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A CORRECTION ON TANGENTBOOST ALGORITHM

ONUR TOKA AND MERAL CETIN

Abstract. TangentBoost is a robust boosting algorithm. The method combines loss function and weak classifiers. In addition, TangentBoost gives penalties not only misclassification but also true classification margin in order to get more stable classifiers. Despite the fact that the method is good one in object tracking, propensity scores are obtained improperly in the algorithm. The problem causes mislabeling of observations in the statistical classification. In this paper, there is a correction proposal for TangentBoost algorithm. After the correction on the algorithm, there is a simulation study for the new algorithm. The results show that correction on the algorithm is useful for binary classification.

1. Introduction

Classification, in other words supervised learning, is a procedure that obtain a classifier based on a training dataset. The observed classifier determines which class the observation belongs to. High accuracy in the testing dataset means that the classifier is better one. Risk classification, cancer detection, object detection, outlier detection, image classification are some applied areas in classification methods. Over the last decade, many statistical methods have been applied including linear regression, logistic regression (LR), neural networks (NNet), Naive Bayes (NB), k-nearest neighbor (kNN), Support Vector Machine (SVM), boosting methods and other approaches [1, 2]. The methods are usually based on optimization problems comprised loss functions. While advanced methods minimize misclassification not only using loss functions but also using the distance between different classes' inputs such as SVM, boosting methods classify inputs according to sum of some weak classifiers [3].

Boosting is a general method to improve the performance of weak learners. Boosting algorithms are iteratively methods and the weak classifiers are obtained in each iterations. Then, combining weak classifiers is a way of determining the

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propensity scores and class labels at the end of the iteration steps [1]. Despite usefulness, boosting algorithms have some limitations. The common problem is unbounded growth with negative margin for boosting algorithm. Thus, outliers and contaminated part in training data can spoil the classifiers in boosting methods. Great advances have been achieved to make more robust boosting algorithms in the last decade [4, 5, 6, 7, 8].

In [9], loss functions are argued with regard to give unbounded penalty values. To solve this problem, they gave some important information between probability elicitation and bayes consistency and they formalized a new way to obtain bayes consistent loss function. After arguing that robust loss function should penalize both large positive and negative margin, they proposed a new loss function, Tangent loss, and a new boosting algorithm TangentBoost. Although the method is better one in object tracking, probabilities (p) assign class label improperly because of $p \in \left(-\frac{\pi}{2} + .5, \frac{\pi}{2} + .5\right)$ [10].

In this study, for TangentBoost algorithm, propensity score is redefined in order to get accurate weights and class labels properly. Section 2 reviewed binary classification, loss functions and concerned boosting methods in binary case. In Section 3, robust loss properties, Tangent loss function and the correction were given. In addition, importance of weights and class assign probabilities with the correction were showed. In Section 4, simulation results were given.

2. Boosting Algorithms in Binary Classification

Binary classification is one of the most encountered methods in applications. Spam mail detection, pattern characterization, diagnosis, digit recognition, signal recognition are some application phases of binary classification. The basic logic is to find classifier that can assign observations to two classifiers well according to inputs. Let consider g maps a inputs vector $x \in X$ to label $y \in \{\,-1, 1\}$. The classifier function $f : \mathrm{X} \to \mathbb{R}$ is the predictor of class label by the way of $g\left(x\right) = \operatorname{sign}\left[f\left(x\right)\right]$. Loss function is defined as below:

$$L\left(f\left(x\right), y\right) = L\left(f\left(x\right) y\right), \quad f\left(x\right) \in \mathbb{R}, \quad y \in \{-1, 1\} \tag{2.1}$$

The predictor is $g\left(x\right) = \operatorname{sign}\left[f\left(x\right)\right]$ and $f\left(x\right) > 0$, case assigns to 1 and -1 otherwise. Combining information $f\left(x\right)$ and y from the Equation (2.1), it is seen that $f\left(x\right) y < 0$ means misclassification and $f\left(x\right) y > 0$ means accurate classification. The quantity of $f\left(x\right)$ identifies the distance from the case to the classifier. Therefore, minimizing Equation (2.1) is affected not only misclassification but also large margin from the classifier. To get robust classifier, loss functions, which also give penalty to large positive margin, have been investigated [8, 9].

Especially in boosting methods, minimizing loss function value is an important task. The most common loss functions are exponential loss and logistic loss that are defined as Equation (2.2) and Equation (2.3):

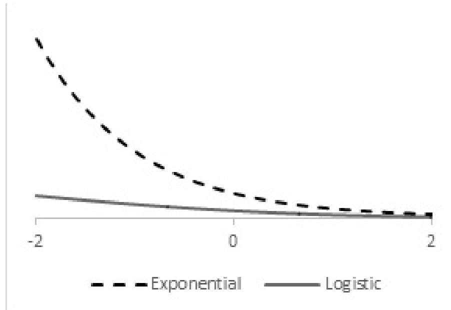

FIGURE 1. Exponential and Logistic Loss

$$L_{Exp}(y, f(x)) = \exp(-f(x)\,y) \tag{2.2}$$

$$L_{Log}(y, f(x)) = \log(1 + \exp(-f(x)\,y)) \tag{2.3}$$

Changing loss functions in the algorithms is a way of obtaining new boosting algorithm. The penalty values for misclassification are changed by using different loss functions. For instance, exponential loss increases penalty values very rapidly than logistic loss though exponential and logistic losses grow unbounded. Logistic loss is also unbounded but its increase is not as rapid as the exponential loss. In addition, exponential loss gives less penalty values than logistic loss in accurate classification, but both functions' penalty values for large positive loss value are zero. It is also examined in Figure 1. The mention differences cause different weighting for training data. Using loss functions, lots of boosting algorithm are proposed. AdaBoost is popular and the first algorithm that could adapt to the weak learners (See [11] for algorithm and the method). LogitBoost was proposed similarly. The main difference is that LogitBoost utilizes logistic loss to weight the data points, while AdaBoost utilizes exponential loss (See LogitBoost algorithm in [12]). On the other hand, unbounded increment of penalty value reveals the overfitting problem. Therefore, bounded loss functions and its boosting algorithms have been proposed in the few years [13, 14]. TangentBoost is an alternative loss function and the method has bounded loss function. In the next section, the algorithm and the correction on the algorithm are given.

3. TangentBoost and the Correction

Robust boosting algorithms obtain classifiers without being affected by outliers. In training data, some mislabeled (outliers) and contaminated observations may affect the classifier. It is usually pointed out that outliers may easily spoil classical boosting algorithms such as AdaBoost, RealBoost [15]. As a result, classifiers can be improper and their generalization ability may not be good. To make classifiers more stable, some researchers proposed robust boosting algorithms [13, 14, 16, 17, 18].

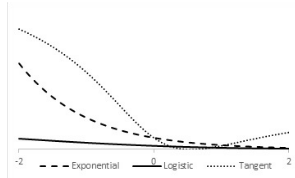

FIGURE 2. Exponential, Logistic and Tangent Loss

TangentBoost is also robust boosting algorithm that combines squared risk function and Tangent link function.

The idea behind TangentBoost algorithm is probability elicitation and conditional risk minimization [19]. The connection between risk minimization and probability elicitation has been studied in [9]. The results showed that if maximal reward function has equality with the formula $J(\eta) = J(1-\eta)$, the classifier f is invertible and has symmetry $f^{-1}(-v) = 1 - f^{-1}(v)$, then new link function and reward function are a way of obtaining a new loss function by using Equation (3.1):

$$\phi(v) = -J\left[f^{-1}(v)\right] - \left(1 - f^{-1}(v)\right) J^{'}\left[f^{-1}(v)\right] \tag{3.1}$$

After theoretical properties, from the tangent link $(f(\eta) = \tan(\eta - .5))$ and the risk function $C_\phi^* = 4\eta(1-\eta)$, tangent loss function is given in Equation (3.2)[9]:

$$\phi(v) = \left(2\tan^{-1}(v) - 1\right)^{-1} \tag{3.2}$$

Tangent loss function arranges more penalties to positive margin than the other loss functions. It is clear from the Figure 2, unlike classical loss functions; tangent loss function penalizes not only negative margin but also positive margin. Penalizing large positive margin limit the effect of observations which are very far from classifier though it is accurate classified. TangentBoost algorithm is adapted with the similar way of LogitBoost (See LogitBoost codes in [20] and [21]). However, probability of class label is not proper because of $p \in \left(-\frac{\pi}{2} + .5, \frac{\pi}{2} + .5\right)$ in TangentBoost algorithm [10]. To solve this problem, propensity scores are reduced to interval $[0, 1]$ by using formula $p = \frac{\tan^{-1}(f) - \tan^{-1}(-\infty)}{\tan^{-1}(\infty) - \tan^{-1}(-\infty)}$ instead of $p = \tan^{-1}(f) - .5$. TangentBoost algorithm with the correction is given as below [22].

In the algorithm, after initialization the values, weights and $z_i^{(m)}$are calculated by formula obtained Tangent loss function. In the second loop, reweighted least squares obtain the most important variable for the first iteration. Using the most important variable and its linear regression prediction, classifier function, weights, propensity scores are updated. The algorithm continues during the iterations. After the last iteration, the classifier function describes the class labels.

Probabilities for assigning class label, in another saying propensity scores, are limited to between zero and one with the correction. When the propensity score is around zero or one, it means class label of observation is clear and weight of observation is around zero. That is, if propensity score is enough to define class label, the weight starts to decline and concerning observation will not be very important in the next iteration. On the other hand, if propensity score is around 0.5, observation is near to classifier. As a result, the weights start to increase and the observation around the classifier will be more important in the next iteration. After defining best variable for each iteration via iteratively reweighted least squares, then it is easy to find classifiers for all iterations. At the end of the algorithm, sign of combining classifier or the propensity scores decide class labels. Additionally, TangentBoost is one of the alternative boosting method that produce propensity score like logistic regression. Separating propensity scores more than two labels is aimed to obtain multiclass label [23]. The method becomes comparable to logistic regression with the statistical correction on propensity scores. Furthermore, classifying observations will become more stable with the correction.

In summary, correction on TangentBoost can be good process to obtain classifier that not been affected by outliers in training data. In the simulation design, it will be seen how TangentBoost can obtain better classifier than classical most-known methods in the presence of outliers and mislabeled data.

Algorithm: TangentBoost Algorithm with the correction on p

Inputs: Training data set $D = \{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}$, where y is class label $\{-1, 1\}$for observations x and number M for weak learners.
Initial Values: Class label probabilities $\eta^1(x_i) = .5$ and the classifier $\hat{f}_1(x) = 0$
Loop 1. $m = 1, 2, \cdots, M$
Calculate the ${z_i}^{(m)}$ and weights for all observations given formula below:
For label $y = 1$, ${z_i}^{(m)} = -(\eta - 1)\left(1 + \tan^2(\eta - .5)\right)$
For label $y = -1$, ${z_i}^{(m)} = -\eta\left(1 + \tan^2(\eta - .5)\right)$
For the weights, ${w_i}^{(m)} = \eta^{(m)}(x_i)\left(1 - \eta^{(m)}(x_i)\right)$, ${w_i}^{(m)} = {w_i}^{(m)} / \sum {w_i}^{(m)}$
Loop 2.
Minimize LS problem below to select the most important variable with the given equation where $\langle q(x_i)\rangle_m = \sum_i {w_i}^{(m)} q(x_i)$ for each $k = 1, 2, \cdots, K$.
$a_{\phi_k} = \frac{\langle 1\rangle_w \langle \phi_k(x_i) z_i\rangle_w - \langle \phi_k(x_i)\rangle_w \langle z_i\rangle_w}{\langle 1\rangle_w \left\langle \phi_k^2(x_i)\right\rangle_w - \langle \phi_k(x_i)\rangle_w^2}$
$b_{\phi_k} = \frac{\left\langle \phi_k(x_i)^2\right\rangle_w \langle z_i\rangle_w - \langle \phi_k(x_i)\rangle_w \langle \phi_k(x_i) z_i\rangle_w}{\langle 1\rangle_w \left\langle \phi_k^2(x_i)\right\rangle_w - \langle \phi_k(x_i)\rangle_w^2}$
End of Loop 2.
Obtain important variable k^*given formula:
$k^* = \arg \min_k \sum_i w_i^{(m)} \left(z_i - a_{\phi_k}\phi_k(x_i) - b_{\phi_k}\right)^2$
Obtain classifier and also probability score for all observation
$\hat{f}^{(m+1)}(x_i) = \hat{f}^{(m)}(x_i) + \left(a_{\phi_k}\phi_k(x_i) + b_{\phi_k}\right)$
$\eta^{(m+1)}(x_i) = \frac{\tan^{-1}\left(\hat{f}^{(m)}(x_i)\right) - \tan^{-1}(-\infty)}{\tan^{-1}(\infty) - \tan^{-1}(-\infty)}$
End of Loop 1.

Define class label with the given formula below:
$h(x) = \mathrm{sgn}\left(\hat{f}^{(M)}(x_i)\right)$
End of Algorithm.

4. Simulation Study

There is a simulation study to compare TangentBoost and classical boosting algorithms in real datasets. There are three different datasets. The datasets are obtained from UCI Machine Learning Repository [24] and they are king gaming [25], qualitative bankruptcy [26] and credit approval datasets [27]. There is some basic information about datasets in Table 1.

Two of dataset's labels are completely separable from each other's. However, there is only one set, credit approval, which has a linearly inseparable data structure. These datasets were included to vary number of observations, number of variables and class label proportions.

Table 1. Some Information about Real Dataset Example

Dataset	# of Variable	# of Observation	Ratio of Class=1
QB	7	250	57.2 %
KG	37	3196	52.2 %
CA	15	690	50.4 %

QB: Qualitative Bankruptcy, KG: King Gaming, CA: Credit Approval

In Table 2, when training part is 70% and 80% of data, when number of iteration is 40, means of overall accuracy in both training and testing parts are given for 250 repetitions. TangentBoost algorithm had similar results in training and testing part for all datasets. There were not any dramatically decreasing from training to testing accuracy scores. On the other hand, while all other boosting algorithm gave impressive result for completely separable datasets, there were not any significant differences between classical algorithms and TangentBoost in testing accuracy scores. Moreover, there were dramatically decreasing all classical boosting algorithms' scores from training datasets to testing datasets while there was not any differences in TangentBoost algorithm. To summarize the results, TangentBoost will not useful in completely separable dataset without mislabeling while the method may be useful almost separable data. Accuracy results of classical boosting methods easily decreased in testing data when the training data are not completely separable. Logistic and exponential loss functions are incapable to preserve stability of general accuracy rate in CA testing data as seen from Table 2 .

To clarify the robustness of TangentBoost in the presence of mislabeled observations, different proportions of mislabeled observations were obtained on qualitative bankruptcy and credit approval datasets. In Table 3, when training part is 70%, when number of iteration is 40, means of overall accuracy in testing parts are given

Table 2. Mean of the Overall Accuracy in Real Datasets for TangentBoost and some Classical Boosting Methods

Dataset		TB	RB-Exp	GB-Exp	RB-Log	GB-Log
Training	% of Data					
QB	0.7	1.0000	1.0000	1.0000	1.0000	1.0000
	0.8	0.9995	1.0000	1.0000	1.0000	1.0000
KG	0.7	0.9386	0.9893	0.9904	0.9913	0.9911
	0.8	0.9386	0.9848	0.9857	0.9865	0.9863
CA	0.7	0.8693	0.9207	0.9429	0.9273	0.9393
	0.8	0.8697	0.9206	0.9432	0.9346	0.9473
Testing						
QB	0.3	0.9961	0.9922	0.9947	0.9947	0.9962
	0.2	0.9962	0.9940	0.9954	0.9952	0.9968
KG	0.3	0.9375	0.9844	0.9853	0.9861	0.9858
	0.2	0.9374	0.9807	0.9814	0.9824	0.9821
CA	0.3	0.8617	0.8671	0.8665	0.8613	0.8618
	0.2	0.8569	0.8620	0.8632	0.8641	0.8634

QB: Qualitative Bankruptcy, KG: King Gaming, CA: Credit Approval
TB: TangentBoost RB-Exp: RealBoost with exponential loss; GB-Exp: GentleBoost with exponential loss RB-Log: RealBoost with logistic loss; GB-Log: GentleBoost with logistic loss

for 250 repetitions. TangentBoost was better than the methods in the presence of mislabeled observations in testing part as seen in Table 3 and Figure 3.

Table 3. Mean of the Overall Accuracy in Real Datasets for TangentBoost and some Classical Boosting Methods in the presence of mislabeled data

Dataset		TB	RB-Exp	GB-Exp	RB-Log	GB-Log
Training	% of mislabeled					
QB	0.05	0.9934	0.9873	0.9909	0.9894	0.9925
QB	0.10	0.9925	0.9869	0.9893	0.9861	0.9897
QB	0.15	0.9898	0.9852	0.9850	0.9858	0.9867
CA	0.05	0.8617	0.8659	0.8662	0.8651	0.8593
CA	0.10	0.8609	0.8609	0.8585	0.8595	0.8586
CA	0.15	0.8583	0.8542	0.8504	0.8523	0.8523

QB: Qualitative Bankruptcy, CA: Credit Approval
TB: TangentBoost RB-Exp: RealBoost with exponential loss; GB-Exp: GentleBoost with exponential loss RB-Log: RealBoost with logistic loss; GB-Log: GentleBoost with logistic loss

To summarize simulation results, TangentBoost can be good robust procedure in the presence of outliers in training data. The method cannot been spoiled by contaminated part in training dataset. However, it is not as well as other classical methods in separable dataset as been expected. Adding mislabeled in separable data uncovered that TangentBoost is better than classical ones. Simulation on real data indicates that the algorithm is a useful method when training data set has mislabeled observations.

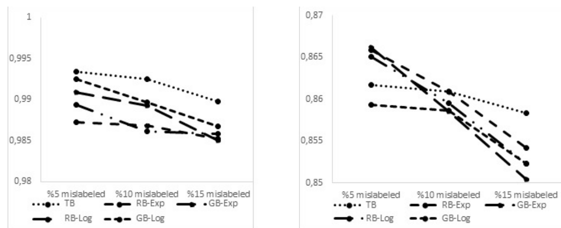

FIGURE 3. Accuracy scores of the methods according to mislabeled proportion (Left).Quality Bankruptcy testing data (Right).Credit Approval testing data

5. Results

In this study, TangentBoost algorithm is given with a correction. Outliers or contaminated part in training data may be problem in boosting algorithm. Especially, outliers in boosting algorithms can influence weak classifiers very easily. To overcome this problem, robust boosting algorithms are effective methods. TangentBoost with the correction is quite useful if there are outliers or contaminated part near the classifier in almost linearly separable data.

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ON LOCALLY UNIT REGULARITY CONDITIONS FOR ARBITRARY LEAVITT PATH ALGEBRAS

TUFAN ÖZDIN

Abstract. Let Γ be a graph, K be any field and S be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right L-module. In this paper, we give defination of the left locally unit regular ring. We show that (1) if S is locally unit regular, then L is locally unit regular, (2) if L is morphic and image projective then S is left morphic, (3) S is a directly finite ring then L is directly finite, (4) if S is an exchange ring then L is directly finite and if L is a direct finite ring then L is an exchange ring.

1. Introduction

Throughout this article Γ will denote a directed graph, K will denote an arbitrary field and the Leavitt path algebras (shortly LPAs) of Γ with coefficients in K will denoted $L := L_K(\Gamma)$.

LPAs can be regarded as the algebraic counterparts of the graph C^*-algebras, the descendants from the algebras investigated by Cuntz in [6]. LPAs can be viewed as a broad generalization of the algebras constructed by Leavitt in [11] to produce rings without the Invariant Basis Number property. LPAs associated to directed graphs were introduced in [4, 1]. These $L_K(\Gamma)$ are algebras associated to directed graphs and are the algebraic analogs of the Cuntz-Krieger graph C^*-algebras [15].

Let Γ be a graph and K a field. In [3], G. Abrams and K. M. Rangaswamy showed how definition of von Neumann regular ring (recall that a ring R is von Neumann regular if for every $a \in R$ there exists $b \in R$ such that $a = aba$) is extended to locally unit regular ring and in [3, Theorem 2] if Γ is arbitrary graph, $L_K(\Gamma)$ is locally unit regular if and only if Γ is acyclic. This article is organized as follows. In Section 2, we recall some preliminaries about LPAs which we need in the next section. In Section 3, for the ring S of endomorphism ring of $L_K(\Gamma)$ (viewed as a right $L_K(\Gamma)$-module), we prove that: (1) if S is locally unit regular,

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then L is locally unit regular, (2) if L is morphic and image projective then S is left morphic, (3) if S is a directly finite ring then L is directly finite, (4) if S is an exchange ring then L is directly finite and if L is a direct finite ring then L is an exchange ring.

2. Definitions and Preliminaries

We recall some graph-theoretic concepts, the definition and standard examples of LPAs.

Definition 1. *A (directed) graph $\Gamma = (V, E, r, s)$ consist of two set V and E (with no restriction on their cardinals) together with maps $r, s : E \to V$. The elements of V are called vertices and the elements of E edges. For $e \in E$, the vertices $s(e)$ and $r(e)$ are called the source and range of e. If $s^{-1}(v)$ is a finite set for every $v \in V$, then the graph is called row-finite. If V is finite and Γ is row finite, then E must necessarily be finite as well; in this case we say simply that Γ is finite.*

A vertex which emits (receives) no edges is called a sink (source). A vertex v is called an infinite emitter if $s^{-1}(v)$ is an infinite set. A vertex v is a bifurcation if $s^{-1}(v)$ has at least two elements. A path p in a graph Γ is a finite sequence of edges $p = e_1...e_n$ such that $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n-1$. In this case, $s(p) = s(e_1)$ and $r(p) = r(e_n)$ are the source and range of p, respectively, and n is the length of p. We view the elements of V as paths of length 0.

A path $p = e_1...e_n$ is said to be closed path based at v if $s(p) = v = r(p)$. If p is an closed path in Γ and $s(e_i) \neq s(e_i)$ for all $i \neq j$, then p is said to be a cycle. A cycle consisting of just one edge is called a loop. A graph which contains no cycles is called acyclic. A graph Γ is said to be no-exit if no vertex of any cycle is a bifurcation.

Definition 2. *(LPAs of Arbitrary Graph)*

For an arbitrary graph Γ and a field K, the Leavitt path K-algebra of Γ, denoted by $L_K(\Gamma)$, is the K-algebra generated by the set $V \cup E \cup \{e^* | e \in E\}$ with the following relations,

1. $v_i v_j = \delta_{v_i, v_j} v_i$ *for every* $v_i, v_j \in V$
2. $s(e)e = e = er(e)$ *for all* $e \in E$.
3. $r(e)e^* = e^* = e^* s(e)$ *for all* $e \in E$.
4. *(CK1)* $e^* f = \delta_{e,f} r(e)$ *for all* $e, f \in E$.
5. *(CK2)*$v = \sum_{\{e \in E, s(e) = v\}} ee^*$ *for every* $v \in V$ *that is neither a sink nor an infinite emitter.*

The first three relations are the path algebra relations. The last two are the so-called Cuntz-Krieger relations. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $p = e_1...e_n$ is a path in Γ, we write p^* for the element $e_n^*...e_1^*$ of $L_K(\Gamma)$. With this notation, the LPA $L_K(\Gamma)$ can be viewed as a $K-$vector space span of $\{pq^* \mid p, q \text{ are paths in } \Gamma\}$. (Recall that the elements of V are viewed as paths of length 0, so that this set includes elements of the form v with $v \in V$.)

If Γ is a finite graph, then $L_K(\Gamma)$ is unital with $\sum_{v\in V} v = 1_{L_K(\Gamma)}$; otherwise, $L_K(\Gamma)$ is a ring with a set of local units consisting of sums of distinct vertices of the graph.

Many well-known algebras can be realized as the LPAs of a graph. The most basic graph configuration is shown below (the isomorphism for can be found in [1]).

Example 1. *The ring of Laurent polynomials $K[x, x^{-1}]$ is the LPA of the graph given by a single loop graph.*

We will now outline some easily derivable basic facts about the endomorphism ring S of $L := L_K(\Gamma)$. Let Γ be any graph and K be any field. Denote by S the unital ring $End(L_L)$. Then we may identify L with subring of S, concretely, the following is a monomorphism of rings:

$$\begin{aligned} \phi : L &\to End(L_L) \\ x &\mapsto \lambda_x \end{aligned}$$

where $\lambda_x : L \to L$ is the left multiplication by x, i.e., for every $y \in L$, $\lambda_x(y) = xy$ which is a homomorphism of right $L-$module. The map ϕ is also a monomorphism because given a nonzero $x \in L$ there exists an idempotent $u \in L$ such that $xu = x$, hence $0 \neq x = \lambda_x(u)$.

3. Results

According to Abrams and Rangaswamy [3]:

• A (possibly nonunital) ring R is called a *ring with local units* if, for each finite subset S of R, there exists an idempotent e of R such that $S \subseteq eRe$;

• If R is a ring with local units then R is called *locally unit regular* if for each $a \in R$ there is an idempotent (a local unit) v and local inverses u, u' such that $uu' = v = u'u$, $va = a = av$ and $aua = a$ (see [3, Definition 6]).

Theorem 1. *Let Γ be an arbitrary graph, K be any field and S be the endomorphism ring of $L := L_K(\Gamma)$.*

(1) *If S is locally unit regular, then L is locally unit regular. Moreover L is regular.*
(2) *If L is locally unit regular, then vLv is locally unit regular for every non zero idempotent v of L.*

Proof. (1) Take $x \in L$. Since S is local unit regular, there exists an idempotent $e \in S$ such that $\lambda_x \in eSe$ and elements $f, g \in eSe$ such that $fg = e = gf$ and $\lambda_x f \lambda_x = \lambda_x$. Choose an idempotent $u \in L$ such that $x\lambda_{e(u)} = x = \lambda_{e(u)}x$ so $x \in \lambda_{e(u)} L \lambda_{e(u)}$. Note that,

$$\lambda_{f(u)}\lambda_{g(u)} = \lambda_{e(u)} = \lambda_{g(u)}\lambda_{f(u)}$$

and

$$\lambda_x = \lambda_x\lambda_{f(x)} = \lambda_x\lambda_{f(ux)} = \lambda_x\lambda_{f(u)}\lambda_x.$$

Since $f \in eSe$, there exists $h \in S$ such that $f = ehe$. Then $f(u) = e(u)h(u)e(u)$, so

$$\lambda_{f(u)} = \lambda_{e(u)h(u)e(u)} = \lambda_{e(u)}\lambda_{h(u)}\lambda_{e(u)}$$

and we get $\lambda_{f(u)} \in \lambda_{e(u)}L\lambda_{e(u)}$. Similarly $\lambda_{g(u)} \in \lambda_{e(u)}L\lambda_{e(u)}$. Hence L is locally unit regular.

(2) Take any $a \in vLv$. Since L is locally unit regular, there exist an idempotent e and local inverses u, u' such that $ea = a = ae$, $uu' = e = u'u$ and $aua = a$. As $ea = a$ and $av = a$ which imply $vea = va = a = ea$ and $eav = ea$ respectively, we get $ea = eav = vea$. Now $ea \in vLv$, which implies $e \in vLv$. Then $ve = e = ev$. Let $e^* = vev$, $h = vuv$ and $h' = veu'ev$. Note that

$$e^*e^* = (vev)(vev) = vevev = veevv = vev = e^* \in vLv$$
$$hh' = (vuv)(veu'ev) = vuveu'ev = vueu'ev = veuu'^*$$
$$h'h = (veu'ev)(vuv) = veu'evuv = veu'euv = veu'^*$$
$$aha = a(vuv)a = vauav = vav = a,$$

which imply vLv is locally unit regular. □

Definition 3. *A ring R is dependent if, for each $a, b \in R$, there are $s, t \in R$, not both zero, such that $sa + tb = 0$.*

Let Γ be an arbitrary graph, K be any field and S be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right L-module. If S is dependent so is L. In fact, suppose S is dependent and $a, b \in L$. Then there are elements $f, g \in S$, not both zero, such that $f\lambda_a + g\lambda_b = 0$. If u_1 and u_2 are local units in L satisfying $u_1a = a = au_1$ and $u_2b = b = bu_2$, then

$$f\lambda_a = f\lambda_{u_1a} = f\lambda_{u_1}\lambda_a = \lambda_{f(u_1)}\lambda_a$$

and

$$g\lambda_b = g\lambda_{u_2b} = g\lambda_{u_2}\lambda_b = \lambda_{g(u_2)}\lambda_b.$$

Now

$$\begin{aligned} 0 &= f\lambda_a + g\lambda_b \\ &= \lambda_{f(u_1)}\lambda_a + \lambda_{g(u_2)}\lambda_b, \end{aligned}$$

and hence L is dependent.

In the literature on von-Neumann regular rings, various conditions have been shown to characterize the subclass of unit regular rings. In [8, Theorem 6], Ehrlich showed that every unit regular ring R is dependent. In [10, Corollary 10], Henriksen shows that not all dependent regular rings are unit regular. The following observation gives one more such condition for dependent rings.

Theorem 2. *If $L_K(\Gamma)$ is locally unit regular, then it is dependent.*

Proof. Let $L_K(\Gamma)$ be locally unit regular and let some elements provide locally unit regular condition in the definition. Take $a, b \in L_K(\Gamma)$. If both a and b have local inverses in $L_K(\Gamma)$, then there exist u_1 and u_2 in $L_K(\Gamma)$ such that $u_1 a = v$ and $u_2 b = v$ for local unit v in $L_K(\Gamma)$. So, we get $sa + tb = 0$, where $s = u_1$ and $t = -u_2$. If one of the elements, say a, has no local inverse in $L_K(\Gamma)$, by definition of locally unit regularity, then we can write $aua = a \Rightarrow aua = va \Rightarrow (au - v)a = 0$. Now we get $au - v \neq 0$. Assume $au - v = 0$. So $au = v$, it is a contradiction. Then, for $s = (au - v) \neq 0$ and $t = 0$, which implies $sa + tb = 0$. $\square$

Definition 4. *Let R be a ring with local units. We call R left (right) locally unit regular ring if for each $a \in R$ there exist an idempotent $v \in R$ and left (right) local inverses u, u' such that $u'u = v$ $(uu' = v)$, $va = a$ $(av = a)$ and $aua = a$.*

Definition 5. *([12]) Let M be a right R-module, and let $S = End_R(M)$. Then M is called is a d-Rickart (or dual Rickart) module if the image in M of any single element of S is a direct summand of M. Clearly, R_R a d-Rickart module iff R is a regular ring.*

Definition 6. *Given paths $p, q \in \Gamma$, we say that q is an initial segment of p if $p = qm$ for some path $m \in \Gamma$. It is well known that, given non-zero paths pq^* and mn^* in $L_K(\Gamma)$, q is an initial segment of m if and only if $(pq^*)(mn^*) \neq 0$.*

Theorem 3. *Let Γ be a graph, K be any field and S be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right L-module. The following conditions are equivalent.*

1. *S is left locally unit regular.*
2. *S is regular and, for all paths $x, y \in L$, $Sx = Sy$ implies x is an initial segment of y.*
3. *L is dual-Rickart and, for all paths $x, y \in L$, $Sx = Sy$ implies x is an initial segment of y.*

Proof. (1) $\Rightarrow$ (2) Assume that S is left locally unit regular. Hence S is regular and L is left locally unit regular by Theorem 1. Let $x, y \in L$ be two paths. Then there exist an idempotent $v \in L$ and left local inverses $v_1, v_2 \in L$ such that $vy = y$, $v_2 v_1 = v$ and $y = yv_1 y$. If $Sx = Sy$, then $x = f(y)$ for some $f \in S$. Now $y = yv_1 y$ implies $f(y) = f(yv_1 y)$, and so $x = \underbrace{f(yv_1)}_{\in L} y$. Hence x is an initial segment of y.

(2) $\Rightarrow$ (3) This follows from [17, Corollary 3.2].

(3) $\Rightarrow$ (1) Assume that L is dual-Rickart. Then $f(L)$ is a direct summand of L, where $f \in S$. Let e be an idempotent in S with $f(L) = eL$. Let $x \in L$. Then there exists $y \in L$ such that $f(x) = e(y)$. Now

$$(ef)(x) = e(f(x)) = e(e(y)) = e(y) = f(x),$$

which implies $ef = f$. Let h be the left inverse of f and $g = fe$. Then $gh = e$ and $fhf = f$. $\square$

Definition 7. *([13]) An endomorphism α of a module M is called morphic if $M/M\alpha \cong \mathrm{Ker}(\alpha)$, equivalently there exists $\beta \in \mathrm{End}(M)$ such that $M\beta = \mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\beta) = M\alpha$ by [13, Lemma 1]. The module M is called a morphic module if every endomorphism is morphic. If R is a ring, an element a in R is called left morphic if right multiplication $\cdot a :_R R \to_R R$ is a morphic endomorphism, that is if $R/Ra \cong l(a)$. The ring itself is called a left morphic ring if every element is left morphic, that is if ${}_RR$ is a morphic module.*

Note that if S is dependent then $L_K(\Gamma)$ is morphic by [14, Corollary 3.5].

Theorem 4. *Let Γ be any graph and let K be any field. If $L_K(\Gamma)$ is left morphic and regular ring then $L_K(\Gamma)$ is left locally unit regular ring.*

Proof. Let $L_K(\Gamma) = L$ be left morphic and regular ring. Then each $a \in L$ is both regular and morphic. So, there exist an $x \in L$ such that $a = axa$ and for some $b \in L$, $La = ann(b)$ and $Lb = ann(a)$. Let $u = xax + b$. Then $a = aua$. To see that u is left local inverse, since L has local units, choose an idempotent $v \in L$ such that $va = a$. Then we get, $0 = va - a = va - axa = (v - ax)a$, so $v - ax \in ann(a) = Lb$ and there exists an element $y \in L$ such that $v - ax = yb$. We take $u' = a + y(v - ax)$. We show that $u'u = v$:

$$\begin{aligned} u'u &= (a + y(v - ax))(xax + b) \\ &= axax + ab + y(v - ax)xax + y(v - ax)b \\ &= ax + ab + yvxax - yxaxax + yvb - yxab \\ &= ax + yb = v \end{aligned}$$

Hence $L = L_K(\Gamma)$ is left locally regular ring. □

Theorem 5. *Let Γ be a graph, K be any field and S be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right L-module. If $L_K(\Gamma)$ is morphic and image projective then S is left morphic.*

Proof. Let $L := L_K(\Gamma)$ be morphic and image projective. Given any $\alpha \in S$, since L is morphic, we may choose an $\beta \in S$ such that, $L\alpha = ker(\beta)$ and $L\beta = ker(\alpha)$. Since $\alpha\beta = 0$, $S\alpha \subset ann^S(\beta)$. Conversely, if $\gamma \in ann^S(\beta)$ then $\gamma\beta = 0$ so $L\gamma \subset ker(\beta) = L\alpha$ and hence $\gamma \in S\alpha$ because L is image projective. Thus $S\alpha = ann^S(\beta)$. We may see $S\beta = ann^S(\alpha)$ in the same way. Hence S is left morphic. □

Definition 8. *([16, Definition 4.1]) If R is a ring with local units then R is called directly finite if for each $x, y \in R$ there is an idempotent u such that $xu = x = ux$ and $yu = y = uy$, we have that $xy = u$ implies $yx = u$.*

Theorem 6. *Let Γ be a graph, K be any field and S be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right L-module. If S is a directly finite ring then $L_K(\Gamma)$ is directly finite.*

Proof. Take any x, y in $L_K(\Gamma)$. Since S is a direct finite ring, there is an idempotent ε in S such that $\lambda_x\varepsilon = \lambda_x = \varepsilon\lambda_x$ and $\lambda_y\varepsilon = \lambda_y = \varepsilon\lambda_y$, we have that $\lambda_x\lambda_y = \varepsilon$ implies $\lambda_y\lambda_x = \varepsilon$. For an idempotent u with $xu = x = ux$ and $yu = y = uy$,

$$\lambda_x\lambda_y = \varepsilon \Rightarrow \lambda_x\lambda_y\lambda_x = \varepsilon\lambda_x \Rightarrow \lambda_x = \varepsilon\lambda_{uv} \Rightarrow \lambda_x = \lambda_{\varepsilon(u)}\lambda_x$$

$$\lambda_x = \varepsilon\lambda_x = \lambda_{\varepsilon(x)} = \lambda_{\varepsilon(xu)} = \lambda_{\varepsilon(x)}\lambda_{\varepsilon(u)} = \varepsilon\lambda_x\lambda_{\varepsilon(u)}$$

So, $\lambda_x\lambda_{\varepsilon(u)} = \lambda_x = \lambda_{\varepsilon(u)}\lambda_x$. Similarly $\lambda_y\lambda_{\varepsilon(u)} = \lambda_y = \lambda_{\varepsilon(u)}\lambda_y$. Assume that, $\lambda_x\lambda_y = \lambda_{\varepsilon(u)}$. We then see that $\lambda_y\lambda_x = \lambda_{\varepsilon(u)}$.

$$\lambda_y\lambda_x = \lambda_y\lambda_{\varepsilon(u)}\lambda_x = \lambda_y\lambda_x\lambda_{\varepsilon(u)} = \varepsilon\lambda_{\varepsilon(u)} = \lambda_{\varepsilon^2(u)} = \lambda_{\varepsilon(u)},$$

as desired. □

Ones hopes that if $L_K(\Gamma)$ is directly finite then $L_K(\Gamma)$ is locally unit regular but this is not true. Because $K[x, x^{-1}]$ is a commutative Leavitt path algebra (of the graph with one vertex and one loop) clearly directly finite. But it is not von Neumann regular ring.

Corollary 1. *Let Γ be a graph, K be any field and S be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right L-module. If S is a directly finite ring, then Γ is no exit.*

Proof. Let S be a directly finite ring. Then $L_K(\Gamma)$ is a directly finite ring. So, by [16, Proposition 4.3], Γ is no exit. □

Definition 9. *R is said to be a (left) exchange ring if for any direct decomposition $A = M \oplus N = \oplus_{i\in I} A_i$ of any left R-module A, where $R \cong M$ as left R-modules and I is a finite set, there always exist submodules B_i of A_i such that $A = M \oplus (\oplus_{i\in I} B_i)$.*

Theorem 7. *Let Γ be an infinite graph, K be any field and S be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right L-module. Then*

(1) *If S is an exchange ring then L is directly finite.*
(2) *If L is a direct finite ring then L is an exchange ring.*

Proof. (1) Let S be an exchange ring. Then, by [5, Proposition 2.10], $L_K(\Gamma)$ is an exchange ring. For every $x, y \in L$ and an idempotent $u \in L$ such that $xu = x = ux$ and $yu = y = uy$ we have that $xy = u$. We show that $yx = u$. Since L is an exchange ring, there exist $r, s \in L$ such that $u = rx = s + x - sx$. So, $u = rx \Rightarrow uy = rxy \Rightarrow y = ru \Rightarrow yx = rux = rx = u$, as desired.

(2) Let L be a direct finite ring. For any $x, y \in L$ and an idempotent $u \in L$ such that $xu = x = ux$ and $yu = y = uy$ we have that $xy = u$ implies $yx = u$. We show that L is an exchange ring. For any $x \in L$ taking $r = y$ and $s = u$, we get $u = rx = s + x - sx$. So, L is an exchange ring. □

Corollary 2. *Let Γ be infinite graph, K be any field and S be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right L-module. Then the following conditions are equivalent.*

(1) *S is an exchange ring.*
(2) *$L_K(\Gamma)$ is an exchange ring.*
(3) *$L_K(\Gamma)$ is a directly finite ring.*
(4) *E is no exit*

Proof. (1) $\Leftrightarrow$ (2) This is [5, Proposition 2.10].
(2) $\Leftrightarrow$ (3) This follows from Theorem 7 (1) and Theorem 7 (2).
(3) $\Leftrightarrow$ (4) This is [16, Teorem 4.12]. $\square$

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FIRST AND SECOND ACCELERATION POLES IN LORENTZIAN HOMOTHETIC MOTIONS

HASAN ES

Abstract. In this paper, using matrix methods, we obtained rotation pole in one-parameter motion on the Lorentzian plane homothetic motions and pole orbits, accelerations and combinations of accelerations, first and second in acceleration poles. Moreover, some new theorems are given.

1. Introduction

In Lorentzian plane, a general planar motion as given by

$$\begin{aligned} y_1 &= x\cosh\varphi + y\sinh\varphi + a \\ y_2 &= x\sinh\varphi + y\cosh\varphi + b \end{aligned} \tag{1.1}$$

If θ, a and b are given by the functions of time parameter t, then this motions is called as one parameter motion [2] . One parameter planar motion given by (1.1) can be written in the form

$$\begin{pmatrix} Y \\ 1 \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

or

$$Y = AX + C, \quad Y = [y_1\ y_2]^T, \quad X = [x\ y]^T, \quad C = [a\ b]^T \tag{1.2}$$

where $A \in SO(2)$, and Y and X are the position vectors of the same point B, respectively, for the fixed and moving systems, and C is the translation vector [2]. By taking the derivatives with respect to t in (1.2), we get

$$\dot{Y} = \dot{A}X + A\dot{X} + \dot{C} \tag{1.3}$$

$$V_a = V_f + V_r \tag{1.4}$$

where the velocities $V_a = \dot{Y}, V_f = \dot{A}X + \dot{C}, V_r = A\dot{X}$ are called absolute, sliding, and relative velocities of the points B, respectively [1]. the solution of the equation $V_f = 0$ gives us the pole points on the moving plane. The locus of these points is

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called the moving pole curve, and correspondingly the locus of pole points on the fixed plane is called the fixed pole curve [1]. by taking the derivatives with respect to t in (1.3),we get

$$\ddot{Y} = \ddot{A}X + 2\dot{A}\dot{X} + A\ddot{X} + \ddot{C} \tag{1.5}$$

$$b_a = b_r + b_c + b_f, \tag{1.6}$$

where the velocities

$$b_a = \ddot{Y}, \tag{1.7}$$

$$b_f = \ddot{A}X + \ddot{C}, \tag{1.8}$$

$$b_r = A\ddot{X}, \tag{1.9}$$

$$b_c = 2\dot{A}\dot{X}, \tag{1.10}$$

are called absolute acceleration, sliding acceleration, relative acceleration and Coriolis accelerations, respectively [1]. The solution of the equation

$$\ddot{A}X + \ddot{C} = 0 \tag{1.11}$$

gives the acceleration pole of the motion [1]

2. HOMOTHETIC MOTION IN LORENTZIAN PLANE

Definition 2.1. The transformation given by the matrix

$$F = \begin{pmatrix} hA & C \\ 0 & 1 \end{pmatrix}$$

is called Homothetic motion in L^2 here $h = hI_2$ is a scalar matrix, $A \in SO(2)$ and $C \in \mathbb{R}^2_1$ [1].

Definition 2.2. Let $J \subset \mathbb{R}$ be an open interval let $O \in \mathbb{J}$. The transformation $F(t) : L^2 \longrightarrow \mathbb{L}^2$ given by

$$F(t) = \begin{pmatrix} h(t)A & C(t) \\ 0 & 1 \end{pmatrix}$$

is called one-parameter homothetic motion in L^2,where the function $h : J \longrightarrow \mathbb{R}$,the matrix $A \in SO(2)$ and the 2×1 type matrix C are differentiable with respect to [1].Since h is scalar we have $B^{-1} = h^{-1}A^{-1} = \frac{1}{h}A^T$ for $X \in L^2$, the geometric plane of the points is a curve in L^2. We will denote this curve by

$$Y(t) = B(t)X(t) + C(t) \tag{2.1}$$

differentiating with respect to t we obtain

$$\frac{dY}{dt} = \frac{dB}{dt}X + B\frac{dX}{dt} + \frac{dC}{dt}. \tag{2.2}$$

Definition 2.3. Equation of the general motion in L^2

$$Y(t) = B(t)X(t) + C(t) \tag{2.3}$$

where $A = A(t) \in SO(2)$ and $C = C(t) \in \mathbb{R}^2_1$ [1].Differentiating this equation with respect to t we have

$$\frac{dY}{dt} = \frac{dB}{dt}X + B\frac{dX}{dt} + \frac{dC}{dt}. \tag{2.4}$$

Here $V_a = \frac{dY}{dt}$, $V_r = B\frac{dX}{dt}$ and $V_f = \frac{dB}{dt}X + \frac{dC}{dt}$ are called absolute velocity, relative velocity and sliding velocity of the motion, respectively[3]. We denote motions in L^2 by $\frac{L}{\acute{L}}$ where $\acute{L}$ is fixed plane and L is the moving plane with respect to $\acute{L}$. If the matrix A and C are the functions of the parameter $t \in \mathbb{R}$ this motion is called a one parameter motion and denoted by $B_1 = \frac{L}{\acute{L}}$ [1].

Definition 2.4. The velocity vector of the point X with respect to the Lorentzian plane L (moving space) i.e. the vectorial velocity of X while it is drawing its orbit in L is called relative velocity of the point X and denoted by V_r [1].

Definition 2.5. The velocity vector of the point X with respect to the fixed plane $\acute{L}$ is called the absolute velocity of X and denoted by V_a. Thus we obtain the relation

$$V_a = V_f + V_r$$

If X is a fixed point in the moving plane L , since $V_r = 0$, then we have $V_a = V_f$. The quality (**??**) is said to be the velocity law the motion $B_1 = \frac{L}{\acute{L}}$ [1].

3. POLES OF ROTATING AND ORBIT

The point in which the sliding velocity V_f at each moment t of a fixed point X in L in the one-parameter homothetic motion $B_1 = \frac{L}{\acute{L}}$ are fixed points in moving and fixed plane. These points are called the pole points of the motion.

Theorem 3.1. *In a motion $B_1 = \frac{L}{\acute{L}}$ whose angular velocity is non zero, there exists a unique point which is fixed in both planes at every moment t.*

Proof. Since the point $X \in L$ is fixed in L then $V_r = 0$ and since X is also fixed in $\acute{L}$ then $V_f = 0$. Hence for this type of points if $V_f = 0$ then

$$\dot{B}X + \dot{C} = 0 \tag{3.1}$$

and

$$X = -\dot{B}^{-1}\dot{C} \tag{3.2}$$

Indeed,since

$$B = \begin{pmatrix} h\cosh\varphi & h\sinh\varphi \\ h\sinh\varphi & h\cosh\varphi \end{pmatrix}$$

and

$$\dot{B} = \begin{pmatrix} \dot{h}\cosh\varphi + h\dot{\varphi}\sinh\varphi & \dot{h}\sinh\varphi + h\dot{\varphi}\cosh\varphi \\ \dot{h}\sinh\varphi + h\dot{\varphi}\cosh\varphi & \dot{h}\cosh\varphi + h\dot{\varphi}\sinh\varphi \end{pmatrix}$$

then

$$C = [a\ b]^T, \tag{3.3}$$

implies that

$$\dot{C} = [\dot{a}\ \dot{b}]^T \tag{3.4}$$

and

$$det\dot{B} = \dot{h}^2 - h^2\dot{\varphi}^2 \neq 0. \tag{3.5}$$

Thus $\dot{B}$ is regular and

$$\dot{B}^{-1} = \frac{1}{\dot{h}^2 - h^2\dot{\varphi}^2} \begin{pmatrix} \dot{h}\cosh\varphi + h\dot{\varphi}\sinh\varphi & -(\dot{h}\sinh\varphi + h\dot{\varphi}\cosh\varphi) \\ -(\dot{h}\sinh\varphi + h\dot{\varphi}\cosh\varphi) & \dot{h}\cosh\varphi + h\dot{\varphi}\sinh\varphi \end{pmatrix}$$

Hence there exists a unique solution X of the equation $V_f = 0$. This point X is called pole point in moving plane. For this reason (3.2) leads to

$$X = -\dot{B}^{-1}\dot{C} \tag{3.6}$$

$$P = X = \frac{1}{h^2\dot{\varphi}^2 - \dot{h}^2} \begin{pmatrix} \dot{a}(\dot{h}\cosh\varphi + h\dot{\varphi}\sinh\varphi) - \dot{b}(\dot{h}\sinh\varphi + h\dot{\varphi}\cosh\varphi) \\ -\dot{a}(\dot{h}\sinh\varphi + h\dot{\varphi}\cosh\varphi) + \dot{b}(\dot{h}\cosh\varphi + h\dot{\varphi}\sinh\varphi) \end{pmatrix}$$

$$P = \frac{1}{M} \begin{pmatrix} (\dot{a}\dot{h} - \dot{b}h\dot{\varphi})\cosh\varphi + (\dot{a}h\dot{\varphi} - \dot{b}\dot{h})\sinh\varphi \\ (-\dot{a}h\dot{\varphi} + \dot{b}\dot{h})\cosh\varphi) + (-\dot{a}\dot{h} + \dot{b}h\dot{\varphi})\sinh\varphi \end{pmatrix}$$

where $h^2\dot{\varphi}^2 - \dot{h}^2 = M$ and the pole point in the fixed plane is

$$\acute{P} = BP + C$$

setting these values in their planes and calculating we have

$$Y = \acute{P} = \frac{1}{M} \begin{pmatrix} h\dot{h}\dot{a} - h^2\dot{b}\dot{\varphi} \\ h\dot{h}\dot{b} - h^2\dot{a}\dot{\varphi} \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

or as a vector

$$Y = \acute{P} = (\frac{1}{M}(h\dot{h}\dot{a} - h^2\dot{b}\dot{\varphi}) + a, \frac{1}{M}(h\dot{h}\dot{b} - h^2\dot{a}\dot{\varphi}) + b) \tag{3.7}$$

Here we assume that $\dot{\varphi(t)} \neq 0$ for all t. That is, angular velocity is not zero. In this case there exists a unique pole points in each of the moving and fixed planes of each moment t.

Corollary 1. *If $\varphi(t) = t$,then we obtain*

$$X = P = \frac{1}{h^2 - \dot{h}^2} \begin{pmatrix} (\dot{a}\dot{h} - \dot{b}h)\cosh\varphi + (\dot{a}h - \dot{b}\dot{h})\sinh\varphi) \\ (-\dot{a}h + \dot{b}\dot{h})\cosh\varphi) + (-\dot{a}\dot{h} + \dot{b}h)sinh\varphi) \end{pmatrix}$$

Corollary 2. *If $\varphi(t) = t$ and $h(t) = 1$,then we obtain*

$$X = P = \begin{pmatrix} \dot{a}\sinh\varphi - \dot{b}\cosh\varphi) \\ -\dot{a}\cosh\varphi + \dot{b}\sinh\varphi) \end{pmatrix}$$

Corollary 3. *If $\varphi(t) = t$,then we obtain*

$$\acute{P} = (\frac{1}{h^2 - \dot{h}^2}(h\dot{h}\dot{a} - h^2\dot{b}\dot{\varphi}) + a, \frac{1}{h^2 - \dot{h}^2}(h\dot{h}\dot{b} - h^2\dot{a}\dot{\varphi}) + b) \tag{3.8}$$

Corollary 4. *If $\varphi(t) = t$ and $h(t) = 1$, then we obtain*

$$\acute{P} = (-\dot{b} + a, -\dot{a} + b) \tag{3.9}$$

Definition 3.2. The point $P = (p_1, p_2)$ is called the instantaneous rotation center or the pole at moment t of the one parameter Euclidean motion $B_1 = \frac{L}{\acute{L}}$ [2]

Theorem 3.3. *The following relation exists between the pole ray from the pole P to the point X, and the sliding velocity vector V_f at each moment t.*

$$h < V_f, \acute{P}Y >= \dot{h}\|\acute{P}Y\| \tag{3.10}$$

Proof. The pole point in the moving plane

$$Y = BX + C. \tag{3.11}$$

implies that

$$X = B^{-1}(Y - C) \tag{3.12}$$

$$V_f = \dot{B}X + \dot{C} \tag{3.13}$$

and

$$\dot{B}X + \dot{C} = 0, \tag{3.14}$$

Leads to

$$X = P = -\dot{B}^{-1}\dot{C}, \tag{3.15}$$

Now Let's find pole points in the fixed plane. Then we have from equation $Y = BX + C$

$$Y = BX + C, \tag{3.16}$$

$$Y = \acute{P} = B(-\dot{B}^{-1}\dot{C}) + C, \tag{3.17}$$

Hence, we get

$$\acute{P} - C = -B\dot{B}^{-1}\dot{C}, \tag{3.18}$$

$$\dot{C} = -\dot{B}B^{-1}(\acute{P} - C). \tag{3.19}$$

If we substitute this values in the equation $V_f = \dot{B}X + \dot{C}$, we have $V_f = \dot{B}B^{-1}\acute{P}Y$. Now let us calculate the value of $\dot{B}B^{-1}\acute{P}Y$ here since $\acute{P}Y = (y_1 - p_1, y_2 - p_2)$ then

$$V_f = (\frac{\dot{h}}{h}(y_1 - p_1) - \dot{\varphi}(y_2 - p_2), \dot{\varphi}(y_1 - p_1) + \frac{\dot{h}}{h}(y_2 - p_2)), \tag{3.20}$$

hence we obtain

$$< V_f, \acute{P}Y >= \frac{\dot{h}}{h}[(y_1 - p_1)^2 - (y_2 - p_2)^2], \tag{3.21}$$

$$< V_f, \acute{P}Y >= \frac{\dot{h}}{h}\|\acute{P}Y\|^2, \tag{3.22}$$

on the other hand we know that

$$h < V_f, \acute{P}Y >= \dot{h}\|\acute{P}Y\|^2 \tag{3.23}$$

Corollary 5. *The pole ray from the pole P to the point X, when the scalar matrix h is constant, is perpendicular to the sliding velocity vector V_f at each instant moment t.*

Corollary 6. *There is a relation among the pole ray from the pole P to the point X, the sliding velocity vector V_f, and angular velocity $\dot{\varphi(t)} \neq 0$ at each moment t.*

$$h(t) = exp\left(\int \frac{< V_f, \acute{P}Y >}{\|\acute{P}Y\|}dt\right). \tag{3.24}$$

Theorem 3.4. *The length of the sliding velocity vector V_f is*

$$\|V_f\| = \sqrt{|(\left(\frac{\dot{h}}{h}\right)^2 - \dot{\varphi}^2)|}\|P'Y\| \tag{3.25}$$

Proof.

$$V_f = (\frac{\dot{h}}{h}(y_1 - p_1) + \dot{\varphi}(y_2 - p_2), \dot{\varphi}(y_1 - p_1) + \frac{\dot{h}}{h}(y_2 - p_2)), \tag{3.26}$$

hence

$$\|V_f\| = \sqrt{|(\left(\frac{\dot{h}}{h}\right)^2 - \dot{\varphi}^2)|}\|P'Y\|. \tag{3.27}$$

Corollary 7. *If the scalar matrix is h is constant, then the length of the sliding velocity vector is*

$$\|V_f\| = |\dot{\varphi}|\|P'Y\| \tag{3.28}$$

Corollary 8. *There is a relation among the pole ray from the pole P to the point X, the sliding velocity vector V_f, and angular velocity $\dot{\varphi(t)} \neq 0$ at each moment t.*

$$h(t) = exp\left(\int \sqrt{|(\left(\frac{\|V_f\|}{|\|P'Y\|}\right)^2 + \dot{\varphi}^2)|}dt\right). \tag{3.29}$$

Definition 3.5. In Lorentzian motion $B_1 = \frac{L}{\acute{L}}$, the geometric place of the pole points P in the moving plane L is called the moving pole curve of the motion $B_1 = \frac{L}{\acute{L}}$ and is denoted by (P). the geometric place of the pole points P in the fixed plane $\acute{L}$ is called fixed and is denoted by $\acute{P}$ [2] .

Theorem 3.6. *The velocity on the curve (P) and $(\acute{P})$ of every moment t of the rotating pol P which draws the pole curves in the fixed and moving planes are equal to each other. In other words, two curves are always tangent to each other* [2] .

Proof. The velocity of the point $X \in L$ while drawing the curve (P) is V_r and the velocity of this point while drawing the curve $(\acute{P})$ is V_a. Since $V_f = 0$ then $V_a = V_r$.

Theorem 3.7. *If two curves α and $\acute{\alpha}$ are tangent to each other of each moment t and if length of the ways ds and ds' of the point drawing these two curves at moment dt on these curves are the same then α and $\acute{\alpha}$ are said to be revolving by sliding on each other. Herehis the coefficient of rolling* [2].

Theorem 3.8. *In the one parameter planer Lorentzian motion $B_1 = \frac{L}{\acute{L}}$ the moving pole curve (P) of the plane L revolves by sliding on the fixed pole curve $(\acute{P})$ of the plane $\acute{L}$* [1] .

Proof. Acording to the definition of ray element of a curve ray of (P) is $ds = \|V_r\|$ and those of (P) is $ds' = \|V_a\|$ .Since for (P) and $(\acute{P})$, $V_a = V_r$ then $ds = hds'$. According to this theorem we way define a Lorentzian motion without mentioning the time. A Lorentzian motion $B_1 = \frac{L}{\acute{L}}$ is obtained by a moving pol curve (P) of L revolving without sliding on a fixed pol curve $(\acute{P})$.

Definition 3.9. Absolute acceleration vector of the point X with respect to the fixed Lorentzian plane $\acute{L}$ is V_a. This vector is denoted by b_a. Since $V_a = \dot{Y}$ then $b_a = \dot{V} = \ddot{Y}$ [2].

Definition 3.10. Let X be a fixed point the moving Lorentzian plane L. The acceleration vector of the point X with respect to the fixed Lorentzian plane $\acute{L}$ is called as sliding acceleration vector and denoted by b_f. Since in the acceleration of the sliding acceleration X is a fixed point of E,then $b_f = \dot{V}_f = \ddot{B}X + \ddot{C}$ [2].

4. ACCELERATIONS AND UNION OF ACCELERATIONS

Assume that the Minkowski motion $B_1 = \frac{L}{\acute{L}}$ of the moving Lorentzian plane L with respect to the fixed Lorentzian plane $\acute{L}$ exists. In this motion, let us consider a point X moving with respect to the plane L,and thus moving respect to the plane $\acute{L}$. We had obtained the velocity formulas concerning the motion of X, now we will obtain the acceleration formulas the acceleration of the point X.

Definition 4.1. The vector $b_r = \dot{V}_r = \ddot{B}X$ which is obtained by differentiating the relative velocity vector $V_r = B\dot{X}$ of the point X with respect to the moving plane L is called the relative acceleration vector of X in L and denote by b_r.Since when taking the derivative X is considered as a moving point in L,the matrix A is taken as constant [2].

Theorem 4.2. *Let X be a point in the moving Lorentzian plane which moves with respect to a parameter t. Hence we have that*

Theorem 4.3.

$$b_a = b_f + b_c + b_r \tag{4.1}$$

Here $b_c = 2\dot{B}\dot{X}$ is called Corilois acceleration [1].

Corollary 9. *If a point $X \in L$ is constant,then the sliding acceleration of the point X is equal to the absolute acceleration of X.*

Proof. Note that

$$V_a = \dot{B}X + B\dot{X} + \dot{C} \tag{4.2}$$

differentiating the both sides we have

$$\dot{V_a} = \ddot{B}X + 2\dot{B}\dot{X} + B\ddot{X} + \dot{C} \tag{4.3}$$

since the point X is constant its derivatives zero. Hence

$$\dot{V_a} = \ddot{B}X + \ddot{C} = b_f. \tag{4.4}$$

Theorem 4.4. *We have the following relation between the Coriolis acceleration vector b_c and relative velocity vector V_r.*

$$< b_c, V_r >= 2h\dot{h}(\dot{x_1}^2 - \dot{x_2}^2) \tag{4.5}$$

Proof. Since $b_c = 2\dot{B}\dot{X} =$, $V_r = B\dot{X}$. Then

$$< b_c, V_r >= 2h\dot{h}(\dot{x_1}^2 - \dot{x_2}^2) \tag{4.6}$$

Corollary 10. *If h is a constant,then Coriolis acceleration b_c is perpendicular to the relative velocity vector V_r at each instant moment t.*

5. FIRST AND SECOND ACCELERATION POLES

The solution of the equation $\dot{V_f} = 0$ gives the first order acceleration pole. $V_f = \ddot{B}X + \ddot{C} = 0$ implies $X = -\ddot{B}^{-1}\ddot{C}$. Now calculating the matrices $-\ddot{B}^{-1}$ and $\ddot{C}$ and setting these in $X = P_1 = -\ddot{B}^{-1}\ddot{C}$ we obtain

$$X = P_1 = \frac{-1}{k}\begin{pmatrix} \ddot{a}(m\cosh\varphi + n\sinh\varphi) - \ddot{b}(m\sinh\varphi + n\cosh\varphi) \\ -\ddot{a}(m\sinh\varphi + n\cosh\varphi) + \ddot{b}(m\cosh\varphi + n\sinh\varphi) \end{pmatrix}$$

Let $k = (\ddot{h} + h\dot{\varphi}^2)^2 - (2\dot{h}\dot{\varphi} + h\ddot{\varphi})^2$, $k \neq 0$, $m = \ddot{h} + h\dot{\varphi}^2$, $n = 2\dot{h}\dot{\varphi} + h\ddot{\varphi}$. Here P_1 is called first order pole curve in the moving plane. Denoting the pole curve in the fixed plane by $\acute{P_1}$ we get

$$\acute{P_1} = BP_1 + C \tag{5.1}$$

Hence

$$\acute{P_1} = (\frac{1}{k}(-\ddot{a}hm + \ddot{b}hn) + a, \frac{1}{k}(\ddot{a}hn - \ddot{b}hm) + b) \tag{5.2}$$

Corollary 11. *If $\varphi(t)=t$,then we obtain*

$$X=P_1=\frac{-1}{(\ddot{h}+h)^2-4(\dot{h})^2}\begin{pmatrix}(\ddot{a}\ddot{h}-2\ddot{b}\dot{h}+\ddot{a}h)\cosh\varphi-(\ddot{b}\ddot{h}-2\ddot{a}\dot{h}+\ddot{b}h)\sinh\varphi)\\(\ddot{b}\ddot{h}-2\ddot{a}\dot{h}+\ddot{b}h)\cosh\varphi)-(\ddot{a}\ddot{h}-2\ddot{b}\dot{h}+\ddot{a}h)sinh\varphi)\end{pmatrix}$$

Corollary 12. *If $\varphi(t)=t$ and $h(t)=1$, then we obtain*

$$P_1=(-\ddot{a}\cosh\varphi+\ddot{b}\sinh\varphi,-\ddot{b}\cosh\varphi+\ddot{a}\sinh\varphi) \tag{5.3}$$

Corollary 13. *If $\varphi(t)=t$, then we obtain*

$$\acute{P}_1=\frac{-1}{(\ddot{h}+h)^2-4(\dot{h})^2}(-\ddot{a}h(\ddot{h}+h)+\ddot{b}h(2\dot{h}),\ddot{a}h(2\ddot{h})-\ddot{b}h(\ddot{h}+h))+(a,b) \tag{5.4}$$

Corollary 14. *If $\varphi(t)=t$ and $h(t)=1$, then we obtain*

$$\acute{P}_1=(-\ddot{a}+a,-\ddot{b}+b) \tag{5.5}$$

The solution of the equation $\ddot{V}_f=0$ gives the second order acceleration pole. $\ddot{V}_f=\dddot{B}X+\dddot{C}=0$ implies $X=-\dddot{B}^{-1}\dddot{C}$. Now calculating the matrices $\dddot{B}^{-1}$ and $\dddot{C}$ and setting these in $X=-\dddot{B}^{-1}\dddot{C}$ we get

$$X=P_2=\frac{-1}{A^2-B^2}\begin{pmatrix}\ddddot{a}(A\cosh\varphi+B\sinh\varphi)-\dddot{b}(A\sinh\varphi+B\cosh\varphi)\\-\dddot{a}(A\sinh\varphi+B\cosh\varphi)+\dddot{b}(A\cosh\varphi+B\sinh\varphi)\end{pmatrix}$$

The pole curve in the fixed plane is obtained as

$$\acute{P}_2=(\frac{-1}{A^2-B^2}(\ddddot{a}hA-\dddot{b}hB)+a,\frac{-1}{A^2-B^2}(-\ddddot{a}hB+\dddot{b}hA)+b) \tag{5.6}$$

Let us

$$A=(3h\dot{\varphi}\ddot{\varphi}+3\dot{h}\dot{\varphi}^2+\dddot{h}),B=(h\dot{\varphi}^3+3\dot{h}\ddot{\varphi}+h\dddot{\varphi}+3\ddot{h}\dot{\varphi}) \tag{5.7}$$

Corollary 15. *If $\varphi(t)=t$, then we obtain*

$$X=P_2=\frac{-1}{T}\begin{pmatrix}(\ddddot{a}\dddot{h}-3\dddot{b}\ddot{h}+3\ddddot{a}\dot{h}-\dddot{b}h)\cosh\varphi+(-\dddot{b}\dddot{h}+3\ddddot{a}\ddot{h}-3\dddot{b}\dot{h}+\ddddot{a}h)\sinh\varphi\\(-\ddddot{a}\dddot{h}+3\dddot{b}\ddot{h}-3\ddddot{a}\dot{h}+\dddot{b}h)\sinh\varphi+(\dddot{b}\dddot{h}-3\ddddot{a}\ddot{h}+3\dddot{b}\dot{h}-\ddddot{a}h)\cosh\varphi)\end{pmatrix}$$

where $T=(3\dot{h}+\dddot{h})^2-(h+3\ddot{h})^2$.

Corollary 16. *If $\varphi(t)=t$ and $h(t)=1$, then we obtain*

$$P_2=(-\dddot{b}\cosh\varphi+\ddddot{a}\sinh\varphi,\dddot{b}\sinh\varphi-\ddddot{a}\cosh\varphi) \tag{5.8}$$

Corollary 17. *If $\varphi(t)=t$, then we obtain*

$$\acute{P}_2=(\frac{-1}{T}(\ddddot{a}h(3\dot{h}+\dddot{h})-\dddot{b}h(h+3\ddot{h}),-\ddddot{a}h(h+3\ddot{h})+\dddot{b}h(3\dot{h}+\dddot{h}))+(a,b) \tag{5.9}$$

where $T=(3\dot{h}+\dddot{h})^2-(h+3\ddot{h})^2$.

Corollary 18. *If $\varphi(t)=t$ and $h(t)=1$, then we obtain*

$$\acute{P}_2=(-\dddot{b}+a,-\ddddot{a}+b) \tag{5.10}$$

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(WEAKLY) $n-$NIL CLEANNESS OF THE RING $\mathbb{Z}_m$

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Abstract. Let R be an associative ring with identity. For a positive integer $n \geqslant 2$, an element $a \in R$ is called $n-$potent if $a^n = a$. We define R to be (weakly) $n-$nil clean if every element in R can be written as a sum (a sum or a difference) of a nilpotent and an $n-$potent element in R. This concept is actually a generalization of weakly nil clean rings introduced by Danchev and McGovern, [6]. In this paper, we completely determine all $n, m \in \mathbb{N}$ such that the ring of integers modulo m, $\mathbb{Z}_m$ is (weakly) $n-$nil clean.

1. Introduction

Let R be an associative ring with identity. Throughout this text, the notations $U(R)$, $J(R)$, $Id(R)$ and $N(R)$ will stand for the set of units, the Jacobson radical, the set of idempotents and the set of nilpotents of R, respectively. Following [14], we define an element r of a ring R to be clean if there is an idempotent $e \in R$ and a unit $u \in R$ such that $r = u + e$. A clean ring is defined to be one in which every element is clean. Similarly, an element r in a ring R is said to be nil clean if $r = e + b$ for some idempotent $e \in R$ and a nilpotent element $b \in R$. A ring R is nil clean if each element of R is nil clean. In [2], Breaz, Danchev and Zhou defined a ring R to be weakly nil clean if each element $r \in R$ can be written as $r = b + e$ or $r = b - e$ for $b \in N(R)$ and $e \in Id(R)$. We refer the reader to [8, 1, 3, 5, 7, 4, 2] for a survey on nil clean and weakly nil clean rings.

For $a \in R$ and a positive integer $n \geqslant 2$, we say that a is $n-$potent if $a^n = a$. Moreover, a is called (weakly) $n-$nil clean if it is a sum (a sum or a difference) of $n-$potent element and a nilpotent element in R. We define R to be (weakly) $n-$nil clean if every element in R is (weakly) $n-$nil clean. Weakly $n-$clean rings are defined in a similar way. Obviously, the (weakly) $2-$nil clean rings are the same as the (weakly) nil clean rings. R is called a generalized nil clean if every element in R is $n-$nil clean for some $n \in \mathbb{N}$. The class of $n-$nil clean and generalized nil clean rings were firstly defined and studied in [9] by Hirano, Tominaga and Yaqub

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in 1988. Some Other authors called generalized nil clean rings as weak periodic rings. A ring R is called periodic if for every $x \in R$, there are distinct integers m and k such that $x^m = x^k$. It is proved that a periodic ring is weak periodic and that the converse is true if in any expansion $r = b + s$ for potent s and $b \in N(R)$, we have $bs = sb$.

