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On the paranormed binomial sequence spaces

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Abstract

In this paper the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ which are the generalization of the classical Maddox's paranormed sequence spaces have been introduced and proved that the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ are linearly isomorphic to spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\ell(p)$, respectively. Besides this, the $\alpha-,\beta-$ and $\gamma-$duals of the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, and $b^{r,s}(p)$ have been computed, their bases have been constructed and some topological properties of these spaces have been studied. Finally, the classes of matrices $(b_0^{r,s}(p):\mu)$, $(b_c^{r,s}(p):\mu)$ and $(b^{r,s}(p):\mu)$ have been characterized, where μ is one of the sequence spaces ℓ_∞, c and c_0 and derives the other characterizations for the special cases of μ.

1. Introduction

We shall denote the space of all real-valued sequences by w as a classical notation. Any vector subspace of w is called a sequence space. The spaces ℓ_∞, c and c_0 are the most common and frequently used spaces which are all bounded, convergent and null sequences, respectively. Also bs, cs, ℓ_1 and ℓ_p notations are used for the spaces of all bounded, convergent, absolutely and $p-$absolutely convergent series, respectively, where $1<p<\infty$.

First, we point out the concept of a paranorm. A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g: X \to \mathbb{R}$ such that $g(\theta)=0, g(x)=g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n-\alpha| \to 0$ and $g(x_n-x)\to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that (p_k) be a bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $L=\max\{1,H\}$. Then, the linear spaces $\ell_\infty(p), c(p), c_0(p)$ and $\ell(p)$ were defined by Maddox [19] (see also Simons [21] and Nakano [20]) as follows:

$$\ell_\infty(p)=\{x=(x_k)\in w: \sup_{k\in\mathbb{N}} |x_k|^{p_k}<\infty\},$$

$$c(p)=\{x=(x_k)\in w: \lim_{k\to\infty} |x_k-l|^{p_k}=0 \text{ for some } l\in\mathbb{R}\},$$

$$c_0(p)=\{x=(x_k)\in w: \lim_{k\to\infty} |x_k|^{p_k}=0\}$$

$$\ell(p)=\left\{x=(x_k)\in w: \sum_k |x_k|^{p_k}<\infty\right\},$$

which are the complete spaces paranormed by

$$g_1(x)=\sup_{k\in\mathbb{N}} |x_k|^{p_k/L} \Longleftrightarrow \inf p_k>0 \text{ and } g_2(x)=\left(\sum_k |x_k|^{p_k}\right)^{1/L},$$

respectively. For convenience in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $\mathscr{F}$ and $\mathbb{N}_k$, we shall denote the collection of all finite subsets of $\mathbb{N}$ and the set of all $n\in\mathbb{N}$ such that $n\geq k$, respectively. We shall assume throughout that $p_k^{-1}+(p_k')^{-1}=1$ provided $1<\inf p_k \leq H<\infty$.

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Let λ,μ be any two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n,k\in\mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A:\lambda\to\mu$, if for every sequence $x=(x_k)\in\lambda$, the sequence $Ax=\{(Ax)_n\}$, the $A-$transform of x, is in μ, where

$$(Ax)_n=\sum_k a_{nk}x_k,\ (n\in\mathbb{N}). \tag{1.1}$$

By $(\lambda:\mu)$, we denote the class of all matrices A such that $A:\lambda\to\mu$. Thus, $A\in(\lambda:\mu)$ if and only if the series on the right-hand side of (1.1) converges for each $n\in\mathbb{N}$ and every $x\in\lambda$, and we have $Ax=\{(Ax)_n\}_{n\in\mathbb{N}}\in\mu$ for all $x\in\mu$. A sequence x is said to be $A-$summable to α if Ax converges to α which is called the $A-$limit of x.

2. The sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$

In this section, we define the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$, and prove that $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ are the complete paranormed linear spaces.
For a sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by

$$\lambda_A=\{x=(x_k)\in w: Ax\in\lambda\}. \tag{2.1}$$

In [7], Choudhary and Mishra have defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that $S-$transforms are in $\ell_{(p)}$, where $S=(s_{nk})$ is defined by

$$s_{nk}=\begin{cases} 1 & , \quad 0\le k\le n,\\ 0 & , \quad k>n.\end{cases}$$

Başar and Altay [3] have studied the space $bs(p)$ which is formerly defined by Başar in [4] as the set of all series whose sequences of partial sums are in $\ell_\infty(p)$. More recently, Altay and Başar have studied the sequence spaces $r^t(p), r^t_\infty(p)$ in [1] and $r^t_c(p), r^t_0(p)$ in [2] which are derived by the Riesz means from the sequence spaces $\ell(p),\ell_\infty(p),c(p)$ and $c_0(p)$ of Maddox, respectively.
With the notation of (2.1), the spaces $\overline{\ell(p)}, bs(p), r^t(p), r^t_\infty(p), r^t_c(p)$ and $r^t_0(p)$ may be redefined by

$$\overline{\ell(p)}=[\ell(p)]_S, bs(p)=[\ell_\infty(p)]_S, r^t(p)=[\ell(p)]^t_R$$

$$r^t_\infty(p)=[\ell_\infty(p)]^t_R, r^t_c(p)=[c(p)]^t_R, r^t_0(p)=[c_0(p)]^t_R.$$

In [8], Demiriz and Çakan have defined the sequence spaces $e^r_0(u,p)$ and $e^r_c(u,p)$ which consists of all sequences such that $E^{r,u}$- transforms are in $c_0(p)$ and $c(p)$, respectively $E^{r,u}=\{e^r_{nk}(u)\}$ is defined by

$$e^r_{nk}(u)=\begin{cases} \binom{n}{k}(1-r)^{n-k}r^k u_k & , \quad (0\le k\le n),\\ 0 & , \quad (k>n)\end{cases}$$

for all $k,n\in\mathbb{N}$ and $0<r<1$.
In [5] and [6], the Binomial sequence spaces $b_0^{r,s}$, $b_c^{r,s}$, $b_\infty^{r,s}$ and $b_p^{r,s}$, which are the matrix domains of Binomial mean $B^{r,s}$ in the sequence spaces c_0, c, ℓ_∞ and ℓ_p, respectively, are introduced, some inclusion relations and Schauder basis for the spaces $b_0^{r,s}$, $b_c^{r,s}$, $b_\infty^{r,s}$ and $b_p^{r,s}$ are given, and the $\alpha-,\beta-$ and $\gamma-$ duals of those spaces are determined. For more papers related to sequence spaces and matrix domains of different infinite matrices one can see [13, 12] and references therein. The main purpose of this paper is to introduce the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ which are the set of all sequences whose $B^{r,s}-$transforms are in the spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\ell(p)$, respectively; where $B^{r,s}$ denotes the matrix $B^{r,s}=\{b^{r,s}_{nk}\}$ defined by

$$b^{r,s}_{nk}=\begin{cases} \frac{1}{(s+r)^n}\binom{k}{n}s^{n-k}r^k & , \quad 0\le k\le n,\\ 0 & , \quad k>n,\end{cases}$$

where $sr>0$. Also, we have constructed the basis and computed the $\alpha-,\beta-$ and $\gamma-$duals and investigated some topological properties of the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$.
Following Choudhary and Mishra [7], Başar and Altay [3], Altay and Başar [1, 2], Demiriz [8], Kirişçi [14, 15], Candan and Güneş [16] and Ellidokuzoğlu and Demiriz [9], we define the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$, as the sets of all sequences such that $B^{r,s}-$transforms of them are in the spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\ell(p)$, respectively, that is,

$$b_0^{r,s}(p)=\left\{x=(x_k)\in w: \lim_{n\to\infty}\left|\frac{1}{(s+r)^n}\sum_{k=0}^n\binom{n}{k}s^{n-k}r^kx_k\right|^{p_n}=0\right\},$$

$$b_c^{r,s}(p)=\left\{x=(x_k)\in w: \exists l\in\mathbb{C}\ni\lim_{n\to\infty}\left|\frac{1}{(s+r)^n}\sum_{k=0}^n\binom{n}{k}s^{n-k}r^kx_k-l\right|^{p_n}=0\right\},$$

$$b_\infty^{r,s}(p)=\left\{x=(x_k)\in w: \sup_{n\in\mathbb{N}}\left|\frac{1}{(s+r)^n}\sum_{k=0}^n\binom{n}{k}s^{n-k}r^kx_k\right|^{p_n}<\infty\right\},$$

$$b^{r,s}(p)=\left\{x=(x_k)\in w: \sum_n\left|\frac{1}{(s+r)^n}\sum_{k=0}^n\binom{n}{k}s^{n-k}r^kx_k\right|^{p_n}<\infty\right\}.$$

In the case $(p_n) = e = (1,1,1,...)$, the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ are, respectively, reduced to the sequence spaces $b_0^{r,s}$, $b_c^{r,s}$, $b_\infty^{r,s}$ and $b_p^{r,s}$ which are introduced by Bişgin [5, 6]. With the notation of (2.1), we may redefine the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ as follows:

$$b_0^{r,s}(p) = [c_0(p)]_{B^{r,s}},\ b_c^{r,s}(p) = [c(p)]_{B^{r,s}}, b_\infty^{r,s}(p) = [\ell_\infty(p)]_{B^{r,s}} \text{ and } b^{r,s}(p) = [\ell(p)]_{B^{r,s}}.$$

Define the sequence $y = \{y_n(r,s)\}$, which will be frequently used, as the $B^{r,s}-$transform of a sequence $x = (x_k)$, i.e.,

$$y_n(r,s) := \frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k x_k;\ \text{for all } k\in\mathbb{N}. \tag{2.2}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *$b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b_\infty^{r,s}(p)$ are the complete linear metric space paranormed by g, defined by*

$$g(x) = \sup_{n\in\mathbb{N}}\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k x_k\right|^{p_n/L}. \tag{2.3}$$

In addition, $b^{r,s}(p)$ is the complete linear metric space paranormed by h, defined by

$$h(x) = \left(\sum_{n=0}^{\infty}\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k x_k\right|^{p_n}\right)^{1/M}. \tag{2.4}$$

Proof. First, we give the proof for $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b_\infty^{r,s}(p)$. Since the proof is similar for $b_c^{r,s}(p)$ and $b_\infty^{r,s}(p)$, we give the proof only for the space $b_0^{r,s}(p)$. The linearity of $b_0^{r,s}(p)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $x,z\in b_0^{r,s}(p)$ (see Maddox [18, p.30])

$$\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k(x_k+z_k)\right|^{p_n/L} \le \left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k x_k\right|^{p_n/L} + \left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k z_k\right|^{p_n/L} \tag{2.5}$$

and for any $\alpha\in\mathbb{R}$ (see [21])

$$|\alpha|^{p_n} \le \max\{1,|\alpha|^L\} = K. \tag{2.6}$$

Using (2.6) inequality, we get

$$\begin{aligned}\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k(\alpha x_k)\right|^{p_n/L} &= |\alpha|^{p_n/L}\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k x_k\right|^{p_n/L}\\ &\le K^{1/L}\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k x_k\right|^{p_n/L}\end{aligned}$$

for $x\in b_0^{r,s}(p)$. This shows the space $b_0^{r,s}(p)$ is a linear space.
Now we will see that g is a paranorm on $b_0^{r,s}(p)$. It is clear that $g(\theta)=0$ and $g(x)=g(-x)$ for all $x\in b_0^{r,s}(p)$.
Let $\{x^n\}$ be any sequence of the points $x^n\in b_0^{r,s}(p)$ such that $g(x^n-x)\to 0$ and (α_n) also be any sequence of scalars such that $\alpha_n\to\alpha$. Then, since the inequality

$$g(x^n)\le g(x)+g(x^n-x)$$

holds by the subadditivity of g, $\{g(x^n)\}$ is bounded and we thus have

$$\begin{aligned}g(\alpha_n x^n-\alpha x) &= \sup_{k\in\mathbb{N}}\left|\frac{1}{(s+r)^k}\sum_{j=0}^{k}\binom{k}{j}s^{k-j}r^j(\alpha_n x_j^n-\alpha x_j)\right|^{p_k/L}\\ &\le |\alpha_n-\alpha|g(x^n)+|\alpha|g(x^n-x)\end{aligned} \tag{2.7}$$

which tends to zero as $n\to\infty$. This means that the scalar multiplication is continuous. Hence, g is a paranorm on the space $b_0^{r,s}(p)$.

It remains to prove the completeness of the space $b_0^{r,s}(p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $b_0^{r,s}(p)$, where $x^i=\{x_0^{(i)},x_1^{(i)},x_2^{(i)},\ldots\}$. Then, for a given $\varepsilon>0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$g(x^i-x^j)<\frac{\varepsilon}{2}$$

for all $i,j>n_0(\varepsilon)$. Using the definition of g we obtain for each fixed $k\in\mathbb{N}$ that

$$|(B^{r,s}x^i)_k-(B^{r,s}x^j)_k|^{p_k/L}\le\sup_{k\in\mathbb{N}}|(B^{r,s}x^i)_k-(B^{r,s}x^j)_k|^{p_k/L}<\frac{\varepsilon}{2} \tag{2.8}$$

for every $i,j>n_0(\varepsilon)$ which leads to the fact that $\{(B^{r,s}x^0)_k,(B^{r,s}x^1)_k,(B^{r,s}x^2)_k,\ldots\}$ is a Cauchy sequence of real numbers for every fixed $k\in\mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(B^{r,s}x^i)_k\to(B^{r,s}x)_k$ as $i\to\infty$. Using these infinitely many limits $(B^{r,s}x)_0,(B^{r,s}x)_1,\ldots$, we define the sequence $\{(B^{r,s}x)_0,(B^{r,s}x)_1,\ldots\}$. From (2.8) with $j\to\infty$, we have

$$|(B^{r,s}x^i)_k-(B^{r,s}x)_k|^{p_k/L}\le\frac{\varepsilon}{2}\ (i,j>n_0(\varepsilon)) \tag{2.9}$$

for every fixed $k\in\mathbb{N}$. Since $x^i=\{x_k^{(i)}\}\in b_0^{r,s}(p)$ for each $i\in\mathbb{N}$, there exists $k_0(\varepsilon)\in\mathbb{N}$ such that

$$|(B^{r,s}x^i)_k|^{p_k/L}<\frac{\varepsilon}{2} \tag{2.10}$$

for every $k\ge k_0(\varepsilon)$ and for each fixed $i\in\mathbb{N}$. Therefore, taking a fixed $i>n_0(\varepsilon)$ we obtain by (2.9) and (2.10) that

$$|(B^{r,s}x)_k|^{p_k/L}\le|(B^{r,s}x)_k-(B^{r,s}x^i)_k|^{p_k/L}+|(B^{r,s}x^i)_k|^{p_k/L}<\frac{\varepsilon}{2}$$

for every $k>k_0(\varepsilon)$. This shows that $x\in b_0^{r,s}(p)$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $b_0^{r,s}(p)$ is complete and this concludes the proof.
Now lets show that, $b^{r,s}(p)$ is the complete linear metric space paranormed by h defined by (2.4). It is easy to see that the space $b^{r,s}(p)$ is linear with respect to the coordinate-wise addition and scalar multiplication. Therefore, we first show that it is a paranormed space with the paranorm h defined by (2.4).

It is clear that $h(\theta)=0$ where $\theta=(0,0,0,\ldots)$ and $h(x)=h(-x)$ for all $x\in b^{r,s}(p)$.

Let $x,y\in b^{r,s}(p)$; then by Minkowski's inequality we have

$$\begin{aligned}
h(x+y)&=\left(\sum_{k=0}^{\infty}\left|\frac{1}{(s+r)^k}\sum_{j=0}^{k}\binom{k}{j}s^{k-j}r^j(x_j+y_j)\right|^{p_k}\right)^{1/M}\\
&=\left(\sum_{k=0}^{\infty}\left[\left|\frac{1}{(s+r)^k}\sum_{j=0}^{k}\binom{k}{j}s^{k-j}r^j(x_j+y_j)\right|^{p_k/M}\right]^{M}\right)^{1/M}\\
&\le\left(\sum_{k=0}^{\infty}\left|\frac{1}{(s+r)^k}\sum_{j=0}^{k}\binom{k}{j}s^{k-j}r^jx_j\right|^{p_k}\right)^{1/M}+\left(\sum_{k=0}^{\infty}\left[\frac{1}{(s+r)^k}\sum_{j=0}^{k}\binom{k}{j}s^{k-j}r^jy_j\right|^{p_k}\right)^{1/M}\\
&=h(x)+h(y)
\end{aligned}\tag{2.11}$$

and for any $\alpha\in\mathbb{R}$ we immediately see that

$$|\alpha|^{p_k}\le\max\{1,|\alpha|^M\}. \tag{2.12}$$

Let $\{x^n\}$ be any sequence of the points $x^n\in b^{r,s}(p)$ such that $h(x^n-x)\to0$ and (λ_n) also be any sequence of scalars such that $\lambda_n\to\lambda$. We observe that

$$h(\lambda_nx^n-\lambda x)\le h[(\lambda_n-\lambda)(x^n-x)]+h[\lambda(x^n-x)]+h[(\lambda_n-\lambda)x]. \tag{2.13}$$

It follows from $\lambda_n\to\lambda\,(n\to\infty)$ that $|\lambda_n-\lambda|<1$ for all sufficiently large n; hence

$$\lim_{n\to\infty}h[(\lambda_n-\lambda)(x^n-x)]\le\lim_{n\to\infty}h(x^n-x)=0. \tag{2.14}$$

Furthermore, we have

$$\lim_{n\to\infty}h[\lambda(x^n-x)]\le\max\{1,|\lambda|^M\}\lim_{n\to\infty}h(x^n-x)=0. \tag{2.15}$$

Also, we have

$$\lim_{n\to\infty}h[(\lambda_n-\lambda)x)]\le\lim_{n\to\infty}|\lambda_n-\lambda|h(x)=0. \tag{2.16}$$

Then, we obtain from (2.13), (2.14), (2.15) and (2.16) that $h(\lambda_nx^n-\lambda x)\to0$, as $n\to\infty$. This shows that h is a paranorm on $b^{r,s}(p)$.
Now, we show that $b^{r,s}(p)$ is complete. Let $\{x^n\}$ be any Cauchy sequence in the space $b^{r,s}(p)$, where $x^n=\{x_0^{(n)},x_1^{(n)},x_2^{(n)},\ldots\}$. Then, for a given $\varepsilon>0$, there exists a positive integer $n_0(\varepsilon)$ such that $h(x^n-x^m)<\varepsilon$ for all $n,m>n_0(\varepsilon)$. Since for each fixed $k\in\mathbb{N}$ that

$$|(B^{r,s}x^n)_k-(B^{r,s}x^m)_k|\le\left[\sum_k|(B^{r,s}x^n)_k-(B^{r,s}x^m)_k|^{p_k}\right]^{\frac{1}{M}}=h(x^n-x^m)<\varepsilon \tag{2.17}$$

for every $n,m>n_0(\varepsilon)$, $\{(B^{r,s}x^0)_k,(B^{r,s}x^1)_k,(B^{r,s}x^2)_k,\ldots\}$ is a Cauchy sequence of real numbers for every fixed $k\in\mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(B^{r,s}x^n)_k\to(B^{r,s}x)_k$ as $n\to\infty$. Using these infinitely many limits $(B^{r,s}x)_0,(B^{r,s}x)_1,\ldots$, we define the sequence $\{(B^{r,s}x)_0,(B^{r,s}x)_1,\ldots\}$. For each $K\in\mathbb{N}$ and $n,m>n_0(\varepsilon)$

$$\left[\sum_{k=0}^{K}|(B^{r,s}x^n)_k-(B^{r,s}x^m)_k|^{p_k}\right]^{\frac{1}{M}}\le h(x^n-x^m)<\varepsilon. \tag{2.18}$$

By letting $m,K\to\infty$, we have for $n>n_0(\varepsilon)$ that

$$h(x^n-x)=\left[\sum_k |(B^{r,s}x^n)_k-(B^{r,s}x)_k|^{p_k}\right]^{\frac{1}{M}}<\varepsilon. \tag{2.19}$$

This shows that $x^n-x\in b^{r,s}(p)$. Since $b^{r,s}(p)$ is a linear space, we conclude that $x\in b^{r,s}(p)$; it follows that $x^n\to x$, as $n\to\infty$ in $b^{r,s}(p)$, thus we have shown that $b^{r,s}(p)$ is complete. □

Note that the absolute property does not hold on the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$, since there exists at least one sequence in the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$ and such that $g(x)\neq g(|x|)$, where $|x|=(|x_k|)$. This says that $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$ are the sequence spaces of non-absolute type.

Theorem 2.2. *The sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$, $b_\infty^{r,s}(p)$ and $b^{r,s}(p)$ are linearly isomorphic to the spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\ell(p)$, respectively, where $0<p_k\le H<\infty$.*

Proof. To avoid repetition of similar statements, we give the proof only for $b_0^{r,s}(p)$. We should show the existence of a linear bijection between the spaces $b_0^{r,s}(p)$ and $c_0(p)$. With the notation of (2.2), define the transformation T from $b_0^{r,s}(p)$ to $c_0(p)$ by $x\mapsto y=Tx$. The linearity of T is trivial. Furthermore, it is obvious that $x=\theta$ whenever $Tx=\theta$, and hence T is injective.
Let $y\in c_0(p)$ and define the sequence

$$x_k=\frac{1}{r^k}\sum_{j=0}^{k}\binom{k}{j}(-s)^{k-j}(s+r)^j y_j;\quad (k\in\mathbb{N}).$$

Then, we have

$$\begin{aligned}
(B^{r,s}x)_n&=\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k x_k\\
&=\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\sum_{j=0}^{k}\binom{k}{j}(-s)^{k-j}(s+r)^j y_j\\
&=\frac{1}{(s+r)^n}\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j}s^{n-k}(-s)^{k-j}(s+r)^j\right)y_j\\
&=\frac{1}{(s+r)^n}\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\binom{n}{j}\binom{n-j}{k-j}(-1)^{k-j}s^{n-j}(s+r)^j\right)y_j\\
&=\frac{1}{(s+r)^n}\sum_{j=0}^{n}\binom{n}{j}s^{n-j}(s+r)^j\left(\sum_{k=j}^{n}\binom{n-j}{k-j}(-1)^{k-j}\right)y_j\\
&=\frac{1}{(s+r)^n}\sum_{j=0}^{n}\binom{n}{j}s^{n-j}(s+r)^j\delta_{nk}y_j\\
&=\frac{1}{(s+r)^n}\binom{n}{n}s^{n-n}(s+r)^n 1y_n\\
&=y_n.
\end{aligned}$$

Thus, we have that $x\in b_0^{r,s}(p)$ and consequently T is surjective. Hence, T is a linear bijection and this says that the spaces $b_0^{r,s}(p)$ and $c_0(p)$ are linearly isomorphic, as was desired. □

3. The basis for the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$

Let (λ,g) be a paranormed space. Recall that a sequence (β_k) of the elements of λ is called a basis for λ if and only if, for each $x\in\lambda$, there exists a unique sequence (α_k) of scalars such that

$$g\left(x-\sum_{k=0}^{n}\alpha_k\beta_k\right)\to 0 \text{ as } n\to\infty.$$

The series $\sum\alpha_k\beta_k$ which has the sum x is then called the expansion of x with respect to (β_n), and written as $x=\sum\alpha_k\beta_k$. Since it is known that the matrix domain λ_A of a sequence space λ has a basis if and only if λ has a basis whenever $A=(a_{nk})$ is a triangle (cf. [11, Remark 2.4]), we have the following. Because of the isomorphism T is onto, defined in the proof of Theorem 2.2, the inverse image of the basis of those spaces $c_0(p)$, $c(p)$ and $\ell(p)$ are the basis of the new spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$, respectively. Therefore, we have the following:

Theorem 3.1. *Let $\lambda_k=(B^{r,s}x)_k$ for all $k\in\mathbb{N}$ and $0<p_k\le H<\infty$. Define the sequence $b^{(k)}=\{b^{(k)}\}_{k\in\mathbb{N}}$ of the elements of the space $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$ by*

$$b_n^{(k)}=\begin{cases}\frac{1}{r^n}\binom{n}{k}(-s)^{n-k}(s+r)^k & , \quad n\ge k\\ 0 & , \quad 0\le k<n\end{cases}$$

for every fixed $k\in\mathbb{N}$. Then

(a) *The sequence* $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $b_0^{r,s}(p)$*, and any* $x \in b_0^{r,s}(p)$ *has a unique representation of the form*

$$x = \sum_k \lambda_k b^{(k)}.$$

(b) *The set* $\{e, b^{(1)}(r), b^{(2)}(r), \ldots\}$ *is a basis for the space* $b_c^{r,s}(p)$*, and any* $x \in b_c^{r,s}(p)$ *has a unique representation of the form*

$$x = le + \sum_k [\lambda_k - l] b^{(k)},$$

where $l = \lim_{k\to\infty}(B^{r,s}x)_k$.

(c) *The sequence* $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $b^{r,s}(p)$*, and any* $x \in b^{r,s}(p)$ *has a unique representation of the form*

$$x = \sum_k \lambda_k b^{(k)}.$$

4. The $\alpha-, \beta-$ and $\gamma-$duals of the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$

In this section, we state and prove the theorems determining the $\alpha-, \beta-$ and $\gamma-$duals of the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$ of non-absolute type.

We shall firstly give the definition of $\alpha-, \beta-$ and $\gamma-$duals of sequence spaces and after quoting the lemmas which are needed in proving the theorems given in Section 4.
The set $S(\lambda, \mu)$ defined by

$$S(\lambda,\mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x = (x_k) \in \lambda\} \tag{4.1}$$

is called the multiplier space of the sequence spaces λ and μ. One can easily observe for a sequence space ν with $\lambda \supset \nu \supset \mu$ that the inclusions

$$S(\lambda,\mu) \subset S(\nu,\mu) \text{ and } S(\lambda,\mu) \subset S(\lambda,\nu)$$

hold. With the notation of (4.1), the alpha-, beta- and gamma-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \lambda^\beta = S(\lambda, cs) \text{ and } \lambda^\gamma = S(\lambda, bs).$$

The alpha-, beta- and gamma-duals of a sequence space are also referred as Köthe- Toeplitz dual, generalized Köthe-Toeplitz dual and Garling dual of a sequence space, respectively.

For to give the alpha-, beta- and gamma-duals of the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$ of non-absolute type, we need the following lemma:

Lemma 4.1. *[10, $q_n = 1$] Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold*

(i) $A \in (c_o(p) : \ell(q))$ *if and only if*

$$\sup_{K\in\mathscr{F}} \sum_n \left| \sum_{k\in K} a_{nk} M^{-1/p_k} \right| < \infty, \quad \exists M \in \mathbb{N}_2. \tag{4.2}$$

(ii) $A \in (c(p) : \ell(q))$ *if and only if (4.2) holds and*

$$\sum_n \left| \sum_k a_{nk} \right| < \infty. \tag{4.3}$$

(iii) $A \in (c_0(p) : c(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| M^{-1/p_k} < \infty, \ \exists M \in \mathbb{N}_2, \tag{4.4}$$

$$\exists (\alpha_k) \subset \mathbb{R} \ni \lim_{n\to\infty} |a_{nk} - \alpha_k| = 0 \textit{ for all } k \in \mathbb{N}, \tag{4.5}$$

$$\exists (\alpha_k) \subset \mathbb{R} \ni \sup_{n\in\mathbb{N}} \sum_k |a_{nk} - \alpha_k| M^{-1/p_k} < \infty, \ \exists M \in \mathbb{N}_2. \tag{4.6}$$

(iv) $A \in (c(p) : c(q))$ *if and only if (4.4), (4.5), (4.6) hold and*

$$\exists \alpha \in \mathbb{R} \ni \lim_{n\to\infty} \left| \sum_k a_{nk} - \alpha \right| = 0. \tag{4.7}$$

(v) *$A \in (c_o(p) : \ell_\infty(q))$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| M^{-1/p_k} < \infty, \ \exists M \in \mathbb{N}_2. \tag{4.8}$$

(vi) *$A \in (c(p) : \ell_\infty(q))$ if and only if (4.8) holds and*

$$\sup_n \left| \sum_k a_{nk} \right| < \infty, \ \exists M \in \mathbb{N}_2. \tag{4.9}$$

(vii) *$A \in (\ell(p) : \ell_1)$ if and only if*

(a) *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then*

$$\sup_{N\in\mathscr{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in N} a_{nk} \right|^{p_k} < \infty. \tag{4.10}$$

(b) *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, there exists an integer $M > 1$ such that*

$$\sup_{N\in\mathscr{F}} \sum_k \left| \sum_{n\in N} a_{nk} M^{-1} \right|^{p_k'} < \infty. \tag{4.11}$$

Lemma 4.2. *[17] Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold*

(i) *$A \in (\ell(p) : \ell_\infty)$ if and only if*

(a) *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then,*

$$\sup_{n,k\in\mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{4.12}$$

(b) *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, there exists an integer $M > 1$ such that*

$$\sup_{n\in\mathbb{N}} \sum_k \left| a_{nk} M^{-1} \right|^{p_k'} < \infty. \tag{4.13}$$

(ii) *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(p) : c)$ if and only if (4.12) and (4.13) hold, and*

$$\lim_{n\to\infty} a_{nk} = \beta_k, \ \forall k \in \mathbb{N}. \tag{4.14}$$

Theorem 4.3. *Let $K \in \mathscr{F}$ and $K^* = \{k \in \mathbb{N} : n \geq k\} \cap K$ for $K \in \mathscr{F}$. Define the sets $T_1^r(p)$, T_2^r, $T_3(p)$ and $T_4(p)$ as follows:*

$$T_1(p) = \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{K\in\mathscr{F}} \sum_n \left| \sum_{k\in K^*} c_{nk} M^{-1/p_k} \right| < \infty \right\},$$

$$T_2 = \left\{ a = (a_k) \in w : \sum_n \left| \sum_{k=0}^{n} c_{nk} \right| \text{ exists for each } n \in \mathbb{N} \right\},$$

$$T_3(p) = \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{N\in\mathscr{F}} \sum_k \left| \sum_{n\in N} c_{nk} M^{-1} \right|^{p_k'} < \infty, \right\},$$

$$T_4(p) = \left\{ a = (a_k) \in w : \sup_{N\in\mathscr{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in N} c_{nk} \right|^{p_k} < \infty \right\},$$

where the matrix $C = (c_{nk})$ defined by

$$c_{nk} = \begin{cases} \frac{1}{r^n} \sum_{k=0}^{n} \binom{n}{k} (-s)^{n-k} (s+r)^k a_n & , \quad 0 \leq k \leq n, \\ 0 & , \quad k \geq n. \end{cases} \tag{4.15}$$

Then, $[b_0^{r,s}(p)]^\alpha = T_1(p)$, $[b_c^{r,s}(p)]^\alpha = T_1(p) \cap T_2$ and

$$[b^{r,s}(p)]^\alpha = \begin{cases} T_3(p) & 1 < p_k \leq H < \infty, \forall k \in \mathbb{N}, \\ T_4(p) & 0 < p_k \leq 1, \forall k \in \mathbb{N}. \end{cases} \tag{4.16}$$

Proof. We chose the sequence $a = (a_k) \in w$. We can easily derive that with the (2.2) that

$$a_n x_n = \frac{1}{r^n} \sum_{k=0}^{n} \binom{n}{k} (-s)^{n-k} (s+r)^k a_n y_k = (Cy)_n, \quad (n \in \mathbb{N}) \tag{4.17}$$

for all $k, n \in \mathbb{N}$, where $C = (c_{nk})$ defined by (4.15). It follows from (4.17) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in b_0^{r,s}(p)$ if and only if $Cy \in \ell_1$ whenever $y \in c_0(p)$. This means that $a = (a_n) \in [b_0^{r,s}(p)]^\alpha$ if and only if $C \in (c_0(p) : \ell_1)$. Then, we derive by (4.2) with $q_n = 1$ for all $n \in \mathbb{N}$ that $[b_0^{r,s}(p)]^\alpha = T_1^r(p)$.
Using the (4.3) with $q_n = 1$ for all $n \in \mathbb{N}$ and (4.17), the proof of the $[b_c^{r,s}(p)]^\alpha = T_1^r(p) \cap T_2$ can also be obtained in a similar way. Also, using the (4.10),(4.11) and (4.17), the proof of the

$$[b^{r,s}(p)]^\alpha = \begin{cases} T_3(p) & 1 < p_k \leq H < \infty, \forall k \in \mathbb{N}, \\ T_4(p) & 0 < p_k \leq 1, \forall k \in \mathbb{N}, \end{cases}$$

can also be obtained in a similar way. □

Theorem 4.4. *The matrix $D=(d_{nk})$ is defined by*

$$d_{nk}=\begin{cases}\sum_{j=k}^{n}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^{k}a_j & , \quad (0\leq k\leq n)\\ 0 & , \quad (k>n)\end{cases} \tag{4.18}$$

for all $k,n\in\mathbb{N}$. Define the sets $T_5(p)$, T_6, $T_7(p)$, T_8, $T_9(p)$, T_{10} and $T_{11}(p)$ as follows:

$$T_5(p)=\bigcup_{M>1}\left\{a=(a_k)\in w:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}|d_{nk}|M^{-1/p_k}<\infty\right\},$$

$$T_6=\left\{a=(a_k)\in w:\lim_{n\to\infty}|d_{nk}| \text{ exists for each } k\in\mathbb{N}\right\},$$

$$T_7(p)=\bigcup_{M>1}\left\{a=(a_k)\in w:\exists(\alpha_k)\subset\mathbb{R}\ni\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}|d_{nk}-\alpha_k|M^{-1/p_k}<\infty\right\},$$

$$T_8=\left\{a=(a_k)\in w:\lim_{n\to\infty}\sum_{k=0}^{n}|d_{nk}| \text{ exists}\right\},$$

$$T_9(p)=\bigcup_{M>1}\left\{a=(a_k)\in w:\sup_{n\in\mathbb{N}}\sum_{k}\left|d_{nk}M^{-1}\right|^{p_k'}<\infty\right\},$$

$$T_{10}=\left\{a=(a_k)\in w:\lim_{n\to\infty}d_{nk} \text{ exists for each } k\in\mathbb{N}\right\},$$

$$T_{11}(p)=\left\{a=(a_k)\in w:\sup_{n,k\in\mathbb{N}}|d_{nk}|^{p_k}<\infty\right\}.$$

Then, $[b_0^{r,s}(p)]^{\beta}=T_5(p)\cap T_6\cap T_7(p)$, $[b_c^{r,s}(p)]^{\beta}=[b_0^{r,s}(p)]^{\beta}\cap T_8$ and

$$[b^{r,s}(p)]^{\beta}=\begin{cases}T_9(p)\cap T_{10} & , \quad 1<p_k\leq H<\infty,\forall k\in\mathbb{N},\\ T_{10}\cap T_{11}(p) & , \quad 0<p_k\leq 1,\forall k\in\mathbb{N}.\end{cases} \tag{4.19}$$

Proof. We give the proof again only for the space $b_0^{r,s}(p)$. Consider the equation

$$\begin{aligned}\sum_{k=0}^{n}a_kx_k&=\sum_{k=0}^{n}\left[\frac{1}{r^k}\sum_{j=0}^{k}\binom{k}{j}(-s)^{k-j}(s+r)^{j}y_j\right]a_k\\ &=\sum_{k=0}^{n}\left[\sum_{j=k}^{n}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^{k}a_j\right]y_k=(Dy)_n,\end{aligned} \tag{4.20}$$

where $D=(d_{nk})$ defined by (4.18). Thus, we deduce from (4.20) that $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in b_0^{r,s}(p)$ if and only if $Dy\in c$ whenever $y=(y_k)\in c_0(p)$. That is to say that $a=(a_k)\in[b_0^{r,s}(p)]^{\beta}$ if and only if $D\in(c_0(p):c)$. Therefore, we derive from (4.4),(4.5) and (4.6) with $q_n=1$ for all $n\in\mathbb{N}$ that $[b_0^{r,s}(p)]^{\beta}=T_5(p)\cap T_6\cap T_7(p)$.
Using the (4.4),(4.5), (4.6) and (4.7) with $q_n=1$ for all $n\in\mathbb{N}$ and (4.20), the proofs of the $[b_c^{r,s}(p)]^{\beta}=[b_0^{r,s}(p)]^{\beta}\cap T_8$ can also be obtained in a similar way. Also, using the (4.12),(4.13), (4.14) and (4.20), the proofs of the

$$[b^{r,s}(p)]^{\beta}=\begin{cases}T_9(p)\cap T_{10} & , \quad 1<p_k\leq H<\infty,\forall k\in\mathbb{N},\\ T_{10}\cap T_{11}(p) & , \quad 0<p_k\leq 1,\forall k\in\mathbb{N}\end{cases}$$

can also be obtained in a similar way. □

Theorem 4.5. *Define the set T_{12} by*

$$T_{12}=\left\{a=(a_k)\in w:\sup_{n}\left|\sum_{k}a_{nk}\right|<\infty\right\}.$$

Then, $[b_0^{r,s}(p)]^{\gamma}=T_5(p)$, $[b_c^{r,s}(p)]^{\gamma}=[b_0^{r,s}(p)]^{\gamma}\cap T_{12}$ and

$$[b^{r,s}(p)]^{\gamma}=\begin{cases}T_8(p) & , \quad 1<p_k\leq H<\infty,\forall k\in\mathbb{N},\\ T_{10}(p) & , \quad 0<p_k\leq 1,\forall k\in\mathbb{N}.\end{cases}$$

Proof. This is obtained in the similar way used in the proof of Theorem 4.4. □

5. Certain matrix mappings on the sequence spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$

In this section, we characterize some matrix mappings on the spaces $b_0^{r,s}(p)$, $b_c^{r,s}(p)$ and $b^{r,s}(p)$.
We known that, if $b_0^{r,s}(p) \cong c_0(p)$, $b_c^{r,s}(p) \cong c(p)$ and $b^{r,s}(p) \cong \ell(p)$, we can say: The equivalence "$x \in b_0^{r,s}(p), b_c^{r,s}(p)$ or $b^{r,s}(p)$ if and only if $y \in c_0(p), c(p)$ or $\ell(p)$" holds.
In what follows, for brevity, we write,

$$\tilde{a}_{nk} := \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_{nj}$$

for all $k, n \in \mathbb{N}$.

Theorem 5.1. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation*

$$e_{nk} := \tilde{a}_{nk} \tag{5.1}$$

for all $k, n \in \mathbb{N}$ and μ be any given sequence space. Then,

(i) *$A \in (b_0^{r,s}(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_0^{r,s}(p)\}^{\beta}$ for all $n \in \mathbb{N}$ and $E \in (c_0(p) : \mu)$.*
(ii) *$A \in (b_c^{r,s}(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_c^{r,s}(p)\}^{\beta}$ for all $n \in \mathbb{N}$ and $E \in (c(p) : \mu)$.*
(iii) *$A \in (b^{r,s}(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b^{r,s}(p)\}^{\beta}$ for all $n \in \mathbb{N}$ and $E \in (\ell(p) : \mu)$.*

Proof. We prove only part of (i). Let μ be any given sequence space. Suppose that (5.1) holds between $A = (a_{nk})$ and $E = (e_{nk})$, and take into account that the spaces $b_0^{r,s}(p)$ and $c_0(p)$ are linearly isomorphic.
Let $A \in (b_0^{r,s}(p) : \mu)$ and take any $y = (y_k) \in c_0(p)$. Then $EB^{r,s}$ exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in T_5(p) \cap T_6$ which yields that $\{e_{nk}\}_{k\in\mathbb{N}} \in c_0(p)$ for each $n \in \mathbb{N}$. Hence, Ey exists and thus

$$\sum_k e_{nk} y_k = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$.
We have that $Ey = Ax$ which leads us to the consequence $E \in (c_0(p) : \mu)$.
Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_0^{r,s}(p)\}^{\beta}$ for each $n \in \mathbb{N}$ and $E \in (c_0(p) : \mu)$ hold, and take any $x = (x_k) \in b_0^{r,s}(p)$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} \binom{j}{k} (-r)^{j-k} (1-r)^{-(j+1)} a_{nj} \right] y_k$$

for all $n \in \mathbb{N}$, that $Ey = Ax$ and this shows that $A \in (b_0^{r,s}(p) : \mu)$. This completes the proof of part of (i). □

Theorem 5.2. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} := \frac{1}{(s+r)^n} \sum_{j=0}^{n} \binom{n}{j} s^{n-j} r^j a_{jk} \text{ for all } k, n \in \mathbb{N}. \tag{5.2}$$

Let μ be any given sequence space. Then,

(i) *$A \in (\mu : b_0^{r,s}(p))$ if and only if $B \in (\mu : c_0(p))$.*
(ii) *$A \in (\mu : b_c^{r,s}(p))$ if and only if $B \in (\mu : c(p))$.*
(iii) *$A \in (\mu : b^{r,s}(p))$ if and only if $B \in (\mu : \ell(p))$.*

Proof. We prove only part of (i). Let $z = (z_k) \in \mu$ and consider the following equality.

$$\sum_{k=0}^{m} b_{nk} z_k = \sum_{j=n}^{\infty} \binom{j}{n} (1-r)^{n+1} r^{j-n} \left(\sum_{k=0}^{m} a_{jk} z_k \right) \quad \text{for all } m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $(Bz)_n = \{B^{r,s}(Az)\}_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Az \in b_0^{r,s}(p)$ whenever $z \in \mu$ if and only if $Bz \in c_0(p)$ whenever $z \in \mu$. This completes the proof of part of (i). □

Of course, Theorems 5.1 and 5.2 have several consequences depending on the choice of the sequence space μ. Whence by Theorem 5.1 and Theorem 5.2, the necessary and sufficient conditions for $(b_0^{r,s}(p) : \mu)$, $(\mu : b_0^{r,s}(p))$, $(b_c^{r,s}(p) : \mu)$, $(\mu : b_c^{r,s}(p))$ and $(b^{r,s}(p) : \mu)$, $(\mu : b^{r,s}(p))$ may be derived by replacing the entries of C and A by those of the entries of $E = C\{B^{r,s}\}^{-1}$ and $B = B^{r,s}A$, respectively; where the necessary and sufficient conditions on the matrices E and B are read from the concerning results in the existing literature.
The necessary and sufficient conditions characterizing the matrix mappings between the sequence spaces of Maddox are determined by Grosse-Erdmann [10]. Let N and K denote the finite subset of $\mathbb{N}$, L and M also denote the natural numbers. Prior to giving the theorems, let us suppose that (q_n) is a non-decreasing bounded sequence of positive numbers and consider the following conditions:

$$\lim_n |a_{nk}|^{q_n} = 0, \text{ for all } k. \tag{5.3}$$

$$\forall L, \exists M \ni \sup_n L^{1/q_n} \sum_k |a_{nk}| M^{-1/p_k} < \infty, \tag{5.4}$$

$$\lim_n |\sum_k a_{nk}|^{q_n} = 0, \tag{5.5}$$

$$\forall L, \sup_n \sup_{k \in K_1} \left| a_{nk} L^{1/q_n} \right|^{p_k} < \infty, \tag{5.6}$$

$$\forall L, \exists M \ni \sup_n \sum_{k \in K_2} |a_{nk} L^{1/q_n} M^{-1}|^{p'_k} < \infty, \tag{5.7}$$

$$\forall M, \lim_n \left(\sum_k |a_{nk}| M^{1/p_k} \right)^{q_n} = 0, \tag{5.8}$$

$$\forall M, \sup_n \sum_k |a_{nk}| M^{1/p_k} < \infty, \tag{5.9}$$

$$\forall M, \sup_K \sum_n \left| \sum_{k \in K} a_{nk} M^{1/p_k} \right|^{q_n} < \infty. \tag{5.10}$$

Lemma 5.3. *Let $A = (a_{nk})$ be an infinite matrix. Then*

(i) $A = (a_{nk}) \in (c_0(p) : \ell_\infty(q))$ *if and only if (4.8) holds.*
(ii) $A = (a_{nk}) \in (c(p) : \ell_\infty(q))$ *if and only if (4.8) and (4.9) hold.*
(iii) $A = (a_{nk}) \in (\ell(p) : \ell_\infty)$ *if and only if (4.12) and (4.13) hold.*
(iv) $A = (a_{nk}) \in (c_0(p) : c(q))$ *if and only if (4.4), (4.5) and (4.6) hold.*
(v) $A = (a_{nk}) \in (c(p) : c(q))$ *if and only if (4.4), (4.5), (4.6) and (4.7) hold.*
(vi) $A = (a_{nk}) \in (\ell(p) : c)$ *if and only if (4.12), (4.13) and (4.14) hold.*
(vii) $A = (a_{nk}) \in (c_0(p) : c_0(q))$ *if and only if (5.3) and (5.4) hold.*
(viii) $A = (a_{nk}) \in (c(p) : c_0(q))$ *if and only if (5.3), (5.4) and (5.5) hold.*
(ix) $A = (a_{nk}) \in (\ell(p) : c_0(q))$ *if and only if (5.3), (5.6) and (5.7) hold.*
(x) $A = (a_{nk}) \in (\ell_\infty(p) : c_0(q))$ *if and only if (5.8) holds.*
(xi) $A = (a_{nk}) \in (\ell_\infty(p) : c(q))$ *if and only if (5.9) holds.*
(xii) $A = (a_{nk}) \in (\ell_\infty(p) : \ell(q))$ *if and only if (5.10) holds.*
(xiii) $A = (a_{nk}) \in (c_0(p) : \ell(q))$ *if and only if (4.2) holds.*
(xiv) $A = (a_{nk}) \in (c(p) : \ell(q))$ *if and only if (4.2) and (4.4) hold.*

Corollary 5.4. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i) $A \in (b_0^{r,s}(p) : \ell_\infty(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_0^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.8) holds with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.
(ii) $A \in (b_0^{r,s}(p) : c_0(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_0^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (5.3) and (5.4) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (b_0^{r,s}(p) : c(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_0^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.4), (4.5) and (4.6) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.

Corollary 5.5. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i) $A \in (b_c^{r,s}(p) : \ell_\infty(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_c^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.8) and (4.9) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.
(ii) $A \in (b_c^{r,s}(p) : c_0(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_c^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (5.3), (5.4) and (5.5) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (b_c^{r,s}(p) : c(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_c^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.4), (4.5), (4.6) and (4.7) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.

