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On the Remainder Term of Some Bivariate Approximation Formulas Based on Linear and Positive Operators

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ABSTRACT. The paper is a survey concerning representations for the remainder term of Bernstein-Schurer-Stancu and respectively Stancu (based on factorial powers) bivariate approximation formulas, using bivariate divided differences. As particular cases the remainder terms of bivariate Bernstein-Stancu, Schurer and classical Bernstein bivariate approximation formulas are obtained. Finally, one presents some mean value properties, similar to those of the remainder term of classical Bernstein univariate approximation formula.

Keywords: Bernstein-Schurer-Stancu bivariate operator, Stancu bivariate operator, Bivariate divided difference, Bivariate approximation formula, Remainder term

2010 Mathematics Subject Classification: 41A80, 41A36.

1. INTRODUCTION

Denote by $\mathbb{N}$ the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $p \in \mathbb{N}$ be given and α, β be real parameters satisfying the condition $0 \leq \alpha \leq \beta$. The Bernstein-Schurer-Stancu operator is defined [6] for any $m \in \mathbb{N}$, any $f \in C[0, 1+p]$ and any $x \in [0, 1+p]$ by:

$$\widetilde{S}_{m,p}^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right) \tag{1.1}$$

where

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}, \tag{1.2}$$

are the fundamental Schurer polynomials [33].

Note that the multiparameter operator (1.1) contains as particular cases the Schurer operator [33] (for $\alpha = \beta = 0$), the Bernstein-Stancu operator [36] (for $p = 0$) and of course the classical Bernstein operator [18] (for $p = 0$ and $\alpha = \beta = 0$). Many of its approximation properties were investigated in [9].

Let $p, q \in \mathbb{N}_0$ be given and let $\alpha, \beta, \gamma, \delta$ be real parameters such that $0 \leq \alpha \leq \beta$, $0 \leq \gamma \leq \delta$.

Using the method of parametric extensions [21], [5], in [7] was introduced the Bernstein-Schurer-Stancu bivariate operator, given by

$$\widetilde{S}_{m,p,n,q}^{(\alpha,\beta,\gamma,\delta)}(f;x,y) = \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x) \widetilde{p}_{n,j}(y) f\left(\frac{k+\alpha}{m+\beta}, \frac{j+\gamma}{n+\delta}\right) \tag{1.3}$$

while in [8] was considered its GBS associated operator, given by

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$$\widetilde{U}^{(\alpha,\beta,\gamma,\delta)}_{m,p,n,q}(f;x,y) = \sum_{k=0}^{m+p}\sum_{j=0}^{n+q}\widetilde{p}_{m,k}(x)\widetilde{q}_{n,j}(y)\left\{f\left(\frac{k+\alpha}{m+\beta},y\right)+f\left(x,\frac{j+\gamma}{n+\delta}\right)-f\left(\frac{k+\alpha}{m+\beta},\frac{j+\gamma}{n+\delta}\right)\right\}. \tag{1.4}$$

Approximation properties of operator (1.3) and (1.4) were established in [7], [9], [13].

In Section 2 we recall the results from [12, 13] regarding the remainder term of the bivariate Bernstein-Schurer-Stancu bivariate approximation formula. At end of this section, we discuss about the remainder term of GBS Bernstein approximation formula [11].

Suppose α is a non-negative parameter which may depends only of $m \in \mathbb{N}$. In 1968, Stancu [36] introduced the operator $P^{\langle\alpha\rangle} : C[0,1] \to C[0,1]$ given by

$$P^{\langle\alpha\rangle}_m(f;x) = \sum_{k=0}^{m} p^{\langle\alpha\rangle}_{m,k}(x)\ f\left(\frac{k}{m}\right), \tag{1.5}$$

where $p^{\langle\alpha\rangle}_{n,k}(x)$ are Stancu's fundamental polynomials expressed by means of factorial power $t^{[m,h]} = t(t-h)\dots(t-(m-1)h)$, $t^{[0,h]} = 1$, by

$$p^{\langle\alpha\rangle}_{m,k}(x) = \binom{m}{k}\frac{x^{[k,-\alpha]}(1-x)^{[m-k,-\alpha]}}{1^{[m,-\alpha]}}, \tag{1.6}$$

for any $x \in [0,1]$, $m \in \mathbb{N}$ and $k \in \{0,1,\dots,m\}$.

The operator (1.5) is the Stancu operator (or Stancu operator based on factorial power).

Let $m,n \in \mathbb{N}$ be given and let $\alpha = \alpha(m), \beta = \beta(n)$ be real parameters. Using the method of parametric extension in [25,26] was obtained the Stancu operator and its GBS-Stancu associated operator, given respectively by

$$P^{\langle\alpha,\beta\rangle}_{m,n}(f;x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p^{\langle\alpha\rangle}_{m,k}(x)p^{\langle\beta\rangle}_{n,j}(y)f\left(\frac{k}{m},\frac{i}{n}\right) \tag{1.7}$$

and

$$U^{\langle\alpha,\beta\rangle}_{m,n}(f;x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p^{\langle\alpha\rangle}_{m,k}(x)p^{\langle\beta\rangle}_{n,j}(y)\left\{f\left(\frac{i}{m},y\right)+f\left(x,\frac{i}{n}\right)-f\left(\frac{i}{m},\frac{i}{n}\right)\right\}. \tag{1.8}$$

The corresponding approximation formulas were studied in the recently papers [25,26] due to Miclăuş, where the remainders of mentioned formula were expressed using bivariate divided differences.

Coming back to the classical Bernstein operator $B_m : C[0,1] \to C[0,1]$ given by

$$B_m(f;x) = \sum_{k=0}^{m} p_{m,k} f\left(\frac{k}{m}\right) \tag{1.9}$$

for any $f \in C[0,1]$, $x \in [0,1]$ and $m \in \mathbb{N}$, recall that Aramă [4], proved that the remainder term of the univariate Bernstein approximation formula

$$f = B_m(f) + R_m(f) \tag{1.10}$$

can be represented under the form

$$R_m(f;x) = -\frac{x(1-x)}{m}\ [\xi_1,\xi_2,\xi_3;f] \tag{1.11}$$

for any $x \in [0,1]$, where $0 \le \xi_1 < \xi_2 < \xi_3 \le 1$.

In Section 3 we will present analogous results for the remainder terms of bivariate Bernstein-Schurer-Stancu and respectively Stancu bivariate approximation formulas.

2. The remainder term of the bivariate Bernstein-Schurer-Stancu approximation formula

We start by recalling some results regarding the divided differences which will be used in the paper. Suppose that $I \subset \mathbb{R}$ is an interval of the real axis and $x_1, x_2 \in I$ such that $x_1 \neq x_2$. The divided difference of f with respect the distinct knots x_1, x_2 is defined by

$$[x_1,x_2;f] = \frac{f(x_2)-f(x_1)}{x_2-x_1}. \tag{2.12}$$

If $x_0, x_1, \ldots, x_n \in I$ are distinct knots and $f : I \to \mathbb{R}$ is given, then the n-th order divided difference of f with respect the mentioned knots is defined by the recurrence relation

$$[x_0,x_1,\ldots,x_m;f] = \frac{[x_1,\ldots,x_m;f]-[x_0,\ldots,x_{m-1};f]}{x_m-x_0}. \tag{2.13}$$

Note that the divided differences were intensively studied by Popoviciu [30]. Interesting properties of the divided differences were obtained by Ionescu [22] and Ivan [23].

Let $I, J \subset \mathbb{R}$ be intervals, $f : I \times J \to \mathbb{R}$ be bounded and $(x_1,y_1),(x_2,y_2) \in I \times J$ such that $x_1 \neq x_2, y_1 \neq y_2$. The bivariate divided difference of f with respect the knots (x_1,y_1), (x_2,y_2) is defined [10] by

$$\left[\begin{array}{c} x_1,x_2 \\ y_1,y_2 \end{array} ;f\right] = \frac{f(x_2,y_2)-f(x_1,y_2)-f(x_2,y_1)+f(x_1,y_1)}{(x_2-x_1)(y_2-y_1)}. \tag{2.14}$$

Other equivalent definitions were given in the monographs by Ionescu [22] and Ivan [23].

In the definition of the bivariate divided difference the number of abscissas is in general not equal with the number of coordinates. For example (see [10]), we have that

$$\left[\begin{array}{c} x_1,x_2 \\ y_1 \end{array} ;f\right] = \frac{f(x_2,y_1)-f(x_1,y_1)}{x_2-x_1} \tag{2.15}$$

and

$$\left[\begin{array}{c} x_1,x_2,x_3 \\ y_1 \end{array} ;f\right] = \frac{1}{x_3-x_2}\left(\left[\begin{array}{c} x_2,x_3 \\ y_1 \end{array} ;f\right] - \left[\begin{array}{c} x_1,x_2 \\ y_1 \end{array} ;f\right]\right) \tag{2.16}$$

where $x_1, x_2, x_3 \in I$ are distinct.

For the bivariate divided difference with respect the distinct knots $(x_i, y_j) \in I \times J$ $(i = \overline{0,m}, j = \overline{0,n})$ the following recurrence formula [10]

$$\left[\begin{array}{l} x_0, x_1, \ldots, x_m \\ y_0, y_1, \ldots, y_n \end{array}; f\right] = \tag{2.17}$$

$$= \frac{\left[\begin{array}{l} x_1, \ldots, x_m \\ y_1, \ldots, y_n \end{array}; f\right] - \left[\begin{array}{l} x_0, \ldots, x_{m-1} \\ y_1, \ldots, y_n \end{array}; f\right] - \left[\begin{array}{l} x_1, \ldots, x_m \\ y_0, \ldots, y_{n-1} \end{array}; f\right] + \left[\begin{array}{l} x_0, \ldots, x_{n-1} \\ y_0, \ldots, y_{m-1} \end{array}; f\right]}{(x_m - x_0)(y_n - y_0)}$$

holds.

Using the above mentioned properties of the bivariate divided differences, in [13] was established the following.

Theorem 2.1. *The remainder term of the bivariate Bernstein-Schurer-Stancu approximation formula*

$$f = \widetilde{S}^{(\alpha,\beta,\gamma,\delta)}_{m,p,n,q}(f) + \widetilde{R}^{(\alpha,\beta,\gamma,\delta)}_{m,p,n,q}(f) \tag{2.18}$$

can be represented under the form

$$\widetilde{R}^{(\alpha,\beta,\gamma,\delta)}_{m,p,n,q}(f; x, y) = S_1 + S_2 + S_3, \tag{2.19}$$

where

$$\begin{aligned} S_1 = {} & \frac{(\beta - p)x - \alpha}{m + \beta} \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x) \widetilde{p}_{n,j}(y) \left[\begin{array}{c} x, \frac{k+\alpha}{m+\beta} \\ \frac{j+\gamma}{n+\delta} \end{array}; f\right] \\ & - \frac{x(1-x)(m+p)}{(m+\beta)^2} \sum_{k=0}^{m+p-1} \sum_{j=0}^{n+q} \widetilde{p}_{m-1,k}(x) \widetilde{p}_{n,j}(y) \left[\begin{array}{c} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} \\ \frac{j+\gamma}{n+\delta} \end{array}; f\right]; \end{aligned} \tag{2.20}$$

$$\begin{aligned} S_2 = {} & \frac{(\delta - q)y - \gamma}{n + \delta} \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x) \widetilde{p}_{n,j}(y) \left[\begin{array}{c} \frac{k+\alpha}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta} \end{array}; f\right] \\ & - \frac{y(1-y)(n+q)}{(n+\delta)^2} \sum_{k=0}^{m+p} \sum_{j=0}^{n+q-1} \widetilde{p}_{m,k}(x) \widetilde{p}_{n-1,j}(y) \left[\begin{array}{c} \frac{k+\alpha}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\gamma+1}{n+\delta} \end{array}; f\right]; \end{aligned} \tag{2.21}$$

$$\begin{aligned} S_3 = {} & xy(1-x)(1-y) \frac{(m+p)(n+q)}{(m+\beta)^2(n+\delta)^2} \\ & \times \sum_{k=0}^{m+p-1} \sum_{j=0}^{n+q-1} \widetilde{p}_{m-1,k}(x) \widetilde{p}_{n-1,j}(y) \left[\begin{array}{c} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\gamma+1}{n+\delta} \end{array}; f\right] \\ & - \frac{m+p}{(m+\beta)^2(n+\delta)} x(1-x)\{(\delta - q)y - \gamma\} \\ & \times \sum_{k=0}^{m+p-1} \sum_{j=0}^{n+q} \widetilde{p}_{m-1,k}(x) \widetilde{p}_{n,j}(y) \left[\begin{array}{c} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta} \end{array}; f\right] \\ & - \frac{n+q}{(m+\beta)(n+\delta)^2} y(1-y)\{(\beta - p)x - \alpha\} \end{aligned} \tag{2.22}$$

$$\times \sum_{k=0}^{m+p} \sum_{j=0}^{n+q-1} \widetilde{p}_{m,k}(x)\widetilde{p}_{n-1,j}(y) \left[\begin{array}{c} x, \frac{k+\alpha}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\gamma+1}{n+\delta} \end{array} ; f \right]$$
$$+ \frac{1}{(m+\beta)(n+\gamma)}\{(\beta-p)x-\alpha\}\{(\delta-q)y-\gamma\}$$
$$\times \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y) \left[\begin{array}{c} x, \frac{k+\alpha}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta} \end{array} ; f \right].$$

Remark 2.1. For $\alpha = \beta = \gamma = \delta = 0$ one obtains that the Bernstein-Schurer-Stancu operator reduces to the Schurer bivariate operator, given by

$$\widetilde{B}_{m,p,n,q}(f;x,y) = \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x)\,\widetilde{p}_{n,j}(y) f\left(\frac{k}{m+p}, \frac{j}{n+q}\right). \tag{2.23}$$

In this case we get the Schurer bivariate approximation formula

$$f = \widetilde{B}_{m,p,n,q}(f) + \widetilde{R}_{m,p,n,q}(f). \tag{2.24}$$

Applying the Theorem 2.1 for $\alpha = \beta = \gamma = \delta = 0$, it follows

Corollary 2.1. *The remainder term of the Schurer bivariate approximation formula* (2.18) *can be represented under the form:*

$$\widetilde{R}_{m,p,n,q}(f) = S_1 + S_2 + S_3 \tag{2.25}$$

where

$$S_1 = -\frac{px}{m} \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x)\,\widetilde{p}_{n,j}(y) \left[\begin{array}{c} x, \frac{k}{m} \\ \frac{j}{n} \end{array} ; f \right] - \frac{x(1-x)(m+p)}{m^2} \sum_{k=0}^{m+p-1} \sum_{j=0}^{n+q} \widetilde{p}_{m-1,k}(x)\,\widetilde{p}_{n,j}(y) \left[\begin{array}{c} x, \frac{k}{m}, \frac{k+1}{m} \\ \frac{j}{n} \end{array} ; f \right]; \tag{2.26}$$

$$S_2 = -\frac{qy}{n} \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x)\,\widetilde{p}_{n,j}(y) \left[\begin{array}{c} \frac{k}{m} \\ y, \frac{j}{n} \end{array} ; f \right] - \frac{y(1-y)(n+q)}{n^2} \sum_{k=0}^{m+p} \sum_{j=0}^{n+q-1} \widetilde{p}_{m,k}(x)\,\widetilde{p}_{n-1,j}(y) \left[\begin{array}{c} \frac{k}{m} \\ y, \frac{j}{n}, \frac{j+1}{n} \end{array} ; f \right]; \tag{2.27}$$

$$S_3 = \frac{xy(1-x)(1-y)}{mn} \sum_{k=0}^{m+p-1} \sum_{j=0}^{n+q-1} \widetilde{p}_{m-1,k}(x)\, \widetilde{p}_{n-1,j}(y) \begin{bmatrix} x, \frac{k}{m}, \frac{k+1}{m} & \\ y, \frac{j}{n}, \frac{j+1}{n} & ; f \end{bmatrix} \tag{2.28}$$
$$+ \frac{(m+p)q}{m^2 n} xy(1-x) \sum_{k=0}^{m+p-1} \sum_{j=0}^{n+q} \widetilde{p}_{m-1,k}(x)\, \widetilde{p}_{n,j}(y) \begin{bmatrix} x, \frac{k}{m}, \frac{k+1}{m} & \\ y, \frac{j}{n} & ; f \end{bmatrix}$$
$$+ \frac{(n+q)p}{mn^2} xy(1-y) \sum_{k=0}^{m+p} \sum_{j=0}^{n+q-1} \widetilde{p}_{m,k}(x)\, \widetilde{p}_{n-1,j}(y) \begin{bmatrix} x, \frac{k}{m} & \\ y, \frac{j}{n}, \frac{j+1}{n} & ; f \end{bmatrix}$$
$$+ \frac{pq}{mn} xy \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x)\, \widetilde{p}_{n,j}(y) \begin{bmatrix} x, \frac{k}{m} & \\ y, \frac{j}{n} & ; f \end{bmatrix}.$$

Remark 2.2. For $p = q = 0$, the bivariate Bernstein-Schurer-Stancu bivariate operator (2.25) reduces to the bivariate Bernstein-Stancu operator, given by

$$S_{m,n}^{(\alpha,\beta,\gamma,\delta)}(f;x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x)\, p_{n,j}(y) f\left(\frac{k+\alpha}{m+\beta}, \frac{j+\gamma}{n+\delta}\right). \tag{2.29}$$

Consequently it follows the Bernstein-Stancu bivariate approximation formula

$$f = S_{m,n}^{(\alpha,\beta,\gamma,\delta)}(f) + R_{m,n}^{(\alpha,\beta,\gamma,\delta)}(f). \tag{2.30}$$

Regarding the remainder term of (2.29), applying the Theorem 2.1 for $p = q = 0$, it follows:

Corollary 2.2. *The remainder term of (2.29) can be expressed under the form:*

$$R_{m,n}^{(\alpha,\beta,\gamma,\delta)}(f;x,y) = S_1 + S_2 + S_3, \tag{2.31}$$

where

$$S_1 = \frac{\beta x - \alpha}{m+\beta} \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x)\, p_{n,j}(y) \begin{bmatrix} x, \frac{k+\alpha}{m+\beta} & \\ \frac{j+\gamma}{n+\delta} & ; f \end{bmatrix} \tag{2.32}$$
$$- \frac{mx(1-x)}{(m+\beta)^2} \sum_{k=0}^{m-1} \sum_{j=0}^{n} p_{m-1,k}(x)\, p_{n,j}(y) \begin{bmatrix} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} & \\ \frac{j+\gamma}{n+\delta} & ; f \end{bmatrix};$$

$$S_2 = \frac{\delta y - \gamma}{n+\delta} \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x)\, p_{n,j}(y) \begin{bmatrix} \frac{k+\alpha}{m+\beta} & \\ y, \frac{j+\gamma}{n+\delta} & ; f \end{bmatrix} \tag{2.33}$$
$$- \frac{ny(1-y)}{(n+\delta)^2} \sum_{k=0}^{m} \sum_{j=0}^{n-1} p_{m,k}(x) p_{n-1,j}(y) \begin{bmatrix} \frac{k+\alpha}{m+\beta} & \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\gamma+1}{n+\delta} & ; f \end{bmatrix};$$

$$S_3 = \frac{mn\,xy(1-x)(1-y)}{(m+\beta)^2(n+\delta)^2} \sum_{k=0}^{m-1}\sum_{j=0}^{n-1} p_{m-1,k}(x)\,p_{n-1,j}(y) \begin{bmatrix} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} & ; f \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\gamma+1}{n+\delta} & \end{bmatrix} \tag{2.34}$$
$$- \frac{m}{(m+\beta)^2(n+\delta)}\, x(1-x)(\delta y-\gamma) \sum_{k=0}^{m-1}\sum_{j=0}^{n} p_{m-1,k}(x)\,p_{n,j}(y) \begin{bmatrix} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} & ; f \\ y, \frac{j+\gamma}{n+\delta} & \end{bmatrix}$$
$$- \frac{n}{(m+\beta)(n+\delta)^2}\, y(1-y)(\beta x-\alpha) \sum_{k=0}^{m}\sum_{j=0}^{n-1} p_{m,k}(x)p_{n-1,j}(y) \begin{bmatrix} x, \frac{k+\alpha}{m+\beta} & ; f \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\gamma+1}{n+\delta} & \end{bmatrix}$$
$$+ \frac{1}{(m+\beta)(n+\gamma)}\,(\beta x-\alpha)(\delta y-\gamma) \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)\,p_{n,j}(y) \begin{bmatrix} x, \frac{k+\alpha}{m+\beta} & ; f \\ y, \frac{j+\gamma}{n+\delta} & \end{bmatrix}.$$

