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Approximation by Baskakov-Szász-Stancu Operators Preserving Exponential Functions

MURAT BODUR*, ÖVGÜ GÜREL YILMAZ, AND ALI ARAL

ABSTRACT. The purpose of this paper is to construct a general class of operators which has known Baskakov-Szász-Stancu that preserving constant and $e^{2ax}, a > 0$ functions. We scrutinize a uniform convergence result and analyze the asymptotic behavior of our operators, as well. Finally, we discuss the convergence of corresponding sequences in exponential weighted spaces and make a comparison about which one approximates better between classical Baskakov-Szász-Stancu operators and the recent operators.

Keywords: Baskakov-Szász-Stancu operators, Exponential functions, Quantitative results, Weighted approximation.

2010 Mathematics Subject Classification: 41A25, 41A36.

1. INTRODUCTION

Approximation theory is one of the crucial subjects that is used by researchers. It is separated into many fields one of which is positive linear operators that play a key role in approximation theory. It has been the inspiration for so many mathematicians from the past. For years, many publications related to the approximation theory has made and has still being studied, too.

One of the remarkable work has been done related to the positive linear operators from King [16] in 2003. King described the modified Bernstein operators which preserve for $e_i(t) = t^i$, $i = 0, 2$ test functions and examined their approximation properties. He accomplished to take an attention in a short time from researchers who perform approximation theory. Since that time, lots of researchers have put forth many relevant studies on this issue. Numerous articles can be given interrelated with Kings research ([4–7], [9], [18]).

In 2016 Acar et al. [1] investigated approximation properties of Szász-Mirakyan operators which preserving constant and $e^{2ax},\ a > 0$. In that paper the rate of convergence of this generalization was obtained by means of the modulus of continuity. They also presented and proved theorems related on shape preserving properties.

Later this idea was applied to some other well known linear positive operators, such as Szász-Durrmeyer [8], Szász-Mirakyan-Kantorovich [13], Baskakov-Szász-Mirakyan [12], Philips Operators [14], Baskakov Operators [19] and the subject is still continue to be relevant.

In this study, inspired by the main paper [1], we constructed a new family of linear positive operators by using Baskakov-Szász-Stancu operators which is based on preserving exponential functions.

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*Corresponding author: M. Bodur; bodur@ankara.edu.tr
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2. Construction of the Operators

For $f \in C[0,\infty)$ and $k,n \in \mathbb{N}$ the Baskakov-Szász type operators was proposed by Gupta and Srivastava [?] as,

$$M_n(f;x) = n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}f(t)dt.$$

In 2015, Mishra et al. considered Stancu type generalization of Baskakov-Szász operators [17] like as,

$$B_n^{\alpha,\beta}(f;x) = n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}(t)f(\frac{nt+\alpha}{n+\beta})dt. \tag{2.1}$$

Here, two parameters α and β satify the condition $0 \le \alpha \le \beta$. We consider the following modified form of Baskakov-Szász-Stancu operators

$$M_n^{\alpha,\beta}(f;x) = n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{(\theta_n(x))^k}{(1+\theta_n(x))^{n+k}}\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}f(\frac{nt+\alpha}{n+\beta})dt. \tag{2.2}$$

We interest in investigated operators preserving e_0 and e^{2ax}. Suppose these operators (2.2) preserve e^{2ax}, then

$$\begin{aligned}
M_n^{\alpha,\beta}(e^{2at};x) &= n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{(\theta_n(x))^k}{(1+\theta_n(x))^{n+k}}\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}e^{2a(nt+\alpha)/n+\beta}dt \\
&= ne^{2a\alpha/n+\beta}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{(\theta_n(x))^k}{(1+\theta_n(x))^{n+k}}\frac{n^k}{k!}\int_0^{\infty} e^{-nt(n+\beta-2a)/(n+\beta)}t^k dt \\
&= \frac{n+\beta}{n+\beta-2a}e^{2a\alpha/n+\beta}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{(n+\beta)\theta_n(x)}{n+\beta-2a}\right)^k\left(\frac{1}{1+\theta_n(x)}\right)^{n+k} \\
&= \frac{n+\beta}{n+\beta-2a}e^{2a\alpha/n+\beta}\left(\theta_n(x)(1-\frac{n+\beta}{n+\beta-2a})+1\right)^{-n}.
\end{aligned}$$

Taking into account $M_n^{\alpha,\beta}(e^{2at};x) = e^{2ax}$, then we can find without hesitation

$$\theta_n(x) = \frac{n+\beta-2a}{2a}\left(1-\left(\frac{n+\beta-2a}{n+\beta}e^{2a(x(n+\beta)-\alpha)/n+\beta}\right)^{-1/n}\right). \tag{2.3}$$

Seemingly the function which is demonstrated $\theta_n(x)$ satisfies the situtation

$$\theta_n(x) = \left(M_n^{\alpha,\beta}(e^{2at};x)\right)^{-1}\circ e^{2ax}.$$

3. Auxiliary Results

In this part, we shall present the moments and the central moments of the operators (2.2) which will be necessary to prove our main results.

Lemma 3.1. *Let* $e_i(t) := t^i,\ i = 0, 1, 2$. *Then the Baskakov-Szász-Stancu operators* $M_n^{\alpha,\beta}$ *satisfies*

$$\begin{aligned} M_n^{\alpha,\beta}(e_0;x) &= 1, \\ M_n^{\alpha,\beta}(e_1;x) &= \frac{n\theta_n(x)+1+\alpha}{n+\beta}, \\ M_n^{\alpha,\beta}(e_2;x) &= \frac{n(n+1)}{(n+\beta)^2}(\theta_n(x))^2 + \frac{4n+2n\alpha}{(n+\beta)^2}\theta_n(x) + \frac{2+2\alpha+\alpha^2}{(n+\beta)^2}. \end{aligned}$$

Lemma 3.2. *Let* $\mu_{n,r}^{\alpha,\beta}(x) = M_n^{\alpha,\beta}((t-x)^r, x),\ r = 0, 1, 2, ...$ *Then by considering Lemma (3.1), we have*

$$\begin{aligned} \mu_{n,0}^{\alpha,\beta} &= 1, \\ \mu_{n,1}^{\alpha,\beta} &= \frac{n\theta_n(x)+1+\alpha}{n+\beta} - x, \\ \mu_{n,2}^{\alpha,\beta} &= \frac{n(n+1)}{(n+\beta)^2}(\theta_n(x))^2 + \frac{4n+2n\alpha}{(n+\beta)^2}\theta_n(x) + \frac{2+2\alpha+\alpha^2}{(n+\beta)^2} - 2x\frac{n\theta_n(x)+1+\alpha}{n+\beta} + x^2. \end{aligned}$$

Respectively, limits hold

$$\lim_{n\to\infty} n\left(\frac{n\theta_n(x)+1+\alpha}{n+\beta} - x\right) = -ax(x+2)$$

and

$$\lim_{n\to\infty} n\left(\left(\frac{n\theta_n(x)+1+\alpha}{n+\beta} - x\right)^2 + \left(\frac{n\theta_n(x)}{n+\beta}\right)^2 + 2n\theta_n(x) + 1\right) = x(x+2).$$

4. Main Results

In this main section, we would like to show that the constructed operators are meticulously discussed linked with a uniform convergence result and a quantitative estimate. We debate the convergence of corresponding sequences in exponential weighted spaces.

Here, we will take $C^*[0,\infty)$ the class of real-valued continuous functions f, possessing finite limit for x sufficiently large and equipped with the uniform norm.

Theorem A. [15] *Take into account a sequence of positive linear operators* $L_n : C^*[0,\infty) \to C^*[0,\infty)$ *and set*

$$||L_n(e_0) - 1||_{[0,\infty)} = \alpha_n^*,$$

$$\left|\left|L_n(e^{-t}) - e^{-x}\right|\right|_{[0,\infty)} = \beta_n^*,$$

$$\left|\left|L_n(e^{-2t}) - e^{-2x}\right|\right|_{[0,\infty)} = \gamma_n^*,$$

then for each $f \in C^*[0,\infty)$

$$||L_n(f) - f||_{[0,\infty)} \le \alpha_n^* \,||f||_{[0,\infty)} + (2+\alpha_n^*)\omega^*(f; \sqrt{\alpha_n^* + 2\beta_n^* + \gamma_n^*}).$$

where the modulus of continuity is defined as

$$\omega^*(f;\delta) := \sup_{\substack{x,t\ge 0 \\ |e^{-x}-e^{-t}|\le\delta}} |f(t) - f(x)|.$$

Now, we can apply Theorem A for Baskakov-Szász-Stancu operators.

Theorem 4.1. *For each function $f \in C^*[0,\infty)$, we possess*

$$||M_n^{\alpha,\beta} f - f||_{[0,\infty)} \leq 2\omega^*(f;\sqrt{2\beta_n^* + \gamma_n^*}),$$

where

$$||M_n^{\alpha,\beta}(e^{-t}) - e^{-x}||_{[0,\infty)} = \beta_n^*,$$

$$||M_n^{\alpha,\beta}(e^{-2t}) - e^{-2x}||_{[0,\infty)} = \gamma_n^*.$$

Proof. According the definition of the operators, since they preserve the constants, we reach

$$||M_n^{\alpha,\beta}(e_0) - 1||_{[0,\infty)} = \alpha_n^* = 0$$

and for $f(t) = e^{-t}$, we get

$$\begin{aligned} M_n^{\alpha,\beta}(e^{-t};x) &= n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{(\theta_n(x))^k}{(1+\theta_n(x))^{n+k}}\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}e^{-(nt+\alpha)/n+\beta}dt \\ &= \frac{n+\beta}{n+\beta+1}e^{-\alpha/n+\beta}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{(n+\beta)\theta_n(x)}{n+\beta+1}\right)^k\left(\frac{1}{1+\theta_n(x)}\right)^{n+k} \\ &= \frac{n+\beta}{n+\beta+1}e^{-\alpha/n+\beta}\left(\frac{\theta_n(x)}{n+\beta+1}+1\right)^{-n}. \end{aligned} \tag{4.4}$$

Using Maple to make a calculation of the right hand side which is found (4.4), we observe

$$M_n^{\alpha,\beta}\left(e^{-t};x\right) = e^{-x} + \frac{e^{-x}(2a+1)(x^2+2x)}{2n} + \mathcal{O}(\frac{1}{n^2}),$$

$$||M_n^{\alpha,\beta}(e^{-t}) - e^{-x}||_{[0,\infty)} = \frac{2(2a+1)}{ne^2} + \frac{2a+1}{ne} + \mathcal{O}(\frac{1}{n^2}) = \beta_n^*.$$

Also, for $f(t) = e^{-2t}$, we have

$$\begin{aligned} M_n^{\alpha,\beta}(e^{-2t};x) &= n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{(\theta_n(x))^k}{(1+\theta_n(x))^{n+k}}\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}e^{-2(nt+\alpha)/n+\beta}dt \\ &= \frac{n+\beta}{n+\beta+2}e^{-2\alpha/n+\beta}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{(n+\beta)\theta_n(x)}{n+\beta+2}\right)^k\left(\frac{1}{1+\theta_n(x)}\right)^{n+k} \\ &= \frac{n+\beta}{n+\beta+2}e^{-2\alpha/n+\beta}\left(\frac{2\theta_n(x)}{n+\beta+2}+1\right)^{-n}. \end{aligned} \tag{4.5}$$

If the procedure applied for previous equality is performed again, we receive

$$M_n^{\alpha,\beta}\left(e^{-2t};x\right) = e^{-2x} + \frac{e^{-2x}(a+1)(2x^2+4x)}{n} + \mathcal{O}(\frac{1}{n^2}),$$

$$||M_n^{\alpha,\beta}(e^{-2t}) - e^{-2x}||_{[0,\infty)} = \frac{2(a+1)}{n}(\frac{1}{e^2}+\frac{1}{e}) + \mathcal{O}(\frac{1}{n^2}) = \gamma_n^*.$$

Here, β_n^* and γ_n^* tend to zero as $n \to \infty$ so this completes the proof. □

Now, we will analyze the asymptotic behavior of given operators $M_n^{\alpha,\beta}$ with Voronovskaya-type theorem.

Theorem 4.2. *Let $f, f'' \in C^*[0,\infty)$ then for any $x \in [0,\infty)$ we have*

$$\begin{aligned} & \left|n[M_n^{\alpha,\beta}(f;x) - f(x)] + ax(x+2)f'(x) - \frac{x(x+2)}{2}f''(x)\right| \\ \leq \quad & |p_n||f'(x)| + |q_n||f''(x)| + 2(2q_n + x(x+2) + r_n)\omega^*\big(f'', \frac{1}{\sqrt{n}}\big), \end{aligned}$$

where

$$\begin{aligned} p_n &= n\mu_{n,1}^{\alpha,\beta}(x) + ax(x+2), \\ q_n &= 2^{-1}(n\mu_{n,2}^{\alpha,\beta}(x) - x(x+2)), \\ r_n &= n^2\sqrt{M_n^{\alpha,\beta}((e^{-x} - e^{-t})^4;x)}\sqrt{\mu_{n,4}^{\alpha,\beta}(x)}. \end{aligned}$$

Proof. By Taylor's expansion of f for some fixed x,

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + h(t,x)(t-x)^2, \tag{4.6}$$

where

$$h(t,x) = \frac{f''(\eta) - f''(x)}{2}$$

and η is a number lying between x and t. Applying $M_n^{\alpha,\beta}$ to both sides of the above identity and using the linearity of the operators and Lemma (3.2), we get

$$\begin{aligned} & \left|n[M_n^{\alpha,\beta}(f;x) - f(x)] + ax(x+2)f'(x) - \frac{x(x+2)}{2}f''(x)\right| \\ \leq \quad & |p_n||f'(x)| + |q_n||f''(x)| + |nM_n^{\alpha,\beta}(h(t,x)(t-x)^2;x)|. \end{aligned}$$

Obviously it is enough to consider the last term of the inequality $|nM_n^{\alpha,\beta}(h(t,x)(t-x)^2;x)|$. Taking into consideration of Holhoş's paper (see [15]), it can be written

$$|f(t) - f(x)| \leq \big(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\big)\omega^*(f,\delta), \quad \delta > 0.$$

For more details on $\omega^*(\cdot,\delta)$, we would like to give a reference [15] to the reader. Trivially we can write,

$$|h(t;x)| \leq \big(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\big)\omega^*(f'',\delta).$$

And also using the fact that,

$$|h(t;x)| \leq \begin{cases} 2\omega^*(f'',\delta) & , \quad |e^{-x} - e^{-t}| < \delta \\ 2\big(\frac{(e^{-x}-e^{-t})^2}{\delta^2}\big)\omega^*(f'',\delta) & , \quad |e^{-x} - e^{-t}| \geq \delta, \end{cases}$$

we arrive at

$$|h(t;x)| \leq 2\big(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\big)\omega^*(f'',\delta).$$

Applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & |n(M_n^{\alpha,\beta}h(t,x)(t-x)^2;x)| \\ \leq \quad & 2n\omega^*(f'',\delta)\mu_{n,2}^{\alpha,\beta}(x)^{\alpha}(x) + \frac{2n}{\delta^2}\omega^*(f'',\delta)\sqrt{M_n^{\alpha,\beta}((e^{-x} - e^{-t})^4;x)}\sqrt{\mu_{n,4}^{\alpha,\beta}(x)}. \end{aligned}$$

Lastly, choosing $\delta=\frac{1}{\sqrt{n}}$ and with some simple calculations yield

$$\begin{aligned} & \left|n[M_n^{\alpha,\beta}(f;x)-f(x)]+ax(x+2)f'(x)-\frac{x(x+2)}{2}f''(x)\right| \\ \leq\ & |p_n||f'(x)|+|q_n||f''(x)|+2\omega^*\big(f'',\frac{1}{\sqrt{n}}\big)(2q_n+x(x+2)+r_n). \end{aligned}$$

Hence, the proof is completed. □

Corollary 4.1. *Let $f,f''\in C^*[0,\infty)$. Then the inequality*

$$\lim_{n\to\infty} n[M_n^{\alpha,\beta}(f;x)-f(x)]=-ax(x+2)f'(x)+\frac{x(x+2)}{2}f''(x) \tag{4.7}$$

holds for any $x\in[0,\infty)$.

Corollary 4.2. *Let $f\in C^2[0,\infty)$ be an decreasing and convex function. Then there exists $n_0\in\mathbb{N}$ such that for $n\geq n_0$, we have $f(x)<M_n^{\alpha,\beta}(f;x)$ for all $x\in[0,\infty)$.*

Presently, we are looking for the behavior of the operators on some weighted spaces. As reported in Gadziev's paper [10], set $\varphi(x)=1+e^{2ax}$, $x\in\mathbb{R}^+$ and turn the following weighted spaces over in our mind:

$$\begin{aligned} B_\varphi(\mathbb{R}^+) &= \{f:\mathbb{R}^+\to\mathbb{R}:|f(x)|\leq M_f\varphi(x),\ x\geq 0\}, \\ C_\varphi(\mathbb{R}^+) &= \{C(\mathbb{R}^+)\cap B_\varphi(\mathbb{R}^+)\}, \\ C_\varphi^k(\mathbb{R}^+) &= \{f\in C_\varphi(\mathbb{R}^+):\lim_{x\to\infty}\frac{f(x)}{\varphi(x)}=k_f\}, \end{aligned}$$

where M_f, k_f are constants depending on f. All three spaces are normed spaces with the norm

$$||f||_\varphi=\sup_{x\in\mathbb{R}^+}\frac{|f(x)|}{\varphi(x)}.$$

For any $f\in C_\varphi(\mathbb{R}^+)$, the inequality

$$\left|\left|M_n^{\alpha,\beta}(f)\right|\right|_\varphi\leq||f||_\varphi$$

holds and we complete that $M_n^{\alpha,\beta}(f)$ maps $C_\varphi(\mathbb{R}^+)$ to $C_\varphi(\mathbb{R}^+)$.

Theorem 4.3. *For each function $f\in C_\varphi^k(\mathbb{R}^+)$*

$$\lim_{n\to\infty}\left|\left|M_n^{\alpha,\beta}(f)-f\right|\right|_\varphi=0.$$

Proof. Using the general result shown in [10], the following three conditional approximations are sufficient.

$$\lim_{n\to\infty}\left|\left|M_n^{\alpha,\beta}(e^{\nu a\cdot})-e^{\nu a\cdot}\right|\right|_\varphi=0,\qquad \nu=0,1,2. \tag{4.8}$$

We know that for the given operator which is represented with $M_n^{\alpha,\beta}$, $M_n^{\alpha,\beta}(e_0)=1$ and $M_n^{\alpha,\beta}(e^{2at})=e^{2ax}$ occurs. Presently, if we take into consideration the situation for $\nu=1$

$$M_n^{\alpha,\beta}(f;x)=n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{(\theta_n(x))^k}{(1+\theta_n(x))^{n+k}}\int_0^\infty e^{-nt}\frac{(nt)^k}{k!}e^{a(nt+\alpha)/n+\beta}dt$$

and also on the assumption that the simple calculations are made, we reach

$$M_n^{\alpha,\beta}\left(e^{at}\right)=\frac{n+\beta}{n+\beta-a}e^{a\alpha/n+\beta}\left(1-\frac{-a\theta_n(x)}{n+\beta-a}\right)^{-n}. \tag{4.9}$$

Keeping an account of $\theta_n(x)$ and computing (4.9) with Maple,

$$\begin{aligned} M_n^{\alpha,\beta}\left(e^{at}\right) &= e^{ax} + \frac{e^{ax}\left(- ax^2 - \frac{4a^2x^2}{8} - a\alpha - a\right) + a\alpha e^{ax} + ae^{ax}}{n} + \mathcal{O}(\frac{1}{n^2}) \\ &= e^{ax} - \frac{e^{ax}a^2x(x+2)}{2n} + \mathcal{O}(\frac{1}{n^2}). \end{aligned}$$

Conclusively,

$$M_n^{\alpha,\beta}\left(e^{at}\right) - e^{ax} = -\frac{e^{ax}a^2x(x+2)}{2n} + \mathcal{O}(\frac{1}{n^2})$$

and

$$\frac{M_n^{\alpha,\beta}\left(e^{at}\right) - e^{ax}}{1+e^{2ax}} = \frac{-a^2x(x+2)e^{ax}}{2n(1+e^{2ax})} + \mathcal{O}(\frac{1}{n^2}).$$

And this circumstance guarantees uniform continuity. Since $M_n^{\alpha,\beta}(e_0) = 1$ and $M_n^{\alpha,\beta}(e^{2at}) = e^{2ax}$, the conditions (4.8) are implemented for $\nu = 0$ and $\nu = 2$. Hence, the proof is completed. □

Now, we desire to demonstrate that our modified operators approximate better than classical Baskakov-Szász-Stancu operators. This part, we take into consideration of article which is Aral et al [3]. Last theorem which would like to be given as below:

Theorem 4.4. *Let $f \in C^2[0,\infty)$. Assume that there exists $n_0 \in \mathbb{N}$, such that*

$$f(x) \leq M_n^{\alpha,\beta}(f;x) \leq B_n^{\alpha,\beta}(f;x), \text{ for all } n \geq n_0,\ x \in (0,\infty). \tag{4.10}$$

Then

$$\frac{x(x+2)}{2}f''(x) \geq (ax^2+2ax+1)f'(x) \geq 0,\ x \in (0,\infty). \tag{4.11}$$

Particularly $f'(x) \geq 0$ and $f''(x) \geq 0$.

Contrarily, if (4.11) holds with strict inequalities at a given point $x \in (0,\infty)$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$f(x) < M_n^{\alpha,\beta}(f;x) < B_n^{\alpha,\beta}(f;x).$$

Proof. From (4.10) we have that

$$0 \leq n(M_n^{\alpha,\beta}(f;x) - f(x)) \leq n(B_n^{\alpha,\beta}(f;x) - f(x)), \text{ for all } n \geq n_0,\ x \in (0,\infty).$$

Considering an asymptotic formula which is held by classical Baskakov-Szász-Stancu operators [17],

$$\lim_{n\to\infty} n(B_n^{\alpha,\beta}(f;x) - f(x)) = (1+\alpha-\beta x)f'(x) + \frac{x(x+2)}{2}f''(x). \tag{4.12}$$

Thanks to (4.10), we also accept that $1+\alpha-\beta x \geq 0$. Combining (4.7) and (4.12)

$$0 \leq (ax^2+2ax+1+\alpha-\beta x)f'(x) \leq \frac{x(x+2)}{2}f''(x),$$

we can reach (4.11) easily.

Contrarily, if (4.11) holds with strict inequalities for a given $x \in (0,\infty)$ then

$$0 < (ax^2+2ax+1+\alpha-\beta x)f'(x) < \frac{x(x+2)}{2}f''(x)$$

employing asymptotic formulas for modified operators (4.7) and the classical operators (4.12), we have the desired result. □

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ANKARA UNIVERSITY
DEPARTMENT OF MATHEMATICS
06100, ANKARA, TURKEY
E-mail address: bodur@ankara.edu.tr

ANKARA UNIVERSITY
DEPARTMENT OF MATHEMATICS
06100, ANKARA, TURKEY
E-mail address: ogurel@ankara.edu.tr

KIRIKKALE UNIVERSITY
DEPARTMENT OF MATHEMATICS
71450, KIRIKKALE, TURKEY
E-mail address: aliaral73@yahoo.com

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Differences of Operators of Lupaş Type

VIJAY GUPTA*

ABSTRACT. In the present article, we study the approximation of difference of operators and find the quantitative estimates for the difference of Lupaş operators with Lupaş-Szász operators and Lupaş-Kantorovich operators in terms of modulus of continuity. Also, we find the quantitative estimate for the difference of Lupaş-Kantorovich operators and Lupaş-Szász operators.

Keywords: Difference of operators, Lupaş operators, Lupaş-Szász operators, Lupaş-Kantorovich operators, Modulus of continuity.

2010 Mathematics Subject Classification: 41A25, 41A30.

1. INTRODUCTION

Approximation for linear positive operators to functions in real and complex setting is an active area of research amongst researchers. Several new operators have been constructed in last six decades and their approximation behaviours have been studied. Concerning approximation properties of linear positive operators the convergence is one of the important aspects, several methods and techniques have been applied to get the direct results in ordinary and simultaneous approximation, we mention some of the recent work viz. [1–4], [8–10], [12] etc.

Acu-Rasa [5] and Aral et al [7] established some interesting results for the difference of operators in order to generalize the problem posed by A. Lupaş [16] on polynomial differences. Some of the results on this topic are compiled in the recent book by Gupta et al [14].

Very recently the author in [11], provided a general result for the difference of operators and applied the result to Szász type operators. We consider here the Lupaş operators and its variants and find the quantitative estimates for the differences of such operators. A. Lupaş [16] proposed a discrete operators, which for $f \in C[0, \infty)$, are defined as

$$L_n(f, x) \quad := \quad \sum_{k=0}^{\infty} l_{n,k}(x) F_{n,k}(f), \tag{1.1}$$

where $F_{n,k} : D \to \mathbb{R}$ be positive linear functional defined on a subspace D of $C[0, \infty)$ and

$$l_{n,k}(x) = 2^{-nx} \frac{(nx)_k}{k!\, 2^k}, \quad F_{n,k}(f) = f\left(\frac{k}{n}\right)$$

It was observed that these operators are linear and positive and preserve linear functions.