Corollary 5.6. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i) $A \in (b^{r,s}(p) : \ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.12) and (4.13) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.
(ii) $A \in (b^{r,s}(p) : c_0(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (5.3), (5.6) and (5.7) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (b^{r,s}(p) : c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b^{r,s}(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.12), (4.13) and (4.14) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

Corollary 5.7. *Let $A = (a_{nk})$ be an infinite matrix and b_{nk} be defined by (5.2). Then, following statements hold:*

(i) $A \in (\ell_\infty(q) : b_0^{r,s}(p))$ *if and only if (5.8) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(ii) $A \in (c_0(q) : b_0^{r,s}(p))$ *if and only if (5.3) and (5.4) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (c(q) : b_0^{r,s}(p))$ *if and only if (5.3), (5.4) and (5.5) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

Corollary 5.8. *Let $A = (a_{nk})$ be an infinite matrix and b_{nk} be defined by (5.2). Then, following statements hold:*

(i) $A \in (\ell_\infty(q) : b_c^{r,s}(p))$ *if and only if (5.9) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(ii) $A \in (c_0(q) : b_c^{r,s}(p))$ *if and only if (4.4), (4.5) and (4.6) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (c(q) : b_c^{r,s}(p))$ *if and only if (4.4), (4.5), (4.6) and (4.7) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

Corollary 5.9. *Let $A = (a_{nk})$ be an infinite matrix and b_{nk} be defined by (5.2). Then, following statements hold:*

(i) $A \in (\ell_\infty(q) : b^{r,s}(p))$ *if and only if (5.10) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(ii) $A \in (c_0(q) : b^{r,s}(p))$ *if and only if (4.2) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.
(iii) $A \in (c(q) : b^{r,s}(p))$ *if and only if (4.2) and (4.4) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

References

[1] B. Altay, F. Başar, *On the paranormed Riesz sequence spaces of non-absolute type*, Southeast Asian Bull. Math., 26, 701-715 (2002).
[2] B. Altay, F. Başar, *Some paranormed Riesz sequence spaces of non-absolute type*, Southeast Asian Bull. Math., 30, 591-608 (2006).
[3] F. Başar, B. Altay, *Matrix mappings on the space $bs(p)$ and its $\alpha-,\beta-$ and $\gamma-$duals*, Aligarh Bull. Math., 21(1), 79-91 (2002).
[4] F. Başar, *Infinite matrices and almost boundedness*, Boll. Un. Mat. Ital., 6(7), 395-402 (1992).
[5] M. C. Bişgin, *The binomial sequence spaces of nonabsolute type*, J. Inequal. Appl. 309 (2016).
[6] M. C. Bişgin, *The binomial sequence spaces which include the spaces ℓ_p and ℓ_∞ and geometric properties*, J. Inequal. Appl. 304 (2016).
[7] B. Choudhary, S. K. Mishra, *On Köthe-Toeplitz duals of certain sequence spaces and their matrix transformations*, Indian J. Pure Appl. Math., 24(5), 291-301 (1993).
[8] S. Demiriz, C. Çakan, *On Some New Paranormed Euler Sequence Spaces and Euler Core*, Acta Math. Sin.(Eng. Ser.), 26(7), 1207-1222 (2010).
[9] S. Demiriz, H. B. Ellidokuzoğlu, *On The Paranormed Taylor Sequence Spaces*, Konuralp Journal Of Mathematics, 4(2), 132-148 (2016).
[10] K. G. Grosse-Erdmann, *Matrix transformations between the sequence spaces of Maddox*. J. Math. Anal. Appl., 180, 223-238 (1993).
[11] A. Jarrah and E. Malkowsky, *BK spaces, bases and linear operators*, Rend. Circ. Mat. Palermo, 52(2), 177-191 (1990).
[12] E.E. Kara and M. İlkhan, *On some Banach sequence spaces derived by a new band matrix*, Br. J. Math. Comput. Sci., 9(2), 141-159 (2015).
[13] E.E. Kara and M. İlkhan, *Some properties of generalized Fibonacci sequence spaces*, Linear Multilinear Algebra, 64(11), 2208-2223 (2016).
[14] M. Kirişci, *On the Taylor sequence spaces of nonabsulate type which include the spaces c_0 and c*, J. Math. Anal., 6(2), 22-35 (2015).
[15] M. Kirişci, *The application domain of infinite matrices with algorithms*, Univ. J. Math. Appl., 1(1), 1-9 (2018).
[16] M. Candan and A. Güneş, *Paranormed sequence space of non-absolute type founded using generalized difference matrix*, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci. 85(2), 269-276 (2015).
[17] C. G. Lascarides and I. J. Maddox, *Matrix transformations between some classes of sequences*, Proc.Camb. Phil. Soc., 68, 99-104 (1970).
[18] I.J. Maddox, Elements of Functional Analysis, second ed., The University Press, Cambridge, 1988.
[19] I. J. Maddox, *Paranormed sequence spaces generated by infinite matrices*, Proc. Camb. Phios. Soc., 64, 335-340 (1968).
[20] H. Nakano, *Modulared sequence spaces*, Proc. Jpn. Acad., 27(2), 508-512 (1951).
[21] S. Simons, *The sequence spaces $\ell(p_v)$ and $m(p_v)$*. Proc. London Math. Soc., 15(3), 422-436 (1965).

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Explicit limit cycles of a class of Kolmogorov system

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Abstract

A class of Kolmogorov differential system is introduced. It is shown that under suitable assumptions on degrees and parameters, algebraic limit cycles can occur. we propose an easy algorithm to test the existence of limit cycles and we give them explicit expressions.

1. Introduction

The second part of sixteenth problem of Hilbert rases the question of the maximum number and the mutual position of limit cycles of the differential system:

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P(x,y) \\ \dot{y} = \dfrac{dy}{dt} = Q(x,y) \end{cases} \tag{1.1}$$

where P and Q are two polynomials of any degree.
Many mathematical models in population dynamics, frequently involve the systems of ordinary differential equations having the form

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = xF(x,y), \\ \dot{y} = \dfrac{dy}{dt} = yG(x,y), \end{cases} \tag{1.2}$$

$x(t)$ and $y(t)$ represent the population density of two species at time t, and $F(x,y)$, $G(x,y)$ are the capita growth rate of each specie, usually, such systems are called Kolmogorov systems.
Kolmogorov models are widely used in ecology to describe the interaction between two populations, and a limit cycle corresponds to an equilibrium state of the system.
In mathematical modeling of ecological systems and population dynamics, more mathematicians and scientists were attracted to the subject and several results have been published, May[15], Kuang and Freedman[12] , X. Huang, Y. Wang, A.Cheng [11], and others.
When $F(x,y)$ and $G(x,y)$ are polynomials of degrees ≥ 2, limit cycles can occur and there is an extensive literature dealing with their existence, number and stability (see for instance Lloyd, Pearson, Sáez and Szántó[13], X.C. Huang, L. Zhu [9], X.C.Huang[10], S.Boqian and L.Demeng [5], K.S. Cheng [6],...), but to our knowledge, the exact analytic expressions of the limit cycles for a given kolmogorov system is still unknown except for simple and specific cases. This paper is a contribution in this direction, to determine the number of limit cycles and to give their explicit form.
Motivated by the recent publication of some research papers exhibiting planar polynomial systems with one or more algebraic limit cycles analytically given (see for instance A. Bendjeddou and R. Cheurfa[1], [2], S.Benyoucef, A, Barbach and A.Bendjeddou, [3], S. Benyoucef

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and A.Bendjeddou[4], S.Chengbin, S.Boqian[7], Lloyd, Pearson, Sáez and Szántó[14], Peng Yue-hui[17]), we will study the existence and the number of limit cycles of a class of Kolmogorov system, and give their explicit form.
Mainly based on the work of Article[4], we will enter some disruption on the differential system, adding other terms and changing the parameters.

2. Some useful notions

Let recall some useful notions.
For $U \in \mathbb{R}[x,y]$, the algebraic curve $U=0$ is called an invariant curve of the polynomial system (1.2), if for some polynomial $K \in \mathbb{R}[x,y]$ called the cofactor of the algebraic curve, we have

$$xF(x,y)\frac{\partial U}{\partial x}+yG(x,y)\frac{\partial U}{\partial y}=KU \tag{2.1}$$

The curve $\Gamma=\left\{(x,y)\in\mathbb{R}^2 : U(x,y)=0\right\}$ is nonsingular of system (1.2) if it is without singular points. The singular points or the equilibrium points of system (1.2) satisfy

$$\begin{cases} xF(x,y)=0 \\ yG(x,y)=0 \end{cases} \tag{2.2}$$

If the curve Γ is nonsingular of system (1.2), the equilibrium points of the system are contained either in its unbounded components or are located on the curve $K=0$.
A limit cycle $\gamma=\{(x(t),y(t)),t\in[0,T]\}$, is a $T-$periodic solution isolated with respect to all other possible periodic solutions of the system.
The $T-$periodic solution γ is an hyperbolic limit cycle if $\int_0^T div(\gamma(t))dt$ is different from zero [16].
We construct here a multi-parameter planar differential system admitting the components of curve

$$\Gamma=\left\{(x,y)\in\mathbb{R}^2 : x^{2n}+bx^n+cy^{2m}+y^m\left(d+fx^n\right)+h=0,\ (n,m)\in\mathbb{N}^*\times\mathbb{N}^*\right\} \tag{2.3}$$

as hyperbolic limit cycles if some conditions on the parameters are satisfied.
Note that our study is not restricted to the realistic quadrant $\left\{(x,y)\in\mathbb{R}^2 : x>0,y>0\right\}$, but it covers all the domain $\mathbb{R}^2$.

3. The main result

As a main result, we have the following theorem

Theorem 3.1. *The system*

$$\begin{cases} \dot{x}=x\left((yP(x)+V(y))\left(ax^{2n}+bx^n+cy^{2m}+dy^m+fx^ny^m+h\right)-my^m\left(d+2cy^m+fx^n\right)\right) \\ \dot{y}=y\left((xQ(y)+W(x))\left(ax^{2n}+bx^n+cy^{2m}+dy^m+fx^ny^m+h\right)+nx^n\left(b+2ax^n+fy^m\right)\right) \end{cases} \tag{3.1}$$

where a,c are positive real, b,d are negative real, f is negative real such as $f^2<4ac$, h satisfied

$$\max\left\{\frac{b^2}{4a},\frac{d^2}{4c}\right\}<h<\frac{bdf-b^2c-ad^2}{f^2-4ac},\quad (n,m)\in\mathbb{N}^*\times\mathbb{N}^* \tag{3.2}$$

$P(x)$ and $Q(y)$ are odd polynomial functions of any degree with positive coefficients, $V(x)$ and $W(y)$ are analytic functions.
This system admits
One limit cycle when n and m are odd numbers.
Two limit cycles when one of numbers n and m is odd and the other is even.
Four limit cycles when n and m are even numbers .
The limit cycles are hyperbolics represented by the curve Γ.

Proof. We will prove that Γ is nonsingular composed of ovals and it is an invariant curve of system (3.1), and $\int_0^T div(\Gamma)dt\neq 0$. (see for instance Perko [16])
i) The curve Γ is non singular of system composed of ovals.
We recall that the curve Γ is non-singular of system (3.1) if the following system has no real solution.

$$\begin{cases} ax^{2n}+bx^n+cy^{2m}+dy^m+fx^ny^m+h=0 \\ nyx^n\left(b+2ax^n+fy^m\right)=0 \\ -mxy^m\left(d+2cy^m+fx^n\right)=0 \end{cases} \tag{3.3}$$

Note that the curve Γ does not intersect the axes, $cy^{2m}+dy^m+h\neq 0$ because $h>\frac{d^2}{4c}$, and $ax^{2n}+bx^n+h\neq 0$ because $h>\frac{b^2}{4a}$, then the possible critical points on Γ are: $A_1\left(-\sqrt[n]{\frac{2bc-df}{f^2-4ac}},-\sqrt[m]{\frac{2ad-bf}{f^2-4ac}}\right)$, $A_2\left(-\sqrt[n]{\frac{2bc-df}{f^2-4ac}},\sqrt[m]{\frac{2ad-bf}{f^2-4ac}}\right)$, $A_3\left(\sqrt[n]{\frac{2bc-df}{f^2-4ac}},-\sqrt[m]{\frac{2ad-bf}{f^2-4ac}}\right)$, $A_4\left(\sqrt[n]{\frac{2bc-df}{f^2-4ac}},\sqrt[m]{\frac{2ad-bf}{f^2-4ac}}\right)$ when m and n are even numbers.
A_4 when m and n are odd numbers.
A_2,A_4 when m is odd and n is even.

A_3, A_4 when m is even and n is odd.
As $h \neq \frac{1}{f^2-4ac}\left(bdf - ad^2 - b^2c\right)$ and $f^2 \neq 4ac$, then all the points A_1, A_2, A_3, A_4 are not on Γ, and the curve Γ is nonsingular.
Now, we prove that Γ is composed by ovals.
We consider the equation (2.3) as

$$cz^2 + (d + fx^n)z + \left(ax^{2n} + bx^n + h\right) = 0 \text{ where } z = y^m \tag{3.4}$$

The discriminant $\Delta = \left(f^2 - 4ac\right)s^2 + 2(df - 2bc)s + d^2 - 4ch$ where $s = x^n$
$\Delta'_\Delta = 4b^2c^2 - 4bcdf - 16ahc^2 + 4acd^2 + 4hcf^2$
We note that if $h > \frac{b^2c^2 - bdf + ad^2}{4ac - f^2}$, $\Delta'_\Delta < 0$, and as $f^2 < 4ac$ then $\Delta < 0$, and there is no real solution of equation (2.3).
If $h = \frac{b^2c^2 - bdf + ad^2}{4ac - f^2}$, $\Delta'_\Delta = 0$, and $\Delta = \left(f^2 - 4ac\right)\left(s - \frac{2bc - df}{f^2 - 4ac}\right)^2 < 0$, with $s \neq \frac{2bc - df}{f^2 - 4ac}$, and there is no real solution of equation (2.3).
If $h < \frac{b^2c^2 - bdf + ad^2}{4ac - f^2}$, $\Delta'_\Delta > 0$ and we have

$$s_1 = \frac{1}{4ac - f^2}\left(2\sqrt{b^2c^2 - 4ahc^2 + acd^2 - bcdf + hcf^2} - 2bc + df\right)$$

$$s_2 = \frac{1}{4ac - f^2}\left(-2\sqrt{b^2c^2 - 4ahc^2 + acd^2 - bcdf + hcf^2} - 2bc + df\right)$$

The expression

$$\frac{1}{4ac - f^2}\left(2\sqrt{b^2c^2 - 4ahc^2 + acd^2 - bcdf + hcf^2} - 2bc + df\right)$$

is positive,
and if $\frac{d^2}{4c} < h$, then the expression $\frac{1}{4ac - f^2}\left(-2\sqrt{b^2c^2 - 4ahc^2 + acd^2 - bcdf + hcf^2} - 2bc + df\right)$ is also positive.
We distinguish the following cases

1. n is odd number
   The equation $\Delta = 0$ admits two real solutions

$$x_1 = \sqrt[n]{\frac{1}{4ac - f^2}\left(2\sqrt{b^2c^2 - 4ahc^2 + acd^2 - bcdf + hcf^2} - 2bc + df\right)}$$

$$x_2 = \sqrt[n]{\frac{1}{4ac - f^2}\left(-2\sqrt{b^2c^2 - 4ahc^2 + acd^2 - bcdf + hcf^2} - 2bc + df\right)}$$

   For $x \in [x_1, x_2]$ the discriminant Δ is positive and the equation (2.3) admits two real solutions

$$z_1 = \frac{-(d + fx^n) - \sqrt{\left(f^2 - 4ac\right)x^{2n} + 2(df - 2bc)x^n + d^2 - 4ch}}{2c}$$

$$z_2 = \frac{-(d + fx^n) + \sqrt{\left(f^2 - 4ac\right)x^{2n} + 2(df - 2bc)x^n + d^2 - 4ch}}{2c}$$

   .
   As $c > 0$ and $h > \frac{b^2}{4a}$, z_1, z_2 are positive.
   If m is odd number, they are two real solutions of equation (2.3) that depend on y which are

$$y_1 = \sqrt[m]{\frac{-(d + fx^n) - \sqrt{\left(f^2 - 4ac\right)x^{2n} + 2(df - 2bc)x^n + d^2 - 4ch}}{2c}}$$

$$y_2 = \sqrt[m]{\frac{-(d + fx^n) + \sqrt{\left(f^2 - 4ac\right)x^{2n} + 2(df - 2bc)x^n + d^2 - 4ch}}{2c}}$$

   $x \to y_1$ is decreasing function when $x \in]x_1, x_0[$ and an increasing function when $x \in]x_0, x_2[$
   $x \to y_2$ is an increasing function when $x \in]x_1, x'_0[$ and a decreasing function when $x \in]x'_0, x_2[$

where $x_0 = \sqrt[n]{-\frac{1}{a(-4ac+f^2)}\left(f\sqrt{a^2d^2+ab^2c-4a^2ch+af^2h-abdf}-2abc+adf\right)}$

$$x_0' = \sqrt[n]{\frac{1}{a\left(-4ac+f^2\right)}\left(f\sqrt{a^2d^2+ab^2c-4a^2ch+af^2h-abdf}+2abc-adf\right)}$$

on the other hand

$$y_1(x_1) = y_2(x_1) = \sqrt[m]{\frac{1}{c\left(f^2-4ac\right)}\left(f\sqrt{b^2c^2-bcdf-4ahc^2+acd^2+hcf^2}+2acd-bcf\right)}$$

$$y_1(x_2) = y_2(x_2) = \sqrt[m]{-\frac{1}{c\left(f^2-4ac\right)}\left(f\sqrt{b^2c^2-bcdf-4ahc^2+acd^2+hcf^2}-2acd+bcf\right)}$$

then Γ is composed of an oval in the area $D_1 = \left\{(x,y)\in\mathbb{R}^2, x_1 < x < x_2,\ y_1\left(x_0\right) < y < y_2\left(x_0'\right)\right\}$.
D_1 is in the realistic quadrant.
If m is even number they are four real solutions of equation (2.3) that depend on y which are $y_1,\, y_2,\, -y_1,\, -y_2$ with the same process as before, we conclude that the curve Γ is composed of two ovals, one is in the area D_1 and the other is in the area $D_2 = \left\{(x,y)\in\mathbb{R}^2, x_1 \le x \le x_2,\ -y_2\left(x_0'\right) \le y \le -y_1\left(x_0\right)\right\}$.

2. n is even number
The equation $\Delta = 0$ admits four real solutions $x_1, x_2, -x_1, -x_2$
For $x \in [-x_2, -x_1] \cup [x_1, x_2]$ the discriminant Δ is positive and the equation (2.3) admits two real solutions z_1, z_2.
If m is odd number, they are two real solutions of equation (2.3) that depend on y wich are

$$y_1 = \sqrt[m]{\frac{-\left(d+fx^n\right)-\sqrt{\left(f^2-4ac\right)x^{2n}+2\left(df-2bc\right)x^n + d^2-4ch}}{2c}},$$

$$y_2 = \sqrt[m]{\frac{-\left(d+fx^n\right)+\sqrt{\left(f^2-4ac\right)x^{2n}+2\left(df-2bc\right)x^n + d^2-4ch}}{2c}}$$

If m is even number, they are four real solutions of equation (2.3) that depend on y

$$y_1,\, y_2,\, -y_1,\, -y_2$$

As before, we conclude that, when m is odd number, the curve Γ is composed of two ovals, one is in the area D_1, and the other is in the area $D_3 = \left\{(x,y)\in\mathbb{R}^2, -x_2 \le x \le -x_1,\ y_1\left(-x_0\right) \le y \le y_2\left(-x_0'\right)\right\}$.
When m is even number, the curve Γ is composed of four ovals in the areas D_1, D_2, D_3, D_4
$D_4 = \left\{(x,y)\in\mathbb{R}^2, -x_2 \le x \le -x_1,\ -y_2\left(-x_0'\right) \le y \le -y_1\left(-x_0\right)\right\}$
ii) Γ is an invariant curve of system (3.1)

$$\frac{\partial U}{\partial x}\dot{x}+\frac{\partial U}{\partial y}\dot{y} = \left(nx^n\left(yP(x)+V(y)\right)\left(b+2ax^n+fy^m\right)+my^m\left(xQ(y)+W(x)\right)\left(d+2cy^m+fx^n\right)\right)U \tag{3.5}$$

the cofactor is

$$K(x,y) = nx^n\left(yP(x)+V(y)\right)\left(b+2ax^n+fy^m\right)+my^m\left(xQ(y)+W(x)\right)\left(d+2cy^m+fx^n\right) \tag{3.6}$$

iii) $\int_0^T div(\Gamma)dt \neq 0$
Note that

$$\int_0^T div(\Gamma)dt = \int_0^T K(x(t),y(t))dt \tag{3.7}$$

see for instance Giacomini & Grau[8].

$$\begin{aligned}
\int_0^T K(x(t),y(t))dt &= \int_0^T nx^n\left(yP(x)+V(y)\right)\left(b+2ax^n+fy^m\right)dt \\
&+ \int_0^T my^m\left(xQ(y)+W(x)\right)\left(d+2cy^m+fx^n\right)dt \\
&= \oint_\Gamma \frac{nx^n\left(yP(x)+V(y)\right)\left(b+2ax^n+fy^m\right)}{nyx^n\left(b+2ax^n+fy^m\right)}dy - \oint_\Gamma \frac{my^m\left(xQ(y)+W(x)\right)\left(d+2cy^m+fx^n\right)}{mxy^m\left(d+2cy^m+fx^n\right)}dx \\
&= \oint_\Gamma \left(P(x)+\frac{V(y)}{y}\right)dy - \oint_\Gamma \left(Q(y)+\frac{W(x)}{x}\right)dx
\end{aligned}$$

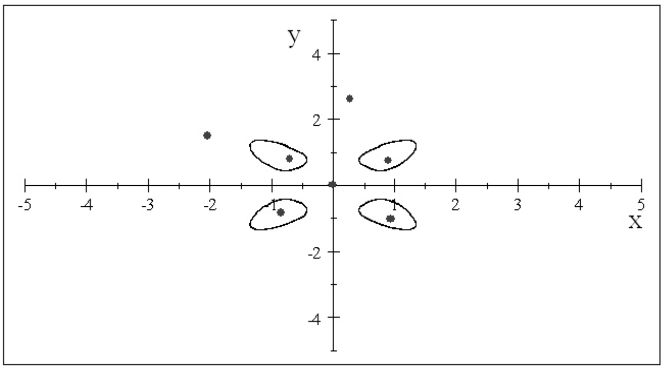

Figure 5.1: Limit cycles of system(5.1) with singular points

by applying the GREEN formula

$$\oint_{\Gamma}\left(P(x)+\frac{V(y)}{y}\right)dy-\oint_{\Gamma}\left(Q(y)+\frac{W(x)}{x}\right)dx=\int\int_{int(\Gamma)}\left(\frac{d(P(x))}{dx}+\frac{d\left(Q(y)\right)}{dy}\right)dxdy \tag{3.8}$$

where $int(\Gamma)$ denotes the interior of Γ (the region bounded by Γ).
As $P(x)$ and $Q(y)$ are odd polynomial functions with positive coefficients then $P'(x)$ and $Q'(y)$ are pair polynomial functions with all the coefficients are positive. We conclude that $\forall(x,y)\in int\ \ (\Gamma)\ ,\ (P'(x)+Q'(y))>0$
then $\int_0^T K(x(t),y(t))dt$ is nonzero.

□

Remark 3.2. *We can generalize Theorem 3.1 such $P(x)$ and $Q(y)$ are analytic functions, in this case we must add the condition $\forall(x,y)\in int\ \ (\Gamma)\,,\,P'(x)+Q'(y)$ is nonzero.*

Remark 3.3. *If the study is restricted to the realistic quadrant $\left\{(x,y)\in\mathbb{R}^2 : x>0,y>0\right\}$, then the theorem remains true with odd degrees n and m, in this case the system (3.1) admits one limit cycle in the realistic quadrant.*

4. Algorithm

* Reading data of system (3.1)
Reading degrees m and n such that $m>0$ and $n>0$.
Reading coefficients a,b,c,d,f , h such that $a>0, b<0, c>0, d<0,\ f<0\ and\ f^2<4ac$.
Enter the odd polynomial functions $P(x)$ and $Q(y)$ with positive coefficients.
Enter the analytic functions $V(x)$ and $W(y)$.
* Existence of limit cycles
If $\max\left\{\frac{b^2}{4a},\frac{d^2}{4c}\right\}<h<\frac{bdf-b^2c-ad^2}{f^2-4ac}$ then
if m and n are odd numbers there is one limit cycle
if m and n are even numbers there are four limit cycles
if one of numbers m and n is odd and the other is even there are two limit cycles
else
there is not limit cycle
endif
* Explicit form of limit cycle
when there exist limit cycle its explicit form is $x^{2n}+bx^n+cy^{2m}+y^m\left(d+fx^n\right)+h=0$

5. Examples

Example 5.1. *Let $m=n=2$, $a=2$, $b=-2$, $c=2$, $d=-2$, $f=-2$, $h=1$, $P(x)=x$, $Q(y)=y$, $V(y)=y+1$, $W(x)=x-1$*

$$\left\{\begin{array}{l}\dot{x}=x\left(\begin{array}{c}2x^5y+2x^4y+2x^4-2x^3y^3-2x^3y-2x^2y^3+2x^2y^2-2x^2y-2x^2\\+2xy^5-2xy^3+xy+2y^5-6y^4-2y^3+2y^2+y+1\end{array}\right)\\ \dot{y}=y\left(\begin{array}{c}2x^5y+2x^5+6x^4-2x^3y^3-2x^3y^2-2x^3y-2x^3-2x^2y^2-2x^2+\\2xy^5+2xy^4-2xy^3-2xy^2+xy+x-2y^4+2y^2-1\end{array}\right)\end{array}\right. \tag{5.1}$$

The system (5.1) admits four limit cycles represented by the curve $2x^4-2x^2+2y^4+y^2\left(-2-2x^2\right)+1=0$, and it has seven singular points, three are saddle points, two are stable focus and two are instable focus. The limit cycles in the first and third quadrant around stable focus and limit cycles in the second and fourth quadrant around unstable focus. Figure (5.1)

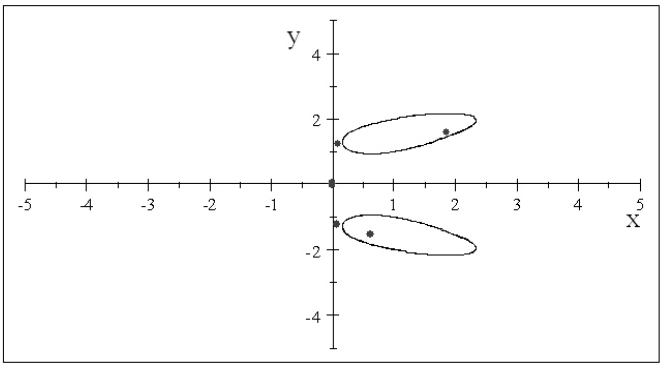

Figure 5.2: Limit cycles of system(5.2) with singular points

Example 5.2. *Let* $m=2$, $n=1$, $a=3$, $b=-2$, $c=1$, $d=-3$, $f=-2$, $h=3$, $P(x)=x^3$, $Q(y)=y^3$, $V(y)=-1$, $W(x)=1$

$$\left\{\begin{array}{l} \dot{x}=x\left(3x^5y-2x^4y^3-2x^4y+x^3y^5-3x^3y^3+3x^3y-3x^2+6xy^2+2x-5y^4+9y^2-3\right) \\ \dot{y}=y\left(3x^3y^3-2x^2y^5-2x^2y^3+9x^2+xy^7-3xy^5+3xy^3-4xy^2-4x+y^4-3y^2+3\right) \end{array}\right. \tag{5.2}$$

The system (5.2) *admits two limit cycles represented by the curve* $3x^2-2x+y^4-3y^2-2xy^2+3=0$, *and it has five singular points, three are saddle points, one is a stable focus, one is an unstable focus, the limit cycle in first quadrant encloses a stable focus, and the other encloses an instable focus. Figure* (5.2)

Example 5.3. *Let* $m=n=1$, $a=2$, $b=-3$, $c=1$, $d=-3$, $f=-1$, $h=3$, $P(x)=\exp(x)$, $Q(y)=\arctan(y)$, $V(y)=y\sin(y)$, $W(x)=x\cos(x)$

$$\left\{\begin{array}{l} \dot{x}=x\left((y\exp(x)+y\sin(y))\left(2x^2-3x+y^2-3y-xy+3\right)-y(2y-x-3)\right) \\ \dot{y}=y\left((x\arctan(y)+x\cos(x))\left(2x^2-3x+y^2-3y-xy+3\right)+x(4x-y-3)\right) \end{array}\right. \tag{5.3}$$

The system (5.3) *admits one limit cycle, represented by the curve* $2x^2-3x+y^2-3y-xy+3=0$, *that enclosed a stable focus, figure* (5.3).
Note that $\frac{d(\exp(x))}{dx}+\frac{d(\arctan(y))}{dy}=e^x+\frac{1}{y^2+1}>0$
and
$\int_0^T K(x,y)dt=\int_{\frac{9}{7}-\frac{2}{7}\sqrt{15}}^{\frac{2}{7}\sqrt{15}+\frac{9}{7}}\int_{\frac{1}{2}x-\frac{1}{2}\sqrt{-7x^2+18x-3}+\frac{3}{2}}^{\frac{1}{2}x+\frac{1}{2}\sqrt{-7x^2+18x-3}+\frac{3}{2}}\left(e^x+\frac{1}{y^2+1}\right)dxdy\simeq 22.55.$

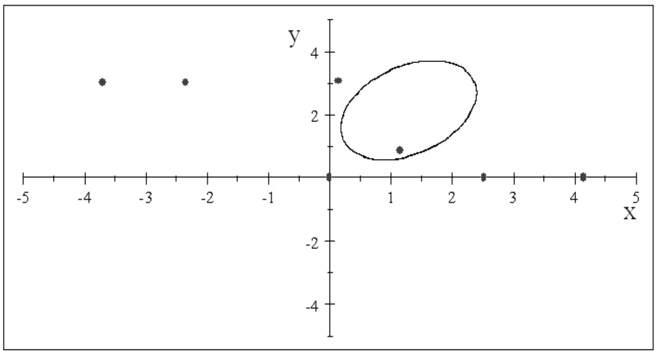

Figure 5.3: Limit cycles of system(5.3) with singular points

6. Conclusion

We proposed in this paper a class of Kolmogorov system, where just choose the parameters satisfying the conditions of Theorem 3.1, we conclude directly that the system has one, two or four limit cycles and we give them explicitly.

References

[1] A. Bendjeddou and R. Cheurfa, *On the exact limit cycle for some class of planar differential systems,* Nonlinear differ. equ. appl. 14 (2007), 491-498.

[2] A. Bendjeddou and R. Cheurfa*, Cubic and quartic planar differential systems with exact algebraic limit cycles,* Elect. J. of Diff. Equ., no15 (2011), 1-12.

[3] S. Benyoucef, A. Barbach, and A. Bendjeddou, A class of Differential system with at most four limit cycles, Annals of applied mathematics, 31, no 4, 2015, 1-9.

[4] S. Benyoucef and A. Bendjeddou, A class of Kolmogorov system with exact algebraic limit cycles, Int.J.of Pure and Applied Mathematics, V103 no 3, 2015, 439-451.

[5] Shen Boqian and Liu Demeng. Existence of limit cycles for a cubic Kolmogorov system with a hyperbolic solution. Northwest Math.16(1), 2000, 91–95.

[6] Cheng K.S, Uniqueness of a limit cycle for a predator-prey system, SIAM J. Math. Anal, 12 (4) (1981),541-548.

[7] SI Chengbin, Shen Boqian. The existence of limit cycles for the Kolmogorov cubic system with a quartic curve solution.J.Sys. Sci.& Math. Scis.28(3) (2008), 334–339.

[8] H. Giacomini, M. Grau, On the stability of limit cycles for planar differential systems, J. of Diff. Equ, v 213 issue 2, 2005, 368-388.

[9] Xun C. Huang and Lemin Zhu, *Limit cycles in a general Kolmogorov model*, Nonlin. Anal. Theo. Meth. and Appl. 60 (2005), 1393-1414.

[10] Huang X.C, Limit cycle in a Kolmogorov-type model, Internat. J. Math. & Math Sci.vol 13 no 3 (1990) 555-566.

[11] X. Huang, Y. Wang, A. Cheng, Limit cycles in a cubic predator–prey differential system, J. Korean Math. Soc. 43 no 4 (2006) 829–843.

[12] Y. Kuang and H.I Freedman, Uniqueness of limit cycles in Gause-type models of Predator-prey systems, Math. Biosci.. 88 (1988), 67-84.

[13] N. G. Lloyd, J. M. Pearson, E Sáez, I. Szántó, *Limit cycles of a Cubic Kolmogorov System,* Appl. Math. Lett. vol 9 no1, (1996) pp 15-18.

[14] N. G. Lloyd, J. M. Pearson, E. Sáez, and I. Szántó, *A cubic Kolmogorov system with six limit cycles*, International Journal Computers and Mathematics with Applications 44 (2002), 445-455.

[15] R.M May , Limit cycles in predator-prey communities, Science 177 (1972), 900-902.

[16] L. Perko, *Differential equations and dynamical systems,* Third edition. Texts in Applied Mathematics, 7. Springer-Verlag, New York, 2001.

[17] Peng Yue-hui. Limit Cycles in a Class of Kolmogorov Model with Two Positive equilibrium Points. Natural Science journal of Xiangtan University,Vol. 32 No.4 Dec.2010, 10-15.

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A note on hyperbolic quaternions

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Abstract

In this work, we introduce hyperbolic quaternions and their algebraic properties. Moreover, we express Euler's and De Moivre's formulas for hyperbolic quaternions.

1. Introduction

Real quaternions were introduced by Hamilton (1805-1865) in 1843 as he looked for ways of extending complex numbers to higher spatial dimensions. So the set of real quaternions can be represented as [1]

$$H = \{q = a_0 + a_1 i + a_2 j + a_3 k : \ a_0, a_1, a_2, a_3 \in R \ and \ i, j, k \notin R\},$$

where

$$i^2 = j^2 = k^2 = -1, \ ij = k - ji, \ jk = i = -kj, \ ki = j = -ik.$$

From these ruled it follows immediately that multiplication of real quaternions is not commutative. The roots of a real quaternions were given by Niven [7] and Brand [8] proved De Moivre's theorem and used it to find nth roots of a real quaternion. Cho [2] generalized Euler's formula and De Moivre's formula for real quaternions. Also, he showed that there are uncountably many unit quaternions satisfying $x^n = 1$ for $n \geq 3$. Using De Moivre's formula to find roots of real quaternion is more useful way.
After the discovery of real quaternions by Hamilton, MacFarlane [4] in 1981 introduced the set of hyperbolic quaternions. The hyperbolic quaternions are not commutative like real quaternions. But the set of hyperbolic quaternions contains zero divisors [6].
In this work, we express Euler and De moivre's formulas for hyperbolic quaternions after we give some algebraic properties of hyperbolic quaternions.

2. Hyperbolic Quaternions

A set of hyperbolic quaternions are denoted by

$$K = \{q = a_0 + a_1 i + a_2 j + a_3 k : \ a_0, a_1, a_2, a_3 \in R \ and \ i, j, k \notin R\} \tag{2.1}$$

where

$$i^2 = j^2 = k^2 = 1, \ ij = k - ji, \ jk = i = -kj, \ ki = j = -ik. \tag{2.2}$$

A hyperbolic quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$ is pieced into two parts with scalar piece $S(q) = a_0$ and vectorial piece $\overrightarrow{V}(q) = a_1 i + a_2 j + a_3 k$. We also write $q = S(q) + \overrightarrow{V}(q)$. Let a hyperbolic quaternion be $q_n = a_n + a_n i + a_n j + a_n k$ for $n = 0, 1$. Addition and subtraction of a hyperbolic quaternions is defined by

$$\begin{aligned} q_0 \pm q_1 &= (a_0 + b_0 i + c_0 j + d_0 k) \pm (a_1 + b_1 i + c_1 j + d_1 k) \\ &= (a_0 \pm a_1) + (b_0 \pm b_1) i + (c_0 \pm c_1) j + (d_0 \pm d_1) k. \end{aligned} \tag{2.3}$$

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Scalar multiplication of a hyperbolic quaternion is defined by

$$\lambda q=\lambda(a+bi+cj+dk)=\lambda a+\lambda bi+\lambda cj+\lambda dk \tag{2.4}$$

for any $\lambda\in R$. Then, the set K is a vector space over R.
Multiplication of hyperbolic quaternions are defined by

$$\begin{aligned} q_0q_1 &= (a_0+b_0i+c_0j+d_0k)(a_1+b_1i+c_1j+d_1k)\\ &=(a_0a_1+b_0b_1+c_0c_1+d_0d_1)+(a_0b_1+b_0a_1+c_0d_1-d_0c_1)i\\ &+(a_0c_1-b_0d_1+c_0a_1+d_0b_1)j+(a_0d_1+b_0c_1-c_0b_1+d_0a_1)k. \end{aligned} \tag{2.5}$$

Equation (2.5) can be represented by means of matrix multiplication. The representation as a 4×4 real matrix is

$$\begin{pmatrix} a_0 & b_0 & c_0 & d_0\\ b_0 & a_0 & -d_0 & c_0\\ c_0 & d_0 & a_0 & -b_0\\ d_0 & -c_0 & b_0 & a_0 \end{pmatrix} \tag{2.6}$$

which is a useful way to compute quaternion multiplication

$$\begin{pmatrix} r_0\\ r_1\\ r_2\\ r_3 \end{pmatrix}=\begin{pmatrix} a_0 & b_0 & c_0 & d_0\\ b_0 & a_0 & -d_0 & c_0\\ c_0 & d_0 & a_0 & -b_0\\ d_0 & -c_0 & b_0 & a_0 \end{pmatrix}\begin{pmatrix} a_1\\ b_1\\ c_1\\ d_1 \end{pmatrix} \tag{2.7}$$

where $pq=r_0+r_1i+r_2j+r_3k$.
Unlike the real quaternions, the hyperbolic quaternion is not associative due to $(ij)j\neq i(jj)$. Moreover, it is not commutative. The conjugate of a hyperbolic quaternion is defined by

$$\overline{q}=a-bi-cj-dk. \tag{2.8}$$

The conjugate of hyperbolic quaternions satisfies the properties $\overline{(\overline{p})}=p$ and $\overline{(pq)}=\overline{q}\,\overline{p}$ for $p,q\in K$.
The scalar and vector parts of $q\in K$ are defined

$$S(q)=\frac{1}{2}(q+\overline{q}),\quad \overrightarrow{V}(q)=\frac{1}{2}(q-\overline{q}). \tag{2.9}$$

We note that

$$\begin{aligned} S(\overline{q})&=\tfrac{1}{2}(\overline{q}+q)=S(q),\\ S(p+q)&=S(p)+S(q). \end{aligned} \tag{2.10}$$

Let p and q be hyperbolic quaternions. Then, inner product of them is defined to be the real number

$$\langle p,q\rangle=S(p\overline{q}). \tag{2.11}$$

As is easily verified, the following properties are satisfied
(I) $\langle p,q\rangle=S(p\overline{q})=S(q\overline{p})=\langle q,p\rangle$,
(II) $\langle p,q+r\rangle=S\left(p\,\overline{(q+r)}\right)=S(p\overline{q}+p\overline{r})=\langle p,q\rangle+\langle p,r\rangle$,
(III) $\alpha\langle p,q\rangle=S((\alpha p)\,\overline{q})=\langle \alpha p,q\rangle=S\left(p\overline{(\alpha q)}\right)=\langle p,\alpha q\rangle,\lambda\in R$
(IV) $\langle q,q\rangle=S(q\overline{q})=a^2-b^2-c^2-d^2$.
Thus the inner product defined here is a symmetric bilinear form but is not positive definite. The inner product defines the norm of $q=a+bi+cj+dk\in K$ as follows

$$N(q)=\langle q,q\rangle=q\overline{q}=a^2-b^2-c^2-d^2. \tag{2.12}$$

The norm is real-valued function and the norm of a hyperbolic quaternions satisfies the properties $N(\overline{q})=N(q)$. But $N(pq)\neq N(p)N(q)$.
Let $p=p^{\alpha}e_{\alpha},\ q=q^{\alpha}e_{\alpha}$ be hyperbolic quaternions where $e_{\alpha}\in\{1,i,j,k\}$. Then the relations:

$$\langle p,q\rangle=S(p\overline{q})=\eta_{\alpha\beta}p^{\alpha}q^{\beta} \tag{2.13}$$

defines metric $\eta_{\alpha\beta}$. To obtain its components explicitly, we choose $p=e_{\alpha},\ q=e_{\beta}$ (for particular α,β). Then

$$p=\delta^{\alpha\mu}e_{\mu}\ ,\ \ q=\delta^{\beta\nu}e_{\nu} \tag{2.14}$$

where $\delta^{\alpha\beta}$ is kronecker delta. That is

$$p^{\mu}=\delta^{\alpha\mu},\ q^{\nu}=\delta^{\beta\nu} \tag{2.15}$$

therefore

$$\langle p,q\rangle=\frac{1}{2}\left[e_{\alpha}\overline{e_{\beta}}+e_{\beta}\overline{e_{\alpha}}\right] \tag{2.16}$$

also

$$\langle p,q\rangle = \eta_{\mu\nu}\delta^{\alpha\mu}\delta^{\beta\nu} = \eta_{\alpha\beta} \tag{2.17}$$

and so

$$\eta_{\alpha\beta} = \frac{1}{2}\left[e_\alpha\overline{e_\beta} + e_\beta\overline{e_\alpha}\right] \tag{2.18}$$

giving, after a short calculation, we reach

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.19}$$

which is the usual flat-space metric of Minkowski space.

3. Euler's and De Moivre's Formulas for a Hyperbolic Quaternions

In this section, we express Euler's formula and De Moivre's formula for hyperbolic quaternions and examine roots of hyperbolic quaternion with respect to the norm of the hyperbolic quaternions.
Every hyperbolic quaternion $q = a + bi + cj + dk$ $(N(q) > 0)$ can be written in the form

$$q = \sqrt{N(q)}\,(\cosh\phi + w\sinh\phi) \tag{3.1}$$

where $\cosh\phi = \frac{|a|}{\sqrt{N(q)}}$, $\sinh\phi = \frac{\sqrt{b^2+c^2+d^2}}{\sqrt{N(q)}}$, $w = \frac{1}{\sqrt{b^2+c^2+d^2}}(bi + cj + dk)$ is unit hyperbolic quaternion and $w^2 = ww = 1$. Since $w^2 = 1$, we have

$$e^{w\phi} = \left(1 + \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \dots\right) + w\left(\phi + \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \dots\right) = \cosh\phi + w\sinh\phi. \tag{3.2}$$

Moreover, this can be shown using another method. In following manner

$$\begin{aligned} q = (\cosh\phi + w\sinh\phi) &\Rightarrow dq = (\sinh\phi + w\cosh\phi)\,d\phi \\ &\Rightarrow dq = w(\cosh\phi + w\sinh\phi)\,d\phi = wqd\phi. \end{aligned} \tag{3.3}$$

Thus, we get

$$\int\frac{dq}{q} = \int wd\phi \Rightarrow \ln q = w\phi \Rightarrow q = e^{w\phi} = (\cosh\phi + w\sinh\phi). \tag{3.4}$$

Now let's prove De Moivre's formula for hyperbolic quaternion.

Theorem 3.1. *Let $q = \sqrt{N(q)}\,(\cosh\phi + w\sinh\phi)$, where ϕ is a real number and $w^2 = 1$. Then*

$$q^n = \left(\sqrt{N_q}\right)^n(\cosh\phi + w\sinh\phi)^n \qquad = \left(\sqrt{N_q}\right)^n(\cosh(n\phi) + w\sinh(n\phi)) \tag{3.5}$$

for every integer n.

Proof. We use induction on positive integers n. Assume that

$$q^n = \left(\sqrt{N_q}\right)^n(\cosh(n\phi) + w\sinh(n\phi))$$

holds. Then,

$$\begin{aligned} q^{n+1} &= \left(\sqrt{|N_q|}\right)^n(\cosh(n\phi) + w\sinh(n\phi))\left(\sqrt{|N_q|}\right)(\cosh\phi + w\sinh\phi) \\ &= \left(\sqrt{|N_q|}\right)^{n+1}\begin{bmatrix} (\cosh(n\phi)\cosh\phi + \sinh(n\phi)\sinh\phi) \\ +w(\cosh(n\phi)\sinh\phi + \sinh(n\phi)\cosh\phi) \end{bmatrix} \\ &= \left(\sqrt{|N_q|}\right)^{n+1}\left[(\cosh(\phi(n+1))) + w(\sinh(\phi(n+1)))\right]. \end{aligned}$$

Hence, the formula is true. Moreover, since

$$\begin{aligned} q^{-1} &= \left(\sqrt{|N_q|}\right)^{-1}(\cosh\phi - w\sinh\phi) \\ q^{-n} &= \left(\sqrt{|N_q|}\right)^{-n}(\cosh(n\phi) - w\sinh(n\phi)) \\ &= \left(\sqrt{|N_q|}\right)^{-n}(\cosh(-n\phi) + w\sinh(-n\phi)), \end{aligned}$$

the formula holds for all integer.
If the power series definition

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \tag{3.6}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \tag{3.7}$$

is used for hyperbolic quaternion w, then we obtain

$$\cosh w = \cos I \tag{3.8}$$

$$\sinh w = -wI \sin I \tag{3.9}$$

where I is complex operator. $\square$

Every hyperbolic quaternion $q = a + bi + cj + dk$, $(N(q) < 0)$ can be written in the form

$$q = \sqrt{|N(q)|}\,(\sinh\phi + w\cosh\phi) \tag{3.10}$$

where $\sinh\phi = \frac{a}{\sqrt{|N(q)|}}$, $\cosh\phi = \frac{1}{\sqrt{|N(q)|}}\sqrt{b^2+c^2+d^2}$ and $w = \frac{1}{\sqrt{b^2+c^2+d^2}}(bi+cj+dk)$ is unit hyperbolic quaternion.

Theorem 3.2. *Let $q = \sqrt{|N(q)|}\,(\sinh\phi + w\cosh\phi)$, $(N(q) < 0)$ be hyperbolic quaternion. Then*

$$q^n = \begin{cases} \left(\sqrt{|N(q)|}\right)^n (\sinh n\phi + w\cosh n\phi), & n \text{ is odd} \\ \left(\sqrt{|N(q)|}\right)^n (\cosh n\phi + w\sinh n\phi), & n \text{ is even.} \end{cases}$$

The proof can be bypassed since it can be proved in same manner of the proof of the Theorem 3.1.

3.1. The roots of a Hyperbolic Quaternions

Theorem 3.3. *Let $p = \sqrt{N(p)}\,(\cosh\phi + w\sinh\phi)$. Then the equation $q^n = p$ has only one root:*

$$q = \sqrt[2n]{N(p)}\left(\cosh\left(\frac{\phi}{n}\right) + w\sinh\left(\frac{\phi}{n}\right)\right) \tag{3.11}$$

in the hyperbolic quaternions which $N(q) > 0$.