Remark 2.3. When $p = q = 0$, $\alpha = \beta = \gamma = \delta = 0$, one obtains the classical Bernstein bivariate operator, namely

$$B_{m,n}(f;x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)\,p_{n,j}(y) f\left(\frac{k}{m}, \frac{j}{n}\right). \tag{2.35}$$

The associated Bernstein bivariate approximation formula is

$$f = B_{m,n}(f) + R_{m,n}(f), \tag{2.36}$$

and, as consequence of Theorem 2.1, for its remainder term we have the following:

Corollary 2.3. *The remainder term of* (2.35) *can be represented under the form:*

$$R_{m,n}(f;x,y) = S_1 + S_2 + S_3, \tag{2.37}$$

where

$$S_1 = -\frac{x(1-x)}{m} \sum_{k=0}^{m-1}\sum_{j=0}^{n} p_{m-1,k}(x)\,p_{n,j}(y) \begin{bmatrix} x, \frac{k}{m}, \frac{k+1}{m} & ; f \\ \frac{j}{n} & \end{bmatrix}, \tag{2.38}$$

$$S_2 = -\frac{y(1-y)}{n} \sum_{k=0}^{m}\sum_{j=0}^{n-1} p_{m,k}(x)\,p_{n-1,j}(y) \begin{bmatrix} \frac{k}{m} & ; f \\ y, \frac{j}{n}, \frac{j+1}{n} & \end{bmatrix}, \tag{2.39}$$

$$S_3 = \frac{xy(1-x)(1-y)}{mn} \sum_{k=0}^{m-1}\sum_{j=0}^{n-1} p_{m-1,k}(x)\,p_{n-1,j}(y) \begin{bmatrix} x, \frac{k}{m}, \frac{k+1}{m} & ; f \\ y, \frac{j}{n}, \frac{j+1}{n} & \end{bmatrix}. \tag{2.40}$$

Applying the mean value theorem for bivariate divided differences, in [13] was proved the following:

Theorem 2.2. *Let be $f \in C^{(2,2)}([0,1+p]\times[0,1+q])$. Then there exists a constant M depending on $f, \alpha, \beta, \gamma, \delta$, such that for any $(x,y) \in [0,1]\times[0,1]$ and any $m,n \in \mathbb{N}$, the following*

$$\left|\widetilde{R}^{(\alpha,\beta,\gamma,\delta)}_{m,p,n,q}(f;x,y)\right| \le \left(\frac{9m+p}{8m^2} + \frac{9n+q}{8n^2} + \frac{(9m+p)(9n+q)}{64m^2n^2}\right) M \tag{2.41}$$

holds.

Remark 2.4. From (2.40) similar estimations for the remainders of Schurer, Bernstein-Stancu and respectively Bernstein bivariate approximation formulas can be derived.

Let now $U_{m,n}$ be the GBS Bernstein operator, given by

$$U_{m,p}(f;x,y)=\sum_{k=0}^{m}\sum_{j=0}^{n}p_{m,k}(x)p_{n,j}(y)\left\{f\left(\frac{k}{m},y\right)+f\left(x,\frac{j}{n}\right)-f\left(\frac{k}{m},\frac{j}{n}\right)\right\}$$

(it is obtained from the operator (1.4), for $\alpha=\beta=\gamma=\delta=p=q=0$).

Considering the GBS Bernstein approximation formula

$$f=U_{m,n}(f)+R_{m,n}(f) \tag{2.42}$$

in [12] we proved the following:

Theorem 2.3. *The remainder term of* (2.42) *can be expressed under the form*

$$R_{m,n}(f;x,y)=\frac{xy(1-x)(1-y)}{mn}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1}p_{m-1,k}(x)\,p_{n-1,j}(y)\begin{bmatrix} x, \frac{k}{m}, \frac{k+1}{m} & \\ & ;f \\ y, \frac{j}{n}, \frac{j+1}{n} & \end{bmatrix}. \tag{2.43}$$

Applying the mean value theorem for, bivariate divided differences, in [12] was proved

Theorem 2.4. *Suppose $f\in C^{(1,1)}([0,1]\times[0,1])$ is a function for which exist $\frac{\partial^4 f}{\partial x^2\partial y^2}$ on $[0,1]\times[0,1]$, bounded on $[0,1]\times[0,1]$. Then, the following inequalities*

$$|R_{m,n}(f;x,y)|\le\frac{xy(1-x)(1-y)}{4mn}M(f)\le\frac{1}{64mn}M(f) \tag{2.44}$$

hold, for any $(x,y)\in[0,1]\times[0,1]$, where

$$M(f)=\sup_{(x,y)\in[0,1]\times[0,1]}\left|\frac{\partial^4 f}{\partial x^2\partial y^2}(x,y)\right|. \tag{2.45}$$

3. Mean value results for the remainder term of some bivariate approximation formulas

We start by recalling some notions regarding the convex functions of higher order.

In his Ph. Thesis (June 1933) Popoviciu [30] introduced the notion of m-th order convexity for univariate real valued functions as follow:

Definition 3.1. Let $I\subseteq\mathbb{R}$ be an interval . The function $f\in\mathbb{R}^I$ is m-th order convex (non-concave, polynomial, non-convex, concave) on I if and only if for each $(m+2)$ distinct points $x_1,x_2,\dots,x_{m+2}\in I$ one of the following inequalities hold true

$$[x_1,x_2,\dots,x_{m+2};f]>0\ (\ge 0,=0,\le 0,<0), \tag{3.46}$$

where the brackets denote divided difference.

Note that monotonous strictly increasing is convex of 1-th order wheil a strictly convex function in the usual sense is convex of 2-th order. Among many others, in [30] was established the following:

Theorem 3.5. *Suppose $a,b\in\mathbb{R}$ such that $a<b$ and $A\in C^{\#}[a,b]$ is a linear functional defined on $[a,b]$. Suppose that*

(i) $A(1)=A(x)=\dots=A(x^m)=0,\ A(x^{m+1})\neq 0;$

(ii) $A(g) \neq 0$ for each $g \in C[a,b]$ convex of m-*th order.*

Then, for each $f \in C[a,b]$ *there exist* $(m+2)$ *distinct points* $a \leq \xi_1 < \xi_2 < \ldots < \xi_{m+2} \leq b$ *such that*

$$A(f) = K[\xi_1, \xi_2, \ldots, \xi_{m+2}; f], \tag{3.47}$$

where K *is a constant independently on* f.

Applying Popoviciu's theorem, Aramă [4] established the following mean value result for the remainder term of Bernstein univariate approximation formula.

Theorem 3.6. *Let* $B_m : C[0,1] \to C[0,1]$ *be the Bernstein operator. The remainder term of the Bernstein univariate approximation formula*

$$f = B_m(f) + R_m(f), \tag{3.48}$$

can be expressed under the form

$$R_m(f;x) = -\frac{x(1-x)}{m}[\xi_1, \xi_2, \xi_3; f], \tag{3.49}$$

where $0 \leq \xi_1 < \xi_2 < \xi_3 \leq 1$.

Suppose $I, J \subseteq \mathbb{R}$ *are compact intervals,* $C(I \times J) = \{f \in \mathbb{R}^{I \times J} | \, f \text{continuos on} \quad I \times J\}$, $C^{\#}(I \times J) = \{A : C(I \times J) \to \mathbb{R} | \, A - \text{linear}\}$.

Our goal is to obtain a representation of a linear functional $A \in C^{\#}(I \times J)$ associated to functions $f \in C[a,b]$ which satisfy some special conditions.

Let $D \subseteq \mathbb{R}^2$ be a convex set, $m, n \in \mathbb{N}, (x_i, y_i) \in D$ for each $i = \overline{1,m}$, $j = \overline{1,n}$ and $f \in R^D$ be a bivariate bounded real valued functions. Recall that the bivariate divided difference of f on the distinct points (x_i, y_i) is given by

$$\begin{bmatrix} x_1, \ldots, x_m \\ y_1, \ldots, y_n \end{bmatrix} ; f \Bigg] = \Big[y_1, \ldots, y_n; [x_1, \ldots, x_m; f]\Big]. \tag{3.50}$$

The notion of $(m,n)-th$ order bivariate divided difference was introduced by Popoviciu [30] and then studied by many others (see for example [10, 11], [22, 23]).

The notion of $(m,n) - th$ order bivariate convex function was introduced by Popoviciu [30] in the following:

Definition 3.2. The function $f \in \mathbb{R}^D$ is $(m,n)-th$ order convex (non concave, polynomial, non convex, concave) if and only if for each distinct points (x_i, y_i) $(i \in \{1, \ldots, m+2\}, \; j \in \{1, \ldots, n+2, \})$ one of the following inequalities

$$\left[\begin{matrix} x_1, \ldots, x_{m+2} \\ y_1, \ldots, y_{n+2} \end{matrix} ; f\right] > 0 \; (\geq 0, = 0, \leq 0, < 0), \tag{3.51}$$

holds.

Example 3.1. A monotonous increasing function $f \in \mathbb{R}^D$ is convex of $(0,0)$-th order, a strictly convex function in the usual sense is convex of $(1,1)$-th order in the sense of definition (3.47), etc.

The analogous of Theorem 3.2 for the bivariate case is the following

Theorem 3.7. *Suppose $I, J \subseteq \mathbb{R}$ are compact intervals and denote*

$$C^{\#}(I,J) = \{A : C(I \times J) \to \mathbb{R} | \, A - \text{linear}\}.$$

If the linear functional A satisfies the conditions:

(i) $A(x^i y^j) = 0, \ (\forall)\ i \in \{0,1,\ldots,m\}, (\forall)\ j \in \{0,1,\ldots,n\}$;

(ii) $A(x^{m+1}y^{n+1}) \neq 0$;

(iii) $A(g) \neq 0$ *for each* $g \in C(I \times J)$ *convex of* $(m,n)-th$ *order, then, for any* $f \in C(I \times J)$ *there exist the distinct points* $(\xi_i\, \eta_j)$ $(i \in \{1,\ldots,m+2\},\ \ j \in \{1,\ldots,n+2,\})$ *such that*

$$A(f) = K \begin{bmatrix} \xi_1,\ldots,\xi_{m+2} \\ \eta_1,\ldots,\eta_{n+2} \end{bmatrix} ; f \Bigg], \tag{3.52}$$

where K is a constant independently of f.

Proof. From the hypothesis (iii) it follows that $A(g) \neq 0$ for each $g \in C(I \times J)$ concave of (m,n)-th order. Let $f \in C(I \times J)$ be arbitrarily chosen and let $g \in C(I \times J)$ be given by

$$g(x,y) = f(x,y) - \frac{A(f)}{A(x^{m+1}g^{n+1})} x^{m+1} g^{n+1}, \tag{3.53}$$

for any $(x,y) \in I \times J$.

From (3.53) it follows that $A(g) = 0$, and consequently g is not convex and not concave of (m,n)-th order on $I \times J$. According to Definition 3.2, there exist the distinct points $(\xi_i\, \eta_j) \in I \times J$ $(i \in \{1,\ldots,m+2\},\ \ j \in \{1,\ldots,n+2,\})$ such that

$$\left[\begin{matrix} \xi_1,\ldots,\xi_{m+2} \\ \eta_1,\ldots,\eta_{n+2} \end{matrix} ; f \right] = 0. \tag{3.54}$$

Taking the linearity of bivariate divided difference into account, from (3.53) and (3.54) are obtains

$$\left[\begin{matrix} \xi_1,\xi_2,\ldots,\xi_{m+2} \\ \eta_1,\eta_2,\ldots,\eta_{n+2} \end{matrix} ; f \right] - \frac{A(f)}{A(x^{m+1}g^{n+1})} \left[\begin{matrix} \xi_1,\ldots,\xi_{m+2} \\ \eta_1,\ldots,\eta_{n+2} \end{matrix} ; x^{m+1}g^{n+1} \right] = 0. \tag{3.55}$$

But it is well known (see [10] or [29]) that

$$\left[\begin{matrix} \xi_1,\ldots,\xi_{m+2} \\ \eta_1,\ldots,\eta_{n+2} \end{matrix} ; x^{m+1}g^{n+1} \right] = 1. \tag{3.56}$$

Taking (3.56) into account, from (3.55) one arrives to the desired result with $K = A(x^{m+1}g^{n+1})$. □

Application 3.1. Consider the bivariate Bernstein approximation formula

$$f = B_{m,n}(f) + R_{m,n}(f), \tag{3.57}$$

where $B_{m,n} : C([0,1] \times [0,1]) \to C([0,1] \times [0,1])$ is the Bernstein bivariate operator, i.e.

$$B_{m,n}(f;x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(\frac{k}{m}, \frac{i}{n}\right). \tag{3.58}$$

It is well known (see for example [17]) that $B_{m,n}(f;x,y) > f(x,y)$ for any $(x,y) \in [0,1]\times[0,1]$ and any $f \in C([0,1] \times [0,1])$, convex of $(1,1)$-th order on $[0,1] \times [0,1]$.

Let $(x_0, y_0) \in [0,1]\times[0,1]$ be arbitrarily chosen and define the functional $A \in C^{\#}([0,1]\times[0,1])$ by

$$A(f) = f(x_0,y_0) - B_{m,n}(f;x_0,y_0). \tag{3.59}$$

From the properties of the bivariate Bernstein operator it follows

$$A(x_0^i y_0^j) = 0, \text{ for } i \in \{0,1\},\ j \in \{0,1\} \tag{3.60}$$

$$A(x_0^2 y_0^2) \neq 0; \tag{3.61}$$

$$A(g) \neq 0 \text{ for any (1,1)-th order convex function.} \tag{3.62}$$

Applying the Theorem 3.2, there exist the distinct point $(\xi_i\, \eta_j) \in [0,1]\times[0,1], i \in \{1,2,3\}, j \in \{1,2,3\}$ such that

$$A(f) = A(x_0^2 y_0^2) \begin{bmatrix} \xi_1, \xi_2, \xi_3 & ; f \\ \eta_1, \eta_2, \eta_3 & \end{bmatrix}. \tag{3.63}$$

But $A(f) = R_{m,n}(f)$ and taking into account that

$$A(x_0^2 y_0^2) = R_{m,n}(x_0^2 y_0^2) = \frac{x_0 y_0 (1-x_0)(1-y_0)}{mn}, \tag{3.64}$$

from (3.64) one arrives to

$$R_{m,n}(f;x_0,y_0) = \frac{x_0 y_0 (1-x_0)(1-y_0)}{mn} \begin{bmatrix} \xi_1, \xi_2, \xi_3 & ; f \\ \eta_1, \eta_2, \eta_3 & \end{bmatrix}. \tag{3.65}$$

Because $(x_0,y_0) \in [0,1] \times [0,1]$ was arbitrarily chosen, it follows (3.64) holds in each point $(x,y) \in [0,1] \times [0,1]$

Application 3.2. Consider the Stancu univariate operator based on factorial powers $P_m^{\langle\alpha\rangle}$: $C[0,1] \to C[0,1]$ given by

$$P_m^{\langle\alpha\rangle}(f;x) = \sum_{k=0}^{m} p_{m,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{m}\right), \tag{3.66}$$

where $\alpha = \alpha(m) \geq 0$ and

$$p_{m,k}^{\langle\alpha\rangle} = \binom{m}{k} \frac{x^{[k,-\alpha]}(1-x)^{[m-k,-\alpha]}}{1^{[m,-\alpha]}}. \tag{3.67}$$

In [26] Miclăuş proved that if, f is convex of 1-th order on $[0,1]$, the sequence $\{P_m^{\langle\alpha\rangle}(f;x)\}_{m\in\mathbb{N}}$ is monotonous decreasing and $P_m^{\langle\alpha\rangle}(f;x) > f(x)$, $(\forall)\, x \in [0,1]$. Using the method of parametric extension, Miclăuş [27] constructed the bivariate Stancu operator based on factorial powers $P_{m,n}^{\langle\alpha,\beta\rangle} : C([0,1]\times[0,1]) \to C\left([0,1] \times [0,1]\right)$, given by

(3.68) $$P_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y)=\sum_{k=0}^{m}\sum_{j=0}^{n}p_{m,k}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)f\left(\frac{k}{m},\frac{j}{n}\right).$$

In the same paper [27] was studied the remainder term of the bivariate Stancu approximation formula

(3.69) $$f=P_{m,n}^{\langle\alpha,\beta\rangle}(f)+R_{m,n}^{\langle\alpha,\beta\rangle}(f),$$

and was obtained an representation of the remainder term using bivariate divided differences.

Let now $(x_0,y_0)\in[0,1]\times[0,1]$ arbitrarily chosen and define the linear functional $A\in C^{\#}([0,1]\times[0,1])$ by

(3.70) $$A(f)=f(x_0,y_0)-P_{m,n}^{\langle\alpha,\beta\rangle}(f;x_0,y_0).$$

From (3.70) we get the following:

(i) $A(x_0^iy_0^j)=0$, for $i\in\{0,1\}$, $j\in\{0,1\}$;

(ii) $A(x_0^2y_0^2)\neq 0$;

(iii) $A(g)\neq 0$ for any $g\in C([0,1]\times[0,1])$ convex of (1,1)-th order.

Consequently, from Theorem 3.2 it follows there exist the distinct points $(\xi_i\,\eta_j)\in[0,1]\times[0,1], i\in\{1,2,3\}, j\in\{1,2,3\}$ such that

(3.71) $$A(f)=A(x_0^2y_0^2)\left[\begin{array}{l}\xi_1,\xi_2,\xi_3\\ \eta_1,\eta_2,\eta_3\end{array};f\right].$$

Taking into account that

(3.72) $$A(x_0^2y_0^2)=\frac{1+\alpha m}{m(1+\alpha)}\cdot\frac{1+\beta n}{n(1+\beta)}x_0y_0(1-x_0)(1-y_0)\left[\begin{array}{l}\xi_1,\xi_2,\xi_3\\ \eta_1,\eta_2,\eta_3\end{array};f\right],$$

and $(x_0,y_0)\in[0,1]\times[0,1]$ was arbitrarily chosen, we get that the remainder term of (3.69) can be expressed under the form

(3.73) $$R_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y)=\frac{1+\alpha m}{m(1+\alpha)}\cdot\frac{1+\beta n}{n(1+\beta)}xy\,(1-x)(1-y)\left[\begin{array}{l}\xi_1,\xi_2,\xi_3\\ \eta_1,\eta_2,\eta_3\end{array};f\right],$$

for any $(x,y)\in[0,1]\times[0,1]$, where $0\le\xi_1<\xi_2<\xi_3\le 1$ and $0\le\eta_1<\eta_2<\eta_3\le 1$.

