is said to be the torsion of h.
Let h be a horizontal endomorphism. If $\mathscr{X}^h(T\mathscr{M}):=Imh$, then $\mathscr{X}(T\mathscr{M})=\mathscr{X}^h(T\mathscr{M})\oplus\mathscr{X}^v(T\mathscr{M})$ and $\mathscr{X}^h(T\mathscr{M})$ is called the supermodule of horizontal supervector fields. $v:=(id-h):\mathscr{X}(T\mathscr{M})\mapsto\mathscr{X}(T\mathscr{M})$, is the vertical projection on $\mathscr{X}^v(T\mathscr{M})$ along $\mathscr{X}^h(T\mathscr{M})$. Also, we have $hoJ=0$ and $Joh=J$.

Definition 2.3. *A morphism $\mathscr{F}:\mathscr{X}(T\mathscr{M})\mapsto\mathscr{X}(T\mathscr{M})$ is said to be an almost complex structure on $\mathscr{M}$ if $\mathscr{F}^2=-1$.*

Definition 2.4. *A supervector field S on $T\mathscr{M}$ is a superspray if*

$$J(S)=y_i\frac{\partial}{\partial y_i}+\theta_\alpha\frac{\partial}{\partial\theta_\alpha}. \tag{2.6}$$

When the coefficients of a superspray S are homogeneous of degree 2, we say that S is a homogeneous superspray.
If S is a homogeneous superspray and C the Liouville supervector field, then $[C,S]=S$. It is not difficult to show that if h is a horizontal endomorphism on $\mathscr{M}$ and S' an arbitrary superspray then $S:=hS'$ is also a superspray on $\mathscr{M}$. It satisfies the relation $h[C,S]=S$. So S is called the superspray associated to h.
A generalized Lagrange superspace is a pair $GL^{m,n}=(\mathscr{M},g(x,y,;\eta,\theta))$, where $g(x,y;\eta,\theta)$ is a distinguished tensor field on $T\mathscr{M}^o=T\mathscr{M}-\{0\}$, supersymmetric of superrank (m,n) . A Lagrange superspace is defined as a particular case of generalize Lagrange superspace when the distinguished tensor field on $\mathscr{M}$ can be expressed as

$$g_{ij}=\frac{1}{2}\frac{\partial^2 L}{\partial y_i\partial y_j},\ g_{i\beta}=\frac{1}{2}\frac{\partial^2 L}{\partial y_i\partial\theta_\beta},\ g_{\alpha j}=\frac{1}{2}\frac{\partial^2 L}{\partial\theta_\alpha\partial y_j},\ g_{\alpha\beta}=\frac{1}{2}\frac{\partial^2 L}{\partial\theta_\alpha\partial\theta_\beta} \tag{2.7}$$

where $L:T\mathscr{M}\mapsto B_L$, is a superfunction called a Lagrangian on $\mathscr{M}$ (see [15]).
Locally, L is regular if and only if the matrix

$$g=\begin{bmatrix} g_{ij} & g_{i\beta} \\ g_{\alpha j} & g_{\alpha\beta}\end{bmatrix}$$

is invertible. For example, if $L=F^2$, where F will be defined in the following definition, then L is a regular Lagrangian. In this case L is a homogeneous superfunction of degree 2.
To define a (super) metric on a supermanifold, We consider the base manifold M of a vector superbundle $\mathscr{E}=(E,\pi_E,\mathscr{M})$ to be a connected and paracompact manifold.

Definition 2.5. *A metric structure on the total space E of a vector superbundle $\mathscr{E}$ is a supersymmetric, second order, covariant supertensor field g which in every point $u\in\mathscr{E}$ is given by nondegenerate supermatrix $g_{ab}=g(\partial_a,\partial_b)$ (with nonvanishing superdeterminant, $\det g\neq 0$).*

Definition 2.6. *A function $F:T\mathscr{M}\to B_L$ is called a Finsler metric (see [15]) if the following conditions are satisfied:*
(1) The restriction of F to $T\mathscr{M}^o=T\mathscr{M}-\{0\}$ is of the class G^∞ and F is only supersmooth on the image of the null cross–section in the tangent supermanifold to M.
(2) $F(x,\lambda y;\eta,\lambda\theta)=\lambda F(x,y;\eta,\theta)$, where λ is a real positive number.
(3) The restriction of F to the even subspace of $T\mathscr{M}^o$ is a positive function.
(4) If we put

$$g_{ij}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y_i\partial y_j},\ g_{i\beta}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y_i\partial\theta_\beta},\ g_{\alpha j}=\frac{1}{2}\frac{\partial^2 F^2}{\partial\theta_\alpha\partial y_j},\ g_{\alpha\beta}=\frac{1}{2}\frac{\partial^2 F^2}{\partial\theta_\alpha\partial\theta_\beta} \tag{2.8}$$

then

$$g=\begin{bmatrix} g_{ij} & g_{i\beta} \\ g_{\alpha j} & g_{\alpha\beta}\end{bmatrix}$$

is invertible .

A pair $(\mathscr{M},F)$ is called a Finsler Supermanifold.
It is obvious that Finsler superspaces form a particular class of Lagrange superspaces with Lagrangian $\mathscr{L}=F^2$.

Definition 2.7. *The dynamics of a system $(T\mathscr{M},\omega,L)$, associated to a Lagrangian $L\in T\mathscr{M}$, is given by a supervector field $X\in\mathscr{X}(T\mathscr{M})$ satisfying the equation*

$$i_X\omega=-dL \tag{2.9}$$

where $\omega=dd_JL$.

It is shown that the Euler-Lagrange supervector field is a superspray [13].

Theorem 2.8. *([13]) On any Finsler supermanifold $(\mathscr{M},F)$, there is a homogeneous superspray*

$$S=y_j\frac{\partial}{\partial x_j}+\theta_\beta\frac{\partial}{\partial\eta_\beta}-2G^j(x,y;\eta,\theta)\frac{\partial}{\partial y_j}-2G^\beta(x,y;\eta,\theta)\frac{\partial}{\partial\theta_\beta}$$

where

$$G^j=\frac{1}{4}g^{jm}(y^k\frac{\partial^2F^2}{\partial x_k\partial y_m}-\frac{\partial^2F^2}{\partial\eta_\alpha\partial y_m}\theta_\alpha-\frac{\partial F^2}{\partial x_m})$$

$$-\frac{1}{4}g^{m\beta}(y^j\frac{\partial^2F^2}{\partial x_j\partial\theta_\gamma}+\frac{\partial^2F^2}{\partial\eta_\mu\partial\theta_\gamma}\theta_\mu-\frac{\partial F^2}{\partial\eta_\gamma}) \tag{2.10}$$

and

$$G^\beta=\frac{1}{4}g^{\beta m}(y^k\frac{\partial^2F^2}{\partial x_k\partial y_m}-\frac{\partial^2F^2}{\partial\eta_\alpha\partial y_m}\theta_\alpha-\frac{\partial F^2}{\partial x_m})$$

$$+\frac{1}{4}g^{\beta\gamma}(y^j\frac{\partial^2F^2}{\partial x_j\partial\theta_\gamma}+\frac{\partial^2F^2}{\partial\eta_\mu\partial\theta_\gamma}\theta_\mu-\frac{\partial F^2}{\partial\eta_\gamma}). \tag{2.11}$$

We call this superspray the **canonical superspray of a Finsler metric**.
Let $(\mathscr{M},F)$ be a Finsler supermanifold and consider $T\mathscr{M}^o = T\mathscr{M} - \{0\}$ and denote by $VT\mathscr{M}^o$ the vertical superbundle over $T\mathscr{M}^o$. It is easy to show that a Finsler metric F allows to define a (super) metric g on the vertical superbundle $VT\mathscr{M}^o$, by setting $L = F^2$ and

$$g(JX,JY) = \omega(JX,Y) \tag{2.12}$$

for $X,Y \in T(TM)$. So the coefficients of this metric are superfunctions defined in (2.8).
If h is a horizontal endomorphism on $\mathscr{M}$ and $v = id - h$, g can be extended to $T\mathscr{M}$ by putting

$$G(X,Y) = g(JX,JY) + g(vX,vY),$$

where J is the vertical endomorphism.

3. A Horizontal endomorphism

We are now in position to define a horizontal endomorphism which is conservative and torsion-free. To do it we need to define a supervector 1-form $[J,X]$, where J is a vector 1-form and X a supervector field. Since J is a vector form of degree 0, for each supervector field Y on $T\mathscr{M}$ we have

$$\begin{aligned}[J,X]Y &= (-1)^{|X||Y|}\left(Y^i[\frac{\partial}{\partial y_i},X] + Y^\alpha[\frac{\partial}{\partial \theta_\alpha},X]\right)\\ &- (-1)^{|X||Y|}\left(Y(X^i)\frac{\partial}{\partial y_i} + Y(X^\alpha)\frac{\partial}{\partial \theta_\alpha}\right).\end{aligned}$$

An easy computation shows that

$$[J,X]Y = (-1)^{|X||Y|}[JY,X] - (-1)^{|X||Y|}J[Y,X]. \tag{3.1}$$

Theorem 3.1. *(1) Any superspray S generates a torsion-free horizontal endomorphism*

$$h = \frac{1}{2}(id + [J,S]), \tag{3.2}$$

where id is the identity map on $T(TM)$. The horizontal lift of a supervector field X on $\mathscr{M}$ is

$$X^h := hX^c = \frac{1}{2}(X^c + [X^v,S]). \tag{3.3}$$

(2) A superspray associated to h is

$$S_h = \frac{1}{2}(S + [C,S]). \tag{3.4}$$

If S is a homogeneous superspray, then $S_h = S$.
(3) The torsion of h vanishes.

Proof. (1) First, we show that h is a horizontal endomorphism. So let X be a homogeneous supervector field on $\mathscr{M}$. Since S is an even supervector field, thus

$$\begin{aligned}h(X^v) &= \frac{1}{2}\Big(X^v - J\Big\{X^i(\frac{\partial}{\partial x_i} - 2\frac{\partial G^j}{\partial y_i}\frac{\partial}{\partial y_j} - 2\frac{\partial G^\beta}{\partial y_i}\frac{\partial}{\partial \theta_\beta})\\ &+ X^\alpha(\frac{\partial}{\partial \eta_\alpha} - 2\frac{\partial G^i}{\partial \theta_\alpha}\frac{\partial}{\partial y_i} - 2\frac{\partial G^\beta}{\partial \theta_\alpha}\frac{\partial}{\partial \theta_\beta})\Big\} - y_j(\frac{\partial X^i}{\partial x_j}\frac{\partial}{\partial y_i} + \frac{\partial X^\alpha}{\partial x_j}\frac{\partial}{\partial \theta_\alpha})\\ &- \theta^\beta(\frac{\partial X^i}{\partial \eta_\beta}\frac{\partial}{\partial y_i} + \frac{\partial X^\alpha}{\partial \eta_\beta}\frac{\partial}{\partial \theta_\beta})\Big) = \frac{1}{2}(X^v - X^i\frac{\partial}{\partial y_i} - X^\alpha\frac{\partial}{\partial \theta_\alpha}) = 0.\end{aligned}$$

This shows that $X^v(T\mathscr{M}) \subset kerh$.
Now, let $Y \in kerh$, then

$$0 = 2h(Y) = Y + [JY,S] - J[Y,S],$$

so $Y = -[JY,S] + J[Y,S]$. If we compute JY, it follows that

$$JY = -J[JY,S] = 0.$$

Thus $kerh \subset X^v(T\mathscr{M})$ and therefore $X^v(T\mathscr{M}) = kerh$.
It is clear that for any supervector field $X^v \in \mathscr{X}(T\mathscr{M})$, we have $h^2(X^v) = 0$. On the other hand

$$\begin{aligned}h^2(X^c) &= \frac{1}{2}\Big(hX^c + h[JX^c,S] - hoJ[X^c,S]\Big)\\ &= \frac{1}{2}\Big(hX^c + h[X^v,S]\Big) = hX^c.\end{aligned}$$

This shows that on $\mathscr{X}(T\mathscr{M})$ we have $h^2 = h$.

(2) If $\tilde{S}$ is an arbitrary superspray on $\mathscr{M}$ and h is the horizontal endomorphism defined by (3.2), then $Joh(\tilde{S}) = C$. So $S_h = h(\tilde{S})$ is a superspray.
Now let $\tilde{S}$ has the local form

$$\tilde{S} = y_i\frac{\partial}{\partial x_i} + \theta_\alpha\frac{\partial}{\partial \eta_\alpha} - 2\tilde{G}^i\frac{\partial}{\partial y_i} - 2\tilde{G}^\alpha\frac{\partial}{\partial \theta_\alpha}.$$

It is not difficult to show that $J[\tilde{S}, S] = -S + \tilde{S}$. If S is a homogeneous superspray, i.e. G^i and G^α are superfunctions of degree two, then $[C,S] = S$ and

$$h(\tilde{S}) = \frac{1}{2}(\tilde{S} + [J\tilde{S}, S] - J[\tilde{S}, S]) = S.$$

(3) We begin this part of proof with the definition of horizontal endomorphism h, thus we have

$$[J,h] = \frac{1}{2}[J,id] + \frac{1}{2}[J,[J,S]].$$

It is clear that $[J,id] = 0$, so we show that $[J,[J,S]] = 0$. Note that in this case J is an even 1-vector valued form and S an even supervector field. From the Bianchi identities for the lie superalgebra of vector-valued forms, we have

$$(-1)^{1.0}[J,[J,S]] + (-1)^{1.1}[J,[S,J]] + (-1)^{0.1}[S,[J,J]] = 0.$$

Apply (3.1) to $[S,J]$, we see that $[S,J] = -[J,S]$. Since $[J,J] = 0$, therefore $[J,[J,S]] = 0$ and the torsion of h is zero. □

Lemma 3.2. *If h is the horizontal endomorphism defined by (3.2), then there is a unique almost complex structure $\mathscr{F}$ on $T\mathscr{M}$ such that*

$$\mathscr{F}oJ = h, \quad \mathscr{F}oh = -J.$$

Proof. If we use the above conditions, it is easy to see that $\mathscr{F}$ permutes the vertical and horizontal superspaces if and only if

$$\begin{aligned} \mathscr{F}(\frac{\partial}{\partial x_i}) &= -\frac{\partial}{\partial y_i} + N_i^j\frac{\delta}{\delta x_j} + N_i^\alpha\frac{\delta}{\delta \eta_\alpha} \quad , \quad \mathscr{F}(\frac{\partial}{\partial y_i}) = \frac{\delta}{\delta x_i}, \\ \mathscr{F}(\frac{\partial}{\partial \eta_\alpha}) &= -\frac{\partial}{\partial \theta_\alpha} + N_\alpha^i\frac{\delta}{\delta x_i} + N_\alpha^\beta\frac{\delta}{\delta \eta_\beta} \quad , \quad \mathscr{F}(\frac{\partial}{\partial \theta_\alpha}) = \frac{\delta}{\delta \eta_\alpha}. \end{aligned}$$

For example $\mathscr{F}oJ = h$ implies that $\mathscr{F}oJ(\frac{\partial}{\partial x_i}) = \frac{\delta}{\delta x_i}$, so $\mathscr{F}(\frac{\partial}{\partial y_i}) = \frac{\delta}{\delta x_i}$. Similarly, $\mathscr{F}(\frac{\partial}{\partial \theta_\alpha}) = \frac{\delta}{\delta \eta_\alpha}$. Also $\mathscr{F}oh = -J$ implies that $\mathscr{F}(\frac{\delta}{\delta x_i}) = -\frac{\partial}{\partial y_i}$, so $\mathscr{F}(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial y_i} + N_i^j\frac{\delta}{\delta x_j} + N_i^\alpha\frac{\delta}{\delta \eta_\alpha}$. □

Definition 3.3. *With respect to the (super) metric G on $T\mathscr{M}$, we define the Kahler form*

$$K(X,Y) = G(X,JY) - G(JX,Y). \tag{3.5}$$

Theorem 3.4. *Let h be a horizontal endomorphism defined by (3.2). So*

$$i_v\omega = K.$$

Proof. The canonical expression of the vertical projection $v = 1 - h$ is

$$v = (N_i^j\frac{\partial}{\partial y_j} + N_i^\beta\frac{\partial}{\partial \theta_\beta})\otimes dx_i - (N_\alpha^i\frac{\partial}{\partial y_i} + N_\alpha^\beta\frac{\partial}{\partial \theta_\beta})\otimes d\eta_\alpha + \frac{\partial}{\partial y_i}\otimes dy_i - \frac{\partial}{\partial \theta_\alpha}\otimes d\theta_\alpha.$$

A long but standard computation shows that

$$\begin{aligned} i_v\omega &= \frac{\partial^2 L}{\partial y_j\partial y_i}N_k^j dx_k\wedge dx_i - \frac{\partial^2 L}{\partial y_j\partial y_i}N_\alpha^j d\eta_\alpha\wedge dx_i + \frac{\partial^2 \mathscr{L}}{\partial y_j\partial y_i}dy_j\wedge dx_i \\ &- (-1)^L\left\{\frac{\partial^2 L}{\partial \theta_\alpha\partial y_i}N_k^\alpha dx_k\wedge dx_i + \frac{\partial^2 L}{\partial \theta_\alpha\partial y_i}N_\alpha^\alpha d\eta_\beta\wedge dx_i + \frac{\partial^2 \mathscr{L}}{\partial \theta_\alpha\partial y_i}d\theta_\alpha\wedge dx_i\right\} \\ &- (-1)^L\left\{\frac{\partial^2 L}{\partial y_i\partial \theta_\alpha}N_j^i dx_j\wedge d\eta_\alpha - \frac{\partial^2 L}{\partial y_i\partial \theta_\alpha}N_\beta^i d\eta_\beta\wedge d\eta_\alpha + \frac{\partial^2 \mathscr{L}}{\partial y_i\partial \theta_\alpha}dy_i\wedge d\eta_\alpha\right\} \\ &- \left\{\frac{\partial^2 L}{\partial \theta_\beta\partial \theta_\alpha}N_i^\beta dx_i\wedge d\eta_\alpha + \frac{\partial^2 L}{\partial \theta_\beta\partial \theta_\alpha}N_\gamma^\beta d\eta_\gamma\wedge d\eta_\alpha + \frac{\partial^2 \mathscr{L}}{\partial \theta_\beta\partial \theta_\alpha}d\theta_\beta\wedge d\eta_\alpha\right\}. \end{aligned}$$

Now, it is easy to check that for two supervector fields $X, Y \in \mathscr{X}(T\mathscr{M})$, we have

$$(i_v\omega)(X,Y) = \omega(vX,Y) + \omega(X,vY). \tag{3.6}$$

Since $\omega(X,vY) = -(-1)^{XY}\omega(vX,Y)$ so

$$\begin{aligned} (i_v\omega)(X,Y) &= g(vX,JY) - (-1)^{XY}g(vY,JX) = \{g(JvX,JJY)\} \\ &+ g(vvX,vJY) - (-1)^{XY}\{g(vvY,vJX) + g(JvY,JJX)\} \\ &= G(vX,JY) - (-1)^{XY}G(vY,JX) \end{aligned}$$

But $G(vX,JY) = G(X,JY)$, thus $(i_v\omega)(X,Y) = K(X,Y)$. □

Definition 3.5. *A nonlinear connection is called Lagrangian if the horizontal superspace is Lagrangian with respect to the 2-form* $\omega = dd_J L$, *i.e. if* $\omega(hX, hY) = 0$ *for any* $X, Y \in \mathscr{X}(T\mathscr{M})$.

An easy computation will show that if a nonlinear connection is Lagrangian then $i_h\omega = \omega$. So from the above proposition we have

$$2\omega = i_{id}\omega = i_h\omega + i_v\omega$$

therefore $K = \omega$.

Theorem 3.6. *Consider a regular homogeneous Lagrangian L and N a Lagrangian connection. There exist a unique horizontal endomorphism h on* $\mathscr{M}$ *such that*
(i) h is conservative, i.e. $d_h L = 0$,
(ii) h is torsion-free,
(iii) The tension of h is zero, i.e. $[h, C] = 0$.
Explicitly, h is given by

$$h = \frac{1}{2}(id + [J, S]) \tag{3.7}$$

where S is the canonical superspray of a Finsler metric.

Proof. Let $(x_i; \eta_\alpha)$ be local coordinates on $\mathscr{M}$ and $(x_i, y_i; \eta_\alpha, \theta_\alpha)$ the corresponding local coordinates on $T\mathscr{M}$. It should be mentioned that we assume L is a homogeneous Lagrangian superfunction of degree $K > 1$ with respect to (y, θ). We proved before that $h = \frac{1}{2}(id + [J, S])$ is a torsion-free horizontal endomorphism. Given the local forms of $h = dx_i \otimes \frac{\delta}{\delta x_i} + d\eta_\alpha \otimes \frac{\delta}{\delta \eta_\alpha}$ and $C = y_i \frac{\partial}{\partial y_i} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}$ and using the method used in Lemma 3.5, it is easy to see that $[h, C] = 0$. To complete the proof, we only need to prove that $d_h L = 0$. Let S be the canonical superspray introduced in theorem 2.8. As we mentioned earlier, S is even supervector field, so for any supervector field X on $T\mathscr{M}$, we have $(i_S\omega)(X) = \omega(S, X)$. Since $K = \omega$ thus

$$\begin{aligned}(i_S\omega)(X) &= G(S, JX) - G(JS, X) = -g(vC, vX)\\ &= -g(vC, J\mathscr{F}X) = -\omega(C, \mathscr{F}X).\end{aligned}$$

Now, we show that for any homogeneous supervector field $X \in \mathscr{X}(T\mathscr{M})$, $\omega(X, \mathscr{F}X) = i_v dL(X)$. So if X has a local form $X = X^i \frac{\partial}{\partial x_i} + \bar{X}^i \frac{\partial}{\partial y_i} + X^\alpha \frac{\partial}{\partial \eta_\alpha} + \bar{X}^\alpha \frac{\partial}{\partial \theta_\alpha}$, then we have

$$\begin{aligned}(i_v dL)(X) &= \frac{\partial L}{\partial y_i}\left(N^i_k dx_k - N^i_\alpha d\eta_\alpha + dy_i\right)(X)\\ &\quad -(-1)^{|L|}\frac{\partial L}{\partial \theta_\alpha}\left(N^\alpha_i dx_i + N^\alpha_\beta d\eta_\beta + d\theta_\alpha\right)(X)\\ &= \frac{\partial L}{\partial y_i}\left(N^i_k X^k - (-1)^{|X|} N^i_\alpha X^\alpha + \bar{X}^i\right)\\ &\quad -(-1)^{|L|}\frac{\partial L}{\partial \theta_\alpha}\left(N^\alpha_i X^i + (-1)^{|X|} N^\alpha_\beta X^\beta + (-1)^{|X|}\bar{X}^\alpha\right).\end{aligned}$$

One can easily check that $\omega(X, \mathscr{F}X) = i_v dL(X)$. Now, $i_S\omega = -dL$, because S is the canonical superspray and $dL = d_v L + d_h L$ then $d_h L = 0$. □

Let h be the horizontal endomorphism (3.2), the horizontal differential operator is defined by

$$d_h L(X) := dL(hX),$$

where X is a homogeneous supervector field on $\mathscr{M}$.
The horizontal covariant derivatives of a Lagrange superfunction L with respect to even or odd coordinates are denoted respectively by $L_{|i} = \frac{\delta L}{\delta x_i}$ and $L_{|\alpha} = \frac{\delta L}{\delta \eta_\alpha}$. In the following theorem, we use the canonical superspray to have a local expression for the horizontal covariant derivative of a Lagrange superfunction.

Theorem 3.7. *Let h be the horizontal endomorphism (3.2). The horizontal covariant derivatives of a Lagrange superfunction L are*

$$L_{|i} = \frac{1}{2}\frac{\partial(S(L))}{\partial y_i}, \tag{3.8}$$

$$L_{|\alpha} = \frac{1}{2}\frac{\partial(S(L))}{\partial \theta_\alpha} \tag{3.9}$$

Proof. First we compute the right hand of the above formulas. Then we have

$$\begin{aligned}\frac{\partial(S(L))}{\partial y_i} &= \frac{\partial L}{\partial x_i} + y_j\frac{\partial^2 L}{\partial y_i \partial x_j} + \theta_\alpha \frac{\partial^2 L}{\partial y_i \partial \eta_\alpha} - 2N^j_i \frac{\partial L}{\partial y_j}\\ &\quad - 4G^j \frac{\partial^2 L}{\partial y_i \partial y_j} - 2N^\alpha_i \frac{\partial L}{\partial \theta_\alpha} - 4G^\alpha \frac{\partial^2 L}{\partial y_i \partial \theta_\alpha},\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\frac{\partial(S(L))}{\partial\theta_\alpha} &= \frac{\partial L}{\partial\eta_\alpha}+y_j\frac{\partial^2 L}{\partial\theta_\alpha\partial x_j}-\theta_\beta\frac{\partial^2 L}{\partial\theta_\alpha\partial\eta_\beta}-2N^j_\alpha\frac{\partial L}{\partial y_j}\\ &- \quad 4G^j\frac{\partial^2 L}{\partial\theta_\alpha\partial y_j}-2N^\beta_\alpha\frac{\partial L}{\partial\theta_\beta}+4G^\beta\frac{\partial^2 L}{\partial\theta_\alpha\partial\theta_\beta}.\end{aligned} \tag{3.11}$$

If we now replace the superfunctions G^i and G^α with (2.10) and (2.11) respectively, then some terms of (3.10) and (3.11) cancel with some terms of the replaced sentences and the only terms that survive are $L_{|i}=2\frac{\delta L}{\delta x_i}$ and $L_{|\alpha}=2\frac{\delta L}{\delta\eta_\alpha}$, and the theorem is proved. □

From the above theorem we found a condition under which the horizontal differential of a Lagrangian L is vanishes. In other words we found that $S(L)=0$ implies $d_hL=0$.

In the previous theorem, we showed that if L is a homogeneous superfunction then there exist a unique horizontal endomorphism h on $\mathscr{M}$ such that $d_hL=0$. In the following, we will show that this result is not true for an arbitrary Lagrangian L. We will find non homogeneous Lagrangian superfunctions for which $d_hL\neq 0$.

Let $\mathscr{M}$ be a Riemannian supermanifold with a supermetric $\tilde{g}$. In the standard local coordinate system (x,η) in $\mathscr{M}$, $\tilde{g}$ is expressed in the form

$$\tilde{g}=\tilde{g}_{ij}dx_i\otimes dx_j+\tilde{g}_{i\alpha}dx_i\otimes d\eta_\alpha+\tilde{g}_{\alpha i}d\eta_\alpha\otimes dx_i+\tilde{g}_{\alpha\beta}d\eta_\alpha\otimes d\eta_\beta$$

where $\tilde{g}_{ij},\tilde{g}_{i\alpha}$ and $\tilde{g}_{\alpha\beta}$ are superfunctions on $\mathscr{M}$ and $\tilde{g}_{ij}=\tilde{g}_{ji},\tilde{g}_{\alpha\beta}=-\tilde{g}_{\beta\alpha},\tilde{g}_{i\alpha}=\tilde{g}_{\alpha i}$. The superfunction

$$L(x,y,\eta,\theta)=\tilde{g}_{ij}(x,\eta)y_iy_j+\tilde{g}_{i\alpha}y_i\theta_\alpha+\tilde{g}_{\alpha\beta}\theta_\alpha\theta_\beta \tag{3.12}$$

is a regular Lagrangian on $T\mathscr{M}$.

Now we are ready to introduce a Lagrangian superfunction which is not homogeneous and its horizontal differential is not zero. To construct this superfunction, let L be the superfunction (3.12) and ϕ an even homogeneous superfunction on the supermanifold $\mathscr{M}$, then

$$L'=L(x,y,\eta,\theta)+\frac{\partial\phi}{\partial x_i}(x,\eta)y_i+\frac{\partial\phi}{\partial\eta_\alpha}(x,\eta)\theta_\alpha \tag{3.13}$$

is a regular Lagrangian on $T\mathscr{M}$. Using (2.9), it is easy to check that the Cartan 2-forms associated to the superfunctions L and L' are equal (see [8]), then the canonical superspray associated to these superfunctions are equal (see (2.10) and (2.11)). On the other hand, in the definition of the endomorphism (3.7) we see that it depends on the canonical superspray, so we conclude that L and L' have the same horizontal endomorphism.

In local coordinates, let $X=X^i\frac{\partial}{\partial x_i}+\bar{X}^i\frac{\partial}{\partial y_i}+X^\alpha\frac{\partial}{\partial\eta_\alpha}+\bar{X}^\alpha\frac{\partial}{\partial\theta_\alpha}$ be a homogeneous supervector field on $T\mathscr{M}$. We have showed that $d_hL=0$, so

$$\begin{aligned} d_hL'(X) &= d_h\left(\frac{\partial\phi}{\partial x_i}(x,\eta)y_i+\frac{\partial\phi}{\partial\eta_\alpha}(x,\eta)\theta_\alpha\right)(X)\\ &= d\left(\frac{\partial\phi}{\partial x_i}y_i+\frac{\partial\phi}{\partial\eta_\alpha}\theta_\alpha)\right)(h(X))\\ &= \left(\frac{\partial^2\phi}{\partial x_j\partial x_i}y_i-\frac{\partial\phi}{\partial x_k}N^k_j+\frac{\partial^2\phi}{\partial x_j\partial\eta_\alpha}\theta_\alpha-\frac{\partial\phi}{\partial\eta_\beta}N^\beta_j\right)X^j\\ &\quad -(-1)^{|X|}\left(\frac{\partial^2\phi}{\partial\eta_\beta\partial x_i}y_i-\frac{\partial\phi}{\partial x_j}N^j_\beta+\frac{\partial^2\phi}{\partial\eta_\beta\partial\eta_\alpha}\theta_\alpha+\frac{\partial\phi}{\partial\eta_\gamma}N^\gamma_\beta\right)X^\beta\end{aligned}$$

Now we need to get the coefficients of X^j and X^β in the last equation to be nonzero. We can do this using a linear type of the superfunction ϕ in x and η. Then $d_hL'\neq 0$.

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Chebyshev Wavelet collocation method for solving a class of linear and nonlinear nonlocal boundary value problems

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Abstract

This study proposes the Chebyshev Wavelet Colocation method for solving a class of rth-order Boundary-Value Problems (BVPs) with nonlocal boundary conditions. This method is an extension of the Chebyshev wavelet method to the linear and nonlinear BVPs with a class of nonlocal boundary conditions. In this study, the method is tested on second and fourth-order BVPs and approximate solutions are compared with the existing methods in the literature and analytical solutions. The proposed method has promising results in terms of the accuracy.

1. Introduction

Many physical phenomena may be modelled by differential equations with nonlocal boundary conditions. Therefore, they have a great attention for researchers of mathematics and physics. Nonlocal conditions occur when values of a function on the boundary are depended on values inside the domain or when direct measurements on the boundary are not taken. These problems with nonlocal boundary conditions are found in many problems such as population dynamics, the process of heat conduction, control theory, theory of elastic stability, evolution equation for species population densities, image processing, porous media flow and turbulence [1, 2]. Henderson et al[3] considered uniqueness questions for certain nonlocal boundary value problems for the nth-order linear differential equation. Xue [4] studied the existence of integral solutions for nonlinear differential equations with nonlocal initial conditions in Banach spaces. Babak [5] investigated the uniqueness and existence of nonlocal initial problems for a system of nonlinear parabolic equations weakly coupled with ordinary differential equations. Liang et al [6] established some new theorems about the existence and uniqueness of solutions for semilinear integrodifferential equations with nonlocal initial conditions. Geng et al [7] gave an effective method for solving nonlocal fractional boundary value problems based on the reproducing kernel theory. Zhou et al [8] discussed the nonlocal Cauchy problem for the fractional evolution equations. All methods given here such as, Finite Difference Method (FDM) [9], Shooting Method [10, 11], Adomian Decomposition Method (ADM) [12], Variational Iteration Method (VIM) [13], Homotopy Analysis Method (HAM) [14], Sinc-Collocation Method (SCM) [15], Differential Transform Method (DTM) [16], Optimal Homotopy Asymptotic Method (OHAM) [17], combination of the VIM and the Homotopy Perturbed Method (HPM) [18], Reproducing Kernel Method (RKM) [19, 20], Monotone Iterative [21] and a spectral method based on operational matrices of Bernstein polynomials using collocation method [22] were used to solve multi-point BVPs. Tzanetis et al [23] studied a nonlocal problem modelling Ohmic heating with variable thermal conductivity including an analysis of the asymptotic behaviour and the blow-up of solutions. Bogoya et al [24] studied a nonlocal diffusion model analogous to heat equation with Neumann boundary conditions and proved an existence and uniqueness of solutions. Pao [25] studied some dynamical property of a reaction-diffusion equation with nonlocal boundary condition. Pao et al [26, 27] investigated a class of fourth-order nonlinear and semilinear elliptic boundary value problem with nonlocal boundary condition.

The Legendre and Chebyshev wavelets operational matrixes of integration and product operation matrix have been introduced in [28, 29, 30, 31]. Our analyses show that there are some disadvantages in applying Legendre wavelet and Chebyshev wavelet. In[32, 33], these disadvantages are eliminated by Çelik with the Chebyshev Wavelet Collocation Method.

This study presents a Chebyshev Wavelet Collocation Method for the solution of the rth-order linear and nonlinear BVPs given in the following form:

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$$y^{(r)}(x)=\sum_{i=1}^{r}A_i(x)\frac{d^{r-i}y(x)}{dx^{r-i}}+g(x) \tag{1.1}$$

$$y^{(r)}(x)=F\left(x,y(x),y'(x),\ \dots,\ y^{(r-1)}(x)\right) \tag{1.2}$$

with the nonlocal boundary conditions

$$\begin{cases} y^{(i-1)}(x_j)=b_{i,j},\ 1\le i\le m_j,\ 1\le j\le \beta \\ y(x_{\beta+1})-y(x_{\beta+2})=b_r \end{cases} \tag{1.3}$$

where $m_1, m_2, \dots, m_\beta$ are positive integers satisfied $m_1+m_2+\ \dots\ +m_\beta=r-1$ and $a<x_1<x_2<\ \dots\ <x_{\beta+2}<b,\ b_{i,j},\ b_r$ are real numbers. The uniqueness of the BVP in Eqs. (1.1), (1.3) has been discussed in [4].

Chebyshev wavelet collocation method is based on the approximation by the truncated Chebyshev wavelets series. By using the Chebyshev collocation points, algebraic equation system has been obtained. The coefficients of the Chebyshev wavelet series can be found from the solution of the algebraic equation system. The method is applied to the linear and nonlinear boundary value problems with nonlocal boundary conditions. Calculations demonstrated that the accuracy of the Chebyshev wavelet collocation method is quite good even for the case of a small number of grid points.

2. Chebyshev Wavelet method

Wavelets have been used in many different fields of science and engineering in recent years. They constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. If the dilation parameter a and the translation parameter b vary continuously, the following family of continuous wavelets can be obtained [34]

$$\psi_{a,b}(x)=|a|^{\ 1/2}\,\psi\left(\frac{x-b}{a}\right),\quad a,b\in R,\ \ a\neq 0. \tag{2.1}$$

Chebyshev wavelets are written as

$$\psi_{nm}(x)=\psi\,(k,n,m,x)$$

where $k=0,\ 1,\ 2,\ \dots,\ \ n=1,\ 2,\ \dots,\ 2^k$, m is degree of Chebyshev polynomials of the first kind and x denotes the normalized time. They are defined on the interval $[0,\ 1)$ by:

$$\psi_{nm}(x)=\begin{cases} \frac{\alpha_m 2^{k/2}}{\sqrt{\pi}}T_m(2^{k+1}x-2n+1), & \frac{n-1}{2^k}\le x<\frac{n}{2^k}, \\ 0 & otherwise \end{cases} \tag{2.2}$$

where

$$\alpha_m=\begin{cases} \sqrt{2} & m=0 \\ 2 & m=1,\ 2,\ \dots \end{cases}$$

and $T_m(2^{k+1}x-2n+1)$ are Chebyshev polynomials of the first kind of degree m orthogonal with respect to the weight function $w_n(x)=w(\,2^{k+1}x-2n+1\,)=\frac{1}{\sqrt{1-(2^{k+1}x-2n+1)^2}}$ on $[-1,\ 1]$ [35].

A function $f(x)\in L^2_w[0,1]$ may be expanded as:

$$f(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}f_{nm}\,\psi_{nm}(x) \tag{2.3}$$

where

$$f_{nm}=\langle f(x),\psi_{nm}(x)\rangle \tag{2.4}$$

and $\langle\,.\,,\,.\,\rangle$ denotes the inner product with weight function $w_n(x)$ in Eq. (2.4).

Truncated form of Eq. (2.3) can be written as:

$$f(x)\cong\sum_{n=1}^{2^k}\sum_{m=0}^{M-1}f_{nm}\,\psi_{nm}(x)=C^T\,\Psi(x) \tag{2.5}$$

where C and $\Psi(x)$ are $2^kM\times 1$ columns vectors given by:

$$C^T=[f_{10},\ f_{11},\ \dots,\ f_{1M-1},\ f_{20},\ \dots,\ f_{2M-1},\ \dots,\ f_{2^k0},\ \dots,\ f_{2^kM-1}\,] \tag{2.6}$$

$$\Psi(x)=[\psi_{10},\ \psi_{11},\ \dots,\ \psi_{1M-1},\ \psi_{20},\ \dots,\ \psi_{2M-1},\ \dots,\ \psi_{2^k0},\ \dots,\ \psi_{2^kM-1}\,]^T \tag{2.7}$$

The integration of the $\psi_{nm}(x)$ given in Eq. (2.2) can be represented as

$$p_{nm}(x)=\int_0^x\psi_{nm}(s)ds \tag{2.8}$$

For $m=0$, $m=1$ and $m>1$, $p_{nm}(x)$ can be obtained as

$$p_{n0}(x)=\begin{cases} 0 & 0\le x<\frac{n-1}{2^k}\\ \frac{\alpha_0 2^{-k/2-1}}{\sqrt{\pi}}\left[T_1(2^{k+1}x-2n+1)+T_0(2^{k+1}x-2n+1)\right] & \frac{n-1}{2^k}\le x<\frac{n}{2^k}\\ \frac{\alpha_0 2^{-k/2}}{\sqrt{\pi}}T_0(2^{k+1}x-2n+1) & \frac{n}{2^k}\le x<1 \end{cases}$$

$$p_{n1}(x)=\begin{cases} 0 & 0\le x<\frac{n-1}{2^k}\\ \frac{\alpha_1 2^{-k/2-3}}{\sqrt{\pi}}\left[T_2(2^{k+1}x-2n+1)-T_0(2^{k+1}x-2n+1)\right] & \frac{n-1}{2^k}\le x<\frac{n}{2^k}\\ 0 & \frac{n}{2^k}\le x<1 \end{cases}$$

$$p_{nm}(x)=\begin{cases} 0 & 0\le x<\frac{n-1}{2^k}\\ \frac{\alpha_m 2^{-k/2-2}}{\sqrt{\pi}}\left[\frac{T_{m+1}(u)-(-1)^{m+1}}{m+1}-\frac{T_{m-1}(u)-(-1)^{m-1}}{m-1}\right] & \frac{n-1}{2^k}\le x<\frac{n}{2^k}\\ \frac{\alpha_m 2^{-k/2-2}}{\sqrt{\pi}}\left[\frac{1-(-1)^{m+1}}{m+1}-\frac{1-(-1)^{m-1}}{m-1}\right] & \frac{n}{2^k}\le x<1 \end{cases}$$

where $u=2^{k+1}x-2n+1$. The integration of the $\Psi(x)$ can be represented as

$$\int_0^x \Psi(s)ds=[p_{10},\ p_{11},\ ...,\ p_{1M-1},\ p_{20},\ ...,\ p_{2M-1},\ ...,\ p_{2^k0},\ ...,\ p_{2^kM-1}]^T=P_1\Psi_1(x) \tag{2.9}$$

where

$$\Psi_1(x)=[\psi_{10},\ \psi_{11},\ ...,\ \psi_{1M},\ \psi_{20},\ ...,\ \psi_{2M},\ ...,\ \psi_{2^k0},\ ...,\ \psi_{2^kM}]^T$$

$$L_1=\begin{bmatrix} 1 & \frac{\sqrt{2}}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ -\frac{\sqrt{2}}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0 & 0 & 0 & 0\\ -\frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-3}}{M-3}-\frac{(-1)^{M-1}}{M-1}\right) & 0 & 0 & 0 & \cdots & -\frac{1}{2(M-3)} & 0 & \frac{1}{2(M-1)} & 0\\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-2}}{M-2}-\frac{(-1)^{M}}{M}\right) & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2(M-2)} & 0 & \frac{1}{2M} \end{bmatrix}$$

$$F_1=\begin{bmatrix} 2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \frac{2\sqrt{2}}{3} & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M-1}}{M-1}-\frac{1-(-1)^{M-3}}{M-3}\right) & 0 & \cdots & 0\\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M}}{M}-\frac{1-(-1)^{M-2}}{M-2}\right) & 0 & \cdots & 0 \end{bmatrix}\quad P_1=\frac{1}{2^{k+1}}\begin{bmatrix} L_1 & F_1 & F_1 & \cdots & F_1 & F_1\\ 0 & L_1 & F_1 & \cdots & F_1 & F_1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & L_1 & F_1\\ 0 & 0 & 0 & \cdots & 0 & L_1 \end{bmatrix}$$

The second integrations of the $\Psi(x)$can be represented as

$$\int_0^x\int_0^{x_1}\Psi(s)dsdx_1=\int_0^x P_1\Psi_1(x_1)dx_1=P_1\int_0^x\Psi_1(x_1)dx_1=P_1P_2\Psi_2(x)$$

The rthintegrations of the $\Psi(x)$can be represented as

$$\int_0^x\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_{r-1}}\Psi(s)\,dsdx_{r-1}dx_{r-2}\cdots dx_1=P_1P_2\cdots P_r\Psi_r(x)$$

where

$$L_r=\begin{bmatrix} 1 & \frac{\sqrt{2}}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ \frac{-\sqrt{2}}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ \frac{-\sqrt{2}}{3} & \frac{-1}{2} & 0 & \frac{1}{6} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-3}}{M-3}-\frac{(-1)^{M-1}}{M-1}\right) & 0 & 0 & 0 & \cdots & \frac{-1}{2(M-3)} & 0 & \frac{1}{2(M-1)} & 0 & \cdots & 0 & 0 & 0\\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-2}}{M-2}-\frac{(-1)^{M}}{M}\right) & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2(M-2)} & 0 & \frac{1}{2M} & \cdots & 0 & 0 & 0\\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-1}}{M-1}-\frac{(-1)^{M+1}}{M+1}\right) & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-1}{2(M-1)} & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-3+r}}{M-3+r}-\frac{(-1)^{M+r-1}}{M+r-1}\right) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2(M-3+r)} & 0 & \frac{1}{2(M-1+r)} \end{bmatrix}$$

$$F_r=\begin{bmatrix} 2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \frac{2\sqrt{2}}{3} & 0 & \cdots & 0\\ \vdots & 0 & \ddots & 0\\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M-1}}{M-1}-\frac{1-(-1)^{M-3}}{M-3}\right) & 0 & \cdots & 0\\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M}}{M}-\frac{1-(-1)^{M-2}}{M-2}\right) & 0 & \cdots & 0\\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M+1}}{M+1}-\frac{1-(-1)^{M-1}}{M-1}\right) & 0 & \cdots & 0\\ \vdots & 0 & \ddots & 0\\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M+r-1}}{M+r-1}-\frac{1-(-1)^{M+r-3}}{M+r-3}\right) & 0 & \cdots & 0 \end{bmatrix} \quad P_r=\frac{1}{2^{k+1}}\begin{bmatrix} L_r & F_r & F_r & \cdots & F_r & F_r\\ 0 & L_r & F_r & \cdots & F_r & F_r\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & L_r & F_r\\ 0 & 0 & 0 & \cdots & 0 & L_r \end{bmatrix}$$

and

$$\Psi_r(x)=[\psi_{10},\ \psi_{11},\ ...,\ \psi_{1M+r-1},\ \psi_{20},\ ...,\ \psi_{2M+r-1},\ ...,\ \psi_{2^k0},\ ...,\ \psi_{2^kM+r-1}\,]^T \tag{2.10}$$

Dimensions of the matrices L_r and F_r are $(M+r-1)\times(M+r)$. Hence P_r has the dimension $2^k(M+r-1)\times 2^k(M+r)$.