In this paper, we focus our attention on the ring $\mathbb{Z}_m$ of integers modulo a positive integer m. We use the well Known Hensel's Lemma to completely determine when the ring $\mathbb{Z}_m$ is (weakly) $n-$nil clean ring for any $m, n \in \mathbb{N}$. Moreover, we determine all $m, n \in \mathbb{N}$ such that every element $r \in \mathbb{Z}_m$ is of the form $r = b \pm s$ where $b \in N(R)$ and $s^n = -s$. Next, we apply our results for some special values of m and n.

In the next section, we study weakly $n-$nil clean rings and introduce some fundamental facts and examples concerning this class of rings. Among many other properties, we determine some conditions on n, R and G under which the group ring RG is (weakly) $n-$nil clean.

2. Weakly $n-$Nil Clean Rings

In this section, we study some of the basic properties of weakly $n-$nil clean rings. Moreover, we give some necessarily examples.

Definition 1. *Let R be a ring and $n \in \mathbb{N}$ where $n \geqslant 2$. An element $r \in R$ is called weakly $n-$nil clean if there exist $b \in N(R)$ and an $n-$potent element s of R such that $r = b + s$ or $r = b - s$. A ring R is called weakly $n-$nil clean if all of its elements are weakly $n-$nil clean.*

For $n \geqslant 2$, let s be an $n-$potent and b be a nilpotent. For $r \in R$, we write $r = b \pm s$ if r is either a sum $b + s$ or a difference $b - s$. Obviously, every $n-$nil clean ring is weakly $n-$nil clean. Since the ring $\mathbb{Z}_6$ is a weakly nil clean ring that is not nil clean, then trivially $\mathbb{Z}_6$ is a weakly $2-$nil clean ring which is not $2-$nil clean. For a non trivial example, one can easily verify that the ring $\mathbb{Z}_3$ is weakly $4-$nil clean but not $4-$nil clean. Moreover, if a ring R is a weakly n-nil clean, then it is weakly n-clean. Indeed, if we let $x \in R$, then $x - 1 = b \pm s$ where $b \in N(R)$ and $s^n = s$. So, $x = (b + 1) \pm s$ where $b + 1 \in U(R)$. The converse is not true in general. For example, simple computations show that the ring $R = T_2(\mathbb{Z}_3)$ is weakly 5-clean which is not weakly 5-nil clean.

Next, we give some properties of the class of weakly $n-$nil clean rings. The proof of the following proposition is straightforward.

Proposition 1. *Let R and S be two rings, $\mu : R \to S$ be a ring epimorphism and $n \geqslant 2$. If R is weakly $n-$nil clean, then S is weakly $n-$nil clean.*

The following Properties (2), (3) and (4) in Corollary 1 are direct consequences of Proposition 1. The proofs of Properties (1) and (5) are similar to that of (weakly) $g(x)$-nil clean appeared in [10, 11, 12].

Corollary 1. *Let R and S be ring and let $n \geqslant 2$. The following hold:*

(1) If I is an ideal in R and R is weakly $n-$nil clean, then R/I is weakly $n-$nil clean. Moreover, the converse holds if I is nil and potent elements lift modulo I.

(2) If the upper triangular matrix ring $T_n(R)$ is weakly $n-$nil clean, then R is weakly $n-$nil clean.

(3) If the skew formal power series $R[[x,\alpha]]$ (or in particular $R[[x]]$) over R is weakly $n-$nil clean, then R is weakly $n-$nil clean.

(4) Let M be an (R,S)-bimodule and $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be the formal triangular matrix ring. If T is weakly $n-$nil clean, then R and S are weakly $n-$nil clean.

(5) If R is commutative and M an R-module. Then the idealization $R(M)$ of R and M is weakly $n-$nil clean if and only if R is weakly $n-$nil clean.

Proposition 2. *Let $R = \prod_{i \in I} R_i$ be a direct product of rings with I is finite and $|I| \geq 2$ and let $n \geqslant 2$. R is weakly $n-$nil clean if and only if there exist $k \in I$ such that R_k is weakly $n-$nil clean and R_j is $n-$nil clean for $j \neq k$.*

Proof. $\Rightarrow$) : For each $i \in I$, R_i is a homomorphic image of $\prod_{i \in I} R_i$ under the projection homomorphism. Hence, R_i is weakly $n-$nil clean by Proposition 1. Without loss of generality, assume that neither R_1 nor R_2 are $n-$nil clean. Then there exist $r_1 \in R_1$ and $r_2 \in R_2$ such that r_1 is not a sum of a nilpotent and an $n-$potent and r_2 is not a difference of a nilpotent and an $n-$potent. Thus (r_1, r_2) is not weakly $n-$nil clean in $R_1 \times R_2$, a contradiction.

$\Leftarrow$) : Assume that R_k is weakly $n-$nil clean for a fixed index $k \in I$. Thus R_j is $n-$nil clean for all $j \neq k$. Let $r = (r_i) \in R$. Then there exist $b_k \in N(R_k)$ and an $n-$potent s_k such that $r_k = b_k + s_k$ or $r_k = b_k - s_k$. If $r_k = b_k + s_k$, then for each $i \in I - \{k\}$, write $r_i = b_i + s_i$ where $b_i \in N(R_i)$ and $s_i^n = s_i$. Therefore, $r = (b_i) + (s_i)$ is a sum of a nilpotent and an $n-$potent. If $r_k = b_k - s_k$, then for each $i \in I - \{k\}$, write $r_i = b_i - s_i$ where $b_i \in N(R_i)$ and $s_i^n = s_i$. Consequently, $r = (b_i) - (s_i)$ is a difference of a nilpotent and an $n-$potent. Therefore, R is weakly $n-$nil clean. □

Definition 2. *Let R be a ring and let $m \in \mathbb{N}$. Then R is said to have the nil m-involution property if for every $r \in R$, we have $r = u + v$ where $u \in 1 \pm N(R)$ and $v^m = 1$.*

We now justify the relation between weakly $n-$nil clean rings and rings with nil $(n-1)$-involution property for an odd $n \in \mathbb{N}$.

Proposition 3. *Let R be a ring and let n be an odd integer with $n \geq 2$. If R has the nil $(n-1)$-involution property, then R is (weakly) $n-$nil clean. If moreover, aR (or Ra) contains no non trivial idempotent for every non unit $a \in R$, then the two statements are equivalent.*

Proof. Suppose R has the nil $(n-1)$-involution property and let $r \in R$. Write $r + 1 = u + v$ where $u \in 1 \pm N(R)$ and $v^{n-1} = 1$. Then $r = (u-1) + v$ where $u - 1 \in N(R)$ and v is clearly an $n-$potent element in R.

Now, we assume that for every non unit $a \in R$, aR (or Ra) contains no non trivial idempotents and suppose R is weakly $n-$nil clean. Let $a \in R$ and write $a - 1 = b \pm s$ where $b \in N(R)$ and $s^n = s$. Then $as^{n-1} = (b+1)s^{n-1} \pm s$ and so $a(1 - s^{n-1}) = (b+1)(1 - s^{n-1}) = u(1 - s^{n-1})$ where $u \in U(R)$. Since clearly $u(1 - s^{n-1})u^{-1} = a(1 - s^{n-1})u^{-1} \in aR$ is an idempotent, then by assumption $u(1 - s^{n-1})u^{-1} = 0$ or $u(1 - s^{n-1})u^{-1} = 1$. Therefore $s^{n-1} = 1$ or $s^{n-1} = 0$. In the last case, we get $s = s^n = 0$ and so $a = b + 1$ is a unit, a contradiction. Thus, $a = (b+1) + (\pm s)$ where $(\pm s)^{n-1} = 1$ since $n - 1$ is even. The case when Ra contains no non trivial idempotent for every non unit $a \in R$ is similar. Therefore, R has the nil $(n-1)$-involution property. $\square$

It is easy to see that the ring $\mathbb{Z}_4$ is a (weakly) $4-$nil clean ring with $a\mathbb{Z}_4$ contains no non trivial idempotent for every $a \in \mathbb{Z}_4$. But, $\mathbb{Z}_4$ does not have the nil 3-involution property. Therefore, the equivalence in Proposition 3 need not be hold for an even integer n.

Let R be a ring and G be a finite cyclic group. In the following Proposition, we determine conditions under which the group ring RG is (weakly) $n-$nil clean. We recall that R is called an $n-$potent ring if $a^n = a$ for every $a \in R$.

Proposition 4. *Let G any cyclic group of order p (prime).*

(1) If R is a Boolean ring, then RG is a $2^{p-1}-$potent ring (and so is (weakly) $2^{p-1}-$nil clean).

(2) If R is a commutative $3-$potent ring of characteristic 3, and $p \neq 3$, then RG is a $3^{p-1}-$potent ring (and so is (weakly) $3^{p-1}-$nil clean).

Proof. (1) See Proposition 3.17 in [10].

(2) Let $G = \left\{1, g, g^2, ..., g^{p-1}\right\}$ where $g^p = 1$ and let $x = a_0 + a_1 g + a_2 g^2 + ... + a_{p-1} g^{p-1} \in RG$. First, we prove by induction that $x^{3^k} = \sum\limits_{i=0}^{p-1} a_i g^{i*(3^k)}$ for all $k \in \mathbb{N}$. Let $k = 1$. Since R is $3-$potent ring of characteristic 3, one can see that $x^3 = a_0 + a_1 g^3 + a_2 g^6 + ... + a_{p-1} g^{3(p-1)} = \sum\limits_{i=0}^{p-1} a_i g^{3i}$. Suppose the result is true for k. Then $x^{3^{k+1}} = (x^3)^{3^k} = \sum\limits_{i=0}^{p-1} a_i (g^3)^{i*(3^k)} = \sum\limits_{i=0}^{p-1} a_i g^{i*(3^{k+1})}$. By Fermat Theorem, $3^{p-1} = 1 + np$ for some integer n. Thus, $x^{3^{p-1}} = \sum\limits_{i=0}^{p-1} a_i g^{i*(3^{p-1})} = \sum\limits_{i=0}^{p-1} a_i g^{i*(1+np)} = \sum\limits_{i=0}^{p-1} a_i g^i = x$. Therefore, RG is a $3^{p-1}-$potent ring. $\square$

By Proposition 4, we conclude that the ring $\mathbb{Z}_2(C_3)$ is (weakly) $4-$nil clean and $\mathbb{Z}_3(C_2)$ is (weakly) $3-$nil clean.

Proposition 5. *Let R be a ring and let $n \geqslant 2$. If R is (weakly) $n-$nil clean, then $J(R)$ is nil.*

Proof. Let $a \in J(R)$. Then $a = b \pm s$ where $b \in N(R)$ and $s^n = s$. If $a = b - s$, then $a + s \in N(R)$. If we choose $m \in \mathbb{N}$ such that $(a+s)^m = 0$, then clearly $s^m \in J(R)$. If $m \lneqq n$, then $s^{n-1} \in J(R)$. Since also $s(1 - s^{n-1}) = 0$ and $1 - s^{n-1} \in U(R)$, then $s = 0$. If $m \geq n$, then we can similarly see that $s = 0$. Hence $a = b \in N(R)$. Similarly, the case $a = b + s$ gives $a \in N(R)$ and so $J(R)$ is nil. □

3. When the ring $\mathbb{Z}_m$ is (weakly) $n-$nil clean

In the main Theorem of this section, we completely determine all $n, m \in \mathbb{N}$ such that the ring $\mathbb{Z}_m$ is (weakly) n-nil clean. We recall that for $m \in \mathbb{N}$, the set of all positive integers less than or equal m that are relatively prime to m is a group under multiplication modulo m. it is denoted by $\mathbb{Z}_m^{\times}$ and is called the group of units modulo m. This group is cyclic if and only if m is equal to 2, 4, p^k, or $2p^k$ where p^k is a power of an odd prime. A generator of this cyclic group is called a primitive root modulo m. The order of $\mathbb{Z}_m^{\times}$ is given by Euler's totient function $\varphi(m)$. It is easy to see that for any prime integer p and any $k \in \mathbb{N}$, $\varphi(p^k) = p^{k-1}(p-1)$. For more details one can see [13].

Lemma 1. *For any $n, k \in \mathbb{N}$, the ring $\mathbb{Z}_{2^k}$ is $n-$nil clean.*

Proof. For any $n \in \mathbb{N}$, at least 0 and 1 are $n-$potent elements in $\mathbb{Z}_{2^k}$. Since $N(\mathbb{Z}_{2^k}) = \{0, 2, 4, ..., 2(2^{k-1} - 1)\}$, then clearly any element in $\mathbb{Z}_{2^k}$ is a sum of a nilpotent and an $n-$potent. □

The following lemma is a special case of the well known Hensel's Lemma.

Lemma 2. *Let $n, k \in \mathbb{N}$ and p be an odd prime integer. Consider the congruence $f(x) \equiv 0(\operatorname{mod} p)$ where $f(x) \in \mathbb{Z}[x]$. If r is a solution of the congruence with $f'(r)$ is not congruent to $0(\operatorname{mod} p)$, then there exists a unique integer s such that $f(s) \equiv 0(\operatorname{mod} p^k)$ and $r \equiv s(\operatorname{mod} p)$.*

In particular, for a prime integer p and $1 \leq m \leq p-1$, let r be a solution of $x^m - 1 \equiv 0(\operatorname{mod} p)$. Then mr^{m-1} is not congruent to $0(\operatorname{mod} p)$. Hence, r corresponds to a unique solution s of $x^m - 1 \equiv 0(\operatorname{mod} p^k)$ such that $r \equiv s(\operatorname{mod} p)$.

The following Lemma is well known in number theory. However, we give the proof for the sake of completeness.

Lemma 3. *Let $n, k \in \mathbb{N}$ and p be any prime integer and let $d = \gcd(n, p^{k-1}(p-1))$. Then*

(1) The polynomial $x^n - 1 \in \mathbb{Z}_{p^k}[x]$ has d solutions in $\mathbb{Z}_{p^k}$.

(2) If $\frac{p^{k-1}(p-1)}{d}$ is even, then the polynomial $x^n + 1 \in \mathbb{Z}_{p^k}[x]$ has d solutions in $\mathbb{Z}_{p^k}$. Otherwise, it has no solutions.

Proof. (1) Consider the cyclic group of units $\mathbb{Z}_{p^k}^{\times}$ with order $\varphi(p^k) = p^{k-1}(p-1)$. Let g be a generator for $\mathbb{Z}_{p^k}^{\times}$ and let $a = g^m \in \mathbb{Z}_{p^k}^{\times}$ be a solution of $x^n \equiv 1(\operatorname{mod} p^k)$. Then $a^n = g^{mn} \equiv 1(\operatorname{mod} p^k)$ and so $p^{k-1}(p-1)$ divides mn. If we let $d =$

$\gcd(n, p^{k-1}(p-1))$, then $\frac{p^{k-1}(p-1)}{d}$ divides m. Therefore, the solution set of $x^n - 1$ in $\mathbb{Z}_{p^k}$ forms a subgroup generated by $g^{\frac{p^{k-1}(p-1)}{d}}$. The result follows since this subgroup is clearly of order d.

(2) Consider again the generator g of the cyclic group of units $\mathbb{Z}_{p^k}^{\times}$. Since $g^{p^{k-1}(p-1)} \equiv 1(\bmod p^k)$, then g must satisfy $g^{\frac{p^{k-1}(p-1)}{2}} \equiv -1(\bmod p^k)$. Hence, $x = g^m$ is a solution of $x^n \equiv -1(\bmod p^k)$ if and only if $p^{k-1}(p-1)$ divides $2mn$ and so $\frac{p^{k-1}(p-1)}{d}$ must divides $2m$. If $\frac{p^{k-1}(p-1)}{d}$ is not even, then $x^n \equiv -1(\bmod p^k)$ has no solutions. However, if $\frac{p^{k-1}(p-1)}{d}$ is even, then $g^{\frac{p^{k-1}(p-1)}{2d}}$ is one solution of $x^n \equiv -1(\bmod p^k)$. The other solutions can be obtained by multiplying by the d solutions of $x^n \equiv 1(\bmod p^k)$. □

Theorem 1. *Let $n, k \in \mathbb{N}$ and p be any odd prime integer. If $d = \gcd(n-1, p^{k-1}(p-1))$, then $\mathbb{Z}_{p^k}$ is $n-$nil clean if and only if $d = p^t(p-1)$ for some $0 \leq t \leq k-1$.*

Proof. To be brief, let S denotes the set of all zeros of $x^n - x$ in $\mathbb{Z}_{p^k}$ and T denotes the set of sums of every element in $N(\mathbb{Z}_{p^k})$ to every element in S.

$\Leftarrow)$: Suppose $d = \gcd(n, p^{k-1}(p-1)) = p-1$. By Lemma 3, The multiplicative group G of roots of unity modulo p^k is of order $p-1$ and so $a^{p-1} \equiv 1(\bmod p^k)$ for all $a \in G$. Now, By Fermat Theorem, any $a \in G$ is also a solution of $x^{p-1} \equiv 1(\bmod p)$. By Lemma 2, the $p-1$ solutions of $x^{p-1} \equiv 1(\bmod p)$ correspond uniquely to the $p-1$ solutions of $x^{p-1} \equiv 1(\bmod p^k)$. Hence, the $p-1$ solutions of $x^{p-1} \equiv 1(\bmod p^k)$ are congruent to $1, 2, ..., p-1$ in some order. Now, $N(\mathbb{Z}_{p^k}) = \{0, p, 2p, ..., (p^{k-1}-1)p\}$ is of order p^{k-1}. If $n_1 + a = n_2 + b$ for some $a, b \in S$ and $n_1, n_2 \in N(\mathbb{Z}_{p^k})$, then $a - b \equiv n_n - n_1 \equiv 0(\bmod p)$. Thus, $a \equiv b(\bmod p)$ which is true only if $a = b = 0$. Therefore, T has exactly $pp^{k-1} = p^k$ distinct elements and $\mathbb{Z}_{p^k}$ is $n-$nil clean. Next, suppose $d = p^t(p-1)$ for some $1 \leq t \leq p^k$. If $a^{p^t(p-1)} \equiv 1(\bmod p^k)$, then $a^{p-1} \equiv (a^{p-1})^{p^t} \equiv 1(\bmod p)$. Again, by Lemma 2, the $p-1$ solutions corresponds uniquely to $p-1$ distinct solutions of $x^{p^t(p-1)} \equiv 1(\bmod p^k)$. Hence, similar to the above argument, we conclude that $\mathbb{Z}_{p^k}$ is $n-$nil clean.

$\Rightarrow)$: Suppose $d = mp^t$ for some $m \mid (p-1)$ with $m \neq p-1$ and $0 \leq t \leq p^{k-1}$. If $t = 0$, then T has at most $(m+1)p^{k-1} \lneqq p^k$ elements and so $\mathbb{Z}_{p^k}$ is not $n-$nil clean. Let $t \gneqq 0$ and consider $x^{mp^t} \equiv 1(\bmod p^k)$. Then $x^m \equiv (x^m)^{p^t} \equiv 1(\bmod p)$ has at most m solutions $(\bmod p)$. By Lemma 2, any solution of $x^{mp^t} \equiv 1(\bmod p^k)$ is congruent to one of the m solutions of $x^m \equiv 1(\bmod p)$. Choose $1 \leq c \leq p-1$ such that c is not a solution of $x^m \equiv 1(\bmod p)$ and suppose $c = a + f$ for some $a \in S$ and $f \in N(\mathbb{Z}_{p^k})$. If $a = 0$, then $c \in N(\mathbb{Z}_{p^k})$, a contradiction. Suppose $a \neq 0$ and let $1 \leq r \leq p-1$ such that $r^m \equiv 1(\bmod p)$ and $a \equiv r(\bmod p)$. Then $c \equiv r + f \equiv r(\bmod p)$ which is a contradiction. Hence, again $\mathbb{Z}_{p^k}$ is not $n-$nil clean. □

Corollary 2. *For any even integer n and odd prime p, the ring $\mathbb{Z}_{p^k}$ is not $n-$nil clean.*

Definition 3. *Let R be a ring and $n \geqslant 2$. R is called (x^n+x)-nil clean if for every $r \in R$, $r = b + s$ where $b \in N(R)$ and $s^n = -s$.*

By direct computations one can easily verify that for any even integer n, R is (x^n+x)-nil clean if and only if R is $n-$nil clean. However for any odd integer n and odd prime integer p, we prove in the following lemma that $\mathbb{Z}_{p^k}$ is never (x^n+x)-nil clean.

Lemma 4. *For any $k \in \mathbb{N}$ and $n \geqslant 2$, the ring $\mathbb{Z}_{2^k}$ is (x^n+x)-nil clean if and only if $\gcd(n-1, 2^k) \neq 2^k$.*

Proof. The proof follows directly by (2) in Lemma 3. □

Theorem 2. *Let p be a prime integer and $k, n \in \mathbb{N}$ where n is odd. Then $\mathbb{Z}_{p^k}$ is never (x^n+x)-nil clean.*

Proof. $\Leftarrow$) : By lemma 3, $x^{n-1} \equiv -1 (\operatorname{mod} p^k)$ has a solution if $\frac{p^{k-1}(p-1)}{d}$ is even where $d = \gcd(n-1, p^{k-1}(p-1))$. Hence, clearly if $d = p^t(p-1)$ for some $0 \leq t \leq k-1$, then $\mathbb{Z}_{p^k}$ is not (x^n+x)-nil clean. Suppose $d = mp^t$ for some $m \mid p-1$ with $m \neq p-1$ and $0 \leq t \leq k-1$. Then $x^m \equiv (x^m)^{p^t} \equiv -1 (\operatorname{mod} p)$. Clearly, this congruence has less than $p-1$ solutions. Thus, as in the proof of the similar case in Theorem 1, we conclude that $\mathbb{Z}_{p^k}$ is not (x^n+x)-nil clean. □

Corollary 3. *Let $m, n, k \in \mathbb{N}$ and write $m = p_1^{r_1} p_2^{r_2} ... p_t^t$ where $p_1, p_2, ..., p_t$ are distinct prime integers. Then the ring $\mathbb{Z}_m$ is $n-$nil clean if and only if for all $i = 1, 2, ..., t$, $\gcd(n-1, p_i^{r_i-1}(p_i-1)) = p_i^l(p_i-1)$ for some $0 \leq l \leq r_i - 1$.*

Proof. We have $\mathbb{Z}_m \simeq \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times ... \times \mathbb{Z}_{p_t^{r_t}}$. By Proposition (2.6) in [10], $\mathbb{Z}_m$ is $n-$nil clean if and only if $\mathbb{Z}_{p_i^{r_i}}$ is $n-$nil clean for all $i = 1, 2, ..., t$. Hence, the result follows by Theorem 1 and Lemma 1. □

As special cases, we have

Corollary 4. *Let $n \in \mathbb{N}$ and consider the ring $\mathbb{Z}_n$. Then*
(1) For any $m \in \mathbb{N}$, $\mathbb{Z}_n$ is $2m-$nil clean if and only if $n = 2^k$ for $k \in \mathbb{N} \cup \{0\}$.
(2) $\mathbb{Z}_n$ is $3-$nil clean if and only if $n = 2^k \times 3^m$ for $k, m \in \mathbb{N} \cup \{0\}$.
(3) $\mathbb{Z}_n$ is $5-$nil clean if and only if $n = 2^k \times 3^m \times 5^l$ for $k, m, l \in \mathbb{N} \cup \{0\}$.
(4) $\mathbb{Z}_n$ is $7-$nil clean if and only if $n = 2^k \times 3^m \times 7^l$ for $k, m, l \in \mathbb{N} \cup \{0\}$.

For $n, m \in \mathbb{N}$, we next clarify when the ring $\mathbb{Z}_m$ is weakly $n-$nil clean.

Theorem 3. *Let $n, k \in \mathbb{N}$, p be any odd prime integer and $d = \gcd(n-1, p^{k-1}(p-1))$. Then*

(1) $\mathbb{Z}_{p^k}$ is weakly $n-$nil clean if and only if $d = p^t(p-1)$ or $d = \frac{p^t(p-1)}{2}$ for some $0 \leq t \leq k-1$.

(2) $\mathbb{Z}_{p^k}$ is weakly $(x^n+x)-$nil clean if and only if $d=\frac{p^t(p-1)}{2}$ for some $0\leq t\leq k-1$.

Proof. (1) $\Leftarrow)$: Let $0\leq t\leq k-1$. If $d=p^t(p-1)$, then $\mathbb{Z}_{p^k}$ is (weakly) $n-$nil clean by Theorem 1. Suppose $d=\frac{p^t(p-1)}{2}$, then for any solution a of $x^{n-1}\equiv 1(\bmod p^k)$, we have $a^{\frac{p-1}{2}}\equiv (a^{\frac{(p-1)}{2}})^{p^t}=a^{\frac{p^t(p-1)}{2}}\equiv 1(\bmod p)$. Clearly, the congruence $x^{\frac{p-1}{2}}\equiv 1(\bmod p)$ has $\frac{p-1}{2}$ solutions. By Lemma 2, those $\frac{p-1}{2}$ solutions correspond uniquely to $\frac{p-1}{2}$ solutions of $x^{\frac{p^t(p-1)}{2}}\equiv 1(\bmod p^k)$. Let T_1 (respectivelyT_2) be the set of all sums (respectively, differences) of each of the $\frac{p-1}{2}$ solutions of $x^{\frac{p^t(p-1)}{2}}\equiv 1(\bmod p^k)$ and each nilpotent in $\mathbb{Z}_{p^k}$. By imitating the proof of Theorem 1, we can see that T_1 (respectivelyT_2) has $(\frac{p-1}{2})p^{k-1}$ distinct elements. Moreover, if $a^{\frac{p^t(p-1)}{2}}\equiv 1(\bmod p^k)$ and $b\in N(\mathbb{Z}_{p^k})$ such that $b+a=b-a$, then $2a=0$ which is a contradiction. Thus, $N(\mathbb{Z}_{p^k})\cup T_1\cup T_2$ contains exactly $(2(\frac{p-1}{2})+1)p^{k-1}=p^k$ distinct elements and so $\mathbb{Z}_{p^k}$ is weakly $n-$nil clean.

$\Rightarrow)$: Suppose $d\neq p^t(p-1)$ and $d\neq\frac{p^t(p-1)}{2}$ for all $0\leq t\leq k-1$. Then $d=mp^t$ for some $m\mid p-1$ where $m\neq p-1$. Hence, either $m=\frac{p-1}{2}$ or $m\lneqq\frac{p-1}{2}$. If $m=\frac{p-1}{2}$, then we get a contradiction. Suppose $m\lneqq\frac{p-1}{2}$ and consider $x^{mp^t}\equiv 1(\bmod p^k)$. Then $x^m\equiv (x^m)^{p^t}\equiv 1(\bmod p)$ has at most m solutions. Since $m\lneqq\frac{p-1}{2}$, then similar to the above argument, the set of all sums or difference of each nilpotent and each solution of x^n-x will not cover $\mathbb{Z}_{p^k}$. Thus, $\mathbb{Z}_{p^k}$ is not weakly $n-$nil clean.

(2) $\Rightarrow)$: If $d=p^t(p-1)$ for some $0\leq t\leq k-1$, then $x^{n-1}\equiv -1(\bmod p^k)$ has no solution and so $\mathbb{Z}_{p^k}$ is not weakly (x^n+x)-nil clean. Suppose $d=mp^t$ where $0\leq t\leq k-1$, $m\neq p-1$ and $m\mid p-1$. If $m\lneqq\frac{p-1}{2}$, then similar to the proof of (1), $\mathbb{Z}_{p^k}$ is also not weakly (x^n+x)-nil clean. Hence, we must have $m=\frac{p-1}{2}$ and $d=\frac{p^t(p-1)}{2}$ for some $0\leq t\leq k-1$.

$\Leftarrow)$: Suppose $d=\frac{p^t(p-1)}{2}$then clearly $x^{\frac{p-1}{2}}\equiv (x^{\frac{p-1}{2}})^{p^t}\equiv -1(\bmod p)$ has $\frac{p-1}{2}$ solutions each of which corresponds uniquely to a solution of $x^{\frac{p^t(p-1)}{2}}\equiv -1(\bmod p^k)$. Define T_1 and T_2 as in (1) for the congruence $x^{\frac{p^t(p-1)}{2}}\equiv -1(\bmod p^k)$, we can similarly see that $N(\mathbb{Z}_{p^k})\cup T_1\cup T_2$ contains exactly p^k distinct elements and so $\mathbb{Z}_{p^k}$ is weakly (x^n+x)-nil clean. $\square$

Corollary 5. *Let $n,k\in\mathbb{N}$, p be any odd prime integer and $d=\gcd(n-1,p^{k-1}(p-1))$. Then $\mathbb{Z}_{p^k}$ is weakly $n-$nil clean that is not $n-$nil clean if and only if $d=\frac{p^t(p-1)}{2}$ for some $0\leq t\leq k-1$.*

For example $\mathbb{Z}_{5^k}$ is a weakly $3-$nil clean that is not $3-$nil clean for any $k\in\mathbb{N}$.

Now, we can use Theorem 3 and Proposition 2 to prove the following corollary.

Corollary 6. *Let $m,n,k\in\mathbb{N}$ and write $m=p_1^{r_1}p_2^{r_2}...p_t^{t}$ where $p_1,p_2,...,p_t$ are distinct prime integers. Then the ring $\mathbb{Z}_m$ is weakly $n-$nil clean if and only if there* *is at most* $1 \leq j \leq t$ *such that for some* $1 \leq l_j \leq r_j - 1$ $\gcd(n-1, p_j^{r_j-1}(p_j - 1)) = p_j^{l_j}(p_j-1)$ *or* $\frac{p_j^{l_j}(p_j-1)}{2}$ *and* $\gcd(n-1, p_i^{r_i-1}(p_i-1)) = p_i^{l_i}(p_i-1)$ *for some* $1 \leq l_i \leq r_i - 1$ *for all* $i \neq j$.

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CESÀRO SUMMABILITY OF INTEGRALS OF FUZZY-NUMBER-VALUED FUNCTIONS

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Abstract. In the present study, we have introduced Cesàro summability of integrals of fuzzy-number-valued functions and given one-sided Tauberian conditions under which convergence of improper fuzzy Riemann integrals follows from Cesàro summability. Also, fuzzy analogues of Schmidt type slow decrease and Landau type one-sided Tauberian conditions have been obtained.

1. Introduction

Given a locally integrable function $f : [0, \infty) \to \mathbb{C}$, *the Cesàro operator* Cf is defined by

$$(Cf)(x) := \frac{1}{x}\int_0^x f(t)dt, \qquad x \in (0, \infty).$$

In classical analysis, the Cesàro operator was investigated from various aspects and a large number of results have appeared recently [1–5]. Titchmarsh [6] also used the operator as a convergence method for divergent integrals and introduced the Cesàro summability of integrals [7, p.11]. Following this introduction, the concept of Cesàro summability of integrals received considerable attention and Tauberian conditions under which Cesàro summable improper integrals converge have been investigated [7–15]. Also, there are studies applying the concept to Fourier integrals [6,16–19].

In the light of the developments mentioned above, establishment of the concept of Cesàro summability of integrals for fuzzy analysis is also of importance for handling divergent integrals of fuzzy-number-valued functions. The concept of integration of fuzzy-number-valued functions has already been introduced by Dubois et al. [20] and studied by many mathematicians [21–24]. Also, in particular, Bede and Gal[25] have proved that there exists a mean value, or a Cesàro sum, for any almost periodic fuzzy-number-valued function and given some applications of these

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functions to fuzzy differential equations and to fuzzy dynamical systems. At this point, approaching the concept of 'mean value' from perspective of summability theory, we define Cesàro summability of integrals of fuzzy-number-valued functions and give various types of convergence conditions for Cesàro summable improper integrals of fuzzy-number-valued functions.

2. Preliminaries

A *fuzzy number* is a fuzzy set on the real axis, i.e. u is normal, fuzzy convex, upper semi-continuous and $\operatorname{supp} u = \overline{\{t \in \mathbb{R} : u(t) > 0\}}$ is compact [26]. We denote the space of fuzzy numbers by E^1. *α-level set* $[u]_\alpha$ of $u \in E^1$ is defined by

$$[u]_\alpha := \begin{cases} \{t \in \mathbb{R} : u(t) \geq \alpha\} & , \quad if \quad 0 < \alpha \leq 1, \\ \overline{\{t \in \mathbb{R} : u(t) > \alpha\}} & , \quad if \quad \alpha = 0. \end{cases}$$

Each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(t) := \begin{cases} 1 & , \quad if \quad t = r, \\ 0 & , \quad if \quad t \neq r. \end{cases}$$

Let $u, v \in E^1$ and $k \in \mathbb{R}$. The addition and scalar multiplication are defined by

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha = [u^-_\alpha + v^-_\alpha, u^+_\alpha + v^+_\alpha], [ku]_\alpha = k[u]_\alpha$$

where $[u]_\alpha = [u^-_\alpha, u^+_\alpha]$, for all $\alpha \in [0,1]$.

Lemma 1. [25] *The following statements hold:*

(i) $\overline{0} \in E^1$ *is neutral element with respect to* $+$, *i.e.,* $u + \overline{0} = \overline{0} + u = u$ *for all* $u \in E^1$.
(ii) *With respect to* $\overline{0}$, *none of* $u \neq \overline{r}$, $r \in \mathbb{R}$ *has opposite in* E^1.
(iii) *For any* $a, b \in \mathbb{R}$ *with* $a, b \geq 0$ *or* $a, b \leq 0$ *and any* $u \in E^1$, *we have* $(a+b)u = au + bu$. *For general* $a, b \in \mathbb{R}$, *the above property does not hold.*
(iv) *For any* $a \in \mathbb{R}$ *and any* $u, v \in E^1$, *we have* $a(u+v) = au + av$.
(v) *For any* $a, b \in \mathbb{R}$ *and any* $u \in E^1$, *we have* $a(bu) = (ab)u$.

The metric D on E^1 is defined as

$$D(u,v) := \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha) := \sup_{\alpha \in [0,1]} \max\{|u^-_\alpha - v^-_\alpha|, |u^+_\alpha - v^+_\alpha|\}.$$

where d is the Hausdorff metric.

Proposition 1. [25] *Let* $u, v, w, z \in E^1$ *and* $k \in \mathbb{R}$. *Then,*

(i) (E^1, D) *is a complete metric space.*
(ii) $D(ku, kv) = |k| D(u,v)$.
(iii) $D(u+v, w+v) = D(u,w)$.
(iv) $D(u+v, w+z) \leq D(u,w) + D(v,z)$.
(v) $|D(u,\overline{0}) - D(v,\overline{0})| \leq D(u,v) \leq D(u,\overline{0}) + D(v,\overline{0})$.

Partial ordering relation on E^1 is defined as follows:

$$u \preceq v \Longleftrightarrow [u]_\alpha \preceq [v]_\alpha \Longleftrightarrow u_\alpha^- \leq v_\alpha^- \text{ and } u_\alpha^+ \leq v_\alpha^+ \text{ for all } \alpha \in [0,1].$$

We say a fuzzy number u is negative if and only if $u(t) = 0$ for all $t \geq 0$ (see [27]).
Combining the results of Lemma 6 in [28], Lemma 5 in [29], Lemma 3.4, Theorem 4.9 in [30] and Lemma 14 in[31], following Lemma is obtained.

Lemma 2. *Let $u, v, w, e \in E^1$ and $\varepsilon > 0$. The following statements hold:*

(i) *$D(u,v) \leq \varepsilon$ if and only if $u - \overline{\varepsilon} \preceq v \preceq u + \overline{\varepsilon}$*
(ii) *If $u \preceq v + \overline{\varepsilon}$ for every $\varepsilon > 0$, then $u \preceq v$.*
(iii) *If $u \preceq v$ and $v \preceq w$, then $u \preceq w$.*
(iv) *If $u \preceq w$ and $v \preceq e$, then $u + v \preceq w + e$.*
(v) *If $u + w \preceq v + w$ then $u \preceq v$.*

Definition 1. *A fuzzy-number-valued function $f : [a,b] \to E^1$ is said to be continuous at $x_0 \in [a,b]$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $D(f(x), f(x_0)) < \varepsilon$ whenever $x \in [a,b]$ with $|x - x_0| < \delta$. If f is continuous at each $x \in [a,b]$, then we say f is continuous on $[a,b]$.*

Definition 2. [32] *A fuzzy-valued function $f : [a,b] \to E^1$ is called Riemann integrable on $[a,b]$, if there exists $I \in E^1$ with the property : $\forall\varepsilon > 0$, $\exists\delta > 0$ such that for any division of $[a,b]$ $d : a = x_0 < x_1 < \cdots < x_n = b$ of norm $v(d) < \delta$, and for any points $\xi_i \in [x_i, x_{i+1}]$ $i = \overline{0, n-1}$, we have*

$$D\left(\sum_{i=0}^{n-1} f(\xi_i)(x_{i+1} - x_i), I\right) < \varepsilon.$$

Then $I = \int\limits_a^b f(x)dx$.

Theorem 1. [32] *If the fuzzy-number-valued function $f : [a,b] \to E^1$ is continuous (with respect to the metric D) and for each $x \in [a,b]$, $f(x)$ has the parametric representation*

$$[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)],$$

then $\int\limits_a^b f(x)dx$ exists, belongs to E^1 and is parametrized by

$$\left[\int_a^b f(x)dx\right]_\alpha = \left[\int_a^b f_\alpha^-(x)dx, \int_a^b f_\alpha^+(x)dx\right].$$

Using the results of Anastassiou [22] we have

Theorem 2. *If $f : [a,b] \to E^1$ and $g : [a,b] \to E^1$ are continuous then*

(i) $\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$ *where* α *and* β *are real numbers.*

(ii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ *where* $a < c < b$.

(iii) *The function* $F : [a, b] \to \mathbb{R}_+$ *defined by* $F(x) = D(f(x), g(x))$ *is continuous on* $[a, b]$ *and*

$$D\left(\int_a^b f(x)dx, \int_a^b g(x)dx\right) \le \int_a^b F(x)dx.$$

(iv) $\int_a^x f(t)dt$ *is a continuous function in* $x \in [a, b]$.

(v) $\int_a^b f(x)dx \preceq \int_a^b g(x)dx$ *whenever* $f(x) \preceq g(x)$ *for all* $x \in [a, b]$.

Definition 3. *Suppose* $f(x)$ *is a fuzzy-number-valued function defined on the unbounded interval* $[a, \infty)$. *Then we define*

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided the limit on the right-hand side exists in E^1, *in which case we say the integral converges and is equal to the value of limit. Otherwise, we say the integral diverges.*

3. Main Results

Definition 4. *Let* $f : [0, \infty) \to E^1$ *be a continuous fuzzy-number-valued function and* $s(t) = \int_0^t f(x)dx$. *The Cesàro means of* $s(t)$ *are defined by*

$$\sigma(t) = \frac{1}{t}\int_0^t s(u)du, \qquad t \in (0, \infty). \tag{3.1}$$

The integral

$$\int_0^\infty f(x)dx \tag{3.2}$$

is said to be Cesàro summable to a fuzzy number L *if* $\lim_{t\to\infty} \sigma(t) = L$. *The value of this limit is said to be the Cesàro sum of the integral.*

Theorem 3. *If the integral (3.2) converges to a fuzzy number* L, *then (3.1) also converges to* L.

Proof. Let

$$\lim_{t\to\infty} s(t) = \int_0^\infty f(x)dx = L$$

for some $L \in E^1$. Then given any $\varepsilon > 0$ there exists $t_0 > 0$ such that $D(s(t), L) < \frac{\varepsilon}{2}$ whenever $t \geq t_0$ and there exists $M > 0$ such that $D(s(t), L) < M$ whenever $t < t_0$. So we have

$$\begin{aligned}
D(\sigma(t), L) &= D\left(\frac{1}{t}\int_0^t s(u)du, L\right) \\
&= D\left(\frac{1}{t}\int_0^t s(u)du, \frac{1}{t}\int_0^t Ldu\right) \\
&= \frac{1}{t}D\left(\int_0^t s(u)du, \int_0^t Ldu\right) \\
&\leq \frac{1}{t}\int_0^t D(s(u), L)du \\
&= \frac{1}{t}\int_0^{t_0} D(s(u), L)du + \frac{1}{t}\int_{t_0}^t D(s(u), L)du \\
&\leq \frac{t_0 M}{t} + \frac{\varepsilon}{2}\frac{(t - t_0)}{t} < \frac{t_0 M}{t} + \frac{\varepsilon}{2}
\end{aligned}$$

Since $\lim\limits_{t\to\infty} \frac{t_0 M}{t} = 0$, there exists $t_1 > 0$ such that $\left|\frac{t_0 M}{t}\right| < \frac{\varepsilon}{2}$ whenever $t \geq t_1$. So there exists $t_2 = \max\{t_0, t_1\}$ such that

$$D(\sigma(t), L) < \varepsilon$$

whenever $t \geq t_2$. This completes the proof. $\square$

By the following example it can be easily seen that the converse statement of Theorem 3 is not true in general.

Example 1. *Take the fuzzy-number-valued function $f : [0, \infty) \to E^1$ such that*

$$(f(x))(t) = \begin{cases} (t - \cos x).(x + 1)^2, & \text{if} \quad \cos x \leq t \leq \cos x + \frac{1}{(1+x)^2}, \\ 2 - (t - \cos x).(x + 1)^2, & \text{if} \quad \cos x + \frac{1}{(1+x)^2} \leq t \leq \cos x + \frac{2}{(1+x)^2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then f is continuous and

$$f_\alpha^-(x) = \cos x + \frac{\alpha}{(x+1)^2} \quad , \quad f_\alpha^+(x) = \cos x + \frac{2-\alpha}{(x+1)^2}$$

$$\int_0^t f_\alpha^-(x)dx = sint + \alpha\left(1 - \frac{1}{t+1}\right) \quad , \quad \int_0^t f_\alpha^+(x)dx = sint + (2-\alpha)\left(1 - \frac{1}{t+1}\right)$$

Obviously $\int\limits_0^\infty f(x)dx$ is divergent. To calculate Cesàro mean, considering (3.1) we have

$$\begin{aligned}
\sigma_\alpha^-(t) &= \frac{1}{t}\int_0^t s_\alpha^-(u)du = \frac{1}{t}\int_0^t \left(\int_0^u f_\alpha^-(x)dx\right) du = -\frac{\cos t}{t} + \frac{1}{t} + \alpha\left(1 - \frac{\ln(t+1)}{t}\right) \\
\sigma_\alpha^+(t) &= \frac{1}{t}\int_0^t s_\alpha^+(u)du = \frac{1}{t}\int_0^t \left(\int_0^u f_\alpha^+(x)dx\right) du = -\frac{\cos t}{t} + \frac{1}{t} + (2-\alpha)\left(1 - \frac{\ln(t+1)}{t}\right).
\end{aligned}$$

So we get

$$\left.\begin{array}{l}\lim_{t\to\infty}\sigma_\alpha^-(t)=\alpha\\ \lim_{t\to\infty}\sigma_\alpha^+(t)=2-\alpha\end{array}\right\}\Longrightarrow [L]_\alpha=[\alpha,2-\alpha] \text{ and } \lim_{t\to\infty}D(\sigma(t),L)=0$$

Then $\int\limits_0^\infty f(x)dx$ *is Cesàro summable to fuzzy number* L *such that*

$$L(t)=\begin{cases} t & \text{if} \quad 0\le t\le 1,\\ 2-t & \text{if} \quad 1\le t\le 2,\\ 0 & \quad\text{otherwise.}\end{cases}$$

We need the following Lemma for the proofs of our main results.

Lemma 3. *If* s *be a continuous fuzzy-number-valued function then for every* $\lambda>1$

$$\frac{1}{\lambda t-t}\int_t^{\lambda t}s(x)dx+\frac{1}{\lambda-1}\sigma(t)=\sigma(\lambda t)+\frac{1}{\lambda-1}\sigma(\lambda t) \tag{3.3}$$

and for every $0<\ell<1$

$$\frac{1}{t-\ell t}\int_{\ell t}^{t}s(x)dx+\frac{\ell}{1-\ell}\sigma(\ell t)=\sigma(t)+\frac{\ell}{1-\ell}\sigma(t). \tag{3.4}$$

Proof. Let s be a continuous fuzzy-number-valued function. Then for every $\lambda>1$ we have

$$\begin{aligned}\sigma(\lambda t)+\frac{1}{\lambda-1}\sigma(\lambda t) &= \frac{\lambda}{\lambda-1}\sigma(\lambda t)\\ &= \frac{\lambda}{\lambda-1}\frac{1}{\lambda t}\int_0^{\lambda t}s(x)dx\\ &= \frac{1}{(\lambda-1)t}\left\{\int_0^t s(x)dx+\int_t^{\lambda t}s(x)dx\right\}\\ &= \frac{1}{\lambda-1}\sigma(t)+\frac{1}{t(\lambda-1)}\int_t^{\lambda t}s(x)dx\end{aligned}$$

by Lemma 1 and Theorem 2. On the other hand for every $0<\ell<1$, using Lemma 1 and Theorem 2 again, we get

$$\begin{aligned}
\sigma(t)+\frac{\ell}{1-\ell}\sigma(t) &= \frac{1}{1-\ell}\sigma(t)\\
&= \frac{1}{1-\ell}\frac{1}{t}\int_0^t s(x)dx\\
&= \frac{1}{1-\ell}\frac{1}{t}\left\{\int_0^{\ell t} s(x)dx+\int_{\ell t}^t s(x)dx\right\}\\
&= \frac{\ell}{1-\ell}\frac{1}{\ell t}\int_0^{\ell t} s(x)dx+\frac{1}{t(1-\ell)}\int_{\ell t}^t s(x)dx\\
&= \frac{\ell}{1-\ell}\sigma(\ell t)+\frac{1}{t-\ell t}\int_{\ell t}^t s(x)dx.
\end{aligned}$$

So equalities (3.3) and (3.4) are satisfied. □

As a result of Lemma 3 we conclude the following lemma.

Lemma 4. *If integral (3.2) is Cesàro summable to a fuzzy number L, then for every $\lambda>1$*

$$\lim_{t\to\infty}\frac{1}{\lambda t-t}\int_t^{\lambda t} s(x)dx=L \tag{3.5}$$

and for every $0<\ell<1$

$$\lim_{t\to\infty}\frac{1}{t-\ell t}\int_{\ell t}^{t} s(x)dx=L. \tag{3.6}$$

Now we give Tauberian conditions under which convergence of the improper integral follows from Cesàro summability.

Theorem 4. *Let fuzzy-number-valued function $f:[0,\infty)\to E^1$ be continuous. If integral (3.2) is Cesàro summable to a fuzzy number L, then it converges to L if and only if for every $\varepsilon>0$ there exist $t_0\geq 0$ and $\lambda>1$ such that for $t>t_0$*

$$\frac{1}{\lambda t-t}\int_t^{\lambda t} s(x)dx\succeq s(t)-\overline{\varepsilon} \tag{3.7}$$

and another $0<\ell<1$ such that

$$\frac{1}{t-\ell t}\int_{\ell t}^{t} s(x)dx\preceq s(t)+\overline{\varepsilon}. \tag{3.8}$$

Proof. Necessity. Let the integral (3.2) converge to L. Using inequality

$$D\left(\frac{1}{\lambda t-t}\int_t^{\lambda t} s(x)dx, s(t)\right)\leq D\left(\frac{1}{\lambda t-t}\int_t^{\lambda t} s(x)dx, L\right)+D(L,s(t)),$$

if we consider the equality (3.5) in Lemma 4 then for $\lambda > 1$ we obtain

$$\lim_{t\to\infty} D\left(\frac{1}{\lambda t - t}\int_t^{\lambda t} s(x)dx, s(t)\right) = 0.$$

For $0 < \ell < 1$, validity of (3.8) can also be obtained analogously by using the equality (3.6) of Lemma 4.

Sufficiency. Assume that integral (3.2) is Cesàro summable to L and (3.7), (3.8) are satisfied. By (3.7), there exist $t_1 \geq 0$ and $\lambda > 1$ such that for $t > t_1$

$$\frac{1}{\lambda t - t}\int_t^{\lambda t} s(x)dx \succeq s(t) - \frac{\overline{\varepsilon}}{3}\cdot$$

Besides since

$$\lim_{t\to\infty} D\left(\frac{1}{\lambda - 1}\sigma(t), \frac{1}{\lambda - 1}\sigma(\lambda t)\right) = 0,$$

there exists $t_2 \geq 0$ such that for $t > t_2$

$$D\left(\frac{1}{\lambda - 1}\sigma(t), \frac{1}{\lambda - 1}\sigma(\lambda t)\right) \leq \frac{\varepsilon}{3}\cdot$$

So by (i) of Lemma 2 we get that

$$\frac{1}{\lambda - 1}\sigma(t) - \frac{\overline{\varepsilon}}{3} \preceq \frac{1}{\lambda - 1}\sigma(\lambda t) \preceq \frac{1}{\lambda - 1}\sigma(t) + \frac{\overline{\varepsilon}}{3}.$$

Also, since $\lim\limits_{t\to\infty} \sigma(\lambda t) = L$, there exists $t_3 \geq 0$ such that $D(\sigma(\lambda t), L) \leq \frac{\varepsilon}{3}$ for $t > t_3$, meaning

$$L - \frac{\overline{\varepsilon}}{3} \preceq \sigma(\lambda t) \preceq L + \frac{\overline{\varepsilon}}{3}\cdot$$

Then considering the equality (3.3) , there exists $t_4 = \max\{t_1, t_2, t_3\}$ such that for $t > t_4$

$$s(t) - \frac{\overline{\varepsilon}}{3} + \frac{1}{\lambda - 1}\sigma(t) \preceq L + \frac{\overline{\varepsilon}}{3} + \frac{1}{\lambda - 1}\sigma(t) + \frac{\overline{\varepsilon}}{3}\cdot$$

So by (v) of Lemma 2, for $t > t_4$ we have

$$s(t) \preceq L + \overline{\varepsilon}. \tag{3.9}$$

On the other hand, if we consider the condition (3.8), equality (3.4), Lemma 2 and proceed in a similar way as that above, we get that there exists a $t_4^* \geq 0$ such that for $t > t_4^*$

$$s(t) \succeq L - \overline{\varepsilon}. \tag{3.10}$$

Then combining inequalities (3.9) and (3.10), we obtain

$$L - \overline{\varepsilon} \preceq s(t) \preceq L + \overline{\varepsilon}$$

whenever $t > \max\{t_4, t_4^*\}$ and this completes the proof. $\square$

Definition 5. *A fuzzy-number-valued function $s(x)$ is said to be slowly decreasing if for every $\varepsilon > 0$ there exist $t_0 \geq 0$ and $\lambda > 1$ such that*

$$s(x) \succeq s(t) - \overline{\varepsilon}$$

whenever $t_0 < t < x \leq \lambda t$.

Remark 1. *Fuzzy-number-valued function $s(x)$ is slowly decreasing if and only if the family of real valued functions $\{s_\alpha^-(x) \mid \alpha \in [0,1]\}$ and $\{s_\alpha^+(x) \mid \alpha \in [0,1]\}$ are equi-slowly decreasing i.e. $\forall \varepsilon > 0$ there exist $t_0 \geq 0$ and $\lambda > 1$ such that for all $\alpha \in [0,1]$*

$$s_\alpha^-(x) - s_\alpha^-(t) \geq -\varepsilon \quad \textit{and} \quad s_\alpha^+(x) - s_\alpha^+(t) \geq -\varepsilon \qquad \textit{whenever} \quad t_0 < t < x \leq \lambda t.$$

Lemma 5. *If the fuzzy-number-valued function $s(x)$ is slowly decreasing, then for every $\varepsilon > 0$ there exist $t_0 \geq 0$ and $0 < \lambda < 1$ such that for every $t > t_0$*

$$s(t) \succeq s(x) - \overline{\varepsilon} \qquad \textit{whenever} \qquad \lambda t < x \leq t. \tag{3.11}$$

Proof. The proof of the lemma is done by contradiction method. Assume that the fuzzy-number-valued function $s(x)$ is slowly decreasing and there exists $\varepsilon_0 > 0$ such that for all $0 < \lambda < 1$ and $t_0 \geq 0$ there exist real numbers x and $t > t_0$ for which

$$s(t) \not\succeq s(x) - \overline{\varepsilon}_0 \qquad \text{whenever} \qquad \lambda t < x \leq t.$$

Therefore, there exists $\alpha_0 \in [0,1]$ such that

$$s_{\alpha_0}^-(t) < s_{\alpha_0}^-(x) - \varepsilon_0 \quad \textit{or} \quad s_{\alpha_0}^+(t) < s_{\alpha_0}^+(x) - \varepsilon_0. \tag{3.12}$$

At this point we recall the reformulated condition of Móricz [9] for a slowly decreasing real valued function f such that

$$\lim_{\lambda \to 1^-} \liminf_{t \to \infty} \min_{\lambda t \leq x \leq t} [f(t) - f(x)] \geq 0. \tag{3.13}$$

No matter which case we choose in (3.12), one of the real valued functions $s_{\alpha_0}^-(t)$ and $s_{\alpha_0}^+(t)$ does not satisfy the condition (3.13). So at least one of them is not slowly decreasing which contradicts the hypothesis that fuzzy-number-valued function $s(x)$ is slowly decreasing. □

It is clear that if function s is slowly decreasing then conditions (3.7) and (3.8) are satisfied by (i) and (v) of Theorem 2. So next corollary immediately follows:

Corollary 1. *If f is a continuous fuzzy-number-valued function such that integral (3.2) is Cesàro summable to a fuzzy number L and its integral function $s(t)$ is slowly decreasing, then the integral (3.2) converges to L.*

Theorem 5. *Let f be a continuous fuzzy-number-valued function on $[0,\infty)$. If there exist negative constant fuzzy number u and a real number $x_0 \geq 0$ such that*

$$xf(x) \succeq u \quad \textit{for} \quad x > x_0, \tag{3.14}$$

then fuzzy-number-valued function $s(t) = \int_0^t f(x)dx$ is slowly decreasing.

Proof. Let $xf(x) \succeq u$ be satisfied under the given conditions on u and x_0 in the theorem. Then for $x > x_0$ we have

$$xf_\alpha^-(x) \geq u_\alpha^- \geq u_0^- \quad , \quad xf_\alpha^+(x) \geq u_\alpha^+ \geq u_1^+ \geq u_0^- .$$

For the sake of simplicity let take $u_0^- = -H$ where $H > 0$. Then

$$xf_\alpha^-(x) \geq -H \Rightarrow f_\alpha^-(x) \geq -\frac{H}{x} \quad , \quad xf_\alpha^+(x) \geq -H \Rightarrow f_\alpha^+(x) \geq -\frac{H}{x}$$

are satisfied. Then for $x_0 < t < x \leq \lambda t$ when $\lambda > 1$, we have

$$s_\alpha^-(x) - s_\alpha^-(t) = \int_t^x f_\alpha^-(u)du \geq -H \int_t^x \frac{du}{u} = -H \ln \frac{x}{t} \geq -H \ln \lambda$$

and

$$s_\alpha^+(x) - s_\alpha^+(t) = \int_t^x f_\alpha^+(u)du \geq -H \int_t^x \frac{du}{u} = -H \ln \frac{x}{t} \geq -H \ln \lambda .$$

Choosing $\lambda = e^{\varepsilon/H}$, we get the inequalities

$$s_\alpha^-(x) \geq s_\alpha^-(t) - \varepsilon \quad , \quad s_\alpha^+(x) \geq s_\alpha^+(t) - \varepsilon$$

and then $s(x) \succeq s(t) - \overline{\varepsilon}$ holds whenever $x_0 < t < x \leq \lambda t$. $\square$

Example 2. *Let the fuzzy-number-valued function $f : [0, \infty) \to E^1$ be given as*

$$(f(x))(t) = \begin{cases} \frac{t}{2-\sin x} \quad , & \text{if} \quad 0 \leq t \leq 2 - \sin x, \\ 2 - \frac{t}{2-\sin x} \quad , & \text{if} \quad 2 - \sin x \leq t \leq 2(2 - \sin x). \end{cases}$$

Then

$$f_\alpha^-(x) = (2 - \sin x)\alpha \quad , \quad f_\alpha^+(x) = (2 - \sin x)(2 - \alpha).$$

Since $f_\alpha^\pm(x) \geq 0$ holds for each $\alpha \in [0, 1]$ and $x > 0$, we have

$$xf_\alpha^-(x) \geq 0 \quad , \quad xf_\alpha^+(x) \geq 0$$

which means that $xf(x) \succeq \overline{0}$. So $s(t)$ is slowly decreasing.

As a result of Theorem 5 the following one-sided Tauberian condition is obtained.

Corollary 2. *If f is a continuous fuzzy-number-valued function such that integral (3.2) is Cesàro summable to a fuzzy number L and condition (3.14) is satisfied, then the integral (3.2) converges to L.*

We note that one may extend Cesàro summability method to continuous fuzzy-number-valued functions and give analogs of Theorem 3–4, Corollary 1 for Cesàro summability of fuzzy-number-valued functions. The proofs are done identically by replacing integral function s with general continuous fuzzy-number-valued function in corresponding proofs and hence omitted.

Definition 6. *A continuous fuzzy-number-valued function* $f : [0, \infty) \to E^1$ *is said to be Cesàro summable to a fuzzy number* L *if*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(x)dx = L.$$

Theorem 6. *Let* f *be a continuous fuzzy-number-valued function. If* $\lim_{t\to\infty} f(t) = L$*, then* f *is Cesàro summable to fuzzy number* L*.*

Theorem 7. *If a continuous fuzzy-number-valued function* f *is Cesàro summable to a fuzzy number* L*, then* $\lim_{t\to\infty} f(t) = L$ *if and only if for every* $\varepsilon > 0$ *there exist* $t_0 \geq 0$ *and* $\lambda > 1$ *such that for* $t > t_0$

$$\frac{1}{\lambda t - t} \int_t^{\lambda t} f(x)dx \succeq f(t) - \overline{\varepsilon}$$

and another $0 < \ell < 1$ *such that*

$$\frac{1}{t - \ell t} \int_{\ell t}^{t} f(x)dx \preceq f(t) + \overline{\varepsilon}.$$

Theorem 8. *If a continuous fuzzy-number-valued function* f *is Cesàro summable to a fuzzy number* L *and* f *is slowly decreasing, then* $\lim_{t\to\infty} f(t) = L$*.*

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PORTFOLIO OPTIMIZATION UNDER PARAMETER UNCERTAINTY USING THE RISK AVERSION FORMULA

SIBEL ACIK KEMALOGLU, GULTAC EROGLU INAN, AND AYSEN APAYDIN

Abstract. The Markowitz portfolio optimization model has certain difficulties in practise since real data are rarely certain. The robust optimization is a recently developed method that is used to overcome the uncertainty situation. The technique has been recently suggested in the portfolio selection problems. In this study, two kinds of portfolio optimization problems are presented: (i) the risk aversion portfolio optimization problem based on the classical Markowitz framework, and (ii) the max-min counterpart problem based on the robust optimization framework. In the application, the two models are performed on a real-world data set obtained from BIST (Borsa Istanbul). Numerical results show that the objective function values of the classical solution and the robust solution are similar to each other. It can be said that the robust model, which works as well as the classical model in the uncertainty situations, can be used instead of the classical model and also that the optimal solution obtained in the uncertainty situation is robust to parameter perturbation.

1. INTRODUCTION

The main objective of the portfolio optimization problem is to choose the optimal portfolio with minimum variance from the set of all possible portfolios for any given level of expected return. Markowitz [22] formulated the first mathematical model for portfolio selection in the literature. After Markowitz, Sharpe [26] developed the Capital Asset Pricing Model (CAPM) and then Linter [19] and Mossin [24] used the Markowitz theory in their studies. In the literature, there are various other portfolio optimization methods developed in the context of the portfolio theory besides Markowitz, such as safety-first models, elliptical distributions, value at risk-based optimization, maximizing the performance measures EVA and RAROC and modelling the uncertainty of input parameters [10].

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The Markowitz mean-variance portfolio optimization is a well-known investment theory that is widely used in allocating the assets. Its biggest influence can be seen on the practice of portfolio management. The theory is focused on evaluating and managing the risks and returns of a portfolio of investments. This is highly advantageous as the resulting "optimized" portfolio will either have the same expected return with fewer risks than before or a higher expected return with the same level of risk. The Markowitz mean-variance optimization problem has several alternative formulations that are used in practical applications. One of these alternative formulations is using a risk aversion coefficient in the model, which is called the risk aversion formulation. This study handles the risk aversion formulation of the classical Markowitz model.

Although the Markowitz model is successful in the theory, there are various challenges of the model. The parameter uncertainty is an important issue in the optimization problems. In the Markowitz model, the uncertainty in the market parameters affects the optimal solution of the problem. Thus, the results cannot be reliable enough. There are numerous studies in the literature to overcome the difficulties of the Markowitz model: Chopra and Ziemba [8] studied the estimated parameters. Broadie and Chopra [6] used the estimation errors in their study. Chopra [7] and Frost and Savarino [12, 13] presented a method related to the portfolio weights. Chopra et al. [9] used the James-Stein estimator for the means, Klein and Bawa [16], Frost and Savarino [12], and Black and Litterman [5] used the Bayesian estimation of means and covariances [19].

An underlying assumption of Markowitz's model is that the precise estimates of μ_i and σ_{ij} have been obtained. Consequently, μ_i and σ_{ij} are treated as known constants; however, asset returns are variable. It is reasonable to conclude that a model which treats returns as known constants will produce a portfolio whose realized return is different from the optimal portfolio return given by the objective function value. In particular, when the realized asset returns are less than the estimates used to optimize the model, the realized portfolio return will be less than the optimal portfolio return given by the objective. Therefore, it is worthwhile exploring the alternative frameworks, such as the robust optimization, for application to the portfolio selection problem. Although the distributions of asset returns are uncertain, in the robust optimization framework, it may be asserted that $\mu$and σ, or both, belong to an uncertainty set, the bounds of which can be defined [15].

The aim of the method is to obtain a solution that is robust to the parameter uncertainty and estimation errors. In this framework, the ***robust counterpart*** of the original problem is handled. The robust problem is in fact the ***worst-case*** formulation of the original problem.

The first studies in the robust optimization framework are given in the studies of Ben-Tal and Nemirovski [2], [3], [4]. The first study handled the robust approach for linear programming. The other studies introduced the robust framework for convex programming. In these studies, it is assumed that the model parameters are unknown, but they are bounded and belong to the specific uncertainty sets defined by historical knowledge. The aim of the robust (worst-case) approach is to obtain the optimal solution of the model which is robust to the parameter uncertainty and the *worst-case* situation. The ***robust counterpart*** of the original problem is handled in the robust model.

Goldfarb and Iyengar handled various robust portfolio selection problems in their study, such as the robust mean-variance portfolio selection, the robust minimum variance problem, the robust maximum return problem, the robust maximum Sharpe Ratio problem, and the robust value at risk problem [14].

There are many latest references in the literature about the robust portfolio selection problem. Wang and Cheng [28] considered the robust portfolio selection problem which has a data uncertainty described by the (p, w)-norm in the objective function. Balbás A., Balbás B. and Balbás R. [1] handled portfolio selection problems under risk and ambiguity. Yu X. [29] developed a multi-period mean-variance model where the model parameters change according to a market with Markov random regime switching. Nalan G. and Canakoglu E. [25] considered a portfolio selection problem under temperature uncertainty. They introduced stochastic and robust portfolio optimization models using weather derivatives. Lotfi S., Salahi M. and Mehrdoust F. [20] used the robust optimization approach to address the ambiguity in the conditional value-at-risk minimization model.

The aim of this study is to introduce the risk aversion portfolio selection problem under the input parameter μ uncertainty. This problem is called the (*maxmin*) robust counterpart of the risk aversion problem. Moreover, it is aimed to obtain the optimal portfolio (the optimal solution of the robust problem) under this uncertainty and to compare the solution with the classical risk aversion solution.

In Inan [16] and Inan [17], the robust optimization approach is studied on the portfolio optimization problem. Numerical results showed that the classical optimal solution and the robust optimal solution gave similar values to the objective function. As a result, the optimal solution obtained in the uncertainty case is robust to the uncertainty case. The finding in the study is consistent with these studies.

The rest of this paper is organized as follows: In Section 2, the Markowitz portfolio optimization model and another alternative model, the risk aversion problem, is introduced. Section 3 presents the robust portfolio optimization method. The (max-min) robust counterpart of the problem is given. Finally, the max-min problem is converted into the classical maximum problem by the Lagrange method. In Section 4, a numerical example of the model with a real data set is handled. The data is taken from BIST (Borsa Istanbul). In Section 5, some conclusions in certain and uncertain situations are given.

2. MARKOWITZ MEAN-VARIANCE PORTFOLIO OPTIMIZATION PROBLEM

Harry Markowitz published his study and formed the basis for the ***mean-variance optimization*** "Portfolio Selection" in 1952. He suggested that investors should create the optimal portfolio based on the balance between the expected return and the risk. In the Markowitz portfolio model, the returns are defined as the mean vector, and the risk is defined as the variance of return. The model uses the optimization and probability methods together under uncertainty. The model comprises the return matrix, the mean vector and variance-covariance matrix components.

Suppose that an investor has a portfolio comprised of n risky assets, denoted as S_i The return of the security S_i is defined as R_i, and the weight of the i.security in the portfolio is defined as X_i.

The model can be created in two frameworks: (i) minimizing the risk of the portfolio for a certain level of expected return, (ii) maximizing the return of the portfolio for a certain level of risk.

The first model is given as,

$$\min X^t \Sigma X$$

$$\begin{aligned} \mu^t X &\geq \alpha \\ \sum_{i=1}^{n} X_i &= 1 \\ X_i &\geq 0, \quad i = 1, ..., n \end{aligned} \tag{2.1}$$

The second model is given as,

$$\max \mu^t X$$

$$\begin{aligned} X^t \Sigma X &\leq \beta \\ \sum_{i=1}^{n} X_i &= 1 \\ X_i &\geq 0, \quad i = 1, ..., n \end{aligned} \tag{2.2}$$

where α, β are constant, which are called the level degree. The descriptions of the model components are given as follows:

$(R_{k1}, ..., R_{kn})^t$ represents the n kinds of returns at time k $(k = 1, ..., m)$, where R_{ki} is the return of i.securities, $i = 1, ..., n,\ k = 1, ..., m$. The total data matrix is represented as,

$$\begin{bmatrix} R_{11} & ... & R_{1n} \\ \vdots & & \vdots \\ R_{m1} & ... & R_{mn} \end{bmatrix} \tag{2.3}$$

The return vector is denoted as $R = [R_1...R_n]^t$ in m period, it contains the expected value (mean) of each security. The expected vector of R is denoted as;

$$\mu = [\mu_1...\mu_n]^t$$

The input parameters μ and Σ are not certain. It is very difficult to estimate the correct values of these parameters. In the Markowitz model, the estimates of these parameters are used as follows:

$$\mu = [\mu_1...\mu_n]^t \tag{2.4}$$

$\mu = \left[\sum_{k=1}^{m} \frac{R_{k1}}{m} ... \sum_{k=1}^{m} \frac{R_{kn}}{m}\right]^t$ and the covariance matrix is given by,

$$\Sigma = \begin{bmatrix} \sigma_{11} & ... & \sigma_{1N} \\ \cdots & & \cdots \\ \sigma_{N1} & \cdots & \sigma_{NN} \end{bmatrix} \tag{2.5}$$

Here; $\sigma_{ij}$is the covariance between asset i and asset j.

The corresponding variance is given as,

$$\sigma_{ij}{}^2 = \sum_{k=1}^{m} \frac{(R_{ki} - \mu_i)\left(R_{kj} - \mu_j\right)}{m-1} \tag{2.6}$$

Thus, the random return vector R is represented by the (μ, Σ), [25].

There are two different definitions of R. One of them is the random vector. In finance applications, one should use the adjusted (from splits and dividends) stock prices to make calculations. However, it is difficult to obtain the adjusted stock prices from the splits and dividends, so only the closing prices are used in the study.