Proof. Assume that $q = \sqrt{N(q)}\,(\cosh x + w\sinh x)$ is a root of the equation $q^n = p$. From theorem 3.1, we have $q^n = \left(\sqrt{N(q)}\right)^n (\cosh(nx) + w\sinh(nx))$. Thus, $x = \frac{\phi}{n}$ and $|N(q)| = \sqrt[n]{|N(p)|}$. Then, $q = \sqrt[2n]{N(p)}\left(\cosh\left(\frac{\phi}{n}\right) + w\sinh\left(\frac{\phi}{n}\right)\right)$ is a root of the equation. If we suppose that there are two roots satisfying the equality, we obtain that these roots must be equal to each other. $\square$

Example 3.4. *We find the roots of the equation $q^2 = \sqrt{3} + i + j$. Here $p = \sqrt{3} + i + j$ is a hyperbolic quaternion such that $N_p = 1$. Then, p can be written as*

$$p = \cosh\left(\ln\left(\sqrt{3}+\sqrt{2}\right)\right) + w\sinh\left(\ln\left(\sqrt{3}+\sqrt{2}\right)\right)$$

where $w = \frac{1}{\sqrt{2}}(i+j)$. From theorem 3.3, the root of the equation

$$q^2 = \cosh\left(\ln\left(\sqrt{3}+\sqrt{2}\right)\right) + w\sinh\left(\ln\left(\sqrt{3}+\sqrt{2}\right)\right)$$

is as follows

$$q = \cosh\left(\frac{\ln\left(\sqrt{3}+\sqrt{2}\right)}{2}\right) + w\sinh\left(\frac{\ln\left(\sqrt{3}+\sqrt{2}\right)}{2}\right).$$

Theorem 3.5. *Let $p = \sqrt{|N(p)|}\,(\sinh\phi + w\cosh\phi)$ be hyperbolic quaternion. Then the solution of hyperbolic quaternion $q^n = p$*
1. doesn't exist if n is an even number,
2. has only one root $q = \sqrt[2n]{|N(p)|}\left(\sinh\frac{\phi}{n} + w\cosh\frac{\phi}{n}\right)$ if n is an odd number
in the hyperbolic quaternions which $N(q) < 0$,

Proof. If n is an even number, the norm of the n^{th} power of hyperbolic quaternion will be positive and in this case there is no solution. So, let $q = \sqrt{|N(q)|}\,(\sinh x + w\cosh x)$ be root of the equation $q^n = p$ such that n is an odd number. Then

$$q^n = \sqrt{|N(q)|^n}\,(\sinh nx + \cosh nx) = \sqrt{|N(p)|}\,(\sinh\phi + \cosh\phi)$$

and we get $x = \frac{\phi}{n}$ and $|N(q)| = \sqrt[n]{|N(p)|}$. $\square$

Example 3.6. *We find the roots of the equation* $q^3 = 1+\sqrt{2}j$. *Here* $p = 1+\sqrt{2}j$ *is a hyperbolic quaternion such that* $N(p) = 1$. *Then* p *can be written as*

$$p = \sinh\left(\ln\left(1+\sqrt{2}\right)\right) + w\cosh\left(\ln\left(1+\sqrt{2}\right)\right)$$

where $w = j$. *From theorem 3.4 the root of the equation*

$$q^3 = \sinh\left(\ln\left(1+\sqrt{2}\right)\right) + w\cosh\left(\ln\left(1+\sqrt{2}\right)\right)$$

is

$$q = \sinh\left(\frac{\ln\left(1+\sqrt{2}\right)}{3}\right) + w\cosh\left(\frac{\ln\left(1+\sqrt{2}\right)}{3}\right).$$

References

[1] Hamilton, W.R., Elements of Quaternions, London Longmans Green, 1866.
[2] Cho, E., De Moivre's Formula for Quaternions, Appl. Math. Lett. **6** (1998), 33-35.
[3] Özdemir, M., The Roots of a Split Quaternion, Appl. Math. Lett. **22** (2009), 258-263.
[4] MacFarlane, A., Hyperbolic Quaternions, Proc. Roy. Soc. Edinburgh, 1900, pp. 169-181.
[5] Ward, JP., Quaternions and Cayley Numbers Algebra and Applications, Boston(SPA), Kluwer Academic, 1997.
[6] Demir, S., Tanışlı, M., Candemir, N., Hyperbolic Quaternions Formulation of Electromacnetism, Adv. Appl. Clifford Algebras, **20** (2010), 547-563.
[7] Niven I., The Roots of a Quaternion, Amer. Math. Monthly 449 (6) (1942) 386-388.
[8] Brand L., The Roots of a Quaternion, Amer. Math. Monthly 49 (8) (1942) 519–520.

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Generalized Zagreb index of some dendrimer structures

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Abstract

Chemical graph theory, is a branch of mathematical chemistry which deals with the nontrivial applications of graph theory to solve molecular problem. A chemical graph is represent a molecule by considering the atoms as the vertices and bonds between them as the edges. A topological index is a graph based molecular descriptor, which is graph theoretic invariant characterising some physicochemical properties of chemical compounds. Dendrimers are generally large, complex, and hyper branched molecules synthesized by repeatable steps with nanometre scale measurements. In this paper, we study the (a,b)-Zagreb index of some regular dendrimers and hence obtain some vertex degree based topological indices.

1. Introduction

A molecule in chemical graph theory generally represented by graph $G=(V(G),E(G))$ where $V(G)$ denote the vertex set and $E(G)$ is the edge set of G, the vertices are consider as atoms of the molecule and edges are bonds between them. The degree of a vertex $v\in V(G)$ is the number of those vertices in G such that which are adjacent to v and is denoted as $d_G(v)$. A topological index of a graph is the real number obtain from that graph numerically and is same for graph isomorphism. Study of various topological indices for chemical structures of various molecules play an important role in medical and pharmaceutical fields to predicting biological activity of new molecules and drugs. Dendrimers is a type of macromolecules that could be synthesized from monomers by reproducible procedures. Generally dendrimers are large, complex and hyper branch with multiple functional groups on the surface. Dendrimer was first introduced in 1985 by D.A. Tomalia et al. [1]. Now a days more than forty families of dendrimers are present which are carries unique properties. These specific properties make dendrimers suitable for various applications in medical and industrial technology. Dendrimers are used in vitro diagnostic cardiac testing, as contrast agents for magnetic resonance. Magnetic resonance imaging (MRI) is a diagnostic process to producing anatomical images of organs and blood vessels. Recently, U. Ahmad et al. studied the atom-bond connectivity indices of certain families of dendrimers in [2], Y. Bashir et al. studied forgotten topological index of some dendrimers structure in [3]. In this paper, we derived the exact expressions of the generalized Zagreb index or (a,b)-Zagreb index of some regular dendrimers and hence as a special case we obtain some important degree based topological indices such as Zagreb indices, forgotten topological index, redefined Zagreb index, general first Zagreb index, general Randić index, symmetric division deg index from using our derived results. Gutman and Trinajestić in a paper, "to study the total π-electron energy (ε) of carbon atoms" introduced the Zagreb indices in 1972 [4] and are defined as

$$M_1(G)=\sum_{v\in V(G)} d_G(v)^2=\sum_{uv\in E(G)}[d_G(u)+d_G(v)]$$

and

$$M_2(G)=\sum_{uv\in E(G)} d_G(u)d_G(v).$$

We refer our reader to [5, 6], for some recent study about these indices. The "forgotten topological index" or F-index was introduced by Gutman and Trinajestić [4], in the same paper where Zagreb indices were introduced and is defined as

$$F(G)=\sum_{v\in V(G)} d_G(v)^3=\sum_{uv\in E(G)}[d_G(u)^2+d_G(v)^2].$$

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For further study about this index we refer our reader to [7, 8, 9]. The redefined Zagreb index was first introduced in 2013 by Ranjini et al. [10] and is defined as

$$ReZM(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)[d_G(u)+d_G(v)].$$

For some recent study about this index we refer our reader to [11, 12]. Li and Zheng was introduced the general Zagreb index in [13], and is defined as

$$M^{\alpha}(G) = \sum_{u\in V(G)} d_G(u)^{\alpha}$$

where, $\alpha \neq 0,\ 1$ *and* $\alpha \in \mathbb{R}$. Clearly, when $\alpha = 2$ we get first Zagreb index and when $\alpha = 3$ it gives the F-index. In 2001, Gutman and Lepović generalized the Randić index in [14] and is defined as

$$R_a = \sum_{uv\in E(G)} \{d_G(u).d_G(v)\}^a$$

where, $a \neq 0,\ a \in \mathbb{R}$. The Symmetric division deg index of a graph is defined as

$$SDD(G) = \sum_{uv\in E(G)} [\frac{d_G(u)}{d_G(v)} + \frac{d_G(v)}{d_G(u)}].$$

For further study about this index, we refer our reader to [15, 16, 17]. Based on some well known vertex degree based topological indices Azari et al. [18], in 2011 introduced a generalized version of vertex degree based topological index, named as generalized Zagreb index or the (a,b)-Zagreb index and is defined as

$$Z_{a,b}(G) = \sum_{uv\in E(G)} (d_G(u)^a d_G(v)^b + d_G(u)^b d_G(v)^a).$$

We refer our reader to [19, 20, 21], for further study about this index. It is shown that in table 1 all the topological indices discussed previously in this paper, are derived from this (a,b)-Zagreb index for some particular values of a and b.

Table 1: Relations between (a,b)-Zagreb index with some other topological indices:

Topological index	Corresponding (a,b)-Zagreb index
First Zagreb index $M_1(G)$	$Z_{1,0}(G)$
Second Zagreb index $M_2(G)$	$\frac{1}{2}Z_{1,1}(G)$
Forgotten topological index $F(G)$	$Z_{2,0}(G)$
Redefined Zagreb index $ReZM(G)$	$Z_{2,1}(G)$
General first Zagreb index $M^a(G)$	$Z_{a-1,0}(G)$
General Randić index R_a	$\frac{1}{2}Z_{a,a}$
Symmetric division deg index $SDD(G)$	$Z_{1,-1}(G)$

2. Main Results

In this section, we derived generalized Zagreb index of some dendrimers. First, we consider the regular dendrimer $G[n]$ with exactly n generations. The edge sets of dendrimer $G[n]$ are divided into three parts and are shown as follows:

$$E_1(G[n]) = \{e = uv \in E(G[n]) : d_{G[n]}(u) = 2 \text{ and } d_{G[n]}(v) = 2\}$$
$$E_2(G[n]) = \{e = uv \in E(G[n]) : d_{G[n]}(u) = 2 \text{ and } d_{G[n]}(v) = 3\}$$
$$E_3(G[n]) = \{e = uv \in E(G[n]) : d_{G[n]}(u) = 2 \text{ and } d_{G[n]}(v) = 1\}$$

note that, $|E_1(G[n])| = (2^{n+3}-5)$, $|E_2(G[n])| = (3\times 2^{n+1}-6)$, $|E_3(G[n])| = 2^{n+1}$. The two dimensional structure of a regular dendrimer $G[n]$ with 6-levels is shown in figure 1.

Theorem 2.1. *The (a,b)-Zagreb index of the regular dendrimer $G[n]$ is given by*

$$Z_{a,b}(G[n]) = (2^{n+3}-5).2^{a+b+1} + (3\times 2^{n+1}-6)(2^a.3^b+2^b.3^a) + 2^{n+1}(2^a+2^b). \tag{2.1}$$

Proof. Applying the definition of (a,b)-Zagreb index, we get

$$\begin{aligned} Z_{a,b}(G[n]) &= \sum_{uv\in E(G[n])} (d_{G[n]}(u)^a d_{G[n]}(v)^b + d_{G[n]}(u)^b d_{G[n]}(v)^a) \\ &= \sum_{uv\in E_1(G[n])} (2^a2^b+2^b2^a) + \sum_{uv\in E_2(G[n])} (2^a3^b+2^b3^a) + \sum_{uv\in E_3(G[n])} (2^a1^b+2^b1^a) \\ &= |E_1(G[n])|(2^a2^b+2^b2^a) + |E_2(G[n])|(2^a3^b+2^b3^a) + |E_3(G[n])|(2^a1^b+2^b1^a) \\ &= (2^{n+3}-5).2^{a+b+1} + (3\times 2^{n+1}-6)(2^a3^b+2^b3^a) + 2^{n+1}(2^a+2^b). \end{aligned}$$

Hence, the theorem. □

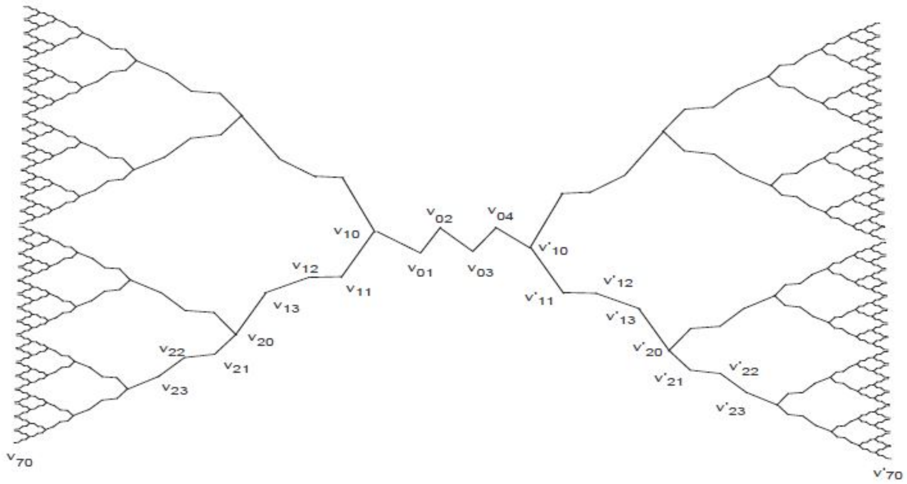

Figure 2.1: The two dimensional structure of regular dendrimer $G[n]$ for n=6.

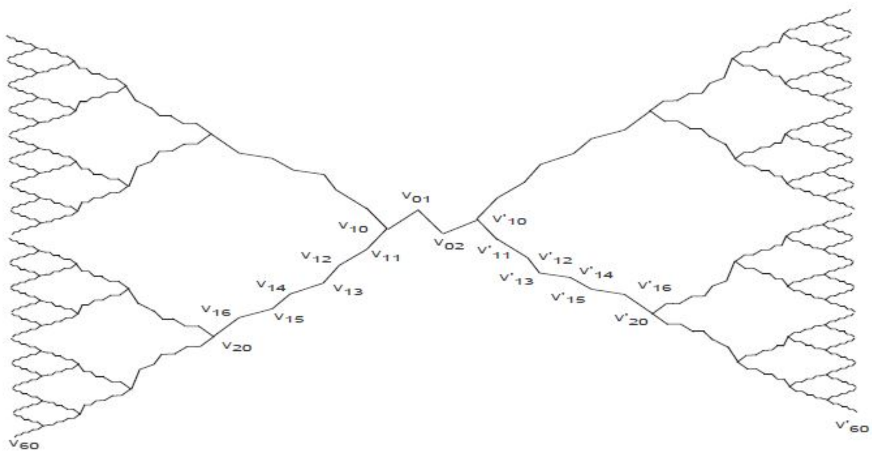

Figure 2.2: The two dimensional structure of regular dendrimer $H[n]$ for $n = 5$.

Corollary 2.2. *Using equation 2.1, the following results follows:*

$$
\begin{aligned}
(i)\ \ M_1(G[n]) &= Z_{1,0}(G[n]) = 4.2^{n+3} + 18.2^{n+1} - 50,\\
(ii)\ \ M_2(G[n]) &= \frac{1}{2}Z_{1,1}(G[n]) = 4.2^{n+3} + 20.2^{n+1} - 56,\\
(iii)\ \ F(G[n]) &= Z_{2,0}(G[n]) = 8.2^{n+3} + 44.2^{n+1} - 118,\\
(iv)\ \ ReZM(G[n]) &= Z_{2,1}(G[n]) = 16.2^{n+3} + 96.2^{n+1} - 260,\\
(v)\ \ M^a(G[n]) &= Z_{a-1,0}(G[n]) = (2^{n+3} - 5).2^a + (3 \times 2^{n+1} - 6)(2^{a-1} + 3^{a-1}) + 2^{n+1}(2^{a-1} + 1),\\
(vi)\ \ R_a(G[n]) &= \frac{1}{2}Z_{a,a}(G[n]) = (2^{n+5} - 5).2^{2a} + (3 \times 2^{n+1} - 6).2^a.3^a + 2^{n+1}.2^a,\\
(vii)\ \ SDD(G[n]) &= Z_{1,-1}(G[n]) = 2.2^{n+3} + 18.2^n - 23.
\end{aligned}
$$

Now, we consider the regular dendrimer $H[n]$ where, n is the steps of growth. The edge sets of $H[n]$ can be partitioned as follows:

$$E_1(H[n]) = \{e = uv \in E(H[n]) : d_{H[n]}(u) = 2 \text{ and } d_{H[n]}(v) = 2\}$$

$$E_2(H[n]) = \{e = uv \in E(H[n]) : d_{H[n]}(u) = 2 \text{ and } d_{H[n]}(v) = 3\}$$

$$E_3(H[n]) = \{e = uv \in E(H[n]) : d_{H[n]}(u) = 2 \text{ and } d_{H[n]}(v) = 1\}$$

note that, $|E_1(H[n])| = (5 \times 2^{n+2} - 19)$, $|E_2(H[n])| = (3 \times 2^{n+1} - 6)$, $|E_3(H[n])| = 2^{n+1}$. The two dimensional structure of $H[n]$ with 5-levels is shown in figure 2.

Theorem 2.3. *The (a,b)-Zagreb index of the regular dendrimer $H[n]$ is given by*

$$Z_{a,b}(H[n]) = (5 \times 2^{n+2} - 19).2^{a+b+1} + (3 \times 2^{n+1} - 6)(2^a.3^b + 2^b.3^a) + 2^{n+1}(2^a + 2^b). \tag{2.2}$$

Proof. Applying the definition of (a,b)-Zagreb index, we get

$$\begin{aligned}
Z_{a,b}(H[n]) &= \sum_{uv\in E(H[n])}(d_{H[n]}(u)^a d_{H[n]}(v)^b + d_{H[n]}(u)^b d_{H[n]}(v)^a)\\
&= \sum_{uv\in E_1(H[n])}(2^a2^b+2^b2^a) + \sum_{uv\in E_2(H[n])}(2^a3^b+2^b3^a) + \sum_{uv\in E_3(H[n])}(2^a1^b+2^b1^a)\\
&= |E_1(H[n])|(2^a2^b+2^b2^a) + |E_2(H[n])|(2^a3^b+2^b3^a) + |E_3(H[n])|(2^a1^b+2^b1^a)\\
&= (5\times 2^{n+2}-19).2^{a+b+1} + (3\times 2^{n+1}-6)(2^a3^b+2^b3^a) + 2^{n+1}(2^a+2^b).
\end{aligned}$$

Hence, the theorem. □

Corollary 2.4. *From equation 2.2, the following results follows:*

$$\begin{aligned}
(i)\ M_1(H[n]) &= Z_{1,0}(H[n]) = 20.2^{n+2}+18.2^{n+1}-106,\\
(ii)\ M_2(H[n]) &= \frac{1}{2}Z_{1,1}(H[n]) = 20.2^{n+2}+20.2^{n+1}-112,\\
(iii)\ F(H[n]) &= Z_{2,0}(H[n]) = 40.2^{n+2}+44.2^{n+1}-230,\\
(iv)\ ReZM(H[n]) &= Z_{2,1}(H[n]) = 80.2^{n+2}+96.2^{n+1}-484,\\
(v)\ M^a(H[n]) &= Z_{a-1,0}(H[n]) = (5\times 2^{n+2}-19).2^a + (3\times 2^{n+1}-6)(2^{a-1}+3^{a-1}) + 2^{n+1}(2^{a-1}+1),\\
(vi)\ R_a(H[n]) &= \frac{1}{2}Z_{a,a}(H[n]) = (5\times 2^{n+2}-19).2^{2a} + (3\times 2^{n+1}-6).2^a.3^a + 2^{n+1}.2^a,\\
(vii)\ SDD(H[n]) &= Z_{1,-1}(H[n]) = 10.2^{n+2}+18.2^n-51.
\end{aligned}$$

Now, we obtained the (a,b)-Zagreb index for the porphyrin dendrimer D_nP_n with n-layers. Here $n=2^m$ $(m\geq 2)$ denote the steps of growth. Note that total number of vertices in D_nP_n is $(96n-10)$ and $(105n-11)$ edges. The edge sets of D_nP_n are divided as follows:

$$\begin{aligned}
E_1(D_nP_n) &= \{e=uv\in E(D_nP_n) : d_{D_nP_n}(u)=1 \ and\ d_{D_nP_n}(v)=3\}\\
E_2(D_nP_n) &= \{e=uv\in E(D_nP_n) : d_{D_nP_n}(u)=1 \ and\ d_{D_nP_n}(v)=4\}\\
E_3(D_nP_n) &= \{e=uv\in E(D_nP_n) : d_{D_nP_n}(u)=2 \ and\ d_{D_nP_n}(v)=2\}\\
E_4(D_nP_n) &= \{e=uv\in E(D_nP_n) : d_{D_nP_n}(u)=2 \ and\ d_{D_nP_n}(v)=3\}\\
E_5(D_nP_n) &= \{e=uv\in E(D_nP_n) : d_{D_nP_n}(u)=3 \ and\ d_{D_nP_n}(v)=3\}\\
E_6(D_nP_n) &= \{e=uv\in E(D_nP_n) : d_{D_nP_n}(u)=3 \ and\ d_{D_nP_n}(v)=4\}
\end{aligned}$$

where, $|E_1(D_nP_n)|=2n$, $|E_2(D_nP_n)|=24n$, $|E_3(D_nP_n)|=10n-5$, $|E_4(D_nP_n)|=48n-6$, $|E_5(D_nP_n)|=13n$, $|E_6(D_nP_n)|=8n$. The figure of porphyrin dendrimer D_nP_n with 16-layers is shown in figure 3.

Theorem 2.5. *The (a,b)-Zagreb index of regular dendrimer D_nP_n is given by*

$$Z_{a,b}(D_nP_n) = 2n(3^a+3^b)+24n(4^a+4^b)+(10n-5).2^{a+b+1}+(48n-6)(2^a3^b+2^b3^a)+26n.3^{a+b}+8n(3^a4^b+3^b4^a). \quad (2.3)$$

Proof. From definition of (a,b)-Zagreb index, we get

$$\begin{aligned}
Z_{a,b}(D_nP_n) &= \sum_{uv\in E(D_nP_n)}(d_{D_nP_n}(u)^a d_{D_nP_n}(v)^b + d_{D_nP_n}(u)^b d_{D_nP_n}(v)^a)\\
&= \sum_{uv\in E_1(D_nP_n)}(1^a3^b+1^b3^a) + \sum_{uv\in E_2(D_nP_n)}(1^a4^b+1^b4^a) + \sum_{uv\in E_3(D_nP_n)}(2^a2^b+2^b2^a) + \sum_{uv\in E_4(D_nP_n)}(2^a3^b+2^b3^a)\\
&\quad + \sum_{uv\in E_5(D_nP_n)}(3^a3^b+3^b3^a) + \sum_{uv\in E_6(D_nP_n)}(3^a4^b+3^b4^a)\\
&= |E_1(D_nP_n)|(1^a3^b+1^b3^a) + |E_2(D_nP_n)|(1^a4^b+1^b4^a) + |E_3(D_nP_n)|(2^a2^b+2^b2^a) + |E_4(D_nP_n)|(2^a3^b+2^b3^a)\\
&\quad + |E_5(D_nP_n)|(3^a3^b+3^b3^a) + |E_6(D_nP_n)|(3^a4^b+3^b4^a)\\
&= 2n(3^a+3^b)+24n(4^a+4^b)+(10n-5).2^{a+b+1}+(48n-6)(2^a3^b+2^b3^a)+26n.3^{a+b}+8n(3^a4^b+3^b4^a).
\end{aligned}$$

Hence, the theorem. □

Corollary 2.6. *Using equation 2.3, we obtain following results as follows:*

$$\begin{aligned}
(i)\ M_1(D_nP_n) &= Z_{1,0}(D_nP_n) = 542n-50,\\
(ii)\ M_2(D_nP_n) &= \frac{1}{2}Z_{1,1}(D_nP_n) = 643n-56,\\
(iii)\ F(D_nP_n) &= Z_{2,0}(D_nP_n) = 1666n-118,\\
(iv)\ ReZM(D_nP_n) &= Z_{2,1}(D_nP_n) = 3010n-260,\\
(v)\ M^a(D_nP_n) &= Z_{a-1,0}(D_nP_n) = 2n(3^{a-1}+1)+24n(4^{a-1}+1)+(10n-5).2^a+(48n-6)(2^{a-1}+3^{a-1})+26n.3^{a-1}+8n(3^{a-1}+4^{a-1}),\\
(vi)\ R_a(D_nP_n) &= \frac{1}{2}Z_{a,a}(D_nP_n) = 2n.3^a+24n.4^a+(10n-5).2^{2a}+(48n-6).2^a.3^a+13n.3^{2a},\\
(vii)\ SDD(D_nP_n) &= Z_{1,-1}(D_nP_n) = \frac{826}{3}.n-23.
\end{aligned}$$

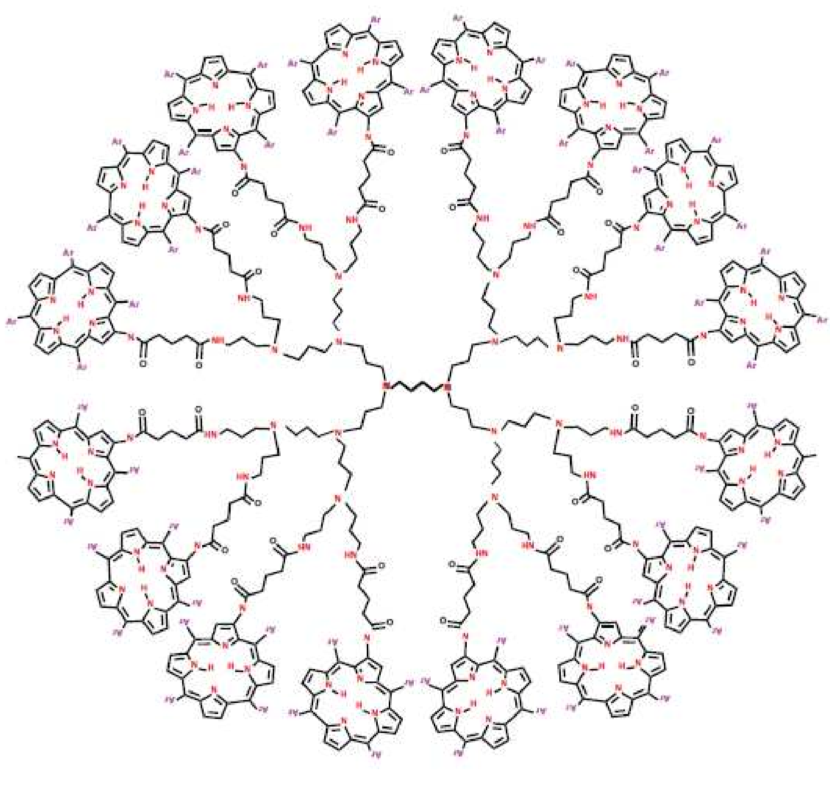

Figure 2.3: Molecular structure of porphyrin dendrimer $D_{16}P_{16}$.

finally, we obtained (a,b)-Zagreb index of Zinc-porphyrin DPZ_n here n is defined the steps of growth $(n \geq 1)$. The Zinc-porphyrin DPZ_n consists four similar branches and contains a central core. The total number of vertices in DPZ_n are $(56 \times 2^n - 7)$ and $(64 \times 2^n - 4)$ number of edges. The edge set of DPZ_n is partitioned as follows:

$$E_1(DPZ_n) = \{e = uv \in E(DPZ_n) : d_{DPZ_n}(u) = 2 \text{ and } d_{DPZ_n}(v) = 2\}$$
$$E_2(DPZ_n) = \{e = uv \in E(DPZ_n) : d_{DPZ_n}(u) = 2 \text{ and } d_{DPZ_n}(v) = 3\}$$
$$E_3(DPZ_n) = \{e = uv \in E(DPZ_n) : d_{DPZ_n}(u) = 3 \text{ and } d_{DPZ_n}(v) = 3\}$$
$$E_4(DPZ_n) = \{e = uv \in E(DPZ_n) : d_{DPZ_n}(u) = 3 \text{ and } d_{DPZ_n}(v) = 4\}$$

note that, $|E_1(DPZ_n)| = 16 \times 2^n - 4$, $|E_2(DPZ_n)| = 40 \times 2^n - 16$, $|E_3(DPZ_n)| = 8 \times 2^n + 12$, $|E_4(DPZ_n)| = 4$. The figure of Zinc-porphyrin DPZ_n with 4 layers is shown in figure 4.

Theorem 2.7. *For (DPZ_n), the (a,b)-Zagreb index is*

$$Z_{a,b}(DPZ_n) = (16 \times 2^n - 4).2^{a+b+1} + (40 \times 2^n - 16)(2^a.3^b + 2^b.3^a) + (8 \times 2^n + 12)2.3^{a+b} + 4(3^a4^b + 3^b4^a). \quad (2.4)$$

Proof. Using the concept of (a,b)-Zagreb index, we get

$$\begin{aligned}
Z_{a,b}(DPZ_n) &= \sum_{uv \in E(DPZ_n)} (d_{DPZ_n}(u)^a d_{DPZ_n}(v)^b + d_{DPZ_n}(u)^b d_{DPZ_n}(v)^a) \\
&= \sum_{uv \in E_1(DPZ_n)} (2^a2^b + 2^b2^a) + \sum_{uv \in E_2(DPZ_n)} (2^a3^b + 2^b3^a) + \sum_{uv \in E_3(DPZ_n)} (3^a3^b + 3^b3^a) + \sum_{uv \in E_4(DPZ_n)} (3^a4^b + 3^b4^a) \\
&= |E_1(DPZ_n)|(2^a2^b + 2^b2^a) + |E_2(DPZ_n)|(2^a3^b + 2^b3^a) + |E_3(DPZ_n)|(3^a3^b + 3^a3^b) + |E_4(DPZ_n)|(3^a4^b + 4^a3^b) \\
&= (16 \times 2^n - 4).2^{a+b+1} + (40 \times 2^n - 16)(2^a.3^b + 2^b.3^a) + (8 \times 2^n + 12)2.3^{a+b} + 4(3^a4^b + 3^b4^a).
\end{aligned}$$

Which is the desired result. ☐

Corollary 2.8. *From equation 2.4, we derived the following results,*

$$\begin{aligned}
(i)\ M_1(DPZ_n) &= Z_{1,0}(DPZ_n) = 312.2^n + 4, \\
(ii)\ M_2(DPZ_n) &= \frac{1}{2} Z_{1,1}(DPZ_n) = 376.2^n + 60, \\
(iii)\ F(DPZ_n) &= Z_{2,0}(DPZ_n) = 792.2^n + 76, \\
(iv)\ ReZM(DPZ_n) &= Z_{2,1}(DPZ_n) = 1888.2^n + 440, \\
(v)\ M^a(DPZ_n) &= Z_{a-1,0}(DPZ_n) = (16 \times 2^n - 4).2^a + (40 \times 2^n - 16)(2^{a-1} + 3^{a-1}) + (8 \times 2^n + 12).2.3^{a-1} + 4(3^{a-1} + 4^{a-1}), \\
(vi)\ R_a(DPZ_n) &= \frac{1}{2} Z_{a,a}(DPZ_n) = (16 \times 2^n - 4).2^{2a} + (40 \times 2^n - 16)2^a.3^a + (8 \times 2^n + 12).3^{2a} + 4.3^a.4^a), \\
(vii)\ SDD(DPZ_n) &= Z_{1,-1}(DPZ_n) = \frac{404}{3}.2^n - \frac{31}{3}.
\end{aligned}$$

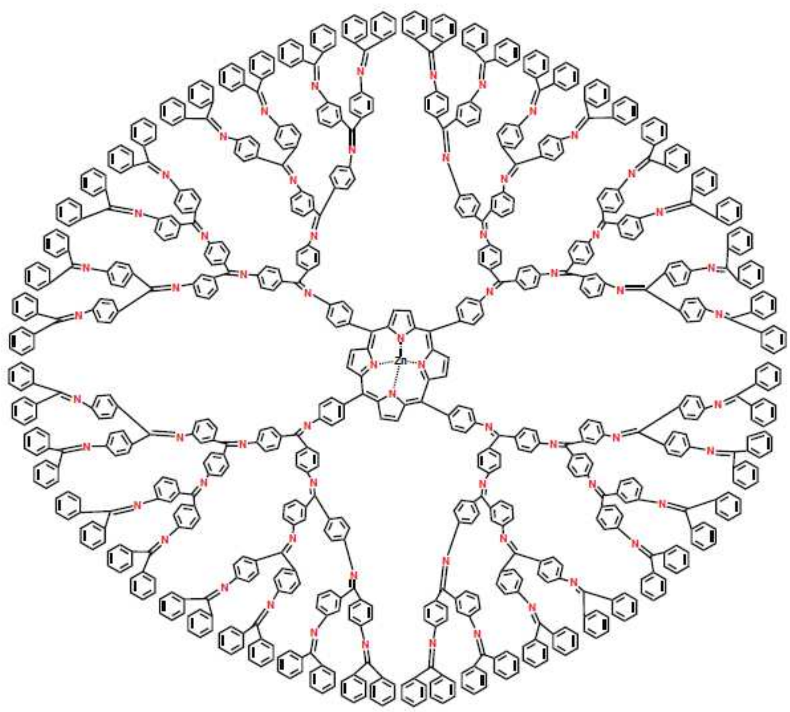

Figure 2.4: Molecular structure of dendrimer zinc porphyrin DPZ_4.

3. Conclusions

In this study, we obtain some closed expressions of the (a,b)-Zagreb index of some regular dendrimers such as $G[n]$, $H[n]$, porphyrin dendrimers D_nP_n and the Zinc-porphyrin DPZ_n and hence obtain some other important degree based topological indices for some particular values of a and b from our derived results. For further study the (a,b)-Zagreb index of some other chemical structures can be computed.

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References

[1] D.A. Tomalia, H. Baker, J.R. Dewald, M. Hall, G. Kallos, S. Martin, J. Roeek, J. Ryder, and P. Smith, *A new class of polynomials: starburst-dendritic macromolecules*, Polym. J., **17**, (1985), 117-132.
[2] U. Ahmad, S. Ahmad, and R. Yousaf, *Computation of Zagreb and atom–bond connectivity indices of certain families of dendrimers by using automorphism group action*, J. Serb. Chem. Soc., **82**, (2),(2017), 151-162.
[3] Y. Bashir, A. Aslam, M. Kamran, M.I. Qureshi, A. Jahangir, M. Rafiq, N. Bibi, and N. Muhammad, *On forgotten topological indices of some dendrimers structure*, Molecules, **22**, (867), (2017), 1-8.
[4] I. Gutman, N. Trinajestić, *Graph theory and molecular orbitals total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett., **17**, (1972), 535-538.
[5] N. De, *The vertex Zagreb indices of some graphs operations*, Carpathian Math. Publ., **8**, (2), (2016), 215-223.
[6] P. Sarkar, N. De, and A. Pal, *The Zagreb indices of graphs based on new operations related to the join of graphs*, J. Int. Math. Virtual Inst., **7**,(2017), 181-209.
[7] P. Sarkar, N. De., and A. Pal, *F-index of graphs based on new operations related to the join of graphs*, arXiv:1709.06301v1.
[8] N. De, *On molecular topological properties of TiO2 nanotubes*, J. Nanoscience, **2016**, (2016), DOI: 1028031.
[9] S. Akhtar, M. Imran, *Computing the forgotten topological index of four operations on graphs*, AKCE Int. J. Graphs Comb., **14**, (1), (2017), 70-79.
[10] P.S. Ranjini, V. Lokesha, and A. Usha, *Relation between phenylene and hexagonal squeeze using harmonic index*, Int. J. Graph Theory, **1**,(2013), 116-121.
[11] B. Zhao, J. Gan, and H. Wu, *Redefined Zagreb indices of some nano structures*, Appl. Math. Nonlinear Sci., **1**, (1), (2016), 291-300.
[12] R.P. Kumar, D.S. Nandappa, and M.R.R. Kanna, Redefined zagreb, Randić, Harmonic, GA indices of graphene, *Int. J. Math. Anal.*, **11**, (10), (2017), 493-502.
[13] X. Li, J. Zheng, *A unified approach to the extremal trees for different indices*, MATCH Commun. Math. Comput. Chem., **54**, (2005), 195-208.
[14] I. Gutman, M. Lepović, *Choosing the exponent in the definition of the connectivity index*, J. Serb. Chem. Soc., **66**,(9),(2001), 605-611.
[15] V. Lokesha, T. Deepika, *Symmetric division deg index of tricyclic and tetracyclic graphs*, Int. J. Sci. Eng. Res, **7**, (5), (2016), 53-55.
[16] V. Alexander, *Upper and lower bounds of symmetric division deg index*, Iran. J. Math. Chem., **52**, (2014), 91-98.
[17] C.K. Gupta, V. Lokesha, B.S. Shwetha, and P.S. Ranjini, *Graph operations on the symmetric division deg index of graphs*, Palestine. J. Math., **6**, (1), (2017), 280-286.
[18] M. Azari, A. Iranmanesh, *Generalized Zagreb index of graphs*, Studia Univ. Babes-Bolyai., **56**,(3), (2011), 59-70.
[19] M. R. Farahani, *The generalized Zagreb index of circumcoronene series of benzenoid*, J. Appl. Phys. Sci. Int., **3**,(3), (2015), 99-105.
[20] P. Sarkar, N. De, and A. Pal, *The (a,b)-Zagreb index of nanostar dendrimers*, preprint.
[21] M. R. Farahani, M. R. R. Kanna, *Generalized Zagreb index of V-phenylenic nanotubes and nanotori*, J. Chem. Pharm. Res., **7**,(11), (2015), 241-245.

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General helices that lie on the sphere S^{2n} in Euclidean space E^{2n+1}

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Abstract

In this work, we give two methods to generate general helices that lie on the sphere S^{2n} in Euclidean (2n+1)-space E^{2n+1}.

1. Introduction

In Euclidean 3-space E^3, the condition for a curve to lie on a sphere (spherical curve) is usually given in the form

$$\frac{k_2}{k_1}+\left(\frac{1}{k_2}\left(\frac{1}{k_1}\right)'\right)'=0$$

where $k_1>0$ and $k_2\neq 0$ [8]. The integral form of the above equation was given in [2] as

$$\frac{1}{k_1}=A\cos\int k_2\,ds+B\sin\int k_2\,ds.$$

Besides, researchers gave different characterizations about spherical curves by using the equations above [9, 10].
In E^3, general helices are defined by the property that their tangent makes a constant angle with a fixed direction in every point. In this paper we use this definition for higher dimensions too. But, the general helix notion in $\mathbb{R}^3$ can be generalized to higher dimensions in many ways. In [7], the same definition is proposed but in $\mathbb{R}^n$. The definition of a general helix is more restrictive in [5]; the fixed direction makes a constant angle with all vectors of the Frenet frame. It is easy to check that this definition only works in odd dimensions [3]. Moreover, in the same paper, it is proven that this definition is equivalent to the fact that the ratios $\frac{k_1}{k_2}, \frac{k_3}{k_4},\dots,\frac{k_{n-4}}{k_{n-3}}, \frac{k_{n-2}}{k_{n-1}}$, where curvatures k_i are constants. This statement is related with the Lancret theorem for general helices in $\mathbb{R}^3$.
If a general helix lies on S^n, we call it spherical helix. This topic have become an active research area in recent years. In [6], Monterde studied constant curvature ratio curves (ccr-curves) for which all the ratios $\frac{k_1}{k_2}, \frac{k_3}{k_4},\dots$ are constant. He found explicit examples of spherical ccr-curves that lie on S^2 with non-constant curvatures. He showed that a ccr-curve on S^2 is a general helix. After that in [1], authors presented some necessary and sufficient conditions for a curve to be a slant helix in Euclidean n-space. They gave an example for a slant helix in E^5 whose tangent indicatrix is a spherical helix that lie on S^4.
In literature, there are studies about spherical helices in E^3 and there is only one example when $n\geq 4$ [1]. By means of the papers mentioned above, the goal of this paper is to find methods for generating spherical helices that lies on S^{2n} in E^{2n+1}.

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2. Basic concepts

The real vector space R^n with standard inner product and the standart orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ is given by

$$< X, Y >= \sum_{i=1}^{n} x_i y_i$$

for each $X = (x_1, x_2, ..., x_n)$, $Y = (y_1, y_2, ..., y_n) \in R^n$. In particular, the norm of a vector $X \in R^n$ is given by $||X||^2 =< X, X >$.
Let $\alpha : I \subset R \to E^n$ be a regular curve in E^n and $\{V_1, V_2, ..., V_n\}$ be the moving Frenet frame along the curve α, where V_i $(i = 1, 2, ..., n)$ denote ith Frenet vector field. Then, the Frenet formulas are given by

$$\left\{ \begin{array}{c} V_1'(t) = \nu(t) k_1(t)\, V_2\ (t) \\ V_i'(t) = \nu(t)(-k_{i-1}(t)\, V_{i-1}\ (t) + k_i(t)\, V_{i+1}\ (t)), \quad i = 2, 3, ..., n-1 \\ V_n'(t) = -\nu(t) k_{n-1}(t)\, V_{n-1}\ (t) \end{array} \right. \tag{2.1}$$

where $\nu(t) = ||d\alpha(t)/dt|| = ||\alpha'(t)||$ and k_i $(i = 1, 2, ..., n-1)$ denote the ith curvature function of the curve [4].

Definition 2.1. *The curve $\alpha : I \subset R \to E^n$ is called general helix if its tangent vector V_1 makes a constant angle with a fixed direction [7].*

A sphere of center $P = (p_1, p_2, ..., p_n) \in E^n$ and radius $R > 0$ is the surface

$$S^n(P, R) = \left\{ (x_1, x_2, ..., x_n) \in E^n | \quad (x_1 - p_1)^2 + \cdots + (x_n - p_n)^2 = R^2 \right\}.$$

When, P is the origin of E^n and $R = 1$, we denote this with S^n, that is,

$$S^n = \left\{ (x_1, x_2, ..., x_n) \in E^n | \quad {x_1}^2 + \cdots + {x_n}^2 = 1 \right\}.$$

3. Spherical helices in E^{2n+1}

Now, we give two theorems to generate general helices that lie on $S^{2n} \subset E^{2n+1}$. To reach our goal; First, we use W-curves, i.e. a curve which has constant Frenet curvatures [3].

Theorem 3.1. *Let,*

$$\gamma(s) = \left(\gamma_1(s), \gamma_2(s), \ldots, \gamma_{2n}(s), \sqrt{1 - R^2} \right) \subset S^{2n-1}(P, R) \subset S^{2n} \subset E^{2n+1}$$

be a unit speed W-curve with the Frenet vector fields $\{u_1, u_2, \ldots, u_{2n}\}$ and the curvatures $\{k_1, k_2, \ldots, k_{2n-1}\}$ where $P = \left(0, 0, \ldots, 0, \sqrt{1 - R^2}\right) \in E^{2n+1}$, $R = 1/a$, $a > 1$. Then, $\alpha(s) = \sin(s)\gamma(s) + \cos(s)u_1(s)$ with the Frenet vector fields $\{V_1, V_2, \ldots, V_{2n}\}$ is a general helix that lies on S^{2n}.

Proof. With straightforward calculations, it is clear that

$$\|\alpha\| = 1,$$

then α is a spherical curve which lies on S^{2n}.
We know $\langle \gamma, e_{2n+1} \rangle = \sqrt{1 - R^2}$. If we take derivatives of this equation with respect to s, we have

$$\langle u_i, e_{2n+1} \rangle = 0, \quad i = 1, 2, ..., 2n. \tag{3.1}$$

Since $\langle \gamma - P, \gamma - P \rangle = R^2$, for $i = 1, 2, ..., n$ we also have

$$\begin{aligned} \langle u_{2i-1}, \gamma \rangle &= 0, \\ \langle u_{2i}, \gamma \rangle &= \frac{-\prod_{j=0}^{i-1} k_{2i}}{\prod_{j=1}^{i} k_{2i-1}} \end{aligned} \tag{3.2}$$

where $k_0 = 1$. Then, by using Equations (3.1) and (3.2), we can write

$$\gamma = P + \lambda_1 u_1 + \lambda_2 u_3, \cdots + \lambda_n u_{2n-1}$$

So,

$$\langle V_1, e_{2n+1} \rangle = \sqrt{\frac{1 - R^2}{{k_1}^2 - 1}}.$$

This completes the proof.

Corollary 3.2. *If*

$$\gamma(s)=\frac{R}{\sqrt{n}}\left(\sum_{j=1}^{n}\sin\left(c_j s\right)e_{2j-1}+\sum_{j=1}^{n}\cos\left(c_j s\right)e_{2j}\right)+\sqrt{1-R^2}\,e_{2n+1} \tag{3.3}$$

where $R=\left(\dfrac{n}{\sum_{j=1}^{n}{c_j}^2}\right)^{1/2}$, $a=\left(\dfrac{\sum_{j=1}^{n}{c_j}^2}{n}\right)^{1/2}>1$ *and* $c_i\neq c_j$, $1\leq i<j\leq n$.
Then, $\alpha(s)=\sin(s)\gamma(s)+\cos(s)u_1(s)$ *is a general helix that lies on* S^{2n} *in* E^{2n+1}.

Example 3.3. *If we take* $c_1=2$, $c_2=4$, *and* $n=2$ *in Corollary 3.2 we have*

$$\begin{aligned}
P&=\left(0,0,0,0,\frac{1}{\sqrt{10}}\right),\\
R&=\frac{3}{\sqrt{10}},\\
\gamma(s)&=\left(\frac{\sin(2s)}{2\sqrt{5}},\frac{\cos(2s)}{2\sqrt{5}},\frac{\sin(4s)}{2\sqrt{5}},\frac{\cos(4s)}{2\sqrt{5}},\frac{3}{\sqrt{10}}\right)\subset S^3\left(P,R\right)\subset S^4,\\
v_1(s)&=\left(\frac{\cos(2s)}{\sqrt{5}},-\frac{\sin(2s)}{\sqrt{5}},\frac{2\cos(4s)}{\sqrt{5}},-\frac{2\sin(4s)}{\sqrt{5}},0\right).
\end{aligned}$$

Then, γ *is a unit speed spherical W-curve with the curvatures*

$$\begin{aligned}
k_1&=2\sqrt{\frac{17}{5}},\\
k_2&=\frac{12}{\sqrt{85}},\\
k_3&=4\sqrt{\frac{5}{17}},\\
k_4&=0.
\end{aligned}$$

Therefore, we get

$$\alpha(s)=\left(\frac{\cos^3(s)}{\sqrt{5}},-\frac{3\sin(s)+\sin(3s)}{4\sqrt{5}},\frac{2\cos^3(s)(3\cos(2s)-2)}{\sqrt{5}},-\frac{5\sin(3s)+3\sin(5s)}{4\sqrt{5}},\frac{3\sin(s)}{\sqrt{10}}\right)$$

with the tangent vector field

$$V_1(s)=\left(-\frac{\sin(s)\cos(s)}{\sqrt{7}},-\frac{(\cos(s)+\cos(3s))\sec(s)}{4\sqrt{7}},\frac{5(\sin(s)-\sin(3s))\cos(s)}{\sqrt{7}},-\frac{5(\cos(3s)+\cos(5s))\sec(s)}{4\sqrt{7}},\frac{1}{\sqrt{14}}\right)$$

where $\|\alpha\|=1$.