3. Chebyshev Wavelet collocation method for BVPs with nonlocal conditions

Consider Eq. (1.1) or Eq. (1.2) with the nonlocal boundary conditions

$$\begin{cases} y^{(i-1)}(x_j)=b_{i,j},\ 1\le i\le m_j,\ 1\le j\le\beta\\ y(x_{\beta+1})-y(x_{\beta+2})=b_r \end{cases}$$

We assume that $y^{(r)}(x)$ can be expanded in terms of truncated Chebyshev wavelet series as

$$y^{(r)}(x)=\sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(x)=C^T\Psi(x). \tag{3.1}$$

By successively integrating Eq. (3.1) from 0 to x, the following equations are obtained

$$y^{(r-1)}(x)=\int_0^x C^T\Psi(s)ds+y^{(r-1)}(0)=C^TP_1\Psi_1(x)+y^{(r-1)}(0) \tag{3.2}$$

$$y^{(r-2)}(x)=C^TP_1P_2\Psi_2(x)+xy^{(r-1)}(0)+y^{(r-2)}(0) \tag{3.3}$$

$$y^{(r-3)}(x)=C^TP_1P_2P_3\Psi_3(x)+\frac{x^2}{2}y^{(r-1)}(0)+xy^{(r-2)}(0)+y^{(r-3)}(0) \tag{3.4}$$

$$\vdots$$

$$y^{(m_j)}(x)=C^TP_1P_2P_3\ldots P_{r-m_j}\Psi_{r-m_j}(x)+\sum_{s=1}^{r-m_j}\frac{x^{s-1}}{(s-1)!}y^{(s+m_j-1)}(0) \tag{3.5}$$

Theorem 3.1. *Chebyshev wavelet expression for zth-order derivatives of unknown function $y(x)$ satisfying nonlocal boundary conditions*

$$\begin{cases} y^{(i-1)}(x_j)=b_{i,j},\ 1\le i\le m_j,\ 1\le j\le\beta\\ y(x_{\beta+1})-y(x_{\beta+2})=b_r \end{cases}$$

are given as for $z=0,1,\ ...,m_j-1$:

$$\begin{aligned} y^{(z)}(x)=C^T&\left(P_1P_2\ldots P_{r-z}\Psi_{r-z}(x)-\sum_{h=1}^{m_j-z}\frac{(x-x_j)^{h-1}}{(h-1)!}P_1P_2\ldots P_{r+1-z-h}\Psi_{r+1-z-h}(x_j)\right)\\ &+\sum_{s=2}^{r-m_j}\left(\frac{x^{s-1+m_j-z}}{(s-1+m_j-z)!}-\sum_{u=1}^{m_j-z}\frac{(x-x_j)^{u-1}}{(u-1)!}\frac{x_j^{m_j-z+s-u}}{(m_j-z+s-u)!}\right)y^{(s+m_j-1)}(0)\\ &+\frac{(x-x_j)^{m_j-z}}{(m_j-z)!}y^{(m_j)}(0)+\sum_{w=1}^{m_j-z}\frac{(x-x_j)^{w-1}}{(w-1)!}b_{z+w,j} \end{aligned} \tag{3.6}$$

$y^{(m_j)}(0),\ y^{(m_j+1)}(0),\ ...,\ y^{(r-1)}(0)$ *in Eq. (3.6) can be obtained the following algebraic equations system*

$$\begin{aligned} &\sum_{s=2}^{r-m_j}\left(\frac{x_{k+2}^{s-1+m_j}-x_{k+1}^{s-1+m_j}}{(s-1+m_j)!}-\sum_{u=1}^{m_j}\frac{(x_{k+2}-x_j)^u-(x_{k+1}-x_j)^u}{u!}\frac{x_j^{m_j+s-u}}{(m_j+s-u)!}\right)y^{(s+m_j-1)}(0)\\ &+\left(\frac{(x_{k+2}-x_j)^{m_j}-(x_{k+1}-x_j)^{m_j}}{m_j!}\right)y^{(m_j)}(0)\ =-b_r-\sum_{w=1}^{m_j}\left(\frac{(x_{k+2}-x_j)^{w-1}-(x_{k+1}-x_j)^{w-1}}{(w-1)!}\right)b_{w,j}\\ &-C^T\left(P_1\ldots P_r\Psi_r(x_{k+2})-P_1P_2\ldots P_r\Psi_r(x_{k+1})-\sum_{h=1}^{m_j}\left(\frac{(x_{k+2}-x_j)^{h-1}-(x_{k+1}-x_j)^{h-1}}{(h-1)!}P_1\ldots P_{r+1-h}\Psi_{r+1-h}(x_j)\right)\right) \end{aligned}$$

where $m_1+m_2+\ ...\ +m_\beta=r-1$.

Proof. By successively integrating Eq. (3.5) from x_j to x and using boundary conditions

$$y^{(i-1)}(x_j)=b_{i,j},\ 1\le i\le m_j,\ 1\le j\le\beta$$

the following expressions are obtained:

$$\begin{aligned}y^{(m_j-1)}(x)=C^T P_1P_2P_3\dots P_{r-m_j+1}\left(\Psi_{r-m_j+1}(x)-\Psi_{r-m_j+1}(x_j)\right)\\ +(x-x_j)y^{(m_j)}(0)+\textstyle\sum_{s=2}^{r-m_j}\frac{x^s-x_j^s}{s!}y^{(s+m_j-1)}(0)+b_{m_j,j}\end{aligned}$$

$$\begin{aligned}y^{(m_j-2)}(x)=C^T\left(P_1\dots P_{r-m_j+2}\left(\Psi_{r-m_j+2}(x)-\Psi_{r-m_j+2}(x_j)\right)-(x-x_j)P_1\dots P_{r-m_j+1}\Psi_{r-m_j+1}(x_j)\right)\\ +\tfrac{(x-x_j)^2}{2}y^{(m_j)}(0)+\textstyle\sum_{s=2}^{r-m_j}\left(\frac{x^{s+1}-x_j^{s+1}}{(s+1)!}-\frac{(x-x_j)x_j^s}{s!}\right)y^{(s+m_j-1)}(0)+(x-x_j)b_{m_j,j}+b_{m_j-1,j}\end{aligned}$$

$\vdots$

$$\begin{aligned}y^{(z)}(x)=C^T\left(P_1P_2\dots P_{r-z}\Psi_{r-z}(x)-\textstyle\sum_{h=1}^{m_j-z}\frac{(x-x_j)^{h-1}}{(h-1)!}P_1P_2\dots P_{r+1-z-h}\Psi_{r+1-z-h}(x_j)\right)\\ +\textstyle\sum_{s=2}^{r-m_j}\left(\frac{x^{s-1+m_j-z}}{(s-1+m_j-z)!}-\sum_{u=1}^{m_j-z}\frac{(x-x_j)^{u-1}}{(u-1)!}\frac{x_j^{m_j-z+s-u}}{(m_j-z+s-u)!}\right)y^{(s+m_j-1)}(0)\\ +\tfrac{(x-x_j)^{m_j-z}}{(m_j-z)!}y^{(m_j)}(0)+\textstyle\sum_{w=1}^{m_j-z}\frac{(x-x_j)^{w-1}}{(w-1)!}b_{z+w,j}\end{aligned}$$

where $z=0,\ 1,\ \dots,\ m_j-1$. This is Eq. (3.6) given in Theorem 3.1. For $z=0$, the following equation can be obtained.

$$\begin{aligned}y(x)=C^T\left(P_1P_2\dots P_r\Psi_r(x)-\textstyle\sum_{h=1}^{m_j}\frac{(x-x_j)^{h-1}}{(h-1)!}P_1P_2\dots P_{r+1-h}\Psi_{r+1-h}(x_j)\right)\\ +\textstyle\sum_{s=2}^{r-m_j}\left(\frac{x^{s-1+m_j}}{(s-1+m_j)!}-\sum_{u=1}^{m_j}\frac{(x-x_j)^u}{u!}\frac{x_j^{m_j+s-u}}{(m_j+s-u)!}\right)y^{(s+m_j-1)}(0)\\ +\tfrac{(x-x_j)^{m_j}}{m_j!}y^{(m_j)}(0)+\textstyle\sum_{w=1}^{m_j}\frac{(x-x_j)^{w-1}}{(w-1)!}b_{w,j}\end{aligned}$$

If boundary condition $y(x_{\beta+1})-y(x_{\beta+2})=b_r$can be satisfied, the following equation can be obtained as:

$$\begin{aligned}\textstyle\sum_{s=2}^{r-m_j}\left(\frac{x_{k+2}^{s-1+m_j}-x_{k+1}^{s-1+m_j}}{(s-1+m_j)!}-\sum_{u=1}^{m_j}\frac{(x_{k+2}-x_j)^u-(x_{k+1}-x_j)^u}{u!}\frac{x_j^{m_j+s-u}}{(m_j+s-u)!}\right)y^{(s+m_j-1)}(0)\\ +\left(\frac{(x_{k+2}-x_j)^{m_j}-(x_{k+1}-x_j)^{m_j}}{m_j!}\right)y^{(m_j)}(0)\ =-b_r-\textstyle\sum_{w=1}^{m_j}\left(\frac{(x_{k+2}-x_j)^{w-1}-(x_{k+1}-x_j)^{w-1}}{(w-1)!}\right)b_{w,j}\\ -C^T\left(P_1\dots P_r\Psi_r(x_{k+2})-P_1P_2\dots P_r\Psi_r(x_{k+1})-\textstyle\sum_{h=1}^{m_j}\left(\frac{(x_{k+2}-x_j)^{h-1}-(x_{k+1}-x_j)^{h-1}}{(h-1)!}P_1\dots P_{r+1-h}\Psi_{r+1-h}(x_j)\right)\right)\end{aligned}$$

where $m_1+m_2+\ \dots\ +m_\beta=r-1$. □

Conclusion 3.2. *If $\beta=1$ then $j=1$ and $m_1=r-1$ are obtained. Hence*

$$\begin{aligned}y(x)=C^T\left(P_1P_2\dots P_r\Psi_r(x)-\textstyle\sum_{h=1}^{r-1}\frac{(x-x_1)^{h-1}}{(h-1)!}P_1P_2\dots P_{r+1-h}\Psi_{r+1-h}(x_1)\right)\\ +\tfrac{(x-x_1)^{r-1}}{(r-1)!}y^{(r-1)}(0)+\textstyle\sum_{w=1}^{r-1}\frac{(x-x_1)^{w-1}}{(w-1)!}b_{w,1}\end{aligned}$$

is obtained, where $y^{(t-1)}(0)$ is obtain as

$$\begin{aligned}\left(\frac{(x_3-x_1)^{r-1}-(x_2-x_1)^{r-1}}{(r-1)!}\right)y^{(r-1)}(0)\ =-b_r-\textstyle\sum_{w=1}^{r-1}\left(\frac{(x_3-x_1)^{w-1}-(x_2-x_2)^{w-1}}{(w-1)!}\right)b_{w,1}\\ -C^T\left(P_1\dots P_r\Psi_r(x_3)-P_1P_2\dots P_r\Psi_r(x_2)-\textstyle\sum_{h=1}^{r-1}\left(\frac{(x_3-x_1)^{h-1}-(x_2-x_j)^{h-1}}{(h-1)!}P_1\dots P_{r+1-h}\Psi_{r+1-h}(x_1)\right)\right)\end{aligned}$$

Replacing (3.1)-(3.6) into Eq. (1.1) or Eq. (1.2), we have linear or nonlinear algebraic equations respectively. The collocation points can be taken as $2^{k+1}x_{ni}-2n+1=\cos\frac{((M+1)-i)\pi}{(M+1)}$ or

$$x_{ni}=\frac{1}{2^{k+1}}\left(2n-1+\cos\frac{((M+1)-i)\pi}{(M+1)}\right),i=1,\ 2,\ \dots,M,n=1,\ 2,\ \dots,\ 2^k \tag{3.7}$$

which are also called the turning points of $T_{M+1}(2^{k+1}x-2n+1)$. Substituting the Chebyshev collocation points into linear or nonlinear algebraic equations, a discretizised form of the vectors $\Psi(x_{ni})$,$\Psi_1(x_{ni})$ and $\Psi_r(x_{ni})$ can be obtained. Hence, we obtain linear or nonlinear algebraic equations systems. By solving linear or nonlinear algebraic equation systems, we can find the coefficients of the Chebyshev wavelet series that satisfied differential equation and its initial or boundary conditions.

4. Error analysis

For error analysis of Chebyshev wavelet method, the following Lemma and Theorems are given.

Lemma 4.1. (See [36]) *If the Chebyshev wavelet expansion of a continuous function $f(x)$ converges uniformly, then the Chebyshev wavelet expansion converges to a function $f(x)$.*

Theorem 4.2. (See [36]) *A function $f(x) \in L^2_\omega([0,1])$ with bounded second derivative $|f''(x)| \leq N$, can be expanded as an infinite sum of Chebyshev wavelets, and the series converges uniformly to $f(x)$. That is*

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_{nm}\psi_{nm}(x).$$

Since the truncated Chebyshev wavelets series

$$C^T\Psi(x) = \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)$$

is an approximate solution of given problem and $y(x)$ is an exact solution, an error function $f(x)$ can be given as:

$$E(x) = \left|y(x) - C^T\Psi(x)\right|.$$

The error bound of the approximate solution obtained by using truncated Chebyshev wavelets series is given by the following theorem.

Theorem 4.3. (See [37]) *Suppose that $y(x) \in C^m[0,1]$ and $C^T\Psi(x)$ is the approximate solution of problem using the Chebyshev wavelets method. Then the error bound can be obtained as follows:*

$$E(x) \leq \left\| \frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} \left|y^{(m)}(x)\right| \right\|^2.$$

5. Numerical results

Example 5.1. *Consider the forth order linear boundary value problem [19]*

$$\begin{cases} y^{(4)}(x) - e^x y'''(x) + y(x) = 1 - e^x\cosh(x) + 2\sinh(x), & 0 \leq x \leq 1 \\ y\left(\frac{1}{4}\right) = 1 + \sinh\left(\frac{1}{4}\right), \quad y'\left(\frac{1}{4}\right) = \cosh\left(\frac{1}{4}\right), \\ y''\left(\frac{1}{4}\right) = \sinh\left(\frac{1}{4}\right), \quad y\left(\frac{1}{2}\right) - y\left(\frac{3}{4}\right) = \sinh\left(\frac{1}{2}\right) - \sinh\left(\frac{3}{4}\right). \end{cases} \tag{5.1}$$

with analytic solution $y(x) = 1 + \sinh(x)$. It is assumed that $y^4(x)$ can be expanded in terms of truncated Chebyshev wavelet series as

$$y^{(4)}(x) = \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(x) = C^T\Psi(x) \tag{5.2}$$

By integrating this equation from 0 *to xand using boundary condition,*

$$y'''(x) = \int_0^x C^T\Psi(s)ds + y'''(0) = C^T P_1\Psi_1(x) + y'''(0) \tag{5.3}$$

is obtained. By integrating this equation three times from $\frac{1}{4}$ to x and using boundary conditions, following equations are obtained.

$$y''(x) = C^T\left(P_1P_2\Psi_2(x) - P_1P_2\Psi_2(\tfrac{1}{4})\right) + (x-\tfrac{1}{4})y'''(0) + \sinh(\tfrac{1}{4})$$

$$y'(x) = C^T P_1P_2\left(P_3\Psi_3(x) - P_3\Psi_3(\tfrac{1}{4}) - (x-\tfrac{1}{4})\Psi_2(\tfrac{1}{4})\right) + \frac{(x-\frac{1}{4})^2}{2}y'''(0) + (x-\tfrac{1}{4})\sinh(\tfrac{1}{4}) + \cosh(\tfrac{1}{4})$$

$$\begin{aligned} y(x) = C^T&\left(P_1P_2P_3P_4\Psi_4(x) - P_1P_2P_3P_4\Psi_4(\tfrac{1}{4}) - (x-\tfrac{1}{4})P_1P_2P_3\Psi_3(\tfrac{1}{4}) - \tfrac{(x-\frac{1}{4})^2}{2}P_1P_2\Psi_2(\tfrac{1}{4})\right) \\ &+ \tfrac{(x-\frac{1}{4})^3}{3!}y'''(0) + \tfrac{(x-\frac{1}{4})^2}{2!}\sinh(\tfrac{1}{4}) + (x-\tfrac{1}{4})\cosh(\tfrac{1}{4}) + 1 + \sinh(\tfrac{1}{4}) \end{aligned} \tag{5.4}$$

By using boundary condition $y\left(\frac{1}{2}\right) - y\left(\frac{3}{4}\right) = \sinh\left(\frac{1}{2}\right) - \sinh\left(\frac{3}{4}\right)$, $y'''(0)$is obtained as:

$$\begin{aligned} y'''(0) = \tfrac{384}{7}C^T&\left(P_1P_2P_3P_4\Psi_4(\tfrac{1}{2}) - P_1P_2P_3P_4\Psi_4(\tfrac{3}{4}) + \tfrac{1}{4}P_1P_2P_3\Psi_3(\tfrac{1}{4}) + \tfrac{3}{32}P_1P_2\Psi_2(\tfrac{1}{4})\right) \\ &- \tfrac{384}{7}\left(\sinh(\tfrac{1}{2}) - \sinh(\tfrac{3}{4}) + \tfrac{1}{4}\cosh(\tfrac{1}{4}) + \tfrac{3}{32}\sinh(\tfrac{1}{4})\right) \end{aligned} \tag{5.5}$$

Hence, replacing Eq. (5.5) *into the Eqs.* (5.3) *and* (5.4)*, we have*

Table 1: Comparisons of the absolute errors of [19] and proposed method for various values of M, k and x

x	$M=4,k=0$	$M=4,k=1$	$M=4,k=2$	$M=8,k=0$	$M=16,k=0$	$\|y-y_{101}\|$[19]	$\|y-y_{151}\|$ [19]
0	1.57537 e-8	4.63339 e-10	2.61472 e-11	6.32711 e-15	1e-19	0.0000255	1.13356E−6
0.1	2.38097 e-9	1.04070 e-10	5.93744 e-12	3.38417 e-15	1e-19	4.53581E−6	2.01715E−7
0.2	3.88574 e-11	3.13685 e-12	2.31892 e-13	2.12340 e-16	3e-19	1.32679E−7	5.90784E−9
0.3	5.15129 e-12	1.90014 e-12	1.48048 e-14	2.20110 e-17	2e-19	9.91385E−8	4.39712E−9
0.4	9.21096 e-10	2.86581 e-11	8.51610 e-13	4.11361 e-15	3e-19	1.90635E−6	8.46552E−8
0.5	5.55451 e-9	1.46005 e-10	5.70554 e-12	8.99446 e-15	1e-19	5.92446E−6	2.63147E−7
0.6	1.28454 e-8	4.31240 e-10	1.38779 e-11	8.47493 e-15	2e-19	9.75828E−6	4.33469E−7
0.7	1.30931 e-8	3.93193 e-10	1.42221 e-11	7.79239 e-15	2e-19	9.32982E−6	4.14438E−7
0.8	9.97851 e-9	3.11259 e-10	1.24207 e-11	1.01678 e-14	2e-19	5.99989E−7	2.67207E−8
0.9	7.17676 e-8	2.30928 e-9	1.02985 e-10	8.93500 e-15	0	0.0000265	1.17736E−6
1.0	1.81832 e-7	6.63219 e-9	3.16336 e-10	6.03590 e-15	0	0.0000765	3.39732E−6

$$\begin{aligned} y'''(x) = C^T &\left(P_1\Psi_1(x) + \tfrac{384}{7}C^T P_1P_2\left(P_3P_4\Psi_4(\tfrac{1}{2}) - P_3P_4\Psi_4(\tfrac{3}{4}) + \tfrac{1}{4}P_3\Psi_3(\tfrac{1}{4}) + \tfrac{3}{32}\Psi_2(\tfrac{1}{4})\right)\right) \\ &- \tfrac{384}{7}\left(\sinh(\tfrac{1}{2}) - \sinh(\tfrac{3}{4}) + \tfrac{1}{4}\cosh(\tfrac{1}{4}) + \tfrac{3}{32}\sinh(\tfrac{1}{4})\right) \end{aligned} \tag{5.6}$$

$$\begin{aligned} y(x) = C^T &\left(\begin{array}{l} P_1P_2P_3P_4\Psi_4(x) - P_1P_2P_3P_4\Psi_4(\frac{1}{4}) - (x-\frac{1}{4})P_1P_2P_3\Psi_3(\frac{1}{4}) - \frac{(x-\frac{1}{4})^2}{2}P_1P_2\Psi_2(\frac{1}{4}) \\ +\frac{384(x-\frac{1}{4})^3}{42}\left(P_1P_2P_3P_4\Psi_4(\frac{1}{2}) - P_1P_2P_3P_4\Psi_4(\frac{3}{4}) + \frac{1}{4}P_1P_2P_3\Psi_3(\frac{1}{4}) + \frac{3}{32}P_1P_2\Psi_2(\frac{1}{4})\right) \end{array}\right) \\ &-\tfrac{384(x-\frac{1}{4})^3}{42}\left(\sinh(\tfrac{1}{2}) - \sinh(\tfrac{3}{4}) + \tfrac{1}{4}\cosh(\tfrac{1}{4}) + \tfrac{3}{32}\sinh(\tfrac{1}{4})\right) + \tfrac{(x-\frac{1}{4})^2}{2}\sinh(\tfrac{1}{4}) \\ &+(x-\tfrac{1}{4})\cosh(\tfrac{1}{4}) + 1 + \sinh(\tfrac{1}{4}) \end{aligned} \tag{5.7}$$

Replacing Eqs. (5.2), (5.6) *and* (5.7) *into Eq.* (5.1)*, we have*

$$\begin{aligned} C^T &\left(\begin{array}{l} \Psi(x) + P_1P_2P_3P_4\left(\Psi_4(x) - \Psi_4(\frac{1}{4})\right) - (x-\frac{1}{4})P_1P_2P_3\Psi_3(\frac{1}{4}) - \frac{(x-\frac{1}{4})^2}{2}P_1P_2\Psi_2(\frac{1}{4}) - e^xP_1\Psi_1(x) \\ +\frac{384}{7}\left(\frac{(x-\frac{1}{4})^3}{6} - e^x\right)\left(P_1P_2P_3P_4\Psi_4(\frac{1}{2}) - P_1P_2P_3P_4\Psi_4(\frac{3}{4}) + \frac{1}{4}P_1P_2P_3\Psi_3(\frac{1}{4}) + \frac{3}{32}P_1P_2\Psi_2(\frac{1}{4})\right) \end{array}\right) \\ &= \tfrac{384}{7}\left(\tfrac{(x-\frac{1}{4})^3}{6} - e^x\right)\left(\sinh(\tfrac{1}{2}) - \sinh(\tfrac{3}{4}) + \tfrac{1}{4}\cosh(\tfrac{1}{4}) + \tfrac{3}{32}\sinh(\tfrac{1}{4})\right) - \tfrac{(x-\frac{1}{4})^2}{2}\sinh(\tfrac{1}{4}) \\ &\quad -(x-\tfrac{1}{4})\cosh(\tfrac{1}{4}) - \sinh(\tfrac{1}{4}) - e^x\cosh(x) + 2\sinh(x) \end{aligned} \tag{5.8}$$

Algebraic equation system achieved in Eq. (5.8) *by using Chebyshev collocation points can be solved and the coefficients C^T in Eq.* (5.7) *which is satisfied differential equation and whose boundary conditions can be obtained. Table 1 shows the absolute errors for $M=4,\ k=0$, $M=4,\ k=1$, $M=4,\ k=2$, $M=8,\ k=0$ and $M=16,\ k=0$. As can be seen in Table 1, the results obtained by the proposed method are superior from Reproducing Kernel Method [19] for small grid points such as $M=4,\ k=0$.*

Example 5.2. *Consider the second order nonlinear boundary value problem [21].*

$$\begin{cases} y''(x) + \frac{\sqrt{2}}{8}y(x) + \frac{1}{32}y^2(x) = \frac{32}{9}x^2 - \frac{16}{3}x, & 0 \le x \le 1 \\ y(0) = 0, \qquad y(1) - y\left(\frac{1}{2}\right) = 0. \end{cases} \tag{5.9}$$

It is assumed that $y''(x)$ can be expanded in terms of truncated Chebyshev wavelet series as

$$y''(x) = \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(x) = C^T\Psi(x)$$

Similar process given in Example 5.1, the following equations can be obtained

$$y(x) = C^T\left(P_1P_2\Psi_2(x) - 2xP_1P_2\Psi_2(1) + 2xP_1P_2\Psi_2(\tfrac{1}{2})\right) \tag{5.10}$$

$$\begin{aligned} &C^T\left(\Psi(x) + \tfrac{\sqrt{2}}{8}P_1P_2\Psi_2(x) - \tfrac{x\sqrt{2}}{4}P_1P_2\Psi_2(1) + \tfrac{x\sqrt{2}}{4}P_1P_2\Psi_2(\tfrac{1}{2})\right) \\ &+\tfrac{1}{32}\left(C^T\left(P_1P_2\Psi_2(x) - 2xP_1P_2\Psi_2(1) + 2x(x-\tfrac{1}{4})P_1P_2\Psi_2(\tfrac{1}{2})\right)\right)^2 - \tfrac{32}{9}x^2 + \tfrac{16}{3}x = 0 \end{aligned} \tag{5.11}$$

Nonlinear algebraic equation system achieved from Eq. (5.11) *by using collocation points can be solved and the coefficients C^T in Eq.* (5.10) *which is satisfied differential equation and whose boundary conditions are obtained. Table 2 shows the approximate solutions for $M=4,\ k=0$, $M=4,\ k=1$, $M=4,\ k=2$, $M=8,\ k=0$ and $M=16,\ k=0$. As can be seen in Table 2 that the results obtained by the proposed method are satisfied the boundary condition $y(1) - y\left(\frac{1}{2}\right) = 0$. When number of grid points increase, the precisions of approximate solutions increase. The results obtained by the proposed method for small grid points such as $M=4,\ k=0$ are also superior from Monotone Iterative Method [21] not satisfying the boundary condition $y(1) - y\left(\frac{1}{2}\right) = 0$ exactly*

Table 2: Approximate solutions of proposed method for various values of M, k and x and [21]

x	$M=4,k=0$	$M=4,k=1$	$M=4,k=2$	$M=8,k=0$	$M=16,k=0$	[21] Third Term
0	0.260929 e-19	0.949067 e-21	0.157524 e-21	0.260974 e-19	0.260974 e-19	0
0.1	0.1035508429	0.1035501266	0.1035502791	0.1035502920	0.1035502920	0.1040764497
0.2	0.2019974623	0.2019955359	0.2019958847	0.2019959102	0.2019959103	0.2027652843
0.3	0.2908872022	0.2908870108	0.2908875266	0.2908875633	0.2908875632	0.2916751511
0.4	0.3664960100	0.3664989187	0.3664995714	0.3664996188	0.3664996189	0.3671584606
0.5	0.4258337040	0.4258381953	0.4258390441	0.4258391011	0.4258391014	0.4263068077
0.6	0.4666492402	0.4666527782	0.4666535640	0.4666536253	0.4666536254	0.4669454120
0.7	0.4874359792	0.4874368797	0.4874377503	0.4874378143	0.4874378143	0.4876278426
0.8	0.4874369540	0.4874368797	0.4874377503	0.4874378140	0.4874378143	0.4876319741
0.9	0.4666501363	0.4666527782	0.4666535639	0.4666536252	0.4666536254	0.4669578065
1.0	0.4258337040	0.4258381953	0.4258390441	0.4258391011	0.4258391014	0.4263274652

Table 3: Approximate solutions of proposed method for various values of M, k and x and [17]

x	$M=4,k=0$	$M=4,k=2$	$M=16,k=0$	$M=16,k=1$	[17] OHAM Second Order	[17] HPM Second Order
0	0.188293 e-19	0.734306 e-23	0.188296 e-19	0.129858 e-20	0	0
0.1	0.0656100870	0.0656099762	0.0656099772	0.0656099772	0.0656099707	0.0655919115
0.2	0.1209706322	0.1209703634	0.1209703654	0.1209703653	0.1209703640	0.1209353047
0.3	0.1658758980	0.1658757275	0.1658757303	0.1658757303	0.1658757339	0.1658256598
0.4	0.2001594201	0.2001594622	0.2001594656	0.2001594656	0.2001594697	0.2000971743
0.5	0.2236942481	0.2236943874	0.2236943913	0.2236943913	0.2236943923	0.2236233202
0.6	0.2363931859	0.2363932068	0.2363932109	0.2363932109	0.2363932086	0.2363172683
0.7	0.2382090332	0.2382088205	0.2382088245	0.2382088245	0.2382088217	0.2381321777
0.8	0.2291348258	0.2291344944	0.2291344982	0.2291344982	0.2291344982	0.2290613518
0.9	0.2092040768	0.2092038847	0.2092038882	0.2092038882	0.2092038920	0.2091382608
1.0	0.1784910170	0.1784909168	0.1784909199	0.1784909199	0.1784909250	0.1784364302

Example 5.3. *Consider the second order nonlinear boundary value problem [17]*

$$\begin{cases} y''(x)+\frac{3}{8}y(x)+\frac{2}{1089}\left(y'(x)\right)^2+1=0, & 0\le x\le 1 \\ y(0)=0, \qquad y(1)-y\left(\frac{1}{3}\right)=0. \end{cases} \tag{5.12}$$

It is assumed that $y''(x)$ can be expanded in terms of truncated Chebyshev wavelet series as

$$y''(x)=\sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(x)=C^T\Psi(x)$$

Similar process given in Example 5.1, the following equations can be obtained

$$y(x)=C^T\left(P_1P_2\Psi_2(x)-\frac{3x}{2}P_1P_2\Psi_2(1)+\frac{3x}{2}P_1P_2\Psi_2(\tfrac{1}{3})\right) \tag{5.13}$$

$$\begin{aligned} &C^T\left(\Psi(x)+\frac{3}{8}P_1P_2\Psi_2(x)-\frac{9x}{16}P_1P_2\Psi_2(1)+\frac{9x}{16}P_1P_2\Psi_2(\tfrac{1}{3})\right) \\ &+\frac{2}{1089}\left(C^T\left(P_1\Psi_1(x)-\frac{3}{2}P_1P_2\Psi_2(1)+\frac{3}{2}P_1P_2\Psi_2(\tfrac{1}{3})\right)\right)^2+1=0 \end{aligned} \tag{5.14}$$

Nonlinear algebraic equation system achieved from Eq. (5.14) by using collocation points can be solved and the coefficients C^T in Eq. (5.13) satisfied differential equation and whose boundary conditions are obtained. Table 3 shows the approximate solutions for $M=4,\ k=0$, $M=4,\ k=2,\ M=16,\ k=0$ and $M=16,\ k=1$. As can be seen in Table 3, the precisions of approximate solutions obtained by the proposed method increase when number of grid points increase. The results obtained by the proposed method for small grid points such as $M=4,\ k=0$ are superior from Homotopy Perturbation Method and Optimal Homotopy Asymptotic Method in [17].

Example 5.4. *Consider the forth order nonlinear boundary value problem*

$$\begin{cases} y^{(4)}(x)-\sin(x)y''(x)+y(x)+\sin(y(x))=1+\sin(1+\sin(x))+(2+\sin(x))\sin(x), & 0\le x\le 1 \\ y\left(\frac{1}{4}\right)=1+\sin\left(\frac{1}{4}\right), \qquad y'\left(\frac{1}{4}\right)=\cos\left(\frac{1}{4}\right), \\ y''\left(\frac{1}{4}\right)=-\sin\left(\frac{1}{4}\right), \qquad y\left(\frac{1}{2}\right)-y\left(\frac{3}{4}\right)=\sin\left(\frac{1}{2}\right)-\sin\left(\frac{3}{4}\right). \end{cases} \tag{5.15}$$

with analytic solution $y(x)=1+\sin(x)$.

Table 4: The absolute errors of proposed method for various values of M, k and x

x	$M=4,k=0$	$M=4,k=1$	$M=4,k=2$	$M=8,k=0$	$M=16,k=0$
0	1.305827 e-8	4.55634 e-10	2.69793 e-11	6.48270 e-15	1.0 e-19
0.1	1.889004 e-9	9.89746 e-11	5.96046 e-12	3.14499 e-15	7.0 e-21
0.2	2.754407 e-11	2.85912 e-12	2.25583 e-13	1.87940 e-16	6.0 e-20
0.3	8.825138 e-12	1.59099 e-12	1.50140 e-14	1.88610 e-16	1.0 e-20
0.4	8.503094 e-10	2.02999 e-11	8.22951 e-13	3.39057 e-15	3.0 e-20
0.5	4.775323 e-9	1.04675 e-10	5.14514 e-12	6.96137 e-15	3.0 e-20
0.6	1.061068 e-8	3.18029 e-10	1.20212 e-11	5.91620 e-15	0
0.7	1.058432 e-8	2.81752 e-10	1.20549 e-11	5.57247 e-15	1.3 e-19
0.8	6.632389 e-9	2.09432 e-10	8.86424 e-12	8.34323 e-15	7.0 e-20
0.9	4.857164 e-8	1.52893 e-9	7.46368 e-11	8.29286 e-15	1.4 e-19
1.0	1.148666 e-7	4.21506 e-9	2.18685 e-10	8.17725 e-15	1.5 e-19

Similar process given in Example 5.1, the following equations can be obtained

$$\begin{aligned} y(x) = C^T &\left(\begin{array}{l} P_1P_2P_3P_4\Psi_4(x) - P_1P_2P_3P_4\Psi_4(\tfrac{1}{4}) - (x-\tfrac{1}{4})P_1P_2P_3\Psi_3(\tfrac{1}{4}) - \frac{(x-\frac{1}{4})^2}{2}P_1P_2\Psi_2(\tfrac{1}{4}) \\ + \frac{384(x-\frac{1}{4})^3}{42}\left(P_1P_2P_3P_4\Psi_4(\tfrac{1}{2}) - P_1P_2P_3P_4\Psi_4(\tfrac{3}{4}) + \tfrac{1}{4}P_1P_2P_3\Psi_3(\tfrac{1}{4}) + \tfrac{3}{32}P_1P_2\Psi_2(\tfrac{1}{4})\right) \end{array} \right) \\ &- \frac{384(x-\frac{1}{4})^3}{42}\left(\sin(\tfrac{1}{2}) - \sin(\tfrac{3}{4}) + \tfrac{1}{4}\cos(\tfrac{1}{4}) - \tfrac{3}{32}\sin(\tfrac{1}{4})\right) - \frac{(x-\frac{1}{4})^2}{2}\sin(\tfrac{1}{4}) + (x-\tfrac{1}{4})\cos(\tfrac{1}{4}) + 1 + \sin(\tfrac{1}{4}) \end{aligned} \tag{5.16}$$

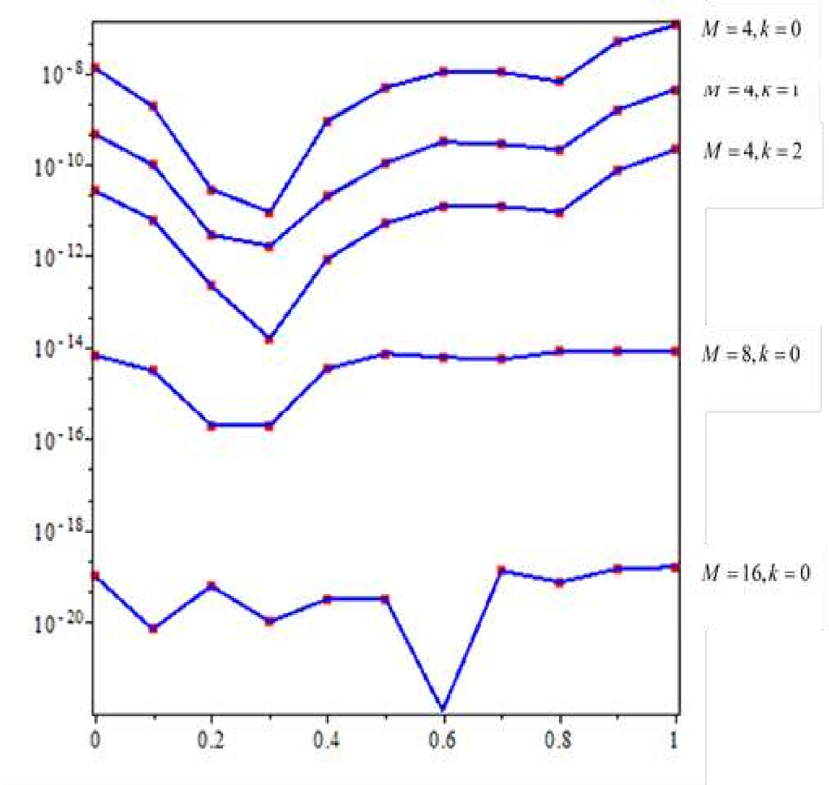
$$\begin{aligned} &C^T \left(\begin{array}{l} \frac{384}{7}\left(\frac{(x-\frac{1}{4})^3}{6} - (x-\tfrac{1}{4})\sin(x)\right)\left(P_1P_2P_3P_4\Psi_4(\tfrac{1}{2}) - P_1P_2P_3P_4\Psi_4(\tfrac{3}{4}) + \tfrac{1}{4}P_1P_2P_3\Psi_3(\tfrac{1}{4}) + \tfrac{3}{32}P_1P_2\Psi_2(\tfrac{1}{4})\right) \\ +\Psi(x) + P_1P_2P_3P_4\left(\Psi_4(x) - \Psi_4(\tfrac{1}{4})\right) - (x-\tfrac{1}{4})P_1P_2P_3\Psi_3(\tfrac{1}{4}) - \frac{(x-\frac{1}{4})^2}{2}P_1P_2\Psi_2(\tfrac{1}{4}) \\ -\sin(x)P_1P_2\left(\Psi_2(x) - \Psi_2(\tfrac{1}{4})\right) \end{array} \right) \\ &+\sin \left(\begin{array}{l} C^T \left(\begin{array}{l} P_1P_2P_3P_4\Psi_4(x) - P_1P_2P_3P_4\Psi_4(\tfrac{1}{4}) - (x-\tfrac{1}{4})P_1P_2P_3\Psi_3(\tfrac{1}{4}) - \frac{(x-\frac{1}{4})^2}{2}P_1P_2\Psi_2(\tfrac{1}{4}) \\ + \frac{384(x-\frac{1}{4})^3}{42}\left(P_1P_2P_3P_4\Psi_4(\tfrac{1}{2}) - P_1P_2P_3P_4\Psi_4(\tfrac{3}{4}) + \tfrac{1}{4}P_1P_2P_3\Psi_3(\tfrac{1}{4}) + \tfrac{3}{32}P_1P_2\Psi_2(\tfrac{1}{4})\right) \end{array} \right) \\ - \frac{384(x-\frac{1}{4})^3}{42}\left(\sin(\tfrac{1}{2}) - \sin(\tfrac{3}{4}) + \tfrac{1}{4}\cos(\tfrac{1}{4}) - \tfrac{3}{32}\sin(\tfrac{1}{4})\right) + \left(1 - \frac{(x-\frac{1}{4})^2}{2}\right)\sin(\tfrac{1}{4}) + (x-\tfrac{1}{4})\cos(\tfrac{1}{4}) + 1 \end{array} \right) \\ &- \frac{384}{7}\left(\frac{(x-\frac{1}{4})^3}{6} - (x-\tfrac{1}{4})\sin(x)\right)\left(\sin(\tfrac{1}{2}) - \sin(\tfrac{3}{4}) + \tfrac{1}{4}\cos(\tfrac{1}{4}) - \tfrac{3}{32}\sin(\tfrac{1}{4})\right) + (x-\tfrac{1}{4})\cos(\tfrac{1}{4}) \\ &+ \left(1 + \sin(x) - \frac{(x-\frac{1}{4})^2}{2}\right)\sin(\tfrac{1}{4}) - \sin(1+\sin(x)) - (2+\sin(x))\sin x = 0 \end{aligned} \tag{5.17}$$

Nonlinear algebraic equation system achieved from Eq. (5.17) by using collocation points can be solved and the coefficients C^T in Eq. (5.16) satisfied differential equation and whose boundary conditions are obtained. Table 4 shows the absolute errors for $M=4,\ k=0$, $M=4,\ k=1$, $M=4,\ k=2$, $M=8,\ k=0$ and $M=16,\ k=0$. As can be seen in Table 4 and Fig. 1, absolute errors tend to zero when number of grid points increase. The results obtained by the proposed method for small grid points such as $M=4,\ k=0$ are superior.

6. Conclusion

Chebyshev wavelet collocation method has been applied to the one linear and three nonlinear nonlocal boundary value problems. Approximate and exact solutions of examples are correspondingly compared. For Example 1, the comparisons of the absolute errors given in Table 1, it is clear that the results obtained by the proposed method are better than Reproducing Kernel Method [19]. Numerical results of Example 2 which is given in Table 2 are stable when number of grid points increase. It can be seen that results of proposed method for small grid points such as $M=4$, $k=0$ are superior to the results of Monotone Iterative Method [21] as given in Table 2. Numeric solutions of Example 3 for various values of M and k are given in Table 3. The precisions of approximate solutions obtained by the proposed method increase when number of grid points increase as can be seen in Table 3. For small grid points such as$M=4$, $k=0$, the results of proposed method are superior to the results of Homotopy Perturbation Method and Optimal Homotopy Asymptotic Method in [17]. Absolute errors of Example 4 are given in Table 4 and Fig 1 for various values of M and k. As can be seen from Table 4 and Fig 1, absolute errors tend to zero when number of grid points increase and the proposed method is highly efficient and accurate. All of the calculations in this study have been made by the Maple program. Newton Raphson method has been used to solve nonlinear algebraic equation systems. These calculations demonstrate that the accuracy of the Chebyshev wavelet collocation method is quite good even for small number of grid points. In proposed method, there are no complex integrals or methodology. Applications of this method are very simple. It is also very convenient for solving the initial, boundary and nonlocal boundary value problems since the initial, boundary and nonlocal conditions are automatically taken in the solution. In addition, it can be concluded that the proposed method is reliable, simple, fast, minimal computation costs, flexible, and convenient alternative method.

Figure 5.1: The absolute errors of Example 5.4 for various values of M and k



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Establishing the existence of Hilfer fractional pantograph equations with impulses

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Abstract

In [1], the authors established the existence of a class of fractional differential equations of a complex order. In this note, we derive some sufficient conditions for the existence of solutions to a class of Hilfer fractional pantograph equations with impulsive effect. Further, using the techniques of nonlinear functional analysis, we establish appropriate conditions and results to discuss various kinds of Ulam-Hyers stability.

1. Preface

Fractional calculus is an emerging field in applied mathematics that deals with derivatives and integrals of arbitrary orders. For their applications and details note, one can refer to [8, 13, 17]. Due to the properties involved in Hilfer fractional type of derivatives (introduced by Hilfer [8]) in the sense that it generalizes the Riemann-Liouville (R-L) and Liouville-Caputo (L-C) fractional derivatives, a lot of studies have been done on it, including the existence and uniqueness of solutions to such differential equations (DEs) involving Hilfer fractional derivative (HFD); see [4, 7, 20, 21], and references therein.
It is well known that the pantograph equations (PEs) arises in quite different fields of pure and applied mathematics and have been investigated extensively. Recently, due to its importance in many applied fields and playing an extremely important role in explaining many different phenomena, for details see [3, 6, 10, 19, 22].
Recently impulsive DEs have been considered by many authors due to their significant applications in various fields of science and technology. For detail study, see [2, 14, 15, 16, 18, 24]. Due to its large number of applications, this area has been received great importance and remarkable attention from the researchers.
In 1940, Ulam posed the following problem about the stability of functional equations: Under what conditions does there exist an additive mapping near an approximately additive mapping? In the following year, Hyers gave an answer to the problem of Ulam for additive functions defined on Banach spaces, [8]. That is why the name of this stability is Ulam-Hyers (U-H) stability. Later on, Hyers results are extended by many mathematicians. The stability analysis is extremely helpful in numerous applications, for example, numerical analysis and optimization, where it is very tough to find the exact solution of a nonlinear problem. The aforementioned stability has very recently attracted the attention of researchers; we refer the reader to some papers [11, 12, 23]. Because of, fractional order system may have more attractive feature over the integer order system.
Consider the PEs with impulsive condition given by

$$\begin{cases} D^{\alpha,\beta}x(t) = f(t,x(t),x(\lambda t)), \quad t \in J' := J \setminus \{t_1,t_2,...,t_m\}, J=[a,b], \ t \notin t_k, \\ \Delta I^{1-\gamma}x(t)|_{t=t_k} = \psi_k(x(t_k)), \quad t=t_k, \quad k=1,2,...,m, \\ I^{1-\gamma}x(t)|_{t=a} = x_a, \quad \gamma = \alpha+\beta-\alpha\beta, \end{cases} \tag{1.1}$$

where $D^{\alpha,\beta}(0<\alpha<1, 0\le\beta\le1)$ is the Hilfer fractional derivative of orders α and type β. Here, $0<\lambda<1$ the function $f: J\times R\times R \to R$ is continuous, $I_k : R \to R$, and $a \in R$, $a=t_0<t_1<...<t_m<t_{m+1}=b$, $\Delta I^{1-\gamma}x(t)|_{t=t_k} = I^{1-\gamma}x(t_k^+) - I^{1-\gamma}x(t_k^-)$, $I^{1-\gamma}x(t_k^+) = \lim_{h\to0+}x(t_k+h)$ and $I^{1-\gamma}x(t_k^-) = \lim_{h\to0-}x(t_k+h)$ represent the right and left limits of $x(t)$ at $t=t_k$.

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The paper is organized as follows: In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. The proofs of soultion existence and uniqueness are given in Section 3. Finally, stability is proved in Section 4.