The alternative model that combines the risk and the return of the objective function can be created using the coefficient of risk aversion. The risk aversion formulation problem is defined as,

$$\max\left(\mu^{'} X - \lambda X^{'} \Sigma X\right)$$

$$X^{'} l = 1, \quad l = [1, 1, ..., 1] \tag{2.7}$$

where, λ is the risk aversion coefficient. When the investor is exposed to the uncertainty situation, the risk aversion coefficient can be used to reduce that uncertainty. If λ is large, the aversion to the risk is high. For example, the risk-averse investor might make an investment in treasury bonds that have low but guaranteed expected returns. Otherwise, if λ is small, the aversion to the risk is low. For example, the risk-loving investor might make an investment in stocks, the options of which have high expected returns but also high risks.

3. ROBUST PORTFOLIO OPTIMIZATION PROBLEM

In spite of the theoretical success of the mean-variance model, practitioners have shied away from this model. The solution of optimization problems is often very sensitive to perturbations in the parameters of the problem. Since the estimates of the market parameters are subject to statistical errors, they are very sensitive to the perturbations in the inputs. The results of the optimization problems may not be very reliable. There are a number of discussions on how to decrease or eliminate the possibility of using incorrect inputs for the optimization problem. Various aspects of this phenomenon have been extensively studied in the literature on portfolio selection.

Michaud [23] proposed to use the technique of resampling. In his study, he suggested resampling the input parameters from a confidence region and then averaging the cumulative portfolios that were obtained by each pair of sampling data. The main idea is that if resampling was performed enough times, the averaged optimal portfolio would be more stable and less sensitive to the perturbations in the inputs. But when the amount of assets becomes large, this method is not useful and efficient [21].

The robust optimization is the one of the aspects in the portfolio selection problems [14]. In the robust approach, the ***worst-case*** formulation of the original optimization problem, called the ***robust counterpart*** of the problem, is handled. The robust counterpart of the classical risk aversion model is used in this study.

In [11], the (*maxmin*) ***robust counterpart*** of the risk aversion model is given as

$$\max_{X} \min_{\mu \in U_\delta(\hat{\mu})} \left(\mu^{'} X - \lambda X^{'} \Sigma X \right)$$

$$X^{'} l = 1, \quad l = [1, 1, ..., 1] \tag{3.1}$$

In the problem, it is assumed that the expected return vector $\mu$is unknown but belongs to the specific uncertainty set $U_\delta(\hat{\mu})$. Many special uncertainty sets are defined for the uncertain parameters in the literature. In this study, the uncertainty set for μ is taken as

$$U_\delta(\hat{\mu}) = \{\mu / (\mu - \hat{\mu})' (\Sigma_\mu)^{-1} (\mu - \hat{\mu}\ ) \leq \delta^2\} \tag{3.2}$$

where the parameters of the model are defined as follows:

$\hat{\mu}$:The estimated expected return vector

μ :The true expected return vector

$\Sigma_\mu = \frac{1}{T}\Sigma$, Estimation error covariance matrix

T : Return data observations for N assets.

δ : small number ($\delta > 0$)

The aim of the problem is to determine the weight vector X, which is robust to the uncertainty and the worst-case realization of the μ parameter.

For solving the robust (*maxmin*) problem easily, the problem is converted to the standard *maximum* optimization problem as follows:

Firstly, the uncertainty set is written as a constraint, then the problem can be written as

$$min_{\mu}\left(\mu^{'}X-\lambda X^{'}\Sigma X\right)$$

$$(\mu-\hat{\mu})^{'}(\Sigma_{\mu})^{-1}(\mu-\hat{\mu})\leq\delta^{2} \tag{3.3}$$

To solve this problem, the Lagrangian method can be used. The Lagrangian of the problem takes the form,

$$L(\mu,\gamma)=\mu^{'}X-\lambda X^{'}\Sigma X-\gamma\left(\delta^{2}-(\mu-\hat{\mu})^{'}(\Sigma_{\mu})^{-1}(\mu-\mu)\right) \tag{3.4}$$

The optimal values of μ and γ are obtained by the first order condition as

$$\mu^{*}=\hat{\mu}-\frac{1}{2\gamma}\Sigma_{\mu}X \tag{3.5}$$

$$\gamma^{*}=\frac{1}{2\delta}\sqrt{X'\Sigma_{\mu}X} \tag{3.6}$$

Finally, by substituting the expressions in the Lagrangian form, the robust problem is obtained as

$$\max\left(\mu^{'}X-\lambda X^{'}\Sigma X-\delta\sqrt{X'\Sigma_{\mu}X}\right)$$

$$X^{'}l=1. \tag{3.7}$$

4. APPLICATION

In this section, the robust portfolio selection approach, which was originally presented in the study by Fabozzi et al., is suggested. The data set is taken as the daily closing prices of nine securities that cycled in BIST 100 between 20.08.2013 and 20.08.2015. In the study, the daily stock price is chosen instead of the monthly stock price because the number of monthly stock price, which is 24 (for two years), may not be enough for the application.

The securities taken from the automotive sector belong to **Balat, Asuzu, Daos, Karsn, Tmsn, Froto, Toaso, Ttrak,** and **Otkar.** The returns of the securities were calculated according to the expression $ln\,(P_t/P_{t-1})$ of the closing prices. Here;

P_t : Closing prices of t. day

The average vector μ, the variance covariance matrix Σ, and the estimation error covariance matrix Σ_M are calculated on the returns. Here;

$$\Sigma_M=\frac{1}{T}\Sigma$$

T :Return data observations for N assets

In this study, T is given as 502 days between the designated dates (20.08.2013–20.08.2015). The return vector μ, the Σ variance covariance matrix Σ and the estimation error covariance matrix Σ_μ are obtained as follows. Note that the values in the variance covariance matrix Σ_μ are multiplied by 1000.

$$\mu = \begin{bmatrix} 0.00000620 \\ -0.00157000 \\ 0.00088650 \\ 0.00018563 \\ 0.00063045 \\ 0.00069976 \\ 0.00043500 \\ 0.00091629 \\ 0.00054343 \end{bmatrix}$$

$$\Sigma = \begin{matrix}
0.000623 & 0.000047 & 0.000363 & 0.000214 & 0.000249 & 0.000291 & 0.000283 & 0.000238 & 0.000200 \\
0.000047 & 0.000804 & 0.000076 & 0.000024 & 0.000119 & 0.000047 & 0.000102 & 0.000053 & 0.000039 \\
0.000363 & 0.000076 & 0.000869 & 0.000306 & 0.000324 & 0.000338 & 0.000340 & 0.000396 & 0.000213 \\
0.000214 & 0.000024 & 0.000306 & 0.000429 & 0.000146 & 0.000216 & 0.000226 & 0.000281 & 0.000137 \\
0.000249 & 0.000119 & 0.000324 & 0.000146 & 0.000607 & 0.000254 & 0.000297 & 0.000188 & 0.000158 \\
0.000291 & 0.000047 & 0.000338 & 0.000216 & 0.000254 & 0.000674 & 0.000337 & 0.000243 & 0.000213 \\
0.000283 & 0.000102 & 0.000340 & 0.000226 & 0.000297 & 0.000337 & 0.000765 & 0.000239 & 0.000177 \\
0.000238 & 0.000053 & 0.000396 & 0.000281 & 0.000188 & 0.000243 & 0.000239 & 0.000598 & 0.000149 \\
0.000200 & 0.000039 & 0.000213 & 0.000137 & 0.000158 & 0.000213 & 0.000177 & 0.000149 & 0.000378
\end{matrix}$$

$$\Sigma_\mu = \begin{matrix}
0.001246 & 0.000093 & 0.000725 & 0.000429 & 0.000497 & 0.000582 & 0.000566 & 0.000476 & 0.000400 \\
0.000093 & 0.001607 & 0.000152 & 0.000048 & 0.000239 & 0.000094 & 0.000204 & 0.000105 & 0.000077 \\
0.000725 & 0.000152 & 0.001739 & 0.000611 & 0.000648 & 0.000676 & 0.000680 & 0.000791 & 0.000426 \\
0.000429 & 0.000048 & 0.000611 & 0.000858 & 0.000293 & 0.000433 & 0.000451 & 0.000562 & 0.000275 \\
0.000497 & 0.000239 & 0.000648 & 0.000293 & 0.001215 & 0.000507 & 0.000595 & 0.000376 & 0.000315 \\
0.000582 & 0.000094 & 0.000676 & 0.000433 & 0.000507 & 0.001348 & 0.000675 & 0.000486 & 0.000426 \\
0.000566 & 0.000204 & 0.000680 & 0.000451 & 0.000595 & 0.000675 & 0.001529 & 0.000479 & 0.000354 \\
0.000476 & 0.000105 & 0.000791 & 0.000562 & 0.000376 & 0.000486 & 0.000479 & 0.001195 & 0.000298 \\
0.000400 & 0.000077 & 0.000426 & 0.000275 & 0.000315 & 0.000426 & 0.000354 & 0.000298 & 0.000756
\end{matrix}$$

The classical model and the robust model, which have been defined in Section 2 and Section 3, are handled in the application. The models are given as follows:

The classical risk aversion portfolio optimization problem

$$\max\left(\mu^{'}X - \lambda X^{'}\Sigma X\right)$$

$$X^{'}l = 1, \quad l = [1, 1, ..., 1]$$

The robust problem

$$max\left(\mu^{'}X - \lambda X^{'}\Sigma X - \delta\sqrt{X^{'}\Sigma_\mu X}\right)$$

$$X^{'}l = 1$$

In the first case, the problem (2.7) is solved for the different 20 values of λ, which is chosen by the information given in the Risk Aversion Formula by Fabozzi et al. [11]. The λ is chosen with an increase of 0.2. The [0,4] interval can be divided into smaller pieces so the number of λ can be increased.

For the different values of λ, the movement of the expected return, the variance and the objective function value are seen in the related Figure 1, Figure 2 and Figure 3.The figures show that when the value of λ increases (The aversion to risk is high–the risk-aversion investor), the values of the expected return, the variance and the objective function decrease. For the small values of λ (the aversion to risk

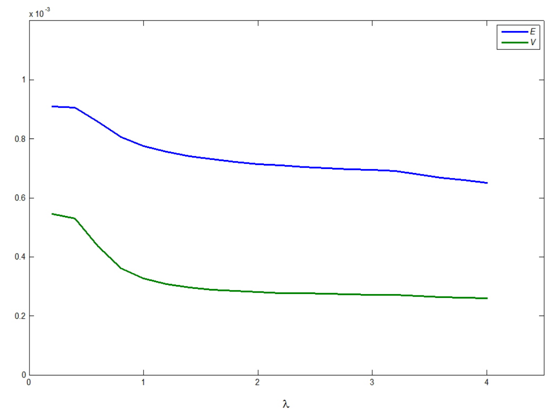

FIGURE 1. The movement of E : *Expected Value* and V : *Variance* for the classical problem

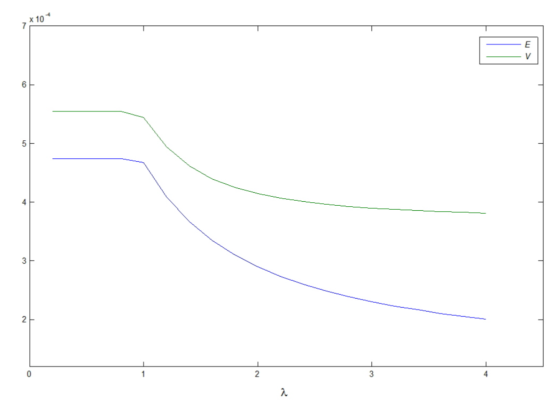

FIGURE 2. The movement of E and V for the robust problem

is low–the risk-lover investor), the values of the expected return, the variance and the objective function increase. In this case, it can be said that if the investor prefers the high expected return, the high risk must be considered.

In the robust case, when the expected return μ parameter is robust, the aim is to show the movement of the optimal solution to the parameter uncertainty. For this aim, the robust problem, which is given in (3.7), is solved for different λ and δ values.

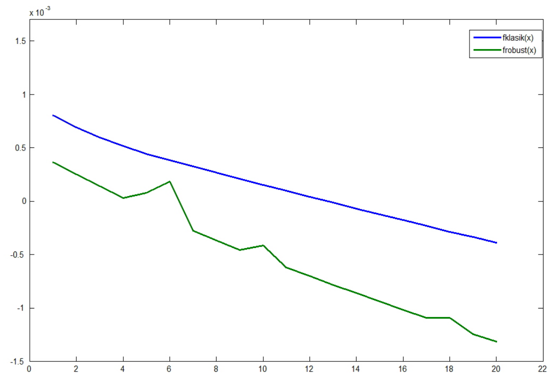

FIGURE 3. The movements of the objective functions for the classical problem and the robust problem

Table 1. The differences between the objective function values for the classical and the robust problem

$\boldsymbol{\lambda}$	$\mathbf{f\_classical}(\mathbf{x}) - \mathbf{f\_robust}(\mathbf{x})$
0.2	0.0004380
0.4	0.0004408
0.6	0.0004546
0.8	0.0004869
1.0	0.0003630
1.2	0.0002022
1.4	0.0006047
1.6	0.0006361
1.8	0.0006652
2.0	0.0005666
2.2	0.0007187
2.4	0.0007442
2.6	0.0007688
2.8	0.0007929
3.0	0.0008167
3.2	0.0008400
3.4	0.0008634
3.6	0.0008103
3.8	0.0009110
4.0	0.0009252

If there is a need to compare the solution of the robust problem with the solution of the classical problem, it is observed that for the same λ values, the expected return of the classical problem is larger than that of the robust problem. At the same time, its variance is smaller than the variance of the robust problem. In this situation, an investor should prefer the classical model.

On the other hand, the performances of portfolio models are measured by the Sharpe Ratio (SR) method. The Sharpe Ratio is defined as

$$SR = \frac{E\left(r_p\right)}{\sqrt{Var\left(r_p\right)}}.$$

High SR values mean high performance. The Sharpe Ratio values obtained for the classical and the robust problem are presented in Table 2.

Table 2. The Sharpe ratio values for the classical and the robust problem.

Sharpe ratio (Classical)	**Sharpe ratio (Robust)**
0.039004	0.020146
0.039332	0.020146
0.041081	0.020146
0.042440	0.020146
0.042931	0.020017
0.043053	0.018400
0.043017	0.017066
0.042916	0.015955
0.042789	0.015024
0.042655	0.014241
0.042526	0.013568
0.042404	0.012994
0.042285	0.012495
0.042081	0.012059
0.041943	0.011673
0.042081	0.011328
0.041514	0.011022
0.041113	0.010750
0.040747	0.010504
0.040390	0.010279

It is seen that the classical Sharpe Ratio values are higher than the robust Sharpe Ratio values. Therefore, it can be said that the performance of the classical problem is better than the performance of the robust problem. However, in the uncertainty situations, it is recommended that the robust problem be used, which works as well as the classical problem.

5. CONCLUSION

The portfolio optimization model of Harry Markowitz can be created in two frameworks, which minimize the risk of the portfolio for a certain level of expected return and maximize the return of the portfolio for a certain level of risk. In spite of the theoretical success of the mean-variance model, practitioners have avoided this model. The alternative model that combines the risk and the return of the objective function can be created using the coefficient of risk aversion. The solution of optimization problems is often very sensitive to perturbations in the parameters of the problem. In the literature, there are many alternative methods suggested to overcome the parameter perturbations. The robust optimization is one of the most commonly used models in the uncertainty case.

The results show that when the value of λ increases, the values of the expected return, the variance and the objective function decrease. It means that the aversion to risk is high here, so it can be said that the investor is the risk-aversion investor. For the small values of λ, on the other hand, the values of the expected return, the variance and the objective function increase. Here, the aversion to risk is low, so it means that the investor is the risk-lover investor. In this case, it can be said that if the investor prefers the high expected return, the high risk must be taken into consideration.

If the solution of the robust problem is to be compared with the solution of the classical problem, it is observed that for the same λ values, the expected return of the classical problem is larger than the robust problem. At the same time, its variance is smaller than the robust problem variance. In this situation, an investor should prefer the classical model. The classical solution obtained in the certainty situation and the solution obtained in the uncertainty situation give similar values at the objective function. Consequently, it can be said that the optimal solution in the uncertainty situation is robust to ambiguity of the parameter μ. The robust model, which works as well as the classical model in the uncertainty situations, can be used instead of the classical model.

Finally, the performances of portfolio models are measured by the Sharpe Ratio (SR) method. It is seen that the classical Sharpe Ratio values are higher than the robust Sharpe ratio values; however, the robust problem, which works well to overcome uncertainty, should be preferred in the uncertainty situations.

In the future studies, the problem can be solved for different values of $\lambda$and δ. The data was taken from the automotive sector in this study. In order to improve the study, it is possible to investigate other sectors. In the modelling part, the short-selling is forbidden. However, in theory it is possible to allow short-selling. Hence, allowing the sort-selling will be studied in the future. The solutions can be compared for all these situations.

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MODIFICATIONS OF KNUTH RANDOMNESS TESTS FOR INTEGER AND BINARY SEQUENCES

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Abstract. Generating random numbers and random sequences that are indistinguishable from truly random sequences is an important task for cryptography. To measure the randomness, statistical randomness tests are applied to the generated numbers and sequences. Knuth test suite is the one of the first statistical randomness suites. This suite, however, is mostly for real number sequences and the parameters of the tests are not given explicitly.

In this work, we review the tests in Knuth Test Suite. We give test parameters in order for the tests to be applicable to integer and binary sequences and make suggestions on the choice of these parameters. We clarify how the probabilities used in the tests are calculated according to the parameters and provide formulas to calculate the probabilities. Also, some tests, like Permutation Test and Max-of-t-test, are modified so that the test can be used to test integer sequences. Finally, we apply the suite on some widely used cryptographic random number sources and present the results.

1. Introduction

Random numbers have an important role in various areas. From daily life cryptographic applications like cell phone, SSL [1] to military communication random numbers are vital. The quality of the random number generator is vital for the security level of the application. For example, if the key used in an encryption algorithm is not random, that is some bits of the key can be guessed with a probability higher than $\frac{1}{2}$, then the complexity for obtaining the ciphertext will be easier than the claimed security of the algorithm. Therefore, generating random numbers and random sequences that are indistinguishable from a truly random sequence is an important task. Random numbers are generated either from a deterministic or an non-deterministic generator. The term *random number generator(RNG)* generally refers to the non-deterministic random number generators. There are various true random number generators actively sold in the market [2, 3]. The

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deterministic random number generators are called *pseudo-random number generators (PRNGs)* [4, 5]. For some reasons like regenerating the random number or the efficiency of the generator, the PRNGs are preferred over RNGs. Among with the advantages PRNGs are weaker than RNGs in terms of randomness of the output as they are deterministic. Therefore, the PRNGs should be tested to measure how their outputs are close to the the outputs of the RNGs. For this purpose, PRNGs are subject to statistical randomness tests.

A statistical randomness test compares a specific property of the sequence to that of a truly random sequence and produces an output value which indicates the randomness of the sequence. For example, in a random bit sequence, the number of ones and the number of zeros should be equal or close to each other. Frequency test [6] checks if the number of occurrences of ones and zeros within the sequence are as expected from a truly random sequence.

A single test is not enough to conclude randomness of a PRNG. The generator should be tested by various statistical randomness tests, each of which inspects a different aspect of a random sequence. Therefore, various tests are gathered together to form a test suite and applied to sequences. Knuth [7], NIST [6], Diehard [8], Dieharder [9], TestU01 [10] are examples of tests suites in the literature.

Knuth is one of the first researchers who published a test suite consisting of 11 tests in his book [7]. In this suite, the underlying theory of tests for real number sequences are given. Some of these tests are intended to be applicable to integer sequences as well. However, assumptions for real number sequences are not suitable for integer sequences and causes problems when testing these sequences. For example, Permutation Test assumes any successive terms cannot be equal and all the test probabilities are given under this assumption but the equality occurs with a non-negligible probability for integer sequences. In order to the make the suite suitable for integer and binary sequences, new combinatorial calculations should be made. Moreover, even if one tests a real number sequence, the test parameters like sequence length, alphabet size, block size and the like, are not given for most of the tests in the suite. Therefore, besides new calculations, corresponding test parameters should be given for each test for the suite to be applicable.

In this paper, we calculate the test probabilities for binary and integer sequences by considering the abovementioned problems. Moreover, we calculate χ^2 probabilities for all tests to have a similar evaluation approach with Knuth. We also give test parameters, necessary sequence lengths and corresponding probability values, regarding efficiency and applicability. As a result, we modify 9 tests of Knuth Test Suite so that the modified tests are applicable to binary sequences.

The paper is organized as follows. In Section 2 the notation used in the paper and preliminary information about the primitives used in the calculations are given. Then, in Section 3, the details of the tests are given. In Section 4 the application results are presented. Finally, Section 5 concludes the paper.

2. Preliminaries

In Knuth Test Suite, integer valued sequences are considered. However, in order to use Knuth Test suite for cryptographic purposes we consider binary sequences in the following manner. Assume that a binary sequence, S of length l, and a block size b are given. Then, partition the sequence into non-overlapping blocks of size b, and discard the remaining terms, if any. Each block is considered as base 2 representation of an integer in $\{0,1,\dots,2^b-1\}$. In this way, we obtain an integer sequence of length $l_b = \left\lfloor \frac{l}{2^b} \right\rfloor$ where the elements are from an alphabet of size $d = 2^b$. In other words,

$$S = s_1 s_2 \dots s_l,\ s_i \in \mathcal{A},\ \text{for } 1 \le i \le l,\ \text{and } \mathcal{A} = \{0,1,\dots,d-1\}.$$

For instance if the binary sequence

$$S = 10010100100111101$$

is given and the alphabet size for the test is 8 (or block size b is 3), then the sequence should be converted to 3-bit integer sequence:

$$S' = (100)_2(101)_2(001)_2(001)_2(111)_2 01 = 4,5,1,1,7.$$

Note that the partitioning is non-overlapping for all the tests mentioned in this paper. It is also trivial to convert any integer sequence to the d-bit integer sequence.

Some tests partition the sequence into blocks of t consecutive elements and consider the distribution of the blocks. In this case, n denotes the number of blocks.

$$\begin{aligned} S &= (s_1 s_2 \dots s_t)(s_{t+1} \dots s_{2t}) \dots (s_{(n-1)t+1} \dots s_{nt}) \\ &= b_1 b_2 \dots b_n \end{aligned}$$

Moreover, some tests need to apply operations on the sequence multiple times.

Knuth evaluates the sequences using χ^2 goodness-of-fit test which compares the observations to the expected values using k bins [7]. The observed number of elements in each bin is compared to the expected number of elements. In order to apply χ^2 properly, each bin should have at least 5 elements. The test outputs a p-value which is the probability of getting the observed results given that the sequence is random. To decide if a sequence passes a test or fails, a limit called significance level, α, is specified. If the p-value is greater than or equal to α, the sequence is said to pass the test. In statistical randomness testing, generally, α is chosen to be 0.01 or 0.05.

In the probability calculations of some tests, the Stirling numbers of the second kind is used. Stirling numbers of the second kind is the number of ways to partition a set of g elements into h non-empty subsets and denoted by $\left\{ \begin{matrix} \mathrm{g} \\ \mathrm{h} \end{matrix} \right\}$. The Stirling number of the second kind $\left\{ {g \atop h} \right\}$ can be computed as

$$\left\{ {g \atop h} \right\} = \frac{1}{h!} \sum_{j=0}^{h} (-1)^{h-j} \binom{h}{j} j^n.$$

3. Knuth's Statistical Randomness Tests

In this chapter, the tests in the Knuth test suite is investigated in details. For some tests, major changes are proposed without changing the approach followed by Knuth. Moreover, we propose test parameters that are not given in [7] for all the tests mentioned in this work.

We cover all the tests in Knuth test suite except the Run Test and the Serial Correlation Test. In the Run Test, it is assumed that the successive elements cannot be equal. For real number sequences this assumption is reasonable, however, for integer sequences the successive elements can be equal with a non-negligible probability. Without this assumption, the required computations are quite difficult and the modification of run test, unlike other tests, is beyond the scope of this paper. Yet, there is an ongoing work to modify the run test for integer and binary sequences. The Serial Correlation Test, on the other hand, does not output a p-value and the output of this test is not comparable to the outputs of the other tests.

3.1. **Equidistribution (Frequency) Test.** Equidistribution test checks if number of occurrences of each element $a \in \mathcal{A}$ are as expected from a random sequence. Knuth proposed two methods to apply this test;

(1) Use the Kolmogorov-Smirnov test with $F(x) = x$ for $0 \leq x < d$.
(2) For each element a, $0 \leq a < d$, count the number of times a appeared in the sequence and then apply the χ^2 test with degree of freedom $k = d - 1$, where the expected probability of each bin is $p_a = \frac{1}{d}$.

In this work, we proceed considering the second method. In [7], no parameters are given for the alphabet size and the length of the sequence. In order to apply the χ^2 test properly, the size of the alphabet should be chosen accordingly with the length of the sequence. For example, if S is 128 bits, then d, the size of the alphabet, should be at most 4. Otherwise, the expected number of elements in each bin cannot exceed 5 and χ^2 test cannot be applied. In fact, for each bin to have at least 5 elements, we should have $l \cdot \frac{1}{d} \geq 5$, ie. $l \geq 5d$. Since each element is of size $log_2 d$ bit, the length of the sequence should be at least $5d\ log_2 d$ bits. Leaving a safe distance, Table 1 can be used to decide on the alphabet size d for a given sequence size.

The following is an example on how to apply the test and calculate the p-value. Let $S = 10001010110111110100100110110010$, with $l_b = 32$. According to Table 1, the alphabet size should be 4 ie. each element is 2-bit. Then, the counters for 2-bit

l_b	$l_b \leq 20$	$20 < l_b \leq 80$	$80 < l_b \leq 240$	$240 < l_b \leq 640$	$640 < l_b \leq 1600$
d	2	4	8	16	32

TABLE 1. Sequence Bit Length-Alphabet Size Table for Equidistribution Test

elements are $\#00 : 3, \#01 : 3, \#10 : 6, \#11 : 4$. Alternatively, one can convert the sequence into a 2-bit integer sequence $S' = 2, 0, 2, 2, 3, 1, 3, 3, 1, 0, 2, 1, 2, 3, 0, 2$ and count the number of occurrences of each element. The test value can be computed as

$$\begin{aligned}
\chi^2 &= \sum_{i=1}^{k} \frac{(Observed_i - Expected_i)^2}{Expected_i} \\
&= \sum_{i=1}^{4} \frac{(Observed_i - 4)^2}{4} \\
&= 0.25 + 0.25 + 1 + 0 \\
&= 1.5
\end{aligned}$$

The p-value for degree of freedom $k = 3$ and the test value 1.5 is 0.6822. Assuming the significance level of $\alpha = 0.01$, the sequence passes the Equidistribution Test.

3.2. **Serial Test.** In Knuth test Suite, Serial test is an Equidistribution Test for pairs and hence it is equivalent of Equidistribution Test with alphabet size d^2. It checks whether the pairs of elements are equally distributed within the tested sequence or not. The test is proposed as follows: partition the sequence into non-overlapping subsequences of size two: $S_2 = (s_1, s_2), (s_3, s_4) \ldots (s_{2n-1}, s_{2n})$. Then, for each possible pair (q, r) with $0 \leq q, r < d$, count the number of occurrences of the pair (q, r) and apply χ^2 goodness-of-fit test with $d^2 - 1$ degrees of freedom and $\frac{1}{d^2}$ expected probability for each bin. Since there are $\frac{l}{2}$ pairs and each bin has the same probability, for χ^2 to be applicable, the inequality $\frac{l}{2}\frac{1}{d^2} \geq 5$ should be satisfied, which gives $l \geq 10d^2$. Therefore, the length of the sequence should be at least $10d^2\ log_2 d$ bits.

The suggested parameters for the Serial Test are given in Table 2.

l_b	$l_b \leq 80$	$80 < l_b \leq 480$	$480 < l_b \leq 2880$	$2880 < l_b \leq 15360$
d	2	4	8	16

TABLE 2. Sequence Bit Length-Alphabet Size Table for Serial Test

This test can be extended to triples or quadruples easily, however, l should be large enough or d should be taken small in order to get reasonable number of triples/quadruples.

3.3. **Gap Test.** This test examines the distribution of the lengths of the gaps among the elements of a specified set within the sequence. To apply the test, first, a subset $\mathbb{U}$ of $\mathcal{A}$ is fixed. Then, the number of gaps between the elements of $\mathbb{U}$ in the sequence S are counted according to their lengths. For example, assume $\mathcal{A} = \{0, 1, \ldots, 7\}$, $S = 7, 2, 4, 6, 2, 5, 2, 7, 4, 5, 6, 0, 7, 4, 1, 1, 7, 0, 4, 1$ and let $\mathbb{U} = \{a \mid a < 4, a \in \mathcal{A}\}$. If we mark the elements of $\mathbb{U}$ we get $S = 7, \mathbf{2}, 4, 6, \mathbf{2}, 5, \mathbf{2}, 7, 4, 5, 6, \mathbf{0}, 7, 4, \mathbf{1}, \mathbf{1}, 7, \mathbf{0}, 4, \mathbf{1}, 6$. The gaps between the elements of $\mathbb{U}$ are of length 2, 1, 4, 2, 0, 1, 1 in order. The number of gaps of size zero is 1, size one is 3, size two is 2 and size four is 1. Finally, the observed distribution of the length of the gaps are compared to the expected distribution applying χ^2 goodness-of -fit test and a p-value is obtained.

The following algorithm gives the expected probabilities of the length of the gaps.

Theorem 1. *Let $\mathcal{A}$ be an alphabet of size d and $\mathbb{U}$ be any nonempty subset of $\mathcal{A}$. Let S be a random sequence of elements of $\mathcal{A}$ and let $s_i \in \mathbb{U}$ for some i. Then, the probability that $s_{i+k} \notin \mathbb{U}$ for $k = 1, 2, .., r$ is*

$$p_r = \left(1 - \frac{|\mathbb{U}|}{d}\right)^r \frac{|\mathbb{U}|}{d}.$$

Proof. In order for a gap of length r to occur, after an element of $\mathbb{U}$, r elements from the set $\mathcal{A} \backslash \mathbb{U}$ should follow and to terminate the gap there must follow an element from $\mathbb{U}$:

$$u \underbrace{v \ldots v}_{\mathrm{r}}\ u,\ \ u \in \mathbb{U}, v \in \mathcal{A} \backslash \mathbb{U}$$

Since an element from $\mathbb{U}$ will appear in the sequence with probability $\frac{|\mathbb{U}|}{d}$ the probability of the length of the gap to be r is $(1 - p_u)^r p_u$. □

For the example above, $p_u = \frac{4}{8}$. Therefore, the probabilities of the length of the gaps being 0, 1, 2 and 3 are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and $\frac{1}{16}$ respectively. So, since the number of total gaps is 7, the expected number of gaps are $\frac{7}{2}, \frac{7}{4}, \frac{7}{8}$ and $\frac{17}{16}$ for 0, 1, 2 and 3. Applying the χ^2 test with the expected and observed values we get the p-value as 0.183255.

For short sequences as above the probability of long gaps will be very small. On the other hand, for long sequences the number of lengths will be too many to handle. Therefore, it is a good idea to limit the number of lengths as $r = 0, 1, \ldots j - 1$ and $r \geq j$ for a proper j. The probability of the length of a gap to be greater than or equal to j is $(1 - p_u)^j$ as after the first j elements from the set $\mathcal{A} \backslash \mathbb{U}$, no matter next element belongs to $\mathbb{U}$ or not the size of the gap will be greater than or equal to j.

One should choose j, $\mathbb{U}$ and l so that, p_j and p_r, for $r = 0, 1, \ldots, j-1$, enables the application of χ^2 test. That is, the number of gaps of length r, for $r = 0, 1, \ldots, j-1$ and $r \geq j$ should be at least 5.

For example, considering $d = 256$, if one chooses $|\mathbb{U}| = 4$, then the probability of a gap of length 0 becomes $\frac{4}{256} = 0.015625$. In order to expect at least 5 gaps of length 0, the total number of total gaps should be at least $5 \cdot \frac{1}{0.015625} = 320$. g gaps require $g+1$ elements from $\mathbb{U}$, therefore, for 320 gaps one needs 321 elements from $\mathbb{U}$. Since $|\mathbb{U}| = 4$, on average 4 elements from $\mathbb{U}$ will occur in 256 elements in the sequence. Therefore, for 320 gaps one needs a sequence of 19968 elements that is 159744 bits. Since the probabilities for longer gaps will be smaller, the required sequence length will be longer.

However, considering $|\mathbb{U}| = 16$ with $d = 256$ one gets more applicable results. In this case

$$\begin{aligned}
p &= \frac{16}{256} = \frac{1}{16} \\
p_0 &= \frac{1}{16} = 0.062500 \\
p_1 &= \frac{1}{16}\frac{240}{256} = 0.058593 \\
p_2 &= \frac{1}{16}\left(\frac{240}{256}\right)^2 = 0.054931 \\
p_3 &= \frac{1}{16}\left(\frac{240}{256}\right)^3 = 0.051498 \\
p_4 &= \frac{1}{16}\left(\frac{240}{256}\right)^4 = 0.048279 \\
p_{>4} &= \left(\frac{240}{256}\right)^5 = 0.724196.
\end{aligned}$$

Since the lowest probability is p_4, about $\lceil \frac{5}{0.048279} \rceil = 104$ gaps needed for χ^2 to be applicable. This makes 1680 elements and a 13440 bit sequence will be long enough which is more feasible than $|\mathbb{U}| = 4$ case. So, one can use the gap test with $d = 256$, $l > 13440$ bits, $|\mathbb{U}| = 16$, for instance $\mathbb{U} = \{x | x < 16\}$, and given probabilities above.

For shorter sequences, one may take $|\mathbb{U}|$ larger and consider less χ^2 bins. For instance, for a sequence of 1200 bits, take $|\mathbb{U}| = 64$, and consider the bins for $r = 0, 1, 2, 3$ and $r > 3$.

$$\begin{aligned}
p &= \frac{64}{256} = \frac{1}{4} \\
p_0 &= \frac{1}{4} = 0.25 \\
p_1 &= \frac{1}{4}\frac{192}{256} = 0.187500
\end{aligned}$$

$$\begin{aligned} p_2 &= \frac{1}{4}\left(\frac{192}{256}\right)^2 = 0.140625 \\ p_3 &= \frac{1}{4}\left(\frac{192}{256}\right)^3 = 0.105468 \\ p_{>3} &= \left(\frac{240}{256}\right)^4 = 0.316406. \end{aligned}$$

3.4. **Poker Test.** This test checks if the distribution of the number of distinct elements in a t-tuple is as expected from a random sequence. In [7], Knuth considers n groups of non-overlapping t successive elements and counts the number of t-tuples containing exactly r distinct elements where $r = 1, 2, \ldots, t$. The probability of a t-tuple to have exactly r distinct elements is as follows.

Theorem 2. *Let $\mathcal{A}$ be an alphabet of size d and $a_1 a_2 \ldots a_t$ be a randomly chosen t-tuple from $\mathcal{A}^t$. Let $\mathbb{U} = \{a_1, \ldots, a_t\} \supseteq \mathcal{A}$. Then for each r, $1 < r \leq t$, the probability that $\mathbb{U}$ contains r distinct elements is*

$$Pr(|\mathbb{U}| = r) = \frac{d(d-1)\cdots(d-r+1)}{d^t} \left\{ \begin{matrix} t \\ r \end{matrix} \right\}$$

where $\left\{ \begin{matrix} a \\ b \end{matrix} \right\}$ is the Stirling number of the second kind.

Proof.

$$\begin{aligned} Pr(|\mathbb{U}| = r) &= \frac{\text{choosing r distinct elements out of d}}{\text{All possible } t\text{-tuples}} \left\{ \begin{matrix} \text{Number of ways to} \\ \text{partition t-tuple} \\ \text{into r subsets} \end{matrix} \right\} \\ &= \frac{d(d-1)\cdots(d-r+1)}{d^t} \left\{ \begin{matrix} t \\ r \end{matrix} \right\} \end{aligned}$$

□

One should choose d and t carefully in order for the test to be applicable to variety of sizes. If we choose $d = 256$ as the above tests, unless selecting t very large which will result in need for a very long sequence, the probabilities for $r = 1, 2, \ldots, t-2$ will be very small. This will lead to small number of bins in χ^2 test and, also, will increase the necessary length of the sequence to have at least 5 elements in each bin. In that case, for the alphabet size a divisor or a multiple of 8 will be a good choose for implementation purposes since one byte corresponds to 8 bits. So, we choose 4-bit alphabet, ie. $d = 16$, with $t = 8$. Using these parameters, the probabilities p_r

can be calculated as

$$\begin{aligned}
p_1 &= 3.7 \times 10^{-9} \approx 0 \text{ since the number of blocks will be smaller than } 10^9 \\
p_2 &= 0.000007 \\
p_3 &= 0.000756 \\
p_4 &= 0.017299 \\
p_5 &= 0.128143 \\
p_6 &= 0.357091 \\
p_7 &= 0.375885 \\
p_8 &= 0.120820.
\end{aligned}$$

The χ^2 test will be applied with 5 bins where the first bin is "less than 5 distinct elements" and other "r distinct elements" each composes a bin: second bin covers "5 distinct elements", third bin is composed of "6 distinct elements" and so on. Since the least probable case, "less than 5 distinct elements", has probability 0.018062, in order to apply χ^2 one needs $\lceil \frac{5}{0.018062} \rceil = 277$ blocks of 8 4-bit elements which means one needs at least 8864 bit sequence.

3.5. **Coupon Collector Test.** Coupon Collector test examines the sequence by the length of the subsequences that have a complete set of alphabet elements. Starting from the first sequence element, one traces the sequence until all the alphabet elements are covered and records the length of the subsequence. For example let $\mathcal{A} = \{0, 1, 2, 3\}$ and $S = 1, 0, 2, 1, 2, 0, 3, 3, \ldots$. Marking the first occurrences of alphabet elements, $S = \mathbf{1}, \mathbf{0}, 1, \mathbf{2}, 2, 0, \mathbf{3}, 3, \ldots$, it is seen that the length of the shortest subsequence containing all the alphabet elements is 7. Then, resuming from the following element, again, finds the length of the subsequence covering all the alphabet elements and so on. When all the sequence is traced, the length of the subsequences are compared to those of a random sequence.The expected probability for a subsequence of length c that covers all the elements in the alphabet is given below.

Theorem 3. *Let $\mathcal{A}$ be an alphabet of size d. The probability that all elements of $\mathcal{A}$ appears in a sequence $a_1 a_2 \ldots a_c$, but not in $a_1 a_2 \ldots a_{c_1}$ is*

$$p_c = \frac{d!}{d^c} \left\{ \begin{matrix} c-1 \\ d-1 \end{matrix} \right\},$$

and the probability that the subsequences is of length greater than or equal to c is

$$p_{\geq c} = 1 - \frac{d!}{d^{c-1}} \left\{ \begin{matrix} c-1 \\ d \end{matrix} \right\}.$$

Proof. Now notice that, since the last element completes the collection, it should not appear previously in the subsequence. That is, this element only occurs one and its the last position. Fixing the last element, we left with a subsequence of length

$c-1$, containing $d-1$ distinct elements. The number of distinct such sequences is equal to the number of onto functions from a set of size $c-1$ to a set of size $d-1$, which is $(d-1)!\left\{ {c-1 \atop d-1} \right\}$. Considering the last element is chosen from a set of size d, the number of distinct subsequences containing all d elements is $d(d-1)!\left\{ {c-1 \atop d-1} \right\}$. Since there are overall d^c subsequences, the probability of such a subsequence is

$$p_c = \frac{d!}{d^c} \left\{ {c-1 \atop d-1} \right\}.$$

The probability of a subsequence of length greater than or equal to c is the complement of the probability that a sequence of length $c-1$ containing all d elements in any order. This includes all subsequences containing d distinct elements from a subsequence of length d to a subsequence of length $c-1$. The probability of a subsequence of length $c-1$ containing d distinct elements is equal to the number of onto functions from a $c-1$-element set to a d-element set. So, the probability of such a subsequence is $p_{\bar{c}} = \frac{d!}{d^{c-1}} \left\{ {c-1 \atop d} \right\}$. Therefore, the probability of a subsequence of length greater than or equal to c containing d distinct elements is $1 - p_{\bar{c}} = 1 - \frac{d!}{d^{c-1}} \left\{ {c-1 \atop d} \right\}$. □

When considering the $d = 256$ again, computing the Stirling numbers becomes infeasible. Therefore, we need to decrease the alphabet size. Similar to the Poker Test case, the best candidate for d is 16. For the case $d = 16$, the bin values and the probabilities are given below where p_{i-j} is the probability that the length of the sequence covering all the alphabet elements is between i and j, inclusive.

$$\begin{aligned}
p_{16-34} &= 0.107625 \\
p_{35-38} &= 0.085983 \\
p_{39-42} &= 0.100841 \\
p_{43-46} &= 0.104948 \\
p_{47-50} &= 0.100590 \\
p_{51-54} &= 0.090983 \\
p_{55-59} &= 0.096727 \\
p_{\geq 60} &= 0.312300
\end{aligned}$$

One can apply an 8-bin χ^2 goodness-of-fit test using the above probabilities. Since the lowest probability is $p_{35-38} = 0.085983$, the number of collections should be at least $\lceil \frac{5}{0.085983} \rceil = 59$. In the worst case, each subsequence containing a collection is at most 60 elements long, or one can stop searching for a collection after 60th element as the bin for 60 and any length longer then 60 are the same. Therefore, the sequence is 3540 elements long which is corresponding to 14160 bits.

3.6. **Permutation Test.** The Knuth Permutation Test focuses on the frequencies of the the arrangements of the elements within a block. Each block can be arranged in different ways considering the lexicographic ordering. For example, (4 3 0 1) and (9 7 4 5) have the same lexicographic ordering. Test compares the observed frequencies of the arrangements to the expected frequencies for a random sequence.

First, the sequence is divided into blocks of size t. In [7], Knuth assumes the sequence is a real number sequence and it is not expected to have a repetition within a block. It is assumed that each block can be arranged in one of $t!$ permutations. Counting the frequencies of each permutation, one can apply a χ^2 test with bin probability $\frac{1}{t!}$ for each bin. However, it is very likely that in an integer sequence there will be elements that will appear more than once within a block. In order to have an integer sequence that does not likely to contain repetitions within t-element blocks, the elements should be very large which makes the sequence too long. Another idea is to reduce the size of the blocks which in turn reduce the sensitivity of the test.s

Here, we propose another method to check the frequencies of the permutations without changing the notions in [7]. Again consider $d = 256$ and let t=4. The probability of occurring 4 distinct elements within a block is $\frac{256}{256}\frac{255}{256}\frac{254}{256}\frac{253}{256} = 0.976729$. Each 24 permutation of 4 distinct elements can occur with probability $p = \frac{0.976729989}{24} = 0.040697$ and repetition within a block occurs with probability $1 - 0.976729 = 0.023270$. So, applying the χ^2 test with 25 bins, 24 bins for non-repeating blocks and one for repeating blocks one can compare the sequence to a random sequence. To apply the χ^2 test one needs at least $\lceil \frac{5}{0.023270} \rceil = 215$ blocks of 4 elements, therefore, the length of the sequence must be at least $215 \cdot 4 \cdot log_2\, 256 = 6880$ bits.

3.7. **Max-of-t Test.** In [7], the Max-of-t Test is proposed to test the maximal elements within blocks of size t in order to check for randomness. The proposed test partitions the sequence into non-overlapping blocsk of t, and applies the Kol-mogorov-Smirnov test to the maximal elements of the sequences. However, Kolmogorov-Smirnov test is applied for examining a random sample from some unknown distribution to see the normality of the sample and it is less powerful than χ^2 goodness-of-fit test. Another option given in [7] is applying the Equidistribution Test to the maximal elements. Yet, the probabilities of maximal element to be 0 or $d-1$ are not equal. Therefore, one should consider each probability while applying the Equidistribution Test. Setting the parameters d and t, we find the probabilities of the maximum element to be exactly m within a block of t and to be smaller than or equal to m. This way one can apply χ^2 test with given probabilities and bin values.

Theorem 4. *Let $\mathcal{A}$ be an alphabet of size d. Then the the probability of maximum element to be less then or equal to m in a block of t terms is*

$$p_{(max \leq m)} = \frac{(m+1)^t}{d^t}.$$

Proof. Including "0", there are $m+1$ numbers less than or equal to m. In order for the maximum of t elements to be less than or equal to m, each of t elements can be one of $m+1$ numbers, ie. there are $(m+1)^t$ such blocks of t. Therefore, the probability of maximum element to be less then or equal to m is

$$p_{(max \leq m)} = \frac{(m+1)^t}{d^t}.$$

□

Moreover, the probability of maximum to be exactly m is

$$\begin{aligned} p_{(max=m)} &= p_{(max \leq m)} - p_{(max \leq m-1)} \\ &= \frac{(m+1)^t - m^t}{d^t} \end{aligned}$$

Again considering $d = 256$ and $t = 4$, one can use the bin values given in Table 3. For the χ^2-test to be applicable the least probable bin, last bin in this case, should

m	Bin Probability
$m \leq 170$	0.199078601
$171 \leq m \leq 203$	0.204158801
$204 \leq m \leq 225$	0.204161350
$226 \leq m \leq 242$	0.204431504
$243 \leq m$	0.188169744

TABLE 3. Bin boundaries and probabilities for Max-of-t Test

have at least 5 elements. Therefore, there should be $\lceil \frac{5}{0.188169744} \rceil = 28$ blocks of 4 8-bit elements which sums up to 896 bits. So the sequence should be at least 896 bits to apply the Max-of-t test.

3.8. **Collision Test.** Collision test checks if the number of collisions in predefined parts of the sequences is as expected from a random sequences. In this test, the number of collisions are counted and the result is compared to the expected number of collisions.

The idea is similar to throwing balls into urns: if a ball lands in a nonempty urn, a collision is said to occur. If there are m urns and n balls then the probability of c collisions can be calculated as follows.

Theorem 5. *If n balls are thrown into m urns at random, the probability of occuring exactly c collisions is*

$$P\{C=c\} = \frac{m(m-1)\cdots(m-(n-c-1))}{m^n}\left\{\begin{matrix} n \\ n-c \end{matrix}\right\}. \tag{1}$$

Proof. In order for exactly c collisions to occur, first, n balls should land in $n-c$ distinct urns guaranteeing the number of collisions does not exceed c. There are $m(m-1)\cdots(m-(n-c-1))$ ways to choose $n-c$ urns out of m^n. Now each of $n-c$ urns have a single ball in it. Then, the remaining c balls can land in any of these urns, urns containing a single ball, in any order. For instance all the remaining c balls can land in the same urn or each ball may land in distinct urns. This is the partitioning of n balls into nonempty $n-c$ subsets, which is the Stirling number of the second kind, $\left\{\begin{matrix} n \\ n-c \end{matrix}\right\}$. Therefore, the probability of c collisions is

$$P\{C=c\} = \frac{m(m-1)\cdots(m-(n-c-1))}{m^n}\left\{\begin{matrix} n \\ n-c \end{matrix}\right\}.$$

□

For the randomness test, similarly, if the specified portions of two sequences are equal, a collision is said to occur and the probability in Equation 1 also applies to the test. In this case, the number of urns is the number of all possible subsequences in the predefined portion of the sequence. For example, consider the first 10 bits of the sequences. The number of "urns" is all possible 10 bit subsequences which is 2^{10}. The balls correspond to the distinct sequences to be tested.

Knuth suggests taking $m = 2^{20}$ and $n = 2^{14}$ which means taking 2^{20} sequences and counting the collisions in the predefined 14 bits of these sequences. For the sake of simplicity, one can take the first 14 bits or the last 14 bits of the sequence, but any set of fixed 14 bits of the sequence can be selected to inspect the collisions.

For the suggestions of Knuth, $m = 2^{20}$ and $n = 2^{14}$, the probabilities of collisions are given in Table 4. After counting the collisions in 2^{20} sequences, if the number of collisions is less than or equal to 101, the However, in this setting, one can just

# of Collisions	≤ 101	≤ 108	≤ 119	≤ 126	≤ 134	≤ 145	≤ 153
Probability	0.009	0.043	0.244	0.476	0.742	0.946	0.989

TABLE 4. Bin boundaries and probabilities for Collision Test

get a very inaccurate idea about the sequence by finding the interval in which the number of collisions lies. Therefore, applying the test on a series of sequences and getting a convenient result becomes inapplicable. In order to overcome this problem in a similar way with the previous tests, we calculate the collision probabilities and construct χ^2 bins. Using the bins one can apply χ^2 goodness-of-fit test and produce a p-value. The boundaries of the bins for $m = 2^{20}$ and $n = 2^{14}$ case are given in Table 5.

Moreover, taking 2^{20} distinct sequences is outside the scope of testing the randomness of a sequence. In fact, it is in the scope of testing a random number generator. Therefore, it is more convenient to partition the sequence into blocks instead of taking distinct sequences. For the given probabilities, in order to apply a proper χ^2 test, the number of experiments should be at least $\lceil \frac{5}{0.88373} \rceil = 56$. So, instead of taking a set of 2^{20} distinct sequences, one needs to partition the sequence into $56 \cdot 2^{20}$ blocks of 14 bits which suggests a sequence of 822083584 bits. In this case, one should divide the sequence into 56 subsequences, partition each subsequence into 2^{20} blocks and count the number of collisions in each subsequence. An

# of Collision	Probability
0-113	0.106253
114-118	0.109894
119-121	0.088373
122-124	0.100719
125-127	0.106608
128-130	0.104977
131-133	0.096322
124-137	0.106367
138-142	0.091574
143-16384	0.088913

TABLE 5. Collision Test χ^2 bin probabilities for $m = 2^{20}$ and $n = 2^{14}$

alternative case for shorter sequences is taking $m = 2^{16}$ and $n = 2^{10}$. In this case, to apply the χ^2 test, the number of experiments should be at least $\lceil \frac{5}{0.141034} \rceil = 36$. Therefore, one needs $36 \cdot 2^{16}$ blocks of length 10 bits which makes 23592960 bits. Table 6 shows the boundaries and the probabilities for $m = 2^{16}$ and $n = 2^{10}$ case. Birthday Spacing Test

# of Collision	Probability
0-5	0.192924
6-7	0.259222
8	0.141034
9-10	0.223346
11-1024	0.177158

TABLE 6. Collision Test χ^2 bin probabilities for $m = 2^{16}$ and $n = 2^{10}$

The Birthday Spacing Test examines the randomness of the sequence by checking the number of equal differences between selected sequence elements. In this test, a number of sequence elements are selected, sorted, and the differences between each consecutive element are calculated. Then, the number of equal differences are compared to the expected number of equal differences. For example, let $S = 9, 5, 6, 1, 16, 24, 2, 13, 34, 29$ and consider the $4^{th}, 5^{th}, 9^{th}$ and 10^{th} elements: 1, 16, 34, 29. Sorting the elements we get S'=1, 16, 29, 34. The differences between the elements are $G = 16 - 1, 29 - 16, 34 - 29$ ie., $G = 15, 13, 15$. There are two equal differences, which means one collision occurs in differences. The test resembles the collision test and throwing balls into urns phenomenon with days of the year as urns and birthdays as balls. Since the elements of the alphabet are considered as the days of the year and the sequence elements are the birthdays, the name of the test is the birthday spacing test.

Knuth suggests to use $m = 2^{25}$ days for $n = 512$ birthdays. This setting, for bit sequences, is corresponding to taking 512 elements of 25 bits each, computing the differences between the consecutive elements. The probabilities for the number of colliding differences are given in Table 7. Using these probabilities one can apply a χ^2 test for goodness-of-fit.

# of Equal Spacings	0	1	2	3 or more
Probability	0.368801	0.369035	0.183471	0.078692

TABLE 7. The probabilities for Birthday Spacing Test

Similar to the Collision Test, in order to test the sequence, instead of taking distinct sequences, we take a sequence and partition the sequence according to the bit length of the "birthdays". In order to apply the χ^2 test properly, one needs to make $\lceil \frac{5}{0.078692} \rceil = 64$ experiments each needs 2^{25} blocks of 9 bits long. Therefore, one needs $2^{25} \cdot 64 \cdot 9 \approx 2^{34}$ bits of data. In [7], advises to repeat the process 1000 times instead of 64 which increases the data size to 2^{40} assuming each sequence is 9 bits long.

4. Application

In this section we present the results of Knuth Test suite on various sequences. The primary aim of the section is to show the applicability of the suite on integer, and therefore on binary, sequences.

We applied the suite on $\pi, e, \sqrt{2}, log(2)$ and Riemann Zeta function $\zeta(3)$. For these numbers, we excluded the integer parts and test the sequence of 1.000.000 digits to the right of the decimal point. Moreover, we generate sequences, that have the same size with the previous sequences, by concatenating the SHA-256 [11] and MD-5 [12] hash values of successive integers starting from 0. Another sequence is generated by using the "random" utility of C#. Then, we generate a new sequence by giving a 1% "1" bias to this sequence. This way, test our parameters for frequency related tests. When testing the suite, we apply some tests twice with distinct parameters. The test parameters can be found in Table 8.

The results can be seen in Table 9. According to these results, all the non-biased sequences can be considered to be random. For the biased sequence, Frequency, Serial, Gap and Max-of-t tests output p-values less than 0.01 indicating the non-randomness as expected.

Test	Parameters
Frequency 1	$d = 256$
Frequency 2	$d = 2^{24}$
Serial	$d = 256$
Gap	$d = 256, \lvert\mathbb{U}\rvert = 16$
Poker	$d = 16, t = 4$
Coupon Coll	$d = 16$
Max-Of-t 1	$d = 256, t = 4$
Max-Of-t 2	$d = 2^{16}, t = 6$
Permutation 1	$d = 256, t = 4$
Permutation 2	$d = 2^{16}, t = 5$
Collision	$m = 2^{16}, n = 2^{10}$
Birthday Sp.	$m = 2^{25}, n = 512$

TABLE 8. Application Test Parameters

	PI	E	Sqrt(2)	Log(2)	Golden Ratio	Zeta(3)	MD5	SHA256	C# Random	C# Biased
Frequency 1	0,940520	0,174365	0,401369	0,551351	0,039588	0,046532	0,942153	0,509073	0,261939	0
Frequency 2	0,964781	0,261258	0,030199	0,931099	0,570073	0,506992	0,265583	0,221097	0,847772	0,001599
Serial	0,384719	0,052247	0,702899	0,980707	0,131627	0,024494	0,993911	0,231856	0,752165	0
Gap	0,709093	0,305874	0,440585	0,754035	0,348360	0,661038	0,723083	0,444013	0,979131	0
Poker	0,699648	0,956847	0,741170	0,560399	0,422498	0,957892	0,933983	0,355740	0,385174	0,002392
Coupon Coll	0,325971	0,213433	0,621810	0,853074	0,560253	0,837512	0,228078	0,519568	0,188181	0,139930
Max-Of-t 1	0,055390	0,267757	0,551455	0,599732	0,701230	0,150366	0,264187	0,576693	0,312611	0
Max-Of-t 2	0,101844	0,567233	0,665188	0,657665	0,765619	0,548888	0,351747	0,809745	0,020687	0
Permutation 1	0,118599	0,413592	0,388108	0,901025	0,953106	0,365188	0,347413	0,048359	0,559867	0,715372
Permutation 2	0,123178	0,639895	0,905754	0,937968	0,951257	0,069539	0,182614	0,591102	0,379035	0,214872
Collision	0,230030	0,640728	0,935599	0,769927	0,435727	0,698075	0,042924	0,044239	0,564757	0,343109
Birthday Sp.	0,042169	0,935038	0,249442	0,450414	0,060426	0,934736	0,135028	0,054025	0,764874	0,856611

TABLE 9. Test results of Knuth Test Suite for some mathematical constants and sequences

5. CONCLUSION

Knuth Test Suite [7] is one of the first statistical randomness test suites. The suite is well formed and the statistical basis of the test is well established. However, the suite is designed primarily to test real number sequences. The assumption given in the suite, that the tests could be applied to the integer sequences misses some points and some tests cannot be applied to integer sequences.

Moreover, the tester is assumed to have a knowledge over statistics and combinatorics that the test parameters and probability calculations are not given excluding one or two exceptions.

In this work, we review all the tests in Knuth Test Suite and excluding the Run Test and the Serial Correlation Test, we give test parameters in order for the tests to be applicable to integer sequences and make suggestions on the choice of these parameters. We clarify how the probabilities used in the tests are calculated according to the parameters and provide users to calculate the probabilities they need without any knowledge of statistics or combinatorics.

Also, some tests, like Permutation Test and Max-of-t-test, are reviewed so that the test can be used for integer sequences.

Finally, we apply the suite on some widely used cryptographic random number sources and present the results.

As a future work, the relations between Knuth Test Suite and NIST Test Suite will be investigated.

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GAUSSIAN PADOVAN AND GAUSSIAN PELL- PADOVAN SEQUENCES

DURSUN TAŞCI

Abstract. In this paper, we extend Padovan and Pell- Padovan numbers to Gaussian Padovan and Gaussian Pell-Padovan numbers, respectively. Moreover we obtain Binet-like formulas,generating functions and some identities related with Gaussian Padovan numbers and Gaussian Pell-Padovan numbers.

1. Introduction

Horadam [3] in 1963 and Berzsenyi [2] in 1977 defined complex Fibonacci numbers. Horadam introduced the concept the complex Fibonacci numbers as the Gaussian Fibonacci numbers.

Padovan sequence is named after Richard Padovan [7] and Atasonav K., Dimitrov D., Shannon A. and Kritsana S. [1, 4, 5, 6] studied Padovan sequence and Pell-Padovan sequence.

The Padovan sequence is the sequence of integers P_n defined by the initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

$$P_n = P_{n-2} + P_{n-3} \quad \text{for all } n \geq 3.$$

The first few values of P_n are $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37$.

Pell-Padovan sequence is defined by the initial values $R_0 = R_1 = R_2 = 1$ and the recurrence relation

$$R_n = 2R_{n-2} + R_{n-3} \quad \text{for all } n \geq 3.$$

The first few values of Pell-Padovan numbers are $1, 1, 1, 3, 3, 7, 9, 17, 25, 43, 67,$ $111, 177, 289$.

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2. Gaussian Padovan Sequences

Firstly we give the definition of Gaussian Padovan sequence.

Definition 2.1. The Gaussian Padovan sequence is the sequence of complex numbers GP_n defined by the initial values $GP_0 = 1,\ GP_1 = 1+i,\ GP_2 = 1+i$ and the recurrence relation

$$GP_n = GP_{n-2} + GP_{n-3} \quad \text{for all } n \geq 3.$$

The first few values of GP_n are $1,\ 1+i,\ 1+i,\ 2+i,\ 2+2i,\ 3+2i,\ 4+3i, 5+4i,\ 7+5i,\ 9+7i$.

The following theorem is related with the generating function of the Gaussian Padovan sequence.

Theorem 2.2. *The generating function of the Gaussian Padovan sequence is*

$$g(x) = \frac{1+(1+i)\ x+i\ x^2}{1-x^2-x^3}.$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} GP_n x^n = GP_0 + GP_1 x + GP_2 x^2 + \cdots + GP_n x^n + \cdots$$

be the generating function of the Gaussian Padovan sequence. On the other hand, since

$$x^2 g(x) = GP_0 x^2 + GP_1 x^3 + GP_2 x^4 + \cdots + GP_{n-2} x^n + \cdots$$

and

$$x^3 g(x) = GP_0 x^3 + GP_1 x^4 + GP_2 x^5 + \cdots + GP_{n-3} x^n + \cdots$$

we write

$$\begin{aligned}(1-x^2-x^3)g(x) &= GP_0 + GP_1 x + (GP_2 - GP_0)x^2 + (GP_3 - GP_1 - GP_0)x^3 \\ &\quad + \cdots + (GP_n - GP_{n-2} - GP_{n-3})x^n + \cdots\end{aligned}$$

Now consider $GP_0 = 1,\ GP_1 = 1+i,\ GP_2 = 1+i$ and $GP_n = GP_{n-2} + GP_{n-3}$. Thus, we obtain

$$(1-x^2-x^3)g(x) = 1+(1+i)x+i\ x^2$$

or

$$g(x) = \frac{1+(1+i)\ x+i\ x^2}{1-x^2-x^3}.$$

So, the proof is complete. □

Now we give Binet-like formula for the Gaussian Padovan sequence.

Theorem 2.3. *Binet-like formula for the Gaussian Padovan sequence is*

$$GP_n = \left(a + i\frac{a}{r_1}\right) r_1^n + \left(b + i\frac{b}{r_2}\right) r_2^n + \left(c + i\frac{c}{r_3}\right) r_3^n$$

where

$$a=\frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}, b=\frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)}, c=\frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)}$$

and r_1, r_2, r_3 are the roots of the equation $x^3-x-1=0$.

Proof. It is easily seen that

$$GP_n = P_n + iP_{n-1}.$$

On the other hand, we know that the Binet-like formula for the Padovan sequence is

$$P_n=\frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}r_1^n+\frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)}r_2^n+\frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)}r_3^n.$$

So, the proof is easily seen. □

Theorem 2.4.

$$\sum_{j=0}^{n} GP_j = GP_n + GP_{n+1} + GP_{n+2} - 2(1+i).$$

Proof. By the definition of Gaussian Padovan sequence recurrence relation

$$GP_n = GP_{n-2} + GP_{n-3}$$

and

$$\begin{aligned}
GP_0 &= GP_2 - GP_{-1}\\
GP_1 &= GP_3 - GP_0\\
GP_2 &= GP_4 - GP_1\\
&\vdots\\
GP_{n-2} &= GP_n - GP_{n-3}\\
GP_{n-1} &= GP_{n+1} - GP_{n-2}\\
GP_n &= GP_{n+2} - GP_{n-1}
\end{aligned}$$

Then we have

$$\sum_{j=0}^{n} GP_j = GP_n + GP_{n+1} + GP_{n+2} - GP_{-1} - GP_0 - GP_1.$$

Now considering $GP_{-1} = i, GP_0 = 1$ and $GP_1 = 1+i$, we write

$$\sum_{j=0}^{n} GP_j = GP_n + GP_{n+1} + GP_{n+2} - 2 - 2i.$$

and so the proof is complete. □

Now we investigate the new property of Gaussian Padovan numbers in relation with Padovan matrix formula. We consider the following matrices:

$$Q_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ K_3 = \begin{bmatrix} 1+i & 1+i & 1 \\ 1+i & 1 & i \\ 1 & i & 1 \end{bmatrix}$$

and

$$M_3^n = \begin{bmatrix} GP_{n+2} & GP_{n+1} & GP_n \\ GP_{n+1} & GP_n & GP_{n-1} \\ GP_n & GP_{n-1} & GP_{n-2} \end{bmatrix}.$$

Theorem 2.5. *For all* $n \in Z^+$*,we have*

$$Q_3^n K_3 = M_3^n.$$

Proof. The proof is easily seen that using the induction on n. □

Theorem 2.6. *If*

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

then we have

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1+i \\ 1+i \end{bmatrix} = \begin{bmatrix} GP_n \\ GP_{n+1} \\ GP_{n+2} \end{bmatrix}.$$

Proof. The proof can be seen by mathematical induction on n. □

3. Gaussian Pell-Padovan Sequence

As well known Pell-Padovan sequence is defined by the recurrence relation

$$R_n = 2R_{n-2} + R_{n-3},\ n \geq 3$$

and initial values are $R_0 = R_1 = R_2 = 1$.

Now we define Gaussian Pell-Padovan sequence.

Definition 3.1. The Gaussian Pell-Padovan sequence is defined by the recurrence relation

$$GR_n = 2GR_{n-2} + GR_{n-3},\ n \geq 3$$

and initial values are $GR_0 = 1 - i, GR_1 = 1 + i, GR_2 = 1 + i$.

The first few values of GR_n are $1-i,\ 1+i,\ 1+i,\ 3+i,\ 3+3i,\ 7+3i,\ 9+7i,\ 17+9i$.

Theorem 3.2. *The generating function of Gaussian Pell-Padovan sequence is*

$$f(x) = \frac{1 - i + (1+i)x + (-1+3i)x^2}{1 - 2x^2 - x^3}.$$

Proof. Let

$$f(x)=\sum_{n=0}^{\infty} GR_n x^n$$

be the generating function of the Gaussian Pell-Padovan sequence. In this case, we have

$$2x^2 f(x)=2GR_0x^2+2GR_1x^3+2GR_2x^4+\cdots+2GR_{n-2}x^n+\cdots$$

and

$$x^3 f(x)=GR_0x^3+GR_1x^4+GR_2x^5+\cdots+GR_{n-3}x^n+\cdots$$

so we obtain

$$\begin{aligned}(1-2x^2-x^3)f(x) &= GR_0+GR_1x+(GR_2-2GR_0)x^2+(GR_3-2GR_1-GR_0)x^3\\ &\quad +\cdots+(GR_n-2GR_{n-2}-GR_{n-3})x^n+\cdots .\end{aligned}$$

On the other hand, since $GR_0=1-i,\ GR_1=1+i,\ GR_2=1+i$ and $GR_n=2GR_{n-2}+GR_{n-3}$, then we have

$$f(x)=\frac{1-i+(1+i)x+(-1+3i)x^2}{1-2x^2-x^3}$$

which is desired. □

Theorem 3.3. *The Binet-like formula of Gaussian Pell-Padovan sequence is*

$$GR_n=\frac{2}{\sqrt{5}}\left[\alpha-1+i\left(1-\frac{1}{\alpha}\right)\right]\alpha^n-\frac{2}{\sqrt{5}}\left[\beta-1+i\left(1-\frac{1}{\beta}\right)\right]\beta^n+(i-1)\gamma^n$$

where

$$\alpha=\frac{1+\sqrt{5}}{2},\beta=\frac{1-\sqrt{5}}{2},\gamma=1$$

are roots of the equation $x^3-2x-1=0$.

Proof. The Binet-like formula of Pell-Padovan sequence is given by

$$R_n=2\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}-2\frac{\alpha^n-\beta^n}{\alpha-\beta}+\gamma^{n+1}.$$

Now consider

$$GR_n=R_n+iR_{n-1}$$

so the proof is easily seen. □

Theorem 3.4. $\sum_{j=0}^{n} GR_j=\frac{1}{2}\left[(-1-3i)-GR_{n+1}+GR_{n+2}+GR_{n+3}\right].$

Proof. We find that

$$\sum_{j=0}^{n} R_j=\frac{1}{2}(-1-R_{n+1}+R_{n+2}+R_{n+3})$$

and

$$\sum_{j=0}^{n} R_{j-1} = \frac{1}{2}(-3 - 2R_n - R_{n+1} + R_{n+2} + R_{n+3}).$$

Since

$$GR_n = R_n + iR_{n-1}$$

we have

$$\sum_{j=0}^{n} GR_j = \sum_{j=0}^{n} R_j + i\sum_{j=0}^{n} R_{j-1}$$

So the theorem is proved. □

Theorem 3.5. $\sum_{j=1}^{n} GR_{2j} = R_{2n+1} + iR_{2n} - (n+1) + i(n-1)$.

Proof. If we consider the following equalities, then the proof is seen:

$$\begin{aligned}\sum_{j=1}^{n} R_{2j} &= R_{2n+1} - (n+1)\\ \sum_{j=1}^{n} R_{2j-1} &= R_{2n} + (n-1)\end{aligned}$$

and

$$\sum_{j=1}^{n} GR_{2j} = \sum_{j=1}^{n} R_{2j} + i\sum_{j=1}^{n} R_{2j-1}$$

□

Theorem 3.6. $\sum_{j=1}^{n} \binom{n}{j} GR_j = GR_{2n} + (1-i)$.

Proof. Considering the following equalities:

$$\begin{aligned}\sum_{j=1}^{n} \binom{n}{j} R_j &= R_{2n} + 1\\ \sum_{j=1}^{n} \binom{n}{j} R_{j-1} &= R_{2n-1} - 1\end{aligned}$$

and

$$\sum_{j=1}^{n} \binom{n}{j} GR_j = \sum_{j=1}^{n} \binom{n}{j} R_j + i\sum_{j=1}^{n} \binom{n}{j} R_{j-1}$$

then the proof is easily seen. □

Now we shall give the new properties of Gaussian Pell-Padovan numbers relation with Pell-Padovan matrix.

Theorem 3.7. *If we take the following matrices*

$$Q_3 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ K_3 = \begin{bmatrix} 1+i & 1+i & 1-i \\ 1+i & 1-i & -1+3i \\ 1-i & -1+3i & 3-5i \end{bmatrix}$$

and

$$S_3^n = \begin{bmatrix} GR_{n+2} & GR_{n+1} & GR_n \\ GR_{n+1} & GR_n & GR_{n-1} \\ GR_n & GR_{n-1} & GR_{n-2} \end{bmatrix}.$$

then

$$Q_3^n.K_3 = S_3^n \text{ for all } n \in \mathbb{Z}^+.$$

Theorem 3.8. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}^n \begin{bmatrix} 1-i \\ 1+i \\ 1+i \end{bmatrix} = \begin{bmatrix} GR_n \\ GR_{n+1} \\ GR_{n+2} \end{bmatrix}$ *for all* $n \in Z^+$.