By means of Theorem 3.1 and Corollary 3.2 we can give a new theorem.

Theorem 3.4. *Let* $\alpha: I\subset R\to E^{2n+1}$

$$\alpha(t)=(\alpha_1(t),\alpha_2(t),\ldots,\alpha_{2n+1}(t))$$

be a regular curve given by

$$\alpha_{2i-1}(t)=\frac{1}{\left(\sum_{j=1}^{n}{c_j}^2\right)^{1/2}}\left(c_i\cos\left(t\right)\cos\left(c_it\right)+\sin\left(t\right)\sin\left(c_it\right)\right),$$

$$\alpha_{2i}(t)=\frac{1}{\left(\sum_{j=1}^{n}{c_j}^2\right)^{1/2}}\left(\cos\left(c_it\right)\sin\left(t\right)-c_i\cos\left(t\right)\sin\left(c_it\right)\right),$$

for $i=1,2,\ldots n$ *and*

$$\alpha_{2n+1}(t)=\left(1-\frac{n}{\sum_{j=1}^{n}{c_j}^2}\right)^{1/2}\sin\left(t\right)$$

where $c_1,c_2,\ldots,c_n>1$ *with* $c_i\neq c_j$, $1\leq i<j\leq n$. *Then,* α *is a general helix which lies on* S^{2n}.

Proof. With straightforward calculations, we easily have

$$||\alpha(t)||=1,$$

$$||\alpha'(t)||=\left(\frac{\left(\sum_{i=1}^{n}c_i{}^2\right)\left(1-\frac{n}{\sum_{i=1}^{n}c_i{}^2}\right)}{\sum_{i=1}^{n}\left(c_i{}^4-c_i{}^2\right)}\right)^{1/2}\cos(t),$$

and

$$V_1(t)=\left(\sum_{i=1}^{n}\left(c_i{}^4-c_i{}^2\right)\right)^{-1/2}\left(\sum_{i=1}^{n}\left(1-c_i{}^2\right)\left(\sin(c_it)e_{2i-1}-\cos(c_it)e_{2i}\right)+\left(\sum_{i=1}^{n}c_i{}^2\left(1-\frac{n}{\sum_{i=1}^{n}c_i{}^2}\right)\right)^{1/2}e_{2n+1}\right).$$

Therefore, we have

$$\langle V_1(t),e_{2n+1}\rangle=\left(\sum_{i=1}^{n}\left(c_i{}^4-c_i{}^2\right)\right)^{-1/2}\left(\sum_{i=1}^{n}c_i{}^2\left(1-\frac{n}{\sum_{i=1}^{n}c_i{}^2}\right)\right)^{1/2}.$$

This completes the proof. □

Example 3.5. *If we take $n=4$ and $c_1=\sqrt{2}$, $c_2=\sqrt{3}$, $c_3=2$ in Theorem 3.4 we have,*

$$\alpha(t)=\left(\frac{\sqrt{2}}{3}\cos(t)\cos\left(\sqrt{2}t\right)+\frac{1}{3}\sin(t)\sin\left(\sqrt{2}t\right),\right.$$

$$\frac{1}{3}\cos\left(\sqrt{2}t\right)\sin(t)-\frac{\sqrt{2}}{3}\cos(t)\sin\left(\sqrt{2}t\right),$$

$$\frac{1}{\sqrt{3}}\cos(t)\cos\left(\sqrt{3}t\right)+\frac{1}{3}\sin(t)\sin\left(\sqrt{3}t\right),$$

$$\frac{1}{3}\cos\left(\sqrt{3}t\right)\sin(t)-\frac{1}{\sqrt{3}}\cos(t)\sin\left(\sqrt{3}t\right),$$

$$\frac{2}{3}\cos(t)\cos(2t)+\frac{1}{3}\sin(t)\sin(2t),$$

$$\frac{1}{3}\cos(2t)\sin(t)-\frac{2}{3}\cos(t)\sin(2t),$$

$$\left.\sqrt{\frac{2}{3}}\sin(t)\right),$$

$$||\alpha(t)||=1,$$

$$||\alpha'(t)||=\frac{2\sqrt{5}\cos(t)}{3},$$

$$V_1(t)=\left(-\frac{\sin\left(\sqrt{2}t\right)}{2\sqrt{5}},-\frac{\cos\left(\sqrt{2}t\right)}{2\sqrt{5}},-\frac{\sin\left(\sqrt{3}t\right)}{\sqrt{5}},-\frac{\cos\left(\sqrt{3}t\right)}{\sqrt{5}},-\frac{3\sin(2t)}{2\sqrt{5}},-\frac{3\cos(2t)}{2\sqrt{5}},\sqrt{\frac{3}{10}}\right)$$

and

$$\langle V_1(t),e_7\rangle=\sqrt{\frac{3}{10}}.$$

Therefore, from Definition 2.1, α is a spherical helix.

□

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References

[1] Ali, Ahmad T., Turgut, M.: Some characterizations of slant helices in the Euclidean space E^n, *Hacettepe Journal of Mathematics and Statistics*, 39, 327-336, (2010).
[2] Breuer, S. and Gottlieb, D.: Explicit characterization of spherical curves, *Proc. Am. Math. Soc.*, 274, 126–127, (1972).
[3] Camci, C. Ilarslan, K. Kula, L. and Hacisalihoglu, H.H.: Harmonic cuvature and general helices, *Chaos Solitons & Fractals*, 40, 2590-2596, (2009).
[4] Gluck, H.: Higher curvatures of curves in Euclidean space, *Amer. Math. Monthly* 73, 699-704, (1966).
[5] Hayden, H. A.: On a general helix in a Riemannian n-space, *Proc. London Math. Soc.* 2, 37-45, (1931).
[6] Monterde, J.,: Curves with constant curvature ratios, *Bol. Soc. Mat. Mexicana 3a*, 13/1, 177–186, (2007).
[7] Romero-Fuster, M.C., Sanabria-Codesal, E.: Generalized helices, twistings and flattenings of curves in n-space. *Mat. Cont.*, 17 , 267-280, (1999).
[8] Struik, D.J.: Lectures on Classical Differential Geometry, Dover, New-York, (1988).
[9] Wong Y.C.,: A global formulation of the condition for a curve to lie in a sphere, *Monatsch Math*, 67, 363–365, (1963).
[10] Wong Y.C.,: On a explicit characterization of spherical curves, *Proc. Am. Math. Soc.*, 34, 239–242, (1972).

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Nonexistence of global solutions to system of semi-linear fractional evolution equations

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Abstract

In this research we are interested to Cauchy problem for system of semi-linear fractional evolution equations.
Some authors were concerned with studying of global existence of solutions for the hyperbolic nonlinear equations with a damping term. Our goal is to extend some results obtained by the authors, by studying the system of semi-linear hyperbolic equations with fractional damping term and fractional Laplacian .
Thanks to the test functions method, we prove the nonexistence of nontrivial global weak solutions to the problem.

1. Introduction

in this paper we are concerned with the following Cauchy problem:

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\beta_1}{2}} u + D^{\alpha_1}_{0|t} u = f(t,x)\,|u|^{p_1}\,|v|^{q_1}, & (t,x) \in (0,+\infty)\times\mathbb{R}^N \\ v_{tt} + (-\Delta)^{\frac{\beta_2}{2}} v + D^{\alpha_2}_{0|t} v = g(t,x)\,|u|^{p_2}\,|v|^{q_2}, & (t,x) \in (0,+\infty)\times\mathbb{R}^N \end{cases} \tag{1.1}$$

subjected to the conditions

$$u(0,x) = u_0(x) \geq 0,, \quad u_t(0,x) = u_1(x) \geq 0,$$

$$v(0,x) = v_0(x) \geq 0,, \quad v_t(0,x) = v_1(x) \geq 0,.$$

where $p_1 \geq 0, q_2 \geq 0, p_2 > 1, q_1 > 1, 0 < \alpha_i < 1 \leq \beta_i \leq 2, i = 1,2$ are constants. $D^{\alpha_i}_{0/t}$ denotes the derivatives of order α_i in the sense of Caputo and $(-\Delta)^{\frac{\beta_i}{2}}$ is the fractional power of the $(-\Delta)$.
The integral representation of the fractional Laplacian in the N-dimensional space is

$$(-\Delta)^{\beta/2}\psi(x) = -c_N(\beta)\int_{\mathbb{R}^N} \frac{\psi(x+z)-\psi(x)}{|z|^{N+\beta}}dz, \quad \forall x \in \mathbb{R}^N, \tag{1.2}$$

where $c_N(\beta) = \Gamma((N+\beta)/2)/(2\pi^{N/2+\beta}\Gamma(1-\beta/2))$, and Γ denotes the gamma function (see [16]).
Note that The fractional Laplacian $\left((-\Delta)^{\beta/2}\right)$ with $1 \leq \beta \leq 2$ is a pseudo-differential operator defined by:

$$(-\Delta)^{\beta/2}u(x) = \mathscr{F}^{-1}\{|\zeta|^{\beta}\mathscr{F}(u)(\zeta)\}(x) \;\; \forall x \in \mathbb{R}^N,$$

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where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are Fourier transform and inverse Fourier transform, respectively.
The functions f and g are non-negatives and assumed to satisfy the conditions

$$f(t,x) \geq C_1 t^{\nu_1} |x|^{\mu_1}, g(t,x) \geq C_2 t^{\nu_2} |x|^{\mu_2}, \text{ where } \nu_i \geq 0,\ \mu_i \geq 0,\ i=1,2. \tag{1.3}$$

The problem of global existence of solutions for nonlinear hyperbolic equations with a damping term have been studied by many researchers in several contexts (see [4], [8], [9] , [12], [18], [20]), for example, the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & (t,x) \in (0,\infty) \times \mathbb{R}^N \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

Todorova-Yordanov [18] showed that, if $p_c < p \leq \frac{n}{n-1}$, then (1.4) admits a unique global solution, and they proved that if $1 < p < 1 + \frac{2}{n}$, then the solution u blows up in a finite time.
Fino-Ibrahim and Wehbe [4] generalized the results of Ogawa-Takeda [12] by proving the blow-up of solutions of (1.4) under weaker assumptions on the initial data and they extended this results to the critical case $p_c = 1 + \frac{2}{n}$.
Qi. Zhang [20] studied the case $1 < p < 1 + \frac{2}{n}$, when $\int u_i(x)dx > 0, i = 0,1$, he proved that global solution of (1.4) does not exist. Therefore, he showed that $p = 1 + \frac{2}{n}$ belongs to the blow-up case.
A. Hakem [8] treated the same type of (1.4), then he extended this result to the case of a system :

$$\begin{cases} u_{tt} + -\Delta u + g(t) u_t = |v|^p, & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ v_{tt} + -\Delta v + f(t) v_t = |u|^q, & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x) \\ v(0,x) = v_0(x), \ v_t(0,x) = v_1(x), \end{cases} \tag{1.5}$$

$g(t)$ and $f(t)$ are functions behaving like t^β and t^α, respectively, where $0 \leq \beta, \alpha < 1$.
Hakem [8] showed that, if

$$\frac{N}{2} \leq \frac{1}{pq-1} \max\left[1 - \beta + p(1-\alpha), 1 - \alpha + q(1-\beta)\right] - \max\left(\alpha, \beta\right),$$

then the problem (1.5) has only the trivial solution.
By combining the works of the above authors with those of Kirane *et al.*[10] and Escobido *et al.*[2], we were able to prove a nonexistence result to (1.1) in the weak formulation.

2. Preliminaries

Let us start by introducing the definitions concerning fractional derivatives in the sense of Caputo and the weak local solution to problem (1.1).

Definition 2.1. *Let $0 < \alpha < 1$ and $\zeta' \in L^1(0,T)$. The left-sided and respectively right-sided Caputo derivatives of order α for ζ are defined as:*

$$D^{\alpha}_{0|t} \zeta(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\zeta'(s)}{(t-s)^\alpha} ds,$$

and

$$D^{\alpha}_{t|T} \zeta(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{\zeta'(s)}{(s-t)^\alpha} ds,$$

where Γ denotes the gamma function (see [13] p 79).

Definition 2.2. *Let $Q_T = (0,T) \times \mathbb{R}^N$, $0 < T < +\infty$.*
We say that $(u,v) \in \left(L^1_{loc}(Q_T)\right)^2$ is a local weak solution to problem (1.1) on Q_T,
if $(f u^{p_1} v^{q_1}, g u^{p_2} v^{q_2}) \in \left(L^1_{loc}(Q_T)\right)^2$, and it satisfies

$$\begin{aligned} &\int_{Q_T} f |u|^{p_1} |v|^{q_1} \zeta_1 \, dx dt + \int_{\mathbb{R}^N} u_0(x) \zeta_1(0,x) \, dx + \int_{\mathbb{R}^N} u_1(x) \zeta_1(0,x) \, dx - \int_{\mathbb{R}^N} u_0(x) \zeta_{1_t}(0,x) \, dx \\ &\qquad = \int_{Q_T} u \zeta_{1_{tt}} \, dx dt + \int_{Q_T} u D^{\alpha_1}_{t|T} \zeta_1 \, dx dt + \int_{Q_T} u (-\Delta)^{\frac{\beta_1}{2}} \zeta_1 \, dx dt. \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\int_{Q_T} g |u|^{p_2} |v|^{q_2} \zeta_2 \, dx dt + \int_{\mathbb{R}^N} v_0(x) \zeta_2(0,x) \, dx + \int_{\mathbb{R}^N} v_1(x) \zeta_2(0,x) \, dx - \int_{\mathbb{R}^N} v_0(x) \zeta_{2_t}(0,x) \, dx \\ &\qquad = \int_{Q_T} v \zeta_{2_{tt}} \, dx dt + \int_{Q_T} v D^{\alpha_2}_{t|T} \zeta_2 \, dx dt + \int_{Q_T} v (-\Delta)^{\frac{\beta_2}{2}} \zeta_2 \, dx dt. \end{aligned} \tag{2.2}$$

for all test function $\zeta_j \in C^{2,2}_{t,x}(Q_T)$ such as $\zeta_j \geq 0$ and $\zeta_j(T,x) = \zeta_{j_t}(T,x) = \zeta_{j_t}(0,x) = 0,\ j = 1,2$

(see [3] p 5501).

Remark 2.3. *To get the definition 2.2, we multiplying the first equation in (1.1) by ζ_1 and the second equation by ζ_2, integrating by parts on $Q_T=(0,T)\times\mathbb{R}^N$ and using the definition 2.1*

The integrals in the above definition are supposed to be convergent.
If in the definition $T=+\infty$, the solution (u,v) is called global.
Now, we recall the following integration by parts formula:

$$\int_0^T \phi(t)(D^{\alpha}_{0|t}\psi)(t)dt=\int_0^T (D^{\alpha}_{t|T}\phi)(t)\psi(t)dt,$$

(see [17], p 46).

3. Main results

We now in position to announce our result.

Theorem 3.1. *Let $p_2>1, q_1>1, 0<\alpha_i<1\le\beta_i\le 2, i=1,2,$ and*

$$\mathscr{A}:=\frac{\alpha_1+\dfrac{\alpha_2}{p_2}-\left(1-\dfrac{1}{p_2q_1}\right)-\dfrac{1}{p_2}\left(\mu_2\dfrac{\alpha_1}{\beta_1}+\nu_2\right)-\dfrac{1}{p_2q_1}\left(\mu_1\dfrac{\alpha_2}{\beta_2}+\nu_1\right)}{\dfrac{\alpha_1}{\beta_1\tilde{p}_2}+\dfrac{\alpha_2}{\beta_2p_2\tilde{q}_1}}$$

and

$$\mathscr{B}:=\frac{\alpha_2+\dfrac{\alpha_1}{q_1}-\left(1-\dfrac{1}{p_2q_1}\right)-\dfrac{1}{q_1}\left(\mu_1\dfrac{\alpha_2}{\beta_2}+\nu_1\right)-\dfrac{1}{p_2q_1}\left(\mu_2\dfrac{\alpha_1}{\beta_1}+\nu_2\right)}{\dfrac{\alpha_2}{\beta_2\tilde{q}_1}+\dfrac{\alpha_1}{\beta_1q_1\tilde{p}_2}}$$

*where $p_2\tilde{p}_2=p_2+\tilde{p}_2$, $q_1\tilde{q}_1=q_1+\tilde{q}_1$,
and the conditions (1.3) are fulfilled.
If*

$$N\le\max\{\mathscr{A};\mathscr{B}\},$$

then the problem (1.1) admits no nontrivial global weak solutions.

Proof. We notice that, in all steps of proof , $C>0$ is a real positive number which may change from line to line.
Set $\zeta_j(t,x)=\Phi\left(\dfrac{t^2+|x|^{2\theta_j}}{R^2}\right)$, $j=1,2$ such as Φ is a decreasing function $C_0^2(\mathbb{R}^+)$, satisfies

$$0\le\Phi\le1 \text{ and } \Phi(r)=\begin{cases}1 & \text{if } 0\le r\le1,\\ 0 & \text{if } r\ge2.\end{cases}$$

Where $R>0$, $\theta_1=\beta_1/\alpha_1$ and $\theta_2=\beta_2/\alpha_2$ (see [10]).
Multiplying the first equation of (1.1) by ζ_1 and integrating by parts on
$Q_T=(0,T)\times\mathbb{R}^N$, we get

$$\begin{aligned}\int_{Q_T} f|u|^{p_1}|v|^{q_1}\zeta_1\,dx\,dt&+\int_{\mathbb{R}^N}u_0(x)\zeta_1(0,x)\,dx+\int_{\mathbb{R}^N}u_1(x)\zeta_1(0,x)\,dx-\int_{\mathbb{R}^N}u_0(x)\zeta_{1t}(0,x)\,dx\\&=\int_{Q_T}u\zeta_{1tt}\,dx\,dt-\int_{Q_T}uD^{\alpha_1}_{0|t}\zeta_1\,dx\,dt+\int_{Q_T}u(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\,dx\,dt.\end{aligned}\tag{3.1}$$

It is clear that $\zeta_{j_t}(t,x)=2R^{-2}t\Phi'\left(\dfrac{t^2+|x|^{2\theta_j}}{R^2}\right)$, consequently $\zeta_{j_t}(0,x)=0$, thus

$$\begin{aligned}\int_{Q_T} f|u|^{p_1}|v|^{q_1}\zeta_1\,dx\,dt&+\int_{\mathbb{R}^N}u_0(x)\zeta_1(0,x)\,dx+\int_{\mathbb{R}^N}u_1(x)\zeta_1(0,x)\,dx\\&=\int_{Q_T}u\zeta_{1tt}\,dx\,dt+\int_{Q_T}uD^{\alpha_1}_{t|T}\zeta_1\,dx\,dt+\int_{Q_T}u(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\,dx\,dt.\end{aligned}\tag{3.2}$$

Hence,

$$\int_{Q_T} f|u|^{p_1}|v|^{q_1}\zeta_1\,dx\,dt\le\int_{Q_T}|u|\,|\zeta_{1tt}|\,dx\,dt+\int_{Q_T}|u|\left|D^{\alpha_1}_{t|T}\zeta_1\right|dx\,dt+\int_{Q_T}|u|\left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|dx\,dt.\tag{3.3}$$

We have also

$$\int_{Q_T} g|u|^{p_2}|v|^{q_2}\zeta_2\,dx\,dt\le\int_{Q_T}|v|\,|\zeta_{2tt}|\,dx\,dt+\int_{Q_T}|v|\left|D^{\alpha_2}_{t|T}\zeta_2\right|dx\,dt+\int_{Q_T}|v|\left|(-\Delta)^{\frac{\beta_2}{2}}\zeta_2\right|dx\,dt.\tag{3.4}$$

To estimate

$$\int_{Q_T} |u|\,|\zeta_{1tt}|\,dx\,dt,$$

we observe that it can be rewritten as

$$\int_{Q_T} |u|\,|\zeta_{1tt}|\,dx\,dt = \int_{Q_T} |u|\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{1}{p_2}}\,|\zeta_{1tt}|\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2}}\,dx\,dt.$$

Using Hölder's inequality, we obtain

$$\int_{Q_T} |u|\,|\zeta_{1tt}|\,dx\,dt \le \left(\int_{Q_T} |u|^{p_2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}}\left(\int_{Q_T} |\zeta_{1tt}|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}}.$$

Proceeding as above, we have

$$\begin{aligned}\int_{Q_T} |u|\left|D^{\alpha_1}_{t|T}\zeta_1\right|dx\,dt \le& \left(\int_{Q_T} |u|^{p2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}}\\ &\times\left(\int_{Q_T} \left|D^{\alpha_1}_{t|T}\zeta_1\right|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}},\end{aligned}$$

and

$$\begin{aligned}\int_{Q_T} |u|\left|(-\Delta)^{\frac{\beta_1}{2}}\,\zeta_1\right|dx\,dt \le& \left(\int_{Q_T} |u|^{p_2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}}\\ &\times\left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_1}{2}}\,\zeta_1\right|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}}.\end{aligned}$$

Finally, we infer

$$\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt \le \left(\int_{Q_T} |u|^{p_2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}}\mathscr{K}_1, \tag{3.5}$$

where

$$\begin{aligned}\mathscr{K}_1 =& \left(\int_{Q_T} |\zeta_{1tt}|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}} + \left(\int_{Q_T} \left|D^{\alpha_1}_{t|T}\zeta_1\right|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}}\\ &+\left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_1}{2}}\,\zeta_1\right|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}}.\end{aligned}$$

Arguing as above we have likewise

$$\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt \le \left(\int_{Q_T} |v|^{q_1}\,(f\,|u|^{p_1}\,\zeta_1)\,dx\,dt\right)^{\frac{1}{q_1}}\mathscr{K}_2, \tag{3.6}$$

where

$$\begin{aligned}\mathscr{K}_2 =& \left(\int_{Q_T} |\zeta_{2tt}|^{\frac{q_1}{q_1-1}}\,(f\,|u|^{p_1}\,\zeta_1)^{\frac{-1}{q_1-1}}\,dx\,dt\right)^{\frac{q_1-1}{q_1}} + \left(\int_{Q_T} \left|D^{\alpha_2}_{t|T}\zeta_2\right|^{\frac{q_1}{q_1-1}}\,(f\,|u|^{p_1}\,\zeta_1)^{\frac{-1}{q_1-1}}\,dx\,dt\right)^{\frac{q_1-1}{q_1}}\\ &+\left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_2}{2}}\,\zeta_2\right|^{\frac{q_1}{q_1-1}}\,(f\,|u|^{p_1}\,\zeta_1)^{\frac{-1}{q_1-1}}\,dx\,dt\right)^{\frac{q_1-1}{q_1}}.\end{aligned}$$

Using inequalities (3.5) and (3.6), it yield

$$\left(\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \le \mathscr{K}_1\mathscr{K}_2^{\frac{1}{p_2}}. \tag{3.7}$$

similarly, we get

$$\left(\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \leq \mathscr{K}_2\mathscr{K}_1^{\frac{1}{q_1}}. \tag{3.8}$$

Now, in $\mathscr{K}_1$ we consider the scale of variables:

$$t=\tau R, \quad x=yR^{\frac{\alpha_1}{\beta_1}},$$

while in $\mathscr{K}_2$ we use:

$$t=\tau R, \quad x=yR^{\frac{\alpha_2}{\beta_2}},$$

and use the fact that

$$dxdt=R^{\left(\frac{N\alpha_1}{\beta_1}+1\right)}dyd\tau,\ \zeta_{itt}=R^{-2}\zeta_{i\tau\tau},\ D^{\alpha_i}_{0|t}\zeta_{it}=R^{-\alpha_i}D^{\alpha_i}_{0|\tau R}\zeta_{i\tau},$$

$(-\Delta)_x^{\frac{\beta_i}{2}}\zeta_i=R^{-\alpha_i}(-\Delta)_y^{\frac{\beta_i}{2}}\zeta_i,\ i=1,2,$
we arrive at

$$\left(\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \leq C\Big[R^{\gamma_1}+R^{\gamma_2}+R^{\gamma_3}\Big]\times\Big[R^{\lambda_1}+R^{\lambda_2}+R^{\lambda_3}\Big]^{\frac{1}{p_2}}, \tag{3.9}$$

similarly, we have

$$\left(\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \leq C\Big[R^{\lambda_1}+R^{\lambda_2}+R^{\lambda_3}\Big]\times\Big[R^{\gamma_1}+R^{\gamma_2}+R^{\gamma_3}\Big]^{\frac{1}{q_1}}, \tag{3.10}$$

where
$$\begin{cases}\gamma_1=\left(\frac{N\alpha_1}{\beta_1}+1\right)\left(\frac{p_2-1}{p_2}\right)-2-\left(\frac{\mu_2\alpha_1}{\beta_1}+\nu_2\right)\frac{1}{p_2}\\[2ex] \gamma_2=\left(\frac{N\alpha_1}{\beta_1}+1\right)\left(\frac{p_2-1}{p_2}\right)-\alpha_1-\left(\frac{\mu_2\alpha_1}{\beta_1}+\nu_2\right)\frac{1}{p_2}\\[2ex] \gamma_3=\left(\frac{N\alpha_1}{\beta_1}+1\right)\left(\frac{p_2-1}{p_2}\right)-\alpha_1-\left(\frac{\mu_2\alpha_1}{\beta_1}+\nu_2\right)\frac{1}{p_2}\end{cases}$$

and
$$\begin{cases}\lambda_1=\left(\frac{N\alpha_2}{\beta_2}+1\right)\left(\frac{q_1-1}{q_1}\right)-2-\left(\frac{\mu_1\alpha_2}{\beta_2}+\nu_1\right)\frac{1}{q_1}\\[2ex] \lambda_2=\left(\frac{N\alpha_2}{\beta_2}+1\right)\left(\frac{q_1-1}{q_1}\right)-\alpha_2-\left(\frac{\mu_1\alpha_2}{\beta_2}+\nu_1\right)\frac{1}{q_1}\\[2ex] \lambda_3=\left(\frac{N\alpha_2}{\beta_2}+1\right)\left(\frac{q_1-1}{q_1}\right)-\alpha_2-\left(\frac{\mu_1\alpha_2}{\beta_2}+\nu_1\right)\frac{1}{q_1}\end{cases}$$

we observe that $\gamma_1<\gamma_2=\gamma_3$ and $\lambda_1<\lambda_2=\lambda_3$, hence

$$\left(\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \leq CR^{\gamma_2+\frac{\lambda_2}{p_2}} \tag{3.11}$$

and

$$\left(\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \leq CR^{\lambda_2+\frac{\gamma_2}{q_1}}. \tag{3.12}$$

with the fact that

$$\frac{1}{p_2}+\frac{1}{\tilde{p}_2}=1 \ \text{ and } \ \frac{1}{q_1}+\frac{1}{\tilde{q}_1}=1 \tag{3.13}$$

by a simple computation,

$$\gamma_2+\frac{\lambda_2}{p_2}=N\Big(\frac{\alpha_1}{\beta_1\tilde{p}_2}+\frac{\alpha_2}{\beta_2p_2\tilde{q}_1}\Big)-\Big(\alpha_1+\frac{\alpha_2}{p_2}\Big)+\frac{1}{\tilde{p}_2}+\frac{1}{p_2\tilde{q}_1}+\frac{1}{p_2}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)+\frac{1}{p_2q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)$$

and

$$\lambda_2+\frac{\gamma_2}{q_1}=N\Big(\frac{\alpha_2}{\beta_2\tilde{q}_1}+\frac{\alpha_1}{\beta_1q_1\tilde{p}_2}\Big)-\Big(\alpha_2+\frac{\alpha_1}{q_1}\Big)+\frac{1}{\tilde{q}_1}+\frac{1}{q_1\tilde{p}_2}+\frac{1}{q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)+\frac{1}{p_2q_1}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)$$

also, using (3.13) we have

$$\frac{1}{\tilde{p}_2}+\frac{1}{p_2\tilde{q}_1}=1-\frac{1}{p_2}+\frac{1}{p_2\tilde{q}_1}=1-\frac{1}{p_2}\Big(1-\frac{1}{\tilde{q}_1}\Big)=1-\frac{1}{p_2q_1}$$

and

$$\frac{1}{\tilde{q}_1}+\frac{1}{q_1\tilde{p}_2}=1-\frac{1}{q_1}+\frac{1}{q_1\tilde{p}_2}=1-\frac{1}{q_1}\Big(1-\frac{1}{\tilde{p}_2}\Big)=1-\frac{1}{p_2q_1}$$

we obtain

$$\gamma_2+\frac{\lambda_2}{p_2}=N\Big(\frac{\alpha_1}{\beta_1\tilde{p}_2}+\frac{\alpha_2}{\beta_2p_2\tilde{q}_1}\Big)-\Big(\alpha_1+\frac{\alpha_2}{p_2}\Big)+1-\frac{1}{p_2q_1}+\frac{1}{p_2}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)+\frac{1}{p_2q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)$$

and

$$\lambda_2+\frac{\gamma_2}{q_1}=N\Big(\frac{\alpha_2}{\beta_2\tilde{q}_1}+\frac{\alpha_1}{\beta_1q_1\tilde{p}_2}\Big)-\Big(\alpha_2+\frac{\alpha_1}{q_1}\Big)+1-\frac{1}{p_2q_1}+\frac{1}{q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)+\frac{1}{p_2q_1}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)$$

We conclude that

- If $\gamma_2+\dfrac{\lambda_2}{p_2}<0$, it yield

$$N<\frac{\alpha_1+\dfrac{\alpha_2}{p_2}-\Big(1-\dfrac{1}{p_2q_1}\Big)-\dfrac{1}{p_2}\Big(\mu_2\dfrac{\alpha_1}{\beta_1}+\nu_2\Big)-\dfrac{1}{p_2q_1}\Big(\mu_1\dfrac{\alpha_2}{\beta_2}+\nu_1\Big)}{\dfrac{\alpha_1}{\beta_1\tilde{p}_2}+\dfrac{\alpha_2}{\beta_2p_2\tilde{q}_1}}.$$

  Then the right hand side of (3.11) goes to 0, when R tends to infinity, while the left hand side converge to

$$\left(\int_{Q_T}f\,|u|^{p_1}\,|v|^{q_1}\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}}.$$

  This implies that $v\equiv0$ or $u\equiv0$.
  Similarly, if $\lambda_2+\dfrac{\gamma_2}{q_1}<0$, it yield

$$N<\frac{\alpha_2+\dfrac{\alpha_1}{q_1}-\Big(1-\dfrac{1}{p_2q_1}\Big)-\dfrac{1}{q_1}\Big(\mu_1\dfrac{\alpha_2}{\beta_2}+\nu_1\Big)-\dfrac{1}{p_2q_1}\Big(\mu_2\dfrac{\alpha_1}{\beta_1}+\nu_2\Big)}{\dfrac{\alpha_2}{\beta_2\tilde{q}_1}+\dfrac{\alpha_1}{\beta_1q_1\tilde{p}_2}},$$

  by using also (3.12) to proceeding as above, we obtain $u\equiv0$ or $v\equiv0$.
- If $\gamma_2+\frac{\lambda_2}{p_2}=0$, we get

$$\int_{\mathbb{R}^+\times\mathbb{R}^N}f\,|u|^{p_1}\,|v|^{q_1}\,dx\,dt<+\infty.$$

  Using again Hölder's inequality, we obtain

$$\int_{Q_T}g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt\le\left(\int_{B_R}|v|^{q_1}\,(f\,|u|^{p_1}\,\zeta_1)\,dx\,dt\right)^{\frac{1}{q_1}}\mathscr{K}_2,$$

  where

$$B_R=\Big\{(t,x)\in\mathbb{R}^+\times\mathbb{R}^N;\ R^2\le t^2+|x|^{2\theta_1}\le 2R^2\Big\}.$$

  Since,

$$\int_{\mathbb{R}^+\times\mathbb{R}^N}f\,|u|^{p_1}\,|v|^{q_1}\,dx\,dt<+\infty,$$

  we get

$$\lim_{R\to+\infty}\int_{B_R}f\,|u|^{p_1}\,|v|^{q_1}\,dx\,dt=0,$$

  hence, we infer that

$$\int_{\mathbb{R}^+\times\mathbb{R}^N}g\,|u|^{p_2}\,|v|^{q_2}\,dx\,dt=0,$$

  this implies that $v\equiv0$ or $u\equiv0$.
  Similarly, if $\lambda_2+\frac{\gamma_2}{q_1}=0$, proceeding as above, we infer that $u\equiv0$ or $v\equiv0$.
  We deduce that no global weak solution is possible other than the trivial one, which ends the proof.

□

Remark 3.2. *In the case* $\alpha_i=1,\ \beta_i=2,\nu_i=\mu_i=0,p_1=q_2=0,$
$i=1,2$*, we recover the case who studied by A. Hakem (see [8]), when* $\alpha=\beta=0$.

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References

[1] M. BERBICHE, A. HAKEM, *Necessary conditions for the existence and sufficient conditions for the nonexistence of solutions to a certain fractional telegraph equation.* Memoirs on Differential Equations and Mathematical physics. vol 56, 2012, 37-55.
[2] M. ESCOBEDO & H. A. LEVINE, *Critical blow up and global existence numbers of a weakly coupled system of reaction-diffusion equation.* Arch. Rational. Mech. Anal. 129 (1995),47-100.
[3] A. Z. FINO, *Critical exponent for damped wave equations with nonlinear memory.* Nonlinear Analysis 74 (2011) 5495-5505.
[4] A. Z. FINO, H. IBRAHIM & A. WEHBE, *A blow-up result for a nonlinear damped wave equation in exterior domain: The critical case*, Computers & Mathematics with Applications, Volume 73, Issue 11, (2017), pp. 2415-2420.
[5] H. FUJITA, *On the blowing up of solutions of the problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci.Univ. Tokyo **13** (1966), 109 - 124.
[6] M. GUEDDA & M. KIRANE, *Local and global nonexistence of solutions to semilinear evolution equations*. Electronic Journal of Differential Equations, Conference **09** (2002), pp. 149-160.
[7] B. GUO, X. PU & F. HUANG, *Fractional Partial Differential Equations and Their Numerical Solutions*. World Scientific Publishing Co. Pte. Ltd. Beijing, China (2011).
[8] A. HAKEM, *Nonexistence of weak solutions for evolution problems on* $\mathbb{R}^N$, Bull. Belg. Math. Soc. **12** (2005), 73-82.
[9] A. HAKEM & M. BERBICHE, *On the blow-up of solutions to semi-linear wave models with fractional damping* . IAENG International Journal of Applied Mathematics, (2011) 41:3, IJAM-41-3-05.
[10] M. KIRANE, Y. LASKRI & N.-E.TATAR, *Critical exponents of fujita type for certain evolution equations and systems with spation-temporal fractional derivatives*.J. Math. Anal. Appl. 312 (2005) 488-501.
[11] W. MINGXIN, *Global existence and finite time blow up for a reaction-diffusion system*. Z. Angew. Math. Phys. 51 (2000) 160-167.
[12] T. OGAWA & H. TAKIDA, *Non-existence of weak solutions to nonlinear damped wave equations in exterior domains*, J. Nonliniear analysis **70** (2009), 3696-3701.
[13] I. PODLUBNY, *Fractional differential equations*. Mathematics in Science and Engineering, vol 198, Academic Press, New York, 1999.
[14] S.I. POHOZAEV & A. TESEI, Nonexistence of Local Solutions to Semilinear Partial Differential Inequalities, Nota Scientifica 01/28, Dip. Mat. Universitá "La Sapienza", Roma (2001).
[15] S. POHOZAEV & L. VERON, *Blow up results for nonlinear hyperbolic enequalities*, Ann. Scuola Norm. Sup. Pisa CI. Sci. (4) Vol. **XXIX** (2000), pp. 393-420.
[16] C. POZRIKIDIS, *The fractional Laplacian*, Taylor & Francis Group, LLC /CRC Press, Boca Raton (USA), (2016).
[17] S. G. SAMKO, A. A. KILBAS & O. I. MARICHEV, *Fractional integrals and derivatives: Theory and applications*. Gordan and Breach Sci. Publishers, Yverdon, 1993.
[18] G.TODOROVA & B.YORDANOV, *Critical Exponent for a Nonlinear Wave Equation with Damping*. Journal of Differential Equations **174**, 464-489 (2001).
[19] Y. YAMAUCHI, *Blow-up results for a reaction-deffusion system* , Methods Appl. Anal. **13** (2006), 337 - 350.
[20] Q. S. ZHANG, *A blow up result for a nonlinear wave equation with damping: the critical case*, C. R. Acad.Sci. paris, Volume **333**, no.2, (2001), 109-114.
[21] S-MU. ZHENG, *Nonlinear evolution equations*,
Chapman & Hall/CRC Press, Florida (USA), (2004).

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Existence and uniqueness of an inverse problem for a second order hyperbolic equation

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Abstract

In this paper, an initial boundary value problem for a second order hyperbolic equation is considered. Giving an additional condition, a time-dependent coefficient multiplying a linear term is determined and existence and uniqueness theorem for small times is proved. The finite difference method is proposed for solving the inverse problem numerically.

1. Introduction and problem formulation

Many physical models include unknown coefficients in the equation and the solution of the inverse problems consisting of the identification of these coefficients has become a very popular area of research in recent years. The inverse problems for hyperbolic equations present significant value for physical applications such as vibrating string, seismology, geophysics etc.

Consider the problem for the vibrating string with arbitrary initial conditions

$$v_{tt} = c^2 v_{xx} + a(t)v(x,t) + s(x,t), \quad (x,t) \in D_T, \tag{1.1}$$

$$v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ 0 \le x \le 1, \tag{1.2}$$

where $D_T = \{(x,t) : \ 0 < x < 1, \ 0 \le t \le T\}$ for some fixed $T > 0$, $c = \frac{\Upsilon}{\rho}$ known as phase velocity of the wave motion, Υ is the force of tension exerted on the string, ρ is the mass of density, $s(x,t)$ is external forcing function, $v = v(x,t)$ represents the wave displacement at position x and time t and the functions $v_0(x)$ and $v_1(x)$ are wave modes or kinks and velocity, respectively. In this paper, we will get $c = 1$ for simplicity.

For a given function $a(t)$, the problem of finding $v(x,t)$ from the equation (1.1) with the initial condition (1.2) and the boundary condition

$$v(0,t) = b(t), \ v_x(0,t) = v_x(1,t), \ 0 \le t \le T, \tag{1.3}$$

is called direct (forward) problem. The boundary condition $v(0,t) = b(t)$ occurs if the left end of string is attached to a spring-mass system and $b(t)$ is the position of the mass at the left end.

If $a(t)$ is unknown, finding the pair of solution $\{a(t), v(x,t)\}$ of the problem (1.1)-(1.3) with the additional condition

$$v(1,t) = r(t), \ 0 \le t \le T, \tag{1.4}$$

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is called inverse problem.
The inverse problems for the hyperbolic equation with different boundary conditions and space dependent coefficients are studied in [7, 10, 14, 16] and more recently in [5, 15]. The inverse problem for the hyperbolic equation with time dependent coefficient with integral condition is investigated in [11].
For the some numerical aspects of initial and initial-boundary value problems of the hyperbolic equations is considered for direct problem in [3], for finding unknown source term in [2, 9], for finding space dependent potential in [1, 4] and for finding time dependent source function of time-fractional wave equation in [17].
The article is organized as following: In Section 2, we rewrite the problem with homogeneous boundary conditions by a simple transformation and present auxiliary spectral problem of this problem and its properties. In Section 3, the series expansion method in terms of eigenfunctions converts the new inverse problem to a fixed point problem in a suitable Banach space. Under some consistency, regularity conditions on initial and boundary data the existence and uniqueness of the solution of the inverse problem is shown by the way that the fixed point problem has unique solution for small T . In Section 4, we solve the inverse problem numerically by applying finite difference method. We also present numerical example to illustrate the behaviour of the proposed method.

2. Auxiliary spectral problem

Since the boundary condition (1.3) is non-homogeneous, we introduce a new variable $u(x,t)=v(x,t)-b(t)$. Then, from Eqs.(1.1)-(1.4), it is easy to see that $u(x,t)$ satisfies the following problem:

$$u_{tt}=u_{xx}+a(t)u(x,t)+f(x,t),\ \ (x,t)\in D_T, \tag{2.1}$$

$$u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x),\ 0\le x\le 1, \tag{2.2}$$

$$u(0,t)=0,\ u_x(0,t)=u_x(1,t),\ 0\le t\le T, \tag{2.3}$$

$$u(1,t)=h(t),\ 0\le t\le T, \tag{2.4}$$

where $f(x,t)=s(x,t)+a(t)b(t)-b''(t)$, $\varphi(x)=v_0(x)-b(0)$, $\psi(x)=v_1(x)-b'(0)$ and $h(t)=r(t)-b(t)$.
We attempt to apply the Fourier method of eigenfunction expansion to the problem (2.1)-(2.4). Auxiliary spectral problem of the problem (2.1)-(2.3) is

$$\begin{aligned} &X''(x)+\lambda X(x)=0,\ 0\le x\le 1,\\ &X(0)=0,\ X'(0)=X'(1). \end{aligned} \tag{2.5}$$

The problem (2.5) has eigenvalues $\lambda_k=(\mu_k)^2=(2\pi k)^2$, $k=0,1,2,...$ and corresponding eigenfunctions

$$X_0(x)=x,\ X_{2k-1}(x)=x\cos\mu_k x,\ X_{2k}(x)=\sin\mu_k x,\ k=1,2,.... \tag{2.6}$$

It is known from [6] that the spectral problem (2.5) is not self-adjoint. Then we have to consider the adjoint spectral problem. The adjoint spectral problem of (2.5) is

$$\begin{aligned} &Y''(x)+\lambda Y(x)=0,\ 0\le x\le 1,\\ &Y'(1)=0,\ Y(0)=Y(1). \end{aligned} \tag{2.7}$$

This problem has the same eigenvalues as in the problem (2.5) and corresponding eigenfunctions

$$Y_0(x)=2,\ Y_{2k-1}(x)=4\cos\mu_k x,\ Y_{2k}(x)=4(1-x)\sin\mu_k x,\ k=1,2,.... \tag{2.8}$$

The systems (2.6) and (2.8) arise in [6] for the solution of non-local boundary value problem in heat equation.
It is easy to verify that the systems (2.6) and (2.8) are bi-orthonormal on [0,1], i.e.

$$\big(X_i(x),Y_j(x)\big)=\int_0^1 X_i(x)Y_j(x)dx=\begin{cases}0,\ i\neq j,\\ 1,\ i=j.\end{cases}$$

Moreover the system (2.6) forms a basis in $L_2[0,1]$ and they are also Riesz basis in $L_2[0,1]$. Then any function $g(x)\in L_2[0,1]$ is expanded in bi-orthogonal series

$$g(x)=\sum_{k=0}^{\infty} g_k X_k(x)$$

where $g_k = \int_0^1 g(x)Y_k(x)dx$, $k = 0,1,2,...$ and the estimate

$$r\|g(x)\|^2_{L_2[0,1]} \le \sum_{k=0}^{\infty} g_k^2 \le R\|g(x)\|^2_{L_2[0,1]}$$

is valid for any function $g(x) \in L_2[0,1]$, here $r = 3/4$, $R = 16$.
Let us introduce the functional space [12]

$$B^3_{2,T} = \left\{ u(x,t) = \sum_{k=0}^{\infty} u_k(t)X_k(x) : \ u_k(t) \in C[0,T], \ J_T(u) = \|u_o(t)\|_{C[0,T]} \right.$$
$$\left. + \left(\sum_{k=1}^{\infty} (\mu_k^3 \|u_{2k}(t)\|_{C[0,T]})^2 \right)^{1/2} + \left(\sum_{k=1}^{\infty} (\mu_k^3 \|u_{2k-1}(t)\|_{C[0,T]})^2 \right)^{1/2} < +\infty \right\},$$

with the norm $\|u(x,t)\|_{B^3_{2,T}} \equiv J_T(u)$ which is related with the Fourier coefficients of the function $u(x,t)$ by the eigenfunctions $X_k(x)$, $k = 0,1,2,....$ It is shown in [8] that $B^3_{2,T}$ is Banach space. Obviously $E^3_T = B^3_{2,T} \times C[0,T]$ of the vector function $z(x,t) = \{a(t), u(x,t)\}$ with the norm $\|z(x,t)\|_{E^3_T} = \|a(t)\|_{C[0,T]} + \|u(x,t)\|_{B^3_{2,T}}$ is also Banach space.