2. Prerequisites

In this section, we recall some preliminaries materials required in this paper from. Consider the following space

$$PC[J,R] = \left\{x : J \to R : x(t) \in C(t_k,t_{k+1}], k = 0,...,m;\ \text{there exists } x(t_k^+) \text{ and } x(t_k^-)\right\}.$$

Now we consider the weighted space $PC_\gamma[J,R]$,

$$PC_\gamma[J,R] = \left\{x : (t-a)^\gamma x(t)|_{t\in[t_k,t_{k+1}]} \in C[t_k,t_{k+1}], k = 0,...,m \text{ where } 0 \le \gamma < 1\right\}.$$

Obviously, which is a Banach space with norm

$$\|x\|_{PC_\gamma} = \sup_{t\in(t_k,t_{k+1}]} \{(t-t_k)^\gamma x(t)\}, k = 0,...,m.$$

Definition 2.1. *[13] The R-L fractional integral of order $\alpha > 0$ of function $f : [0,\infty) \to R$ can be written as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds.$$

Definition 2.2. *[13] The R-L fractional derivative of order $\alpha > 0$ of a continuous function $f : [0,\infty) \to R$ can be written as*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{\alpha-n+1} f(s)ds,$$

provided that the right side is pointwise defined on $[0,\infty)$.

Definition 2.3. *[13] The L-C fractional derivative of order $\alpha > 0$ of a continuous function $f : [0,\infty) \to R$ can be written as*

$${}^C D^\alpha f(t) = D^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0)\right], \ t > 0, n-1 < \alpha < n.$$

Definition 2.4. *[8] The HFD of order $0 < \alpha < 1$ and $0 \le \beta \le 1$ of function $f(t)$ is defined by*

$$D^{\alpha,\beta} f(t) = (I^{\beta(1-\alpha)} D(I^{(1-\beta)(1-\alpha)} f))(t).$$

The GRL fractional derivative is considered as an interpolation between the R-L and L-C fractional derivative and the relations are given below.

Remark 2.5. *(i) Operator $D^{\alpha,\beta}$ also can be written as*

$$D^{\alpha,\beta} = (I^{\beta(1-\alpha)} D(I^{(1-\beta)(1-\alpha)})) = I^{\beta(1-\alpha)} D^\gamma, \quad \gamma = \alpha + \beta - \alpha\beta.$$

(ii) If $\beta = 0$, then $D^{\alpha,\beta} = D^{\alpha,0}$ is called R-L fractional derivative.
(iii) If $\beta = 1$, then $D^{\alpha,\beta} = I^{1-\alpha} D$ is called L-C fractional derivative.

Lemma 2.6. *[4] If $\alpha > 0$ and $\beta > 0$, there exists*

$$\left[I^\alpha (t)^{\beta-1}\right](x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} x^{\beta+\alpha-1},$$

and

$$\left[D^\alpha (t)^{\alpha-1}\right](x) = 0, \quad 0 < \alpha < 1.$$

Lemma 2.7. *[4] If $\alpha > 0$ and $\beta > 0$ and $f \in L^1(a,b]$, there exists the following properties*

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t),$$

and

$$D^\alpha I^\alpha f(t) = f(t).$$

Lemma 2.8. *[4] Let $\gamma = \alpha + \beta - \alpha\beta$ where $0 < \alpha < 1$ and $0 \le \beta \le 1$. Let $f : J \times R \to R$ be a function such that $f(\cdot,(\cdot)) \in C_{1-\gamma}[a,b]$ for any $x \in C_{1-\gamma}[a,b]$. If $x \in C^\gamma_{1-\gamma}[a,b]$, then x satisfies*

$$\begin{aligned} D^{\alpha,\beta} x(t) &= f(t,x(t)), \quad t \in (a,b] \\ I^{1-\gamma} x(a) &= x_a. \end{aligned}$$

if and only if x satisfies

$$x(t) = \frac{x_a}{\Gamma(\gamma)} (t-a)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s,x(s))ds \tag{2.1}$$

Lemma 2.9. *Let $f : J \to R$ be continuous. A function $x \in PC_{1-\gamma}[J,R]$ is a solution of the fractional differential equation*

$$\begin{aligned} D^{\alpha,\beta}x(t) &= f(t), \quad t \in J' \\ I^{1-\gamma}x(t_i) &= x_{t_i}, \end{aligned}$$

if and only if x is a solution of the integral equation

$$x(t) = \frac{x_{t_i}}{\Gamma(\gamma)}(t-a)^{\gamma-1} - \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)\Gamma(1-\beta(1-\alpha))}\int_a^{t_i}(t_i-s)^{(1-\beta(1-\alpha))-1}f(s)ds + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds. \tag{2.2}$$

Next, we shall give the definitions and the criteria of U-H stability and U-H-R stability for PEs with impulsive effect under Hilfer fractional derivative. Let ε be a positive number and $\varphi : J \to R^+$ be a continuous function, for every $t \in J'$ and $k = 1,2,...,m$, we have the following inequalities

$$\begin{cases} \left|D^{\alpha,\beta}z(t) - f(t,z(t),z(\lambda t))\right| \leq \varepsilon, \\ \left|\Delta I^{1-\gamma}z(t)|_{t=t_k} - \psi_k(z(t_k))\right| \leq \varepsilon, \end{cases} \tag{2.3}$$

$$\begin{cases} \left|D^{\alpha,\beta}z(t) - f(t,z(t),z(\lambda t))\right| \leq \varepsilon\varphi(t), \\ \left|\Delta I^{1-\gamma}z(t)|_{t=t_k} - \psi_k(z(t_k))\right| \leq \varepsilon\varphi(t), \end{cases} \tag{2.4}$$

$$\begin{cases} \left|D^{\alpha,\beta}z(t) - f(t,z(t),z(\lambda t))\right| \leq \varphi(t), \\ \left|\Delta I^{1-\gamma}z(t)|_{t=t_k} - \psi_k(z(t_k))\right| \leq \varphi(t), \end{cases} \tag{2.5}$$

Definition 2.10. *Eq. (1.1) is U-H stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in PC_{1-\gamma}[J,R]$ of the inequality (2.3) there exists a solution $x \in PC_{1-\gamma}[J,R]$ of Eq. (1.1) with*

$$|z(t) - x(t)| \leq C_f\, \varepsilon, \quad t \in J.$$

Definition 2.11. *Eq. (1.1) is generalized U-H stable if there exist $\varphi \in PC_{1-\gamma}[J,R^+]$, $\varphi_f(0) = 0$ such that for each solution $z \in PC_{1-\gamma}[J,R]$ of the inequality (2.3) there exists a solution $x \in PC_{1-\gamma}[J,R]$ of Eq. (1.1) with*

$$|z(t) - x(t)| \leq \varphi_f\, \varepsilon, \quad t \in J.$$

Definition 2.12. *Eq. (1.1) is U-H-R stable with respect to $\varphi \in PC_{1-\gamma}[J,R^+]$ if there exists a real number $C_f > 0$ such that for each solution $z \in PC_{1-\gamma}[J,R]$ of the inequality (2.4) there exists a solution $x \in PC_{1-\gamma}[J,R]$ of Eq. (1.1) with*

$$|z(t) - x(t)| \leq C_f\, \varepsilon\varphi(t), \quad t \in J.$$

Definition 2.13. *Eq. (1.1) is generalized U-H-R stable with respect to $\varphi \in PC_{1-\gamma}[J,R^+]$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $z \in PC_{1-\gamma}[J,R]$ of the inequality (2.5) there exists a solution $x \in PC_{1-\gamma}[J,R]$ of Eq. (1.1) with*

$$|z(t) - x(t)| \leq C_{f,\varphi}\varphi(t), \quad t \in J.$$

Remark 2.14. *A function $z \in PC_{1-\gamma}[J,R]$ is a solution of the inequality*

$$\left|D^{\alpha,\beta}z(t) - f(t,z(t),z(\lambda t))\right| \leq \varepsilon,$$

if and only if there exist a function $g \in PC_{1-\gamma}[J,R]$ and a sequence g_k, $k = 1,2,...,m$ (which depend on z) such that

(i) $|g(t)| \leq \varepsilon,\ |g_k| < \varepsilon.$
(ii) $D^{\alpha,\beta}z(t) = f(t,z(t),z(\lambda t)) + g(t).$
(iii) $\Delta I^{1-\gamma}z(t)|_{t_k} = \psi_k(z(t_k)) + g_k.$
(iv) Let $0 < \alpha < 1,\ 0 \leq \beta \leq 1$, if z is solution of the inequality (2.3) then z is a solution of the following integral inequality

$$\left|z(t) - \frac{x_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} - \frac{\sum\limits_{0<t_k<t}\psi_k(z(t_k))}{\Gamma(\gamma)}(t-a)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,z(s),z(\lambda s))ds\right| \leq \varepsilon\left(m + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right).$$

Lemma 2.15. *[25] Let $a(t)$ be a nonnegative function locally integrable on $a \leq t < b$ for some $b \leq \infty$, and let $g(t)$ be a nonnegative, nondecreasing continuous function defined on $a \leq t < b$, such that $g(t) \leq K$ for some constant K. Further let $x(t)$ be a nonnegative locally integrable on $a \leq t < b$ function satisfying*

$$|x(t)| \leq a(t) + g(t)\int_a^t (t-s)^{\alpha-1}x(s)ds, \ \ t \in [a,b)$$

with some $\alpha > 0$. Then

$$|x(t)| \leq a(t) + \int_a^t \left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}\right]a(s)ds, \ \ a \leq t < b.$$

Remark 2.16. *Under the hypothesis of Lemma 2.15 let $a(t)$ be a nondecreasing function on $[0,T)$. Then $y(t) \leq a(t)E_\alpha(g(t)\Gamma(\alpha)t^\alpha)$, where E_α is the Mittag-Leffler function defined by*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)},\ z \in C,\ Re(\alpha) > 0.$$

Lemma 2.17. *[24] Let $x \in PC_{1-\gamma}(J,R)$ satisfies the following inequality*

$$|x(t)| \leq c_1 + c_2 \int_0^t (t-s)^{\alpha-1} |x(t)|\, ds + \sum_{0<t_k<t} \psi_k |x(t_k)|,$$

where c_1 is a nonnegative, continuous and nondecreasing function and c_2, ψ_i are constants. Then

$$|x(t)| \leq c_1 \left(1 + \psi E_\alpha(c_2\Gamma(\alpha)t^\alpha)^k E_\alpha(c_2\Gamma(\alpha)t^\alpha\right)\ for\ t \in (t_k.t_{k+1}],$$

where $\psi = \sup\{\psi_k : k = 1,2,3,...,m\}$.

Theorem 2.18. *[5](Schaefer's Fixed Point Theorem) Let K be a Banach space and let $N : K \to K$ be completely continuous operator. If the set $\{x \in K : x = \delta Nx\ for\ \ some\ \delta \in (0,1)\}$ is bounded, then N has a fixed point.*

Theorem 2.19. *[5](Banach Fixed Point Theorem) Suppose Q be a non-empty closed subset of a Banach space E. Then any contraction mapping N from Q into itself has a unique fixed point.*

3. Existence of at least one solution

In this section, we investigate the existence and uniqueness of solution to the proposed problem. We need the following lemma to establish our main results.

Lemma 3.1. *Let $f : J \times R \times R \to R$ be continuous. A function x is a solution of the fractional integral equation*

$$x(t) = \frac{x_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{\sum\limits_{0<t_k<t} \psi_k(x(t_k))}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s,x(s),x(\lambda s))ds \tag{3.1}$$

if and only if x is a solution of the problem (1.1).

Let us introduce the following assumptions which are used hereafter.

(H1) Let $f : J \times R \times R \to R$ be a continuous function and there exists a positive constant $\ell > 0$, such that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq \ell(|x_1 - y_1| + |x_2 - y_2|),\ \text{for all}\ x_1,x_2,y_1,y_2 \in R.$$

(H2) Let $f : J \times R \times R \to R$ is completely continuous function and there exists a function $p \in L^1$ such that

$$|f(t,x,y)| \leq p(t),\ \forall\, t \in J,\ x,y \in R.$$

(H3) Let the functions $\psi_k : R \to R$ are continuous and there exists a constant $\ell_k^* > 0$, such that

$$|\psi_k(x) - \psi_k(y)| \leq \ell_k^* |x-y|,\ \text{for all}\ x,y \in R,\ k = 1,2,...,m.$$

(H4) Let the functions $\psi_k : R \to R$ are continuous and there exists a constant $p^* > 0$, such that

$$|\psi_k(x)| \leq p^*(t),\ \text{for all}\ x \in R,\ k = 1,2,...,m.$$

(H5) : There exists an increasing finctions $\varphi \in PC_{1-\gamma}[J,R^+]$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$,

$$I^\alpha \varphi(t) \leq \lambda_\varphi \varphi(t).$$

Theorem 3.2 (Existence)**.** *Assume that [H1] - [H4] are satisfied. Then, Eq.(1.1) has at least one solution.*

Proof. Consider the operator $N : PC_{1-\gamma}[J,R] \to PC_{1-\gamma}[J,R]$. The operator form of integral equation (3.1) is written as follows

$$x(t) = Nx(t),$$

where

$$Nx(t) = \frac{x_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{\sum\limits_{0<t_k<t} \psi_k(x(t_k))}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s,x(s),x(\lambda s))ds. \tag{3.2}$$

First, we prove that the operator N defined by (3.2) verifies the conditions of Theorem 2.18.
Claim 1: The operator N is continuous. Let x_n be a sequence such that $x_n \to x$ in $PC_{1-\gamma}[J,R]$. Then for each $t \in J$,

$$\begin{aligned}\left|(N(x_n)(t) - N(x)(t))(t-a)^{1-\gamma}\right| &\leq \frac{1}{\Gamma(\gamma)} \sum_{0<t_k<t} |\psi_k(x_n(t_k)) - \psi_k(x(t_k))| \\ &+ \frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |f(s,x_n(s),x_n(\lambda s)) - f(s,x(s),x(\lambda s))|\, ds.\end{aligned}$$

since f is continuous, then we have

$$\|N(x_n)(t)-N(x)(t)\|_{PC_{1-\gamma}} \to 0 \text{ as } n\to\infty.$$

This proves the continuity of N.

Claim 2: The operator N maps bounded sets into bounded sets in $PC_{1-\gamma}[J,R]$. Indeed, it is enough to show that for $r>0$, there exists a positive constant l such that

$$B_r=\left\{x\in PC_{1-\gamma}[J,R] : \|x\|_{PC_{1-\gamma}}\le r\right\},$$

we have $\|N(x)\|_{PC_{1-\gamma}}\le l$.

$$\begin{aligned}
\left|(Nx)(t)(t-a)^{1-\gamma}\right| &\le \frac{|x_a|}{\Gamma(\gamma)}+\frac{\sum\limits_{0<t_k<t}|\psi_k(x(t_k))|}{\Gamma(\gamma)}+\frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}|f(s,x(s),x(\lambda s))|\,ds\\
&\le \frac{|x_a|}{\Gamma(\gamma)}+\frac{(t-a)^{\gamma-1}\sum\limits_{0<t_k<t}\left|(t-a)^{1-\gamma}p^*(t)\right|}{\Gamma(\gamma)}+\frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}|p(s)|\,ds\\
&\le \frac{|x_a|}{\Gamma(\gamma)}+\frac{m(t-a)^{\gamma-1}}{\Gamma(\gamma)}\|p^*\|_{PC_{1-\gamma}}+\frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)}(t-a)^{\alpha+\gamma-1}B(\gamma,\alpha)\|p\|_{PC_{1-\gamma}}\\
&\le \frac{|x_a|}{\Gamma(\gamma)}+\frac{m(b-a)^{\gamma-1}}{\Gamma(\gamma)}\|p^*\|_{PC_{1-\gamma}}+\frac{1}{\Gamma(\alpha)}(b-a)^{\alpha}B(\gamma,\alpha)\|p\|_{PC_{1-\gamma}}
\end{aligned}$$

$$\|(Nx)(t)\|_{PC_{1-\gamma}}:=l.$$

That is N is bounded.

Claim 3: The operator N maps bounded sets into equicontinuous set of $PC_{1-\gamma}[J,R]$. Let $t_1,t_2\in J, t_1>t_2, B_r$ be a bounded set of $PC_{1-\gamma}[J,R]$ as in Claim 2, and $x\in B_r$. Then,

$$\begin{aligned}
\left|(t_1-a)^{1-\gamma}(Nx)(t_1)-(t_2-a)^{1-\gamma}(Nx)(t_2)\right| \le & \left|\frac{\sum\limits_{0<t_k<t_1}\psi_k(x(t_k))}{\Gamma(\gamma)}+\frac{(t_1-a)^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_1}(t_1-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\right|\\
& - \left|\frac{\sum\limits_{0<t_k<t_2}\psi_k(x(t_k))}{\Gamma(\gamma)}-\frac{(t_2-a)^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_2}(t_2-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\right|.
\end{aligned}$$

As $t_1\to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Claim 1 - Claim 3 together with Arzelä-Ascoli theorem, we can conclude that $N: PC_{1-\gamma}[J,R]\to PC_{1-\gamma}[J,R]$ is continuous and completely continuous.

It is continuous and bounded from Claim 1 - Claim 3. Now, it remains to show that the set

$$\omega=\left\{x\in PC_{1-\gamma}[J,R] : x=\tau N(x), 0<\tau<1\right\}$$

is bounded set.

Let $x\in\omega,\ x=\tau N(x)$ for some $0<\tau<1$. Thus for each $t\in J$. We have

$$x(t)=\tau\left[\frac{x_a}{\Gamma(\gamma)}(t-a)^{\gamma-1}+\frac{\sum\limits_{0<t_k<t}\psi_k(x(t_k))}{\Gamma(\gamma)}(t-a)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\right].$$

This shows that the set ω is bounded. As a consequence of Theorem 2.18, we deduce that N has a fixed point which is a solution of problem (1.1). □

Theorem 3.3. *(Uniqueness) Assume that [H1] and [H3] are satisfied. If*

$$\rho=\left(\frac{m\ell^*}{\Gamma(\gamma)}(b-a)^{1-\gamma}+\frac{2\ell}{\Gamma(\alpha)}B(\gamma,\alpha)(b-a)^{\alpha}\right)<1, \tag{3.3}$$

then, the Eq. (1.1) *has a unique solution.*

Proof. Consider the operator $N: PC_{1-\gamma}[J,R]\to PC_{1-\gamma}[J,R]$. The equivalent integral equation (3.1) which can be written in the operator form

$$Nx(t)=\frac{x_a}{\Gamma(\gamma)}(t-a)^{\gamma-1}+\frac{\sum\limits_{0<t_k<t}\psi_k(x(t_k))}{\Gamma(\gamma)}(t-a)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds. \tag{3.4}$$

By Lemma 3.2, it is clear that the fixed points of N are solutions of Eq. (1.1).
Let $x,y \in PC_{1-\gamma}[J,R]$ and $t \in J$, then we have

$$\begin{aligned}\left|(t-a)^{1-\gamma}(Nx(t)-Ny(t))\right| &\leq \frac{1}{\Gamma(\gamma)}\sum_{0<t_k<t}|\psi_k(x(t_k))-\psi_k(y(t_k))| + \frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|f(s,x(s),x(\lambda s))-f(s,y(s),y(\lambda s))|\,ds \\ &\leq (t-a)^{1-\gamma}\frac{m\ell^*}{\Gamma(\gamma)}\|x-y\|_{PC_{1-\gamma}} + \frac{2\ell(t-a)^{1-\gamma}}{\Gamma(\alpha)}B(\gamma,\alpha)(t-a)^{\alpha+\gamma-1}\|x-y\|_{PC_{1-\gamma}} \\ &\leq \left(\frac{m\ell^*}{\Gamma(\gamma)}(b-a)^{1-\gamma} + \frac{2\ell}{\Gamma(\alpha)}B(\gamma,\alpha)(b-a)^{\alpha}\right)\|x-y\|_{PC_{1-\gamma}} \\ &= \rho\|x-y\|_{PC_{1-\gamma}}.\end{aligned}$$

This yields that N has a unique fixed point which is solution of Eq. (1.1). □

4. U-H stability analysis

In this section, we obtain stability results for the proposed problem.

Theorem 4.1. *The assumptions [H1], [H3], [H5] and (3.3) hold. Then, Eq.(1.1) is generalized U-H-R stable.*

Proof. Let z be solution of inequality (2.5) and by Theorem 3.3 there x is unique solution of the problem

$$\begin{aligned} D^{\alpha,\beta}x(t) &= f(t,x(t),x(\lambda t)),\\ \Delta I^{1-\gamma}x(t)|_{t=t_k} &= \psi_k(x(t_k^-)),\\ I^{1-\gamma}x(a) &= I^{1-\gamma}z(a) = x_a.\end{aligned}$$

Then, we have

$$x(t) = \frac{x_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{\sum\limits_{0<t_k<t}\psi_k(x(t_k))}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds$$

By differentiating inequality (2.5), for each $t \in (t_k,t_{k+1}]$, we have

$$\begin{aligned}&\left|z(t) - \tfrac{x_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} - \frac{\sum\limits_{0<t_k<t}\psi_k(z(t_k))}{\Gamma(\gamma)}(t-a)^{\gamma-1} - \tfrac{1}{\Gamma(\alpha)}\textstyle\int_a^t (t-s)^{\alpha-1}f(s,z(s),z(\lambda s))ds\right| \\ &\leq \left|\frac{\sum\limits_{0<t_k<t} g_k}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \tfrac{1}{\Gamma(\alpha)}\textstyle\int_a^t (t-s)^{\alpha-1}\varphi(t)ds\right| \\ &\leq m\varphi(t) + \lambda_\varphi\varphi(t) \\ &\leq (m+\lambda_\varphi)\varphi(t).\end{aligned}$$

Hence for each $t \in (t_k,t_{k+1}]$, it follows

$$\begin{aligned}|z(t)-x(t)| &\leq \left|z(t) - \frac{x_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} - \frac{\sum\limits_{0<t_k<t}\psi_k(x(t_k))}{\Gamma(\gamma)}(t-a)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\right| \\ &\leq \left|z(t) - \frac{x_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} - \frac{\sum\limits_{0<t_k<t}\psi_k(z(t_k))}{\Gamma(\gamma)}(t-a)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,z(s),z(\lambda s))ds\right| \\ &+ \frac{\sum\limits_{0<t_k<t}|\psi_k(z(t_k))-\psi_k(x(t_k))|}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|f(s,x(s),x(\lambda s))-f(s,x(s),x(\lambda s))|\,ds \\ &\leq (m+\lambda_\varphi)\varphi(t) + \frac{m\ell^*}{\Gamma(\gamma)}(t-a)^{\gamma-1}|z(t)-x(t)| + \frac{2\ell}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|z(t)-x(t)|\,ds\end{aligned}$$

By Lemma 2.17, there exists a constant $K>0$ independent of $\lambda_\varphi\varphi(t)$ such that

$$|z(t)-x(t)| \leq K(m+\lambda_\varphi)\varphi(t) := C_{f,\varphi}\varphi(t).$$

Thus, Eq.(1.1) is generalized U-H-R stable. □

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Best proximity points for weak $\mathscr{MT}$-cyclic Kannan contractions

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Abstract

In this paper, we introduce a notion of weak $\mathscr{MT}$-cyclic Kannan contractions with respect to a $\mathscr{MT}$-function φ and then we shall prove some new convergent and existence theorems of best proximity point theorems for these contractions in uniformly Banach spaces.

1. Introduction

Let A and B be nonempty subsets of a Banach space E. A map T on $A\cup B$ into $A\cup B$ is called a *cyclic* mapping if $T(A)\subset B$ and $T(B)\subset A$. Let $T: A\cup B\to A\cup B$ be a cyclic map. For any nonempty subsets A and B of E, let $dist(A,B)=\inf\{||x-y||: x\in A, y\in B\}$. A point $x\in A\cup B$ is called to be a best proximity point for T if $||x-Tx||=dist(A,B)$.
In [2] A. Anthony Eldred and P. Veeramani introduced cyclic contraction mappings and then in a uniformly convex Banach space a theorem was established which ensures the existence of a best proximity point of cyclic contractions. Afterward, in these spaces, C. Di Bari et al. in [13] introduced the notion of cyclic Meir-Keeler contractions and proved the existence of a best proximity point for cyclic Meir-Keeler contractions in the case of two sets. After this, this result was generalized for p sets by S. Karpagam, Sushama Agrawal [11]. In [4] a new class of maps was introduced, called cyclic φ-contraction which contains the cyclic contractions maps as a subclass and for this type of contractive conditions, in uniformly convex Banach spaces, results of best proximity points were obtained. Many authors have been investigated the existence, uniqueness and convergence of iterates to the best proximity point under weaker assumptions over T; see [1]-[5], [8], [10]-[14], [16]-[18], and [22]-[24] and their references. See also [25, 26].
The notion of weak cyclic Kannan contractions (see below definition) was introduced by M. A. Petric [14]; see also [21]-[23].

Definition 1.1. *[14] Let A and B be nonempty subsets of a metric space (X,d). If a map $T: A\cup B\to A\cup B$ satisfies*

(i) $T(A)\subset B$ and $T(B)\subset A$;
(ii) there exists a $\alpha\in[0,\frac{1}{2})$ such that

$$d(Tx,Ty)\le \alpha[d(x,Tx)+d(y,Ty)]+(1-2\alpha)dist(A,B) \text{ for any } x\in A \text{ and } y\in B,$$

then T is called a weak cyclic Kannan contraction on $A\cup B$.

The existence and convergence theorems of best proximity points in uniformly convex Banach spaces is proved as follows:

Theorem 1.2. *[14] Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Let $T: A\cup B\to A\cup B$ be a weak cyclic Kannan contraction map. Then*

(i) T has a unique best proximity point z in A.
(ii) The sequence $\{T^{2n}x\}$ converges to z for any starting point $x\in A$.
(iii) z is the unique fixed point of T^2.
(iv) Tz is a best proximity point of T in B.

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2. Preliminaries

Definition 2.1. *[6, 7, 20] A function $\varphi : [0,\infty) \to [0,1)$ is said to be a $\mathcal{MT}$-function if it satisfies Mizoguchi-Takahashi's condition (i.e. $\limsup_{s\to t+0} \varphi(s) < 1$ for all $t \in [0,\infty)$).*

Obviously, if $\varphi : [0,\infty) \to [0,1)$ is a nondecreasing or nonincreasing function, then φ is a $\mathcal{MT}$-function. So, in particular, if $\varphi : [0,\infty) \to [0,1)$ is defined by $\varphi(t) = c$, where $c \in [0,1)$, then φ is a $\mathcal{MT}$-function. It is known that $\varphi : [0,\infty) \to [0,1)$ is a $\mathcal{MT}$-function *if and only if* for each $t \in [0,\infty)$, there exist $r_t \in [0,1)$ and $\varepsilon_t > 0$ such that $\varphi(s) \le r_t$ for all $s \in [t, t+\varepsilon_t)$. For more details, one can see Remark 2.5 in [7].
Note that if φ is a $\mathcal{MT}$-function then clearly $\psi := \frac{\varphi}{2-\varphi}$ is a $\mathcal{MT}$-function.
The notion of $\mathcal{MT}$-cyclic contraction with respect to a $\mathcal{MT}$-function φ (see below definition) is introduced by W.-S. Du et al [8] that contain cyclic contractions as a subclass. Some new existence and convergence theorems of iterates of best proximity points for $\mathcal{MT}$-cyclic contractions has been proved.

Lemma 2.2. *[2] Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*
(i) $||z_n - y_n|| \to dist(A,B)$.
(ii) For every $\varepsilon > 0$ there exists N_0 such that for all $m > n \ge N_0$, $||x_m - y_n|| \le dist(A,B) + \varepsilon$.
Then, for every $\varepsilon > 0$ there exists N_1 such that for all $m > n \ge N_1$, $||x_m - z_n|| \le \varepsilon$.

Lemma 2.3. *[2] Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*
(i) $||x_n - y_n|| \to dist(A,B)$.
(ii) $||z_n - y_n|| \to dist(A,B)$.
Then $||x_n - z_n|| \to 0$.

In this paper, we first define weak $\mathcal{MT}$-cyclic Kannan contractions with respect to a $\mathcal{MT}$-function φ and then we generalized Theorem P for these contractions in uniformly convex Banach spaces.

3. Main results

Definition 3.1. *Let A and B be nonempty subsets of a metric space (X,d). If a map $T : A\cup B \to A\cup B$ satisfies*

(MTK1) $T(A) \subset B$ and $T(B) \subset A$;
(MTK2) there exists a $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that

$$d(Tx,Ty) \le \frac{1}{2}\varphi(d(x,y))[d(x,Tx)+d(y,Ty)] + (1-\varphi(d(x,y)))dist(A,B) \text{ for any } x\in A \text{ and } y \in B,$$

then T is called a weak $\mathcal{MT}$-cyclic Kannan contraction with respect to φ on $A\cup B$.

Remark 3.2. *It is obvious that (MTK2) implies that for any $x \in A$ and $y \in B$, T satisfies $d(Tx,Ty) - dist(A,B) \le \frac{1}{2}\varphi(d(x,y))[d(x,Tx) + d(y,Ty) - 2dist(A,B)] \le 0$ and so $d(Tx,Ty) \le d(x,y)$, for any $x \in A$ and $y \in B$.*

In the case that $dist(A,B) = 0$, we can obtain the following theorem that generalize Kannan theorem [19] and Theorem 2 in [14].

Theorem 3.3. *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that $A\cap B \ne \emptyset$ and $T : A\cup B \to A\cup B$ be a weak $\mathcal{MT}$-cyclic Kannan contraction with respect to φ such that*

$$d(Tx,Ty) \le \frac{1}{2}\varphi(d(x,y))[d(x,Tx)+d(y,Ty)] \text{ for any } x\in A \text{ and } y\in B. \tag{3.1}$$

Then T has a unique fixed point z in $A\cap B$.

Proof. Suppose that x is an arbitrary point in A. Then by (3.1), we have

$$d(T^n x, T^{n+1}x) \le \frac{1}{2}\varphi(d(T^{n-1}x, T^n x))[d(T^{n-1}x, T^n x) + d(T^n x, T^{n+1}x)],$$

so,

$$d(T^n x, T^{n+1}x) \le \psi(d(T^{n-1}x, T^n x))d(T^{n-1}x, T^n x), \tag{3.2}$$

where $\psi := \frac{\varphi}{2-\varphi}$; by Definition 2.1 ψ is a $\mathcal{MT}$-function, so $\psi(t) < 1$ for any $t > 0$; therefore we have,

$$d(T^n x, T^{n+1}x) < d(T^{n-1}x, T^n x),$$

for any $n \in N$. Thus the sequence $\{d(T^n x, T^{n+1}x)\}$ is decreasing in $[0,\infty)$. Then

$$t_0 := \lim_{n\to\infty} d(T^n x, T^{n+1}x) = \inf_{n\to\infty} d(T^n x, T^{n+1}x) \ge 0. \tag{3.3}$$

Since ψ is a $\mathcal{MT}$-function, there exist $r_{t_0} \in [0,1)$ and $\varepsilon_{t_0} > 0$ such that $\psi(s) \le r_{t_0}$ for all $s \in [t_0, t_0+\varepsilon_{t_0})$. By (3.3), there exists $\ell \in N$, such that

$$t_0 \le d(T^n x, T^{n+1}x) < t_0 + \varepsilon_{t_0}$$

for all $n \in N$ with $n \ge \ell$. Hence $\psi(d(T^n x, T^{n+1}x)) \le r_{t_0}$ for all $n \ge \ell$. Let

$$\lambda := \max\{\psi(d(T^1x, T^2x)), \psi(d(T^2x, T^3x)), \cdots, \psi(d(T^{\ell-1}x, T^{\ell}x)), r_{t_0}\}.$$

Then

$$0 \le \psi(d(T^n x, T^{n+1}x)) \le \lambda < 1 \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

Now, by (3.2) and (3.4), we have $d(T^n x, T^{n+1}x) \le \lambda d(T^{n-1}x, T^n x)$ and by induction, we conclude that $d(T^n x, T^{n+1}x) \le \lambda^n d(x, Tx)$, for any $n \in N$.
Now, if $m > n$,

$$\begin{aligned} d(T^n x, T^m x) &\le d(T^n x, T^{n+1}x) + \cdots + d(T^{m-1}x, T^m x) \\ &\le \lambda^n d(x, Tx) + \cdots + \lambda^{m-1} d(x, Tx) \\ &\le \frac{\lambda^n}{1-\lambda} d(x, Tx), \end{aligned}$$

Since $\lambda \in (0,1)$, $\lim_{n\to\infty} \lambda^n = 0$. Thus, $\{T^n x\}$ is a Cauchy sequence. Since A is closed, there exists $z \in A$ such that

$$\lim_{n\to\infty} d(T^n x, z) = 0. \tag{3.5}$$

Now, we show that $Tz = z$.
By (3.2), we have $d(Tz, T^{n+1}x) \le \psi(d(z, T^n x))d(z, T^n x)$, and so

$$\lim_{n\to\infty} d(Tz, T^{n+1}x) = 0. \tag{3.6}$$

Hence, by (3.5), (3.6) and Lemma 2.3, $d(Tz, z) = 0$, or $Tz = z$. We prove z is unique. Let v be another point such that $Tv = v$. Then by (3.1),

$$d(v, z) = d(Tv, Tz) \le \frac{1}{2}\varphi(d(v, z))[d(Tv, v) + d(z, Tz)] = 0.$$

So, $v = z$. □

For the main results of this paper we need the following lemma.

Lemma 3.4. *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of uniformly convex Banach space X and $T : A \cup B \to A \cup B$ cyclic map with respect to $\mathscr{MT}$-function φ satisfying*

$$||Tx - T^2x|| \le \varphi(||x - Tx||)||x - Tx|| + (1 - \varphi(||x - Tx||))dist(A, B) \tag{3.7}$$

for all $x \in A \cup B$. Then
(i) $\lim_{n\to\infty} ||T^n x - T^{n+1}x|| = dist(A, B)$ for all $x \in A \cup B$.
(ii) $\lim_{n\to\infty} ||T^{2n}x - T^{2n\pm2}x|| = 0$ for all $x \in A \cup B$.
(iii) z is a best proximity point if and only if z is a fixed point of T^2.

Proof. First we prove (i). This proof follows similar patterns as Theorem 2.1 in [8]. We include the proof for completeness reasons. Let $x \in A \cup B$ be given. Clearly, $dist(A, B) \le ||T^n x - T^{n+1}x||$ for all $n \in \mathbb{N}$. If there exists $j \in \mathbb{N}$ such that $T^j x = T^{j+1}x \in A \cap B$, then $\lim_{n\to\infty} ||T^n x - T^{n+1}x|| = 0$ and $dist(A, B) = 0$; therefore (i). So it suffices to consider the case $x_{n+1} \ne x_n$ for all $n \in \mathbb{N}$. By Remark 3.2, it is easy to see that the sequence $\{||T^n x - T^{n+1}x||\}$ is nonincreasing in $(0, \infty)$ and so it is convergent. Set

$$\widehat{t} := \lim_{n\to\infty} ||T^n x - T^{n+1}x||. \tag{3.8}$$

Since φ is a $\mathscr{MT}$-function, there exist $r_{\widehat{t}} \in [0,1)$ and $\varepsilon_{\widehat{t}} > 0$ such that $\varphi(s) \le r_{\widehat{t}}$ for all $s \in [\widehat{t}, \widehat{t} + \varepsilon_{\widehat{t}})$. By (3.8), there exists $\ell \in N$, such that

$$\widehat{t} \le ||T^n x - T^{n+1}x|| < \widehat{t} + \varepsilon_{\widehat{t}}$$

for all $n \in N$ with $n \ge \ell$. Hence $\varphi(||T^n x - T^{n+1}x||) \le r_{\widehat{t}}$ for all $n \ge \ell$. Let

$$\lambda := \max\{\varphi(||T^1x - T^2x||), \varphi(||T^2x - T^3x||), \cdots, \varphi(||T^{\ell-1}x - T^{\ell}x||), r_{\widehat{t}}\}.$$

Then $0 \le \varphi(||T^n x - T^{n+1}x||) \le \lambda < 1$ for all $n \in N$. If $x \in A$, then, by $(MTK1)$, we have $T^{2n-1}x \in A$ and $T^{2n}x \in B$ for all $n \in N$. Notice first that $(MTK2)$ implies that

$$||Tx - T^2x|| \le \varphi(||x - Tx||)||x - Tx|| + (1 - \varphi(||x - Tx||))dist(A, B) \le \lambda||x - Tx|| + dist(A, B)$$

and

$$\begin{aligned}
||T^3x - T^4x|| &\leq \varphi(||T^2x - T^3x||)||T^2x - T^3x|| + (1 - \varphi(||T^2x - T^3x||))dist(A,B) \\
&\leq \varphi(||T^2x - T^3x||)\,[\lambda||x - Tx|| + dist(A,B)] + (1 - \varphi(||T^2x - T^3x||))dist(A,B) \\
&= \varphi(||T^2x - T^3x||)\lambda||x - Tx|| + dist(A,B) \\
&\leq \lambda^2||x - Tx|| + dist(A,B).
\end{aligned}$$

Hence, by induction, one can obtain

$$dist(A,B) \leq ||T^{n+1}x - T^{n+2}x|| \leq \lambda^n||x - Tx|| + dist(A,B) \text{ for all } n \in \mathbb{N} \tag{3.9}$$

Since $\lambda \in (0,1)$, $\lim_{n\to\infty} \lambda^n = 0$. Using (3.8) and (3.9), we obtain $\lim_{n\to\infty} ||T^n x - T^{n+1}x|| = dist(A,B)$. So (i) is proved.
To see (ii), let $x \in A \cup B$. By using (i), we have $\lim_{n\to\infty} ||T^{2n}x - T^{2n-1}x|| = dist(A,B)$ and $\lim_{n\to\infty} ||T^{2n-2}x - T^{2n-1}x|| = dist(A,B)$. Lemma 2.3 concludes that

$$\lim_{n\to\infty} ||T^{2n}x - T^{2n-2}x|| = 0,$$

for any $x \in A \cup B$. In the same way, from $\lim_{n\to\infty} ||T^{2n}x - T^{2n+1}x|| = dist(A,B)$, $\lim_{n\to\infty} ||T^{2n+2}x - T^{2n+1}x|| = dist(A,B)$ and Lemma 2.3 we can obtain

$$\lim_{n\to\infty} ||T^{2n}x - T^{2n+2}x|| = 0,$$

for any $x \in A \cup B$.
Now we prove (iii). Let z be a fixed point of T^2 but it is not a best proximity point of T, i.e. $dist(A,B) \leq ||z - Tz||$. Then by (3.1) we have

$$\begin{aligned}
||z - Tz|| &= ||T^2z - Tz|| \leq \varphi(||z - Tz||)||z - Tz|| + (1 - \varphi(||z - Tz||))dist(A,B) \\
&< \varphi(||z - Tz||)||z - Tz|| + (1 - \varphi(||z - Tz||))||z - Tz|| = ||z - Tz||,
\end{aligned}$$

a contradiction.
Now, if z is a best proximity point of T i.e. $||z - Tz|| = dist(A,B)$ then from (3.1) we have $||T^2z - Tz|| = dist(A,B)$. So by Lemma 2.3, $T^2z = z$ which shows that (iii) is true. □

The following lemma can be obtained immediately from Lemma 3.4.

Lemma 3.5. *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of uniformly convex Banach space X and $T : A \cup B \to A \cup B$ cyclic map. Suppose that there exists a nondecreasing (or nonincreasing) function $\tau : [0,\infty) \to [0,1)$ such that*

$$||Tx - T^2x|| \leq \tau(||x - Tx||)||x - Tx|| + (1 - \tau(||x - Tx||))dist(A,B) \text{ for any } x \in A \text{ and } y \in B.$$

Then (i) $\lim_{n\to\infty} ||T^n x - T^{n+1}x|| = dist(A,B)$ for all $x \in A \cup B$.
(ii) $\lim_{n\to\infty} ||T^{2n}x - T^{2n\pm 2}x|| = 0$ for all $x \in A \cup B$.
(iii) z is a best proximity point if and only if z is a fixed point of T^2.

The following result is indeed proved in [8], but we give the proof for the sake of completeness.

Theorem 3.6. *[8] Let A and B be nonempty subsets of a metric space X and $T : A \cup B \to A \cup B$ be a cyclic map. Let $x_1 \in A$ be given. Define an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. Suppose that*

(i) $d(Tx,Ty) \leq d(x,y)$ for any $x \in A$ and $y \in B$;
(ii) $\{x_{2n-1}\}$ has a convergent subsequence in A;
(iii) $\lim_{n\to\infty} d(x_n, x_{n+1}) = dist(A,B)$.

Then there exists $v \in A$ such that $d(v,Tv) = dist(A,B)$.

Proof. Since T is a cyclic map and $x_1 \in A$, $x_{2n-1} \in A$ and $x_{2n} \in B$ for all $n \in \mathbb{N}$. By (ii), $\{x_{2n-1}\}$ has a convergent subsequence $\{x_{2n_k-1}\}$ and $x_{2n_k-1} \to v$ as $k \to \infty$ for some $v \in A$. Since

$$dist(A,B) \leq d(v, x_{2n_k}) \leq d(v, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}) \text{ for all } k \in \mathbb{N},$$

it follows from $\lim_{n\to\infty} d(v, x_{2n_k-1}) = 0$ and the condition (iii) that $\lim_{n\to\infty} d(v, x_{2n_k}) = dist(A,B)$. By (i), we have

$$dist(A,B) \leq d(Tv, x_{2n_k+1}) \leq d(v, x_{2n_k}) \text{ for all } k \in \mathbb{N},$$

which implies $d(v,Tv) = dist(A,B)$. □

In the following theorem we prove a new existence theorem for weak $\mathscr{MT}$-cyclic Kannan contractions.

Theorem 3.7. *Let (X,d) be a metric space, let A and B be nonempty subsets of X. Let $T : A \cup B \to A \cup B$ be a weak $\mathscr{MT}$-cyclic Kannan contraction with respect to a $\mathscr{MT}$-function φ. Let $x \in A$ such that the sequence $\{T^{2n}x\}$ has a convergent subsequence in A. Then there exists a unique point $z \in A$ such that $d(z,Tz) = dist(A,B)$.*

Proof. The existence of best proximity point z by using of Lemma 3.4 and Theorem 3.3 is concluded. We prove z is unique. Let v be another point such that $d(v,Tv) = dist(A,B)$. Then by Lemma 3.4 we have $v = T^2v$. If $d(v,Tz) > dist(A,B)$, then by $(MTK2)$ we have

$$\begin{aligned} d(v,Tz) &= d(T^2v,Tz) \\ &\leq \frac{1}{2}\varphi(d(Tv,Tz))[d(Tv,v)+d(z,Tz)]+(1-\varphi(d(Tv,Tz)))dist(A,B) \\ &< \varphi(d(Tv,Tz))dist(A,B)+(1-\varphi(d(Tv,Tz)))dist(A,B) = dist(A,B). \end{aligned}$$

So, $d(v,Tz) = dist(A,B)$. On the other hand $d(z,Tz) = dist(A,B)$. Hence by Lemma 2.3 we have $d(z,v) = 0$ or $z = v$. □

For weak $\mathscr{MT}$-cyclic Kannan contractions, we establish the following convergence theorem, which is our main result in this paper.

Theorem 3.8. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Let $T : A\cup B \to A\cup B$ be a weak $\mathscr{MT}$-cyclic Kannan contraction with respect to a $\mathscr{MT}$-function φ. Then*

(i) T has a unique best proximity point z in A.
(ii) The sequence $\{T^{2n}x\}$ converges to z for any starting point $x \in A$.
(iii) z is the unique fixed point of T^2.
(iv) Tz is a best proximity point of T in B.