We note that for the proofs Theorem 3.7 and Theorem 3.8 are used induction on n.

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COMMUNICATIONS
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A THEOREM ON WEIGHTED APPROXIMATION BY SINGULAR INTEGRAL OPERATORS

OZGE OZALP GULLER AND ERTAN IBIKLI

Abstract. In this paper, pointwise approximation of functions $f \in L_{1,\varphi}(\mathbb{R})$ by the convolution type singular integral operators given in the following form:

$$L_\lambda(f;x) = \int_{\mathbb{R}} f(t)\, K_\lambda(t-x)\, dt,\ x \in \mathbb{R},\ \lambda \in \Lambda \subset \mathbb{R}_0^+,$$

is studied. Here, $L_{1,\varphi}(\mathbb{R})$ denotes the space of all measurable functions f for which $\left|\frac{f}{\varphi}\right|$ is integrable on $\mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$ is a corresponding weight function.

1. Introduction

The purpose of approximation theory is the approximation of functions by simply calculated functions. This theory is one of the most fundamental and important arm of mathematical analysis. The Weierstrass approximation theorem says that every continuous function defined on a closed and bounded interval of real numbers can be uniformly approximated by polynomials. Also, this well-known theorem plays significant role in the development of analysis. Then, Bernstein also proved Weierstrass's theorem by describing specific approximate polynomials known as Bernstein polynomials in the literature. Bernstein polynomials were changed by Kantorovich in order to approximate to the integrable functions. These polynomials and the generalizations were studied in [2], [8] and [11].

Taberski [21] studied the pointwise approximation of integrable functions and the approximation properties of derivatives of integrable functions in $L_1\langle -\pi, \pi\rangle$, where $\langle -\pi, \pi\rangle$ is an arbitrary closed, semi-closed or open interval, by a two parameter

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family of convolution type singular integral operators of the form:

$$T_\lambda(f;x) = \int_{-\pi}^{\pi} f(t)\, K_\lambda(t-x)\, dt,\ x \in \langle -\pi, \pi \rangle,\ \lambda \in \Lambda \subset \mathbb{R}_0^+, \tag{1}$$

where $K_\lambda(t)$ is the kernel satisfying appropriate assumptions for all $\lambda \in \Lambda$ and Λ is a given set of non-negative indices with accumulation point λ_0.

Then, based on Taberski's indicated analysis, Gadjiev [10] and Rydzewska [16] proved some theorems concerning the pointwise convergence and the order of pointwise convergence of the operators of type (1) at a generalized Lebesgue point and $\mu-$generalized Lebesgue point of $f \in L_1(-\pi,\pi)$, respectively.

Further, the results of Taberski [21], Gadjiev [10] and Rydzewska [16] were extended by Karsli and Ibikli [12]. They proved some theorems for the more general integral operators defined by

$$T_\lambda(f;x) = \int_a^b f(t)\, K_\lambda(t-x)\, dt,\ x \in \langle a, b \rangle, \lambda \in \Lambda \subset \mathbb{R}_0^+.$$

Here, $f \in L_1 \langle a,b \rangle$, where $\langle a,b \rangle$ is an arbitrary interval in $\mathbb{R}$ such as $[a,b]$, (a,b), $[a,b)$ or $(a,b]$. As concerns the study of integral operators in several settings, the reader may see also, e.g., [13], [18], [23], [24], [25], [26] and [27].

The main aim of this paper is to investigate the pointwise convergence of convolution type singular integral operators in the following form:

$$L_\lambda(f;x) = \int_{\mathbb{R}} f(t)\, K_\lambda(t-x)\, dt,\ x \in \mathbb{R},\ \lambda \in \Lambda \subset \mathbb{R}_0^+, \tag{2}$$

where $L_{1,\varphi}(\mathbb{R})$ is the space of all measurable functions f for which $\left|\frac{f}{\varphi}\right|$ is integrable on $\mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$ is a corresponding weight function, at a common μ-generalized Lebesgue point of $\frac{f}{\varphi}$ and φ. In this paper, we studied a theorem of the Faddeev type similar to that of Taberski [19].

The paper is organized as follows: First, we introduce the fundamental definitions in the sequel of Introduction part. In Section 2, we prove the existence of the operators of type (2). Later, we present a theorem concerning the pointwise convergence of $L_\lambda(f;x)$ to $f(x_0)$ whenever x_0 is a common $\mu-$generalized Lebesgue point of $\frac{f}{\varphi}$ and φ.

Consequently, given that linear integral operators have become important tools in many areas, including the theory of Fourier series and Fourier integrals, approximation theory and summability theory, it is possible to use this article in the mathematical theorem.

Now, we introduce the main definitions used in this paper.

Definition 1. *A point $x_0 \in \langle a, b\rangle$ is called $\mu-$generalized Lebesgue point of the function $f \in L_1 \langle a, b\rangle$, if*

$$\lim_{h\to 0}\left(\frac{1}{\mu(h)}\int_0^h |f(t+x_0)-f(x_0)|\,dt\right)=0,$$

where the function $\mu : \mathbb{R} \to \mathbb{R}$ is increasing and absolutely continuous on $[0, b-a]$ and $\mu(0) = 0$. Here, also holds when the integral is taken from $-h$ to 0 [12] and [16].

Definition 2. *(Class A_φ) Let $\Lambda \subset \mathbb{R}_0^+$ be an index set and $\lambda_0 \in \Lambda$ be an accumulation point of it. Let the weight function $\varphi : \mathbb{R} \to \mathbb{R}^+$ be bounded on arbitrary bounded subsets of $\mathbb{R}$ and satisfies the following inequality:*

$$\varphi(t+x) \le \varphi(t)\varphi(x), \qquad x, t \in \mathbb{R}.$$

Suppose that there exists a function $K_\lambda^* : \mathbb{R} \to \mathbb{R}^+$ such that the following conditions hold there:

a) $\|\varphi K_\lambda^*\|_{L_1(\mathbb{R})} \le M < \infty$, for all $\lambda \in \Lambda$.
b) For every $\xi > 0$,

$$\lim_{\lambda\to\lambda_0}\sup_{\xi\le|t|}[\varphi(t)K_\lambda^*(t)] = 0.$$

c) For every $\xi > 0$,

$$\lim_{\lambda\to\lambda_0}\int_{\xi\le|t|}\varphi(t)K_\lambda^*(t)dt = 0.$$

d)

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)}\left|\frac{1}{\varphi(x_0)}\int_{\mathbb{R}}\varphi(t)K_\lambda(t-x)dt-1\right| = 0.$$

e) For any $\lambda \in \Lambda, K_\lambda(t)$ satisfies the following inequality:

$$|K_\lambda(t)| \le K_\lambda^*(t)$$

and there exists $\delta_0 > 0$ such that $K_\lambda^*(t)$ is non-decreasing on $(-\delta_0, 0]$ and non-increasing on $[0, \delta_0)$ for any $\lambda \in \Lambda$.

If the above conditions are satisfied, then the function $K_\lambda : \mathbb{R} \to \mathbb{R}$ belongs to *class A_φ*.

Throughout this paper, we suppose that the kernel $K_\lambda(t)$ belongs to *class A_φ*.

2. Main Theorem

Definition 3. *Let $L_{1,\varphi}(\mathbb{R})$ is the space of all measurable functions for which $\left|\frac{f(t)}{\varphi(t)}\right|$ is integrable on $\mathbb{R}$. Here $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a weight function and the norm in this space is given by the equality:*

$$\|f\|_{L_{1,\varphi}(\mathbb{R})} = \int\limits_{\mathbb{R}} \left|\frac{f(t)}{\varphi(t)}\right| dt.$$

Throughout this paper we suppose that the weight function $\varphi : \mathbb{R} \to \mathbb{R}^+$ [13].

The following lemma gives the existence of the operators defined by (2).

Lemma 1. *Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a weight function. If $f \in L_{1,\varphi}(\mathbb{R})$, then $L_\lambda(f;x)$ defines a continuous transformation from $L_{1,\varphi}(\mathbb{R})$ to $L_{1,\varphi}(\mathbb{R})$.*

Proof. By the linearity of the operator $L_\lambda(f;x)$, it is sufficient to show that the expression

$$\|L_\lambda\|_1 = \sup_{f\neq 0} \frac{\|L_\lambda(f;x)\|_{L_{1,\varphi}(\mathbb{R})}}{\|f\|_{L_{1,\varphi}(\mathbb{R})}}$$

remains bounded. Now, using Fubini's Theorem (see, e.g., [7]), we can write

$$\begin{aligned}
\|L_\lambda(f;x)\|_{L_{1,\varphi}(\mathbb{R})} &= \int\limits_{\mathbb{R}} \frac{1}{\varphi(x)} \left| \int\limits_{\mathbb{R}} f(t)\frac{\varphi(t)}{\varphi(t)} K_\lambda(t-x)dt \right| dx \\
&\leq \int\limits_{\mathbb{R}} \frac{1}{\varphi(x)} \left(\int\limits_{\mathbb{R}} \left| f(t+x)\frac{\varphi(t+x)}{\varphi(t+x)} K_\lambda(t) \right| dt \right) dx \\
&\leq \int\limits_{\mathbb{R}} |K_\lambda(t)| \left(\int\limits_{\mathbb{R}} \left| \frac{f(t+x)}{\varphi(t+x)} \right| \left| \frac{\varphi(t)\varphi(x)}{\varphi(x)} \right| dx \right) dt \\
&\leq \int\limits_{\mathbb{R}} \varphi(t) K_\lambda^*(t)dt \int\limits_{\mathbb{R}} \left| \frac{f(t+x)}{\varphi(t+x)} \right| dx \\
&\leq M \|f\|_{L_{1,\varphi}(\mathbb{R})}.
\end{aligned}$$

Thus, the proof is completed. □

The following theorem gives a pointwise convergence of the integral operators of type (2) at a common $\mu-$generalized Lebesgue point of $f \in L_{1,\varphi}(\mathbb{R})$ and the weight function $\varphi : \mathbb{R} \to \mathbb{R}^+$.

Theorem 1. *If x_0 is a common $\mu-$generalized Lebesque point of functions $f \in L_{1,\varphi}(\mathbb{R})$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$, then*

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} L_\lambda(f;x) = f(x_0),$$

on any set Z *on which the function*

$$\sup_{t\in N_\delta(x_0)} \varphi(t)\left\{2K_\lambda^*(0)\mu(|x_0-x|)+\int\limits_{N_\delta(x_0)} K_\lambda^*(t-x)\left|\{\mu(|x_0-t|)\}_t^{'}\right|dt\right\}$$

is bounded as (x,λ) *tends to* (x_0,λ_0), *where* $N_\delta(x_0)=(x_0-\delta,x_0+\delta)$.

Proof. Suppose that x_0 is a $\mu-$generalized Lebesque point of function $f\in L_{1,\varphi}(\mathbb{R})$.
Set $E=|L_\lambda(f;x)-f(x_0)|$. According to condition (d), we shall write

$$\begin{aligned}
E &= |L_\lambda(f;x)-f(x_0)| \\
&= \left|\int\limits_{\mathbb{R}} f(t)K_\lambda(t-x)dt-f(x_0)\right| \\
&\leq \int\limits_{\mathbb{R}}\left|\frac{f(t)}{\varphi(t)}-\frac{f(x_0)}{\varphi(x_0)}\right|\varphi(t)\left|K_\lambda(t-x)\right|dt \\
&\quad +\left|\frac{f(x_0)}{\varphi(x_0)}\right|\left|\int\limits_{\mathbb{R}}\varphi(t)K_\lambda(t-x)dt-\varphi(x_0)\right| \\
&= I_1+I_2.
\end{aligned}$$

By condition (d) of class A_φ, $I_2\to 0$ as $(x,\lambda)\to(x_0,\lambda_0)$. Now, we investigate the integral I_1 i.e:

$$\begin{aligned}
I_1 &= \left\{\int\limits_{\mathbb{R}\setminus N_\delta(x_0)}+\int\limits_{N_\delta(x_0)}\right\}\left|\frac{f(t)}{\varphi(t)}-\frac{f(x_0)}{\varphi(x_0)}\right|\varphi(t)\left|K_\lambda(t-x)\right|dt \\
&= I_{11}+I_{12}.
\end{aligned}$$

The following inequality holds for the integral I_{11} i.e:

$$\begin{aligned}
I_{11} &= \int\limits_{\mathbb{R}\setminus N(x_0)}\left|\frac{f(t)}{\varphi(t)}-\frac{f(x_0)}{\varphi(x_0)}\right|\varphi(t)\left|K_\lambda(t-x)\right|dt \\
&\leq \int\limits_{\mathbb{R}\setminus N(x_0)}\left|\frac{f(t+x)}{\varphi(t+x)}-\frac{f(x_0)}{\varphi(x_0)}\right|\varphi(t+x)\left|K_\lambda(t)\right|dt \\
&\leq \sup_{\xi\leq|t|}\left[\varphi(t)K_\lambda^*(t)\right]\varphi(x)\left\|f\right\|_{L_{1,\varphi}(\mathbb{R})}+\left|\frac{f(x_0)}{\varphi(x_0)}\right|\varphi(x)\int\limits_{\xi\leq|t|}\varphi(t)K_\lambda^*(t)dt.
\end{aligned}$$

According to conditions (c) and (d) of class A_φ, $I_{11} \to 0$ as $\lambda \to \lambda_0$. Next, we can show that I_{12} tends to zero as $(x,\lambda) \to (x_0,\lambda_0)$ on $N_\delta(x_0)$.

$$\begin{aligned}
I_{12} &= \int\limits_{N_\delta(x_0)} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t) \left| K_\lambda(t-x) \right| dt \\
&= \left\{ \int\limits_{x_0-\delta}^{x_0} + \int\limits_{x_0}^{x_0+\delta} \right\} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t) \left| K_\lambda(t-x) \right| dt \\
&\leq \sup_{t \in N_\delta(x_0)} \varphi(t) \left\{ \int\limits_{x_0-\delta}^{x_0} + \int\limits_{x_0}^{x_0+\delta} \right\} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \left| K_\lambda(t-x) \right| dt \\
&= \sup_{t \in N_\delta(x_0)} \varphi(t) \left\{ I_{121} + I_{122} \right\}.
\end{aligned}$$

Let us consider first the integral I_{121}. By definition of $\mu-$generalized lebesgue point for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int\limits_{x_0-h}^{x_0} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| dt < \varepsilon \mu(h)$$

for all $0 < h \leq \delta < \delta_0$. Define the new function as

$$F(t) = \int\limits_{t}^{x_0} \left| \frac{f(u)}{\varphi(u)} - \frac{f(x_0)}{\varphi(x_0)} \right| du. \tag{2.1}$$

Then, for every t satisfying $0 < x_0 - t \leq \delta$ we have

$$|F(t)| \leq \varepsilon \mu(x_0 - t). \tag{2.2}$$

Hence, by (2.1) we can write

$$\begin{aligned}
|I_{121}| &= \left| \int\limits_{x_0-\delta}^{x_0} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \left| K_\lambda(t-x) \right| dt \right| \\
&= \left| (LS) \int\limits_{x_0-\delta}^{x_0} \left| K_\lambda(t-x) \right| d\left[-F(t) \right] \right|,
\end{aligned}$$

where (LS) denotes Lebesgue-Stieltjes integral. Applying integration by parts method to the Lebesgue-Stieltjes integral, we have

$$|I_{121}| \leq |F(x_0-\delta)| \left| K_\lambda(x_0-\delta-x) \right| + \int\limits_{x_0-\delta}^{x_0} |F(t)| \left| (d_t \left| K_\lambda(t-x) \right|) \right|.$$

According to (2.2) and condition (e) of class A_φ, we obtain

$$|I_{121}| \leq \varepsilon\mu(\delta)K_\lambda^*(x_0-\delta-x)+\varepsilon\int_{x_0-\delta}^{x_0}\mu(x_0-t)\left|(d_tK_\lambda^*(t-x))\right|.$$

Now, we define the variations:

$$A(t)=\begin{cases}\bigvee\limits_{x_0-x-\delta}^{t}K_\lambda^*(s) & , \quad x_0-x-\delta<t\leq x_0-x\\ \quad 0 & , \qquad t=x_0-x-\delta.\end{cases} \tag{2.3}$$

Taking above variations and applying integration by parts method to last inequality, we get

$$\begin{aligned}|I_{121}| &\leq \varepsilon\mu(\delta)K_\lambda^*(x_0-\delta-x)+\varepsilon\int_{x_0-x-\delta}^{x_0-x}\{\mu(x_0-x-t)\}_t^{'}A(t)dt\\ &= \varepsilon(i_1+i_2).\end{aligned}$$

Let us consider the integral i_2. Write

$$\begin{aligned}i_2 &= \int_{x_0-x-\delta}^{x_0-x}\{\mu(x_0-x-t)\}_t^{'}A(t)dt\\ &= \left\{\int_{x_0-x-\delta}^{0}+\int_{0}^{x_0-x}\right\}\{\mu(x_0-x-t)\}_t^{'}A(t)dt\\ &= i_{21}+i_{22}.\end{aligned}$$

From (2.3), we shall write

$$\begin{aligned}i_{21} &= \int_{x_0-x-\delta}^{0}\left[\bigvee_{x_0-x-\delta}^{t}K_\lambda^*(s)\right]\{\mu(x_0-x-t)\}_t^{'}dt\\ &= \int_{x_0-x-\delta}^{0}\left[K_\lambda^*(t)-K_\lambda^*(x_0-x-\delta)\right]\{\mu(x_0-x-t)\}_t^{'}dt\end{aligned} \tag{2.4}$$

and

$$\begin{aligned} i_{22} &= \int_0^{x_0-x} \left[\bigvee_{x_0-x-\delta}^{t} K_\lambda^*(s) \right] \{\mu(x_0-x-t)\}_t^{'} \, dt \\ &= \int_0^{x_0-x} \left[\bigvee_{x_0-x-\delta}^{0} K_\lambda^*(s) + \bigvee_0^t K_\lambda^*(s) \right] \{\mu(x_0-x-t)\}_t^{'} \, dt \\ &= \int_0^{x_0-x} (2K_\lambda^*(0) - K_\lambda^*(x_0-x-\delta) - K_\lambda^*(t)) \{\mu(x_0-x-t)\}_t^{'} \, dt. \quad (2.5) \end{aligned}$$

Combining (2.4) and (2.5), we obtain

$$\begin{aligned} i_2 &= i_{21} + i_{22} \\ &\leq -2K_\lambda^*(0)\mu(x_0-x) - K_\lambda^*(x_0-x-\delta)\mu(\delta) \\ &\quad + \int_{x_0-x-\delta}^{x_0-x} K_\lambda^*(t) \{\mu(x_0-x-t)\}_t^{'} \, dt. \end{aligned}$$

Thus

$$\begin{aligned} |I_{121}| &\leq \varepsilon (i_1 + i_2) \\ &\leq 2\varepsilon K_\lambda^*(0)\mu(x_0-x) + \varepsilon \int_{x_0-x-\delta}^{x_0-x} K_\lambda^*(t) \{\mu(x_0-x-t)\}_t^{'} \, dt \\ &\leq 2\varepsilon K_\lambda^*(0)\mu(x_0-x) + \varepsilon \int_{x_0-\delta}^{x_0} K_\lambda^*(t-x) \{\mu(x_0-t)\}_t^{'} \, dt. \quad (2.6) \end{aligned}$$

We can use a similar method for estimating I_{122}. Then we find the inequality

$$|I_{122}| \leq \varepsilon \int_{x_0}^{x_0+\delta} K_\lambda^*(t-x) \{\mu(t-x_0)\}_t^{'} \, dt. \quad (2.7)$$

Consequently, from (2.6) and (2.7), we can write the following inequality:

$$\begin{aligned} I_{12} &\leq \sup_{t \in N_\delta(x_0)} \varphi(t) \{I_{121} + I_{122}\} \\ &\leq \varepsilon \sup_{t \in N_\delta(x_0)} \varphi(t) \left[2K_\lambda^*(0)\mu(x_0-x) + \int_{x_0-\delta}^{x_0+\delta} K_\lambda^*(t-x) \left| \{\mu(|x_0-t|)\}_t^{'} \right| dt \right]. \end{aligned}$$

Note that in the above inequality we used the hypothesis of the theorem, i.e., boundedness of the following function:

$$\sup_{t\in N_\delta(x_0)} \varphi(t)\left[2K_\lambda^*(0)\mu(x_0-x)+\int\limits_{x_0-\delta}^{x_0+\delta} K_\lambda^*(t-x)\left|\{\mu(|x_0-t|)\}_t^{'}\right| dt\right].$$

Since the remaining expression is bounded by the hypothesis, $I_{12}\to 0$ as $(x,\lambda)\to(x_0,\lambda_0)$. Thus, we obtain

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} L_\lambda(f;x)=f(x_0)$$

and the proof is completed. □

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THE RECURRENCE SEQUENCES VIA POLYHEDRAL GROUPS

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Abstract. In this paper, we define recurrence sequences by using the relation matrices of the finite polyhedral groups and then, we obtain some of their properties. Also, we obtain the cyclic groups and the semigroups which are produced by the generating matrices when read modulo α and we study the sequences defined modulo α. Then we derive the relationships between the orders of the cyclic groups obtained and the periods of the sequences defined working modulo α. Furthermore, we extend these sequences to groups and obtain the periods of the sequences extended in the finite polyhedral groups case.

1. Introduction

The polyhedral group (p,q,r) for $p,q,r>1$, is defined by the presentation

$$\langle x,y,z \mid x^p = y^q = z^r = xyz = e\rangle$$

or

$$\langle x,y \mid x^p = y^q = (xy)^r = e\rangle.$$

The polyhedral group (p,q,r) is finite if and only if the number

$$k = pqr\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1\right) = qr+rp+pq-pqr$$

is positive, i.e., in the case $(2,2,m)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$. Its order is $2pqr/k$. Using Tietze transformations we may show that $(p,q,r) \approxeq (q,r,p) \approxeq (r,p,q)$.

For more information on these groups, see [4].

Let G be a finite j-generator group and let

$$X = \left\{ (x_1,x_2,\ldots,x_j) \in \underbrace{G\times G\times\cdots\times G}_{j} \mid \langle x_1,x_2,\ldots,x_j\rangle = G \right\}.$$

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We call $(x_1, x_2, \ldots, x_j)$ a generating j-tuple for G.

Let G be the group defined by the finite presentation

$$G = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle.$$

The relation matrix of G is an $m \times n$ matrix where the $(i,j)^{\text{th}}$ entry of the matrix is the sum of the exponents of the generator x_j in the relator r_i.

For detailed information about the relation matrix, see [12].

Example 1.1. *The relation matrix of the group defined by the presentation* $\langle x, y, z \mid x^m = y^2 = z^2 = xyz = e \rangle$ *is*

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [13], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix A be defined by

$$A = [a_{ij}]_{k \times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

Number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper have been studied by many authors [2, 5, 6, 10, 11, 13, 15, 19, 20, 21, 22]. In Section 2, we develop properties of the 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind which are obtained from the matices defined by the aid of the relation matrices of the polyhedral groups $(m, 2, 2)$, $(2, m, 2)$, $(2, 2, m)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$.

In [5, 6, 7, 17], the authors have produced the cyclic groups and the semigroups via some special matrices and then, they have studied the orders of these algebraic structures. In Section 3, we obtain the cyclic groups and the semigroups by using the generating matrices of the 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind when read modulo α and then, we give their miscellaneous properties.

The study of recurrence sequences in groups began with the earlier work of Wall [23] where the ordinary Fibonacci sequences in cyclic groups has been investigated. In the mid eighties, Wilcox extended the problem to abelian groups [24]. Further, the theory has been expanded to some special linear recurrence sequences by several authors; see, for example, [1, 3, 5, 6, 8, 9, 14, 16]. In Section 3, we study the defined sequences modulo α and then, we derive the relationships among the orders of the cyclic groups obtained and the periods of these sequences. Also, in this section, we redefine the 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind by means of the elements of the groups which have two or three generators and then, we examine these sequences in the finite groups. Finally, we obtain the lengths of the periods of the 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind in the polyhedral groups $(m,2,2)$, $(2,m,2)$, $(2,2,m)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$ by using the periods of these sequences with respect to a modulus α, where we consider each one of the sequences in one group such that the sequence is produced by the aid of the presentation of this group.

2. Polyhedral Sequences

We next define the matrices M_1, M_2, M_3, M_4, M_5 and M_6 by using the presentations of the polyhedral groups $(m,2,2)$, $(2,m,2)$, $(2,2,m)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$ in the two generator cases, that is for generating pair (x,y), as follows, respectively:

$$M_u = \begin{bmatrix} \alpha_1 & 0 & 1 \\ 0 & \alpha_2 & 1 \\ \alpha_3 & \alpha_3 & 1 \end{bmatrix},\ (u = 1,2,3,\ \alpha_u = m \text{ and } \alpha_i = 2 \text{ if } i \neq u)$$

and

$$M_v = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ v-1 & v-1 & 1 \end{bmatrix},\ (v = 4,5,6)\,.$$

Similarly, we define the matrices M_1^*, M_2^*, M_3^*, M_4^*, M_5^* and M_6^* by the aid of the presentations of these groups in the three generator cases, that is for generating triple (x, y, z), as follows, respectively:

$$M_u^* = \begin{bmatrix} \alpha_1 & 0 & 0 & 1 \\ 0 & \alpha_2 & 0 & 1 \\ 0 & 0 & \alpha_3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \ (u = 1, 2, 3, \ \alpha_u = m \text{ and } \alpha_i = 2 \text{ if } i \neq u).$$

and

$$M_v^* = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & v-1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \ (v = 4, 5, 6).$$

Note that $\det M_1 = \det M_2 = -4$, $\det M_3 = 4-4m$, $\det M_4 = -9$, $\det M_5 = -14$, $\det M_6 = -19$, $\det M_1^* = \det M_2^* = \det M_3^* = -4$, $\det M_4^* = -3$, $\det M_5^* = -2$ and $\det M_6^* = -1$.

We now define new sequences from the matrices M_k and M_k^*, $(k = 1, \ldots, 6)$ as shown, respectively:

$$a_n^u = \begin{cases} a_{n-1}^u + \alpha_1 a_{n-3}^u & n \equiv 1 \pmod 3, \\ a_{n-2}^u + \alpha_2 a_{n-3}^u & n \equiv 2 \pmod 3, \\ a_{n-3}^u + \alpha_3 a_{n-4}^u + \alpha_3 a_{n-5}^u & n \equiv 0 \pmod 3, \end{cases}$$
$$(u = 1, 2, 3, \ \alpha_u = m \text{ and } \alpha_i = 2 \text{ if } i \neq u),$$

$$a_n^v = \begin{cases} a_{n-1}^v + 2a_{n-3}^v & n \equiv 1 \pmod 3, \\ a_{n-2}^v + 3a_{n-3}^v & n \equiv 2 \pmod 3, \\ a_{n-3}^v + (v-1)\, a_{n-4}^v + (v-1)\, a_{n-5}^v & n \equiv 0 \pmod 3, \end{cases}$$
$$(v = 4, 5, 6)$$

for $n \geq 4$, where $a_1^k = 0$, $a_2^k = 0$, $a_3^k = 1$ and

$$b_n^u = \begin{cases} b_{n-1}^u + \alpha_1 b_{n-4}^u & n \equiv 1 \pmod 4, \\ b_{n-2}^u + \alpha_2 b_{n-4}^u & n \equiv 2 \pmod 4, \\ b_{n-3}^u + \alpha_3 b_{n-4}^u & n \equiv 3 \pmod 4, \\ b_{n-4}^u + b_{n-5}^u + b_{n-6}^u + b_{n-7}^u & n \equiv 0 \pmod 4, \end{cases}$$
$$(u = 1, 2, 3, \ \alpha_u = m \text{ and } \alpha_i = 2 \text{ if } i \neq u),$$

$$b_n^v = \begin{cases} b_{n-1}^v + 2b_{n-4}^v & n \equiv 1 \pmod 4, \\ b_{n-2}^v + 3b_{n-4}^v & n \equiv 2 \pmod 4, \\ b_{n-3}^v + (v-1)\, b_{n-4}^v & n \equiv 3 \pmod 4, \\ b_{n-4}^v + b_{n-5}^v + b_{n-6}^v + b_{n-7}^v & n \equiv 0 \pmod 4, \end{cases} \quad (v = 4, 5, 6)$$

for $n \geq 5$, where $b_1^k = 0$, $b_2^k = 0$, $b_3^k = 0$, $b_4^k = 1$.

The sequences $\{a_n^k\}$ and $\{b_n^k\}$ for $k = 1, \ldots, 6$ are called the 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind, respectively.

By an inductive argument for $n \geq 3$ and $k = 1, \ldots, 6$, we may write

$$(M_k)^n = \begin{bmatrix} m_1^k & m_2^k & a_{3n+1}^k \\ m_3^k & m_4^k & a_{3n+2}^k \\ \lambda_k a_{3n+1}^k & \lambda_k a_{3n+2}^k & a_{3n+3}^k \end{bmatrix},$$
$$(\lambda_1 = \lambda_2 = 2,\ \lambda_3 = m,\ \lambda_4 = 3,\ \lambda_5 = 4,\ \lambda_6 = 5)$$

where

$$m_1^1 = 2a_{3n-2}^1 + ma_{3n+1}^1 + m^{n-1} + \sum_{i=0}^{n-3} m^{n-2-i} a_{8+3i}^1,$$
$$m_1^2 = a_{3n+1}^2 + a_{3n+3}^2 - 2m^{n-2} - 2\sum_{i=0}^{n-3} m^{n-3-i} a_{7+3i}^2,$$
$$m_1^3 = \frac{a_{3n+2}^3 + a_{3n+3}^3 + 2^n}{2},$$
$$m_1^4 = 7.2^{n-2} + 3\sum_{i=0}^{n-3} 2^{n-3-i} a_{7+3i}^4,$$
$$m_1^5 = 2^{n+1} + \sum_{i=0}^{n-3} 2^{n-1-i} a_{7+3i}^5,$$
$$m_1^6 = 9.2^{n-2} + 5\sum_{i=0}^{n-3} 2^{n-3-i} a_{7+3i}^6,$$

$$m_2^1 = 2m^{n-2} + 2\sum_{i=0}^{n-3} m^{n-3-i} a_{8+3i}^1,$$
$$m_2^2 = 2m^{n-2} + 2\sum_{i=0}^{n-3} m^{n-3-i} a_{7+3i}^2,$$
$$m_2^3 = \frac{a_{3n+2}^3 + a_{3n+3}^3 - 2^n}{2},$$
$$m_2^4 = 3^{n-1} + \sum_{i=0}^{n-3} 3^{n-2-i} a_{7+3i}^4,$$
$$m_2^5 = 2^n + \sum_{i=0}^{n-3} 2^{n-1-i} a_{7+3i}^5,$$
$$m_2^6 = 5 \cdot 3^{n-2} + 5\sum_{i=0}^{n-3} 3^{n-3-i} a_{7+3i}^6,$$

$$m_3^1 = m_2^1,\ m_3^2 = m_2^2,\ m_3^3 = m_2^3,\ m_3^4 = m_2^4,\ m_3^5 = m_2^5,\ m_3^6 = m_2^6$$

and

$$\begin{aligned}
m_4^1 &= a_{3n+2}^1 + a_{3n+3}^1 - 2m^{n-2} - 2\sum_{i=0}^{n-3} m^{n-3-i} a_{8+3i}^1, \\
m_4^2 &= 2a_{3n-1}^2 + ma_{3n+2}^2 - m^{n-1} - \sum_{i=0}^{n-3} m^{n-2-i} a_{7+3i}^2, \\
m_4^3 &= m_1^3, \\
m_4^4 &= 3^n + \sum_{i=0}^{n-2} 3^{n-1-i} a_{5+3i}^4, \\
m_4^5 &= 3^n + 4\sum_{i=0}^{n-2} 3^{n-2-i} a_{5+3i}^5, \\
m_4^6 &= 3^n + 5\sum_{i=0}^{n-2} 3^{n-2-i} a_{5+3i}^6.
\end{aligned}$$

Similarly, we obtain the matrices $(M_k^*)^n$ for $n \geq 3$ and $k = 1, \ldots, 6$ by using mathematical induction as shown:

For $k = 1, 2, 3$,

$$(M_k^*)^n = \begin{bmatrix} m_1^{*k} & m_2^{*k} & m_3^{*k} & a_{4n+1}^{*k} \\ m_4^{*k} & m_5^{*k} & m_6^{*k} & a_{4n+2}^{*k} \\ m_7^{*k} & m_8^{*k} & m_9^{*k} & a_{4n+3}^{*k} \\ a_{4n+1}^{*k} & a_{4n+2}^{*k} & a_{4n+3}^{*k} & a_{4n+4}^{*k} \end{bmatrix},$$

where

$$\begin{aligned}
m_1^{*1} &= a_{4n-3}^{*1} + ma_{4n+1}^{*1} - m^{n-1} - \sum_{i=0}^{n-3} m^{n-2-i} a_{10+4i}^{*1}, \; m_1^{*2} = a_{4n-3}^{*2} + 2^n \\
&\quad + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*2}, \\
m_1^{*3} &= a_{4n-3}^{*3} + 2^n + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*3},
\end{aligned}$$

$$\begin{aligned}
m_2^{*1} &= m^{n-2} + \sum_{i=0}^{n-3} m^{n-3-i} a_{10+4i}^{*1}, \; m_2^{*2} = m^{n-2} + \sum_{i=0}^{n-3} m^{n-3-i} a_{9+4i}^{*2}, \; m_2^{*3} = a_{4n-3}^{*3} \\
&\quad + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*3},
\end{aligned}$$

$$m_3^{*1} = m_2^{*1},\, m_3^{*2} = a_{4n-3}^{*2} + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*2},\, m_3^{*3} = m^{n-2} + \sum_{i=0}^{n-3} m^{n-3-i} a_{9+4i}^{*3},$$

$$m_4^{*1} = m_2^{*1},\, m_4^{*2} = m_2^{*2},\, m_4^{*3} = m_2^{*3},$$

$$\begin{aligned} m_5^{*1} &= a_{4n-2}^{*1} + 2^n + \sum_{i=0}^{n-3} 2^{n-2-i} a_{6+4i}^{*1},\, m_5^{*2} \\ &= a_{4n-2}^{*2} + m a_{4n+2}^{*2} - m^{n-1} + \sum_{i=0}^{n-3} m^{n-2-i} a_{9+4i}^{*2},\, m_5^{*3} = m_1^{*3}, \end{aligned}$$

$$m_6^{*1} = a_{4n-2}^{*1} + \sum_{i=0}^{n-3} 2^{n-2-i} a_{6+4i}^{*1},\, m_6^{*2} = m_2^{*2},\, m_6^{*3} = m_3^{*3},$$

$$m_7^{*1} = m_2^{*1},\, m_7^{*2} = m_3^{*2},\, m_7^{*3} = m_3^{*3},$$

$$m_8^{*1} = m_6^{*1},\, m_8^{*2} = m_2^{*2},\, m_8^{*3} = m_3^{*3}$$

and

$$m_9^{*1} = m_5^{*1},\, m_9^{*2} = m_1^{*2},\, m_9^{*3} = a_{4n-1}^{*3} + m a_{4n+3}^{*3} - m^{n-1} + \sum_{i=0}^{n-3} m^{n-2-i} a_{9+4i}^{*3}.$$

For $k = 4, 5, 6$,

$$(M_4^*)^n = \begin{bmatrix} a_{4n-3}^{*4} + 2^n + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*4} & a_{4n+2}^{*4} - a_{4n+1}^{*4} & a_{4n+2}^{*4} - a_{4n+1}^{*4} & a_{4n+1}^{*4} \\ a_{4n+2}^{*4} - a_{4n+1}^{*4} & a_{4n-2}^{*4} + 3^n + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*4} & a_{4n-2}^{*4} + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*4} & a_{4n+2}^{*4} \\ a_{4n+2}^{*4} - a_{4n+1}^{*4} & a_{4n-2}^{*4} + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*4} & a_{4n-2}^{*4} + 3^n + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*4} & a_{4n+3}^{*4} \\ a_{4n+1}^{*4} & a_{4n+2}^{*4} & a_{4n+3}^{*4} & a_{4n+4}^{*4} \end{bmatrix},$$

$$(M_5^*)^n = \begin{bmatrix} a_{4n-3}^{*5} + 2^n + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*5} & a_{4n+2}^{*5} - a_{4n+1}^{*5} & a_{4n-3}^{*5} + \sum_{i=0}^{n-3} 4^{n-2-i} a_{5+4i}^{*5} & a_{4n+1}^{*5} \\ a_{4n+2}^{*5} - a_{4n+1}^{*5} & a_{4n-2}^{*5} + 3^n + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*5} & a_{4n+3}^{*5} - a_{4n+2}^{*5} & a_{4n+2}^{*5} \\ a_{4n-3}^{*5} + \sum_{i=0}^{n-3} 4^{n-2-i} a_{5+4i}^{*5} & a_{4n+3}^{*5} - a_{4n+2}^{*5} & a_{4n-1}^{*5} + 4^n + \sum_{i=0}^{n-3} 4^{n-2-i} a_{7+4i}^{*5} & a_{4n+3}^{*5} \\ a_{4n+1}^{*5} & a_{4n+2}^{*5} & a_{4n+3}^{*5} & a_{4n+4}^{*5} \end{bmatrix}$$

and

$$(M_6^*)^n = \begin{bmatrix} a_{4n-3}^{*6} + 2^n + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*6} & a_{4n+2}^{*6} - a_{4n+1}^{*6} & a_{4n-1}^{*6} + \sum_{i=0}^{n-3} 2^{n-2-i} a_{7+4i}^{*6} & a_{4n+1}^{*6} \\ a_{4n+2}^{*6} - a_{4n+1}^{*6} & a_{4n-2}^{*6} + 3^n + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*6} & a_{4n-2}^{*6} + \sum_{i=0}^{n-3} 5^{n-2-i} a_{6+4i}^{*6} & a_{4n+2}^{*6} \\ a_{4n-1}^{*6} + \sum_{i=0}^{n-3} 2^{n-2-i} a_{7+4i}^{*6} & a_{4n-2}^{*6} + \sum_{i=0}^{n-3} 5^{n-2-i} a_{6+4i}^{*6} & a_{4n-1}^{*6} + 5^n + \sum_{i=0}^{n-3} 5^{n-2-i} a_{7+4i}^{*6} & a_{4n+3}^{*6} \\ a_{4n+1}^{*6} & a_{4n+2}^{*6} & a_{4n+3}^{*6} & a_{4n+4}^{*6} \end{bmatrix}.$$

It is well-known that the Simpson formula for a recurrence sequence can be obtained from the determinant of its generating matrix. For example, the Simpson formula for the sequence $\{a_n^3\}$ is

$$\begin{aligned}(4-4m)^n &= \left(a_{3n+2}^3 + a_{3n+3}^3 + 2^n\right)\left(-\frac{m}{2}\left(a_{3n+1}^3\right)^2 - \frac{m}{2}\left(a_{3n+2}^3\right)^2\right) + \\ &\quad m a_{3n+1}^3\left(\left(a_{3n+2}^3\right)^2 + a_{3n+2}^3 a_{3n+3}^3 - 2^n a_{3n+2}^3\right) + 2^n a_{3n+3}^3\left(a_{3n+3}^3 + a_{3n+2}^3\right).\end{aligned}$$

It is easy to see that the characteristic equations of the sequences $\{a_n^k\}$ and $\{b_n^k\}$, $(k = 1, \ldots, 6)$ do not have multiple roots; that is, each of the eigenvalues of the matrices M_k and M_k^* is distinct.

Let $\{x_1^k, x_2^k, x_3^k\}$ and $\{x_1^k, x_2^k, x_3^k, x_4^k\}$ be the sets of the eigenvalues of the matrices M_k and M_k^* for $k = 1, \ldots, 6$, respectively and let $V_k^{(u)}$ be a $(u+2)\times(u+2)$ Vandermonde matrix as follows:

$$V_k^{(u)} = \begin{bmatrix} \left(x_1^k\right)^{u+1} & \left(x_2^k\right)^{u+1} & \cdots & \left(x_{u+2}^k\right)^{u+1} \\ \left(x_1^k\right)^{u} & \left(x_2^k\right)^{u} & \cdots & \left(x_{u+2}^k\right)^{u} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

where $u = 1, 2$. Suppose now that

$$W_k^i = \begin{bmatrix} \left(x_1^k\right)^{n+u+2-i} \\ \left(x_2^k\right)^{n+u+2-i} \\ \vdots \\ \left(x_{u+2}^k\right)^{n+u+2-i} \end{bmatrix}$$

and $V_{k,j}^{(u,i)}$ İS a $(u+2)\times(u+2)$ matrix obtained from $V_k^{(u)}$ by replacing the jth column of $V_k^{(u)}$ by W_k^i. This yields the Binet-type formulas for the sequences $\{a_n^k\}$ and $\{b_n^k\}$, namely.

Theorem 2.1. *For $k = 1, \ldots, 6$,*

$$m_{ij}^{(k,n)} = \frac{\det V_{k,j}^{(1,i)}}{\det V_k^{(1)}} \text{ and } m_{ij}^{*(k,n)} = \frac{\det V_{k,j}^{(2,i)}}{\det V_k^{(2)}},$$

where $(M_k)^n = m_{ij}^{(k,n)}$ and $(M_k^*)^n = m_{ij}^{*(k,n)}$.

Proof. Since the eigenvalues of the matrices M_k and M_k^* are are distinct, these matrices are diagonalizable. Let

$$D^{(1,k)} = \operatorname{diag}\left(x_1^k, x_2^k, x_3^k\right) \text{ and } D^{(2,k)} = \operatorname{diag}\left(x_1^k, x_2^k, x_3^k, x_4^k\right),$$

then it is easy to see that $M_k V_k^{(1)} = V_k^{(1)} D^{(1,k)}$ and $M_k^* V_k^{(2)} = V_k^{(2)} D^{(2,k)}$. Since the matrices $V_k^{(1)}$ and $V_k^{(2)}$ are invertible, $\left(V_k^{(1)}\right)^{-1} M_k V_k^{(1)} = D^{(1,k)}$ and $\left(V_k^{(2)}\right)^{-1} M_k^* V_k^{(2)} =$

$D^{(2,k)}$. Thus, the matrices M_k and M_k^* are similar to $D^{(1,k)}$ and $D^{(2,k)}$, respectively. So, we get $(M_k)^n V_k^{(1)} = V_k^{(1)} \left(D^{(1,k)}\right)^n$ and $(M_k^*)^n V_k^{(2)} = V_k^{(2)} \left(D^{(2,k)}\right)^n$ for $n \geq 1$.

Then we can write the following linear system of equations:

$$\begin{cases} m_{i1}^{(k,n)} \left(x_1^k\right)^2 + m_{i2}^{(k,n)} \left(x_1^k\right) + m_{i3}^{(k,n)} = \left(x_1^k\right)^{n+3-i} \\ m_{i1}^{(k,n)} \left(x_2^k\right)^2 + m_{i2}^{(k,n)} \left(x_2^k\right) + m_{i3}^{(k,n)} = \left(x_2^k\right)^{n+3-i} \\ m_{i1}^{(k,n)} \left(x_3^k\right)^2 + m_{i2}^{(k,n)} \left(x_3^k\right) + m_{i3}^{(k,n)} = \left(x_3^k\right)^{n+3-i} \end{cases}, \ (1 \leq i, j \leq 3)$$

and

$$\begin{cases} m_{i1}^{*(k,n)} \left(x_1^k\right)^3 + m_{i2}^{*(k,n)} \left(x_1^k\right)^2 + m_{i3}^{*(k,n)} \left(x_1^k\right) + m_{i4}^{*(k,n)} = \left(x_1^k\right)^{n+4-i} \\ m_{i1}^{*(k,n)} \left(x_2^k\right)^3 + m_{i2}^{*(k,n)} \left(x_2^k\right)^2 + m_{i3}^{*(k,n)} \left(x_2^k\right) + m_{i4}^{*(k,n)} = \left(x_2^k\right)^{n+4-i} \\ m_{i1}^{*(k,n)} \left(x_3^k\right)^3 + m_{i2}^{*(k,n)} \left(x_3^k\right)^2 + m_{i3}^{*(k,n)} \left(x_3^k\right) + m_{i4}^{*(k,n)} = \left(x_3^k\right)^{n+4-i} \\ m_{i1}^{*(k,n)} \left(x_4^k\right)^3 + m_{i2}^{*(k,n)} \left(x_4^k\right)^2 + m_{i3}^{*(k,n)} \left(x_4^k\right) + m_{i4}^{*(k,n)} = \left(x_4^k\right)^{n+4-i} \end{cases}, \ (1 \leq i, j \leq 4).$$

Therefore, we obtain

$$m_{ij}^{(k,n)} = \frac{\det V_{k,j}^{(1,i)}}{\det V_k^{(1)}} \text{ and } m_{ij}^{*(k,n)} = \frac{\det V_{k,j}^{(2,i)}}{\det V_k^{(2)}}$$

for $k = 1, \ldots, 6$. □

3. The Cyclic Groups and The Semigroups via The Matrices M_k and M_k^*

Given an integer matrix $A = [a_{ij}]$, $A \,(\operatorname{mod} \alpha)$ means that all entries of A are modulo α, that is, $A \,(\operatorname{mod} \alpha) = (a_{ij} \,(\operatorname{mod} \alpha))$. Let us consider the set $\langle A \rangle_\alpha = \left\{A^i \,(\operatorname{mod} \alpha) \mid i \geq 0\right\}$. If $\gcd(\alpha, \det A) = 1$, then $\langle A \rangle_\alpha$ is a cyclic group; if $\gcd(\alpha, \det A) \neq 1$, then $\langle A \rangle_\alpha$ is a semigroup. Let the notation $|\langle A \rangle_\alpha|$ denote the order of the set $\langle A \rangle_\alpha$.

We next consider the orders of the cyclic groups and the semigroups generated by the matrices M_k and M_k^* for $k = 1, \ldots, 6$.

Theorem 3.1. *Let p be a prime and let $\langle G \rangle_{p^n}$ be any of the cyclic groups of $\langle M_k \rangle_{p^n}$ and $\langle M_k^* \rangle_{p^n}$ for $k = 1, \ldots, 6$ and $n \in N$. If i is the largest positive integer such that $\left|\langle G \rangle_{p^i}\right| = \left|\langle G \rangle_p\right|$, then $\left|\langle G \rangle_{p^j}\right| = p^{j-i} \left|\langle G \rangle_p\right|$. In particular, if $\left|\langle G \rangle_{p^2}\right| \neq \left|\langle G \rangle_p\right|$, then $\left|\langle G \rangle_{p^j}\right| = p^{j-1} \left|\langle G \rangle_p\right|$.*

Proof. Let us consider the cyclic group $\langle M_1 \rangle_{p^n}$. Then $\gcd(p, 4) = 1$ that is, p is an odd prime. Suppose that u is positive integer and $\left|\langle M_1 \rangle_{p^n}\right|$ is denoted by $\circ(p^n)$. Since $(M_1)^{\circ\left(p^{u+1}\right)} \equiv I \left(\operatorname{mod} p^{u+1}\right)$, $(M_1)^{\circ\left(p^{u+1}\right)} \equiv I \,(\operatorname{mod} p^u)$ where I is a 3×3

identity matrix. Thus, we show that $\circ(p^u)$ divides $\circ\left(p^{u+1}\right)$. Furthermore, if we denote

$$(M_1)^{\circ(p^u)} = I + \left(m_{ij}^{(u)} \cdot p^u\right),$$

then by the binomial expansion, we have

$$(M_1)^{\circ(p^u)\cdot p} = \left(I + \left(m_{ij}^{(u)} \cdot p^u\right)\right)^p = \sum_{r=0}^{p} \binom{p}{r} \left(m_{ij}^{(u)} \cdot p^u\right)^r \equiv I \,(\operatorname{mod} p^u).$$

So we get that $\circ\left(p^{u+1}\right)$ is divisible by $\circ\left(p^{u+1}\right) \cdot p$. Then, $\circ\left(p^{u+1}\right) = \circ(p^u)$ or $\circ\left(p^{u+1}\right) = \circ\left(p^{u+1}\right) \cdot p$. It is clear that the latter holds if and only if there exists an integer $m_{ij}^{(u)}$ which is not divisible by p. Since i is the largest positive integer such that $\circ\left(p^i\right) = \circ(p)$ we have $\circ\left(p^{i+1}\right) \neq \circ\left(p^i\right)$, which yields that there exists an integer $m_{ij}^{(u)}$ such that $p \nmid m_{ij}^{(u)}$. So we find that $\circ\left(p^{i+2}\right) \neq \circ\left(p^{i+1}\right)$. To complete the proof we use an inductive method on i.

There are similar proofs for the other cyclic groups which are obtained as the above. □

Theorem 3.2. *Let α be an positive integer and let $\langle G\rangle_\alpha$ be any of the cyclic groups of $\langle M_k\rangle_\alpha$ and $\langle M_k^*\rangle_\alpha$ for $k = 1, \ldots, 6$. If α has the prime factorization $\alpha = \prod_{j=1}^{t} p_j^{e_j}$, $(t \geq 1)$, then*

$$\left|\langle G\rangle_\alpha\right| = lcm\left[\left|\langle G\rangle_{p_1^{e_1}}\right|, \left|\langle G\rangle_{p_2^{e_2}}\right|, \ldots, \left|\langle G\rangle_{p_t^{e_t}}\right|\right].$$

Proof. Let us consider the cyclic group $\langle M_4^*\rangle_\alpha$, then $\gcd(\alpha, 3) = 1$. Suppose that $\left|\langle M_4^*\rangle_{p_j^{e_j}}\right| = v_j$ for $j = 1, \ldots, t$ and $\left|\langle M_4^*\rangle_\alpha\right| = v$. Then by $(M_4^*)^n$, we can write

$$\begin{aligned}
a_{4v_j-3}^{*4} + 2^{v_j} + \sum_{i=0}^{v_j-3} 2^{v_j-2-i} a_{5+4i}^{*4} &\equiv a_{4v_j-2}^{*4} + 3^{v_j} + \sum_{i=0}^{v_j-3} 3^{v_j-2-i} a_{6+4i}^{*4} \equiv a_{4v_j+4}^{*4} \\
&\equiv 1 \left(\operatorname{mod} p_j^{ej}\right), \\
a_{4v_j-2}^{*4} + \sum_{i=0}^{v_j-3} 3^{v_j-2-i} a_{6+4i}^{*4} &\equiv a_{4v_j+1}^{*4} \equiv a_{4v_j+2}^{*4} \equiv a_{4v_j+3}^{*4} \equiv 0 \left(\operatorname{mod} p_j^{e_j}\right)
\end{aligned}$$

and

$$\begin{aligned}
a_{4v-3}^{*4} + 2^{v} + \sum_{i=0}^{v-3} 2^{v-2-i} a_{5+4i}^{*4} &\equiv a_{4v-2}^{*4} + 3^{v} + \sum_{i=0}^{v-3} 3^{v-2-i} a_{6+4i}^{*4} \equiv a_{4v+4}^{*4} \equiv 1 \,(\operatorname{mod} \alpha), \\
a_{4v-2}^{*4} + \sum_{i=0}^{v-3} 3^{v-2-i} a_{6+4i}^{*4} &\equiv a_{4v+1}^{*4} \equiv a_{4v+2}^{*4} \equiv a_{4v+3}^{*4} \equiv 0 \,(\operatorname{mod} \alpha).
\end{aligned}$$

This implies that $(M_4^*)^v$ is of the form $\lambda \cdot (M_4^*)^{v_j}$, $(\lambda \in N)$ for all values of j. Thus it is verified that $v = \text{lcm}[v_1, v_2, \ldots, v_t]$.

There are similar proofs for the other cyclic groups which are obtained as the above. □

We have the following useful results for the orders of the semigroups generated by the matrices M_k and M_k^* from $(M_k)^n$ and $(M_k^*)^n$.

Corollary 3.3. *Let $\alpha = 2^\eta$ and $m = 2^\mu$ such that $\eta, \mu \in N$ and $1 \le \mu \le \eta$. Then the orders of the semigroups $\langle M_k \rangle_\alpha$ for $k = 1, 2, 3$ are as follows:*

(i). If $\eta = \mu = 1$, then $|\langle M_k \rangle_\alpha| = 1$.

(ii). If $\eta \ge 2$ and $\mu = \eta$ or $\mu = \eta - 1$, then $|\langle M_k \rangle_\alpha| = \eta$.

(iii). If $\eta \ge 3$ and $\mu = \eta - i$ such that $2 \le i \le \eta - 1$, then $|\langle M_k \rangle_\alpha| = \eta + 2^{i-1} - 1$.

Corollary 3.4. *Let $m \equiv 1 \pmod 4$ or $m \equiv 2 \pmod 4$ and let $\eta \in N$. Then the orders of the semigroups $\langle M_3 \rangle_{2^\eta}$ are as follows:*

i. If $m \equiv 1 \pmod 4$, then

$$|\langle M_3 \rangle_{2^\eta}| = \begin{cases} 2 & \textit{for } \eta = 1, \\ 4 & \textit{for } \eta = 2, \\ 2^{\eta-2} + \eta & \textit{for } \eta \ge 3. \end{cases}$$

ii. If $m \equiv 2 \pmod 4$, then

$$|\langle M_3 \rangle_{2^\eta}| = \begin{cases} 1 & \textit{for } \eta = 1, \\ 2 & \textit{for } \eta = 2, \\ 2^{\eta-2} + \eta - 1 & \textit{for } \eta \ge 3. \end{cases}$$

Corollary 3.5. *Let $\eta \in N$. Then the orders of the semigroups $\langle M_4 \rangle_{3^\eta}$, $\langle M_5 \rangle_{2^\eta}$, $\langle M_5 \rangle_{7^\eta}$ and $\langle M_6 \rangle_{19^\eta}$ are as follows:*

$$|\langle M_4 \rangle_{3^\eta}| = \begin{cases} 2 \cdot 3^{\eta-1} + \eta - 1 & \textit{for } \eta = 1, \\ 2 \cdot 3^{\eta-1} + \eta - 2 & \textit{for } \eta \ge 2, \end{cases}$$

$$|\langle M_5 \rangle_{2^\eta}| = \begin{cases} 1 & \textit{for } \eta = 1, \\ 2^{\eta-1} + \eta - 1 & \textit{for } \eta \ge 2, \end{cases}$$

$$|\langle M_5 \rangle_{7^\eta}| = 48 \cdot 7^{\eta-1} + \eta - 1$$

and

$$|\langle M_6 \rangle_{19^\eta}| = 20 \cdot 19^{\eta-1} + \eta - 1.$$

Corollary 3.6. *Let $\eta \in N$. Then the orders of the semigroups $\langle M_k^* \rangle_{2^\eta}$ for $k = 1, 2, 3$ are as follows:*

(i). If $m \equiv 0 \pmod 4$, then

$$|\langle M_k^* \rangle_{2^\eta}| = \begin{cases} 3 & \textit{for } \eta = 1, \\ 7 & \textit{for } \eta = 2, \\ 2^{\eta-1} + 2^{\eta-2} + \eta - 1 & \textit{for } \eta \ge 3. \end{cases}$$

(ii). If $m \equiv 2 \pmod 4$, then $\left|\langle M_k^* \rangle_{2^\eta}\right| = 2^{\eta-1} + 2^{\eta-2} + \eta - 1$.

(iii). If m is odd, then $\left|\langle M_k^* \rangle_{2^\eta}\right| = 3\eta + 1$.

Corollary 3.7. *Let $\eta \in N$. Then the orders of the semigroups $\langle M_4^* \rangle_{3^\eta}$ and $\langle M_5^* \rangle_{2^\eta}$ are as follows:*

$$|\langle M_4^* \rangle_{3^\eta}| = 26 \cdot 3^{\eta-1} + \eta - 1$$

and

$$|\langle M_5^* \rangle_{2^\eta}| = 4\eta.$$

By an inductive argument for $n \geq 1$, we obtain

$$(M_1)^n = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_6 & x_7 & x_8 \end{bmatrix}, \quad (M_2)^n = \begin{bmatrix} x_4 & x_2 & x_5 \\ x_2 & x_1 & x_3 \\ x_7 & x_6 & x_8 \end{bmatrix}$$

and

$$\begin{aligned} (M_1^*)^n &= \begin{bmatrix} y_1 & y_2 & y_2 & y_3 \\ y_2 & y_4 & y_5 & y_6 \\ y_2 & y_5 & y_4 & y_6 \\ y_3 & y_6 & y_6 & y_7 \end{bmatrix}, \ (M_2^*)^n = \begin{bmatrix} y_4 & y_2 & y_5 & y_6 \\ y_2 & y_1 & y_2 & y_3 \\ y_5 & y_2 & y_4 & y_6 \\ y_6 & y_3 & y_6 & y_7 \end{bmatrix}, \\ (M_3^*)^n &= \begin{bmatrix} y_4 & y_5 & y_2 & y_6 \\ y_5 & y_4 & y_2 & y_6 \\ y_2 & y_2 & y_1 & y_3 \\ y_6 & y_6 & y_3 & y_7 \end{bmatrix}, \end{aligned}$$

where $x_i, y_j \in N$ such that $i = 1, \ldots, 8$ and $j = 1, \ldots, 7$. Thus, we have the following results

$$a_{3n+1}^1 = a_{3n+2}^2,\, a_{3n+2}^1 = a_{3n+1}^2,\, a_{3n+3}^1 = a_{3n+3}^2$$

and

$$\begin{aligned} b_{4n+1}^1 &= b_{4n+2}^2 = b_{4n+3}^3,\, b_{4n+2}^1 = b_{4n+1}^2 = b_{4n+1}^3,\, b_{4n+4}^1 = b_{4n+4}^2 = b_{4n+4}^3, \\ b_{4n+2}^1 &= b_{4n+3}^1,\, b_{4n+1}^2 = b_{4n+3}^2,\, b_{4n+1}^3 = b_{4n+2}^3 \end{aligned}$$

and hence

$$|\langle M_1 \rangle_\alpha| = |\langle M_2 \rangle_\alpha|\,,\ |\ \langle M_1^* \rangle_\alpha| = |\ \langle M_2^* \rangle_\alpha| = |\ \langle M_2^* \rangle_\alpha|$$

for every positive integer α.

4. The Polyhedral Sequences in Groups

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence.

Reducing 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind by a modulus α, then we get the repeating sequences, respectively denoted by

$$\left\{a_n^k(\alpha)\right\} = \left\{a_1^k(\alpha), a_2^k(\alpha), \ldots, a_i^k(\alpha), \ldots\right\}$$

and

$$\left\{b_n^k(\alpha)\right\} = \left\{b_1^k(\alpha), b_2^k(\alpha), \ldots, b_i^k(\alpha), \ldots\right\},$$

where $a_i^k(\alpha) = a_i^k \pmod{\alpha}$, $b_i^k(\alpha) = b_i^k \pmod{\alpha}$ and $k = 1, \ldots, 6$. The recurrence relations in the sequences $\left\{a_n^k(\alpha)\right\}$, $\left\{b_n^k(\alpha)\right\}$ and $\left\{a_n^k\right\}$, $\left\{b_n^k\right\}$ are the same, respectively.

Theorem 4.1. *For $k = 1, \ldots, 6$, the sequences $\left\{a_n^k(\alpha)\right\}$, $\left\{b_n^k(\alpha)\right\}$ are periodic.*

Proof. Let us consider the 4-step polyhedral sequence of the first kind $\left\{b_n^1(\alpha)\right\}$ as an example. Let $X = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid 0 \leq x_i \leq \alpha - 1\}$. Since there are α^7 distinct 7-tuples of elements of Z_α, at least one of the 7-tuples appears twice in the sequence $\left\{b_n^1(\alpha)\right\}$. Therefore, the subsequence following this 7-tuple repeats; that is the sequence is periodic.

There are similar proofs for the other sequences which are defined as the above. $\square$

We next denote the periods of the sequences $\left\{a_n^k(\alpha)\right\}$ and $\left\{b_n^k(\alpha)\right\}$ by $l_{a^k}(\alpha)$ and $l_{b^k}(\alpha)$, respectively.

Example 4.1. *For $m = 2$, the sequence $\left\{b_n^1(3)\right\}$ is*

$$\{0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, \ldots\}$$

and thus $l_{b^1}(3) = 8$.

Theorem 4.2. *Let α be an positive integer and let $\left\{x_n^k(\alpha)\right\}$ be any of the sequences of $\left\{a_n^k(\alpha)\right\}$, $\left\{b_n^k(\alpha)\right\}$ for $k = 1, \ldots, 6$. If α has the prime factorization $\alpha = \prod_{j=1}^{t} p_j^{e_j}$, $(t \geq 1)$ and $(\alpha, \det M) = 1$ where M is generating matrix of the sequence that is, $M = M_k$ or $M = M_k^*$, then*

$$l_{x^k}(\alpha) = lcm\left[l_{x^k}(p_1^{e_1}), l_{x^k}(p_2^{e_2}), \ldots, l_{x^k}(p_t^{e_t})\right].$$

Proof. Let us consider the 3-step polyhedral sequence of the fourth kind $\left\{a_n^4(\alpha)\right\}$ as an example. Since $l_{a^4}\left(p_j^{e_j}\right)$ is the length of the period of the sequence $\left\{a_n^k\left(p_j^{e_j}\right)\right\}$, this sequence repeats only after blocks of length $u \cdot l_{a^4}\left(p_j^{e_j}\right)$, $(u \in N)$. Since also $l_{a^4}(\alpha)$ is the length of the period of $\left\{a_n^k(\alpha)\right\}$, the sequence $\left\{a_n^k\left(p_j^{e_j}\right)\right\}$ repeats after $l_{a^4}(\alpha)$ terms for all values j. Thus, $l_{a^4}(\alpha)$ is of the form $u \cdot l_{a^4}\left(p_j^{e_j}\right)$ for all values j, and since any such number gives a period of $l_{a^4}(\alpha)$, we find that $l_{a^4}(\alpha) = lcm\left[l_{a^4}(p_1^{e_1}), l_{a^4}(p_2^{e_2}), \ldots, l_{a^4}(p_t^{e_t})\right]$.

There are similar proofs for the other sequences which are defined as the above. $\square$

Since

$$(M_k)^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{3n+1}^k \\ a_{3n+2}^k \\ a_{3n+3}^k \end{bmatrix}$$

and

$$\left(M_k^*\right)^n \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_{4n+1}^k \\ b_{4n+2}^k \\ b_{4n+3}^k \\ b_{4n+4}^k \end{bmatrix},$$

it is clear that $l_{a^k}(\alpha) = 3 \cdot \left|\langle M_k \rangle_\alpha\right|$ and $l_{b^k}(\alpha) = 4 \cdot \left|\langle M_k^* \rangle_\alpha\right|$ when $(\det M, \alpha) = 1$ where $M = M_k$ or $M = M_k^*$ for $k = 1, \ldots, 6$.

We next redefine the sequences $\left\{a_n^k\right\}$ and $\left\{b_n^k\right\}$ by means of the elements of the groups which have two or three generators.

Definition 4.1. *Let G be a 2-generator group. For a generating pair (x, y), we define the polyhedral 3-orbits of the first, second, third, fourth, fifth and sixth kind by:*

$$s_n^u = \begin{cases} \left(s_{n-3}^u\right)^{\alpha_1} s_{n-1}^u & n \equiv 1 \,(\mathrm{mod}\, 3)\,, \\ \left(s_{n-3}^u\right)^{\alpha_2} s_{n-2}^u & n \equiv 2 \,(\mathrm{mod}\, 3)\,, \\ \left(s_{n-5}^u\right)^{\alpha_3} \left(s_{n-4}^u\right)^{\alpha_3} s_{n-3}^u & n \equiv 0 \,(\mathrm{mod}\, 3)\,, \end{cases},$$
$$(u = 1, 2, 3,\ \alpha_u = m \text{ and } \alpha_i = 2 \text{ if } i \neq u)$$

$$s_n^v = \begin{cases} \left(s_{n-3}^v\right)^2 s_{n-1}^v & n \equiv 1 \,(\mathrm{mod}\, 3)\,, \\ \left(s_{n-3}^v\right)^3 s_{n-2}^v & n \equiv 2 \,(\mathrm{mod}\, 3)\,, \\ \left(s_{n-5}^v\right)^{v-1} \left(s_{n-4}^v\right)^{v-1} s_{n-3}^v & n \equiv 0 \,(\mathrm{mod}\, 3)\,, \end{cases} \quad (v = 4, 5, 6)$$

for $n \geq 4$, with initial conditions $s_1^k = x$, $s_2^k = y$, $s_3^k = y$, $(k = 1, \ldots, 6)$.

For a generating pair (x, y), the polyhedral 3-orbits of the first, second, third, fourth, fifth and sixth kind are denoted by $O_{(x,y)}^{3,1}(G)$, $O_{(x,y)}^{3,2}(G)$, $O_{(x,y)}^{3,3}(G)$, $O_{(x,y)}^{3,4}(G)$, $O_{(x,y)}^{3,5}(G)$ and $O_{(x,y)}^{3,6}(G)$, respectively.

Definition 4.2. *Let G be a 3-generator group. For a generating triple (x, y, z), we define the polyhedral 4-orbits of the first, second, third, fourth, fifth and sixth kind by:*

$$r_n^u = \begin{cases} \left(r_{n-4}^u\right)^{\alpha_1} r_{n-1}^u & n \equiv 1 \,(\mathrm{mod}\, 4)\,, \\ \left(r_{n-4}^u\right)^{\alpha_2} r_{n-2}^u & n \equiv 2 \,(\mathrm{mod}\, 4)\,, \\ \left(r_{n-4}^u\right)^{\alpha_3} r_{n-3}^u & n \equiv 3 \,(\mathrm{mod}\, 4)\,, \\ r_{n-7}^u r_{n-6}^u r_{n-5}^u r_{n-4}^u & n \equiv 0 \,(\mathrm{mod}\, 4)\,, \end{cases} \quad \left(u = 1, 2, 3,\ \alpha_u = m \text{ and } \alpha_i = 2 \text{ if } i \neq u\right),$$

$$r_n^v = \begin{cases} \left(r_{n-4}^v\right)^2 r_{n-1}^v & n \equiv 1 \,(\mathrm{mod}\, 4)\,, \\ \left(r_{n-4}^v\right)^3 r_{n-2}^v & n \equiv 2 \,(\mathrm{mod}\, 4)\,, \\ \left(r_{n-4}^v\right)^{(v-1)} r_{n-3}^v & n \equiv 3 \,(\mathrm{mod}\, 4)\,, \\ r_{n-7}^v r_{n-6}^v r_{n-5}^v r_{n-4}^v & n \equiv 0 \,(\mathrm{mod}\, 4)\,, \end{cases} \quad (v = 4, 5, 6)$$

for $n \geq 5$, with initial conditions $r_1^k = x$, $r_2^k = y$, $r_3^k = z$, $r_4^k = z$, $(k = 1, \ldots, 6)$.

For a generating triple (x, y, z), the polyhedral 4-orbits of the first, second, third, fourth, fifth and sixth kind are denoted by $O^{4,1}_{(x,y,z)}(G)$, $O^{4,2}_{(x,y,z)}(G)$, $O^{4,3}_{(x,y,z)}(G)$, $O^{4,4}_{(x,y,z)}(G)$, $O^{4,5}_{(x,y,z)}(G)$ and $O^{4,6}_{(x,y,z)}(G)$, respectively.

Theorem 4.3. *The polyhedral* 3*-orbits and* 4*-orbits of the first, second, third, fourth, fifth and sixth kind of a finite group G are periodic.*

Proof. Let us consider the polyhedral 3-orbit of the first kind $O^{3,1}_{(x,y)}(G)$ as an example. Suppose that n is the order of G. Since there are n^5 distinct 5-tuples of elements of G, at least one of the 5-tuples appears twice in the sequence $O^{3,1}_{(x,y)}(G)$. Therefore, the subsequence following this 5-tuple repeats. Because of the repetition, the sequence is periodic. □

We denote the lengths of the periods of the orbits $O^{3,k}_{(x,y)}(G)$ and $O^{4,k}_{(x,y,z)}(G)$ by $LO^{3,k}_{(x,y)}(G)$ and $LO^{4,k}_{(x,y,z)}(G)$ for $k = 1, \ldots, 6$, respectively.