3. Existence and uniqueness of the solution

The pair $\{a(t), u(x,t)\}$ from the class $C[0,T] \times C^2(\overline{D}_T)$ for which the conditions (2.1)-(2.4) are satisfied, is called a classical solution of the inverse problem (2.1)-(2.4).
Since the system (2.6) forms Riesz basis and the systems (2.6),(2.8) are bi-orthogonal in $L_2[0,1]$ and the function $a(t)$ is time dependent, seeking the solution of the problem (2.1)-(2.4) in the following form is suitable:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)X_k(x), \tag{3.1}$$

where $u_k(t) = \int_0^1 u(x,t)Y_k(x)dx$, $k = 0,1,2,....$
For an arbitrary $a(t) \in C[0,T]$, the solution of the problem (2.1)-(2.4) can be written as

$$u(x,t) = u_0(t)X_0(x) + \sum_{k=1}^{\infty} u_{2k-1}(t)X_{2k-1}(x) + \sum_{k=1}^{\infty} u_{2k}(t)X_{2k}(x).$$

By using the Fourier's method, it is easy to obtain that $u_k(t)$, $k = 0,1,2,...$ should be satisfies the equations:

$$u_0''(t) = a(t)u_0(t) + f_0(t), \tag{3.2}$$

$$u_{2k-1}''(t) + \mu_k^2 u_{2k-1}(t) = a(t)u_{2k-1}(t) + f_{2k-1}(t), \ k = 1,2,..., \tag{3.3}$$

$$u_{2k}''(t) + \mu_k^2 u_{2k}(t) = a(t)u_{2k}(t) + f_{2k}(t) - 2\mu_k u_{2k-1}(t), \ k = 1,2,..., \tag{3.4}$$

$$u_k(0) = \varphi_k, \ \ u_k'(0) = \psi_k, \ k = 0,1,2,..., \tag{3.5}$$

where $f_k(t) = \int_0^1 f(x,t)Y_k(x)dx$, $\varphi_k = \int_0^1 \varphi(x)Y_k(x)dx$, $\psi_k = \int_0^1 \psi(x)Y_k(x)dx$, $k = 0,1,2,....$
Solving the problem (3.2)-(3.5), we obtain

$$u_0(t) = \varphi_0 + t\psi_0 + \int_0^t (t-\tau)F_0(\tau;u,a)d\tau, \ 0 \le t \le T, \tag{3.6}$$

$$u_{2k-1}(t) = \varphi_{2k-1}\cos\mu_k t + \frac{1}{\mu_k}\psi_{2k-1}\sin\mu_k t + \frac{1}{\mu_k}\int_0^t F_{2k-1}(\tau;u,a)\sin\mu_k(t-\tau)d\tau, \ k = 1,2,..., \tag{3.7}$$

$$\begin{aligned} u_{2k}(t) = {} & \varphi_{2k}\cos\mu_k t + \frac{1}{\mu_k}\psi_{2k}\sin\mu_k t + \frac{1}{\mu_k}\int_0^t F_{2k}(\tau;u,a)\sin\mu_k(t-\tau)d\tau \\ & -t\varphi_{2k-1}\sin\mu_k t - \frac{1}{\mu_k}\psi_{2k-1}\left[\frac{1}{\mu_k}\sin\mu_k t - t\cos\mu_k t\right] \\ & -\frac{2}{\mu_k}\int_0^t\int_0^\tau F_{2k-1}(\xi;u,a)\sin\mu_k(\tau-\xi)d\xi\sin\mu_k(t-\tau)d\tau, \ k = 1,2,..., \end{aligned} \tag{3.8}$$

where $F_k(t;u,a) = a(t)u_k(t) + f_k(t)$, $k = 0,1,2,....$

Substituting (3.6)-(3.8) into (3.1),

$$
\begin{aligned}
u(x,t) &= \left(\varphi_0 + t\psi_0 + \int_0^t (t-\tau)F_0(\tau;u,a)d\tau\right)X_0(x) \\
&+ \sum_{k=1}^{\infty}\left(\varphi_{2k-1}\cos\mu_k t + \frac{1}{\mu_k}\psi_{2k-1}\sin\mu_k t + \frac{1}{\mu_k}\int_0^t F_{2k-1}(\tau;u,a)\sin\mu_k(t-\tau)d\tau\right)X_{2k-1}(x) \\
&+ \sum_{k=1}^{\infty}\left(\varphi_{2k}\cos\mu_k t + \frac{1}{\mu_k}\psi_{2k}\sin\mu_k t + \frac{1}{\mu_k}\int_0^t F_{2k}(\tau;u,a)\sin\mu_k(t-\tau)d\tau - t\varphi_{2k-1}\sin\mu_k t\right. \\
&\left. - \frac{1}{\mu_k}\psi_{2k-1}\left[\frac{1}{\mu_k}\sin\mu_k t - t\cos\mu_k t\right] - \frac{2}{\mu_k}\int_0^t\int_0^\tau F_{2k-1}(\xi;u,a)\sin\mu_k(\tau-\xi)d\xi\sin\mu_k(t-\tau)d\tau\right)X_{2k}(x). \qquad (3.9)
\end{aligned}
$$

Consider $x=1$ in the equation (2.1) and by using the over-determination condition (2.4), we obtain

$$
a(t) = \frac{1}{h(t)}\left[h''(t) - f(1,t) + \sum_{k=1}^{\infty}\mu_k^2\left(\varphi_{2k-1}\cos\mu_k t + \frac{\psi_{2k-1}}{\mu_k}\sin\mu_k t + \frac{1}{\mu_k}\int_0^t F_{2k-1}(\tau;u,a)\sin\mu_k(t-\tau)d\tau\right)\right]. \qquad (3.10)
$$

Thus, we get the equalities of the pair $\{a(t),u(x,t)\}$.
Let us denote $z=[a(t),u(x,t)]^T$ and consider the operator equation

$$
z = \Phi(z). \qquad (3.11)
$$

The operator Φ is determined in the set of functions z and has the form $[\phi_0,\phi_1]^T$, where

$$
\phi_0(z) = \frac{1}{h(t)}\left[h''(t) - f(1,t) + \sum_{k=1}^{\infty}\mu_k^2\left(\varphi_{2k-1}\cos\mu_k t + \frac{\psi_{2k-1}}{\mu_k}\sin\mu_k t + \frac{1}{\mu_k}\int_0^t F_{2k-1}(\tau;u,a)\sin\mu_k(t-\tau)d\tau\right)\right], \qquad (3.12)
$$

$$
\begin{aligned}
\phi_1(z) &= \left(\varphi_0 + t\psi_0 + \int_0^t (t-\tau)F_0(\tau;u,a)d\tau\right)X_0(x) \\
&+ \sum_{k=1}^{\infty}\left(\varphi_{2k-1}\cos\mu_k t + \frac{1}{\mu_k}\psi_{2k-1}\sin\mu_k t + \frac{1}{\mu_k}\int_0^t F_{2k-1}(\tau;u,a)\sin\mu_k(t-\tau)d\tau\right)X_{2k-1}(x) \\
&+ \sum_{k=1}^{\infty}\left(\varphi_{2k}\cos\mu_k t + \frac{1}{\mu_k}\psi_{2k}\sin\mu_k t + \frac{1}{\mu_k}\int_0^t F_{2k}(\tau;u,a)\sin\mu_k(t-\tau)d\tau - t\varphi_{2k-1}\sin\mu_k t\right. \\
&\left. - \frac{1}{\mu_k}\psi_{2k-1}\left[\frac{1}{\mu_k}\sin\mu_k t - t\cos\mu_k t\right] - \frac{2}{\mu_k}\int_0^t\int_0^\tau F_{2k-1}(\xi;u,a)\sin\mu_k(\tau-\xi)d\xi\sin\mu_k(t-\tau)d\tau\right)X_{2k}(x). \qquad (3.13)
\end{aligned}
$$

Let us demonstrate that Φ maps E_T^3 onto itself continuously. In other words, we need to show $\phi_0(z)\in C[0,T]$ and $\phi_1(z)\in B_{2,T}^3$ for arbitrary $z=[a(t),u(x,t)]^T$ with $a(t)\in C[0,T]$, $u(x,t)\in B_{2,T}^3$.
We will use the following assumptions on the data of problem (2.1) - (2.4):

(A_1) $\varphi(x)\in C^3[0,1]$, $\varphi(0)=\varphi''(0)=0$, $\varphi'(0)=\varphi'(1)$,

(A_2) $\psi(x)\in C^2[0,1]$, $\psi(0)=0$, $\psi'(0)=\psi'(1)$,

(A_3) $h(t)\in C^2[0,T]$, $h(0)=\varphi(1)$, $h'(0)=\psi(1)$,

(A_4) $f(x,t)\in C(\overline{D}_T)$, $f_x, f_{xx}\in C[0,1]$, $\forall t\in[0,T]$, $f(0,t)=0$, $f_x(0,t)=f_x(1,t)$.

First, let us show that $\phi_0(z)\in C[0,T]$. Under the assumptions (A_1)-(A_4), we obtain from (3.12)

$$
|\phi_0(z)| \le \frac{1}{|h(t)|}\left[\left|h''(t)\right| + |f(1,t)| + \sum_{k=1}^{\infty}\left(\frac{1}{\mu_k}|\alpha_{2k-1}| + \frac{1}{\mu_k}|\beta_{2k-1}| + \frac{1}{\mu_k}\int_0^T\left[|\gamma_{2k-1}(t)| + \mu_k^2|a(t)||u_{2k-1}(t)|\right]dt\right)\right],
$$

where $\varphi_{2k-1}=\frac{1}{\mu_k^3}\alpha_{2k-1}$, $\psi_{2k-1}=\frac{1}{\mu_k^2}\beta_{2k-1}$, $f_{2k-1}(t)=\frac{1}{\mu_k^2}\gamma_{2k-1}(t)$, $\alpha_{2k-1}=-4\int_0^1\varphi'''(x)\sin(\mu_k x)dx$, $\beta_{2k-1}=4\int_0^1\psi''(x)\sin(\mu_k x)dx$, $\gamma_{2k-1}(t)=4\int_0^1 f_{xx}(x,t)\sin(\mu_k x)dx$.
Using Cauchy-Schwartz and Bessel inequalities, we derive from the last inequality

$$
\|\phi_0(t)\|_{C[0,T]} \le R_1(T) + R_2(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}, \qquad (3.14)
$$

where $R_1(T)=\frac{1}{\|h(t)\|_{C[0,T]}}(\|h''(t)\|_{C[0,T]}+\|f(1,t)\|_{C[0,T]}+C_0(\|\varphi'''(x)\|_{L_2[0,1]}+\|\psi''(x)\|_{L_2[0,1]}+T\|f_{xx}(x,t)\|_{L_2(D_T)}))$, $R_2(T)=\frac{2C_1T}{\|h(t)\|_{C[0,T]}}$, $C_0=\left(\sum_{k=1}^{\infty}\frac{1}{\mu_k^2}\right)^{1/2}$, $C_1=\left(\sum_{k=1}^{\infty}\frac{1}{\mu_k^4}\right)^{1/2}$. Thus $\phi_0(z)$ is continuous in $[0,T]$.
Now, let us verify that $\phi_1(z)\in B_{2,T}^3$, i.e.

$$J_T(\phi_1)=\left\|\phi_{1,0}(t)\right\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}(\mu_k^3\left\|\phi_{1,2k}(t)\right\|_{C[0,T]})^2\right)^{1/2}+\left(\sum_{k=1}^{\infty}(\mu_k^3\left\|\phi_{1,2k-1}(t)\right\|_{C[0,T]})^2\right)^{1/2}<+\infty,$$

where $\phi_{1,0}(t)$, $\phi_{1,2k}(t)$ and $\phi_{1,2k-1}(t)$ are the equal to the right hand side of $u_0(t)$, $u_{2k}(t)$ and u_{2k-1} as in (3.6)-(3.8), respectively. After some manipulations under the assumptions (A$_1$)-(A$_4$),

$$\|\phi_1\|_{B_{2,T}^3}\leq\widetilde{R}_1(T)+\widetilde{R}_2(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}, \tag{3.15}$$

where $\widetilde{R}_1(T)=\|\varphi(x)\|_{C[0,1]}+T\|\psi(x)\|_{C[0,1]}+2T^{3/2}\|f(x,t)\|_{C(D_T)}+(\sqrt{2}+4T)\|\varphi'''(x)\|_{L_2[0,1]}+(4+4T)\|\psi''(x)\|_{L_2[0,1]}$
$+(2\sqrt{2T}+8T^{3/2})\|f_{xx}(x,t)\|_{L_2(D_T)}+2\|\varphi'''(x)(1-x)-3\varphi''(x)\|_{L_2[0,1]}+2\|\psi''(x)(1-x)-2\psi'(x)\|_{L_2[0,1]}+2T^{1/2}\|f_{xx}(x,t)(1-x)-2f_x(x,t)\|_{L_2(D_T)}$,
$\widetilde{R}_2(T)=\max(T^2,2C_0T,4\sqrt{2}C_0T^2)$.
Since $J_T(\phi_1)<+\infty$, ϕ_1 is belongs to the space $B_{2,T}^3$. Now, let z_1 and z_2 be any two elements of E_T^3. We know that $\|\Phi(z_1)-\Phi(z_2)\|_{E_T^3}=\|\phi_0(z_1)-\phi_0(z_2)\|_{C[0,T]}+\|\phi_1(z_1)-\phi_2(z_2)\|_{B_{2,T}^3}$. Here $z_i=[a^i(t),u^i(x,t)]^T$, $i=1,2$.
Under the assumptions (A$_1$)-(A$_4$), we obtain

$$\|\Phi(z_1)-\Phi(z_2)\|_{E_T^3}\leq A(T)C(a^1,u^2)\|z_1-z_2\|_{E_T^3},$$

where $A(T)=R_2(T)+\widetilde{R}_2(T)$ and $C(a^1,u^2)$ is the constant includes the norms of $\left\|a^1(t)\right\|_{C[0,T]}$ and $\left\|u^2(x,t)\right\|_{B_{2,T}^3}$.
For sufficiently small T, $0<A(T)<1$. This implies that the operator Φ is contraction mapping which maps E_T^3 onto itself continuously. Then according to Banach fixed point theorem there exists a unique solution of (3.11).
Thus, we proved the following theorem:

Theorem 3.1. *Let the assumptions (A$_1$)-(A$_4$) be satisfied. Then, the inverse problem (2.1)-(2.4) has unique solution for small T.*

Note that $s(x,t)=f(x,t)-a(t)b(t)-b''(t)$, $v_0(x)=\varphi(x)+b(0)$, $v_1(x)=\psi(x)+b'(0)$ and $r(t)=h(t)+b(t)$ then under the following assumptions:

$(\widetilde{A}_1)$ $v_0(x)\in C^3[0,1]$, $v_0(0)=b(0)$, $v_0''(0)=0$, $v_0'(0)=v_0'(1)$,
$(\widetilde{A}_2)$ $v_1(x)\in C^2[0,1]$, $v_1(0)=b'(0)$, $v_1'(0)=v_1'(1)$,
$(\widetilde{A}_3)$ $r(t)\in C^2[0,T]$, $r(0)=v_0(1)$, $r'(0)=v_1(1)$,
$(\widetilde{A}_4)$ $s(x,t)\in C(\overline{D}_T)$, $s_x,s_{xx}\in C[0,1]$, $\forall t\in[0,T]$, $s(0,t)=-a(t)b(t)+b''(t)$, $s_x(0,t)=s_x(1,t)$,

the problem (1.1)-(1.4) has a unique classical solution $\{r(t),v(x,t)\}$ for small T where $v(x,t)=u(x,t)+b(t)$.

4. Numerical solution of the problem

In this section,we describe the numerical method applied to the inverse initial boundary value problem (1.1)-(1.4). The discrete form of our problem is as follows: We divide the domain $(0,1)\times(0,T)$ into nx and nt subintervals of equal length hx and ht, where $hx=1/nx$ and $ht=T/nt$, respectively. We denote by $V_j^n:=V(x_j,t_n)$, $a^n:=a(t_n)$ and $s_j^n:=s(x_j,t_n)$, where $x_j=jhx$, $t_n=nht$ for $j=0,...,nx$, $n=0,...,nt$.

Then, a central difference approximation to the equations (1.1)-(1.3) at the mesh points (x_j,t_n) is

$$V_j^{n+1}=k^2V_{j+1}^n+2(1-k^2)V_j^n+k^2V_{j-1}^n-V_j^{n-1}+(ht)^2(a^nV_j^n+s_j^n),\ j=1,...,nx-1,\ n=1,...,nt-1, \tag{4.1}$$

$$V_j^0=(v_0)_j,\ \ j=0,...,nx,\ \frac{V_j^1-V_j^{-1}}{2ht}=(v_1)_j,\ j=1,...,nx-1, \tag{4.2}$$

$$V_0^n=b^n,\ \frac{V_{nx}^n-V_{nx-1}^n}{hx}=\frac{V_1^n-V_0^n}{hx},\ n=0,...,nt, \tag{4.3}$$

where $k=\frac{ht}{hx}$. Equation (4.1) represents an explicit finite difference method which is stable for $k\leq 1$. Putting $n=0$ in the equation (4.1) and using (4.2), we obtain

$$V_j^1=\frac{1}{2}(k^2(v_0)_{j+1}+2(1-k^2)(v_0)_j+k^2(v_0)_{j-1}+2ht(v_1)_j+(ht)^2(a^0(v_0)_j+s_j^0)),j=1,...,nx-1. \tag{4.4}$$

Consider (1.4) in the equation (1.1) at $x = 1$, we obtain

$$a(t) = \frac{r''(t) - v_{xx}(1,t) - s(1,t)}{r(t)}.$$

After discretizing last equation, we have

$$a^n = \frac{(r^{n+1} - 2r^n + r^{n-1})/(ht)^2 - (V^n_{nx} - 2V^n_{nx-1} + V^n_{nx-2})/(hx)^2 - s^n_{nx}}{r^n}, \ n = 1,...,nt-1, \tag{4.5}$$

$$a^{nt} = \frac{(r^{nt} - 2r^{nt-1} + r^{nt-2})/(ht)^2 - (V^{nt}_{nx} - 2V^{nt}_{nx-1} + V^{nt}_{nx-2})/(hx)^2 - s^{nt}_{nx}}{r^{nt}}, \tag{4.6}$$

$$a^0 = \frac{(r^2 - 2r^1 + r^0)/(ht)^2 - (V^0_{nx} - 2V^0_{nx-1} + V^0_{nx-2})/(hx)^2 - s^0_{nx}}{r^0}. \tag{4.7}$$

Now let us consider (4.5)-(4.7) in (4.1), we obtain the system with respect to V^n_j , $j = 0,...,nx$, $n = 0,...,nt$ which can be solved explicitly. Then using the calculated values of V^n_{nx-2} in (4.5)-(4.7), we obtain the values of a^n , $n = 0,...,nt$.

Example 4.1. *Consider the inverse problem (1.1)-(1.4) with the input data*

$$\begin{aligned} s(x,t) &= (1 + 2\pi x - \sin 2\pi x)\exp(t) - 1 - 2\pi x + \sin 2\pi x, \\ v_0(x) &= 1 + 2\pi x - \sin 2\pi x, \ v_1(x) = 1 + 2\pi x - \sin 2\pi x, \\ b(t) &= \exp(t), \ r(t) = (1 + 2\pi)\exp(t), \ x \in [0,1], \ t \in [0,1]. \end{aligned}$$

It is easy to check that the conditions $\left(\widetilde{A}_1\right) - \left(\widetilde{A}_4\right)$ are satisfied. Then according to Theorem 3.1 the solution of the inverse problem exists and unique. In fact, it can easily be checked by direct substitution that the analytical solution is given by

$$\{a(t), v(x,t)\} = \{1/\exp(t), (1 + 2\pi x - \sin 2\pi x)\exp(t)\}.$$

Figure 4.1 shows the exact and numerical solution of $\{a(t), v(x,t)\}$ for $nt = 128$ and $nx = 64$. Next, we investigate the stability of numerical solution with respect to the noisy over-determination data (1.4), denoted by the function $r_\gamma(t) = r(t)(1 + \gamma\theta)$ where γ is the percentage of noise and θ are random variables generated from a uniform distribution in the interval $[-0.5, 0.5]$ which are generated using rand command in MATLAB. Figs. 4.2, 4.3 show the exact and numerical solutions of $\{a(t), v(nx/2,t)\}$ when the input data (1.4) is contaminated by $\gamma = 1\%, 3\%$ and 5% noise. Figs. 4.4, 4.5 show the exact and numerical solutions of $\{a(t), v(nx/2,t)\}$ obtained after mollification, when the input data (1.4) is contaminated by $\gamma = 1\%, 3\%$ and 5% noise. This mollification procedure has been performed using MATLAB version of the computational program supplied by D. A. Murio in [13]. From these figures it can be seen that the application of the mollification to stabilize the noisy function $r_\gamma(t)$, produce stable numerical solutions for $\{a(t), v(nx/2,t)\}$.

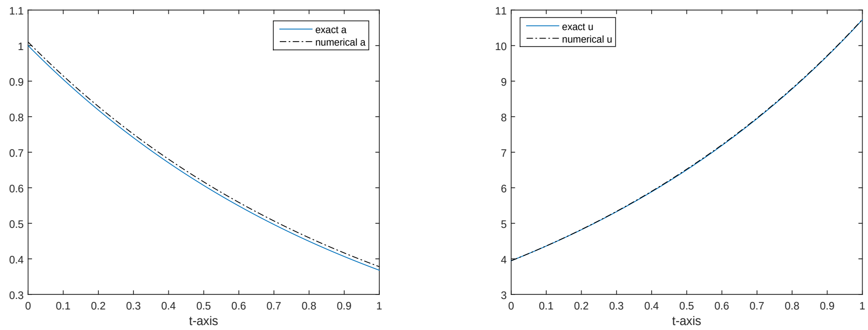

Figure 4.1: Exact and numerical solution of the problem (1)-(4) for example 4.1.

5. Conclusion

The inverse problems for linear wave equations connected with recovery of the coefficient are scarce. The paper considers the inverse problem of recovering a time-dependent coefficient in an initial boundary value problem for a wave equation with a non-homogeneous boundary condition. The series expansion method in terms of eigenfunctions of a Sturm-Liouville problem converts the considered inverse problem to a fixed point problem in a suitable Banach space. Under some conditions on the data, the existence and uniqueness of inverse problem is shown by using the Banach fixed point theorem. Numerically, the inverse problem has been discretized by using finite difference method, which has been solved using the MATLAB. Numerical results show that accurate, and stable solutions have been obtained.

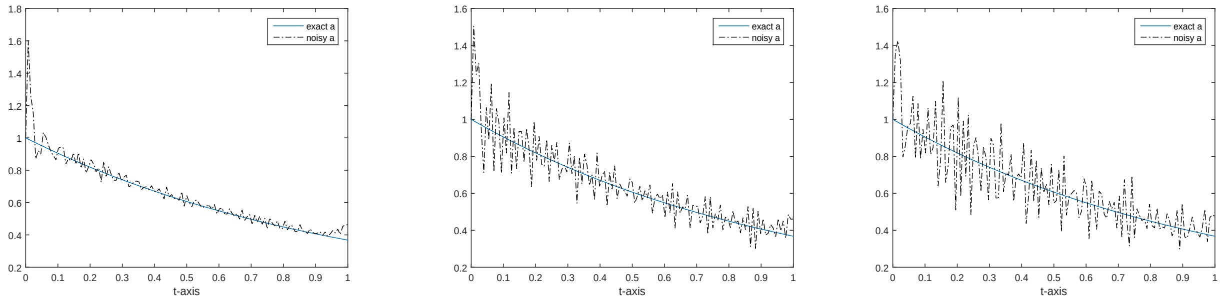

Figure 4.2: Exact and numerical coefficient solution of the problem (1)-(4) for example 4.1 with 1%, 3% and 5% noise.

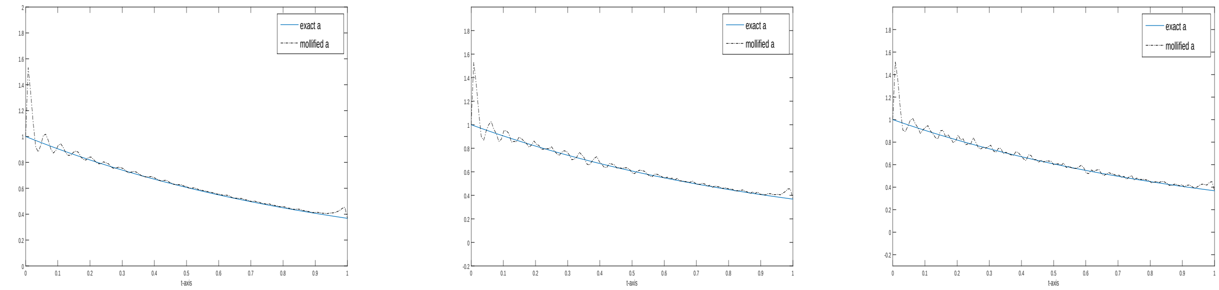

Figure 4.3: Exact and numerical coefficient solution of the problem (1)-(4) for example 4.1 after mollification with 1%, 3% and 5% noise.

References

[1] L. Beilina, M. V. Klibanov, "A globally convergent numerical method for a coefficient inverse problem." SIAM Journal on Scientific Computing 31.1 (2008): 478-509.
[2] J. R. Cannon, P. DuChateau, "An inverse problem for an unknown source term in a wave equation." SIAM Journal on Applied Mathematics 43.3 (1983): 553-564.
[3] M. Dehghan, "On the solution of an initial-boundary value problem that combines Neumann and integral condition for the wave equation." Numerical Methods for Partial Differential Equations 21.1 (2005): 24-40.
[4] S. O. Hussein, D. Lesnic, M. Yamamoto, "Reconstruction of space-dependent potential and/or damping coefficients in the wave equation." Computers & Mathematics with Applications 74.6 (2017): 1435-1454.
[5] O. Imanuvilov, M. Yamamoto, "Global uniqueness and stability in determining coefficients of wave equations." Comm. Part. Diff. Equat., 26 (2001), 1409– 1425.
[6] N. I. Ionkin, "The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition." Differ. Uravn., 1977, Volume 13, Number 2, 294–304
[7] V. Isakov, Inverse problems for partial differential equations. Applied mathematical sciences. New York (NY): Springer; 2006.
[8] K. I. Khudaverdiyev, A. G. Alieva, "On the global existence of solution to one-dimensional fourth order nonlinear Sobolev type equations." Appl. Math. Comput. 217 (2010), no. 1, 347-354.
[9] D. Lesnic, S. O. Hussein, B. T. Johansson, "Inverse space-dependent force problems for the wave equation." Journal of Computational and Applied Mathematics 306 (2016): 10-39.
[10] Z. Lin, R. P. Gilbert, "Numerical algorithm based on transmutation for solving inverse wave equation." Mathematical and computer modelling 39.13 (2004): 1467-1476.
[11] Y. Megraliev, Q. N. Isgenderova, "Inverse boundary value problem for a second-order hyperbolic equation with integral condition of the first kind." Problemy Fiziki, Matematiki i Tekhniki (Problems of Physics, Mathematics and Technics) 1 (2016): 42-47.
[12] Y. T. Mehraliyev, "On the identification of a linear source for the second order elliptic equation with integral condition." Tr. Inst. Mat., 2013, Volume 21, Number 2, 128–141
[13] D.A. Murio, Mollification and space marching, in: K.A.Woodbury (Ed.), Inverse Engineering Handbook, CRC Press, Boca Raton, Florida, 2002, pp. 219-326.
[14] G. K. Namazov, Inverse Problems of the Theory of Equations of Mathematical Physics, Baku, Azerbaijan, 1984. (in Russian).
[15] A. I. Prilepko, D. G. Orlovsky, I. A. Vasin, Methods for solving inverse problems in mathematical physics. Vol. 231, Pure and AppliedMathematics. New York (NY): Marcel Dekker; 2000.
[16] V.G. Romanov, Inverse Problems of Mathematical Physics, VNU Science Press BV, Utrecht, Netherlands, 1987.
[17] K. Šišková, M. Slodička, "Recognition of a time-dependent source in a time-fractional wave equation." Applied Numerical Mathematics 121 (2017): 1-17.

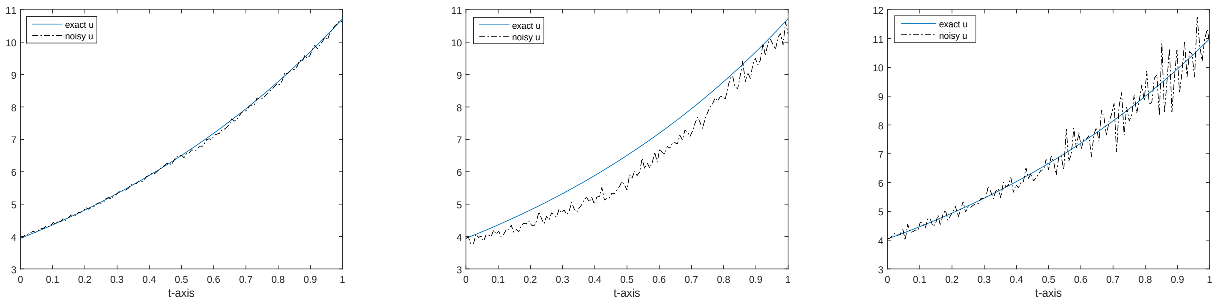

Figure 4.4: Exact and numerical $u(x,t)$ solution of the problem (1)-(4) for example 1 with 1%, 3% and 5% noise.

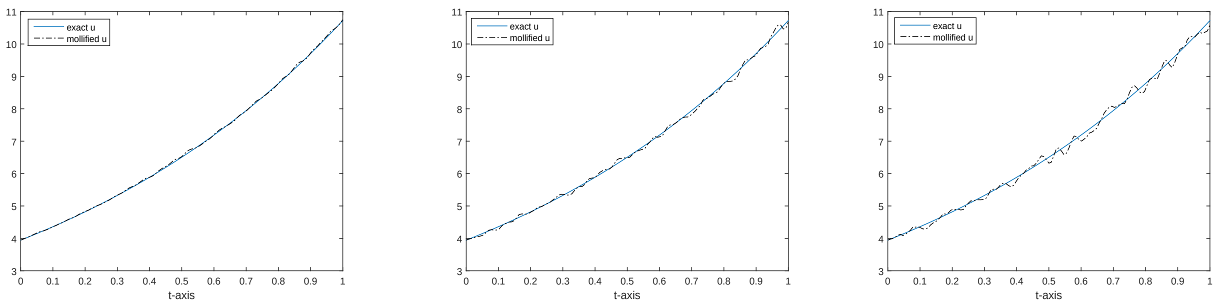

Figure 4.5: Exact and numerical $u(x,t)$ solution of the problem (1)-(4) for example 1 after mollification with 1%, 3% and 5% noise.

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Multiple solutions for a class of superquadratic fractional Hamiltonian systems

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Abstract

In this paper, we are concerned with the existence of solutions for a class of fractional Hamiltonian systems

$$\begin{cases} {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u)(t) + L(t)u(t) = \nabla W(t,u(t)),\ t \in \mathbb{R} \\ u \in H^{\alpha}(\mathbb{R},\ \mathbb{R}^N), \end{cases}$$

where ${}_tD^{\alpha}_{\infty}$ and ${}_{-\infty}D^{\alpha}_t$ are the Liouville-Weyl fractional derivatives of order $\frac{1}{2} < \alpha < 1$, $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function and $W(t,x) \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$. Applying a Symmetric Mountain Pass Theorem, we prove the existence of infinitely many solutions for (1) when L is not required to be either uniformly positive definite or coercive and $W(t,x)$ satisfies some weaker superquadratic conditions at infinity in the second variable but does not satisfy the well-known Ambrosetti-Rabinowitz superquadratic growth condition.

1. Introduction.

Fractional differential equations both ordinary and partial ones have attracted extensive attentions because of their applications in mathematical modeling of processes in physics, mechanics, control theory, viscoelasticity, electro chemistry, bioengineering, economics and others. Therefore, the theory of fractional differential equations is an area intensively developed during the last decades [1], [11]. The monographs [13], [17], [18] enclose a review of methods of solving fractional differential equations, which are an extension of procedures from differential equations theory.
Recently, many results were obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by using techniques of Nonlinear Analysis, such as fixed point theory [5], [25], topological degree theory [6], [19], comparison methods [14], [24], and so on.
It should be noted that critical point theory and variational methods serve as effective tools in the study of integer-order differential equations. The underlying idea in this approach rest on finding critical points for suitable energy functional defined on an appropriate function space. During the last three decades, the critical point theory has been developed into a wonderful tool for investigating the existence criteria for the solutions of differential equations with variational structures, for example see [15], [19] and the references cited therein.
Motivated by the classical works in [15], [19], for the first time, the author [10] showed that critical point theory and variational methods are an effective approach to tackle the existence of solutions for the following fractional boundary value problem

$$\begin{cases} {}_tD^{\alpha}_{T}({}_0D^{\alpha}_t u)(t) = \nabla W(t,u(t)),\ t \in [0,T] \\ u(0) = u(T), \end{cases}$$

where $\frac{1}{2} < \alpha < 1$, $W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ with derivative $\nabla W(t,x) = \frac{\partial W}{\partial x}(t,x)$, and obtained the existence of at least one nontrivial solution. Inspired by this work, Torres [20] considered the following fractional Hamiltonian system

$$\mathscr{FHS} \qquad \begin{cases} {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u)(t) + L(t)u(t) = \nabla W(t,u(t)),\ t \in \mathbb{R} \\ u \in H^{\alpha}(\mathbb{R},\ \mathbb{R}^N), \end{cases}$$

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where $\frac{1}{2} < \alpha < 1$, $W(t,x)$ is as above and L satisfies
(1.1) $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a positive definite symmetric matrix-valued function, and there exists an $l \in C(\mathbb{R}, \mathbb{R}_+^*)$ such that $l(t) \longrightarrow +\infty$ as $|t| \longrightarrow \infty$ and

$$L(t)x.x \geq l(t)\,|x|^2\,,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Assuming that $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ satisfies the well-known Ambrosetti-Rabinowitz superquadratic condition $(\mathscr{AR})$ and some other suitable conditions, the author [20] showed that the fractional Hamiltonian system $(\mathscr{FHS})$ possesses at least one nontrivial solution using the Mountain Pass Theorem. Since then, the existence and multiplicity of solutions of problem $(\mathscr{FHS})$ via critical point theory have been investigated in many papers [3,4,7,8,16,20-23,25-28].
Recently, Mèndez and Torres [16] proved the existence of multiple solutions for the fractional Hamiltonian system $(\mathscr{FHS})$ when the potential W satisfies some subquadratic conditions at infinity and the matrix-valued function L satisfies the following noncoercive conditions
(L_1) $L(t)$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R}, \mathbb{R})$ such that

$$\inf_{t\in\mathbb{R}} l(t) > 0 \textit{ and } L(t)x.x \geq l(t)\,|x|^2\,,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(L_2) There exists a constant $r_0 > 0$ such that

$$\lim_{|s|\longrightarrow\infty} meas(\{t \in]s - r_0, s + r_0[/ L(t) < bI_N\}) = 0,\ \forall b > 0,$$

where *meas* denotes the Lebesgue's measure on $\mathbb{R}$. The above conditions on L guarantee the compactness of Sobolev embedding. Besides, in all the above mentioned papers, the potential W is required to be subquadratic or to satisfy the Ambrosetti-Rabinowitz superquadratic condition $(\mathscr{AR})$ at infinity.
The aim of this paper is to study the existence of infinitely many solutions for $(\mathscr{FHS})$, when the function L is unnecessarily positive definite or coercive, and the potential W satisfies some superquadratic conditions at infinity, weaker than the $(\mathscr{AR})-$condition. More precisely, let $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ be such that for all $r > 0$, ∇W is bounded in $\mathbb{R} \times B_r(0)$, we make the following hypotheses:
(L) The smallest eigenvalue of $L(t)$ is bounded from below;

$$(W_1) \qquad W(t,0) = 0 \textit{ and } \nabla W(t,x) = o(|x|),\ \textit{as}\ |x| \longrightarrow 0\ \textit{uniformly on}\ t \in \mathbb{R};$$

$$(W_2) \qquad \lim_{|x|\to+\infty} \frac{|W(t,x)|}{|x|^2} = +\infty,\ \forall t \in \mathbb{R},$$

and

$$(W_3) \qquad W(t,x) \geq 0,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \textit{ with } |x| \geq R_0; W(t,-x) = W(t,x),\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W_4) There exist $g \in L^1(\mathbb{R})$ and constants b_0, $c_0 > 0$ and $\nu \in]0,2[$ such that

$$\widehat{W}(t,x) = \frac{1}{2}\nabla W(t,x).x - W(t,x) \geq \begin{cases} g(t),\ \forall t \in \mathbb{R},\ |x| \leq R_0 \\ b_0\,|x|^\nu\,,\ \forall t \in \mathbb{R},\ |x| \geq R_0; \end{cases}$$

$$|W(t,x)| \leq c_0\,|x|^{2-\nu}\,\widehat{W}(t,x),\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \textit{ with } |x| \geq R_0;$$

(W_5) There exist constants $\mu > 2$ and $\rho_0 > 0$ such that

$$\mu W(t,x) \leq \nabla W(t,x).x + \rho_0\,|x|^2\,,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Our main results read as follows.

Theorem 1.1. *Assume that (L), (L_2) and $(W_1)-(W_4)$ are satisfied. Then $(\mathscr{FHS})$ possesses infinitely many nontrivial solutions.*

Theorem 1.2. *Assume that (L), (L_2), $(W_1)-(W_3)$ and (W_5) are satisfied. Then $(\mathscr{FHS})$ possesses infinitely many nontrivial solutions.*

Remark 1.3.
1. *In our results, $L(t)$ is unnecessarily required to be either uniformly positive definite or coercive. For example $L(t) = (t^2 sin^2 t - 1)I_N$, where I_N is the identity matrix, satisfies (L) and (L_2), but it does satisfy neither Theorem 1.1 nor Theorem 1.2.*
2. *Let $W(t,x) = a(t)\,|x|^2\,ln(\frac{1}{2} + |x|)$, where a is a continuous bounded function with positive lower bound. Then an easy computation shows that W satisfies the superquadratic conditions $(W_1)-(W_4)$. However, W does not satisfy the $(\mathscr{AR})-$condition.*

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. Section 3 is devoted to the proofs of our results.

2. Preliminaries

In this Section, for the reader's convenience, first we will recall some facts about the fractional calculus on the whole real axis. On the other hand, we will give some preliminaries lemmas for using in the sequel.

2.1. Liouville-Weyl fractional calculus

The Liouville-Weyl fractional integrals of order $0<\alpha<1$ on the whole axis $\mathbb{R}$ are defined as (see [12], [13], [18])

$$ {}_{-\infty}I_t^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-x)^{\alpha-1}u(x)dx, $$

and

$$ {}_tI_{\infty}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_{t}^{\infty}(x-t)^{\alpha-1}u(x)dx. $$

The Liouville-Weyl fractional derivatives of order $0<\alpha<1$ on the whole axis $\mathbb{R}$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals (see [12], [13], [18])

$$ {}_{\infty}D_t^{\alpha}u(t)=\frac{d}{dt}({}_{-\infty}I_t^{1-\alpha}u)(t), \tag{2.1} $$

and

$$ {}_tD_{\infty}^{\alpha}u(t)=-\frac{d}{dt}({}_tI_{\infty}^{1-\alpha}u)(t). \tag{2.2} $$

The definitions of 2.1 and 3.2 may be written in an alternative form as follows

$$ {}_{-\infty}D_t^{\alpha}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{u(t)-u(t-x)}{x^{\alpha+1}}dx, $$

and

$$ {}_tD_{\infty}^{\alpha}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{u(t)-u(t+x)}{x^{\alpha+1}}dx. $$

We establish the Fourier transform properties of the fractional integral and fractional differential operators. Recall that the Fourier transform $\widehat{u}$ of u is defined by

$$ \widehat{u}(s)=\int_{-\infty}^{\infty}e^{-ist}u(t)dt. $$

Let u be defined on $\mathbb{R}$. Then the Fourier transform of the Liouville-Weyl integrals and differential operators satisfies (see [12,13])

$$ \widehat{{}_{-\infty}I_t^{\alpha}u}(s)=(is)^{-\alpha}\widehat{u}(s), $$
$$ \widehat{{}_tI_{\infty}^{\alpha}u}(s)=(-is)^{-\alpha}\widehat{u}(s), $$
$$ \widehat{{}_{-\infty}D_t^{\alpha}u}(s)=(is)^{\alpha}\widehat{u}(s), $$
$$ \widehat{{}_tD_{\infty}^{\alpha}u}(s)=(-is)^{\alpha}\widehat{u}(s). $$

Next, we present some properties for Liouville-Weyl fractional integrals and derivatives on the real axis, which were proved in [12]. Denote by $L^p(\mathbb{R},\mathbb{R}^N)$ $(1\leq p<\infty)$, the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norms

$$ \|u\|_{L^p}=\left(\int_{\mathbb{R}}|u(t)|^p\,dt\right)^{\frac{1}{p}}, $$

and $L^{\infty}(\mathbb{R},\mathbb{R}^N)$ the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^N$ equipped with the norm

$$ \|u\|_{\infty}=esssup\left\{|u(t)|\,/t\in\mathbb{R}\right\}. $$

2.2. Fractional derivative spaces

In order to establish the variational structure which enables us to reduce the existence of solutions of $(\mathscr{FHS})$ to find critical points of the corresponding functional, it is necessary to construct the appropriate functional spaces.
For $\alpha>0$, define the semi-norm

$$ |u|_{I_{-\infty}^{\alpha}}=\|{}_{-\infty}D_t^{\alpha}u\|_{L^2} $$

and the norm

$$ \|u\|_{I_{-\infty}^{\alpha}}=\left(\|u\|_{L^2}+|u|^2_{I_{-\infty}^{\alpha}}\right)^{\frac{1}{2}}, $$

and let

$$ I_{-\infty}^{\alpha}=\overline{C_0^{\infty}(\mathbb{R},\mathbb{R}^N)}^{\|.\|\|_{I_{-\infty}^{\alpha}}} $$

where $C_0^{\infty}(\mathbb{R},\mathbb{R}^N)$ denotes the space of infinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}^N$ with vanishing property at infinity.
Now, we can define the fractional Sobolev space $H^{\alpha}(\mathbb{R},\mathbb{R}^N)$ in terms of the Fourier transform. Choose $0<\alpha<1$, define the semi-norm

$$ |u|_{\alpha}=\left\||s|^{\alpha}\widehat{u}\right\|_{L^2} $$

and the norm

$$\|u\|_\alpha = (\|u\|_{L^2} + |u|_\alpha^2)^{\frac{1}{2}},$$

and let

$$H^\alpha(\mathbb{R},\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^N)}^{\|\cdot\|_\alpha}.$$

Moreover, we note that a function $u \in L^2(\mathbb{R},\mathbb{R}^N)$ belongs to $I^\alpha_{-\infty}$ if and only if

$$|s|^\alpha \widehat{u} \in L^2(\mathbb{R},\mathbb{R}^N).$$

Especially, we have

$$|u|_{I^\alpha_{-\infty}} = \left\| |s|^\alpha \widehat{u} \right\|_{L^2}.$$

Therefore, $I^\alpha_{-\infty}$ and $H^\alpha(\mathbb{R},\mathbb{R}^N)$ are equivalent with equivalent semi-norms and norms. Analogous to $I^\alpha_{-\infty}$, we introduce I^α_∞. Define the semi-norm

$$|u|_{I^\alpha_\infty} = \|{}_tD^\alpha_\infty u\|_{L^2}$$

and the norm

$$\|u\|_{I^\alpha_\infty} = (\|u\|_{L^2} + |u|^2_{I^\alpha_\infty})^{\frac{1}{2}},$$

and let

$$I^\alpha_\infty = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^N)}^{\|\cdot\|\|_{I^\alpha_\infty}}$$

Then $I^\alpha_{-\infty}$ and I^α_∞ are equivalent with equivalent semi-norms and norms.
Let $C(\mathbb{R},\mathbb{R}^N)$ denotes the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^N$. Then we obtain the following Sobolev lemma.

Lemma 2.1. *[[21], Theorem 2.1]. If $\alpha > \frac{1}{2}$, then $H^\alpha(\mathbb{R},\mathbb{R}^N) \subset C(\mathbb{R},\mathbb{R}^N)$, and there exists a constant $C = C_\alpha$ such that*

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} |u(t)| \le C_\alpha \|u\|_\alpha, \forall u \in H^\alpha(\mathbb{R},\mathbb{R}^N).$$

Remark 2.2. *From Lemma 2.1, we know that if $u \in H^\alpha(\mathbb{R},\mathbb{R}^N)$ with $\frac{1}{2} < \alpha < 1$, then $u \in L^p(\mathbb{R},\mathbb{R}^N)$ for all $p \in]2,\infty[$, because*

$$\int_\mathbb{R} |u(t)|^p\, dt \le \|u\|_\infty^{p-2} \|u\|_{L^2}^2.$$

In this section, we assume the L satisfies the following condition

$$(L_0) \qquad L(t)x.x \ge |x|^2,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N$$

and we introduce the following fractional space

$$X^\alpha = \left\{ u \in H^\alpha(\mathbb{R},\mathbb{R}^N) / \int_\mathbb{R} L(t)u(t).u(t)dt < \infty \right\}.$$

Then X^α is a Hilbert space with the inner product

$$< u,v >_{X^\alpha} = \int_\mathbb{R} [{}_{-\infty}D^\alpha_t u(t).{}_{-\infty}D^\alpha_t v(t) + L(t)u(t).v(t)]dt$$

and the corresponding norm

$$\|u\|^2_{X^\alpha} = < u,u >_{X^\alpha}.$$

It is easy to see that X^α is continuously embedded in $H^\alpha(\mathbb{R},\mathbb{R}^N)$ and in $L^2(\mathbb{R},\mathbb{R}^N)$. In fact, for $u \in X^\alpha$, we have

$$\|u\|^2_{X^\alpha} = \int_\mathbb{R} [|{}_{-\infty}D^\alpha_t u(t)|^2 + L(t)u(t).u(t)]dt \ge \int_\mathbb{R} [|{}_{-\infty}D^\alpha_t u(t)|^2\, |u(t)|^2]dt = \|u\|^2_{H^\alpha} \ge \|u\|^2_{L^2}.$$

For $p \in]2,\infty[$, we have by Remark 2.2

$$\|u\|^p_{L^p} = \int_\mathbb{R} |u(t)|^p\, dt \le \|u\|_\infty^{p-2} \|u\|^2_{L^2} \le C_\alpha^{p-2} \|u\|^p_{X^\alpha}.$$

Hence for all $p \in [1,\infty]$, there exists a constant $\eta_p > 0$ such that

$$\|u\|^p_{L^p} \le \eta_p \|u\|^p_{X^\alpha}. \tag{2.3}$$

The main difficulty in dealing with the existence of infinitely many solutions for $(\mathscr{FHS})$ is the lack of compactness of the Sobolev embedding. To overcome this difficulty under the assumptions of Theorems 1.1 and 1.2, we employ the following compact embedding lemma.

Lemma 2.3. *[16] Assume (L_0) and (L_2) are satisfied. Then X^α is compactly embedded in $L^2(\mathbb{R},\mathbb{R}^N)$.*

Remark 2.4. *From Remark 2.2 and Lemma 2.3, it is easy to verify that the embedding of X^α in $L^p(\mathbb{R},\mathbb{R}^N)$ is also compact for $p \in]2,\infty[$.*

To study the critical points of the variational functional associated with $(\mathscr{FHS})$, we need to recall the Symmetric Mountain Pass Theorem [19].

Definition 2.5. *Let X be a Banach space with the norm $\|.\|$, we say that $\Phi \in C^1(X,\mathbb{R})$ satisfies the*
a) $(PS)_c$-condition, $c \in \mathbb{R}$, if any sequence $(u_n) \subset X$ satisfying

$$\Phi(u_n) \longrightarrow c \text{ and } \Phi'(u_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

possesses a convergent subsequence,
b) $(C)_c-$condition, $c \in \mathbb{R}$, if any sequence $(u_n) \subset X$ satisfying

$$\Phi(u_n) \longrightarrow c \text{ and } \left\|\Phi'(u_n)\right\|(1+\|u_n\|) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

possesses a convergent subsequence.