Proof. We divide the proof of theorem into two cases:
case 1: $dist(A,B) = 0$.
For proof of this case see Theorem 3.6.
case 2: $dist(A,B) \neq 0$. Let x be an arbitrary point in A. Since T is a weak $\mathscr{MT}$-cyclic Kannan contraction, by part (i) of Lemma 3.4, $\lim_{n\to\infty} ||T^{2n}x-T^{2n+1}x|| = dist(A,B)$.
Now, we claim that for every $\varepsilon > 0$ there exists $N_0 \in N$ such that for $m > n > N_0$,

$$||T^{2m}x-T^{2n+1}x|| < dist(A,B)+\varepsilon.$$

Hence by Lemma 2.3 and for given $\varepsilon > 0$ there exists $N_1 \in N$ such that for $n > N_1$,

$$||T^{2m}x-T^{2n}x|| \leq \varepsilon;$$

it follows that $\{T^{2n}x\}$ is a Cauchy sequence and so there exists $z \in A$ such that $T^{2n}x \to z$ as $n \to \infty$. Using Theorem 3.6, z is a unique best proximity point of T in A.
Lemma 3.4-(iii) concludes that z is a unique fixed point of T^2, since z is unique.
Now, we prove the claim. Suppose not. Then there exists $\varepsilon > 0$ such that for any $k \in N$ there exists $m_k > n_k > k$ such that

$$||T^{2m_k}x-T^{2n_k+1}x|| \geq dist(A,B)+\varepsilon.$$

We can assume that m_k is minimal index such that $||T^{2m}x-T^{2n+1}x|| \geq dist(A,B)+\varepsilon$ but $||T^hx-T^{2n_k+1}x|| < dist(A,B)+\varepsilon, h \in \{2n_{k+1},\cdots,2m_k-1\}$. We have

$$dist(A,B)+\varepsilon \leq ||T^{2m_k}x-T^{2n_k+1}x|| \leq ||T^{2m_k}x-T^{2m_k-2}x||+||T^{2m_k-2}x-T^{2n_k+1}x||.$$

Using part of (ii) in Lemma 3.4 concludes that $||T^{2m_k}x-T^{2m_k-2}x|| \to 0$ as $k \to \infty$ this implies $\lim_k ||T^{2m_k}x-T^{2n_k+1}x|| = dist(A,B)+\varepsilon$.
Now,

$$||T^{2m_k}x-T^{2n_k+1}x|| \leq ||T^{2m_k}x-T^{2m_k+2}x||+||T^{2m_k+2}x-T^{2n_k+3}x||+||T^{2n_k+3}x-T^{2n_k+1}x|| = ||T^{2m_k+2}x-T^{2n_k+3}x||.$$

So, $dist(A,B)+\varepsilon \leq \limsup_{n\to\infty} ||T^{2m_k+2}x-T^{2n_k+3}x||$. But,

$$\begin{aligned} ||T^{2m_k+2}x-T^{2n_k+3}x|| &\leq \frac{1}{2}\varphi(||T^{2m_k+1}x-T^{2n_k+2}x||)[||T^{2m_k+1}x-T^{2m_k+2}x|| \\ &+ ||T^{2n_k+2}x-T^{2n_k+3}x||](1-\varphi(||T^{2m_k+1}x-T^{2n_k+2}x||))dist(A,B). \end{aligned}$$

Hence, by "$\limsup$" from the above inequality, as $(n \to \infty)$, we conclude that $dist(A,B)+\varepsilon \leq dist(A,B)$ and so $\varepsilon \leq 0$, a contradiction.
Now we prove (iv). z is best proximity point of T and so $||z-Tz|| = dist(A,B)$. Since T is a weak $\mathscr{MT}$-cyclic Kannan contraction, we have

$$||Tz-T^2z|| \leq \frac{1}{2}\varphi(||z-Tz||)[||z-Tz||+||Tz-T^2z||]+(1-\varphi(||z-Tz||))dist(A,B),$$

and so

$$dist(A,B) \leq ||Tz-T^2z|| \leq \frac{\varphi(||z-Tz||)}{2-\varphi(||z-Tz||)}||z-Tz||+\frac{2-2\varphi(||z-Tz||)}{2-\varphi(||z-Tz||)}dist(A,B) = dist(A,B).$$

Therefore $||Tz-T^2z|| = dist(A,B)$, i.e. Tz is best proximity point of T in B. This complete the proof. □

The following theorem can be obtain immediately from Lemma 3.5 and Theorem 3.7.

Theorem 3.9. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Let* $T : A\cup B \to A\cup B$ *be a a cyclic map. Suppose that there exists a nondecreasing (or nonincreasing) function* $\tau : [0,\infty) \to [0,1)$ *such that*

$$||Tx-Ty|| \leq \frac{1}{2}\tau(||x-y||)||x-y|| + (1-\tau(||x-y||))dist(A,B) \text{ for any } x\in A \text{ and } y\in B.$$

Then

(i) T *has a unique best proximity point* z *in* A.
(ii) The sequence $\{T^{2n}x\}$ *converges to* z *for any starting point* $x\in A$.
(iii) z *is the unique fixed point of* T^2.
(iv) Tz *is a best proximity point of* T *in* B.

Remark 3.10. *In Theorems 3.6 and 3.8 if we define* $\varphi(t) = c$, *where* $c\in[0,1)$ *and for all* $t\in[0,\infty)$, *then* φ *is a* $\mathscr{MT}$*-function and so we can obtain Theorems P [12] as the special cases.*

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Characterizations of slant and spherical helices due to pseudo-Sabban frame

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Abstract

In this paper, we investigate that under which conditions of the geodesic curvature of unit speed curve γ that lies on S_1^2 or H^2, the curve α which is obtained by using γ, is a spherical helix or slant helix in Minkowski 3-space.

1. Introduction

There are several studies in literature examining methodology to use spherical curves to construct some specialized curves. For example, Izuyama and Takeuchi [7], defined a way to construct Bertrand curves from the spherical curve whose spherical evolute coincides with the spherical Darboux image of the Bertrand curve. In addition to this paper, Encheva and Georgiev [4] showed a way to construct all *Frenet curves* $(\kappa > 0)$ by the following formula

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

where b is a constant number, a is a constant vector, γ is a unit speed curve on S^2 with the *Sabban frame* and $k : I \to R$ is a function of class C^1. Moreover, they showed that the spherical curve γ is a circle if and only if the corresponding *Frenet curves* are cylindrical helices. Previously, we have found some characterizations to construct spherical helices and slant helices in Euclidean space by using these methods [2].

This paper is organized in the following way. In section 2 basic concepts of Minkowski 3-space R_1^3 are given. In section 3, spherical helices in R_1^3 are discussed by indicating some examples. Similarly, in section 4, slant helices in R_1^3 are examined.

2. Basic Concepts

Let us consider the Minkowski 3-space R_1^3 with the Lorentzian inner product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in {R_1}^3$. The pseudo-norm of a vector x is given by $\|x\| = \sqrt{|\langle x, x\rangle|}$.
In the space R_1^3, the Lorentzian cross-product is defined as follows

$$x \wedge y = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2, \quad x_3y_1 - x_1y_3, \quad x_2y_1 - x_1y_2).$$

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It's clearly seen that the cross-product has the following properties [3],

$$\begin{array}{ll} (i) & x\wedge y=-(y\wedge x) \\ (ii) & \langle x\wedge y,z\rangle = det(x,y,z) \\ (iii) & x\wedge(y\wedge z)=\langle x,y\rangle z-\langle x,z\rangle y \\ (iv) & \langle x\wedge y,x\wedge y\rangle = (\langle x,y\rangle)^2-\langle x,x\rangle\langle y,y\rangle \\ (v) & \langle x\wedge y,x\rangle = 0 \quad , \quad \langle x\wedge y,y\rangle = 0 \end{array}$$

where $x,y,z\in {R_1}^3$.
A vector $x\in {R_1}^3$ is called spacelike if $\langle x,x\rangle>0$ or $x=0$, timelike if $\langle x,x\rangle<0$, lightlike if $\langle x,x\rangle=0$ and $x\neq 0$ [8].
In [8], the hyperbolic plane (resp. pseudosphere) center $q\in R_1^3$ and of radius $r>0$ are defined by,

$$H^2(r;q)=\left\{x=(x_1,x_2,x_3)\in R_1^3:\langle x-q,x-q\rangle=-r^2,x_3-q_3>0\right\},$$

$$S_1^2(r;q)=\left\{(x_1,x_2,x_3)\in R_1^3:\langle x-q,x-q\rangle=r^2\right\}.$$

When $r=1$ and p is the origin, the *hyperbolic plane* is denoted by H^2 and the *pseudosphere* is denoted by S_1^2. ,
In this paper, when a helix lies on $H^2(r;q)$ or $S_1^2(r;q)$, we call it spherical curve.
Given a regular curve $\alpha(t):I\subset R\to R_1^3$. We say that α is spacelike (resp. timelike, lightlike) at t if $\alpha'(t)$ is a spacelike (resp. timelike, lightlike) vector. The curve α is called spacelike (resp. timelike, lightlike) if it is for any $t\in I$ [8].
A non-lightlike curve $\alpha:I\subset R\longrightarrow {E_1}^3$ is said to be parametrized by the pseudo arclength parameter s, if $|\langle\alpha'(s),\alpha'(s)\rangle|=1$. In this case, we call α is a unit speed curve.
For a unit speed non-lightlike curve α with a spacelike or timelike normal vector $N(s)$, the Frenet formulae are given in [8]. It's easy to calculate the formulae for arbitrary speed non-lightlike curves as follows.
If α is a timelike curve,

$$\begin{bmatrix}T'\\N'\\B'\end{bmatrix}=\begin{bmatrix}0&\kappa\nu&0\\\kappa\nu&0&\tau\nu\\0&-\tau\nu&0\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix} \tag{2.1}$$

If α is a spacelike curve with a spacelike normal vector $N(t)$,

$$\begin{bmatrix}T'\\N'\\B'\end{bmatrix}=\begin{bmatrix}0&\kappa\nu&0\\-\kappa\nu&0&\tau\nu\\0&\tau\nu&0\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix} \tag{2.2}$$

If α is a spacelike curve with a timelike normal vector $N(t)$,

$$\begin{bmatrix}T'\\N'\\B'\end{bmatrix}=\begin{bmatrix}0&\kappa\nu&0\\\kappa\nu&0&\tau\nu\\0&\tau\nu&0\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix} \tag{2.3}$$

where

$$\kappa=\frac{\|\alpha'\wedge\alpha''\|}{\|\alpha'\|^3},\tau=\frac{det(\alpha',\alpha'',\alpha''')}{\|\alpha'\wedge\alpha''\|^2},\nu=\sqrt{|\langle\alpha',\alpha'\rangle|}. \tag{2.4}$$

In the formulae above, we denote unit tangent vector with $T(t)$, unit binormal vector with $B(t)$, unit normal vector with $N(t)$.
A regular timelike or spacelike curve α is a helix, if τ/κ is a constant function.
For a unit speed curve α in R_1^3, slant helix characterization is given in [1]. Also, some characterizations of Lorentzian unit speed curves which lies on H^2 or S_1^2 were investigated in [9, 10, 11, 12]. With the help of these papers, we easily have the Lemmas for arbitrary speed curves below.

Lemma 2.1. *Let α be a timelike curve in R_1^3. Then, α is a slant helix if and only if either one of the next two functions*

$$\frac{\kappa^2}{\nu\left(\tau^2-\kappa^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \quad or \quad \frac{\kappa^2}{\nu\left(\kappa^2-\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \tag{2.5}$$

is constant everywhere $\tau^2-\kappa^2$ does not vanish.

Lemma 2.2. *Let α be a spacelike curve in R_1^3 with a spacelike normal vector. Then, α is a slant helix if and only if either one of the next two functions*

$$\frac{\kappa^2}{\nu\left(\tau^2-\kappa^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \quad or \quad \frac{\kappa^2}{\nu\left(\kappa^2-\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \tag{2.6}$$

is constant everywhere $\tau^2-\kappa^2$ does not vanish.

Lemma 2.3. *Let α be a spacelike curve in R_1^3 with a timelike normal vector. Then, α is a slant helix if and only if the function*

$$\frac{\kappa^2}{\nu\left(\tau^2+\kappa^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \tag{2.7}$$

is constant.

Lemma 2.4. *Let α be a spacelike curve in R_1^3 with a spacelike normal vector. Image of α lies on the pseudosphere (resp. hyperbolic plane) of radius r and center q if and only if*

$$\frac{1}{\kappa^2}-\left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)^2=\pm r^2(resp.) \tag{2.8}$$

where $r>0\in R, \kappa\neq 0, \tau\neq 0$.

Lemma 2.5. *Let α be a timelike curve in R_1^3. Image of α lies on the pseudosphere of radius r and center q if and only if*

$$\frac{1}{\kappa^2}+\left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)^2=r^2 \tag{2.9}$$

where $r>0\in R, \kappa\neq 0, \tau\neq 0$.

Lemma 2.6. *Let α be a spacelike curve in R_1^3 with a timelike normal vector. Image of α lies on the hyperbolic plane of radius r and center q if and only if*

$$\frac{-1}{\kappa^2}+\left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)^2=-r^2 \tag{2.10}$$

where $r>0\in R, \kappa\neq 0, \tau\neq 0$.

Let γ be a non-lightlike unit speed spherical curve with the arc-length parameter s and denote $\gamma'=t$ where $\gamma'=d\gamma/ds$. If we set a vector $p=\gamma\wedge t$, by definition we have an orthonormal frame $\{\gamma,t,p\}$. This frame is called the pseudo-Sabban frame of γ [5, 6]. Thus, we have the following Lemma .

Lemma 2.7. *Let $\gamma(s)$ be a unit speed spherical curve in R_1^3, then*
(i) If γ is a timelike curve on S_1^2 then,

$$\begin{aligned}\gamma' &= t\\ t' &= k_g p+\gamma\\ p' &= k_g t\end{aligned} \tag{2.11}$$

(ii) If γ is a spacelike curve on S_1^2, then

$$\begin{aligned}\gamma' &= t\\ t' &= -k_g p-\gamma\\ p' &= -k_g t\end{aligned} \tag{2.12}$$

(iii) If γ is a spacelike curve on H^2, then

$$\begin{aligned}\gamma' &= t\\ t' &= k_g p+\gamma\\ p' &= -k_g t\end{aligned} \tag{2.13}$$

where $k_g=det\left(\gamma,t,t'\right)$ the geodesic curvature of curve γ.

3. Spherical helices on $S_1^2(r;p)$ and $H^2(r;p)$

Let us take the curve

$$\alpha(s)=b\int e^{\int k(s)ds}\gamma(s)ds+a \tag{3.1}$$

at [4]. If we make the neccessary calculations, we have

$$\begin{aligned}\alpha'(s) &= be^{\int k(s)ds}\gamma(s),\\ \alpha''(s) &= be^{\int k(s)ds}\left(k(s)\gamma(s)+\gamma'(s)\right),\\ \alpha'''(s) &= be^{\int k(s)ds}\left(\left(k^2(s)+k'(s)\right)\gamma(s)+2k(s)\gamma'(s)+\gamma''(s)\right).\end{aligned} \tag{3.2}$$

If we calculate κ, τ, and ν of the curve α by using the equations at (2.4) and (3.2), we find

$$\begin{aligned}\kappa(s) &= \frac{1}{be^{\int k(s)ds}},\\ \tau(s) &= \frac{k_g(s)}{be^{\int k(s)ds}},\\ \nu(s) &= be^{\int k(s)ds}.\end{aligned} \tag{3.3}$$

It's easy to see

$$\begin{aligned} \langle \alpha'(s), \alpha'(s) \rangle &= b^2 e^{2\int k(s)ds} \langle \gamma(s), \gamma(s) \rangle, \\ T(s) &= \gamma(s), \\ T'(s) &= t(s). \end{aligned} \tag{3.4}$$

So, we can say if γ is a unit speed spacelike curve which lies on S_1^2, then α is a spacelike curve with a spacelike normal vector N.
If γ is a unit speed spacelike curve which lies on H^2, then α is a timelike curve with a spacelike normal vector N.
If γ is a unit speed timelike curve which lies on S_1^2 then α is a spacelike curve with a timelike normal vector N.
Now, we want to show, under which circumstances the curve α at equation (3.1) is a spherical helix on $S_1^2(r;p)$.

Theorem 3.1. *If the curve γ is a unit speed spacelike curve with a constant geodesic curvature, which lies on S_1^2, the curve α defined by (3.1) is a spherical helix which lies on the pseudosphere of the radius $|bd|$ and of the center origin if and only if the function $k(s) = k_g tanh\left[(k_g)(s-c)\right]$ where $b,c,d \in R$.*

Proof. From (3.2), (3.3), and (3.4), we know the curve

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

is a spacelike curve with a spacelike normal vector $N(s)$. So we need to use (2.8). Let's take the derivate of (2.8) with respect to s. Then, we have

$$\left(\frac{1}{\nu}\left[\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right]' - \frac{\tau}{\kappa}\right)(s) = 0$$

By putting (3.3) in this equation, we have

$$\left(\frac{1}{be^{\int kds}}\left[\frac{1}{k_g}\left(be^{\int kds}\right)'\right]' - k_g\right)(s) = 0$$

$$k'(s) + k^2(s) = {k_g}^2.$$

If we solve this differential equation, we have

$$k(s) = k_g tanh\left[(k_g)(s-c)\right]$$

Conversely, if we take $k(s) = k_g tanh\left[(k_g)(s-c)\right]$ in (14), then

$$\int k(s)ds = \int k_g tanh\left[(k_g)(s-c)\right]ds.$$

Let $u = k_g(s-c) = k_g s - k_g c$ then $k_g ds = du$, by using these equations

$$\begin{aligned} \int k(s)ds &= \int \tanh u du \\ &= \ln\cosh u + \ln d \\ &= \ln\left[d\cosh\left(k_g(s-c)\right)\right] \end{aligned}$$

we have

$$\begin{aligned} \alpha(s) &= b\int e^{\int k(s)ds}\gamma(s)ds + a \\ &= b\int e^{\ln\left[d\cosh\left(k_g(s-c)\right)\right]}\gamma(s)ds + a \\ &= b\int d\cosh\left(k_g(s-c)\right)\gamma(s)ds + a \end{aligned}$$

where $c,d \in R$.
Now, we must show that curve α is spherical. If we use (2.8) to do it, we have

$$\begin{aligned} r^2 &= \left(\left(\frac{1}{\kappa^2} - \left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)\right)^2\right)(s) \\ &= \left(b^2 e^{2\int kds}\left(1 - \frac{k^2}{k_g^2}\right)\right)(s) \\ &= b^2 d^2\cosh^2\left(k_g(s-c)\right)\left(\frac{1}{\cosh^2\left(k_g(s-c)\right)}\right) \\ &= b^2 d^2. \end{aligned}$$

Therefore, it can be said that the curve α lies on S_1^2 which has a radius $|bd|$. □

Now, we can give another theorem.

Theorem 3.2. *If the curve γ is a unit speed spacelike curve with a constant geodesic curvature, which lies on H^2, the curve α defined by (3.1) is a spherical helix which lies on the pseudosphere of the radius $|bd|$ and of the center origin if and only if the function $k(s) = k_g tan\left[\left(k_g\right)(s-c)\right]$ where $b,c,d \in R$.*

Proof. By using (2.9) instead of (2.8) in Theorem 3.1, the proof is similar. □

Theorem 3.3. *If the curve γ is a unit speed timelike curve with a constant geodesic curvature, which lies on S_1^2, the curve α defined by (3.1) is a spherical helix which lies on the hyperbolic plane of the radius $|bd|$ and of the center origin if and only if the function $k(s) = k_g tanh\left[\left(k_g\right)(s-c)\right]$ where $b,c,d \in R$.*

Proof. By using (2.10) instead of (2.8) in Theorem 3.1, the proof is similar. □

Example 3.4. *Let's take $\gamma(s) = \left\{\sqrt{2}\cos\left(s/\sqrt{2}\right), \sqrt{2}\sin\left(s/\sqrt{2}\right), 1\right\}$, we know that γ is a spacelike curve on S_1^2 with the geodesic curvature $\sqrt{2}$. Then due to Theorem 3.1,*

$$k(s) = k_g tanh\left[\left(k_g\right)(s-c)\right]$$

and

$$\alpha(s) = b\int d\cosh\left(k_g(s-c)\right)\gamma(s)ds + a$$

where $b,c,d \in R$. If we take $b = 2, c = 0, d = 1$; then, we have

$$\alpha_1(s) = 2\cosh\left(s/\sqrt{2}\right)\sin\left(s/\sqrt{2}\right) + 2\cos\left(s/\sqrt{2}\right)\sinh\left(s/\sqrt{2}\right)$$
$$\alpha_2(s) = -2\cos\left(s/\sqrt{2}\right)\cosh\left(s/\sqrt{2}\right) - 2\sin\left(s/\sqrt{2}\right)\sinh\left(s/\sqrt{2}\right)$$
$$\alpha_3(s) = 2\sqrt{2}\sinh\left(s/\sqrt{2}\right)$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ and $a = (0,0,0)$

Example 3.5. *Let's take $\gamma(s) = \left\{cos(s), sin(s), \sqrt{2}\right\}$, we know that γ is a spacelike curve on H^2 with the geodesic curvature $\sqrt{2}$. Then, due to Theorem 3.2,*

$$k(s) = k_g tan\left[\left(k_g\right)(s-c)\right]$$

and

$$\alpha(s) = b\int d\cos\left(k_g(s-c)\right)\gamma(s)ds + a$$

where $b,c,d \in R$. If we take $b = 2, c = 0, d = 1$; then, we have

$$\alpha_1(s) = -2\cos\left(\sqrt{2}s\right)\sin(s) + 2\sqrt{2}\cos(s)\sin\left(\sqrt{2}s\right)$$
$$\alpha_2(s) = 2\cos(s)\cos\left(\sqrt{2}s\right) + 2\sqrt{2}\sin(s)\sin\left(\sqrt{2}s\right)$$
$$\alpha_3(s) = 2\sin\left(\sqrt{2}s\right)$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ and $a = (0,0,0)$

Example 3.6. *Let's take $\gamma(s) = \left\{\frac{1}{\sqrt{3}}\cosh\left(\sqrt{3}s\right), \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\sinh\left(\sqrt{3}s\right)\right\}$, we know that γ is a timelike curve on S_1^2 with the geodesic curvature $\sqrt{2}$. Then, due to Theorem 3.3,*

$$k(s) = k_g tanh\left[\left(k_g\right)(s-c)\right]$$

and

$$\alpha(s) = b\int d\cosh\left(k_g(s-c)\right)\gamma(s)ds + a$$

where $b,c,d \in R$. If we take $b = 2, c = 0, d = 1$; then, we have

$$\alpha_1(s) = -2\sqrt{\tfrac{2}{3}}\cosh\left(\sqrt{3}s\right)\sinh\left(\sqrt{2}s\right) + 2\cosh\left(\sqrt{2}s\right)\sinh\left(\sqrt{3}s\right)$$
$$\alpha_2(s) = \frac{2\sinh\left(\sqrt{2}s\right)}{\sqrt{3}}$$
$$\alpha_3(s) = 2\cosh\left(\sqrt{2}s\right)\cosh\left(\sqrt{3}s\right) - 2\sqrt{\tfrac{2}{3}}\sinh\left(\sqrt{2}s\right)\sinh\left(\sqrt{3}s\right)$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ and $a = (0,0,0)$

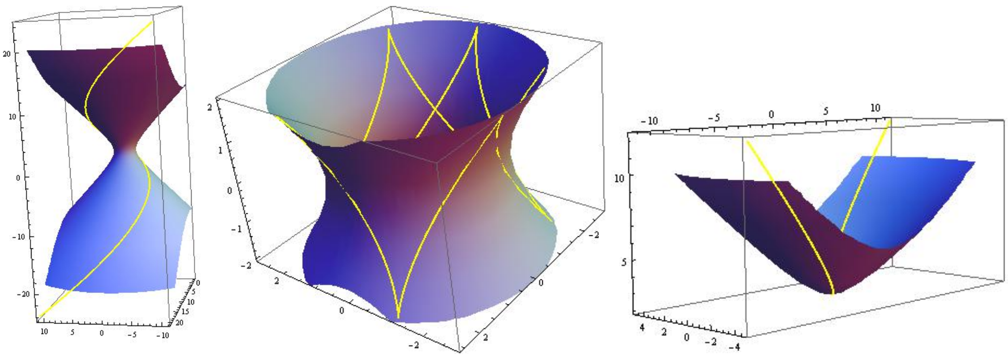

Figure 3.1: Spherical Helices (Resp. Example 1,2, and 3)

4. Constructing slant helices from unit speed spherical curves

In this section, we want to give some characterizations about slant helices.

Theorem 4.1. *Let $\gamma(s)$ be a unit speed spacelike curve on S_1^2; b,m,n be constant numbers; and a be a constant vector. The geodesic curvature of $\gamma(s)$ satisfies*

$$k_g{}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}$$

if and only if

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

is a spacelike slant helix with a spacelike normal vector.

Proof. Let, for γ

$$k_g{}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}. \tag{4.1}$$

From (3.2), (3.3), and (3.4), we know α is a spacelike curve with a spacelike normal vector N. So; from (2.6), the geodesic curvature of the spherical image of the principal normal indicatrix of α is as follows

$$\sigma(s) = \left(\frac{\kappa^2}{\nu\left(\kappa^2-\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s)$$
$$= \left(\frac{\frac{1}{\nu^2}}{\nu\left(\frac{1}{\nu^2}-\frac{k_g{}^2}{\nu^2}\right)^{3/2}}k_g{}'\right)(s).$$

So, we have

$$\sigma(s) = \frac{k_g{}'(s)}{\left(1-k_g{}^2(s)\right)^{3/2}} \tag{4.2}$$

Now, let's take $u(s) = ms+n$, then we have (4.1)

$$k_g{}^2(s) = \frac{u^2(s)}{1+u^2(s)}. \tag{4.3}$$

If we take the derivates of the both sides of (4.3) with respect to s, we have

$$2k_g(s)k_g{}'(s) = \left(\frac{2uu'\left(1+u^2\right)-\left(2uu'\right)u^2}{\left(1+u^2\right)^2}\right)(s)$$
$$k_g(s)k_g{}'(s) = \left(\frac{uu'}{\left(1+u^2\right)^2}\right)(s)$$

$$k_g{}'(s) = \left(\left(\frac{uu'}{\left(1+u^2\right)^2}\right)\left(\varepsilon\sqrt{\frac{1+u^2}{u^2}}\right)\right)(s) \tag{4.4}$$

where $\varepsilon = \pm 1$. Putting (4.3) and (4.4) in (4.3), we have

$$\begin{aligned}\sigma(s) &= \frac{k_g{}'(s)}{\left(1-k_g{}^2(s)\right)^{3/2}}\\ &= \left(\varepsilon \frac{\sqrt{1+u^2}uu'}{|u|\left(1+u^2\right)^2}\left(1+u^2\right)^{3/2}\right)(s)\\ &= \varepsilon \frac{ms+n}{|ms+n|}m\\ &= \varepsilon m\end{aligned}$$

which is constant.

Conversely, let $\alpha(s)$ be a spacelike slant helix, then the geodesic curvature of the spherical image of the principal normal indicatrix of α is a constant function. So, we can take

$$\sigma(s) = \left(\frac{\kappa^2}{\nu\left(\kappa^2-\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s) = m$$

where $m \in R$. Therefore, from (4.2)

$$\begin{aligned}m &= \left(\frac{\kappa^2}{\nu\left(\kappa^2-\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s)\\ &= \frac{k_g{}'(s)}{\left(1-k_g{}^2(s)\right)^{3/2}}\end{aligned}$$

If we solve this differential equation, we have

$$\frac{k_g(s)}{\sqrt{1-k_g{}^2(s)}} = ms+n$$

where $n \in R$. Then,

$$k_g{}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}.$$

□

Theorem 4.2. *Let $\gamma(s)$ be a unit speed spacelike curve on H^2; b,m,n be constant numbers; and a be a constant vector. The geodesic curvature of $\gamma(s)$ satisfies*

$$k_g{}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}$$

if and only if

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

is a timelike slant helix with a spacelike normal vector.

Proof. By using (2.5) instead of (2.6) in Theorem 4.1, the proof is similar. □

Theorem 4.3. *Let $\gamma(s)$ be a unit speed timelike curve on S_1^2; b,m,n be constant numbers; and a be a constant vector. The geodesic curvature of $\gamma(s)$ satisfies*

$$k_g{}^2(s) = \frac{(ms+n)^2}{1-(ms+n)^2}$$

if and only if

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

is a spacelike slant helix with a timelike normal vector.

Proof. By using (2.7) instead of (2.6) in Theorem 4.1, the proof is similar. □

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Quantum metrics on noncommutative spaces

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Abstract

We introduce two new algebraic formulations for the notion of 'quantum metric on noncommutative space'. For a compact noncommutative space associated to a unital C*-algebra, our quantum metrics are elements of the spatial tensor product of the C*-algebra with itself. We consider some basic properties of these new objects, and state some connections with the Rieffel theory of compact quantum metric spaces.

1. Introduction

There are at least two mathematically rigorous algebraic formulations for the notion of 'quantum (noncommutative) metric space' in the literature. The famous one is due to Rieffel, and the other has been recently introduced by G. Kuperberg and N. Weaver. Following some ideas from Connes [1, Chapter VI] in noncommutative Riemannian geometry [2], Rieffel has introduced the notions of 'compact quantum metric space' and 'quantum Hausdorff-Gromov distance' [6, 7, 8]. In his theory, a compact quantum metric space $\mathfrak{q}$ is identified with the state space of a unital C*-algebra $\mathscr{A}$ (or more generally, with the state space of an order unit space) together with a weak*-compatible metric which must be induced by a 'Lipschitz seminorm' on $\mathscr{A}$ via Monge-Kantorovich's formula. Thus, in the Rieffel theory, the role of quantum metrics is played by Lipschitz seminorms. In Kuperberg-Weaver theory [4] the noncommutative space is distinguished by a von Neumann algebra $\mathscr{M} \subseteq \mathbf{L}(\mathscr{H})$ and the role of quantum metric is played by a specific one-parameter family $\{\mathscr{V}_t\}_{t\geq 0}$ of weak closed operator systems in $\mathbf{L}(\mathscr{H})$ such that $\mathscr{V}_0 = \mathscr{M}'$. This construction also can be characterized by a specific "quantum distance function" between projections of the von Neumann algebra $\mathscr{M}\bar{\otimes}\mathbf{L}(\ell^2)$.
The notion of 'quantum metric' recently has been considered by many authors. See [5, 9, 10, 11] and references therein. In this note, we introduce two new models for 'quantum metrics on noncommutative spaces'. Our formulations are natural translations of the concept of '(ordinary) metric' into noncommutative geometric language. In Section 2, we give our first model for quantum metrics based on 'atomic representation' of C*-algebras. We also consider some basic properties of this model. In Section 3, we show that there is no quantum metric of the first model on 'noncommutative two point space'. In Section 4, we consider a relation between our first model and the Rieffel's model of quantum metrics. In Section 5, we introduce our second model of quantum metrics.

2. The first new model of quantum metrics

For preliminaries on C* and von Neumann algebras we refer the reader to [3] or [12]. Let $\mathscr{X}$ be a compact metrizable space. A function ρ is a compatible metric on $\mathscr{X}$ if and only if $\rho \in \mathbf{C}(\mathscr{X}^2) = \mathbf{C}(\mathscr{X})\check{\otimes}\mathbf{C}(\mathscr{X})$ and the following five conditions are satisfied for $x, y, z \in \mathscr{X}$.

(1) $\rho(x,y) \geq 0$.
(2) $\rho(x,x) = 0$.
(3) If $\rho(x,y) = 0$, then $x = y$.
(4) $\rho(x,y) = \rho(y,x)$.
(5) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$.

Let $\mathscr{A}$ be a unital C*-algebra. Suppose that an element $\rho \in \mathscr{A}\check{\otimes}\mathscr{A}$ deserves to be called a compatible metric on $\mathfrak{q}\mathscr{A}$. Then, ρ must satisfy the following analogues of (1),(4),(5).

(1′) $\rho \in (\mathscr{A}\check{\otimes}\mathscr{A})^+$.

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(4′) $\mathfrak{F}(\rho)=\rho$, where $\mathfrak{F}:\mathscr{A}\check{\otimes}\mathscr{A}\to\mathscr{A}\check{\otimes}\mathscr{A}$ denotes the flip.
(5′) $\mathfrak{M}(\rho)\leq\rho\otimes 1+1\otimes\rho$, where $\mathfrak{M}:\mathscr{A}\check{\otimes}\mathscr{A}\to\mathscr{A}\check{\otimes}\mathscr{A}\check{\otimes}\mathscr{A}$ denotes the *-morphism that puts 1 in the mid position, i.e. $\mathfrak{M}(a\otimes b):=a\otimes 1\otimes b$ $(a,b\in\mathscr{A})$.

There are many ways to state the noncommutative analogues of (2) and (3). But, it seems that the most effective and applicable way is as follows. Let $\pi:\mathscr{A}\to\mathbf{L}(\mathscr{H})$ denote a representation for $\mathscr{A}$ by bounded operators on a Hilbert space $\mathscr{H}$. We say that π is an *atomic* representation if there is a family $\{\pi_i:\mathscr{A}\to\mathbf{L}(\mathscr{H}_i)\}$ of pairwise inequivalent irreducible representations of $\mathscr{A}$ such that $\pi=\oplus\pi_i$. (Note that our atomic representations are special cases of the atomic representations defined in [12].) Then, it follows from [3, Corollary 10.3.9] that the enveloping von Neumann algebra $\pi(\mathscr{A})''$ is equal to $\oplus_i\mathbf{L}(\mathscr{H}_i)$. Let $\pi:\mathscr{A}\to\mathbf{L}(\mathscr{H})$ be a faithful atomic representation of $\mathscr{A}$. We consider $\mathscr{A}$ as a subalgebra of $\mathbf{L}(\mathscr{H})$ and write $\mathscr{A}''$ for $\pi(\mathscr{A})''$. The *characteristic function of the diagonal* (w.r.t. π) of $\mathfrak{q}\mathscr{A}\times\mathfrak{q}\mathscr{A}$ is denoted by P_δ and defined to be the supremum of the family of all projections of the form $p\otimes p$ in $\mathscr{A}''\bar{\otimes}\mathscr{A}''=(\mathscr{A}\check{\otimes}\mathscr{A})''\subseteq\mathbf{L}(\mathscr{H}\bar{\otimes}\mathscr{H})$ such that $p\in\mathscr{A}''$ is a minimal projection. (In the classical case that $\mathscr{A}=\mathbf{C}(\mathscr{X})$, if we choose π to be the *reduced atomic representation* then $\pi(\mathbf{C}(\mathscr{X}))''$ is isomorphic to $\ell^\infty(\mathscr{X})$ and P_δ is identified with the usual characteristic function of the diagonal of $\mathscr{X}\times\mathscr{X}$.) The analogues of (2) and (3) are as follows.

(2′) $\rho P_\delta=P_\delta\rho=0$.
(3′) Let $\mathscr{H}_\delta$ denote the the image of the projection P_δ in $\mathscr{H}\bar{\otimes}\mathscr{H}$. Then, 0 is not an eigenvalue of the operator $\rho|_{\mathscr{H}_\delta^\perp}\in\mathbf{L}(\mathscr{H}_\delta^\perp)$.

Definition 2.1. *Let $\mathscr{A}$ be a unital C*-algebra and let $\pi:\mathscr{A}\to\mathbf{L}(\mathscr{H})$ be a faithful atomic representation. A (compatible) quantum metric w.r.t. π on $\mathfrak{q}\mathscr{A}$ is an element $\rho\in\mathscr{A}\check{\otimes}\mathscr{A}$ satisfying (1′)-(5′). In this case, we call $(\mathscr{A},\rho,\pi)$ a compact quantum metric space.*

Let $(\mathscr{A},\rho,\pi)$ be a compact quantum metric space. Comparing with the classical case, it is natural that we consider the value $\|\rho\|$ as the diameter of ρ. It is clear that if $(\mathscr{X},\rho)$ is an ordinary compact metric space then $(\mathbf{C}(\mathscr{X}),\rho,\pi)$ is a compact quantum metric space where π is an arbitrary atomic representation of $\mathbf{C}(\mathscr{X})$. (Indeed, it is easily checked that for any of such representation $\pi(\mathbf{C}(\mathscr{X}))''$ is isomorphic to $\ell^\infty(\mathscr{X}_0)$ where $\mathscr{X}_0$ is a dense subspace of $\mathscr{X}$.)
Similar to the case of ordinary metric spaces, we have the following three theorems.

Theorem 2.2. *Let ρ_1 and ρ_2 be quantum metrics on $\mathfrak{q}\mathscr{A}$ w.r.t. the same representation π of $\mathscr{A}$. Then, for every positive real number r, $\rho_1+r\rho_2$ is a quantum metric on $\mathfrak{q}\mathscr{A}$ w.r.t. π.*

Proof. Straightforward. □

Theorem 2.3. *Let $(\mathscr{A}_1,\rho_1,\pi_1)$ and $(\mathscr{A}_2,\rho_2,\pi_2)$ be compact quantum metric spaces and let r be a real number not less than $2^{-1}\max(\|\rho_1\|,\|\rho_2\|)$. Then, $(\mathscr{A}_1\oplus\mathscr{A}_2,\rho,\pi_1\oplus\pi_2)$ is a compact quantum metric space where $\rho=\rho_1+\rho_2+r1_{\mathscr{A}_1\check{\otimes}\mathscr{A}_2}+r1_{\mathscr{A}_2\check{\otimes}\mathscr{A}_1}$.*

Proof. The conditions (1′) and (4′) are trivial for ρ. (2′) and (3′) follows from the fact that any minimal projection in $(\pi_1\oplus\pi_2)(\mathscr{A}_1\oplus\mathscr{A}_2)''=\pi_1(\mathscr{A}_1)''\oplus\pi_2(\mathscr{A}_2)''$ is a minimal projection of $\pi_1(\mathscr{A}_1)''$ or of $\pi_2(\mathscr{A}_2)''$. Let $\mathfrak{M}$,$\mathfrak{M}_1$,$\mathfrak{M}_2$ denote the corresponding morphisms as in (5′) respectively for $\mathscr{A}_1\oplus\mathscr{A}_2$,$\mathscr{A}_1$,$\mathscr{A}_2$. With the notation $1_{ijk}:=1_{\mathscr{A}_i\check{\otimes}\mathscr{A}_j\check{\otimes}\mathscr{A}_k}$ we have,

$$\mathfrak{M}(\rho_1)\leq\mathfrak{M}_1(\rho_1)+2r1_{121},\qquad\mathfrak{M}(\rho_2)\leq\mathfrak{M}_2(\rho_2)+2r1_{212}.$$

It follows that,

$$\begin{array}{rcl}\mathfrak{M}(\rho)&=&\mathfrak{M}(\rho_1)+\mathfrak{M}(\rho_2)+r1_{112}+r1_{122}+r1_{211}+r1_{221}\\ &\leq&\mathfrak{M}_1(\rho_1)+\mathfrak{M}_2(\rho_2)+2r1_{121}+2r1_{212}+r1_{112}+r1_{122}+r1_{211}+r1_{221}\\ &\leq&\rho_1\otimes 1_1+1_1\otimes\rho_1+\rho_2\otimes 1_2+1_2\otimes\rho_2+2r1_{121}+2r1_{212}+r1_{112}+r1_{122}+r1_{211}+r1_{221}\\ &\leq&\rho_1\otimes 1_1+\rho_1\otimes 1_2+\rho_2\otimes 1_1+\rho_2\otimes 1_2+r1_{121}+r1_{122}+r1_{211}+r1_{212}\\ &+&1_1\otimes\rho_1+1_2\otimes\rho_1+1_1\otimes\rho_2+1_2\otimes\rho_2+r1_{112}+r1_{212}+r1_{121}+r1_{221}\\ &=&\rho\otimes 1+1\otimes\rho.\end{array}$$

□

Theorem 2.4. *Let $(\mathscr{A}_1,\rho_1,\pi_1)$ and $(\mathscr{A}_2,\rho_2,\pi_2)$ be compact quantum metric spaces. Then, $(\mathscr{A}_1\check{\otimes}\mathscr{A}_2,\rho,\pi_1\otimes\pi_2)$ is a compact quantum metric space where*

$$\rho=(\rho_1\otimes 1_{\mathscr{A}_2\check{\otimes}\mathscr{A}_2}+1_{\mathscr{A}_1\check{\otimes}\mathscr{A}_1}\otimes\rho_2)\in(\mathscr{A}_1\check{\otimes}\mathscr{A}_1)\check{\otimes}(\mathscr{A}_2\check{\otimes}\mathscr{A}_2)\cong(\mathscr{A}_1\check{\otimes}\mathscr{A}_2)\check{\otimes}(\mathscr{A}_1\check{\otimes}\mathscr{A}_2).$$

Proof. Straightforward. □

3. A non-example

In this section, we show that there is no quantum metric on the *two-point noncommutative space* $\mathfrak{q}\mathbf{M}_2$. Let $\mathscr{A}=\mathbf{M}_2$ be the algebra of complex 2×2 matrices. Then, $\pi=\mathrm{id}$ is an atomic representation of $\mathscr{A}$ on $\mathscr{H}=\mathbb{C}^2$. Let $\{e_1,e_2\}$ (resp. $\{f_1,\cdots,f_4\}$) denote the Euclidean basis of $\mathbb{C}^2$ (resp. $\mathbb{C}^4$). We identify $\mathbb{C}^2\otimes\mathbb{C}^2$ with $\mathbb{C}^4$ via $e_1\otimes e_1\mapsto f_1$, $e_2\otimes e_2\mapsto f_4$, $e_1\otimes e_2\mapsto f_2$, $e_2\otimes e_1\mapsto f_3$. Then, $\mathbf{M}_2\otimes\mathbf{M}_2$ is identified with $\mathbf{M}_4$ via,

$$\sum_{i,j,k,\ell}\lambda_{ijk\ell}1_{ij}\otimes 1_{k\ell}\mapsto\begin{pmatrix}\lambda_{1111}&\lambda_{1112}&\lambda_{1211}&\lambda_{1212}\\ \lambda_{1121}&\lambda_{1122}&\lambda_{1221}&\lambda_{1222}\\ \lambda_{2111}&\lambda_{2112}&\lambda_{2211}&\lambda_{2212}\\ \lambda_{2121}&\lambda_{2122}&\lambda_{2221}&\lambda_{2222}\end{pmatrix}.$$

With these identifications, P_δ is the projection onto the linear subspace generated by f_1, f_4, f_2+f_3, and hence,

$$P_\delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Suppose that $\rho \in \mathbf{M}_4$ satisfies $(1')$-$(4')$. Then, ρ must be of the form,

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

for some real number $\lambda > 0$. The 8×8 matrix $M = \rho \otimes 1 + 1 \otimes \rho - \mathfrak{M}(\rho)$ is equal to,

$$M = \lambda \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For any vector $X = (x_1, \cdots, x_8) \in \mathbb{R}^8$ we have,

$$\lambda^{-1}\langle MX, X\rangle = (x_3 - x_2 - x_5)^2 + (x_3^2 - x_2^2 - x_5^2) + (x_6 - x_4 - x_7)^2 + (x_6^2 - x_4^2 - x_7^2).$$

Thus M is not positive and hence ρ does not satisfy $(5')$.
Although we just mentioned a negative result on the existence of quantum metrics but it seems that there must be a huge class of quantum metrics on $\mathfrak{q}\mathbf{M}_n$ for $n \geq 3$.

Question 3.1. *Does there exist a quantum metric on $\mathfrak{q}\mathbf{M}_n$ for some $n \geq 3$?*

4. Some relation between our first model and Rieffel's model of quantum metrics

In this section, we consider some relations between our first model of 'compact quantum metric space' and the model introduced by Rieffel [7]. Let $(\mathscr{A}, \rho, \pi)$ be a compact quantum metric space. We are able to define a new seminorm on $\mathscr{A}$ which generalizes the Lipschitz seminorm for continuous functions on an ordinary metric space. Let $\mathscr{H}$ denote the Hilbert space of π and let $\mathscr{H}_\delta$ be as in $(3')$. Let ρ^{-1} denote the inverse of the operator $\rho|_{\mathscr{H}_\delta^\perp} \in \mathbf{L}(\mathscr{H}_\delta^\perp)$. For any $a \in \mathscr{A}$, the Lipschitz seminorm $\|a\|_{Lip}$ are defined to be the (possibly infinite) value $\|(a \otimes 1 - 1 \otimes a)\rho^{-1}\|$, that is the operator norm of $(a \otimes 1 - 1 \otimes a)\rho^{-1}$ as an operator from the image of $\rho|_{\mathscr{H}_\delta^\perp}$ into $\mathscr{H}\bar{\otimes}\mathscr{H}$. For $a, b \in \mathscr{A}$ with $ab = ba$ the Leibnitz rule is satisfied:

$$\begin{aligned} \|ab\|_{Lip} &= \|(ab \otimes 1 - 1 \otimes ab)\rho^{-1}\| \\ &= \|(ab \otimes 1 - a \otimes b + a \otimes b - 1 \otimes ab)\rho^{-1}\| \\ &= \|[(a \otimes 1)(b \otimes 1 - 1 \otimes b) + (a \otimes 1 - 1 \otimes a)(1 \otimes b)]\rho^{-1}\| \\ &\leq \|(a \otimes 1)(b \otimes 1 - 1 \otimes b)\rho^{-1}\| + \|(1 \otimes b)(a \otimes 1 - 1 \otimes a)\rho^{-1}\| \\ &\leq \|a\|\|b\|_{Lip} + \|a\|_{Lip}\|b\| \end{aligned}$$

Also, it is clear that for any normal element a we have $\|a\|_{Lip} = \|a^*\|_{Lip}$. The seminorm $\|\cdot\|_{Lip}$ gives rise to a semimetric on the state space $S(\mathscr{A})$ of $\mathscr{A}$ via Monge-Kantorovich formula:

$$d(\phi, \psi) := \sup_{a^* = a, \|a\|_{Lip} \leq 1} |\langle \phi - \psi, a\rangle| \qquad (\phi, \psi \in S(\mathscr{A})).$$

We give an upper bound for $d(\phi, \psi)$ in the case that ϕ and ψ are some special pure states of $\mathscr{A}$: Let π be the direct sum of $\{\pi_i : \mathscr{A} \to \mathbf{L}(\mathscr{H}_i)\}$. Suppose that $i \neq j$, and let v and w be two unit vectors respectively in $\mathscr{H}_i$ and $\mathscr{H}_j$. Let ϕ and ψ be pure states on $\mathscr{A}$ defined respectively by $a \mapsto \langle \pi_i(a)v, v\rangle$ and $a \mapsto \langle \pi_j(a)w, w\rangle$. Let a be a self-adjoint element of $\mathscr{A}$ with $\|a\|_{Lip} \leq 1$. Since $v \otimes w \in \mathscr{H}_\delta^\perp$, we have $\|a(v) \otimes w - v \otimes a(w)\| \leq \|\rho(v \otimes w)\|$. Thus,

$$\begin{aligned} |\langle \phi - \psi, a\rangle|^2 &= \langle a(v), v\rangle^2 + \langle a(w), w\rangle^2 - 2\langle a(v), v\rangle\langle a(w), w\rangle \\ &\leq \langle a(v), a(v)\rangle + \langle a(w), a(w)\rangle - 2\langle a(v), v\rangle\langle a(w), w\rangle \\ &= \langle a(v), a(v)\rangle\langle w, w\rangle + \langle a(w), a(w)\rangle\langle v, v\rangle - 2\langle a(v), v\rangle\langle a(w), w\rangle \\ &= \|a(v) \otimes w - v \otimes a(w)\|^2 \leq \|\rho(v \otimes w)\|^2. \end{aligned}$$

This shows that $d(\phi, \psi) \leq \|\rho(v \otimes w)\|$.