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*Corresponding author: V. Gupta; vijaygupta2001@hotmail.com
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Remark 1.1. *For the Lupaş operators, we have* $F_{n,k}(f) = f\left(\frac{k}{n}\right)$ *such that* $F_{n,k}(e_0) = 1, b^{F_{n,k}} := F_{n,k}(e_1)$ *If we denote* $\mu_r^{F_{n,k}} = F_{n,k}(e_1 - b^{F_{n,k}}e_0)^r, r \in \mathbb{N}$, *then by simple computation, we have*

$$\mu_2^{F_{n,k}} := F_{n,k}(e_1 - b^{F_{n,k}}e_0)^2 = 0$$

$$\mu_4^{F_{n,k}} := F_{n,k}(e_1 - b^{F_{n,k}}e_0)^4 = 0$$

Remark 1.2. *The The moments of Lupaş operators with* $e_r(t) = t^r, r \in \mathbb{N} \cup \{0\}$ *are given by*

$$\begin{aligned}
L_n(e_0, x) &= 1,\\
L_n(e_1, x) &= x,\\
L_n(e_2, x) &= x^2 + \frac{2x}{n},\\
L_n(e_3, x) &= x^3 + \frac{6x^2}{n} + \frac{6x}{n^2},\\
L_n(e_4, x) &= x^4 + \frac{12x^3}{n} + \frac{36x^2}{n^2} + \frac{26x}{n^3},\\
L_n(e_5, x) &= x^5 + \frac{20x^4}{n} + \frac{120x^3}{n^2} + \frac{250x^2}{n^3} + \frac{150x}{n^4},\\
L_n(e_6, x) &= x^6 + \frac{30x^5}{n} + \frac{300x^4}{n^2} + \frac{1230x^3}{n^3} + \frac{2040x^2}{n^4} + \frac{1082x}{n^5}.
\end{aligned}$$

2. Difference of Operators

Let $C_B[0,\infty)$ be the class of bounded continuous functions defined on the interval $[0,\infty)$ equipped with the norm $||.|| = \sup_{x\in[0,\infty)} |f(x)| < \infty$. Let us consider another operator V_n having the same Lupaş basis $l_{n,k}(x)$ such that

$$V_n(f, x) = \sum_{k=0}^{\infty} l_{n,k}(x) G_{n,k}(f),$$

where $G_{n,k} : D \to \mathbb{R}$. Following [11], we have the following quantitative general result.

Theorem 2.1. *[11] Let* $f^{(s)} \in C_B[0,\infty), s \in \{0,1,2\}$ *and* $x \in [0,\infty)$, *then for* $n \in \mathbb{N}$, *we have*

$$|(L_n - V_n)(f, x)| \leq ||f''||\alpha(x) + \omega(f'', \delta_1)(1 + \alpha(x)) + 2\omega(f, \delta_2(x)),$$

where

$$\alpha(x) = \frac{1}{2}\sum_{k=0}^{\infty} l_{n,k}(x)(\mu_2^{F_{n,k}} + \mu_2^{G_{n,k}})$$

and

$$\delta_1^2 = \frac{1}{2}\sum_{k=0}^{\infty} l_{n,k}(x)(\mu_4^{F_{n,k}} + \mu_4^{G_{n,k}}), \delta_2^2 = \sum_{k=0}^{\infty} l_{n,k}(x)(b^{F_{n,k}} - b^{G_{n,k}})^2.$$

We now establish quantitative estimates for the difference of Lupaş operators with the Lupaş-Kantorovich operators and Lupaş-Szász operators.

2.1. **Lupaş and Lupaş-Kantorovich operators.** In [6] Agratini proposed the Kantorovich variant of the Lupaş operators as

$$K_n(f,x) := \sum_{k=0}^{\infty} l_{n,k}(x) G_{n,k}(f) = n \sum_{k=0}^{\infty} l_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \tag{2.2}$$

where

$$G_{n,k}(f) = n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt.$$

Below, we present the quantitative estimate for difference of Lupaş and Lupaş-Kantorovich operators.

Theorem 2.2. *Let $f^{(s)} \in C_B[0,\infty)$, $s \in \{0,1,2\}$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have*

$$|(K_n - L_n)(f,x)| \leq \frac{1}{24n^2}||f''|| + \omega\left(f'', \frac{1}{4\sqrt{10}n^2}\right)\left(1 + \frac{1}{24n^2}\right) + 2\omega\left(f, \frac{1}{2n}\right).$$

Proof. Following Theorem 2.1, by simple computation, we have

$$b^{G_{n,k}} = G_{n,k}(e_1) = \frac{2k+1}{2n}$$

and

$$\begin{aligned}
\mu_2^{G_{n,k}} &:= G_{n,k}(e_1 - b^{G_{n,k}} e_0)^2 \\
&= G_{n,k}(e_2) + \left(\frac{2k+1}{2n}\right)^2 - 2G_{n,k}(e_1)\left(\frac{2k+1}{2n}\right) \\
&= \frac{3k^2+3k+1}{3n^2} - \left(\frac{2k+1}{2n}\right)^2 \\
&= \frac{1}{12n^2}.
\end{aligned}$$

Next, using Remark 1.2, we have

$$\alpha(x) := \frac{1}{2}\sum_{k=0}^{\infty} l_{n,k}(x)(\mu_2^{F_{n,k}} + \mu_2^{G_{n,k}}) = \frac{1}{24n^2}.$$

Further,

$$\begin{aligned}
\mu_4^{G_{n,k}} &:= G_{n,k}(e_1 - b^{G_{n,k}} e_0)^4 \\
&= G_{n,k}(e_4) - 4G_{n,k}(e_3)\left(\frac{2k+1}{2n}\right) + 6G_{n,k}(e_2)\left(\frac{2k+1}{2n}\right)^2 \\
&- 4G_{n,k}(e_1)\left(\frac{2k+1}{2n}\right)^3 + G_{n,k}(e_0)\left(\frac{2k+1}{2n}\right)^4 \\
&= \frac{5k^4+10k^3+10k^2+5k+1}{5n^4} - 4\frac{4k^3+6k^2+4k+1}{4n^3}\left(\frac{2k+1}{2n}\right) \\
&+ 6\frac{3k^2+3k+1}{3n^2}\left(\frac{2k+1}{2n}\right)^2 - 3\left(\frac{2k+1}{2n}\right)^4 = \frac{1}{80n^4}.
\end{aligned}$$

Then using Remark 1.1 and above equality, we get

$$\delta_1^2(x) = \frac{1}{2}\sum_{k=0}^{\infty} l_{n,k}(x)(\mu_4^{F_{n,k}} + \mu_4^{G_{n,k}}) = \frac{1}{160n^4}.$$

and by using Remark 1.2, we have

$$\begin{aligned}\delta_2^2(x) &= \sum_{k=0}^{\infty} l_{n,k}(x)(b^{F_{n,k}} - b^{G_{n,k}})^2\\ &= \sum_{k=0}^{\infty} l_{n,k}(x)\left[\frac{k}{n} - \frac{2k+1}{2n}\right]^2\\ &= \frac{1}{4n^2}.\end{aligned}$$

This completes the proof of the theorem. □

2.2. **Lupaş and Lupaş-Szász operators.** The Lupaş-Szász operators are defined as

$$S_n(f;x) = n\sum_{k=1}^{\infty} l_{n,k}(x)\int_0^{\infty} s_{n,k-1}(t)f(t)dt + l_{n,0}(x)f(0), \tag{2.3}$$

where the Szász basis function is defined as $s_{n,k}(t) = \frac{e^{-nt}(nt)^k}{k!}$.

If we denote

$$H_{n,k}(f) = n\int_0^{\infty} s_{n,k-1}(t)f(t)dt, 0 \leq k < \infty, H_{n,0}(f) = f(0)$$

then the operators (2.3) take the following form:

$$S_n(f,x) = \sum_{k=0}^{\infty} l_{n,k}(x)H_{n,k}(f).$$

We present below the quantitative estimate for difference of Lupaş and Lupaş-Szász operators.

Theorem 2.3. *Let $f^{(s)} \in C_B[0,\infty), s \in \{0,1,2\}$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have*

$$|(S_n - L_n)(f,x)| = ||f''||\frac{x}{2n} + \omega\left(f'', \sqrt{\frac{3x^2}{2n^2} + \frac{6x}{n^3}}\right)\left(1 + \frac{x}{2n}\right).$$

Proof. By simple computation, we have

$$b^{H_{n,k}} = H_{n,k}(e_1) = \frac{k}{n}.$$

Also, we have

$$\begin{aligned}\mu_2^{H_{n,k}} &:= H_{n,k}(e_1 - b^{H_{n,k}}e_0)^2\\ &= H_{n,k}(e_2) + \left(\frac{k}{n}\right)^2 - 2H_{n,k}(e_1)\left(\frac{k}{n}\right)\\ &= \frac{k(k+1)}{n^2} - \frac{k^2}{n^2} = \frac{k}{n^2}.\end{aligned}$$

Next, using Remark 1.1, we have

$$\alpha(x) := \frac{1}{2}\sum_{k=0}^{\infty} l_{n,k}(x)(\mu_2^{F_{n,k}} + \mu_2^{H_{n,k}}) = \frac{x}{2n}.$$

and

$$\begin{aligned}
\mu_4^{H_{n,k}} &:= H_{n,k}(e_1 - b^{H_{n,k}}e_0)^4 \\
&= H_{n,k}(e_4) - 4H_{n,k}(e_3)\left(\frac{k}{n}\right) + 6H_{n,k}(e_2)\left(\frac{k}{n}\right)^2 \\
&- 4H_{n,k}(e_1)\left(\frac{k}{n}\right)^3 + H_{n,k}(e_0)\left(\frac{k}{n}\right)^4 \\
&= \frac{3k^2+6k}{n^4}.
\end{aligned}$$

Then by Remark 1.1, we have

$$\begin{aligned}
\delta_1^2(x) &= \frac{1}{2}\sum_{k=0}^{\infty} l_{n,k}(x)(\mu_4^{F_{n,k}} + \mu_4^{H_{n,k}}) \\
&= \sum_{k=0}^{\infty} l_{n,k}(x)\frac{3k^2+6k}{2n^4} \\
&= \frac{3x^2}{2n^2} + \frac{6x}{n^3}.
\end{aligned}$$

and by using above identities, we have

$$\delta_2^2(x) = \sum_{k=0}^{\infty} l_{n,k}(x)(b^{F_{n,k}} - b^{H_{n,k}})^2 = 0.$$

This completes the proof of the theorem. □

2.3. **Lupaş-Kantorovich and Lupaş-Szász operators.**

Theorem 2.4. *Let $f^{(s)} \in C_B[0,\infty)$, $s \in \{0,1,2\}$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have*

$$\begin{aligned}
|(S_n - K_n)(f,x)| &= ||f''||\left(\frac{1}{24n^2} + \frac{x}{2n}\right) + 2\omega\left(f, \frac{1}{2n}\right) \\
&+ \omega\left(f'', \sqrt{\frac{1}{160n^4} + \frac{3x^2}{2n^2} + \frac{6x}{n^3}}\right)\left(1 + \frac{1}{24n^2} + \frac{x}{2n}\right).
\end{aligned}$$

Proof. By previous subsections, we have

$$b^{G_{n,k}} = G_{n,k}(e_1) = \frac{2k+1}{2n}, \quad b^{H_{n,k}} = H_{n,k}(e_1) = \frac{k}{n}.$$

$$\mu_2^{G_{n,k}} = \frac{1}{12n^2}, \quad \mu_2^{H_{n,k}} = \frac{k}{n^2}$$

and

$$\mu_4^{G_{n,k}} = \frac{1}{80n^4}, \quad \mu_2^{H_{n,k}} = \frac{3k^2+6k}{n^4}.$$

Thus, we have

$$\alpha(x) := \frac{1}{2}\sum_{k=0}^{\infty} l_{n,k}(x)(\mu_2^{G_{n,k}} + \mu_2^{H_{n,k}}) = \frac{1}{24n^2} + \frac{x}{2n}$$

$$\begin{aligned}\delta_1^2(x) &= \frac{1}{2}\sum_{k=0}^{\infty} l_{n,k}(x)(\mu_4^{G_{n,k}} + \mu_4^{H_{n,k}})\\ &= \sum_{k=0}^{\infty} l_{n,k}(x)\left[\frac{1}{160n^4} + \frac{3k^2+6k}{2n^4}\right]\\ &= \frac{1}{160n^4} + \frac{3x^2}{2n^2} + \frac{6x}{n^3}.\end{aligned}$$

and by using above identities, we have

$$\delta_2^2(x) = \sum_{k=0}^{\infty} l_{n,k}(x)(b^{G_{n,k}} - b^{H_{n,k}})^2 = \frac{1}{4n^2}.$$

The result follows by combining above estimates as in Theorem 2.1. □

Remark 2.3. *In [13] Gupta et al and [15] Gupta-Yadav also considered Lupaş-Beta type operators, the difference estimates can be obtained analogously, the analysis is different we can discuss them elsewhere.*

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Netaji Subhas Institute of Technology
Department of Mathematics
Sector 3 Dwarka, New Delhi 110078, India
E-mail address: vijaygupta2001@hotmail.com

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A Short Survey on the Recent Fixed Point Results on b-Metric Spaces

ERDAL KARAPINAR*

ABSTRACT. The aim of this short survey is to collect and combine basic notions and results in the fixed point theory in the context of b-metric spaces. It is also aimed to show that there are still enough rooms for several researchers in this interesting direction and a huge application potential.

Keywords: b-Metric space, Fixed point

2010 Mathematics Subject Classification: 47H10, 54H25, 46J10, 46J15.

1. INTRODUCTION AND PRELIMINARIES

The notion of distance is as old as the history of humanity and it was first properly formulated by Euclid. Basically, Euclidean distance is defined to measure the space (or gap, or interval) between two points as the length of the straight line segment connecting them. Indeed, the notion of metric, axiomatically formulated by Maurice Fréchet [38], is a generalization form of the Euclid distance. On the other hand, the name is due to Felix Hausdorff [40].

It is evident that the notion of the metric is the corner stone of the the field of real analysis, complex analysis and functional analysis Taking the key role of the notion of the metric in mathematics and hence in quantitative sciences, it has been extended and generalized in several distinct directions by many authors. Consequently, several version, adaptation, extension and generalization of metric has been reported in the literature, for instance, 2-metric, D-metric, G-metric, S-metric, set-valued metric, fuzzy metric, symmetric, quasi-metric, partial metric, b-metric, ultrametric, dislocated metric, modular metric, Hausdorff metric, cone metric, multiplicative metric, and so on. It is worthy of note that not all these generalizations are real generalization, see e.g. [4,9,36,37,46,55,76].

Clearly, it is not possible to consider all these notions in a short survey. In this work, we restrict ourselves on the merging of one of the most interesting generalization of a notion of metric, namely b-metric. Before state the definition of b-metric, we recall the notion of (standard) metric for the sake of self-containment.

Definition 1.1. *For a nonempty set M, a (standard) metric is a function $m : M \times M \to \mathbb{R}_0^+ = [0, \infty)$ such that*

(M_0) $m(x,y) \geq 0$ *(nonnegativity),*
(M_1) $x = y \Rightarrow m(x,y) = 0$ *(self-distance),*
(M_2) $m(x,y) = 0 \Rightarrow x = y$ *(indistancy),*
(M_3) $m(x,y) = m(y,x)$ *(symmetry), and*
(M_4) $m(x,y) \leq m(x,z) + m(z,y)$ *(triangularity),*

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*Corresponding author: E. Karapınar; erdalkarapinar@yahoo.com
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for all $x, y, z \in M$. Here, the ordered pair (M, m) is called a (standard) metric space.

Indeed, the notion of the metric can be expressed in two axioms, as follows.

$(a1)$ $x = y \Rightarrow m(x, y) = 0$ (self-distance),
$(a2)$ $m(x, y) \leq m(x, z) + m(y, z)$ (triangularity),

for all $x, y, z \in M$. It is clear that (M_0)-(M_4) are obtained from $(a1)$ and $(a2)$. On the other hand, we separately state the axioms to explain and emphasize the nature how one can attempt to generalize the notion of standard metric. For instance, the axioms (M_0), (M_2)-(M_4) yield dislocated metric (also known as metric-like), the axioms (M_0)-(M_3) provide the notion of symmetric. It is clear that the removing any conditions from (M_0)-(M_4) propose a new notion.

In this study, we focus on an interesting generalization of the standard metric, so-called, b-metric. This metric was popular after the interesting papers of Czerwik [34, 35] and it has been attracted attention of the several researchers. Indeed, this notion was considered earlier by different authors, e.g. Bourbaki [29], Bakhtin [17], Heinhonen [44], Berinde [18] and so on.

What follows we recall the notion of b-metric.

Definition 1.2. *([17], [35]) Let M be a set and let $s \geq 1$ be a given real number. A function $d : M \times M \to \mathbb{R}_0^+$ is said to be a b-metric if the following conditions are satisfied:*

(bM_o) $d(x, y) \geq 0$ (nonnegativity),
(bM_1) $x = y \Rightarrow d(x, y) = 0$ (self-distance),
(bM_2) $d(x, y) = 0 \Rightarrow x = y$ (indistancy),
(bM_3) $d(x, y) = d(y, x)$, (symmetry),
(bM_4) $d(x, z) \leq s[d(x, y) + d(y, z)]$, (weakened triangularity).

for all $x, y, z \in M$. Furthermore, the ordered pair (M, d) is called a b-metric space. We abbreviate the concept of the b-metric space as bMS.

As it is expected that each b-metric forms a metric by letting $s = 1$. On the other hand, the converse is not case.

Example 1.1. *(See e.g. [29].) Let $M = L^p[0, 1]$ be the collections of all real functions $x(t)$ such that $\int_0^1 |x(t)|^p dt < \infty$, where $t \in [0, 1]$ and $0 < p < 1$. For the function $d : M \times M \to \mathbb{R}_0^+$ defined by*

$$b(x, y) := (\int_0^1 |x(t) - y(t)|^p dt)^{1/p}, \text{ for each } x, y \in L^p[0, 1],$$

the ordered pair (M, b) forms a b-metric space with $s = 2^{1/p}$.

Example 1.2. *Let X be a set with the cardinal $card(X) \geq 3$. Suppose that $M = X_1 \cup X_2$ is a partition of X such that $card(X_1) \geq 2$. Let $s > 1$ be arbitrary. Then, the functional $d : M \times M \to [0, \infty)$ defined by:*

$$d(x, y) := \begin{cases} 0, & x = y \\ 2s, & x, y \in M_1 \\ 1, & \textit{otherwise.} \end{cases}$$

is a b-metric on X with coefficient $s > 1$.

Example 1.3. *(See e.g. [29].) Let $p \in (0, 1)$ and let*

$$M = l_p(\mathbb{R}) = \left\{ x = \{x_n\} \subset \mathbb{R} \textit{ such that } \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Define $d(x,y) : M \times M \to [0,\infty)$ *by*

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}.$$

Then (X,d) *is a b-metric space with* $s = 2^{1/p}$.

The special case of the example above can be the following:

Example 1.4. *Let* $M = \mathbb{R}$. *The function* $d : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ *defined as*

$$d(x,y) = |x-y|^2, \tag{1.1}$$

is a b-metric on $\mathbb{R}$. *Clearly, the first two conditions are satisfied. For the third condition, we have*

$$\begin{aligned} |x-y|^2 &= |x-z+z-y|^2 = |x-z|^2 + 2|x-z||z-y| + |z-y|^2 \\ &\leq 2[|x-z|^2 + |z-y|^2], \end{aligned}$$

since

$$2|x-z||z-y| \leq |x-z|^2 + |z-y|^2.$$

Example 1.5. *Let* $M=\{0,1,2\}$ *and* $d : M \times M \to \mathbb{R}_+$ *such that* $d(0,1) = d(1,0) = d(0,2) = d(2,0) = 1, d(1,2) = d(2,1) = \alpha \geq 2, d(0,0) = d(1,1) = d(2,2) = 0$. *Then*

$$d(x,y) \leq \frac{\alpha}{2}\left[d(x,z) + d(z,y)\right], \text{ for } x,y,z \in M.$$

Example 1.6. *Let* E *be a Banach space and* 0_E *be the zero vector of* E. *Let* P *be a cone in* E *with* $int(P) \neq \emptyset$ *and* $\preceq$ *be a partial ordering with respect to* P. *Let* X *be a non-empty set. Suppose the mapping* $d : X \times X \to E$ *satisfies:*

- $(M1)$ $0 \preceq d(x,y)$ *for all* $x,y \in X$,
- $(M2)$ $d(x,y) = 0$ *if and only if* $x = y$,
- $(M3)$ $d(x,y) \preceq d(x,z) + d(z,y)$, *for all* $x,y \in X$,
- $(M4)$ $d(x,y) = d(y,x)$ *for all* $x,y \in X$,

then d is called cone metric on X, *and the pair* (X,d) *is called a cone metric space (CMS).*

Let E *be a Banach space and* P *be a normal cone in E with the coefficient of normality denoted by* K. *Let* $D : X \times X \to [0,\infty)$ *be defined by* $D(x,y) = ||d(x,y)||$, *where* $d : X \times X \to E$ *is a cone metric space. Then* (X,D) *is a b-metric space with constant* $s := K \geq 1$.

The basic topological properties (convergence, completeness, etc.) have been observed by the mimic of the standard metric versions. Next, we recollect some essential notions together with the basic observations. Each b-metric d on a non-empty set M have a topology τ_d that was generated by the family of open balls

$$B_d(x,\varepsilon) = \{y \in M : |d(x,y) - d(x,x)| < \varepsilon, \} \text{ for all } x \in M \text{ and } \varepsilon > 0.$$

In the frame of the b-metric (M,d), a given sequence $\{x_n\}$ converges to a point $x \in M$ if the following limit exists

$$\lim_{n\to\infty} d(x_n,x) = 0.$$

As it is expected, a sequence $\{x_n\}$ is said to be Cauchy if the following limit

$$L = \lim_{n\to\infty} d(x_n,x_m) = 0. \tag{1.2}$$

Furthermore, a pair (M,d) is called complete b-metric space if for each Cauchy sequence $\{x_n\}$ is convergent, that is, there is some $x \in M$ such that

$$\lim_{n\to\infty} d(x_n,x) = 0 = \lim_{n\to\infty} d(x_n,x_m). \tag{1.3}$$

Let (M, d_1) and (K, d_2) be b-metric spaces. A mapping $T : M \to K$ is called continuous if

$$\lim_{n\to\infty} d_1(x_n, x) = 0 = \lim_{n,m\to\infty} d_1(x_n, x_m),$$

then we have

$$\lim_{n\to\infty} d_2(Tx_n, Tx) = 0 = \lim_{n,m\to\infty} d_2(Tx_n, Tx_m).$$

Definition 1.3. *Let (M, d) be a b-metric space and S be a subset of M. We say S is open subset of M, if for all $x \in M$ there exists $r > 0$ such that $B_d(x, r) \subseteq S$. Also, $F \subseteq X$ is a closed subset of M if $(M \backslash F)$ is a open subset of M.*

A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \to 0$, $n \to \infty$, for any $t \in [0, \infty)$. We denote by Φ, the class of the comparison function $\varphi : [0, \infty) \to [0, \infty)$. For more details and examples, see e.g. [20, 71]. Among them, we recall the following essential result.

Lemma 1.1. *(Berinde [20], Rus [71]) If $\varphi : [0, \infty) \to [0, \infty)$ is a comparison function, then:*

(1) *each iterate φ^k of φ, $k \geq 1$, is also a comparison function;*
(2) *φ is continuous at 0;*
(3) *$\varphi(t) < t$, for any $t > 0$.*

Later, Berinde [20] introduced the concept of (c)-*comparison function* in the following way.

Definition 1.4. *(Berinde [20]) A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a (c)-comparison function if*

(c_1) *φ is increasing,*

(c_2) *there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.*

The notion of a (c)-comparison function was improved as a (b)-comparison function by Berinde [19] in order to extend some fixed point results to the class of b-metric space.

Definition 1.5. *(Berinde [19]) Let $s \geq 1$ be a real number. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a (b)-comparison function if the following conditions are fulfilled*

(1) *φ is monotone increasing;*
(2) *there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.*

We denote by Ψ_b for the class of (b)-comparison function $\varphi : [0, \infty) \to [0, \infty)$. It is evident that the concept of (b)-comparison function reduces to that of (c)-comparison function when $s = 1$.

The following lemma has a crucial role in the proof of our main result.

Lemma 1.2. *(Berinde [18]) If $\varphi : [0, \infty) \to [0, \infty)$ is a (b)-comparison function, then we have the following*

(1) *the series $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;*

(2) *the function* $b_s : [0, \infty) \to [0, \infty)$ *defined by* $b_s(t) = \sum_{k=0}^{\infty} s^k \varphi^k(t),\ t \in [0, \infty)$, *is increasing and continuous at* 0.

Remark 1.1. *By the Lemma 1.2, we conclude that every* (b)*-comparison function is a comparison function and hence, by the Lemma 1.1, any* (b)*-comparison function* ϕ *satisfies* $\phi(t) < t$.