Remark 3.5. In the case when $\alpha=\beta=0$ one obtains the Bernstein bivariate approximation formula (3.57) and (3.73) is reduced to (3.65).

Application 3.3. Let $p\in\mathbb{N}_0$. In 1962, Schurer in [33] introduced the linear operator $\widetilde{B}_{m,p}:C([0,1+p])\to C([0,1])$, given by

(3.74) $$\widetilde{B}_{m,p}(f;x)=\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)f\left(\frac{k}{m}\right),$$

where $\widetilde{p}_{m,k}(x)$ are the fundamental Schurer's polynomials, defined by

(3.75) $$\widetilde{p}_{m,k}(x)=\binom{m+p}{k}x^k(1-x)^{m+p-k}.$$

Note that the operator (3.74) is positive only on $C[0,1]$ and the sequence $\{\widetilde{B}_{m,p}(f)\}_{m\in\mathbb{N}}$ convergences uniformly to $f \in C[0,1+p]$ only on $[0,1]$.

Muraru [28] proved that if $f \in C[0,1+p]$ is convex of 1-th order, the sequence of Schurer's polynomials is monotonous decreasing and $B_m(f;x) > f(x)$, for any $x \in [0,1]$.

More at this, taking into account that

$$\widetilde{B}_{m,p}(1;x) = 1, \quad \widetilde{B}_{m,p}(t;x) = \left(1+\frac{p}{m}\right)x, \tag{3.76}$$

it follows (applying Theorem 3.1) that the remainder term of the Schurer approximation formula

$$f = \widetilde{B}_{m,p}(f) + \widetilde{R}_{m,p}(f), \tag{3.77}$$

can be represented under the form

$$\widetilde{R}_{m,p}(f) = -\frac{m+p}{x}[\xi_1,\xi_2;f], \tag{3.78}$$

where $0 \le \xi_1 < \xi_2 < \xi_3 \le 1$.

In [7], using the method of parametric extensions [21], [5] was constructed the bivariate Schurer operator $\widetilde{B}_{m,p,n,q} : C([0,1+p]\times[0,1+q]) \to C([0,1+p]\times[0,1+q])$, given by

$$\widetilde{B}_{m,p,n,q}(f;x,y) = \sum_{k=0}^{m+p}\sum_{j=0}^{n+q} \widetilde{p}_{m,p}(x)\widetilde{p}_{n,q}(y) f\left(\frac{k}{m},\frac{j}{n}\right). \tag{3.79}$$

It is immediately that $\widetilde{B}_{m,p,n,q}$ is monotonous decreasing on $[0,1]\times[0,1]$ for each $f \in C([0,1+p]\times[0,1+q])$ monotonous strictly increasing and $\widetilde{B}_{m,p,n,q}(f;x,y) > f(x,y)$ for any $(x,y) \in [0,1]\times[0,1]$.

Consequently, the remainder term of the Schurer bivariate approximation formula

$$f = \widetilde{B}_{m,p,n,q}(f;x,y) + \widetilde{R}_{m,p,n,q}(f;x,y), \tag{3.80}$$

can be expressed under the form

$$\widetilde{R}_{m,p,n,q}(f;x,y) = \left(1+\frac{p}{n}\right)\left(1+\frac{q}{n}\right)xy\begin{bmatrix}\xi_1,\xi_2 & \\ \eta_1,\eta_2 & ;f\end{bmatrix}, \tag{3.81}$$

for any $(x,y) \in [0,1]\times[0,1]$, where $0 \le \xi_1 < \xi_2 \le 1$ and $0 \le \eta_1 < \eta_2 \le 1$.

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Approximation of Modified Jakimovski-Leviatan-Beta Type Operators

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ABSTRACT. In the present paper, we define Jakimovski-Leviatan type modified operators. We study some approximation results for these operators. We also determine the order of convergence in terms of modulus of continuity, Lipschitz functions, Peetre's K-functional, second order modulus of continuity and weighted modulus of continuity.

Keywords: Jakimovski-Leviatan operators, Korovkin's theorem, Modulus of continuity, Rate of convergence, K-functional, Weighted space

2010 Mathematics Subject Classification: 41A10, 41A25, 41A36.

1. INTRODUCTION AND PRELIMINARIES

Appell polynomials were introduced in 1880 (see [4]). In 1969, Jakimovski and Leviatan introduced an operators P_n by using Appell polynomials [7]. The Appell polynomials are defined by the identity as follows:

$$S(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k, \tag{1.1}$$

for an analytic function in the disk $|u| < r$ $(r > 1)$ and $p_n(x) = \sum_{i=0}^{n} a_i \frac{x^{n-i}}{(n-i)!}$ $(n \in \mathbb{N})$ taken $S(u) = \sum_{n=0}^{\infty} a_n u^n$, $S(1) \neq 0$. An exponential type the class of functions considerable on the semi-axis and satisfy the property $|f(x)| \leq \kappa e^{\gamma x}$, for some finite constants κ, $\gamma > 0$ and denote the set of such functions by $E[0,\infty)$. The sequence of infinite sum of the operators P_n is convergent and well-defined which are considered by the authors as follows [7]:

$$P_n(f;x) = \frac{e^{-nx}}{S(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.2}$$

for all $n \in \mathbb{N}$, where $n > \frac{\alpha}{\log r}$. In case of $\frac{a_n}{S(1)} \geq 0$ for all $n \in \mathbb{N}$, Wood [20] proved that the operator P_n is positive on $[0;1)$. For more results see also [13], [11] and [6]. They established that $\lim_{n\to\infty} P_n(f;x) \to f(x)$, uniformly in each compact subset of $[0,1)$.

If $S(1) = 1$ in (1.2) we get $p_n(x) = \frac{x^n}{n!}$, and we recover the well-known classical Favard-Szász operators defined in 1950 by

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \tag{1.3}$$

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In the last quarter of twentieth century, the quantum calculus (also known as q- calculus) was studied in [8,12] (see [3,14,15,18]).

2. Construction of Operators and Auxiliary results

In this paper, we define a Beta integral type modification of Jakimovski-Leviatan operators. We also introduce modified Jakimovski-Leviatan-Stancu type operators and obtain better approximation results. For $x \in [0,\infty), \quad p_r(x) \geq 0$ and $S(1) \neq 0$, we define

$$J_n^*(f;x) = \frac{e^{-nx}}{S(1)} \sum_{r=0}^{\infty} P_r(nx) \frac{1}{B(r+1,n)} \int_0^\infty \frac{t^r}{(1+t)^{r+n+1}} f(t)\mathrm{d}t, \tag{2.4}$$

Lemma 2.1. *If we take $e_i = t^{i-1}$ for $i = 1,2,3$. Let $J_n^*(\cdot\,;\cdot)$ be the operators given by (2.4). Then for all $x \in [0,\infty), \quad p_r(x) \geq 0$ and $S(1) \neq 0$, we have the following identities:*

(1) $J_n^*(e_1;x) = 1,$

(2) $J_n^*(e_2;x) = \left(\frac{n}{n-1}\right) x + \frac{1}{n-1}\left(\frac{S'(1)}{S(1)} + 1\right),$

(3) $J_n^*(e_3;x) = \frac{n^2}{(n-2)(n-1)}x^2 + \frac{2n}{(n-2)(n-1)}\left(\frac{S'(1)}{S(1)} + 2\right)x + \frac{1}{(n-2)(n-1)}\left(\frac{S''(1)}{S(1)} + \frac{S'(1)}{S(1)} + 2\right).$

Proof. We can easily see that

$$\sum_{r=0}^{\infty} P_r(nx) = S(1)e^{nx}, \tag{2.5}$$

$$\sum_{r=0}^{\infty} rP_r(nx) = \left(S'(1) + nS(1)x\right)e^{nx}, \tag{2.6}$$

$$\sum_{r=0}^{\infty} r^2 P_r(nx) = \left(S''(1) + 2nS'(1)x + S'(1) + n^2S(1)x^2\right)e^{nx}. \tag{2.7}$$

(1) By taking $f = e_1$

$$\begin{aligned} J_n^*(e_1;x) &= \frac{e^{-nx}}{S(1)} \sum_{r=0}^{\infty} P_r(nx) \frac{1}{B(r+1,n)} \int_0^\infty \frac{t^r}{(1+t)^{r+n+1}} \mathrm{d}t, \\ &= \frac{e^{-nx}}{S(1)} \sum_{r=0}^{\infty} P_r(nx) \frac{B(r+1,n)}{B(r+1,n)} \\ &= 1. \end{aligned}$$

(2) By taking $f = e_2$

$$\begin{aligned}
J_n^*(e_2;x) &= \frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^{\infty}\frac{t^{r+1}}{(1+t)^{r+n+1}}\mathrm{d}t,\\
&= \frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{B(r+2,n-1)}{B(r+1,n)}\\
&= \frac{r+1}{n-1}\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{B(r+1,n)}{B(r+1,n)}\\
&= \frac{1}{n-1}+\frac{1}{n-1}\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}rP_r(nx)\\
&= \frac{1}{n-1}+\frac{n}{n-1}\left(x+\frac{1}{n}\frac{S'(1)}{S(1)}\right).
\end{aligned}$$

(3) By taking $f = e_3$

$$\begin{aligned}
J_n^*(e_2;x) &= \frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^{\infty}\frac{t^{r+2}}{(1+t)^{r+n+1}}\mathrm{d}t,\\
&= \frac{1}{(n-2)(n-1)}\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)(r^2+3r+2)\\
&= \frac{2}{(n-2)(n-1)}+\frac{3}{(n-2)(n-1)}\left(\frac{S'(1)}{S(1)}+nx\right)\\
&+ \frac{1}{(n-2)(n-1)}\left(\frac{S''(1)}{S(1)}+2nx\frac{S'(1)}{S(1)}+\frac{S'(1)}{S(1)}+nx+n^2x^2\right).
\end{aligned}$$

□

Lemma 2.2. *Take $\eta_j = (e_i - x)^j$ for $i = 2, j = 1,2$. Let $J_n^*(\cdot\,;\cdot)$ be the operators given by (2.4). Then for all $x \in [0,\infty)$, $p_r(x) \geq 0$ and $S(1) \neq 0$, we have the following identities:*

1° $J_n^*(\eta_1;x) = \frac{x}{n} + \frac{1}{n-1}\left(\frac{S'(1)}{S(1)}+1\right)$;

2° $J_n^*(\eta_2;x)$
$= \frac{(n+2)}{(n-2)(n-1)}x^2 + \frac{2n}{(n-2)(n-1)}\left(\frac{2}{n}\left(\frac{S'(1)}{S(1)}\right)+1\right)x + \frac{1}{(n-2)(n-1)}\left(\frac{S''(1)}{S(1)}+\frac{S'(1)}{S(1)}+2\right)x.$

Let $\alpha,\beta \in \mathbb{R}$ such that $0 \leq \alpha < \beta$. Then for $x \in [0,\infty)$, $p_r(x) \geq 0$, and $S(1) \neq 0$, we define

$$J_n^{\alpha,\beta}(f;x) = \frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^{\infty}\frac{t^r}{(1+t)^{r+n+1}}f\left(\frac{nt+\alpha}{n+\beta}\right)\mathrm{d}t, \tag{2.8}$$

Lemma 2.3. *Take $e_i = t^{i-1}$ for $i = 1,2,3$. Let $J_n^{\alpha,\beta}(\cdot\,;\cdot)$ be the operators given by (2.8). Then for all $x \in [0,\infty)$, $p_r(x) \geq 0$ and $S(1) \neq 0$, we have the following identities:*

(1) $J_n^{\alpha,\beta}(e_1;x) = 1$

(2) $J_n^{\alpha,\beta}(e_2;x) = \frac{n^4}{(n+\beta)(n-1)}x + \frac{n}{(n+\beta)(n-1)}\left(\frac{S'(1)}{S(1)}+1\right) + \frac{\alpha}{n+\beta}$

(3) $J_n^{\alpha,\beta}(e_3;x) = \frac{n^2}{(n+\beta)^2(n-2)(n-1)}x^2 + \frac{2n^2}{(n+\beta)^2(n-1)}\left\{\frac{n}{n-2}\left(\frac{S'(1)}{S(1)}+2\right)+\alpha\right\}x$
$+ \frac{n^2}{(n+\beta)^2(n-2)(n-1)}\left(\frac{S''(1)}{S(1)}+\frac{S'(1)}{S(1)}+2\right) + \frac{2n\alpha}{(n+\beta)^2(n-1)}\left(\frac{S'(1)}{S(1)}+1\right) + \frac{\alpha^2}{(n+\beta)^2}.$

3. Main Results

We obtain the Korovkin type and weighted Korovkin type approximation theorems for the operators defined by (2.8).

Let $C_B[0,\infty)$ be the set of all bounded and continuous functions on $[0,\infty)$, which is a linear normed space with

$$\|f\|_{C_B} = \sup_{x\geq 0} |f(x)|.$$

Let

$$C_\zeta[0,\infty) := \left\{ f \in C[0,\infty) \, : \, |f(t)| \leq M(1+t)^\zeta \text{ for some } M > 0 \right\},$$

and

$$H := \left\{ f \in C[0,\infty) : \frac{f(x)}{1+x^2} \quad \text{is convergent as } x \to \infty \right\}.$$

Theorem 3.1. *Let $x \in [0,\infty)$, $f \in C_\zeta[0,\infty) \cap H$ with $\zeta \geq 2$. Then for $p_r(x) \geq 0$, $S(1) \neq 0$, the operators $J_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ defined by (2.8) satisfy*

$$\lim_{n\to\infty} J_n^{\alpha,\beta}(f;x) \to f(x)$$

uniformly on each compact subset of $[0,\infty)$.

Proof. The proof is based on Lemma 2.3 and well known Korovkin's theorem regarding the convergence of a sequence of linear positive operators. So it is enough to prove the conditions

$$\lim_{n\to\infty} J_n^{\alpha,\beta}((e_i;x) = x^{i-1}, \quad i = 1,2,3 \quad \text{as } n \to \infty$$

uniformly on $[0,\infty]$.
Clearly $\frac{1}{n} \to 0$, $\ (n \to \infty)$ we have

$$\lim_{n\to\infty} J_n^{\alpha,\beta}(e_2;x) = x, \quad \lim_{n\to\infty} J_n^{\alpha,\beta}(e_3;x) = x^2.$$

This completes the proof. □

In the space $[0,\infty)$, following Gadžiev [9,10,17], we recall the weighted spaces of the functions for which the analogous of the Korovkin theorem holds (see also [1,5,19]) .

Let $x \to \phi(x)$ be a continuous and strictly increasing function and $\varrho(x) = 1 + \phi^2(x)$, $\lim_{x\to\infty} \varrho(x) = \infty$. Let $B_\varrho[0,\infty)$ be a set of functions defined on $[0,\infty)$ and satisfying

$$|f(x)| \leq M_f \varrho(x),$$

where M_f is a constant depending only on f . Its subset of continuous functions will be denoted by $C_\varrho[0,\infty)$, i.e., $C_\varrho[0,\infty) = B_\varrho[0,\infty) \cap C[0,\infty)$. It is well known that a sequence of linear positive operators $\{J_n^{\alpha,\beta}\}_{n\geq 1}$ maps $C_\varrho[0,\infty)$ into $B_\varrho[0,\infty)$ if and only if

$$|L_n(\varrho;x)| \leq K\varrho(x),$$

where $x \in [0,\infty)$ and K is a positive constant. Note that $B_\varrho[0,\infty)$ is a normed space with the norm

$$\|f\|_\varrho = \sup_{x\geq 0} \frac{|f(x)|}{\varrho(x)}.$$

Finally, let $C_\varrho^0[0,\infty)$ be a subset of $C_\varrho[0,\infty)$ such that the limit

$$\lim_{n\to\infty} \frac{f(x)}{\varrho(x)} = K_f$$

exists and is finite.

Let $B[0,1]$ be the space of all bounded functions on $[0,1]$ and $C[0,1]$ be the space of all functions f continuous on $[0,1]$ equipped with norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|, \quad f \in C[0,1].$$

The famous Korovkin's theorems state as follows:

Theorem 3.2 (cf. [16]). *Let $\{L_n\}_{n\geq 1}$ be the sequence of linear positive operators acting from $C[0,1]$ into $B[0,1]$. Then*

$$\lim_{n\to\infty} \|L_n(t^k;x) - x^k\|_\infty = 0 \ (k = 0,1,2),$$

if and only if for all $f \in C[0,1]$

$$\lim_{n\to\infty} \|L_n(f(t);x) - f\|_\infty = 0.$$

Theorem 3.3. *Let $\{J_n^{\alpha,\beta}\}_{n\geq 1}$ be the sequence of linear positive operators acting from $C_\varrho[0,\infty)$ into $B_\varrho[0,\infty)$ satisfies the conditions*

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(\varphi^{i-1}(t);x) - \varphi^{i-1}(x)\|_\varrho = 0 \ (i = 1,2,3)$$

then for any function $f \in C_\varrho^0[0,\infty)$,

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(f(t);x) - f\|_\varrho = 0.$$

Proof. For the completeness, we give some sketch of the proof for the version which will be used in our next result. Consider $\varphi(x) = x, \quad \varrho(x) = 1 + x^2$, and

$$\|J_n^{\alpha,\beta}(e_i;x) - x^{i-1}\|_\varrho = \sup_{x\geq 0} \frac{|\ J_n^{\alpha,\beta}(e_i;x) - x^{i-1}\ |}{1+x^2}.$$

Then for $i = 1,2,3$ it is easily proved that

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(e_i;x) - x^{i-1}\|_\varrho = 0.$$

Hence by using the above Theorem 3.2, for any function $f \in C_\varrho^0(\mathbb{R}^+)$, we get

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(f(t);x) - f\|_\varrho = 0.$$

□

Theorem 3.4. *Let $x \in [0,\infty), \quad f \in C_\varrho^0[0,\infty)$ with $\varrho(x) = 1 + x^2$. Then for $p_r(x) \geq 0, \quad S(1) \neq 0$, we have*

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(f;x) - f\|_\varrho \to 0.$$

Proof. Using Theorem 3.3 for $\varphi(x) = x$ and $\varrho(x) = 1 + x^2$, we consider

$$\|J_n^{\alpha,\beta}(e_i;x) - x^{i-1}\|_\varrho = \sup_{x\geq 0} \frac{|\ J_n^{\alpha,\beta}(e_i;x) - x^{i-1}\ |}{1+x^2},$$

for $i = 1,2,3$.
According to Lemma 2.3 for $i = 1$, it is obvious that $|\ J_n^{\alpha,\beta}(e_1;x) - 1\ |\to 0$, and therefore

$$\lim_{n\to\infty} \|J_n^{\alpha,\beta}(e_1;x) - 1\|_\varrho = 0.$$

For $i = 2$

$$\begin{aligned}\sup_{x\geq 0}\frac{\mid J_n^{\alpha,\beta}(e_2;x)-t\mid}{1+x^2} &\leq \left|\frac{n^2}{(n+\beta)(n-1)}-1\right|\sup_{x\geq 0}\frac{x}{1+x^2}\\ &+ \left|\frac{n}{(n+\beta)(n-1)}\left(\frac{S'(1)}{S(1)}+1\right)+\frac{\alpha}{n+\beta}\right|\sup_{x\geq 0}\frac{1}{1+x^2}.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\|J_n^{\alpha,\beta}(e_2;x)-x\|_{\varrho}=0.$$

For $i = 3$

$$\begin{aligned}\sup_{x\geq 0}\frac{J_n^{\alpha,\beta}(e_3;x)-x^2\mid}{1+x^2} &\leq \left|\frac{n^4}{(n+\beta)^2(n-2)(n-1)}-1\right|\sup_{x\geq 0}\frac{x^2}{1+x^2}\\ &+ \left|\frac{2n^2}{(n+\beta)^2(n-2)(n-1)}\left\{\frac{n}{n-2}\left(\frac{S'(1)}{S(1)}+2\right)+\alpha\right\}\right|\sup_{x\geq 0}\frac{x}{1+x^2}\\ &+ \left|\frac{n^2}{(n+\beta)^2(n-2)(n-1)}\left(\frac{S''(1)}{S(1)}+\frac{S'(1)}{S(1)}+2\right)\right.\\ &+ \left.\frac{2n\alpha}{(n+\beta)^2(n-1)}\left(\frac{S'(1)}{S(1)}\right)+\frac{\alpha^2}{(n+\beta)^2}\right|\sup_{x\geq 0}\frac{1}{1+x^2}.\end{aligned}$$

Hence we have

$$\lim_{n\to\infty}\|J_n^{\alpha,\beta}(e_3;x)-x^2\|_{\varrho}=0.$$

Which completes the proof of Korovkin's type theorem. □

4. Rate of Convergence

Here we calculate the rate of convergence of operators (2.8) by means of modulus of continuity and Lipschitz type functions.