5. The second new model of quantum metrics

As we saw above, the most problematic part of the definition of a quantum metric is the translation of conditions (2) and (3). We now translate these conditions in another way where there is no using of enveloping von Neumann algebras. Let $\mathscr{A}$ be a unital *spatially continuous multiplication* C*-algebra, that means the multiplication of $\mathscr{A}$, $m : a\otimes b \mapsto ab$, is continuous w.r.t. spatial tensor norm (e.g. $\mathscr{A}$ is abelian or finite dimensional). For $\rho \in \mathscr{A}\check{\otimes}\mathscr{A}$ satisfying ($1'$), consider the following conditions.

($2''$) $m(\rho) = 0$.
($3''$) For every positive element $\nu \in \mathscr{A}\check{\otimes}\mathscr{A}$ with $m(\nu) = 1$ and $\mathfrak{F}(\nu) = \nu$, the element $\rho + \nu$ is invertible in $\mathscr{A}\check{\otimes}\mathscr{A}$.

In the case that $\mathscr{A} = \mathbf{C}(\mathscr{X})$, it is easily checked that these conditions coincide with (2),(3).

Definition 5.1. *Let $\mathscr{A}$ be a unital spatially continuous multiplication C*-algebra. An element $\rho \in \mathscr{A}\check{\otimes}\mathscr{A}$ which satisfies ($1'$),($2''$), ($3''$),($4'$),($5'$) is called an algebraic (compatible) quantum metric on $\mathfrak{q}\mathscr{A}$. In this case, $(\mathscr{A},\rho)$ is called an algebraic compact quantum metric space.*

Theorem 5.2. *Let $(\mathscr{A}_1,\rho_1)$ and $(\mathscr{A}_2,\rho_2)$ be algebraic compact quantum metric spaces. Then, $(\mathscr{A}_1\oplus\mathscr{A}_2,\rho)$ is an algebraic compact quantum metric space where ρ is as in Theorem 2.3.*

Proof. We only show that ρ satisfies ($3''$). The other conditions are easily checked. Let $\mathscr{A} := \mathscr{A}_1\oplus\mathscr{A}_2$. Let m, m_1, m_2 denote respectively the multiplications of $\mathscr{A},\mathscr{A}_1,\mathscr{A}_2$, and let $\mathfrak{F},\mathfrak{F}_1,\mathfrak{F}_2$ denote the corresponding flips as in ($4'$). We have $\mathscr{A}\check{\otimes}\mathscr{A} = \oplus_{i,j=1,2}\mathscr{A}_i\check{\otimes}\mathscr{A}_j$. Let ν be a positive element of $\mathscr{A}\check{\otimes}\mathscr{A}$ with $m(\nu) = 1$ and $\mathfrak{F}(\nu) = \nu$. Let $\nu_{ij} \in \mathscr{A}_i\check{\otimes}\mathscr{A}_j$ be such that $\nu = \sum \nu_{ij}$. Then, ν_{ij} is positive and we have $m_i(\nu_{ii}) = 1_{\mathscr{A}_1}$ and $\mathfrak{F}_i(\nu_{ii}) = \nu_{ii}$. It follows that $\rho_i + \nu_{ii}$ is invertible in $\mathscr{A}_i\check{\otimes}\mathscr{A}_i$, and $r1_{\mathscr{A}_i}\otimes 1_{\mathscr{A}_j} + \nu_{ij}$ is invertible in $\mathscr{A}_i\check{\otimes}\mathscr{A}_j$ for $i \neq j$. Thus, $\rho + \nu$ is invertible in $\mathscr{A}\check{\otimes}\mathscr{A}$. □

Theorem 5.3. *Let $(\mathscr{A}_1,\rho_1)$ and $(\mathscr{A}_2,\rho_2)$ be algebraic compact quantum metric spaces such that $\mathscr{A}_1$ is commutative. Then, $(\mathscr{A}_1\check{\otimes}\mathscr{A}_2,\rho)$ is an algebraic compact quantum metric space where ρ is as in Theorem 2.4.*

Proof. We only show that ρ satisfies ($3''$). The other conditions are easily checked. Let m, m_2 denote respectively the multiplications of $\mathscr{A},\mathscr{A}_2$, and let $\mathfrak{F},\mathfrak{F}_2$ denote the corresponding flips as in ($4'$). Let $\mathscr{X}$ denote the Gelfand spectrum of $\mathscr{A}_1$. Thus, $\mathscr{A}_1 \cong \mathbf{C}(\mathscr{X})$ and $\mathscr{A}_1\check{\otimes}\mathscr{A}_2 \cong \mathbf{C}(\mathscr{X},\mathscr{A}_2)$, the algebra of $\mathscr{A}_2$ valued continuous functions on $\mathscr{X}$. Let $\nu \in \mathbf{C}(\mathscr{X}\times\mathscr{X},\mathscr{A}_2\check{\otimes}\mathscr{A}_2)$ be a positive element with $m(\nu) = 1$ and $\mathfrak{F}(\nu) = \nu$. Then, for every $x,y \in \mathscr{X}$, $\nu(x,y)$ is positive, $m_2(\nu(x,y)) = 1_{\mathscr{A}_2}$, and $\mathfrak{F}_2(\nu(x,y)) = \nu(x,y)$. Thus, $\rho_2 + \nu(x,y)$ is invertible in $\mathscr{A}_2\check{\otimes}\mathscr{A}_2$. It follows that $1_{\mathscr{A}_1\check{\otimes}\mathscr{A}_1}\otimes\rho_2 + \nu$ (which is equal to the function $(x,y) \mapsto \rho_2 + \nu(x,y)$) is invertible. Thus, $\rho + \nu$ is also invertible. □

The main gap in our work is the lack of a nonclassical example:

Problem 5.4. *Give an example of a nonclassical (algebraic) quantum metric.*

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Some new Pascal sequence spaces

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Abstract

The main purpose of the present paper is to study of some new Pascal sequence spaces p_∞, p_c and p_0. New Pascal sequence spaces p_∞, p_c and p_0 are found as BK-spaces and it is proved that the spaces p_∞, p_c and p_0 are linearly isomorphic to the spaces l_∞, c and c_0 respectively. Afterward, α-, β- and γ-duals of these spaces p_c and p_0 are computed and their bases are consructed. Finally, matrix the classes $(p_c : l_p)$ and $(p_c : c)$ have been characterized.

1. Preliminaries, background and notation

By w, we shall denote the space all real or complex valued sequences. Any vector subspace of w is called a sequence space. We shall write l_∞, c, and c_0 for the spaces of all bounded, convergent and null sequence are given by $l_\infty = \left\{ x = (x_k) \in w : \sup_{k\to\infty} |x_k| < \infty \right\}$, $c = \left\{ x = (x_k) \in w : \lim_{k\to\infty} x_k \text{ exists} \right\}$ and $c_0 = \left\{ x = (x_k) \in w : \lim_{k\to\infty} x_k = 0 \right\}$. Also by bs, cs, l_1 and l_p we denote the spaces of all bounded, convergent, absolutely convergent and p-absolutely convergent series, respectively.

A sequence space λ with a linear topology is called an K-space provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the set of complex field and $\mathbb{N} = \{0, 1, 2, ...\}$. An K-space λ is called an FK- space provided λ is a complete linear metric space. An FK-space provided whose topology is normable is called a BK- space [1].

Let X, Y be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we write $Ax = ((Ax)_n)$, the A-transform of x, if $A_n(x) = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. If $x \in X$ implies that $Ax \in Y$, then we say that A defines a matrix transformation from X into Y and denote it by $A : X \to Y$. By $(X : Y)$ we denote the class of all infinite matrices A such that $A : X \to Y$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞.

Let F denote the collection of all finite subsets on $\mathbb{N}$ and $K, \mathbb{N} \subset F$. The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.1}$$

which is a sequence space.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method was used by authors [2, 3, 4, 5, 6, 7, 8]. They introduced the sequence spaces $(c_0)_{T^r} = t_0^r$ and $(c)_{T^r} = t_c^r$ in [2], $(c_0)_{E^r} = e_0^r$ and $(c)_{E^r} = e_c^r$ in [3], $(c_0)_C = \bar{c}_0$ and $c_C = \bar{c}$ in [4], $(l_p)_{E^r} = e_p^r$ in [5], $(l_\infty)_{R^t} = r_\infty^t$, $c_{R^t} = r_c^t$ and $(c_0)_{R^t} = r_0^t$ in [6], $(l_p)_C = X_p$ in [7] and $(l_p)_{N_q}$ in [8] where T^r, E^r, C, R^t and N_q denote the Taylor, Euler, Cesaro, Riesz and Nörlund means, respectively.

Following [2, 3, 4, 5, 6, 7, 8], this way, the purpose of this paper is to introduce the new Pascal sequence spaces p_∞, p_c and p_0 and derive some results related to those sequence spaces. Furthermore, we have constructed the basis and computed the α-, β- and γ-duals of the spaces p_∞, p_c and p_0. Finally, we have characterized the matrix mappings from the space p_c to l_p and from the space p_c to c.

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2. The Pascal matrix of inverse formula and Pascal sequence spaces

Let P denote the Pascal means defined by the Pascal matrix [9] as is defined by

$$P=[p_{nk}]=\begin{cases}\binom{n}{n-k},(0\leq k\leq n)\\ 0,\ (k>n)\end{cases},(n,k\in\mathbb{N})$$

and the inverse of Pascal's matrix $P_n=[p_{nk}]$ [10] is given by

$$P^{-1}=[p_{nk}]^{-1}=\begin{cases}(-1)^{n-k}\binom{n}{n-k},(0\leq k\leq n)\\ 0,(k>n)\end{cases},(n,k\in\mathbb{N}). \tag{2.1}$$

There is some interesting properties of Pascal matrix. For example; we can form three types of matrices: symmetric, lower triangular, and upper triangular, for any integer $n>0$. The symmetric Pascal matrix of order n is defined by

$$S_n=(s_ij)=\binom{i+j-2}{j-1}i,\ j=1,2,....,n. \tag{2.2}$$

We can define the lower triangular Pascal matrix of order n by

$$L_n=(l_{ij})=\begin{cases}\binom{i-1}{j-1},(0\leq j\leq i)\\ 0,\qquad (j>i)\end{cases}, \tag{2.3}$$

and the upper triangular Pascal matrix of order n is defined by

$$U_n=(u_{ij})=\begin{cases}\binom{j-1}{i-1},(0\leq i\leq j)\\ 0,\qquad (j>i)\end{cases}. \tag{2.4}$$

We notice that $U_n=(L_n)^T$, for any positive integer n.
i. Let S_n be the symmetric Pascal matrix of order n defined by (2.1), L_n be the lower triangular Pascal matrix of order n defined by (2.3), and U_n be the upper triangular Pascal matrix of order n defined by (2.4), then $S_n=L_nU_n$ and $det(S_n)=1$ [11].
ii. Let A and B be $n\times n$ matrices. We say that A is similar to B if there is an invertible $n\times n$ matrix P such that $P^{-1}AP=B$ [12].
iii. Let S_n be the symmetric Pascal matrix of order n defined by (2.2), then S_n is similar to its inverse S_n^{-1} [11].
iv. Let L_n be the lower triangular Pascal matrix of order n defined by (2.3), then $L_n^{-1}=((-1)^{i-j}l_{ij})$ [13].
We wish to introduce the Pascal sequence spaces p_∞, p_c and p_0, as the set of all sequences such that P-transforms of them are in the spaces l_∞, c and c_0, respectively, that is

$$p_\infty=\left\{x=(x_k)\in w:\sup_n\left|\sum_{k=0}^{n}\binom{n}{n-k}x_k\right|<\infty\right\},$$

$$p_c=\left\{x=(x_k)\in w:\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{n-k}x_k\text{ exists}\right\}$$

and

$$p_0=\left\{x=(x_k)\in w:\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{n-k}x_k=0\right\}.$$

With the notation of (1.1), we may redefine the spaces p_∞, p_c and p_0 as follows:

$$p_\infty=(l_\infty)_P,\ p_c=(c)_P\text{ and }p_0=(c_0)_P. \tag{2.5}$$

If λ is an normed or paranormed sequence space, then matrix domain λ_P is called an Pascal sequence space. We define the sequence $y=(y_n)$ which will be frequently used, as the P-transform of a sequence $x=(x_n)$ i.e.,

$$y_n=\sum_{k=0}^{n}\binom{n}{n-k}x_k,\quad (n\in N). \tag{2.6}$$

It can be shown easily that p_∞, p_c and p_0 are linear and normed spaces by the following norm:

$$\|x\|_{p_0}=\|x\|_{p_c}=\|x\|_{p_\infty}=\|Px\|_{l_\infty}. \tag{2.7}$$

Theorem 2.1. *The sequence spaces p_∞, p_c and p_0 endowed with the norm (2.7) are Banach spaces.*

Proof. Let sequence $\{x^t\} = \{x_0^{(t)}, x_1^{(t)}, x_2^{(t)}, \ldots\}$ at p_∞ a Cauchy sequence for every fixed $t \in \mathbb{N}$. Then, there exists an $n_0 = n_0(\varepsilon)$ for every $\varepsilon > 0$ such that $\|x^t - x^r\|_\infty < \varepsilon$ for all t, $r > n_0$. Hence, $|P(x^t - x^r)| < \varepsilon$ for all t, $r > n_0$ and for each $k \in \mathbb{N}$.
Therefore, $\{Px_k^t\} = \{(Px^0)_k, (Px^1)_k, (Px^2)_k, \ldots\}$ is a Cauchy sequence in the set of complex numbers $\mathbb{C}$. Since $\mathbb{C}$ is complete, it is convergent say $\lim_{t\to\infty} (Px^t)_k = (Px)_k$ and $\lim_{m\to\infty} (Px^m)_k = (Px)_k$ for each $k \in \mathbb{N}$. Hence, we have

$$\lim_{m\to\infty} \left|Px_k^t - x_k^m\right| = \left|P\left(x_k^t - x_k\right) - P\left(x_k^m - x_k\right)\right| \leq \varepsilon \text{ for all } n \geq n_0.$$

This implies that $\|x^t - x^m\| \to \infty$ for t, $m \to \infty$. Now, we should that $x \in p_\infty$. We have

$$\|x\|_\infty = \|Px\|_\infty = \sup_n \left|\sum_{k=0}^{n} \binom{n}{n-k} x_k\right| = \sup_n \left|\sum_{k=0}^{n} \binom{n}{n-k} (x_k - x_k^t + x_k^t\right|$$

$$\leq \sup_n \left|P\left(x_k^t - x_k\right)\right| + \sup_n \left|Px_k^t\right|$$

$$\leq \left\|x^t - x\right\|_\infty + \left|Px_k^t\right| < \infty$$

for t, $k \in \mathbb{N}$. This implies that $x = (x_k) \in p_\infty$. Thus, p_∞ the space is a Banach space with the norm (2.7). It can be shown that p_0 and p_c are closed subspaces of p_∞ which leads us to the consequence that the spaces p_0 and p_c are also the Banach spaces with the norm (2.7). Furthermore, since p_∞ is a Banach space with continuous coordinates, i.e., $\left\|P\left(x_k^t - x\right)\right\|_\infty \to \infty$ imples $\left|P\left(x_k^t - x_k\right)\right| \to \infty$ for all $k \in \mathbb{N}$, it is also a BK-space. □

Theorem 2.2. *The sequence spaces p_∞, p_c and p_0 are linearly isomorphic to the spaces l_∞, c and c_0 respectively, i.e $p_\infty \cong l_\infty$, $p_c \cong c$ and $p_0 \cong c_0$.*

Proof. To prove the fact $p_0 \cong c_0$, we should show the existence of a linear bijection between the spaces p_0 and c_0. Consider the transformation T defined, with the notation (2.6), from p_0 to c_0. The linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.
Let $y \in c_0$. We define the sequence $x = (x_k)$ as follows:

$$x_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{k-i} y_i.$$

Then

$$\lim_{n\to\infty} (Px)_n = \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{n-k} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{k-i} y_i = \lim_{n\to\infty} y_n = 0.$$

Thus, we have that $x \in p_0$. In addition, note that

$$\|x\|_{p_0} = \sup_{n\in\mathbb{N}} \left|\sum_{k=0}^{n} \binom{n}{n-k} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{k-i} y_i\right| = \sup_{n\in\mathbb{N}} |y_n| = \|y\|_{c_0} < \infty.$$

Consequently, T is surjective and is norm preserving. Hence, T is a linear bijection which therefore says us that the spaces p_0 to c_0 are linearly isomorphic. In the same way, it can be shown that p_c and p_∞ are linearly isomorphic to c and l_∞, respectively, and so we omit the detail. □

Before giving the basis of of the sequence spaces p_c and p_0, we define the Schauder basis. A sequence $(b_n)_{n\in\mathbb{N}}$ in a normed sequence space λ is called a Schauder basis (or briefly basis) [14], if for every $x \in \lambda$ there is a unique sequence (α_n) of scalars such that

$$\lim_{n\to\infty} \|x - (\alpha_0 x_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n)\| = 0.$$

In the following theorem, we shall give the Schauder basis for the spaces p_c and p_0.

Theorem 2.3. *Let $k \in \mathbb{N}$ a fixed natural number and $b^{(k)} = \left\{b_n^{(k)}\right\}_{n\in\mathbb{N}}$ where*

$$b_n^{(k)} = \begin{cases} 0, & (0 \leq n < k) \\ (-1)^{n-k}\binom{n}{n-k}, & (n \geq k \end{cases}.$$

Then the following assertions are true:
i. *The sequence $\left\{b_n^{(k)}\right\}$ is a basis for the space p_0 and every $x \in p_0$ has a unique representation of the from $x = \sum_k \lambda_k b^{(k)}$ where $\lambda_k = (Px)_k$ for all $k \in \mathbb{N}$.*

ii. *The set $\left\{e, b^{(0)}, b^{(1)}, \ldots, b^{(k)}, \ldots\right\}$ is a basis for the space p_c and every $x \in p_c$ has a unique representation of the form $x = le + \sum_k (\lambda_k - l) b^{(k)}$, where $l = \lim_{k\to\infty} (Px)_k$ and $\lambda_k = (Px)_k$ for all $k \in \mathbb{N}$.*

3. The $\alpha-$, $\beta-$ and $\gamma-$ duals of the spaces p_∞, p_c and p_0

In this section, we state and prove the theorems determining the α-, β- and γ-duals of the sequence spaces p_∞, p_c and p_0. For the sequence spaces X and Y define the set $S(X,Y)$ by

$$S(X,Y)=\{z=(z_k)\in w : xz=(x_kz_k)\in Y \text{ for all } x\in X\}.$$

The α-, β- and γ-duals of the sequence spaces λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by Garling [15] , by $\lambda^\alpha=S(\lambda,l_1)$, $\lambda^\beta=S(\lambda,cs)$ and $\lambda^\gamma=S(\lambda,bs)$. We shall begin with the Lemmas due to Stieglitz and Tietz [16], which are needed in the proof of the Theorems 3.4-3.6.

Lemma 3.1. *$A\in(c_0:l_1)=(c:l_1)$ if and only if*

$$\sup_{K\in F}\sum_n\left|\sum_{k\in K}a_{nk}\right|<\infty. \tag{3.1}$$

Lemma 3.2. *$A\in(c_0:c)$ if and only if*

$$\sup_n\sum_k|a_{nk}|<\infty, \tag{3.2}$$

$$\lim_{n\to\infty}a_{nk}=\alpha_k,\ (k\in\mathbb{N}). \tag{3.3}$$

Lemma 3.3. *$A\in(c_0:l_\infty)$ if and only if (3.2) holds.*

Theorem 3.4. *The $\alpha-$ dual of the sequence spaces p_∞, p_c and p_0 is the set*

$$D=\left\{a=(a_k)\in w:\sup_{K\in F}\sum_n\left|\sum_{k\in K}(-1)^{n-k}\binom{n}{n-k}a_n\right|<\infty\right\}.$$

Proof. Let $a=(a_n)\in w$ and consider the matrix B whose rows are the products of the rows of the matrix P^{-1} and sequence $a=(a_n)$. Bearing in mind the relation (2.3), we immediately derive that

$$a_nx_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{n-k}a_ny_k=\sum_{k=0}^{n}b_{nk}y_k=(By)_n,\ (n\in\mathbb{N}). \tag{3.4}$$

Therefore by (3.4) we observe that that $ax=(a_nx_n)\in l_1$ whenever $x\in p_\infty$, p_c and p_0 if and only if $By\in l_1$ whenever $y\in l_\infty$, c, and c_0. Then, we derive by Lemma 3.1 that

$$\sup_{K\in F}\sum_n\left|\sum_{k\in K}(-1)^{n-k}\binom{n}{n-k}a_n\right|<\infty$$

which yields the consequences that $\{p_\infty\}^\alpha=\{p_c\}^\alpha=\{p_0\}^\alpha=D$. □

Theorem 3.5. *Consider the sets D_1, D_2 and D_3 defined as follows:*

$$D_1=\left\{a=(a_k)\in w:\ \sup_{n\in N}\sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right|<\infty\right\},$$

$$D_2=\left\{a=(a_k)\in w:\sum_{i=k}^{\infty}(-1)^{i-k}\binom{i}{i-k}a_i \text{ exists for each } k\in\mathbb{N}\right\},$$

and

$$D_3=\left\{a=(a_k)\in w:\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i \text{ exists}\right\}.$$

Then $\{p_0\}^\beta=D_1\cap D_2$, $\{p_c\}^\beta=D_1\cap D_2\cap D_3$ and $\{p_\infty\}^\beta=D_2\cap D_3$.

Proof. We give the proof only for the space p_0. Since the proof may be given by a similar way for the spaces p_c and p_∞, we omit it. Consider the equation

$$\sum_{k=0}^{n}a_kx_k=\sum_{k=0}^{n}\left[\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}y_i\right]a_k=\sum_{k=0}^{n}\left[\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right]y_k=(Dy)_n, \tag{3.5}$$

where

$$D=(d_{nk})=\left\{\begin{array}{cc}\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i, & (0\le k\le n)\\ 0 & ,(k>n)\end{array}\right. ,(n,k\in\mathbb{N}). \tag{3.6}$$

Thus, we deduce from Lemma 3.2 with (3.5) that $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in p_0$ if and only if $Dy\in c$ whenever $y=(y_k)\in c_0$. Therefore, using relations (3.2) and (3.3), we conclude that $\lim_{n\to\infty}d_{nk}$ exists fo each $k\in\mathbb{N}$ and

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right|<\infty$$

which shows that $\{p_0\}^{\beta}=D_1\cap D_2$. □

Theorem 3.6. *The $\gamma-$ dual of the sequence spaces p_∞, p_c and p_0 are D_1.*

Proof. We give the proof only for the space p_0. Consider the equality

$$\begin{aligned}\left|\sum_{k=0}^{n}a_kx_k\right| &= \left|\sum_{k=0}^{n}a_k\left[\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{k-i}y_i\right]\right|\\ &= \left|\sum_{k=0}^{n}\left[\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right]y_k\right|\\ &\le \sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right||y_k|.\end{aligned}$$

Taking supremum over $n\in\mathbb{N}$, we get

$$\begin{aligned}\sup_{n\in N}\left|\sum_{k=0}^{n}a_kx_k\right| &\le \sup_{n\in N}\left(\sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right||y_k|\right)\\ &\le \|y\|_{c_0}\sup_{n}\left(\sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right|\right)\le\infty.\end{aligned}$$

This means that $a=(a_k)\in\{p_0\}^{\gamma}$. Hence,

$$D_1\subset\{p_0\}^{\gamma}. \tag{3.7}$$

Conversely, let $a=(a_k)\in\{p_0\}^{\gamma}$ and $x\in p_0$. Then one can easily see that

$$\left(\sum_{k=0}^{n}\left[\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right]y_k\right)\in l_\infty$$

whenever $ax=(a_kx_k)\in bs$. This implies that the matrix D given at the (3.6) is in the class $(c_0:l_\infty)$. Hence, the condition

$$\sup_{n}\left(\sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_i\right|\right)<\infty$$

is satisfied, which implies that $a=(a_k)\in D_1$. In other words,

$$\{p_0\}^{\gamma}\subset D_1. \tag{3.8}$$

Therefore, by combining inclusions (3.7) and (3.8), we estahlish that the γ-dual of the sequence spaces p_0 is D_1, which completes the proof. □

4. Some matrix mappings related to Pascal sequence spaces

Lemma 4.1. *[16, p. 57] The matrix mappings between BK-spaces are continuous.*

Lemma 4.2. *[16, p. 128]* $A\in\left(c:l_p\right)$ *if and only if*

$$\sup_{K\in F}\sum_n\left|\sum_{k\in K}a_{nk}\right|^p<\infty,\ 1\le p<\infty. \tag{4.1}$$

Theorem 4.3. $A\in\left(p_c:l_p\right)$ *if and only if the following conditions are satisfied: For* $1\le p<\infty$*,*

$$\sup_{K\in F}\sum_k\left|\sum_{k\in K}\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}\right|^p<\infty\,, \tag{4.2}$$

$$\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}\text{ exists for all }k,\ n\in\mathbb{N}, \tag{4.3}$$

$$\sum_k\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}\text{ converges for all }n\in\mathbb{N}, \tag{4.4}$$

$$\sup_{m\in N}\sum_{k=0}^{m}\left|\sum_{i=k}^{m}(-1)^{i-k}\binom{i}{i-k}a_{ni}\right|<\infty\,,\ n\in\mathbb{N}, \tag{4.5}$$

and for $p=\infty$*, conditions (4.3) and (4.5) are satisfied and*

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}\right|<\infty. \tag{4.6}$$

Proof. Let $1\le p<+\infty$. Assume that conditions (4.2) - (4.6) are satisfied and take any $x\in p_c$. Then $(a_{nk})\in(p_c)^\beta$ for all $k,n\in\mathbb{N}$, which implies that Ax exists. We define the matrix $G=(g_{nk})$ with

$$g_{nk}=\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}$$

for all $k,n\in\mathbb{N}$. Then, since condition (4.1) is satisfied for the matrix G, we have $G\in\left(c:l_p\right)$. Now consider the following equality obtained from the s. th partial sum of the series $\sum_k a_{nk}x_k$:

$$\sum_{k=0}^{s}a_{nk}x_k=\sum_{k=0}^{s}\sum_{i=k}^{s}(-1)^{i-k}\binom{i}{i-k}a_{ni}y_k,\ m,n\in\mathbb{N}. \tag{4.7}$$

Therefore, we derive from (4.7) as $s\to\infty$ that

$$\sum_{k=0}^{\infty}a_{nk}x_k=\sum_{k=0}^{\infty}\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{i-k}a_{ni}y_k,\ n\in\mathbb{N}. \tag{4.8}$$

Whence taking l_p-norm we get

$$\|Ax\|_{l_p}=\|Gy\|_{l_p}<\infty. \tag{4.9}$$

This means that $A\in\left(p_c:l_p\right)$. Now let $p=\infty$. Assume that conditions (4.2) - (4.6) are satisfied and take any $x\in p_c$. Then $(a_{nk})\in(p_c)^\beta$ for all $k,\ n\in\mathbb{N}$, which implies that Ax exists. Whence taking l_∞-norm (4.8)

$$\|Ax\|_{l_\infty}=\sup_{n\in N}\left|\sum_k g_{nk}\right|\le\|y\|_{l_\infty}\sup_{n\in N}\sum_k|g_{nk}|<\infty.$$

Then, we have $A\in(p_c:l_\infty)$.
Conversely, assume that $A\in\left(p_c:l_p\right)$. Then, since p_c and l_p are BK-spaces, it follows from Lemma 4 that there exists a real constant $K>0$ such that

$$\|Ax\|_{l_p}=K\,\|x\|_{h_c} \tag{4.10}$$

for all $x\in p_c$. Since inequality (4.10) also holds for the sequence

$$x=(x_k)=\sum_{k\in F}b^{(k)}\in p_c,$$

where

$$b^{(k)}=\{b_n^{(\ k)}\}=\left\{\begin{array}{c}0,\ (0\leq n<k)\\(-1)^{n-k}\binom{n}{n-k},\ (n\geq k\end{array}\right.$$

for every fixed $k\in\mathbb{N}$. We have

$$\|Ax\|_{l_p}=\left[\sum_n\left|\sum_{k\in F}\sum_{i=k}^n(-1)^{i-k}\binom{i}{i-k}a_{ni}\right|^p\right]^{\frac{1}{p}}\leq K\|x\|_{p_c}=K,$$

which shows the necessity of (4.2). □

Theorem 4.4. *$A\in(p_c:c)$ if and only if conditions (4.3), (4.5) and (4.6) are satisfied,*

$$\lim_{n\to\infty}\sum_{i=k}^n(-1)^{i-k}\binom{i}{i-k}a_{ni}=\alpha_k \text{ for all } k\in\mathbb{N} \tag{4.11}$$

and

$$\lim_{n\to\infty}\sum_k\sum_{i=k}^n(-1)^{i-k}\binom{i}{i-k}a_{ni}=\alpha. \tag{4.12}$$

Proof. Assume that A satisfies conditions (4.3), (4.5), (4.6), (4.11) and (4.12). Let us take an arbitrary an $x=(x_k)$ in p_c such that $x_k\to l$ as $k\to\infty$. Then Ax exists, and it is trivial that the sequence $y=(y_k)$ associated with the sequence $x=(x_k)$ by relation (2.3) belongs to c and is such that $y_k\to l$ as $k\to\infty$. At this stage, it follows from (4.11) and (4.6) that

$$\sum_{j=0}^k|\alpha_j|\leq\sup_{n\in N}\sum_j\left|\sum_{i=k}^n(-1)^{i-k}\binom{i}{i-k}a_{ni}\right|<\infty$$

for every $n\in\mathbb{N}$. This yield $\alpha_n\in l_1$. Considering (4.8), we write

$$\sum_k a_{nk}x_k=\sum_k\sum_{i=k}^n(-1)^{i-k}\binom{i}{i-k}a_{ni}\left(y_k-l\right)+l\sum_k\sum_{i=k}^n(-1)^{i-k}\binom{i}{i-k}a_{ni}y_k. \tag{4.13}$$

In this situation, letting $n\to\infty$ in (4.13), we establish that the first term on the right-hand side tends to $\sum_k\alpha_k(y_k-l)$ by (4.6) and(4.11), and the second term tends to $l\alpha$ by (4.11). Taking these facts into account, we deduce from (4.13) as $n\to\infty$ that

$$(Ax)_n\to\sum_k\alpha_k(y_k-l)+l\alpha$$

which shows that $A\in(p_c:c)$.
Conversely, assume that $A\in(p_c:c)$. Then, since the inclusion $c\subset l_\infty$ holds the necessity of (4.3), (4.5) and (4.6) is immediately obtained from

$$\sup_n\sum_k\left|\sum_{i=k}^n(-1)^{i-k}\binom{i}{i-k}a_{ni}\right|<\infty.$$

To prove the necessity of (4.11) consider the sequence $x=b^{(k)}=\left\{b_n^{(k)}\right\}_{n\in\mathbb{N}}$ in p_c. Where

$$b^{(k)}=\{b_n^{(\ k)}\}=\left\{\begin{array}{c}0,\ (0\leq n<k)\\(-1)^{n-k}\binom{n}{n-k},\ (n\geq k\end{array}\right.$$

for every fixed $k\in\mathbb{N}$. Since Ax exists and belongs to c for every $x\in p_c$, one can easily see that

$$Ab^{(k)}=\left\{\sum_{i=k}^n(-1)^{i-k}\binom{i}{i-k}a_{ni}\right\}_{n\in\mathbb{N}}$$

for each $k\in\mathbb{N}$, which yields the necessity of (4.11).
Similarly, by setting $x=e=(1,1,...)$ in (4.8), we obtain

$$Ax=\left\{\sum_k\sum_{i=k}^n(-1)^{i-k}\binom{i}{i-k}a_{ni}\right\}_{n\in\mathbb{N}},$$

which belongs to the space c, and this shows the necessity of (4.12). This step conludes the proof.

□

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A comparison study for solving systems of high-order ordinary differential equations with constants coefficients by exponential Legendre collocation method

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Abstract

In this article we are interested to study the use of the Legendre exponential (EL) collocation method to solve systems of high order linear ordinary differential equations with constant coefficients. The method transforms the system of differential equations and the conditions given by matrix equations with constant coefficients a new system of equations that corresponds to the system of linear algebraic equations which can be solved . Numerical problems are given to illustrate the validity and applicability of the method. For obtaining the approximate solution Maple software is used.

1. Introduction

Legendre polynomials are one of the most important special functions, which are widely used in numerical analysis[6]. The Legendrre polynomials are orthogonal with respect to the weight function 1 on the interval $[-1;1]$ and the recurrence relations is

$$L_0(x) = 1,\, L_1(x) = x,\, L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x),\, n \geqslant 1.$$

One of the applications of Legendre polynomials is the solution of ordinary differential equations with boundary conditions with collocation points. Under a transformation that maps the interval $[-1;1]$ into a semi-infinite domain $[0;\infty)$, we applied spectral methods to solve problems on semi-infinite intervals[3, 1, 8, 9]. In their studies, the basis functions called exponential Legendre (EL) functions $E_n(t)$ are orthogonal in $[-1;1]$. The EL functions are defined as

$$E_0(t) = 1,\, E_1(t) = 1 - 2e^{-t}$$

$$E_{n+1}(t) = \frac{2n+1}{n+1}(1-2e^{-t})E_n(t) - \frac{n}{n+1}E_{n-1}(t),\, n \geqslant 1.$$

Recently, we reported a new operational matrix of derivatives of EL functions for solving ODEs in semi infinite domains. In this paper we applied the matrix of derivative mentioned in [3] to solve systems of ordinary differential equations defined on the whole range by means of collocation method. The organization of this paper is as follows. In Section 2, Preliminaries introduced while in Section 3 Properties of the exponential Legendre (EL) functions are presented. In Section 4, we formulated the fundamental matrix relation based on collocation Points. In Section 5, method of solution is presented. Section 6 contains numerical illustrations.

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2. Preliminaries

In this paper, we considered a system of k linear differential equations with variable coefficients of the mth order in the form

$$\sum_{n=0}^{m}\sum_{j=1}^{k} p_{ij}^{n}(t)y_j^{(n)}(t) = f_i(t),\ i = 1,2,......,k. \tag{2.1}$$

This system can be written as follow

$$\sum_{i=0}^{m} P_i(t)Y^{(i)}(t) = F(t), \tag{2.2}$$

where the $p_{ij}^{n}(t)$ and $f_i(t)$ are well defined functions on the interval $[0,\infty)$ where the matrices $P_i(t)$, $Y^{(i)}(t)$ and $F(t)$ on the form

$$P_i(t) = \begin{pmatrix} p_{11}^{i} & p_{12}^{i} & \cdots & p_{1k}^{i} \\ p_{21}^{i} & p_{22}^{i} & \cdots & p_{2k}^{i} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ p_{k1}^{i} & p_{k2}^{i} & \cdots & p_{kk}^{i} \end{pmatrix}, \quad Y^{(i)}(t) = \begin{pmatrix} y_1^{(i)}(t) \\ y_2^{(i)}(t) \\ \vdots \\ \vdots \\ y_k^{(i)}(t) \end{pmatrix}, \quad F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ \vdots \\ f_k(t) \end{pmatrix}$$

We consider the above system under the mixed condition defined as

$$\sum_{n=0}^{m-1} a_i Y^{(i)}(a) + b_i Y^{(i)}(b) + c_i Y^{(i)}(c) = \lambda,\ 0 \leqslant a \leqslant c \leqslant b < \infty, \tag{2.3}$$

where a_i, b_i, c_i and λ are real valued vectors, and $a \to 0$, $b \to \infty$.

3. The exponential Legendre functions

In this section we list some properties of the EL functions.

3.1. Orthogonality of EL functions

The weight function $\omega(t)$ corresponding to the EL function, such that they are orthogonal in the interval $[0,\infty)$ is given by $2e^{-t}$, with orthogonal condition

$$\int_0^\infty E_n(t)E_m(t)\omega(t)dt = \frac{2}{2n+1}\delta_{nm},$$

where δ_{nm} is the Kronecker function.

3.2. Function expansion in terms of EL functions

A function $f(t)$ is well defined over the interval $[0,\infty)$ and can be expanded in terms of EL functions as

$$f(t) = \sum_{n=0}^{\infty} a_n E_n(t), \tag{3.1}$$

where

$$a_n = \frac{2n+1}{2}\int_0^\infty E_n(t)f(t)\omega(t)dt.$$

If the summation in expression (3.1) is truncated to N where $N < \infty$ it takes the following form

$$f(t) \cong \sum_{n=0}^{N} a_n E_n(t),$$

the $(k)th-$order derivative of $f(t)$ can be written as

$$f^{(k)}(t) \cong \sum_{n=0}^{N} a_n (E_n(t))^{(k)}, \tag{3.2}$$

where $(E_n(t))^{(0)} = E_n(t)$.

3.3. The operational matrix

The new representation of EL functions is presented as follows.
If we use the expression $v(t) = 1 - 2e^{-t}$ in the EL functions, we can express it explicitly in terms of powers of $v(t)$ as

$$E_n(t) = \sum_{k=0}^{[n2]} q_k^{(n)} v^{n-2k}(t),$$

where

$$q_k^{(n)} = (-1)^k \frac{1}{2^n} \binom{n}{k} \binom{2n-2k}{n}, \quad n \geqslant 2k,$$

and $[n2]$ denotes the integer part of the value $\frac{n}{2}$.
From previous relation with simple modification we can define:
if n is even number

$$E_{2l}(t) = \sum_{j=0}^{l} (-1)^{l-j} \frac{1}{2^{2l}} \binom{2l}{l-j} \binom{2l+2j}{2l} v^{2j}(t),$$

if n is odd number

$$E_{2l+1}(t) = \sum_{j=0}^{l} (-1)^{l-j} \frac{1}{2^{2l+1}} \binom{2l+1}{l-j} \binom{2l+2j+2}{2l+1} v^{2j+1}(t).$$

Form above relations we can deduce general matrix form of EL functions as

$$E(t) = V(t)L^T, \tag{3.3}$$

where $E(t)$ and $V(t)$ are two matrices of the form:

$$E(t) = \begin{bmatrix} E_0(t) & E_1(t) & \cdots\cdots & E_N(t) \end{bmatrix},$$

$$V(t) = \begin{bmatrix} v^0(t) & v(t) & \cdots\cdots & v^N(t) \end{bmatrix},$$

and $v^0(t) = 1, \quad v^1(t) = 1 - 2e^{-t}, \quad v^2(t) = (1-2e^{-t})^2, \cdots\cdots, v^N(t) = (1-2e^{-t})^N,$
and L is a matrix given by

$$\mathrm{L}= \begin{pmatrix} \binom{0}{0}\binom{0}{0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2}\binom{1}{0}\binom{2}{1} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{2^2}\binom{2}{1}\binom{2}{2} & 0 & \frac{1}{2^2}\binom{2}{0}\binom{4}{2} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{2^3}\binom{3}{1}\binom{4}{3} & 0 & \frac{1}{2^3}\binom{3}{0}\binom{6}{3} & \cdots & 0 & 0 \\ & & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & & & & \\ (-1)^l \frac{1}{2^{2l}}\binom{2l}{l} & 0 & (-1)^{l-1}\frac{1}{2^{2l}}\binom{2l}{l-1}\binom{2l+2}{2l} & 0 & \cdots & \frac{1}{2^{2l}}\binom{4l}{2l} & 0 \\ 0 & (-1)^l \frac{1}{2^{2l+1}}\binom{2l+1}{l}\binom{2l+2}{2l+1} & 0 & (-1)^{l-1}\frac{1}{2^{2l}}\binom{2l}{l-1}\binom{2l+4}{2l} & \cdots & 0 & \frac{1}{2^{2l+1}}\binom{4l+2}{2l+1} \end{pmatrix}$$

Now, from (3.3) we can obtain the k^{th} derivative of matrix $E(t)$ as:

$$\begin{gathered} E^{(0)}(t) = V(t)L^T \\ E^{(1)}(t) = V^{(1)}(t)L^T \\ E^{(2)}(t) = V^{(2)}(t)L^T \\ \vdots \\ \vdots \end{gathered}$$

then, by induction the k^{th}-order derivative of the matrix $E(t)$ defined as:

$$E^{(k)}(t) = V^{(k)}(t)L^T, \tag{3.4}$$

the equation (3.4) represents the new operational matrix of derivatives of the EL functions.

4. Fundamental matrix relations

Let us define the collocation points, so that $0 \leqslant t_s < \infty$, as

$$t_s = \frac{1+cos(\frac{s\pi}{N})}{1-cos(\frac{s\pi}{N})}$$

and at the boundaries$(s=0, \quad s=N)$, $t_0 \to \infty, \quad t_N \to 0$, since the EL functions are convergent at both boundaries 0 and ∞, namely their values are ± 1, the appearance of infinity in the collocation points does not cause a loss or divergence in the method.
We assume that the solution $y_i(t)$ of (2.1) can be expressed in the form (3.2), which is a truncated Legendre series in terms of EL functions. Then $y_i(t)$ and its derivative $y_i^{(j)}(t)$ can be written in the matrix form as

$$y_i(t) = E(t)A_i,$$

and

$$y_i^{(j)}(t) = E^{(j)}(t)A_i, \quad i = 1,2,......,k, \quad j = 0,1,2,.......,m. \tag{4.1}$$

where

$$A_i = [a_{i0}, a_{i1}........., a_{iN}]^T.$$

By substituting relation (3.4) into (4.1), we obtain

$$y_i^{(j)}(t) = V^{(j)}(t)L^T A_i, \quad j = 0,1,2,.......,m.$$

So, the matrix $y^{(i)}(t)$ defined as a column matrix that is formed of i^{th} derivatives of unknown functions, can be expressed by

$$y^{(i)}(t) = V^{(i)}(t)L^T A \tag{4.2}$$

where $V^{(i)}(t)$, L^T are two size matrix $k \times k$,

$$V^{(i)}(t) = \begin{pmatrix} V^{(i)}(t) & 0 & \cdots & 0 \\ 0 & V^{(i)}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V^{(i)}(t) \end{pmatrix}, L^T = \begin{pmatrix} L^T & 0 & \cdots & 0 \\ 0 & L^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L^T \end{pmatrix}, \quad A = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_k \end{pmatrix}_{k\times 1}.$$

By putting the collocation points t_s in (4.2), we have the system

$$y^{(i)}(t_s) = V^{(i)}(t_s)L^T A, \tag{4.3}$$

the system (4.3) can be written in the matrix form as

$$Y^{(i)} = \widetilde{V}^{(i)} L^T A$$

where

$$\widetilde{V}^{(i)} = \begin{pmatrix} V^{(i)}(t_0) \\ V^{(i)}(t_1) \\ \vdots \\ V^{(i)}(t_N) \end{pmatrix},$$

then, the equation (2.2) becomes as follows

$$\sum_{i=0}^{m} \widetilde{P}_i \widetilde{V}^{(i)} L^T A = F. \tag{4.4}$$

Next, we can obtain the corresponding matrices form for the conditions by using the relation (2.3), we have the fundamental matrix equation corresponding to the mixed conditions (2.3) as

$$\sum_{i=0}^{m-1} \left[a_i V^{(i)}(a) + b_i V^{(i)}(b) + c_i V^{(i)}(c) \right] L^T A = \alpha. \tag{4.5}$$

5. Collocation method

The fundamental matrix (4.4) for equation (2.1) corresponding to system of $(N+1)$ algebraic equations for the $(N+1)$ unknown coefficients $a_{0,}, a_1, ..., a_N$. We can write equation (4.4) as

$$HA = M. \tag{5.1}$$

So that $H = (h_{d,t}) = \sum_{i=0}^{m} \widetilde{P}_i \widetilde{V}^{(i)} L^T, \quad d,t = 1,2,....., k(N+1).$
We can obtain the matrix form for the mixed conditions by means of (4.5)

$$RA = [\alpha_i], \tag{5.2}$$

and

$$R = \sum_{i=0}^{m-1} \left[a_i V^{(i)}(a) + b_i V^{(i)}(b) + c_i V^{(i)}(c) \right] L^T.$$

To obtain the solution of Eq(2.1) under the conditions (2.3) we replace the rows of matrices(5.2) by the last m rows of the matrix (5.1) Then, we have the required augmented matrix as

$$[\widetilde{R}, \widetilde{F}] = \begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,k(N+1)} & ; & f_1(t_0) \\ h_{2,1} & h_{2,2} & \cdots & h_{2,k(N+1)} & ; & f_2(t_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{k,1} & h_{k,2} & \cdots & h_{k,k(N+1)} & ; & f_k(t_0) \\ h_{k+1,1} & h_{k+1,2} & \cdots & h_{k+1,k(N+1)} & ; & f_1(t_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{k(N-m+1),1} & h_{k(N-m+1),2} & \cdots & h_{k(N-m+1),k(N+1)} & ; & f_k(t_{N-m}) \\ r_{1,1} & r_{1,2} & \cdots & r_{1,k(N+1)} & ; & \alpha_1 \\ r_{2,1} & r_{2,2} & \cdots & r_{2,k(N+1)} & ; & \alpha_2 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ r_{mk,1} & r_{mk,2} & \cdots & r_{mk,k(N+1)} & ; & \alpha_{mk} \end{pmatrix}$$

or the corresponding matrix equation

$$\widetilde{R}A = \widetilde{F},$$

then we can write

$$A = (\widetilde{R})^{-1} \widetilde{F}.$$

6. Illustrative examples

In this section, we demonstrate the effectiveness of the proposed Legendre exponential function method with numerical examples.