We will now address the lengths of the periods of the polyhedral 3-orbits and 4-orbits of the first, second, third, fourth, fifth and sixth kind of finite polyhedral groups as applications of the results obtained.

Theorem 4.4. *The orbit $O^{3,1}_{(x,y)}((m,2,2))$ is a simply periodic sequence and $LO^{3,1}_{(x,y)}((m,2,2)) = 6i$ where i is the least positive integer such that $(-2)^i \equiv 1 \pmod{m}$ and*
$\left[(-2)^i + (-2)^{i-1} + \cdots + (-2)^3\right] + 2 \equiv 0 \pmod{m}$.

Proof. We first note that the polyhedral group $(m, 2, 2)$ of order $2m$ is presented in the 2-generator case by

$$\left\langle x, y \mid x^m = y^2 = (xy)^2 = e \right\rangle.$$

The sequence $O^{3,1}_{(x,y)}((m,2,2))$ is

$$x, y, y, y, y, x^2y, \ldots.$$

Using the above, the sequence becomes:

$$\begin{aligned}
s^1_1 &= x,\ s^1_2 = y,\ s^1_3 = y,\ s^1_4 = y,\ s^1_5 = y,\ s^1_6 = x^2y, \ldots, \\
s^1_{6i+1} &= x^{(-2)^i},\ s^1_{6i+2} = s^1_{6i+3} = s^1_{6i+4} = s^1_{6i+5} = x^{-\left[(-2)^i+(-2)^{i-1}+\cdots+(-2)^3\right]-2}y, \\
s^1_{6i+6} &= x^{-\left[(-2)^{i+1}+(-2)^i+\cdots+(-2)^3\right]-2}y, \ldots.
\end{aligned}$$

So we need the smallest positive integer i such that

$$(-2)^i = um + 1 \text{ and } \left[(-2)^i + (-2)^{i-1} + \cdots + (-2)^3\right] + 2 = vm \text{ for } u, v \in N.$$

Thus the proof is complete. □

Theorem 4.5.

$$LO^{3,2}_{(x,y)}\left((2,m,2)\right)=l_{a^2}\left(m\right), LO^{3,3}_{(x,y)}\left((2,2,m)\right)=3, LO^{3,4}_{(x,y)}\left((2,3,3)\right)=18,$$
$$LO^{3,5}_{(x,y)}\left((2,3,4)\right)=9, LO^{3,6}_{(x,y)}\left((2,3,5)\right)=21$$

and

$$LO^{4,1}_{(x,y,z)}\left((m,2,2)\right)=LO^{4,2}_{(x,y,z)}\left((2,m,2)\right)=\begin{cases}4 & \text{if } m \text{ is odd,}\\ 12 & \text{if } m \text{ is even,}\end{cases}$$
$$LO^{4,3}_{(x,y,z)}\left((2,2,m)\right)=l_{b^3}\left(m\right), LO^{4,4}_{(x,y,z)}\left((2,3,3)\right)=104,$$
$$LO^{4,5}_{(x,y,z)}\left((2,3,4)\right)=4, LO^{4,6}_{(x,y,z)}\left((2,3,5)\right)=248.$$

Proof. Let us consider the polyhedral 4-orbit of the third kind of the polyhedral group $(2,2,m)$, $O^{4,3}_{(x,y,z)}\left((2,2,m)\right)$ as an example. The sequence $O^{4,3}_{(x,y,z)}\left((2,2,m)\right)$ is

$$x,y,z,z,z,z,z,z,z^3,z^3,z,z^4,z^{10},z^{10},z^4,z^{11},\ldots.$$

Using the above, the sequence becomes:

$$\begin{aligned} r^3_5 &= z=z^{b^3_5},\ r^3_6=z=z^{b^3_6},\ r^3_7=z=z^{b^3_7},\ r^3_8=z=z^{b^3_8},\ldots,\\ r^3_{4i+1} &= z^{b^3_{4i+1}},\ r^3_{4i+2}=z^{b^3_{4i+2}},\ r^3_{4i+3}=z^{b^3_{4i+3}},\ r^3_{4i+4}=z^{b^3_{4i+4}},\ldots. \end{aligned}$$

Since the order of z is m, it is easy to see that the length of the period of the orbit $O^{4,3}_{(x,y,z)}\left((2,2,m)\right)$ is $l_{b^3}\left(m\right)$.

There are similar proofs for the other orbits. □

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SEMI-SLANT SUBMANIFOLDS OF (k, μ)- CONTACT MANIFOLD

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Abstract. In the present paper, we study semi-slant submanifolds of (k, μ)-contact manifold and give conditions for the integrability of invariant and slant distributions which are involved in the definition of semi-slant submanifold. Further, we show the totally geodesicity of such distributions.

1. Introduction

The geometry of slant submanifolds was initiated by Chen [6] as a natural generalization of both holomorphic and totally real submanifolds. Since then many geometers have studied such slant immersions in almost Hermitian manifolds. The contact version of slant immersions was introduced by Lotta [11]. Latter, Cabrerizo et al., [3] studied and characterized slant submanifolds of K-contact and Sasakian manifolds and have given several examples of such immersions.

In 1994, Papaghiuc [12] has introduced the notion of semi-slant submanifolds of almost Hermitian manifolds. Cabrerizo et al., [4] extended the study of semi-slant submanifolds to the setting of almost contact metric manifolds. They worked out the integrability conditions of the distributions involved on these submanifolds and studied the geometrical significance of these distributions. Motivated by these studies of the above authors [4, 9, 12], in the present paper we extend the study of the semi-slant submanifolds of (k, μ)-contact manifold, which consist of both Sasakian as well as non-Sasakian cases and are introduced in 1995 by Blair, Koufogiorgos and Papantoniou [2]. Hence it is worth studying and is a generalization of [4].

The paper is organized as follows: In section-2, we recall the notion of (k, μ)-contact manifold and some basic results of submanifolds, which are used for further study. Section-3 is devoted to study semi-slant submanifolds of (k, μ)-contact manifold. Lastly, in section-4 we consider totally umbilical and totally contact umbilical semi-slant submanifolds of (k, μ)-contact manifold and find the necessary conditions to be totally geodesic.

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2. Preliminaries

A contact manifold is a $C^\infty-(2n+1)$ manifold $\tilde{M}^{2n+1}$ equipped with a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere on $\tilde{M}^{2n+1}$. Given a contact form η it is well known that there exists a unique vector field ξ, called the characteristic vector field of η, such that $\eta(\xi)=1$ and $d\eta(X,\xi)=0$ for every vector field X on $\tilde{M}^{2n+1}$. A Riemannian metric g is said to be associated metric if there exists a tensor field ϕ of type (1,1) such that

$$\begin{aligned}\phi^2 &= -I+\eta\otimes\xi,\quad \eta(\xi)=1,\quad \phi\xi=0,\quad \eta\cdot\phi=0, &(2.1)\\ g(\phi X,\phi Y) &= g(X,Y)-\eta(X)\eta(Y),\quad g(X,\xi)=\eta(X), &(2.2)\\ g(X,\phi Y) &= -g(\phi X,Y), &(2.3)\end{aligned}$$

for all vector fields $X,Y\in T\tilde{M}$. Then the structure (ϕ,ξ,η,g) on $\tilde{M}^{2n+1}$ is called a contact metric structure and the manifold $\tilde{M}^{2n+1}$ equipped with such a structure is called a contact metric manifold [1].

Now we define a $(1,1)$ tensor field h by $h=\frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ denotes the Lie differentiation, then h is symmetric and satisfies $h\phi=-\phi h$. Further, a q-dimensional distribution on a manifold M is defined as a mapping D on M which assigns to each point $p\in M$, a q-dimensional subspace D_p of T_pM.
The (k,μ)-nullity distribution of a contact metric manifold $\tilde{M}(\phi,\xi,\eta,g)$ is a distribution

$$\begin{aligned}N(k,\mu):p\rightarrow N_p(k,\mu) &= \{Z\in T_pM:\tilde{R}(X,Y)Z\\ &= k[g(Y,Z)X-g(X,Z)Y]+\mu[g(Y,Z)hX-g(X,Z)hY]\},\end{aligned}$$

for all $X,Y\in T\tilde{M}$. Hence if the characteristic vector field ξ belongs to the (k,μ) nullity distribution, then we have

$$\tilde{R}(X,Y)\xi=k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY]. \tag{2.4}$$

The contact metric manifold satisfying the relation (2.4) is called (k,μ) contact metric manifold [2]. It consists of both k-nullity distribution for $\mu=0$ and Sasakian for $k=1$. In (k,μ)-contact manifold the following relation holds:

$$(\tilde{\nabla}_X\phi)(Y)=g(X+hX,Y)\xi-\eta(Y)(X+hX), \tag{2.5}$$

for all $X,Y\in T\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$. We also have on (k,μ)-contact manifold $\tilde{M}$

$$\tilde{\nabla}_X\xi=-\phi X-\phi hX. \tag{2.6}$$

Let M be a submanifold of a (k,μ)-contact manifold $\tilde{M}$, we denote by the same symbol g the induced metric on M. Let TM be the set of all vector fields tangent to M and $T^\perp M$ is the set of all vector fields normal to M. Then, the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_XY=\nabla_XY+\sigma(X,Y),\quad \tilde{\nabla}_XV=-A_VX+\nabla^\perp_XV, \tag{2.7}$$

for any $X, Y \in TM$, $V \in T^{\perp}M$, where ∇ (resp. $\nabla^{\perp}$) is the induced connection on the tangent bundle TM (resp. normal bundle $T^{\perp}M$) [7]. The shape operator A is related to the second fundamental form σ of M by

$$g(A_V X, Y) = g(\sigma(X, Y), V). \tag{2.8}$$

Now, for any $x \in M$, $X \in T_x M$ and $V \in T_x^{\perp} M$, we put

$$\phi X = TX + FX, \;\; \phi V = tV + fV, \tag{2.9}$$

where TX (resp. FX) is the tangential (resp. normal) component of ϕX, and tV (resp. fV) is the tangential (resp. normal) component of ϕV. The relation (2.9) gives rise to an endomorphism $T : T_x M \to T_x M$ whose square (T^2) will be denoted by Q. The tensor fields on M of type $(1,1)$ determined by these endomorphisms will be denoted by the same letters T and Q respectively. From (2.3) and (2.9)

$$g(TX, Y) + g(X, TY) = 0, \tag{2.10}$$

for each $X, Y \in TM$. The covariant derivatives of the tensor fields T, Q and F are defined as

$$(\nabla_X T)Y = \nabla_X TY - T(\nabla_X Y), \tag{2.11}$$
$$(\nabla_X Q)Y = \nabla_X QY - Q(\nabla_X Y), \tag{2.12}$$
$$(\nabla_X F)Y = \nabla_X FY - F(\nabla_X Y). \tag{2.13}$$

Using (2.5), (2.6), (2.7), (2.9), (2.11), and (2.12), we obtain

$$(\nabla_X T)Y = A_{FY}X + t\sigma(X, Y) + g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.14}$$
$$(\nabla_X F)Y = -\sigma(X, TY) + f\sigma(X, Y). \tag{2.15}$$

3. Semi-slant submanifolds of a (k, μ)-contact manifold

As a generalization of slant and CR-submanifolds, Papaghiuc [12] introduced the notion of semi-slant submanifolds of an almost Hermitian manifolds. Cabrerizo et al., [4] gave the contact version of semi-slant submanifold and they obtained several interesting results. The purpose of the present section is to study semi-slant submanifolds of a (k, μ)-contact manifold.

A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be a slant submanifold if for any $x \in M$ and any $X \in T_x M$, the Wirtinger's angle, the angle between ϕX and $T_x M$, is constant $\theta \in [0, 2\pi]$. Here the constant angle θ is called the slant angle of M in $\tilde{M}$. The invariant submanifolds are slant submanifolds with slant angle 0 and anti-invariant submanifolds are slant submanifolds with slant angle $\frac{\pi}{2}$. A slant submanifold is called proper, if it is neither invariant nor anti-invariant. Recently, we have defined and studied slant submanifolds of a (k, μ)-contact manifold in [13].

A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be a semi-slant submanifold of $\tilde{M}$ [4] if there exist two orthogonal distributions D_1 and D_2 on M such that:

(i) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2 \oplus <\xi>$.
(ii) The distribution D_1 is an invariant distribution, i.e., $\phi(D_1) = D_1$.
(iii) The distribution D_2 is slant with slant angle $\theta \neq 0$.
In particular, if $\theta = \frac{\pi}{2}$, then a semi-slant submanifold reduces to a semi-invariant submanifold. On a semi-slant submanifold M, for any $X \in TM$, we write

$$X = P_1X + P_2X + \eta(X)\xi, \tag{3.1}$$

where $P_1X \in D_1$ and $P_2X \in D_2$. Now by equations (2.9) and (3.1)

$$\phi X = \phi P_1X + TP_2X + FP_2X. \tag{3.2}$$

Then, it is easy to see that

$$\phi P_1X = TP_1X, \;\; FP_1X = 0, \quad TP_2X \in D_2. \tag{3.3}$$

Thus

$$TX = \phi P_1X + TP_2X \;\; and \;\; FX = FP_2X. \tag{3.4}$$

Let ν denote the orthogonal complement of ϕD_2 in $T^{\perp}M$ i.e., $T^{\perp}M = \phi D_2 \oplus \nu$. Then it is easy to observe that ν is an invariant subbundle of $T^{\perp}M$.

Now, we are in a position to workout the integrability conditions of the distributions D_1 and D_2 on a semi-slant submanifold of a (k,μ)-contact manifold.

Lemma 3.1. *Let M be a semi-slant submanifold of a (k,μ)-contact manifold $\tilde{M}$, then*

$$g([X,Y],\xi) = 2g(\phi X,Y) + g(Y,\phi hX) - g(X,\phi hY), \tag{3.5}$$

for any $X,Y \in D_1 \oplus D_2$.

The assertion can be proved by using the fact that $\nabla_X\xi = -\phi X - \phi hX$ for $X \in D_1$ and (2.3). Since for any $X \in D_1$
$g([X,\phi X],\xi) \neq 0$, we have

Corollary 3.1. *Let M be a semi-slant submanifold of a (k,μ)-contact manifold $\tilde{M}$ such that dim$(D_1) \neq 0$. Then, the invariant distribution D_1 is not integrable.*

Now for the slant distribution, we have

Theorem 3.1. *Let M be a semi-slant submanifold of a (k,μ)-contact manifold $\tilde{M}$. Then the slant distribution D_2 is integrable if and only if slant angle of D_2 is $\frac{\pi}{2}$ i.e., M is semi-invariant submanifold.*

Proof. For any $Z,W \in D_2$, by (3.5) we have

$$g([Z,W],\xi) = 2g(TZ,W) + g(W,ThZ) - g(Z,ThW).$$

If D_2 is integrable, then $T \mid D_2 \equiv 0$ and so $\theta = \frac{\pi}{2}$. Hence M is a semi-invariant submanifold.
Conversely, if sla$(D_2) = \frac{\pi}{2}$, then $\phi Z = FZ$ for each $Z \in D_2$ and by equations (2.5) and (2.7)

$$\phi\nabla_ZW + \phi\sigma(Z,W) = -A_{FZ}W + \nabla^{\perp}_ZFW - g(Z+hZ,W)\xi,$$

for each $Z, W \in D_2$. Interchanging Z and W in the above equation and subtracting the obtained relation from the same, we obtain

$$\phi[Z,W] = A_{FZ}W - A_{FW}Z + \nabla^{\perp}_{Z}FW - \nabla^{\perp}_{W}FZ - g(hZ,W)\xi + g(hW,Z)\xi. \quad (3.6)$$

Further, by using equations (2.3), (2.7) and (2.8) in (2.5), it is easy to obtain that

$$A_{FZ}W = A_{FW}Z, \quad (3.7)$$

for each $Z, W \in D_2$. In view of (3.5), (2.1) and (3.7), equation (3.6) yields

$$[Z,W] = \phi(\nabla^{\perp}_{Z}FW - \nabla^{\perp}_{W}FZ). \quad (3.8)$$

The right hand side of the above lies in D_2 because on using equations (2.5), (2.7) and (2.10), we observe that

$$g(V, \nabla^{\perp}_{W}FZ) = -g(A_{\phi V}W, Z)$$

for all $V \in \nu$ and $Z, W \in D_2$. This shows that

$$g(\nabla^{\perp}_{Z}FW - \nabla^{\perp}_{W}FZ, V) = 0.$$

i.e., $\nabla^{\perp}_{Z}FW - \nabla^{\perp}_{W}FZ$ lies in FD_2 for each $Z, W \in D_2$, and thus from equation (3.8), $[Z,W] \in D_2$. $\square$

Now, for $Y \in D_1 \oplus D_2$, by equation (2.5), we have

$$\tilde{\nabla}_{\xi}\phi Y = \phi\tilde{\nabla}_{\xi}Y.$$

In particular, for $Y \in D_1$, the above equation yields

$$\nabla_{\xi}\phi Y = \phi\nabla_{\xi}Y.$$

This implies $\nabla_{\xi}Y \in D_1$ for any $Y \in D_1$.

The above observation together with the fact that $\sigma(X,\xi) = 0$ for $X \in D_1$ yields

Lemma 3.2. *On a semi-slant submanifold M of a (k,μ)-contact manifold $\tilde{M}$,*

$$[X,\xi] \in D_1 \;\; \text{and} \;\; [Z,\xi] \in D_2$$

for any $X \in D_1$ and $Z \in D_2$.

Lemma 3.3. *Let M be a semi-slant submanifold of a (k,μ)-contact manifold $\tilde{M}$. Then, for any $X, Y \in TM$, we have*

$$P_1(\nabla_X \phi P_1 Y) + P_1(\nabla_X TP_2Y) = \phi P_1(\nabla_X Y) + P_1 A_{FP_2Y}X - \eta(Y)P_1X \quad (3.9)$$

Proof. By using equations (2.1), (2.7), (3.1), (3.2) and (3.3) we obtain

$$\begin{aligned}
&\nabla_X \phi P_1 Y + \sigma(\phi P_1 Y, X) + \nabla_X TP_2 Y + \sigma(TP_2Y, X) - A_{FP_2Y}X + \nabla^{\perp}_{X}FP_2Y \\
&= \phi P_1 \nabla_X Y + TP_2 \nabla_X Y + FP_2 \nabla_X Y + t\sigma(X,Y) + f\sigma(X,Y) \\
&+ g(X + hX, Y)\xi - \eta(Y)P_1(X + hX) - \eta(Y)P_2(X + hX) - \eta(Y)\eta(X)\xi.
\end{aligned}$$

Equating the components of D_1 we get (3.9). $\square$

Proposition 3.2. *Let M be a semi-slant submanifold of (k,μ)-contact manifold $\tilde{M}$. Then*
(i) $D_1\oplus <\xi>$ is integrable if and only if

$$\sigma(X,\phi Y)=\sigma(Y,\phi X); \tag{3.10}$$

(ii) $D_2\oplus <\xi>$ is integrable if and only if

$$P_1(\nabla_Z TW - A_{NW}Z - \nabla_W TZ + A_{NZ}W)=0; \tag{3.11}$$

for any $X,Y\in D_1$ and $Z,W\in D_2$.

Proof. Now, for any $X,Y\in D_1\oplus <\xi>$ and $V\in T^\perp M$

$$g(\tilde{\nabla}_X\phi Y-\tilde{\nabla}_Y\phi X,V)=g(\sigma(X,\phi Y)-\sigma(\phi X,Y),V),$$

after simplification, we get

$$g((\tilde{\nabla}_X\phi)Y-(\tilde{\nabla}_Y)\phi X+\phi[X,Y],V)=g(\sigma(X,\phi Y)-\sigma(\phi X,Y),V).$$

Now using (2.5) and (3.2), we obtain

$$g(FP_2[X,Y],V)=g(\sigma(X,\phi Y)-\sigma(\phi X,Y),V).$$

Removing inner product, we get

$$FP_2[X,Y]=\sigma(X,\phi Y)-\sigma(\phi X,Y). \tag{3.12}$$

Hence, if $D_1\oplus <\xi>$ is integrable then (3.10) holds directly from (3.12).
Conversely, by using (3.10), it is easy to prove that

$$\sigma(X,\phi Y)-\sigma(Y,\phi X)=\sigma(P_1X,\phi P_1Y)-\sigma(P_1Y,\phi P_1X)=0,$$

for any $X,Y\in D_1\oplus <\xi>$. Thus, by applying (3.12) it follows that $FP_2[X,Y]=0$. So, we can easily deduce that $P_2[X,Y]$ must vanish. Since D_2 is a slant distribution with nonzero slant angle. Hence $[X,Y]\in D_1\oplus <\xi>$ and statement (i) holds.
With regards to statement (ii), by virtue of (3.9) we have

$$\phi P_1[Z,W]=P_1(\nabla_Z TW-\nabla_W TZ-A_{FW}Z+A_{FZ}W).$$

for any $Z,W\in D_2\oplus <\xi>$. Hence (3.11) holds if and only if

$$\phi P_1[Z,W]=0, \tag{3.13}$$

for any $Z,W\in D_2\oplus <\xi>$. But it can be showed that (3.13) is equivalent to $D_2\oplus <\xi>$ being an integrable distribution. □

The Nijenhuis tensor field S of the tensor T is given by

$$S(X,Y)=[TX,TY]+T^2[X,Y]-T[TX,Y]-T[X,TY],$$

for $X,Y\in TM$. In particular, for $X\in D_1$ and $Z\in D_2$, the above equation on simplification takes the form

$$S(X,Z)=(\nabla_{TX}T)Z-(\nabla_{TZ}T)X+T(\nabla_Z T)X-T(\nabla_X T)Z.$$

Using (2.14) the above equation becomes

$$S(X,Z) = A_{FZ}TX + t\sigma(TX,Z) - t\sigma(TZ,X) - T(A_{FZ}X). \tag{3.14}$$

Theorem 3.3. *If the invariant distribution D_1 on a semi-slant submanifold M of a (k,μ)-contact manifold $\tilde{M}$ is integrable and its leaves are totally geodesic in M, then*
(i) $\sigma(D_1, D_1) \in \nu$,
(ii) $S(D_1, D_2) \in D_2$.

Proof. By hypothesis, for any X, Y in D_1 and Z in D_2

$$g(\nabla_X Y, Z) = 0,$$

and therefore by Gauss formula, we have

$$g(\phi\tilde{\nabla}_X Y, \phi Z) = 0.$$

The above equation on making use of equations (2.5), (2.7) and (2.9) yields

$$g(\sigma(X, \phi Y), FZ) = 0.$$

This proves statement (i). To prove statement (ii), use (3.14) to get

$$g(S(X,Z),Y) = g(A_{FZ}TX + t\sigma(TX,Z) - t\sigma(TZ,X) - TA_{FZ}X, Y).$$

The right hand side of the above equation is zero in view of statement (i) and thus (ii) is established. □

Next for the slant distribution, we have:

Theorem 3.4. *If the slant distribution D_2 on a semi-slant submanifold M of a (k,μ)-contact manifold $\tilde{M}$ is integrable and its leaves are totally geodesic in M, then*
(i) $\sigma(D_1, D_2) \in \nu$,
(ii) $S(D_1, D_2) \in D_1$.

Proof. By hypothesis,

$$g(\nabla_Z W, \phi X) = 0,$$

for any $Z, W \in D_2$ and $X \in D_1$. By applying (2.5), (2.7) and (2.9)

$$g(\sigma(X,Z), FW) = 0.$$

That proves (i). Now by using equation (3.14)

$$g(S(X,Z),W) = g(A_{FZ}TX + t\sigma(TX,Z) - t\sigma(TZ,X) - TA_{FZ}X, W),$$

for $X \in D_1$ and $Z, W \in D_2$. The right hand side of the above equation is zero by part (i). This proves (ii) and the theorem. □

Example: For any $\theta \in [0, \frac{\pi}{2}]$

$$x(u_1,u_2,u_3,u_4,u_5) = (u_1, 0, u_3, 0, u_2, 0, u_4 cos\theta, u_4 sin\theta, u_5)$$

defines a five dimensional semi-slant submanifold M, with slant angle θ, in R^9 with its usual (k,μ)-contact structure (ϕ_0, ξ, η, g) [13]. Further,

$$\begin{aligned} e_1 &= 2(\frac{\partial}{\partial x_1} + x_5\frac{\partial}{\partial t}); \qquad e_2 = 2\frac{\partial}{\partial x_5}; \qquad e_3 = 2(\frac{\partial}{\partial x_3} + x_7\frac{\partial}{\partial t}); \\ e_4 &= cos\theta(2\frac{\partial}{\partial x_7} + sin\theta(2\frac{\partial}{\partial x_8}); \qquad e_5 = \frac{\partial}{\partial t} = \xi, \end{aligned} \tag{3.15}$$

form a local orthonormal frame of TM. If we define the distribution $D_1 =< e_1, e_2 >$ and $D_2 =< e_3, e_4 >$, then it is easy to check that the distribution D_1 is invariant under ϕ and D_2 is slant with slant angle θ. That is M is semi-slant submanifold.

4. Totally umbilical submanifolds of (k,μ)-contact manifold

Definition 1. A submanifold M is said to be totally umbilical submanifold if its second fundamental form satisfies

$$\sigma(X,Y) = g(X,Y)H,$$

for all $X, Y \in TM$, where H is the mean curvature vector.

To investigate totally umbilical submanifolds of a (k,μ)-contact manifold, we first establish the following preliminary result.

Proposition 4.5. *Let M be a semi-slant submanifold of a (k,μ)-contact manifold $\tilde{M}$ with $\sigma(X,TX) = 0$ for each $X \in D_1\oplus < \xi >$. If $D_1\oplus < \xi >$ is integrable then each of its leaves are totally geodesic in M as well as in $\tilde{M}$.*

Proof. For $X \in D_1\oplus < \xi >$, by equation (2.15)

$$(\nabla_X F)X = -\sigma(X,TX) + f\sigma(X,X),$$

by using (2.13) and the fact that $FX = 0$ for each $X \in D_1$, we get

$$F\nabla_X X = f\sigma(X,X). \tag{4.1}$$

Now, making use of Proposition 3.2 and the assumption that $\sigma(X,TX) = 0$, we obtain $\sigma(X,TY) = 0$ i.e., $\sigma(X,Y) = 0$ for each $X, Y \in D_1\oplus < \xi >$. This proves that the leaves of $D_1\oplus < \xi >$ are totally geodesic in $\tilde{M}$. Thus by (4.1), we obtain that $\nabla_X Y \in D_1\oplus < \xi >$ i.e., the leaves of $D_1\oplus < \xi >$ are totally geodesic in M. □

As an immediate consequence of the above, we have

Corollary 4.2. *Let M be a totally umbilical semi-slant submanifold of a (k,μ)-contact manifold $\tilde{M}$. If $D_1\oplus < \xi >$ is integrable, then each of its leaves are totally geodesic in M as well as in $\tilde{M}$.*

Definition 2. [10] A submanifold M of an almost contact metric manifold is said to be totally contact umbilical submanifold if

$$\sigma(X,Y) = g(\phi X, \phi Y)K + \eta(Y)\sigma(X,\xi) + \eta(X)\sigma(Y,\xi),$$

for all $X, Y \in TM$, where K is a normal vector field on M. If $K = 0$ then M is said to be a totally contact geodesic submanifold. For a submanifold of a (k, μ)-contact manifold, the condition for totally contact umbilicalness reduces to

$$\sigma(X, Y) = g(\phi X, \phi Y)K.$$

Theorem 4.6. *Let M be a totally contact umbilical semi-slant submanifold of a (k, μ)-contact manifold $\tilde{M}$, with $dim(D_1) \neq 0$. Then the mean curvature vector is a global section of FD_2.*

Proof. Let $X \in D_1$ be a unit vector field and $V \in \nu$, then

$$g(H, V) = g(\sigma(X, X), V) = g(\tilde{\nabla}_X \phi X, \phi V) = g(\sigma(X, \phi X, \phi V)) = 0$$

$\Longrightarrow H \in FD_2$. □

In view of Theorem 4.6, we have the following:

Theorem 4.7. *A totally contact umbilical semi-slant submanifold of a (k, μ)-contact manifold is totally contact geodesic if the invariant distribution D_1 is integrable.*

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WRONSKIAN SOLUTIONS OF (2+1) DIMENSIONAL NON-LOCAL ITO EQUATION

YAKUP YILDIRIM AND EMRULLAH YAŞAR

Abstract. In this work, the Wronskian determinant technique is performed to (2+1)-dimensional non-local Ito equation in the bilinear form. First, we obtain some sufficient conditions in order to show Wronskian determinant solves the (2+1)-dimensional non-local Ito equation. Second, rational solutions, soliton solutions, positon solutions, negaton solutions and their interaction solutions were deduced by using the Wronskian formulations

1. Introduction

The nonlinear evolution equations (NLEEs) model abundant physical processes which occur in the nature. Therefore, investigating and obtaining solutions of these type equations have an extremely important place in nonlinear science. In this context, in the literature a plenty of analytic and numerical methods were developed such as inverse scattering transform, Hirota bilinear method, the Riccati equation expansion method, the sine–cosine method, the $\tanh - \operatorname{sech}$ method, G'/G expansion method, Adomian decomposition method, He's variational principle, Lie symmetry method and many more ([1],[3]-[6]-[7], [8],[14], [19]-[20], [22]-[23]).

Nowadays, besides to above aforementioned methods, the Wronskian determinant method ([5], [15]) depending upon Hirota bilinear forms has a wide range of impact and applicability on the NLEES. Wronskian determinant technique is a important tool to get exact solutions to the corresponding Hirota bilinear equations of the NLEE equations.

In [11], we observe that there is a bridge between Wronskian solutions and generalized Wronskian solutions. It gives us a way to obtain generalized Wronskian solutions simply from Wronskian determinants. The basic idea was used to generate positons, negatons and their interaction solutions through the Wronskian formulation.

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It is demonstrated in [12] that for each type of Jordan blocks of the coefficient matrix $J(\lambda_{ij})$, there exist special sets of eigenfunctions. These functions were used to generate rational solutions, solitons, positons, negatons, breathers, complexitons and their interaction solutions. The obtained solution formulas of the representative systems allow us to construct more general Wronskian solutions than rational solutions, positons, negatons, complexitons and their interaction solutions.

As stated in [13], integrable equations can have three different kinds of explicit exact transcendental function solutions: negatons, positons and complexitons. Solitons are usually a specific class of negatons. Roughly speaking, negatons and positons are solutions which involve exponential functions and trigonometric functions of space variables, respectively, and they are all associated with real eigenvalues of the associated spectral problems. But complexitons are different solutions which involve both exponential and trigonometric functions of space variables, and they are associated with complex eigenvalues of the associated spectral problems. Interaction solutions among negatons, positons, rational solutions and complexitons are a class of much more general and complicated solutions to soliton equations, in the category of elementary function solutions.

The generalized (2+1) dimensional non-local Ito equation

$$u_{tt}+u_{xxxt}+3\left(2u_xu_t+uu_{xt}\right)+3u_{xx}\left(\int u_t\ dx\right)+au_{yt}+bu_{xt}=0. \tag{1}$$

was firstly studied by Ito for generalizing the bilinear Korteweg-de Vries (KdV) equation [9]. To get rid of the integral operator, we use the transformation

$$u=v_x$$

to cast (1) into the following equation

$$v_{xtt}+v_{xxxxt}+3\left(2v_{xx}v_{xt}+v_xv_{xxt}\right)+3v_{xxx}v_t+av_{xyt}+bv_{xxt}=0. \tag{2}$$

We observe increasing interest for Eq.(2) in the literature ([2], [4], [18],[21]). For instance in [21], Wazwaz obtains single soliton solutions and periodic solutions of Eq.(2) by tanh-coth method. He also constructs multiple-soliton solutions of sech-squared type by using Hirota bilinear method. In [2], Adem constructs multiple wave solutions of Eq.(2) by exploiting the multiple exp-function algorithm.

To solve Eq.(2) we can get dependent variable v by

$$v=\alpha\left(\ln f\right)_x\sim\left\{\begin{array}{l} v=\alpha w_x\\ w=\ln f\end{array}\right. \tag{3}$$

where $f(x,y,t)$ is an unknown real function which will be determined. Substituting Eq.(3) into Eq. (2), we have

$$\begin{aligned}\alpha w_{xxtt}+\alpha w_{xxxxxt}+3\left(2\alpha^2w_{xxx}w_{xxt}+\alpha^2w_{xx}w_{xxxt}\right)+3\alpha^2w_{xxxx}w_{xt}\\ +\alpha aw_{xxyt}+\alpha bw_{xxxt}=0,\end{aligned} \tag{4}$$

which can be integrated twice with respect to x to give

$$\alpha w_{tt} + \alpha w_{xxxt} + 3\alpha^2 w_{xt} w_{xx} + \alpha a w_{yt} + \alpha b w_{xt} = C, \tag{5}$$

where C is the constant of integration.

If we get

$$6\alpha = 3\alpha^2, \alpha = 2,$$

then (5) can be written as

$$w_{tt} + w_{xxxt} + 6w_{xt} w_{xx} + a w_{yt} + b w_{xt} = C. \tag{6}$$

Substituting $w = \ln f$ into Eq. (6), we get

$$\frac{f_{tt}}{f} - \frac{f_t^2}{f^2} + \frac{f_{xxxt}}{f} - \frac{f_{xxx} f_t}{f^2} - \frac{3f_{xxt} f_x}{f^2} + \frac{3f_{xx} f_{xt}}{f^2} + \frac{a f_{yt}}{f} - \frac{a f_y f_t}{f^2} + \frac{b f_{xt}}{f} - \frac{b f_x f_t}{f^2} = C. \tag{7}$$

Substituting $C = 0$ into Eq. (7) and employing Hitora derivative operators [8] we obtain the Hitora bilinear form of Eq.(2) as

$$\left(D_t^2 + D_x^3 D_t + a D_y D_t + b D_x D_t\right) f.f$$

$$= f(f_{xxxt} + f_{tt} + a f_{yt} + b f_{xt}) + 3f_{xx} f_{xt} - f_t^2 - f_{xxx} f_t - 3f_{xxt} f_x - a f_y f_t - b f_x f_t. \tag{8}$$

In this work, our intention is to present the generalized Wronskian solutions of the Eq. (2). The generalized Wronskian solutions are obtained through Wronskian solutions. The generalized Wronskian solutions can be viewed as Wronskian solutions. Solitons are examples of Wronskian solutions, and positons and negatons are examples of generalized Wronskian solutions ([11]-[10]).

The paper is organized as follows. In Section 2, the Wronskian determinant solution is deduced for Hirota bilinear form corresponding to Eq. (2). In Section 3, using Wronskian formulation rational solutions, solitons, positons, negatons and their interaction solutions are presented. Lastly, conclusions are given in Section 4.

2. **Wronskian formulation**

We first present notation to be used and recall the definitions and theorems that appear in ([5],[15]-[17]).

The solutions determined by $v = 2(\ln f)_x$ with $f = |\widehat{N-1}|$ and

$$W(\phi_1, \phi_2, , , , , \phi_n) = (\widehat{N-1}; \Phi) = |\widehat{N-1}| = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & .. & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & .. & \phi_2^{(N-1)} \\ . & . & . & . \\ . & . & . & . \\ \phi_N^{(0)} & \phi_N^{(1)} & .. & \phi_N^{(N-1)} \end{vmatrix}, \ N \geq 1, \tag{9}$$

where

$$\Phi = (\phi_1, \phi_2, , , , , \phi_n)^T, \quad \phi_i^{(0)} = \phi_i \ , \quad \phi_i^{(j)} = \frac{\partial^j}{\partial x^j} \phi_i \ , \quad j \geq 1 \ , \quad 1 \leqslant i \leqslant N \ . \tag{10}$$

to the Eq. (2) will be called Wronskian solutions ([5],[15] and [17]). Now, we give the following important properties on determinants ([17]).

Property 1. If D is $N*(N-2)$ matrix, and a,b,c,d are n-dimensional column vectors then,

$$|D,a,b|\,|D,c,d|-|D,a,c|\,|D,b,d|+|D,a,d|\,|D,b,c|=0\ . \tag{11}$$

Property 2. If $a_j(j=1,...,n)$ is an n-dimensional column vector, and $b_j(j=1,...,n)$ is a real constant different form zero then

$$\sum_{i=1}^{N} b_i\,|a_1,a_2,....,a_N|=\sum_{j=1}^{N}|a_1,a_2,....,ba_j,....,a_N|\,, \tag{12}$$

where $ba_j=(b_1a_{1j},b_2a_{2j},.....,b_Na_{Nj})^T$.

Property 3.

$$\begin{aligned}|\widehat{N-1}|\sum_{i=1}^{N}\lambda_{ii}(t)\left(\sum_{i=1}^{N}\lambda_{ii}(t)|\widehat{N-1}|\right)&=|\widehat{N-1}|(|\widehat{N-5},N-3,N-2,N-1,N|\\&-|\widehat{N-4},N-2,N-1,N+1|-|\widehat{N-3},N-1,N+2||\\&+2|\widehat{N-3},N,N+1|+|\widehat{N-2},N+3).\end{aligned} \tag{13}$$

Now, we present a set of sufficient conditions consisting of systems of linear partial differential equations which guarantees that the Wronskian determinant solves the Eq. (2) in the bilinear form (8). Upon solving the linear conditions, the resulting Wronskian formulations bring solution formulas, which can yield rational solutions, solitons, negatons, positons and interaction solutions. Also, positons, negatons and their interaction solutions are called the generalized Wronskian solutions ([11]).

Theorem 1. *Assuming that $\phi_i=\phi_i(x,y,t)$ (where $i=1,2,...,N$) satisfies the following linear partial differential equations (LPDEs)*

$$\phi_{i,xx}=\sum_{j=1}^{N}\lambda_{ij}(t)\phi_j\ , \tag{14}$$

$$\phi_{i,t}=m\phi_{i,x}\ , \tag{15}$$

$$\phi_{i,y}=n\phi_{i,xxx}+k\phi_{i,x} \tag{16}$$

with

$$n=-\frac{4}{a}\ ,\ m=-(b+ak)$$

then $f=|\widehat{N-1}|$ defined by (9) solves the bilinear Eq. (8).

Proof. Considering (9), we can obtain the following derivatives

$$f=|\widehat{N-1}|$$

$$f_x=|\widehat{N-2},N|$$

$$f_{xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|$$
$$f_{xxx} = |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|.$$

In addition, keeping in mind the conditions of (15)-(16), we can produce that

$$f_t = m|\widehat{N-2}, N|$$
$$f_{xt} = m(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|)$$
$$f_{tt} = m^2(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|)$$
$$f_y = n|\widehat{N-4}, N-2, N-1, N| - n|\widehat{N-3}, N-1, N+1| + n|\widehat{N-2}, N+2| + k|\widehat{N-2}, N|$$
$$f_{yt} = mn|\widehat{N-5}, N-3, N-2, N-1, N| - mn|\widehat{N-3}, N, N+1| + mn|\widehat{N-2}, N+3|$$
$$+mk|\widehat{N-3}, N-1, N| + mk|\widehat{N-2}, N+1|$$
$$f_{xxt} = m(|\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|)$$
$$f_{xxxt} = m(|\widehat{N-5}, N-3, N-2, N-1, N| + 3|\widehat{N-4}, N-2, N-1, N+1| + 2|\widehat{N-3}, N, N+1|$$
$$+3|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|)$$

Therefore, we can compute all terms in Eq.(8) such as

$$\begin{aligned} 3f_{xx}f_{xt} &= 3m(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|)(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|) \\ &= 3m(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|)^2 \\ &= 3m(|\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N| + 2|\widehat{N-3}, N-1, N|)^2 \\ &= 3m(|\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N|)^2 + 12m|\widehat{N-3}, N-1, N||\widehat{N-2}, N+1|, \end{aligned} \tag{17}$$

$$ff_{xxxt} = m|\widehat{N-1}|(|\widehat{N-5}, N-3, N-2, N-1, N| + 3|\widehat{N-4}, N-2, N-1, N+1|$$
$$+2|\widehat{N-3}, N, N+1| + 3|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|),$$
$$ff_{tt} = m^2|\widehat{N-1}(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|),$$
$$aff_{yt} = a|\widehat{N-1}|(mn|\widehat{N-5}, N-3, N-2, N-1, N| - mn|\widehat{N-3}, N, N+1|$$
$$+mn|\widehat{N-2}, N+3| + mk|\widehat{N-3}, N-1, N| + mk|\widehat{N-2}, N+1|),$$
$$bff_{xt} = bm|\widehat{N-1}|(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|),$$

$$\begin{aligned} f(f_{xxxt} + f_{tt} + af_{yt} + bf_{xt}) &= |\widehat{N-1}|((m+amn)\,|\widehat{N-5}, N-3, N-2, N-1, N| \\ &+3m|\widehat{N-4}, N-2, N-1, N+1| + (2m-amn)\,|\widehat{N-3}, N, N+1| \\ &+3m|\widehat{N-3}, N-1, N+2| + (m+amn)\,|\widehat{N-2}, N+3| + \left(m^2+amk+bm\right)|\widehat{N-3}, N-1, N| \\ &+ \left(m^2+amk+bm\right)|\widehat{N-2}, N+1|). \end{aligned} \tag{18}$$

We can obtain from Eq. (17) and Eq. (18) (Property 3)

$$m + amn = -3m$$
$$n = -\frac{4}{a}$$

and

$$m^2 + amk + bm = 0$$

$$m = -(b + ak).$$

Then, Eq. (8) can be rewritten as the following

$$f(f_{xxxt} + f_{tt} + af_{yt} + bf_{xt}) = -3m|\widehat{N-1}|(|\widehat{N-5}, N-3, N-2, N-1, N|$$

$$-|\widehat{N-4}, N-2, N-1, N+1| - 2|\widehat{N-3}, N, N+1| - |\widehat{N-3}, N-1, N+2|$$
$$+|\widehat{N-2}, N+3|)$$

$$= -3m(|\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N|)^2 + 12m|\widehat{N-3}, N, N+1||\widehat{N-1}| \quad (19)$$

and

$$-f_t^2 = -m^2|\widehat{N-2}, N|^2$$

$$-f_{xxx}f_t = -m|\widehat{N-2}, N|(|\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|)$$

$$-3f_{xxt}f_x = -3m|\widehat{N-2}, N|(|\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|)$$

$$-af_yf_t = -am|\widehat{N-2}, N|(n|\widehat{N-4}, N-2, N-1, N| - n|\widehat{N-3}, N-1, N+1| + n|\widehat{N-2}, N+2|$$

$$+k|\widehat{N-2}, N|) - bf_xf_t = -bm|\widehat{N-2}, N||\widehat{N-2}, N| = -bm|\widehat{N-2}, N|^2$$

$$-f_t^2 - f_{xxx}f_t - 3f_{xxt}f_x - af_yf_t - bf_xf_t = -12m|\widehat{N-3}, N-1, N+1||\widehat{N-2}, N| \quad (20)$$

After substituting of the Eq. (17),(19) and (20) into (8) we obtain the following Plücker relation:

$$\left(D_t^2 + D_x^3D_t + aD_yD_t + bD_xD_t\right)ff = 12m|\widehat{N-3}, N-1, N||\widehat{N-2}, N+1|$$

$$+12m|\widehat{N-3}, N, N+1||\widehat{N-1}| - 12m|\widehat{N-3}, N-1, N+1||\widehat{N-2}, N|$$

As result of Property 1, we get

$$12m|\widehat{N-3}, N-1, N||\widehat{N-2}, N+1| + 12m|\widehat{N-3}, N, N+1||\widehat{N-1}|$$
$$- 12m|\widehat{N-3}, N-1, N+1||\widehat{N-2}, N| = 0.$$

□

This demonstrates that $f = |\widehat{N-1}|$ solves the bilinear Eq. (8). The corresponding solution of Eq. (2) is

$$v = 2\left(\ln f\right)_x = \frac{2f_x}{f} = 2\frac{|\widehat{N-2}, N|}{|\widehat{N-1}|}$$

3. Wronskian solutions of Eq.(2)

In this section, new exact solutions including rational solutions, soliton solutions, positon solutions, negaton solutions and their interaction solutions are formally derived to Eq.(8) ([11]-[10]).

The Jordan form of a real matrix

$$A = \begin{bmatrix} J(\lambda_1) & & & & 0 \\ 1 & J(\lambda_2) & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & J(\lambda_m) \end{bmatrix}_{n\mathrm{x}n}$$

has the following type of block:

$$J(\lambda_i) = \begin{bmatrix} \lambda_i & & & & 0 \\ 1 & \lambda_i & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & \lambda_i \end{bmatrix}_{k_i\mathrm{x}k_i}$$

This type of block has the real eigenvalue λ_i.

3.1. Rational solutions.

Let's assume that $J(\lambda_1)$ is

$$J(\lambda_1) = \begin{bmatrix} \lambda_1 & & & & 0 \\ 1 & \lambda_1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & \lambda_1 \end{bmatrix}_{k_1\mathrm{x}k_1}$$

If the eigenvalue $\lambda_1 = 0$, then $J(\lambda_1)$ becomes to the following form:

$$\begin{bmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & 0 \end{bmatrix}_{k_1\mathrm{x}k_1}$$

Then the conditions (14)-(16), convert to

$$\phi_{1,xx} = 0 \ , \ \phi_{i+1,xx} = \phi_i \ , \ \phi_{i,t} = -(b+ak)\phi_{i,x},$$
$$\phi_{i,y} = -\frac{4}{a}\phi_{i,xxx} + k\phi_{i,x} \ , \ i \geq 1 \quad (21)$$

If we can obtain the functions of $\phi_i (i \geq 1)$ from Eq.(21) then

$$v = 2\partial_x \ln W(\phi_1, \phi_2, ..., \phi_{k_1})$$

is called a rational Wronskian solution of order k_1.

After solving

$$\phi_{1,xx} = 0\ ,\quad \phi_{1,t} = -(b + ak)\phi_{1,x}\ ,\quad \phi_{1,y} = -\frac{4}{a}\phi_{1,xxx} + k\phi_{1,x}$$

we get

$$\phi_1 = c_1\,(x + ky - (b + ak)t) + c_2.$$

where c_1, c_2 and $k \neq 0$ are all real constants.

Similarly, by solving

$$\phi_{i+1,xx} = \phi_i\ ,\ \phi_{i+1,t} = -(b + ak)\phi_{i+1,x}\ ,\quad \phi_{i+1,y} = -\frac{4}{a}\phi_{i+1,xxx} + k\phi_{i+1,x}\ ,\ i \geq 1\ ,$$

then zero,first and second order rational solutions can be achieved.

1) Zero-order: When $c_1 = 1, c_2 = 0, \phi_1 = x + ky - (b + ak)t$, we have the corresponding Wronskian determinant $f = W(\phi_1) = x + ky - (b + ak)t$, and the associated rational Wronskian solution of zero-order:

$$v = 2\partial_x \ln W(\phi_1) = \frac{2}{x + ky - (b + ak)t} \tag{22}$$

2) First-order: When $c_1 = 1, c_2 = 0, \phi_1 = x + ky - (b + ak)t$, we have $\phi_2 = \frac{(x+ky-(b+ak)t)^3}{6} - \frac{4y}{a}$ and the corresponding Wronskian determinant $f = W(\phi_1, \phi_2) = \frac{(x+ky-(b+ak)t)^3}{3} + \frac{4y}{a}$, and the associated rational Wronskian solution of first-order

$$v = 2\partial_x \ln W(\phi_1, \phi_2) = \frac{2\,(x + ky - (b + ak)t)^2}{\frac{(x+ky-(b+ak)t)^3}{3} + \frac{4y}{a}} \tag{23}$$

3) Second-order: When $\phi_1 = x + ky - (b + ak)t,\ \phi_2 = \frac{(x+ky-(b+ak)t)^3}{6} - \frac{4y}{a}$, we have $\phi_3 = \frac{(x+ky-(b+ak)t)^5}{120} - \frac{2y(x+ky-(b+ak)t)^2}{a}$ and the corresponding Wronskian determinant $f = W(\phi_1, \phi_2, \phi_3) = \frac{(x+ky-(b+ak)t)^6}{45} + \frac{4y(x+ky-(b+ak)t)^3}{3a} - \frac{16y^2}{a^2}$, and the associated rational Wronskian solution of second-order

$$v = 2\partial_x \ln W(\phi_1, \phi_2, \phi_3) = \frac{\frac{4(x+ky-(b+ak)t)^5}{15} + \frac{8y(x+ky-(b+ak)t)^2}{a}}{\frac{(x+ky-(b+ak)t)^6}{45} + \frac{4y(x+ky-(b+ak)t)^3}{3a} - \frac{16y^2}{a^2}} \tag{24}$$

3.2. **Solitons, negatons and positons.** If the eigenvalue $\lambda_1 \neq 0$, $J(\lambda_1)$ becomes to the following form

$$\begin{bmatrix} \lambda_1 & & & & & 0 \\ 1 & \lambda_1 & & & & \\ & \cdot & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & & & & 1 & \lambda_1 \end{bmatrix}_{k_1 \mathrm{x} k_1}$$

We start from the eigenfunction $\phi_1(\lambda_1)$, which is determined by

$$(\phi_1(\lambda_1))_{xx} = \lambda_1 \phi_1(\lambda_1)\ ,\ (\phi_1(\lambda_1))_t = -(b+ak)(\phi_1(\lambda_1))_x\ ,$$
$$(\phi_1(\lambda_1))_y = -\frac{4}{a}(\phi_1(\lambda_1))_{xxx} + k(\phi_1(\lambda_1))_x \tag{25}$$

General solutions to this system in two cases of $\lambda_1 > 0$ and $\lambda_1 < 0$ are

$$\begin{aligned} \phi_1(\lambda_1) &= C_1 \sinh\left(\sqrt{\lambda_1}\left(x+ky-(b+ak)t-\frac{4y\lambda_1}{a}\right)\right) \\ &\quad +C_2\cosh\left(\sqrt{\lambda_1}\left(x+ky-(b+ak)t-\frac{4y\lambda_1}{a}\right)\right) \end{aligned} \tag{26}$$

when $\lambda_1 > 0$,

$$\begin{aligned} \phi_1(\lambda_1) &= C_3 \cos\left(\sqrt{-\lambda_1}\left(x+ky-(b+ak)t-\frac{4y\lambda_1}{a}\right)\right) \\ &\quad -C_4\sin\left(\sqrt{-\lambda_1}\left(x+ky-(b+ak)t-\frac{4y\lambda_1}{a}\right)\right) \end{aligned} \tag{27}$$

when $\frac{ak}{4} < \lambda_1 < 0$ respectively, where C_1, C_2, C_3 and C_4 are arbitrary real constants.

1) Solitons: The $n-$soliton solution is a special $n-$negaton:

$$v = 2\partial_x \ln W(\phi_1, \phi_2, ..., \phi_n)$$

with ϕ_i given by

$$\phi_i = \cosh\left(\sqrt{\lambda_i}\left(x+ky-(b+ak)t-\frac{4y\lambda_i}{a}\right)+\gamma_i\right),\quad i \text{ odd},$$
$$\phi_i = \sinh\left(\sqrt{\lambda_i}\left(x+ky-(b+ak)t-\frac{4y\lambda_i}{a}\right)+\gamma_i\right),\quad i \text{ even},$$

where $0 < \lambda_1 < \lambda_2 .... < \lambda_n$ and γ_i $(1 \leq i \leq n)$ are arbitrary real constants.

Zero-order:

$$\begin{aligned} v &= 2\partial_x \ln W(\phi_1) = 2\partial_x \ln\left(\cosh\left(\sqrt{\lambda_1}\left(x+ky-(b+ak)t-\frac{4y\lambda_1}{a}\right)+\gamma_1\right)\right) \\ &= 2\sqrt{\lambda_1}\tanh(\theta_1) \end{aligned} \tag{28}$$

$$\begin{aligned} v &= 2\partial_x \ln W(\phi_1) = 2\partial_x \ln\left(\sinh\left(\sqrt{\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1\right)\right) \\ &= 2\sqrt{\lambda_1}\coth(\theta_1) \end{aligned} \tag{29}$$

where $\theta_1 = \sqrt{\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1, \quad \lambda_1 > 0$

First-order:

$$\begin{aligned} v &= 2\partial_x \ln W(\cosh(\phi_1), \sinh(\phi_2)) \\ &= \frac{2(\lambda_1-\lambda_2)(\sinh(\theta_1+\theta_2)-\sinh(\theta_1-\theta_2))}{\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)\cosh(\theta_1+\theta_2)-\left(\sqrt{\lambda_1}+\sqrt{\lambda_2}\right)\cosh(\theta_1-\theta_2)} \end{aligned} \tag{30}$$

where $\theta_i = \sqrt{\lambda_i}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_i}{a}\right)+\gamma_i, \quad \lambda_i > 0, \quad i = 1, 2.$

2) Positons: We obtain two special positon solutions as the following

$$v = 2\partial_x \ln W(\phi, \partial_\lambda\phi, ..., \partial_\lambda^{k-1}\phi)$$

$$\phi(\lambda) = \cos\left(\sqrt{-\lambda}\left(x+ky-(b+ak)\,t-\frac{4y\lambda}{a}\right)+\gamma\right) \qquad \lambda < 0,$$

$$\phi(\lambda) = \sin\left(\sqrt{-\lambda}\left(x+ky-(b+ak)\,t-\frac{4y\lambda}{a}\right)+\gamma\right) \qquad \lambda < 0.$$

Zero-order:

$$\begin{aligned} v &= 2\partial_x \ln W(\phi_1) = 2\partial_x \ln\left(\cos\left(\sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1\right)\right) \\ &= -2\sqrt{-\lambda_1}\tan(\theta_3) \end{aligned} \tag{31}$$

$$\begin{aligned} v &= 2\partial_x \ln W(\phi_1) = 2\partial_x \ln\left(\sin\left(\sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1\right)\right) \\ &= 2\sqrt{-\lambda_1}\cot(\theta_3) \end{aligned} \tag{32}$$

where $\theta_3 = \sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1$

First-order:

$$v = 2\partial_x \ln W\left(\cos(\theta), \partial_{\lambda_1}\cos(\theta)\right) = \frac{4\sqrt{-\lambda_1}\,(1+\cos(2\theta))}{2\sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{12y\lambda_1}{a}\right)+\sin(2\theta)} \tag{33}$$

where $\theta = \sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1.$

3) Negatons: We obtain two special negaton solutions as the following

$$v = 2\partial_x \ln W(\phi, \partial_\lambda\phi, ..., \partial_\lambda^{k-1}\phi)$$

$$\phi = \cosh\left(\sqrt{\lambda}\left(x+ky-(b+ak)\,t-\frac{4y\lambda}{a}\right)+\gamma\right)$$

$$\phi = \sinh\left(\sqrt{\lambda}\left(x + ky - (b + ak)\,t - \frac{4y\lambda}{a}\right) + \gamma\right)$$

where $\lambda > 0$ and γ is an arbitrary constant.

First-order:

$$v = 2\partial_x \ln W\left(\cosh(\theta), \partial_{\lambda_1}\cosh(\theta)\right) = \frac{4\sqrt{\lambda_1}\left(1 + \cosh(2\theta)\right)}{2\sqrt{\lambda_1}\left(x + ky - (b + ak)\,t - \frac{12y\lambda_1}{a}\right) + \sinh(2\theta)} \tag{34}$$

where $\theta = \sqrt{\lambda_1}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_1}{a}\right) + \gamma_1$

3.3. **Interaction solutions.** A Wronskian solution $v = 2\partial_x \ln W(\phi_1(\lambda), \phi_2(\lambda), ..., \phi_k(\lambda); \psi_1(\mu), ..., \psi_l(\mu))$ will be called as Wronskian interaction solution between two solutions determined by the two sets of eigenfunctions

$$(\phi_1(\lambda), \phi_2(\lambda), ..., \phi_k(\lambda); \psi_1(\mu), ..., \psi_l(\mu)) \tag{35}$$

Moreover, one can generate more general Wronskian interaction solutions for instance using the rational solutions, negatons and positons.

Now, our aim is to demonstrate some special Wronskian interaction solutions. First, we consider the following eigenfunctions:

$$\phi_{\text{rational}} = x + ky - (b + ak)\,t$$

$$\phi_{\text{soliton}} = \cosh\left(\sqrt{\lambda_1}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_1}{a}\right)\right)$$

$$\phi_{\text{positon}} = \cos\left(\sqrt{-\lambda_2}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_2}{a}\right)\right)$$

where $\lambda_1 > 0,\ \lambda_2 < 0$ are constants.

We get the following Wronskian interaction determinants using the rational, a single soliton and a single positon solutions

$$W(\phi_{\text{rational}}, \phi_{\text{soliton}}) = \sqrt{\lambda_1}\left(x + ky - (b + ak)\,t\right)\sinh(\theta_1) - \cosh(\theta_1) \tag{36}$$

$$W(\phi_{\text{rational}}, \phi_{\text{positon}}) = -\sqrt{-\lambda_2}\left(x + ky - (b + ak)\,t\right)\sin(\theta_2) - \cos(\theta_2) \tag{37}$$

$$W(\phi_{\text{soliton}}, \phi_{\text{positon}}) = -\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2) - \sqrt{\lambda_1}\sinh(\theta_1)\cos(\theta_2) \tag{38}$$

where $\theta_1 = \sqrt{\lambda_1}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_1}{a}\right)$, $\theta_2 = \sqrt{-\lambda_2}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_2}{a}\right)$

Then, the corresponding Wronskian interaction solutions are

$$v = 2\partial_x \ln W(\phi_{\text{rational}}, \phi_{\text{soliton}}) = \frac{2\sqrt{\lambda_1}\left(x + ky - (b + ak)\,t\right)\cosh(\theta_1)}{\sqrt{\lambda_1}\left(x + ky - (b + ak)\,t\right)\sinh(\theta_1) - \cosh(\theta_1)} \tag{39}$$

$$v = 2\partial_x \ln W(\phi_{\text{rational}}, \phi_{\text{positon}}) = \frac{-2\lambda_2\left(x + ky - (b + ak)\,t\right)\cos(\theta_2)}{\sqrt{-\lambda_2}\left(x + ky - (b + ak)\,t\right)\sin(\theta_2) + \cos(\theta_2)} \tag{40}$$

$$v = 2\partial_x \ln W(\phi_{\text{soliton}}, \phi_{\text{positon}}) = \frac{2(\lambda_1 - \lambda_2)\cosh(\theta_1)\cos(\theta_2)}{\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2) + \sqrt{\lambda_1}\sinh(\theta_1)\cos(\theta_2)} \tag{41}$$

where $\theta_1 = \sqrt{\lambda_1}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_1}{a}\right)$, $\theta_2 = \sqrt{-\lambda_2}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_2}{a}\right)$

The following is one Wronskian interaction determinant and solution involving the three eigenfunctions.

$$\begin{aligned} W(\phi_{\text{rational}}, \phi_{\text{soliton}}, \phi_{\text{positon}}) &= (x + ky - (b + ak)\,t) \\ &\quad \times \left(\lambda_2\sqrt{\lambda_1}\sinh(\theta_1)\cos(\theta_2) + \lambda_1\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2)\right) \\ &\quad + (\lambda_1 - \lambda_2)\cosh(\theta_1)\cos(\theta_2) = p \end{aligned} \tag{42}$$

$$v = 2\partial_x \ln W(\phi_{\text{rational}}, \phi_{\text{soliton}}, \phi_{\text{positon}}) = \frac{2q}{p} \tag{43}$$

where

$$q = (x + ky - (b + ak)\,t)\sqrt{-\lambda_1\lambda_2}\,(\lambda_1 - \lambda_2)\sinh(\theta_1)\sin(\theta_2) + \lambda_1\sqrt{\lambda_1}\sinh(\theta_1)\cos(\theta_2) + \lambda_2\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2)$$

$$\theta_1 = \sqrt{\lambda_1}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_1}{a}\right), \quad \theta_2 = \sqrt{-\lambda_2}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_2}{a}\right)$$

4. Conclusions

In summary, based on Hirota's bilinear method, we have used Wronskian determinant method to construct exact solutions of (2+1) dimensional nonlocal Ito equation. The performance of this method is reliable and effective and gives more important physical solutions including solitons, negatons and positons. Some of the results are in agreement with the results obtained in the previous literature, and also new results are formally developed. We hope that the obtained solutions can be used in numerical schemes as initial values and they may be of significant importance for the explanation of some special physical phenomenas.

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ON UNIVALENCE OF INTEGRAL OPERATORS

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Abstract. In this paper we consider functions of ψ_λ and we define integral operators denoted by $F_{\beta,\lambda}$ and $G_{\beta,\lambda}$ using by ψ_λ, then we proved sufficient conditions for univalence of these integral operators.

1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by S the subclass of A consisting of the functions $f \in A$ which are univalent in U.

Let ψ_λ defined by $\psi_\lambda(z) = (1-\lambda) f(z) + \lambda z f^{'}(z)$ for $z \in U, f \in A$ and $0 \leq \lambda \leq 1$. We consider the integral operators

$$F_{\beta,\lambda}(z) = \left[\beta \int_0^z u^{\beta-1} \psi_\lambda^{'}(u)\, du\right]^{\frac{1}{\beta}} \qquad (z \in U), \tag{1.1}$$

$$G_{\beta,\lambda}(z) = \int_0^z \left[\psi_\lambda^{'}(u)\right]^{\beta} du \quad (z \in U) \tag{1.2}$$

for $\psi_\lambda \in A, 0 \leq \lambda \leq 1$ and for some complex numbers β. In the present paper, we obtain new univalence conditions for the integral operators $F_{\beta,\lambda}$ and $G_{\beta,\lambda}$ to be in the class S.

Recently the problem of univalence of some generalized integral operators have discussed by many authors such as: (see [2]-[8], [10],[14]-[16])

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2. Preliminary Results

To discuss our problems for univalence of integral operators $F_{\beta,\lambda}$ and $G_{\beta,\lambda}$, we recall here some results.

Theorem 1. *Let $\alpha \in \mathbb{C}, \operatorname{Re}\alpha > 0$ and $f \in A$. If*

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in U$, then for any complex number β, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$, the function

$$F_\beta(z) = \left[\beta\int_0^z u^{\beta-1}f'(u)\,du\right]^{\frac{1}{\beta}}$$

is in the class S [12].

Theorem 2. *Let $f \in A$. If for all $z \in U$*

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

then the function f is univalent in U [1].

Theorem 3. *If the function g is regular and $|g(z)| < 1$ in U, then for all $\eta \in U$ and $z \in U$ the following inequalities hold:*

$$\left|\frac{g(\eta)-g(z)}{1-\overline{g(z)}g(\eta)}\right| \leq \left|\frac{\eta-z}{1-\overline{z}\eta}\right| \tag{2.1}$$

and

$$\left|g'(z)\right| \leq \frac{1-|g(z)|^2}{1-|z|^2}.$$

In here, the equalities hold only in the case $g(z) = \varepsilon\frac{z+u}{1+\overline{u}z}$ where $|\varepsilon| = 1$ and $|u| < 1$ [9].

Remark 1. *For $z = 0$ and all $\eta \in U$, from inequality (2.1) we obtain*

$$\left|\frac{g(\eta)-g(0)}{1-\overline{g(0)}g(\eta)}\right| \leq |\eta|$$

and, hence

$$|g(\eta)| \leq \frac{|\eta|+|g(0)|}{1+|g(0)|\,|\eta|}.$$

Considering $g(0) = a$ and $\eta = z$, then

$$|g(z)| \leq \frac{|z|+|a|}{1+|a|\,|z|}$$

for all $z \in U$ [9].

Theorem 4. *Let β be a complex number,* $\operatorname{Re}\beta \geq 1$ *and* $f \in A$, $\frac{f(z)}{z} \neq 0$ *for all* $z \in U$*. If there exist a constant* $K \in (0, m(r)]$, *where*

$$m(r) = \frac{1 - 2|a_2| r\left(1 - r^2\right) + \sqrt{\left[1 - 2|a_2| r\left(1 - r^2\right)\right]^2 + 8|a_2| r^3\left(1 - r^2\right)}}{2r^2\left(1 - r^2\right)}$$

$r = |z|, r \in (0, 1)$ *such that*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq K$$

for all $z \in U^* = U - \{0\}$*, then the function*

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)\, du\right]^{\frac{1}{\beta}}$$

is regular and univalent in U^* [11].

Theorem 5. *Let* $\beta \in \mathbb{C}$ *and* $g \in A$*. If*

$$\left|\frac{g''(z)}{g'(z)}\right| < 1$$

for all $z \in U$ *and the constant* $|\beta|$ *satisfies the condition*

$$|\beta| \leq \frac{1}{\max\limits_{|z| \leq 1}\left[\left(1 - |z|^2\right)|z|\,\frac{|z| + 2|a_2|}{1 + 2|a_2||z|}\right]}$$

then the function

$$G_\beta(z) = \int_0^z \left[g'(u)\right]^\beta du$$

is univalent in U [13].

3. Main Results

Theorem 6. *Let* $\beta \in \mathbb{C}$, $\operatorname{Re}\beta \geq 1$ *and* ψ_λ *a regular function in* U, $\frac{\psi_\lambda(z)}{z} \neq 0$ *for all* $z \in U$*. If there exist a constant* $K \in (0, m(r)]$, *where*

$$m(r) = \frac{1 - 2(1+\lambda)|a_2| r\left(1 - r^2\right) + \sqrt{\left[1 - 2(1+\lambda)|a_2| r\left(1 - r^2\right)\right]^2 + 8(1+\lambda)|a_2| r^3\left(1 - r^2\right)}}{2r^2\left(1 - r^2\right)} \tag{3.1}$$

$r = |z|, r \in (0, 1)$ *such that*

$$\left|\frac{\psi_\lambda''(z)}{\psi_\lambda'(z)}\right| \leq K$$

for all $z \in U^*$*, then the function (1.1) is regular and univalent in* U^**.*

Proof. Let's consider the function $g(z) = \frac{1}{K}\frac{\psi_\lambda''(z)}{\psi_\lambda'(z)}$ where K is a real positive constant. Applying Theorem3 and Remark1 to the function g, we obtain

$$\left|\frac{1}{K}\frac{\psi_\lambda''(z)}{\psi_\lambda'(z)}\right| \le \frac{|z| + \frac{2(1+\lambda)|a_2|}{K}}{1 + \frac{2(1+\lambda)|a_2|}{K}|z|}, \quad z \in U^*$$

and hence, we have

$$\left(1 - |z|^2\right)\left|\frac{z\psi_\lambda''(z)}{\psi_\lambda'(z)}\right| \le K\left(1 - |z|^2\right)|z|\frac{|z| + \frac{2(1+\lambda)|a_2|}{K}}{1 + \frac{2(1+\lambda)|a_2|}{K}|z|}. \tag{3.2}$$

Let's consider the inequality

$$K \le \frac{1}{\left(1 - |z|^2\right)|z|\frac{|z| + \frac{2(1+\lambda)|a_2|}{K}}{1 + \frac{2(1+\lambda)|a_2|}{K}|z|}}. \tag{3.3}$$

Considering $|z| = r, r \in (0,1)$ and $2|a_2| = p, p > 0$, the inequality (3.3) becomes

$$K \le \frac{K + (1+\lambda)pr}{(1-r^2)r[Kr + (1+\lambda)p]}. \tag{3.4}$$

We note that

$$\left(1 - r^2\right)r[Kr + (1+\lambda)p] > 0 \tag{3.5}$$

for every $K > 0, p > 0, r \in (0,1)$ and $0 \le \lambda \le 1$. Using (3.5) the inequality (3.4) becomes

$$r^2\left(1 - r^2\right)K^2 + \left[(1+\lambda)pr\left(1 - r^2\right) - 1\right]K - (1+\lambda)pr \le 0.$$

Let us consider the equation

$$r^2\left(1 - r^2\right)K^2 + \left[(1+\lambda)pr\left(1 - r^2\right) - 1\right]K - (1+\lambda)pr = 0, \tag{3.6}$$

with the unknown K. From (3.6) we obtain

$$K_{1,2} = \frac{1 - (1+\lambda)pr\left(1 - r^2\right) \pm \sqrt{\left[1 - (1+\lambda)pr(1-r^2)\right]^2 + 4(1+\lambda)pr^3(1-r^2)}}{2r^2(1-r^2)}. \tag{3.7}$$

For every $p > 0$, $r \in (0,1)$ and $0 \le \lambda \le 1$ the following inequality holds

$$\left[1 - (1+\lambda)pr\left(1 - r^2\right)\right]^2 + 4(1+\lambda)pr^3\left(1 - r^2\right) > 0. \tag{3.8}$$

Using (3.7) and (3.8) it results that K_1, K_2 are real solutions. Considering $a = 1 - r^2$, $a \in (0,1)$ and $b = pr$, $b > 0$ from (3.7) we get

$$K_{1,2} = \frac{1 - (1+\lambda)ab \pm \sqrt{\left[1 - (1+\lambda)ab\right]^2 + 4(1+\lambda)ab(1-a)}}{2a(1-a)}. \tag{3.9}$$

□

We have the following cases:

Case 1. *For* $|a_2| > \frac{1}{2(1+\lambda)r(1-r^2)}$ *it results that* $1-(1+\lambda)\,ab<0$, *so that*

$$K_1 = \frac{1-(1+\lambda)\,ab - \sqrt{[1-(1+\lambda)\,ab]^2 + 4\,(1+\lambda)\,ab\,(1-a)}}{2a\,(1-a)}$$

is real negative solution. Clearly,

$$K_2 = \frac{1-(1+\lambda)\,ab + \sqrt{[1-(1+\lambda)\,ab]^2 + 4\,(1+\lambda)\,ab\,(1-a)}}{2a\,(1-a)}$$

is real positive solution. In this case, for $K \in (0, K_2]$ *the inequality (3.3) is verified.*

Case 2. *For* $|a_2| < \frac{1}{2(1+\lambda)r(1-r^2)}$ *it results that* $1-(1+\lambda)\,ab>0$.