Let $f \in C_B[0,\infty]$ be the space of all bounded and uniformly continuous functions on $[0,\infty)$ and $x \geq 0$. Then for $\delta > 0$, the modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by

(4.9) $$\omega(f,\delta)=\sup_{|t-x|\leq\delta}|f(t)-f(x)\mid,\quad t\in[0,\infty).$$

It is known that $\lim_{\delta\to 0+}\omega(f,\delta)=0$ for $f\in C_B[0,\infty)$ and for any $\delta>0$ one has

(4.10) $$|f(t)-f(x)|\leq\left(\frac{|t-x|}{\delta}+1\right)\omega(f,\delta).$$

Take $\mu_j=(e_i-x)^j$ for $i=2, j=1,2$ and in the sequel we use the following notations:

(4.11) $$\delta_n^{\alpha,\beta}=\sqrt{J_n^{\alpha,\beta}(\mu_2;x)},$$

Here

$$J_n^{\alpha,\beta}(\mu_j;x) = \begin{cases} \left(\frac{n^2}{(n+\beta)(n-1)}-1\right)x+\frac{n}{(n+\beta)(n-1)}\left(\frac{S'(1)}{S(1)}+1\right)+\frac{\alpha}{n+\beta} \\ \text{for } j=1,\ 0<\alpha<\beta,\ \ \alpha,\beta\in\mathbb{R} \\ \left(\frac{n^4}{(n+\beta)^2(n-2)(n-1)}-\frac{2n^2}{(n+\beta)(n-1)}+1\right)x^2 \\ +\left[\frac{2n^2}{(n+\beta)^2(n-1)}\left\{\frac{n}{n-2}\left(\frac{S'(1)}{S(1)}+2\right)+\alpha\right\}\right. \\ \left.-\frac{2n}{(n+\beta)(n-1)}\left(\frac{S'(1)}{S(1)}+1\right)+\frac{2\alpha}{n+\beta}\right]x \\ +\frac{n^2}{(n+\beta)^2(n-2)(n-1)}\left(\frac{S''(1)}{S(1)}+\frac{S'(1)}{S(1)}+2\right) \\ +\frac{2n\alpha}{(n+\beta)^2(n-1)}\left(\frac{S'(1)}{S(1)}+1\right)+\frac{\alpha^2}{(n+\beta)^2} \\ \text{for } j=2,\ 0<\alpha<\beta,\ \ \alpha,\beta\in\mathbb{R} \end{cases}$$

when $\alpha=\beta=0$, then $\delta_n^{\alpha,\beta}$ is reduced to $\delta_n^* = \sqrt{J_n^*(\eta_2;x)}$.

Theorem 4.5. *For $x\in[0,\infty)$, $f\in C_B[0,\infty)$ the operators $J_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ defined by (2.8) satisfying:*

$$|J_n^{\alpha,\beta}(f;x)-f(x)| \leq 2\omega\left(f;\delta_n^{\alpha,\beta}\right), \tag{4.12}$$

where $n\in\mathbb{N}$, $p_r(x)\geq 0$, $S(1)\neq 0$ and $\delta_n^{\alpha,\beta}$ is defined in (4.11).

Proof. For our sequence of positive linear operators $\{J_n^{\alpha,\beta}(.;.)\}$ we have

$$\begin{aligned} J_n^{\alpha,\beta}(f;x)-f(x) &= J_n^{\alpha,\beta}(f;x)-f(x)J_n^{\alpha,\beta}(1;x) \\ &= J_n^{\alpha,\beta}\left(f(t)-f(x);x\right) \\ &\leq J_n^{\alpha,\beta}\left(|\,f(t)-f(x)\,|;x\right), \end{aligned}$$

since $J_n^{\alpha,\beta}(1;x)=1$. From (4.9) and (4.10) easily we get

$$\begin{aligned} |J_n^{\alpha,\beta}(f;x)-f(x)| &\leq J_n^{\alpha,\beta}\left(1+\frac{|\,t-x\,|}{\delta};x\right)\omega(f;\delta) \\ &= \left(1+\frac{1}{\delta}J_n^{\alpha,\beta}(|\,t-x\,|;x)\right)\omega(f;\delta). \end{aligned}$$

Cauchy-Schwarz inequality give us

$$J_n^{\alpha,\beta}(|\,t-x\,|;x) \leq J_n^{\alpha,\beta}(1;x)^{\frac{1}{2}}J_n^{\alpha,\beta}\left((t-x)^2;x\right)^{\frac{1}{2}}$$

so that

$$|\,J_n^{\alpha,\beta}(f;x)-f(x)\,| \leq \left(1+\frac{1}{\delta}J_n^{\alpha,\beta}\left((t-x)^2;x\right)^{\frac{1}{2}}\right)\omega(f;\delta). \tag{4.13}$$

Finally, putting $\delta=\delta_n^{\alpha,\beta}=\sqrt{J_n^{\alpha,\beta}(\mu_2;x)}$ we get the assertion. □

Remark 4.1. *Choosing $\delta = \frac{1}{n+\beta}$ in (4.13) we obtain the following estimate*

$$|J_n^{\alpha,\beta}(f;x) - f(x)| \leq \left(1 + (n+\beta)\delta_n^{\alpha,\beta}\right)\omega\left(f; \frac{1}{n+\beta}\right), \tag{4.14}$$

where δ_n^* defined in (4.11).

Remark 4.2. *For $\alpha = \beta = 0$ the corresponding estimate for the sequence of positive linear operators $\{J_n^{\alpha,\beta}\}$ is reduced to $\{J_n^*\}$ defined by (2.4) which can take the form as*

$$|J_n^*(f;x) - f(x)| \leq 2\omega\left(f; \delta_n^*\right), \tag{4.15}$$

where $\delta_n^* = \sqrt{J_n^*\left(\eta_2; x\right)}$.

Now we give the rate of convergence of the operators $J_n^{\alpha,\beta}(f;x)$ defined in (2.8) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$. Let $f \in C_B[0,\infty)$, $M > 0$ and $0 < \nu \leq 1$. The class $Lip_M(\nu)$ is defined as

$$Lip_M(\nu) = \left\{f : |f(\zeta_1) - f(\zeta_2)| \leq M|\zeta_1 - \zeta_2|^\nu \;\; (\zeta_1, \zeta_2 \in [0,\infty))\right\}. \tag{4.16}$$

Theorem 4.6. *Suppose $x \in [0,\infty)$, $f \in Lip_M(\nu)$ with $(M > 0,\ 0 < \nu \leq 1)$. Then operators $J_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ defined by (2.8) satisfying:*

$$|J_n^{\alpha,\beta}(f;x) - f(x)| \leq M\left(\delta_n^{\alpha,\beta}\right)^{\nu/2},$$

where $\delta_n^{\alpha,\beta}$ is defined in (4.11).

Proof. Use (4.16) and apply Hölder's inequality

$$\begin{aligned}
|J_n^{\alpha,\beta}(f;x) - f(x)| &\leq |J_n^{\alpha,\beta}(f(t) - f(x);x)\,| \\
&\leq J_n^{\alpha,\beta}\left(|f(t) - f(x)|;x\right) \\
&\leq MJ_n^{\alpha,\beta}\left(|t-x|^\nu;x\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&|J_n^{\alpha,\beta}(f;x) - f(x)| \\
&\leq M\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^\infty \frac{t^r}{(1+t)^{r+n+1}}|t-x|^\nu \mathrm{d}t \\
&= M\frac{e^{-nx}}{S(1)}\left(\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\right)^{\frac{2-\nu}{2}} \\
&\times \left(P_r(nx)\frac{1}{B(r+1,n)}\right)^{\frac{\nu}{2}}\int_0^\infty \frac{t^r}{(1+t)^{r+n+1}}|t-x|^\nu \mathrm{d}t \\
&\leq M\left(\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^\infty \frac{t^r}{(1+t)^{r+n+1}}\mathrm{d}t\right)^{\frac{2-\nu}{2}} \\
&\times \left(\frac{e^{-nx}}{S(1)}\sum_{r=0}^{\infty}P_r(nx)\frac{1}{B(r+1,n)}\int_0^\infty \frac{t^r}{(1+t)^{r+n+1}}|t-x|^2 \mathrm{d}t\right)^{\frac{\nu}{2}} \\
&= MJ_n^{\alpha,\beta}\left(\mu_2;x\right)^{\frac{\nu}{2}}.
\end{aligned}$$

This completes the proof. □

Let

$$C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}, \tag{4.17}$$

with the norm

$$\|g\|_{C_B^2[0,\infty)} = \|g\|_{C_B[0,\infty)} + \|g'\|_{C_B[0,\infty)} + \|g''\|_{C_B[0,\infty)}, \tag{4.18}$$

where

$$\|g\|_{C_B[0,\infty)} = \sup_{x\in[0,\infty)} |\, g(x)|. \tag{4.19}$$

Theorem 4.7. *Let $x \in [0,\infty)$ and $J_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ be the operator defined by (2.8). Then for any $g \in C_B^2[0,\infty)$, we have*

$$|J_n^{\alpha,\beta}(f;x) - f(x)| \leq \frac{1}{2}\delta_n^{\alpha,\beta}(2+\delta_n^{\alpha,\beta})\|g\|_{C_B^2[0,\infty)},$$

where $n \in \mathbb{N}$, $\ p_r(x) \geq 0,\ S(1) \neq 0$ and $\delta_n^{\alpha,\beta}$ is defined in (4.11).

Proof. Let $g \in C_B^2[0,\infty)$. Then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(t) = g(x) + g'(x)(t-x) + g''(\psi)\frac{(t-x)^2}{2},$$

which follows

$$|g(t) - g(x)| \leq M_1|t-x| + \frac{1}{2}M_2(t-x)^2,$$

where by using the result of (4.18) and (4.19) we have

$$M_1 = \sup_{x\in[0,\infty)} |g'(x)| = \|g'\|_{C_B[0,\infty)} \leq \|g\|_{C_B^2[0,\infty)},$$

$$M_2 = \sup_{x\in[0,\infty)} |g''(x)| = \|g''\|_{C_B[0,\infty)} \leq \|g\|_{C_B^2[0,\infty)},$$

again from 4.18, we have

$$|g(t) - g(x)| \leq \left(|t-x| + \frac{1}{2}(t-x)^2\right)\|g\|_{C_B^2[0,\infty)}.$$

Since

$$|J_n^{\alpha,\beta}(g,x) - g(x)| = |J_n^{\alpha,\beta}(g(t)-g(x);x)| \leq J_n^{\alpha,\beta}(|g(t)-g(x)|;x),$$

and also

$$J_n^{\alpha,\beta}\left(|t-x|;x\right) \leq J_n^{\alpha,\beta}\left((t-x)^2;x\right)^{\frac{1}{2}} = \delta_n^{\alpha,\beta}$$

we get

$$\begin{aligned} |J_n^{\alpha,\beta}(g;x) - g(x)| &\leq \left(J_n^{\alpha,\beta}(|t-x|;x) + \frac{1}{2}J_n^{\alpha,\beta}((t-x)^2;x)\right)\|g\|_{C_B^2[0,\infty)} \\ &\leq \frac{1}{2}\delta_n^{\alpha,\beta}(2+\delta_n^{\alpha,\beta})\|g\|_{C_B^2[0,\infty)}. \end{aligned}$$

This completes the proof. □

The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf_{C_B^2[0,\infty)} \left\{ \left(\|f-g\|_{C_B[0,\infty)} + \delta \|g''\|_{C_B^2[0,\infty)} \right) : g \in \mathcal{W}^2 \right\}, \tag{4.20}$$

where

$$\mathcal{W}^2 = \{ g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty) \}. \tag{4.21}$$

There exits a positive constant $C > 0$ such that $K_2(f,\delta) \leq C\omega_2(f,\delta^{1/2}),\quad \delta > 0$, where the second order modulus of continuity is given by

$$\omega_2(f,\delta^{1/2}) = \sup_{0<h<\delta^{1/2}} \sup_{x\in\mathbb{R}^+} |f(x+2h) - 2f(x+h) + f(x)|. \tag{4.22}$$

Theorem 4.8. *Suppose $x \in [0,\infty), n \in \mathbb{N}$ and $f \in C_B[0,\infty)$. Then the operators $J_n^{\alpha,\beta}(\cdot\,;\cdot)$ defined by (2.8) satisfying*

$$|J_n^{\alpha,\beta}(f;x) - f(x)| \leq 2M \left\{ \omega_2\left(f; \sqrt{\Delta_n^{\alpha,\beta}}\right) + \min(1, \Delta_n^{\alpha,\beta}) \|f\|_{C_B[0,\infty)} \right\},$$

where M is a positive constant, $p_r(x) \geq 0$, $\quad S(1) \neq 0$ and $\Delta_n^{\alpha,\beta} = \frac{(2+\delta_n^{\alpha,\beta})\delta_n^{\alpha,\beta}}{4}$.

Proof. As previous we easily conclude that

$$\begin{aligned} |J_n^{\alpha,\beta}(f;x) - f(x)| &\leq |J_n^{\alpha,\beta}(f-g;x)| + |J_n^{\alpha,\beta}(g;x) - g(x)| + |f(x) - g(x)|, \\ &\leq 2\|f-g\|_{C_B[0,\infty)} + \frac{\delta_n^{\alpha,\beta}}{2}(2+\delta_n^{\alpha,\beta})\|g\|_{C_B^2[0,\infty)}, \\ &\leq 2\left(\|f-g\|_{C_B[0,\infty)} + \frac{\delta_n^{\alpha,\beta}}{4}(2+\delta_n^{\alpha,\beta})\|g\|_{C_B^2[0,\infty)} \right). \end{aligned}$$

By taking infimum over all $g \in C_B^2[0,\infty)$ and by using (4.20), we get

$$|J_n^{\alpha,\beta}(f;x) - f(x)| \leq 2K_2\left(f; \frac{\delta_n^{\alpha,\beta}(2+\delta_n^{\alpha,\beta})}{4}\right).$$

Now for an absolute constant $M > 0$ in [2] we use the relation

$$K_2(f;\delta) \leq M\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta)\|f\|\}.$$

This completes the proof. □

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On the Bézier Variant of the Srivastava-Gupta Operators

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ABSTRACT. In the present paper, we introduce the Bézier variant of the Srivastava-Gupta operators, which preserve constant as well as linear functions. Our study focuses on a direct approximation theorem in terms of the Ditzian-Totik modulus of smoothness, respectively the rate of convergence for differentiable functions whose derivatives are of bounded variation.

Keywords: Srivastava-Gupta operators, Genuine operators, Hypergeometric series, Rate of convergence, Bounded variation.

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1. INTRODUCTION

Srivastava-Gupta [19] presented the following summation-integral type operators defined as follows:

$$G_{n,c}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} p_{n+c,k-1}f(t)dt + p_{n,0}(x,c)f(0), \tag{1.1}$$

where

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x),$$

with the following special cases:

(1) If $c = 0$ and $\phi_{n,c}(x) = e^{-nx}$, then we get $p_{n,k}(x,0) = e^{-nx}\dfrac{(nx)^k}{k!}$,

(2) $c = \mathbb{N}$ and $\phi_{n,c}(x) = (1+cx)^{-n/c}$, then we obtain $p_{n,k}(x,0) = \dfrac{(n/c)_k}{k!}\dfrac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}}$,

(3) If $c = -1$ and $\phi_{n,c}(x) = (1-x)^n$, then $p_{n,k}(x,-1) = \binom{n}{k}x^k(1-x)^{n-k}$.

Gupta [12] introduced the general class of Durrmeyer type operators and studied some direct results. In [16], the authors considered the Bézier variant of the operators (1.1) and established the estimate of the rate of convergence of these operators for functions of bounded variation. Kajla and Acar [17] constructed mixed hybrid operators and established quantitative Voronovskaja type theorems, local approximation theorems and weighted approximation properties for these operators. Verma and Agrawal [23] presented the generalized form of the operators (1.1) and obtained some approximation properties for these operators. Acar et al. [3] proposed Stancu type generalization of the operators (1.1) and studied the rate of convergence for functions having derivatives of bounded variation and also discussed the simultaneous approximation for these operators. Recently, Neer et al. [18] introduced the Bézier variant of the

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operators which is proposed by Yadav [22] and obtained several approximation properties.

Gupta [11] introduced a modification of the operators (1.1) as

$$U_{n,c}(f;x) = (n+2c)\sum_{k=1}^{\infty} p_{n+c,k}(x,c)\int_0^{\infty} p_{n+3c,k-1}(t,c)f(t)dt + p_{n+c,0}(x,c)f(0). \tag{1.2}$$

It is important to note here that these operators preserve constant as well as linear functions. The $r^{th}(r \in \mathbb{N})$ order moments are given by

$$U_{n,c}(e_r,x) = \begin{cases} \dfrac{x\Gamma\left((n/c)-r+2\right)\Gamma(r+1)}{\Gamma\left((n/c)+1\right)c^{r-1}}\ {}_2F_1\left(\frac{n}{c}+2, 1-r; 2; -cx\right), & \text{for } c = \mathbb{N}\cup\{-1\}, \\ \dfrac{(nx)r!}{n^r}\ {}_1F_1\left(1-r; 2; -nx\right), & \text{for } c = 0. \end{cases}$$

Srivastava and Gupta [20] got the rate of convergence for the Bézier variant of the Bleimann Butzer and Hahn operators for functions with bounded variation. In 2007, Guo et al. [15] studied Baskakov-Bézier operators and established direct, inverse and equivalence approximation theorems with the help of Ditzian-Totik modulus of smoothness. Very recently, Agrawal et al. [5] introduced mixed hybrid operators for which they got direct results and the rate of convergence for differentiable functions whose derivatives are of bounded variation. Many other interesting Bézier type operators were studied by several researchers, cf. [1,2,4,6,7,9,10,13,14, 21,24,25].