Lemma 2.6. *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional space. Suppose that $\Phi \in C^1(X,\mathbb{R})$ satisfies the Palais-Smale condition and*

$$(a) \qquad \Phi(0) = 0,\ \Phi(-u) = \Phi(u),\ \forall u \in X;$$

$$(b) \qquad There exist constants \rho, \alpha > 0 such that \Phi_{|\partial B_\rho \cap Z} \geq \alpha;$$

$$(c) \qquad For any finite dimensional subspace \widetilde{E} \subset X, there is R = R(\widetilde{E}) > 0 such that \Phi(u) \leq 0 on \widetilde{E} \setminus B_R, where B_R = \{u \in X/\, \|u\| < R\}.$$

Then Φ possesses an unbounded sequence of critical values.

Remark 2.7. *As shown in [2], a deformation lemma can be proved with $(C)_c-$condition replacing the $(PS)_c-$condition, and it turns out that Lemma 2.3 still holds true with the $(C)_c-$condition instead of the $(PS)_c-$condition.*

3. Proof of theorems

From (L), (W_1) and (W_2), we know that there exists a positive constant d_0 such that $L(t)+2d_0 I_N \geq I_N$ for all $t \in \mathbb{R}$. Let $\overline{L}(t) = L(t)+2d_0 I_N$ and $\overline{W}(t,x) = W(t,x)+d_0|x|^2$. Consider the following fractional Hamiltonian system

$$(\overline{FHS}) \qquad \begin{cases} {}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u)(t)+\overline{L}(t)u(t) = \nabla\overline{W}(t,u(t)),\ t \in \mathbb{R} \\ u \in H^\alpha(\mathbb{R},\ \mathbb{R}^N), \end{cases}$$

then $(\overline{\mathscr{FHS}})$ is equivalent to $(\mathscr{FHS})$. Moreover, it is easy to check that the hypotheses $(W_1)-(W_5)$ still hold for $\overline{W}$ provided that those hold for W, and $\overline{L}$ satisfies the conditions (L_0), (L_2). Hence, in what follows, we always assume without loss of generality that L satisfies (L_0) instead of (L).

Consider the variational functional Φ associated to $(\mathscr{FHS})$:

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}}[|{}_{-\infty}D^\alpha_t u(t)|^2+L(t)u(t).u(t)]dt - \int_{\mathbb{R}} W(t,u)dt$$

defined on the space X^α introduced in Section 2. In the following, to simplify the notation, we will note the norm $\|.\|_{X^\alpha}$ of X^α by $\|.\|$.

Lemma 3.1. *Under assumptions (L_0), (L_2) and (W_1), the functional*

$$\psi(u) = \int_{\mathbb{R}} W(t,u)dt$$

is continuously differentiable on X^α and

$$\psi'(u)v = \int_{\mathbb{R}} \nabla W(t,u).vdt,\ \forall u,v \in X^\alpha. \tag{3.1}$$

Proof. By (W_1), there exist constants a_0, $R_0>0$ such that

$$|\nabla W(t,x)| \le a_0 |x|, \ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N \ with \ |x| \le R_0. \tag{3.2}$$

For any given $u \in X^\alpha$, we know that $u \in H^\alpha(\mathbb{R},\mathbb{R}^N)$ and hence there exists a constant $T_0>0$ such that

$$|u(t)| \le \frac{R_0}{2}, \ \forall |t| \ge T_0. \tag{3.3}$$

By (2.3), for any $v \in X^\alpha$ with $\|v\| \le \frac{R_0}{2\eta_\infty}$, we have

$$\|v\|_{L^\infty} \le \frac{R_0}{2}. \tag{3.4}$$

Combining (3.2)-(3.4) and (W_1), by the Mean Value Theorem and Hölder's inequality, for any $T \ge T_0$ and $v \in X^\alpha$ with $\|v\| \le \frac{R_0}{2\eta_\infty}$, one has

$$\begin{aligned}\left|\int_{|t|\ge T}[W(t,u+v)-W(t,u)-\nabla W(t,u).v]dt\right| &= \left|\int_{|t|\ge T}\int_0^1[\nabla W(t,u+sv)-\nabla W(t,u)].vdsdt\right| \\ &\le 2a_0\int_{|t|\ge T}(|u|+|v|)\,|v|\,dt \le 2a_0\Big(\int_{|t|\ge T}(|u|+|v|)^2dt\Big)^{\frac12}\|v\|_{L^2} \\ &\le 2a_0\eta_2\Big[\Big(\int_{|t|\ge T}|u|^2\,dt\Big)^{\frac12}+\eta_2\|v\|\Big]\|v\|.\end{aligned}$$

Since $u \in L^2(\mathbb{R})$, for any $\varepsilon>0$, there exist $0<\alpha_1<\frac{R_0}{2\eta_\infty}$ and $T_\varepsilon \ge T_0$ such that for all $v \in X^\alpha$ with $\|v\| \le \alpha_1$

$$2a_0\eta_2\Big[\Big(\int_{|t|\ge T_\varepsilon}|u|^2\,dt\Big)^{\frac12}+\eta_2\|v\|\Big] \le \frac{\varepsilon}{2}. \tag{3.5}$$

It is well known that the functional

$$\psi_\varepsilon(u) = \int_{[-T_\varepsilon,T_\varepsilon]} W(t,u)dt \tag{3.6}$$

is continuously differentiable on $H^1([-T_\varepsilon,T_\varepsilon],\mathbb{R}^N)$. Thus, since X^α is compactly embedded in $H^\alpha(\mathbb{R})$, there exists a constant $\alpha_2>0$ such that for all $\|v\| \le \alpha_2$

$$\left|\int_{[-R_\varepsilon,R_\varepsilon]}[W(t,u+v)-W(t,u)-\nabla W(t,u).v]dt\right| \le \frac{\varepsilon}{2}\|v\|. \tag{3.7}$$

Taking $\alpha=\min\{\alpha_1,\alpha_2\}$, then (3.5)-(3.7) imply

$$\left|\int_{\mathbb{R}}[W(t,u+v)-W(t,u)-\nabla W(t,u).v]dt\right| \le \varepsilon\|v\|$$

for all $v \in X^\alpha$ with $\|v\| \le \alpha$. Therefore, ψ is differentiable on X^α and satisfies (3.1).
It remains to prove that ψ' is continuous. Let $u_n \longrightarrow u$ in X^α. By Hölder's inequality, we have

$$\begin{aligned}\left\|\psi'(u_n)-\psi'(u)\right\|_{E'} &= \sup_{\|v\|=1}\left|\psi'(u_n)v-\psi'(u)v\right| \\ &= \sup_{\|v\|=1}\left|\int_{\mathbb{R}}[\nabla W(t,u_n)-\nabla W(t,u)].vdt\right| \\ &\le \sup_{\|v\|=1}\Big(\int_{\mathbb{R}}|\nabla W(t,u_n)-\nabla W(t,u)|^2dt\Big)^{\frac12}\|v\|_{L^2} \\ &\le \eta_2\Big(\int_{\mathbb{R}}|\nabla W(t,u_n)-\nabla W(t,u)|^2dt\Big)^{\frac12}.\end{aligned} \tag{3.8}$$

Lemma 2.3 implies that $u_n \longrightarrow u$ in $L^2(\mathbb{R})$. Let M be a positive constant such that $\|u_n\|_{L^2} \le M$ for all integer n. By (W_1), for any $\varepsilon>0$, there exists a constant $0<r<R_0$ such that for all $t \in \mathbb{R}$ and $|x| \le r$

$$|\nabla W(t,x)| \le \frac{\varepsilon}{2(M+\|u\|_{L^2})}|x|. \tag{3.9}$$

Due to (3.9) and the facts that $u \in H^\alpha(\mathbb{R})$ and $u_n \longrightarrow u$ in $L^\infty(\mathbb{R})$, there exists $R_\varepsilon > R_0$ and $N_1 \in \mathbb{N}$ such that for all $|t| \ge R_\varepsilon$ and $n \ge N_1$

$$|\nabla W(t,u_n(t))| \le \frac{\varepsilon}{2(M+\|u\|_{L^2})}|u_n(t)|$$

and

$$|\nabla W(t,u(t))| \le \frac{\varepsilon}{2(M+\|u\|_{L^2})}|u(t)|.$$

Thus

$$\left(\int_{|t|\geq T_\varepsilon}|\nabla W(t,u_n)-\nabla W(t,u)|^2\,dt\right)^{\frac{1}{2}} \leq (\int_{|t|\geq T_\varepsilon}|\nabla W(t,u_n)|^2\,dt)^{\frac{1}{2}}+(\int_{|t|\geq T_\varepsilon}|\nabla W(t,u)|^2\,dt)^{\frac{1}{2}}$$
$$\leq \frac{\varepsilon}{2(M+\|u\|_{L^2})}(\|u_n\|_{L^2}+\|u\|_{L^2})$$
$$\leq \frac{\varepsilon}{2}. \tag{3.10}$$

Observing that $u_n \longrightarrow u$ in $L^\infty(\mathbb{R})$, then by Lebesgue's Dominated Convergence Theorem, we have

$$\left(\int_{[-T_\varepsilon,T_\varepsilon]}|\nabla W(t,u_n)-\nabla W(t,u)|^2\,dt\right)^{\frac{1}{2}} \longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

Hence, there is $N_2\in\mathbb{N}$ such that for all $n\geq N_2$

$$(\int_{[-T_\varepsilon,T_\varepsilon]}|\nabla W(t,u_n)-\nabla W(t,u)|^2\,dt)^{\frac{1}{2}} \leq \frac{\varepsilon}{2}, \tag{3.11}$$

which together with (3.10) implies that for all $n\geq \max\{N_1,N_2\}$

$$(\int_{\mathbb{R}}|\nabla W(t,u_n)-\nabla W(t,u)|^2\,dt)^{\frac{1}{2}} \leq \varepsilon.$$

Combining this with (3.8) implies that $\psi'(u_n)\longrightarrow\psi'(u)$ as $n\longrightarrow\infty$ and then $\psi\in C^1(X^\alpha,\mathbb{R})$. The proof of Lemma 3.1 is completed. □

From Lemma 3.1, we deduce that $\Phi\in C^1(X^\alpha,\mathbb{R})$ and

$$\Phi'(u)v=<u,v>-\int_{\mathbb{R}}\nabla W(t,u).vdt,\ \forall u,v\in X^\alpha.$$

Moreover, if $u\in X^\alpha$ is a critical point of Φ, we have

$${}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u)(t)=-L(t)u(t)+\nabla W(t,u(t))$$

which implies that u is a solution of $(\mathscr{FHS})$.

Lemma 3.2. *Under assumptions (L_0), (L_2), (W_1) and (W_2), for any finite dimensional subspace $\widetilde{E}\subset X^\alpha$, there is $R=R(\widetilde{E})>0$ such that*

$$\Phi(u)\leq 0,\ \forall u\in\widetilde{E},\ \|u\|\geq R. \tag{3.12}$$

Proof. We will prove the following

$$\Phi(u)\longrightarrow -\infty \text{ as } \|u\|\longrightarrow\infty,\ u\in\widetilde{E}. \tag{3.13}$$

Arguing indirectly, assume that there exists a sequence $(u_n)\subset\widetilde{E}$ with $\|u_n\|\longrightarrow\infty$ and a constant $M>0$ such that $\Phi(u_n)\geq -M$ for all $n\in\mathbb{N}$. Set $v_n=\frac{u_n}{\|u_n\|}$, then $\|v_n\|=1$. Passing to a subsequence if necessary, we may assume that $v_n\rightharpoonup v$ in X^α. Since $\widetilde{E}$ is finite dimensional, we have $v_n\longrightarrow v$ in $\widetilde{E}$ and $v_n\longrightarrow v$ a.e. on $\mathbb{R}$. It follows that $\|v\|=1$. For $0\leq a<b$, let

$$\Omega_n(a,b)=\{t\in\mathbb{R}/a\leq|u_n(t)|<b\}$$

$$A=\{t\in\mathbb{R}/v(t)\neq 0\}.$$

Since $v\neq 0$, then $meas(A)>0$. For a.e. $t\in\mathbb{R}$, we have $\lim_{n\longrightarrow\infty}|u_n(t)|=\infty$, hence $t\in\Omega_n(R_0,\infty)$ for n large enough. Since $v_n(t)\longrightarrow v(t)$ a.e. $t\in\mathbb{R}$, we have $\chi_{\Omega_n(R_0,\infty)}(t)\,|v_n(t)|\longrightarrow|v(t)|$ a.e. $t\in A$, where χ_Ω denotes the characteristic function of Ω. Hence, it follows from (W_1), (W_2) and Fatou's Lemma that

$$0=\lim_{n\longrightarrow\infty}\frac{-M}{\|u_n\|^2}\leq\lim_{n\longrightarrow\infty}\frac{\Phi(u_n)}{\|u_n\|^2}=\lim_{n\longrightarrow\infty}[\frac{1}{2}-\int_{\mathbb{R}}\frac{W(t,u_n)\,|v_n|^2}{|u_n|^2}dt]$$
$$=\lim_{n\longrightarrow\infty}[\frac{1}{2}-\int_{\Omega_n(0,R_0)}\frac{W(t,u_n)\,|v_n|^2}{|u_n|^2}dt-\int_{\Omega_n(R_0,\infty)}\frac{W(t,u_n)\,|v_n|^2}{|u_n|^2}dt]$$
$$\leq\limsup_{n\longrightarrow\infty}[\frac{1}{2}+\frac{a_0}{2}\int_{\mathbb{R}}|v_n|^2\,dt-\int_{\Omega_n(R_0,\infty)}\frac{W(t,u_n)\,|v_n|^2}{|u_n|^2}dt]$$
$$\leq\frac{1}{2}+\frac{a_0}{2}\eta_2^2-\liminf_{n\longrightarrow\infty}\int_{\Omega_n(R_0,\infty)}\frac{W(t,u_n)\,|v_n|^2}{|u_n|^2}dt$$
$$\leq\frac{1}{2}+\frac{a_0}{2}\eta_2^2-\int_{\mathbb{R}}\liminf_{n\longrightarrow\infty}\frac{W(t,u_n)\,|v_n|^2}{|u_n|^2}\chi_{|\Omega_n(R_0,\infty)}(t)dt=-\infty \tag{3.14}$$

which is a contradiction. Hence (3.13) and then (3.12) is verified. The proof of Lemma 3.2 is completed. □

Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of X^α and define $X_j = \mathbb{R}e_j$

$$Y_k = \oplus_{j=1}^{k} X_j,\ Z_k = \overline{\oplus_{j=k+1}^{\infty} X_j},\ k \in \mathbb{N}.$$

Lemma 3.3. *Suppose (L_0) and (L_2) hold. Then for any $p \in [2,\infty]$*

$$l_p(k) = \sup_{u\in Z_k, \|u\|=1} \|u\|_{L^p} \longrightarrow 0 \text{ as } k \longrightarrow \infty. \tag{3.15}$$

Proof. It is clear that $0 < l_p(k+1) \le l_p(k)$, so that $l_p(k) \longrightarrow \bar{l}_p$ as $k \longrightarrow \infty$. For every $k \ge 1$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_{L^p} > \frac{1}{2} l_p(k)$. For any $v \in X^\alpha$, let $v = \sum_{i=1}^{\infty} v_i e_i$. By the Cauchy-Schwartz inequality, one has

$$\begin{aligned} |< u_k, v >| &= \left| < u_k, \sum_{i=1}^{\infty} v_i e_i > \right| = \left| < u_k, \sum_{i=k+1}^{\infty} v_i e_i > \right| \\ &\le \|u_k\| \left\| \sum_{i=k+1}^{\infty} v_i e_i \right\| \le \sum_{i=k+1}^{\infty} |v_i| \, \|e_i\| \longrightarrow 0 \text{ as } k \longrightarrow \infty \end{aligned} \tag{3.16}$$

which implies that $u_k \rightharpoonup 0$. Without loss of generality, Lemma 2.3 implies that $u_k \longrightarrow 0$ in $L^2(\mathbb{R})$. Thus we have proved that $\bar{l}_p = 0$. The proof of Lemma 3.3 is completed. □

By (3.15), we can choose an integer $m \ge 1$ such that

$$\|u\|_{L^2} \le \frac{1}{2a_0} \|u\|,\ \forall u \in Z_m. \tag{3.17}$$

In the following, we will apply Lemma 3.1 with $Y = Y_m$ and $Z = Z_m$.

Lemma 3.4. *Under assumptions (L_0), (L_2) and (W_1), there exist constants $\rho, \alpha > 0$ such that $\Phi_{|\partial B_\rho \cap Z} \ge \alpha$.*

Proof. If $\|u\| = \frac{R_0}{\eta_\infty}$, then by (2.3), we have $\|u\|_{L^\infty} \le R_0$. Hence, it follows from (W_1) that

$$W(t,u(t)) \le \frac{a_0}{2} |u(t)|^2,\ \forall u \in X^\alpha,\ \|u\| = \frac{R_0}{\eta_\infty}. \tag{3.18}$$

Combining (3.16) with (3.17) yields for all $u \in Z$ with $\|u\| = \frac{R_0}{\eta_\infty} = \rho$

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t,u)dt \ge \frac{1}{2}\|u\|^2 - \frac{a_0}{2}\int_{\mathbb{R}} |u|^2 dt \ge \frac{1}{2}\|u\|^2 - \frac{a_0}{2}\|u\|_{L^2}^2 \\ &\ge \frac{1}{4}\|u\|^2 = \frac{1}{4}(\frac{R_0}{\eta_\infty})^2 = \alpha. \end{aligned} \tag{3.19}$$

The proof of Lemma 3.4 is completed.

□

Proof of Theorem 1.1 By assumptions (W_1) and (W_3), it is clear that

$$\Phi(0) = 0 \text{ and } \Phi(-u) = \Phi(u),\ \forall u \in X^\alpha. \tag{3.20}$$

Thus the condition (a) of Lemma 3.1 is satisfied. Lemmas 3.2, 3.4 imply that conditions (b) and (c) of Lemma 3.1 are satisfied. It remains to prove that Φ satisfies the $(C)_c-$condition.

Lemma 3.5. *Assume that (L_0), (L_2), (W_1), (W_2) and (W_4) are satisfied. Then Φ verifies the $(C)_c-$condition for all $c \in \mathbb{R}$.*

Proof. Let (u_n) be a $(C)_c$ sequence, that is

$$\Phi(u_n) \longrightarrow c \text{ and } \left\|\Phi'(u_n)\right\|(1+\|u_n\|) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.21}$$

Firstly, we prove that (u_n) is bounded in X^α. Arguing by contradiction, suppose that $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$. Observe that for n large enough

$$c+1 \ge \Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n = \int_{\mathbb{R}} \widehat{W}(t,u_n)dt. \tag{3.22}$$

It follows from (3.21) that

$$\frac{1}{2} \le \limsup_{n\longrightarrow\infty} \int_{\mathbb{R}} \frac{|W(t,u_n)|}{\|u_n\|^2} dt. \tag{3.23}$$

Passing to a subsequence if necessary, we may assume that $v_n \rightharpoonup v$ in X^α. Then by Lemma 2.3, without loss of generality, we have $v_n \longrightarrow v$ in $L^2(\mathbb{R})$ and $v_n \longrightarrow v$ a.e. on $\mathbb{R}$.
If $v = 0$, then $v_n \longrightarrow 0$ in $L^2(\mathbb{R})$ and $v_n \longrightarrow 0$ a.e. on $\mathbb{R}$. Hence, it follows from (W_1) that

$$\int_{\Omega_n(0,R_0)} \frac{|W(t,u_n)|}{\|u_n\|^2} dt = \int_{\Omega_n(0,R_0)} \frac{|W(t,u_n)|}{|u_n|^2} |v_n|^2 dt \leq \frac{a_0}{2} \int_{\Omega_n(0,R_0)} |v_n|^2 dt \leq \frac{a_0}{2} \|v_n\|_{L^2}^2 \longrightarrow 0 \; as \; n \longrightarrow \infty. \tag{3.24}$$

From (W_4) and (3.22), one has

$$\begin{aligned}
\int_{\Omega_n(R_0,\infty)} \frac{|W(t,u_n)|}{\|u_n\|^2} dt &= \int_{\Omega_n(R_0,\infty)} \frac{|W(t,u_n)|}{|u_n|^2} |v_n|^2 dt \\
&\leq \frac{c_0 \|v_n\|_{L^\infty}^{2-\nu}}{\|u_n\|^\nu} \int_{\Omega_n(R_0,\infty)} \widehat{W}(t,u_n) dt \\
&\leq \frac{c_0 \|v_n\|_{L^\infty}^{2-\nu}}{\|u_n\|^\nu} [1 + c - \int_{\Omega_n(0,R_0)} \widehat{W}(t,u_n) dt] \\
&\leq \frac{c_0 \eta_\infty^{2-\nu}}{\|u_n\|^\nu} [1 + c - \int_{\Omega_n(0,R_0)} g(t) dt] \\
&\leq \frac{c_0 \eta_\infty^{2-\nu}}{\|u_n\|^\nu} [1 + c + \int_{\mathbb{R}} |g(t)| dt] \longrightarrow 0 \; as \; n \longrightarrow \infty.
\end{aligned} \tag{3.25}$$

Combining (3.22) with (3.23) yields

$$\int_{\mathbb{R}} \frac{|W(t,u_n)|}{\|u_n\|^2} dt = \int_{\Omega_n(0,R_0)} \frac{|W(t,u_n)|}{\|u_n\|^2} dt + \int_{\Omega_n(R_0,\infty)} \frac{|W(t,u_n)|}{\|u_n\|^2} dt \longrightarrow 0 \; as \; n \longrightarrow \infty$$

which contradicts (3.21).
If $v \neq 0$. By a similar fashion as for (3.14), we can get a contradiction. Therefore, (u_n) is bounded in X^α.
Next, we prove that (u_n) possesses a convergent subsequence. Without loss of generality, we can assume by Remark 2.4 that $u_n \longrightarrow u$ in $L^2(\mathbb{R})$. Using Hölder's inequality, (W_1) and the fact that ∇W is bounde'd in $\mathbb{R} \times B_r(0)$ for all $r > 0$, we can show that

$$\int_{\mathbb{R}} [\nabla W(t,u_n) - \nabla W(t,u)].(u_n - u) dt \longrightarrow 0 \; as \; n \longrightarrow \infty. \tag{3.26}$$

Observe that

$$\|u_n - u\|^2 = (\Phi'(u_n) - \Phi'(u))(u_n - u) + \int_{\mathbb{R}} [\nabla W(t,u_n) - \nabla W(t,u)].(u_n - u) dt. \tag{3.27}$$

It is clear that

$$(\Phi'(u_n) - \Phi'(u))(u_n - u) \longrightarrow 0 \; as \; n \longrightarrow \infty. \tag{3.28}$$

Combining (3.24) – (3.26), we get $\|u_n - u\| \longrightarrow 0$ as $n \longrightarrow \infty$. The proof of Lemma 3.5 is completed. □

Consequently, Lemma 2.3 together with Remark 2.2 imply that Φ possesses an unbounded sequence of critical points. Therefore $(\mathscr{FHS})$ possesses infinitely many solutions. The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2

Lemma 3.6. *Under assumptions (L_0), (L_2), $(W_1) - (W_3)$ and (W_5), Φ satisfies the $(C)_c$−condition for all $c \in \mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ and $(u_n) \subset X^\alpha$ satisfying (3.19). First, we prove that (u_n) is bounded in X^α. Arguing by contradiction, suppose that $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$ and $\|v_n\| \leq \eta_p \|v_n\|_{L^p}$ for $2 \leq p \leq \infty$. By (W_5), one has for n large enough

$$\begin{aligned}
c + 1 \geq \Phi(u_n) - \frac{1}{\mu} \Phi'(u_n) u_n &= \frac{\mu - 2}{2\mu} \|u_n\|^2 + \int_{\mathbb{R}} [\frac{1}{\mu} \nabla W(t,u_n).u_n - W(t,u_n)] dt \\
&\geq \frac{\mu - 2}{2\mu} \|u_n\|^2 - \frac{\rho_0}{\mu} \|u_n\|_{L^2}^2,
\end{aligned}$$

which implies

$$\frac{\mu - 2}{2\rho_0} \leq \lim \sup_{n \longrightarrow \infty} \|v_n\|_{L^2}^2. \tag{3.29}$$

Taking a subsequence if necessary, we may assume that $v_n \longrightarrow v$ in $L^2(\mathbb{R})$ and $v_n \longrightarrow v$ a.e. on $\mathbb{R}$. Hence, it follows from (3.27) that $v \neq 0$. By a similar fashion as for (3.14), we can get a contradiction. Therefore (u_n) is bounded in X^α. The rest of the proof is the same as that in Lemma 3.5 and the proof of Lemma 3.6 is completed. □

We conclude as in the proof of Theorem 1.1 that Φ possesses an unbounded sequence of critical points and the proof of Theorem 1.2 is completed.

4. Conclusion

Using the variational methods and critical point theory, we proved that the fractional Hamiltonian system $(\mathscr{F}\mathscr{H}\mathscr{S})$ possesses infinitely many nontrivial solutions, where L is neither uniformly positive definite nor coercive and W does not satisfy the classical superquadratic growth conditions like the well-known Ambrosetti-Rabinowitz superquadratic condition. Recent results in the literature are generalized and significantly improved.

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References

[1] O. Agrawal, J. Tenreiro Machado, J. Sabatier, *Fractional derivatives and their applications*, Springer-Verlag, Berlin, 2004;
[2] T. Bartolo, V. Benci, D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity*, Nonlinear Analysis, Vol. 7 , No. 9 (1983) 981-1012;
[3] N. Nyamoradi, A. Alsaedi, B. Ahmad, Y. Zou, *Multiplicity of homoclinic solutions for fractional Hamiltonian systems with subquadratic potential*, Entropy 2017, 19,50,1-24;
[4] N. Nyamoradi, A. Alsaedi, B. Ahmad, Y. Zou, *Variational approach to homoclinic solutions for fractional Hamiltonian systems*, J. Optim. Theory Appl. 2017;
[5] Z. Bai, H. L*ü*, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. (2005), 311, 495-505;
[6] Z. Bai, Y. Zhang, *The existence of solutions for a fractional multi-point boundary value problem*, Computers and Mathematics with Applications 2010, 69, 2364-2372;
[7] P. Chen, X. He, X.H. Tang, *Infinitely many solutions for a class of Hamiltonian systems via critical point theory*, Math. Meth. Appl. Sci. 2016, 39, 1005-1019;
[8] Y. Li, B. Dai, *Existence and multiplicity of nontrivial solutions for Liouville-Weyl fractional nonlinear Schrödinger equation*, RA SAM (2017);
[9] W. Jiang, *The existence of solutions for boundary value problems of fractional differential equations at resonance*, Nonlinear Analysis (2011), 74, 1987-1994;
[10] F. Jiao, Y. Zhou, *Existence results for fractional boundary value problem via critical point theory*, Intern. Journal of Bif. and Chaos, 22, No. 4 (2012), 1-17;
[11] R. Hiffer, *Applications of fractional calculus in physics*, World Science, Singapore, 2000;
[12] S. G. Samko, A.A Kilbas, O.I. Marichev, *Fractional integrals and derivatives, Theory and applications*, Gordon and Breach, Switzerland 1993;
[13] A.A. Kilbas, H.M. Srivastawa, J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematical Studies; Vol. 204, Singapore 2006;
[14] S. Liang, J. Zhang, *Positive solutions for boundary value problems of nonlinear fractional differential equations*, Nonlinear Analysis, 2009, 71, 5545-5550;
[15] J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences, Springer, Berlin, 1989;
[16] A. Mèndez, C. Torres, *Multiplicity of solutions for fractional Hamiltonian systems with Liouville-Weyl fractional derivative*, arXiv: 1409.0765v1[math-ph] 2 Sep. 2014;
[17] K. Miller, B. Ross, *An introduction to differential equations*, Wiley and Sons, New York, 1993;
[18] I. Pollubny, *Fractional differential equations*, Academic Press, 1999;
[19] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. in Math., Vol. 65, American Mathematical Society, Providence, RI, 1986;
[20] K. Tang, *Multiple homoclinic solutions for a class of fractional Hamiltonian systems*, Progr. Fract. DIff. Appl. 2, , No. 4 (2016), 265-276;
[21] C. Torres, *Existence of solutions for fractional Hamiltonian systems*, Electr. J. DIff. Eq., Vol. 2013 (2013), No. 259, 1-12;
[22] C. Torres Ledesma, *Existence of solutions for fractional Hamiltonian systems with nonlinear derivative dependence in* $\mathbb{R}$, J. Fractional Calculus and Applications; Vol. 7 (2) (2016) 74-87;
[23] X. Wu, Z. Zhang, *Solutions for perturbed fractional Hamiltonian systems without coercive conditions*, Boundary Value Problems (2015) 2015: 149, 1-12;
[24] S. Zhang, *Existence of solutions for the fractional equations with nonlinear boundary conditions*, Computers and Mathematics with Applications (2011), 61, 1202-1208;
[25] S. Zhang, *Existence of solutions for a boundary value problems of fractional differential equations at resonance*, Nonlinear Analysis (2011): 74, 1987-1994;
[26] Z. Zhang, R. Yuan, *Existence of solutions to fractional Hamiltonian systems with combined nonlinearities*, Electr. J. Diff. Eq., Vol. 2016 (2016) No. 40, 1-13;
[27] Z. Zhang, R. Yuan, *Solutions for subquadratic fractional Hamiltonian systems without coercive conditions*, Math. Meth. Appl. Sci. (2014) 37, 2934-2945;
[28] Z. Zhang, R. Yuan, *Variational approach to solutions for a class of fractional Hamiltonian systems*, Math. Meth. Appl. Sci. (2014) 37, 1873-1883;

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Geometry of bracket-generating distributions of step 2 on graded manifolds

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Abstract

A Z_2-graded analogue of bracket-generating distribution is given. Let $\mathscr{D}$ be a distribution of rank (p,q) on an (m,n)-dimensional graded manifold $\mathscr{M}$, we attach to $\mathscr{D}$ a linear map F on $\mathscr{D}$ defined by the Lie bracket of graded vector fields of the sections of $\mathscr{D}$. Then $\mathscr{D}$ is a bracket-generating distribution of step 2, if and only if F is of constant rank $(m-p,n-q)$ on $\mathscr{M}$.

1. Introduction

A smooth distribution $D \subset TM$ is said to be bracket-generating if all iterated brackets among its sections generate the whole tangent space to the manifold M, [1, 8]. D is a bracket-generating distribution of step 2 if $D^2 = TM$, where $D^2 = D + [D,D]$. Bejancu showed that a distribution of rank $k < m = dimM$ is a bracket-generating distribution of step 2, if and only if, the curvature of D is of constant rank $m-k$ on M, [1].

In this paper, a Z_2-graded analogue of bracket-generating distribution of step 2 is given. Some differences arise in the graded case due to the presence of odd generators. Given a distribution $\mathscr{D}$ of rank (p,q) on an (m,n)-dimensional graded manifold $\mathscr{M}$, we attach to $\mathscr{D}$ a linear map F on $\mathscr{D}$ defined by the Lie bracket of graded vector fields of the sections of $\mathscr{D}$. Then $\mathscr{D}$ is a bracket-generating distribution of step 2, if and only if F is of constant rank $(m-p,n-q)$ on $\mathscr{M}$. In particular, if $rank\mathscr{D}(z) = (m-1,n)$, then for the linear map $F = F_0 + F_1$ associated to $\mathscr{D}$, $F_0 \neq 0$ and if $rank\mathscr{D}(z) = (m,n-1)$, then $F_1 \neq 0$ on $\mathscr{M}$.

2. Preliminaries

Let M be a topological space and let $\mathscr{O}_M$ be a sheaf of super $\mathbb{R}$-algebras with unity. A graded manifold of dimension (m,n) is a ringed space $\mathscr{M} = (M, \mathscr{O}_M)$ which is locally isomorphic to $\mathbb{R}^{m|n}$, (see [6]).

Let $\mathscr{M}$ and $\mathscr{N}$ be graded manifolds. Let $\phi : M \mapsto N$ be a continuous map such that $\phi^* : \mathscr{O}_N \longrightarrow \mathscr{O}_M$ takes $\mathscr{O}_N(V)$ into $\mathscr{O}_M(\phi^{-1}(V))$ for each open set $V \subset N$, then we say that $\Phi = (\phi, \phi^*) : \mathscr{M} \longrightarrow \mathscr{N}$ is a morphism between $\mathscr{M}$ and $\mathscr{N}$.

Let A be a super $\mathbb{R}$-algebra, $\varphi \in End_{\mathbb{R}}A$ is called a derivation of A, if for all $a,b \in A$,

$$\varphi(ab) = \varphi(a).b + (-1)^{|\varphi||a|}a.\varphi(b), \tag{2.1}$$

where for a homogeneous element x of some graded object, $|x| \in \{0,1\}$ denotes the parity of x (see [6]).

A vector field on $\mathscr{M}$ is a derivation of the sheaf $\mathscr{O}_M$. Let $U \subset M$ be an open subset, the $\mathscr{O}_M(U)-$super module of derivations of $\mathscr{O}_M(U)$ is defined by

$$T\mathscr{M}(U) := Der(\mathscr{O}_M(U)).$$

The $\mathscr{O}_M-$module $T\mathscr{M}$ is locally free of dimension (m,n) and is called the tangent sheaf of $\mathscr{M}$. A vector field is a section of $T\mathscr{M}$.

If $\Omega^1(\mathscr{M}) := T^*\mathscr{M}$ be the dual of the tangent sheaf of a graded manifold $\mathscr{M}$, then it is the sheaf of super $\mathscr{O}_M$-modules and

$$\Omega^1(\mathscr{M}) := Hom(T\mathscr{M}, \mathscr{O}_M). \tag{2.2}$$

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It is called the cotangent sheaf of a graded manifold $\mathscr{M}$, and the sections of $\Omega^1(\mathscr{M})$ are called super differential 1-forms [2, 6].
Let $\mathscr{M}=(M,\mathscr{O}_M)$ be an (m,n)-dimensional graded manifold and $\mathscr{D}$ be a distribution of rank (p,q) $(p<m,q<n)$ on $\mathscr{M}$. Then for each point $x\in M$ there is an open subset U over which any set of generators $\{D_i,D\mu|1\leq i\leq p,1\leq\mu\leq q\}$ of the module $\mathscr{D}(U)$ can be enlarged to a set

$$\left\{C_a,D_i,D_\mu,C\alpha\;\middle|\;{}_{p+1\leq a\leq m}^{1\leq i\leq p}\ \textit{and}\ {}_{q+1\leq\alpha\leq n}^{1\leq\mu\leq q}\ {}_{|D_i|=0}^{|C_a|=0}\ \textit{and}\ {}_{|C_\alpha|=1}^{|D_\mu|=1}\right\}$$

of free generators of $Der\mathscr{O}_M$, [3].
We attach to $\mathscr{D}$ a sequence of distributions defined by,

$$\mathscr{D}\subset\mathscr{D}^2\subset\ldots\subset\mathscr{D}^r\subset\ldots\subset Der\mathscr{O}_M,$$

with

$$\mathscr{D}^2=\mathscr{D}+[\mathscr{D},\mathscr{D}],\ldots,\mathscr{D}^{r+1}=\mathscr{D}^r+[\mathscr{D},\mathscr{D}^r],$$

where

$$[\mathscr{D},\mathscr{D}^r]=\operatorname{span}\{[X,Y]:X\in\mathscr{D},Y\in\mathscr{D}^r\}.$$

As in the classical case, we say that $\mathscr{D}$ is a bracket-generating distribution, if there exists an $r\geq 2$ such that $\mathscr{D}^r=Der\mathscr{O}_M$. In this case r is called the step of the distribution $\mathscr{D}$.
Suppose that $X,Y\in\mathscr{D}$ and consider the linear map on $\mathscr{D}$ as follows:

$$F(X,Y)=-(-1)^{|X||Y|}[X,Y]\bmod\mathscr{D}. \tag{2.3}$$

With respect to the above local basis $\{D_i,C_a,D_\mu,C_\alpha\}$ of $Der\mathscr{O}_M$, if

$$\begin{aligned}
[D_i,D_j]&=D^k_{ij}D_k+D^d_{ij}C_d+\tilde{D}^\nu_{ij}D_\nu+\tilde{D}^\gamma_{ij}C_\gamma,\\
[D_i,D_\xi]&=D^k_{i\xi}D_k+D^d_{i\xi}C_d+\tilde{D}^\nu_{i\xi}D_\nu+\tilde{D}^\gamma_{i\xi}C_\gamma,\\
[D_\mu,D_j]&=D^k_{\mu j}D_k+D^d_{\mu j}C_d+\tilde{D}^\nu_{\mu j}D_\nu+\tilde{D}^\gamma_{\mu j}C_\gamma,\\
[D_\mu,D_\xi]&=D^k_{\mu\xi}D_k+D^d_{\mu\xi}C_d+\tilde{D}^\nu_{\mu\xi}D_\nu+\tilde{D}^\gamma_{\mu\xi}C_\gamma,
\end{aligned}$$

then, by using (2.3), we conclude that

$$\begin{aligned}
F(D_j,D_i)&=D^d_{ij}C_d+\tilde{D}^\gamma_{ij}C_\gamma\ \bmod\mathscr{D},\\
F(D_\xi,D_i)&=D^d_{i\xi}C_d+\tilde{D}^\gamma_{i\xi}C_\gamma\ \bmod\mathscr{D},\\
F(D_j,D_\mu)&=D^d_{\mu j}C_d+\tilde{D}^\gamma_{\mu j}C_\gamma\ \bmod\mathscr{D},\\
F(D_\xi,D_\mu)&=D^d_{\mu\xi}C_d+\tilde{D}^\gamma_{\mu\xi}C_\gamma\ \bmod\mathscr{D}.
\end{aligned} \tag{2.4}$$

Each component D^a_{bc} of F is a superfunction on U.
Let $\overline{U}$ be an open subset of M such that $U\cap\overline{U}\neq\emptyset$. If we change the basis of $Der\mathscr{O}_M(U\cap\overline{U})$ to $\{\overline{D}_i,\overline{C}_a,\overline{D}_\mu,\overline{C}_\alpha\}$ then we have

$$\begin{aligned}
\overline{D}_j&=f^i_jD_i+f^\mu_jD_\mu,\\
\overline{D}_\nu&=f^i_\nu D_i+f^\mu_\nu D_\mu,\\
\overline{C}_b&=f^i_bD_i+g^a_bC_a+f^\mu_bD_\mu+g^\alpha_bC_\alpha,\\
\overline{C}_\beta&=f^i_\beta D_i+g^a_\beta C_a+f^\mu_\beta D_\mu+g^\alpha_\beta C_\alpha,
\end{aligned}$$

where

$$\begin{bmatrix}f^i_j & f^\mu_j\\ f^i_\nu & f^\mu_\nu\end{bmatrix}\ \text{and}\ \begin{bmatrix}g^a_b & g^\alpha_b\\ g^a_\beta & g^\alpha_\beta\end{bmatrix},$$

are nonsingular supermatrices of smooth functions on $U\cap\overline{U}$. Both of these matrices are even. With respect to the basis $\{\overline{D}_j,\overline{C}_b,\overline{D}_\nu,\overline{C}_\beta\}$ on $\overline{U}$, if $\{\overline{D}^b_{kh},\overline{D}^\beta_{kh},\ldots,\overline{D}^\beta_{\xi\rho}\}$ are the local components of F, then we have

$$\begin{bmatrix}\overline{D}^b_{kh} & \overline{D}^\beta_{kh}\\ \overline{D}^b_{\xi h} & \overline{D}^\beta_{\xi h}\\ \overline{D}^b_{k\rho} & \overline{D}^\beta_{k\rho}\\ \overline{D}^b_{\xi\rho} & \overline{D}^\beta_{\xi\rho}\end{bmatrix}\begin{bmatrix}g^a_b & g^\alpha_b\\ g^a_\beta & g^\alpha_\beta\end{bmatrix}=\begin{bmatrix}f^j_h & 0 & f^\mu_h & 0\\ 0 & f^\mu_\rho & 0 & f^j_\rho\\ f^j_\rho & 0 & f^\mu_\rho & 0\\ 0 & f^\mu_h & 0 & f^j_h\end{bmatrix}\begin{bmatrix}f^i_k & 0 & f^\nu_k & 0\\ 0 & f^\nu_\xi & 0 & -f^i_\xi\\ 0 & -f^\nu_k & 0 & f^i_k\\ f^i_\xi & 0 & f^\nu_\xi & 0\end{bmatrix}\begin{bmatrix}D^a_{ij} & D^\alpha_{ij}\\ D^a_{\nu\mu} & D^\alpha_{\nu\mu}\\ D^a_{\nu j} & D^\alpha_{\nu j}\\ D^a_{i\mu} & D^\alpha_{i\mu}\end{bmatrix} \tag{2.5}$$

Since $\begin{bmatrix} f^i_j & f^\mu_j \\ f^i_\nu & f^\mu_\nu \end{bmatrix}$ is invertible at $x \in U \cap \overline{U}$, we see that $\begin{bmatrix} f^i_j & 0 \\ 0 & f^\mu_\nu \end{bmatrix}$ is invertible and from (2.5) we conclude that if

$$D(x) = \begin{bmatrix} D^{p+q+1}_{1\,2} & D^{p+q+1}_{1\,3} & \dots & D^{p+q+1}_{1\,p+q} & D^{p+q+1}_{2\,3} & \dots & D^{p+q+1}_{2\,p+q} & \dots & D^{p+q+1}_{p+q-1\;p+q} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ D^{m+n}_{1\,2} & D^{m+n}_{1\,3} & \dots & D^{m+n}_{1\,p+q} & D^{m+n}_{2\,3} & \dots & D^{m+n}_{2\,p+q} & \dots & D^{m+n}_{p+q-1\;p+q} \end{bmatrix}(x)$$

then rank $D(x) = \mathrm{rank}\overline{D}(x)$.
Now we can define the rank of F, which is related to its coefficients matrix. Before doing this, in view of (2.4), we note that the submatrices

$$\begin{bmatrix} D^a_{ij}(x) & D^a_{\mu\nu}(x) \\ \tilde{D}^\alpha_{ij}(x) & \tilde{D}^\alpha_{\mu\nu}(x) \end{bmatrix} \mathit{and} \begin{bmatrix} D^a_{i\mu}(x) & D^a_{\mu i}(x) \\ \tilde{D}^\alpha_{i\mu}(x) & \tilde{D}^\alpha_{\mu i}(x) \end{bmatrix}$$

are even and odd respectively. The rank of the first submatrix can be defined but for the second submatrix, since $D^a_{\mu i}(x)$ and $\tilde{D}^\alpha_{i\mu}(x)$ are even, we consider the matrix $\begin{bmatrix} D^a_{\mu i}(x) & D^a_{i\mu}(x) \\ \tilde{D}^\alpha_{\mu i}(x) & \tilde{D}^\alpha_{i\mu}(x) \end{bmatrix}$ to define its rank. Now set

$$r := rank \begin{bmatrix} D^a_{ij}(x) & D^a_{\mu\nu}(x) \\ \tilde{D}^\alpha_{ij}(x) & \tilde{D}^\alpha_{\mu\nu}(x) \end{bmatrix} \mathit{and} \quad s := rank \begin{bmatrix} D^a_{\mu i}(x) & D^a_{i\mu}(x) \\ \tilde{D}^\alpha_{\mu i}(x) & \tilde{D}^\alpha_{i\mu}(x) \end{bmatrix},$$

where $i, j = 1, ..., p, a = p+1, .., m$ and $\mu, \nu = 1, ..., q, \alpha = q+1, ..., n$. Thus we define

$$rankF(x) = (r, s).$$

If $(q_{\bar{a}}, \xi_{\bar{\mu}})$ are local supercoordinates on a coordinate neighborhood U of $x \in M$, $(\bar{a} = 1, ..., m, \bar{\mu} = 1, ..., n)$, then $\mathscr{D}$ is locally given by the graded 1-forms

$$\phi_{\bar{b}} = \phi^{\bar{a}}_{\bar{b}} dq_{\bar{a}} + \tilde{\phi}^{\bar{\mu}}_{\bar{b}} d\xi_{\bar{\mu}} = 0, \quad \bar{b} = 1, ..., p$$
$$\phi_{\bar{\alpha}} = \phi^{\bar{a}}_{\bar{\alpha}} dq_{\bar{a}} + \tilde{\phi}^{\bar{\mu}}_{\bar{\alpha}} d\xi_{\bar{\mu}} = 0, \quad \bar{\alpha} = 1, ..., q.$$

Since $\mathscr{D}$ is a distribution of rank (p, q), we may assume that the submatrices $(\phi^{\bar{a}}_{\bar{b}}), 1 \le \bar{a}, \bar{b} \le p$, and $(\tilde{\phi}^{\bar{\mu}}_{\bar{\alpha}}), 1 \le \bar{\alpha}, \bar{\mu} \le q$ are invertible. Let the matrix $\psi = (\psi^\bullet_\bullet)$ denotes the inverse of the matrix $\begin{pmatrix} \phi^{\bar{a}}_{\bar{b}} & \tilde{\phi}^{\bar{\mu}}_{\bar{b}} \\ \phi^{\bar{a}}_{\bar{\alpha}} & \tilde{\phi}^{\bar{\mu}}_{\bar{\alpha}} \end{pmatrix}$, $1 \le \bar{a}, \bar{b} \le p, 1 \le \bar{\alpha}, \bar{\mu} \le q$ and suppose

$$\bar{\phi}_{\bar{a}} = \psi^{\bar{b}}_{\bar{a}} \phi_{\bar{b}} + \tilde{\phi}^{\bar{\mu}}_{\bar{a}} \phi_{\bar{\mu}}, \quad \bar{\phi}_{\bar{\alpha}} = \phi^{\bar{b}}_{\bar{\alpha}} \phi_{\bar{b}} + \tilde{\phi}^{\bar{\mu}}_{\bar{\alpha}} \phi_{\bar{\mu}}.$$

Therefore, the new notation

$$y_a = q_a, x_i = q_i, i = 1, ..., p, \quad a = p+1, ..., m,$$
$$\zeta_\alpha = \xi_\alpha, \eta_\mu = \xi_\mu, \mu = 1, ..., q, \quad \alpha = q+1, ..., n,$$

for the coordinates, may be performed to bring the local basis of $\Omega^1(\mathscr{M})$ into the form $\{dx_i, d\eta_\mu, dy_a + r^a_i dx_i + r^a_\mu d\eta_\mu, d\zeta_\alpha + r^\alpha_i dx_i + r^\alpha_\mu d\eta_\mu\}$. It is easy to check that

$$\frac{\delta}{\delta x_i} := \frac{\partial}{\partial x_i} - r^a_i \frac{\partial}{\partial y_a} - r^\alpha_i \frac{\partial}{\partial \zeta_\alpha}, \ i = 1, \dots, p,$$
$$\frac{\delta}{\delta \eta_\mu} := \frac{\partial}{\partial \eta_\mu} + r^a_\mu \frac{\partial}{\partial y_a} - r^\alpha_\mu \frac{\partial}{\partial \zeta_\alpha}, \ \mu = 1, \dots, q, \tag{2.6}$$

are (respectively even and odd) generators of $\mathscr{D}$ on U and $\{\delta/\delta x_i, \delta/\delta\eta_\mu, \partial/\partial y_a, \partial/\partial\zeta_\alpha\}$ is a local basis for $Der(\mathscr{O}_M(U))$, (see also [4, 5]). With respect to this basis, if we put

$$F(\frac{\delta}{\delta x_j}, \frac{\delta}{\delta x_i}) = F_i{}^a{}_j \frac{\partial}{\partial y_a} + \tilde{F}_i{}^\alpha{}_j \frac{\partial}{\partial \zeta_\alpha} \bmod \mathscr{D},$$
$$F(\frac{\delta}{\delta x_j}, \frac{\delta}{\delta \eta_\nu}) = F_\nu{}^a{}_j \frac{\partial}{\partial y_a} + \tilde{F}_\nu{}^\alpha{}_j \frac{\partial}{\partial \zeta_\alpha} \bmod \mathscr{D},$$
$$F(\frac{\delta}{\delta \eta_\mu}, \frac{\delta}{\delta x_i}) = F_i{}^a{}_\mu \frac{\partial}{\partial y_a} + \tilde{F}_i{}^\alpha{}_\mu \frac{\partial}{\partial \zeta_\alpha} \bmod \mathscr{D},$$
$$F(\frac{\delta}{\delta \eta_\mu}, \frac{\delta}{\delta \eta_\nu}) = F_\nu{}^a{}_\mu \frac{\partial}{\partial y_a} + \tilde{F}_\nu{}^\alpha{}_\mu \frac{\partial}{\partial \zeta_\alpha} \bmod \mathscr{D}, \tag{2.7}$$

then by using (2.3) and (2.6), we deduce that

$$
\begin{aligned}
&F_{i\ j}^{\ a}\frac{\partial}{\partial y_a}+\tilde{F}_{i\ j}^{\ \alpha}\frac{\partial}{\partial \zeta_\alpha}=[\frac{\delta}{\delta x_i},\frac{\delta}{\delta x_j}]=(\frac{\delta r_i^a}{\delta x_j}-\frac{\delta r_j^a}{\delta x_i})\frac{\partial}{\partial y_a}+(\frac{\delta r_i^\alpha}{\delta x_j}-\frac{\delta r_j^\alpha}{\delta x_i})\frac{\partial}{\partial \zeta_\alpha}\mod \mathscr{D},\\
&F_{\nu\ j}^{\ a}\frac{\partial}{\partial y_a}+\tilde{F}_{\nu\ j}^{\ \alpha}\frac{\partial}{\partial \zeta_\alpha}=[\frac{\delta}{\delta \eta_\nu},\frac{\delta}{\delta x_j}]=(-\frac{\delta r_\nu^a}{\delta x_j}-\frac{\delta r_j^a}{\delta \eta_\nu})\frac{\partial}{\partial y_a}+(\frac{\delta r_\nu^\alpha}{\delta x_j}-\frac{\delta r_j^\alpha}{\delta \eta_\nu})\frac{\partial}{\partial \zeta_\alpha}\mod \mathscr{D},\\
&F_{i\ \mu}^{\ a}\frac{\partial}{\partial y_a}+\tilde{F}_{i\ \mu}^{\ \alpha}\frac{\partial}{\partial \zeta_\alpha}=[\frac{\delta}{\delta x_i},\frac{\delta}{\delta \eta_\mu}]=(\frac{\delta r_i^a}{\delta \eta_\mu}+\frac{\delta r_\mu^a}{\delta x_i})\frac{\partial}{\partial y_a}+(\frac{\delta r_i^\alpha}{\delta \eta_\mu}-\frac{\delta r_\mu^\alpha}{\delta x_i})\frac{\partial}{\partial \zeta_\alpha}\mod \mathscr{D},\\
&F_{\nu\ \mu}^{\ a}\frac{\partial}{\partial y_a}+\tilde{F}_{\nu\ \mu}^{\ \alpha}\frac{\partial}{\partial \zeta_\alpha}=[\frac{\delta}{\delta \eta_\nu},\frac{\delta}{\delta \eta_\mu}]=(\frac{\delta r_\nu^a}{\delta \eta_\mu}+\frac{\delta r_\mu^a}{\delta \eta_\nu})\frac{\partial}{\partial y_a}+(-\frac{\delta r_\nu^\alpha}{\delta \eta_\mu}-\frac{\delta r_\mu^\alpha}{\delta \eta_\nu})\frac{\partial}{\partial \zeta_\alpha}\mod \mathscr{D}.
\end{aligned}
\tag{2.8}
$$

Now let us consider a distribution $\mathscr{D}$ of corank one on $\mathscr{M}$. For each $z\in M$, there are two cases.