We denote with Ψ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$ It is clear that if $\Psi \subset \Phi$ (see e.g. [43]) and hence, by Lemma 1.1 (3), for $\psi \in \Psi$ we have $\psi(t) < t$, for any $t > 0$.

In this short survey, we collect the interesting fixed point theorems for single valued mapping in the frame of complete b-metric space. This survey can be considered the collection the attractive results in [3, 11, 24].

2. Fixed point of α-ψ-contractive mappings

We start this section by recalling the definition of α-ψ-contractive and α-orbital admissible mappings introduced in [75].

Definition 2.6. *(Samet et al. [75]) Let* (M, d) *be a metric space and* $T : M \to M$ *be a given mapping. We say that* T *is an* α-ψ*-contractive mapping if there exist two functions* $\alpha : M \times M \to [0, \infty)$ *and* $\psi \in \Psi$ *such that*

$$\alpha(x, y)d(T(x), T(y)) \le \psi(d(x, y)),\ \text{for all } x, y \in M. \tag{2.4}$$

Remark 2.2. *If* $T : M \to M$ *satisfies the Banach contraction principle, then* T *is an* α-ψ*-contractive mapping, where* $\alpha(x, y) = 1$ *for all* $x, y \in M$ *and* $\psi(t) = kt$ *for all* $t \ge 0$ *and some* $k \in [0, 1)$.

Definition 2.7. *(Samet et al. [75]) Let* $T : M \to M$ *and* $\alpha : M \times M \to [0, \infty)$. *We say that* T *is* α*-admissible if*

$$x, y \in M,\ \ \alpha(x, y) \ge 1 \Longrightarrow \alpha(T(x), T(y)) \ge 1.$$

Let $\mathcal{F}_T(X)$ be the class of fixed points of a self-mapping T defined on a non-empty set X, that is, $\mathcal{F}_T(X) = \{x \in M : T(x) = x\}$.

Example 2.7. *(Samet et al. [75]) Let* $M = (0, +\infty)$. *Define* $T : M \to M$ *and* $\alpha : M \times M \to [0, \infty)$ *by*

(1) $T(x) = \ln(x)$, *for all* $x \in M$ *and* $\alpha(x, y) = \begin{cases} 2, & \text{if } x \ge y; \\ 0, & \text{if } x < y. \end{cases}$

Then T *is* α*-admissible.*

(2) $T(x) = \sqrt{x}$, *for all* $x \in M$ *and* $\alpha(x, y) = \begin{cases} e^{x-y}, & \text{if } x \ge y; \\ 0, & \text{if } x < y. \end{cases}$

Then T *is* α*-admissible.*

Example 2.8. *Let* $(M, \preceq)$ *be a partially ordered set and* d *be a metric on* X *such that* (M, d) *is complete. Let* $T : M \to M$ *be a nondecreasing mapping with respect to* $\preceq$, *that is* $x, y \in M,\ x \preceq y \Longrightarrow Tx \preceq Ty$. *Suppose that there exists* $x_0 \in M$ *such that* $x_0 \preceq Tx_0$. *Define the mapping* $\alpha : M \times M \to [0, \infty)$ *by*

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x \preceq y \text{ or } x \succeq y, \\ 0 \text{ otherwise}. \end{cases}$$

Then, T *is* α*-admissible. Since there exists* $x_0 \in M$ *such that* $x_0 \preceq Tx_0$, *we have* $\alpha(x_0, Tx_0) \ge 1$. *On the other hand, for all* $x, y \in M$, *from the monotone property of* T, *we have*

$$\alpha(x, y) \ge 1 \Longrightarrow x \succeq y \text{ or } x \preceq y \Longrightarrow Tx \succeq Ty \text{ or } Tx \preceq Ty \Longrightarrow \alpha(Tx, Ty) \ge 1.$$

Thus T is $\alpha-$admissible.

Theorem 2.1. *(Samet et al. [75]) Let (M, d) be a complete metric space and $T : M \to M$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in M$ such that $\alpha(x_0, T(x_0)) \geq 1$;*
(iii) *T is continuous.*

Then, T has a fixed point, that is, there exists $x^* \in \mathcal{F}_T(X)$.

Theorem 2.2. *(Samet et al. [75]) Let (M, d) be a complete metric space and $T : M \to M$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in M$ such that $\alpha(x_0, T(x_0)) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then, T has a fixed point, that is, there exists $x^* \in \mathcal{F}_T(X)$.

In what follows we recollect the concept of triangular α-admissible mapping.

Definition 2.8. *[52] A self-mapping $T : M \to M$ is called triangular α-admissible if*

(T_1) T is $\alpha-$ admissible,
(T_1) $\alpha(x, z) \geq 1, \;\; \alpha(z, y) \geq 1 \Longrightarrow \alpha(x, y) \geq 1, \;\; x, y, z \in M.$

First of all, we refine the notion of α-admissible mapping by proposing the notion of α-orbital admissible as follows.

Definition 2.9. *[68] Let $T : M \to M$ be a self-mapping and $\alpha : M \times M \to [0, \infty)$ be a function. Then T is said to be α-orbital admissible if*

$$(T3) \;\; \alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1.$$

Analogously, we refine the notion of triangular α-admissible mapping by proposing the notion of triangular α-orbital admissible in the following way.

Definition 2.10. *[68] Let $T : M \to M$ be a self-mapping and $\alpha : M \times M \to [0, \infty)$ be a function. Then, T is said to be triangular α-orbital admissible if T is α-orbital admissible and*

$$(T4) \;\; \alpha(x, y) \geq 1 \text{ and } \alpha(y, Ty) \geq 1 \Rightarrow \alpha(x, Ty) \geq 1.$$

As it was mentioned in [68], each α-admissible mapping is an α-orbital admissible mapping and each triangular α-admissible mapping is a triangular α-orbital admissible mapping. The converse is false, see e.g. ([68], Example 7).

Definition 2.11. *[68] Let (M, d) be a b-metric space and $\alpha : X \times M \to M$ be a function. X is said α-regular, if for every sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ with $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Lemma 2.3. *[68] Let $T : M \to M$ be a triangular α-orbital admissible mapping. Assume that there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}_0$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

First we give the following definition as a generalization of Definition 2.6.

Definition 2.12. *Let (M,d) be a b-metric space and $T: M \to M$ be a given mapping. We say that T is an α-ψ-contractive mapping of type-(b) if there exist two functions $\alpha: M \times M \to [0,\infty)$ and $\psi \in \Psi_b$ such that*

$$\alpha(x,y)d(T(x),T(y)) \leq \psi(d(x,y)), \text{ for all } x,y \in X. \tag{2.5}$$

Our first main result is the following.

Theorem 2.3. *Let (M,d) be a complete b-metric space with constant $s>1$. Let $T: M \to M$ be an α-ψ-contractive mapping of type-(b) satisfying the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in M$ such that $\alpha(x_0, T(x_0)) \geq 1$;*
(iii) *T is continuous.*

Then the fixed point equation (3.15) has a solution, that is, there exists $x^* \in \mathcal{F}_f(X)$.

Proof. Let $x_0 \in M$ such that $\alpha(x_0,T(x_0)) \geq 1$ (such a point exists from condition (ii)). Define the sequence $\{x_n\}$ in X by

$$x_{n+1} = T(x_n), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x^* = x_n$ is a fixed point for T and the proof finishes. Hence we assume that

$$x_n \neq x_{n+1} \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.6}$$

Since T is α-orbital admissible, we have:

$$\alpha(x_0,x_1) = \alpha(x_0,T(x_0)) \geq 1 \Longrightarrow \alpha(T(x_0),T(x_1)) = \alpha(x_1,x_2) \geq 1.$$

By induction, we get

$$\alpha(x_n,x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.7}$$

Applying the inequality (2.5) with $x = x_{n-1}$ and $y = x_n$, and using (2.7), we obtain:

$$d(x_n,x_{n+1}) = d(T(x_{n-1}),T(x_n)) \leq \alpha(x_{n-1},x_n)d(T(x_{n-1}),T(x_n)) \leq \psi(d(x_{n-1},x_n)).$$

By induction, we get

$$d(x_n,x_{n+1}) \leq \psi^n(d(x_0,x_1)), \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.8}$$

From (2.8) and using the triangular inequality, for all $p \geq 1$, we have:

$$\begin{aligned}
d(x_n,x_{n+p}) &\leq sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + \ldots + s^{p-2}d(x_{n+p-3},x_{n+p-2}) \\
&\quad + s^{p-1}d(x_{n+p-2},x_{n+p-1}) + s^p d(x_{n+p-1},x_{n+p}) \\
&\leq s\psi^n(d(x_0,x_1)) + s^2\psi^{n+1}(d(x_0,x_1)) + \ldots + s^{p-2}\psi^{n+p-3}(d(x_0,x_1)) \\
&\quad + s^{p-1}\psi^{n+p-2}(d(x_0,x_1)) + s^{p-1}\psi^{n+p-1}(d(x_0,x_1)) \\
&= \tfrac{1}{s^{n-1}}[s^n\psi^n(d(x_0,x_1)) + s^{n+1}\psi^{n+1}(d(x_0,x_1)) + \ldots + s^{n+p-2}\psi^{n+p-2}(d(x_0,x_1)) \\
&\quad + s^{n+p-1}\psi^{n+p-1}(d(x_0,x_1))].
\end{aligned}$$

Denoting $S_n = \sum_{k=0}^{n} s^k \psi^k(d(x_0,x_1))$, $n \geq 1$ we obtain:

$$d(x_n,x_{n+p}) \leq \frac{1}{s^{n-1}}[S_{n+p-1} - S_{n-1}], \; n \geq 1, \; p \geq 1. \tag{2.9}$$

Due to the assumption (2.6) together with Lemma 1.2, we conclude that the series $\sum_{k=0}^{n} s^k\psi^k(d(x_0,x_1))$ is convergent. Thus there exists $S = \lim_{n\to\infty} S_n \in [0,\infty)$. Regarding $s \geq 1$ and by (2.9), we obtain that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in the b-metric space (M,d). Since (M,d) is complete, there exists $x^* \in M$ such that $x_n \to x^*$ as $n \to \infty$. From the continuity of T, it follows that $x_{n+1} = T(x_n) \to T(x^*)$ as $n \to \infty$. By the uniqueness of the limit, we get $x^* = T(x^*)$, that is, x^* is a fixed point of T. □

In the following theorem, we able omit the continuity hypothesis of T by adding a new condition.

Theorem 2.4. *Let (M,d) be a complete b-metric space with constant $s > 1$. Let $T : M \to M$ be an α-ψ-contractive mapping of type-(b) satisfying the following conditions:*

(i) *T is α-orbital admissible;*
(ii) *there exists $x_0 \in M$ such that $\alpha(x_0,T(x_0)) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then $\alpha(x_n,x) \geq 1$ for all n.*

Then the fixed point equation (3.15) has a solution.

Proof. Following the proof of Theorem 2.3, we know that $\{x_n\}$ is a Cauchy sequence in the complete b-metric space (M,d). Then, there exists $x^* \in M$ such that $x_n \to x^*$ as $n \to \infty$. On the other, hand from (2.7) and the hypothesis (iii), we have

$$\alpha(x_n,x^*) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{2.10}$$

Now, using the triangular inequality, (2.5) and (2.10), we get

$$\begin{aligned} d(T(x^*),x^*) &\leq s[d(T(x^*),T(x_n)) + d(x_{n+1},x^*)] \\ &\leq s[\alpha(x_n,x^*)d(T(x^*),T(x_n)) + d(x_{n+1},x^*)] \\ &\leq s[\psi(d(x_n,x^*)) + d(x_{n+1},x^*)]. \end{aligned}$$

Letting $n \to \infty$, since ψ is continuous at $t = 0$, we obtain $d(T(x^*),x^*) = 0$, that is $x^* = T(x^*)$. □

To assure the uniqueness of the fixed point, we will consider the following hypothesis.

$$(H): \quad \text{for all } x,y \in M, \text{ there exists } z \in M \text{ such that } \alpha(x,z) \geq 1 \text{ and } \alpha(y,z) \geq 1.$$

Theorem 2.5. *Adding condition (H) to the hypotheses of Theorem 2.3 (resp. Theorem 2.4) we obtain uniqueness of the fixed point of T.*

Proof. Suppose that x^* and y^* are two fixed point of T. From (H), there exists $z \in M$ such that

$$\alpha(x^*,z) \geq 1 \quad \text{and} \quad \alpha(y^*,z) \geq 1. \tag{2.11}$$

Since T is α-orbital admissible, from (2.11), we get

$$\alpha(x^*,T^n(z)) \geq 1 \quad \text{and} \quad \alpha(y^*,T^n(z)) \geq 1. \tag{2.12}$$

Using (2.12) and (2.5), we have

$$\begin{aligned} d(x^*,T^n(z)) = d(T(x^*),T(T^{n-1}(z))) &\leq \alpha(x^*,T^{n-1}(z))d(T(x^*),T(T^{n-1}(z))) \\ &\leq \psi(d(x^*,T^{n-1}(z))). \end{aligned}$$

This imply that

$$d(x^*,T^n(z)) \leq \psi^n(d(x^*,z)), \text{ for all } n \in \mathbb{N}.$$

Then, letting $n \to \infty$, we have

$$T^n(z) \to x^*. \tag{2.13}$$

Similarly, using (2.12) and (2.5), we get

$$T^n(z) \to y^* \text{ as } n \to \infty. \tag{2.14}$$

Using (2.13) and (2.14), the uniqueness of the limit gives us $x^* = y^*$. This finishes the proof. □

Remark 2.3. *Theorem 2.1 (respectively, Theorem 2.2) can be derived from Theorem 2.3 (respectively, Theorem 2.4) by taking $s = 1$. Consequently, all results in [75] can be considered as a corollaries of our main results.*

3. Ulam-Hyers stability results through the fixed point problems

Definition 3.13. *Let (M, d) be a metric space and $T : M \to M$ be an operator. By definition, the fixed point equation*

$$x = T(x) \tag{3.15}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is increasing, continuous at 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each $w^* \in M$ an ε-solution of the fixed point equation (3.15), i.e. w^* satisfies the inequality

$$d(w^*, T(w^*)) \leq \varepsilon \tag{3.16}$$

there exists a solution $x^* \in M$ of the equation (3.15) such that

$$d(w^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) = c \cdot t$, for each $t \in \mathbb{R}_+$, then the fixed point equation (3.15) is said to be Ulam-Hyers stable.

For Ulam-Hyers stability results in the case of fixed point problems see M. F. Bota-Boriceanu, A. Petruşel [23], V. L. Lazăr [60], I. A. Rus [70], I. A. Rus [72].

Regarding the Ulam-Hyers stability problem the ideas given in T. P. Petru, A. Petruşel and J.-C. Yao [67] allow us to obtain the following result.

Theorem 3.6. *Let (M, d) be a complete b-metric space with constant $s > 1$. Suppose that all the hypotheses of Theorem 2.5 hold and additionally that the function $\beta : [0, \infty) \to [0, \infty)$, $\beta(r) := r - s\psi(r)$ is strictly increasing and onto. Then*

(a) *the fixed point equation (3.15) is generalized Ulam-Hyers stable.*
(b) *$Fix(T) = \{x^*\}$ and if $x_n \in M$, $n \in \mathbb{N}$ are such that $d(x_n, T(x_n)) \to 0$, as $n \to \infty$, then $x_n \to x^*$, as $n \to \infty$, i.e. the fixed point equation (3.15) is well posed.*
(c) *If $g : M \to M$ is such that there exists $\eta \in [0, \infty)$ with*

$$d(T(x), g(x)) \leq \eta, \text{ for all } x \in M,$$

then

$$y^* \in Fix(g) \Longrightarrow d(x^*, y^*) \leq \beta^{-1}(s \cdot \eta).$$

Proof. (a) Since $T : M \to M$ is a Picard operator, so $Fix(T) = \{x^*\}$. Let $\varepsilon > 0$ and $w^* \in M$ be a solution of (3.16), i.e,

$$d(w^*, T(w^*)) \leq \varepsilon.$$

Since T is α-ψ-contractive mapping of type-(b) and since $x^* \in Fix(T)$, from (H) there exists $w^* \in M$ such that $\alpha(x^*, w^*) \geq 1$, we obtain:

$$d(x^*, w^*) = d(T(x^*), w^*) \leq s[d(T(x^*), T(w^*)) + d(T(w^*), w^*)]$$

$$\leq s[\alpha(x^*, w^*)d(T(x^*), T(w^*)) + \varepsilon] \leq s[\psi(d(x^*, w^*)) + \varepsilon].$$

Therefore,

$$\beta(d(x^*, w^*)) := d(x^*, w^*) - s\psi(d(x^*, w^*)) \leq s \cdot \varepsilon \Longrightarrow d(x^*, w^*) \leq \beta^{-1}(s \cdot \varepsilon).$$

Consequently, the fixed point equation (3.15) is generalized Ulam-Hyers stable.

(b) Since T is α-ψ-contractive mapping of type-(b) and since $x^* \in Fix(T)$, from (H) there exists $x_n \in M$ such that $\alpha(x^*, x_n) \geq 1$, we obtain:

$$d(x_n, x^*) \leq s[d(x_n, T(x_n)) + d(T(x_n), x^*)] = s[d(x_n, T(x_n)) + d(T(x_n), T(x^*))]$$
$$\leq s[d(x_n, T(x_n)) + \alpha(x_n, x^*)d(T(x_n), T(x^*))] \leq s[d(x_n, T(x_n)) + \psi(d(x_n, x^*))].$$

Therefore

$$\beta(d(x_n, x^*)) := d(x_n, x^*) - s\psi(d(x_n, x^*)) \leq sd(x_n, T(x_n)) \to 0 \text{ as } n \to \infty$$
$$\Longrightarrow d(x_n, x^*) \to 0 \text{ as } n \to \infty \Longrightarrow x_n \to x^*, \text{ as } n \to \infty.$$

So, the fixed point equation (3.15) is well posed.

(c) Since T is α-ψ-contractive mapping of type-(b) and since $x^* \in Fix(T)$, from (H) there exists $x \in M$ such that $\alpha(x^*, x) \geq 1$, we obtain:

$$d(x, x^*) \leq s[d(x, T(x)) + d(T(x), x^*)] = s[d(x, T(x)) + d(T(x), T(x^*))]$$
$$\leq s[d(x, T(x)) + \alpha(x, x^*)d(T(x), T(x^*))] \leq s[d(x, T(x)) + \psi(d(x, x^*))].$$

Therefore

$$\beta(d(x, x^*)) := d(x, x^*) - s\psi(d(x, x^*)) \leq s \cdot d(x, T(x)).$$

So, we have the following estimation

$$d(x, x^*) \leq \beta^{-1}(s \cdot d(x, T(x))). \tag{3.17}$$

Writing (3.17) for $x := y^*$ we get:

$$d(x^*, y^*) \leq \beta^{-1}(s \cdot d(y^*, T(y^*))) = \beta^{-1}(s \cdot d(s(y^*), T(y^*))) \Longrightarrow d(x^*, y^*) \leq \beta^{-1}(s \cdot \eta).$$

□

4. Non unique fixed points on b-metric spaces

In this section, inspired by the well-known non-unique fixed point of Ćirić, we state and prove some new non-unique fixed point theorems in the setting of b-metric spaces. Our results improve the existence results in the literature, see e.g. [33,49,50,65]. We shall start to this section by recalling the notion of orbitally continuous.

Definition 4.14. *A mapping T on b-metric space (M, d) is said to be orbitally continuous if $\lim_{i\to\infty} T^{n_i}(x) = z$ implies $\lim_{i\to\infty} T(T^{n_i}(x)) = Tz$. A b-metric space (M, d) is called T-orbitally complete if every Cauchy sequence of the form $\{T^{n_i}(x)\}_{i=1}^{\infty}$, $x \in M$ converges in (M, d).*

Remark 4.4. *It is evident that orbital continuity of T yields orbital continuity of T^m for any $m \in \mathbb{N}$.*

Theorem 4.7. *Let T be an orbitally continuous self-map on the T-orbitally complete b-metric space (M, d). If there is $\psi \in \Psi$ such that*

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \leq \psi(d(x, y)), \tag{4.18}$$

for all $x, y \in M$, then for each $x_0 \in M$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. For an arbitrary $x \in M$, we shall construct an iterative sequence $\{x_n\}$ as follows:

$$x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}. \tag{4.19}$$

We suppose that

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{4.20}$$

Indeed, if for some $n \in \mathbb{N}$ we have the inequality $x_n = Tx_{n-1} = x_{n-1}$, then, the proof is completed. By substituting $x = x_{n-1}$ and $y = x_n$ in the inequality (4.18), we derive that

$$\begin{aligned} &\min\{d(Tx_{n-1}, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &\qquad - \min\{d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\} \leq \psi(d(x_{n-1}, x_n)). \end{aligned} \tag{4.21}$$

It implies that

$$\min\{d(x_n, x_{n+1}), d(x_n, x_{n-1})\} \leq \psi(d(x_{n-1}, x_n)). \tag{4.22}$$

Since $\psi(t) < t$ for all $t > 0$, the case $d(x_n, x_{n-1}) \leq \psi(d(x_{n-1}, x_n))$ is impossible. Thus, we have

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)). \tag{4.23}$$

Applying Remark 1.1 recurrently, we find that

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1})) \leq \cdots \leq \psi^n(d(x_0, x_1)). \tag{4.24}$$

By Lemma 1.2, we deduce that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{4.25}$$

In what follow we shall prove that the sequence $\{x_n\}$ is Cauchy.

Consider $d(x_n, x_{n+k})$ for $k \geq 1$. By using the triangle inequality $(b3)$ again and again, we get the following approximation

$$\begin{array}{rcl} d(x_n, x_{n+k}) & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+k})] \\ & \leq & sd(x_n, x_{n+1}) + s\{s[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+k})]\} \\ & = & sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+k}) \\ & \vdots & \\ & \leq & sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \dots \\ & + & s^{k-1} d(x_{n+k-2}, x_{n+k-1}) + s^{k-1} d(x_{n+k-1}, x_{n+k}) \\ & \leq & sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \dots \\ & + & s^{k-1} d(x_{n+k-2}, x_{n+k-1}) + s^k d(x_{n+k-1}, x_{n+k}), \end{array} \tag{4.26}$$

since $s \geq 1$. Combining (4.24) and (4.26) we derive that

$$\begin{array}{rcl} d(x_n, x_{n+k}) & \leq & s\psi^n(d(x_0, x_1)) + s^2\psi^{n+1} d(x_0, x_1) + \dots \\ & + & s^{k-1}\psi^{n+k-2}(d(x_0, x_1)) + s^k\psi^{n+k-1}(d(x_0, x_1)) \\ & = & \dfrac{1}{s^{n-1}}[s^n\psi^n(d(x_0, x_1)) + s^{n+1}\psi^{n+1} d(x_0, x_1) + \dots \\ & + & s^{n+k-2}\psi^{n+k-2}(d(x_0, x_1)) + s^{n+k-1}\psi^{n+k-1}(d(x_0, x_1))]. \end{array} \tag{4.27}$$

Consequently, we have

$$d(x_n, x_{n+k}) \leq \frac{1}{s^{n-1}}\left[P_{n+k-1} - P_{n-1}\right], \quad n \geq 1, k \geq 1, \tag{4.28}$$

where $P_n = \sum_{j=0}^{n} s^j \psi^j(d(x_0, x_1))$, $n \geq 1$. From Lemma 1.2, the series $\sum_{j=0}^{\infty} s^j \psi^j(d(x_0, x_1))$ is convergent and since $s \geq 1$, upon taking limit $n \to \infty$ in (4.28) we get

$$\lim_{n\to\infty} d(x_n, x_{n+k}) \leq \lim_{n\to\infty} \frac{1}{s^{n-1}} [P_{n+k-1} - P_{n-1}] = 0. \tag{4.29}$$

We conclude that the sequence $\{x_n\}$ is Cauchy in (M, d).

Owing to the construction $x_n = T^n x_0$ and the fact that (X, p) is T-orbitally complete, there is $z \in M$ such that $x_n \to z$. Due to the orbital continuity of T, we conclude that $x_n \to Tz$. Hence $z = Tz$ which terminates the proof. □

Corollary 4.1. *Let T be an orbitally continuous self-map on the T-orbitally complete b-metric space (M, d). If there exists $k \in [0, 1)$ such that*

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \leq kd(x, y), \tag{4.30}$$

for all $x, y \in M$, then for each $x_0 \in M$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

If we take $s = 1$ in the previous corollary, we get the famous non-unique fixed point theorem of Ćirić.

Corollary 4.2. [Non-unique fixed point theorem of Ćirić [33]] Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (M, d). If there is $k \in [0, 1)$ such that

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \leq kd(x, y),$$

for all $x, y \in M$, then for each $x_0 \in M$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Remark 4.5. *Regarding the Example 1.6, we deduce that the analog of Ćirić non-unique fixed point theorem, Corollary 4.2, in the setting of cone metric space with normal cone, is still valid (see e.g. [50]).*

Theorem 4.8. *Let T be an orbitally continuous self-map on the T-orbitally complete b-metric space (M, d).*

Suppose there exist real numbers a_1, a_2, a_3, a_4, a_5 and a self mapping $T : M \to M$ satisfies the conditions

$$0 \leq \frac{a_4 - a_2}{a_1 + a_2} < 1, \quad a_1 + a_2 \neq 0, \; a_1 + a_2 + a_3 > 0 \text{ and } 0 \leq a_3 - a_5 \tag{4.31}$$

$$a_1 d(Tx, Ty) + a_2\big[d(x, Tx) + d(y, Ty)\big] + a_3[d(y, Tx) + d(x, Ty)] \leq a_4 d(x, y) + a_5 d(x, T^2x) \tag{4.32}$$

hold for all $x, y \in M$. Then, T has at least one fixed point.