For $\theta \geq 1$, we present the Bézier variant of the operators $U_{n,c}f$ defined by

$$U_{n,c}^{(\theta)}(f;x) = (n+2c)\sum_{k=1}^{\infty} Q_{n,k}^{(\theta)}(x,c)\int_0^{\infty} p_{n+3c,k-1}(t,c)f(t)dt + Q_{n,0}^{(\theta)}(x,c)f(0), \tag{1.3}$$

where $Q_{n,k}^{(\theta)}(x,c) = (J_{n,k}(x,c))^{\theta} - (J_{n,k+1}(x,c))^{\theta}$, with $J_{n,k}(x,c) = \sum_{j=k}^{\infty} p_{n+c,j}(x,c)$. For $\theta = 1$, the operators $U_{n,c}^{(\theta)}f$ reduce to the operators $U_{n,c}f$.
Alternatively we may rewrite the operators (1.3) as

$$U_{n,c}^{(\theta)}(f;x) = \int_0^{\infty} P_{n,\theta,c}(x,t)f(t)dt, \quad x \in [0,\infty), \tag{1.4}$$

where

$$P_{n,\theta,c}(x,t) = (n+2c)\sum_{k=1}^{\infty} Q_{n,k}^{(\theta)}(x,c)p_{n+3c,k-1}(t,c) + Q_{n,0}^{(\theta)}(x,c)\delta(t),$$

$\delta(t)$ being the Dirac-delta function.

The aim of this paper is to introduce the Bézier variant (1.3) of the Srivastava-Gupta operators, which preserve linear functions. Our further study focuses on a direct approximation theorem in terms of the Ditzian-Totik modulus of smoothness, respectively the rate of convergence for differential functions whose derivatives are of bounded variation on every finite subinterval of $(0, \infty)$, for the presented operators (1.3).

2. Auxiliary results

Throughout this paper, C denotes a positive constant independent of n and x, not necessarily the same at each occurrence. For these new operators (1.3) we establish some auxiliary results. The monomials $e_k(x) = x^k$, for $k \in \mathbb{N}_0$ called test functions play an important role in uniform approximation by linear positive operators.

Lemma 2.1. *For any $n \in \mathbb{N}$, the images of test functions by Gupta operators (1.2) are given by*

$$U_{n,c}(e_0; x) = 1, \quad U_{n,c}(e_1; x) = x, \quad U_{n,c}(e_2; x) = x^2 + \frac{2x(1+cx)}{n}.$$

Consequently,

$$U_{n,c}\left((t-x)^2; x\right) = \frac{2x(1+cx)}{n}. \tag{2.5}$$

Lemma 2.2. *Let f be a real-valued function continuous and bounded on $[0, \infty)$, with $\|f\| = \sup_{x \in [0,+\infty)} |f(x)|$, then $|U_{n,c}(f)| \leq \|f\|$.*

Lemma 2.3. *Let f be a real-valued function continuous and bounded on $[0, \infty)$ and $\theta \geq 1$, then $|U_{n,c}^{(\theta)}(f)| \leq \theta \|f\|$.*

Proof. Applying the well known property $|a^\alpha - b^\alpha| \leq \alpha |a - b|$, with $0 \leq a, b \leq 1$, $\alpha \geq 1$ and the definition of $Q_{n,k}^{(\theta)}(x, c)$, we have

$$0 < (J_{n,k}(x,c))^\theta - (J_{n,k+1}(x,c))^\theta \leq \theta(J_{n,k}(x,c) - J_{n,k+1}(x,c)) = \theta p_{n+c,k}(x). \tag{2.6}$$

Hence, from the definition of $U_{n,c}^{(\theta)}(f)$ operators and Lemma 2.2, we get

$$|U_{n,c}^{(\theta)}(f)| \leq \theta |U_{n,c}(f)| \leq \theta \|f\|.$$

□

Remark 2.1. *We have*

$$\begin{aligned} U_{n,c}^{(\theta)}(f; x)(e_0; x) &= \sum_{k=0}^{\infty} Q_{n,k}^{(\theta)}(x, c) = [J_{n,0}(x, c)]^\theta \\ &= \left[\sum_{j=0}^{\infty} p_{n+c,j}(x) \right]^\theta = 1. \end{aligned}$$

In order to present our further results, we recall from [8] the definitions of the Ditizian-Totik modulus of smoothness. Let $\varphi(x) = \sqrt{x(1+cx)}$, then

$$\omega_\varphi(f, t) = \sup_{0 < h \leq t} \sup_{x \pm h\varphi(x)/2 \geq 0} \left\{ \left| f\left(x + \frac{h\varphi(x)}{2}\right) - f\left(x - \frac{h\varphi(x)}{2}\right) \right| \right\},$$

and the appropriate Peetre's K-functional is defined by

$$\overline{K}_\varphi(f, t) = \inf_{g \in V_\varphi} \{\|f - g\| + t\|\varphi g'\|\}, \quad t > 0,$$

where $V_\varphi = \{g \in C[0,+\infty) \,|\, g \in AC_{\text{loc}}[0,+\infty), \|\varphi g'\| < \infty\}$. According to Th. 3.1.2, [8], it is well known that $\overline{K}_\varphi(f,t) \sim \omega_\varphi(f,t)$, which means that there exists a constant $M > 0$, such that

$$M^{-1}\omega_\varphi(f,t) \le \overline{K}_\varphi(f,t) \le M\omega_\varphi(f,t). \tag{2.7}$$

3. Direct Theorem

Now we are able to prove the following direct approximation theorem in terms of Ditzian-Totik modulus of smoothness.

Theorem 3.1. *Let $f \in C_B[0,\infty)$ and $\theta \ge 1$, then for any $x \in [0,\infty)$, we have*

$$\left|U_{n,c}^{(\theta)}(f;x) - f(x)\right| \le C\omega_\varphi\left(f, \frac{\varphi(x)}{\sqrt{n}}\right), \tag{3.8}$$

where C is an absolute constant.

Proof. By the definition of $\overline{K}_\varphi(f,t)$ and the relation (2.7), for fixed n, x, we can choose $g = g_{n,x} \in V_\varphi$ such that

$$||f-g|| + \frac{1}{\sqrt{n}}||\varphi g'|| + \frac{1}{n}||g'|| \le \omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right). \tag{3.9}$$

Using Remark 2.1, we can write

$$\begin{aligned} |\, U_{n,c}^{(\theta)}(f) - f \,| &\le |\, U_{n,c}^{(\theta)}(f-g;x) \,| + |f-g| + |\, U_{n,c}^{(\theta)}(g;x) - g(x) \,| \\ &\le C||f-g|| + |\, U_{n,c}^{(\theta)}(g;x) - g(x) \,| \,. \end{aligned} \tag{3.10}$$

We only need to estimate the second term in the above relation. We will have to split the estimate into two domains, i.e. $x \in F_n^c = [0, 1/n]$ and $x \in F_n = (1/n, \infty)$.
Using the representation $g(t) = g(x) + \int_x^t g'(u)du$, we get

$$\left|U_{n,c}^{(\theta)}(g;x) - g(x)\right| = \left|U_{n,c}^{(\theta)}\left(\int_x^t g'(u)du; x\right)\right|. \tag{3.11}$$

If $x \in F_n = (1/n, \infty)$, then $U_{n,c}^{(\theta)}\left((t-x)^2; x\right) \sim \frac{2\theta}{n}\varphi^2(x)$. We have

$$\left|\int_x^t g'(u)du\right| \le ||\varphi g'|| \left|\int_x^t \frac{1}{\varphi(u)}du\right|. \tag{3.12}$$

For any $x, t \in (0,\infty)$, we find that

$$\begin{aligned} \left|\int_x^t \frac{1}{\varphi(u)}du\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1+cu)}}du\right| \\ &\le \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{(1+cu)}}\right)du\right| \\ &\le 2\left(\sqrt{t} - \sqrt{x} + \frac{\sqrt{(1+ct)} - \sqrt{(1+cx)}}{c}\right) \\ &= 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{(1+ct)} + \sqrt{(1+cx)}}\right) \\ &< 2|t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{(1+cx)}}\right) \\ &\le \frac{2(c+1)}{\sqrt{c(c-1)}}\frac{|t-x|}{\varphi(x)}. \end{aligned} \tag{3.13}$$

Combining (3.11)-(3.13) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |U_{n,c}^{(\theta)}(g;x) - g(x)| &< \frac{2(c+1)}{\sqrt{c(c-1)}}||\varphi g'||\varphi^{-1}(x)U_{n,c}^{(\theta)}(|t-x|;x) \\ &\leq \frac{2(c+1)}{\sqrt{c(c-1)}}||\varphi g'||\varphi^{-1}(x)\Big(U_{n,c}^{(\theta)}((t-x)^2;x)\Big)^{1/2} \\ &\leq \frac{2(c+1)}{\sqrt{c(c-1)}}||\varphi g'||\varphi^{-1}(x)\Big(\theta\, U_{n,c}((t-x)^2;x)\Big)^{1/2}. \end{aligned}$$

Now applying the relation (2.5), we get

$$|U_{n,c}^{(\theta)}(g;x) - g(x)| < C\frac{||\varphi g'||}{\sqrt{n}}. \tag{3.14}$$

For $x \in F_n^c = [0, 1/n]$, $U_{n,c}^{(\theta)}\left((t-x)^2;x\right) \sim \frac{2\theta}{n^2}$ and

$$\left|\int_x^t g'(u)du\right| \leq ||g'||\,|t-x|.$$

Therefore, using Cauchy-Schwarz inequality we have

$$|U_{n,c}^{(\theta)}(g;x) - g(x)| \leq ||g'||U_{n,c}^{(\theta)}(|t-x|;x) \leq C||g'||\frac{\sqrt{2\theta}}{\sqrt{n}} < \frac{C}{n}||g'||. \tag{3.15}$$

From (3.14) and (3.15), we have

$$|U_{n,c}^{(\theta)}(g;x) - g(x)| < C\left(\frac{||\varphi g'||}{\sqrt{n}} + \frac{1}{n}||g'||\right). \tag{3.16}$$

Using $\overline{K_\varphi}(f,t) \sim \omega_\varphi(f,t)$ and (3.9), (3.10), (3.16), we get the desired relation (3.8). This completes the proof of the theorem.

□

4. Rate of Convergence

Let $f \in DBV_\gamma(0,\infty)$, $\gamma \geq 0$, be the class of differentiable functions defined on $(0,\infty)$, whose derivatives f' are of bounded variation on every finite subinterval of $(0,\infty)$ and $|f(t)| \leq Mt^\gamma$, for all $t > 0$ and some $M > 0$. The functions $f \in DBV_\gamma(0,\infty)$, could be represented as

$$f(x) = \int_0^x g(t)dt + f(0),$$

where g is a function of bounded variation on each finite subinterval of $(0,\infty)$.

Lemma 4.4. *Let $x \in (0,\infty)$, then for $\theta \geq 1$ and sufficiently large n, we have*

i) $\zeta_{n,\theta,c}(x,y) = \int_0^y P_{n,\theta,c}(x,t)dt \leq \frac{\theta\rho}{n}\frac{\varphi^2(x)}{(x-y)^2}, \quad 0 \leq y < x,$

ii) $1 - \zeta_{n,\theta,c}(x,z) = \int_z^\infty P_{n,\theta,c}(x,t)dt \leq \frac{\theta\rho}{n}\frac{\varphi^2(x)}{(z-x)^2}, \quad x < z < \infty,$

where $\rho \geq 2$.

Proof.

$i)$ Using Lemma 2.3 and (2.5), we get

$$\zeta_{n,\theta,c}(x,y)=\int_0^y P_{n,\theta,c}(x,t)dt\leq\int_0^y\left(\frac{x-t}{x-y}\right)^2 P_{n,\theta,c}(x,t)dt$$
$$\leq U_{n,c}^{(\theta)}((t-x)^2;x)\,(x-y)^{-2}\leq\theta U_{n,c}((t-x)^2;x)(x-y)^{-2}$$
$$\leq\frac{\theta\rho}{n}\frac{\varphi^2(x)}{(x-y)^2},\qquad 0\leq y<x.$$

$ii)$ The second relation can be proved analogously. □

Theorem 4.2. *Let $f\in DBV_\gamma(0,\infty)$, $\theta\geq 1$ and $\bigvee_a^b(f_x')$ be the total variation of f_x' on $[a,b]\subset(0,\infty)$. Then, for every $x\in(0,\infty)$ and sufficiently large n, we have*

$$\left|U_{n,c}^{(\theta)}(f;x)-f(x)\right|\leq\frac{\sqrt{\theta}}{\theta+1}\left|f'(x+)+\theta f'(x-)\right|\sqrt{\frac{\rho}{n}}\varphi(x)+\sqrt{\frac{\rho}{n}}\,\varphi(x)\frac{\theta^{3/2}}{\theta+1}\left|f'(x+)-f'(x-)\right|$$
$$+\frac{\theta\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_{x-x/k}^{x}(f_x')+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}(f_x')$$
$$+\frac{\theta\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_{x}^{x+x/k}(f_x')+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+x/\sqrt{n}}(f_x'),$$

where $\rho\geq 2$ and the auxiliary function f_x' is defined by

$$f_x'(t)=\begin{cases} f'(t)-f'(x-), & 0\leq t<x\\ 0, & t=x\\ f'(t)-f'(x+), & x<t\leq 1.\end{cases}$$

Proof. Since $\displaystyle\int_0^\infty P_{n,\theta,c}(x,t)dt=U_{n,c}^{(\theta)}(e_0;x)=1$, we can write

$$U_{n,c}^{(\theta)}(f;x)-f(x)=\int_0^\infty\left(f(t)-f(x)\right)P_{n,\theta,c}(x,t)dt$$
$$(4.17)\qquad =\int_0^\infty\left(\int_x^t f'(u)du\right)P_{n,\theta,c}(x,t)dt.$$

Using definition of the function f_x', for any $f\in DBV_\gamma(0,\infty)$, it follows

$$f'(t)=\frac{1}{\theta+1}\Big(f'(x+)+\theta f'(x-)\Big)+f_x'(t)+\frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\left(\operatorname{sgn}(t-x)+\frac{\theta-1}{\theta+1}\right)$$
$$(4.18)\qquad +\delta_x(t)\left(f'(x)-\frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\right),$$

where

$$\delta_x(t)=\begin{cases}1\ , & x=t\\ 0\ , & x\neq t.\end{cases}$$

It is clear that

$$\int_0^\infty P_{n,\theta,c}(x,t)\int_x^t\left(f'(x)-\frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\right)\delta_x(u)dudt=0.$$

Using the definition of operators (1.4), then simple computations lead us to

$$E_1 = \int_0^\infty \Big(\int_x^t \frac{1}{\theta+1}\Big(f'(x+)+\theta f'(x-)\Big)du\Big) P_{n,\theta,c}(x,t)dt$$
$$= \frac{1}{\theta+1}\Big|f'(x+)+\theta f'(x-)\Big| \int_0^\infty |t-x| P_{n,\theta,c}(x,t)dt$$

(4.19)

$$\leq \frac{1}{\theta+1}\Big(f'(x+)+\theta f'(x-)\Big)\Big(U_{n,c}^{(\theta)}((e_1-x)^2;x)\Big)^{1/2} \leq \frac{\sqrt{\theta}}{\theta+1}\Big|f'(x+)+\theta f'(x-)\Big|\sqrt{\frac{\rho}{n}}\varphi(x)$$

and

$$E_2 = \int_0^\infty \Big(\int_x^t \frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\Big(\text{sgn}(u-x)+\frac{\theta-1}{\theta+1}\Big)du\Big) P_{n,\theta,c}(x,t)dt$$
$$\leq \frac{\theta}{\theta+1}\Big|f'(x+)-f'(x-)\Big| \int_0^\infty |t-x|\, P_{n,\theta,c}(x,t)dt = \frac{\theta}{\theta+1}\Big|f'(x+)-f'(x-)\Big| U_{n,c}^{(\theta)}\,(|t-x|\,;x)$$

(4.20)

$$\leq \frac{\theta}{\theta+1}\Big|f'(x+)-f'(x-)\Big|\Big(U_{n,c}^{(\theta)}\,((e_1-x)^2;x)\Big)^{1/2} \leq \frac{\theta^{3/2}}{\theta+1}\Big|f'(x+)-f'(x-)\Big|\sqrt{\frac{\rho}{n}}\varphi(x).$$

Involving the relations (4.17)–(4.20), we obtain the following estimate

$$\Big|U_{n,c}^{(\theta)}(f;x)-f(x)\Big| \leq |A_{n,\theta,c}(f'_x,x)+B_{n,\theta,c}(f'_x,x)| + \frac{\sqrt{\theta}}{\theta+1}\Big|f'(x+)+\theta f'(x-)\Big|\sqrt{\frac{\rho}{n}}\varphi(x)$$

(4.21)

$$+\frac{\theta^{3/2}}{\theta+1}\,|f'(x+)-f'(x-)|\,\sqrt{\frac{\rho}{n}}\,\varphi(x),$$

where

$$A_{n,\theta,c}(f'_x,x) = \int_0^x \Big(\int_x^t f'_x(u)du\Big)P_{n,\theta,c}(x,t)dt$$

and

$$B_{n,\theta,c}(f'_x,x) = \int_x^\infty \Big(\int_x^t f'_x(u)du\Big)P_{n,\theta,c}(x,t)dt.$$

For a complete proof of the theorem, it remains to estimate the terms $A_{n,\theta,c}(f'_x,x)$ and $B_{n,\theta,c}(f'_x,x)$. Since $\int_a^b d_t\zeta_{n,\theta,c}(x,t) \leq 1$, for all $[a,b] \subseteq (0,\infty)$, using integration by parts and applying Lemma 4.4 with $y = x-(x/\sqrt{n})$, it follows

$$|A_{n,\theta,c}(f'_x,x)| = \Big|\int_0^x \Big(\int_x^t f'_x(u)du\Big)d_t\zeta_{n,\theta,c}(x,t)\Big| = \Big|\int_0^x \zeta_{n,\theta,c}(x,t)f'_x(t)dt\Big|$$
$$\leq \Big(\int_0^y + \int_y^x\Big)\,|f'_x(t)|\,|\zeta_{n,\theta,c}(x,t)|\,dt$$
$$\leq \theta\frac{\rho\varphi^2(x)}{n}\int_0^y \bigvee_t^x (f'_x)(x-t)^{-2}dt + \int_y^x \bigvee_t^x (f'_x)dt$$
$$\leq \theta\frac{\rho\varphi^2(x)}{n}\int_0^y \bigvee_t^x (f'_x)(x-t)^{-2}dt + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^x (f'_x).$$

Taking $u=x/(x-t)$ into account, we get

$$\theta\frac{\rho\varphi^2(x)}{n}\int_0^{x-x/\sqrt{n}}(x-t)^{-2}\bigvee_t^x(f_x')dt=\theta\frac{\rho(1+cx)}{n}\int_1^{\sqrt{n}}\bigvee_{x-x/u}^{x}(f_x')du$$
$$\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\int_k^{k+1}\bigvee_{x-x/u}^{x}(f_x')du\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_{x-x/k}^{x}(f_x').$$

Hence, we reach the following estimation

$$|A_{n,\theta,c}(f_x',x)|\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_{x-x/k}^{x}(f_x')+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}(f_x'). \tag{4.22}$$