Let's prove that $K_1<0$. Supposing that $K_1>0$, we obtain $4\,(1+\lambda)\,ab\,(1-a)<0$ the fact which is false. It results that $K_1<0$. We note that $K_2>0$, and the inequality (3.3) is verified for $K \in (0, K_2]$.

Case 3. *For* $|a_2| = \frac{1}{2(1+\lambda)r(1-r^2)}$ *using (3.9) we obtain*

$$K_{1,2} = \frac{\pm\sqrt{(1+\lambda)\,ab\,(1-a)}}{a\,(1-a)}$$

and the inequality (3.3) is verified only for $K \in (0, K_2]$ *where*

$$K_2 = \frac{\sqrt{(1+\lambda)\,ab\,(1-a)}}{a\,(1-a)}.$$

Considering equality (3.1) in conclusion for $|a_2|$, r *stable and* $K \in (0, m\,(r)]$, *the inequality (3.3) is verified and using (3.2) it results that*

$$\left(1-|z|^2\right)\left|\frac{z\psi_\lambda^{''}(z)}{\psi_\lambda^{'}(z)}\right| \le 1, z \in U^*. \tag{3.10}$$

From (3.10) and Theorem1 in the case $\alpha = 1$ *we obtain that the function* $F_{\beta,\lambda}(z)$ *is regular and univalent in* U^*.

Theorem 7. *Let* β *be a complex number and the function* $\psi_\lambda \in A, \psi_\lambda(z) = (1-\lambda)\,f(z) + \lambda z f^{'}(z)$ *for* $f \in A$ *and* $0 \le \lambda \le 1$. *If*

$$\left|\frac{\psi_\lambda^{''}(z)}{\psi_\lambda^{'}(z)}\right| < 1 \tag{3.11}$$

for all $z \in U$ *and the constant* $|\beta|$ *satisfies the condition*

$$|\beta| \le \frac{1}{\max\limits_{|z|\le 1}\left[\left(1-|z|^2\right)|z|\,\frac{|z|+2(1+\lambda)|a_2|}{1+2(1+\lambda)|a_2||z|}\right]} \tag{3.12}$$

then the function $G_{\beta,\lambda}$ *is univalent in* U.

Proof. The function $G_{\beta,\lambda}$ defined by (1.2) is regular in U. Let us consider the function

$$p(z) = \frac{1}{|\beta|} \frac{G''_{\beta,\lambda}(z)}{G'_{\beta,\lambda}(z)} \tag{3.13}$$

where the constant $|\beta|$ satisfies the inequality (3.12). The function p is regular in U and from (1.2) and (3.13) we have

$$p(z) = \frac{\beta}{|\beta|} \frac{\psi''_{\lambda}(z)}{\psi'_{\lambda}(z)}. \tag{3.14}$$

Using (3.14) and (3.11) we obtain

$$|p(z)| < 1$$

for all $z \in U$ and $|p(0)| = 2(1+\lambda)|a_2|$. When Remark1 applied to the function p, it gives

$$\frac{1}{|\beta|} \frac{G''_{\beta,\lambda}(z)}{G'_{\beta,\lambda}(z)} \le \frac{|z| + 2(1+\lambda)|a_2|}{1 + 2(1+\lambda)|a_2||z|} \tag{3.15}$$

for all $z \in U$. From (3.15) we get

$$\left(1 - |z|^2\right) \left| \frac{zG''_{\beta,\lambda}(z)}{G'_{\beta,\lambda}(z)} \right| \le |\beta| \left(1 - |z|^2\right) |z| \frac{|z| + 2(1+\lambda)|a_2|}{1 + 2(1+\lambda)|a_2||z|}$$

for all $z \in U$. Hence we have

$$\left(1 - |z|^2\right) \left| \frac{zG''_{\beta,\lambda}(z)}{G'_{\beta,\lambda}(z)} \right| \le |\beta| \max_{|z| \le 1} \left(1 - |z|^2\right) |z| \frac{|z| + 2(1+\lambda)|a_2|}{1 + 2(1+\lambda)|a_2||z|}. \tag{3.16}$$

From (3.16) and (3.12) we obtain

$$\left(1 - |z|^2\right) \left| \frac{zG''_{\beta,\lambda}(z)}{G'_{\beta,\lambda}(z)} \right| \le 1$$

for all $z \in U$. From Theorem2, it follows that the function $G_{\beta,\lambda}$ defined by (1.2) is univalent in U. □

Remark 2. *Taking $\lambda = 0$ in Theorem6 and Theorem7, we obtain Theorem4 and Theorem5, respectively.*

If we take $\lambda = 1$ in Theorem6 and Theorem7, we have the following corollaries.

Corollary 1. *Let β be a complex number, $\operatorname{Re}\beta \ge 1$ and ψ_1 a regular function in $U, \psi_1(z) = zf'(z)$ and $\frac{\psi_1(z)}{z} \ne 0$ for all $z \in U$. If there exist a constant $K \in (0, m(r)]$, where*

$$m(r) = \frac{1 - 4|a_2| r \left(1 - r^2\right) + \sqrt{\left[1 - 4|a_2| r \left(1 - r^2\right)\right]^2 + 16|a_2| r^3 \left(1 - r^2\right)}}{2r^2 \left(1 - r^2\right)},$$

$r = |z|, r \in (0,1]$ *such that*

$$\left|\frac{\psi_1^{''}(z)}{\psi_1^{'}(z)}\right| = \left|\frac{f''(z)}{f'(z)}\right| \leq K$$

for all $z \in U^*$, *then the function*

$$F_{\beta,1}(z) = \left[\beta \int_0^z u^{\beta-1}\psi_1^{'}(u)\, du\right]^{\frac{1}{\beta}}$$

is regular and univalent in U^*.

Corollary 2. *Let* β *be a complex number and the function* $\psi_1(z) = zf^{'}(z)$ *where* $f \in A$. *If*

$$\left|\frac{\psi_1^{''}(z)}{\psi_1^{'}(z)}\right| < 1$$

for all $z \in U$ *and the constant* $|\beta|$ *satisfies the condition*

$$|\beta| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\left(1-|z|^2\right)|z|\,\frac{|z|+4|a_2|}{1+4|a_2||z|}\right]}$$

then the function

$$G_{\beta,1}(z) = \int_0^z \left[\psi_1^{'}(u)\right]^{\beta} du$$

is univalent in U.

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A CONVERGENCE THEOREM IN GENERALIZED CONVEX CONE METRIC SPACES

BIROL GUNDUZ

ABSTRACT. The aim of this work is to establish convergence theorem of a new iteration process for a finite family of I-asymptotically quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings in generalized convex cone metric spaces. Our result is valid in the whole space, whereas the results given in [4, 5] are valid in a nonempty convex subset of a convex cone metric space. Our convergence results generalize and refine not only result of Gunduz [6] but also results of Lee [4, 5] and Temir [9].

1. Introduction

Fixed point theory plays an important role in applications of many branches of mathematics and applied sciences. The study of metric fixed point theory has been at the centre of vigorous research activity. There has been a number of generalizations of the usual notion of a metric space. One such generalization is a cone metric space introduced and studied by Huang and Zhang [2], in 2007. The idea of cone metric spaces is based on replacing the set of real numbers by an ordered Banach space in definition of metric spaces. Huang and Zhang [2] modified definitions of some concepts such as convergence of sequences, Cauchy sequences, and completeness in this space. They also proved some fixed point theorems of contractive mappings on complete cone metric spaces using assumption of the normality of a cone. After that a series of articles have been dedicated to existence and uniqueness of fixed point of different type mappings in cone metric spaces. In [4], Lee introduced the concept of convex cone metric spaces by combining idea of cone metric space and convex metric space defined by Takahashi [1], and started iterative approximation of fixed points of nonlinear mappings. Gunduz [7] studied convergence of a new multistep iteration for a finite family of asymptotically quasi-nonexpansive mappings in convex cone metric spaces. Result of Gunduz [7]

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is valid in the whole space, whereas the results of Lee [4, 5] are valid in a nonempty convex subset of a convex cone metric spaces.

The aim of this work is to study convergence of a new iteration process for a finite family of I-asymptotically quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings in generalized convex cone metric spaces. Our convergence results generalize and refine not only result of Gunduz [6] but also result of paper given in his references.

Throughout this article, we use the notation $F(T)$ for the set of fixed points of a mapping T and $F := (\bigcap_{i=1}^{r} F(T_i)) \cap (\bigcap_{i=1}^{r} F(I_i))$ for the set of common fixed points of two finite families of mappings $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$, where J is set of first r natural numbers.

2. Preliminaries

In this section, we need to recall some basic notations, definitions, and necessary results and examples from existing literature.

In 1970, Takahashi [1] introduced the concept of convexity in a metric space (X, d) as follows.

Definition 1. [1] *A convex structure in a metric space (X, d) is a mapping $W : X^2 \times [0, 1] \to X$ satisfying, for all $x, y, u \in X$ and all $\lambda \in [0, 1]$,*

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

Let E be a normed vector space, then the following definitions can be found in [2].

Definition 2. [2] *A nonempty subset P of E is called a cone if P is closed, $P \neq \{\theta\}$, for $a, b \in \mathbb{R}^+ = [0, \infty)$ and $x, y \in P$, $ax + by \in P$ and $P \cap \{-P\} = \{\theta\}$. We define a partial ordering $\preceq$ in E as $x \preceq y$ if $y - x \in P$. $x \ll y$ indicates that $y - x \in intP$ and $x \prec y$ means that $x \preceq y$ but $x \neq y$. A cone P is said to be solid if its interior $intP$ is nonempty. A cone P is said to be normal if there exists a positive number k such that for $x, y \in P$, $\theta \preceq x \preceq y$ implies $\|x\| \leq k \|y\|$ or equivalently, if $(\forall n)$ $x_n \preceq y_n \preceq z_n$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x$ imply $\lim_{n\to\infty} y_n = x$. The least positive number k is called the normal constant of P.*

It is clear that $k \geq 1$. There exist cones which are not normal.

Example 1. [3] *Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = 1$. Then $\theta \preceq x_n \preceq y_n$, and $\lim_{n\to\infty} y_n = \theta$, but $\|x_n\| = \max_{t\in[0,1]} \left|\frac{t^n}{n}\right| + \max_{t\in[0,1]} \left|t^{n-1}\right|$ $= \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. Thus P is a nonnormal cone.*

Definition 3. [2] *Let X be a nonempty set. A mapping $d : X \times X \to (E, P)$ is called a cone metric if (i) for $x, y \in X$, $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ iff $x = y$, (ii) for $x, y \in X$, $d(x, y) = d(y, x)$ and (iii) for $x, y, z \in X$, $d(x, y) \preceq d(x, z) + d(z, y)$.*

A nonempty set X with a cone metric $d : X \times X \to (E, P)$ is called a cone metric space denoted by (X, d), where P is a solid normal cone.

Since each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty)$, the concept of a cone metric space is more general than that of a metric space.

Example 2. [2]*Let $E = \mathbb{R}^2$, $P = \left\{(x; y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\right\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space with normal cone P where $k = 1$.*

Definition 4. *A sequence $\{x_n\}$ in a cone metric space (X, d) is said to converge to $x \in X$ and is denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ (as $n \to \infty$) if for any $c \in intP$, there exists a natural number N such that for all $n > N$, $c - d(x_n, x) \in intP$. A sequence $\{x_n\}$ in (X, d) is called a Cauchy sequence if for any $c \in intP$, there exists a natural number N such that for all $n, m > N$, $c - d(x_n, x_m) \in intP$. A cone metric space (X, d) is said to be complete if every Cauchy sequence converges.*

In other words, $\{x_n\}$ is said to converge to x, if there exists a natural number N such that $d(x_n, x) \ll c$ for all $n > N$ and for any $c \in E$ with $\theta \ll c$. $\{x_n\}$ is called a Cauchy sequence in X, if there exists a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m > N$ and for any $c \in E$ with $\theta \ll c$.

Proposition 1. [2] *Let $\{x_n\}$ be a sequence in a cone metric space (X, d) and P be a normal cone. Then*

(1) *$\{x_n\}$ converges to x in X if and only if $d(x_n, x) \to \theta$ (as $n \to \infty$) in E.*
(2) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ (as $n, m \to \infty$) in E.*

Definition 5. [4] *Let (X, d) be a cone metric space. A mapping $W : X^2 \times [0, 1] \to X$ is called a convex structure on X if $d(W(x, y, \lambda), u) \preceq \lambda d(x, u) + (1 - \lambda) d(y, u)$ for all $x, y, u \in X$ and λ in $[0, 1]$. A cone metric space (X, d) with a convex structure W is called a convex cone metric space and denoted as (X, d, W). A nonempty subset C of a convex cone metric space (X, d, W) is said to be convex if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$.*

Example 3. *Let (X, d) be a cone metric space as in Example 2. If $W(x, y; \lambda) = \lambda x + (1 - \lambda) y$, then (X, d) is a convex cone metric space. Hence, this concept is more general than that of a convex metric space.*

Definition 1 can be extended as follows: A mapping $W : X^3 \times [0, 1]^3 \to X$ is said to be a convex structure on X, if it satisfies the following condition: For any $(x, y, z; a, b, c) \in X^3 \times [0, 1]^3$ with $a + b + c = 1$, and $u \in X$:

$$d(u, W(x, y, z; a, b, c)) \leq ad(u, x) + bd(u, y) + cd(u, z).$$

If (X, d) is a metric space with a convex structure W, then (X, d) is called a generalized convex metric space. A nonempty subset C of a generalized convex metric space X is said to be convex if $W(x,y,z;a,b,c) \in C$, $\forall (x,y,z) \in C^3$, $\forall (a,b,c) \in [0,1]^3$ with $a+b+c=1$.

Every linear normed space is a generalized convex metric space with a convex structure $W(x,y,z;a,b,c) = ax+by+cz$, for all $x,y,z \in X$ and $a,b,c \in [0,1]$ with $a+b+c=1$. But there exist some convex metric spaces which can not be embedded into any linear normed spaces (see, Gunduz and Akbulut [8]).

Considering generalized convex metric space together with cone metric space, any one can be defined generalized convex cone metric spaces as follow:

Definition 6. [4] *Let (X,d) be a cone metric space. A mapping $W : X^3 \times [0,1]^3 \to X$ is called a convex structure on X if $d(u, W(x,y,z;a,b,c)) \preceq ad(u,x)+bd(u,y)+cd(u,z)$ for all $x,y,z,u \in X$ and $a,b,c \in [0,1]$ with $a+b+c=1$. A cone metric space (X,d) with a convex structure W is called a generalized convex cone metric space and denoted as (X,d,W). A nonempty subset C of a generalized convex cone metric space (X,d,W) is said to be convex if $W(x,y,z;a,b,c) \in C$ for all $x,y,z, \in C$ and $a,b,c \in [0,1]$ with $a+b+c=1$.*

Remark 1. *If we take $E=\mathbb{R}$, $P=[0,+\infty)$ and $\|.\| = |.|$, then generalized convex cone metric spaces coincide with generalized convex metric spaces.*

Now we give definition of some mappings which will be used later.

Definition 7. *Let (X,d) be a cone metric space with a solid cone P and $T, I : (X,d) \to (X,d)$ be two mapping. The mapping T is said to be*

(1) *asymptotically nonexpansive if there exists $u_n \in [1,\infty)$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} u_n = 1$ such that*

$$d(T^n x, T^n y) \preceq u_n d(x,y) \text{ for all } x,y \in X \text{ and } n \in \mathbb{N}.$$

(2) *asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists $u_n \in [1,\infty)$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} u_n = 1$ such that*

$$d(T^n x, p) \preceq u_n d(x,p) \text{ for all } x \in X,\ p \in F(T) \text{ and } n \in \mathbb{N}.$$

(3) *I-asymptotically nonexpansive if there exists a sequence $\{v_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} v_n = 0$ such that*

$$d(T^n x, T^n y) \preceq (1+v_n) d(I^n x, I^n y)$$

for all $x,y \in X$ and $n \geq 1$.

(4) *I-asymptotically quasi nonexpansive if $F(T) \cap F(I) \neq \emptyset$ and there exists a sequence $\{v_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} v_n = 0$ such that*

$$d(T^n x, p) \preceq (1+v_n) d(I^n x, p)$$

for all $x \in X$ and $p \in F(T) \cap F(I)$ and $n \geq 1$.

(5) *I-uniformly Lipschitz if there exists $\Gamma > 0$ such that*

$$d(T^n x, T^n y) \preceq \Gamma d(I^n x - I^n y),\ x,y \in X \text{ and } n \geq 1.$$

Remark 2. *From the above definition, it follows that if $F(T)$ is nonempty, then an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. Also, an I-asymptotically nonexpansive mapping is I-uniformly Lipschitz with the Lipschitz constant $\Gamma = \sup\{1+v_n : n \geq 1\}$ and an I-asymptotically nonexpansive mapping with $F(T) \cap F(I) \neq \emptyset$ is I-asymptotically quasi nonexpansive. However, the converse of these claims are not true in general. It is easy to see that if I is identity mapping, then I-asymptotically nonexpansive mappings and I-asymptotically quasi nonexpansive mappings coincide with asymptotically nonexpansive mappings and asymptotically quasi nonexpansive mappings, respectively.*

In [6], Gunduz used the Ishikawa iteration process with error terms to prove some convergence results in a convex metric space. We can modify his process in accordance with our purpose as follow:

Let (X,d) be a generalized convex cone metric space with convex structure W, $\{T_i : i \in J\} : X \to X$ be a finite family of I_i-asymptotically quasi-nonexpansive mappings and $\{I_i : i \in J\} : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings. Suppose that $\{u_n\}$ and $\{v_n\}$ are two bounded sequences (with respect to cone metric d) in X and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}, \{\hat{\gamma}_n\}$ are six sequences in $[0,1]$ such that $\alpha_i + \beta_n + \gamma_n = 1 = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n$ for $n \in \mathbb{N}$. For any given $x_1 \in X$, iteration process $\{x_n\}$ defined by,

$$\begin{aligned} x_{n+1} &= W\left(x_n, I_i^n y_n, u_n; \alpha_n, \beta_n, \gamma_n\right), \\ y_n &= W\left(x_n, T_i^n x_n, v_n; \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n\right),\ n \geq 1, \end{aligned} \tag{2.1}$$

where $n = (k-1)r + i,\ i = i(n) \in J$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Thus, (2.1) can be expressed in the following form:

$$\begin{aligned} x_{n+1} &= W\left(x_n, I_{i(n)}^{k(n)} y_n, u_n; \alpha_n, \beta_n, \gamma_n\right), \\ y_n &= W\left(x_n, T_{i(n)}^{k(n)} x_n, v_n; \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n\right),\ n \geq 1. \end{aligned}$$

Let's give with a proposition.

Lemma 1. [10] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying*

$$\sum_{n=0}^{\infty} b_n < \infty,\ \sum_{n=0}^{\infty} c_n < \infty,\ a_{n+1} = (1+b_n)\, a_n + c_n,\ n \geq 0.$$

Then

i) $\lim_{n\to\infty} a_n$ *exists,*

ii) if either $\liminf_{n\to\infty} a_n = 0$ *or* $\limsup_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Using the steps in the proof of [6, Proposition 1.9.], we can prove easily the next proposition which plays a key role in the proof of our main result.

Proposition 2. *Let (X,d) be a generalized convex cone metric space with a solid cone P and convex structure W, $\{T_i : i \in J\} : X \to X$ be a finite family of I_i-asymptotically quasi-nonexpansive mappings, and $\{I_i : i \in J\} : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F := (\bigcap_{i=1}^r F(T_i)) \cap (\bigcap_{i=1}^r F(I_i)) \neq \emptyset$. Then, there exist a point $p \in F$ and sequences $\{k_n\}, \{l_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_n = \lim_{n\to\infty} l_n = 0$ such that*

$$d(T_i^n x, p) \preceq (1+k_n) d(I_i^n x, p) \text{ and } d(I_i^n x, p) \preceq (1+l_n) d(x,p)$$

for all $x \in K$, for each $i \in I$.

We now prove convergence theorem of the iterative scheme (2.1) in generalized convex cone metric spaces.

Theorem 1. *Let (X,d,W) be a generalized convex cone metric space with a cone metric $d : X \times X \to (E,P)$, where P is a solid normal cone with the normal constant k. Let $\{T_i : i \in J\} : X \to X$ be a finite family of I_i-asymptotically quasi-nonexpansive mappings and $\{I_i : i \in J\} : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} k_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$ and $\{x_n\}$ is as in (2.1) with $\{\gamma_n\}, \{\hat{\gamma}_n\}$ satisfying $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \hat{\gamma}_n < \infty$. (i) If $\{x_n\}$ converges to a point in F, then $\liminf_{n\to\infty} d(x_n, F) = \theta$. (ii) $\{x_n\}$ converges to a point in F, if X is complete and $\liminf_{n\to\infty} d(x_n, F) = \theta$.*

Proof. We prove only (ii), since (i) is obvious. Let $p \in F$. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences with respect to cone metric d in X, there exists $M \succeq \theta$ such that $\max\left\{\sup_{n\geq 1} d(u_n,p), \sup_{n\geq 1} d(v_n,p)\right\} \preceq M$. Considering Proposition 2 and (2.1), we have

$$\begin{aligned}
d(y_n,p) &= d\left(W\left(x_n, T_i^n x_n, v_n; \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n\right), p\right) \\
&\preceq \hat{\alpha}_n d(x_n,p) + \hat{\beta}_n d(T_i^n x_n, p) + \hat{\gamma}_n d(v_n,p) \\
&\preceq \hat{\alpha}_n d(x_n,p) + \hat{\beta}_n (1+k_n) d(I_i^n x_n, p) + \hat{\gamma}_n M \\
&\preceq \hat{\alpha}_n d(x_n,p) + \hat{\beta}_n (1+k_n)(1+l_n) d(x_n,p) + \hat{\gamma}_n M \\
&\preceq \left(1 + \hat{\beta}_n (k_n + l_n + k_n l_n)\right) d(x_n,p) + \hat{\gamma}_n M \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned}
d(x_{n+1},p) &= d\left(W\left(x_n, I_i^n y_n, u_n; \alpha_n, \beta_n, \gamma_n\right), p\right) \\
&\preceq \alpha_n d(x_n,p) + \beta_n d(I_i^n y_n, p) + \gamma_n d(u_n,p) \\
&\preceq \alpha_n d(x_n,p) + \beta_n (1+l_n) d(y_n,p) + \gamma_n M. \qquad (3.2)
\end{aligned}$$

Substituting (3.1) into (3.2),

$$\begin{aligned}
d(x_{n+1},p) &\preceq \alpha_n d(x_n,p)+\beta_n(1+l_n)\,d(y_n,p)+\gamma_n M \\
&\preceq \alpha_n d(x_n,p)+\beta_n(1+l_n)\left(1+\hat{\beta}_n(k_n+l_n+k_nl_n)\right)d(x_n,p) \\
&\quad +\beta_n(1+l_n)\,\hat{\gamma}_n M+\gamma_n M \\
&\preceq \alpha_n d(x_n,p)+\beta_n(1+l_n)\,d(x_n,p) \\
&\quad +\beta_n(1+l_n)\,\hat{\beta}_n(k_n+l_n+k_nl_n)\,d(x_n,p) \\
&\quad +\left(\beta_n(1+l_n)\,\hat{\gamma}_n+\gamma_n\right)M \\
&\preceq \left[1+\beta_nl_n+\beta_n\hat{\beta}_n(1+l_n)(k_n+l_n+k_nl_n)\right]d(x_n,p) \\
&\quad +\left(\beta_n(1+l_n)\,\hat{\gamma}_n+\gamma_n\right)M.
\end{aligned}$$

Thus we obtain

$$d(x_{n+1},p)\preceq[1+\kappa_n]\,d(x_n,p)+t_nM \tag{3.3}$$

where $\kappa_n=\beta_nl_n+\beta_n\hat{\beta}_n(1+l_n)(k_n+l_n+k_nl_n)$ and $t_n=(\beta_n(1+l_n)\,\hat{\gamma}_n+\gamma_n)$ with $\sum_{n=1}^{\infty}\kappa_n<\infty$ and $\sum_{n=1}^{\infty}t_n<\infty$. Hence, by the normality of P, we have for the normal constant $k>0$

$$\|d(x_{n+1},F)\|\le k[1+\kappa_n]\,\|d(x_n,F)\|+kt_n\|M\| \tag{3.4}$$

Lemma 1 and (3.4) imply that the $\lim_{n\to\infty}\|d(x_n,F)\|$ exists.
Now $\liminf_{n\to\infty}\|d(x_n,F)\|=0$ implies $\lim_{n\to\infty}\|d(x_n,F)\|=0$.
Next, we show that the sequence $\{x_n\}$ is a Cauchy sequence. Taking into account that the inequality $1+x\le e^x$ for all $x\ge 0$, and (3.4), therefore we have

$$\|d(x_{n+1},p)\|\le k\exp\{\kappa_n\}\,\|d(x_n,p)\|+k\|M\|\,t_n. \tag{3.5}$$

Hence, for any positive integers n,m, from (3.5) it follows that

$$\begin{aligned}
\|d(x_{n+m},p)\| &\le k_1\exp\{\kappa_{n+m-1}\}\,\|d(x_{n+m-1},p)\|+k_1t_{n+m-1}\|M\| \\
&\le k_1\exp\{\kappa_{n+m-1}\}\,[k_2\exp\{\kappa_{n+m-2}\}\,\|d(x_{n+m-2},p)\| \\
&\quad +k_2t_{n+m-2}\|M\|]+k_1t_{n+m-1}\|M\| \\
&= k_1k_2\exp\{\kappa_{n+m-1}\}\exp\{\kappa_{n+m-2}\}\,\|d(x_{n+m-2},p)\| \\
&\quad +k_1k_2\exp\{\kappa_{n+m-1}\}\,t_{n+m-2}\|M\|+k_1t_{n+m-1}\|M\| \\
&\le \cdots \\
&\le \prod_{j=1}^{m}k_j\exp\left\{\sum_{i=n}^{n+m-1}\kappa_i\right\}\|d(x_n,p)\| \\
&\quad +\prod_{j=1}^{m}k_j\exp\left\{\sum_{i=n}^{n+m-1}\kappa_i\right\}\sum_{i=n}^{n+m-1}t_i\|M\| \\
&\le BG\,\|d(x_n,p)\|+BG\sum_{i=n}^{n+m-1}t_i\|M\|,
\end{aligned}$$

where $B=\prod_{j=1}^{m}k_j$, $G=\exp\left\{\sum_{i=n}^{n+m-1}\kappa_i\right\}<\infty$ and k_i is corresponding normal constant for $i=1,2,\ldots,m$.

Since $\lim_{n\to\infty}\|d(x_n,F)\|=0$ and $\sum_{n=1}^{\infty}t_n<\infty$, for any given positive real number ε, there exists a natural number $N_0\in N$ such that $\|d(x_n,F)\|\leq\frac{\varepsilon}{2(1+BG)}$ and $\sum_{n=1}^{\infty}t_n<\frac{\varepsilon}{2BG\|M\|}$ for $n\geq N_0$. In particular, there exist a point $p_1\in F$ such that $\|d(x_n,p_1)\|\leq\frac{\varepsilon}{2(1+BG)}$ for $n\geq N_0$. Consequently, for any $n\geq n_0$ and for all $m\geq 1$ we have

$$\begin{aligned}\|d(x_{n+m},x_n)\| &\leq \|d(x_{n+m},p_1)\|+\|d(x_n,p_1)\|\\ &\leq (1+BG)\|d(x_n,p_1)\|+BG\sum_{i=n}^{n+m-1}t_i\|M\|\\ &\leq (1+BG)\frac{\varepsilon}{2(1+BG)}+BG\frac{\varepsilon}{2BG\|M\|}\|M\|=\varepsilon.\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X, therefore, it converges to some point q in the complete space X.

Finally, we show that $q\in F$. Let $\{q_n\}$ be a sequence in F such that $q_n\to q$. Since

$$\begin{aligned}d(q,T_iq) &\preceq d(q,q_n)+d(q_n,I_iq)\\ &= d(q,q_n)+d(Iq_n,I_iq)\\ &\preceq d(q,q_n)+(1+l_n)d(q_n,q),\end{aligned}$$

taking limit in above inequality, we have $q\in\bigcap_{i=1}^{r}F(I_i)$ for all $i\in I$. Similarly, $q\in\bigcap_{i=1}^{r}F(T_i)$. So $q\in F$, which means that F is closed. Since $d(q,F)=d(\lim_{n\to\infty}x_n,F)=\lim_{n\to\infty}d(x_n,F)=\theta$ by Propostion 1 (i), we have $q\in F$. In other words, $\{x_n\}$ converges to a common fixed point in F. □

Remark 3. *We get Theorem 2.2. of Gunduz* [6] *restricting the normed linear space (E,P) to a real number system $(\mathbb{R},[0,\infty))$ from Theorem 1. Additionally to this restriction taking the metric space (X,d) to a Banach space with $W(x,y,z;\alpha,\beta,\gamma)=\alpha x+\beta y+\gamma z$, and $\gamma_n=\hat{\gamma}_n=0$ for all $n\in\mathbb{N}$, we get a generalization of corresponding result of Temir* [9].

Remark 4. *We want to point out that our theorem generalizes the result of Temir* [9] *in two ways: (i) from a closed convex subset of Banach spaces to general setup of generalized convex cone metric space, (ii) a finite family of I_i-asymptotically nonexpansive mappings to a finite family of I_i-asymptotically quasi-nonexpansive mappings.*

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SELFADJOINT SINGULAR DIFFERENTIAL OPERATORS FOR FIRST ORDER

PEMBE IPEK AND ZAMEDDIN I. ISMAILOV

Abstract. The parametrization of all selfadjoint extensions of the minimal operator generated by first order linear symmetric singular differential-operator expression in the Hilbert space of vector-functions defined at the right semi-axis has been given. To this end we use the Calkin-Gorbachuk method. Finally, the structure of spectrum set of such extensions is researched.

1. Introduction

It is known that fundamental question on the parametrization of selfadjoint extensions of the linear closed densely defined with equal deficiency indices symmetric operators in a Hilbert space has been investigated by J. von Neumann [11] and M. H. Stone [10] firstly. Applications of these results to any scaler linear even order symmetric differential operators and representation of all selfadjoint extensions in terms of boundary conditions have been investigated by I. M. Glazman-M. G. Krein- M. A. Naimark (see [5,8]). In mathematical literature there is co-called Calkin-Gorbachuk method (see [6,9]).

The motivation of this paper originates from the interesting researches of W. N. Everitt, L. Markus, A. Zettl, J. Sun, D. O'Regan, R. Agarwal [2,3,4,12] in scaler cases. Throughout this paper A. Zettl's and J. Suns's view about these topics is to be taken into consideration [12]. A selfadjoint ordinary differential operator in a Hilbert space is generated by two things:
(1) a symmetric (formally selfadjoint) differential expression;
(2) a boundary condition which consists selfadjoint differential operators.
And also the geometrical place in plane of the spectrum of given selfadjoint differential operator is one of the important questions of this theory.

In this work in Section 3 the representation of all selfadjoint extensions of the symmetric singular differential operator, generated by first order symmetric

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differential-operator expression (for the definition see [4]) in the Hilbert spaces of vector-functions defined at the semi-axis in terms of boundary conditions are described. In Section 4 the structure of spectrum of these selfadjoint extensions is investigated.

2. Statement of the Problem

Let us H is a separable Hilbert space and $a \in R$. In the Hilbert space $L^2(H,(a,\infty))$ consider the following differential-operator expression in a form (for scaler case see [4])

$$l(u) = i\rho u' + \frac{1}{2} i\rho' u + Au,$$

where:
(1) $\rho : (a,\infty) \to (0,\infty)$;
(2) $\rho \in AC_{loc}(a,\infty)$;
(3) $\int\limits_a^\infty \dfrac{ds}{\rho(s)} < \infty$;
(4) $A^* = A : D(A) \subset H \to H$.

By standard way the minimal operator L_0 corresponding to differential-operator expression $l(\,\cdot\,)$ in $L^2(H,(a,\infty))$ can be defined (see [7]). The operator $L = (L_0)^*$ is called the maximal operator corresponding to $l(\,\cdot\,)$ in $L^2(H,(a,\infty))$ (see [7]).

It is clear that

$$\begin{aligned} D(L) &= \{u \in L^2(H,(a,\infty)) : l(u) \in L^2(H,(a,\infty)\}, \\ D(L_0) &= \{u \in D(L) : (\sqrt{\rho}u)(a) = (\sqrt{\rho}u)(\infty) = 0\}. \end{aligned}$$

In this case the operator L_0 is symmetric and is not maximal in $L^2(H,(a,\infty))$.

In this paper, firstly the represention of all selfadjoint extensions of the minimal operator L_0 will be described. Secondly, structure of the spectrum of these extensions shall be researched.

In special case when $H = C$ the similar questions was investigated in [4] using the Glazman-Krein-Naimark method.

In left and right semi-infinitive intervals case the similar problems have been surveyed in [1].

3. Description of Selfadjoint Extensions

In this section, the general representation of selfadjoint extensions of the minimal operator L_0 will be investigated by using the Calkin-Gorbachuk method.

Firstly, let us prove the following proposition.

Lemma 1. *The deficiency indices of the operator L_0 is in form* $(m(L_0), n(L_0)) = (dimH, dimH)$.

Proof. For the simplicity of calculations it will be taken $A = 0$. It is clear that the general solutions of following differential equations

$$i\rho(t)u'_{\pm}(t) + \frac{1}{2}i\rho'(t)u_{\pm}(t) \pm iu_{\pm}(t) = 0,$$

in the $L^2(H,(a,\infty))$ are in forms

$$u_{\pm}(t) = exp\left(\mp\int_c^t \frac{2 \pm \rho'(s)}{2\rho(s)}ds\right)f,\ f \in H,\ t > a,\ c > a.$$

From these representations, we have

$$\begin{aligned}
\|u_+\|^2_{L^2(H,(a,\infty))} &= \int_a^\infty \|u_+(t)\|^2_H dt \\
&= \int_a^\infty exp\left(-\int_c^t \frac{2+\rho'(s)}{\rho(s)}ds\right) dt\|f\|^2_H \\
&= \int_a^\infty \frac{\rho(c)}{\rho(t)} exp\left(-\int_c^t \frac{2}{\rho(s)}ds\right) dt\|f\|^2_H \\
&= \frac{\rho(c)}{2}\int_a^\infty exp\left(-\int_c^t \frac{2}{\rho(s)}ds\right) d\left(\int_c^t \frac{2}{\rho(s)}ds\right)\|f\|^2_H \\
&= \frac{\rho(c)}{2}\left[exp\left(-\int_c^a \frac{2}{\rho(s)}ds\right) - exp\left(-\int_c^\infty \frac{2}{\rho(s)}ds\right)\right]\|f\|^2_H < \infty.
\end{aligned}$$

Consequently $m(L_0) = dim\ ker(L + iE) = dimH$.

On the other hand it is clear that for any $f \in H$ the solution

$$
\begin{aligned}
\|u_-\|^2_{L^2(H,(a,\infty))} &= \int_a^\infty \|u_-(t)\|^2_H dt \\
&= \int_a^\infty exp\left(\int_c^t \frac{2-\rho'(s)}{\rho(s)}ds\right) dt\|f\|^2_H \\
&= \int_a^\infty \frac{\rho(c)}{\rho(t)} exp\left(\int_c^t \frac{2}{\rho(s)}ds\right) dt\|f\|^2_H \\
&= \frac{\rho(c)}{2}\int_a^\infty exp\left(\int_c^t \frac{2}{\rho(s)}ds\right) d\left(\int_c^t \frac{2}{\rho(s)}ds\right)\|f\|^2_H \\
&= \frac{\rho(c)}{2}\left[exp\left(\int_c^\infty \frac{2}{\rho(s)}ds\right) - exp\left(\int_c^a \frac{2}{\rho(s)}ds\right)\right]\|f\|^2_H < \infty.
\end{aligned}
$$

It follows from that $n(L_0) = dim\ ker(L - iE) = dim H$. This completes the proof of theorem consequently, the minimal operator L_0 has at least one selfadjoint extensions (see [6]).

Definition 1. *Let H be any Hilbert space and $S : D(S) \subset H \to H$ be a closed densely defined symmetric operator in the Hilbert space H having equal finite or infinite deficiency indices. A triplet (H, γ_1, γ_2), where H is a Hilbert space, γ_1 and γ_2 are linear mappings from $D(S^*)$ into H, is called a space of boundary values for the operator S if for any $f, g \in D(S^*)$*

$$(S^*f, g)_{\mathcal{H}} - (f, S^*g)_{\mathcal{H}} = (\gamma_1(f), \gamma_2(g))_{\mathbf{H}} - (\gamma_2(f), \gamma_1(g))_{\mathbf{H}}$$

while for any $F_1, F_2 \in H$, there exists an element $f \in D(S^*)$ such that $\gamma_1(f) = F_1$ and $\gamma_2(f) = F_2$.

Lemma 2. *The triplet (H, γ_1, γ_2),*

$$
\begin{aligned}
&\gamma_1 : D(L) \to H,\ \gamma_1(u) = \frac{1}{\sqrt{2}}((\sqrt{\rho}u)(\infty) - (\sqrt{\rho}u)(a)), \\
&\gamma_2 : D(L) \to H,\ \gamma_2(u) = \frac{1}{i\sqrt{2}}((\sqrt{\rho}u)(\infty) + (\sqrt{\rho}u)(a)),\ u \in D(L)
\end{aligned}
$$

is a space of boundary values of the minimal operator L_0 in $L^2(H, (a, \infty))$.

Proof. In this case the direct calculations show for arbitrary $u, v \in D(L)$ that

$$\begin{aligned}
(Lu,v)_{L^2(H,(a,\infty))} - (u,Lv)_{L^2(H,(a,\infty))} &= (i\rho u' + \frac{1}{2} i\rho' u + Au, v)_{L^2(H,(a,\infty))} \\
&\quad -(u, i\rho v' + \frac{1}{2} i\rho' v + Av)_{L^2(H,(a,\infty))} \\
&= (i\rho u', v)_{L^2(H,(a,\infty))} + \frac{1}{2}(i\rho' u, v)_{L^2(H,(a,\infty))} \\
&\quad -(u, i\rho v')_{L^2(H,(a,\infty))} - (u, \frac{1}{2} i\rho' v)_{L^2(H,(a,\infty))} \\
&= i\Big[(\rho u', v)_{L^2(H,(a,\infty))} + (\rho' u, v)_{L^2(H,(a,\infty))} \\
&\quad +(\rho u, v')_{L^2(H,(a,\infty))}\Big] \\
&= i\Big[((\rho u)', v)_{L^2(H,(a,\infty))} + (\rho u, v')_{L^2(H,(a,\infty))}\Big] \\
&= i\,((\rho u, v))'_{L^2(H,(a,\infty))} \\
&= i\,((\sqrt{\rho} u, \sqrt{\rho} v))'_{L^2(H,(a,\infty))} \\
&= i\,[((\sqrt{\rho} u)(\infty), (\sqrt{\rho} v)(\infty))_H \\
&\quad - ((\sqrt{\rho}) u(a), (\sqrt{\rho}) v(a))_H] \\
&= (\gamma_1(u), \gamma_2(v))_H - (\gamma_2(u), \gamma_1(v))_H.
\end{aligned}$$

Now for any given elements $f,\ g \in H$, let us find the function $u \in D(L)$ satisfying

$$\gamma_1(u) = \frac{1}{\sqrt{2}}((\sqrt{\rho}u)(\infty) - (\sqrt{\rho}u)(a)) = f \text{ and } \gamma_2(u) = \frac{1}{i\sqrt{2}}((\sqrt{\rho}u)(\infty) + (\sqrt{\rho}u)(a)) = g.$$

From this

$$(\sqrt{\rho}u)(\infty) = (ig+f)/\sqrt{2} \ \text{ and } \ (\sqrt{\rho}u)(a) = (ig-f)/\sqrt{2}$$

is obtained.

If we choose the function u in following form

$$u(t) = \frac{1}{\sqrt{\rho(t)}}(1 - e^{a-t})(ig+f)/\sqrt{2} + \frac{1}{\sqrt{\rho(t)}} e^{a-t}(ig-f)/\sqrt{2},$$

$u \in D(L),\ \gamma_1(u) = f$ and $\gamma_2(u) = g$.

Finally, using the method given in [6], we can introduce the following result.

Theorem 1. *If $\widetilde{L}$ is a selfadjoint extension of the minimal operator L_0 in $L^2(H,(a,\infty))$, then it is generated by the differential-operator expression $l(\cdot)$ and boundary condition*

$$(\sqrt{\rho}u)(\infty) = W(\sqrt{\rho}u)(a),$$

where $W : H \to H$ is a unitary operator. Moreover, the unitary operator W in H is determined uniquely by the extension $\widetilde{L}$, i.e. $\widetilde{L} = L_W$ and vice versa.

Proof. It is known from [6] or [9] that all selfadjoint extensions of the minimal operator L_0 are described by differential-operator expression $l(\cdot)$ and the boundary condition

$$(V - E)\gamma_1(u) + i(V + E)\gamma_2(u) = 0,$$

where $V : H \to H$ is a unitary operator. So from Lemma 2, we have

$$(V - E)\left((\sqrt{\rho}u)(\infty) - (\sqrt{\rho}u)(a)\right) + (V + E)\left((\sqrt{\rho}u)(\infty) + (\sqrt{\rho}u)(a)\right) = 0.$$

Hence, we obtain

$$(\sqrt{\rho}u)(a) = -V(\sqrt{\rho}u)(\infty).$$

Choosing $W = -V^{-1}$ in last boundary condition, we have

$$(\sqrt{\rho}u)(\infty) = W(\sqrt{\rho}u)(a).$$

4. The Spectrum of the Selfadjoint Extensions

In this section the structure of the spectrum of the selfadjoint extensions L_W of the minimal operator L_0 in $L^2(H,(a,\infty))$ will be investigated.

First of all let us prove the following result.

Theorem 2. *The spectrum of any selfadjoint extension L_W is in form*

$$\sigma(L_W) = \left\{\lambda \in \mathbb{C} : \lambda = \left(\int_a^\infty \frac{ds}{\rho(s)}\right)^{-1} (2n\pi - arg\mu),\ n \in \mathbb{Z},\ \mu \in \sigma\left(Wexp\left(-iA\int_a^\infty \frac{ds}{\rho(s)}\right)\right)\right\}.$$

Proof. Consider the following problem to spectrum of the extension L_W

$$l(u) = \lambda u + f,\quad u,\ f \in L^2(H,(a,\infty)),\ \lambda \in \mathbb{R},$$

$$(\sqrt{\rho}u)(\infty) = W(\sqrt{\rho}u)(a),$$

that is,

$$i\rho(t)u'(t) + \frac{1}{2}i\rho'(t)u(t) + Au(t) = \lambda u(t) + f(t),\ t > a,$$

$$(\sqrt{\rho}u)(\infty) = W(\sqrt{\rho}u)(a).$$

The general solution of the last differential equation is in the following form

$$\begin{aligned} u(t;\lambda) &= \sqrt{\frac{\rho(c)}{\rho(t)}}exp\left(i(A-\lambda E)\int_c^t \frac{ds}{\rho(s)}\right) f_\lambda \\ &+ \frac{i}{\sqrt{\rho(t)}}\int_t^\infty \exp\left(i(A-\lambda E)\int_s^t \frac{d\tau}{\rho(\tau)}\right)\frac{f(s)}{\sqrt{\rho(s)}}ds,\ f_\lambda \in H,\ t > a,\ c > a. \end{aligned}$$

In this case

$$\|\sqrt{\frac{\rho(c)}{\rho(t)}}exp\left(i(A-\lambda E)\int_c^t \frac{ds}{\rho(s)}\right) f_\lambda\|^2_{L^2(H,(a,\infty))} = \rho(c)\int_a^\infty \frac{dt}{\rho(t)}\|f_\lambda\|^2_H < \infty$$

and

$$
\begin{aligned}
& \| \frac{i}{\sqrt{\rho(t)}} \int\limits_t^\infty \exp\left(i(A-\lambda E) \int\limits_s^t \frac{d\tau}{\rho(\tau)} \right) \frac{f(s)}{\sqrt{\rho(s)}} ds \|^2_{L^2(H,(a,\infty))} \\
= & \int\limits_a^\infty \frac{1}{\rho(t)} \| \int\limits_t^\infty \exp\left(i(A-\lambda E) \int\limits_s^t \frac{d\tau}{\rho(\tau)} \right) \frac{f(s)}{\sqrt{\rho(s)}} ds \|^2_H dt \\
\leq & \int\limits_a^\infty \frac{1}{\rho(t)} \left[\int\limits_t^\infty \| \exp\left(i(A-\lambda E) \int\limits_s^t \frac{d\tau}{\rho(\tau)} \right) \|_H \frac{\|f(s)\|_H}{\sqrt{\rho(s)}} ds \right]^2 dt \\
\leq & \int\limits_a^\infty \frac{1}{\rho(t)} \left(\int\limits_t^\infty \frac{ds}{\rho(s)} \right) \left(\int\limits_t^\infty \|f(s)\|^2_H ds \right) dt \\
\leq & \int\limits_a^\infty \frac{1}{\rho(t)} \left(\int\limits_a^\infty \frac{ds}{\rho(s)} \right) \left(\int\limits_a^\infty \|f(s)\|^2_H ds \right) dt \\
= & \int\limits_a^\infty \frac{dt}{\rho(t)} \int\limits_a^\infty \frac{ds}{\rho(s)} \|f(s)\|^2_{L^2(H,(a,\infty))} ds \\
= & \left(\int\limits_a^\infty \frac{dt}{\rho(t)} \right)^2 \|f\|^2_{L^2(H,(a,\infty))} < \infty.
\end{aligned}
$$

Hence for $u(\,\cdot\,,\lambda) \in L^2(H,(a,\infty))$ for $\lambda \in R$. From this and boundary condition, we have

$$
\begin{aligned}
& \left(exp\left(-i\lambda \int\limits_a^\infty \frac{ds}{\rho(s)} \right) - W exp\left(-iA \int\limits_a^\infty \frac{ds}{\rho(s)} \right) \right) exp\left(iA \int\limits_c^\infty \frac{ds}{\rho(s)} \right) exp\left(-i\lambda \int\limits_c^a \frac{ds}{\rho(s)} \right) f_\lambda \\
= & \frac{i}{\sqrt{\rho(c)}} W \int\limits_a^\infty \exp\left(i(A-\lambda) \int\limits_s^a \frac{d\tau}{\rho(\tau)} \right) \frac{f(s)}{\sqrt{\rho(s)}} ds
\end{aligned}
$$

In order to get $\lambda \in \sigma(L_W)$, the necessary and sufficient condition is

$$
exp\left(-i\lambda \int\limits_a^\infty \frac{ds}{\rho(s)} \right) = \mu \in \sigma \left(W exp\left(-iA \int\limits_a^\infty \frac{ds}{\rho(s)} \right) \right)
$$

Consequently,

$$
\lambda \int\limits_a^\infty \frac{ds}{\rho(s)} = 2n\pi - arg\mu, \ n \in \mathbb{Z},
$$

that is,

$$\lambda = \left(\int_a^\infty \frac{ds}{\rho(s)} \right)^{-1} (2n\pi - arg\mu), \ n \in \mathbb{Z}.$$

This completes proof of theorem.

Example. All selfadjoint extensions L_φ of the minimal operator L_0 generated by differential expression

$$l(u) = it^2 \frac{\partial u(t,x)}{\partial t} + itu(t,x) + Au,$$

$$A : D(A) \subset L^2(0,1) \to L^2(0,1),$$

where $Av(t) = -\dfrac{\partial^2 v(t)}{\partial t^2}$,

$$D(A) = \left\{ u \in W_2^2(0,1) : \ v(0) = v(1), \ v'(0) = v'(1) \right\},$$

in the Hilbert space $L^2((1,\infty)\times(0,1))$ in terms of boundary conditions are described by following form

$$(tu(t,x))(\infty) = e^{i\varphi}(tu(t,x))(1), \ \varphi \in [0, 2\pi), \ x \in (0,1).$$

Moreover, the spectrum of such extension is

$$\sigma(L_\varphi) = \left\{ \lambda \in \mathbb{C} : \lambda = 2n\pi + (\varphi - \alpha), \ n \in \mathbb{Z}, \ \alpha \in \sigma(A) \right\}.$$

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COMMUNICATIONS
SERIES A1

C*-ALGEBRA-VALUED S-METRIC SPACES

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Abstract. In this study, we present the concept of a C*-algebra-valued S-metric space. We prove Banach contraction principle in this space. Finally, we prove a common fixed point theorem in C*-algebra-valued S-metric spaces defining new notions such as L-condition and k-contraction.

1. Introduction

As we have known, Banach contraction principle has very useful structure. For this reason, it has been used in various areas such as modern analysis, applied mathematics and fixed point theory. The main goal of researchers is to obtain new results in different metric spaces. On the other hand, coupled fixed point theorems have been given in different metric spaces [12, 23, 31].

The notion of S-metric space was presented by Sedghi et al. [24]. Then, Chouhan [6] proved a common unique fixed point theorem for expansive mappings in S-metric space. Sedghi and Dung [25] proved a general fixed point theorem in S-metric spaces.

Hieu et al. [11] gave a fixed point theorem for a class of maps depending on another map on S-metric spaces. Afra [2] introduced double contractive mappings. For other important papers related to S-metric spaces, see [1, 7, 8, 9, 10, 26].

After studying the operator-valued metric spaces in [17], Ma et al. [18] introduced the concept of C*-valued metric spaces and give a fixed point theorem for C*-valued contraction mappings. In [19], C*-algebra-valued b-metric spaces were presented and some applications related to operator and integral equations were given. Coincidence and common fixed point theorems for two mappings in complete C*-algebra-valued metric spaces were proved in [22].

Batul and Kamran [5] generalized the notion of C*-valued contraction mappings and established a fixed point theorem for such mappings. In [29], Caristi's fixed point theorem was given for C*-algebra-valued metric spaces. Kamran et al. [14]

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gave the Banach contraction principle in C*-algebra-valued b-metric spaces with application. Bai [4] presented coupled fixed point theorems in C*-algebra-valued b-metric spaces. For other works, see [3, 13, 15, 21, 28, 30, 32, 33].

In this work, we introduce C*-algebra-valued S-metric spaces and prove Banach contraction principle. We also prove a coupled fixed point theorem in C*-algebra-valued S-metric spaces. For this purpose, we give some definitions such as coupled fixed point, L-condition and k-contraction.

2. Preliminaries

In this section, we give some basic definitions and theorems from [18] which will be used later. Throughout this paper, $\mathbb{A}$ will denote a unital C*-algebra with a unit I. An involution on $\mathbb{A}$ is a conjugate linear map $a \mapsto a^*$ on $\mathbb{A}$ such that

$$a^{**} = a \quad \text{and} \quad (ab)^* = b^* a^*$$

for all $a, b \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that

$$\|a^*\| = \|a\| \quad (\forall a \in A).$$

A C*-algebra is a Banach $*$-algebra such that $\|a^* a\| = \|a\|^2$.

Set $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. An element $x \in \mathbb{A}$ is said to be a positive element, denoted by $x \succeq \theta$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset \mathbb{R}_+ = [0, \infty)$, where $\sigma(x)$ is the spectrum of x. A partial ordering $\preceq$ on $\mathbb{A}_h$ can be defined with these positive elements as follows:

$$x \preceq y \quad \text{if and only if} \quad y - x \succeq \theta,$$

where θ means the zero element in $\mathbb{A}$. The set $\{x \in \mathbb{A} : x \succeq \theta\}$ will be denoted by $\mathbb{A}_+$.

When $\mathbb{A}$ is a unital C*-algebra, then for any $x \in \mathbb{A}_+$ we have $x \preceq I \iff \|x\| \leq 1$ and $|x| = (x * x)^{\frac{1}{2}}$.

Definition 2.1. [18]. Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies the following:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta \iff x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is called a C*-algebra-valued metric space.

It is obvious that C*-algebra-valued metric spaces generalize the concept of metric spaces, replacing the set of real numbers by $\mathbb{A}_+$.

Definition 2.2. [18]. Let $(X, \mathbb{A})$ be a C*-algebra-valued metric space. Suppose that $\{x_n\} \subset X$ and $x \in X$.

(i) If for any $\varepsilon > 0$, there is N such that for all $n > N$, $\|d(x_n, x)\| \leq \varepsilon$, then $\{x_n\}$ is said to be convergent with respect to $\mathbb{A}$ and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote it by $\lim_{n\to\infty} x_n = x$.

(ii) If for any $\varepsilon > 0$, there is N such that for all $n, m > N$, $\|d(x_n, x_m)\| \leq \varepsilon$, then $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$.

(iii) We say that $(X, \mathbb{A}, d)$ is a complete C*-algebra-valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Example 2.3. [18]. Let $X = \mathbb{R}$ and $\mathbb{A} = M_2(\mathbb{R})$. Define

$$d(x, y) = diag(|x - y|, \alpha|x - y|),$$

where $x, y \in \mathbb{R}$ and $\alpha \geq 0$ is a constant. d is a C*-algebra-valued metric and $(X, M_2(\mathbb{R}), d)$ is a complete C*-algebra-valued metric space by the completeness of $\mathbb{R}$.

Definition 2.4. [18]. Suppose that $(X, \mathbb{A}, d)$ is a C*-algebra-valued metric space. We call a mapping $T : X \to X$ is a C*-algebra-valued contractive mapping on X, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \preceq A^* d(x, y) A$$

for all $x, y \in A$.

Theorem 2.5. [18]. *If $(X, \mathbb{A}, d)$ is a complete C*-algebra-valued metric space and T is a contractive mapping, there exists a unique fixed point in X.*

Definition 2.6. [18]. Let X be a nonempty set. We call a mapping T is a C*-algebra-valued expansion mapping on X, if $T : X \to X$ satisfies:

(1) $T(X) = X$;
(2) $d(Tx, Ty) \succeq A^* d(x, y) A$, $\forall x, y \in X$,

where $A \in \mathbb{A}$ is an invertible element and $\|A^{-1}\| < 1$.

Theorem 2.7. [18]. *Let $(X, \mathbb{A}, d)$ be a complete C*-algebra-valued metric space. Then for the expansion mapping T, there exists a unique fixed point in X.*

Lemma 2.8. [18]. *Suppose that $\mathbb{A}$ is a unital C*-algebra with a unit I.*

(1) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I - a)^{-1}\| < 1$;*
(2) *Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$;*
(3) *by $\mathbb{A}'$ we denote the set*

$$\{a \in \mathbb{A} : ab = ba, \ \forall b \in \mathbb{A}\}.$$

Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$ and $I - a \in \mathbb{A}'_+$ is an invertible operator, then

$$(I - a)^{-1} b \succeq (I - a)^{-1} c.$$

Theorem 2.9. [18]. *Let $(X, \mathbb{A}, d)$ be a complete C^*-valued metric space. Suppose the mapping $T : X \to X$ satisfies for all $x, y \in X$*

$$d(Tx, Ty) \preceq A(d(Tx, y) + d(Ty, x)),$$

where $A \in \mathbb{A}_+^{'}$ and $\|A\| < \frac{1}{2}$. Then there exists a unique fixed point in X.

On the other hand, we need to recall the definition of S-metric spaces.

Definition 2.10. [24]. Let X be a non-empty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(i) $S(x, y, z) \geq 0$;
(ii) $S(x, y, z) = 0$ if and only if $x = y = z$;
(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

3. Main Results

In this section, we introduce C*-algebra-valued S-metric spaces and give some results on this new space.

Definition 3.1. Let X be a nonempty set. Suppose the mapping $S : X \times X \times X \to \mathbb{A}$ satisfies the following conditions for each $x, y, z, a \in X$:

(i) $S(x, y, z) \succeq \theta$;
(ii) $S(x, y, z) = \theta$ if and only if $x = y = z$;
(iii) $S(x, y, z) \preceq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called a C*-algebra-valued S-metric and $(X, \mathbb{A}, S)$ is called a C*-algebra-valued S-metric space.

Example 3.2. Let $\mathbb{A} = M_2(\mathbb{R})$ be all 2×2-matrices with the usual operations of addition, scalar multiplication and matrix multiplication. It is clear that

$$\|A\| = (\sum_{i,j=1}^{2} |a_{i,j}|^2)^{\frac{1}{2}}$$

defines a norm on $\mathbb{A}$ where $A = (a_{ij}) \in \mathbb{A}$. $* : \mathbb{A} \to \mathbb{A}$ defines an involution on $\mathbb{A}$ where $A^* = A$. Then $\mathbb{A}$ is a C*-algebra [27]. For $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{A}$, a partial order on $\mathbb{A}$ can be given as follows:

$$A \preceq B \quad \Leftrightarrow \quad (a_{ij} - b_{ij}) \leq 0 \ \text{ for all } i, j = 1, 2.$$

If we define on $\mathbb{A}$

$$S(x, y, z) = \begin{bmatrix} d(x, z) + d(y, z) & 0 \\ 0 & d(x, z) + d(y, z) \end{bmatrix},$$

then it is a C*-algebra-valued S-metric space.

Lemma 3.3. *In a C^*-algebra-valued S-metric space, we have $S(x, x, y) = S(y, y, x)$.*

Proof. By the condition (*iii*) of C*-algebra-valued S-metric, we obtain

$$S(x,x,y) \preceq S(x,x,x) + S(x,x,x) + S(y,y,x) = S(y,y,x)$$

and

$$S(y,y,x) \preceq S(y,y,y) + S(y,y,y) + S(x,x,y) = S(x,x,y).$$

Thus we get $S(x,x,y) = S(y,y,x)$. □

Definition 3.4. Let $(X, \mathbb{A}, S)$ be a C*-algebra-valued S-metric space.

(i) A sequence $\{x_n\}$ in X converges to $x \in X$ with respect to $\mathbb{A}$ if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence with respect to $\mathbb{A}$ if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $S(x_n, x_n, x_m) \prec \varepsilon$ for each $n, m \succeq N$.

(iii) We say that $(X, \mathbb{A}, S)$ is a complete C*-algebra-valued S-metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Example 3.5. Let $X = \mathbb{R}$, $\mathbb{A} = \mathbb{R}^2$ and $S(x,y,z) = (|x-z| + |y-z|, 0)$ be a C*-algebra valued S-metric space. Consider a sequence $(x_n) = (\frac{1}{n})$. Since

$$S(x_n, x_n, x_m) = (2|\frac{1}{n} - \frac{1}{m}|, 0) \leq (2\big(|\frac{1}{n}| + |\frac{1}{m}|\big), 0) \overset{n,m\to\infty}{\rightarrow} (0,0),$$

(x_n) is a Cauchy sequence. On the other hand, (x_n) converges to $0 \in X$ because

$$S(x_n, x_n, 0) = (2|\frac{1}{n}|, 0) \overset{n\to\infty}{\rightarrow} (0,0).$$

Definition 3.6. Let $(X, \mathbb{A}, S)$ be a C*-algebra-valued S-metric space. A map $T : X \to X$ is said to be C*-algebra-valued contractive mapping on X, if there exists $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$S(Tx, Tx, Ty) \preceq A^* S(x,x,y) A \tag{3.1}$$

for all $x, y \in X$.

Example 3.7. Let $X = [0,1]$ and $\mathbb{A} = M_2(\mathbb{R})$ with $\|A\| = \max\{a_1, a_2, a_3, a_4\}$, where a_i's are the entries of A. Then $(X, \mathbb{A}, S)$ is a C*-algebra-valued S-metric space, where

$$S(x,y,z) = \begin{bmatrix} |x-z| + |y-z| & 0 \\ 0 & |x-z| + |y-z| \end{bmatrix},$$

and partial ordering on $\mathbb{A}$ is given by

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \succeq \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \quad \Leftrightarrow \quad a_i \geq b_i, \text{ for } i = 1,2,3,4.$$

Define a map $T : X \to X$ by $T(x) = \frac{x}{4}$. Since

$$\begin{aligned}
S(Tx, Tx, Ty) &= S(\frac{x}{4}, \frac{x}{4}, \frac{y}{4}) \\
&= \begin{bmatrix} \frac{1}{2}|x-y| & 0 \\ 0 & \frac{1}{2}|x-y| \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} |x-y| & 0 \\ 0 & |x-y| \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= A^* S(x, x, y) A,
\end{aligned}$$

where $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\|A\| = \frac{1}{\sqrt{2}} < 1$, T is a C*-algebra-valued contractive mapping.

We now prove the Banach's contraction principle for C*-algebra-valued S-metric spaces.

Theorem 3.8. *Let $(X, \mathbb{A}, S)$ be a complete C*-algebra-valued S-metric space and $T : X \to X$ be a C*-algebra-valued contractive mapping. Then T has a unique fixed point $x_0 \in X$.*

Proof. Let's first prove the existence. We choose $x \in X$ and show that $\{T^n(x)\}$ is a Cauchy sequence with respect to $\mathbb{A}$. Using induction, we obtain the following:

$$\begin{aligned}
S(x_n, x_n, x_{n+1}) &= S(T^n(x), T^n(x), T^{n+1}(x)) \\
&\preceq A^* S(T^{n-1}(x), T^{n-1}(x), T^n(x)) A \\
&\preceq (A^*)^2 S(T^{n-2}(x), T^{n-2}(x), T^{n-1}(x)) A^2 \\
&\vdots \\
&\preceq (A^*)^n S(x, x, T(x)) A^n
\end{aligned}$$

for $n = 0, 1, \ldots$. Therefore for $m > n$, we get

$$\begin{aligned}
S(x_n, x_n, x_m) &= S(T^n(x), T^n(x), T^m(x)) \\
&\preceq 2\sum_{i=n}^{m-2} S(T^i(x), T^i(x), T^{i+1}(x)) + S(T^{m-1}(x), T^{m-1}(x), T^m(x)) \\
&\preceq 2\sum_{i=n}^{m-2} (A^*)^i S(x, x, Tx) A^i + (A^*)^{m-1} S(x, x, T(x)) A^{m-1} \\
&= 2\sum_{i=n}^{m-2} (A^*)^i B^{\frac{1}{2}} B^{\frac{1}{2}} A^i + (A^*)^{m-1} B^{\frac{1}{2}} B^{\frac{1}{2}} A^{m-1} \\
&= 2\sum_{i=n}^{m-2} (B^{\frac{1}{2}} A^i)^* (B^{\frac{1}{2}} A^i) + (B^{\frac{1}{2}} A^{m-1})^* (B^{\frac{1}{2}} A^{m-1}) \\
&= 2\sum_{i=n}^{m-2} |B^{\frac{1}{2}} A^i|^2 + |B^{\frac{1}{2}} A^{m-1}|^2 \\
&\leq \|2\sum_{i=n}^{m-2} |B^{\frac{1}{2}} A^i|^2 + |B^{\frac{1}{2}} A^{m-1}|^2\| I \\
&\leq 2\sum_{i=n}^{m-2} \|B^{\frac{1}{2}}\|^2 \|A^i\|^2 I + \|B^{\frac{1}{2}}\|^2 \|A^{m-1}\|^2 I \\
&\leq 2\|B^{\frac{1}{2}}\|^2 \sum_{i=n}^{m-2} \|A\|^{2i} I + \|B^{\frac{1}{2}}\|^2 \|A\|^{2m-2} I \\
&\leq 2\|B^{\frac{1}{2}}\|^2 \frac{\|A\|^{2n}}{1 - \|A\|} I + \|B^{\frac{1}{2}}\|^2 \|A\|^{2m-2} I \\
&\xrightarrow{m,n\to\infty} \theta
\end{aligned}$$

where $B = S(x, x, Tx)$. So $\{T^n(x)\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, S)$, there exists an element $x_0 \in X$ with $\lim_{n\to\infty} T^n(x) = x_0$. Since

$$\begin{aligned}
\theta &\preceq S(Tx_0, Tx_0, x_0) \\
&= S(Tx_0, Tx_0, Tx_n) + S(Tx_0, Tx_0, Tx_n) + S(x_0, x_0, x_n) \\
&\preceq A^* S(x_0, x_0, x_n) A + A^* S(x_0, x_0, x_n) A + S(x_0, x_0, x_n) \\
&\overset{n\to\infty}{\to} \theta,
\end{aligned}$$

we conclude that $Tx_0 = x_0$, i.e., x_0 is a fixed point of T.

Finally we show the uniqueness. Assume that there exists $u, v \in X$ with $u = T(u)$ and $v = T(v)$. Since T is a C*-algebra-valued contractive mapping, we have

$$\theta \preceq S(u, u, v) = S(Tu, Tu, Tv) \preceq A^* S(u, u, v) A.$$

On the other hand, since $\|A\| < 1$, we obtain

$$\begin{aligned}0 &\leq \|S(u,u,v)\| = \|S(Tu,Tu,Tv)\| \\ &\leq \|A^*S(u,u,v)A\| \\ &\leq \|A^*\|\|S(u,u,v)\|\|A\| \\ &= \|A\|^2\|S(u,u,v)\| \\ &< \|S(u,u,v)\|.\end{aligned}$$

But this is impossible. So $S(u,u,v) = \theta$ and $u = v$ which implies that the fixed point is unique. □

Example 3.9. Let X, $\mathbb{A}$, S and T be as in Example 3.7. T satisfies the hypothesis of Theorem 3.8. So 0 is the unique fixed point of T.

Definition 3.10. Let $(X,\mathbb{A},S)$ be a C*-algebra-valued S-metric space. An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

Definition 3.11. Let $(X,\mathbb{A},S)$ be a C*-algebra-valued S-metric space. An element $(x,y) \in X\times X$ is called a coupled coincidence point of the mappings $F : X\times X \to X$ and $g : X \to X$ if $F(x,y) = gx$ and $F(y,x) = gy$.

Definition 3.12. [1]. Let X be a nonempty set. We say the mappings $F : X\times X \to X$ and $g : X \to X$ satisfy the L-condition if $gF(x,y) = F(gx,gy)$ for all $x,y \in X$.

Definition 3.13. Let $(X,\mathbb{A},S)$ be a C*-algebra-valued S-metric space. We say the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the k-contraction if

$$S(F(x,y),F(x,y),F(z,w)) \preceq kA^*[S(gx,gx,gz) + S(gy,gy,gw)]A \tag{3.2}$$

with respect to $\mathbb{A}$ for all $x,y,z,w,u,v \in X$.