Proof. Take $x_0 \in M$ be arbitrary. Construct a sequence $\{x_n\}$ as follows:

$$x_{n+1} := Tx_n \quad n = 0, 1, 2, ... \tag{4.33}$$

When we substitute $x = x_n$ and $y = x_{n+1}$ on the inequality (4.32), it implies that

$$\begin{aligned} a_1 d(Tx_n, Tx_{n+1}) + a_2\big[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\big] + a_3[d(x_{n+1}, Tx_n) + d(x_n, Tx_{n+1})] \\ \leq a_4 d(x_n, x_{n+1}) + a_5 d(x_n, T^2x_n) \end{aligned} \tag{4.34}$$

for all a_1, a_2, a_3, a_4, a_5 that satisfy (4.31). Due to (4.33), the statement (4.34) turns into

$$\begin{aligned} a_1 d(x_{n+1}, x_{n+2}) + a_2\big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\big] + a_3[d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})] \\ \leq a_4 d(x_n, x_{n+1}) + a_5 d(x_n, x_{n+2}). \end{aligned} \tag{4.35}$$

By a simple calculation, one can get

$$(a_1 + a_2)d(x_{n+1}, x_{n+2}) + (a_3 - a_5)d(x_n, x_{n+2}) \leq (a_4 - a_2)d(x_n, x_{n+1}) \tag{4.36}$$

which implies

$$d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}), \tag{4.37}$$

where $k = \frac{a_4 - a_2}{a_1 + a_2}$. Due to (4.31), we have $0 \leq k < 1$. Taking account of (4.37), we get inductively

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq k^n d(x_0, x_1). \tag{4.38}$$

We shall prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

$$\begin{aligned} d(x_n, x_{n+p}) &\leq s \cdot d(x_n, x_{n+1}) + s^2 \cdot d(x_{n+1}, x_{n+2}) + \ldots + s^{p-2} \cdot d(x_{n+p-3}, x_{n+p-2}) + \\ &+ s^{p-1} \cdot d(x_{n+p-2}, x_{n+p-1}) + s^p \cdot d(x_{n+p-1}, x_{n+p}) \\ &\leq s \cdot k^n \cdot d(x_0, x_1) + s^2 \cdot k^{n+1} \cdot d(x_0, x_1) + \ldots + \\ &+ s^{p-2} \cdot k^{n+p-3} \cdot d(x_0, x_1) + s^{p-1} \cdot k^{n+p-2} \cdot d(x_0, x_1) + \\ &+ s^p \cdot k^{n+p-1} \cdot d(x_0, x_1) \\ &= \frac{1}{s^n \cdot k} \cdot \big[s^{n+1} \cdot k^{n+1} \cdot d(x_0, x_1) + \ldots + s^{n+p-1} \cdot k^{n+p-1} \cdot d(x_0, x_1) + \\ &+ s^{n+p} \cdot k^{n+p} \cdot d(x_0, x_1)\big] \\ &\leq \frac{1}{s^n \cdot k} \cdot \big[s^{n+1} \cdot k^{n+1} \cdot d(x_0, x_1) + \ldots + s^{n+p} \cdot k^{n+p} \cdot d(x_0, x_1)\big] \\ &= \frac{1}{s^n \cdot k} \cdot \sum_{i=n+1}^{n+p} s^i \cdot k^i \cdot d(x_0, x_1) \\ &< \frac{1}{s^n k} \cdot \sum_{i=n+1}^{\infty} s^i \cdot k^i \cdot d(x_0, x_1). \end{aligned}$$

The precedent inequality is

$$d(x_n, x_{n+p}) < \frac{1}{s^n k} \cdot \sum_{i=n+1}^{\infty} s^i \cdot k^i \cdot d(x_0, x_1). \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

As in the proof of previous theorem, regarding the construction $x_n = T^n x_0$ together with the fact that (X, p) is T-orbitally complete, there is $z \in M$ such that $x_n \to z$. Again by the orbital continuity of T, we deduce that $x_n \to Tz$. Hence $z = Tz$. □

Theorem 4.8 is still valid in the context of standard metric space.

Corollary 4.3. *(See [49]) Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (M, d).*

Suppose there exist real numbers a_1, a_2, a_3, a_4, a_5 and a self mapping $T : M \to M$ satisfies the conditions

$$0 \leq \frac{a_4 - a_2}{a_1 + a_2} < 1, \;\; a_1 + a_2 \neq 0, \;\; a_1 + a_2 + a_3 > 0 \text{ and } 0 \leq a_3 - a_5 \tag{4.39}$$

$$a_1 d(Tx, Ty) + a_2\big[d(x, Tx) + d(y, Ty)\big] + a_3[d(y, Tx) + d(x, Ty)] \leq a_4 d(x, y) + a_5 d(x, T^2 x) \tag{4.40}$$

hold for all $x, y \in M$. Then, T has at least one fixed point.

Remark 4.6. *As we discuss in Remark 4.5, we obtain the analog of Theorem 4.8 in the context of cone metric spaces. More precisely, again taking Example 1.6 into account, one can derive that Corollary 4.3 is also still fulfilled in the setting of cone metric space with normal cone (see e.g. [49]).*

Theorem 4.9. *Let T be an orbitally continuous self-map on the T-orbitally complete b-metric space (M,d). Suppose that there exists $\psi \in \Psi$ such that*

$$\frac{P(x,y)-Q(x,y)}{R(x,y)} \leq \psi(d(x,y)), \tag{4.41}$$

for all $x,y \in M$, where

$$\begin{aligned} P(x,y) &= \min\{d(Tx,Ty)d(x,y), d(x,Tx)d(y,Ty)\}, \\ Q(x,y) &= \min\{d(x,Tx)d(x,Ty), d(y,Ty)d(Tx,y)\}, \\ R(x,y) &= \min\{d(x,Tx), d(y,Ty)\}. \end{aligned}$$

with $R(x,y) \neq 0$. Then, for each $x_0 \in M$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. As in the proof of Theorem 4.7, we shall construct an iterative sequence $\{x_n\}$, for an arbitrary initial value $x \in M$:

$$x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}. \tag{4.42}$$

As it is discussed in the proof of Theorem 4.7, we suppose

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{4.43}$$

By substituting $x = x_{n-1}$ and $y = x_n$ in the inequality (4.41), we derive that

$$\frac{P(x_{n-1},x_n)-Q(x_{n-1},x_n)}{R(x_{n-1},x_n)} \leq \psi(d(x_{n-1},x_n)), \tag{4.44}$$

where

$$\begin{aligned} P(x_{n-1},x_n) &= \min\{d(Tx_{n-1},Tx_n)d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)\}, \\ Q(x_{n-1},x_n) &= \min\{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n), d(x_n,Tx_n)d(Tx_{n-1},x_n)\}, \\ R(x_{n-1},x_n) &= \min\{d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n)\}. \end{aligned}$$

Due to axioms of b-metric space , we find that

$$\frac{d(x_n,x_{n+1})d(x_{n-1},x_n)}{\min\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}} \leq \psi(d(x_{n-1},x_n)), \tag{4.45}$$

If $R(x_{n-1},x_n) = d(x_n,x_{n+1})$, then, the inequality (4.45) turns into

$$d(x_{n-1},x_n) \leq \psi(d(x_{n-1},x_n)) < d(x_{n-1},x_n), \tag{4.46}$$

which is a contraction, since $\psi(t) < t$ for all $t > 0$. Accordingly, we deduce that

$$d(x_n,x_{n+1}) \leq \psi(d(x_{n-1},x_n)). \tag{4.47}$$

Applying Remark 1.1 recurrently, we find that

$$d(x_n,x_{n+1}) \leq \psi(d(x_{n-1},x_n)) \leq \psi^2(d(x_{n-2},x_{n-1})) \leq \cdots \leq \psi^n(d(x_0,x_1)). \tag{4.48}$$

By Lemma 1.2, we deduce that

$$\lim_{n\to\infty} d(x_{n+1},x_n) = 0. \tag{4.49}$$

The rest of the proof is a verbatim repetition of the related lines in the proof of Theorem 4.7. □

Corollary 4.4. *Let T be an orbitally continuous self-map on the T-orbitally complete b-metric space (M,d). Suppose that there exists $k \in [0,1)$ such that*

$$\frac{P(x,y)-Q(x,y)}{R(x,y)} \leq kd(x,y), \tag{4.50}$$

for all $x,y \in M$, where

$$\begin{aligned} P(x,y) &= \min\{d(Tx,Ty)d(x,y), d(x,Tx)d(y,Ty)\}, \\ Q(x,y) &= \min\{d(x,Tx)d(x,Ty), d(y,Ty)d(Tx,y)\}, \\ R(x,y) &= \min\{d(x,Tx), d(y,Ty)\}. \end{aligned}$$

with $R(x,y) \neq 0$. Then, for each $x_0 \in M$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Corollary 4.5. [Nonunique fixed point of Achari [1]] Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (M,d). Suppose that there exists $k \in [0,1)$ such that

$$\frac{P(x,y)-Q(x,y)}{R(x,y)} \leq kd(x,y), \tag{4.51}$$

for all $x,y \in M$, where

$$\begin{aligned} P(x,y) &= \min\{d(Tx,Ty)d(x,y), d(x,Tx)d(y,Ty)\}, \\ Q(x,y) &= \min\{d(x,Tx)d(x,Ty), d(y,Ty)d(Tx,y)\}, \\ R(x,y) &= \min\{d(x,Tx), d(y,Ty)\}. \end{aligned}$$

with $R(x,y) \neq 0$. Then, for each $x_0 \in M$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Theorem 4.10. *Let T be an orbitally continuous self-map on the T-orbitally complete b-metric space (M,d). Suppose that there exists $k \in [0,1)$ such that*

$$m(x,y) - n(x,y) \leq kd(x,Tx)d(y,Ty), \tag{4.52}$$

for all $x,y \in M$, where

$$\begin{aligned} m(x,y) &= \min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), [d(y,Ty)]^2\}, \\ n(x,y) &= \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \end{aligned}$$

with $R(x,y) \neq 0$. Then, for each $x_0 \in M$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. By following the lines in the proof of Theorem 4.7, we shall formulate an recursive sequence $\{x_n\}$, for an arbitrary initial value $x \in M$:

$$x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}. \tag{4.53}$$

Regarding the analysis in the proof of Theorem 4.7, we assume that

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{4.54}$$

By replacing $x = x_{n-1}$ and $y = x_n$ in the inequality (4.52), we observe that

$$m(x_{n-1},x_n) - n(x_{n-1},x_n) \leq kd(x_{n-1},Tx_{n-1})d(x_n,Tx_n), \tag{4.55}$$

where

$$\begin{aligned} m(x_{n-1},x_n) &= \min\{[d(Tx_{n-1},Tx_n)]^2, d(x_{n-1},x_n)d(Tx_{n-1},Tx_n), [d(x_n,Tx_n)]^2\}, \\ n(x_{n-1},x_n) &= \min\{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n), d(x_{n-1},Tx_n)d(x_n,Tx_{n-1})\}. \end{aligned}$$

By utilizing the above inequality, we get that

$$m(x_{n-1},x_n) \leq kd(x_{n-1},x_n)d(x_n,x_{n+1}), \tag{4.56}$$

where $m(x_{n-1},x_n)=\min\{[d(x_n,x_{n+1})]^2, d(x_{n-1},x_n)d(x_n,x_{n+1})\}$. Notice that the case $m(x_{n-1},x_n)=d(x_{n-1},x_n)d(x_n,x_{n+1})$ is impossible. Indeed, in this case, since $\psi(t)<t$ for all $t>0$, the inequality (4.56) turns into

$$d(x_{n-1},x_n)d(x_n,x_{n+1}) \le kd(x_{n-1},x_n)d(x_n,x_{n+1}) \tag{4.57}$$

It is a contradiction since $k<1$. Appropriately, we infer that

$$[d(x_n,x_{n+1})]^2 \le kd(x_{n-1},x_n)d(x_n,x_{n+1}) \tag{4.58}$$

which is equivalent to

$$d(x_n,x_{n+1}) \le kd(x_{n-1},x_n). \tag{4.59}$$

Recurrently, we find that

$$d(x_n,x_{n+1}) \le k^n d(x_0,x_1). \tag{4.60}$$

The rest of the proof is a verbatim repetition of the related lines in the proof of Theorem 4.8. □

Theorem 4.8 is still valid in the context of standard metric space.

Corollary 4.6. [Nonunique fixed point of Pachpatte [65]] Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (M,d). Suppose that there exists $k\in[0,1)$ such that

$$m(x,y)-n(x,y) \le kd(x,Tx)d(y,Ty), \tag{4.61}$$

for all $x,y\in M$, where

$$\begin{aligned} m(x,y) &= \min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), [d(y,Ty)]^2\},\\ n(x,y) &= \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \end{aligned}$$

with $R(x,y)\neq 0$. Then, for each $x_0\in M$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Remark 4.7. *One can deduce the analog of Theorem 4.10 in the context of cone metric spaces as it mentioned in Remark 4.5.*

5. On generalized $\alpha-\psi$-Geraghty contractive mapping

Now, we are ready to state and prove our main results. Let Ψ be set of all increasing and continuous functions $\psi:[0,\infty)\to[0,\infty)$ with $\psi^{-1}(\{0\})=\{0\}$. Let $\mathcal{F}$ be the family of all nondecreasing functions $\beta:[0,\infty)\to[0,\frac{1}{s})$ which satisfy the condition

$$\lim_{n\to\infty}\beta(t_n)=\frac{1}{s} \text{ implies } \lim_{n\to\infty}t_n=0, \tag{5.62}$$

for some $s\ge 1$.

Definition 5.15. *Let (M,d) be a b-metric space and $T:M\to M$ be a self-map. We say that T is a generalized $\alpha-\psi$-Geraghty contractive mapping whenever there exist $\alpha:M\times M\to[0,\infty)$ and some $L\ge 0$ such that for*

$$E(x,y)=\max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\} \tag{5.63}$$

$$\text{and } N(x,y)=\min\{d(x,Tx), d(y,Tx)\}, \tag{5.64}$$

we have

$$\alpha(x,y)\psi(s^3 d(Tx,Ty)) \le \beta(\psi(E(x,y)))\psi(E(x,y)) + L\phi(N(x,y)), \tag{5.65}$$

for all $x,y\in M$, where $\beta\in\mathcal{F}$ and $\psi,\phi\in\Psi$.

Remark 5.8. *Since the functions belonging to $\mathcal{F}$ are strictly smaller than $\frac{1}{s}$, the expression $\beta(\psi(E(x,y)))$ in (5.65) can be estimated as*

$$\beta(\psi(E(x,y))) < \frac{1}{s} \text{ for any } x, y \in M \text{ with } x \neq y.$$

Theorem 5.11. *Let (M,d) be a complete b-metric space and $T : M \to M$ be a generalized $\alpha - \psi$-Geraghty contractive mapping such that*
(i) T is triangular α-orbital admissible;
(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then T has a fixed point.

Proof. Let $x_0 \in M$ be such that $\alpha(x_0, Tx_0) \geq 1$. We construct an iterative sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}_0.$$

If there exists an n_0 such that $Tx_{n_0} = x_{n_0}$ for some n_0, then x_{n_0} is a fixed point of T which completes the proof. Thus, without loss of generality, we assume that

$$x_n \neq x_{n+1} \text{ for all } n \in \mathbb{N}_0. \tag{5.66}$$

The mapping T is triangular α-orbital admissible, by Lemma 2.3, we have

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{5.67}$$

By taking $x = x_{n-1}$ and $y = x_n$ in the inequality (5.65) together with the inequality (5.67) and regarding that ψ is an increasing function, we obtain

$$\begin{aligned} \psi(d(x_n, x_{n+1})) = \psi(d(Tx_{n-1}, Tx_n)) &\leq \alpha(x_{n-1}, x_n)\psi(s^3 d(Tx_{n-1}, Tx_n)) \\ &\leq \beta(\psi(M(x_{n-1}, x_n)))\psi(M(x_{n-1}, x_n)) + L\phi(N(x_{n-1}, x_n)), \end{aligned} \tag{5.68}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2s}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2s}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s}\} \end{aligned}$$

and

$$\begin{aligned} N(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_{n-1})\} \\ &= \min\{d(x_{n-1}, x_n), d(x_n, x_n)\} = 0. \end{aligned} \tag{5.69}$$

Since

$$\frac{d(x_{n-1}, x_{n+1})}{2s} \leq \frac{s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{2s} \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},$$

then we get

$$M(x_{n-1}, x_n) \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \tag{5.70}$$

Taking (5.70) and (5.69) into account, (5.68) yields that

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \psi(s^3 d(x_n, x_{n+1})) \leq \alpha(x_{n-1}, x_n)\psi(s^3 d(x_n, x_{n+1})) \\ &\leq \beta(\psi(M(x_{n-1}, x_n))\psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \end{aligned} \tag{5.71}$$

If for some $n \in \mathbb{N}$, we have $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then by (5.71) and Remark 5.8, we get

$$\psi(d(x_n, x_{n+1})) \leq \beta(\psi(M(x_{n-1}, x_n))\psi(d(x_n, x_{n+1}) < \frac{1}{s}\psi(d(x_n, x_{n+1}) < \psi(d(x_n, x_{n+1}), \tag{5.72}$$

which is a contradiction. Thus, from (5.71) we conclude that

$$\psi(d(x_n, x_{n+1})) \leq \beta(\psi(M(x_{n-1}, x_n)))\psi(d(x_{n-1}, x_n)) < \frac{1}{s}\psi(d(x_{n-1}, x_n)) < \psi(d(x_{n-1}, x_n)), \tag{5.73}$$

for all $n \in \mathbb{N}$. Hence $\{\psi(d(x_n, x_{n+1}))\}$ is a non-negative decreasing sequence. Since ψ is increasing, so the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing. Consequently, there exists $\delta \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \delta$. We claim that $\delta = 0$. Suppose, on the contrary that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \delta > 0. \tag{5.74}$$

Since $s \geq 1$, the inequality (5.73) can be estimated as

$$\frac{1}{s}\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) \leq \beta(\psi(M(x_{n-1}, x_n)))\psi(d(x_{n-1}, x_n)). \tag{5.75}$$

Regarding (5.66), the inequality (5.75) implies that

$$\frac{1}{s}\frac{\psi(d(x_n, x_{n+1}))}{\psi(d(x_{n-1}, x_n))} \leq \beta(\psi(M(x_{n-1}, x_n))) < \frac{1}{s}.$$

It yields that $\lim_{n\to\infty} \beta(\psi(M(x_{n-1}, x_n))) = \frac{1}{s}$. Since $\beta \in \mathcal{F}$, then $\lim_{n\to\infty} \psi(M(x_{n-1}, x_n)) = 0$. We deduce that

$$\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0.$$

Thus, regarding the fact that $d(x_n, x_{n+1}) \to \delta$ and the continuity of ψ, we derive that $\psi(\delta) = 0$. Since $\psi^{-1}(\{0\}) = \{0\}$, so$\delta = 0$, which is a contradiction. Thus, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{5.76}$$

Now, we claim that

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0.$$

Assume on the contrary that exist $\varepsilon > 0$ and subsequences $\{x_{m_i}\}, \{x_{n_i}\}$ of $\{x_n\}$ with $n_i > m_i \geq i$ such that

$$d(x_{m_i}, x_{n_i}) \geq \varepsilon. \tag{5.77}$$

Additionally, corresponding to m_i, we may choose n_i such that it is the smallest integer satisfying (5.77) and $n_i > m_i \geq i$. Thus, we have

$$d(x_{m_i}, x_{n_i-1}) < \varepsilon. \tag{5.78}$$

From (5.77) and the triangle inequality, we obtain

$$\begin{aligned} \varepsilon \leq d(x_{n_i}, x_{m_i}) &\leq sd(x_{n_i}, x_{n_{i+1}}) + sd(x_{n_{i+1}}, x_{m_i}) \\ &\leq sd(x_{n_i}, x_{n_{i+1}}) + s^2 d(x_{n_{i+1}}, x_{m_{i+1}}) + s^2 d(x_{m_{i+1}}, x_{m_i}). \end{aligned} \tag{5.79}$$

Letting $i \to \infty$ and regarding (5.76), the inequality (5.79) yields that

$$\frac{\varepsilon}{s^2} \leq \limsup_{i\to\infty} d(x_{n_{i+1}}, x_{m_{i+1}}). \tag{5.80}$$

By Lemma 2.3, recall that

$$\alpha(x_{m_i}, x_{n_i}) \geq 1. \tag{5.81}$$

Consequently, by (5.65) we have

$$\begin{aligned} \psi(d(x_{n_{i+1}}, x_{m_{i+1}})) &= \psi(d(Tx_{n_i}, Tx_{m_i})) \\ &\leq \psi(s^3\, d(Tx_{n_i}, Tx_{m_i})) \leq \alpha(x_{m_i}, x_{n_i})\psi(s^3 d(Tx_{n_i}, Tx_{m_i})) \\ &\leq \beta(\psi(M(x_{n_i}, x_{m_i})))\psi(M(x_{n_i}, x_{m_i})) + L\phi(d(x_{m_i}, Tx_{n_i}))), \end{aligned} \tag{5.82}$$

where

(5.83)

$$\begin{aligned} M(x_{n_i}, x_{m_i}) &= \max\{d(x_{n_i}, x_{m_i}), d(x_{n_i}, Tx_{n_i}), d(x_{m_i}, Tx_{m_i}), \frac{d(x_{n_i}, Tx_{m_i}) + d(x_{m_i}, Tx_{n_i})}{2s}\} \\ &= \max\{d(x_{n_i}, x_{m_i}), d(x_{n_i}, x_{n_{i+1}}), d(x_{m_i}, x_{m_{i+1}}), \frac{d(x_{n_i}, x_{m_{i+1}}) + d(x_{m_i}, x_{n_{i+1}})}{2s}\}, \end{aligned}$$

and

$$N(x_{n_i}, x_{m_i}) = \min\{d(x_{n_i}, Tx_{n_i}), d(x_{m_i}, Tx_{n_i})\} = \min\{d(x_{n_i}, x_{n_i+1}), d(x_{m_i}, x_{n_i+1})\}.$$

Notice that

(5.84)

$$\frac{d(x_{n_i}, x_{m_{i+1}}) + d(x_{m_i}, x_{n_{i+1}})}{2s} \leq \frac{s[d(x_{n_i}, x_{m_i}) + d(x_{m_i}, x_{m_{i+1}})] + s[d(x_{m_i}, x_{n_i}) + d(x_{n_i}, x_{n_{i+1}})]}{2s}$$

and

$$d(x_{n_i}, x_{m_i}) \leq s[d(x_{n_i}, x_{n_i-1}) + d(x_{n_i-1}, x_{m_i})] < sd(x_{n_i}, x_{n_i-1}) + s\varepsilon. \tag{5.85}$$

Taking (5.78), (5.84) and (5.85) into account, we find that

$$\limsup_{i\to\infty} M(x_{n_i}, x_{m_i}) \leq s\varepsilon, \text{ and} \tag{5.86}$$

$$\lim_{i\to\infty} N(x_{n_i}, x_{m_i}) = 0. \tag{5.87}$$

By taking the upper limit as $i \to \infty$ and regarding the condition $(T4)$ together with the expressions (5.80), (5.86) and (5.87), the inequality (5.82) becomes

$$\begin{aligned} \frac{1}{s}\psi(s\varepsilon) \leq \psi(s\varepsilon) &\leq \limsup_{i\to\infty} \psi(s^3\, d(x_{n_{i+1}}, x_{m_{i+1}})) \\ &\leq \limsup_{i\to\infty} \alpha(x_{m_i}, x_{n_i}))\psi(s^3 d(x_{n_{i+1}}, x_{m_{i+1}})) \\ &= \limsup_{i\to\infty} \alpha(x_{m_i}, x_{n_i})\psi(s^3 d(Tx_{n_i}, Tx_{m_i})) \\ &\leq \limsup_{i\to\infty} [\beta(\psi(M(x_{n_i}, x_{m_i})))\psi(M(x_{n_i}, x_{m_i})) + L\phi(N(d(x_{n_i}, x_{m_i})))] \\ &\leq \psi(s\varepsilon) \limsup_{i\to\infty} \beta(\psi(M(x_{n_i}, x_{m_i}))) \\ &\leq \frac{1}{s}\psi(s\varepsilon). \end{aligned}$$

Then $\limsup_{i\to\infty} \beta(\psi(M(x_{n_i}, x_{m_i}))) = \frac{1}{s}$. Due to the fact $\beta \in \mathcal{F}$, we have

$$\limsup_{i\to\infty} \psi(M(x_{n_i}, x_{m_i})) = 0.$$

Thus, we conclude that

$$\lim_{i\to\infty} \psi(d(x_{n_i}, x_{m_i})) = 0.$$

Therefore, by continuity of ψ and the fact that $\psi^{-1}(\{0\}) = \{0\}$, so

$$\lim_{i\to\infty} d(x_{n_i}, x_{m_i}) = 0, \tag{5.88}$$

which is a contradiction with respect to (5.77). We deduce that $\{x_n\}$ is a Cauchy sequence in (M, d). Since (M, d) is a complete b-metric space, there exists $x^* \in M$ such that $\lim_{n\to\infty} x_n = x^*$. The mapping T is continuous and it is obvious that $Tx^* = x^*$. □

We replace the continuity of the mapping T in the above theorem by a suitable condition on X.