Example 6.1. *Consider the system*

$$\begin{cases} x' - x - 8y = -12e^{-2t} \\ y' - 2x - y = \frac{15}{4} - 9e^{-t} - \frac{9}{2}e^{-2t} \end{cases}, \quad 0 \leqslant t < \infty \tag{6.1}$$

with the conditions $x(0) = 4$, $y(\infty) = \frac{1}{4}$, where the exact solution

$$\begin{cases} x(t) = -2 + 6e^{-t} \\ y(t) = \frac{1}{4} - \frac{3}{2}e^{-t} + \frac{3}{2}e^{-2t} \end{cases}$$

for this example we have

$$k = 2, \quad m = 1, \quad f_1(t) = -12e^{-2t}, \quad f_2(t) = -\frac{9}{2}e^{-2t} - 9e^{-t} + \frac{15}{4},$$

and

$$P_0 = \begin{pmatrix} -1 & -8 \\ -2 & -1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, for $N = 2$, the collocation points are $t_0 \to \infty$, $t_1 \to 1$, $t_2 \to 0$, and the fundamental matrix is

$$\left[\widetilde{P}_0 \widetilde{V}^{(0)} + \widetilde{P}_1 \widetilde{V}^{(1)} \right] L^T A = F,$$

where $\widetilde{P}_0$, $\widetilde{P}_1$, $\widetilde{V}^{(0)}$ and $\widetilde{V}^{(1)}$ are matrices of order (6×6) given as:

$$\widetilde{P}_0=\begin{pmatrix}-1&-8&0&0&0&0\\-2&-1&0&0&0&0\\0&0&-1&-8&0&0\\0&0&-2&-1&0&0\\0&0&0&0&-1&-8\\0&0&0&0&-2&-1\end{pmatrix},\ \widetilde{P}_1=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\ L^T=\begin{pmatrix}1&0&-\frac{1}{2}&0&\frac{3}{8}&0\\0&1&0&-\frac{3}{2}&0&\frac{15}{8}\\0&0&\frac{3}{2}&0&-\frac{15}{4}&0\\0&0&0&\frac{5}{2}&0&-\frac{35}{4}\\0&0&0&0&\frac{35}{8}&0\\0&0&0&0&0&\frac{63}{8}\end{pmatrix}$$

$$\widetilde{V}^{(0)}=\begin{pmatrix}1&1&1&0&0&0\\0&0&0&1&1&1\\1&(1-2e^{-1})&(1-2e^{-1})^2&0&0&0\\0&0&0&1&(1-2e^{-1})&(1-2e^{-1})^2\\1&-1&1&0&0&0\\0&0&0&1&-1&1\end{pmatrix},$$

$$\widetilde{V}^{(1)}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&(2e^{-1})&(4e^{-1}-8e^{-2})&0&0&0\\0&0&0&0&(2e^{-1})&(4e^{-1}-8e^{-2})\\0&2&-4&0&0&0\\0&0&0&0&2&-4\end{pmatrix},$$

and $F=\begin{bmatrix}0 & \frac{15}{4} & -12e^{-2} & -\frac{9}{2}e^{-2}-9e^{-1}+\frac{15}{4} & -12 & -\frac{39}{4}\end{bmatrix}$, *and the augmented matrix for the conditions with* $N=2$ *is* $\begin{bmatrix}1 & -1 & 1 & 0 & 0 & 0 & ; 4\end{bmatrix}$ *for the first condition* $x(0)=4$, *and for the other condition* $y(\infty)=\frac{1}{4}$, *the augmented matrix is* $\begin{bmatrix}0 & 0 & 0 & 1 & 1 & 1 & ; \frac{1}{4}\end{bmatrix}$.
After the augmented matrices of the system and condition are computed, we obtain the solution

$$A=\begin{bmatrix}1 & -3 & 0 & 0 & 0 & \frac{1}{4}\end{bmatrix}$$

therefore, we find the approximate solutions as

$$\begin{cases} x(t)=E_0-3E_1=1-3(1-2e^{-t})=1-3+6e^{-t}=-2+6e^{-t} \\ y(t)=0E_0+0E_1+\frac{1}{4}E_2=\frac{1}{4}\left[\frac{3}{2}(1-2e^{-t})^2-\frac{1}{2}\right]=\frac{1}{4}-\frac{3}{2}e^{-t}+\frac{3}{2}e^{-2t}\end{cases}$$

Example 6.2. *Consider the second order system of two equations*

$$\begin{cases} y''+x''-x=90e^{-3t}-24e^{-2t}+\frac{23}{4}e^{-t}-3 \\ y''-2x'=90e^{-3t}-108e^{-2t}+\frac{119}{4}e^{-t}\end{cases},\quad 0\leqslant t<\infty$$

with the conditions $x(0)=3,\quad y(\infty)=3,\quad x(\infty)=\frac{1}{2},\quad y(0)=\frac{5}{4}$.
Where the exact solutions $x(t)=3-12e^{-t}+12e^{-2t}$ *and* $y(t)=\frac{1}{2}+\frac{23}{4}e^{-t}-15e^{-2t}+10e^{-3t}$.
In this system, we have

$$k=2,\quad m=2,\quad f_1(t)=90e^{-3t}-24e^{-2t}+\frac{23}{4}e^{-t}-3,\quad f_2(t)=90e^{-3t}-108e^{-2t}+\frac{119}{4}e^{-t}$$

and

$$P_0=\begin{pmatrix}-1&0\\0&0\end{pmatrix},\quad P_1=\begin{pmatrix}0&0\\2&0\end{pmatrix},\quad P_2=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

Then, for $N=4$, *the collocation points are* $t_0\to\infty$, $t_1\to 3+2\sqrt{2}$, $t_2\to 1$, $t_3\to 3-2\sqrt{2}$, $t_4\to 0$, *and the fundamental matrix is*

$$\left[\widetilde{P}_0\widetilde{V}^{(0)}+\widetilde{P}_1\widetilde{V}^{(1)}+\widetilde{P}_2\widetilde{V}^{(2)}\right]L^TA=F,$$

where $\widetilde{P}_0$, $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{V}^{(0)}$, $\widetilde{V}^{(1)}$ *and* $\widetilde{V}^{(2)}$ *are matrices of order* (10×10)

$$\widetilde{P}_0=\begin{pmatrix}-1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&-1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&-1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&-1&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&-1&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix},\ \widetilde{P}_1=\begin{pmatrix}0&0&0&0&0&0&0&0&0&0\\2&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&2&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&2&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&2&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&2&0\end{pmatrix},$$

$$\widetilde{P}_2=\begin{pmatrix}1&1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&1&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&1&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&0&0&1\end{pmatrix},$$

$$L^T=\begin{pmatrix}1&0&-\frac{1}{2}&0&\frac{3}{8}&0&-\frac{15}{6}&0&\frac{35}{128}&0\\0&1&0&-\frac{3}{2}&0&\frac{15}{8}&0&-\frac{35}{16}&0&\frac{315}{128}\\0&0&\frac{3}{2}&0&-\frac{15}{4}&0&\frac{105}{16}&0&-\frac{1260}{128}&0\\0&0&0&\frac{5}{2}&0&-\frac{35}{4}&0&\frac{315}{16}&0&-\frac{4620}{128}\\0&0&0&0&\frac{35}{8}&0&-\frac{35}{16}&0&\frac{6930}{128}&0\\0&0&0&0&0&\frac{63}{8}&0&-\frac{693}{16}&0&\frac{1801}{128}\\0&0&0&0&0&0&\frac{231}{16}&0&-\frac{12012}{128}&0\\0&0&0&0&0&0&0&\frac{429}{16}&0&-\frac{25740}{128}\\0&0&0&0&0&0&0&0&\frac{6435}{128}&0\\0&0&0&0&0&0&0&0&0&\frac{12155}{128}\end{pmatrix}$$

and the augmented matrix for the conditions with $N=4$ is $\begin{bmatrix}1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0;\ 3\end{bmatrix}$ for the first condition $x(0)=3$, and $y(\infty)=3$, the augmented matrix is $\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1;\ 3\end{bmatrix}$, and for the other condition $x(\infty)=\frac{1}{2}$ we have $\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0;\ \frac{1}{2}\end{bmatrix}$ and $y(0)=\frac{5}{4}$ we have the augmented matrix $\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1;\ \frac{5}{4}\end{bmatrix}$. After the augmented matrices of the system and condition are computed, we obtain the solution

$$A=\begin{bmatrix}1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0\end{bmatrix}.$$

Finally, we find the approximate solutions as

$$\begin{cases} x(t)=E_0+2E_2=1+2\left[\frac{3}{2}(1-2e^{-t})^2-\frac{1}{2}\right]=3(1-2e^{-t})^2 \\ y(t)=E_1-\frac{1}{2}E_3=\frac{7}{4}-\frac{7}{4}e^{-t}-\frac{5}{4}(1-2e^{-t})^3 \end{cases}.$$

7. Conclusion

Systems of high order linear differential equations are generally difficult to solve analytically under mixture conditions. In many cases, obtaining approximate solutions is necessary especially if the problem is defined in semi-infinite domain. For this reason, the Legendre exponential collocation method is proposed to obtain an approximate solution of high order linear differential equation. The new definition of EL functions is studied and introduced to solve the system of high order linear ordinary differential equations with constant coefficients. In addition, an interesting feature of this method is to find the exact solutions if the system has an exact solution that is a polynomial exponential function. Examples of problems are used to demonstrate the applicability and effectiveness of the proposed technique.

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Transverse vibration of nonuniform Euler-Bernoulli beams on bounded time scales

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Abstract

In this article, we consider Euler-Bernoulli equation of transverse vibrations of nonuniform beams on bounded time scales $\mathbb{T}$. We will give a description of all maximal dissipative, maximal accretive, self adjoint and other extensions of such operators.

1. Introduction

The theory of symmetric extensions of a symmetric operator in a Hilbert space developed by J. von Neumann [27]. Especially, it plays a central role in spectral problems associated with formally self-adjoint linear differential operators. The problem on the description of all self adjoint extensions of a symmetric operator in terms of abstract boundary conditions was given by Calkin [25]. Later, Rofe- Beketov [28] described self adjoint extensions of a symmetric operator in terms of abstract boundary conditions with aid of linear relations. Bruk [24] and Kochubei [12] are introduced the notion of a space of boundary values. They described all maximal dissipative, accretive, self adjoint extensions of symmetric operators. This problem has been investigated by many mathematicians (see [13]-[20]). For a more comprehensive discussion of extension theory of symmetric operators, the reader is referred to [26].

The theory of time scales unifies continuous and discrete analysis. It was introduced by Hilger (see [1]). Recently, it has received a lot of attention. The study of dynamic equations on time scales has several important applications, e.g., in the study of heat transfer, insect population models, epidemic models stock market, and neural networks (see [1]-[5]).

On the other hand, transverse vibration of nonuniform beams is one of the important problems in mechanical and civil engineering. It has led to several applications in modern engineering, e.g., turbine blade, helicopter blades, satellites structure, even robotic arms etc. It has been studied by many investigators (see [29]-[43]).

In this article, we consider Euler-Bernoulli dynamic equation of transverse vibrations of nonuniform beans on bounded time scales. A space of boundary value is constructed for this operator. It is given a description all maximal dissipative, accretive, self adjoint and other extensions of such operators in terms of boundary conditions.

2. Preliminaries

Now, we recall some necessary fundamental concepts of time scales, and we refer to [8], [9] for more detail.

Definition 2.1. *Let $\mathbb{T}$ be a time scale, i.e., a non-empty closed subset of real numbers $\mathbb{R}$. The forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by*

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}\ \text{where}\ t\in\mathbb{T}$$

and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ is defined by

$$\rho(t)=\sup\{s\in\mathbb{T}:s<t\}\ \text{where}\ t\in\mathbb{T}.$$

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It is convenient to have graininess operators $\mu_\sigma : \mathbb{T} \to [0,\infty)$ *and* $\mu_\rho : \mathbb{T} \to (-\infty, 0]$ *defined by* $\mu_\sigma(t) = \sigma(t) - t$ *and* $\mu_\rho(t) = \rho(t) - t$, *respectively. A point* $t \in \mathbb{T}$ *is left scattered if* $\mu_\rho(t) \neq 0$ *and left dense if* $\mu_\rho(t) = 0$. *A point* $t \in \mathbb{T}$ *is right scattered if* $\mu_\sigma(t) \neq 0$ *and right dense if* $\mu_\sigma(t) = 0$. *We introduce the sets* $\mathbb{T}^k$, $\mathbb{T}_k$, $\mathbb{T}^*$ *which are derived from the time scale* $\mathbb{T}$ *as follows. If* $\mathbb{T}$ *has a left scattered maximum* t_1, *then* $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, *otherwise* $\mathbb{T}^k = \mathbb{T}$. *If* $\mathbb{T}$ *has a right scattered minimum* t_2, *then* $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, *otherwise* $\mathbb{T}_k = \mathbb{T}$. *Finally,* $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

Definition 2.2. *A function* f *on* $\mathbb{T}$ *is said to be* Δ*-differentiable at some point* $t \in \mathbb{T}$ *if there is a number* $f^\Delta(t)$ *such that for every* $\varepsilon > 0$ *there is a neighborhood* $U \subset \mathbb{T}$ *of* t *such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{where } s \in U.$$

Analogously one may define the notion of ∇*-differentiability of some function using the backward jump* ρ. *One can show (see [11])*

$$f^\Delta(t) = f^\nabla(\sigma(t)), \qquad f^\nabla(t) = f^\Delta(\rho(t))$$

for continuously differentiable functions.

Example 2.3. *If* $\mathbb{T} = \mathbb{R}$, *then we have*

$$\sigma(t) = t, \; f^\Delta(t) = f'(t).$$

If $\mathbb{T} = \mathbb{Z}$, *then we have*

$$\sigma(t) = t + 1, \; f^\Delta(t) = \Delta f(t) = f(t+1) - f(t).$$

If $\mathbb{T} = q^{\mathbb{N}_0} = \left\{q^k : q > 1, \; k \in \mathbb{N}_0\right\}$, *then we have*

$$\sigma(t) = qt, \; f^\Delta(t) = \frac{f(qt) - f(t)}{qt - t}.$$

Definition 2.4. *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a function, and* $a, b \in \mathbb{T}$. *If there exists a function* $F : \mathbb{T} \to \mathbb{R}$ *such that* $F^\Delta(t) = f(t)$ *for all* $t \in \mathbb{T}^k$, *then* F *is a* Δ*-antiderivative of* f. *In this case the integral is given by the formula*

$$\int_a^b f(t)\Delta t = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

Analogously one may define the notion of ∇*-antiderivative of some function.*

Let $L^2_\Delta(\mathbb{T}^*)$ be the space of all functions defined on $\mathbb{T}^*$ such that

$$\|f\| := \left(\int_a^b |f(t)|^2 \Delta t\right)^{1/2} < \infty.$$

The space $L^2_\Delta(\mathbb{T}^*)$ is a Hilbert space with the inner product (see [23])

$$(f, g) := \int_a^b f(t)\overline{g(t)}\Delta t, \;\; f, g \in L^2_\Delta(\mathbb{T}^*).$$

3. Main results

Let us consider Euler-Bernoulli dynamic expression of transverse vibrations of nonuniform beams

$$l(y) := \left(EI^*(t)y^{\Delta\nabla}\right)^{\nabla\Delta}(t) - \rho_0 w^2 A^*(t)y(t), \; t \in \mathbb{T}_1 = \mathbb{T}^* \cap (a,b), a < b, \tag{3.1}$$

where y is the transverse displacement, E, ρ_0 and w are Young modulus, mass density, and natural frequency, respectively, $A^*(t)$ and $I^*(t)$ are the area and moment of inertia of current cross-section, respectively; t is the current longitudinal coordinate of the beam, and a and b are the coordinates of the fixed end and the free end of the beam, respectively.

For simplicity of notation, we have

$$\begin{aligned} y^{[0]} &= y, \\ y^{[1]} &= y^\Delta, \\ y^{[2]} &= EI^*(t)y^{\Delta\nabla}, \\ y^{[3]} &= -\left(y^{[2]}\right)^\nabla, \\ y^{[4]} &= -\rho_0 w^2 A^*(t)y - \left(y^{[3]}\right)^\Delta. \end{aligned}$$

Let $y_i, 1 \leq i \leq 4$, be solutions of Eq. (3.1). The Wronskian of y_1, y_2, y_3 and y_4 is defined to be (see [6])

$$W(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1^{[1]} & y_2^{[1]} & y_3^{[1]} & y_4^{[1]} \\ y_1^{[2]} & y_2^{[2]} & y_3^{[2]} & y_4^{[2]} \\ y_1^{[3]} & y_2^{[3]} & y_3^{[3]} & y_4^{[3]} \end{vmatrix}.$$

We will denote by $Dom_{\max}$ the set of all functions $y(t)$ in $L^2_\Delta(\mathbb{T}_1)$ such that first three Δ derivatives are locally Δ absolutely continuous in $\mathbb{T}_1$, and $l(y) \in L^2_\Delta(\mathbb{T}_1)$. We define the maximal operator $L_{\max}$ on $Dom_{\max}$ by the equality $L_{\max} y = ly$.
For every $y, z \in Dom_{\max}$, we have Green's formula

$$\int_a^b (ly)(t)\overline{z(t)}\Delta t - \int_a^b y(t)\overline{(lz)(t)}\Delta t = [y,z]_b - [y,z]_a, \tag{3.2}$$

where $[y,z]_t := y^{[0]}(t)\bar{z}^{[3]}(t) - y^{[3]}(t)\bar{z}^{[0]}(t) + y^{[1]}(t)\bar{z}^{[2]}(t) - y^{[2]}(t)\bar{z}^{[1]}(t)$ (see [6]).
Let $Dom_{\min}$ denote the linear set of all vectors $y \in Dom_{\max}$ satisfying the conditions

$$\begin{aligned} y^{[0]}(a) &= y^{[1]}(a) = y^{[2]}(a) = y^{[3]}(a) = 0, \\ y^{[0]}(b) &= y^{[1]}(b) = y^{[2]}(b) = y^{[3]}(b) = 0. \end{aligned}$$

If we restrict the operator $L_{\max}$ to the set $Dom_{\min}$, then we obtain the minimal operator $L_{\min}$. It is clear that $L^*_{\min} = L_{\max}$, and $L_{\min}$ is a closed symmetric operator (see [6]). Now we recall the following definitions.

Definition 3.1. *A linear operator M (with dense domain $D(M)$) acting on some Hilbert space H is called dissipative (accumulative) if $\Im(Mf,f) \geq 0$ ($\Im(Mf,f) \leq 0$) for all $f \in D(M)$ and maximal dissipative (maximal accumulative) if it does not have a proper dissipative (accumulative) extension (see [14], [16]-[19]).*

Definition 3.2. *A triplet $(\mathbb{H}, \Phi_1, \Phi_2)$ is called a space of boundary values of a closed symmetric operator M on a Hilbert space H if Φ_1 and Φ_2 are linear maps from $D(M^*)$ to H, with equal deficiency numbers and such that:*

i) For every $f, g \in D(M^*)$ we have

$$(M^* f, g)_H - (f, M^* g)_H = (\Phi_1 f, \Phi_2 g)_{\mathbb{H}} - (\Phi_2 f, \Phi_1 g)_{\mathbb{H}};$$

ii) For any $F_1, F_2 \in H$ there is a vector $f \in D(M^*)$ such that $\Phi_1 f = F_1$ and $\Phi_2 f = F_2$ (see [10]).
Let's define by Φ_1, Φ_2 the linear maps from D to $\mathbb{C}^4$ by the formula

$$\Phi_1 y = \begin{pmatrix} -y^{[0]}(a) \\ -y^{[1]}(a) \\ y^{[0]}(b) \\ y^{[1]}(b) \end{pmatrix}, \ \Phi_2 y = \begin{pmatrix} y^{[3]}(a) \\ y^{[2]}(a) \\ y^{[3]}(b) \\ y^{[2]}(b) \end{pmatrix}. \tag{3.3}$$

Now we will state and prove a theorem.

Theorem 3.3. *The triple $(\mathbb{C}^4, \Phi_1, \Phi_2)$ defined by (3.3) is a boundary spaces of the operator $L_{\min}$.*

Proof. For every $y, z \in Dom_{\max}$, we have

$$\begin{aligned} (\Phi_1 y, \Phi_2 z)_{\mathbb{C}^4} - (\Phi_1 z, \Phi_2 y)_{\mathbb{C}^4} &= -y^{[0]}(a)\bar{z}^{[3]}(a) - y^{[1]}(a)\bar{z}^{[2]}(a) \\ &\quad + y^{[0]}(b)\bar{z}^{[3]}(b) + y^{[1]}(b)\bar{z}^{[2]}(b) \\ &\quad - (-\bar{z}^{[0]}(a)y^{[3]}(a) - \bar{z}^{[1]}(a)y^{[2]}(a)) \\ &\quad - (\bar{z}^{[0]}(b)y^{[3]}(b) + \bar{z}^{[1]}(b)y^{[2]}(b)) \\ &= y^{[0]}(b)\bar{z}^{[3]}(b) - \bar{z}^{[0]}(b)y^{[3]}(b) \\ &\quad + y^{[1]}(b)\bar{z}^{[2]}(b) - \bar{z}^{[1]}(b)y^{[2]}(b) \\ &\quad + \bar{z}^{[0]}(a)y^{[3]}(a) - y^{[0]}(a)\bar{z}^{[3]}(a) \\ &\quad - y^{[1]}(a)\bar{z}^{[2]}(a) + \bar{z}^{[1]}(a)y^{[2]}(a) \\ &= [y,z](b) - [y,z](a). \end{aligned}$$

□

From Green's formula (3.2), we obtain the following equation

$$(\Phi_1 y, \Phi_2 z)_{\mathbb{C}^4} - (\Phi_1 z, \Phi_2 y)_{\mathbb{C}^4} = [y,z](b) - [y,z](a) = (L_{\max} y, z) - (y, L_{\max} z).$$

So, we proved the first requirement of the definition of a space of boundary values.

Now, we will prove the second requirement of the definition of a space of boundary values. Let $u=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}, v=\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix}\in\mathbb{C}^4$. Then the vector-valued function

$$y(t)=\alpha_1(t)u_1+\alpha_2(t)v_1+\alpha_3(t)u_2+\alpha_4(t)v_2+\alpha_5(t)u_3+\alpha_6(t)v_3+\alpha_7(t)u_4+\alpha_8(t)v_4,$$

where $\alpha_i(t)\in H\ (i=1,...,8)$ satisfy the conditions

$$\begin{array}{cccc}
\alpha_1^{[0]}(a)=1 & \alpha_1^{[1]}(a)=0 & \alpha_1^{[2]}(a)=0 & \alpha_1^{[3]}(a)=0\\
\alpha_2^{[0]}(a)=0 & \alpha_2^{[1]}(a)=0 & \alpha_2^{[2]}(a)=0 & \alpha_2^{[3]}(a)=1\\
\alpha_3^{[0]}(a)=0 & \alpha_3^{[1]}(a)=-1 & \alpha_3^{[2]}(a)=0 & \alpha_3^{[3]}(a)=0\\
\alpha_4^{[0]}(a)=0 & \alpha_4^{[1]}(a)=0 & \alpha_4^{[2]}(a)=1 & \alpha_4^{[3]}(a)=0\\
\alpha_5^{[0]}(a)=0 & \alpha_5^{[1]}(a)=0 & \alpha_5^{[2]}(a)=0 & \alpha_5^{[3]}(a)=0\\
\alpha_6^{[0]}(a)=0 & \alpha_6^{[1]}(a)=0 & \alpha_6^{[2]}(a)=0 & \alpha_6^{[3]}(a)=0\\
\alpha_7^{[0]}(a)=0 & \alpha_7^{[1]}(a)=0 & \alpha_7^{[2]}(a)=0 & \alpha_7^{[3]}(a)=0\\
\alpha_8^{[0]}(a)=0 & \alpha_8^{[1]}(a)=0 & \alpha_8^{[2]}(a)=0 & \alpha_8^{[3]}(a)=0
\end{array}$$

and

$$\begin{array}{cccc}
\alpha_1^{[0]}(b)=0 & \alpha_1^{[1]}(b)=0 & \alpha_1^{[2]}(b)=0 & \alpha_1^{[3]}(b)=0\\
\alpha_2^{[0]}(b)=0 & \alpha_2^{[1]}(b)=0 & \alpha_2^{[2]}(b)=0 & \alpha_2^{[3]}(b)=0\\
\alpha_3^{[0]}(b)=0 & \alpha_3^{[1]}(b)=0 & \alpha_3^{[2]}(b)=0 & \alpha_3^{[3]}(b)=0\\
\alpha_4^{[0]}(b)=0 & \alpha_4^{[1]}(b)=0 & \alpha_4^{[2]}(b)=0 & \alpha_4^{[3]}(b)=0\\
\alpha_5^{[0]}(b)=-1 & \alpha_5^{[1]}(b)=0 & \alpha_5^{[2]}(b)=0 & \alpha_5^{[3]}(b)=0\\
\alpha_6^{[0]}(b)=0 & \alpha_6^{[1]}(b)=0 & \alpha_6^{[2]}(b)=0 & \alpha_6^{[3]}(b)=1\\
\alpha_7^{[0]}(b)=0 & \alpha_7^{[1]}(b)=1 & \alpha_7^{[2]}(b)=0 & \alpha_7^{[3]}(b)=0\\
\alpha_8^{[0]}(b)=0 & \alpha_8^{[1]}(b)=0 & \alpha_8^{[2]}(b)=1 & \alpha_8^{[3]}(b)=0
\end{array}$$

belongs to the set $Dom_{\max}$ and $\Phi_1 y=u,\ \Phi_2 y=v$

Corollary 3.4. *For any contraction T in $\mathbb{C}^4$ the restriction of the operator $L_{\max}$ to the set of functions $y\in\mathrm{Dom}_{\max}$ satisfying either*

$$(T-I)\Phi_1 y+i(T+I)\Phi_2 y=0 \tag{3.4}$$

or

$$(T-I)\Phi_1 y-i(T+I)\Phi_2 y=0 \tag{3.5}$$

is respectively the maximal dissipative and accretive extension of the operator $L_{\min}$. Conversely, every maximal dissipative (accretive) extension of the operator $L_{\min}$ is the restriction of $L_{\max}$ to the set of functions $y\in\mathrm{Dom}_{\max}$ satisfying (3.4) ((3.5)), and the extension uniquely determines the contraction T. If T is an isometry in $\mathbb{C}^4$, then the conditions (3.4) ((3.5)) describe the maximal symmetric extensions of $L_{\min}$ in $L_\Delta^2(\mathbb{T}_1)$.

The general form of dissipative and accretive extensions of an operator L is given by the conditions

$$\begin{aligned}
T(\Phi_1 y+i\Phi_2 y) &= \Phi_1 y-i\Phi_2 y,\ \Phi_1 y+i\Phi_2 y\in Dom(T),\\
T(\Phi_1 y-i\Phi_2 y) &= \Phi_1 y+i\Phi_2 y,\ \Phi_1 y-i\Phi_2 y\in Dom(T),
\end{aligned} \tag{3.6}$$

respectively, where T is a linear operator with

$$\|Tf\|\le\|f\|,\ f\in Dom(T).$$

4. Conclusion

In this paper, we have considered Euler-Bernoulli equation of transverse vibrations of nonuniform beams on bounded time scales $\mathbb{T}$.. In this context, we have constructed a space of boundary values of the minimal operator and described all maximal dissipative, maximal accretive, self-adjoint extensions of of such operators.

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Differential bubordinations and argument inequalities for certain multivalent functions defined by convolution structure

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Abstract

The main object of the present paper is to investigate certain interesting argument inequalities and differential subordinations properties of multivalent functions associated with a linear operator $D^n_{\lambda,p}(f*g)(z)$ defined by Hadamard product

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1,2,....\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. If f and g are analytic in U, we say that f is subordinate to g, written symbolically as follows:

$$f \prec g \text{ or } f(z) \prec g(z) ,$$

if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function $g(z)$ is univalent in U, then we have the following equivalence (cf., e.g., [4], [13]; see also [14, p. 4]:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f(z) \in A(p)$ given by (1.1), and $g(z) \in A(p)$ defined by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \quad (p \in \mathbb{N}), \tag{1.2}$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g*f)(z) \quad (p \in \mathbb{N}; z \in U). \tag{1.3}$$

For functions $f,g \in A(p)$, we define the following differential operator:

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$$D^0_{\lambda,p}(f*g)(z) = (f*g)(z), \tag{1.4}$$

$$D^1_{\lambda,p}(f*g)(z) = D_{\lambda,p}(f*g)(z) = (1-\lambda)(f*g)(z) + \frac{\lambda z}{p}(f*g)'(z)\ (\lambda \geq 0), \tag{1.5}$$

and (in general)

$$\begin{aligned} D^n_{\lambda,p}(f*g)(z) &= D_{\lambda,p}(D^{n-1}_{\lambda,p}(f*g)(z)) \\ &= z^p + \sum_{k=1}^{\infty}\left(\frac{p+\lambda k}{p}\right)^n a_{k+p}b_{k+p}z^{k+p} \\ (\lambda &\geq 0; p\in\mathbb{N}; n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}). \end{aligned} \tag{1.6}$$

From (1.6) it is easy to verify that

$$\frac{\lambda}{p}z(D^n_{\lambda,p}(f*g)(z))' = D^{n+1}_{\lambda,p}(f*g)(z) - (1-\lambda)D^n_{\lambda,p}(f*g)(z)\ (\lambda>0; n\in\mathbb{N}_0). \tag{1.7}$$

The operator $D^n_{\lambda,p}(f*g)(z)$, when $p=1$, was introduced and studied by Aouf and Mostafa [3].

We observe that the linear operator $D^n_{\lambda,p}(f*g)(z)$ reduces to several interesting operators for different choices of n,λ,p and the function $g(z)$:
(i) For $\lambda=1$ and $g(z)=\frac{z^p}{1-z}$ (or $b_{k+p}=1$), $D^n_{1,p}(f*g)(z)=D^n_p f(z)$, where D^n_p is the p-valent Salagean operator introduced and studied by Kamali and Orhan [9], Orhan and Kiziltunc [17] (see also [2]);
(ii) For $g(z)=\frac{z^p}{1-z}$ (or $b_{k+p}=1$), we have

$$D^n_{\lambda,p}(f*g)(z) = D^n_{\lambda,p}f(z) = z^p + \sum_{k=1}^{\infty}\left(\frac{p+\lambda k}{p}\right)^n a_{k+p}z^{k+p}\ \ (\lambda\geq 0);$$

for $p=1$, the operator D^n_λ is the generalized Sălăgean operator introduced and studied by Al-Oboudi [1]) which in turn contains as special case the Sălăgean operator see [20];
(iii) For $n=0$ and

$$g(z) = z^p + \sum_{k=1}^{\infty}\left[\frac{p+\ell+\lambda k}{p+\ell}\right]^m z^{k+p}\ \ (\lambda\geq 0; p\in\mathbb{N}; \ell,m\in\mathbb{N}_0),$$

we see that $D^0_{\lambda,p}(f*g)(z)=(f*g)(z)=I^m_p(\lambda,\ell)f(z)$, where $I^m_p(\lambda,\ell)$ is the generalized multiplier transformation which was introduced and studied by Cătaş [5], the operator $I^m_p(\lambda,\ell)$, contains as special cases, the multiplier transformation $I^m_p(\ell)$ (see Kumar et al. [11] and Srivastava et al. [23]);
(iv) For $n=0$,

$$g(z) = z^p + \sum_{k=1}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k}.\frac{z^{k+p}}{k!} \tag{1.8}$$

$$\left(\alpha_i\in\mathbb{C}; i=1,\ldots,q; \beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}; j=1,\ldots,s; \atop q\leq s+1; q,s\in\mathbb{N}_0, p\in\mathbb{N}; z\in U\right)$$

and

$$(\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu=0;\theta\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}), \\ \theta(\theta-1)\ldots(\theta+\nu-1) & (\nu\in\mathbb{N};\theta\in\mathbb{C}), \end{cases}$$

we have $D^0_{\lambda,p}(f*g)(z)=(f*g)(z)=H_{p,q,s}(\alpha_1)f(z)$, where $H_{p,q,s}(\alpha_1)$ is the Dziok-Srivastava operator introduced and studied by Dziok and Srivastava [8]. The operator $H_{p,q,s}(\alpha_1)$ contains in turn many interesting operators such as, Carlson and Shaffer linear operator (see [19]), the Ruscheweyh derivative operator (see [10]), the Choi-Saigo-Srivastava operator (see [7]), the Cho-Kwon-Srivastava operator (see [6]), the differeintegral operator (see Srivastava and Aouf [22] and Patel and Mishra [18]) and the Noor integral operator (see Liu and Noor [12]);
(v) For $p=1$ and $g(z)$ of the form (1.8), the operator $D^n_\lambda(f*g)(z)$ inroduced and studied by Selvaraj and Karthikeyan [21].
For $f,g\in A(p),\lambda>0,\delta\geq 0,p\in\mathbb{N}$ and $n\in\mathbb{N}_0$, we define a function $H(z)$ by

$$H(z) = H^n_{\lambda,p,\delta}(f*g)(z) = \left[1-\delta\left(1+\frac{p}{\lambda}-p\right)\right]D^n_{\lambda,p}(f*g)(z) + \delta\frac{p}{\lambda}D^{n+1}_{\lambda,p}(f*g)(z). \tag{1.9}$$

We note that:
(i) For $\lambda=1$ and $g(z)=\frac{z^p}{1-z}$ in (1.9), we obtain

$$H^n_{1,p,\delta}(f*\frac{z^p}{1-z})(z)=G^n_{p,\delta}f(z)=G(z)=(1-\delta)D^n_p f(z)+\delta pD^{n+1}_p f(z); \tag{1.10}$$

(ii) For $g(z)=\frac{z^p}{1-z}$ in (1.9), we obtain

$$\begin{aligned}H^n_{\lambda,p,\delta}(f*\frac{z^p}{1-z})(z) &= K^n_{\lambda,p,\delta}f(z)=K(z)\\ &= \left[1-\delta\left(1+\frac{p}{\lambda}-p\right)\right]D^n_{\lambda,p}f(z)+\delta\frac{p}{\lambda}D^{n+1}_{\lambda,p}f(z).\end{aligned} \tag{1.11}$$

In this paper, we investigate some interesting argument inequalities and differential subordinations properties of the function $H(z)$ given by (1.9). The following lemma will be required in our investigation.

Lemma 1.1. *[15], [16] Let a function $\phi(z)=1+b_1z+\ldots$ be analytic in U and $\phi(z)\neq 0$ $(z\in U)$. If there exists a point $z_0\in U$ such that*

$$|\arg\phi(z)|<\frac{\pi}{2}\beta \quad (|z|<|z_0|) \quad \textit{and} \quad |\arg\phi(z_0)|=\frac{\pi}{2}\beta \quad (0<\beta\leq 1),$$

then we have $z_0\phi^{'}(z_0)/\phi(z_0)=ik\beta$, where

$$\begin{aligned}k &\geq \frac{1}{2}(a+\frac{1}{a}) \quad (\textit{where } \arg\phi(z_0)=\frac{\pi\beta}{2}),\\ k &\leq -\frac{1}{2}(a+\frac{1}{a}) \quad (\textit{where } \arg\phi(z_0)=-\frac{\pi\beta}{2}),\end{aligned}$$

and $(\phi(z_0))^{\frac{1}{\beta}}=\pm ia$ $(a>0)$.

2. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\lambda>0,\delta\geq 0,p\in\mathbb{N},n\in\mathbb{N}_0$ and g(z) is given by (1.2).

Theorem 2.1. *Let $f,g\in A(p)$ and let H be defined by (1.9). If*

$$\left|\arg\left(\frac{H^{(q)}(z)}{z^{p-q}}\right)\right|<\frac{\pi}{2}\beta \quad (z\in U), \tag{2.1}$$

then

$$\left|\arg\left(\frac{\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}}{z^{p-q}}\right)\right|<\frac{\pi}{2}\beta \quad (z\in U),$$

where $0<\beta\leq 1$ and $0\leq q\leq p$.

Proof. Let

$$\phi(z)=\frac{(p-q)!}{p!}\frac{\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}}{z^{p-q}} \quad (z\in U). \tag{2.2}$$

Then $\phi(z)$ is analytic in U, $\phi\neq 0$ for all $z\in U$ and $\phi(z)$ can be written as $\phi(z)=1+b_1z+\ldots$. Since

$$\left(z\left(D^n_{\lambda,p}(f*g)(z)\right)^{'}\right)^{(q)}=q\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}+z\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q+1)}, \tag{2.3}$$

we have from (1.7), (1.9) and (2.3) that

$$\begin{aligned}H^{(q)}(z) &= \left[1-\delta\left(1+\frac{p}{\lambda}-p\right)\right]\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}+\delta\frac{p}{\lambda}\left(D^{n+1}_{\lambda,p}(f*g)(z)\right)^{(q)}\\ &= \left[1-\delta\left(1+\frac{p}{\lambda}-p\right)\right]\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}+\delta\left(z\left(D^n_{\lambda,p}(f*g)(z)\right)^{'}\right)^{(q)}\\ &\quad +\delta\frac{p}{\lambda}(1-\lambda)\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}\\ &= (1-\delta+\delta q)(D^n_{\lambda,p}(f*g)(z))^{(q)}+\delta z(D^n_{\lambda,p}(f*g)(z))^{(q+1)}.\end{aligned}$$

(2.4)

It is easy to see from (2.4) and (2.2) that

$$\begin{aligned} \frac{H^{(q)}(z)}{z^{p-q}} &= (1-\delta+\delta q)\frac{(D^n_{\lambda,p}(f*g)(z))^{(q)}}{z^{p-q}} + \delta \frac{z(D^n_{\lambda,p}(f*g)(z))^{(q+1)}}{z^{p-q}} \\ &= \frac{p!(1-\delta+\delta q)}{(p-q)!}\phi(z) + \frac{\delta p!}{(p-q)!}\left((p-q)\phi(z)+z\phi^{'}(z)\right) \\ &= \frac{p!(1-\delta+\delta p)}{(p-q)!}\left(\phi(z)+\frac{\delta}{1-\delta+\delta p}z\phi^{'}(z)\right). \end{aligned} \tag{2.5}$$

Suppose there exists a point $z_0 \in U$ such that

$$|\arg\phi(z)| < \frac{\pi}{2}\beta \quad (|z|<|z_0|)$$

and

$$|\arg\phi(z_0)| = \frac{\pi}{2}\beta\ .$$

Then, by using Lemma 1.1, we can write that $z_0\phi^{'}(z_0)/\phi(z_0) = ik\beta$ and $(\phi(z_0))^{\frac{1}{\beta}} = \pm ia$ $(a>0)$. Therefore, if $\arg\phi(z_0) = \frac{\pi}{2}\beta$, then by using (2.5), we have

$$\begin{aligned} \frac{H^{(q)}(z_0)}{z_0^{p-q}} &= \frac{p!(1-\delta+\delta p)}{(p-q)!}\phi(z_0)\left(1+\frac{\delta}{1-\delta+\delta p}\frac{z_0\phi^{'}(z_0)}{\phi(z_0)}\right) \\ &= \frac{p!(1-\delta+\delta p)}{(p-q)!}a^{\beta}e^{i\pi\beta/2}\left(1+\frac{\delta}{1-\delta+\delta p}ik\beta\right). \end{aligned}$$

This shows that

$$\begin{aligned} \arg\left(\frac{H^{(q)}(z_0)}{z_0^{p-q}}\right) &= \frac{\pi}{2}\beta + \arg\left(1+\frac{\delta k\beta i}{1-\delta+\delta p}\right) \\ &= \frac{\pi}{2}\beta + \tan^{-1}\left(\frac{\delta k\beta}{1-\delta+\delta p}\right) \\ &\geq \frac{\pi}{2}\beta,\ \ (where\ k \geq \frac{1}{2}(a+\frac{1}{a}) \geq 1), \end{aligned}$$

which contradicts the condition (2.1). Similarly, if $\arg\phi(z_0) = \frac{-\pi\beta}{2}$, then we obtain

$$\arg\left(\frac{H^{(q)}(z_0)}{z_0^{p-q}}\right) \leq -\frac{\pi}{2}\beta\ ,$$

which also contradicts the condition (2.1). Thus, the function $\phi(z)$ satisfies $|\arg\phi(z)| < \frac{\pi\beta}{2}$ $(z\in U)$. This shows that

$$\left|\arg\left(\frac{\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}}{z^{p-q}}\right)\right| < \frac{\pi}{2}\beta \quad (z\in U)\ .$$

This completes the proof of Theorem 2.1. □

Putting $n=0$ and $\lambda=1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let $f,g \in A(p)$ and let Q be defined by*

$$Q(z) = (1-\delta)(f*g)(z) + \delta\frac{z}{p}\left((f*g)(z)\right)^{'}. \tag{2.6}$$

If

$$\left|\arg\left(\frac{Q^{(q)}(z)}{z^{p-q}}\right)\right| < \frac{\pi}{2}\beta \quad (z\in U)\ ,$$

then

$$\left|\arg\left(\frac{((f*g)(z))^{(q)}}{z^{p-q}}\right)\right| < \frac{\pi}{2}\beta \quad (z\in U)\ ,$$

where $0<\beta\leq 1$ and $0\leq q\leq p$.

Theorem 2.3. *Let $f,g \in A(p)$ and let H be defined by (1.9). If*

$$\frac{\left(D^{n}_{\lambda,p}(f*g)(z)\right)^{(q)}}{z^{p-q}} \prec \frac{p!}{(p-q)!}\frac{1+(1-2\alpha)z}{1-z} \quad (z\in U)\,. \tag{2.7}$$

Then

$$\frac{H^{(q)}(z)}{z^{p-q}} \prec \frac{p!(1-\delta+\delta p)}{(p-q)!}\frac{1+(1-2\alpha)z}{1-z} \quad (|z|<\rho)\,, \tag{2.8}$$

where $0\le q\le p, 0\le\alpha<1$, and

$$\rho=\left[1+\left(\frac{\delta}{1-\delta+\delta p}\right)^{2}\right]^{\frac{1}{2}}-\frac{\delta}{1-\delta+\delta p}\,. \tag{2.9}$$

The bound $\rho\in(0,1)$ is the best possible.