Using again the integration by parts and applying Lemma 4.4 with $z=x+x/\sqrt{n}$, it follows

$$|B_{n,\theta,c}(f_x^{'},x)|=\left|\int_x^\infty\left(\int_x^t f_x'(u)du\right)P_{n,\theta,c}(x,t)dt\right|$$
$$=\left|\int_x^z\left(\int_x^t f_x'(u)du\right)d_t(1-\zeta_{n,\theta,c}(x,t))+\int_z^\infty\left(\int_x^t f_x'(u)du\right)d_t(1-\zeta_{n,\theta,c}(x,t))\right|$$
$$=\left|\left[\left(\int_x^t f_x'(u)du\right)(1-\zeta_{n,\theta,c}(x,t))\right]_x^z-\int_x^z f_x'(t)(1-\zeta_{n,\theta,c}(x,t))dt\right.$$
$$\left.+\int_z^\infty\left(\int_x^t f_x'(u)du\right)d_t(1-\zeta_{n,\theta,c}(x,t))\right|$$
$$=\left|\left(\int_x^z f_x'(u)du\right)(1-\zeta_{n,\theta,c}(x,z))-\int_x^z f_x'(t)(1-\zeta_{n,\theta,c}(x,t))dt\right.$$
$$\left.+\left[\left(\int_x^t f_x'(u)du\right)(1-\zeta_{n,\theta,c}(x,t))\right]_z^\infty-\int_z^\infty f_x'(t)(1-\zeta_{n,\theta,c}(x,t))dt\right|$$
$$=\left|\int_x^z f_x'(t)(1-\zeta_{n,\theta,c}(x,t))dt+\int_z^\infty f_x'(t)(1-\zeta_{n,\theta,c}(x,t))dt\right|$$
$$<\theta\frac{\rho\varphi^2(x)}{n}\int_z^\infty\bigvee_x^t(f')_x(t-x)^{-2}dt+\int_x^z\bigvee_x^t(f_x')dt$$
$$\leq\theta\frac{\rho\varphi^2(x)}{n}\int_{x+x/\sqrt{n}}^\infty\bigvee_x^t(f_x')(t-x)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_x^{x+x/\sqrt{n}}(f_x'). \tag{4.23}$$

Taking $u=x/(t-x)$ into account, we get

$$\theta\frac{\rho\varphi^2(x)}{n}\int_{x+x/\sqrt{n}}^\infty\bigvee_x^t(f_x')(t-x)^{-2}dt=\theta\frac{\rho\varphi^2(x)}{xn}\int_0^{\sqrt{n}}\bigvee_x^{x+x/u}(f_x')du$$
$$\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\int_k^{k+1}\bigvee_x^{x+x/u}(f_x')du\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_x^{x+x/k}(f_x'). \tag{4.24}$$

Using the relations (4.23)-(4.24), we get the following estimation

$$|B_{n,\theta,c}(f_x^{'},x)|\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_x^{x+x/k}(f_x')+\frac{x}{\sqrt{n}}\bigvee_x^{x+x/\sqrt{n}}(f_x'). \tag{4.25}$$

The relations (4.21), (4.22) and (4.25) lead us to the desired result. □

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Notes on Some Recent Papers Concerning F-Contractions in b-Metric Spaces

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ABSTRACT. In several recent papers, attempts have been made to apply Wardowski's method of F-contractions in order to obtain fixed point results for single and multivalued mappings in b-metric spaces. In this article, it is shown that in most cases the conditions imposed on respective mappings are too strong and that the results can be obtained directly, i.e., without using most of the properties of auxiliary function F.

Keywords: b-Metric space, F-Contraction, α-Admissible mappings

2010 Mathematics Subject Classification: 4H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

b-metric spaces, as a generalization of metric spaces, were introduced by Bakhtin [3] and Czerwik [6]. If X is a nonempty set and $s \geq 1$ is a fixed real number, a function $b : X \times X \to [0, +\infty)$ is called a b-metric on X with parameter s if the following holds for all $x, y, z \in X$:

(1) $b(x, y) = 0$ if and only if $x = y$,
(2) $b(x, y) = b(y, x)$,
(3) $b(x, z) \leq s[b(x, y) + b(y, z)]$.

Then, (X, b, s) is called a b-metric space.

Further on, several researchers obtained a lot of fixed point and common fixed point results, both for single and multivalued mappings in such spaces.

On the other hand, F-contractions were introduced by Wardowski [19] and several genuine generalizations of Banach Contraction Principle were produced using this concept. In [19], the class $\mathcal{F}$ of all functions $F : (0, +\infty) \to \mathbb{R}$ was used, satisfying the conditions:

(F1) F is strictly increasing,
(F2) $\lim_{t\to+0} F(t) = -\infty$,
(F3) for each sequence $\{t_n\}$ of positive reals with $\lim_{n\to\infty} t_n = 0$ there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} t_n^k F(t_n) = 0$.

In the paper [5], Cosentino et al. attempted to apply Wardowski's method in the context of b-metric spaces, by using the following additional assumption for the class of auxiliary functions that are used (they denoted the new class by $\mathcal{F}_s$):

(F4) if t_n is a sequence of positive reals satisfying $\tau + F(st_n) \leq F(t_{n-1})$ for some $\tau > 0$ and each $n \in \mathbb{N}$, then $\tau + F(s^n t_n) \leq F(s^{n-1} t_{n-1})$ for each $n \in \mathbb{N}$.

Further, their approach was used in the papers [1, 2, 8, 10, 11, 12, 13, 14, 15, 16] to obtain various fixed point results, mostly for multivalued mappings.

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However, as we are going to show using the following result, most of the conditions used in all these articles are too strong. In fact, with these conditions, just the property $(F1)$ of functions $F \in \mathcal{F}$ is sufficient to obtain the desired results.

Lemma 1.1. [18, 9] *Let (X, b, s) be a b-metric space and let $\{x_n\}$ be a sequence in X. If there exists $r \in [0, 1)$ satisfying*

$$b(x_n, x_{n+1}) \leq rb(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N},$$

then $\{x_n\}$ is a Cauchy sequence.

Moreover, we will show that some conditions of admissibility, used in [12] and some other papers, can be replaced by easier ones.

2. Main results

We will assume in this section that $s > 1$ (otherwise, the results are already known).

The notion of α-admissibility of mappings was introduced and used in fixed point results by Samet et al. in [17]. It can be used as a unified approach to problems in spaces endowed with partial order, graph and alike. The notion was modified in several papers; we will use here the following version.

Definition 2.1. [4, Definitions 1.4 and 1.7] *Let X be a non-empty set and $f, g, h : X \to X$ be mappings such that $f(X) \cup g(X) \subseteq h(X)$, and let $\alpha : X \times X \to [0, +\infty)$ be a function. The pair (f, g) is said to be*

(1) partially weakly α-admissible with respect to h if $\alpha(fx, gy) \geq 1$ for all $y \in X$ with $hy = fx$,
(2) triangular partially weakly α-admissible with respect to h if, moreover, $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply that $\alpha(x, y) \geq 1$ for all $x, y, z \in X$.

In [12], the authors modified the previous definition, putting s^2 instead of 1 on the right-hand sides of the respective inequalities (the idea was to use them for mappings acting in b-metric spaces with parameter s). However, it is clear that if one puts $\alpha_1(x, y) = \frac{1}{s^2}\alpha(x, y)$, all of their definitions reduce to the ones from [4]. In particular, [12, Definition 7] reduces to Definition 2.1. Similarly, instead of [12, Definitions 8, 9 and 10], it is enough to use the following ones.

Definition 2.2. *Let (X, b, s) be a b-metric space, $\alpha : X \times X \to [0, +\infty)$ be a function, and f, g be self-mappings on X.*

(1) [7] *The space (X, b, s) is called α-complete if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X.*
(2) [4] *The space (X, b, s) is called α-regular if*
$$\lim_{n\to\infty} x_n = x \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \text{ imply that } \alpha(x_n, x) \geq 1 \text{ for all } n \in \mathbb{N}.$$
(3) [7] *The mapping f is α-b-continuous if, for given x and sequence $\{x_n\}$ in X,*
$$\lim_{n\to\infty} x_n = x \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \text{ implies that } \lim_{n\to\infty} fx_n = fx.$$
(4) [4] *The pair (f, g) is α-compatible if $\lim_{n\to\infty} b(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$.*

As a sample, we formulate and prove first of all an improved version of [12, Theorems 1 and 2] (since the conditions $(F2)$–$(F4)$ for functions $F \in \mathcal{F}$ will not be assumed and conditions of admissibility will be formulated in an easier way); moreover the proof will be much shorter than in [12].

Theorem 2.1. *Let (X, b, s) be an α-complete b-metric space, f, g, S, T be self-mappings on X such that $f(X) \subseteq T(X)$, $g(X) \subset S(X)$ and let $\alpha : X \times X \to [0, +\infty)$. Suppose that*

(1) *there exist $\tau > 0$ and $F : (0, +\infty) \to \mathbb{R}$ satisfying $(F1)$ such that*

$$\tau + F(sb(fx, gy)) \le F(M(x, y)) \tag{2.1}$$

holds for all $x, y \in X$ with $\alpha(x, y) \ge 1$ and $b(fx, gy) > 0$, where

$$M(x, y) = \max\left\{b(Sx, Ty), b(fx, Sx), b(gy, Ty), \frac{b(Sx, gy) + b(fx, Ty)}{2s}\right\},$$

(2) *the pairs (f, S) and (g, T) are α-compatible,*

(3) *the pairs (f, g) and (g, f) are triangular partially weakly α-admissible with respect to T and S, respectively.*

If

$(4')$ *f, g, S, T are α-b-continuous, or*

$(4'')$ *$T(X)$ and $S(X)$ are closed subsets of X and X is α-regular,*

then the pairs (f, S), (g, T) have a common coincidence point $z \in X$. If, moreover, $\alpha(Sz, Tz) \ge 1$, then z is a common fixed point of the mappings f, g, S, T.

Proof. Take an arbitrary point $x_0 \in X$. Since $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, we can form Jungck sequences $\{x_n\}$, $\{y_n\}$ in a standard way, satisfying

$$y_{2n+1} = f(x_{2n}) = T(x_{2n+1}) \text{ and } y_{2n+2} = g(x_{2n+1}) = S(x_{2n+2})$$

for $n = 0, 1, 2, \dots$ Moreover, using the assumption (3), we have that

$$\alpha(Tx_{2n+1}, Sx_{2n+2}) = \alpha(y_{2n+1}, y_{2n+2}) \ge 1 \text{ and } \alpha(Sx_{2n+2}, Tx_{2n+3}) = \alpha(y_{2n+2}, y_{2n+3}) \ge 1,$$

i.e., $\alpha(y_n, y_{n+1}) \ge 1$ for $n = 0, 1, 2, \dots$

Assume that $b(y_n, y_{n+1}) > 0$ for each $n = 0, 1, 2, \dots$ (otherwise the conclusions follow easily). As $\alpha(Sx_{2n}, Tx_{2n+1}) \ge 1$ and $b(fx_{2n}, gx_{2n-1}) > 0$, we get by (2.1) that

$$\tau + F(sb(y_{2n}, y_{2n+1})) \le F(M(y_{2n-1}, y_{2n})), \tag{2.2}$$

and, similarly,

$$\tau + F(sb(y_{2n-1}, y_{2n})) \le F(M(y_{2n-2}, y_{2n-1})). \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\tau + F(sb(y_n, y_{n+1})) \le F(M(y_{n-1}, y_n)), \tag{2.4}$$

for $n = 1, 2, \dots$ However, in a standard way, we have that, in this case, $M(y_{n-1}, y_n) = b(y_{n-1}, y_n)$. Hence, from (2.4), we have

$$F(sb(y_n, y_{n+1})) < \tau + F(sb(y_n, y_{n+1})) \le F(b(y_{n-1}, y_n)),$$

i.e., since F is strictly increasing,

$$b(y_n, y_{n+1}) < \frac{1}{s} b(y_{n-1}, y_n) \quad \text{for all } n \in \mathbb{N}.$$

Since $s > 1$, applying Lemma 1.1, we get that $\{y_n\}$ is a Cauchy sequence in X with $\alpha(y_n, y_{n+1}) \ge 1$. Thus, there exists $z \in X$ such that

$$\lim_{n\to\infty} b(y_{2n+1}, z) = \lim_{n\to\infty} b(Tx_{2n+1}, z) = \lim_{n\to\infty} b(fx_{2n}, z) = 0$$

and

$$\lim_{n\to\infty} b(y_{2n}, z) = \lim_{n\to\infty} b(Sx_{2n}, z) = \lim_{n\to\infty} b(gx_{2n-1}, z) = 0.$$

Hence, $Sx_{2n} \to z$ and $fx_{2n} \to z$ as $n \to \infty$.

The rest of the proof is the same as for [12, Theorems 1 and 2] (note that it does not use further properties of the function F). □

Corollaries 1–6 of the paper [12], formulated in an easier way, follow similarly. The same is true for Theorem 3 of that paper, as well as for the results in ordered b-metric spaces and for b-metric spaces endowed with a graph.

Further on, we will formulate and prove an improved version of [5, Theorem 3.4] (since again just the condition $(F1)$ of function F will be assumed); moreover the proof will again be much shorter than in [5]. First, we recall the following notions.

If (X, b, s) is a b-metric space, $CB(X)$ will denote the family of all non-empty, closed and bounded subsets of X. The Pompeiu-Hausdorff b-metric H on $CB(X)$ is defined by

$$H(C, D) = \max\{\sup_{c\in C} b(c, D), \sup_{d\in D} b(d, C)\},$$

for $C, D \in CB(X)$, where $b(x, A) = \inf_{a\in A} b(x, a)$ for $x \in X$ and $A \in CB(X)$.

Theorem 2.2. *Let (X, b, s) be a complete b-metric space and let $T : X \to CB(X)$. Assume that there exist $\tau > 0$ and a continuous from the right function $F : (0, +\infty) \to \mathbb{R}$ satisfying $(F1)$ such that*

$$2\tau + F(sH(Tx, Ty)) \le F(b(x, y)) \tag{2.5}$$

for all $x, y \in X$, $Tx \ne Ty$. Then T has a fixed point.

Proof. As in the proof of [5, Theorem 3.4] (this part of the proof does not use the conditions $(F2)$–$(F4)$), starting from arbitrary $x_0 \in X$, we can form a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $x_n \notin Tx_n$ and

$$\tau + F(sb(x_{n+1}, x_{n+2})) \le F(b(x_n, x_{n+1}))$$

for all $n = 0, 1, 2, \dots$, and hence

$$F(sb(x_{n+1}, x_{n+2})) < F(b(x_n, x_{n+1})).$$

Using the condition $(F1)$, we get that

$$b(x_{n+1}, x_{n+2}) < \frac{1}{s} b(x_n, x_{n+1}).$$

Since $\frac{1}{s} < 1$, applying Lemma 1.1, we conclude that $\{x_n\}$ is a Cauchy sequence in X and, hence, it converges to some $z \in X$. The proof that $z \in Tz$ is the same as in [5, Theorem 3.4] (this part again does not use any other conditions of function F). □

Open question 1. *Does Theorem 2.1 remain valid if the condition (2.1) is replaced by*

$$\tau + F(b(fx, gy)) \le F(b(Sx, Ty))$$

(which is the case for $s = 1$)? Similarly for Theorem 2.2.

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A Survey on Estimates for the Differences of Positive Linear Operators

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ABSTRACT. We survey some results concerning differences of positive linear operators from Approximation Theory, and present some new results in this direction.

Keywords: Positive linear operators, Inverses of operators, Moduli of smoothness, Estimations of differences of operators

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1. INTRODUCTION

In the recent years the differences of positive linear operators have been investigated from several points of view. The aim of this paper is to survey some of the known results and to present some new ones.

In Section 2 we recall the definitions of some classical operators used in Approximation Theory and some general results concerning their differences. These results are illustrated with the corresponding classical operators. Section 3 is devoted to the differences of the operators constructed with the same fundamental functions, only the functionals in front of them being different. In Section 4 we consider differences of operators from the family U_n^ρ. The case of operators on unbounded intervals is discussed in Section 5. Operators and their derivatives are considered in Section 6, while Section 7 is devoted to discrete operators versus integral operators. Finally, new results concerning operators and their inverses are presented in Section 8.

Throughout the paper $C[0,1]$ stands for the space of all continuous real-valued functions, endowed with the supremum norm and usual ordering.

2. SOME GENERAL RESULTS

The classical Bernstein operators introduced by Bernstein [5] in order to prove Weierstrass's fundamental theorem are given by

$$B_n : C[0,1] \to C[0,1],\ B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \text{ where}$$

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1].$$

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Let $n = 1, 2, 3, \dots$ and $f \in C[0, 1]$. The Beta-type operators $\overline{\mathbb{B}}_n$ were introduced by Lupaş in his German thesis [19] as follows

$$\overline{\mathbb{B}}_n(f; x) := \begin{cases} f(0), \ x = 0, \\ \dfrac{1}{B(nx, n - nx)} \displaystyle\int_0^1 t^{nx-1}(1 - t)^{n-1-nx} f(t)dt, \ 0 < x < 1, \\ f(1), \ x = 1, \end{cases}$$

where $B(\cdot, \cdot)$ is the Euler's Beta function.

The genuine Bernstein-Durrmeyer operators were introduced by Chen [6] and Goodman and Sharma [13] as a composition of Bernstein operators and Beta operators, namely $U_n := B_n \circ \overline{\mathbb{B}}_n$. These are given in explicit form by

$$U_n(f; x) = (1 - x)^n f(0) + x^n f(1) + (n - 1) \sum_{k=1}^{n-1} \left(\int_0^1 f(t) p_{n-2,k-1}(t) dt \right) p_{n,k}(x), \ f \in C[0, 1].$$

Denote by $S_n := \overline{\mathbb{B}}_n \circ B_n$ the Stancu-type operator investigated in [20] and defined as

$$S_n(f; x) = \frac{1}{(n)_n} \sum_{k=0}^{n} \binom{n}{k} (nx)_k (n - nx)_k f\left(\frac{k}{n}\right),$$

where the rising factorial $(x)_k$ is given by $(x)_k = x(x + 1)(x + 2) \dots (x + k - 1)$ with $(x)_0 = 1$.

Let

$$\omega_k(f; h) := \sup \left\{ \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + j\delta) \right| : |\delta| \le h, x, x + j\delta \in I \right\}$$

be the classical k^{th} order modulus of smoothness given for a compact interval I and $h \in \mathbb{R}_+$ and $\tilde{\omega}$ is the least concave majorant of the first order modulus ω_1 given by

$$\tilde{\omega}(f; t) = \sup \left\{ \frac{(t - x)\omega_1(f; y) + (y - t)\omega_1(f; x)}{y - x} : 0 \le x \le t \le y \le b - a, x \ne y \right\}.$$

The study of differences of certain positive and linear operators has as starting point the problem proposed by Lupaş [18], namely the question raised by him was to give an estimate for the difference $U_n - S_n$. A solution for the problem proposed by Lupaş in a more general case was given by Gonska et al. [10] and we recall their results below:

Theorem 2.1. [10] *Let $A, B : C[0, 1] \to C[0, 1]$ be positive linear operators such that*

$$(A - B)\left((e_1 - x)^i; x\right) = 0 \text{ for } i = 0, 1, 2, 3 \text{ and } x \in [0, 1].$$

Then for $f \in C^3[0, 1]$ there holds

$$|(A - B)(f; x)| \le \frac{1}{6}(A + B)\left(|e_1 - x|^3; x\right) \tilde{\omega}\left(f'''; \frac{1}{4} \frac{(A + B)\left((e_1 - x)^4; x\right)}{(A + B)\left(|e_1 - x|^3; x\right)}\right).$$

Theorem 2.2. [10] *If A and B are given as in Theorem 2.1, satisfying $Ae_0 = Be_0 = e_0$, then for all $f \in C[0, 1]$, $x \in [0, 1]$ we have*

$$|(A - B)(f; x)| \le c_1 \omega_4 \left(f; \sqrt[4]{\frac{1}{2}(A + B)\left((e_1 - x)^4; x\right)}\right).$$

Here c_1 is an absolute constant independent of f, x, A and B.