Theorem 5.12. *Let (M, d) be a complete b-metric space and $T : M \to M$ be a generalized $\alpha - \psi$-Geraghty contractive mapping such that*
(i) T is triangular α-orbital admissible;
(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) X is α-regular.
Then T has a fixed point.

Proof. Following the lines in the proof of Theorem 5.11, we conclude that $\lim_{n\to\infty} x_n = x^*$. If X is α-regular, then since $\alpha(x_n, x_{n+1}) \geq 1$, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n_k}, x^*) \geq 1, \tag{5.89}$$

for all k. By triangular inequality

$$\begin{aligned} d(x^*, Tx^*) &\leq sd(x^*, x_{n_k+1}) + sd(x_{n_k+1}, Tx^*) \\ &= sd(x^*, x_{n_k+1}) + sd(Tx_{n_k}, Tx^*). \end{aligned}$$

Letting k tends to infinity

$$d(x^*, Tx^*) \leq \liminf_{k\to\infty} sd(Tx_{n_k}, Tx^*). \tag{5.90}$$

Having $\psi \in \Psi$, (5.89) and (5.90), so

$$\begin{aligned} \psi(s^2 d(x^*, Tx^*)) &\leq \lim_{k\to\infty} \psi(s^3 d(Tx_{n_k}, Tx^*)) \leq \lim_{k\to\infty} \alpha(x_{n_{k+1}}, x^*)\psi(s^3 d(Tx_{n_k}, Tx^*)) \\ &\leq \lim_{k\to\infty} [\beta(\psi(M(x_{n_k}, x^*)))\psi(M(x_{n_k}, x^*)) + L\phi(N(x_{n_k}, x^*))]. \end{aligned} \tag{5.91}$$

We have

$$\begin{aligned} M(x_{n_k}, x^*) &= \max\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})}{2s}\} \\ &= \max\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_{k+1}}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_{k+1}})}{2s}\}, \end{aligned}$$

and

$$\begin{aligned} N(x_{n_k}, x^*) &= \min\{d(x_{n_k}, Tx_{n_k}), d(x^*, Tx_{n_k})\} \\ &= \min\{d(x_{n_k}, x_{n_{k+1}}), d(x^*, x_{n_{k+1}})\}. \end{aligned}$$

Recall that

$$\frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_{k+1}})}{2s} \leq \frac{sd(x_{n_k}, x^*) + sd(x^*, Tx^*) + d(x^*, x_{n_{k+1}})}{2s}.$$

Then, by (5.76), we get that

$$\limsup_{k\to\infty} \frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_{k+1}})}{2s} \le \frac{d(x^*, Tx^*)}{2}.$$

When k tends to infinity, we deduce

$$\lim_{k\to\infty} M(x_{n_k}, x^*) = d(x^*, Tx^*),$$

and

$$\lim_{k\to\infty} N(x_{n_k}, x^*) = 0.$$

Since $\beta(\psi(M(x_{n_k}, x^*))) \le \frac{1}{s}, \forall k \in \mathbb{N}$ so by (5.91)

$$\psi(s^2 d(x^*, Tx^*)) \le \frac{1}{s}\psi(d(x^*, Tx^*)) \le \psi(d(x^*, Tx^*)).$$

Since $\psi \in \Psi$, so the above holds unless $d(x^*, Tx^*) = 0$, that is, $Tx^* = x^*$ and x^* is a fixed point of T. □

For the uniqueness of a fixed point of a generalized $\alpha - \psi$ contractive mapping, we will consider the following hypothesis.

(H) For all $x, y \in \text{Fix}(T)$, either $\alpha(x, y) \ge 1$ or $\alpha(y, x) \ge 1$.

Here, Fix(T) denotes the set of fixed points of T.

Theorem 5.13. *Adding condition (H) to hypotheses of Theorem 5.11 (respectively, Theorem 5.12), we obtain uniqueness of the fixed point of T.*

Proof. Suppose that x^* and y^* are two fixed points of T. Then we have, it is obvious that $M(x^*, y^*) = d(x^*, y^*)$ and $N(x^*, y^*) = 0$. So

$$\begin{aligned}
\psi(d(x^*, y^*)) &\le \psi(s^3 d(Tx^*, Ty^*)) \\
&\le \alpha(x^*, y^*)\psi(s^3 d(Tx^*, Ty^*)) \\
&\le \beta(\psi(M(x^*, y^*)))\psi(M(x^*, y^*)) + L\phi(N(x^*, y^*)) \\
&< \frac{1}{s}\psi(d(x^*, y^*)) \le \psi(d(x^*, y^*)),
\end{aligned}$$

which is contradiction. □

Definition 5.16. *Let (M, d) be a b-metric space and $T : M \to M$ be a self-map. We say that T is a generalized $\alpha{-}\psi$-Geraghty contractive mapping of type (B) whenever there exists $\alpha : M \times M \to [0, \infty)$ such that for*

$$E(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\} \tag{5.92}$$

we have

$$\alpha(x, y)\psi(s^3 d(Tx, Ty)) \le \beta(\psi(E(x, y)))\psi(E(x, y)) \tag{5.93}$$

for all $x, y \in M$, where $\beta \in \mathcal{F}$ and $\psi \in \Psi$.

By verbatim of the proofs of Theorem 5.11, Theorem 5.12 and Theorem 5.13, we get the following results:

Theorem 5.14. *Let (M,d) be a complete b-metric space and $T : M \to M$ be a generalized $\alpha-\psi$-Geraghty contractive mapping of type (B) such that*
(i) T is triangular α-orbital admissible;
(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) either T is continuous or X is α-regular.
Then T has a fixed point.

Theorem 5.15. *Adding condition (H) to hypotheses of Theorem 5.14, we obtain uniqueness of the fixed point of T.*

Example 5.9. *Let X be the set of Lebesgue measurable functions on $[0,1]$ such that*

$$\int_0^1 |x(t)|dt < 1.$$

Define $d : M \times M \to [0,\infty)$ by

$$d(x,y) = (\int_0^1 |x(t) - y(t)|dt)^2.$$

Then, d is a b-metric on X with $s = 2$.
The operator $T : M \to M$ is defined by

$$Tx(t) = \frac{1}{4} ln(1 + |x(t)|).$$

Consider the mappings $\alpha : M \times M \to [0,\infty)$, $\beta : [0,\infty) \to [0, \frac{1}{2})$ and $\psi : [0,\infty) \to [0,\infty)$ defined by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x(t) \geq y(t), \forall t \in [0,1], \\ 0 \text{ otherwise.} \end{cases}$$

$$\psi(t) = t \quad \text{and} \quad \beta(t) = \frac{(ln(1+\sqrt{t}))^2}{2t}.$$

Evidently, $\psi \in \Psi$ and $\beta \in \mathcal{F}$. Moreover, T is a triangular α-orbital admissible mapping and $\alpha(1, T1) \geq 1$.
Now, we shall prove that T is a generalized $\alpha-\psi$-Geraghty contractive mapping. In fact, for all $t \in [0,1]$, we have

$$\sqrt{\alpha(x(t),y(t))\psi(s^3 d(Tx(t),Ty(t)))} \leq \sqrt{2^3(\int_0^1 |Tx(t) - Ty(t)|dt)^2}$$
$$\leq 2\sqrt{2}\int_0^1 |\frac{1}{4} ln(1+|x(t)|) - \frac{1}{4} ln(1+|y(t)|)|dt$$
$$= \frac{1}{\sqrt{2}}\int_0^1 |ln(\frac{1+|x(t)|}{1+|y(t)|})|dt$$
$$= \frac{1}{\sqrt{2}}\int_0^1 |ln(1+\frac{|x(t)|-|y(t)|}{1+|y(t)|})|dt$$
$$\leq \frac{1}{\sqrt{2}}\int_0^1 |ln(1+|x(t)|-|y(t)|)|dt$$

By Lemma 8.4 (given in Appendix), we get

$$\int_0^1 |ln(1+|x(t)|-|y(t)|)|dt \le ln(\int_0^1 (1+|x(t)-y(t)|)dt) = ln(1+\int_0^1 |x(t)-y(t)|dt).$$

Therefore

$$\sqrt{\alpha(x(t),y(t))\psi(s^3 d(Tx(t),Ty(t)))} \le \frac{1}{\sqrt{2}} ln(1+\int_0^1 |x(t)-y(t)|dt) \le \frac{1}{\sqrt{2}} ln(1+\sqrt{d(x,y)}).$$

So, we obtain

$$\alpha(x(t),y(t))\psi(s^3 d(Tx(t),Ty(t))) \le \frac{1}{2}(ln(1+\sqrt{d(x,y)}))^2$$
$$\le \frac{1}{2}(ln(1+\sqrt{E(x,y)}))^2$$
$$= \frac{(ln(1+\sqrt{E(x,y)}))^2}{2E(x,y)}\ E(x,y)$$
$$= \beta(\psi(E(x,y)))\ \ \psi(E(x,y)).$$

Thus, by Theorem 5.14, we see that T has a fixed point.

6. CONSEQUENCES

In this section, we shall demonstrate that several existing results in the literature can be easily concluded from Theorem 5.13.

6.1. **Standard fixed point theorems in b-metric.** By taking $\alpha(x,y) = 1$ in Theorem 5.13, for all $x, y \in M$, we obtain immediately the following fixed point theorem.

Corollary 6.7. *Let (M,d) be a complete b-metric space with $s \ge 1$ and $T : M \to M$ be a mapping on X. If there exists $L \ge 0$ such that for all $x, y \in M$,*

(6.94) $$\psi(s^3 d(Tx,Ty)) \le \beta(\psi(E(x,y)))\psi(E(x,y)) + L\phi(N(x,y)),$$

where $\beta \in \mathcal{F}$, $\psi, \phi \in \Psi$ and

(6.95) $$E(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\},$$

(6.96) $$\text{and } N(x,y) = \min\{d(x,Tx), d(y,Tx)\},$$

then T has a unique fixed point.

By taking $\alpha(x,y) = 1$ in Theorem 5.15, for all $x, y \in M$, we obtain immediately the following fixed point result.

Corollary 6.8. *Let (M,d) be a complete b-metric space with $s \ge 1$ and $T : M \to M$ be a mapping on X such that for all $x, y \in M$,*

(6.97) $$\psi(s^3 d(Tx,Ty)) \le \beta(\psi(E(x,y)))\psi(E(x,y))$$

where $\beta \in \mathcal{F}$, $\psi \in \Psi$ and

(6.98) $$E(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}.$$

Then T has a unique fixed point.

If we put $\alpha(x,y) = 1, \forall x,y \in M$, $L = 0$ and $\psi(t) = t$ in Theorem 5.13, we may state the following result.

Corollary 6.9. *Let (M,d) be a complete b-metric space with $s \geq 1$ and $T: M \to M$ be a mapping on X such that for all $x,y \in M$,*

$$s^3 d(Tx,Ty) \leq \beta(E(x,y))\, E(x,y)$$

where $\beta \in \mathcal{F}$ and

$$E(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}. \tag{6.99}$$

Then, T has a unique fixed point.

If we take $s = 1$ and $\beta(t) = \frac{1}{t+1}$ for $t > 0$ in Corollary 6.9, we deduce the following result.

Corollary 6.10. *Let (M,d) be a complete metric space and $T: M \to M$ be a mapping on X such that for all $x,y \in M$,*

$$d(Tx,Ty) \leq \frac{E(x,y)}{1+E(x,y)}.$$

Then T has a unique fixed point.

6.2. **Fixed point theorems on b-metric spaces endowed with a partial order.** On the last decade, several exciting developments have been reported in the field of existence of fixed point on metric spaces endowed with partial orders see e.g. [64,69,81]. In this section, from Theorem 5.13 (and also from Theorem 5.15), we shall easily conclude some fixed point results on a b-metric space endowed with a partial order. First of all, we recall some basic concepts:

Definition 6.17. *Let $(M,\preceq)$ be a partially ordered set and $T: M \to M$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if*

$$x,y \in M,\ x \preceq y \Longrightarrow Tx \preceq Ty.$$

Definition 6.18. *Let $(M,\preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.*

Definition 6.19. *Let $(M,\preceq)$ be a partially ordered set and d be a b-metric on X. We say that $(M,\preceq,d)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in M$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.*

We have the following result.

Corollary 6.11. *Let $(M,\preceq)$ be a partially ordered set and d be a b-metric on X such that (M,d) is complete . Let $T: M \to M$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist functions $\beta \in \mathcal{F}$, $\psi \in \Psi$ such that*

$$\psi(s^3 d(Tx,Ty)) \leq \beta(\psi(E(x,y)))\psi(E(x,y)) \tag{6.100}$$

and

$$E(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}, \tag{6.101}$$

for all $x,y \in M$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in M$ such that $x_0 \preceq Tx_0$;*

(ii) *T is continuous or $(M, \preceq, d)$ is regular.*

Then T has a fixed point. Moreover, if for all $x, y \in Fix(T)$ either $x \preceq y$ or $y \preceq x$, we have uniqueness of the fixed point.

Proof. Define the mapping $\alpha : M \times M \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x \preceq y \text{ or } x \succeq y, \\ 0 \text{ otherwise.} \end{cases}$$

Clearly, T is a generalized $\alpha - \psi$ contractive mapping, that is,

$$\alpha(x, y)\psi(s^3 d(Tx, Ty)) \leq \beta(\psi(E(x, y)))\psi(E(x, y)),$$

for all $x, y \in M$. From condition (i), we have $\alpha(x_0, Tx_0) \geq 1$. On the other hand, for all $x, y \in M$, from the monotone property of T, we have

$$\alpha(x, y) \geq 1 \Longrightarrow x \succeq y \text{ or } x \preceq y \Longrightarrow Tx \succeq Ty \text{ or } Tx \preceq Ty \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

So T is $\alpha-$admissible. In case of T is continuous, the existence of a fixed point is concluded from Theorem 5.14. Now, assume that $(M, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$. From the regularity hypothesis, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. It yields from the definition of α that $\alpha(x_{n(k)}, x) \geq 1$ for all k. In this case, the existence of a fixed point follows from Theorem 5.14. To prove the uniqueness, let $x, y \in M$. Due to the hypothesis, we have $\alpha(x, y) \geq 1$ and $\alpha(y, x) \geq 1$. Hence, by Theorem 5.15, we conclude the uniqueness of the fixed point. □

The following results are immediate consequences of Corollary 6.11.

Corollary 6.12. *Let $(M, \preceq)$ be a partially ordered set and d be a b-metric on X such that (M, d) is complete . Let $T : M \to M$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist functions $\beta \in \mathcal{F}$ and $\psi \in \Psi$ such that*

$$\psi(s^3 d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)) \tag{6.102}$$

for all $x, y \in M$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in M$ such that $x_0 \preceq Tx_0$;*

(ii) *T is continuous or $(M, \preceq, d)$ is regular.*

Then T has a fixed point. Moreover, if for all $x, y \in Fix(T)$ either $x \preceq y$ or $y \preceq x$, we have uniqueness of the fixed point.

Remark 6.9. *In fact, in all results above, one can take $s = 1$ to conclude the existing results in the literature.*

7. Application

As an application, we consider the following integral equation

$$x(t) = h(t) + \int_0^1 k(t, \xi)T(\xi, x(\xi))d\xi, \quad \forall\, t \in [0, 1]. \tag{7.103}$$

Let Ω denote the class of non-decreasing functions $\omega : [0, \infty) \to [0, \infty)$ verifying

$$(\omega(t))^r \leq t^r\, \omega(t^r), \quad \text{for all } r \geq 1 \quad \text{and } \forall t \geq 0.$$

We will analyze equation (7.103) under the following assumptions:

(a_1) $h : [0, 1] \to \mathbb{R}$ is a continuous function,

(a_2) $T : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $T(t,x) \geq 0$ and there exists $\omega \in \Omega$ such that for all $x, y \in \mathbb{R}$,

$$|T(t,x) - T(t,y)| \leq \omega(|x-y|)$$

with $w(t_n) \to \frac{1}{2^{r-1}}$ as $n \to \infty$ implies that $\lim_{n\to\infty} t_n = 0$,

(a_3) $k : [0,1] \times [0,1] \to \mathbb{R}$ is continuous in $t \in [0,1]$ for every $\xi \in [0,1]$ and is measurable in $\xi \in [0,1]$ for all $t \in [0,1]$ such that $k(t,x) \geq 0$ and

$$\int_0^1 k(t,\xi)d\xi \leq \frac{1}{2^{3-\frac{3}{r}}}.$$

Consider the space $M = C([0,1])$ of continuous functions with the standard metric given by

$$\rho(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)|, \forall x, y \in C([0,1]).$$

Now, for $r \geq 1$, we define

$$d(x,y) = (\rho(x,y))^r = (\sup_{t\in[0,1]} |x(t) - y(t)|)^r = \sup_{t\in[0,1]} |x(t) - y(t)|^r, \forall x, y \in C([0,1]).$$

Note that (M,d) is a complete b-metric space with $s = 2^{r-1}$.

Theorem 7.16. *Under assumptions* $(a_1)-(a_3)$*, the equation (7.103) has a unique solution in* $C([0,1])$*.*

Proof. We consider the operator $T : M \to M$ defined by

$$T(x)(t) = h(t) + \int_0^1 k(t,\xi)T(\xi, x(\xi))d\xi, \quad t \in [0,1].$$

By virtue of our assumptions, T is well defined (this means that if $x \in M$ then $Tx \in M$). Also, for $x, y \in M$, we have

$$\begin{aligned}
|T(x)(t) - T(y)(t)| &= |h(t) + \int_0^1 k(t,\xi)T(\xi,x(\xi))d\xi - h(t) - \int_0^1 k(t,\xi)T(\xi,x(\xi))d\xi| \\
&\leq \int_0^1 k(t,\xi)|T(\xi,x(\xi)) - T(\xi,y(\xi))|d\xi \\
&\leq \int_0^1 k(t,\xi)\,\omega(|x(\xi) - y(\xi)|)d\xi.
\end{aligned}$$

Since the function ω is non-decreasing, so

$$\omega(|x(\xi) - y(\xi)|) \leq \omega(\sup_{t\in[0,1]} |x(\xi) - y(\xi)|) = \omega(\rho(x,y)).$$

Therefore

$$|T(x)(t) - T(y)(t)| \leq \frac{1}{2^{3-\frac{3}{r}}}\omega(\rho(x,y)).$$

Now, we have

$$\begin{aligned}
d(Tx,Ty) &= \sup_{t\in[0,1]} |T(x)(t) - T(y)(t)|^r \\
&\leq [\frac{1}{2^{3-\frac{3}{r}}}\omega(\rho(x,y))]^r \leq \frac{1}{2^{3r-3}} d(x,y)\,\omega(d(x,y)) \\
&\leq \frac{1}{2^{3r-3}}\omega(E(x,y))\,E(x,y),
\end{aligned}$$

that is,

$$s^3 d(Tx,Ty) \leq \beta(E(x,y))\,E(x,y),$$

where $s = 2^{r-1}$ and $\beta(t) = \omega(t)$. Notice that, if $\omega \in \mathcal{F}$, so $\beta \in \mathcal{F}$. By Corollary 6.9, equation (7.103) has a unique solution in $C[0,1]$ and the proof is completed. □

8. APPENDIX

Lemma 8.4. *Let (X,μ) be a measure space such that $\mu(X) = 1$. Take $f \in L^1(X,\mu)$ satisfying the condition $T(x) > 0$ for all $x \in M$. Then, $\ln(f) \in L^1(X,\mu)$ and*

$$\int \ln(f) d\mu \leq \ln(\int f\, d\mu).$$

Proof. Put $g(t) := t - 1 - \ln(t)$ and $h(t) := 1 - \frac{1}{t} - \ln(t)$ for $t > 0$. Then, $g'(t) = 1 - \frac{1}{t}$ and $h'(t) = \frac{1}{t^2} - \frac{1}{t}$. Clearly, notice that

$$g(t) \geq g(1) = 0 \quad \text{and} \quad h(t) \leq h(1) = 0 \quad \forall\, t > 0.$$

We deduce

$$t - 1 \geq \ln(t) \geq 1 - \frac{1}{t} \quad \forall\, t > 0. \tag{8.104}$$

Since T is measurable and $\ln$ is continuous, then $\ln(f)$ is measurable. Now, for all $x \in M$ let $t = \frac{T(x)}{\|f\|_1}$ in (8.104). So, we have

$$1 - \frac{\| f \|_1}{T(x)} \leq \ln(T(x)) - \ln(\| f \|_1) \leq \frac{T(x)}{\| f \|_1} - 1.$$

Since both right hand and left hand of $[\ln(T(x)) - \ln(\| f \|_1)]$ is integrable, so $\ln(T(x)) - \ln(\| f \|_1)$ is integrable. We also have

$$\int (\ln(T(x)) - \ln(\| f \|_1) d\mu \leq \int (\frac{T(x)}{\| f \|_1} - 1) d\mu = 0.$$

Therefore,

$$\int \ln(f)\ d\,\mu \leq \ln(\int f\, d\mu).$$

□

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Atilim University,
Department of Mathematics
06836, İncek, Ankara, Turkey.
E-mail address: erdalkarapinar@yahoo.com

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Approximation Results for Urysohn Type Two Dimensional Nonlinear Bernstein Operators

HARUN KARSLI*

ABSTRACT. In the present work, our aim of this study is generalization and extension of the theory of interpolation of two dimensional functions to functionals or operators by means of Urysohn type nonlinear operators. In accordance with this purpose, we introduce and study a new type of Urysohn type nonlinear operators. In particular, we investigate the convergence problem for nonlinear operators that approximate the Urysohn type operator in two dimensional case. The starting point of this study is motivated by the important applications that approximation properties of certain families of nonlinear operators have in signal-image reconstruction and in other related fields. We construct our nonlinear operators by using a nonlinear form of the kernels together with the Urysohn type operator values instead of the sampling values of the function.

Keywords: Urysohn integral operators, Nonlinear Bernstein operators, Urysohn type two dimensional nonlinear Bernstein operators.

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1. INTRODUCTION

For a function defined on the interval $[0,1]$, the Bernstein operators $(B_n f)$, $n \geq 1$, are defined by

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \quad , \quad n \geq 1, \tag{1.1}$$

where $p_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}$ is the well-known Binomial distribution and called Bernstein basis $(0 \leq x \leq 1)$. These polynomials were introduced by Bernstein [9] in 1912 to give the first constructive proof of the Weierstrass approximation theorem.

For detailed approaches to this operator see the fundamental book of G.G. Lorentz [27].

In his Ph.D. thesis [12] written under the direction of G.G. Lorentz and afterwards in the paper [11] , the famous German mathematician P.L. Butzer considered two dimensional Bernstein polynomials on the square $\square := \{(x,y) : 0 \leq x, y \leq 1\}$ as follows:

$$B_{n,m}(f;x,y) = \sum_{k=0}^{n} \sum_{j=0}^{m} f\left(\frac{k}{n}, \frac{j}{m}\right) p_{n,k}(x)\, p_{m,j}(y)$$

where $p_{n,k}(t) = \begin{pmatrix} n \\ k \end{pmatrix} t^k (1-t)^{n-k}$.

At the beginning, the theory of approximation is strongly related with the linearity of the operators. But, thanks to the approachs of the Polish mathematician Julian Musielak, see [29],

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*Corresponding author: H. Karslı; karsli_h@ibu.edu.tr
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and continuous works of C.Bardaro, G. Vinti and their research group, this theory can be extended to the nonlinear type operators, under some specific assumptions on its kernel functions, see the fundamental book due to Bardaro, Musielak and Vinti [6]. For further reading please see [1]- [5], [13], [14] as well as the monographs [33].

In view of the approaches due to Musielak [29], recently, Karsli-Tiryaki and Altin [23] introduced the following type nonlinear counterpart of the well-known Bernstein operators (1.1);

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \quad , \quad 0 \leq x \leq 1 \quad , \quad n \in \mathbb{N}, \tag{1.2}$$

acting on bounded functions f on the interval $[0, 1]$, where $P_{n,k}$ satisfy some suitable assumptions. They proved some existence and approximation theorems for the nonlinear Bernstein operators.

Many problems in engineering and mechanics can be transformed into two-dimensional integral equations and corresponding two dimensional integral operators. Especially the integral operators of Fredholm, Volterra, Hammerstein and Urysohn type are used frequently when describing real problems which arise from different sciences, such as physics, engineering, mechanics, theory of elasticity, signal-image reconstruction and in the applications of mathematical physics. So, integral operators of various types form an important and unavoidable part of linear and nonlinear functional analysis.