Case1. Let $rank\mathscr{D}(z)=(m-1,n)$. Then there exist a coordinate system $(x_i,t,\eta_\mu), i=1,...,m-1, \mu=1,...,n$, defined in a neighborhood U of z, such that $\mathscr{D}$ is locally given by

$$dt+r_i dx_i+r_\mu d\eta_\mu=0.$$

Case2. Let $rank\mathscr{D}(z)=(m,n-1)$. Then there exist a coordinate system $(x_j,\eta_\nu,\theta), j=1,...,m, \nu=1,...,n-1$ defined in a neighborhood U of z, such that $\mathscr{D}$ is locally given by

$$d\theta+r_j dx_j+r_\nu d\eta_\nu=0.$$

Note that in the first case, (2.8) becomes

$$
\begin{aligned}
&F_{ij}\frac{\partial}{\partial t}=[\frac{\delta}{\delta x_i},\frac{\delta}{\delta x_j}]=(\frac{\delta r_i}{\delta x_j}-\frac{\delta r_j}{\delta x_i})\frac{\partial}{\partial t}\mod \mathscr{D},\\
&F_{\nu j}\frac{\partial}{\partial t}=[\frac{\delta}{\delta \eta_\nu},\frac{\delta}{\delta x_j}]=(-\frac{\delta r_j}{\delta \eta_\nu}-(-1)^{|t|}\frac{\delta r_\nu}{\delta x_j})\frac{\partial}{\partial t}\mod \mathscr{D},\\
&F_{i\mu}\frac{\partial}{\partial t}=[\frac{\delta}{\delta x_i},\frac{\delta}{\delta \eta_\mu}]=(\frac{\delta r_i}{\delta \eta_\mu}+(-1)^{|t|}\frac{\delta r_\mu}{\delta x_i})\frac{\partial}{\partial t}\mod \mathscr{D},\\
&F_{\nu\mu}\frac{\partial}{\partial t}=[\frac{\delta}{\delta \eta_\nu},\frac{\delta}{\delta \eta_\mu}]=((-1)^{|t|}\frac{\delta r_\nu}{\delta \eta_\mu}+(-1)^{|t|}\frac{\delta r_\mu}{\delta \eta_\nu})\frac{\partial}{\partial t}\mod \mathscr{D},
\end{aligned}
\tag{2.9}
$$

where $F_{ij}, F_{\nu j}, F_{i\mu}$ and $F_{\nu\mu}$ are the local components of F with respect to the local basis $\{\delta/\delta x_i, \delta/\delta x_\mu, \partial/\partial t\}$.

3. Bracket-generating distribution of step 2

In this section, we want to find the conditions under which a distribution $\mathscr{D}$ is bracket-generating of step 2. As mentioned in the previous section, we attach to $\mathscr{D}$ a linear map F on $\mathscr{D}$ defined by the Lie bracket of graded vector fields of the sections of $\mathscr{D}$. We will have several types of possibilities for the rank of F. Using this, we find conditions to describe the problem.

Theorem 3.1. *Let $\mathscr{D}$ be a distribution of rank (p,q) $(p<m, q<n)$ on an (m,n)-dimensional graded manifold $\mathscr{M}$ such that*

$$m-p\leq \frac{p(p-1)}{2}+\frac{q(q-1)}{2}, n-q\leq \frac{q(q-1)}{2}, \tag{3.1}$$

Then $\mathscr{D}$ is a bracket-generating distribution of step 2, if and only if, the linear map F associated to $\mathscr{D}$ is of constant rank $(m-p,n-q)$ on $\mathscr{M}$.

Proof. Let $x\in M$. Suppose $\mathscr{D}$ is a bracket-generating distribution of step 2 and let $\{\delta/\delta x_i, \delta/\delta\eta_\mu, \partial/\partial y_a, \partial/\partial\zeta_\alpha\}$ be a basis of $Der\mathscr{O}_M(U)$ in a coordinate neighborhood U of x. Then $rank[\mathscr{D},\mathscr{D}](x)=(m-p,n-q)$. This means that the number of linearly independent graded vector fields of the set $\{[\frac{\delta}{\delta x_i},\frac{\delta}{\delta x_j}],[\frac{\delta}{\delta \eta_\mu},\frac{\delta}{\delta \eta_\nu}], 1\leq i,j\leq p, 1\leq \mu,\nu\leq q\}$, (respectively $\{[\frac{\delta}{\delta x_i},\frac{\delta}{\delta \eta_\mu}], 1\leq i\leq p, 1\leq\mu\leq q\}$) is $m-p$ (respectively $n-q$). Therefore the coefficient matrix, the matrix consisting of the coefficients of the Lie brackets of graded vector fields $\{[\frac{\delta}{\delta x_i},\frac{\delta}{\delta x_j}],[\frac{\delta}{\delta \eta_\mu},\frac{\delta}{\delta \eta_\nu}]\}$ at the point x, denoted by

$$\begin{bmatrix} D_{ij}^a(x) & D_{\mu\nu}^a(x)\\ \tilde{D}_{ij}^\alpha(x) & \tilde{D}_{\mu\nu}^\alpha(x)\end{bmatrix}, {}_{\substack{a=1,\dots,m-p\\ \alpha=1,\dots,n-q}}, (mod\mathscr{D}),$$

having the rank $m-p$, is invertible. Similarly, the coefficient matrix

$$\begin{bmatrix} D_{i\mu}^a(x)\\ \tilde{D}_{i\mu}^\alpha(x)\end{bmatrix}, {}_{\substack{a=1,\dots,m-p\\ \alpha=1,\dots,n-q}}, (\mod \mathscr{D}),$$

the matrix consisting of the coefficients of the Lie brackets of graded vector fields $\{[\frac{\delta}{\delta x_i},\frac{\delta}{\delta \eta_\mu}]\}$ at the point x, has rank $n-q$, (i.e. $n-q=rank\begin{pmatrix}\tilde{D}_{i\mu}^\alpha(x)\\ D_{i\mu}^a(x)\end{pmatrix}$, and this matrix is even). Hence associated with F is the graded vector field, represented by the matrix $\begin{pmatrix}D_{bc}^a(x)\\ \tilde{D}_{ef}^\alpha(x)\end{pmatrix}, (mod\mathscr{D})$, relative to the above basis. It is clear that $rankF(x)=(m-p,n-q)$.

Conversely, suppose that $x \in M$ and $rankF(x) = (m-p, n-q)$ on $\mathscr{M}$. Let $\{\delta/\delta x_i, \delta/\delta\eta_\mu, \partial/\partial y_a, \partial/\partial\zeta_\alpha\}$ be a basis of $Der\mathscr{O}_M(U)$ in a coordinate neighborhood U of x. Consider the coefficient matrix of the graded vector fields $F(\frac{\delta}{\delta x_i}, \frac{\delta}{\delta x_j})$ and $F(\frac{\delta}{\delta\eta_\mu}, \frac{\delta}{\delta\eta_\nu})$, which is even and denoted by

$$\begin{bmatrix} F^a_{ij}(x) & F^a_{\mu\nu}(x) \\ \tilde{F}^\alpha_{ij}(x) & \tilde{F}^\alpha_{\mu\nu}(x) \end{bmatrix}. \tag{3.2}$$

Note that its rank is $m-p$, otherwise F would not be a map of the given rank. Thus there are two non-negative integers r and s such that $r+s = m-p$ and $rank(F^a_{ij}(x)) = r, rank(\tilde{F}^\alpha_{\mu\nu}(x)) = s$. Hence we may assume that the submatrices $G = (F^{a'}_{i'j'}(x)), 1 \le a', j'-1 \le r, i' < j'$ and $J = (\tilde{F}^{\alpha'}_{\mu'\nu'}(x)), 1 \le \alpha', \nu'-1 \le s, \mu' < \nu'$, are both invertible. Therefore, the submatrix,

$$\begin{bmatrix} G & H \\ I & J \end{bmatrix} = \begin{bmatrix} F^{a'}_{i'j'}(x) & F^{a'}_{\mu'\nu'}(x) \\ \tilde{F}^{\alpha'}_{i'j'}(x) & \tilde{F}^{\alpha'}_{\mu'\nu'}(x) \end{bmatrix}, \ {}_{\substack{1\le a', j'-1\le r \\ i'<j'}}, \ {}_{\substack{1\le \alpha', \nu'-1\le s \\ \mu'<\nu'}}, \tag{3.3}$$

is invertible.

Similarly, consider the coefficient matrix of the graded vector fields $F(\frac{\delta}{\delta x_i}, \frac{\delta}{\delta\eta_\mu})$, which is odd and its rank is $n-q$. We denote it by

$$\begin{bmatrix} F^a_{i\mu}(x) \\ \tilde{F}^\alpha_{i\mu}(x) \end{bmatrix}, 1 \le a \le m-p, 1 \le \alpha \le n-q.$$

Since $rank(\tilde{F}^\alpha_{i\mu}(x)) = n-q$, we may assume that the submatrice $(\tilde{F}^\alpha_{i'\mu'}(x)), 1 \le \mu'-1 \le n-q, i' < \mu'$, is invertible. We thus consider

$$\begin{bmatrix} F^a_{i'\mu'}(x) \\ \tilde{F}^\alpha_{i'\mu'}(x) \end{bmatrix}, 1 \le \mu'-1 \le n-q, i' < \mu'. \tag{3.4}$$

Given the matrices (3.3) and (3.4), we may change the generators of $Der\mathscr{O}_\mathscr{M}$ to $\{\delta/\delta x_i, \delta/\delta\eta_\mu, Y_b, Z_\nu\}, b = 1,...,m-p; \nu = 1,...,n-q$, where $Y_b \in \{[\frac{\delta}{\delta x_{i'}}, \frac{\delta}{\delta x_{j'}}], [\frac{\delta}{\delta\eta_{\mu'}}, \frac{\delta}{\delta\eta_{\nu'}}]\}$, with local coefficients $(F^{a'}_{i'j'}(x) \ \ \tilde{F}^{\alpha'}_{i'j'}(x))$ or $(F^{a'}_{\mu'\nu'}(x) \ \ \tilde{F}^{\alpha'}_{\mu'\nu'}(x))$ of the matrix (3.3) and $Z_\nu \in \{[\frac{\delta}{\delta x_{i'}}, \frac{\delta}{\delta\eta_{\mu'}}]\}$, with local coefficients $(F^{a'}_{i'\mu'}(x) \ \ \tilde{F}^{\alpha'}_{i'\mu'}(x))$ of the matrix (3.3). Thus $\mathscr{D}$ is bracket-generating of step 2. □

By using Theorem (3.1) we can easily prove the following theorems.

Theorem 3.2. *Let $\mathscr{M}$ be an (m,n) dimensional graded manifold. Suppose that $\mathscr{D}$ is a distribution of rank $(m-1,n)$. Then $\mathscr{D}$ is bracket-generating of step 2, if and only if, for the linear map $F = F_0 + F_1$ associated to $\mathscr{D}$, $F_0 \neq 0$ on $\mathscr{M}$.*

Proof. Since $rank\mathscr{D}(z) = (m-1,n)$, there exist a coordinate system $(x_i, t, \eta_\mu), i = 1,...,m-1, \mu = 1,...,n$, defined in a neighborhood U of z, such that $\mathscr{D}$ is locally given by $\{\delta/\delta x_i, \delta/\delta\eta_\mu\}$ and $\{\delta/\delta x_i, \delta/\delta\eta_\mu, \partial/\partial t\}$ is a local basis for $Der\mathscr{O}_M$. Therefore, according to the Theorem 3.1, the coefficient matrix,

$$\begin{bmatrix} D^1_{ij}(x) & D^1_{\mu\nu}(x) \end{bmatrix}, \quad (mod\mathscr{D}),$$

has the rank 1. Hence $F_0 \neq 0$. □

Theorem 3.3. *Let $\mathscr{M}$ be an $(m,n)-$dimensional graded manifold. Suppose that $\mathscr{D}$ is a distribution of rank $(m,n-1)$. Then $\mathscr{D}$ is bracket-generating of step 2, if and only if, for the linear map $F = F_0 + F_1$ associated to $\mathscr{D}$, $F_1 \neq 0$ on $\mathscr{M}$.*

Proof. Since $rank\mathscr{D}(z) = (m,n-1)$, there exist a coordinate system $(x_i, \eta_\mu, \theta), i = 1,...,m, \ \mu = 1,...,n-1$, defined in a neighborhood U of z, such that $\mathscr{D}$ is locally given by $\{\delta/\delta x_i, \delta/\delta\eta_\mu\}$ and $\{\delta/\delta x_i, \delta/\delta\eta_\mu, \partial/\partial\theta\}$ is a local basis for $Der\mathscr{O}_M$. Therefore, according to the Theorem 3.1, the coefficient matrix,

$$\begin{bmatrix} \tilde{D}^1_{i\mu}(x) \end{bmatrix}, \quad (\mod \mathscr{D}),$$

has the rank n. Hence $F_1 \neq 0$. □

Theorem 3.4. *Let $\mathscr{M}$ be an (m,n) dimensional graded manifold. Suppose that $\mathscr{D}$ is a distribution of rank $(0,n)$. Then $\mathscr{D}$ is bracket-generating of step 2, if and only if, for the linear map $F = F_0 + F_1$ associated to $\mathscr{D}$, $rankF_0 = m$ on $\mathscr{M}$.*

Proof. The details are the same as those given in the proof of Theorem 3.1. □

Example 3.5. *Consider the graded manifold $\mathscr{M} = R^{3|1}$. Let $(x_i, t, \eta), i = 1,...,2$ be local supercoordinates on a coordinate neighborhood U of $x \in R^3$. Suppose that $\mathscr{D}$ is the distribution spanned by $\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}$ and $\frac{\delta}{\delta\eta}$ where*

$$\frac{\delta}{\delta\eta} = \frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial t}, \quad \frac{\delta}{\delta x_1} = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial t}, \quad \frac{\delta}{\delta x_2} = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial t}.$$

A simple calculation shows that $[\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}] = \frac{\partial}{\partial t}$ and $\{\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}, \frac{\partial}{\partial t}, \frac{\delta}{\delta\eta}\}$ is a basis of $Der\mathscr{O}_{R^3}(U)$. Thus $\mathscr{D}$ is bracket-generating of step 2.

Example 3.6. *Consider the graded manifold* $\mathscr{M} = R^{4|4}$. *Let* $(x_i, \eta_\mu), i, \mu = 1, ..., 4$ *be local supercoordinates on a coordinate neighborhood* U *of* $x \in R^4$. *Suppose that* $\mathscr{D}$ *is the distribution (see [7]) spanned by* $\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_3}$ *and* $\frac{\delta}{\delta\eta_4}$, *where*

$$\begin{aligned}
\frac{\delta}{\delta\eta_1} &= \frac{\partial}{\partial\eta_1} - i\eta_3\frac{\partial}{\partial x_1} - i\eta_4\frac{\partial}{\partial x_2} - \eta_4\frac{\partial}{\partial x_3} - i\eta_3\frac{\partial}{\partial x_4},\\
\frac{\delta}{\delta\eta_2} &= \frac{\partial}{\partial\eta_2} - i\eta_4\frac{\partial}{\partial x_1} - i\eta_3\frac{\partial}{\partial x_2} + \eta_3\frac{\partial}{\partial x_3} + i\eta_4\frac{\partial}{\partial x_4},\\
\frac{\delta}{\delta\eta_3} &= \frac{\partial}{\partial\eta_3} - i\eta_1\frac{\partial}{\partial x_1} - i\eta_2\frac{\partial}{\partial x_2} + \eta_2\frac{\partial}{\partial x_3} - i\eta_1\frac{\partial}{\partial x_4},\\
\frac{\delta}{\delta\eta_4} &= \frac{\partial}{\partial\eta_4} - i\eta_2\frac{\partial}{\partial x_1} - i\eta_1\frac{\partial}{\partial x_2} - \eta_1\frac{\partial}{\partial x_3} + i\eta_2\frac{\partial}{\partial x_4}.
\end{aligned}$$

Here $i = \sqrt{-1}$. *Thus the vector fields* $[\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_1}], [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_2}], [\frac{\delta}{\delta\eta_3}, \frac{\delta}{\delta\eta_3}]$, *and* $[\frac{\delta}{\delta\eta_3}, \frac{\delta}{\delta\eta_4}]$ *are zero and*

$$\begin{aligned}
&[\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_3}] = -2i\frac{\partial}{\partial x_1} - 2i\frac{\partial}{\partial x_4}, \quad [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_4}] = -2i\frac{\partial}{\partial x_2} - 2\frac{\partial}{\partial x_3},\\
&[\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_3}] = -2i\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3}, \quad [\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_4}] = -2i\frac{\partial}{\partial x_1} + 2i\frac{\partial}{\partial x_4}.
\end{aligned}$$

In the notation used in Theorem 3.1, all of the entries $D^a_{ij}, \tilde{D}^\alpha_{ij}, \tilde{D}^\alpha_{\mu\nu}, D^a_{i\mu}, \tilde{D}^\alpha_{i\mu}$ *of the coefficient matrix except* $D^a_{\mu\nu}$ *are zero and*

$$[D^a_{\mu\nu}] = \begin{bmatrix} 0 & -2i & 0 & 0 & -2i & 0\\ 0 & 0 & -2i & -2i & 0 & 0\\ 0 & 0 & -2 & 2 & 0 & 0\\ 0 & -2i & 0 & 0 & +2i & 0 \end{bmatrix}. \tag{3.5}$$

So we have $\mathrm{rank}(D^a_{\mu\nu}) = 4$, *and we conclude from Corollary 3.4, that* $\mathscr{D}$ *is a bracket-generating distribution of step* 2. *By calculation we have*

$$\begin{aligned}
&\frac{1}{4}i((\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_3}]) + (\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_4}]) - 2(\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_1}])) = \frac{\partial}{\partial x_1},\\
&\frac{1}{4}i((\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_3}]) + (\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_4}]) - 2(\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_1}])) = \frac{\partial}{\partial x_2},\\
&\frac{1}{4}((\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_3}]) - (\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_4}])) = \frac{\partial}{\partial x_3},\\
&\frac{1}{4}i((\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_1}, \frac{\delta}{\delta\eta_3}]) - (\frac{\delta}{\delta\eta_1} + [\frac{\delta}{\delta\eta_2}, \frac{\delta}{\delta\eta_4}])) = \frac{\partial}{\partial x_4}.
\end{aligned}$$

Example 3.7. *Let* $\mathscr{M} = R^{3|1}$ *equiped with local supercoordinates* (x_1, x_2, x_3, η) *and* $\mathscr{D}$ *be the distribution spanned by* $\{\frac{\delta}{\delta x_1} = \frac{\partial}{\partial x_1}, \frac{\delta}{\delta x_2} = \frac{\partial}{\partial x_2} + (x_1)^2\frac{\partial}{\partial x_3}, \frac{\delta}{\delta\eta} = \frac{\partial}{\partial\eta}\}$. *In this case we have*

$$\begin{aligned}
[\frac{\delta}{\delta x_1}, \frac{\delta}{\delta\eta}] = [\frac{\delta}{\delta x_2}, \frac{\delta}{\delta\eta}] &= 0,\\
[\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}] &= 2x_1\frac{\partial}{\partial x_3},\\
[\frac{\delta}{\delta x_1}, [\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}]] &= 2\frac{\partial}{\partial x_3}.
\end{aligned}$$

We conclude from Corollary 3.2 that $\mathscr{D}$ *is not bracket-generating of step* 2 *on the whole* $R^{3|1}$. *It is bracket-generating of step* 3.

Example 3.8. *Let* $\mathscr{M} = R^{1|2}$ *equiped with local supercoordinates* (x, η_1, η_2) *and* $\mathscr{D}$ *be the distribution spanned by* $\{\frac{\delta}{\delta x} = \frac{\partial}{\partial x}, \frac{\delta}{\delta\eta_1} = \frac{\partial}{\partial\eta_1} + x\frac{\partial}{\partial\eta_2}\}$. *Then* $[\frac{\partial}{\partial x}, \frac{\delta}{\delta\eta_1}] = \frac{\delta}{\delta\eta_2}$ *and from Corollary 3.3, we see that* $\mathscr{D}$ *is bracket-generating of step* 2 *on* $R^{1|2}$.

References

[1] A. Bejancu, On bracket-generating distributions, Int. Electron. J. Geom. **3** (2010) no. 2, 102 - 107.
[2] O. Goertsches, Riemannian supergeometry, *Math. Z.*, **260** (2008) 557-–593.
[3] J. Monterde and J. Munoz-Masque and O. A. Sanchez-Valenzuela, *Geometric properties of involutive distributions on graded manifolds,* Indag. Mathem., N.S., **8** (1997), 217-246.
[4] S. Vacaru and H. Dehnen, *Locally Anisotropic Structures and Nonlinear Connections in Einstein and Gauge Gravity*, Gen. Rel. Grav., **35** (2003) 209-250.
[5] S. I. Vacaru, *Superstrings in higher order extensions of Finsler Superspaces,* Nucl. Phys. B **494** (1997) no. 3, 590-656.
[6] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction,* Courant Lecture Notes Series, New York, 2004.
[7] P. C. West, *Introduction to supersymmetry and supergravity,* Second Edition, World Scientific Pub Co Inc, 1990.
[8] C. D. Zanet, Generic one-step bracket-generating distributions of rank four, Archivum Mathematicum, 51 (2015), 257 - 264.

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On the convergence of a modified superquadratic method for generalized equations

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Abstract

Let X and Y be Banach spaces. Let Ω be an open subset of X. Suppose that $f : X \to Y$ is Fréchet differentiable in Ω and $\mathscr{F} : X \rightrightarrows 2^Y$ is a set-valued mapping with closed graph. In the present paper, a modified superquadratic method (MSQM) is introduced for solving the generalized equations $0 \in f(x)+\mathscr{F}(x)$, and studied its convergence analysis under the assumption that the second Fréchet derivative of f is Hölder continuous. Indeed, we show that the sequence, generated by MSQM, converges super-quadratically in both semi-locally and locally to the solution of the above generalized equation whenever the second Fréchet derivative of f satisfies a Hölder-type condition.

1. Introduction

Throughout this paper we assume that X and Y are two real or complex Banach spaces and $\Omega \neq \emptyset$ is an open subset of X. Let $f : X \to Y$ be a Fréchet differentiable function on Ω. Further, assume that the first and second Fréchet derivatives of f are denoted by ∇f and $\nabla^2 f$ respectively. Let $\mathscr{F}$ be a set-valued mapping with closed graph acting between Banach space X and the subsets of Y. In this communication, we are interested to approximate the solution of the following generalized equation problem

$$0 \in f(x)+\mathscr{F}(x). \tag{1.1}$$

The inclusions type (1.1), introduced by Robinson [24, 26] as a general tool for describing, analyzing, and solving different problems in a unified manner, have been studied extensively. The inclusion problem (1.1) is an abstract model for variety of problems. When $\mathscr{F} = \{0\}$, (1.1) is an equation. When $\mathscr{F}$ is the positive orthant in $\mathbb{R}^n$, (1.1) is a system of inequalities. When $\mathscr{F}$ is the normal cone to a convex and closed set in X, (1.1) reduces to variational inequalities. When $\mathscr{F} = \partial\psi_C$ is the subdifferential of the function

$$\psi_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise,} \end{cases}$$

(1.1) is reduced to some minimization problems which has been studied by Robinson [25].

To solve (1.1), Dontchev [1] introduced the following classical Newton-type method, for each $k = 0,1,\ldots,$

$$0 \in f(x_k)+\nabla f(x_k)(x_{k+1}-x_k)+\mathscr{F}(x_{k+1}),$$

under the assumptions the set-valued mapping $\mathscr{F}$ is pseudo-Lipschitz and the Fréchet derivative of f is Lipschitz on a neighborhood of the solution of (1.1) and established a quadratic convergence of the method. In his subsequent paper [2], he proved the uniform convergence of the method. By following Dontchev's method, Piétrus [5] obtained a super-linear convergence when the Fréchet derivative of f is Hölder continuous on a neighborhood of the solution of (1.1) and later he [6] established the uniform convergence of this method in this mild differentiability context.

Let $x \in X$. By $\mathscr{D}(x)$, we symbolize the subset of X which is defined by

$$\mathscr{D}(x) := \left\{ d \in X : 0 \in f(x)+\nabla f(x)d+\frac{1}{2}\nabla^2 f(x)d^2+\mathscr{F}(x+d) \right\}.$$

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For finding an approximate solution of (1.1), the extension of Dontchev's indigenous work [3] was done by Geoffroy et al. [14]. Geoffroy and Pietrus [13] introduced the following superquadratic method (see Algorithm 1) for solving the generalized equation (1.1) and showed that it is locally superquadratic convergent:

Algorithm 1 (The Superquadratic Method)

Step 0. Pick $x_0 \in X$ and put $k := 0$.
Step 1. If $0 \in \mathscr{D}(x_k)$, then stop; otherwise, go to Step 2.
Step 2. If $0 \notin \mathscr{D}(x_k)$, choose d_k such that $d_k \in \mathscr{D}(x_k)$.
Step 3. Set $x_{k+1} := x_k + d_k$.
Step 4. Replace k by $k+1$ and go to Step 1.

Note that under some suitable conditions around a solution x^* of the generalized equation (1.1), the authors [13, Theorem 3.1] showed that there exists a neighborhood Ω of x^* such that, for any point in Ω, there exists a sequence generated by Algorithm 1 which is superquadratically convergent to the solution x^*. This implies that the convergence result, established in [13], guarantees the existence of a convergent sequence. Therefore, for any initial point near to a solution, the sequences generated by Algorithm 1 are not uniquely defined and not every generated sequence is convergent. Hence, in view of numerical computation, this kind of methods is not convenient in practical application. This drawback motivates us to propose a method 'so-called' modified superquadratic method (MSQM) as follows:

Algorithm 2 (The Modified Superquadratic Method (MSQM))

Step 0. Pick $\eta \in [1,\infty)$, $x_0 \in X$ and put $k := 0$.
Step 1. If $0 \in \mathscr{D}(x_k)$, then stop; otherwise, go to Step 2.
Step 2. If $0 \notin \mathscr{D}(x_k)$, choose d_k such that $d_k \in \mathscr{D}(x_k)$ and

$$\|d_k\| \leq \eta \text{ dist } (0, \mathscr{D}(x_k)).$$

Step 3. Set $x_{k+1} := x_k + d_k$.
Step 4. Replace k by $k+1$ and go to Step 1.

The difference between Algorithms 1 and 2 is that Algorithm 2 generates at least one sequence and every generated sequence is convergent but this does not appear in Algorithm 1. Since the sequences generated by Algorithm 1 are not uniquely defined, in contrast with Algorithm 1, we can guess that Algorithm 2 is more suitable than Algorithm 1 in numerical computation.
It is remark that if we replace the set $\mathscr{D}(x)$ by

$$\mathscr{S}(x) := \Big\{ d \in X : 0 \in f(x) + \nabla f(x)d + \mathscr{F}(x+d) \Big\},$$

the Algorithm 2 introduced in the present paper will be the same with the Algorithm given in [16, 23].
To solving (1.1), there have a large number of works on semilocal analysis ; see for example [7, 8, 11, 12, 19, 20, 27, 28]. Rashid et al. [16, 23] established semilocal convergence analysis for solving the generalized equation problem (1.1), which was the extension of Dontchev's work in [1]. Rashid [17] introduced a variant of Newton-type Method for solving (1.1) and obtained its semilocal and local convergence results. The same author [18] associated extended Newton-type method for solving a variational inclusion of the form

$$0 \in f(x) + g(x) + \mathscr{F}(x),$$

where $g : X \to Y$ admits first order divided difference and established its semilocal and local convergence results for solving (1.1). As far as we know, there doesn't have any other study on semilocal analysis for the Algorithm 1.
The purpose of this study is to analyze the semilocal convergence for the modified superquadratic method defined by Algorithm 2. The main tool is the Lipschitz-like property of set-valued mappings. The main results are the convergence criteria, established in Sect.3, which, based on the information around the initial point, provides some sufficient conditions ensuring the convergence to a solution of any sequence generated by Algorithm 2. As a consequence, local convergence result for the modified superquadratic method is obtained.
This paper is organized as follows: In Section 2, we recall a few necessary preliminary results. In Section 3, we consider the modified superquadratic method for solving the generalized equation as well as using the concept of Lipchitz-like mappings, we prove the existence of a sequence $\{x_k\}$ generated by Algorithm 2 and show that it is semilocally and locally superquadratic convergent. In the last section, we give a summary of the major results presented in this paper.

2. Preliminary results

Let $x \in X$ and $\mathbb{B}(x,r) = \{y : \|y-x\| \leq r\}$ be denote the closed ball centered at x with radius $r > 0$. Let $\Gamma \colon X \rightrightarrows 2^Y$ be a set-valued mapping. The domain of Γ, denoted by $\operatorname{dom}\Gamma$, is defined by

$$\operatorname{dom}\Gamma := \{x \in X : \ \Gamma(x) \neq \emptyset\}.$$

The inverse and the graph of Γ, denoted by Γ^{-1} and $\operatorname{gph}\Gamma$ respectively, are defined by

$$\Gamma^{-1}(y) := \{x \in X : y \in \Gamma(x)\} \quad \text{for each } y \in Y$$

$$\text{and} \quad \operatorname{gph}\Gamma := \{(x,y) \in X \times Y : y \in \Gamma(x)\}.$$

Let $B \subseteq X$. The distance from a point $x \in X$ to a set B is defined by

$$\operatorname{dist}(x,B) := \inf_{b\in B} \|x-b\|,$$

and the excess from the set A to the set B is defined by

$$e(B,A) = \sup_{x\in B}\{\operatorname{dist}(x,A)\}.$$

The notions of pseudo-Lipschitz and Lipchitz-like set-valued mappings are due to [23]. Aubin [9, 10] introduced these notions and studied extensively.

Definition 2.1. *Let $G : Y \rightrightarrows 2^X$ be a set-valued mapping and let $(\bar{y},\bar{x}) \in \operatorname{gph}G$. Let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $M > 0$. Then the mapping G is said to be*

(a) *Lipschitz-like on $\mathbb{B}(\bar{y},r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x},r_{\bar{x}})$ with constant M if the following inequality holds:*

$$e(G(y_1)\cap\mathbb{B}(\bar{x},r_{\bar{x}}),G(y_2)) \le M\|y_1-y_2\| \quad \textit{for any } y_1,y_2 \in \mathbb{B}(\bar{y},r_{\bar{y}}).$$

(b) *pseudo-Lipschitz around $(\bar{y},\bar{x})$ if there exist constants $a > 0$, $b > 0$ and $M' > 0$ such that G is Lipschitz-like on $\mathbb{B}(\bar{y},b)$ relative to $\mathbb{B}(\bar{x},a)$ with constant M'.*

The following notion of (L, p)-Hölder continuity property is due to [21].

Definition 2.2. *Let $f : X \to Y$ be a Fréchet differentiable function on some neighborhood U of $\bar{x}$ and let $\nabla^2 f$ be the second Fréchet derivative of f on U. Let $p \in [0,1]$ and $L > 0$. Then $\nabla^2 f$ is called (L, p)-Höder continuous on U with constant L if the following condition holds:*

$$\|\nabla^2 f(x_1)-\nabla^2 f(x_2)\| \le L\|x_1-x_2\|^p, \ \text{ for any } x_1,x_2 \in U.$$

The following lemma has taken from [23]. This lemma employs a vital role for proving the convergence analysis.

Lemma 2.3. *Let $G : Y \rightrightarrows 2^X$ be a set-valued mapping and let $(\bar{y},\bar{x}) \in \operatorname{gph} G$. Assume that G is Lipschitz-like on $\mathbb{B}(\bar{y},r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x},r_{\bar{x}})$ with constant M. Then*

$$\operatorname{dist}(x,G(y)) \le M\operatorname{dist}(y,G^{-1}(x))$$

holds for every $x \in \mathbb{B}(\bar{x},r_{\bar{x}})$ and $y \in \mathbb{B}(\bar{y},\frac{r_{\bar{y}}}{3})$ satisfying $\operatorname{dist}(y,G^{-1}(x)) \le \dfrac{r_{\bar{y}}}{3}$.

We would like to finish this section with the following lemma that is known in [4].

Lemma 2.4. *Let $\Phi : X \rightrightarrows 2^X$ be a set-valued mapping. Let $\bar{x} \in X$, $c > 0$ and $0 < r < 1$ be such that*

$$\operatorname{dist}(\bar{x},\Phi(\bar{x})) < c(1-r); \tag{2.1}$$

and

$$e(\Phi(x_1)\cap\mathbb{B}(\bar{x},c),\Phi(x_2)) \le r\|x_1-x_2\| \quad \textit{for any } x_1,x_2 \in \mathbb{B}(\bar{x},c). \tag{2.2}$$

Then Φ has a fixed point in $\mathbb{B}(\bar{x},c)$, that is, there exists $x \in \mathbb{B}(\bar{x},c)$ such that $x \in \Phi(x)$. Moreover, if Φ is single-valued, then the fixed point of Φ in $\mathbb{B}(\bar{x},c)$ is unique.

3. Convergence analysis of MSQM

This section is devoted to prove the existence and convergence of the sequences generated by the modified superquadratic method defined by Algorithm 2. To this end, let $x \in X$ and let us define the mapping T_x by

$$T_x(\cdot) := f(x)+\nabla f(x)(\cdot-x)+\frac{1}{2}\nabla^2 f(x)(\cdot-x)^2+\mathscr{F}(\cdot).$$

Then for the construction of $\mathscr{D}(x)$, we have that

$$\begin{aligned}\mathscr{D}(x) &= \Big\{d \in X : 0 \in T_x(x+d)\Big\}\\ &= \Big\{d \in X : x+d \in {T_x}^{-1}(0)\Big\}.\end{aligned} \tag{3.1}$$

Moreover, for any $v \in X$ and $y \in Y$, the inclusions

$$v \in T_x^{-1}(y) \text{ and } y \in f(x)+\nabla f(x)(v-x)+\frac{1}{2}\nabla^2 f(x)(v-x)^2+\mathscr{F}(v). \tag{3.2}$$

are equivalent. In particular,

$$\bar{x} \in T_{\bar{x}}^{-1}(\bar{y}) \quad \text{for each } (\bar{x},\bar{y}) \in \operatorname{gph}\,(f+\mathscr{F}).$$

The following result is due to [15]. This result establishes the equivalence relation between $(f+\mathscr{F})^{-1}$ and $T_{\bar{x}}^{-1}$.

Lemma 3.1. *Let $f : X \to Y$ be a function and let $(\bar{x},\bar{y}) \in \operatorname{gph}\,(f+\mathscr{F})$. Assume that f is twice differentiable in an open neighborhood Ω of $\bar{x}$ and that its second-order derivative is continuous at $\bar{x}$. Then the following are equivalent:*

(i) *The mapping* $(f+\mathscr{F})^{-1}$ *is pseudo-Lipschitz at* $(\bar{y},\bar{x})$*;*
(ii) *The mapping* $T_{\bar{x}}^{-1}(\cdot)$ *is pseudo-Lipschitz at* $(\bar{y},\bar{x})$.

Let $r_{\bar{x}}>0$, $r_{\bar{y}}>0$ and $(\bar{x},\bar{y})\in \mathrm{gph}\,(f+\mathscr{F})$. Then, the closed graph property of the set-valued mapping $f+\mathscr{F}$ implies that $f+\mathscr{F}$ is continuous at $\bar{x}$ for $\bar{y}$, that is,

$$\lim_{x\to\bar{x}} \mathrm{dist}(\bar{y}, f(x)+\mathscr{F}(x))=0. \tag{3.3}$$

Assume that $\mathbb{B}(\bar{x},r_{\bar{x}})\subseteq \Omega\cap\mathrm{dom}\,\mathscr{F}$. Moreover, by Lemma 3.1 we assume that the mapping $T_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y},r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x},r_{\bar{x}})$ with constant M, that is,

$$e(T_{\bar{x}}^{-1}(y_1)\cap\mathbb{B}(\bar{x},r_{\bar{x}}),T_{\bar{x}}^{-1}(y_2))\le M\|y_1-y_2\| \;\forall\, y_1,\, y_2\in\mathbb{B}(\bar{y},r_{\bar{y}}). \tag{3.4}$$

Let $p\in(0,1]$, $L>0$ and setting

$$\alpha:=\min\Big\{r_{\bar{y}}-\frac{L(3^{p+2}+2^{p+2})r_{\bar{x}}^{p+2}}{(p+1)(p+2)2^{p+2}},\,\frac{r_{\bar{x}}(2^{p+1}-5MLr_{\bar{x}}^{p})}{5M2^{p+2}}\Big\}. \tag{3.5}$$

Then

$$\alpha>0 \text{ if and only if } L<\min\Big\{\frac{2^{p+2}(p+1)(p+2)r_{\bar{y}}}{(3^{p+2}+2^{p+2})r_{\bar{x}}^{p+2}},\,\frac{2^{p+1}}{5Mr_{\bar{x}}^{p}}\Big\}. \tag{3.6}$$

The following lemma plays a vital role for convergence analysis of the modified superquadratic method. The proof is a refinement of the one for [23, Lemma 3.1].