Using the above result the following solution to Lupaş' problem was given in [10]:

Proposition 2.1. *The Beta operators and the Bernstein operators verify*

$$|(U_n - S_n)(f;x)| \leq c_1\omega_4\left(f; \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right).$$

Here c_1 is an absolute constant independent of n, f and x.

Gonska et al. have continued their research on the differences of positive linear operators in [8]-[9]. The inequality of Theorem 2.1 was extended for a more general case as follows:

Theorem 2.3. [9] *Let $A, B : C[0,1] \to C[0,1]$ be positive operators such that*

$$(A-B)\left((e_1-x)^i;x\right) = 0 \text{ for } i = 0,1,\ldots,n \text{ and } x \in [0,1].$$

Then for $f \in C^n[0,1]$ there holds

$$|(A-B)(f;x)| \leq \frac{1}{n!}(A+B)\left(|e_1-x|^n;x\right)\tilde{\omega}\left(f^{(n)}; \frac{1}{n+1}\frac{(A+B)\left(|e_1-x|^{n+1};x\right)}{(A+B)\left(|e_1-x|^n;x\right)}\right).$$

The result from Theorem 2.2 was extended in [9] as follows:

Theorem 2.4. [9] *If A and B are given as in Theorem 2.3, satisfying $Ae_0 = Be_0 = e_0$, then for all $f \in C[0,1]$, $x \in [0,1]$ we have*

$$|(A-B)(f;x)| \leq c_1\omega_{n+1}\left(f; \sqrt[n+1]{\frac{1}{2}(A+B)\left(|e_1-x|^{n+1};x\right)}\right).$$

Here c_1 is an absolute constant independent of f, x, A and B.

In [8] Gonska, Piţul and Raşa applied the above results for some known positive linear operators as the Bernstein operators B_n, the Beta operators $\overline{\mathbb{B}}_n$, the genuine Bernstein-Durrmeyer operators $U_n = B_n \circ \overline{\mathbb{B}}_n$ and the composition of two Bernstein operators $D_n = B_n \circ B_{n+1}$.

Proposition 2.2. [9] *The Bernstein operators and the Beta-type operators verify the following relations:*

$$i) |(B_{n+1}-\overline{\mathbb{B}}_n)(f;x)| \leq \frac{x(1-x)}{n+1}\tilde{\omega}\left(f''; \sqrt{\frac{(n+1)(6nx(1-x)+7)}{18n^2}}\right), f \in C^2[0,1]$$

$$\leq \frac{x(1-x)}{3n\sqrt{n+1}}\sqrt{\frac{6nx(1-x)+7}{2n}}\|f'''\|,\ f \in C^3[0,1];$$

$$ii) |(B_{n+1}-\overline{\mathbb{B}}_n)(f;x)| \leq c\omega_3\left(f; \sqrt[3]{\frac{1}{2}(B_{n+1}+\overline{\mathbb{B}}_n)(|e_1-x|^3;x)}\right)$$

$$\leq c\omega_3\left(f; \sqrt[6]{\frac{x^2(1-x)^2}{n^3}\cdot\frac{6nx(1-x)+7}{n}}\right).$$

Proposition 2.3. [9] *The Bernstein operators and the genuine Bernstein-Durrmeyer operators verify the following relation:*

$$|(B_n - U_n)(f;x)| \leq c\omega_2\left(f; \sqrt{\frac{3x(1-x)}{2n}}\right).$$

Proposition 2.4. [9] *The composition of two Bernstein operators $D_n = B_n \circ B_{n+1}$ and the genuine Bernstein-Durrmeyer operators U_n verify the following relations:*

$$i)|(D_n - U_n)(f;x)| \le \frac{2x(1-x)}{n+1}\tilde{\omega}\left(f''; \sqrt{\frac{(n+1)(8nx(1-x)+13)}{12n^3}}\right), f \in C^2[0,1],$$

$$\le \frac{x(1-x)}{n\sqrt{n+1}}\sqrt{\frac{8nx(1-x)+13}{3n}}\|f'''\|,\ f \in C^3[0,1];$$

$$ii)|(D_n - U_n)(f;x)| \le c\omega_3\left(f; \sqrt[3]{\frac{1}{2}(D_n + U_n)(|e_1 - x|^3; x)}\right)$$

$$\le c\omega_3\left(f; \sqrt[6]{\frac{x^2(1-x)^2}{(n+1)n^3}(24nx(1-x)+39)}\right).$$

3. Differences of positive linear operators with the same fundamental functions

The results mentioned in the previous section are based on the fact that A and B have the same moments up to a certain order. The approach presented in this section involves operators constructed with the same "fundamental functions", but with different functionals in front of them (see [2]). So the difference $A - B$ is controlled by the differences of these functionals.

Let $I \subset \mathbb{R}$ be an interval and $E(I)$ a space of real-valued continuous functions on I containing the polynomials. $E_b(I)$ will be the space of all $f \in E(I)$ with

$$\|f\| := \sup\{|f(x)| : x \in I\} < \infty.$$

For $i \in \mathbb{N}$ let $e_i(x) := x^i, x \in I$. Let $F : E(I) \to \mathbb{R}$ be a positive linear functional such that $F(e_0) = 1$. Set $b^F := F(e_1)$ and

$$\mu_i^F := \frac{1}{i!}F(e_1 - b^F e_0)^i,\ i \in \mathbb{N}.$$

Then $\mu_0^F = 1$, $\mu_1^F = 0$, $\mu_2^F = \frac{1}{2}\left(F(e_2) - (b^F)^2\right) \ge 0$.

Lemma 3.1. [2] *Let $f \in E(I)$ with $f'' \in E_b(I)$. Then*

$$|F(f) - f(b^F)| \le \mu_2^F \|f''\|. \tag{3.1}$$

Lemma 3.2. [2] *Let $f \in E(I)$ with $f^{IV} \in E_b(I)$. Then*

$$\left|F(f) - f(b^F) - \mu_2^F f''(b^F) - \mu_3^F f'''(b^F)\right| \le \mu_4^F \|f^{IV}\|. \tag{3.2}$$

Proposition 3.5. [2] *Let $I = [0,1]$, $f \in C[0,1]$, $\lambda \ge 2\sqrt{\mu_2^F} > 0$. Then*

$$\left|F(f) - f(b^F)\right| \le \frac{3}{2}\omega_2\left(f, \frac{\sqrt{\mu_2^F}}{\lambda}\right)(1+\lambda^2). \tag{3.3}$$

Let K be a set of non-negative integers and $p_k \in C(I)$, $p_k \ge 0$, $k \in K$, such that $\sum_{k\in K} p_k = e_0$. For each $k \in K$ let $F_k : E(I) \to \mathbb{R}$ and $G_k : E(I) \to \mathbb{R}$ be positive linear functionals such that $F_k(e_0) = G_k(e_0) = 1$. Let $D(I)$ be the set of all $f \in E(I)$ for which $\sum_{k\in K} F_k(f)p_k \in C(I)$ and $\sum_{k\in K} G_k(f)p_k \in C(I)$.

Consider the positive linear operators $V : D(I) \to C(I)$ and $W : D(I) \to C(I)$ defined, for $f \in D(I)$, by

$$V(f;x) := \sum_{k\in K} F_k(f)p_k(x) \text{ and } W(f;x) := \sum_{k\in K} G_k(f)p_k(x).$$

Denote $\sigma(x) := \sum_{k\in K} \left(\mu_2^{F_k} + \mu_2^{G_k}\right) p_k(x)$ and $\delta := \sup_{k\in K} \left|b^{F_k} - b^{G_k}\right|$.

Theorem 3.5. [2] *Let $f \in D(I)$ with $f'' \in E_b(I)$. Then*

$$|(V-W)(f;x)| \le \|f''\|\sigma(x) + \omega_1(f,\delta). \tag{3.4}$$

Theorem 3.6. [2] *Suppose that $b^{F_k} = b^{G_k} = b_k$, $k \in K$. Let $f \in D(I)$ with f'', f''', $f^{IV} \in E_b(I)$. Then for each $x \in I$,*

$$|(V-W)(f;x)| \le \|f''\|\alpha(x) + \|f'''\|\beta(x) + \|f^{IV}\|\gamma(x), \tag{3.5}$$

where

$$\alpha(x) := \sum_{k\in K} |\mu_2^{F_k} - \mu_2^{G_k}|p_k(x), \beta(x) := \sum_{k\in K} |\mu_3^{F_k} - \mu_3^{G_k}|p_k(x),$$

$$\gamma(x) := \sum_{k\in K} (\mu_4^{F_k} + \mu_4^{G_k})p_k(x).$$

Theorem 3.7. [2] *Let $I = [0,1]$, $f \in C[0,1]$, $0 < h \le \frac{1}{2}$, $x \in [0,1]$. Then*

$$|(V-W)(f;x)| \le \frac{3}{2}\left(1 + \frac{\sigma(x)}{h^2}\right)\omega_2(f,h) + \frac{5\delta}{h}\omega_1(f,h). \tag{3.6}$$

Theorem 3.8. [2] *Let $I = [0,1]$, $f \in C[0,1]$, $0 < h < 1$, $x \in [0,1]$ and $b^{F_k} = b^{G_k}$, $k \in K$. Then*

$$|(V-W)(f;x)| \le c\left[\left(2 + \frac{\gamma(x)}{h^4}\right)\omega_4(f,h) + \frac{\beta(x)}{h^3}\omega_3(f,h) + \frac{\alpha(x)}{h^2}\omega_2(f,h)\right], \tag{3.7}$$

where c is an absolute constant.

The classical Durrmeyer operators are defined as

$$M_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt, \;\; x \in [0,1],$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ and f is an integrable function on $[0,1]$.

Proposition 3.6. [2] *For Bernstein operators and Durrmeyer operators the following relations hold:*

i) $|(B_n - M_n)(f;x)| \le \sigma(x)\|f''\| + \omega_1\left(f, \frac{1}{n+2}\right)$, *for $f'' \in C[0,1]$;*

ii) $|(B_n - M_n)(f;x)| \le 3\omega_2(f,\sqrt{\sigma(x)}) + \frac{5}{(n+2)\sqrt{\sigma(x)}}\omega_1\left(f,\sqrt{\sigma(x)}\right)$, *for $f \in C[0,1]$,*

where $\sigma(x) = \frac{1}{2(n+3)(n+2)^2}\{x(1-x)n(n-1)+n+1\} \le \frac{1}{8(n+3)}$.

Proposition 3.7. [2] *The Bernstein operators and the genuine Bernstein-Durrmeyer operators verify the following relations*

i) $|(B_n - U_n)(f;x)| \le \sigma(x)\|f''\|, \;\; f'' \in C[0,1]$,

ii) $|(B_n - U_n)(f;x)| \le 3\omega_2(f,\sqrt{\sigma(x)}), \;\; f \in C[0,1]$,

where $\sigma(x) = \frac{x(1-x)(n-1)}{2n(n+1)} \le \frac{1}{8(n+1)}$.

Proposition 3.8. [2] *The composition of two Bernstein operators $D_n := B_n \circ B_{n+1}$ and genuine Bernstein-Durrmeyer operators U_n verify*

i) $|(D_n - U_n)(f;x)| \leq \dfrac{n-1}{n(n+1)} x(1-x)\|f''\|,\ f'' \in C[0,1]$;

ii) $|(D_n - U_n)(f;x)| \leq \dfrac{x(1-x)}{2(n+1)^2}\left(\dfrac{1}{3}\|f^{(3)}\| + \dfrac{1}{8}\|f^{(4)}\|\right),\ f^{(4)} \in C[0,1]$;

iii) $|(D_n - U_n)(f;x)| \leq 3\omega_2\left(f, \sqrt{\dfrac{(n-1)x(1-x)}{n(n+1)}}\right),\ f \in C[0,1]$;

iv) $|(D_n - U_n)(f;x)| \leq c\left[\dfrac{33}{16}\omega_4\left(f, \sqrt[4]{\dfrac{x(1-x)}{(n+1)^2}}\right) + \dfrac{\sqrt[4]{x(1-x)}}{6\sqrt{n+1}}\omega_3\left(f, \sqrt[4]{\dfrac{x(1-x)}{(n+1)^2}}\right)\right], f \in C[0,1]$, *where c is an absolute constant and $n \geq 6$.*

Let K_n be the Kantorovich operators defined in [17] as follows

$$K_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \tag{3.8}$$

Proposition 3.9. [2] *The Bernstein operators and the Kantorovich operators verify the following relations*

i) $|(K_n - B_n)(f;x)| \leq \dfrac{1}{24(n+1)^2}\|f''\| + \omega_1\left(f, \dfrac{1}{2(n+1)}\right),\ f'' \in C[0,1]$;

ii) $|(K_n - B_n)(f;x)| \leq 3\omega_2\left(f, \dfrac{1}{2\sqrt{6}(n+1)}\right) + 5\sqrt{6}\,\omega_1\left(f, \dfrac{1}{2\sqrt{6}(n+1)}\right),\ f \in C[0,1]$.

This result can be extended for a generalized class of Kantorovich-type operators. Let $C_b[0,\infty)$ be the space of all real-valued continuous functions on $[0,\infty)$ with $\|f\| < \infty$ and $V_n : C_b[0,\infty) \to C_b[0,\infty)$, $V_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\varphi_{n,k}(x)$ be a positive linear operator, where $(\varphi_{n,k})_{k\geq 0}$ is a sequence of real-valued functions which verify:

i) $\varphi_{n,k}(x) \geq 0,\ k \geq 0,\quad x \in [0,\infty)$,

ii) $\varphi_{n,k} \in C[0,\infty)$;

iii) $\sum_{k=0}^{\infty} \varphi_{n,k}(x) = 1$.

Let $W_n : C_b[0,\infty) \to C_b[0,\infty)$ be the Kantorovich generalized variant of the operator V_n. Therefore,

$$W_n(f;x) = n\sum_{k=0}^{\infty} \varphi_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt. \tag{3.9}$$

Proposition 3.10. [2] *With the above notation,*

$$|W_n(f;x) - V_n(f;x)| \leq \frac{1}{24n^2}\|f''\| + \omega_1\left(f, \frac{1}{2n}\right),\ f^{(i)} \in C_b[0,\infty),\ i \in \{0,1,2\}.$$

4. Differences of U_n^ρ operators

The class of operators U_n^ρ was introduced in [26] by Păltănea and further investigated by Păltănea and Gonska in [11] and [12].

Let $\rho > 0$ and $n \in \mathbb{N}$. The operators $U_n^\rho : C[0,1] \to \prod_n$ are defined by

$$U_n^\rho(f;x) := \sum_{k=0}^{n} F_k^\rho(f) p_{n,k}(x)$$
$$:= \sum_{k=1}^{n-1} \left(\int_0^1 \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)} f(t)dt \right) p_{n,k}(x) + f(0)(1-x)^n + f(1)x^n,$$

for $f \in C[0,1]$, $x \in [0,1]$.

Remark 4.1. *For $\rho = 1$ and $f \in C[0,1]$, we obtain the genuine Bernstein-Durrmeyer operators, while for $\rho \to \infty$, for each $f \in C[0,1]$ the sequence $U_n^\rho(f;x)$ converges uniformly to the Bernstein polynomial $B_n(f;x)$.*

H. Gonska et al. [12] proved that for $n \geq 1$ and $f \in C[0,1]$,

$$\lim_{\rho \to 0^+} U_n^\rho f = B_1 f, \text{ uniformly on } [0,1]. \tag{4.10}$$

Moreover, the following result was obtained:

Theorem 4.9. [12] *For $U_n^\rho, 0 < \rho < \infty, n \geq 1$, we have*

$$|U_n^\rho f(x) - B_1 f(x)| \leq \frac{9}{4}\omega_2\left(f; \sqrt{\frac{n\rho - \rho}{n\rho + 1}x(1-x)}\right).$$

The following result was obtained with the method presented in [9].

Proposition 4.11. *([28], [30]) Let $f \in C[0,1]$, $n \geq 1$, $\rho, r > 0$, $x \in [0,1]$. The following inequality is verified*

$$|(U_n^\rho - U_n^r)(f;x)| \leq c_1\omega_2\left(f; \sqrt{\frac{1}{2}(U_n^\rho + U_n^r)(|e_1 - x|^2;x)}\right)$$
$$\leq c_1\omega_2\left(f; \sqrt{\frac{1}{2}\frac{2n\rho r + (n+1)(\rho + r) + 2}{(n\rho+1)(nr+1)}x(1-x)}\right).$$

Here c_1 is an absolute constant independent of f, x, ρ and r.

Another result in this direction was obtained in [28] and [30]:

Theorem 4.10. *([28], [30]) Let $f \in C[0,1]$, $n \geq 1$, $\rho, r > 0$, $x \in [0,1]$. Then*

$$|(U_n^\rho - U_n^r)(f;x)| \leq \frac{9}{4}\omega_2\left(f; \sqrt{\frac{(n-1)|\rho - r|}{(n\rho+1)(nr+1)}x(1-x)}\right).$$

In the next statement we give some estimates of the difference $U_n^\rho - U_n^r$ using the results proved in Section 3:

Proposition 4.12. *The following properties hold*

i) $|(U_n^\rho - U_n^r)(f;x)| \leq \dfrac{(n-1)[2 + (\rho + r)n]}{2n(1+\rho n)(1+rn)}x(1-x)\|f''\|, \;\; f'' \in C[0,1];$

ii) $|(U_n^\rho - U_n^r)(f;x)| \le \dfrac{(n-1)|r-\rho|}{2(1+\rho n)(1+rn)} x(1-x)\,\|f''\|$

$$+\frac{1}{3}x(1-x)(n-1)\frac{|r-\rho|[3+n(r+\rho)]}{(1+\rho n)(2+\rho n)(1+rn)(2+rn)}\,\|f'''\|$$

$$+\frac{1}{32}x(1-x)\frac{n^2(\rho^2+r^2)+4n(\rho+r)+6}{(1+\rho n)(3+\rho n)(1+rn)(3+rn)}\|f^{IV}\|,\ f^{(4)}\in C[0,1],$$

$n\rho \ge 6, \quad nr \ge 6;$

iii) $|(U_n^\rho - U_n^r)(f;x)| \le 3\omega_2\left(f, \sqrt{\dfrac{2+(\rho+r)n}{2(1+\rho n)(1+rn)}x(1-x)}\right),\ f\in C[0,1].$

Denote by $[A;B] := AB - BA$ the commutator of two positive linear operators A and B. In [28] the following result concerning the comutator $[U_n^\varrho; U_n^\sigma]$ was obtained:

Theorem 4.11. [28] *For each $f \in C^6[0,1]$ one has*

$$\lim_{n\to\infty} n^3(U_n^\rho U_n^r - U_n^r U_n^\rho)f(x) = \frac{(r-\rho)(\rho+1)(r+1)}{\rho^2 r^2}x(1-x)f^{(4)}(x),$$

uniformly with respect to $x\in[0,1]$.

5. Differences of linear positive operators defined on unbounded interval

Most of the results presented in the previuos sections are given for operators defined over bounded intervals. Very recently, Aral et al.[4] obtained some quantitative results in terms of weighted modulus of continuity for differences of certain positive linear operators defined on unbounded intervals.