In 2000, Demkiv [15] and [16] defined and investigated some properties of the following type one and two dimensional Bernstein operators, which are linear with respect to F defined by (2.4);

$$(B_n F)\, x\,(t) = \int_0^1 \sum_{k=0}^{n} f\left(t, s, \frac{k}{n}\right) p_{n,k}\left(x(s)\right) ds,$$

and

$$(B_{n,m} F)\left(x\left(.\right), y(.)\right) = \int_0^1 \int_0^1 \sum_{i=0}^{n} \sum_{j=0}^{m} f\left(t, z_1, z_2, \frac{i}{n}, \frac{j}{m}\right) p_{n,i}\left(x(z_1)\right) p_{n,j}\left(y(z_2)\right) dz_1 dz_2.$$

In 2017, the author [24] defined the following Urysohn type Meyer-König and Zeller operators;

$$\begin{aligned}
(M_n F)x(t) &= \int_0^1 \left[\sum_{k=0}^{\infty} f\left(t, s, \frac{k}{k+n}\right) m_{n,k}\left(x(s)\right)\right] ds \\
(M_n F)1(t) &= F1(t) = F(1),
\end{aligned}$$

where

$$m_{n,k}\left(x(s)\right) = \binom{n+k-1}{k} \left(x(s)\right)^k \left(1 - x(s)\right)^n,$$

n is a non-negative integer and $0 \leq x(s) < 1$, and obtained some positive results about the convergence problem.

Very recently in [25] and [26], the author defined and investigated the Urysohn type nonlinear Bernstein operators, having the form

$$(NB_nF)x\left(t\right)=\int_0^1\left[\sum_{k=0}^{n}P_{k,n}\left(x(s),f\left(t,s,\frac{k}{n}\right)\right)\right]ds\;\;,\;\;0\leq x(s)\leq 1\;\;,\;\;\;n\in\mathbb{N}.$$

The central issue of this paper is to extend the theory of interpolation to functionals and operators by introducing the Urysohn type nonlinear counterpart of the two dimensional Bernstein operators. Afterwards, we investigate the convergence problem for these nonlinear operators.

Due to this importance, in this paper we will deal with integral operators of the two dimensional Urysohn type:

$$U\left(x(t),y(t)\right)=\int_a^b\int_a^b k(t,s,z,x(s),y(z))dsdz,\qquad t\in\left[a,b\right],$$

where k is a known function and x and y are the unknown functions to be determined.

Let us consider a sequence $NBF=(NB_nF)$ of operators, which we call it Urysohn type nonlinear counterpart of the two dimensional Bernstein operators, having the form:

$$\begin{aligned}\left(NB_nF\right)\left(x\left(t\right),y(t)\right) &= \int_0^1\int_0^1\left[\sum_{k=0}^{n}\sum_{i=0}^{n}P_{k,i,n}\left(x(s),y(z),f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right)\right)\right]dsdz,\\ 0 &\leq x(s),y(z)\leq 1,\quad n\in\mathbb{N},\end{aligned}$$

acting on bounded functions f on $[0,1]^5=[0,1]*[0,1]*[0,1]*[0,1]*[0,1]$, where $P_{k,i,n}$ satisfy some suitable assumptions. In particular, we will put $Dom\ NBF=\bigcap_{n\in\mathbb{N}} Dom\ NB_nF$, where $Dom\ NB_nF$ is the set of all functions $f:[0,1]^5\to\mathbb{R}$ for which the operator is well defined.

2. Preliminaries and auxiliary results

This section is devoted to collecting some definitions and results which will be needed further on.

Here we consider the following type two dimensional Urysohn integral operator,

$$F\left(x(t),y(t)\right)=\int_0^1\int_0^1 f(t,s,z,x(s),y(z))dsdz,\qquad t\in[0,1] \tag{2.3}$$

with unknown kernel f: If such a representation exists, then the kernel function $f(t,s,z,x(.),y(.))$ is called the two dimensional Green's function, which is strongly related to the functions x and y.

Note that in the univariate case, the solution of the following differential equation

$$DG(x,y)=\delta(x-y),$$

represents a Green function $G(x,y)$, here D is a differential operator, δ is the Dirac Delta function and satisfying a boundary condition. Note that

$$\delta(x)=\frac{dH(x)}{dx},$$

is true, where

$$H(x) = \begin{cases} 1 & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$$

is the Heaviside function.

In view of the above relations, we assume that the two dimensional continuous interpolation conditions hold:

$$F\left(x_i(t), y_j(t)\right) = \int_0^1 \int_0^1 f(t, s, z, x_i(s), y_j(z))dsdz, \qquad t \in [0,1] \tag{2.4}$$

where

$$\begin{aligned} x_i(s) &= \frac{i}{n}H(s-\xi); \xi \in [0;1], \\ y_j(z) &= \frac{j}{n}H(z-\varsigma); \varsigma \in [0;1] \end{aligned} \tag{2.5}$$

and $i, j = 0, 1, 2, ...n$.

Taking into account (2.4) and (2.5), by a straightforward calculation the stated identities follow.

$$\begin{aligned} F\left(\frac{i}{n}H(s-\xi), \frac{j}{n}H(z-\varsigma)\right) &= \int_0^1 \int_0^1 f(t, s, z, \frac{i}{n}H(s-\xi), \frac{j}{n}H(z-\varsigma))dsdz \\ &= \int_\varsigma^1 \int_\xi^1 f(t, s, z, \frac{i}{n}, \frac{j}{n})dsdz + \int_0^\varsigma \int_\xi^1 f(t, s, z, \frac{i}{n}, 0)dsdz \\ &+ \int_0^\varsigma \int_0^\xi f(t, s, z, 0, 0)dsdz + \int_\varsigma^1 \int_0^\xi f(t, s, z, 0, \frac{j}{n})dsdz \end{aligned} \tag{2.6}$$

and hence

$$\begin{aligned} \frac{\partial F\left(\frac{i}{n}H(s-\xi), \frac{j}{n}H(z-\varsigma)\right)}{\partial \varsigma} &= -\int_\xi^1 f(t, s, \varsigma, \frac{i}{n}, \frac{j}{n})ds + \int_\xi^1 f(t, s, \varsigma, \frac{i}{n}, 0)ds \\ &+ \int_0^\xi f(t, s, \varsigma, 0, 0)ds - \int_0^\xi f(t, s, \varsigma, 0, \frac{j}{n})ds, \\ \frac{\partial^2 F\left(\frac{i}{n}H(s-\xi), \frac{j}{n}H(z-\varsigma)\right)}{\partial \xi \partial \varsigma} &= f(t, \xi, \varsigma, \frac{i}{n}, \frac{j}{n}) - f(t, \xi, \varsigma, \frac{i}{n}, 0) \\ &+ f(t, \xi, \varsigma, 0, 0) - f(t, \xi, \varsigma, 0, \frac{j}{n}). \end{aligned}$$

Say

$$F_1\left(t, \xi, \varsigma, \frac{i}{n}, \frac{j}{n}\right) := \frac{\partial^2 F\left(\frac{i}{n}H(s-\xi), \frac{j}{n}H(z-\varsigma)\right)}{\partial \xi \partial \varsigma}. \tag{2.7}$$

According to the above definition together with (2.6) and (2.7), it is possible to construct an approximation operator in order to generalize and extend the theory of interpolation of functions to operators.

In view of (1.2) and (2.4), we introduce the following Urysohn type nonlinear Bernstein operators;

$$(2.8)\qquad (NB_nF)\left(x\left(t\right),y(t)\right)=\int_0^1\int_0^1\left[\sum_{k=0}^{n}\sum_{i=0}^{n}P_{k,i,n}\left(x(s),y(z),f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right)\right)\right]dsdz$$

where n is a non-negative integer, $P_{k,i,n}$ satisfy some suitable assumptions.and $0\leq x(s),y(z)\leq 1$.

Now, we assemble the main definitions and notations which will be used throughout the paper.

Let X be the set of all bounded Lebesgue measurable functions $f:[0,1]^5\rightarrow\mathbb{R}_0^+=[0,\infty)$.

Let Ψ be the class of all functions $\psi:\mathbb{R}_0^+\rightarrow\mathbb{R}_0^+$ such that the function ψ is continuous and concave with $\psi(0)=0$, $\psi(u)>0$ for $u>0$.

We now introduce a sequence of functions. Let $\{P_{k,i,n}\}_{n\in\mathbb{N}}$ be a sequence of functions $P_{k,i,n}:[0,1]\,\mathrm{x}\,[0,1]\,\mathrm{x}\,\mathbb{R}\rightarrow\mathbb{R}$ defined by

$$(2.9)\qquad P_{k,i,n}\left(t,l,u\right)=p_{k,n}(t)p_{i,n}(l)H_n(u)$$

for every $t,l\in[0,1],u\in\mathbb{R}$, where $H_n:\mathbb{R}\rightarrow\mathbb{R}$ is such that $H_n(0)=0$ and $p_{k,n}(\bullet)$ is the Bernstein basis. For simplicity we will write

$$P_{k,i,n}\left(t,l\right):=p_{k,n}(t)p_{i,n}(l).$$

In what follows, throughout the paper, we assume that $\mu:\mathbb{N}\rightarrow\mathbb{R}^+$ is an increasing and continuous function such that $\lim_{n\rightarrow\infty}\mu(n)=\infty$.

First of all we assume that the following conditions hold:

a) $H_n:\mathbb{R}\rightarrow\mathbb{R}$ is such that

$$\left|H_n(u)-H_n(v)\right|\leq\psi\left(\left|u-v\right|\right),$$

holds for every $u,v\in\mathbb{R}$, for every $n\in\mathbb{N}$. That is, H_n satisfies a $(L-\Psi)$ Lipschitz condition.

b) Denoting by $r_n(u):=H_n(u)-u$, $u\in\mathbb{R}$ and $n\in\mathbb{N}$, such that for n sufficiently large

$$\sup_u\left|r_n(u)\right|=\sup_u\left|H_n(u)-u\right|\leq\frac{1}{\mu(n)},$$

holds.

Following our announced aim, in this part we recall results regarding the univariate and linear case of the celerated Bernstein polynomials.

Lemma 2.1. *For $(B_nt^s)(x,y)$, $s=0,1,2$, one has*

$$\begin{aligned}(B_n1)(x,y)&=1\\(B_nt)(x,y)&=x\\(B_nt^2)(x,y)&=x^2+\frac{x(1-x)}{n}.\end{aligned}$$

For proof of this Lemma see [27].

By direct calculation, we find the following equalities:

$$(B_n\,(t-x)^2)(x,y)=\frac{x(1-x)}{n}\quad,\quad (B_n\,(t-x))(x,y)=0\,.$$

Lemma 2.2. *For the central moments of order $m\in N_0$*

$$T_{n,m}(x):=\sum_{k=0}^{n}(k-nx)^m\,p_{k,n}\,(x)\,,$$

for each $m=0,1,...$ there is a constant A_m such that

$$0\leq T_{n,2m}(x)\leq A_m n^m.$$

The presented well-known inequality can be found in [17].

3. Convergence property

We now introduce some notations and structural hypotheses, which will be fundamental in proving our convergence theorems.

Let $C[0,1]$ the Banach space of continuous functions $u:[0,1]\to R$ endowed with the norm

$$\|u\|=\sup\{|u(x)|:x\in[0,1]\}.$$

Definition 3.1. *Let $f\in C\left([a,b]^5\right)$ and $\delta>0$ be given. Then the complete modulus of continuity is given by;*

$$\omega\left(\delta\right)=\sup_{\sqrt{(u_1-u_2)^2+(v_1-v_2)^2}\leq\delta}\left|f(t,s,z,u_1,v_1)-f(t,s,z,u_2,v_2)\right|. \tag{3.10}$$

Further on, the first and second partial modulus of continuity are given by

$$\omega_1(\delta_1,0)=\sup_{|u_1-u_2|\leq\delta_1}\left|f(t,s,z,u_1,v_1)-f(t,s,z,u_2,v_1)\right|,$$

$$\omega_2(0,\eta)=\sup_{|v_1-v_2|\leq\eta}\left|f(t,s,z,u_1,v_1)-f(t,s,z,u_1,v_2)\right|.$$

Recall that $\omega\left(f;\delta\right)$ has the following properties;

(i) Let $\lambda\in\mathbb{R}^+$, then $\omega\left(f;\lambda\delta\right)\leq(\lambda+1)\,\omega\left(f;\delta\right)$,

(ii) $\lim\limits_{\delta\to0^+}\omega\left(f;\delta\right)=0$,

(iii) $|f(t,s,z,u_1,v_1)-f(t,s,z,u_2,v_2)|\leq\omega\left(\delta\right)\left(1+\frac{\sqrt{(u_1-u_2)^2+(v_1-v_2)^2}}{\delta}\right)$,

Note that the same properties also hold for partial moduli of continuity.

We are now ready to establish one of the main results of this study:

Theorem 3.1. *Let F be the Urysohn integral operator with $0\leq x(s),y(z)\leq1$. Then (NB_nF) converges to F uniformly in $x,y\in C[0,1]$. That is*

$$\lim_{n\to\infty}\left\|\left(NB_nF\right)\left(x\left(t\right),y(t)\right)-F\left(x\left(t\right),y(t)\right)\right\|_{C([0,1]^2)}=0.$$

Proof. In view of the definition of the operator (2.8), by considering (2.4), (2.9), (2.6) and (2.7), we have

$$\begin{aligned}
&\left|(NB_nF)(x(t),y(t)) - F(x(t),y(t))\right| \\
= &\left|\int_0^1\int_0^1 \left[\sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}\left(x(s),y(z),f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right)\right)\right] dsdz - F(x(t),y(t))\right| \\
\leq &\int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\left|H_n\left(f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right)\right) - H_n\left(f\left(t,s,z,x(s),y(z)\right)\right)\right| dsdz \\
+ &\int_0^1\int_0^1 \sum_{k=0}^{n} P_{k,i,n}(x,y)\left|H_n\left(f\left(t,s,z,x(s),y(z)\right)\right) - f(t,s,z,x(s),y(z))\right| dsdz \\
:= &I_1 + I_2.
\end{aligned}$$

By assumption $b)$ I_2 tends to zero as $n \to \infty$. In fact

$$\begin{aligned}
I_2 &= \int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\left|H_n\left(f\left(t,s,z,x(s),y(z)\right)\right) - f(t,s,z,x(s),y(z))\right| dsdz \\
&\leq \int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\frac{1}{\mu(n)} dsdz \\
&= \frac{1}{\mu(n)},
\end{aligned}$$

which tends to zero as $n \to \infty$. Now, it is sufficient to evaluate the term I_1. Using the definition of the function $F_1(t,s,z,x(s),y(z))$, by concavity of the function ψ, and using Jensen inequality, we obtain

$$\begin{aligned}
I_1 &\leq \int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\,\psi\left(\left|f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right) - f\left(t,s,z,x(s),y(z)\right)\right|\right) dsdz \\
&\leq \psi\left(\int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\left|f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right) - f\left(t,s,z,x(s),y(z)\right)\right| dsdz\right)
\end{aligned}$$

$$\leq \psi \left\{ \int_0^1 \int_0^1 \sum_{k=0}^{n} \sum_{i=0}^{n} P_{k,i,n}(x,y) \left| F_1(t,s,z,x(s),y(z)) - F_1\left(t,s,z,\frac{k}{n},\frac{i}{n}\right) \right| dsdz \right.$$

$$+ \int_0^1 \int_0^1 \left| f(t,s,z,x(s),0) - \sum_{k=0}^{n} p_{k,n}(x(s)) f\left(t,s,z,\frac{k}{n},0\right) \right| dsdz$$

$$\left. + \int_0^1 \int_0^1 \left| f(t,s,z,0,y(z)) - \sum_{i=0}^{n} p_{i,n}(y(z)) f\left(t,s,z,0,\frac{i}{n}\right) \right| dsdz \right\}$$

$$:\leq I_{1,1} + I_{1,2} + I_{1,3}.$$

Let us divide the first term into four parts as;

$$I_{1,1} = \psi \left(\int_0^1 \int_0^1 \sum_{k=0}^{n} \sum_{i=0}^{n} P_{k,i,n}(x,y) \left| F_1(t,s,z,x(s),y(z)) - F_1\left(t,s,z,\frac{k}{n},\frac{i}{n}\right) \right| dsdz \right)$$

$$:\leq I_{1,1,1} + I_{1,1,2} + I_{1,1,3} + I_{1,1,4},$$

where

$$I_{1,1,1} = \psi \left(\int_0^1 \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta_1} \sum_{\left|\frac{i}{n}-y(z)\right|<\delta_2} P_{k,i,n}(x,y) \left| F_1(t,s,z,x(s),y(z)) - F_1\left(t,s,z,\frac{k}{n},\frac{i}{n}\right) \right| dsdz \right),$$

$$I_{1,1,2} = \psi \left(\int_0^1 \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta_1} \sum_{\left|\frac{i}{n}-y(z)\right|\geq\delta_2} P_{k,i,n}(x,y) \left| F_1(t,s,z,x(s),y(z)) - F_1\left(t,s,z,\frac{k}{n},\frac{i}{n}\right) \right| dsdz \right),$$

$$I_{1,1,3} = \psi \left(\int_0^1 \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta_1} \sum_{\left|\frac{i}{n}-y(z)\right|<\delta_2} P_{k,i,n}(x,y) \left| F_1(t,s,z,x(s),y(z)) - F_1\left(t,s,z,\frac{k}{n},\frac{i}{n}\right) \right| dsdz \right),$$

and

$$I_{1,1,4} = \psi \left(\int_0^1 \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta_1} \sum_{\left|\frac{i}{n}-y(z)\right|\geq\delta_2} P_{k,i,n}(x,y) \left| F_1(t,s,z,x(s),y(z)) - F_1\left(t,s,z,\frac{k}{n},\frac{i}{n}\right) \right| dsdz \right).$$

Since $x, y \in C[0,1]$, then there exist $\delta_1, \delta_2 > 0$ such that

$$\left| F_1\left(t, s, z, x(s), y(z)\right) - F_1\left(t, s, z, \frac{k}{n}, \frac{i}{n}\right) \right| < \epsilon$$

holds true when $\left|\frac{k}{n} - x(s)\right| < \delta_1$ and $\left|\frac{i}{n} - y(z)\right| < \delta_2$. So one can easily obtain

$$I_{1,1,1} < \psi\left(\epsilon\right).$$

As to the other terms

$$\left| F_1\left(t, s, z, x(s), y(z)\right) - F_1\left(t, s, z, \frac{k}{n}, \frac{i}{n}\right) \right| \leq 2M$$

holds true for some $M > 0$, when $\left|\frac{k}{n} - x(s)\right| \geq \delta_1$ or $\left|\frac{i}{n} - y(z)\right| \geq \delta_2$.

In view of Lemma 2, we obtain

$$\begin{aligned}
I_{1,1,2} &= \psi\left(\int_0^1\int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta_1} \sum_{\left|\frac{i}{n}-y(z)\right|\geq\delta_2} P_{k,i,n}\left(x,y\right)\left| F_1\left(t, s, z, x(s), y(z)\right) - F_1\left(t, s, z, \frac{k}{n}, \frac{i}{n}\right) \right| dsdz\right) \\
&\leq \psi\left(2M\int_0^1\int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta_1} \sum_{\left|\frac{i}{n}-y(z)\right|\geq\delta_2} \left(\frac{i-ny(z)}{\delta_2}\right)^2 P_{k,i,n}\left(x,y\right) dsdz\right) \\
&\leq \psi\left(2M\int_0^1\int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta_1} \sum_{\left|\frac{i}{n}-y(z)\right|\geq\delta_2} \left(\frac{i-ny(z)}{\delta_2}\right)^2 P_{k,i,n}\left(x,y\right) dsdz\right) \\
&\leq \psi\left(\frac{2M}{\delta_2^2}\frac{A_1}{n}\right).
\end{aligned}$$

Similarly one has

$$I_{1,1,3} \leq \psi\left(\frac{2M}{\delta^2}\frac{A_1}{n}\right),$$

and

$$I_{1,1,4} \leq \psi\left(\frac{2M}{\delta_1^2\delta_2^2}\frac{A_1^2}{n^2}\right).$$

Collecting these estimates we have

$$\left|\left(NB_nF\right)\left(x\left(t\right), y(t)\right) - F\left(x\left(t\right), y(t)\right)\right| \leq \psi\left(\epsilon\right) + \psi\left(\frac{2MA_1}{n\delta_1^2}\right) + \psi\left(\frac{2MA_1}{n\delta_2^2}\right) + \psi\left(\frac{2M}{\delta_1^2\delta_2^2}\frac{A_1^2}{n^2}\right) + \frac{1}{\mu\left(n\right)}.$$

That is

$$\lim_{n\to\infty}\left\|\left(NB_nF\right)\left(x\left(t\right), y(t)\right) - F\left(x\left(t\right), y(t)\right)\right\|_{C([0,1]^2)} = 0.$$

This completes the proof. □

Theorem 3.2. *Let F be the Urysohn integral operator with $x, y \in C[0,1]$, and $0 \leq x(s), y(z) \leq 1$. Then*

$$\left|\left(NB_nF\right)\left(x\left(t\right), y(t)\right) - F\left(x\left(t\right), y(t)\right)\right| \leq 2\psi\left(\omega\left(f;\delta\right)\right) + \frac{1}{\mu\left(n\right)}$$

holds true, where $\delta = \sqrt{\frac{2A_1}{n}}$.

Proof. Clearly one has

$$
\begin{aligned}
& \left|\left(NB_nF\right)\left(x\left(t\right),y(t)\right)-F\left(x\left(t\right),y(t)\right)\right| \\
\leq\ & \int_0^1\int_0^1\sum_{k=0}^{n}\sum_{i=0}^{n}P_{k,i,n}\left(x,y\right)\left|H_n\left(f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right)\right)-H_n\left(f\left(t,s,z,x(s),y(z)\right)\right)\right|dsdz \\
& +\frac{1}{\mu\left(n\right)}
\end{aligned}
$$

$$
(3.11)\ :\quad = I_{n,1}\left(x\right)+\frac{1}{\mu\left(n\right)},
$$

say. Since $x,y\in C[0,1]$ we can re-write (3.11) as follows

$$
\begin{aligned}
I_{n,1}\left(x\right) &\leq \int_0^1\int_0^1\sum_{k=0}^{n}\sum_{i=0}^{n}P_{k,i,n}\left(x,y\right)\psi\left(\left|f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right)-f\left(t,s,z,x(s),y(z)\right)\right|\right)dsdz \\
&\leq \int_0^1\int_0^1\sum_{k=0}^{n}\sum_{i=0}^{n}P_{k,i,n}\left(x,y\right)\psi\left(\omega\left(f;\delta\right)\right)dsdz \\
&\leq \psi\left(\int_0^1\int_0^1\sum_{k=0}^{n}\sum_{i=0}^{n}P_{k,i,n}\left(x,y\right)\omega\left(f;\delta\right)dsdz\right) \\
&\leq \psi\left(\int_0^1\int_0^1\sum_{k=0}^{n}\sum_{i=0}^{n}P_{k,i,n}\left(x,y\right)\left(\frac{\sqrt{\left(\frac{k}{n}-x(s)\right)^2+\left(\frac{i}{n}-y(z)\right)^2}}{\delta}+1\right)\omega\left(f;\delta\right)dsdz\right) \\
&= \psi\left(\omega\left(f;\delta\right)\int_0^1\int_0^1\sum_{k=0}^{n}\sum_{i=0}^{n}P_{k,i,n}\left(x,y\right)\frac{\sqrt{\left(\frac{k}{n}-x(s)\right)^2+\left(\frac{i}{n}-y(z)\right)^2}}{\delta}dsdz\right) \\
&+ \psi\left(\omega\left(f;\delta\right)\int_0^1\int_0^1\sum_{k=0}^{n}\sum_{i=0}^{n}P_{k,i,n}\left(x,y\right)dsdz\right) \\
&\leq \psi\left(\frac{\omega\left(f;\delta\right)}{\delta}\int_0^1\int_0^1\left(\sum_{k=0}^{n}\sum_{i=0}^{n}P_{k,i,n}\left(x,y\right)\left[\left(\frac{k}{n}-x(s)\right)^2+\left(\frac{i}{n}-y(z)\right)^2\right]\right)^{1/2}dsdz\right) \\
&+ \psi\left(\omega\left(f;\delta\right)\right) \\
&\leq \psi\left(\frac{\omega\left(f;\delta\right)}{\delta}\left[\frac{2A_1}{n}\right]^{1/2}\right)+\psi\left(\omega\left(f;\delta\right)\right).
\end{aligned}
$$

Taking into account that $\omega\left(f;\delta\right)$ is the modulus of continuity defined as (3.10). If we choose

$$
\delta=\sqrt{\frac{2A_1}{n}},
$$

then one can obtain the desired estimate, namely,

$$
\left|\left(NB_nF\right)\left(x\left(t\right),y(t)\right)-F\left(x\left(t\right),y(t)\right)\right|\leq 2\psi\left(\omega\left(f;\delta\right)\right)+\frac{1}{\mu\left(n\right)}.
$$

Thus the proof is now complete. □

Theorem 3.3. *Let F be the Urysohn integral operator with $x, y \in C[0,1]$, and $0 \leq x(s), y(z) \leq 1$. Then*

$$\begin{aligned} \left|(NB_nF)(x(t), y(t)) - F(x(t), y(t))\right| &\leq 2\left[\psi\left(\omega_1\left(f;\left[\frac{A_1}{n}\right]^{1/2}\right)\right) + \psi\left(\omega_2\left(f;\left[\frac{A_1}{n}\right]^{1/2}\right)\right)\right] \\ &+ \frac{1}{\mu(n)} \end{aligned}$$

holds true.