Lemma 3.2. *Let* $T_{\bar{x}}^{-1}$ *be a Lipschitz-like mapping on* $\mathbb{B}(\bar{y},r_{\bar{y}})$ *relative to* $\mathbb{B}(\bar{x},r_{\bar{x}})$ *with constant* M*. Let* $p\in(0,1]$ *and* $x\in\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$*. Assume that* ∇f *and* $\nabla^2 f$ *are* (L,p)*-Höder continuous at* $\bar{x}$ *on* $\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$ *with the same constant* L *defined by* (3.6)*. Let* α *be defined in* (3.5) *so that* (3.6) *is satisfied. Then the mapping* T_x^{-1} *is Lipschitz-like on* $\mathbb{B}(\bar{y},\alpha)$ *relative to* $\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$ *with constant* $\frac{5M2^p}{2^{p+1}-5MLr_{\bar{x}}^{p}}$ *i.e.*

$$e(T_x^{-1}(t_1)\cap\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2}),T_x^{-1}(t_2))\le\frac{5M2^p}{2^{p+1}-5MLr_{\bar{x}}^{p}}\|t_1-t_2\| \quad \textit{for every } t_1,\, t_2\in\mathbb{B}(\bar{y},\alpha).$$

Proof. Since α is defined in (3.5) so that (3.6) is satisfied, then it is clear that $\alpha>0$. Now let

$$t_1,\, t_2\in\mathbb{B}(\bar{y},\alpha) \quad \text{and} \quad u'\in T_x^{-1}(t_1)\cap\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2}). \tag{3.7}$$

To complete the proof, it is sufficient to show that there exists $u''\in T_x^{-1}(t_2)$ such that

$$\|u'-u''\|\le\frac{5M2^p}{2^{p+1}-5MLr_{\bar{x}}^{p}}\|t_1-t_2\|.$$

To finish this, we need to verify that there exists a sequence $\{x_k\}\subseteq\mathbb{B}(\bar{x},r_{\bar{x}})$ such that

$$\begin{aligned} t_2\in{}& f(x)+\nabla f(x)(x_{k-1}-x)+\nabla f(\bar{x})(x_k-x_{k-1})\\ &+\frac{1}{2}\nabla^2 f(x)(x_{k-1}-x)^2+\frac{1}{2}\nabla^2 f(\bar{x})\Big((x_k-\bar{x})^2-\\ &(x_{k-1}-\bar{x})^2\Big)+\mathscr{F}(x_k) \end{aligned} \tag{3.8}$$

and

$$\|x_k-x_{k-1}\|\le\frac{5M}{2}\|t_1-t_2\|\Big(\frac{5MLr_{\bar{x}}^{p}}{2^{p+1}}\Big)^{k-2} \tag{3.9}$$

hold for each $k=2,3,4,\ldots$. We proceed by induction on k. Write

$$\begin{aligned} a_i:={}& t_i-f(x)-\nabla f(x)(u'-x)-\frac{1}{2}\nabla^2 f(x)(u'-x)^2+f(\bar{x})\\ &+\nabla f(\bar{x})(u'-\bar{x})+\frac{1}{2}\nabla^2 f(\bar{x})(u'-\bar{x})^2 \quad \text{for each } i=1,2. \end{aligned} \tag{3.10}$$

Note by (3.7) that

$$\|x-u'\|\le\|x-\bar{x}\|+\|\bar{x}-u'\|\le r_{\bar{x}}. \tag{3.11}$$

Furthermore, we have, for (3.10), that

$$\begin{aligned} \|a_i-\bar{y}\|\le{}& \|t_i-\bar{y}\|+\|f(u')-f(x)-\nabla f(x)(u'-x)\\ &-\frac{1}{2}\nabla^2 f(x)(u'-x)^2\|+\|f(u')-f(\bar{x})\\ &-\nabla f(\bar{x})(u'-\bar{x})-\frac{1}{2}\nabla^2 f(\bar{x})(u'-\bar{x})^2\|. \end{aligned} \tag{3.12}$$

If ∇f is (L,p)-Hölder continuous at $\bar{x}$ with constant L, then we have that

$$\begin{aligned}
\|f(x)-f(\bar{x})-\nabla f(\bar{x})(x-\bar{x})\| &= \|\int_0^1 [\nabla f(\bar{x}+t(x-\bar{x}))-\nabla f(\bar{x})](x-\bar{x})dt\| \\
&\leq \int_0^1 \|\nabla f(\bar{x}+t(x-\bar{x}))-\nabla f(\bar{x})\|\|x-\bar{x}\|dt \\
&\leq L\|x-\bar{x}\|^{p+1}\int_0^1 t^p dt \\
&= \frac{L}{p+1}\|x-\bar{x}\|^{p+1}. \qquad (3.13)
\end{aligned}$$

Analogously, if $\nabla^2 f$ is (L,p)-Hölder continuous at $\bar{x}$ with constant L, then we have that

$$\begin{aligned}
\|f(x)-f(\bar{x})-\nabla f(\bar{x})(x-\bar{x})-\frac{1}{2}\nabla^2 f(\bar{x})(x-\bar{x})^2\| &= \|\int_0^1 [\nabla f(\bar{x}+t(x-\bar{x}))-\nabla f(\bar{x})-\nabla^2 f(\bar{x})(\bar{x}+t(x-\bar{x})-\bar{x})](x-\bar{x})dt\| \\
&\leq \int_0^1 \|\nabla f(\bar{x}+t(x-\bar{x}))-\nabla f(\bar{x})-\nabla^2 f(\bar{x})(\bar{x}+t(x-\bar{x})-\bar{x})\|\|x-\bar{x}\|dt \\
&= \int_{\bar{x}}^x \|\nabla f(u)-\nabla f(\bar{x})-\nabla^2 f(\bar{x})(u-\bar{x})\|du \\
&= \int_{\bar{x}}^x \int_0^1 \Big\{\|\nabla^2 f(\bar{x}+s(u-\bar{x}))-\nabla^2 f(\bar{x})\|ds\Big\}\|u-\bar{x}\|du \\
&\leq \int_{\bar{x}}^x \Big\{L\|u-\bar{x}\|^{p+1}\int_0^1 s^p ds\Big\}du \\
&\leq \frac{L}{p+1}\int_{\bar{x}}^x \|u-\bar{x}\|^{p+1}du \\
&= \frac{L}{(p+1)(p+2)}\|x-\bar{x}\|^{p+2}.
\end{aligned}$$

Then from (3.12), using the relations in (3.7), (3.11) and the relation $\alpha \leq r_{\bar{y}} - \dfrac{L(3^{p+2}+2^{p+2})r_{\bar{x}}^{p+2}}{(p+1)(p+2)2^{p+2}}$ by (3.5), we have that

$$\begin{aligned}
\|a_i-\bar{y}\| &\leq \alpha + \frac{L}{(p+1)(p+2)}(\|u'-x\|^{p+2}+\|u'-\bar{x}\|^{p+2}) \\
&\leq \alpha + \frac{L}{(p+1)(p+2)}\Big(r_{\bar{x}}^{p+2}+\frac{r_{\bar{x}}^{p+2}}{2^{p+2}}\Big) \\
&= \alpha + \frac{L(1+2^{p+2})r_{\bar{x}}^{p+2}}{(p+1)(p+2)2^{p+2}} \leq r_{\bar{y}}.
\end{aligned}$$

That is $a_i \in \mathbb{B}(\bar{y},r_{\bar{y}})$ for each $i=1,2$. Define $x_1 := u'$. Then $x_1 \in T_x^{-1}(t_1)$ by (3.7) and it follows from (3.2) that

$$t_1 \in f(x)+\nabla f(x)(x_1-x)+\frac{1}{2}\nabla^2 f(x)(x_1-x)^2+\mathscr{F}(x_1).$$

This can be written in another form as follows:

$$\begin{aligned}
& t_1+f(\bar{x})+\nabla f(\bar{x})(x_1-\bar{x}))+\frac{1}{2}\nabla^2 f(\bar{x})(x_1-\bar{x})^2 \in f(x) \\
& +\nabla f(x)(x_1-x)+\frac{1}{2}\nabla^2 f(x)(x_1-x)^2+\mathscr{F}(x_1)+f(\bar{x}) \\
& +\nabla f(\bar{x})(x_1-\bar{x})+\frac{1}{2}\nabla^2 f(\bar{x})(x_1-\bar{x})^2.
\end{aligned}$$

This, by the definition of a_1, implies that

$$a_1 \in f(\bar{x})+\nabla f(\bar{x})(x_1-\bar{x})++\frac{1}{2}\nabla^2 f(\bar{x})(x_1-\bar{x})^2+\mathscr{F}(x_1).$$

Hence $x_1 \in T_{\bar{x}}^{-1}(a_1)$ by (3.2). This together with (3.7) implies that

$$x_1 \in T_{\bar{x}}^{-1}(a_1)\cap\mathbb{B}(\bar{x},r_{\bar{x}}).$$

Noting that $a_1, a_2 \in \mathbb{B}(\bar{y},r_{\bar{y}})$ and $T_{\bar{x}}^{-1}$ is Lipschitz-like by our assumption. Then it follows from (3.4) that there exists $x_2 \in T_{\bar{x}}^{-1}(a_2)$ such that

$$\|x_2-x_1\| \leq M\|a_1-a_2\| = M\|t_1-t_2\| < \frac{5M}{2}\|t_1-t_2\|.$$

Moreover, by the construction of t_2 and noting $x_1=u'$, we have

$$x_2 \in T_{\bar{x}}^{-1}(a_2) = T_{\bar{x}}^{-1}\big(t_2-f(x)-\nabla f(x)(x_1-x)-\frac{1}{2}\nabla^2 f(x)(x_1-x)^2+f(\bar{x})+\nabla f(\bar{x})(x_1-\bar{x})+\frac{1}{2}\nabla^2 f(\bar{x})(x_1-\bar{x})^2\big),$$

which, together with (3.2), implies that

$$t_2 \in f(x)+\nabla f(x)(x_1-x)+\nabla f(\bar{x})(x_2-x_1)+\frac{1}{2}\nabla^2 f(x)(x_1-x)^2+\frac{1}{2}\nabla^2 f(\bar{x})\Big((x_2-\bar{x})^2-(x_1-\bar{x})^2\Big)+\mathscr{F}(x_2).$$

This shows that (3.8) and (3.9) are hold with generated points x_1,x_2.
Assume that $x_1,x_2,...,x_n$ are obtained so that (3.8) and (3.9) are hold for $k=2,3,\dots,n$. We need to construct x_{n+1} such that (3.8) and (3.9) are also true for $k=n+1$. For this purpose, set

$$\begin{aligned} a_i^n &:= t_2-f(x)-\nabla f(x)(x_{n+i-1}-x)-\frac{1}{2}\nabla^2 f(x)(x_{n+i-1}-x)^2+f(\bar{x})\\ &+\nabla f(\bar{x})(x_{n+i-1}-\bar{x})+\frac{1}{2}\nabla^2 f(\bar{x})(x_{n+i-1}-\bar{x})^2 \quad \text{for each } i=0,1. \end{aligned}$$

Then, for $i=0,1$, we obtain that

$$\begin{aligned} \|a_0^n-a_1^n\| &= \|(\nabla f(x)-\nabla f(\bar{x}))(x_n-x_{n-1})+\frac{1}{2}\nabla^2 f(x)\big((x_n-x)^2-(x_{n-1}-x)^2\big)\\ &-\frac{1}{2}\nabla^2 f(\bar{x})\big((x_n-\bar{x})^2-(x_{n-1}-\bar{x})^2\big)\|\\ &= \|(\nabla f(x)-\nabla f(\bar{x}))(x_n-x_{n-1})+\frac{1}{2}\nabla^2 f(x)\big((x_n-x_{n-1}+x_{n-1}-x)^2\\ &-(x_{n-1}-x)^2\big)-\frac{1}{2}\nabla^2 f(\bar{x})\big((x_n-x_{n-1}+x_{n-1}-\bar{x})^2-(x_{n-1}-\bar{x})^2\big)\|\\ &\le \|\nabla f(x)-\nabla f(\bar{x})\|\|x_n-x_{n-1}\|+\frac{1}{2}\|\nabla^2 f(x)-\nabla^2 f(\bar{x})\|\|x_n-x_{n-1}\|^2\\ &+\|\nabla^2 f(x)(x_{n-1}-x)-\nabla^2 f(\bar{x})(x_{n-1}-\bar{x})\|\|x_n-x_{n-1}\|. \end{aligned}$$

For all $z\in\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$, $x\mapsto\nabla f(x)$, $x\mapsto\nabla^2 f(x)$ and $x\mapsto\nabla^2 f(x)(z-x)$ are (L,p)-Hölder continuous at $\bar{x}$, thus we have that

$$\begin{aligned} \|a_0^n-a_1^n\| &\le L\|x-\bar{x}\|^p\{\|x_n-x_{n-1}\|+\frac{1}{2}\|x_n-x_{n-1}\|^2+\|x_n-x_{n-1}\|\}\\ &\le \frac{Lr_{\bar{x}}^p}{2^p}\big(2\|x_n-x_{n-1}\|+\frac{1}{2}\|x_n-x_{n-1}\|^2\big)\\ &\le \frac{Lr_{\bar{x}}^p}{2^p}\big(2\|x_n-x_{n-1}\|+\frac{1}{2}\|x_n-x_{n-1}\|\big), \text{ if the ball}\\ &\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2}) \text{ is sufficiently small}\\ &= \frac{5Lr_{\bar{x}}^p}{2^{p+1}}\|x_n-x_{n-1}\|. \end{aligned} \tag{3.14}$$

Since $\|x_1-\bar{x}\|\le\frac{r_{\bar{x}}}{2}$ by (3.7) and $\|t_1-t_2\|\le 2\alpha$ by (3.7), it follows from (3.9) that

$$\begin{aligned} \|x_n-\bar{x}\| &\le \sum_{j=2}^{n}\|x_j-x_{j-1}\|+\|x_1-\bar{x}\|\\ &\le 5M\alpha\sum_{j=2}^{n}\Big(\frac{5MLr_{\bar{x}}^p}{2^{p+1}}\Big)^{j-2}+\frac{r_{\bar{x}}}{2}\\ &= \frac{5M\alpha 2^{p+1}}{2^{p+1}-5MLr_{\bar{x}}^p}+\frac{r_{\bar{x}}}{2}. \end{aligned}$$

By (3.5), we have $\alpha\le\dfrac{r_{\bar{x}}(2^{p+1}-5MLr_{\bar{x}}^p)}{5M2^{p+2}}$ and so

$$\|x_n-\bar{x}\|\le r_{\bar{x}}. \tag{3.15}$$

Therefore, we obtain that

$$\|x_n-x\|\le\|x_n-\bar{x}\|+\|\bar{x}-x\|\le\frac{3}{2}r_{\bar{x}}. \tag{3.16}$$

Furthermore, using (3.7) and (3.16), one has that, for each $i=0,1$,

$$\begin{aligned} \|a_i^n-\bar{y}\| &\le \|t_i-\bar{y}\|+\|f(x_{n+i-1})-f(x)-\nabla f(x)(x_{n+i-1}-x)-\frac{1}{2}\nabla^2 f(x)(x_{n+i-1}-x)^2\|\\ &+\|f(x_{n+i-1})-f(\bar{x})-\nabla f(\bar{x})(x_{n+i-1}-\bar{x})-\frac{1}{2}\nabla^2 f(\bar{x})(x_{n+i-1}-\bar{x})^2\|\\ &\le \alpha+\frac{L}{(p+1)(p+2)}\big(\|x_{n+i-1}-x\|^{p+2}+\|x_{n+i-1}-\bar{x}\|^{p+2}\big)\\ &\le \alpha+\frac{L}{(p+1)(p+2)}\Big(\frac{3^{p+2}r_{\bar{x}}^{p+2}}{2^{p+2}}+r_{\bar{x}}^{p+2}\Big)\\ &= \alpha+\frac{L(3^{p+2}+2^{p+2})r_{\bar{x}}^{p+2}}{(p+1)(p+2)2^{p+2}}. \end{aligned}$$

It follows, from the definition of α in (3.5), that $a_i^n \in \mathbb{B}(\bar{y}, r_{\bar{y}})$ for each $i = 0, 1$. Since assumption (3.8) holds for $k = n$, we have

$$t_2 \in f(x) + \nabla f(x)(x_{n-1} - x) + \nabla f(\bar{x})(x_n - x_{n-1}) + \frac{1}{2}\nabla^2 f(x)(x_{n-1} - x)^2 + \frac{1}{2}\nabla^2 f(\bar{x})[(x_n - \bar{x})^2 - (x_{n-1} - \bar{x})^2] + \mathscr{F}(x_n),$$

which can be rewritten as

$$\begin{aligned} t_2 + f(\bar{x}) + \nabla f(\bar{x})(x_{n-1} - \bar{x})) + \frac{1}{2}\nabla^2 f(\bar{x})(x_{n-1} - \bar{x})^2 &\in f(x) + \nabla f(x)(x_{n-1} - x) + \nabla f(\bar{x})(x_n - x_{n-1}) + \frac{1}{2}\nabla^2 f(x)(x_{n-1} - x)^2 \\ &+ \frac{1}{2}\nabla^2 f(\bar{x})[(x_n - \bar{x})^2 - (x_{n-1} - \bar{x})^2] + \mathscr{F}(x_n) + f(\bar{x}) + \nabla f(\bar{x})(x_{n-1} - \bar{x}) + \frac{1}{2}\nabla^2 f(\bar{x})(x_{n-1} - \bar{x})^2. \end{aligned}$$

Then by the definition of a_0^n, we have that $a_0^n \in f(\bar{x}) + \nabla f(\bar{x})(x_n - \bar{x}) + \frac{1}{2}\nabla^2 f(\bar{x})(x_n - \bar{x})^2 + \mathscr{F}(x_n)$. This, together with (3.2) and (3.15), yields that

$$x_n \in T_{\bar{x}}^{-1}(a_0^n) \cap \mathbb{B}(\bar{x}, r_{\bar{x}}).$$

Using (3.4) again, there exists an element $x_{n+1} \in T_{\bar{x}}^{-1}(a_1^n)$ such that

$$\|x_{n+1} - x_n\| \le M\|a_0^n - a_1^n\| \le \frac{5M}{2}\|t_1 - t_2\|\left(\frac{5MLr_{\bar{x}}^p}{2^{p+1}}\right)^{n-1}, \tag{3.17}$$

where the last inequality holds by (3.14). By the definition of a_1^n, we have

$$x_{n+1} \in T_{\bar{x}}^{-1}(a_1^n) = T_{\bar{x}}^{-1}\Big(t_2 - f(x) - \nabla f(x)(x_n - x) - \frac{1}{2}\nabla^2 f(x)(x_n - x)^2 + f(\bar{x}) + \nabla f(\bar{x})(x_n - \bar{x}) + \frac{1}{2}\nabla^2 f(\bar{x})(x_n - \bar{x})^2\Big),$$

which, together with (3.2), implies that

$$t_2 \in f(x) + \nabla f(x)(x_n - x) + \nabla f(\bar{x})(x_{n+1} - x_n) + \frac{1}{2}\nabla^2 f(x)(x_n - x)^2 + \frac{1}{2}\nabla^2 f(\bar{x})\Big((x_{n+1} - \bar{x})^2 - (x_n - \bar{x})^2\Big) + \mathscr{F}(x_{n+1}).$$

This, together with (3.17), completes the induction step and ensure the existence of a sequence $\{x_n\}$ satisfying (3.8) and (3.9). Since $\frac{5MLr_{\bar{x}}^p}{2^{p+1}} < 1$, we see from (3.9) that $\{x_k\}$ is a Cauchy sequence and hence it is convergent, say to u'', that is $u'' := \lim_{k\to\infty} x_k$. Note that $\mathscr{F}$ has closed graph. Then, taking limit in (3.8), we get $t_2 \in f(x) + \nabla f(x)(u'' - x) + \frac{1}{2}\nabla^2 f(x)(u'' - x)^2 + \mathscr{F}(u'')$ and so $u'' \in T_x^{-1}(t_2)$. Moreover,

$$\|u' - u''\| \le \limsup_{n\to\infty} \sum_{k=2}^{n} \|x_k - x_{k-1}\| \le \lim_{n\to\infty} \sum_{k=2}^{n} \frac{5M}{2}\|t_1 - t_2\|\left(\frac{5MLr_{\bar{x}}^p}{2^{p+1}}\right)^{k-2} = \frac{5M2^p}{2^{p+1} - 5MLr_{\bar{x}}^p}\|t_1 - t_2\|.$$

This completes the proof of the Lemma 3.2. □

Before going to demonstrate our main results, we define, for each $x \in X$, the mapping $J_x : X \to Y$ by

$$J_x(\cdot) := f(\bar{x}) + \nabla f(\bar{x})(\cdot - \bar{x}) + \frac{1}{2}\nabla^2 f(\bar{x})(\cdot - \bar{x})^2 - f(x) - \nabla f(x)(\cdot - x) - \frac{1}{2}\nabla^2 f(x)(\cdot - x)^2,$$

and the set-valued mapping $\Phi_x : X \rightrightarrows 2^X$ by

$$\Phi_x(\cdot) = T_{\bar{x}}^{-1}[J_x(\cdot)].$$

Then, for any $x', x'' \in X$, we have that

$$\begin{aligned} \|J_x(x') - J_x(x'')\| &= \|(\nabla f(\bar{x}) - \nabla f(x))(x' - x'') + \frac{1}{2}\nabla^2 f(\bar{x})\Big((x' - \bar{x})^2 - (x'' - \bar{x})^2\Big) - \frac{1}{2}\nabla^2 f(x)\big((x' - x)^2 - (x'' - x)^2\big)\| \\ &= \|(\nabla f(\bar{x}) - \nabla f(x))(x' - x'') + \frac{1}{2}\nabla^2 f(\bar{x})\Big((x' - x'' + x'' - \bar{x})^2 - (x'' - \bar{x})^2\Big) - \frac{1}{2}\nabla^2 f(x)\Big((x' - x'' + x'' - x)^2 - (x'' - x)^2\Big)\| \\ &\le \|\nabla f(\bar{x}) - \nabla f(x)\|\|x' - x''\| + \frac{1}{2}\|\nabla^2 f(\bar{x}) - \nabla^2 f(x)\|\|x' - x''\|^2 + \|\nabla^2 f(\bar{x})(x'' - \bar{x}) - \nabla^2 f(x)(x'' - x)\|\|x' - x''\|. \end{aligned} \tag{3.18}$$

3.1. Superquadratic convergence

This subsection is devoted to study that if $\nabla^2 f$ is (L, p)-Hölder continuous, the sequence generated by Algorithm 2 converges superquadratically to the solution of (1.1). Thus, the main theorem of this study, which gives some sufficient conditions confirming the convergence of the modified superquadratic method with starting point x_0, read as follows:

Theorem 3.3. *Let $p \in (0, 1]$ and $\eta \in (1, \infty)$. Suppose that $T_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant M and that $\nabla^2 f$ is (L, p)-Höder continuous on $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant L. Let α be defined by (3.5) such that (3.6) is hold. Let $\delta > 0$ be such that*

(a) $\delta \le \min\left\{\frac{r_{\bar{x}}}{4}, \frac{2(p+1)(p+2)r_{\bar{y}}}{L(2^{p+3} + 2\cdot 4^{p+2} + 1)}, \frac{1285\alpha}{3\cdot 2^p}, 1\right\}$;

(b) $5(M+1)L\big(\eta 2^p \delta^{p+1} + 4^{4-p}(p+1)(p+2)r_{\bar{x}}^p\big) \le 2^{p+1}(p+1)(p+2)$;

(c) $\|\bar{y}\| < \dfrac{L\delta^{p+2}}{2(p+1)(p+2)}$.

Suppose that $f+\mathscr{F}$ is continuous at $\bar{x}$ for $\bar{y}$, i.e. (3.3) is hold. Then there exists some $\hat{\delta} > 0$ such that any sequence $\{x_n\}$ generated by Algorithm 2 with initial point in $\mathbb{B}(\bar{x},\hat{\delta})$ converges superquadratically to a solution x^* of (1.1).

Proof. According to the continuity of $f+\mathscr{F}$ at $\bar{x}$ for $\bar{y}$ and assumption (c), we can choose $0 < \hat{\delta} \le \delta$ be such that

$$\operatorname{dist}(0, f(x_0)+F(x_0)) \le \frac{L\delta^{p+2}}{2(p+1)(p+2)} \quad \text{for each } x_0 \in \mathbb{B}(\bar{x},\hat{\delta}). \tag{3.19}$$

Setting

$$t := \frac{5\eta ML2^p\delta^{p+1}}{(p+1)(p+2)(2^{p+1}-5MLr_{\bar{x}}^p)}.$$

It follows, from assumption (b), that

$$5ML\big(\eta 2^p\delta^{p+1}+(p+1)(p+2)r_{\bar{x}}^p\big) \le 5(M+1)L\big(\eta 2^p\delta^{p+1}+4^{4-p}(p+1)(p+2)r_{\bar{x}}^p\big) \le 2^{p+1}(p+1)(p+2).$$

The above inequality implies that

$$t \le 1. \tag{3.20}$$

Let $x_0 \in \mathbb{B}(\bar{x},\hat{\delta})$. We use mathematical induction on n to show that Algorithm 2 generates at least one sequence and every sequence $\{x_n\}$ obtained by Algorithm 2 satisfies the following assertions:

$$\|x_n - \bar{x}\| \le 2\delta \tag{3.21}$$

and

$$\|d_n\| \le t\left(\frac{1}{2}\right)^{(p+2)^n}\delta \tag{3.22}$$

for each $n = 0,1,2,\ldots$. Now, define

$$r_x := \frac{9}{(p+1)(p+2)}\Big(ML\|x-\bar{x}\|^{p+2}+(p+1)(p+2)M\|\bar{y}\|\Big) \quad \text{for each } x \in X. \tag{3.23}$$

Because $\eta > 1, p \in (0,1]$ and $\delta \le \dfrac{r_{\bar{x}}}{4}$ by assumption (a), it follows, from assumption (b), that

$$\begin{aligned}
257(M+1)L\delta^{p+1} &= (M+1)L(\delta^{p+1}+4^4\delta^{p+1}) \le (M+1)L(\delta^{p+1}+4^4\delta^{p}) \\
&\le (M+1)L(2^p\eta\delta^{p+1}+4^{4-p}(p+1)(p+2)r_{\bar{x}}^p) \\
&\le \frac{2^{p+1}(p+1)(p+2)}{5},
\end{aligned}$$

which gives

$$ML\delta^{p+1} \le \frac{2^{p+1}(p+1)(p+2)}{1285} \quad \text{and} \quad L\delta^{p+1} \le \frac{2^{p+1}(p+1)(p+2)}{1285}. \tag{3.24}$$

Thus, by $3\cdot 2^p\delta \le 1285\alpha$ in assumption (a) and second inequality in (3.24), we obtain that

$$\|\bar{y}\| < \frac{L\delta^{p+2}}{2(p+1)(p+2)} = \frac{L\delta^{p+1}}{2(p+1)(p+2)}\cdot\delta \le \frac{\alpha}{3} \tag{3.25}$$

(thanks to assumption (c)). Thus, we obtain from (3.23), together with assumption (c), that

$$\begin{aligned}
r_x &< \frac{9}{2(p+1)(p+2)}\Big(2^{p+3}ML\delta^{p+2}+ML\delta^{p+2}\Big) \\
&= \frac{9(2^{p+3}+1)ML}{2(p+1)(p+2)}\delta^{p+2} \\
&= \frac{9(2^{p+3}+1)ML\delta^{p+1}}{2(p+1)(p+2)}\cdot\delta \quad \text{for each } x \in \mathbb{B}(\bar{x},2\delta).
\end{aligned} \tag{3.26}$$

Since $p \in (0,1]$, by the first inequality in (3.24) we have from (3.26) that

$$r_x \le 2\delta.$$

It is clear that $\alpha > 0$ by assumption (a). Then we have from (3.5) that

$$\alpha > 0 \Rightarrow 2^{p+1} - 5MLr_{\bar{x}}^p > 0 \Rightarrow 5MLr_{\bar{x}}^p < 2^{p+1}.$$

Therefore, with the help of above relation we obtain that

$$5\cdot 4^pML\delta^p < 5MLr_{\bar{x}}^p < 2^{p+1} \Rightarrow LM\delta^p < \frac{2^{p+1}}{5\cdot 4^p}. \tag{3.27}$$

Note that, for $n=0$, (3.21) is trivial. To show that the point x_1 exists and (3.22) holds for $n=0$, it suffices to prove that $\mathscr{D}(x_0)\neq\emptyset$. We will do that by applying Lemma 2.4 to the mapping $\Phi := \Phi_{x_0}$. To do this, let us check that both assumptions (2.1) and (2.2) of Lemma 2.4 hold with $c := r_{x_0}$ and $r := \frac{8}{9}$. Here, we note that $\bar{x}\in T_{\bar{x}}^{-1}(\bar{y})\cap\mathbb{B}(\bar{x},\delta)$. Then by the definition of the excess e, we obtain that

$$\begin{aligned}
\operatorname{dist}(\bar{x},\Phi_{x_0}(\bar{x})) &\le e(T_{\bar{x}}^{-1}(\bar{y})\cap\mathbb{B}(\bar{x},\delta),\Phi_{x_0}(\bar{x}))\\
&\le e(T_{\bar{x}}^{-1}(\bar{y})\cap\mathbb{B}(\bar{x},r_{\bar{x}}),T_{\bar{x}}^{-1}[J_{x_0}(\bar{x})]).
\end{aligned}\tag{3.28}$$

By the (L,p)-Hölder continuity property of $\nabla^2 f$ and (3.14), we obtain, for each $x\in\mathbb{B}(\bar{x},2\delta)\subseteq\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$, that

$$\begin{aligned}
\|J_{x_0}(x)-\bar{y}\| &= \|f(\bar{x})+\nabla f(\bar{x})(x-\bar{x})+\frac{1}{2}\nabla^2 f(\bar{x})(x-\bar{x})^2-f(x_0)-\nabla f(x_0)(x-x_0)-\frac{1}{2}\nabla^2 f(x_0)(x-x_0)^2-\bar{y}\|\\
&\le \|f(x)-f(x_0)-\nabla f(x_0)(x-x_0)-\frac{1}{2}\nabla^2 f(x_0)(x-x_0)^2\|+\|f(x)-f(\bar{x})-\nabla f(\bar{x})(x-\bar{x})+\frac{1}{2}\nabla^2 f(\bar{x})(x-\bar{x})^2\|+\|\bar{y}\|\\
&\le \frac{L}{(p+1)(p+2)}\left(\|x-x_0\|^{p+2}+\|x-\bar{x}\|^{p+2}\right)+\|\bar{y}\|.
\end{aligned}\tag{3.29}$$

Because of $\|x_0-\bar{x}\|\le\hat{\delta}\le\delta$, $L(2^{p+3}+2\cdot 4^{p+2}+1)\delta\le 2(p+1)(p+2)r_{\bar{y}}$, $\delta\le 1$ by assumption (a), $\|\bar{y}\|<\dfrac{L\delta^{p+2}}{2(p+1)(p+2)}$ by assumption (c) and second relation in (3.24), (3.29) implies that

$$\begin{aligned}
\|J_{x_0}(x)-\bar{y}\| &\le \frac{L}{(p+1)(p+2)}\left(\|(x-\bar{x})+(\bar{x}-x_0)\|^{p+2}+\|x-\bar{x}\|^{p+2}\right)+\|\bar{y}\|\\
&\le \frac{L}{(p+1)(p+2)}\left((3\delta)^{p+2}+(2\delta)^{p+2}\right)+\frac{L\delta^{p+2}}{2(p+1)(p+2)}\\
&\le \frac{L}{2(p+1)(p+2)}\left(2\cdot 3^{p+2}+2^{p+3}+1\right)\delta^{p+2}\\
&\le \frac{L}{2(p+1)(p+2)}\left(2\cdot 3^{p+2}+2^{p+3}+1\right)\delta,\ \text{ since } \delta^{p+1}\le 1\\
&\le r_{\bar{y}}.
\end{aligned}\tag{3.30}$$

This means that, for each $x\in\mathbb{B}(\bar{x},2\delta)$, $J_{x_0}(x)\in\mathbb{B}(\bar{y},r_{\bar{y}})$. In particular case, putting $x=\bar{x}$ in (3.29). Then we have that

$$\begin{aligned}
\|J_{x_0}(\bar{x})-\bar{y}\| &\le \frac{L}{(p+1)(p+2)}\|\bar{x}-x_0\|^{p+2}+\|\bar{y}\|\\
&\le \frac{3L}{2(p+1)(p+2)}\delta^{p+2}\le\frac{3L}{2(p+1)(p+2)}\delta\\
&\le r_{\bar{y}}.
\end{aligned}\tag{3.31}$$

Hence, by (3.31) and the Lipschitz-like property of $T_{\bar{x}}^{-1}$, we have, from (3.28), that

$$\begin{aligned}
\operatorname{dist}(\bar{x},\Phi_{x_0}(\bar{x})) &\le M\|\bar{y}-J_{x_0}(\bar{x})\|\\
&\le \frac{1}{(p+1)(p+2)}\left(ML\|\bar{x}-x_0\|^{p+2}+(p+1)(p+2)M\|\bar{y}\|\right)\\
&=\left(1-\frac{8}{9}\right)r_{x_0}=c(1-r),
\end{aligned}$$

which shows that the assumption (2.1) of Lemma 2.4 is satisfied.
Next, we show that assumption (2.2) of Lemma 2.4 is satisfied. To do this, let $x',x''\in\mathbb{B}(\bar{x},r_{x_0})$. Then we have that $x',x''\in\mathbb{B}(\bar{x},r_{x_0})\subseteq\mathbb{B}(\bar{x},2\delta)\subseteq\mathbb{B}(\bar{x},r_{\bar{x}})$ by (3.26) and $J_{x_0}(x'),J_{x_0}(x'')\in\mathbb{B}(\bar{y},r_{\bar{y}})$ by (3.30). This, together with the Lipschitz-like property of $T_{\bar{x}}^{-1}$, implies that

$$\begin{aligned}
e(\Phi_{x_0}(x')\cap\mathbb{B}(\bar{x},r_{x_0}),\Phi_{x_0}(x'')) &\le e(\Phi_{x_0}(x')\cap\mathbb{B}(\bar{x},r_{\bar{x}}),\Phi_{x_0}(x''))\\
&= e(T_{\bar{x}}^{-1}[J_{x_0}(x')]\cap\mathbb{B}(\bar{x},r_{\bar{x}}),T_{\bar{x}}^{-1}[J_{x_0}(x'')])\\
&\le M\|J_{x_0}(x')-J_{x_0}(x'')\|.
\end{aligned}$$

Since $\nabla^2 f$ and $\nabla^2 f(\cdot)(z-\cdot)$ are (L,p)-Hölder continuous on $\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$ for all $z\in\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$, then ∇f is also (L,p)-Hölder continuous on $\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$ and for simplicity we take the same constant L. Thus, for the choice of x_0, (3.18) yields that

$$\begin{aligned}
\|J_{x_0}(x')-J_{x_0}(x'')\| &\le \|\nabla f(\bar{x})-\nabla f(x_0)\|\|x'-x''\|+\frac{1}{2}\|\nabla^2 f(\bar{x})-\nabla^2 f(x_0)\|\|x'-x''\|^2+\|\nabla^2 f(\bar{x})(x''-\bar{x})-\nabla^2 f(x_0)(x''-x_0)\|\|x'-x''\|\\
&\le L\|\bar{x}-x_0\|^p\|x'-x''\|+\frac{L}{2}\|\bar{x}-x_0\|^p\|x'-x''\|^2+L\|\bar{x}-x_0\|^p\|x'-x''\|\\
&\le \left(2L+\frac{L}{2}\|x'-x''\|\right)\|\bar{x}-x_0\|^p\|x'-x''\|\\
&\le 2L(\delta^p+\delta^{p+1})\|x'-x''\|.
\end{aligned}$$

Applying the first inequality of (3.24) and (3.27), it follows, from (3.27), that

$$e(\Phi_{x_0}(x')\cap\mathbb{B}(\bar{x},r_{x_0}),\Phi_{x_0}(x''))\leq 2ML(\delta^p+\delta^{p+1})\|x'-x''\|\leq 2^{p+2}\Big(\frac{1}{5\cdot 4^p}+\frac{(p+1)(p+2)}{1285}\Big)\|x'-x''\|$$
$$<\frac{8}{9}\|x'-x''\|=r\|x'-x''\|,\ \text{since } p\in(0,1].$$

This means that the assumption (2.2) of Lemma 2.4 is also satisfied. Thus by Lemma 2.4, we can deduce the existence of a fixed point $\hat{x}_1\in\mathbb{B}(\bar{x},r_{x_0})$ such that $\hat{x}_1\in\Phi_{x_0}(\hat{x}_1)$, which translates to $0\in f(x_0)+\nabla f(x_0)(\hat{x}_1-x_0)+\frac{1}{2}\nabla^2 f(x_0)(\hat{x}_1-x_0)^2+\mathscr{F}(\hat{x}_1)$ and hence $\mathscr{D}(x_0)\neq\emptyset$. Consequently, we can choose $d_0\in\mathscr{D}(x_0)$ such that

$$\|d_0\|\leq\eta\ \operatorname{dist}(0,\mathscr{D}(x_0)). \tag{3.32}$$

Therefore, according to the Algorithm 2, we can say that $x_1:=x_0+d_0$ is defined.
Now, we will show that the assertion (3.22) is also hold for $n=0$. Note by assumption (a) that $x_0\in\mathbb{B}(\bar{x},\hat{\delta})\subseteq\mathbb{B}(\bar{x},\delta)\subseteq\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$. Since $T_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y},r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x},r_{\bar{x}})$, it follows from Lemma 3.2 that $T_{x_0}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y},\alpha)$ relative to $\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$ with constant $\frac{5M2^p}{2^{p+1}-5MLr_{\bar{x}}^p}$. Moreover, (3.19) and (3.25) imply that

$$\operatorname{dist}(0,T_{x_0}(x_0)=\operatorname{dist}(0,f(x_0)+F(x_0))\leq\frac{L\delta^{p+2}}{2(p+1)(p+2)}\leq\frac{\alpha}{3}. \tag{3.33}$$

It has been mentioned earlier that $x_0\in\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$ and by (3.25)) we have $0\in\mathbb{B}(\bar{y},\frac{\alpha}{3})$. This, together with (3.33), implies that Lemma 2.3 is applicable and hence by applying it we have that

$$\operatorname{dist}(x_0,{T_{x_0}}^{-1}(0))\leq\frac{5M2^p}{2^{p+1}-5MLr_{\bar{x}}^p}\operatorname{dist}(0,T_{x_0}(x_0)) \tag{3.34}$$

Applying (3.34), we have from (3.1) that

$$\begin{aligned}\operatorname{dist}(0,\mathscr{D}(x_0))&=\operatorname{dist}(x_0,T_{x_0}^{-1}(0))\\&\leq\frac{5M2^p}{2^{p+1}-5MLr_{\bar{x}}^p}\operatorname{dist}(0,T_{x_0}(x_0)).\end{aligned} \tag{3.35}$$

Using (3.35), (3.33) and then (3.20) in (3.32), we obtain that

$$\begin{aligned}\|x_1-x_0\|=\|d_0\|&\leq\eta\ \operatorname{dist}(0,\mathscr{D}(x_0))\\&\leq\frac{5\eta M2^p}{2^{p+1}-5MLr_{\bar{x}}^p}\operatorname{dist}(0,T_{x_0}(x_0))\\&\leq\frac{5\eta ML2^p\delta^{p+2}}{2(p+1)(p+2)(2^{p+1}-5MLr_{\bar{x}}^p)}=t\Big(\frac{1}{2}\Big)\delta.\end{aligned}$$

This shows that (3.22) is hold for $n=0$.
We assume that the points $x_1,x_2,...,x_k$ are generated by Algorithm 2, and (3.21) and (3.22) are true for $n=0,1,\ldots,k-1$. We show that there exists x_{k+1} such that (3.21) and (3.22) are hold for $n=k$. Since, for each $n\leq k-1$, (3.21) and (3.22) are true and $t\leq 1$ by (3.20), we have the following inequality

$$\begin{aligned}\|x_k-\bar{x}\|&\leq\sum_{i=0}^{k-1}\|x_{i+1}-x_i\|+\|x_0-\bar{x}\|\leq\delta\sum_{i=0}^{k-1}t\left(\frac{1}{2}\right)^{(p+2)^i}+\delta\\&\leq\delta\sum_{i=0}^{k-1}\left(\frac{1}{2}\right)^{(p+2)^i}+\delta\leq\delta+\delta=2\delta.\end{aligned}$$

This shows that (3.21) holds for $n=k$. Finally, we will show that (3.22) holds for $n=k$. Now if we use the same arguments that we did for the case when $n=0$, we can prove that $\mathscr{D}(x_k)\neq\emptyset$ and so by Algorithm 2 we can choose $d_k\in\mathscr{D}(0,x_k)$ such that

$$\|d_k\|\leq\eta\ \operatorname{dist}(0,\mathscr{D}(x_k)),$$

that is, the point x_{k+1} exists. Moreover, we have that $T_{x_k}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y},\alpha)$ relative to $\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$ with constant $\frac{5M2^p}{2^{p+1}-5MLr_{\bar{x}}^p}$.

Therefore, we have that

$$\begin{aligned}
\|x_{k+1}-x_k\| &= \|d_k\| \le \eta\ \mathrm{dist}(0,\mathscr{D}(x_k)) \\
&\le \frac{5\eta M2^p}{2^{p+1}-5MLr_{\bar{x}}^p}\ \mathrm{dist}(0,T_{x_k}(x_k)) \\
&= \frac{5\eta M2^p}{2^{p+1}-5MLr_{\bar{x}}^p}\ \mathrm{dist}(0,f(x_k)+F(x_k)) \\
&\le \frac{5\eta M2^p}{2^{p+1}-5MLr_{\bar{x}}^p}\|f(x_k)-f(x_{k-1})-\nabla f(x_{k-1})(x_k-x_{k-1})-\frac{1}{2}\nabla^2 f(x_{k-1})(x_k-x_{k-1})^2\| \\
&\le \frac{5\eta ML2^p}{(p+1)(p+2)(2^{p+1}-5MLr_{\bar{x}}^p)}\|x_k-x_{k-1}\|^{p+2} \\
&\le \frac{5\eta ML2^p\delta^{p+1}}{(p+1)(p+2)(2^{p+1}-5MLr_{\bar{x}}^p)}\left(t\left(\frac{1}{2}\right)^{(p+2)^{k-1}}\right)^{p+2}\delta \\
&\le t\left(\frac{1}{2}\right)^{(p+2)^k}\delta.
\end{aligned}$$

This implies that (3.22) holds for $n=k$ and therefore the proof of the theorem is completed. □

In the special case when $\bar{x}$ is a solution of (1.1) (that is $\bar{y}=0$ in Theorem 3.3), then we have the following corollary which gives the super-quadratically local convergence result for the modified superquadratic method.

Corollary 3.4. *Let $p\in(0,1]$ and $\eta>1$. Let $\bar{x}$ be the solution of (1.1) and $T_{\bar{x}}^{-1}$ be pseudo-Lipschitz around $(0,\bar{x})$. Suppose that $\nabla^2 f$ is (L,p)-Hölder continuous around $\bar{x}$. Suppose that*

$$\lim_{x\to\bar{x}}\mathrm{dist}(0,f(x)+\mathscr{F}(x))=0.$$

Then there exists some $\hat{\delta}>0$ such that any sequence $\{x_n\}$ generated by Algorithm 2 with initial point in $\mathbb{B}(\bar{x},\hat{\delta})$ converges superquadratically to a solution x^* of (1.1).

Proof. By our assumption, $T_{\bar{x}}^{-1}$ is pseudo-Lipschitz around $(0,\bar{x})$. Then there exist constants r_0, β and M such that $T_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(0,r_0)$ relative to $\mathbb{B}(\bar{x},\beta)$ with constant M. Then, for each $0<r\le\beta$, one has that

$$e(T_{\bar{x}}^{-1}(y_1)\cap\mathbb{B}(\bar{x},r),T_{\bar{x}}^{-1}(y_2)\le M\|y_1-y_2\| \quad \text{for any } y_1,y_2\in\mathbb{B}(0,r_0)$$

i.e. $T_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(0,r_0)$ relative to $\mathbb{B}(\bar{x},r)$ with constant M. Let $L\in(0,1)$ and $r_{\bar{x}}\in(0,\beta)$ be such that $\frac{r_{\bar{x}}}{2}\le r$, $MLr_{\bar{x}}^p\le\frac{2^{p+1}}{5}$ and $r_0-\frac{L(3^{p+2}+2^{p+2})r_{\bar{x}}^{p+2}}{(p+1)(p+2)2^{p+2}}>0$. By the (L,p)-Hölder continuous property of $\nabla^2 f$, for each $x,\ x'\in\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$, we have that

$$\|\nabla^2 f(x)-\nabla^2 f(x')\|\le L\|x-x'\|^p.$$

Choose α so that

$$\alpha:=\min\left\{r_0-\frac{L(3^{p+2}+2^{p+2})r_{\bar{x}}^{p+2}}{(p+1)(p+2)2^{p+2}},\frac{r_{\bar{x}}(2^{p+1}-5MLr_{\bar{x}}^p)}{5M2^{p+2}}\right\}>0,$$

and

$$\min\left\{\frac{r_{\bar{x}}}{4},\frac{2(p+1)(p+2)r_0}{L(2^{p+3}+2\cdot4^{p+2}+1)},\frac{1285\alpha}{3\cdot2^p}\right\}>0.$$

Thus we can choose $0<\delta\le1$ such that

$$\delta\le\min\left\{\frac{r_{\bar{x}}}{4},\frac{2(p+1)(p+2)r_0}{L(2^{p+3}+2\cdot4^{p+2}+1)},\frac{1285\alpha}{3\cdot2^p}\right\}.$$

and

$$5(M+1)L\left(\eta2^p\delta^{p+1}+4^{4-p}(p+1)(p+2)r_{\bar{x}}^p\right)\le2^{p+1}(p+1)(p+2).$$

Now one can easily sees that the assumptions (a)-(c) of Theorem 3.3 are hold. Therefore, to complete the proof of the corollary, we can apply Theorem 3.3 . □

4. Conclusion

The semilocal and local convergence results for the modified superquadratic method are established with $\eta > 1$ under the assumptions that $T_{\bar{x}}^{-1}$ is Lipschitz-like as well as $\nabla^2 f$ is (L, p)-Hölder continuous. This result extends and improves the corresponding one [13]. This result seems new for the generalized equation problem (1.1). According to the main result of this study, we have the following conclusions:

- If $p = 0$, then the Fréchet derivative of f satisfies the continuity condition with constant L and we obtain the quadratic convergence of the modified superquadratic method. In this case the result established in the present paper coincides with the result presented in [22, Theorem 3.1, Corollary 3.1].
- If $p = 1$, then the Fréchet derivative of f satisfies the Lipschitz condition and we obtain the cubic convergence of the modified superquadratic method. In this case the result established in the present paper coincides with the result presented in [22, Theorem 3.2, Corrolary 3.2].

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References

[1] A. L. Dontchev, *Local convergence of the Newton method for generalized equation*, C. R. A. S Paris Ser.I **322** (1996), 327–331.
[2] A. L. Dontchev, *Uniform convergence of the Newton method for Aubin continuous maps*, Serdica Math. J. **22** (1996), 385–398.
[3] A. L. Dontchev, *Local analysis of a Newton-type method based on partial linearization*, Lectures in Applied Mathematics **32** (1996), 295–306.
[4] A. L. Dontchev and W. W. Hager, *An inverse mapping theorem for set-valued maps*, Proc. Amer. Math. Soc. **121** (1994), 481–489.
[5] A. Piétrus, *Generalized equations under mild differentiability conditions*, Rev. R. Acad. Cienc. Exact. Fis. Nat. **94**(1) (2000), 15–18.
[6] A. Piétrus, *Does Newton's method for set-valued maps converges uniformly in mild differentiability context?*, Rev. Colombiana Mat. **34** (2000), 49–56.
[7] C. Li, W. H. Zhang and X. Q. Jin, *Convergence and uniqueness properties of Gauss-Newton's method*, Comput. Math. Appl. **47** (2004), 1057–1067.
[8] J. P. Dedieu and M. H. Kim, *Newton's method for analytic systems of equations with constant rank derivatives*, J. Complexity **18** (2002), 187–209.
[9] J. P. Aubin, *Lipschitz behavior of solutions to convex minimization problems*, Math. Oper. Res. **9** (1984), 87–111.
[10] J. P. Aubin and H. Frankowska, *Set-valued Analysis*, Birkhäuser, Boston, 1990.
[11] J. P. Dedieu and M. Shub, *Newton's method for overdetermined systems of equations*, Math. Comp. **69** (2000), 1099–1115.
[12] J. S. He, J. H. Wang and C. Li, *Newton's method for underdetemined systems of equations under the modified γ-condition*, Numer. Funct. Anal. Optim. **28** (2007), 663–679.
[13] M. Geoffroy and A. Piétrus, *A superquadratic method for solving generalized equations in the Hóder case*, Ricerche di Matematica **LII** (2003), 231–240.
[14] M. Geoffroy, S. Hilout and A. Piétrus, *Acceleration of convergence in Dontchev's iterative method for solving variational inclusions*, Serdica Math. J. **29** (2003), 45–54.
[15] M. Geoffroy, S. Hilout and A. Piétrus, *Stability of a cubically convergent method for generalized equations*, Set-Valued Analysis **14** (2006), 41–54.
[16] M. H. Rashid, *A Convergence Analysis of Gauss-Newton-type Method for Holder Continuous Maps*, Indian Journal of Mathematics **57**(2) (2014), 181–198.
[17] M. H. Rashid, *Convergence Analysis of a Variant of Newton-type Method for Generalized Equations*, International Journal of Computer Mathematics **95**(3) (2018), 584–600.
[18] M. H. Rashid, *On the convergence of extended Newton-type method for solving variational inclusions*, Cogent Mathematics, **1**(1) (2014), DOI 10.1080/23311835.2014.980600.
[19] M. H. Rashid, *Convergence Analysis of Extended Hummel-Seebeck-type Method for Solving Variational Inclusions*, Vietnam Journal of Mathematics **44** (2016), 709–726.
[20] M. H. Rashid, *Extended Newton-type Method and its Convergence Analysis for Nonsmooth Generalized Equations*, J. Fixed Point Theory and Appl. **19** (2017), 1295–1313.
[21] M. H. Rashid, J. H. Wang and C. Li, *Convergence analysis of a method for variational inclusions*, Applicable Analysis **91**(10) (2012), 1943–1956.
[22] M. H. Rashid, M. Z. Ali and A. Pietrus, *Extended Cubic Method and Its Convergence Analysis for Generalized Equations*, Journal of Advances and Applied Mathematics, **3**(3) (2018), 91-108.
[23] M. H. Rashid, S. H. Yu, C. Li and S, Y. Wu, *Convergence analysis of the Gauss-Newton-type method for Lipschitz–like mappings*, J. Optim. Theory Appl. **158**(1) (2013), 216–233.
[24] S. M. Robinson, *Generalized equations and their solutions, part I: basic theory*, Math. Progamming Stud.**10** (1979), 128–141.
[25] S. M. Robinson, *Strong regular generalized equations*, Math. of Oper. Res. **5** (1980), 43–62.
[26] S. M. Robinson, *Generalized equations and their solutions, part II: applications to nonlinear programming*, Math. Programming Stud.**19** (1982), 200–221.
[27] X. B. Xu and C. Li, *Convergence of Newton's method for systems of equations with constant rank derivatives*, J. Comput. Math. **25** (2007), 705–718.
[28] X. B. Xu and C. Li, *Convergence criterion of Newton's method for singular systems with constant rank derivatives*, J. Math. Anal. Appl. **345** (2008), 689–701.