Proof. Set

$$\psi(z)=(1-\gamma)\frac{z}{1-z}+\gamma\frac{z}{(1-z)^{2}} \quad (z\in U)\,,$$

where $\gamma=\frac{\delta}{1-\delta+\delta p}>0$. We need to show that

$$\operatorname{Re}\left\{\frac{\psi(\rho z)}{\rho z}\right\}>\frac{1}{2} \quad (z\in U)\,, \tag{2.10}$$

where $\rho=(1+\gamma^{2})^{\frac{1}{2}}-\gamma$ and $0<\rho<1$. Let $\frac{1}{1-z}=R\,e^{i\theta}$ and $|z|=r<1$. In view of

$$\cos\theta=\frac{1+R^{2}(1-r^{2})}{2R}\;,\; R\ge\frac{1}{1+r}\,,$$

we have

$$\begin{aligned}2\operatorname{Re}\left\{\frac{\psi(z)}{z}-\frac{1}{2}\right\} &= 2(1-\gamma)R\cos\theta+2\gamma R^{2}\cos 2\theta-1\\ &= R^{4}\gamma(1-r^{2})^{2}+R^{2}\left((1-\gamma)(1-r^{2})\right)-2\gamma r^{2})\\ &\ge R^{2}(\gamma(1-r)^{2}+(1-\gamma)(1-r^{2})-2\gamma r^{2})\\ &= R^{2}(1-2\gamma r-r^{2})>0\end{aligned}$$

for $|z|=r<\rho$, which gives (2.10). Thus the function ψ has the integral representation

$$\frac{\psi(\rho z)}{\rho z}=\int\limits_{1x1=1}\frac{d\mu(x)}{1-xz} \quad (z\in U)\,, \tag{2.11}$$

where $\mu(x)$ is a prabability measure on $|x|=1$.
Now letting $\phi(z)$ be in the form (2.2), we see that $\phi(z)=1+b_{1}z+\ldots$ is analytic in U and it follows from (2.7) that

$$\operatorname{Re}\phi(z)>\alpha \quad (0\le\alpha<1; z\in U)\,. \tag{2.12}$$

Since we can write

$$\phi(z)+\gamma z\phi^{'}(z)=\left(\frac{\psi(z)}{z}\right)*\phi(z)\,,$$

it follows from (2.11) that

$$\operatorname{Re}\left\{\phi(\rho z)+\gamma\rho z\phi^{'}(\rho z)\right\}=\operatorname{Re}\left\{\left(\frac{\psi(\rho z)}{\rho z}\right)*\phi(z)\right\}$$

$$=\operatorname{Re}\left\{\int\limits_{1x1=1}\phi(xz)d\mu(x)\right\}>\alpha \quad (z\in U)\,. \tag{2.13}$$

Thus, from (2.3) and (2.13), we conclude that (2.8)) holds. To show that the bound ρ is sharp we take $f,g \in A(p)$ defined by

$$\frac{(p-q)!}{(p)_q}\frac{\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}}{z^{p-q}} = \alpha+(1-\alpha)\frac{1+z}{1-z}.$$

Since

$$\begin{aligned}\frac{(p-q)!}{(p)_q(1-\delta+\delta p)}\frac{H^{(q)}(z)}{z^{p-q}} &= \alpha+(1-\alpha)\frac{1+z}{1-z}+\gamma(1-\alpha)z\left(\frac{1+z}{1-z}\right)' \\ &= \alpha+(1-\alpha)\frac{1+2\gamma z-z^2}{(1-z)^2}=\alpha\end{aligned}$$

for $z=-\rho$, it follows that ρ is sharp. □

Remark 2.4. (i) *Putting* $\lambda=1$ *and* $g(z)=\frac{z^p}{1-z}$ *in the above results we obtain the results for function* $G(z)$ *defined by* (1.10).
(ii) *Putting* $g(z)=\frac{z^p}{1-z}$ *in the above results we obtain the results for function* $K(z)$ *defined by* (1.11).

3. Conclusion

In this paper, three subclasses $H^n_{\lambda,p,\delta}(f*g)(z)$, $G^n_{p,\delta}f(z)$ and $K^n_{\lambda,p,\delta}f(z)$ are introduced and certain interesting argument inequalities and differential subordinations properties are investigated.

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Initial value problems spreadsheet solver using VBA for engineering education

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Abstract

Spreadsheet solver using VBA programming has been designed for solving initial value problems (IVPs), analytically and numerically by all Runge-Kutta (RK) methods including also fifth order with calculation of true percent relative error for corresponding RK method. This solver is user-friendly especially for beginner users of Excel and VBA.

1. Introduction

IVPs arise in any field of science and engineering education such as mechanics, geotechnics, dynamics, chemical kinetics, optimization and stability, et cetera. There are computing approaches; exact solution method and numerical methods for solving these IVPs. Numerical methods are both applicable and practical in solving IVPs in many engineering problems because of the existence of complicated problems in engineering and limitations of exact solution method [1, 2]. Numerical methods yield approximate the solutions of the IVPs, particularly for the nonlinear IVPs.

This study mainly has focussed on numerical solutions followed by Euler and various Runge-Kutta methods for solving single IVPs. These methods progress the solution over step starting from some given initial condition at the initial starting point. To simplify the steps in solving IVPs by RK methods, a tool is used. This tool is a prevalent spreadsheet application, fundamentally called as Excel, also commonly used by professionals for diverse applications in business [3], engineering and science [4]-[6].

Numerical methods in science and engineering may also be implemented in by use of Excel and also VBA. Use of VBA in explicit form Visual Basic for Applications programming capability lurks in the background behind Excel handled in the texts like Lilley and Chapra [2, 7]. In addition to this, a series of studies in literature employed spreadsheet as a calculator or solver to focus on design of solver and calculator for polynomial interpolation [8, 9], solution for systems of linear and nonlinear equations [10, 11], computation of eigenvalues [12, 13], design of spreadsheet calculator for numerical differentiation [14]-[16], spreadsheet solver for solution of partial differential equations [17], a spreadsheet solution of system of initial value problems using fourth-order RK method [18], and fourth-order RK method by spreadsheet [19]. Only the works of Tay et al. [20, 21] include design of spreadsheet calculator for solving system of IVPs using fourth-order RK method and also solving IVPs using fourth-order RK method with use of VBA programming.

In this study, a spreadsheet solver is designed to solve both IVPs by all RK methods and also exact solution method in the spreadsheet environment based on VBA programming. Microsoft Excel 2010 and Microsoft Visual Basics for Applications 7.0 are used during this study. The generation of VBA programming includes three steps. The first step is to develop an user interface input form is designed to acquire the needed information such as initial conditions of independent and dependent variables for each RK method, step size and number of steps. Then a general VBA code for any IVPs is created behind the Solve button in user interface input form. The third step is to generate function files depending on the related IVP and its analytical solution. Once the SOLVE button in user interface input form is clicked, the complete numerical and analytical solutions of the IVP and corresponding true percent relative error will be computed automatically for each order of RK method.

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Examples are presented from various fields of engineering to demonstrate the merits of this unconventional solver design which shields the tedious algorithmic implementation details from the user (such as students and educators) and greatly simplifies solving an IVP using RKSOLVER.
This spreadsheet solver is user-friendly such that users only require to enter initial conditions of independent and dependent variables for each RK method, step size and number of steps at the first step to compute the complete solution of the IVPs automatically without typing any commands in the spreadsheet cells. Here, complete solution of the IVPs means solutions from each order of RK method, exact solutions and also true percent relative errors in terms of comparison with each RK method and exact solutions. So users as educators have an oppurtunity to elucidate students the differences and similarities that exist between each order of RK method and also exact solutions at the same time and be able to comment on the solution of any engineering problem including IVPs correctly. There is no need to know the various derivations of RK methods and memorize the complicated formulations of RK methods. The solver is general and standard for any engineering problem. The main aim of this paper is to design a tool in other words spreadsheet solver which employes both numerical methods: RK methods with fifth order and also analytical methods giving exact solutions with automatically calculated true percent relative errors in solving IVPs at the same time. Therefore this solver is called as IVP spreadsheet solver.

2. Runge Kutta (RK) methods

This section is devoted to solving IVPs of the form given below:

$$\frac{dy}{dx} = f(x,y) \tag{2.1}$$

with the initial value $y(x_0) = y_0$ for the number of points n within the interval $x_0 \leq x \leq x_n$. Here x is the independent variable, y is the dependent variable, f is the function of derivation (in other words slope) and h is the fixed step size. n, the number of steps can be found as $(x_n - x_0)/h$ [1].

1) First-Order RK Method

Euler's Method:

$$y_{i+1} = y_i + hk_1 \tag{2.2}$$

where $k_1 = f(x,y)$

2) Second-Order RK Methods

a) Heun's Method:

$$y_{i+1} = y_i + h(\frac{k_1 + k_2}{2}) \tag{2.3}$$

where $k_2 = f(x_i + h, y_i + hk_1)$

b) Midpoint (Improved Polygon) Method:

$$y_{i+1} = y_i + hk_2 \tag{2.4}$$

where $k_2 = f(x_i + \frac{h}{2}, y_i + \frac{k_1 h}{2})$

c) Ralston's Method:

$$y_{i+1} = y_i + (\frac{k_1 + 2k_2}{3})h \tag{2.5}$$

where $k_2 = f(x_i + \frac{3h}{4}, y_i + \frac{3hk_1}{4})$

3) Third-Order RK Method

$$y_{i+1} = y_i + (\frac{k_1 + 4k_2 + k_3}{6})h \tag{2.6}$$

where $k_2 = f(x_i + \frac{h}{2}, y_i + \frac{k_1 h}{2}), \qquad k_3 = f(x_i + h, y_i - k_1 h + 2k_2 h)$

4) Fourth-Order RK Method

$$y_{i+1} = y_i + (\frac{k_1 + 2k_2 + 2k_3 + k_4}{6})h \tag{2.7}$$

Function f(x, y0, h)
f = y0 / 0.2254
End Function

Table 1: Function module for stress-strain relationship IVP

Function fexact(x, y0, h, i)
fexact = Exp((h * i) / 0.2254)
End Function

Table 2: Function module for exact solution of stress-strain relationship

where $k_2 = f(x_i + \frac{h}{2}, y_i + \frac{k_1 h}{2})$, $k_3 = f(x_i + \frac{h}{2}, y_i + \frac{k_2 h}{2})$, $k_4 = f(x_i + h, y_i + k_3 h)$

5) Fifth-Order RK Method

$$y_{i+1} = y_i + (\frac{7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6}{90})h \tag{2.8}$$

where $k_2 = f(x_i + \frac{h}{4}, y_i + \frac{k_1 h}{4})$, $k_3 = f(x_i + \frac{h}{4}, y_i + \frac{k_1 h}{8} + \frac{k_2 h}{8})$, $k_4 = f(x_i + \frac{h}{2}, y_i - \frac{k_2 h}{2} + k_3 h)$, $k_5 = f(x_i + \frac{3h}{4}, y_i + \frac{3k_1 h}{16} + \frac{9k_4 h}{16})$, and $k_6 = f(x_i + h, y_i - \frac{3k_1 h}{7} + \frac{2k_2 h}{7} + \frac{12k_3 h}{7} - \frac{12k_4 h}{7} + \frac{8k_5 h}{7})$

It should be noted that k's are recurrence relationships. In other words, k_1 appears in the equation for k_2 which appears in the equation for k_3 and so on. Since each k is a functional evaluation, this recurrence makes RK methods efficient for computations [1].

In this work, fifth-order RK method yields the superior results in terms of less error than the other order of RK methods. As the order of RK method increases, convergence to the exact results also increases in terms of less errors.

3. Numerical examples

Numerical examples are presented from various engineering applications.

1) Geotechnical Engineering

To mIVPl the the behavior of soil under the effect of load, it is required to formulate the stress and strain relationship and this is achieved by the following IVP:

$$\frac{d\sigma}{d\varepsilon} = \frac{\sigma}{c_C} \tag{3.1}$$

The exact solution for equation (3.1) is

$$\sigma = e^{\frac{\varepsilon}{c_C}} \tag{3.2}$$

where σ is the stress, ε is the strain of soil and c_C is the compression index and it is 0.2254 for this soil type. Initial conditions are, ε_0 is 0 for independent variable and σ_0 is 1 kPa for dependent variable. Final ε is 1.2 and step size (h) is 0.1. This means that number of steps (n) is 12.

At first, for each numerical example, function modules are prepared for both IVP and exact solution of it respectively. These modules change from example to example. The functions for IVP and exact solution are illustrated in the following tables.

Here x is the independent variable, y0 is the initial dependent variable, i is the counter of steps.

Then equations (2.2) to (2.8) are applied to obtain the solutions by each order of RK method respectively. Besides exact solution of the IVP with true percent relative error for each RK method are also incorporated in the computations.

Finally IVP spreadsheet solver is applied which is discussed in the next section to obtain the complete solutions.

2) Mechanical Engineering

To determine the change in velocity in other words acceleration of a free-falling body to the forces acting on it with considering the air resistance, the following IVP is used:

$$\frac{dv}{dt} = g - \frac{c}{m}v \tag{3.3}$$

The exact solution for equation (3.3), which also gives velocity of the object, is

$$v(t) = \frac{gm}{c}(1 - e^{(-\frac{c}{m})t}) \tag{3.4}$$

where v is the velocity (dependent variable y), t is the time in seconds (indepedent variable x), g is the gravitational constant, 9.8 m/s^2, m is the mass of the object, 68.1 kg and c is the drag coefficient, 12.5 kg/s. Initial conditions are, t_0 is 0 s and v_0 is 0 m/s [1]. Final value of time is 5 s and step size (h) is 0.5. This means that number of steps (n) for computation is 10.

At first, for this example, function modules are written for both IVP and exact solution of it respectively. These functions are illustrated in Table 3 and Table 4 respectively.

Here x is the independent variable corresponding to time, y0 is the initial dependent variable corresponding to velocity.

```
Function f(x, y0, h)
f = 9.8 - ((12.5 / 68.1) * y0)
End Function
```

Table 3: Function module for exact solution yielding velocity

```
Function fexact(x, y0, h,i)
fexact = ((9.8 * 68.1) / 12.5) * (1 - Exp((-12.5 / 68.1) * (h * i)))
End Function
```

Table 4: Function module for exact solution yielding velocity

Like geotechnical engineering example, equations (2.2) to (2.8) are employed to find the solutions by each order of RK method respectively. Besides exact solution of the IVP with true percent relative error for each RK method are also inserted in the computations.
Finally IVP spreadsheet solver is used which is mentioned in the next section to obtain the complete solutions.
3) Chemical Engineering: Mixture Problem
The mixture problem related to a tank containing 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same time. In this problem, it is required to determine the amount of salt after t minutes in this tank. For this reason, the following IVP is employed:

$$\frac{dA}{dt} = \frac{-A}{100} \tag{3.5}$$

$A(t)$ is the amount of salt after t minutes in tank, also the dependent variable is obtained by the following exact solution:

$$A(t) = 15e^{\left(\frac{-t}{100}\right)} \tag{3.6}$$

Initial conditions are, t_0 is 0 min and A_0 is 15 kg. Final value of time is 0.96 min and step size (h) is 0.02. Number of steps (n) for computation is 49.
At first, function modules are formed for both IVP and exact solution of the problem respectively. These functions are displayed in Table 5 and Table 6 respectively.
Here x is the independent variable corresponding to time, y0 is the initial dependent variable corresponding to amount of salt after t minutes in the tank.
Then, equations (2.2) to (2.8) are used to determine the solutions by writing codes for each order of RK method respectively. These codes are standard and valid for any scince and engineering problem including IVP. So there is no need to write cIVP for various problems. Besides exact solution of the IVP with true percent relative error for each RK method are also included in the computations. True percent relative error is in the following form:

$$\varepsilon_T = \left| \frac{ExactResult - ApproximateResult}{ExactResult} \right| \times 100 \tag{3.7}$$

Where Exact Result in other words true result represents the solution obtained by analytically. Approximate Result corresponds with the corresponding solution obtained by numerical methods, any order of RK methods.
Finally IVP spreadsheet solver is employed which is argued in the next section to obtain the complete solutions.

4. IVP spreadsheet solver

Using this IVP spreadsheet solver leads to a macro named RKSOLVER which solves the whole IVP at once completely.
The general procedure for obtaining complete solution of an IVP is composed of some steps. These steps are standard and applicable for any type of IVP.
The first step is to design an user interface input form (userform) called as UserForm4 to enable users to enter required data for solving an IVP completely. The standard form of UserForm4 for any problem is illustrated in Figure 4.1.
The second step is to generate a new tab name as IVP Solver with RKSOLVER macro including codes for solving IVP by both numerically (by each order of RK method) and analytically (gives exact solution). RKSOLVER also provides user to compute true percent relative error for each RK method.
Figure 4.2 illustrates the standard IVP Solver tab with RKSOLVER button. One more variation is to add a button assigned RKSOLVER macro in the spreadsheet. So user is able to run the macro simply by clicking this button. It is sufficient to start the complete solution procedure of IVPs.

```
Function f(x, y0, h)
f = -y0 / 100
End Function
```

Table 5: Function module for IVP of the problem

```
Function fexact(x, y0, h, i)
fexact = 15 * Exp(-(h * i) / 100)
End Function
```

Table 6: Function module for exact solution of the problem

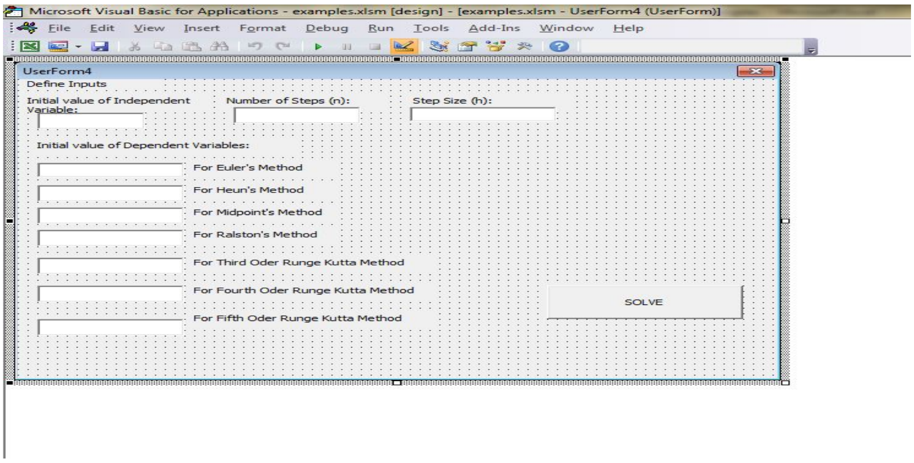


Figure 4.1: The standard userform for all examples

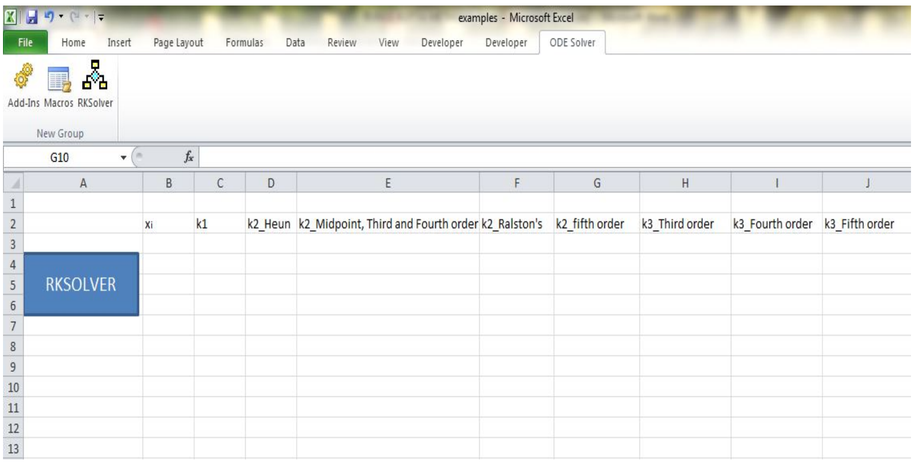


Figure 4.2: The standard IVP Solver tab with RKSOLVER button

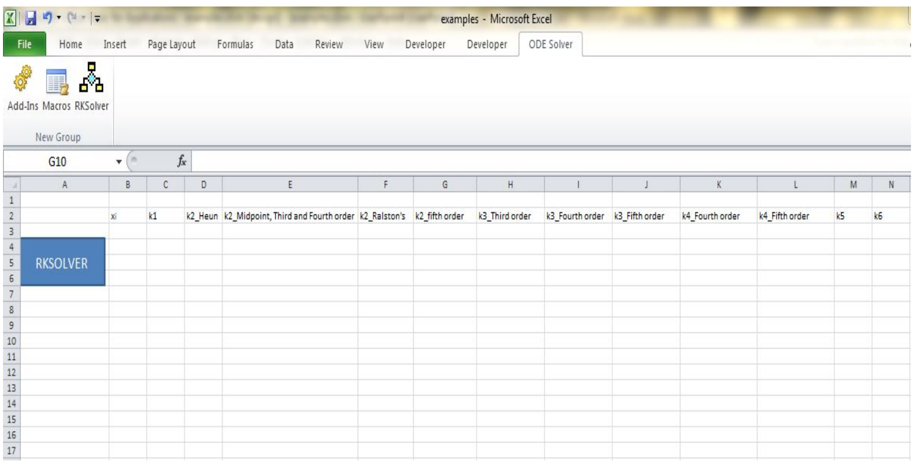


Figure 4.3: The standard blank spreadsheet image with k's (recurrence relationships) titles

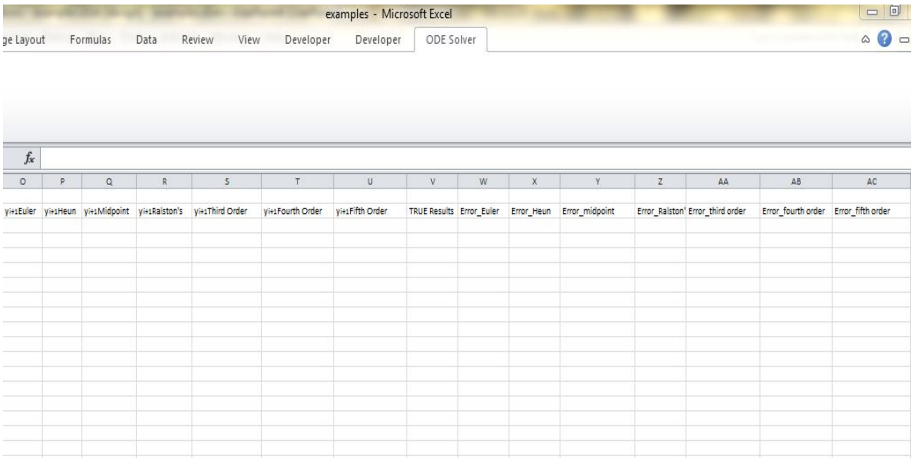


Figure 4.4: The standard blank spreadsheet image with RK results, exact results and error titles

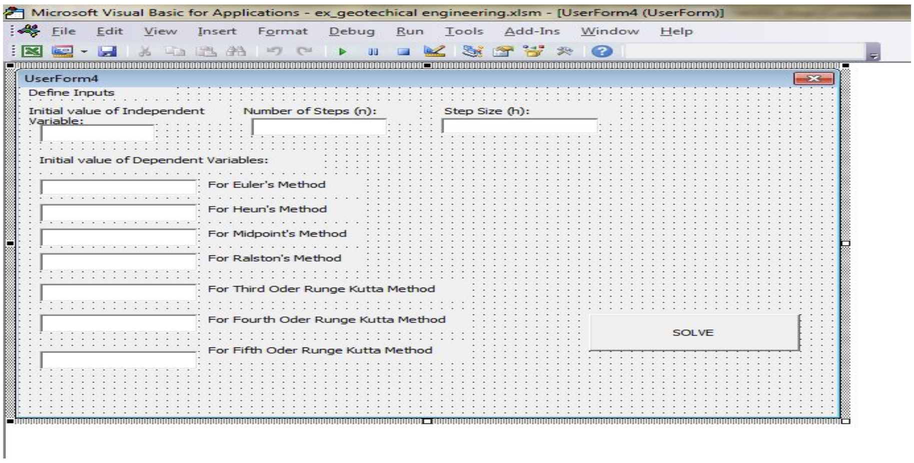


Figure 4.5: Userform for geotechnical engineering example

Then the only thing is to specify sufficient place in spreadsheet cells to make macro fill them with solutions for any IVP examples. For this reason, the titles for k's, RK results, exact results and error titles are written as is the case with Figure 4.3 and Figure 4.4 respectively.
The working procedure for IVP solver namely RKSOLVER is described for each numerical examples (geotechnical engineering, mechanical engineering and chemical engineering). The steps for geotechnical engineering example are illustrated in the Figure 4.5- Figure 4.11.
The first step is to call userform by clicking run in the toolbar or simply clicking RKSOLVER button. The image of this userform for geotechnical engineering example is given in Figure 4.5. This userform is standard for any IVP example.
Due to the fact that initial conditions are different for all IVPs, the filled userform is distinctive for all problems. As is the case with geotechnical engineering example. Userform is filled with initial conditions of the problem in Figure 4.6. Then by clicking SOLVE button in UserForm4; k's, numerical solutions obtained form all RK methods, exact solutions (true solutions) and true percent relative errors can be obtained and displayed as the spreadsheet images in Figure 4.7 to Figure 4.11 respectively.
To Figure 4.10 and Figure 4.11, fifth-order RK method gives the best solution in terms of the least error and best convergence to exact solutions.
Similarly for mechanical engineering, userform is invoked by clicking RKSOLVER in Figure 4.12. Then this form is filled with necessary data as it is shown in Figure 4.13.
By clicking the SOLVE button in userform, computations are performed and given in the spreadsheet images of Figure 4.14 to Figure 4.18. To Figure 4.17 and Figure 4.18, the worst solution is obtained by Euler's method while fifth-order RK method is the best one with the least error and best convergence to the exact solution.
For mixture problem, userform is called by clicking RKSOLVER button in spreadsheet. Figure 4.19 illustrates this process.
Then this userform is filled by entering initial conditions as given in Figure 4.20. Clicking the SOLVE button in userform leads to complete solution of the problem. These solutions are displayed in Figure 4.21 to Figure 4.25.
To Figure 4.24 and Figure 4.25, all RK methods give quite well solutions with convergence to exact results in terms of less errors.

5. Conclusion

An IVP solver with use of RK methods including also the highest order; fifth order has been generated by VBA for the first time in literature. Emphasis was on all types of RK methods usable simultaneously and the solver generated applicable to IVPs for science and engineering problems.

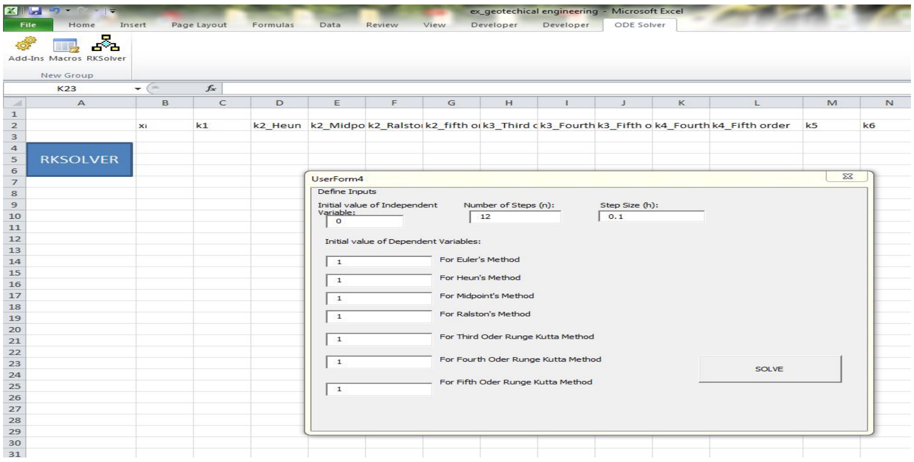
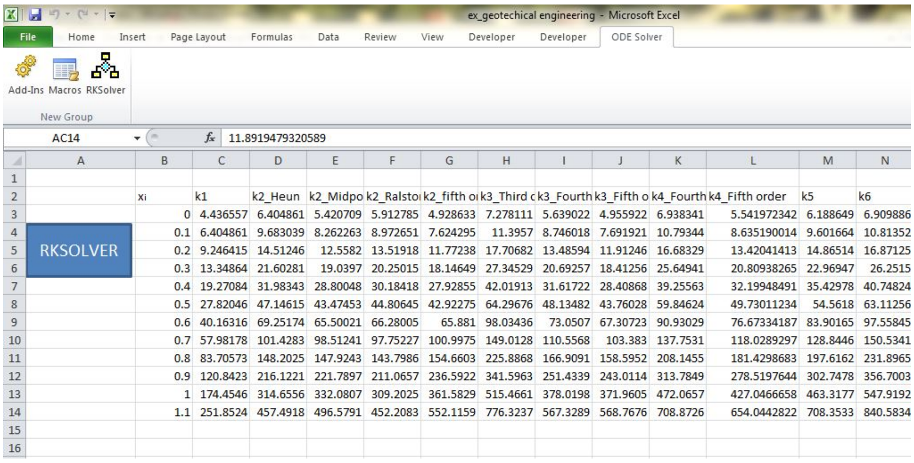
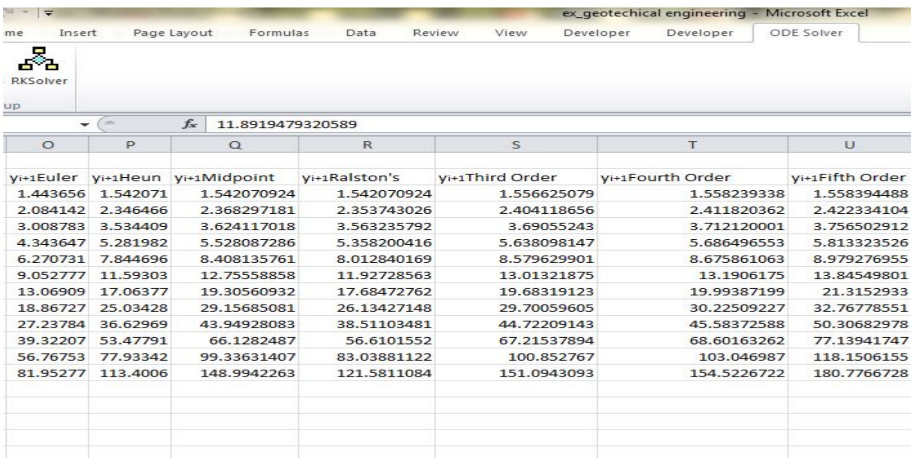


Figure 4.6: Filled userform for geotechnical engineering example



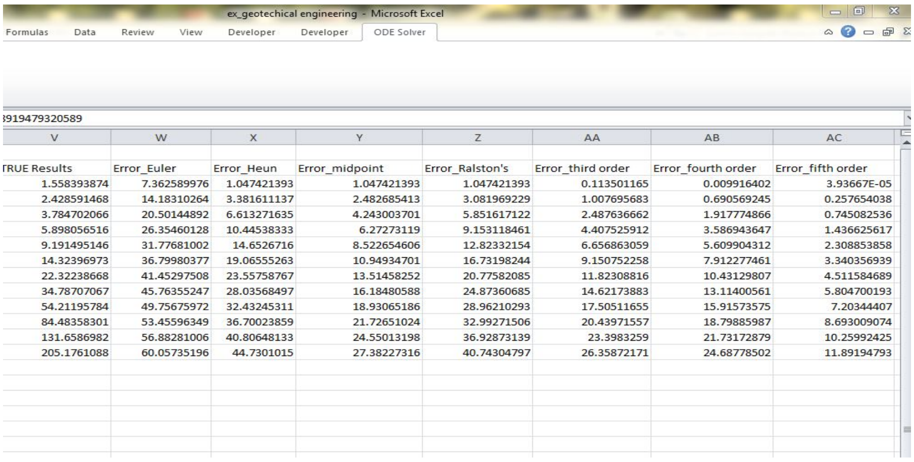
xi	k1	k2_Heun	k2_Midpo	k2_Ralsto	k2_fifth o	k3_Third c	k3_Fourth	k3_Fifth o	k4_Fourth	k4_Fifth order	k5	k6
0	4.436557	6.404861	5.420709	5.912785	4.928633	7.278111	5.639022	4.955922	6.938341	5.541972342	6.188649	6.909886
0.1	6.404861	9.683039	8.262263	8.972651	7.624295	11.3957	8.746018	7.691921	10.79344	8.635190014	9.601664	10.81352
0.2	9.246415	14.51246	12.5582	13.51918	11.77238	17.70682	13.48594	11.91246	16.68329	13.42041413	14.86514	16.87125
0.3	13.34864	21.60281	19.0397	20.25015	18.14649	27.34529	20.69257	18.41256	25.64941	20.80938265	22.96947	26.2515
0.4	19.27084	31.98343	28.80048	30.18418	27.92855	42.01913	31.61722	28.40868	39.25563	32.19948491	35.42978	40.74824
0.5	27.82046	47.14615	43.47453	44.80645	42.92275	64.29676	48.13482	43.76028	59.84624	49.73011234	54.5618	63.11256
0.6	40.16316	69.25174	65.50021	66.28005	65.881	98.03436	73.0507	67.30723	90.93029	76.67334187	83.90165	97.55845
0.7	57.98178	101.4283	98.51241	97.75227	100.9975	149.0128	110.5568	103.383	137.7531	118.0289297	128.8446	150.5341
0.8	83.70573	148.2025	147.9243	143.7986	154.6603	225.8868	166.9091	158.5952	208.1455	181.4298683	197.6162	231.8965
0.9	120.8423	216.1221	221.7897	211.0657	236.5922	341.5963	251.4339	243.0114	313.7849	278.5197644	302.7478	356.7003
1	174.4546	314.6556	332.0807	309.2025	361.5829	515.4661	378.0198	371.9605	472.0657	427.0466658	463.3177	547.9192
1.1	251.8524	457.4918	496.5791	452.2083	552.1159	776.3237	567.3289	568.7676	708.8726	654.0442822	708.3533	840.5834

Figure 4.7: Computation results for k's for geotechnical engineering example



y_{i+1}Euler	y_{i+1}Heun	y_{i+1}Midpoint	y_{i+1}Ralston's	y_{i+1}Third Order	y_{i+1}Fourth Order	y_{i+1}Fifth Order
1.443656	1.542071	1.542070924	1.542070924	1.556625079	1.558239338	1.558394488
2.084142	2.346466	2.368297181	2.353743026	2.404118656	2.411820362	2.422334104
3.008783	3.534409	3.624117018	3.563235792	3.69055243	3.712120001	3.756502912
4.343647	5.281982	5.528087286	5.358200416	5.638098147	5.686496553	5.813323526
6.270731	7.844696	8.408135761	8.012840169	8.579629901	8.675861063	8.979276955
9.052777	11.59303	12.75558858	11.92728563	13.01321875	13.1906175	13.84549801
13.06909	17.06377	19.30560932	17.68472762	19.68319123	19.99387199	21.3152933
18.86727	25.03428	29.15685081	26.13427148	29.70059605	30.22509227	32.76778551
27.23784	36.62969	43.94928083	38.51103481	44.72209143	45.58372588	50.30682978
39.32207	53.47791	66.1282487	56.6101552	67.21537894	68.60163262	77.13941747
56.76753	77.93342	99.33631407	83.03881122	100.852767	103.046987	118.1506155
81.95277	113.4006	148.9942263	121.5811084	151.0943093	154.5226722	180.7766728

Figure 4.8: Computation results for each RK method for geotechnical engineering example



TRUE Results	Error_Euler	Error_Heun	Error_midpoint	Error_Ralston's	Error_third order	Error_fourth order	Error_fifth order
1.558393874	7.362589976	1.047421393	1.047421393	1.047421393	0.113501165	0.009916402	3.93667E-05
2.428591468	14.18310264	3.381611137	2.482685413	3.081969229	1.007695683	0.690569245	0.257654038
3.784702066	20.50144892	6.613271635	4.243003701	5.851617122	2.487636662	1.917774866	0.745082536
5.898056516	26.35460128	10.44538333	6.27273119	9.153118461	4.407525912	3.586943647	1.436625617
9.191495146	31.77681002	14.6526716	8.522654606	12.82332154	6.656863059	5.609904312	2.308853858
14.32396973	36.79980377	19.06555263	10.94934701	16.73198244	9.150752258	7.912277461	3.340356939
22.32238668	41.45297508	23.55758767	13.51458252	20.77582085	11.82308816	10.43129807	4.511584689
34.78707067	45.76355247	28.03568497	16.18480588	24.87360685	14.62173883	13.11400561	5.804700193
54.21195784	49.75675972	32.43245311	18.93065186	28.96210293	17.50511655	15.91573575	7.20344407
84.48358301	53.45596349	36.70023859	21.72651024	32.99271506	20.43971557	18.79885987	8.693009074
131.6586982	56.88281006	40.80648133	24.55013198	36.92873139	23.3983259	21.73172879	10.25992425
205.1761088	60.05735196	44.7301015	27.38227316	40.74304797	26.35872171	24.68778502	11.89194793

Figure 4.9: Computation results for exact results (true results) and true percent relative errors of each RK method for geotechnical engineering example

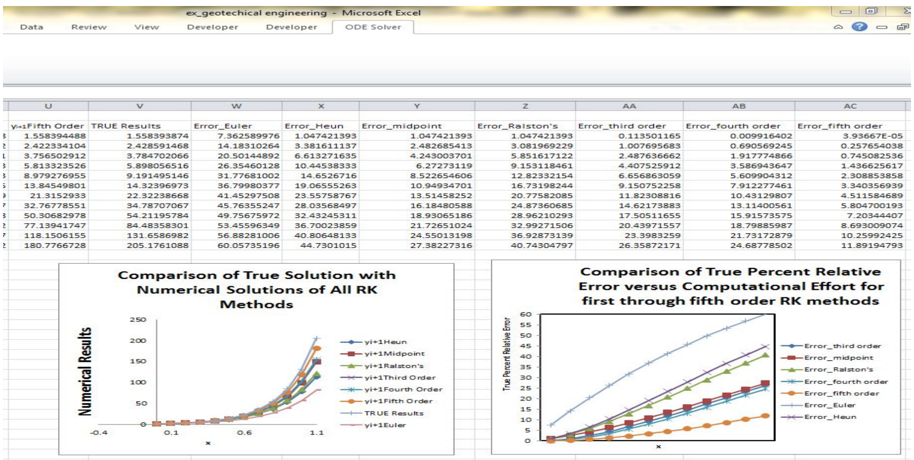


Figure 4.10: Graphical display of the computation results for geotechnical engineering example

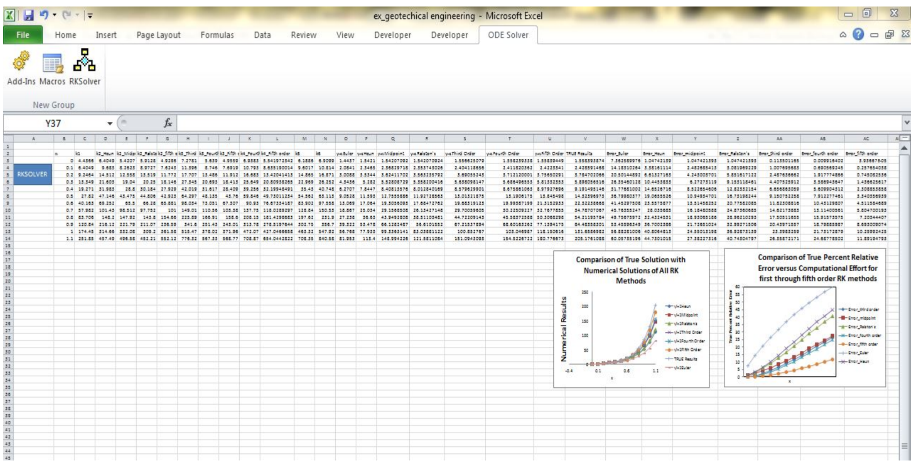


Figure 4.11: The spreadsheet image of full computation results for geotechnical engineering

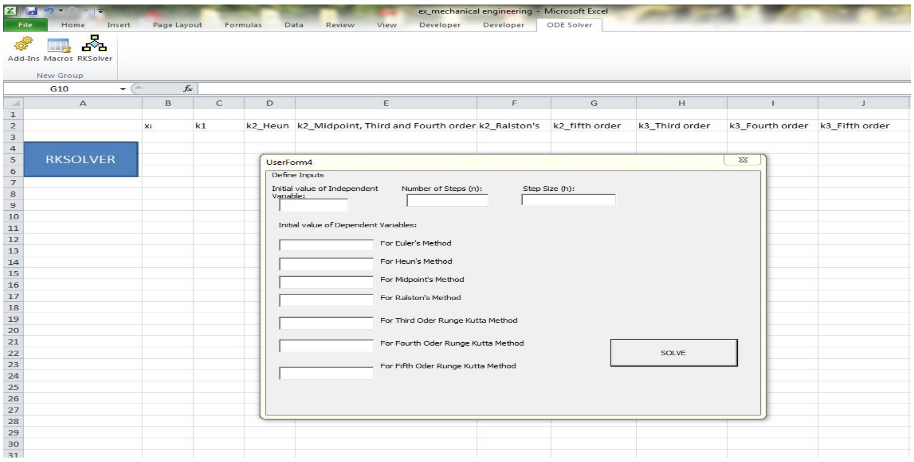


Figure 4.12: Userform in spreadsheet for mechanical engineering example

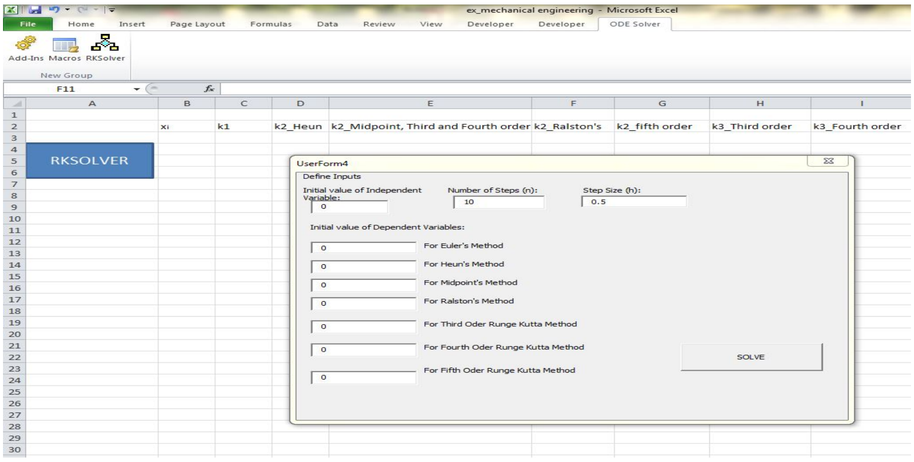


Figure 4.13: Filled userform for mechanical engineering example

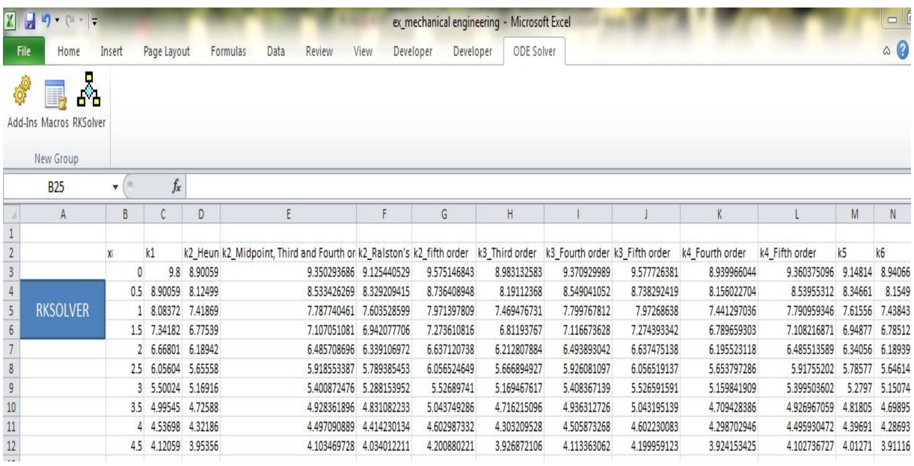
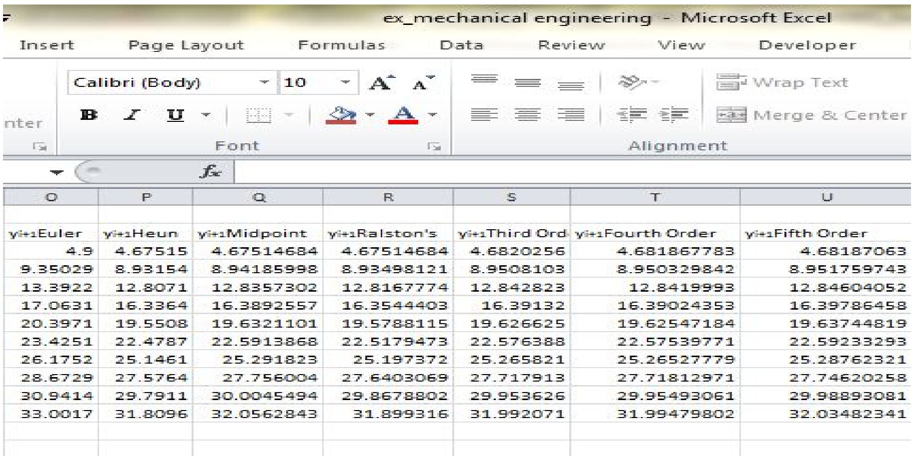
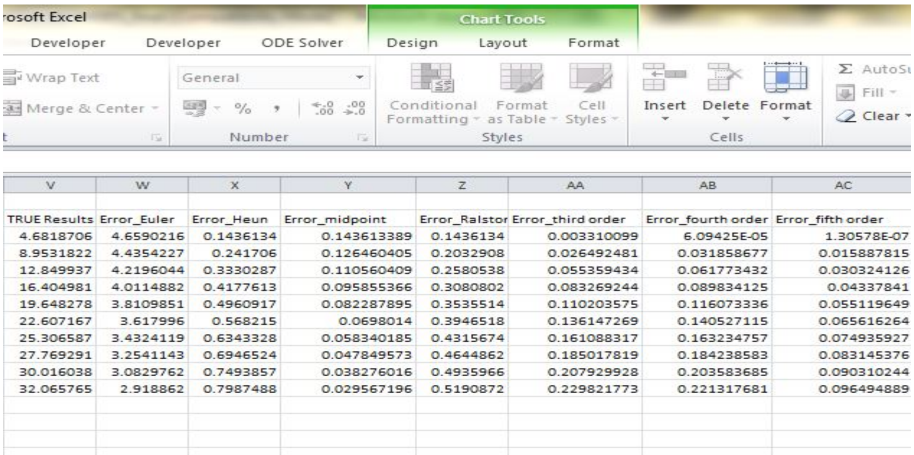