Proof. Clearly one has

$$\begin{aligned} &\left|(NB_nF)(x(t), y(t)) - F(x(t), y(t))\right| \\ \leq\ & \int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\left|H_n\left(f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right)\right) - H_n\left(f\left(t,s,z,x(s),y(z)\right)\right)\right| dsdz + \frac{1}{\mu(n)} \\ =\ & \int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\left|\begin{array}{c} H_n\left(f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right)\right) - H_n\left(f\left(t,s,z,x(s),\frac{i}{n}\right)\right) \\ +H_n\left(f\left(t,s,z,x(s),\frac{i}{n}\right)\right) - H_n\left(f\left(t,s,z,x(s),y(z)\right)\right) \end{array}\right| dsdz + \frac{1}{\mu(n)} \\ \leq\ & \int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\left|H_n\left(f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right)\right) - H_n\left(f\left(t,s,z,x(s),\frac{i}{n}\right)\right)\right| dsdz \\ +\ & \int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\left|H_n\left(f\left(t,s,z,x(s),\frac{i}{n}\right)\right) - H_n\left(f\left(t,s,z,x(s),y(z)\right)\right)\right| dsdz \\ +\ & \frac{1}{\mu(n)} \\ :=\ & I_{n,1}(x) + I_{n,2}(x) + \frac{1}{\mu(n)}, \end{aligned}$$

say. Since $x, y \in C[0,1]$ we can re-write (3.11) as follows: By concavity of the function ψ, and using Jensen inequality, we obtain

$$\begin{aligned} I_{n,1}(x) &= \int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\left|H_n\left(f\left(t,s,z,\frac{k}{n},\frac{i}{n}\right)\right) - H_n\left(f\left(t,s,z,x(s),\frac{i}{n}\right)\right)\right| dsdz \\ &\leq \int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\,\psi\left(\omega_1\left(f;\left|\frac{k}{n} - x(s)\right|\right)\right) dsdz \\ &\leq \psi\left(\int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\,\omega_1\left(f;\left|\frac{k}{n} - x(s)\right|\right) dsdz\right) \end{aligned}$$

Since ψ is non decreasing, then one has

$$\begin{aligned} I_{n,1}(x) &\leq \psi\left(\int_0^1\int_0^1 \sum_{k=0}^{n}\sum_{i=0}^{n} P_{k,i,n}(x,y)\left(\frac{\sqrt{\left(\frac{k}{n}-x(s)\right)^2}}{\delta_1}+1\right)\omega_1(f;\delta_1)\,dsdz\right) \\ &\leq \psi\left(\frac{\omega_1(f;\delta_1)}{\delta_1}\left[\frac{A_1}{n}\right]^{1/2}\right)+\psi\left(\omega_1(f;\delta_1)\right). \end{aligned}$$

Similarly

$$I_{n,1}(x) \leq \psi\left(\frac{\omega_2(f;\eta)}{\eta}\left[\frac{A_1}{n}\right]^{1/2}\right)+\psi\left(\omega_2(f;\eta)\right).$$

If we choose $\delta=\eta=\left[\frac{A_1}{n}\right]^{1/2}$, so we get the desired estimate. □

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Bolu Abant Izzet Baysal University
Faculty of Science and Arts
Department of Mathematics
14280, Gölköy-Bolu, Turkey
E-mail address: karsli_h@ibu.edu.tr

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Approximation Properties of Kantorovich Type Modifications of (p, q)-Meyer-König-Zeller Operators

HONEY SHARMA, RAMAPATI MAURYA*, AND CHEENA GUPTA

ABSTRACT. In this paper, we introduce Kantorovich type modification of (p, q)-Meyer-König-Zeller operators. We estimate rate of convergence of proposed operators using modulus of continuity and Lipschitz class functions. Further, we obtain the statistical convergence and local approximation results for these operators. In the last section, we estimate the rate of convergence of (p, q)-Meyer-König-Zeller Kantorovich operators by means of Matlab programming.

Keywords: (p, q)-Calculus, (p, q)-Meyer-König-Zeller operators, Modulus of continuity, Statistical Convergence, Peetre's K-functional.

2010 Mathematics Subject Classification: 41A25, 41A35.

1. INTRODUCTION

In 1960, Meyer-König and Zeller [22] defined the operators known as Meyer-König-Zeller (MKZ) operators, as follows:

$$\begin{aligned} M_n(f;x) &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\left(\begin{array}{c} n+k \\ k \end{array}\right) x^k(1-x)^{n+1}, \; if\ x \in [0,1), \\ M_n(f;1) &= f(1), \; if\ x=1, \; n \in \mathbb{N}. \end{aligned}$$

Further, Cheney and Sharma [3] modified these operators and introduced a new form of the Meyer-König-Zeller operators, as follows:

$$\begin{aligned} M_n(f;x) &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right)\left(\begin{array}{c} n+k \\ k \end{array}\right) x^k(1-x)^{n+1}, \; if\ x \in [0,1), \\ M_n(f;1) &= f(1), \; if\ x=1, \; n \in \mathbb{N}. \end{aligned}$$

In 2000, T. Trif [21] introduced the q-Meyer-König-Zeller operators for $f \in C[0,1]$, as follows:

$$\begin{aligned} M_{n,q}(f;x) &= \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n+k]_q}\right)\left[\begin{array}{c} n+k \\ k \end{array}\right]_q x^k(1-x)_q^{n+1}, \; if\ x \in [0,1), \\ M_{n,q}(f;1) &= f(1), \; if\ x=1, \; n \in \mathbb{N}. \end{aligned}$$

Further, with a slight modification in these operators, Dǒgru and Duman [5] defined the q-Meyer-König-Zeller operators for $f \in C[0,a]$, $a \in (0,1)$, as follows:

$$M_{n,q}(f;x) = \prod_{s=0}^{n}(1-q^s x)\sum_{k=0}^{\infty} f\left(\frac{q^n[k]_q}{[n+k]_q}\right)\left[\begin{array}{c} n+k \\ k \end{array}\right]_q x^k, \; q \in (0,1), \; n \in \mathbb{N}.$$

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*Corresponding author: R. Maurya; ramapatimaurya@gmail.com
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Recently, Gupta and Sharma [9] introduced the Kantorovich type modification of q-Meyer-König-Zeller operators and studied some of their approximation properties. For detail studies of Meyer-König and Zeller operators, one may refer to [6, 7, 10, 16, 17, 20].
In the recent years, (p, q)-analogue of various linear positive operators were introduced and studied by many researchers [1, 8, 12, 15, 19].
In 2016, Mursaleen et al. [11] introduced the (p, q)-Meyer-König-Zeller operators as follows:

$$M_{n,p,q}(f;x) = \frac{1}{p^{n(n+1)/2}} \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k \\ k \end{array} \right]_{p,q} x^k p^{-kn} \prod_{s=0}^{n} (p^s - q^s x)\, f\left(\frac{p^n [k]_{p,q}}{[n+k]_{p,q}} \right).$$

Motivated by the above mentioned studied on Meyer-König and Zeller operators, in this paper, we introduced Kantorovich type modification of (p, q)-Meyer-König-Zeller operators and discus their approximation properties.
We begin by recalling certain notations of (p, q)-calculus (for more details, see [2, 18]).
Let $0 < q < p \leq 1$. The (p, q)-integer $[n]_{p,q}$ and (p, q)-factorial $[n]_{p,q}!$ are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2....$$

$$[n]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q}.........[n]_{p,q}, & \text{n} \geq 1 \\ 1, & \text{n=0} \end{cases}.$$

For integers $0 \leq k \leq n$, (p, q)-binomial coefficient is defined as

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The (p, q)-binomials expansion is expressed as:

$$(x + y)^n_{p,q} = \prod_{j=0}^{n-1} (p^j x + q^j y).$$

For a function $f : \mathbb{R} \to \mathbb{R}$, the (p, q)-analogue of derivative is defined as

$$D_{p,q}(f(x)) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0$$

and

$$D_{p,q}(f(0)) = \lim_{x \to 0} D_{p,q}(f(x)),$$

provided the limit exists.

Let $f : C[0, a] \to R$, the (p, q)-integration of a function f is defined as

$$\int_0^a f(t) d_{p,q}t = (q - p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f(\frac{p^k}{q^{k+1}} a), \quad when \quad |\frac{p}{q}| < 1,$$

and

$$\int_0^a f(t) d_{p,q}t = (p - q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f(\frac{q^k}{p^{k+1}} a), \quad when \quad |\frac{p}{q}| > 1.$$

2. Construction of operators

In this section, we introduce the Kantorovich type modification of (p,q)-Meyer-König-Zeller operators. We estimate moments and obtain the uniform convergence of operators.
For $0<q<p\leq 1$ and $f\in C[0,1]$, Kantorovich variant of (p,q)-Meyer-König-Zeller operators are defined as follows:

$$\tilde{M}_{n,p,q}(f;x)=\frac{[n+1]_{p,q}}{p^{n(n-1)/2}}\sum_{k=0}^{\infty}m_{n,k}^{p,q}(x)\int_{\frac{[k]_{p,q}}{[n+k]_{p,q}}}^{\frac{[k+1]_{p,q}}{[n+k+1]_{p,q}}}f(p^{n-1}t)d_{p,q}t,$$

where

$$m_{n,k}^{p,q}(x)=\left[\begin{array}{c} n+k+1 \\ k \end{array}\right]_{p,q}p^{-kn}q^{-k}x^kP_{n-1}(x),$$

here, $P_n(x)=\prod_{s=0}^{n}(p^s-q^sx)$.

Remark 2.1. *It can be easily verified that for $p\to 1$, above operators reduces to q-Meyer-König-Zeller-Kantorovich operators defined in* [9].

By using mathematical induction on n, one can verify the following identity:

$$\frac{P_n(x)}{p^{n(n+1)/2}}\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k \\ k \end{array}\right]_{p,q}x^kp^{-kn}=1. \tag{2.1}$$

Further, by using simple computation, we can obtain the following identity:

$$\frac{[k+1]_{p,q}}{[n+k+1]_{p,q}}-\frac{[k]_{p,q}}{[n+k]_{p,q}}=\frac{(pq)^k[n]_{p,q}}{[n+k]_{p,q}[n+k+1]_{p,q}}. \tag{2.2}$$

Lemma 2.1. *For $r=0,1,2,...$ and $n>r$, we have*

$$P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q}\frac{x^kp^{-k(n-r-1)}}{[n+k-1]_{p,q}^{\underline{r}}}=\frac{\prod_{j=1}^{r}(p^{n-j}-q^{n-j}x)}{[n-1]_{p,q}^{\underline{r}}}p^{(n-r)(n-r-1)/2},$$

where $[n-1]_{p,q}^{\underline{r}}=[n-1]_{p,q}[n-2]_{p,q}....[n-r]_{p,q}$.

Proof. By using identity (2.1), lemma can be proved as similar to [9, Lemma 2]. □

Lemma 2.2. *For $r\geq 0$, we have the following inequality*

$$\frac{1}{[n+k+r]_{p,q}}\leq\frac{1}{q^{r+1}[n+k-1]_{p,q}}.$$

Lemma 2.3. *Let $I(f(t))=\int_{\frac{[k]_{p,q}}{[n+k]_{p,q}}}^{\frac{[k+1]_{p,q}}{[n+k+1]_{p,q}}}f(p^{n-1}t)d_{p,q}t$ and $e_i=t^i$ for $i=0,1,2$, we have following identities:*

$$I(e_0)=\frac{(pq)^k[n]_{p,q}}{[n+k]_{p,q}[n+k+1]_{p,q}},$$

$$I(e_1)=\frac{p^{n-1}}{[2]_{p,q}}\frac{(pq)^k[n]_{p,q}}{[n+k]_{p,q}[n+k+1]_{p,q}}\left([k]_{p,q}\left(\frac{q}{[n+k+1]_{p,q}}+\frac{1}{[n+k]_{p,q}}\right)+\frac{p^k}{[n+k+1]_{p,q}}\right),$$

$$I(e_2)=\frac{p^{2(n-1)}}{[3]_{p,q}}\frac{(pq)^k[n]_{p,q}}{[n+k]_{p,q}[n+k+1]_{p,q}}\left([k]_{p,q}^2S_2(n,k)+[k]_{p,q}S_1(n,k)+S_0(n,k)\right).$$

Here,

$$\begin{aligned} S_2(n,k) &= \frac{q^2}{[n+k+1]_{p,q}^2} + \frac{q}{[n+k+1]_{p,q}[n+k]_{p,q}} + \frac{1}{[n+k]_{p,q}^2}, \\ S_1(n,k) &= \frac{2p^k q}{[n+k+1]_{p,q}^2} + \frac{p^k}{[n+k+1]_{p,q}[n+k]_{p,q}}, \\ S_0(n,k) &= \frac{p^{2k}}{[n+k+1]_{p,q}^2}. \end{aligned}$$

Proof. By using definition of (p, q)-integral and identity (2.2), we have

$$\begin{aligned} I(e_2) &= \int_{\frac{[k]_{p,q}}{[n+k]_{p,q}}}^{\frac{[k+1]_{p,q}}{[n+k+1]_{p,q}}} (p^{n-1}t)^2 \, d_{p,q}t \\ &= p^{2(n-1)}(p-q)\left(\left(\frac{[k+1]_{p,q}}{[n+k+1]_{p,q}}\right)^3 - \left(\frac{[k]_{p,q}}{[n+k]_{p,q}}\right)^3\right)\sum_{j=0}^{\infty}\left(\frac{q^j}{p^{j+1}}\right)^3 \\ &= \frac{p^{2(n-1)}}{[3]_{p,q}} \frac{(pq)^k[n]_{p,q}}{[n+k]_{p,q}[n+k+1]_{p,q}} \\ &\quad \left(\frac{[k+1]_{p,q}^2}{[n+k+1]_{p,q}^2} + \frac{[k+1]_{p,q}[k]_{p,q}}{[n+k+1]_{p,q}[n+k]_{p,q}} + \frac{[k]_{p,q}^2}{[n+k]_{p,q}^2}\right) \\ &= \frac{p^{2(n-1)}}{[3]_{p,q}} \frac{(pq)^k[n]_{p,q}}{[n+k]_{p,q}[n+k+1]_{p,q}} \\ &\quad \left(\frac{(p^k+q[k]_{p,q})^2}{[n+k+1]_{p,q}^2} + \frac{(p^k+q[k]_{p,q})[k]_{p,q}}{[n+k+1]_{p,q}[n+k]_{p,q}} + \frac{[k]_{p,q}^2}{[n+k]_{p,q}^2}\right) \\ &= \frac{p^{2(n-1)}}{[3]_{p,q}} \frac{(pq)^k[n]_{p,q}}{[n+k]_{p,q}[n+k+1]_{p,q}} \\ &\quad \left(\frac{(p^{2k}+2p^kq[k]_{p,q}+q^2[k]_{p,q}^2)}{[n+k+1]_{p,q}^2} + \frac{(p^k[k]_{p,q}+q[k]_{p,q}^2)}{[n+k+1]_{p,q}[n+k]_{p,q}} + \frac{[k]_{p,q}^2}{[n+k]_{p,q}^2}\right) \\ &= \frac{p^{2(n-1)}}{[3]_{p,q}} \frac{(pq)^k[n]_{p,q}}{[n+k]_{p,q}[n+k+1]_{p,q}}\Big([k]_{p,q}^2 S_2(n,k) + [k]_{p,q}S_1(n,k) + S_0(n,k)\Big). \end{aligned}$$

Similarly, we can get result for e_0 and e_1. □

Lemma 2.4. *For $e_i = t^i$, here $i = 0, 1, 2$, moments estimate of proposed operators are as follows:*

$$\begin{aligned} &\tilde{M}_{n,p,q}(e_0;x) = 1, \\ &\tilde{M}_{n,p,q}(e_1;x) \le \frac{1}{[2]_{p,q}q}\left(2x + \frac{(p^{n-1}-q^{n-1}x)}{q[n-1]_{p,q}}\right), \\ &\tilde{M}_{n,p,q}(e_1;x) \ge \frac{2x}{[2]_{p,q}q^2}\left(1 - \frac{(1+q)}{p^{n-1}}\left(\frac{p^{n-1}-q^{n-1}x}{[n-1]_{p,q}}\right)\right), \\ &\tilde{M}_{n,p,q}(e_2;x) \le \frac{1}{[3]_{p,q}q}\left(3x^2 + \left(\frac{3x}{q} + \frac{3}{q^2} + \frac{p}{q^3}\frac{(p^{n-2}-q^{n-2}x)}{[n-2]_{p,q}}\right)\frac{(p^{n-1}-q^{n-1}x)}{[n-1]_{p,q}}\right). \end{aligned}$$

Proof. By using definition of (p,q)-Meyer-König-Zeller Kantorovich operators, identity (2.1), Lemma 2.2 and Lemma 2.3, moments of sequence of the opeartors can be estimated as follows: For $e_0 = 1$, we have

$$\begin{aligned}
\tilde{M}_{n,p,q}(e_0;x) &= \frac{[n+1]_{p,q}}{p^{n(n-1)/2}} P_{n-1}(x) \sum_{k=0}^{\infty} m_{n,k}^{p,q}(x) \frac{(pq)^k [n]_{p,q}}{[n+k]_{p,q}[n+k+1]_{p,q}} \\
&= \frac{[n+1]_{p,q}}{p^{n(n-1)/2}} P_{n-1}(x) \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k+1 \\ k \end{array} \right]_{p,q} x^k p^{-kn} q^{-k} \\
&\quad \frac{(pq)^k [n]_{p,q}}{[n+k]_{p,q}[n+k+1]_{p,q}} \\
&= \frac{P_{n-1}(x)}{p^{n(n-1)/2}} \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} x^k p^{-k(n-1)} \\
&= 1.
\end{aligned}$$

For $e_1 = t$, upper bound of moment can be obtained as follows:

$$\begin{aligned}
\tilde{M}_{n,p,q}(e_1;x) &\leq \frac{[n+1]_{p,q}}{p^{n(n-1)/2}} P_{n-1}(x) \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k+1 \\ k \end{array} \right]_{p,q} x^k p^{-kn} q^{-k} \\
&\quad \frac{p^{n-1}}{[2]_{p,q}} \frac{(pq)^k [n]_{p,q}}{[n+k]_{p,q}[n+k+1]_{p,q}} \left(\frac{[k]_{p,q}}{q} \frac{2}{[n+k-1]_{p,q}} + \frac{p^k}{q^2 [n+k-1]_{p,q}} \right) \\
&= \frac{p^{n-1}}{[2]_{p,q} q p^{n(n-1)/2}} \left(2P_{n-1}(x) \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} x^k p^{-k(n-1)} \frac{[k]_{p,q}}{[n+k-1]_{p,q}} \right. \\
&\quad + \left. \frac{P_{n-1}(x)}{q} \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} \frac{x^k p^{-k(n-2)}}{[n+k-1]_{p,q}} \right) \\
&= \frac{p^{n-1}}{[2]_{p,q} q p^{n(n-1)/2}} \left(2P_{n-1}(x) \sum_{k=1}^{\infty} \left[\begin{array}{c} n+k-2 \\ k-1 \end{array} \right]_{p,q} x^k p^{-k(n-1)} \right. \\
&\quad + \left. \frac{P_{n-1}(x)}{q} \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} \frac{x^k p^{-k(n-2)}}{[n+k-1]_{p,q}} \right) \\
&= \frac{p^{n-1}}{[2]_{p,q} q p^{n(n-1)/2}} \left(\frac{2x}{p^{(n-1)}} P_{n-1}(x) \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} x^k p^{-k(n-1)} \right. \\
&\quad + \left. \frac{P_{n-1}(x)}{q} \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} \frac{x^k p^{-k(n-2)}}{[n+k-1]_{p,q}} \right) \\
&= \frac{1}{[2]_{p,q} q} \left(2x + \frac{p^{n-1} - q^{n-1} x}{q[n-1]_{p,q}} \right).
\end{aligned}$$

Lower bound of moment for e_1 can be obtained as follows:

$$\begin{aligned}
\tilde{M}_{n,p,q}(e_1;x) &= \frac{p^{n-1}P_{n-1}(x)}{[2]_{p,q}p^{n(n-1)/2}}\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-1)}\left(\frac{[k+1]_{p,q}}{[n+k+1]_{p,q}}+\frac{[k]_{p,q}}{[n+k]_{p,q}}\right) \\
&= \frac{p^{n-1}P_{n-1}(x)}{[2]_{p,q}p^{n(n-1)/2}}\sum_{k=1}^{\infty}\left[\begin{array}{c} n+k-2 \\ k-1 \end{array}\right]_{p,q} x^k p^{-k(n-1)} \\
&\quad \frac{[n+k-1]_{p,q}}{[k]_{p,q}}\left(\frac{[k+1]_{p,q}}{[n+k+1]_{p,q}}+\frac{[k]_{p,q}}{[n+k]_{p,q}}\right) \\
&= \frac{p^{n-1}P_{n-1}(x)}{[2]_{p,q}p^{n(n-1)/2}}\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^{k+1} p^{-(k+1)(n-1)} \\
&\quad \frac{[n+k]_{p,q}}{[k+1]_{p,q}}\left(\frac{[k+2]_{p,q}}{[n+k+2]_{p,q}}+\frac{[k+1]_{p,q}}{[n+k+1]_{p,q}}\right) \\
&\geq \frac{x\,P_{n-1}(x)}{[2]_{p,q}p^{n(n-1)/2}}\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-1)} \\
&\quad [n+k]_{p,q}\left(\frac{1}{[n+k+2]_{p,q}}+\frac{1}{[n+k+1]_{p,q}}\right) \\
&\geq \frac{2x\,P_{n-1}(x)}{[2]_{p,q}p^{n(n-1)/2}}\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-1)}\left(\frac{[n+k]_{p,q}}{[n+k+2]_{p,q}}\right) \\
&= \frac{2x\,P_{n-1}(x)}{[2]_{p,q}p^{n(n-1)/2}}\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-1)} \\
&\quad \left(\frac{[n+k+2]_{p,q}-qp^{n+k}-p^{n+k+1}}{q^2[n+k+2]_{p,q}}\right) \\
&\geq \frac{2x\,P_{n-1}(x)}{[2]_{p,q}p^{n(n-1)/2}}\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-1)} \\
&\quad \left(\frac{1}{q^2}-\frac{p^k}{q[n+k-1]_{p,q}}-\frac{p^k}{q^2[n+k-1]_{p,q}}\right) \\
&= \frac{2x\,P_{n-1}(x)}{[2]_{p,q}p^{n(n-1)/2}}\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-1)} \\
&\quad \left(\frac{1}{q^2}-\left(\frac{1}{q}+\frac{1}{q^2}\right)\frac{p^k}{[n+k-1]_{p,q}}\right) \\
&= \frac{2x}{[2]_{p,q}q^2p^{n(n-1)/2}}P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-1)} \\
&- \frac{2(1+q)x}{[2]_{p,q}q^2p^{n(n-1)/2}}P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-2)}\frac{1}{[n+k-1]_{p,q}}
\end{aligned}$$

$$= \frac{2x}{[2]_{p,q}q^2} - \frac{2(1+q)x}{[2]_{p,q}q^2p^{n-1}}\left(\frac{p^{n-1}-q^{n-1}x}{[n-1]_{p,q}}\right)$$
$$= \frac{2x}{[2]_{p,q}q^2}\left(1 - \frac{(1+q)}{p^{n-1}}\left(\frac{p^{n-1}-q^{n-1}x}{[n-1]_{p,q}}\right)\right).$$

Finally for $e_2 = t^2$, moments of the operators can be obtained as follows:

$$\tilde{M}_{n,p,q}(e_2;x) = A + B + C.$$

Here,

$$A = \frac{p^{2(n-1)}}{[3]_{p,q}p^{n(n-1)/2}}P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-1)}[k]_{p,q}^2 S_2(n,k),$$
$$B = \frac{p^{2(n-1)}}{[3]_{p,q}p^{n(n-1)/2}}P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-1)}[k]_{p,q} S_1(n,k),$$
$$C = \frac{p^{2(n-1)}}{[3]_{p,q}p^{n(n-1)/2}}P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-1)} S_0(n,k).$$

Using Lemma 2.2, we have

$$S_2(n,k) \leq \frac{3}{q^3[n+k-1]_{p,q}^2}, \tag{2.3}$$
$$S_1(n,k) \leq \frac{3p^k}{q^4[n+k-1]_{p,q}^2}, \tag{2.4}$$
$$S_0(n,k) \leq \frac{p^{2k}}{q^5[n+k-1]_{p,q}^2}. \tag{2.5}$$