Lemma 3.14. *Let $(X,\mathbb{A},S)$ be a C*-algebra-valued S-metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ satisfies the k-contraction for $k \in (0,\frac{1}{2})$. If (x,y) is a coupled coincidence point of the mappings F and g, then*

$$F(x,y) = gx = gy = F(y,x).$$

Proof. We have $gx = F(x,y)$ and $gy = F(y,x)$ because (x,y) is the coupled coincidence point of the mappings F and g. If we assume $gx \neq gy$, then we obtain

$$\begin{aligned}S(gx,gx,gy) &= S(F(x,y),F(x,y),F(y,x)) \\ &\preceq kA^*[S(gx,gx,gy) + S(gy,gy,gx)]A \\ &= 2kA^*S(gx,gx,gy)A\end{aligned}$$

and

$$\begin{aligned}\|S(gx,gx,gy)\| &\leq 2k\|A\|^2\|S(gx,gx,gy)\| \\ &< \|S(gx,gx,gy)\|\end{aligned}$$

by (3.2) and Lemma 3.3. But it is a contradiction. Therefore $gx = gy$ and

$$F(x, y) = gx = gy = F(y, x).$$

$\square$

Theorem 3.15. *Let $(X, \mathbb{A}, S)$ be a C*-algebra-valued S-metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are mappings satisfying k-contraction for $k \in (0, \frac{1}{2})$ and L-condition. If $g(X)$ is continuous with closed range such that $F(X \times X) \subset g(X)$, then there is a unique x in X such that $gx = F(x, x) = x$.*

Proof. Let $x_0, y_0 \in X$. By the fact that $F(X \times X) \subseteq g(X)$, two elements x_1, y_1 could be chosen as follows:

$$gx_1 = F(x_0, y_0) \quad \text{and} \quad gy_1 = F(y_0, x_0).$$

Starting from the pair (x_1, y_1), two sequences $\{x_n\}$ and $\{y_n\}$ in X can be obtained such that

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n).$$

The inequality (3.2) gives the following for $n \in \mathbb{N}$:

$$S(gx_{n-1}, gx_{n-1}, gx_n) \preceq kA^*[S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})]A. \tag{3.3}$$

On the other hand, we get

$$\begin{aligned} F(y_{n-2}, x_{n-2}) = S(gy_{n-1}, gy_{n-1}, gy_n) \preceq & kA^*[S(gy_{n-2}, gy_{n-2}, gy_{n-1}) \\ & + S(gx_{n-2}, gx_{n-2}, gx_{n-1})]A. \end{aligned} \tag{3.4}$$

If we sum (3.3) and (3.4), we get

$$\begin{aligned} S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n) \preceq & 2kA^*[S(gx_{n-2}, gx_{n-2}, gx_{n-1}) \\ & + S(gy_{n-2}, gy_{n-2}, gy_{n-1})]A \end{aligned}$$

for all $n \in \mathbb{N}$. If (3.2) is applied adequately,

$$\begin{aligned} S(gx_n, gx_n, gx_{n+1}) &\preceq 2k^2(A^*)^2[S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})]A^2 \\ &\cdots \\ &\preceq \frac{1}{2}k^n(\sqrt{2}A^*)^n[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)](\sqrt{2}A)^n. \end{aligned}$$

Using the definition of C*-algebra-valued S-metric space and Lemma 3.3,

$$\begin{aligned} S(gx_n, gx_n, gx_m) &\preceq 2\sum_{i=n}^{m-2} S(gx_i, gx_i, gx_{i+1}) + S(gx_{m-1}, gx_{m-1}, gx_m) \\ &\preceq 2\sum_{i=n}^{m-2} \frac{1}{2}k^i(\sqrt{2}A^*)^i[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)](\sqrt{2}A)^i \\ &+ \frac{1}{2}k^{m-1}(\sqrt{2}A^*)^{m-1}[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)](\sqrt{2}A)^{m-1}, \end{aligned}$$

where $m, n \in \mathbb{N}$, $m > n+2$, then we conclude that

$$\|S(gx_n, gx_n, gx_m)\| \leq \sum_{i=n}^{m-2} k^i \|\sqrt{2}A\|^{2i}[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)] \\ + \frac{1}{2}k^{m-1}\|\sqrt{2}A\|^{2m-2}[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)].$$

Since $\|A\| < \frac{1}{\sqrt{2}}$, when $n, m \to \infty$, we get $\|S(gx_n, gx_n, gx_m)\| \to 0$. So $\{gx_n\}$ is a Cauchy sequence. In a similar way, $\{gy_n\}$ is a Cauchy sequence. From the closedness of $g(X)$, $\{gx_n\}$ and $\{gy_n\}$ are convergent to $x \in X$ and $y \in X$. Since g is continuous, $\{g(gx_n)\}$ is convergent to gx and $\{g(gy_n)\}$ is convergent to gy. Since F and g satisfy the L-condition, we get

$$g(gx_{n+1}) = g(F(x_n, y_n)) = F(gx_n, gy_n)$$
$$g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n).$$

This shows that the following inequalities:

$$S(g(gx_{n+1}), g(gx_{n+1}), F(x,y)) \preceq kA^*[S(g(gx_n), g(gx_n), gx) + S(g(gy_n), g(gy_n), gy)]A$$

and

$$\|S(g(gx_{n+1}), g(gx_{n+1}), F(x,y))\| \leq k\|A\|^2\|S(g(gx_n), g(gx_n), gx) + S(g(gy_n), g(gy_n), gy)\|.$$

If we take the limit as $n \to \infty$,

$$\|S(gx, gx, F(x,y))\| \leq k\|A\|^2\|S(gx, gx, gx)\| + \|S(gy, gy, gy)\| = 0.$$

So $gx = F(x,y)$. Similarly, $gy = F(y,x)$. From Lemma 3.14, (x,y) is a coupled coincidence point of the mappings F and g. So $gx = F(x,y) = F(y,x) = gy$. Since

$$S(gx_{n+1}, gx_{n+1}, gx) = S(F(x_n, y_n), F(x_n, y_n), F(x,y)) \\ \preceq kA^*(S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy))A$$

and

$$S(gy_{n+1}, gy_{n+1}, gy) \preceq kA^*(S(gy_n, gy_n, gy) + S(gx_n, gx_n, gx))A,$$

we have

$$S(gx_{n+1}, gx_{n+1}, gx) + S(gy_{n+1}, gy_{n+1}, gy) \preceq 2kA^*(S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy))A$$

and

$$\|S(gx_{n+1}, gx_{n+1}, gx) + S(gy_{n+1}, gy_{n+1}, gy)\| \leq 2k\|A^*\|\|S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)\|\|A\|.$$

Taking the limit as $n \to \infty$, we obtain the following:

$$\|S(x, x, gx) + S(y, y, gy)\| \leq 2k\|A^*\|\|S(x, x, gx) + S(y, y, gy)\|\|A\| \\ = 2k\|A\|^2\|S(x, x, gx) + S(y, y, gy)\|.$$

Since $2k < 1$ and $\|A\| < \frac{1}{\sqrt{2}}$, we have $S(x, x, gx) = 0$ and $S(y, y, gy) = 0$. So $gx = x$ and $gy = y$, that is, $gx = gy = x = y$. As a result, we have $gx = F(x,x) = x$.

To show the uniqueness, assume that there is an element $z \neq x$ in X such that $z = gz = F(z, z)$. We have

$$\begin{aligned} S(x,x,z) &= S(F(x,x),F(x,x),F(z,z)) \\ &\preceq 2kA^*S(gx,gx,gz)A \\ &= 2kA^*S(x,x,z)A. \end{aligned}$$

Since $2k < 1$, $\|A\| < \frac{1}{\sqrt{2}}$ and

$$\|S(x,x,z)\| \leq 2k\|A\|^2\|S(x,x,z)\|,$$

we conclude that $S(x,x,z) = 0$, that is, $x = z$. □

The following corollary can be easily deduced from the Theorem 3.15.

Corollary 3.16. *Let* $(X, \mathbb{A}, S)$ *be a C*-algebra-valued S-metric space. If a mapping* $F : X \times X \to X$ *satisfies the following condition*

$$S(F(x,y),F(u,v),F(z,w)) \preceq kA^*[S(x,u,z) + S(y,v,w)]A$$

with respect to $\mathbb{A}$ *for all* $x, y, z, u, v, w \in X$ *and* $k \in (0, \frac{1}{2})$*, then there exists a unique element* $x \in X$ *such that* $F(x, x) = x$.

4. Conclusion

In this work, we investigate whether there are correspondences of some metric and fixed point properties in S-metric spaces taking the domain set of S-metric function as $\mathbb{A}$ which is a C*-algebra-valued set, and first present C*-algebra-valued S-metric space on the set having this structure using properties of this algebraic notion. This given structure is important in terms of integrating some metric constructions of algebraic topology and fixed point theory.

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ON CERTAIN TOPOLOGICAL INDICES OF NANOSTRUCTURES USING $Q(G)$ AND $R(G)$ OPERATORS

V. LOKESHA, R. SHRUTI, P. S. RANJINI, AND A. SINAN CEVIK

Abstract. The invention of new nanostructures gives a key measurement to industry, electronics, pharmaceutical and biological therapeutics. By considering the importance of this key point, in here we compute the $2D$*-lattice*, *nanotube* and *nanotorus* of $TUC_4C_8[p,q]$ over the graphs $Q(G)$ and $R(G)$ in terms of certain topological indices, namely first, second and third Zagreb indices, hyper Zagreb index and forgotten topological index. These indices are numerical propensity that often characterizes the quantitative structural activity/property/toxicity relationships, and also correlates physico-chemical properties such as boiling point, melting point and stability of respective nanostructures.

1. Introduction and Preliminaries

In the fields of chemical graph theory, molecular topology and mathematical chemistry, a *topological index* is actually a molecular graph invariant which making matches the physio-chemical properties of a molecular graph with a number. Furthermore, in some cases, a topological index known as a *connectivity index* which is a type of a molecular descriptor and is calculated based on the molecular graph of a chemical compound. A large amount of chemical experiments require a determination of the chemical properties of new compounds.

If we enumerate all octagons of $TUC_4C_8[p,q]$ (any cycle C_8) and all quadrangles (any cycle C_4), where p and q denotes number of octagons in a fixed row and column, respectively, of a 2-dimensional lattice (see Figure 1-(a)), then

- the *nanotube* is obtained from the lattice by wrapping it up so that each dangling edge from the left-hand side connects to the right most vertex of the same row (see Figure 1-(b)), or

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- the *nanotorus* is obtained from again the lattice by wrapping it up so that each dangling edge from the left-hand side connects to the right most vertex of the same row and each dangling edge from up side connects to the down most vertex of the same row (see Figure 1-(c)).

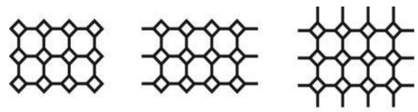

FIGURE 1. (a) $2D$-lattice of $TUC_4C_8[p, q]$; (b) nanotubes of $TUC_4C_8[p, q]$; (c) nanotores of $TUC_4C_8[p, q]$.

Recently, in [10], Hosamani has been computed the topological properties of the line graphs of subdivision graphs of certain nanostructures-II, and also obtained upper bounds for Wiener index of $2D$-lattice, nanotube and nanotorus of $TUC_4C_8[p, q]$. In [14], V. Lokesha et al. established on model graph structure of Alveoli in human lungs in terms of graph operators such as subdivision, double graph, $Q(G)$ and $R(G)$ of certain topological indices. In the same reference, by using $Q(G)$ and $R(G)$, the authors also exhibited the relation between indices. At this point let us remind the graph operators $Q(G)$ and $R(G)$ which are directly related to the main aim of this paper.

- The $Q(G)$ graph is obtained from G by inserting a new vertex into each edge of G and by joining edges those pairs of new vertices which lie on adjacent edges of G.
- The $R(G)$ graph is obtained from G by adding a new vertex corresponding to every edge of G and by joining each new vertex to the end vertices of the edge corresponding to it.

We note that the first and third authors of this paper utilized these above graph operators previously (cf. [14, 19]). On the other hand, Diudea et al. considered the problem of computing topological indices of some chemical graphs related to nanostructurer in joint works (cf. [3, 4, 12]). In addition to these above studies, Ashrafi et al. computed some topological indices of nanotubes in [1, 2], and Nadeem et al. ([16]) obtained the expressions for certain topological indices for the line graph of subdivision graphs of $2D$-lattice, nanotube and nanotorus of $TUC_4C_8[p, q]$. Moreover, Lokesha et al. in [15] studied on nanostructurer in terms of SDD, ABC_4 and GA_5 indices.

Motivated from the above references, in here we aimed to compute the first, second and third Zagreb indices, the hyper Zagreb index and the forgotten topological index of $Q(G)$ and $R(G)$ graphs of $2D$-lattice, nanotube and nanotorus of $TUC_4C_8[p, q]$. To reach the aim, let us recall the following topological indices that will be needed in our results:

The *first* and *second Zagreb indices* were introduced more than thirty years ago by I. Gutman and Trinajstic [8] which are defined as

$$\left.\begin{aligned} M_1(G) &= \sum_{e=uv\in E(G)} [d_G(u)+d_G(v)] \\ \text{and} \qquad & \\ M_2(G) &= \sum_{e=uv\in E(G)} d_G(u).d_G(v) \end{aligned}\right\}, \tag{1.1}$$

where $d_G(u)$ denotes the degree of a vertex u in G. We may refer [9, 11, 13, 17, 18, 20] for more detailed works on Zagreb indices. On the other hand, the *third Zagreb index*

$$M_3(G) = \sum_{e=uv\in E(G)} |d_G(u)-d_G(v)| \tag{1.2}$$

was introduced by Fath-Tabar in [5]. Although this modified version of Zagreb indices has been taken interest since 2011, another modified version, namely the *hyper-Zagreb index* (cf. [21]) had to be created depending on the importance of these indices. The hyper-Zagreb index is defined as

$$HM(G) = \sum_{e=uv\in E(G)} [d_G(u)+d_G(v)]^2 . \tag{1.3}$$

Unfortunately another the degree based graph invariant has not attracted any attention in the literature of mathematical chemistry for more than forty years. In view of this fact, B. Furtula et al. ([6, 7]) named it as *forgotten (topological) index* in 2015 and defined it as

$$F(G) = \sum_{e=uv\in E(G)} [d_G(u)^2+d_G(v)^2] . \tag{1.4}$$

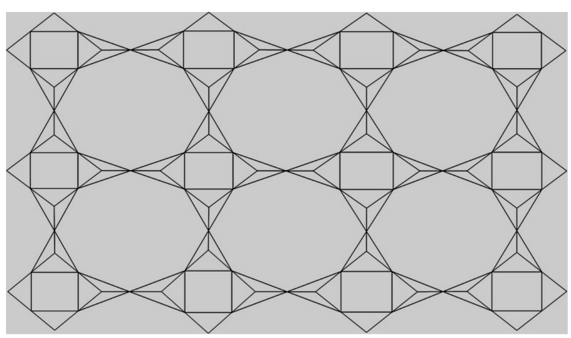

FIGURE 2. $Q(G)$ of 2D-lattice of $TUC_4C_8[p,q]$.

Forthcoming two sections, we shall give the results on the topological indices (indicated in (1.1), (1.2), (1.3) and (1.4)) of the graphs $Q(G)$ and $R(G)$ for $2D$-lattice, nanotube and nanotorus of $TUC_4C_8[p,q]$, respectively.

2. The case on $Q(G)$

In this section, by considering Equations (1.1)-(1.4) and the operator $Q(G)$, we discuss the results on $2D$-lattice, nanotube and nanotorus of $TUC_4C_8[p,q]$.

Theorem 1. *Let H be the $Q(G)$ of 2D-lattice of $TUC_4C_8[p,q]$ (see Figure 2). Then*

$$\begin{aligned} M_1(H) &= 4(p+q)\,[11(q-1)+15]+9q^2(q-6)+18p(6q-5)+9(q+2)+\\ &\quad 18q(5-q)+10pq(q-5)+20(4p+q)+44(p-q)+\\ &\quad (q-2)[8(q-3)-11(2p+q-1)(q-1)]-188\,,\\ M_2(H) &= 18q^2(q-6)+36p(6q-5)+18(q+2)+40q(5-q)+25pq(q-5)+\\ &\quad 50(4p+q)+120(p-q)+100(p+q)+16(q-2)(q-3)+\\ &\quad (q-1)[12(p+q)-30(2p+q-1)(q-2)+36q(5p-6)-\\ &\quad 36(q-2)((p-1)(p-2)+5)]-416 \end{aligned}$$

and

$$\begin{aligned} M_3(H) &= 4(p+q)[(q-1)+5]+3q^2(q-6)+6p(6q-5)+3(q+2)+\\ &\quad 2q(5-q)+4(p-q)-(2p+q-1)(q-1)(q-2)-28\,,\\ HM(H) &= 4(p+q)[121(q-1)+113]+81q^2(q-6)+162p(6q-5)+\\ &\quad 81(q+2)+64(q-2)(q-3)+162q(5-q)+100pq(q-5)+\\ &\quad 200(4p+q)+484(p-q)+144q(q-1)(5p-6)-\\ &\quad (q-1)(q-2)[121(2p+q-1)+(p-1)(p-2)+5]-1740\,,\\ F(H) &= 90p(6q-5)+45(q+2)+32(q-2)(q-3)+82q(5-q)+\\ &\quad 50pq(q-5)+45q^2(q-6)+100(4p+q)+244(p-q)+\\ &\quad 252(p+q)+(q-1)[244(p+q)-61(2p+q-1)(q-2)+\\ &\quad 72q(5p-6)-72(q-2)((p-1)(p-2)+5)]-908. \end{aligned}$$

Proof. For the graph H, there are $p^2(3q-q^2-2)+q^2(7p-12)+3q(p+10)+8p-24$ number of edges. Among these number of edges;

8 edges are of the type $(2,4)$,
$4[(p+q)-2]$ edges are of type $(2,5)$,
$4[(p+q)-2]$ edges are of the type $(3,5)$,
$[q(q^2-6q+12p+1)-10p+2]$ edges are of type $(3,6)$,
(q^2-5q+6) edges are of the type $(4,4)$,
$[2q(5-q)-4]$ edges are of the type $(4,5)$,
$[pq^2-5pq+2(4p+q-4)]$ edges are of the type $(5,5)$,
$(q(8q-q^2-13)+2p(5q-q^2-2)+2)$ edges are of the type $(5,6)$, and
$[pq(8q-14-pq+3p)+q(27-13q)+2p(3-p)-14]$ edges are of the type $(6,6)$.

Now, by taking into account of edge partition and then applying Equations (1.1)-(1.4) to H, we obtain the required results. □

Theorem 2. *Let S be the $Q(G)$ of nanotubes of $TUC_4C_8[p,q]$ (see Figure 3). Then*

$$\begin{aligned} M_1(S) &= 68p+12q+16pq(5-q)-32(2p-1)+44(p-1)+ \\ &\quad (q-1)[9(16p-pq+q-2)+10p(q-4)-22p(q-4)+ \\ &\quad 12(8p+5q+q(p-q)-10)]. \\ M_2(S) &= 140p+18q+30pq(5-q)-60(2p-1)+120(p-1)+ \\ &\quad (q-1)[18(16p-pq+q-2)+25p(q-4)-60p(q-4)+ \\ &\quad 36(8p+5q+q(p-q)-10)]. \end{aligned}$$

and

$$\begin{aligned} M_3(S) &= 12p+4pq(5-q)-8(2q-1)+4(p-1)+ \\ &\quad (q-1)[3(16p-pq+q-2)-2p(q-4)]. \\ HM(S) &= 596p+72q+128pq(5-q)-256(2p-1)+484(p-1)+ \\ &\quad (q-1)[81(16p-pq+q-2)-142p(q-4)+ \\ &\quad 144(8p+5q+q(p-q)-10)]. \\ F(S) &= 316p+36q+68pq(5-q)-136(2p-1)+244(p-1)+ \\ &\quad (q-1)[45(16p-pq+q-2)-72p(q-4)+ \\ &\quad 72(8p+5q+q(p-q)-10)]. \end{aligned}$$

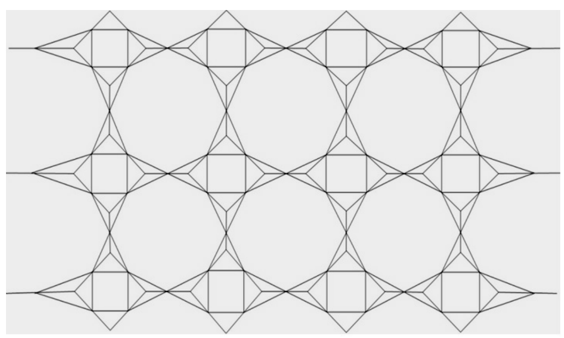

FIGURE 3. $Q(G)$ of nanotube of $TUC_4C_8[p,q]$.

Proof. For the graph S, we have total $[3pq(13-q)+q(7q-q^2-16)-24p+12]$ number of edges. From Figure 3, it can be observed that there are seven partition of those edges such that

$(2,5)$ having $4p$ edges,
$(3,3)$ are of $2q$ edges,
$(3,5)$ are of $[2pq(5-q)-4(2p-1)]$ edges,
$(3,6)$ are of $[17pq-(16p+3q)+q^2(1-p)+2]$ edges,
$(5,5)$ are of $[8p-5pq+pq^2]$ edges,
$(5,6)$ are of $[2(5pq-pq^2-2p-2)]$ edges, and

$(6,6)$ are of $[q(6q-15-q^2)+p(7q+q^2-8)+10]$ edges.
Similarly as in the proof of Theorem 1, by considering the edge partition and also applying Equations (1.1)-(1.4) to S, we get the results. □

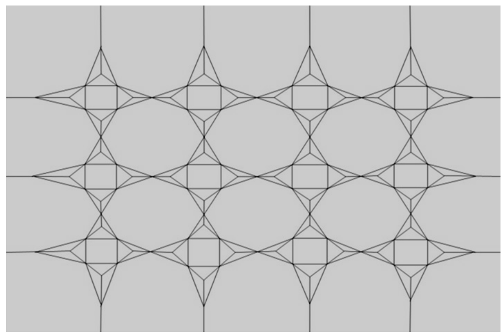

FIGURE 4. $Q(G)$ of nanotorus of $TUC_4C_8[p,q]$.

Theorem 3. *Let K be the $Q(G)$ of nanotorus of $TUC_4C_8[p,q]$ (see Figure 4). Then*

$$M_1(K) = 252pq - 18(p+q)\,, \qquad M_2(K) = 648pq - 90(p+q)\,,$$
$$M_3(K) = 6(p+q+6pq)\,, \qquad HM(K) = 2700pq - 342(p+q)\,,$$
$$F(K) = 1404pq - 162(p+q)\,.$$

Proof. The graph K has $24pq$ number of edges from Figure 4. The partitions of edges are of $(3,3)$, $(3,6)$ and $(6,6)$ having $2(p+q)$, $2(p+q+6pq)$ and $4(3pq-p-q)$ edges, respectively. Similarly as in the final parts of the proofs of Theorems 1 and 2, we obtain the results. □

3. THE CASE ON $R(G)$

With a quite parallel approximation as in Section 2, by considering Equations (1.1)-(1.4) and the operator $R(G)$, we shall present the results on $2D$-lattice, nanotube and nanotorus of $TUC_4C_8[p,q]$.

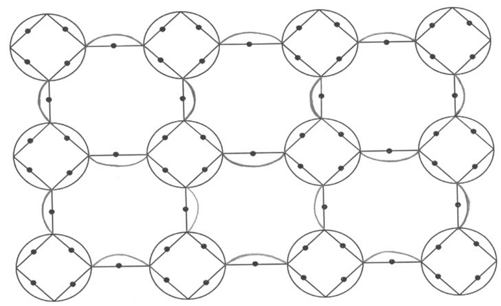

FIGURE 5. $R(G)$ of 2D-lattice of $TUC_4C_8[p,q]$.

Theorem 4. *Let H_1 be the $R(G)$ of 2D-lattice of $TUC_4C_8[p,q]$ (see Figure 5). Then we have the following equations:*

$$\begin{aligned}
M_1(H_1) &= 48(p-q) - 36(p+q) + 40(p+q-2) + \\
&\quad 8pq[9 - 2(q-1)(q-5)] + 80\,, \\
M_2(H_1) &= 72(p-q) - 148(p+q) + 96(p+q-2) + \\
&\quad 24pq[9 - (q-1)(q-5)] + 208\,, \\
M_3(H_1) &= 8(p+q) + 24(p-q) - 8pq(q-1)(q-5) + 8(p+q-2)\,, \\
HM(H_1) &= 384(p-q) + 400(p+q-2) - 576(p+q) + \\
&\quad 16pq[54 - 8(q-1)(q-5)] + 832\,, \\
F(H_1) &= 208(p+q-2) + 16pq[27 - 5(q-1)(q-5)] - 208(p+q) + \\
&\quad 240(p-q) + 416\,.
\end{aligned}$$

Proof. The total number of edges for the graph H_1 is $3(3p-q)+2pq[3-(q-1)(q-5)]$. Among these number of edges, there are

$4(p+q)$ edges are of type $(2,4)$,
$6(p-q) - 2pq(q-1)(q-5)$ edges are of the type $(2,6)$,
4 edges are of type $(4,4)$,
$4(p+q-2)$ edges are of the type $(4,6)$, and
$6pq - 5(p+q) + 4$ edges are of the type $(6,6)$.

As in the proof of Theorem 1, by taking into account of edge partition and then applying Equations (1.1)-(1.4) to H_1, we reached the results as required. □

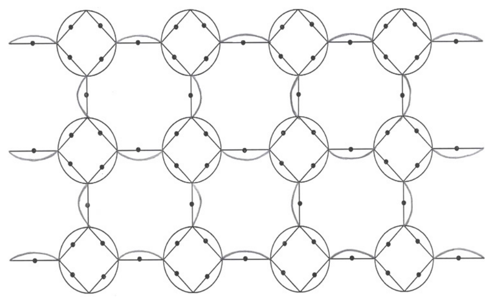

FIGURE 6. $R(G)$ of nanotube of $TUC_4C_8[p,q]$.

Theorem 5. *Let S_1 be the $R(G)$ of nanotubes of $TUC_4C_8[p,q]$ (see Figure 6). Therefore the following bounds are eligible.*

$$\begin{aligned}
M_1(S_1) &= 64p + 8[6p(2q-1)+2q] + 12(6pq-5p+q)\,, \\
M_2(S_1) &= 36(6pq-5p+q) + 128p + 12[6p(2q-1)+2q]\,, \\
M_3(S_1) &= 4[6p(2q-1)+2q] + 16p\,, \\
HM(S_1) &= 544p + 384p(2q-1) + 144(6pq-5p) + 272q\,, \\
F(S_1) &= 40[6p(2q-1)+2q] + 72(6pq+q) - 72p\,.
\end{aligned}$$

Proof. By Figure 6, it is easy to observe that the total number of edges for the graph S_1 is $3[q(6p+1)-p]$. It can be also observed that there are four partition of edges such that

$(2,4)$ having $4p$ edges,

$(2,6)$ are of $6p(2q-1)+2q$ edges,

$(4,6)$ are of $4p$ edges and $(6,6)$ are of $6pq-5p+q$ edges.

With a completely same idea as in the final parts of the proofs of previous theorems, the result follows. □

Theorem 6. *Let K_1 be the $R(G)$ of nanotorus of $TUC_4C_8[p,q]$ (see Figure 7). Then*

$$M_1(K_1) = 28(p+q) + 168pq\,, \quad M_2(K_1) = 60(p+q) + 360pq\,,$$
$$M_3(K_1) = 8(p+q) + 48pq\,, \qquad HM(K_1) = 272(p+q) + 1632pq\,,$$
$$F(K_1) = 152(p+q) + 912pq\,.$$

Proof. By Figure 7, the graph K_1 has $3p+3q+18pq$ number of edges. In here, the partitions of edges are of $(2,6)$ and $(6,6)$ having $2(p+q+6pq)$ and $p+q+6pq$ edges, respectively. Now, by considering the edge partition and then applying Equations (1.1)-(1.4) to K_1, the correctness of the theorem is shown. □

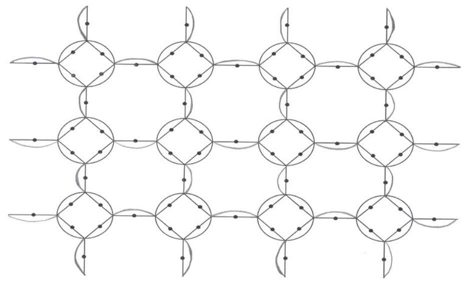

FIGURE 7. $R(G)$ of nanotorus of $TUC_4C_8[p,q]$.

Conclusion 1. *In this paper, we computed the nanostructures through degree-based topological indices over $TUC_4C_8[p,q]$. The topological indices calculated in here can help us to understand the physical features, chemical reactivity, and biological activities of these structures. From this view point, topological indices in graph theory can be regarded as a score function that maps each molecular structure to a real number, and are used as descriptors of the molecular under testing.*

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POTENTIAL OPERATORS ON CARLESON CURVES IN MORREY SPACES

AHMET EROGLU AND IRADA B. DADASHOVA

Abstract. In this paper we study the potential operator $\mathcal{I}^{\alpha}$ in the Morrey space $L_{p,\lambda}$ and the spaces BMO defined on Carleson curves Γ. We prove that for $0 < \alpha < 1$, $\mathcal{I}^{\alpha}$ is bounded from the Morrey space $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$ on simple Carleson curves if (and only if in the infinite simple Carleson curve Γ) $1/p - 1/q = \alpha/(1-\lambda)$, $1 < p < (1-\lambda)/\alpha$, and from the spaces $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$ if (and only if in the infinite case) $1 - \frac{1}{q} = \frac{\alpha}{1-\lambda}$.

1. Introduction

Morrey spaces were introduced by C. B. Morrey [11] in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations. Later, Morrey spaces found important applications to Navier-Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients, and potential theory.

The main purpose of this paper is to establish the boundedness of potential operator $\mathcal{I}^{\alpha}$ in Morrey spaces $L_{p,\lambda}$ defined on Carleson curves Γ. We prove Sobolev-Morrey inequalities for the operator $\mathcal{I}^{\alpha}$. In particular, we get the analog of the theorem by D.R. Adams [1] regarding the inequality for the Riesz potentials in Morrey spaces defined on Carleson curves. We emphasize that in the infinite case of Γ the derived conditions are necessary and sufficient for appropriate inequalities.

Note that the results we obtain here the potential operators are valid not only on Carleson curves, but also in a more general context of metric spaces or homogeneous type spaces at least under the condition $\mu(B(x,r)) \sim r^d$ (see [4, 5, 8, 12]).

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we establish the main result of the paper: We prove that for $0 < \alpha < 1$, $\mathcal{I}^{\alpha}$ is bounded from the Morrey space $L_{p,\lambda}(\Gamma)$ to

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$L_{q,\lambda}(\Gamma)$ on simple Carleson curves if (and only if in the infinite simple Carleson curves) $1/p - 1/q = \alpha/(1-\lambda)$, $1 < p < (1-\lambda)/\alpha$, and from the spaces $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$ if (and only if in the infinite case) $1 - \frac{1}{q} = \frac{\alpha}{1-\lambda}$.

2. Preliminaries

Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le l \le \infty\}$ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ with arc-length measure $\nu(t) = s$, here $l = \nu\Gamma =$ lengths of Γ. We denote

$$\Gamma(t,r) = \Gamma \cap B(t,r),\ t \in \Gamma,\ r > 0,$$

where $B(t,r) = \{z \in \mathbb{C} : |z - t| < r\}$.

A rectifiable Jordan curve Γ is called a Carleson curve if the condition

$$\nu\Gamma(t,r) \le c_0 r$$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ does not depend on t and r. Let $L_p(\Gamma)$, $1 \le p < \infty$ be the space of measurable functions on Γ with finite norm

$$\|f\|_{L_p(\Gamma)} = \left(\int_\Gamma |f(t)|^p d\nu(t) \right)^{1/p}.$$

Let $1 \le p < \infty$,$0 \le \lambda \le 1$. We denote by $L_{p,\lambda}(\Gamma)$ the Morrey space as the set of locally integrable functions f on Γ with the finite norm

$$\|f\|_{L_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma,\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))}.$$

Note that $L_{p,0}(\Gamma) = L_p(\Gamma)$, and if $\lambda < 0$ or $\lambda > 1$, then $L_{p,\lambda}(\Gamma) = \Theta$, where Θ is the set of all functions equivalent to 0 on Γ.

We denote by $WL_{p,\lambda}(\Gamma)$ the weak Morrey space as the set of locally integrable functions f with finite norm

$$\|f\|_{WL_{p,\lambda}(\Gamma)} = \sup_{\beta>0} \beta \sup_{r>0,\, t\in\Gamma} \left(r^{-\lambda} \int_{\{\tau\in\Gamma(t,r):\ |f(\tau)|>\beta\}} d\nu(\tau) \right)^{1/p}.$$

Let $f \in L_1^{loc}(\Gamma)$. The maximal operator $\mathcal{M}$ and the potential operator $\mathcal{I}^\alpha$ on Γ are defined by

$$\mathcal{M}f(t) = \sup_{t>0} |\Gamma(t,r)|^{-1} \int_{\Gamma(t,r)} |f(\tau)| d\nu(\tau),$$

and

$$\mathcal{I}^\alpha f(t) = \int_\Gamma \frac{f(\tau) d\nu(\tau)}{|t-\tau|^{1-\alpha}}, \quad 0 < \alpha < 1,$$

respectively.

Maximal operators and potential operators in various spaces defined on Carleson curves has been widely studied by many authors (see, for example [2, 3, 6, 7, 8, 9, 10, 12]).

N. Samko [12] studied the boundedness of the maximal operator $\mathcal{M}$ defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces $L_{p,\lambda}(\Gamma)$:

Theorem A. *Let Γ be a Carleson curve, $1 < p < \infty$, $0 < \alpha < 1$ and $0 \le \lambda < 1$. Then $\mathcal{M}$ is bounded from $L_{p,\lambda}(\Gamma)$ to $L_{p,\lambda}(\Gamma)$.*

V. Kokilashvili and A. Meskhi [9] studied the boundedness of the potential operator defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces and proved the following:

Theorem B. *Let Γ be a Carleson curve, $1 < p < q < \infty$, $0 < \alpha < 1$, $0 < \lambda_1 < \frac{p}{q}$, $\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$ and $\frac{1}{p} - \frac{1}{q} = \alpha$. Then the operator $\mathcal{I}^\alpha$ is bounded from the spaces $L_{p,\lambda_1}(\Gamma)$ to $L_{q,\lambda_2}(\Gamma)$.*

3. Sobolev-Morrey inequality for potential operator on Carleson curves

In this section we prove Sobolev-Morrey inequalities for the potential operators in Morrey space defined on Carleson curves.

Theorem 1. *Let Γ be a simple Carleson curve, $0 < \alpha < 1$, $0 \le \lambda < 1 - \alpha$ and $1 \le p < \frac{1-\lambda}{\alpha}$.*

1) If $1 < p < \frac{1-\lambda}{\alpha}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the infinite case also necessary for the boundedness of $\mathcal{I}^\alpha$ from $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$.

2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the infinite case also necessary for the boundedness of $\mathcal{I}^\alpha$ from $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$.

Proof. 1) *Sufficiency.* Let Γ be a simple Carleson curve, $0 < \alpha < 1$, $0 \le \lambda < 1 - \alpha$, $f \in L_{p,\lambda}(\Gamma)$ and $1 < p < \frac{1-\lambda}{\alpha}$. Then

$$\mathcal{I}^\alpha f(t) = \left(\int_{\Gamma(t,r)} + \int_{\Gamma \setminus \Gamma(t,r)} \right) f(\tau) |t - \tau|^{\alpha - 1} d\nu(\tau) \equiv A(t,r) + C(t,r). \qquad (1)$$

For $A(t,r)$ we have

$$\begin{aligned} |A(t,r)| &\le \int_{\Gamma(t,r)} |f(\tau)| |t - \tau|^{\alpha - 1} d\nu(\tau) \\ &\le \sum_{j=1}^{\infty} \left(2^{-j} r\right)^{\alpha - 1} \int_{\Gamma(t, 2^{-j+1} r) \setminus \Gamma(t, 2^{-j} r)} |f(\tau)| d\nu(\tau) \\ &\le \sum_{j=1}^{\infty} \left(2^{-j} r\right)^{\alpha - 1} \nu\Gamma(t, 2^{-j+1} r) \, \mathcal{M} f(t) \\ &\le 2 c_0 r^\alpha \, \mathcal{M} f(t) \sum_{j=1}^{\infty} 2^{-j\alpha}. \end{aligned}$$

Hence

$$|A(t,r)| \le C_1 r^{\alpha} \mathcal{M}f(t) \qquad \text{with} \qquad C_1 = \frac{2c_0}{2^{\alpha}-1}. \tag{2}$$

For $C(t,r)$ by the Hölder's inequality we have

$$|C(t,r)| \le \left(\int_{\Gamma\setminus\Gamma(t,r)} |t-\tau|^{-\beta} |f(\tau)|^p d\nu(\tau) \right)^{1/p}$$

$$\times \left(\int_{\Gamma\setminus\Gamma(t,r)} |t-\tau|^{\left(\frac{\beta}{p}+\alpha-1\right)p'} d\nu(\tau) \right)^{1/p'} = J_1 \cdot J_2.$$

Let $\lambda < \beta < 1-\alpha p$. For J_1 we get

$$J_1 = \Big(\sum_{j=0}^{\infty} \int_{\Gamma(t,2^{j+1}r)\setminus\Gamma(t,2^j r)} |f(\tau)|^p |t-\tau|^{-\beta} d\nu(\tau) \Big)^{1/p}$$

$$\le 2^{\frac{\lambda}{p}} r^{\frac{\lambda-\beta}{p}} \|f\|_{L_{p,\lambda}(\Gamma)} \Big(\sum_{j=0}^{\infty} 2^{(\lambda-\beta)j} \Big)^{1/p} = C_2 r^{\frac{\lambda-\beta}{p}} \|f\|_{L_{p,\lambda}(\Gamma)}, \tag{3}$$

where $C_2 = \left(\frac{2^{\beta}}{2^{\beta-\lambda}-1}\right)^{1/p}$.

For J_2 we obtain

$$J_2 = \left(\sum_{j=1}^{\infty} \int_{\Gamma(t,2^{j+1}r)\setminus\Gamma(t,2^j r)} |t-\tau|^{\left(\frac{\beta}{p}+\alpha-1\right)p'} d\nu(\tau) \right)^{1/p'}$$

$$\le \left(\sum_{j=1}^{\infty} \left(2^j r\right)^{\left(\frac{\beta}{p}+\alpha-1\right)p'} \nu\Gamma(t,2^{j+1}r) \right)^{1/p'}$$

$$\le \left(c_0 \sum_{j=1}^{\infty} \left(2^j r\right)^{\left(\frac{\beta}{p}+\alpha-1\right)p'+1} \right)^{1/p'} \le C_3 r^{\frac{\beta}{p}+\alpha-\frac{1}{p}}, \tag{4}$$

where $C_3 = \frac{c_0^{\frac{1}{p'}}}{1-2^{\frac{1-\beta}{p}-\alpha}}$.

Then from (3) and (4) we have

$$|C(t,r)| \le C_4 r^{\frac{\lambda-Q}{p}+\alpha} \|f\|_{L_{p,\lambda}(\Gamma)}, \tag{5}$$

where $C_4 = C_2 \cdot C_3$.

Thus, from (2) and (5) we have

$$|\mathcal{I}^{\alpha} f(t)| \le C_1 r^{\alpha} \mathcal{M}f(t) + C_4 r^{\frac{\lambda-1}{q}} \|f\|_{L_{p,\lambda}(\Gamma)}.$$

Minimizing with respect to r, at $t = \left[(\mathcal{M}f(t))^{-1} \|f\|_{L_{p,\lambda}}\right]^{p/(1-\lambda)}$ we arrive at

$$|\mathcal{I}^\alpha f(t)| \le C_5 \left(\mathcal{M}f(t)\right)^{p/q} \|f\|_{L_{p,\lambda}(\Gamma)}^{1-p/q},$$

where $C_5 = C_1 + C_4$.

Hence, by Theorem B, we have

$$\begin{aligned} \int_{\Gamma(t,r)} |\mathcal{I}^\alpha f(t)|^q \, d\nu(\tau) &\le C_5 \|f\|_{L_{p,\lambda}(\Gamma)}^{q-p} \int_{\Gamma(t,r)} (\mathcal{M}f(t))^p \, d\nu(\tau) \\ &\le C_5 C_{p,\lambda} r^\lambda \|f\|_{L_{p,\lambda}(\Gamma)}^{q-p} \|f\|_{L_{p,\lambda}(\Gamma)}^{p} = C_6 r^\lambda \|f\|_{L_{p,\lambda}(\Gamma)}^{q}, \end{aligned}$$

where $C_6 = C_5 \cdot C_{p,\lambda}$.

Therefore $\mathcal{I}^\alpha f \in L_{q,\lambda}(\Gamma)$ and

$$\|\mathcal{I}^\alpha f\|_{L_{q,\lambda}(\Gamma)} \le C_6 \|f\|_{L_{p,\lambda}(\Gamma)}.$$

Necessity. Let Γ be an infinite simple Carleson curve, $1 < p < \frac{1-\lambda}{\alpha}$ and $\mathcal{I}^\alpha$ bounded from $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$.

Define $f_r(\tau) =: f(r\tau)$. Then

$$\|f_r\|_{L_{p,\lambda}(\Gamma)} = r^{-\frac{1}{p}} \sup_{r_1>0,\, \tau\in\Gamma} \left(r_1^{-\lambda} \int_{\Gamma(t,rr_1)} |f(\tau)|^p d\nu(\tau) \right)^{1/p} = r^{-\frac{1-\lambda}{p}} \|f\|_{L_{p,\lambda}(\Gamma)}$$

and

$$\mathcal{I}^\alpha f_r(t) = r^{-\alpha} \mathcal{I}^\alpha f(rt),$$

$$\begin{aligned} \|\mathcal{I}^\alpha f_r\|_{L_{q,\lambda}(\Gamma)} &= r^{-\alpha} \sup_{r_1>0,\, t\in\Gamma} \left(r_1^{-\lambda} \int_{\Gamma(t,r_1)} |\mathcal{I}^\alpha f(rt)|^q \, d\nu(t) \right)^{1/q} \\ &= r^{-\alpha-\frac{1}{q}} \sup_{r_1>0,\, t\in\Gamma} \left(r_1^{-\lambda} \int_{\Gamma(t,rr_1)} |\mathcal{I}^\alpha f(t)|^q \, d\nu(t) \right)^{1/q} \\ &= r^{-\alpha-\frac{1-\lambda}{q}} \|\mathcal{I}^\alpha f\|_{L_{q,\lambda}(\Gamma)}. \end{aligned}$$

By the boundedness $\mathcal{I}^\alpha$ from $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$

$$\|\mathcal{I}^\alpha f\|_{L_{q,\lambda}(\Gamma)} \le C_{p,q,\lambda} r^{\alpha+\frac{1-\lambda}{q}-\frac{1-\lambda}{p}} \|f\|_{L_{p,\lambda}(\Gamma)},$$

where $C_{p,q,\lambda}$ depends only on p, q and λ.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{1-\lambda}$, then for all $f \in L_{p,\lambda}(\Gamma)$, we have $\|\mathcal{I}^\alpha f\|_{L_{q,\lambda}} = 0$ as $r \to 0$.

Similarly, if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{1-\lambda}$, then for all $f \in L_{p,\lambda}(\Gamma)$, we obtain $\|\mathcal{I}^\alpha f\|_{L_{q,\lambda}(\Gamma)} = 0$ as $r \to \infty$

Therefore $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{1-\lambda}$.

2) *Sufficiency.* Let $f \in L_{1,\lambda}(\Gamma)$. We have

$$\nu\{\tau \in \Gamma(t,r) : |\mathcal{I}^\alpha f(\tau)| > 2\beta\} \le \nu\{\tau \in \Gamma(t,r) : |A(\tau,r)| > \beta\}$$
$$+\nu\{\tau \in \Gamma(t,r) : |C(\tau,r)| > \beta\}.$$

Taking into account inequality (2) and Theorem A we have

$$\nu\{\tau \in \Gamma(t,r) \ : \ |A(\tau,r)| > \beta\} \le \nu\left\{\tau \in \Gamma(t,r) \ : \ \mathcal{M}f(\tau) > \frac{\beta}{C_1 r^\alpha}\right\}$$
$$\le \frac{C_7 r^\alpha}{\beta} \cdot r^\lambda \|f\|_{L_{1,\lambda}(\Gamma)},$$

where $C_7 = C_1 \cdot C_{1,\lambda}$ and thus if $C_4 r^{\frac{\lambda-1}{q}} \|f\|_{L_{1,\lambda}(\Gamma)} = \beta$, then $|C(\tau,r)| \le \beta$ and consequently, $|\{\tau \in \Gamma(t,r) \ : \ |C(\tau,r)| > \beta\}| = 0$.

Finally

$$\nu\{\tau \in \Gamma(t,r) \ : \ |\mathcal{I}^\alpha f(\tau)| > 2\beta\} \le \frac{C_7}{\beta} r^\lambda r^\alpha \|f\|_{L_{1,\lambda}(\Gamma)} = C_8 r^\lambda \left(\frac{\|f\|_{L_{1,\lambda}(\Gamma)}}{\beta}\right)^q,$$

where $C_8 = C_7 \cdot C_4^{q-1}$.

Necessity. Let $\mathcal{I}^\alpha$ bounded from $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$. We have

$$\|\mathcal{I}^\alpha f_r\|_{WL_{q,\lambda}} = \sup_{\beta>0} \beta \sup_{r_1>0,\, \tau\in\Gamma} \left(r_1^{-\lambda} \int_{\{\tau\in\Gamma(t,r_1)\ :\ |\mathcal{I}^\alpha f_r(\tau)|>\beta\}} d\nu(\tau)\right)^{1/q}$$

$$= r^{-\alpha} \sup_{\beta>0} \beta r^\alpha \sup_{r_1>0,\, \tau\in\Gamma} \left(\tau^{-\lambda} \int_{\{\tau\in\Gamma(t,r_1)\ :\ |\mathcal{I}^\alpha f(r\tau)|>\beta r^\alpha\}} d\nu(\tau)\right)^{1/q}$$

$$= r^{-\alpha-\frac{1}{q}} \sup_{\beta>0} \beta r^\alpha \sup_{r_1>0,\, \tau\in\Gamma} \left(r^\lambda (r_1 r)^{-\lambda} \int_{\{\tau\in\Gamma(t,rr_1)\ :\ |\mathcal{I}^\alpha f(\tau)|>\beta r^\alpha\}} d\nu(\tau)\right)^{1/q}$$

$$= r^{-\alpha-\frac{1-\lambda}{q}} \|\mathcal{I}^\alpha f\|_{WL_{q,\lambda}}.$$

By the boundedness $\mathcal{I}^\alpha$ from $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$

$$\|\mathcal{I}^\alpha f\|_{WL_{q,\lambda}} \le C_{1,q,\lambda} r^{\alpha+\frac{1-\lambda}{q}-(1-\lambda)} \|f\|_{L_{1,\lambda}(\Gamma)},$$

where $C_{1,q,\lambda}$ depends only on q and λ.

If $1 < \frac{1}{q} + \frac{\alpha}{1-\lambda}$, then for all $f \in L_{1,\lambda}(\Gamma)$, we have $\|\mathcal{I}^\alpha f\|_{WL_{q,\lambda}} = 0$ as $r \to 0$.

Similarly, if $1 > \frac{1}{q} + \frac{\alpha}{1-\lambda}$, then for all $f \in L_{1,\lambda}(\Gamma)$, we obtain $\|\mathcal{I}^\alpha f\|_{WL_{q,\lambda}} = 0$ as $r \to \infty$. Therefore $1 = \frac{1}{q} + \frac{\alpha}{1-\lambda}$.

Thus Theorem 1 is proved. □

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BLENDING TYPE APPROXIMATION BY BÉZIER-SUMMATION-INTEGRAL TYPE OPERATORS

TUNCER ACAR AND ARUN KAJLA

Abstract. In this note we construct the Bézier variant of summation integral type operators based on a non-negative real parameter. We present a direct approximation theorem by means of the first order modulus of smoothness and the rate of convergence for absolutely continuous functions having a derivative equivalent to a function of bounded variation. In the last section, we study the quantitative Voronovskaja type theorem.

1. Introduction

In 1912 Bernstein introduced the most famous algebraic polynomials $B_n(f;x)$ in approximation theory in order to give a constructive proof of Weierstrass's theorem which is given by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and he proved that if $f \in C[0,1]$ then $B_n(f;x)$ converges uniformly to $f(x)$ in $[0,1]$.

The Bernstein operators have been used in many branches of mathematics and computer science. Since their useful structure, Bernstein polynomials and their modifications have been intensively studied. Among the other we refer the readers to (cf. [2, 3, 12, 8, 23, 27]).

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For $f \in C[0,1]$, Chen et al. [10] introduced a generalization of the Bernstein operators based on a non-negative parameter α $(0 \leq \alpha \leq 1)$ as follows:

$$T_n^{(\alpha)}(f;x) = \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1] \tag{1.1}$$

where

$$p_{n,k}^{(\alpha)}(x) = \left[\binom{n-2}{k}(1-\alpha)x + \binom{n-2}{k-2}(1-\alpha)(1-x) + \binom{n}{k}\alpha x(1-x)\right] x^{k-1}(1-x)^{n-k-1}$$

and $n \geq 2$. They proved the rate of convergence, Voronovskaja type asymptotic formula and Shape preserving properties for these operators. For the special case, $\alpha = 1$, these operators reduce the well-known Bernstein operators.
In [19], Kajla and Acar introduced a sequence of summation-integral type operators as follows:

$$D_n^{(\alpha)}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) \int_0^1 p_{n,k}(t) f(t)dt, \tag{1.2}$$

where $f \in L_1[0,1]$ (the space of all Lebesgue integrable functions on $[0,1]$), $p_{n,k}(t) = \binom{n}{k} t^k(1-t)^{n-k}$ and $p_{n,k}^{(\alpha)}(x)$ is defined as above. In [19], Voronoskaja type asymptotic formula, rate of convergence, local and global convergence results were established for these operators (1.2).

The aim of this paper is to introduce Bézier variant of the operators (1.2) and obtain the direct approximation results. Furthermore we study the rate of convergence for an absolutely continuous function f having a derivative f' equivalent with a function of bounded variation on $[0,1]$ and quantitative Voronovskaja type theorem.

A Bézier curve is a parametric curve frequently used in computer graphics and image processing. These are mainly used in approximation, interpolation, curve fitting etc. Bézier-Bernstein type operators were established by many mathematicians. The pioneer works in this direction are due to [3, 5, 9, 13, 24, 26, 28, 29, 30]. In these works, the direct approximation results were obtained and the rate of convergence for functions of bounded variation were established. The order of approximation of the summation-integral type operators for functions with derivatives of bounded variation is estimated in [1, 4, 6, 7, 14, 15, 16, 17, 18, 21, 20, 22, 25].

For $f \in L_1[0,1]$, we define the Bézier variant of the operators $D_n^{(\alpha)}(f;x)$ as

$$\mathcal{S}_{n,\alpha}^{(\rho)}(f;x) = (n+1)\sum_{k=0}^{n} Q_{n,k,\alpha}^{(\rho)}(x) \int_0^1 p_{n,k}(t) f(t)dt, \quad x \in [0,1], \tag{1.3}$$

where $\rho \geq 1,\ Q_{n,k,\alpha}^{(\rho)}(x) = [J_{n,k,\alpha}(x)]^\rho - [J_{n,k+1,\alpha}(x)]^\rho$ and $J_{n,k,\alpha}(x) = \sum_{j=k}^{n} p_{n,j}^{(\alpha)}(x)$, when $k \leq n$ and 0 otherwise.

Alternatively we may rewrite the operators (1.3) as

$$\mathcal{S}_{n,\alpha}^{(\rho)}(f;x) = \int_0^1 \mathcal{M}_{n,\alpha,\rho}(x,t)f(t)dt,\ \ x \in [0,1], \tag{1.4}$$

where

$$\mathcal{M}_{n,\alpha,\rho}(x,t) = (n+1)\sum_{k=0}^{n} Q_{n,k,\alpha}^{(\rho)}(x)p_{n,k}(t).$$

If $\rho = 1$ then the operators $\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)$ reduce to the operators $D_n^{(\alpha)}(f;x)$. Throughout this article, C denotes a positive constant independent of n and x, not necessarily the same at each occurrence.

To express our results we give the following auxiliary results.

Lemma 1. [19] *Let* $e_i(t) = t^i, i = \overline{0,4}$, *then we have*

(1) $D_n^{(\alpha)}(e_0;x) = 1$;

(2) $D_n^{(\alpha)}(e_1;x) = x + \dfrac{1-2x}{(n+2)}$;

(3) $D_n^{(\alpha)}(e_2;x) = x^2 + \dfrac{2x^2(\alpha-3n-4)}{(n+2)(n+3)} + \dfrac{2x(2n-\alpha+1)}{(n+2)(n+3)} + \dfrac{2}{(n+2)(n+3)}$;

(4) $D_n^{(\alpha)}(e_3;x) = x^3 + \dfrac{6x^3(-n(5+2n-\alpha)-2(1+\alpha))}{(n+2)(n+3)(n+4)} + \dfrac{3x^2(n(3n-2\alpha-1)+10(\alpha-1))}{(n+2)(n+3)(n+4)}$
$+ \dfrac{18x(n-\alpha+1)}{(n+2)(n+3)(n+4)} + \dfrac{6}{(n+2)(n+3)(n+4)}$;

(5) $D_n^{(\alpha)}(e_4;x) = x^4 + \dfrac{x^4(-4(n+3)(16+n(3+5n))+12\alpha(n-3)(n-2))}{(n+2)(n+3)(n+4)(n+5)}$
$+ \dfrac{4x^3(n-2)(n(4n-3\alpha-1)+33(\alpha-1))}{(n+2)(n+3)(n+4)(n+5)} + \dfrac{24x^2\left(n+3n^2+14(\alpha-1)-4n\alpha\right)}{(n+2)(n+3)(n+4)(n+5)}$
$+ \dfrac{48x(2n-3\alpha+3)}{(n+2)(n+3)(n+4)(n+5)} + \dfrac{24}{(n+2)(n+3)(n+4)(n+5)}$.

Hence we get

$$D_n^{(\alpha)}((t-x);x) = \frac{1-2x}{n+2} < \frac{\lambda_1(1-x)}{n+1},\ \ \forall x \in [0,1] \text{ and } \forall n \in \mathbb{N} \tag{1.5}$$

with $\lambda_1 \geq 2$ and

$$D_n^{(\alpha)}((t-x)^2;x) = \frac{2x(1-x)(n-\alpha-2)}{(n+2)(n+3)} + \frac{2}{(n+2)(n+3)}.$$

From [19], we have

$$D_n^{(\alpha)}((t-x)^2;x) < \frac{2}{(n+2)}\gamma_n^2(x), \quad \forall x \in [0,1] \text{ and } \forall n \in \mathbb{N}$$

where $\gamma_n^2(x) = \varphi^2(x) + \frac{1}{(n+2)}$ and $\varphi^2(x) = x(1-x)$. Then we can write

$$\begin{aligned} D_n^{(\alpha)}((t-x)^2;x) &< \frac{\lambda_2 \gamma_n^2(x)}{n+2},\ \lambda_2 \geq 2. &(1.6)\\ D_n^{(\alpha)}((t-x)^4;x) &= \frac{12x^3(x-2)\,(n(n-2\alpha-19)+46\alpha-36)}{(n+2)(n+3)(n+4)(n+5)} + \frac{12x^2\,(n(n-2\alpha-25)+58\alpha-38)}{(n+2)(n+3)(n+4)(n+5)}\\ &\quad + \frac{24x(3n-6\alpha+1)}{(n+2)(n+3)(n+4)(n+5)} + \frac{24}{(n+2)(n+3)(n+4)(n+5)} &(1.7)\end{aligned}$$

Remark 1. *We have*

$$\begin{aligned} \mathcal{S}_{n,\alpha}^{(\rho)}(e_0;x) &= \sum_{k=0}^{n} Q_{n,k,\alpha}^{(\rho)}(x) = [J_{n,0,\alpha}(x)]^\rho \\ &= \left[\sum_{j=0}^{n} p_{n,j}^{(\alpha)}(x)\right]^\rho = 1,\ \textit{since}\ \sum_{j=0}^{n} p_{n,j}^{(\alpha)}(x) = 1. \end{aligned}$$

Lemma 2. [19] *Let $f \in C[0,1]$. Then for $x \in [0,1]$ we have*

$$\| D_n^{(\alpha)}(f) \| \leq \| f \| .$$

2. Direct Estimates

To describe our results, we recall the definitions of the first order modulus of smoothness and the K-functional [11]. Let $\varphi(x) = \sqrt{x(1-x)}, f \in C[0,1]$. The first order modulus of smoothness is given by

$$\omega_\varphi(f;t) = \sup_{0<h\leq t}\left\{ \left| f\left(x + \frac{h\varphi(x)}{2}\right) - f\left(x - \frac{h\varphi(x)}{2}\right)\right|, x \pm \frac{h\varphi(x)}{2} \in [0,1]\right\},$$

and the appropriate Peetre's K-functional is defined by

$$\overline{K}_\varphi(f;t) = \inf_{g\in W_\varphi}\{||f-g|| + t||\varphi g'|| + t^2||g'||\}\ (t>0),$$

where $W_\varphi = \{g : g \in AC_{loc}, ||\varphi g^{'}|| < \infty, ||g^{'}|| < \infty\}$ and $||.||$ is the uniform norm on $C[0,1]$. It is well known that ([11], Thm. 3.1.2) $\overline{K}_\varphi(f;t) \sim \omega_\varphi(f;t)$ which means that there exists a constant $M > 0$ such that

$$M^{-1}\omega_\varphi(f;t) \leq \overline{K}_\varphi(f;t) \leq M\omega_\varphi(f;t). \tag{2.1}$$

Lemma 3. *Let $f \in C[0,1]$. Then, for $x \in [0,1]$, we have*

$$\| \mathcal{S}_{n,\alpha}^{(\rho)}(f) \| \leq \rho \| f \| .$$

Proof. Applying the inequality $| a^\rho - b^\rho | \leq \rho \mid a - b \mid$ with $0 \leq a, b \leq 1, \rho \geq 1$ and from definition of $Q^{(\rho)}_{n,k,\alpha}(x)$, we may write

$$\begin{aligned} 0 &< [J_{n,k,\alpha}(x)]^\rho - [J_{n,k+1,\alpha}(x)]^\rho \leq \rho \,(J_{n,k,\alpha}(x) - J_{n,k+1,\alpha}(x)) \\ &= \rho \; p^{(\alpha)}_{n,k}(x). \end{aligned}$$

Hence from the definition $\mathcal{S}^{(\rho)}_{n,\alpha}(f;x)$ and Lemma 2, we obtain

$$\| \mathcal{S}^{(\rho)}_{n,\alpha}(f) \| \leq \rho \, \| D^{(\alpha)}_n(f) \| \leq \rho \, \| f \| .$$

□

Now we study a direct approximation theorem for the operators $\mathcal{S}^{(\rho)}_{n,\alpha}$.

Theorem 1. *Suppose that f be in $C[0,1]$ and $\varphi(x) = \sqrt{x(1-x)}$ then for every $x \in [0,1)$, we have*

$$| \mathcal{S}^{(\rho)}_{n,\alpha}(f;x) - f(x) | < C\omega_\varphi\left(f; \frac{1}{\sqrt{n+2}}\right), \tag{2.2}$$

where C is a constant independent of n and x.

Proof. By the definition of $\overline{K}_\varphi(f;t)$, for fixed n, x, we can choose $g = g_{n,x} \in W_\varphi$ such that

$$||f-g|| + \frac{1}{\sqrt{n+2}}||\varphi g'|| + \frac{1}{n+2}||g'|| \leq 2\overline{K}_\varphi\left(f; \frac{1}{\sqrt{n+2}}\right). \tag{2.3}$$

Using Remark 1, we can write

$$\begin{aligned} | \mathcal{S}^{(\rho)}_{n,\alpha}(f;x) - f(x) | &\leq | \mathcal{S}^{(\rho)}_{n,\alpha}(f-g;x) | + |f-g| + | \mathcal{S}^{(\rho)}_{n,\alpha}(g;x) - g(x) | \qquad (2.4) \\ &\leq 2||f-g|| + | \mathcal{S}^{(\rho)}_{n,\alpha}(g;x) - g(x) | . \end{aligned}$$

We only need to compute the second term in the above equation. We will have to split the estimate into two domains, i.e. $x \in F^c_n = [0, 1/n]$ and $x \in F_n = (1/n, 1)$. Using the representation $g(t) = g(x) + \int_x^t g'(u)du$, we get

$$\left|\mathcal{S}^{(\rho)}_{n,\alpha}(g;x) - g(x)\right| = \left|\mathcal{S}^{(\rho)}_{n,\alpha}\left(\int_x^t g'(u)du; x\right)\right|. \tag{2.5}$$

If $x \in F_n = (1/n, 1)$ then $\gamma_n(x) \sim \varphi(x)$. We have

$$\left|\int_x^t g'(u)du\right| \leq ||\varphi g'|| \left|\int_x^t \frac{1}{\varphi(u)}du\right|. \tag{2.6}$$

For any $x, t \in (0,1)$, we find that

$$\begin{aligned}
\left|\int_x^t \frac{1}{\varphi(u)}du\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right| \\
&\leq \left|\int_x^t \left(\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{1-u}}\right)du\right| \\
&\leq 2\bigg(\ |\sqrt{t}-\sqrt{x}\ | + |\ \sqrt{1-t}-\sqrt{1-x}\ |\ \bigg) \\
&= 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}}+\frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right) \\
&< 2|t-x|\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right) \\
&\leq \frac{2\sqrt{2}\ |t-x|}{\varphi(x)}. \qquad (2.7)
\end{aligned}$$

Combining (2.5)-(2.7) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|\mathcal{S}_{n,\alpha}^{(\rho)}(g;x)-g(x)| &< 2\sqrt{2}||\varphi g'^{-1}(x)\mathcal{S}_{n,\alpha}^{(\rho)}(|t-x|;x) \\
&\leq 2\sqrt{2}||\varphi g'^{-1}(x)\left(\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\right)^{1/2} \\
&\leq 2\sqrt{2}||\varphi g'^{-1}(x)\left(\rho\ D_n^{(\alpha)}((t-x)^2;x)\right)^{1/2}.
\end{aligned}$$

From (1.6), we get

$$|\mathcal{S}_{n,\alpha}^{(\rho)}(g;x)-g(x)| < \frac{C||\varphi g'||}{\sqrt{n+2}}. \qquad (2.8)$$

For $x \in F_n^c = [0, 1/n]$, $\gamma_n(x) \sim \dfrac{1}{\sqrt{n+2}}$ and

$$\left|\int_x^t g'(u)du\right| \leq ||g'||\ |t-x|.$$

Therefore, using Cauchy-Schwarz inequality we have

$$\begin{aligned}
|\mathcal{S}_{n,\alpha}^{(\rho)}(g;x)-g(x)| &\leq ||g'||\mathcal{S}_{n,\alpha}^{(\rho)}(|t-x|;x) \\
&\leq C||g'||\frac{\gamma_n(x)}{\sqrt{n+2}} < \frac{C}{n+2}||g'||. \qquad (2.9)
\end{aligned}$$

From (2.8) and (2.9), we have

$$|\mathcal{S}_{n,\alpha}^{(\rho)}(g;x)-g(x)| < C\left(\frac{||\varphi g'||}{\sqrt{n+2}}+\frac{1}{n+2}||g'||\right). \qquad (2.10)$$

Using $\overline{K}_\varphi(f;t) \sim \omega_\varphi(f;t)$ and (2.3), (2.4), (2.10), we get the desired relation (2.2). This completes the proof of the theorem. □

3. Rate of Convergence

In this section we would like to obtain the rate of convergence of the operators $\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)$ for an absolutely continuous function f having a derivative f' equivalent to a function of bounded variation on $[0,1]$.

Throughout this section $DBV[0,1]$ will denote the class of all absolutely continuous functions f defined on $[0,1]$ and having on $(0,1)$, a derivative $f^{'}$ equivalent with a function of bounded variation on $[0,1]$. We notice that the functions $f\in DBV[0,1]$ possess a representation

$$f(x)=\int_0^x g(t)dt+f(0),$$

where $g\in BV[0,1]$, i.e., g is a function of bounded variation on $[0,1]$.