Figure 4.14: Computation results for k's for mechanical engineering example



yi+1Euler	yi+1Heun	yi+1Midpoint	yi+1Ralston's	yi+1Third Ord	yi+1Fourth Order	yi+1Fifth Order
4.9	4.67515	4.67514684	4.67514684	4.6820256	4.681867783	4.68187063
9.35029	8.93154	8.94185998	8.93498121	8.9508103	8.950329842	8.951759743
13.3922	12.8071	12.8357302	12.8167774	12.842823	12.8419993	12.84604052
17.0631	16.3364	16.3892557	16.3544403	16.39132	16.39024353	16.39786458
20.3971	19.5508	19.6321101	19.5788115	19.626625	19.62547184	19.63744819
23.4251	22.4787	22.5913868	22.5179473	22.576388	22.57539771	22.59233293
26.1752	25.1461	25.291823	25.197372	25.265821	25.26527779	25.28762321
28.6729	27.5764	27.756004	27.6403069	27.717913	27.71812971	27.74620258
30.9414	29.7911	30.0045494	29.8678802	29.953626	29.95493061	29.98893081
33.0017	31.8096	32.0562843	31.899316	31.992071	31.99479802	32.03482341

Figure 4.15: Computation results for each RK method for mechanical engineering example



TRUE Results	Error_Euler	Error_Heun	Error_midpoint	Error_Ralstor	Error_third order	Error_fourth order	Error_fifth order
4.6818706	4.6590216	0.1436134	0.143613389	0.1436134	0.003310099	6.09425E-05	1.30578E-07
8.9531822	4.4354227	0.241706	0.126460405	0.2032908	0.026492481	0.031858677	0.015887815
12.849937	4.2196044	0.3330287	0.110560409	0.2580538	0.055359434	0.061773432	0.030324126
16.404981	4.0114882	0.4177613	0.095855366	0.3080802	0.083269244	0.089834125	0.04337841
19.648278	3.8109851	0.4960917	0.082287895	0.3535514	0.110203575	0.116073336	0.055119649
22.607167	3.617996	0.568215	0.0698014	0.3946518	0.136147269	0.140527115	0.065616264
25.306587	3.4324119	0.6343328	0.058340185	0.4315674	0.161088317	0.163234757	0.074935927
27.769291	3.2541143	0.6946524	0.047849573	0.4644862	0.185017819	0.184238583	0.083145376
30.016038	3.0829762	0.7493857	0.038276016	0.4935966	0.207929928	0.203583685	0.090310244
32.065765	2.918862	0.7987488	0.029567196	0.5190872	0.229821773	0.221317681	0.096494889

Figure 4.16: Computation results for exact results (true results) and true percent relative errors of each RK method for mechanical engineering example

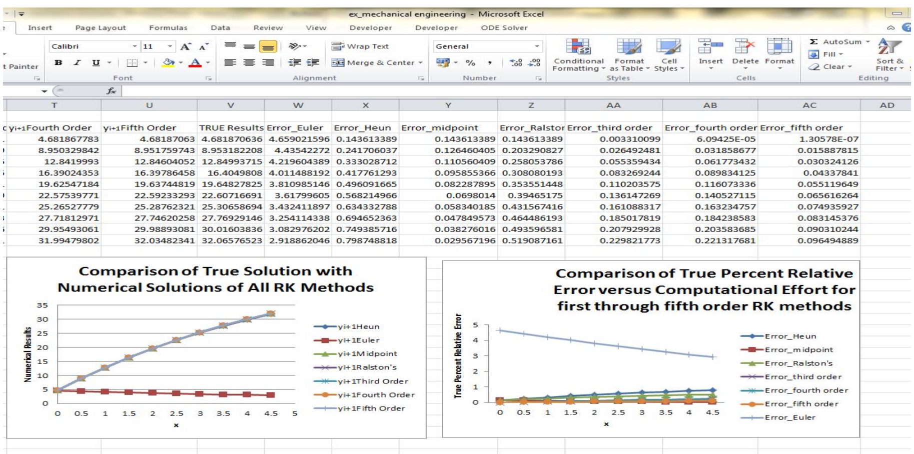


Figure 4.17: Graphical display of the computation results for mechanical engineering example

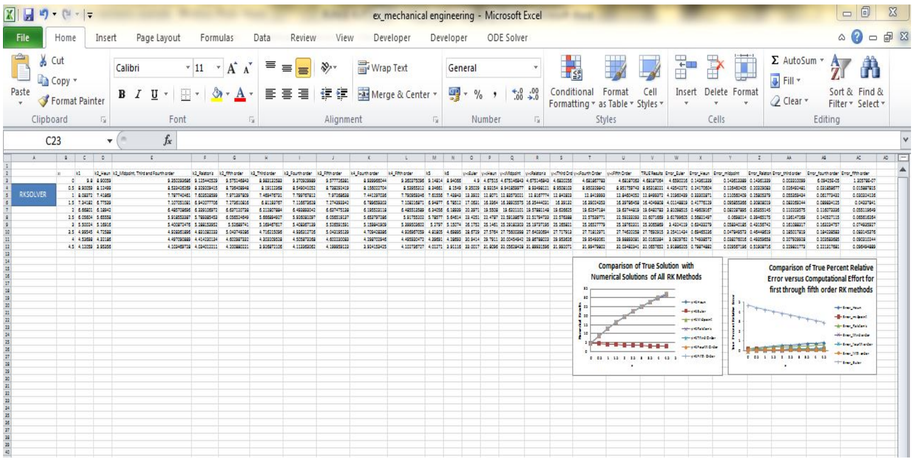


Figure 4.18: The spreadsheet image of full computation results for mechanical engineering example

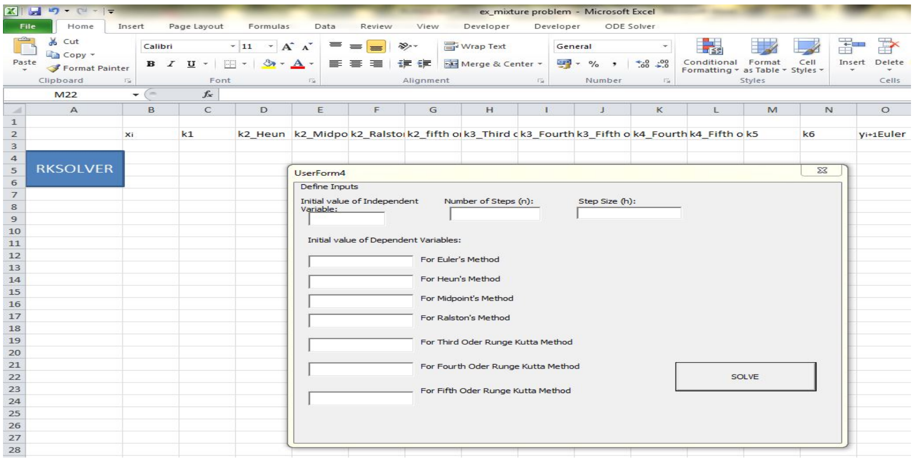


Figure 4.19: Userform in spreadsheet for mixture problem

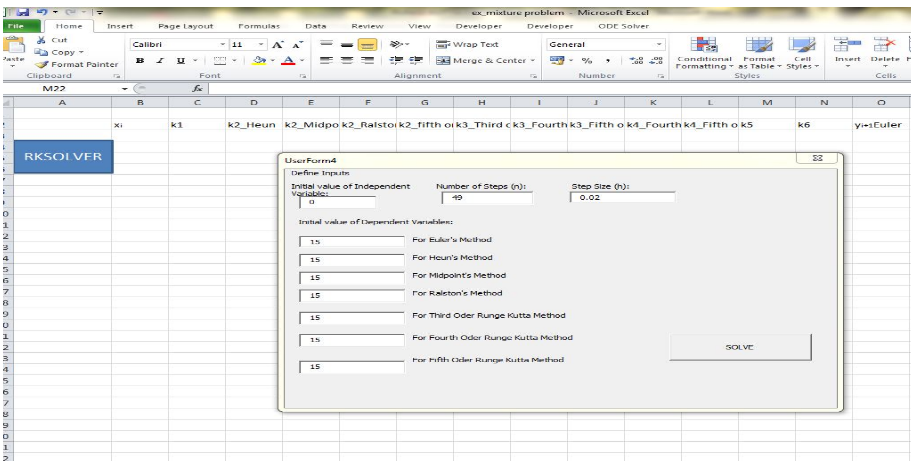


Figure 4.20: Filled userform for mixture problem

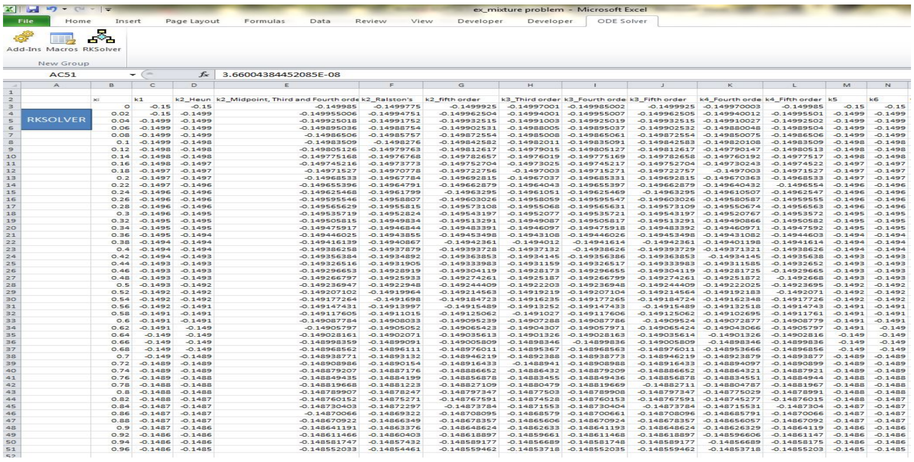


Figure 4.21: Computation results for k's for mixture problem

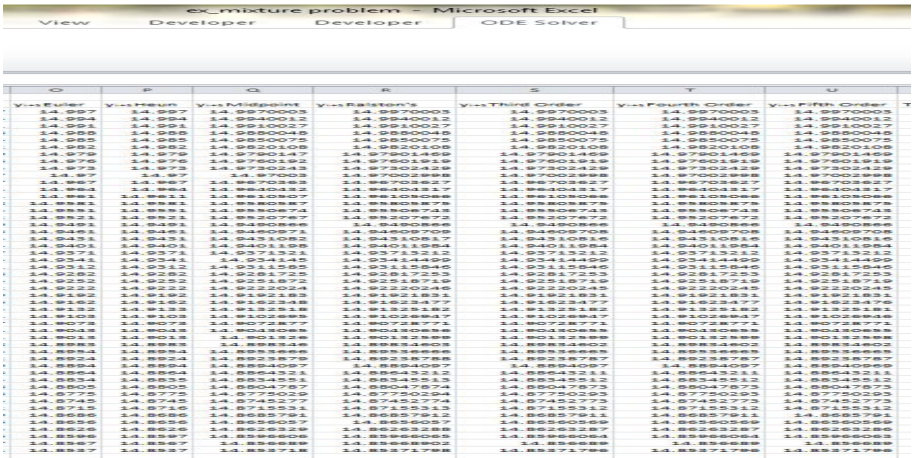


Figure 4.22: Computation results for each RK method for mixture problem

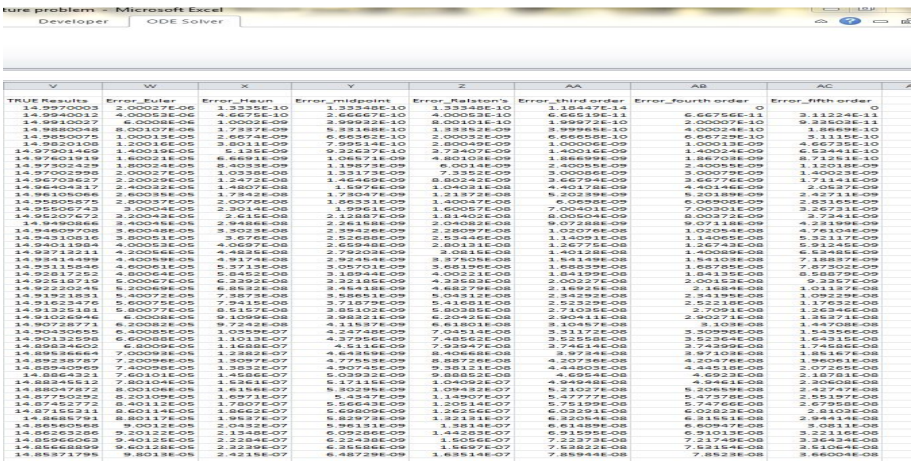


Figure 4.23: Computation results for exact results (true results) and true percent relative errors of each RK method for mixture problem

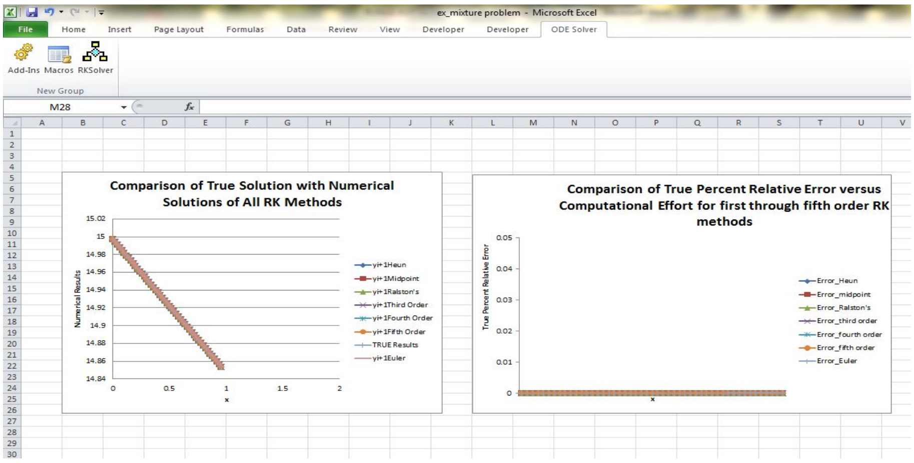


Figure 4.24: Graphical display of the computation results for mixture problem

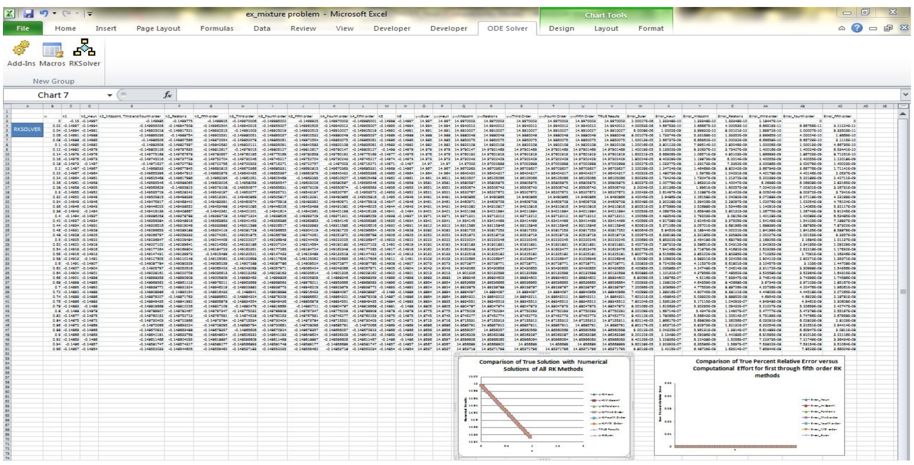


Figure 4.25: The spreadsheet image of full computation results for mixture problem

This spreadsheet solver is so user-friendly that users (students, educators and also beginner users of Excel and VBA) only require to click RKSOLVER button and enter relevant information in userform to perform all computations for the complete solution of IVPs efficiently without typing any commands in the spreadsheet.

It is hoped that this spreadsheet solver can be used as a marking scheme for users who need the complete solutions of IVPs numerically and analytically with comparison of them in terms of error at the same time. Lastly, it is hoped that this spreadsheet solver could serve as not only a numerical IVP tool but also an analytical IVP tool with a comparison of them that is convenient for the community of engineering educators and students.

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Elzaki transform combined with variational iteration method for partial differential equations of fractional order

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Abstract

The idea, which will be communicated through this paper is to make a change to the proposed method by Tarig M. Elzaki [21] and we extend it to solve nonlinear partial differential equations with time-fractional derivative. This document also includes illustrative examples show us how to apply this method, we also show the interest of combining these two methods is the speed of the calculates the terms, and not calculating the Lagrange multipliers.

1. Introduction

The nonlinear differential equations are a type of equations which are difficult to solve with respect to linear differential equations. Therefore, we find that a lot of researchers are working to discover new methods to enable us to solve this kind of equations. These efforts made, which is still ongoing, resulted in the promotion of this research in many methods, among them, we find the homotopy analysis method, Adomian decomposition method, variational iteration method [7, 8, 9] and homotopy perturbation method, which have become known in a large number of researchers in this area. A new option emerged recently, includes the composition of Laplace transform, Sumudu transform, natural transform or Elzaki transform with these methods. Among wich are the Laplace homotopy analysis method [15], homotopy analysis Sumudu transform method [18], modified fractional homotopy analysis transform method [11], Adomian decomposition method coupled with Laplace transform method [16], Sumudu decomposition method for nonlinear equations [5], Elzaki transform decomposition algorithm [13], natural decomposition method [14], variational iteration method coupled with Laplace transform method [4], variational iteration Sumudu transform method [3], Elzaki variational iteration method [21], homotopy perturbation transform method [17], homotopy perturbation Sumudu transform method [10], homotopy perturbation Elzaki transform method [12].
The motivation of this article is to make a change on the method proposed by Elzaki and suggested in [21], and then extend it to solve nonlinear partial differential equations with time-fractional derivative.
The present paper has been organized as follows: In Section 2 some basic definitions of ELzaki transform are montioned. In section 3 we will propose an analysis of the modified method. In section 4 it was presented three examples of application of this method (FEVIM). Finally, the conclusion follows

2. Basic definitions

2.1. Fractional calculus

In this section, we present some basic definitions and properties of fractional calculus [1, 6], and we focus specifically on the definitions of the following concepts: Riemann–Liouville fractional, Caputo fractional derivative, some important results, definition of Elzaki transform and Elzaki transform of fractional derivatives which are used further in this paper.

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Definition 2.1. *Let $\Omega = [\alpha, \beta]$ $(-\infty < \alpha < \beta < +\infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integral $I^\rho h$ of order $\rho \in \mathbb{R}$ $(\rho > 0)$ is defined by*

$$\begin{aligned} (I^\rho h)(\tau) &= \frac{1}{\Gamma(\rho)} \int_0^\tau \frac{h(\zeta) d\zeta}{(\tau - \zeta)^{1-\rho}}, \quad \tau > 0,\ \rho > 0, \\ (I^0 h)(\tau) &= h(\tau), \end{aligned} \tag{2.1}$$

where $\Gamma(x) = \int_0^\infty \tau^{x-1} e^{-\tau} d\tau$, $x > 0$, is called the gamma function of Euler.

Theorem 2.2. *Let $\rho \geqslant 0$ and let $m = [\rho] + 1$. If $h(\tau) \in AC^m[\alpha, \beta]$, then the Caputo fractional derivative $({}^c D^\rho_{0+} h)(\tau)$ exist almost evrywhere on $[\alpha, \beta]$.*
If $\rho \notin \mathbb{N}$, $({}^c D^\rho_\tau h)(\tau)$ is represented by

$$({}^c D^\rho_\tau h)(\tau) = \frac{1}{\Gamma(m - \rho)} \int_0^\tau \frac{h^{(m)}(\zeta) d\zeta}{(\tau - \zeta)^{\rho - m + 1}}, \tag{2.2}$$

where $D = \frac{d}{d\tau}$ and $m = [\rho] + 1$.

Proof. (see [1]). □

Remark 2.3. *The time-fractional derivative in the Caputo's sense, is given by*

$$\begin{aligned} ({}^c D^\rho_\tau w)(\varkappa, \tau) &= \frac{\partial^\rho w(\varkappa, \tau)}{\partial \tau^\rho} \\ &= \begin{cases} \frac{1}{\Gamma(k-\rho)} \int_0^\tau (\tau - \zeta)^{k-\rho-1} \frac{\partial^\rho w(\varkappa, \zeta)}{\partial \zeta^\rho}, & k - 1 < \rho < k, \\ \frac{\partial^k w(\varkappa, \tau)}{\partial \tau^k}, & \rho = k, \end{cases} \end{aligned} \tag{2.3}$$

where $k \in \mathbb{N}^*$ and $\rho \in \mathbb{R}^+$.
(1) Let $\rho > 0$ and let $m = [\rho] + 1$ for $m \notin \mathbb{N}$, $m = \rho$ for $m \in \mathbb{N}$. If $h(\tau) \in AC^m[\alpha, \beta]$, then

$$(I^\rho_{0+}\, {}^c D^\rho_{0+} h)(\tau) = h(\tau) - \sum_{j=0}^{m-1} \frac{h^{(j)}(0)}{j!} \tau^j.$$

(2) $(I^\rho_{0+} x^{\lambda-1})(\tau) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\rho)} \tau^{\lambda+\rho-1}$, $\rho > 0$, $\lambda > 0$.
(3) $({}^c D^\rho_{0+} x^{\lambda-1})(\tau) = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\rho)} t^{\lambda-\beta-1}$, $\alpha > 0$, $\beta > m$.
(4) $({}^c D^\rho_{0+} C)(\tau) = 0$, where C is constant.

2.2. Definitions of Elzaki transform

A new integral transform called Elzaki transform [20] defined for functions of exponential order, is proclaimed. They consider functions in the set G defined by

$$G = \left\{ h(\tau) : \exists Q,\ p_1, p_2 > 0,\ |h(\tau)| < Q e^{\frac{|\tau|}{p_i}},\ if\ \tau \in (-1)^i \times [0, \infty) \right\}.$$

Definition 2.4. *If $h(\tau)$ is function defined for all $\tau \geqslant 0$, its Elzaki transform is defined by $E[h]$*

$$E[h(\tau)] = T(s) = s \int_0^\infty h(\tau) e^{-\frac{\tau}{s}} d\tau. \tag{2.4}$$

Theorem 2.5. *Elzaki transform amplifies the coefficients of the power series function*

$$h(\tau) = \sum_{n=0}^\infty a_n \tau^n, \tag{2.5}$$

on the new integral transform "Elzaki transform", given by

$$E[h(\tau)] = T(v) = \sum_{n=0}^\infty n! a_n v^{n+2}. \tag{2.6}$$

Theorem 2.6. *Let $h(\tau)$ be in G and Let $T_n(v)$ denote Elzaki transform of nth derivative $h^{(n)}(\tau)$ of $h(\tau)$, then for $n \geq 1$,*

$$T_n(v) = \frac{T(v)}{v^n} - \sum_{j=0}^{n-1} v^{2-n+j} h^{(j)}(0). \tag{2.7}$$

By using the integration by parts, Elzaki transform of partial derivative is given as

$$\begin{aligned} E\left(\frac{\partial h(\varkappa,\tau)}{\partial \tau}\right) &= \frac{1}{v} T(\varkappa, v) - v h(\varkappa, 0), \\ E\left(\frac{\partial^2 h(\varkappa,\tau)}{\partial \tau^2}\right) &= \frac{1}{v^2} T(\varkappa, v) - h(\varkappa, 0) - v \frac{\partial h(\varkappa,0)}{\partial \tau} \end{aligned} \tag{2.8}$$

2.3. Elzaki transform of fractional derivatives

To give the formula of Elzaki transform of Caputo fractional derivative, we use the Laplace transform formula for the Caputo fractional derivative [6]

$$L\{\, ({}^c D_\tau^\rho h)(\tau); u\} = u^\rho F(u) - \sum_{i=0}^{n-1} s^{\rho-i-1} f^{(i)}(0),$$

where $n-1 < \rho \leqslant n,\, n \in \mathbb{N}^*$.

Theorem 2.7. *[19] Let G defined as above. With Laplace transform $F(u)$, then the Elzaki transform $T(v)$ of $h(\tau)$ is given by*

$$T(v) = vF(\frac{1}{v}).$$

Theorem 2.8. *Suppose $T(v)$ is the Elzaki transform of the function $h(\tau)$ then*

$$E\{\, ({}^c D_t^\rho h)(\tau), v\} = \frac{T(v)}{v^\rho} - \sum_{i=0}^{n-1} v^{i-\rho+2} h^{(i)}(0). \tag{2.9}$$

Proof. (see [2]). □

3. Fractional Elzaki Variational Iteration Method (FEVIM)

The work that we will do in this paragraph, is to make a change to the method proposed in [21], and we extend to solve nonlinear partial differential equations of order ρ, $(\, n-1 < \rho \leqslant n,\, n = 1, 2, \ldots)$. For this cause, we consider a general nonlinear partial differential equation with time-fractional derivative

$${}^c D_\tau^\rho Z(\varkappa, \tau) + RZ(\varkappa, \tau) + NZ(\varkappa, \tau) = f(\varkappa, \tau), \tag{3.1}$$

subject to the initial conditions

$$\left[\frac{\partial^{n-1} Z(\varkappa, \tau)}{\partial \tau^{n-1}}\right]_{\tau=0} = g_{n-1}(\varkappa), \tag{3.2}$$

where ${}^c D_\tau^\rho = \frac{\partial^\rho}{\partial \tau^\rho}$ is the Caputo fractional derevative, R is the linear differential operator, N represents the general nonlinear differential operator, and $g(\varkappa, \tau)$ is the source term.
Applying Elzaki transform on both sides of (3.1), we obtain

$$E\left[{}^c D_\tau^\rho Z(\varkappa, \tau)\right] + E\left[RZ(\varkappa, \tau)\right] + E\left[NZ(\varkappa, \tau)\right] = E\left[f(\varkappa, \tau)\right]. \tag{3.3}$$

Depending on the properties of Elzaki transform, the equation (3.3) becomes

$$E\left[Z(\varkappa, \tau)\right] = \sum_{i=0}^{n-1} v^{i+2} g_i(\varkappa) + v^\rho E\left[f(\varkappa, \tau)\right] - v^\rho E\left[RZ(\varkappa, \tau) + NZ(\varkappa, \tau)\right]. \tag{3.4}$$

Operating with the inverse Elzaki transform on both sides of (3.4), we obtain

$$Z(\varkappa, \tau) = K(\varkappa, \tau) - E^{-1}\left(v^\rho E\left[RZ(\varkappa, \tau) + NZ(\varkappa, \tau)\right]\right). \tag{3.5}$$

where $K(\varkappa, \tau) = \sum_{i=0}^{n-1} v^{i+2} g_i(\varkappa) + v^\rho E\left[f(\varkappa, \tau)\right]$.
Applying $\frac{\partial}{\partial t}$ on both sides of (3.5), we have

$$\frac{\partial Z(\varkappa,\tau)}{\partial \tau}+\frac{\partial}{\partial \tau}E^{-1}\left(v^{\rho}E\left[RZ(\varkappa,\tau)+NZ(\varkappa,\tau)\right]\right)-\frac{\partial K(\varkappa,\tau)}{\partial \tau}=0. \tag{3.6}$$

According to the variational iteration method [8], we can construct a correct functional as follows

$$Z_{m+1}(\varkappa,\tau)=\ Z_m(\varkappa,\tau)-\int_0^{\tau}\left[\frac{\partial Z_m}{\partial \zeta}+\frac{\partial}{\partial \zeta}E^{-1}\left(v^{\rho}E\left[RZ_m+NZ_m\right]\right)-\frac{\partial K}{\partial \zeta}\right]d\zeta. \tag{3.7}$$

Or alternately

$$Z_{m+1}(\varkappa,\tau)=K(\varkappa,\tau)-E^{-1}\left(v^{\rho}E\left[RZ_m(\varkappa,\tau)+NZ_m(\varkappa,\tau)\right]\right). \tag{3.8}$$

Recall that $Z(\varkappa,\tau)=\lim_{m\to\infty}Z_m(\varkappa,\tau)$.
According to the preceding limit, we can obtain the exact solution if it exists, or we obtain an approximate solution for the considered equation.

4. Applications

In the following examples, we'll apply the method proposed in the previous paragraph to solve nonlinear time-fractional partial differential equations.

Example 4.1. *First, we consider the nonlinear time-fractional partial differential equation*

$$^{c}D_{\tau}^{\rho}Z+ZZ_{\varkappa}-Z_{\varkappa}=0,\ \ 0<\rho\leqslant 1, \tag{4.1}$$

subject to the initial condition

$$Z(\varkappa,0)=\varkappa+1. \tag{4.2}$$

If $\rho=1$, *we obtain*

$$Z_{\tau}+ZZ_{\varkappa}-Z_{\varkappa}=0. \tag{4.3}$$

The exact solution of (4.3), is

$$Z(\varkappa,\tau)=1+\frac{\varkappa}{1+\tau}. \tag{4.4}$$

According to (3.8), we can construct the folowing formula

$$Z_{m+1}(\varkappa,\tau)=\varkappa+1-E^{-1}\left(v^{\rho}E\left[\left(Z_m-1\right)\left(Z_m\right)_{\varkappa}\right]\right). \tag{4.5}$$

Using the iteration formula (4.5), we get

$$\begin{array}{c}Z_0(\varkappa,\tau)=\varkappa+1,\\ Z_1(\varkappa,\tau)=1+\varkappa-\varkappa\frac{\tau^{\rho}}{\Gamma(\rho+1)},\\ Z_2(\varkappa,\tau)=1+\varkappa-\varkappa\frac{\tau^{\rho}}{\Gamma(\rho+1)}+2\varkappa\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}-\varkappa\frac{\Gamma(2\rho+1)}{\Gamma^2(\rho+1)}\frac{\tau^{3\rho}}{\Gamma(3\rho+1)},\\ Z_3(\varkappa,\tau)=1+\varkappa-\varkappa\frac{\tau^{\rho}}{\Gamma(\rho+1)}+2\varkappa\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}-a_1\varkappa\tau^{3\rho}+a_2\varkappa\tau^{4\rho}-a_3\varkappa\tau^{5\rho}\\ +a_4\varkappa\tau^{6\rho}-a_5\varkappa\tau^{7\rho},\\ \vdots\end{array} \tag{4.6}$$

where
$a_1=\left[\frac{4}{\Gamma(2\rho+1)}+\frac{1}{\Gamma^2(\rho+1)}\right]\frac{\Gamma(2\rho+1)}{\Gamma(3\rho+1)}$, $a_2=\left[\frac{4}{\Gamma(\rho+1)\Gamma(2\rho+1)}+\frac{2\Gamma(2\rho+1)}{\Gamma^2(\rho+1)\Gamma(3\rho+1)}\right]\frac{\Gamma(3\rho+1)}{\Gamma(4\rho+1)}$,
$a_3=\left[\frac{2\Gamma(2\rho+1)}{\Gamma^3(\rho+1)\Gamma(3\rho+1)}+\frac{4}{\Gamma^2(2\rho+1)}\right]\frac{\Gamma(4\rho+1)}{\Gamma(5\rho+1)}$, $a_4=\frac{4}{\Gamma^2(\rho+1)\Gamma(3\rho+1)}\times\frac{\Gamma(5\rho+1)}{\Gamma(6\rho+1)}$,
$a_5=\frac{\Gamma^2(2\rho+1)}{\Gamma^4(\rho+1)\Gamma^2(3\rho+1)}\times\frac{\Gamma(6\rho+1)}{\Gamma(7\rho+1)}$.
Recall that, the exact solution of Eq.(4.1) is calculated by

$$Z(\varkappa,\tau)=\lim_{m\to\infty}Z_m(\varkappa,\tau).$$

From (4.6), the approximate solution of (4.1), is

$$\begin{aligned} Z(\varkappa,\tau) &= 1+\varkappa-\varkappa\frac{\tau^{\rho}}{\Gamma(\rho+1)}+2\varkappa\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}-a_1\varkappa\tau^{3\rho}+a_2\varkappa\tau^{4\rho}-a_3\varkappa\tau^{5\rho} \\ &\quad +a_4\varkappa\tau^{6\rho}-a_5\varkappa\tau^{7\rho}\cdots, \end{aligned}$$

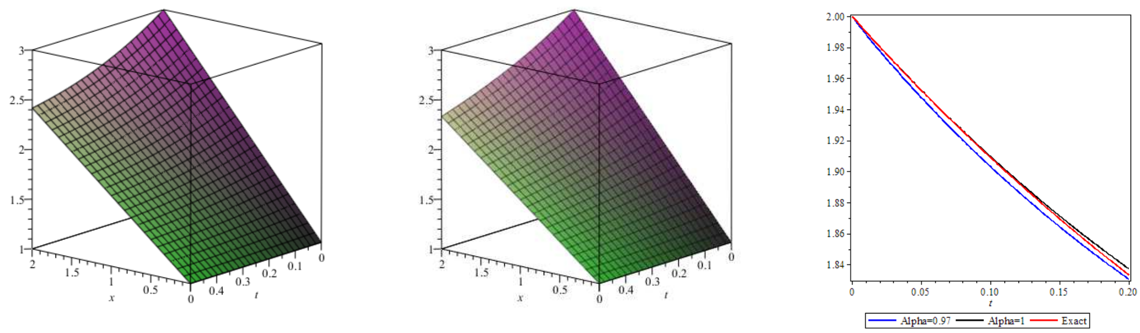
and in the special case $\rho=1$, is

$$Z(\varkappa,\tau)=1+\varkappa\left(1-\tau+\tau^2-\tau^3+\frac{2}{3}\tau^4-\frac{1}{3}\tau^5+\frac{1}{9}\tau^6-\frac{1}{63}\tau^7+\cdots\right).$$

When $m\longrightarrow+\infty$, we get the following exact solution

$$Z(\varkappa,\tau)=1+\frac{\varkappa}{1+\tau},\ \ |\tau|<1.$$

which is an exact solution to the nonlinear partial differential equation.



(a) (b) (c)

Figure 4.1: (a) Exact solution, (b) The approximate solution in the case $\rho=1$, (c) The exact solution and approximate solutions to (4.1) for different values of ρ when $\varkappa=1$. From (c) noted that the graphics have changed his position based on ρ values, if ρ took values closer to 1, we see that the graph corresponding to this value is barely graphical representation of the exact solution.

Example 4.2. *Next, we consider the nonlinear time-fractional partial differential equation*

$$^{c}D_{\tau}^{\alpha}Z+\frac{2}{\tau}ZZ_{\varkappa}=0,\ \tau>0,\ \varkappa\neq0,\ \ 1<\rho\leqslant2, \tag{4.7}$$

with

$$Z(\varkappa,0)=0,\ \ Z_{\tau}(\varkappa,0)=\frac{1}{\varkappa}. \tag{4.8}$$

If $\rho=2$, we obtain

$$Z_{\tau}+\frac{2}{\tau}ZZ_{\varkappa}=0,\ \ \tau>0,\ \ \varkappa\neq0 \tag{4.9}$$

The exact solution of (4.9), is

$$Z(\varkappa,\tau)=\tan\left(\frac{\tau}{\varkappa}\right). \tag{4.10}$$

According to (3.8), we can construct the iteration formula as follows

$$Z_{m+1}=\frac{\tau}{\varkappa}-E^{-1}\left[v^{\rho}E\left[\frac{2}{\tau}Z_m(Z_m)_{\varkappa}\right]\right]. \tag{4.11}$$

Using the iteration formula (4.11), we obtain

$$\begin{gathered} Z_0(\varkappa,\tau)=\tfrac{\tau}{\varkappa}, \\ Z_1(\varkappa,\tau)=\tfrac{\tau}{\varkappa}+\tfrac{2}{\varkappa^3}\tfrac{\tau^{\rho+1}}{\Gamma(\rho+2)}, \\ Z_2(\varkappa,\tau)=\tfrac{\tau}{\varkappa}+\tfrac{2}{\Gamma(\rho+2)}\tfrac{\tau^{\rho+1}}{\varkappa^3}+\tfrac{16}{\Gamma(2\rho+2)}\tfrac{\tau^{2\rho+1}}{\varkappa^5}+\tfrac{24\Gamma(2\rho+2)}{\Gamma^2(\rho+2)\Gamma(3\rho+2)}\tfrac{\tau^{3\rho+1}}{\varkappa^7}, \\ \vdots \end{gathered}$$

The approximate solution is giveb by

$$\begin{aligned} Z(\varkappa,\tau) &= \frac{\tau}{\varkappa}+\frac{2}{\Gamma(\rho+2)}\frac{\tau^{\rho+1}}{\varkappa^3}+\frac{16}{\Gamma(2\rho+2)}\frac{\tau^{2\rho+1}}{\varkappa^5} \\ &\quad +\frac{24\Gamma(2\rho+2)}{\Gamma^2(\rho+2)\Gamma(3\rho+2)}\frac{\tau^{3\rho+1}}{\varkappa^7}+\cdots \end{aligned} \tag{4.12}$$

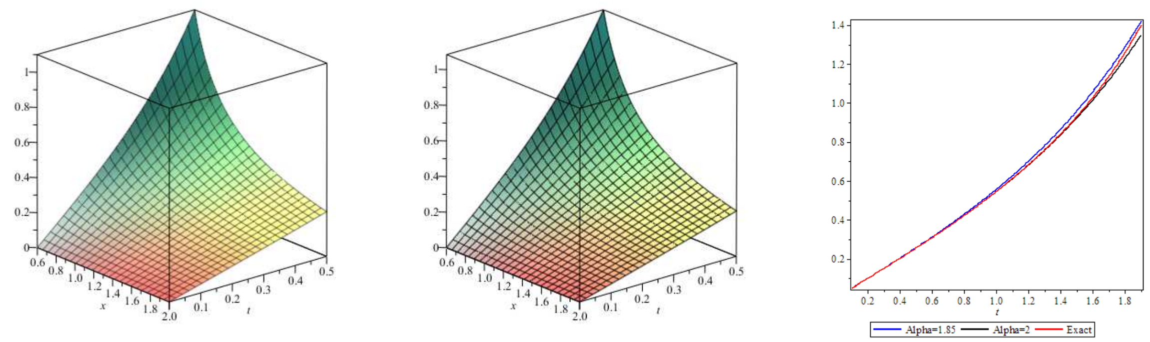
As $\rho\longrightarrow 2$, *we obtain*

$$Z(\varkappa,\tau)=\frac{\tau}{\varkappa}+\frac{1}{3}\left(\frac{\tau}{\varkappa}\right)^3+\frac{2}{15}\left(\frac{\tau}{\varkappa}\right)^5+\frac{1}{63}\left(\frac{\tau}{\varkappa}\right)^7.$$

And in closed form, is given by

$$Z(\varkappa,\tau)=\tan\left(\frac{\tau}{\varkappa}\right),$$

we get the exact solution of (4.7) when $\rho=2$.



(a') (b') (c')

Figure 4.2: (a') Exact solution, (b') the approximate solution in the case $\rho=2$, (c') The exact solution and approximate solutions to (4.7) for different values of ρ when $\varkappa=2$. From (c') noted that the graphics have changed his position based on ρ values, if ρ took values closer to 2, we see that the graph corresponding to this value is barely graphical representation of the exact solution.

Example 4.3. *finally, we consider the nonlinear time-fractional partial differential equation*

$${}^{c}D_{\tau}^{\rho}Z-\frac{3}{8}\left[(Z_{\varkappa\varkappa})^2\right]_{\varkappa}=\frac{3}{2}\tau,\ \ 2<\rho\leqslant 3, \tag{4.13}$$

with

$$Z(\varkappa,0)=\frac{1}{2}\varkappa^2,\ \ Z_{\tau}(\varkappa,0)=\frac{1}{3}\varkappa^3,\ \ Z_{\tau\tau}(\varkappa,0)=0. \tag{4.14}$$

If $\rho=3$, *we obtain*

$$Z_{\tau\tau\tau}-\frac{3}{8}\left[(Z_{\varkappa\varkappa})^2\right]_{\varkappa}=\frac{3}{2}\tau. \tag{4.15}$$

According to (3.8), we can construct the following iteration formula

$$Z_{m+1}=-\frac{1}{2}\varkappa^2+\frac{1}{3}\varkappa^3\tau+\frac{3}{2}\frac{\tau^{\rho+1}}{\Gamma(\rho+2)}-E^{-1}\left(v^{\rho}E\left[-\frac{3}{8}\left[(Z_{m\varkappa\varkappa})^2\right]_{\varkappa}\right]\right). \tag{4.16}$$

Use the (4.16) to get

$$\begin{aligned} Z_0(\varkappa,\tau) &= -\tfrac{1}{2}\varkappa^2 + \tfrac{1}{3}\varkappa^3\tau, \\ Z_1(\varkappa,\tau) &= -\tfrac{1}{2}\varkappa^2 + \tfrac{1}{3}\varkappa^3\tau + 6\varkappa\tfrac{\tau^{\rho+2}}{\Gamma(\rho+3)}, \\ Z_2(\varkappa,\tau) &= -\tfrac{1}{2}\varkappa^2 + \tfrac{1}{3}\varkappa^3\tau + 6\varkappa\tfrac{\tau^{\rho+2}}{\Gamma(\rho+3)}, \\ Z_3(\varkappa,\tau) &= -\tfrac{1}{2}\varkappa^2 + \tfrac{1}{3}\varkappa^3\tau + 6\varkappa\tfrac{\tau^{\rho+2}}{\Gamma(\rho+3)}, \\ &\vdots \end{aligned} \tag{4.17}$$

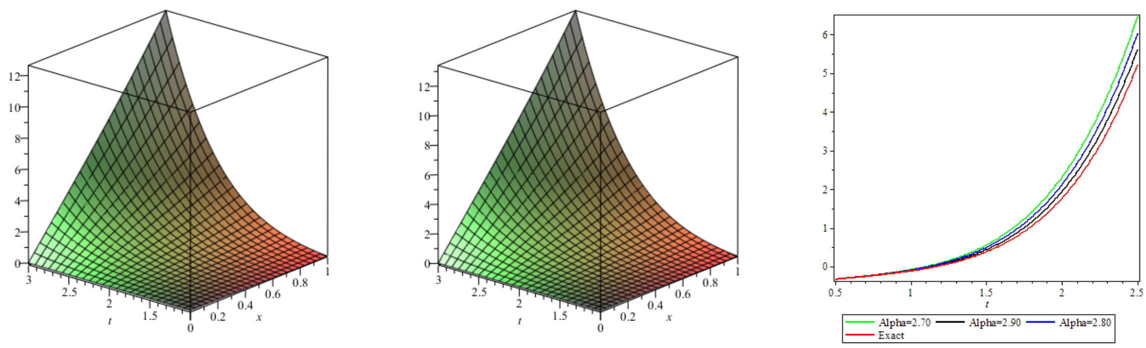
The approximate solution in a series form, is given by

$$Z(\varkappa,\tau) = -\frac{1}{2}\varkappa^2 + \frac{1}{3}\varkappa^3\tau + 6\varkappa\frac{\tau^{\rho+2}}{\Gamma(\rho+3)}. \tag{4.18}$$

As $\rho \longrightarrow 3$, *we obtain the following exact solution*

$$Z(\varkappa,\tau) = \frac{1}{20}\varkappa\tau^5 + \frac{1}{3}\varkappa^3\tau - \frac{1}{2}\varkappa^2.$$

That gives the exact solution of (4.13) when $\rho = 3$.



(a”) (b”) (c”)

Figure 4.3: (a”) Exact solution, (b”)the approximate solution in the case $\rho = 2.90$, (c”) The exact solution and approximate solutions to (4.13) for different values of ρ when $\varkappa = 1$. From (c”) noted that the graphics have changed his position based on ρ values, if ρ took values closer to 3, we see that the graph corresponding to this value is barely graphical representation of the exact solution.

5. Conclusion

Coupling of Variational Iteration Method and Elzaki Transform, to be an effective method for solving nonlinear partial differential equations with time-fractional derivative. The proposed algorithm is suitable for such problems and is very efficient. From the results, it is clear that the FVIETM yields very accurate approximate solutions using only a few iterates. It provides a solution as a more realistic series, which converges rapidly to the exact solution.

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