Using the inequality (2.3), we have

$$A \leq \frac{3p^{2(n-1)}}{[3]_{p,q}q^3p^{n(n-1)/2}}P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^k p^{-k(n-1)}\frac{[k]_{p,q}^2}{[n+k-1]_{p,q}^2}$$
$$= \frac{3p^{2(n-1)}}{[3]_{p,q}q^3p^{n(n-1)/2}}P_{n-1}(x)\sum_{k=1}^{\infty}\left[\begin{array}{c} n+k-2 \\ k-1 \end{array}\right]_{p,q} x^k p^{-k(n-1)}\frac{[k]_{p,q}}{[n+k-2]_{p,q}}$$
$$= \frac{3p^{2(n-1)}}{[3]_{p,q}q^3p^{n(n-1)/2}}P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^{k+1} p^{-(k+1)(n-1)}\frac{[k+1]_{p,q}}{[n+k-1]_{p,q}}$$
$$= \frac{3p^{2(n-1)}}{[3]_{p,q}q^3p^{n(n-1)/2}}P_{n-1}(x)\Bigg(\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^{k+1} p^{-(k+1)(n-1)}\frac{p^k}{[n+k-1]_{p,q}}$$
$$+ \sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} x^{k+1} p^{-(k+1)(n-1)}\frac{q[k]_{p,q}}{[n+k-1]_{p,q}}\Bigg)$$

$$= \frac{3p^{2(n-1)}}{[3]_{p,q}q^3p^{n(n-1)/2}}\Bigg(xp^{-(n-1)}P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q}\frac{x^kp^{-k(n-2)}}{[n+k-1]_{p,q}}$$

$$+ \; qp^{-2(n-1)}x^2P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q}x^kp^{-k(n-1)}\Bigg)$$

$$= \frac{3p^{2(n-1)}}{[3]_{p,q}q^3p^{n(n-1)/2}}\left(xp^{-(n-1)}p^{(n-1)(n-2)/2}\frac{(p^{n-1}-q^{n-1}x)}{[n-1]_{p,q}} + qp^{-2(n-1)}p^{n(n-1)/2}x^2\right)$$

$$= \frac{3x}{[3]_{p,q}q^3}\frac{(p^{n-1}-q^{n-1}x)}{[n-1]_{p,q}} + \frac{3x^2}{[3]_{p,q}q^2}.$$

Again, using the inequality (2.4), we have

$$B \le \frac{3p^{2(n-1)}}{[3]_{p,q}q^4p^{n(n-1)/2}}P_{n-1}(x)\sum_{k=1}^{\infty}\left[\begin{array}{c} n+k-2 \\ k-1 \end{array}\right]_{p,q}x^kp^{-k(n-1)}\frac{p^k}{[n+k-2]_{p,q}}$$

$$= \frac{3p^{2(n-1)}}{[3]_{p,q}q^4p^{n(n-1)/2}}P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q}x^{k+1}p^{-(k+1)(n-1)}\frac{p^{k+1}}{[n+k-1]_{p,q}}$$

$$= \frac{3p^{2(n-1)}}{[3]_{p,q}q^4p^{n(n-1)/2}}\left(xp^{-(n-1)}p^{(n-1)(n-2)/2}\frac{(p^{n-1}-q^{n-1}x)}{[n-1]_{p,q}}\right)$$

$$= \frac{3}{[3]_{p,q}q^4}\frac{(p^{n-1}-q^{n-1}x)}{[n-1]_{p,q}}.$$

Further, using the inequality (2.5), we have

$$C \le \frac{p^{2(n-1)}}{[3]_{p,q}q^5p^{n(n-1)/2}}P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q}\frac{x^kp^{-k(n-3)}}{[n+k-1]^2_{p,q}}$$

$$= \frac{p}{[3]_{p,q}q^5}\frac{\prod_{j=1}^{2}(p^{n-j}-q^{n-j}x)}{[n-1]^2_{p,q}}.$$

Finally,

$$\tilde{M}_{n,p,q}(e_2;x) \le \frac{1}{[3]_{p,q}q^2}\left(3x^2 + \left(\frac{3x}{q} + \frac{3}{q^2} + \frac{p}{q^3}\frac{(p^{n-2}-q^{n-2}x)}{[n-2]_{p,q}}\right)\frac{(p^{n-1}-q^{n-1}x)}{[n-1]_{p,q}}\right).$$

Hence the lemma. □

Lemma 2.5. *For all $x \in [0,1]$ and $0 < q < p \le 1$, central moments of the operators are given by:*

$$\tilde{M}_{n,p,q}((t-x);x) \le \frac{1}{[2]_{p,q}q}\left((2-[2]_{p,q}q)x + \frac{(p^{n-1}-q^{n-1}x)}{q[n-1]_{p,q}}\right),$$

$$\tilde{M}_{n,p,q}((t-x)^2;x) \le \left(1 + \frac{3}{[3]_{p,q}q^2} - \frac{4}{[2]_{p,q}q^2}\right)x^2 + \frac{p^{n-1}-q^{n-1}x}{[n-1]_{p,q}q^2}$$
$$\left(\frac{3x}{[3]_{p,q}} + \frac{3}{[3]_{p,q}q^2} + \frac{4x^2(1+q)}{[2]_{p,q}p^{n-1}} + \frac{p}{q^3}\frac{p^{n-2}-q^{n-2}x}{[n-2]_{p,q}}\right).$$

Now, we give the result for the uniform convergence of operator by means of Bohman-Korovkin type theorem.

Remark 2.2. *For $0 < q < p \leq 1$, by simple computations $\lim_{n\to\infty}[n]_{p,q} = 1/(p-q)$. In order to obtain results for order of convergence of the operator, we take $q_n \in (0,1)$, $p_n \in (q_n, 1]$ such that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} p_n^n = a$ and $\lim_{n\to\infty} q_n^n = b$, so that $\lim_{n\to\infty} \frac{1}{[n]_{p_n,q_n}} = 0$. Such a sequence can always be constructed for example, we can take $q_n = 1 - 1/n$ and $p_n = 1 - 1/2n$, clearly $\lim_{n\to\infty} p_n^n = e^{-1/2}$, $\lim_{n\to\infty} q_n^n = e^{-1}$ and $\lim_{n\to\infty} \frac{1}{[n]_{p_n,q_n}} = 0$.*

Theorem 2.1. *Let $\{p_n\}_n$ and $\{q_n\}_n$ be the sequence as defined in Remark 2.2. Then for each $f \in C[0,1]$, $\tilde{M}_{n,p_n,q_n}(f;x)$ converges uniformly to f on $[0,1]$.*

Proof. By the Bohman-Korovkin theorem [13], to prove uniform convergence of the operators, it is sufficient to show that following equality holds for $i = 0, 1, 2$:

$$\lim_{n\to\infty} \|\tilde{M}_{n,p_n,q_n}(e_i;.) - e_i\| = 0. \tag{2.6}$$

By using moment estimates obtained in Lemma 2.4, equality (2.6) holds directly for $i = 0$. Also,

$$\|\tilde{M}_{n,p_n,q_n}(e_1;.) - e_1\| \leq \frac{1}{[2]_{p_n,q_n} q_n}\left(|2 - [2]_{p_n,q_n} q_n| + \left|\frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n [n-1]_{p_n,q_n}}\right|\right),$$

$$\begin{aligned}\|\tilde{M}_{n,p_n,q_n}(e_2;.) - e_2\| \leq\; & \frac{1}{[3]_{p_n,q_n} q_n^2}\Bigg(|3 - [3]_{p_n,q_n} q_n^2| \\ & + \left|\left(\frac{3}{q_n} + \frac{3}{q_n^2} + \frac{p_n}{q_n^3}\frac{(p_n^{n-2} - q_n^{n-2}x)}{[n-2]_{p_n,q_n}}\right)\frac{(p_n^{n-1} - q_n^{n-1}x)}{[n-1]_{p_n,q_n}}\right|\Bigg).\end{aligned}$$

For $n \to \infty$ and Remark 2.2, equality (2.6) holds for $i = 0, 1, 2$. Hence the theorem. □

3. Rate of Convergence

In this section, we estimate the rate of convergence of proposed operator by means of modulus of continuity and Lipschitz class functions. We also show the statistical convergence of the operator.
Recall the concept of modulus of continuity, the modulus of continuity of $f(x) \in [0,a]$, denoted by $\omega(f,\delta)$, is defined by

$$\omega(f,\delta) = \sup_{|x-y|\leq\delta;\, x,y\in[0,a]} |f(x) - f(y)|.$$

A function $f \in Lip_M(\alpha)$, ($M > 0$ and $0 < \alpha \leq 1$), if the inequality

$$|f(t) - f(x)| \leq M|t - x|^{\alpha},$$

holds for all $t, x \in [0,1]$.

Theorem 3.2. *Let $\{p_n\}_n$ and $\{q_n\}_n$ be the sequence as defined in Remark 2.2. Then*

$$|\tilde{M}_{n,p_n,q_n}(f;x) - f| \leq 2\omega(f, \sqrt{\delta_n}),$$

for all $f \in C[0,1]$, here $\delta_n = \tilde{M}_{n,p_n,q_n}\left((t-x)^2; x\right)$.

Proof. By the linearity and monotonicity of the operators, we get

$$|\tilde{M}_{n,p_n,q_n}(f;x) - f| \leq \tilde{M}_{n,p_n,q_n}(|f(t) - f(x)|; x),$$

also, by property of modulus of continuity (see [14])

$$|f(t) - f(x)| \leq \omega(f,\delta)\left(1 + \frac{1}{\delta^2}(t-x)^2\right),$$

therefore,

$$|\tilde{M}_{n,p_n,q_n}(f;x) - f| \leq \omega(f,\delta)\left(1 + \frac{1}{\delta^2}\tilde{M}_{n,p_n,q_n}((t-x)^2;x)\right).$$

By Lemma 2.5 and Remark 2.2, we can find

$$\lim_{n\to\infty} \tilde{M}_{n,p_n,q_n}\left((t-x)^2;x\right) = 0.$$

So, letting $\delta_n = \tilde{M}_{n,p_n,q_n}\left((t-x)^2;x\right)$ and take $\delta = \sqrt{\delta_n}$, we finally get the result. □

In the following theorem, we compute the rate of convergence by means of the Lipschitz class.

Theorem 3.3. *Let $\{p_n\}_n$ and $\{q_n\}_n$ be the sequence as defined in Remark 2.2. Then for all $f \in Lip_M(\alpha)$, we have*

$$|\tilde{M}_{n,p_n,q_n}(f;x) - f(x)| \leq M\delta_n(x)^{\alpha/2},$$

here $\delta_n(x) = \tilde{M}_{n,p_n,q_n}(|t-x|^2;x)$.

Proof. By using definition of Lipschitz class functions and applying Hölder's inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$, we get

$$\begin{aligned}
|\tilde{M}_{n,p_n,q_n}(f;x) - f(x)| &\leq \tilde{M}_{n,p_n,q_n}(|f(t)-f(x)|;x) \\
&\leq M\tilde{M}_{n,p_n,q_n}(|t-x|^\alpha;x) \\
&\leq M\tilde{M}_{n,p_n,q_n}(|t-x|^2;x)^{\alpha/2}.
\end{aligned}$$

Taking $\delta_n = \tilde{M}_{n,p_n,q_n}(|t-x|^2;x)$, we get the result. □

A sequence $(x_n)_n$ is said to be statistically convergent to a number L, denoted by $st - \lim_n x_n = L$ if, for every $\varepsilon > 0$,

$$\delta\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0,$$

where

$$\delta(S) := \frac{1}{N}\sum_{k=1}^{N} \chi_S(j)$$

is the natural density of set $S \subseteq \mathbb{N}$ and χ_S is the characteristic function of S.
Let $C_B(D)$ represents the space of all continuous functions on D and bounded on entire real line, where D is any interval on real line. It can be easily shown that $C_B(D)$ is a Banach space with supreme norm.

Theorem A. ([5]) *Let $\{L_n\}_n$ be a sequence of positive linear operators from $C_B([a,b])$ into $B([a,b])$, satisfying the condition that*

$$st - \lim_{n\to\infty} \|L_n e_i - e_i\| = 0 \forall i = 0,1,2.$$

Then,

$$st - \lim_{n\to\infty} \|L_n f - f\| = 0 \forall f \in C_B([a,b]).$$

Theorem 3.4. *Let $\{p_n\}_n, \{q_n\}_n$ be sequences such that*

$$st-\lim_{n\to\infty} q_n = 1,\;\; st-\lim_{n\to\infty} {q_n}^n = a,$$
$$st-\lim_{n\to\infty} p_n = 1,\;\; st-\lim_{n\to\infty} {p_n}^n = b.$$

. Then, we have

$$st-\lim_{n\to\infty} \|\tilde{M}_{n,p_n,q_n} f - f\| = 0 \;\; for \;\; all \;\; f \in C_B[0,1].$$

Proof. We use moment estimates obtained in Lemma 2.4, to prove that operator converges statistically for e_i, $i = 0,1,2$. For first moment result is trivial.
For $i = 1,2$, we have

$$\begin{aligned} |\tilde{M}_{n,p_n,q_n}(e_1;.) - e_1| &\le \frac{1}{[2]_{p_n,q_n} q_n}\left(|2 - [2]_{p_n,q_n} q_n| x + \left| \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}} \right| \right), \\ |\tilde{M}_{n,p_n,q_n}(e_2;.) - e_2| &\le \frac{1}{[3]_{p_n,q_n} q_n^2}\left(|3 - [3]_{p_n,q_n} q_n^2| x^2 + \left| \left(\frac{3x}{q_n} + \frac{3}{q_n^2}\right) \frac{(p_n^{n-1} - q_n^{n-1}x)}{[n-1]_{p_n,q_n}} \right| \right. \\ &\quad \left. + \left| \frac{p_n}{q_n^3} \frac{(p_n^{n-1} - q_n^{n-1}x)(p_n^{n-2} - q_n^{n-2}x)}{[n-1]_{p_n,q_n}[n-2]_{p_n,q_n}} \right| \right). \end{aligned}$$

By taking supremum over $x \in [0,1]$ in above inequalities and using $st - \lim_{n\to\infty} \frac{1}{[n]_{p_n,q_n}} = 0$, we get

$$\begin{aligned} st-\lim_{n\to\infty} \|\tilde{M}_{n,p_n,q_n}(e_1;.) - e_1\| &= 0, \\ st-\lim_{n\to\infty} \|\tilde{M}_{n,p_n,q_n}(e_2;.) - e_2\| &= 0. \end{aligned}$$

By Theorem A, we obtain statistical convergence of the operator. □

4. Local Approximation

The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf_{g\in W^2}\{\|f-g\| + \delta\|g''\|\},$$

here $W^2 = \{g \in C[0,1] : g', g'' \in C[0,1]\}$ and norm $\|.\|$ denotes the uniform norm on $C[0,1]$. Further, we have a well-known inequality given by DeVore and Lorentz [4, p. 177, Theorem 2.4], there exists a positive constant $C > 0$ such that $K_2(f,\delta) \le C\omega(f,\delta^{\frac{1}{2}}), \delta > 0$, where ω_2 is known as the second order modulus of continuity, given by

$$\omega_2(f,\delta^{\frac{1}{2}}) = \sup_{0<h<\delta^{\frac{1}{2}}, x\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|.$$

Here, we give some local result for the operators.

Theorem 4.5. *Let $\{p_n\}_n$ and $\{q_n\}_n$ be the sequence as defined in Remark 2.2. Then for all $f \in C[0,1]$, there exists an absolute constant $C > 0$ such that*

$$|\tilde{M}_{n,p_n,q_n}(f;x) - f| \le C\omega_2(f,\delta_n(x)) + \omega(f,\alpha_n(x)),$$

Here,

$$\delta_n(x) = \sqrt{\tilde{M}_{n,p_n,q_n}((t-x)^2;x) + \left(\frac{1}{[2]_{p_n,q_n} q_n}\left(2x + \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right) - x \right)^2},$$

$$\alpha_n(x) = \left| \frac{1}{[2]_{p_n,q_n} q_n}\left((2 - [2]_{p_n,q_n} q_n)x + \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}} \right) \right|.$$

Proof. For $f \in C[0,1]$, we consider

$$M_{n,p_n,q_n}(f;x) = \tilde{M}_{n,p_n,q_n}(f;x) + f(x) - f\left(\frac{1}{[2]_{p_n,q_n}q_n}\left(2x + \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right)\right).$$

Now, using Lemma 2.4, we immediately get

$$M_{n,p_n,q_n}(1;x) = \tilde{M}_{n,p_n,q_n}(1;x) = 1$$

and

$$M_{n,p_n,q_n}(t;x) = \tilde{M}_{n,p_n,q_n}(t;x) + x - \frac{1}{[2]_{p_n,q_n}q_n}\left(2x + \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right) \leq x.$$

By Taylor's formula $g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du$, we get

$$\begin{aligned} M_{n,p_n,q_n}(g(t);x) &= g(x) + g'(x)M_{n,p_n,q_n}((t-x);x) + M_{n,p_n,q_n}\left(\int_x^t (t-u)g''(u)du;x\right) \\ &\leq g(x) + \tilde{M}_{n,p_n,q_n}\left(\int_x^t (t-u)g''(u)du;x\right) \\ &- \int_x^{\frac{1}{[2]_{p_n,q_n}q_n}\left(2x+\frac{(p_n^{n-1}-q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right)} \\ &\left(\frac{1}{[2]_{p_n,q_n}q_n}\left(2x + \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right) - u\right)g''(u)du. \end{aligned}$$

Further, we have

$$\begin{aligned} &|M_{n,p_n,q_n}(g(t);x) - g(x)| \leq \tilde{M}_{n,p_n,q_n}\left(\left|\int_x^t (t-u)g''(u)du\right|;x\right) \\ &+ \left|\int_x^{\frac{1}{[2]_{p_n,q_n}q_n}\left(2x+\frac{(p_n^{n-1}-q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right)} \left|\frac{1}{[2]_{p_n,q_n}q_n}\left(2x + \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right) - u\right| |g''(u)|\, du\right| \\ &\leq \tilde{M}_{n,p_n,q_n}((t-x)^2;x)\, \|g''(x)\| + \left(\frac{1}{[2]_{p_n,q_n}q_n}\left(2x + \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right) - x\right)^2 \|g''(x)\| \\ &= \delta_n^2(x)\|g''\|. \end{aligned}$$

Now, by boundedness of $\tilde{M}_{n,p_n,q_n}$, we get

$$\begin{aligned} |M_{n,p_n,q_n}(f;x)| &\leq \left|\tilde{M}_{n,p_n,q_n}(f;x)\right| + |f(x)| \\ &+ \left|f\left(\frac{1}{[2]_{p_n,q_n}q_n}\left(2x + \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right)\right)\right|, \\ &\leq 3\,\|f\|\,. \end{aligned}$$

Finally, we obtain

$$\left|\tilde{M}_{n,p_n,q_n}(f;x) - f(x)\right|$$

$$= \left|M_{n,p_n,q_n}(f;x) - f(x) + f\left(\frac{1}{[2]_{p_n,q_n}q_n}\left(2x + \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right)\right) - f(x)\right|$$

$$\leq |M_{n,p_n,q_n}(f-g;x)| + |M_{n,p_n,q_n}(g;x) - g(x)| + |g(x) - f(x)|$$

$$+ \left|f\left(\frac{1}{[2]_{p_n,q_n}q_n}\left(2x + \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right)\right) - f(x)\right|$$

$$\leq 4\,\|f-g\| + \delta_n^2(x)\,\|g''(x)\|$$

$$+\omega\left(f, \left|\frac{1}{[2]_{p_n,q_n}q_n}\left((2 - [2]_{p_n,q_n}q_n)x + \frac{(p_n^{n-1} - q_n^{n-1}x)}{q_n[n-1]_{p_n,q_n}}\right)\right|\right).$$

By taking the infimum on the right hand side over all $g \in \mathbb{W}^2$, we get

$$\left|\tilde{M}_{n,p_n,q_n}(f;x) - f(x)\right| \leq 4K_2(f, \delta_n^2(x)) + \omega(f, \alpha_n(x)).$$

Finally, by using the property of K-functional, we obtain

$$\left|\tilde{M}_{n,p_n,q_n}(f;x) - f(x)\right| \leq C\omega_2(f, \delta_n(x)) + \omega(f, \alpha_n(x)).$$

Hence the proof is completed. □

5. Graphical Examples

In this section, we estimate approximation for functions $f(x) = \sin(x)$ (Figure (1)), $f(x) = (x-4/5)(x-2/3)(x-1/4)$ (Figure (2)), $f(x) = (x-2/3)(x-1/4)$ (Figure (3)) and $f(x) = \exp(x)$ (Figure (4)), by (p,q)-Meyer-König-Zeller Kantrovich operators using Matlab programming.

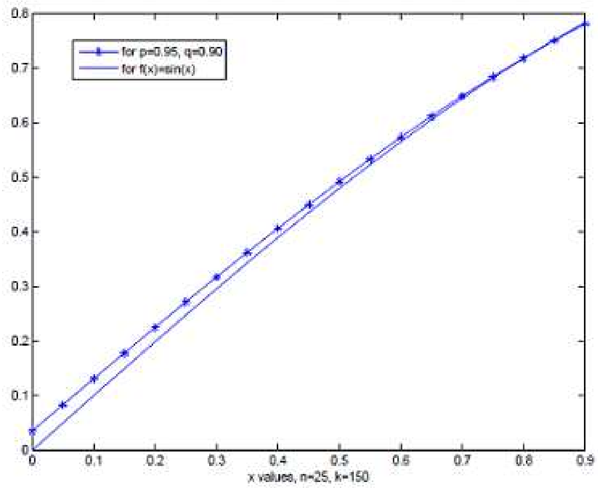

FIGURE 1. $f(x) = \sin(x)$

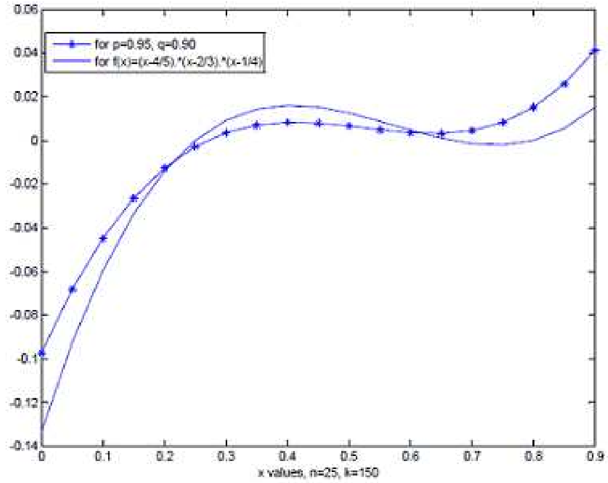

FIGURE 2. $f(x) = (x - 4/5)(x - 2/3)(x - 1/4)$

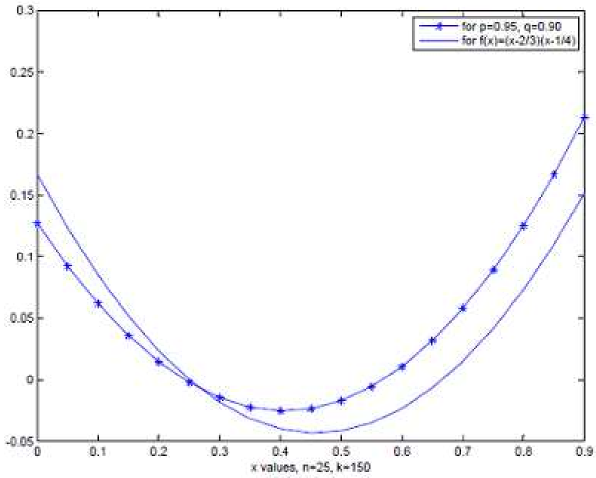

FIGURE 3. $f(x) = (x - 2/3)(x - 1/4)$

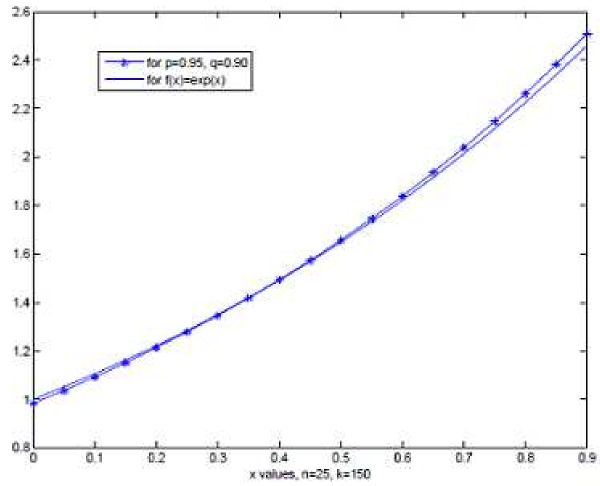

FIGURE 4. $f(x) = \exp(x)$

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GULZAR GROUP OF INSTITUTES,
KHANNA, LUDHIANA, PUNJAB, INDIA
E-mail address: pro.sharma.h@gmail.com

GULZAR GROUP OF INSTITUTES,
KHANNA, LUDHIANA, PUNJAB, INDIA
E-mail address: ramapatimaurya@gmail.com

I. K. GUJRAL PUNJAB TECHNICAL UNIVERSITY,
DEPARTMENT OF MATHEMATICS
KAPURTHALA, PUNJAB, INDIA
E-mail address: guptacheena21@gmail.com