Let $C[0,\infty)$ be the set of all continuous functions f defined on $[0,\infty)$ and $B_2[0,\infty)$ the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \le M(1+x^2)$ for some positive constant M, which may depend only on f. $C_2[0,\infty)$ denotes the subspace of all continuous functions in $B_2[0,\infty)$. Let $C_2^*[0,\infty)$ be the closed subspace of $C_2[0,\infty)$ formed by the functions f for which the limit $\lim\limits_{x\to\infty}\dfrac{f(x)}{1+x^2}$ exists and is finite. $B_2[0,\infty)$ is a linear normed space with the norm

$\|f\|_2 = \sup\limits_{x\ge 0}\dfrac{|f(x)|}{1+x^2}.$

In 2001, Ispir [16] introduced the weighted modulus of continuity as follows:

$$\Omega(f;\delta) = \sup_{|h|<\delta, x\in[0,\infty)}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)},\ \text{for } f\in C_2[0,\infty).$$

Let $e_i(x) = x^i$, $x\in[0,\infty)$, $i\in\mathbb{N}$ and $F: D\to\mathbb{R}$ be a positive linear functional defined on a linear subspace D of $C[0,\infty)$ which contains $C_2[0,\infty)$ and the polynomials up to degree 6, such that $F(e_0)=1$, $b^F := F(e_1)$ and

$$\mu_i^F = F(e_1 - b^F e_0)^i,\ i\in\mathbb{N},\ 0\le i\le 6.$$

Therefore, $\mu_0^F = 1$, $\mu_1^F = 0$ and $\mu_2^F = F(e_2) - (b^F)^2 \ge 0$.

The next estimate concerning the functional F was given in [4].

Lemma 5.3. [4] *Let $f\in C_2[0,\infty)$ with $f''\in C_2^*[0,\infty)$ and $0<\delta\le 1$. Then we have*

$$\left|F(f) - f(b^F)\right| \le \frac{1}{2}\left|f''(b^F)\right|\mu_2^F + 8\left(1+(b^F)^2\right)\Omega(f'';\delta)\left(\mu_2^F + \frac{\mu_6^F}{\delta^4}\right).$$

Let $\mathbb{K}$ be a set of non-negative integers and consider $p_k : [0,\infty) \to [0,\infty)$, $k \in \mathbb{K}$. Denote

$$U(f;x) = \sum_{k\in\mathbb{K}} F_k(f)p_k(x) \text{ and } V(f;x) = \sum_{k\in\mathbb{K}} G_k(f)p_k(x),$$

where $U, V : D \to B_2[0,\infty)$ and $F_k, G_k : D \to \mathbb{R}$ are positive linear functionals such that $F_k(e_0) = 1$ and $G_k(e_0) = 1$. Applying Lemma 5.3 for the functionals F_k and G_k the following result was obtained by Aral et al. [4]:

Theorem 5.12. [4] *Let $f \in C_2[0,\infty)$ with $f'' \in C_2^*[0,\infty)$. Then*

$$|(U-V)(f;x)| \le \frac{1}{2}\|f''\|_2\beta(x) + 8\Omega(f'';\delta_1)(1+\beta(x)) + 16\Omega(f;\delta_2)(\gamma(x)+1),$$

where

$$\beta(x) = \sum_{k\in\mathbb{K}} p_k(x)\left\{\left[1+(b^{F_k})^2\right]\mu_2^{F_k} + \left[1+(b^{G_k})^2\right]\mu_2^{G_k}\right\},$$

$$\gamma(x) = \sum_{k\in\mathbb{K}} p_k(x)\left[1+(b^{F_k})^2\right],$$

$$\delta_1(x) = \left\{\sum_{k\in\mathbb{K}} p_k(x)\left\{\left[1+(b^{F_k})^2\right]\mu_6^{F_k} + \left[1+(b^{G_k})^2\right]\mu_6^{G_k}\right\}\right\}^{1/4},$$

$$\delta_2(x) = \left\{\sum_{k\in\mathbb{K}} p_k(x)\left(1+(b^{F_k})^2\right)\left(b^{F_k}-b^{G_k}\right)^4\right\}^{1/4},$$

and we suppose that $\delta_1(x) \le 1$, $\delta_2(x) \le 1$.

In [4] this result was applied for the sequences of Szász-Mirakyan [31] and Szász-Mirakyan-Durrmeyer operators [23] defined as

$$\mathbb{S}_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)s_{n,k}(x),$$

$$\mathbb{D}_n(f;x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} f(t)s_{n,k}(t)dt,\ \ x \in [0,\infty),$$

where $s_{n,k}(x) = e^{-nx}\dfrac{(nx)^k}{k!}$.

Proposition 5.13. [4] *Let $f \in C_2[0,\infty)$ with $f'' \in C_2^*[0,\infty)$ and $x \ge 0$ is given. Then, for each $n \ge 1$ such that*

$$\delta_1^4(x) := \frac{2}{2n^8}\left(3n^5x^5 + 71n^4x^4 + (3n^4+479n^3)x^3 + (44n^4+1064n^2)x^2 + (123n^3+651n)x + 53n^2 + 53\right) \le 1,$$

$$\delta_2^4(x) := \frac{x(nx+1)+n}{n^5} \le 1,$$

we have

$$|(\mathbb{S}_n - \mathbb{D}_n)(f;x)| \le \frac{1}{2}\|f''\|_2\beta(x) + 8\Omega(f'';\delta_1(x))(1+\beta(x)) + 16\Omega(f;\delta_2(x))\left(\frac{x(nx+1)+n}{n}\right),$$

where $\beta(x) = \dfrac{n^3x^3 + 6n^2x^2 + n^3x + 7nx + n^2}{n^4}$.

6. Estimates for the differences of positive linear operators and their derivatives

It is well-known that de la Vallée-Poussin operator commutes with the derivative. Certainly, this property is not available for the Bernstein operators B_n. This remark motivated us to estimate in terms of moduli of continuity the differences $(L_n f)^{(k)} - L_{n-k}(f^{(k)})$ for certain positive linear operators. In the following we will exemplify for Bernstein operators and Durrmeyer operators.

Theorem 6.13. [3] *For Bernstein operators the following property holds:*

$$\left\|(B_n f)^{(r)} - B_{n-r}\left(f^{(r)}\right)\right\| \le \frac{(r-1)r}{2n}\|f^{(r)}\| + \omega\left(f^{(r)}, \frac{r}{n}\right),\ f \in C^r[0,1],\ r = 0, 1, \dots n.$$

Theorem 6.14. [3] *The Kantorovich operators verify*

$$\left\|(K_n f)^{(r)} - K_{n-r}\left(f^{(r)}\right)\right\| \le \frac{(r+1)r}{2(n+1)}\|f^{(r)}\| + \omega\left(f^{(r)}, \frac{r+1}{n+1}\right),\ f \in C^r[0,1],\ r = 0, 1, \dots n.$$

7. Discrete operators associated with certain integral operators

In 2011, I. Raşa [27] constructed discrete operators associated with certain integral operators using a probabilistic approach.

Let $I_n : C[a,b] \to C[a,b]$, $n \ge 1$, be a sequence of positive linear operators of the form

$$I_n(f;x) = \sum_{k=0}^{n} h_{n,k}(x) A_{n,k}(f),\ f \in C[a,b],\ x \in [a,b],$$

where $h_{n,k} \in C[a,b]$, $h_{n,k} \ge 0$ and

$$A_{n,k}(f) = \int_a^b f(t) d\mu_{n,k}(t),$$

with $\mu_{n,k}$ probability Borel measures on $[a,b]$, $n \ge 1$, $k = 0, 1, \dots, n$.

Raşa [27] introduced the following discrete operator associated with the sequence (I_n)

$$D_n(f;x) = \sum_{k=0}^{n} h_{n,k}(x) f(x_{n,k}),$$

where $x_{n,k} = \displaystyle\int_a^b t d\mu_{n,k}(t)$.

For example, the associated operators of the genuine Bernstein-Durrmeyer operators are Bernstein operators.

D. Mache intoduced the sequence of positive linear operators (see [21], [22])

$$P_n(f;x) := \sum_{k=0}^{n} p_{n,k}(x) T_{n,k}(f),\ n \ge 1,$$

where

$$T_{n,k}(f) := \frac{\int_0^1 f(t) t^{ck+a}(1-t)^{c(n-k)+b} dt}{B(ck+a+1, c(n-k)+b+1)},$$

for $a, b > -1$, $\alpha \ge 0$, $c := c_n := [n^\alpha]$ and B is the Beta function.

Remark 7.2. [27] *The sequence of positive linear operators (P_n) represents a link between the Durrmeyer operators with Jacobi weights $(\alpha = 0)$ and the Bernstein operators $(\alpha \to \infty)$.*

The sequence of positive linear operators (V_n) defined as follows

$$V_n(f;x) := \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{ck+a+1}{cn+a+b+2}\right), \; f \in C[0,1], \;\; x \in [0,1]$$

is associated with the sequence (P_n) (see [27]).

Remark 7.3. [27] *For $a = b = -1$, or $\alpha \to \infty$, we get the classical Bernstein operators. For $\alpha = 0$, the operators V_n reduce to the operators considered by D.D. Stancu in* [29].

Theorem 7.15. [27] *For $n \geq 1$, $x \in [0,1]$, and $f \in C^2[0,1]$ we have*

$$|P_n(f;x) - V_n(f;x)| \leq$$

$$\frac{c^2 n(n-1)x(1-x) + cn(b-a)x + cn(a+1) + (a+1)(b+1)}{2(cn+a+b+2)^2(cn+a+b+3)} \|f''\|.$$

The study on this topic was continued by Heilmann et al. [15]. They associated to an integral operator a discrete one which is conceptually simpler, and study the relations between them.

Let $I \subset \mathbb{R}$ be an interval, H a subspace of $C(I)$ containing $e_i, i = 0,1,2$ and $L : H \to C(I)$ be a positive linear operator such that $Le_0 = e_0$. For $f \in H$ and $A_j : H \to \mathbb{R}$ positive linear functionals such that $A_j(e_0) = 1$ and $p_j \in C(I)$, $p_j \geq 0$, $\sum_{j=0}^{\infty} p_j = e_0$, the following operator was defined in [15]:

$$Lf := \sum_{j=0}^{\infty} A_j(f) p_j.$$

The discrete operator associated with L is defined by

$$D : H \to C(I), \;\; Df := \sum_{j=0}^{\infty} f(b_j) p_j,$$

where $b_j := A_j(e_1)$. The point evaluation functional at b_j is conceptually simpler than A_j. So, from this point of view, D is simpler than L.

Let $VarA_j := A_j(e_2) - b_j^2$, $j \geq 0$. $VarA_j$ shows how far is A_j from the point evaluation at b_j. Define by

$$E(L)(x) := \sum_{j=0}^{\infty} (VarA_j) p_j(x), \; x \in I.$$

The following relation between L and D was obtained in [15]:

$$|Lf(x) - Df(x)| \leq \frac{1}{2} \|f''\|_\infty E(L)(x), \; x \in I.$$

In the above relation $E(L)(x)$ shows how far is L from D.

The discrete operators associated with Baskakov type operators, genuine Baskakov-Durrmeyer type operators, and their Kantorovich modifications were constructed and the above general result was applied in this context.

8. Differences of inverses of positive linear operators

The Voronovskaya type result for the Bernstein operators is well known:

$$\lim_{n\to\infty} n(B_n(f;x)-f(x)) = \frac{x(1-x)}{2}f^{(2)}(x),\ \ f\in C^2[0,1].$$

Moreover, Abel and Ivan [1] obtained the following result

$$\lim_{n\to\infty} n\left[n(B_n(f;x)-f(x)) - \frac{x(1-x)}{2}f^{(2)}(x)\right]$$
$$= \frac{x(1-x)}{24}\left(3x(1-x)f^{(4)}(x)+4(1-2x)f^{(3)}(x)\right),\ f\in C^4[0,1].$$

Similarly, Voronovskaya type results for Beta operators were proved by Gonska et al. [7] as follows

$$\lim_{n\to\infty} n(\overline{\mathbb{B}}_n(f;x)-f(x)) = \frac{x(1-x)}{2}f^{(2)}(x),\ \ f\in C^2[0,1]$$
$$\lim_{n\to\infty} n\left[n(\overline{\mathbb{B}}_n(f;x)-f(x)) - \frac{x(1-x)}{2}f^{(2)}(x)\right]$$
$$= \frac{x(1-x)}{24}\left(3x(1-x)f^{(4)}(x)+8(1-2x)f^{(3)}(x)-12f^{(2)}(x)\right),\ f\in C^4[0,1].$$

Voronovskaja type formulas are usually established for sequences of positive linear operators. Nasaireh et al. [24] obtained some general formulas concerning compositions of operators on Banach spaces, without any assumption of positivity. Let X be a Banach space and $W\subset Z\subset Y$ linear subspaces of X. Let $A,B:Y\to X$; $U,V:Z\to X$ be linear operators. Consider also two sequences of linear operators $P_n:X\to X$, $Q_n:Y\to X$, $n\geq 1$, and suppose that each P_n is bounded.

Theorem 8.16. [24] *(i) Suppose that*

$$\lim_{n\to\infty} P_n x = x,\ x\in X \tag{8.11}$$

and

$$\lim_{n\to\infty} n(P_n y - y) = Ay,\quad \lim_{n\to\infty} n(Q_n y - y) = By,\ y\in Y. \tag{8.12}$$

Then

$$\lim_{n\to\infty} n(P_nQ_n y - y) = Ay+By,\ y\in Y. \tag{8.13}$$

ii) In addition to (8.11) and (8.12), suppose that

$$Bz\in Y, z\in Z,$$

$$\lim_{n\to\infty} n[n(P_n z - z)-Az] = Uz;\ \lim_{n\to\infty} n[n(Q_n z - z)-Bz] = Vz,\ z\in Z.$$

Then

$$\lim_{n\to\infty} n\left[n(P_nQ_n z - z)-Az-Bz\right] = Uz+Vz+ABz,\ z\in Z.$$

Very recently, this result was extended for a more general case by Heilmann et al. [14].

Using Theorem 8.16, Voronovskaya type results for genuine Bernstein-Durrmeyer operators were proved by Nasaireh et al. [24] as follows

$$\lim_{n\to\infty} n(U_n(f;x) - f(x)) = x(1-x)f^{(2)}(x), \; f \in C^2[0,1]$$

$$\lim_{n\to\infty} n\left[n(U_n(f;x) - f(x)) - x(1-x)f^{(2)}(x)\right]$$
$$= \frac{x(1-x)}{2}\left(x(1-x)f^{(4)}(x) + 2(1-2x)f^{(3)}(x) - 2f^{(2)}(x)\right), \; f \in C^4[0,1].$$

Let Π_n be the space of all polynomial functions of degree at most n, defined on $\mathbb{R}$, and $\Pi = \cup_{n\geq 0}\Pi_n$. Nasaireh et al. [24] expressed the inverse of Beta operator as follows:

$$\overline{\mathbb{B}}_n^{-1} : \Pi \to \Pi, n \geq 1, \quad \overline{\mathbb{B}}_n^{-1}(p;x) = \sum_{k=0}^{m} \frac{(n)_k}{n^k}\left[0, -\frac{1}{n}, \ldots, -\frac{k}{n}; p\right] x^k, \; p \in \Pi_m.$$

Using Theorem 8.16 the following Voronovskaya type results for inverses of Beta operators, Bernstein operators and genuine Bernstein-Durrmeyer operators can be obtained (see [24, 25])

Theorem 8.17. *Let $m \geq 0$ and $p_n \in \Pi_m$, $n \geq 1$. Suppose that the sequence (p_n) is uniformly convergent on $[0,1]$ to $p \in \Pi_m$. Then*

i) $\lim_{n\to\infty} n\left(\overline{\mathbb{B}}_n^{-1}(p_n;x) - p_n(x)\right) = -\frac{x(1-x)}{2}p''(x),$

ii) $\lim_{n\to\infty} n\left[n\left(\overline{\mathbb{B}}_n^{-1}(p_n;x) - p_n(x)\right) + \frac{x(1-x)}{2}p_n''(t)\right]$
$= \frac{x(1-x)}{24}\left\{3x(1-x)p^{(4)}(x) + 4(1-2x)p^{(3)}(x)\right\}.$

Theorem 8.18. *Let $m \geq 0$ and $p_n \in \Pi_m$, $n \geq 1$. Suppose that the sequence (p_n) is uniformly convergent on $[0,1]$ to $p \in \Pi_m$. Then*

i) $\lim_{n\to\infty} n\left(B_n^{-1}(p_n;x) - p_n(x)\right) = -\frac{x(1-x)}{2}p''(x),$

ii) $\lim_{n\to\infty} n\left[n\left(B_n^{-1}(p_n;x) - p_n(x)\right) + \frac{x(1-x)}{2}p_n''(t)\right]$
$= \frac{x(1-x)}{24}\left\{3x(1-x)p^{(4)}(x) + 8(1-2x)p^{(3)}(x) - 12p''(x)\right\}.$

Theorem 8.19. *Let $m \geq 0$ and $p_n \in \Pi_m$, $n \geq 1$. Suppose that the sequence (p_n) is uniformly convergent on $[0,1]$ to $p \in \Pi_m$. Then*

i) $\lim_{n\to\infty} n\left(U_n^{-1}(p_n;x) - p_n(x)\right) = -x(1-x)p''(x),$

ii) $\lim_{n\to\infty} n\left[n\left(U_n^{-1}(p_n;x) - p_n(x)\right) + x(1-x)p_n''(x)\right]$
$= \frac{x(1-x)}{2}\left[x(1-x)p^{(4)}(x) + 2(1-2x)p^{(3)}(x) - 2p''(x)\right].$

From the above results one can obtain new estimates concerning certain operators and their inverses.

Proposition 8.14. *Let $p \in \Pi$. Then*

$$\lim_{n\to\infty} n\left(B_n(p;x) - B_n^{-1}(p;x)\right) = x(1-x)p''(x).$$

Proposition 8.15. *Let $p \in \Pi$. Then*

$$\lim_{n\to\infty} n\left[n\left(B_n(p;x) - B_n^{-1}(p;x)\right) - x(1-x)p''(x)\right] = \frac{x(1-x)}{6}\left[3p''(x) - (1-2x)p^{(3)}(x)\right].$$

Proposition 8.16. *Let* $p \in \Pi$. *Then*

$$\lim_{n\to\infty} n\left(\overline{\mathbb{B}}_n(p;x) - \overline{\mathbb{B}}_n^{-1}(p;x)\right) = x(1-x)p''(x).$$

Proposition 8.17. *Let* $p \in \Pi$. *Then*

$$\lim_{n\to\infty} n\left[n\left(\overline{\mathbb{B}}_n(p;x) - \overline{\mathbb{B}}_n^{-1}(p;x)\right) - x(1-x)p''(x)\right] = \frac{x(1-x)}{6}\left[(1-2x)p^{(3)}(x) - 3p^{(2)}(x)\right].$$

Proposition 8.18. *Let* $p \in \Pi$. *Then*

$$\lim_{n\to\infty} n\left[U_n(p;x) - U_n^{-1}(p;x)\right] = 2x(1-x)p''(x).$$

Proposition 8.19. *Let* $p \in \Pi$. *Then*

$$\lim_{n\to\infty} n\left[n\left(U_n(p;x) - U_n^{-1}(p;x)\right) - 2x(1-x)p''(x)\right] = 0.$$

Proposition 8.20. *Let* $p \in \Pi$. *Then*

$$\lim_{n\to\infty} n^2\left(\overline{\mathbb{B}}_n^{-1}(p;x) - B_n^{-1}(p;x)\right) = \frac{x(1-x)}{6}\left\{3p''(x) - (1-2x)p^{(3)}(x)\right\}.$$

Other results of this type can be obtained in a similar way.

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