Lemma 4. *Let $x\in(0,1]$, then for $\rho\geq 1,\lambda_2\geq 2$ and sufficiently large n, we have*

$$(1)\ \vartheta_{n,\alpha,\rho}(x,y)=\int_0^y \mathcal{M}_{n,\alpha,\rho}(x,t)dt<\frac{\rho\lambda_2}{(n+2)}\frac{\gamma_n^2(x)}{(x-y)^2}\ ,\ 0\leq y<x,$$

$$(2)\ 1-\vartheta_{n,\alpha,\rho}(x,z)=\int_z^1 \mathcal{M}_{n,\alpha,\rho}(x,t)dt<\frac{\rho\lambda_2}{(n+2)}\frac{\gamma_n^2(x)}{(z-x)^2},\ x<z<1.$$

Proof. (i) From Lemmas 1 and 2, we get

$$\begin{aligned}\vartheta_{n,\alpha,\rho}(x,y) &= \int_0^y \mathcal{M}_{n,\alpha,\rho}(x,t)dt\leq\int_0^y\left(\frac{x-t}{x-y}\right)^2\mathcal{M}_{n,\alpha,\rho}(x,t)dt\\ &= \mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\ (x-y)^{-2}\leq\rho D_n^{(\alpha)}((t-x)^2;x)\ (x-y)^{-2}\\ &< \frac{\rho\lambda_2}{(n+2)}\frac{\gamma_n^2(x)}{(x-y)^2}.\end{aligned}$$

The proof of (ii) is similar to the proof of (i). Hence it is omitted. □

Theorem 2. *Let $f\in DBV(0,1),\rho\geq 1$ and let $\upsilon_a^b(f_x')$ be the total variation of f_x' on $[a,b]\subset[0,1]$. Then for every $x\in(0,1)$ and for sufficiently large n, we have*

$$\begin{aligned}\left|\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right| &< \frac{1}{\rho+1}\left|f'(x+)+\rho f'(x-)\right|\sqrt{\frac{\rho\lambda_2}{(n+2)}}\ \gamma_n(x)\\ &\quad+\sqrt{\frac{\rho\lambda_2}{(n+2)}}\ \gamma_n(x)\frac{\rho}{\rho+1}\left|f'(x+)-f'(x-)\right|\\ &\quad+\rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}x^{-1}\sum_{k=1}^{[\sqrt{n}\ ]}\upsilon_{x-(x/k)}^x(f_x')+\frac{x}{\sqrt{n}}\upsilon_{x-(x/\sqrt{n})}^x(f_x')\\ &\quad+\rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}(1-x)^{-1}\sum_{k=1}^{[\sqrt{n}\ ]}\upsilon_x^{x+((1-x)/k)}(f_x')+\frac{1-x}{\sqrt{n}}\upsilon_x^{x+((1-x)/\sqrt{n})}(f_x'),\end{aligned}$$

where $\lambda_2 \geq 2$ and the auxiliary function and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+) & x < t \leq 1. \end{cases}$$

Proof. Using the fact that $\int_0^1 \mathcal{M}_{n,\alpha,\rho}(x,t)dt = \mathcal{S}^{(\rho)}_{n,\alpha}(e_0;x) = 1$, we have

$$\begin{aligned} \mathcal{S}^{(\rho)}_{n,\alpha}(f;x) - f(x) &= \int_0^1 [f(t) - f(x)]\mathcal{M}_{n,\alpha,\rho}(x,t)dt \\ &= \int_0^1 \left(\int_x^t f'(u)du \right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt. \qquad (3.1) \end{aligned}$$

From definition of the function f'_x, for any $f \in DBV(0,1)$, we can write

$$\begin{aligned} f'(t) &= \frac{1}{\rho+1}\Big(f'(x+) + \rho f'(x-) \Big) + f'_x(t) \\ &\quad + \frac{1}{2}\Big(f'(x+) - f'(x-) \Big)\Big(sgn(t-x) + \frac{\rho-1}{\rho+1} \Big) \\ &\quad + \delta_x(t)\Big(f'(x) - \frac{1}{2}\Big(f'(x+) + f'(x-) \Big)\Big), \qquad (3.2) \end{aligned}$$

where

$$\delta_x(t) = \begin{cases} 1 \ , & x = t \\ 0 \ , & x \neq t. \end{cases}$$

It is clear that

$$\int_0^1 \mathcal{M}_{n,\alpha,\rho}(x,t) \int_x^t \left[f'(x) - \frac{1}{2}\Big(f'(x+) + f'(x-) \Big) \right] \delta_x(t)dudt = 0.$$

By (1.4) and simple computations, we have

$$\begin{aligned} P_1 &= \int_0^1 \left(\int_x^t \frac{1}{\rho+1}\Big(f'(x+) + \rho f'(x-) \Big) du \right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt \\ &= \frac{1}{\rho+1}\Big| f'(x+) + \rho f'(x-) \Big| \int_0^1 |t-x| \mathcal{M}_{n,\alpha,\rho}(x,t)dt \\ &\leq \frac{1}{\rho+1}\Big| f'(x+) + \rho f'(x-) \Big| \Big(\mathcal{S}^{(\rho)}_{n,\alpha}((t-x)^2;x) \Big)^{1/2} \end{aligned}$$

and

$$\begin{aligned}
P_2 &= \int_0^1 \left(\int_x^t \frac{1}{2}\Big(f'(x+) - f'(x-)\Big)\left(sgn(u-x) + \frac{\rho-1}{\rho+1}\right) du\right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt \\
&= \frac{1}{2}\left(f'(x+) - f'(x-)\right)\Bigg[-\int_0^x \left(\int_t^x \left(sgn(u-x) + \frac{\rho-1}{\rho+1}\right) du\right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt \\
&\quad + \int_x^1 \left(\int_x^t \left(sgn(u-x) + \frac{\rho-1}{\rho+1}\right) du\right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt \Bigg] \\
&\leq \frac{\rho}{\rho+1}\left(f'(x+) - f'(x-)\right) \int_0^1 |t-x|\, \mathcal{M}_{n,\alpha,\rho}(x,t)dt \\
&= \frac{\rho}{\rho+1}\left(f'(x+) - f'(x-)\right) \mathcal{S}_{n,\alpha}^{(\rho)}\left(|t-x|\,; x\right) \\
&\leq \frac{\rho}{\rho+1}\left(f'(x+) - f'(x-)\right) \left(\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2; x)\right)^{1/2}.
\end{aligned}$$

By using (1.6) and considering (3.1), (3.2) we obtain the following estimate

$$\begin{aligned}
\left|\mathcal{S}_{n,\alpha}^{(\rho)}(f;x) - f(x)\right| &< |E_{n,\alpha,\rho}(f'_x, x) + F_{n,\alpha,\rho}(f'_x, x)| \\
&\quad + \frac{1}{\rho+1}\,|f'(x+) + \rho f'(x-)| \sqrt{\frac{\rho\lambda_2}{(n+2)}}\ \gamma_n(x) \\
&\quad + \frac{\rho}{\rho+1}\,|f'(x+) - f'(x-)| \sqrt{\frac{\rho\lambda_2}{(n+2)}}\ \gamma_n(x), \qquad (3.3)
\end{aligned}$$

where

$$E_{n,\alpha,\rho}(f'_x, x) = \int_0^x \left(\int_x^t f'_x(u)du\right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt$$

and

$$F_{n,\alpha,\rho}(f'_x, x) = \int_x^1 \left(\int_x^t f'_x(u)du\right) \mathcal{M}_{n,\alpha,\rho}(x,t)dt.$$

To complete the proof, it is sufficient to estimate the terms $E_{n,\alpha,\rho}(f'_x, x), F_{n,\alpha,\rho}(f'_x, x)$. Since $\int_a^b d_t\vartheta_{n,\alpha,\rho}(x,t) \leq 1$ for all $[a,b] \subseteq [0,1]$, using integration by parts and applying Lemma 4 with $y = x - (x/\sqrt{n})$, we have

$$\begin{aligned}
|E_{n,\alpha,\rho}(f'_x, x)| &= \left| \int_0^x \left(\int_x^t f'_x(u)du\right) d_t\vartheta_{n,\alpha,\rho}(x,t)\right| \\
&= \left| \int_0^x \vartheta_{n,\alpha,\rho}(x,t) f'_x(t)dt\right| \\
&\leq \left(\int_0^y + \int_y^x \right) |f'_x(t)|\, |\vartheta_{n,\alpha,\rho}(x,t)|\, dt \\
&< \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)} \int_0^y \upsilon_t^x(f'_x)(x-t)^{-2}dt + \int_y^x \upsilon_t^x(f'_x)dt \\
&\leq \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)} \int_0^y \upsilon_t^x(f'_x)(x-t)^{-2}dt + \frac{x}{\sqrt{n}}\upsilon_{x-(x/\sqrt{n})}^x(f'_x).
\end{aligned}$$

By the substitution of $u = x/(x-t)$, we get

$$\begin{aligned}
\rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}\int_0^{x-(x/\sqrt{n})}(x-t)^{-2}\upsilon_t^x(f_x')dt &= \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}x^{-1}\int_1^{\sqrt{n}}\upsilon_{x-(x/u)}^x(f_x')du \\
&\leq \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}x^{-1}\sum_{k=1}^{[\sqrt{n}\,]}\int_k^{k+1}\upsilon_{x-(x/u)}^x(f_x')du \\
&< \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}x^{-1}\sum_{k=1}^{[\sqrt{n}\,]}\upsilon_{x-(x/k)}^x(f_x').
\end{aligned}$$

Hence we reach the following result

$$|E_{n,\alpha,\rho}(f_x',x)| < \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}x^{-1}\sum_{k=1}^{[\sqrt{n}\,]}\upsilon_{x-(x/k)}^x(f_x') + \frac{x}{\sqrt{n}}\upsilon_{x-(x/\sqrt{n})}^x(f_x').$$

Using integration by parts and applying Lemma 4 with $z = x + ((1-x)/\sqrt{n})$, we have

$$\begin{aligned}
|F_{n,\alpha,\rho}(f_x',x)| &= \left|\int_x^1\left(\int_x^t f_x'(u)du\right)\mathcal{M}_{n,\alpha,\rho}(x,t)dt\right| \\
&= \left|\int_x^z\left(\int_x^t f_x'(u)du\right)d_t(1-\vartheta_{n,\alpha,\rho}(x,t)) + \int_z^1\left(\int_x^t f_x'(u)du\right)d_t(1-\vartheta_{n,\alpha,\rho}(x,t))\right| \\
&= \left|\left[\left(\int_x^t f_x'(u)du\right)(1-\vartheta_{n,\alpha,\rho}(x,t))\right]_x^z - \int_x^z f_x'(t)(1-\vartheta_{n,\alpha,\rho}(x,t))dt\right. \\
&\quad \left. + \int_z^1\left(\int_x^t f_x'(u)du\right)d_t(1-\vartheta_{n,\alpha,\rho}(x,t))\right| \\
&= \left|\int_x^z f_x'(u)du(1-\vartheta_{n,\alpha,\rho}(x,z)) - \int_x^z f_x'(t)(1-\vartheta_{n,\alpha,\rho}(x,t))dt\right. \\
&\quad \left. + \left[\int_x^t f_x'(u)du(1-\vartheta_{n,\alpha,\rho}(x,t))\right]_z^1 - \int_z^1 f_x'(t)(1-\vartheta_{n,\alpha,\rho}(x,t))dt\right| \\
&= \left|\int_x^z f_x'(t)(1-\vartheta_{n,\alpha,\rho}(x,t))dt + \int_z^1 f_x'(t)(1-\vartheta_{n,\alpha,\rho}(x,t))dt\right| \\
&< \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}\int_z^1\upsilon_x^t(f_x')(t-x)^{-2}dt + \int_x^z\upsilon_x^t(f_x')dt \\
&\leq \rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}\int_{x+((1-x)/\sqrt{n})}^1\upsilon_x^t(f_x')(t-x)^{-2}dt + \frac{(1-x)}{\sqrt{n}}\upsilon_x^{x+((1-x)/\sqrt{n})}(f_x').
\end{aligned}$$

By the substitution of $u = (1-x)/(t-x)$, we get

$$\begin{aligned}
\rho\frac{\lambda_2\gamma_n^2(x)}{(n+2)}\int_{x+((1-x)/\sqrt{n})}^1\upsilon_x^t(f_x')(t-x)^{-2}dt &= \rho\frac{\lambda_2\gamma_n^2(x)}{(1-x)(n+2)}\int_1^{\sqrt{n}}\upsilon_x^{x+((1-x)/u)}(f_x')du \\
&< \rho\frac{\lambda_2\gamma_n^2(x)}{(1-x)(n+2)}\sum_{k=1}^{[\sqrt{n}\,]}\int_k^{k+1}\upsilon_x^{x+((1-x)/u)}(f_x')du \\
&\leq \rho\frac{\lambda_2\gamma_n^2(x)}{(1-x)(n+2)}\sum_{k=1}^{[\sqrt{n}\,]}\upsilon_x^{x+((1-x)/k)}(f_x').
\end{aligned}$$

Thus, we get

$$|F_{n,\alpha,\rho}(f'_x,x)| < \rho\frac{\lambda_2\gamma_n^2(x)}{(1-x)(n+2)}\sum_{k=1}^{[\sqrt{n}\,]}\upsilon_x^{x+((1-x)/k)}(f'_x) + \frac{1-x}{\sqrt{n}}\upsilon_x^{x+((1-x)/\sqrt{n})}(f'_x). \tag{3.4}$$

Collecting the estimates (3.3)-(3.4), we get the required result. This completes the proof of theorem. □

4. QUANTITATIVE VORONOVSKAJA-TYPE THEOREM

Now we are going to study a quantitative Voronovskaja-type result for the operators $\mathcal{S}_{n,\alpha}^{(\rho)}$. This result is given using the first order Ditzian-Totik modulus of smoothness.

Theorem 3. *Let $f \in C^2[0,1]$. Then there hold*

$$\left|\sqrt{n}\left(\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right)\right| \leq \sqrt{2\rho\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{\sqrt{n}}\varphi^2(x) + \frac{C}{\sqrt{n}}\omega_\varphi(x)\left(f'';\frac{2\sqrt{3}}{\sqrt{n}}\varphi(x)\right)+\circ(n^{-1});$$

$$\left|\sqrt{n}\left(\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right)\right| \leq \sqrt{2\rho\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{\sqrt{n}}\varphi^2(x) + \frac{C}{\sqrt{n}}\omega_\varphi(x)\varphi(x)\left(f'';\frac{2\sqrt{3}}{\sqrt{n}}\right)+\circ(n^{-1})$$

Proof. Let $f \in C^2[0,1]$ and $x,t \in [0,1]$. Then Taylor's expansion, we may write

$$f(t)-f(x)=(t-x)f'(x)+\int_x^t (t-u)f''(u)du.$$

Thus,

$$f(t)-f(x)=f'(x)(t-x)-\frac{1}{2}(t-x)^2f''(x)+\int_x^t(t-u)f''(u)du-\int_x^t(t-u)f''(u)du.$$

Operating $\mathcal{S}_{n,\alpha}^{(\rho)}(\cdot;x)$ to both sides of the above relation, we get

$$|\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)| = |f'(x)|\mathcal{S}_{n,\alpha}^{(\rho)}(|t-x|;x)+\frac{1}{2}|f''(x)|\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x) + \mathcal{S}_{n,\alpha}^{(\rho)}\left(\left|\int_x^t|t-u|\;|f''(u)-f''(x)|du\right|;x\right). \tag{4.1}$$

Therefore, $g \in W_\varphi$ we have

$$\left|\int_x^t |t-u|\ |f''(u)-f''(x)|du\right| \leq ||f''-g||(t-x)^2+2||\varphi g'||\varphi^{-1}(x)|t-x|^3.$$

Thus, in view of (4.1), (**??**), (1.7) and using Cauchy-Schwarz inequality, we may write

$$\begin{aligned}
\left|\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right| &\leq |f'(x)|\mathcal{S}_{n,\alpha}^{(\rho)}(|t-x|;x)+\frac{1}{2}|f''(x)|\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\\
&\quad +||f''-g||\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)+2||\varphi g'||\varphi^{-1}(x)\mathcal{S}_{n,\alpha}^{(\rho)}(|t-x|^3;x)\\
&\leq ||f'||\left(\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\right)^{1/2}+\frac{1}{2}||f''||\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)+||f''-g||\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\\
&\quad +2||\varphi g'||\varphi^{-1}(x)\left(\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^2;x)\right)^{1/2}\left(\mathcal{S}_{n,\alpha}^{(\rho)}((t-x)^4;x)\right)^{1/2}\\
&= \sqrt{\frac{2\rho}{(n+2)}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{n+2}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}\\
&\quad +\frac{2\rho}{n+2}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}||f'''||\varphi^{-1}(x)\left\{\varphi^2(x)+\frac{1}{n+2}\right\}\\
&\quad \times\Bigg(\rho\Bigg(\frac{12x^3(x-2)\left(n(n-2\alpha-19)+46\alpha-36\right)}{(n+2)(n+3)(n+4)(n+5)}\\
&\quad +\frac{12x^2\left(n(n-2\alpha-25)+58\alpha-38\right)}{(n+2)(n+3)(n+4)(n+5)}+\frac{24x(3n-6\alpha+1)}{(n+2)(n+3)(n+4)(n+5)}\\
&\quad +\frac{24}{(n+2)(n+3)(n+4)(n+5)}\Bigg)\Bigg)^{1/2}\\
&\leq \sqrt{\frac{2\rho}{(n+2)}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{n+2}\varphi^2(x)\\
&\quad +\frac{2\rho}{n+2}\left\{\varphi^2(x)||f'''||\varphi(x)\frac{2\sqrt{3}}{\sqrt{n}}\right\}+\circ(n^{-3/2}).
\end{aligned}$$

Since $\varphi^2(x)\leq\varphi(x)\leq 1, x\in[0,1]$, we have

$$\begin{aligned}
\left|\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right| &\leq \sqrt{\frac{2\rho}{(n+2)}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{n+2}\varphi^2(x)\\
&\quad +\frac{2\rho}{n+2}\left\{||f'''||\varphi(x)\frac{2\sqrt{3}}{\sqrt{n}}\right\}+\circ(n^{-3/2}). \qquad (4.2)\\
\left|\mathcal{S}_{n,\alpha}^{(\rho)}(f;x)-f(x)\right| &\leq \sqrt{\frac{2\rho}{(n+2)}\left\{\varphi^2(x)+\frac{1}{n+2}\right\}}||f''||+||f''||\frac{\rho}{n+2}\varphi^2(x)\\
&\quad +\frac{2\rho}{n+2}\varphi(x)\left\{||f'''||\frac{2\sqrt{3}}{\sqrt{n}}\right\}+\circ(n^{-3/2}). \qquad (4.3)
\end{aligned}$$

By taking the infimum on the right hand side of the above relations over $g \in W_\phi$, we get

$$\begin{aligned}
\left|\sqrt{n}\left(\mathcal{S}_{n,\alpha}^{(\rho)}(f;x) - f(x)\right)\right| &\leq \sqrt{2\rho\left\{\varphi^2(x) + \frac{1}{n+2}\right\}}||f''|| + ||f''||\frac{\rho}{\sqrt{n}}\varphi^2(x) \\
&\quad + \frac{C}{\sqrt{n}}\overline{K_\varphi}\left(f''; \frac{2\sqrt{3}}{\sqrt{n}}\varphi(x)\right) + \circ(n^{-1}); \\
\left|\sqrt{n}\left(\mathcal{S}_{n,\alpha}^{(\rho)}(f;x) - f(x)\right)\right| &\leq \sqrt{2\rho\left\{\varphi^2(x) + \frac{1}{n+2}\right\}}||f''|| + ||f''||\frac{\rho}{\sqrt{n}}\varphi^2(x) \\
&\quad + \frac{C}{\sqrt{n}}\varphi(x)\overline{K_\varphi}\left(f''; \frac{2\sqrt{3}}{\sqrt{n}}\right) + \circ(n^{-1}).
\end{aligned}$$

Applying $\overline{K_\varphi}(f,t) \sim \omega_\varphi(f,t)$, the theorem is proved. □

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THE RELATION BETWEEN SOFT TOPOLOGICAL SPACE AND SOFT DITOPOLOGICAL SPACE

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Abstract. Conditions related to bounds on the relations between soft spaces appear to be rare in the literature. In this paper, I study the notion of soft ditopology relates to the soft topology. Firstly, the soft ditopology via soft set theory is developed by defining soft ditopological subspace. Secondly, properties concerning to soft interior and soft closure are presented in soft ditopological subspace. In conclusion, soft subspaces of soft topology and soft ditopology being coincident have been proved, whence it is readily inferred that soft ditopological subspace can be obtained from soft topological subspace.

1. Introduction

In 1999, Molodtsov [14] proposed a new approach, viz. soft set theory for modeling vagueness and uncertainties inherent in many related concepts with the theory and the application of soft sets. After this invention, in 2002 and 2003, very readable account of this theory has been given by Maji *et al.* [10, 11] on some mathematical aspects of soft sets and fuzzy soft sets. However, over the last fifteen years, there have been many examples are defined by [1], [4], [12] and [19] in the literature related to soft topology. By using this operations, the theory of soft topological space defined by Shabir and Naz [18] over an initial universe. There are numerous results in the literature relating soft topology. In the course of analyzing the theory of soft topology I learned that the authors [9] and [13] have simultaneously obtained results similar to each other in certain respects. One of the finest work in them, Aygünoglu and Aygün [2] introduced the soft product topology and defined the soft compactness to investigate the behavior of topological structures in soft set theoretic form. In view of this and also considering the importance of topological structure in developing soft set theory, I have introduced in this paper a notion of soft ditopological subspaces. In this connection, it is worth mentioning that some

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significant works have been done on soft ditopological spaces (SDT - Spaces) by Dizman [6] and Senel [16].

They defined basic notions and concepts of soft ditopological spaces such as soft open and closed sets, soft interior, soft closure, soft basis, soft complement and established several properties of these soft notions. Rather than discuss these works in full generality, let us look at a particular situation of this kind: In the work of Dizman [6], the concept of soft ditopological space is introduced with two structures which one related to the property of openness in the space and the other one relayed on the property of closeness in the space. This is a clear contradiction of the fact that if we know the soft topology on a soft set, we can easily get soft open and soft closed sets by complement operation. In that way, in a soft ditopological space that includes same characteristic properties composed with openness and closeness of each other.

In order to make a more comprehensive research, I have decided to develop the theory; considering that rather than studying with the triple structure that is built by the spaces made through one another, it is more beneficial to study a triple structure that contains a different and a new space.

The notion of soft ditopology by Senel [16] is more general than that by Dizman [6]. The concept of soft ditopological (SDT) space on a soft set in [16] with two structures on it is being introduced - a soft topology and a soft subspace topology. The first one is used to describe soft openness properties of a soft topological space while the second one deals with its sub - soft openness properties. This structure enables to study with all soft open sets that can be obtained on a soft set. Therefore, I continue investigating the work of Senel [16] and follow this theory's notations and mathematical formalism.

In this paper, the detailed analysis of SDT - space is carried out in Section 2. In this section, I introduce a new concept called soft ditopological subspace in soft ditopological space by giving two different definitions that are not a consequence of each other. Although, these definitions run along different lines, I prove that only one and the same soft ditopological subspace can be established on the same soft set by giving examples.

I also observe relations of soft ditopological space and soft ditopological subspace in different cases with soft open and soft closed sets. It shows how soft sets in soft ditopological space can preserve their properties in soft ditopological subspaces. In this context, I serve a bridge among soft ditopological space theory and soft ditopological subspace theory.

In the last section, I analyze the relationship between soft topology and soft ditopology. The two characteristics, soft subspace and the soft subspace of a soft subspace, are connected, but the relationship is quite a complex one. Although individually these systems can still be quite complicated, a possibly more tractable task is to describe their possible joint distributions. The aim of this article is to study the relationship between the soft topology and the extent to which soft topology to be soft ditopology. In this paper I wish to renew an interest in the systematic study of the relationships between topological spaces with respect to soft set theory.

2. Preliminaries

Following the works of Molodtsov [14], Maji et al. [10, 11] and Aktas and Cagman [3] some definitions and preliminary results are presented in this section. The following basic properties of soft sets have been given in [2], [5], [14], [15], [16], [17]. Unless otherwise stated, throughout this paper, U refers to an initial universe, E is a set of parameters and $P(U)$ is the power set of U.

Definition 2.1. [5, 14] A soft set f on the universe U is a set defined by

$$f : E \to \mathcal{P}(U) \text{ such that } f(e) = \emptyset \text{ if } e \in E \setminus A \text{ then, } f = f_A$$

Here f is also called an approximate function. A soft set f on the universe U is a set defined by

$$f = \Big\{ \big(e, f(e)\big) : e \in E \Big\}$$

We will identify any soft set f with the function $f(e)$ and we use that concept as interchangeable. Soft sets are denoted by the letters f, g, h, ... and the corresponding functions by $f(e)$, $g(e)$, $h(e)$, ...

Throughout this paper, the set of all soft sets over U will be denoted by $\mathbb{S}$. From now on, for all the undefined concepts about soft sets, we refer to: [5].

Definition 2.2. [5] Let $f \in \mathbb{S}$. Then,
If $f(e) = \emptyset$ for all $e \in E$, then f is called an empty set, denoted by Φ.
If $f(e) = U$ for all $e \in E$, then f is called universal soft set, denoted by $\tilde{E}$.

Definition 2.3. [5] Let $f, g \in \mathbb{S}$. Then,
f is a soft subset of g, denoted by $f \widetilde{\subseteq} g$, if $f(e) \subseteq g(e)$ for all $e \in E$.
f and g are soft equal, denoted by $f = g$, if and only if $f(e) = g(e)$ for all $e \in E$.

Definition 2.4. [5] Let $f, g \in \mathbb{S}$. Then, the intersection of f and g, denoted $f \widetilde{\cap} g$, is defined by

$$(f \widetilde{\cap} g)(e) = f(e) \cap g(e) \text{ for all } e \in E$$

and the union of f and g, denoted $f \widetilde{\cup} g$, is defined by

$$(f \widetilde{\cup} g)(e) = f(e) \cup g(e) \text{ for all } e \in E$$

Definition 2.5. [5] Let $f \in \mathbb{S}$. Then, the soft complement of f, denoted $f^{\widetilde{c}}$, is defined by

$$f^{\widetilde{c}}(e) = U \setminus f(e), \text{ for all } e \in E$$

Definition 2.6. [5] Let $f \in \mathbb{S}$. The power soft set of f is defined by

$$\widetilde{\mathcal{P}}(f) = \{f_i \tilde{\subseteq} f : f_i \in \mathbb{S}, i \in I\}$$

and its cardinality is defined by

$$|\widetilde{\mathcal{P}}(f)| = 2^{\sum_{e \in E} |f(e)|}$$

where $|f(e)|$ is the cardinality of $f(e)$.

Example 2.7. Let $U = \{u_1, u_2, u_3\}$ and $E = \{e_1, e_2\}$. $f \in \mathbb{S}$ and

$$f = \big\{(e_1, \{u_1, u_2\}), (e_2, \{u_2, u_3\})\big\}$$

Then,

$$\begin{array}{rcl} f_1 &=& \{(e_1, \{u_1\})\}, \\ f_2 &=& \{(e_1, \{u_2\})\}, \\ f_3 &=& \{(e_1, \{u_1, u_2\})\}, \\ f_4 &=& \{(e_2, \{u_2\})\}, \\ f_5 &=& \{(e_2, \{u_3\})\}, \\ f_6 &=& \{(e_2, \{u_2, u_3\})\}, \\ f_7 &=& \{(e_1, \{u_1\}), (e_2, \{u_2\})\}, \\ f_8 &=& \{(e_1, \{u_1\}), (e_2, \{u_3\})\}, \\ f_9 &=& \{(e_1, \{u_1\}), (e_2, \{u_2, u_3\})\}, \\ f_{10} &=& \{(e_1, \{u_2\}), (e_2, \{u_2\})\}, \\ f_{11} &=& \{(e_1, \{u_2\}), (e_2, \{u_3\})\}, \\ f_{12} &=& \{(e_1, \{u_2\}), (e_2, \{u_2, u_3\})\}, \\ f_{13} &=& \{(e_1, \{u_1, u_2\}), (e_2, \{u_2\})\}, \\ f_{14} &=& \{(e_1, \{u_1, u_2\}), (e_2, \{u_3\})\}, \\ f_{15} &=& f, \\ f_{16} &=& \Phi \end{array}$$

are all soft subsets of f. So $|\widetilde{\mathcal{P}}(f)| = 2^4 = 16$.

Definition 2.8. [17] The soft set f is called a soft point in $\mathbb{S}$, if for the parameter $e_i \in E$ such that $f(e_i) \neq \emptyset$ and $f(e_j) = \emptyset$, for all $e_j \in E \setminus \{e_i\}$ is denoted by $\left(e_{i_f}\right)_j$ for all $i, j \in \mathbb{N}^+$.

Note that the set of all soft points of f will be denoted by $SP(f)$.

Example 2.9. [17] Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $E = \{e_1, e_2, e_3\}$. $f \in \mathbb{S}$ and

$$f = \Big\{\big(e_1, \{u_1, u_3\}\big), \big(e_2, \{u_2, u_3\}\big), \big(e_3, \{u_1, u_2, u_3\}\big)\Big\}$$

Then the soft points for the parameter e_1 are;

$$\begin{array}{rcl} \left(e_{1_f}\right)_1 &=& \left(e_1, \{u_1\}\right) \\ \left(e_{1_f}\right)_2 &=& \left(e_1, \{u_3\}\right) \\ \left(e_{1_f}\right)_3 &=& \left(e_1, \{u_1, u_3\}\right) \end{array}$$

For the the parameter e_2 one of three occasions can be chosen as soft point likewise;

$$\begin{aligned} \big(e_{2_f}\big)_1 &= \big(e_2, \{u_2\}\big) \\ \big(e_{2_f}\big)_2 &= \big(e_2, \{u_3\}\big) \\ \big(e_{2_f}\big)_3 &= \big(e_2, \{u_2, u_3\}\big) \end{aligned}$$

The soft points for the parameter e_3 are;

$$\begin{aligned} \big(e_{3_f}\big)_1 &= \big(e_3, \{u_1\}\big) \\ \big(e_{3_f}\big)_2 &= \big(e_3, \{u_2\}\big) \\ \big(e_{3_f}\big)_3 &= \big(e_3, \{u_3\}\big) \\ \big(e_{3_f}\big)_4 &= \big(e_3, \{u_1, u_2\}\big) \\ \big(e_{3_f}\big)_5 &= \big(e_3, \{u_1, u_3\}\big) \\ \big(e_{3_f}\big)_6 &= \big(e_3, \{u_2, u_3\}\big) \\ \big(e_{3_f}\big)_7 &= \big(e_3, \{u_1, u_2, u_3\}\big) \end{aligned}$$

Definition 2.10. [2] Let $f \in \mathbb{S}$. A soft topology on f, denoted by $\tilde{\tau}$, is a collection of soft subsets of f having the following properties:

i.: $f, \Phi \in \tilde{\tau}$,
ii.: $\{g_i\}_{i\in I} \subseteq \tilde{\tau} \Rightarrow \tilde{\bigcup}_{i\in I} g_i \in \tilde{\tau}$,
iii.: $\{g_i\}_{i=1}^{n} \subseteq \tilde{\tau} \Rightarrow \tilde{\bigcap}_{i=1}^{n} g_i \in \tilde{\tau}$.

The pair $(f, \tilde{\tau})$ is called a soft topological space.

Example 2.11. Refer to Example 2.7, $\tilde{\tau}^1 = \tilde{\mathcal{P}}(f)$, $\tilde{\tau}^0 = \{\Phi, f\}$ and $\tilde{\tau} = \{\Phi, f, f_2, f_{11}, f_{13}\}$ are soft topologies on f.

Definition 2.12. [2] Let $(f, \tilde{\tau})$ be a soft topological space. Then, every element of $\tilde{\tau}$ is called soft open set. Clearly, Φ and f are soft open sets.

Theorem 2.13. [15] *If $\tilde{F}$ is a collection of soft closed sets in a soft topological space $(f, \tilde{\tau})$, then*

i.: *The universal soft set $\tilde{E}$ is soft closed.*
ii.: *Any intersection of members of $\tilde{F}$ belongs to $\tilde{F}$.*
iii.: *Any finite union of members of $\tilde{F}$ belongs to $\tilde{F}$.*

Remark 2.14. [15] Since $\tilde{E}^{\tilde{c}} = \Phi \in \tilde{\tau}$, $\tilde{E}$ is soft closed. But, Φ and f need not to be soft closed. The following example shows that:

Example 2.15. Consider the topology $\tilde{\tau} = \{\Phi, f, f_2, f_{11}, f_{13}\}$ is defined in Example 2.7. Here, f and Φ are not soft closed sets because $f^{\tilde{c}} = \Big\{\big(e_1, \{u_3\}\big), \big(e_2, \{u_1\}\big)\Big\} \notin \tilde{\tau}$ and $\Phi^{\tilde{c}} = \tilde{E} \notin \tilde{\tau}$.

Theorem 2.16. [15] *Let $(f, \tilde{\tau})$ be a soft topological space and $g \tilde{\subseteq} f$. Then, the collection*

$$\tilde{\tau}_g = \{h \tilde{\cap} g : h \in \tilde{\tau}\}$$

is a soft topology on g and the pair $(g, \tilde{\tau}_g)$ is a soft topological space.

Definition 2.17. [15] Let $(f, \tilde{\tau})$ be a soft topological space and $g \tilde{\subseteq} f$. Then, the collection

$$\tilde{\tau}_g = \{h \tilde{\cap} g : h \in \tilde{\tau}\}$$

is called a soft subspace topology on g and $(g, \tilde{\tau}_g)$ is called a soft topological subspace of $(f, \tilde{\tau})$.

Definition 2.18. [16] Let f be a nonempty soft set over the universe U, $g \tilde{\subseteq} f$, $\tilde{\tau}$ be a soft topology on f and $\tilde{\tau}_g$ be a soft subspace topology on g. Then, $(f, \tilde{\tau}, \tilde{\tau}_g)$ is called a soft ditopological space which is abbreviated as SDT-space.

A pair $\tilde{\delta} = (\tilde{\tau}, \tilde{\tau}_g)$ is called a soft ditopology over f and the members of $\tilde{\delta}$ are said to be $\tilde{\delta}$-soft open in f.

The complement of $\tilde{\delta}$-soft open set is called $\tilde{\delta}$-soft closed set.

Example 2.19. [16] Let us consider all soft subsets on f in the Example 2.7. Let $\tilde{\tau} = \{\Phi, f, f_2, f_{11}, f_{13}\}$ be a soft topology on f. If $g = f_9$, then $\tilde{\tau}_g = \{\Phi, f_5, f_7, f_9\}$, and $(g, \tilde{\tau}_g)$ is a soft topological subspace of $(f, \tilde{\tau})$.
Hence, we get soft ditopology over f as $\tilde{\delta} = \{\Phi, f, f_2, f_5, f_7, f_9, f_{11}, f_{13}\}$.

Definition 2.20. [16] Let $h \tilde{\subseteq} f$. Then, $\tilde{\delta}$ - interior of h, denoted by $(g)^{\circ}_{\tilde{\delta}}$, is defined by

$$(h)^{\circ}_{\tilde{\delta}} = \tilde{\bigcup}\{h : k \tilde{\subset} h, k \text{ is } \tilde{\delta}\text{-soft open}\}$$

The $\tilde{\delta}$ - closure of h, denoted by $(\overline{h})_{\tilde{\delta}}$, is defined by

$$(\overline{h})_{\tilde{\delta}} = \tilde{\bigcap}\{k : h \tilde{\subset} k, k \text{ is } \tilde{\delta}\text{-soft closed}\}$$

Note that $(h)^{\circ}_{\tilde{\delta}}$ is the biggest $\tilde{\delta}$-soft open set that contained in h and $(\overline{h})_{\tilde{\delta}}$ is the smallest $\tilde{\delta}$-soft closed set that containing h.

3. Soft Ditopological Subspace

In this section, the detailed analysis of SDT - space is carried out by introducing a new concept called soft ditopological subspace (SDT-subspace) in SDT-space. I give two different definitions of SDT-subspace that are not a consequence of each other. Although, these definitions run along different lines, I prove that only one and the same soft ditopological subspace can be established on the same soft set using two different definitions.

I also observe relations of soft ditopological space and soft ditopological subspace in different cases with soft open and soft closed sets. It shows how soft sets in soft ditopological space can preserve their properties in soft ditopological subspaces. In this context, I serve a bridge among soft ditopological space theory and soft ditopological subspace theory.

Theorem 3.1. *Let $(f, \tilde{\delta})$ be a SDT-space and $t \tilde{\subseteq} f$. Then, the collection*

$$\tilde{\delta}_t = \{k \tilde{\cap} t : k \in \tilde{\delta}\}$$

is a soft topology on t and the pair $(t, \tilde{\delta}_t)$ is a soft topological space.

Proof: Since $\Phi \tilde{\cap} t = \Phi$ and $f \tilde{\cap} t = t$, then $t, \Phi \in \tilde{\delta}_t$.
Moreover,

$$\tilde{\bigcap}_{i=1}^{n} (k_i \tilde{\cap} t) = \left(\tilde{\bigcap}_{i=1}^{n} k_i\right) \tilde{\cap} t$$

and

$$\tilde{\bigcup}_{i \in I} (k_i \tilde{\cap} t) = \left(\tilde{\bigcup}_{i \in I} k_i\right) \tilde{\cap} t$$

for $\tilde{\delta} = \{k_i \tilde{\subseteq} f : i \in I\}$. Thus, the union of any number of soft sets in $\tilde{\delta}_t$ belongs to $\tilde{\delta}_t$ and the finite intersection of soft sets in $\tilde{\delta}_t$ belongs to $\tilde{\delta}_t$. So, $\tilde{\delta}_t$ is a soft topology on t.

Definition 3.2. Let $(f, \tilde{\delta})$ be a SDT-space and $t \tilde{\subseteq} f$. Then, the collection

$$\tilde{\delta}_t = \{k \tilde{\cap} t : k \in \tilde{\delta}\}$$

is called a soft subspace ditopology on t and $(t, \tilde{\delta}_t)$ is called a soft ditopological subspace of $(f, \tilde{\delta})$.

In order to carry out the construction, I have to make a judicious choice of soft ditopological subspace that given in the below example:

Example 3.3. Let us consider the SDT-space $(f, \tilde{\delta})$ defined in the Example 2.19. If $t = f_8$, then,

$$\begin{aligned}
\Phi \tilde{\cap} f_8 &= \Phi \\
f_2 \tilde{\cap} f_8 &= \Phi \\
f_5 \tilde{\cap} f_8 &= f_5 \\
f_7 \tilde{\cap} f_8 &= f_1 \\
f_9 \tilde{\cap} f_8 &= f_8 \\
f_{11} \tilde{\cap} f_8 &= f_5 \\
f_{13} \tilde{\cap} f_8 &= f_1 \\
f \tilde{\cap} f_8 &= f_8
\end{aligned}$$

Hence, we get soft subspace ditopology on t as $\tilde{\delta}_t = \{\Phi, f_1, f_5, f_8\}$.

In the next definition, I state a new characterization of SDT-subspace topology which seems not to be a consequence of previous SDT-subspace definition made in Definition 3.2. The reason for my attention to obtain a new different definition, is that, in certain circumstances, it provides a way to prove results for SDT-subspaces will be seen in the next section:

Definition 3.4. Let $(f, \tilde{\delta})$ be a SDT-space and $t \tilde{\subseteq} f$. If the collections

$$\tilde{\delta}_t = \{k \tilde{\cap} t : k \in \tilde{\tau}\}$$

and
$$(\tilde{\delta}_g)_t = \{z \,\tilde{\cap}\, t : z \in \tilde{\tau}_g\}$$
are two soft topologies on t, then a SDT-space $(t, \tilde{\delta}_t, (\tilde{\delta}_g)_t)$ is called a SDT-subspace of $(f, \tilde{\delta})$.

This definition is convenient for the induction on soft subspace of a soft subspace that will be used in the next section.

The usefulness and interest of this correspondence of two different definitions about SDT-subspaces made above will of course be enhanced if they are coincident. I now study to get the same soft subspace ditopology on t obtained in Example 3.3 using Definition 3.4:

Example 3.5. Let us consider the SDT-space $(f, \tilde{\delta})$ defined in the Example 2.19 where $g = f_9$ and $\tilde{\delta}_g = \{\Phi, f_5, f_7, f_9\}$. Here we get the collections
$$\tilde{\delta}_t = \{\Phi, f_1, f_5, f_8\}$$
and
$$(\tilde{\delta}_g)_t = \{\Phi, f_1, f_5, f_8\}$$
Hence, we obtain SDT-subspace $(t, \tilde{\delta}_t, (\tilde{\delta}_g)_t) = \{\Phi, f_1, f_5, f_8\}$.

The result above seems appropriate to mention that we obtain the same SDT-subspace topology with different definitions. Although, these definitions run along different lines, it is easy to deduce that I obtain exactly only one and the same soft ditopological subspace on the same soft set.
The reader is cautioned that the notation in Definition 3.2 is coincident with Definition 3.4 and will be used to serve a bridge among soft topological space and soft ditopological spaces in the next section.
On the way, I continue to investigate SDT-subspace properties:

Definition 3.6. A soft ditopological property is said to be hereditary if whenever a soft ditopological space $(f, \tilde{\delta})$ has that property, then so does every soft ditopological subspace of it.

Definition 3.7. Let $(t, \tilde{\delta}_t)$ be a SDT-subspace of a SDT-space $(f, \tilde{\delta})$ and $z \tilde{\subseteq} t$.
z is called a soft open set in SDT-subspace t if $z = m \tilde{\cap} t$ for $m \in \tilde{\delta}$.
So, the members of $\tilde{\delta}_t$ are said to be a soft open sets in SDT-subspace $(t, \tilde{\delta}_t)$.

Theorem 3.8. *Let $(t, \tilde{\delta}_t)$ be a SDT-subspace of a SDT-space $(f, \tilde{\delta})$ and $z \tilde{\subseteq} t$. If $z \in \tilde{\delta}$ then, $z \in \tilde{\delta}_t$.*

Proof: Suppose that $z \in \tilde{\delta}$. Since $z \tilde{\subseteq} t$, $z = z \tilde{\cap} t$. Then, $z \in \tilde{\delta}_t$ by assumption $z \in \tilde{\delta}$.

Theorem 3.9. *Let $(t, \tilde{\delta}_t)$ be a SDT-subspace of a SDT-space $(f, \tilde{\delta})$. Then, the following are equivalent:*

i. $t \in \tilde{\delta}$
ii. $\tilde{\delta}_t \tilde{\subseteq} \tilde{\delta}$.

Proof: $(i) \Rightarrow (ii)$: Let $t \in \tilde{\delta}$. Take as given $\forall z \in \tilde{\delta}_t$. From the Definition 3.2, $z = m \tilde{\cap} t$, where $\exists m \in \tilde{\delta}$. Since $t \in \tilde{\delta}$ and $m \in \tilde{\delta}$ then, $z \in \tilde{\delta}$. Hence $\tilde{\delta}_t \tilde{\subseteq} \tilde{\delta}$.
$(ii) \Rightarrow (i)$: Assume that $\tilde{\delta}_t \tilde{\subseteq} \tilde{\delta}$. Since $t \in \tilde{\delta}_t$ then, $t \in \tilde{\delta}$.

Remark 3.10. A $\tilde{\delta}$-soft open set in a SDT-subspace is not need to be $\tilde{\delta}$-soft open in the SDT-space which is given in the following example:

Example 3.11. Consider the SDT-subspace $(t, \tilde{\delta}_t)$ defined in the Example 3.3. Here, $f_1 \in \tilde{\delta}_t$ but $f_1 \notin \tilde{\delta}$.

4. The Relation Between Soft Topological Space and Soft Ditopological Space

Conditions related to bounds on the relations between soft spaces appear to be rare in the literature, I study how the notion of soft ditopology relates to the soft topology.

The two characteristics, soft ditopological subspace and the soft subspace of a soft topological subspace, are coincided, but the relationship is quite a complex one. Although individually these systems can still be quite complicated, a possibly more tractable task is to describe their possible joint distributions. The relationship between the soft topology and the extent in which soft subspace topology is soft ditopological subspace is studied, obtained joint distributions.

I now exploit the relation to see what else I can say about the relation between soft topology and soft ditopology in the view of subspace topology.

Theorem 4.1. *Let $h \tilde{\subseteq} t \tilde{\subseteq} f$, $(t, \tilde{\delta}_t)$ and $(h, \tilde{\delta}_h)$ be SDT-subspaces of SDT-space $(f, \tilde{\delta})$ and $(h, (\tilde{\delta}_t)_h)$ be a soft subspace of $(t, \tilde{\delta}_t)$. Then,*

$$\tilde{\delta}_h = (\tilde{\delta}_t)_h$$

Proof: Take as given $\forall\, w \in \tilde{\delta}_h$. From the definition of SDT-subspace, $w = w \tilde{\cap} h$, where $w \in \tilde{\delta}$. We obtain that $w \tilde{\cap} t \in \tilde{\delta}_t$. Then, by choosing $w \tilde{\cap} t = y$, $y \tilde{\cap} h \in (\tilde{\delta}_t)_h$ because of $(h, (\tilde{\delta}_t)_h) \subseteq (t, \tilde{\delta}_t)$ and $y \in \tilde{\delta}_t$.
Since $y = w \tilde{\cap} t$ then, $y \tilde{\cap} h = w \tilde{\cap} t \tilde{\cap} h \in (\tilde{\delta}_t)_h$.
$h \tilde{\subseteq} t \Leftrightarrow h = h \tilde{\cap} t$ then, $y \tilde{\cap} h = w \tilde{\cap} h \in (\tilde{\delta}_t)_h$.
Since $w = w \tilde{\cap} h \in (\tilde{\delta}_t)_h$ then, $w \in (\tilde{\delta}_t)_h$.
Hence, we get $\tilde{\delta}_h \subseteq (\tilde{\delta}_t)_h$.
Conversely, assume that $\forall z \in (\tilde{\delta}_t)_h$. From the definition of SDT-subspace, $z = k \tilde{\cap} h$, where $k \in \tilde{\delta}_t$. We obtain that $k = w \tilde{\cap} t$, where $w \in \tilde{\delta}$.
$z = k \tilde{\cap} h = w \tilde{\cap} t \tilde{\cap} h = w \tilde{\cap} h \in \tilde{\delta}_h$, so this completes the proof.

The method of this proof carries over to soft ditopological space satisfying soft topological space's properties via soft subspace topology.

Theorem 4.2. *Let* $(t, \tilde{\delta}_t)$ *and* $(h, \tilde{\delta}_h)$ *be SDT-subspaces of SDT-space* $(f, \tilde{\delta})$ *and* $w \subseteq h \tilde{\cap} t$. *Then,*

$$\tilde{\delta}_w = (\tilde{\delta}_t)_w = (\tilde{\delta}_h)_w$$

Proof: Since the entire argument is based solely upon the Theorem 4.1, the conclusion of the theorem must hold.

I underline that all the aforementioned results in this theorem rely on the conformality of the underlying construction of a relation between soft topological soft ditopological spaces.

Using Theorem 4.1, Theorem 4.2 and Definition 3.2, Definition 3.4 about SDT-subspace and soft topological subspace, I am able to deduce that their soft subtopological spaces are coincident that is proved below:

Remark 4.3. Let $(t, \tilde{\delta}_t)$ be a SDT-subspace of a SDT-space $(f, \tilde{\delta})$ and $t, g \tilde{\subseteq} f$. If we consider the notation:

$$(f, \tilde{\delta}) = (f, \tilde{\tau}, \tilde{\tau}_g)$$

The SDT-subspace of $(f, \tilde{\delta})$ is $(t, \tilde{\delta}_t)$ and the SDT-subspace of $(f, \tilde{\tau}, \tilde{\tau}_g)$ is $(t, \tilde{\tau}_t, (\tilde{\tau}_g)_t)$. If we write the equality of SDT-subspaces;

$$(t, \tilde{\delta}_t) = (t, \tilde{\tau}_t, (\tilde{\tau}_g)_t)$$

From Theorem 4.1, $\tilde{\tau}_t = (\tilde{\tau}_g)_t$, then,

$$(t, \tilde{\delta}_t) = (t, \tilde{\tau}_t)$$

Hence, SDT-subspace and soft topological subspace of t are coincident.

I have shown that soft subspaces of soft topology and soft ditopology are overlapped, whence it is readily inferred that I can obtain soft ditopological subspace via soft topological subspace.

5. Conclusion

The aim of this article is to study the relationship between the soft topology and the extent in which soft topology to be soft ditopology. Also, in this work, soft ditopological subspace on a soft set is defined and its related properties are studied. Then, the relation between soft topology and soft ditopology is presented. The concept of soft topological subspace of soft ditopological subspace have been introduced. Also, several relations of soft ditopological and soft ditopological subspace have been established and their properties with given examples have been compared. All these results present a bridge among soft topological and soft ditopological theory. In the last section, I describe how the notion of soft ditopology relates to the soft topology. A complete explication of the soft ditopological subspace is warranted, as it will likely reveal further clues to the differences between the soft topology and soft ditopology theories.

It considers some of the new results and consequences, which could be useful from the point of view of soft set theory, that were not studied at all. All these findings will provide a base to researchers who want to work in the field of soft ditopology and will help to strengthen the foundations of the theory of soft ditopological spaces.

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A Q-ANALOG OF THE BI-PERIODIC LUCAS SEQUENCE

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Abstract. In this paper, we introduce a q-analog of the bi-periodic Lucas sequence, called as the q-bi-periodic Lucas sequence, and give some identities related to the q-bi-periodic Fibonacci and Lucas sequences. Also, we give a matrix representation for the q-bi-periodic Fibonacci sequence which allow us to obtain several properties of this sequence in a simple way. Moreover, by using the explicit formulas for the q-bi-periodic Fibonacci and Lucas sequences, we introduce q-analogs of the bi-periodic incomplete Fibonacci and Lucas sequences and give a relation between them.

1. Introduction

It is well-known that the classical Fibonacci numbers F_n are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \tag{1.1}$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. The Lucas numbers L_n, which follows the same recursive pattern as the Fibonacci numbers, but begins with $L_0 = 2$ and $L_1 = 1$. There are a lot of generalizations of Fibonacci and Lucas sequences. In [6], Edson and Yayenie introduced a generalization of the Fibonacci sequence, called as bi-periodic Fibonacci sequence, as follows:

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \; n \geq 2 \tag{1.2}$$

with initial values $q_0 = 0$ and $q_1 = 1$, where a and b are nonzero numbers. Note that if we take $a = b = 1$ in $\{q_n\}$, we get the classical Fibonacci sequence. These sequences are emerged as denominators of the continued fraction expansion of the quadratic irrational numbers. For detailed information related to these sequences,

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we refer to [6, 19, 8, 11, 12, 17, 18, 15, 16]. Yayenie [19] gave an explicit formula of q_n as:

$$q_n = a^{\xi(n-1)} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-i}{i} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - i} \tag{1.3}$$

where $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd.

Similar to (1.2), by taking initial conditions $p_0 = 2$ and $p_1 = a$, Bilgici [2] introduced the bi-periodic Lucas numbers as follows:

$$p_n = \begin{cases} bp_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \\ ap_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \ n \geq 2. \tag{1.4}$$

It should also be noted that, it gives the classical Lucas sequence in the case of $a = b = 1$ in $\{p_n\}$. In analogy with (1.3), Tan and Ekin [14] gave the explicit formula of the bi-periodic Lucas numbers as:

$$p_n = a^{\xi(n)} \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\left\lfloor \frac{n}{2} \right\rfloor - i}, \ n \geq 1. \tag{1.5}$$

On the other hand, there are several different q-analogs for the Fibonacci and Lucas sequences [3, 4, 5, 13, 7, 1]. Particularly, Cigler [5] gave the (Carlitz-) q-Fibonacci and q-Lucas polynomials

$$f_n(x,s) = xf_{n-1}(x,s) + q^{n-2} s f_{n-2}(x,s); \ f_0(x,s) = 0, \ f_1(x,s) = 1, \tag{1.6}$$

$$l_n(x,s) = f_{n+1}(x,s) + sf_{n-1}(x,qs); \ l_0(x,s) = 2, \ l_1(x,s) = x, \tag{1.7}$$

respectively.

Additionally, Ramírez and Sirvent [10] introduced a q-analog of the bi-periodic Fibonacci sequence by

$$F_n^{(a,b)}(q,s) = \begin{cases} aF_{n-1}^{(a,b)}(q,s) + q^{n-2} s F_{n-2}^{(a,b)}(q,s), & \text{if } n \text{ is even} \\ bF_{n-1}^{(a,b)}(q,s) + q^{n-2} s F_{n-2}^{(a,b)}(q,s), & \text{if } n \text{ is odd} \end{cases}, n \geq 2 \tag{1.8}$$

with initial conditions $F_0^{(a,b)}(q,s) = 0$ and $F_1^{(a,b)}(q,s) = 1$. They derived the following equality to evaluate the q-bi-periodic Fibonacci sequence:

$$F_n^{(a,b)}(q,s) = \chi_n F_{n-1}^{(a,b)}(q,qs) - qsF_{n-2}^{(a,b)}\left(q,q^2 s\right), \tag{1.9}$$

where $\chi_n := a^{\xi(n+1)} b^{\xi(n)}$. Also, they gave the relationship between the q-bi-periodic Fibonacci sequence and the (Carlitz-) q-Fibonacci polynomials as:

$$F_n^{(a,b)}(q,s) = \left(\sqrt{\frac{a}{b}}\right)^{\xi(n+1)} f_n\left(\sqrt{ab}, s\right). \tag{1.10}$$

By using (1.10), they obtained the explicit formula of the q-bi-periodic Fibonacci sequence as:

$$F_n^{(a,b)}(q,s) = a^{\xi(n-1)} \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - k} q^{k^2} s^k, \qquad (1.11)$$

where $\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]_q!}{[k]_q![n-k]_q!}$ is the q-binomial coefficients with $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]_q! := [1]_q [2]_q \cdots [n]_q$.

Motivated by the Ramirez's results in [10], here we introduce a q-analog of the bi-periodic Lucas sequence, called as the q-bi-periodic Lucas sequence, and give some identities related to the q-bi-periodic Fibonacci and Lucas sequences. Also, we give a matrix representation for the q-bi-periodic Fibonacci sequence which allow us to obtain several properties of this sequence in a simple way. Moreover, by using the explicit formulas for the q-bi-periodic Fibonacci and Lucas sequences, we introduce q-analogs of the bi-periodic incomplete Fibonacci and Lucas sequences and give a relation between them.

2. A q-analog of the bi-periodic Lucas sequence

First, we consider the (Carlitz-) q-Lucas polynomials in (1.7), and define the q-bi-periodic Lucas sequence by means of the (Carlitz-) q-Lucas polynomials.

Definition 1. The q-bi-periodic Lucas sequence is defined by

$$L_n^{(a,b)}(q,s) = \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} l_n\left(\sqrt{ab}, s\right) \qquad (2.1)$$

where $l_n(x,s)$ is the (Carlitz-) q-Lucas polynomials.

The terms of the q-bi-periodic Lucas sequence can be given as:

n	$L_n^{(a,b)}(q,s)$
0	2
1	a
2	$ab + sq + s$
3	$a^2b + as + asq + asq^2$
4	$a^2b^2 + abs + absq + absq^2 + absq^3 + s^2q^2 + s^2q^4$
5	$a^3b^2 + a^2bs + a^2bsq + a^2bsq^2 + a^2bsq^3 + a^2bsq^4$ $+as^2q^2 + as^2q^3 + as^2q^4 + as^2q^5 + as^2q^6$
$\vdots$	$\vdots$

Note that if we take $a = b = x$, we obtain the (Carlitz-) q-Lucas polynomials $l_n(x,s)$.

In the following lemma, we state the q-bi-periodic Lucas sequence in terms of the q-bi-periodic Fibonacci sequence.

Lemma 1. *For any integer $n \geq 0$, we have*

$$L_n^{(a,b)}(q,s) = F_{n+1}^{(a,b)}(q,s) + sF_{n-1}^{(a,b)}(q,qs). \tag{2.2}$$

Proof. By using the definition of the q-bi-periodic Lucas sequence and the relations (1.7) and (1.10), we have

$$\begin{aligned} L_n^{(a,b)}(q,s) &= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} l_n\left(\sqrt{ab},s\right) \\ &= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} \left(f_{n+1}\left(\sqrt{ab},s\right) + sf_{n-1}\left(\sqrt{ab},qs\right)\right) \\ &= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} \left(\sqrt{\frac{b}{a}}\right)^{\xi(n)} \left(F_{n+1}^{(a,b)}(q,s) + sF_{n-1}^{(a,b)}(q,qs)\right) \end{aligned}$$

which gives the desired result. □

Now we give an another relation between the q-bi-periodic Fibonacci sequence and q-bi-periodic Lucas sequence.

Theorem 1. *For any integer $n \geq 0$, we have*

$$\chi_n L_n^{(a,b)}(q,qs) = F_{n+2}^{(a,b)}(q,s) - q^{n+1}s^2 F_{n-2}^{(a,b)}\left(q,q^2s\right) \tag{2.3}$$

where $\chi_n := a^{\xi(n+1)}b^{\xi(n)}$.

Proof. By using the definition of the q-bi-periodic Fibonacci sequence in (1.8) and the relations (2.2) and (1.9), we get

$$\begin{aligned} \chi_n L_n^{(a,b)}(q,qs) &= \chi_n \left(F_{n+1}^{(a,b)}(q,qs) + qsF_{n-1}^{(a,b)}\left(q,q^2s\right)\right) \\ &= F_{n+2}^{(a,b)}(q,s) - qsF_n^{(a,b)}\left(q,q^2s\right) + \chi_n qsF_{n-1}^{(a,b)}\left(q,q^2s\right) \\ &= F_{n+2}^{(a,b)}(q,s) - qs\left(F_n^{(a,b)}\left(q,q^2s\right) - \chi_n F_{n-1}^{(a,b)}\left(q,q^2s\right)\right) \\ &= F_{n+2}^{(a,b)}(q,s) - q^{n+1}s^2F_{n-2}^{(a,b)}\left(q,q^2s\right). \end{aligned}$$

□

If we take $a = b = x$ in (2.3), it reduces to the relation between q-bi-periodic Fibonacci sequence and Lucas polynomials

$$xl_n(x,qs) = f_{n+2}(x,s) - q^{n+1}s^2 f_{n-2}\left(x,q^2s\right)$$

which can be found in [5, Equation (3.15)].

In the following theorem, we give the explicit expression of the q-bi-periodic Lucas sequence $L_n^{(a,b)}(q,s)$. Since we define the incomplete sequences by using its explicit formula, the following theorem play a key role for our further study in the next section.

Theorem 2. *For any integer $n \geq 0$, we have*

$$L_n^{(a,b)}(q,s) = a^{\xi(n)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{[n]}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2-k} s^k. \tag{2.4}$$

Proof. By using the relations (2.2) and (1.11), we have

$$\begin{aligned}
L_n^{(a,b)}(q,s) &= F_{n+1}^{(a,b)}(q,s) + sF_{n-1}^{(a,b)}(q,qs) \\
&= a^{\xi(n)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2} s^k \\
&\quad + a^{\xi(n-2)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \begin{bmatrix} n-2-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - 1 - k} q^{k^2+k} s^{k+1} \\
&= a^{\xi(n)} \Bigg(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2} s^k \\
&\quad + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2-k} s^k \Bigg) \\
&= a^{\xi(n)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(q^k \begin{bmatrix} n-k \\ k \end{bmatrix} + \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} \right) (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2-k} s^k.
\end{aligned}$$

By using the identity

$$q^k \begin{bmatrix} n-k \\ k \end{bmatrix} + \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} = \frac{[n]}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix},$$

we obtain the desired result. □

If we take $a = b = x$ in the above theorem, it reduces to the (Carlitz-) q-Lucas polynomials

$$l_n(x,s) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{[n]}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix} q^{k^2-k} s^k x^{n-2k}$$

which can be found in [5, Equation (3.14)].

Now we give a matrix representation for the q-bi-periodic Fibonacci sequence which can be proven by induction. By using matrix formula, one can obtain several properties of this sequence.

Theorem 3. *For $n \geq 1$, let define the matrix $C(\chi_n, s) := \begin{pmatrix} 0 & 1 \\ s & \chi_n \end{pmatrix}$. Then we have*

$$\begin{aligned} M_n(\chi_n, s) &:= C\left(\chi_n, q^{n-1}s\right) C\left(\chi_{n-1}, q^{n-2}s\right) \cdots C(\chi_2, qs)\, C(\chi_1, s) \\ &= \begin{pmatrix} sF_{n-1}^{(a,b)}(q, qs) & \left(\frac{b}{a}\right)^{\xi(n+1)} F_n^{(a,b)}(q, s) \\ sF_n^{(a,b)}(q, qs) & \left(\frac{b}{a}\right)^{\xi(n)} F_{n+1}^{(a,b)}(q, s) \end{pmatrix}. \end{aligned} \tag{2.5}$$

In the following theorem, we give the q-Cassini formula for the q-bi-periodic Fibonacci sequence by taking the determinant of the both sides of the equation (2.5).

Theorem 4. *For any integer $n > 0$, we have*

$$\begin{aligned} &\left(\frac{b}{a}\right)^{\xi(n)} F_{n-1}^{(a,b)}(q, qs)\, F_{n+1}^{(a,b)}(q, s) - \left(\frac{b}{a}\right)^{\xi(n+1)} F_n^{(a,b)}(q, s)\, F_n^{(a,b)}(q, qs) \\ = \ & (-1)^n s^{n-1} q^{\frac{n(n-1)}{2}}. \end{aligned} \tag{2.6}$$

Note that by taking $a = b = x$, we obtain the result in [5, Equation (3.12)].

Theorem 5. *For any integer $n > 0$, we have*

$$F_{2n}^{(a,b)}(q, s) = \left(\frac{a}{b}\right)^{\xi(n)} q^n s F_{n-1}^{(a,b)}\left(q, q^{n+1}s\right) F_n^{(a,b)}(q, s) + F_n^{(a,b)}(q, q^n s)\, F_{n+1}^{(a,b)}(q, s). \tag{2.7}$$

Proof. Since $M_{m+n}(\chi_n, s) = M_m(\chi_n, q^n s)\, M_n(\chi_n, s)$, if we equate the corresponding entries of each matrices and take $m = n$ in the resulting equality, we get the desired result. □

One can get several properties of the q-bi-periodic Fibonacci sequence by taking proper powers of the matrix in (2.5).

3. $q-$bi-periodic incomplete Fibonacci and Lucas sequences

In this section, we define q-bi-periodic incomplete Fibonacci and Lucas sequences. Let n be a positive integer and l be an integer.

Ramirez [9] defined the bi-periodic incomplete Fibonacci numbers by using the explicit formula of the bi-periodic Fibonacci sequences in (1.3) as:

$$q_n(l) = a^{\xi(n-1)} \sum_{i=0}^{l} \binom{n-1-i}{i} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - i}, \ 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor$$

Similarly, by using the explicit formula of the bi-periodic Lucas sequence in (1.5), Tan and Ekin [14] defined the bi-periodic incomplete Lucas numbers as:

$$p_n(l) = a^{\xi(n)} \sum_{i=0}^{l} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\left\lfloor \frac{n}{2} \right\rfloor - i}, \ 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Analogously, by using the explicit formulas of the q-bi-periodic Fibonacci sequence in (1.11) and the q-bi-periodic Lucas sequence in (2.4), we define the q-bi-periodic incomplete Fibonacci and Lucas sequences as follows.

Definition 2. For any non negative integer n, the q-bi-periodic incomplete Fibonacci and Lucas sequences are defined by

$$F_{n,l}^{(a,b)}(q,s)=a^{\xi(n-1)}\sum_{k=0}^{l}\begin{bmatrix} n-1-k \\ k \end{bmatrix}(ab)^{\left\lfloor \frac{n-1}{2}\right\rfloor -k}q^{k^2}s^k,\ 0\le l\le \left\lfloor \frac{n-1}{2}\right\rfloor \tag{3.1}$$

and

$$L_{n,l}^{(a,b)}(q,s)=a^{\xi(n)}\sum_{k=0}^{l}\frac{[n]}{[n-k]}\begin{bmatrix} n-k \\ k \end{bmatrix}(ab)^{\left\lfloor \frac{n}{2}\right\rfloor -k}q^{k^2-k}s^k,\ 0\le l\le \left\lfloor \frac{n}{2}\right\rfloor, \tag{3.2}$$

respectively.

If we take $l=\left\lfloor \frac{n-1}{2}\right\rfloor$ in (3.1), we obtain the q-bi-periodic Fibonacci sequence, and if we take $l=\left\lfloor \frac{n}{2}\right\rfloor$ in (3.2), we obtain the q-bi-periodic Lucas sequence.

Next, we give non-homogenous recurrence relation for the q-bi-periodic incomplete Fibonacci sequence.

Theorem 6. *For $0\le l\le \frac{n-2}{2}$, the non-linear recurrence relation of the q-bi-periodic incomplete Fibonacci sequence is*

$$F_{n+2,l+1}^{(a,b)}(q,s)=\begin{cases} aF_{n+1,l+1}^{(a,b)}(q,s)+q^n sF_{n,l}^{(a,b)}(q,s), & \text{if } n \text{ is even} \\ bF_{n+1,l+1}^{(a,b)}(q,s)+q^n sF_{n,l}^{(a,b)}(q,s), & \text{if } n \text{ is odd} \end{cases}. \tag{3.3}$$

The relation (3.3) can be transformed into the non-homogeneous recurrence relation

$$F_{n+2,l}^{(a,b)}(q,s)=aF_{n+1,l}^{(a,b)}(q,s)+q^n sF_{n,l}^{(a,b)}(q,s)-a\begin{bmatrix} n-1-l \\ l \end{bmatrix}(ab)^{\left\lfloor \frac{n-1}{2}\right\rfloor -l}q^{n+l^2}s^{l+1} \tag{3.4}$$

for even n, and

$$F_{n+2,l}^{(a,b)}(q,s)=bF_{n+1,l}^{(a,b)}(q,s)+q^n sF_{n,l}^{(a,b)}(q,s)-\begin{bmatrix} n-1-l \\ l \end{bmatrix}(ab)^{\left\lfloor \frac{n-1}{2}\right\rfloor -l}q^{n+l^2}s^{l+1} \tag{3.5}$$

for odd n.

Proof. If n is even, then $\left\lfloor \frac{n+1}{2}\right\rfloor=\left\lfloor \frac{n}{2}\right\rfloor$. By using the Definition (3.1), we can write the RHS of (3.3) as

$$a^{1+\xi(n)}\sum_{k=0}^{l+1}\begin{bmatrix} n-k \\ k \end{bmatrix}(ab)^{\left\lfloor \frac{n}{2}\right\rfloor -k}q^{k^2}s^k$$

$$+q^n sa^{\xi(n-1)}\sum_{k=0}^{l}\begin{bmatrix} n-1-k \\ k \end{bmatrix}(ab)^{\left\lfloor \frac{n-1}{2}\right\rfloor -k}q^{k^2}s^k$$

$$= a \sum_{k=0}^{l+1} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2} s^k + q^n \ a \sum_{k=0}^{l} \begin{bmatrix} n-1-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n-1}{2} \rfloor - k} q^{k^2} s^{k+1}$$

$$= a \sum_{k=0}^{l+1} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2} s^k$$

$$+ q^n \ a \sum_{k=1}^{l+1} \begin{bmatrix} n-k \\ k-1 \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{(k-1)^2} s^k$$

$$= a \sum_{k=0}^{l+1} \left(\begin{bmatrix} n-k \\ k \end{bmatrix} + q^{n-2k+1} \begin{bmatrix} n-k \\ k-1 \end{bmatrix} \right) (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2} s^k \ \ (ab)^{\lfloor \frac{n}{2} \rfloor - k} - 0$$

$$= a \sum_{k=0}^{l+1} \begin{bmatrix} n-k+1 \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2} s^k \ \ (ab)^{\lfloor \frac{n}{2} \rfloor - k}$$

$$= F_{n+2,l+1}^{(a,b)} (q,s) .$$

Also from equation (3.3), we have

$$\begin{aligned} F_{n+2,l}^{(a,b)} (q,s) &= aF_{n+1,l}^{(a,b)} (q,s) + q^n s F_{n,l-1}^{(a,b)} (q,s) \\ &= aF_{n+1,l}^{(a,b)} (q,s) + q^n s F_{n,l}^{(a,b)} (q,s) + q^n s (F_{n,l-1}^{(a,b)} (q,s) - F_{n,l}^{(a,b)} (q,s)) \\ &= aF_{n+1,l}^{(a,b)} (q,s) + q^n s F_{n,l}^{(a,b)} (q,s) - a \begin{bmatrix} n-1-l \\ l \end{bmatrix} (ab)^{\lfloor \frac{n-1}{2} \rfloor - l} q^{n+l^2} s^{l+1}. \end{aligned}$$

If n is odd, the proof is completely analogous. □

Note that the q-bi-periodic Lucas sequence does not satisfy a recurrence like (3.3), since $F_{n+1}^{(a,b)} (q,s)$ and $F_{n+1}^{(a,b)} (q,qs)$ do not satisfy the same recurrence relation.

Finally we give the relationship between the q-bi-periodic incomplete Fibonacci and Lucas sequences as follows:

Theorem 7. *For $0 \le l \le \lfloor \frac{n}{2} \rfloor$, we have*

$$L_{n,l}^{(a,b)} (q,s) = F_{n+1,l}^{(a,b)} (q,s) + F_{n-1,l-1}^{(a,b)} (q,qs) . \tag{3.6}$$

Proof. It can be proved easily by using the definitions (3.1) and (3.2). □

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A GENERALIZATION OF THE PEANO KERNEL AND ITS APPLICATIONS

GÜLTER BUDAKÇI AND HALIL ORUÇ

Abstract. Based on the q-Taylor Theorem, we introduce a more general form of the Peano kernel (q-Peano) which is also applicable to non-differentiable functions. Then we show that quantum B-splines are the q-Peano kernels of divided differences. We also give applications to polynomial interpolation and construct examples in which classical remainder theory fails whereas q-Peano kernel works.

1. Introduction

Recent advances in the quantum B-splines, [4, 6, 17] have given us an opportunity to arise the question if there is a way to link quantum B-splines with a more general Peano kernel. The quantum B-spline functions are piecewise polynomials whose quantum derivatives agree at the joins up to some order. The quantum B-splines are introduced by Simeonov & Goldman [17] to generalize classical B-splines by replacing ordinary derivatives by quantum derivatives. Their work constructs not only a new type of de Boor algorithm but also novel identities via blossoms. Actually the underlying idea in [17] goes back to the work [16] which aimed to find a new form of blossoms to represent q-Bernstein polynomials. Just like classical Bernstein polynomials, the q-Bernstein polynomials possesses remarkable geometric and analytic properties, see [11, 13]. So, our objectives are to extend the Peano kernel and then relate with the quantum B-splines. This extension is important because there are functions whose q-derivatives exist but whose classical derivatives fail to exist. Furthermore it will also lead us to investigate errors in approximations.

The classical Peano kernel theorem provides a useful technique for computing the errors of approximations such as interpolation, quadrature rules and B-splines. The errors are represented by a linear functional that operates on functions

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$f \in C^{n+1}[a,b]$ and annihilates all polynomials of degree at most n.

Namely, if $L(f)=0$ for all $f \in \mathcal{P}_n$, the space of polynomials of degree n, then

$$L(f) = \int_a^b f^{(n+1)}(t) K(x,t) dt,$$

where $K(x,t) = \frac{1}{n!} L\left((x-t)_+^n\right)$.

An important application of this result is the Kowalewski's interpolating polynomial remainder. Let $t_0, t_1, \ldots, t_n \in [a,b]$ be fixed and distinct, and

$$L(f) = f(x) - \sum_{k=0}^{n} f(t_k) l_{nk}(x)$$

where $l_{nk}(x) = \prod_{\substack{v=0 \\ v \neq k}}^{n} \frac{x - t_v}{t_k - t_v}$. If $f \in C^{m+1}[a,b]$, then

$$L(f) = \frac{1}{m!} \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^m f^{(m+1)}(t) dt, \quad \text{for each } \ m = 0, 1, \ldots, n$$

is the error functional, see [7].

This paper is organized as follows: We begin with definitions and properties of the quantum calculus needed for this work. In Section 3, we give the q-Taylor theorem and develop a generalization of the Peano kernel (q-Peano kernel). We present a simple way to find $L(f)$ under the condition in which the kernel has no sign change. Moreover, taking $L(f)$ as divided differences we construct a relation between q-B-splines and q-Peano kernel. Section 4 demonstrates how the q-Peano kernel is used to find the error of Lagrange interpolation. Finally, the error bounds of quadrature formula on the remainder involving q-integration is discussed.

2. Preliminaries

Throughout the paper we consider q as a real fixed parameter. Let us give basic definitions and theorems of the q-calculus that are required in the next sections. For a fixed parameter $q \neq 1$, the q-derivatives are defined by,

$$\begin{aligned} D_q f(t) &= \frac{f(qt) - f(t)}{(q-1)t} \\ D_q^n f(t) &= D_q(D_q^{n-1} f(t)), \quad n \geqslant 2. \end{aligned}$$

Note that q-derivatives are approximations to classical derivatives and if f is a differentiable function, then

$$\lim_{q \to 1} D_q f(x) = Df(x).$$

For polynomials the q-derivative is easy to compute. Indeed it follows easily from the definition of the q-derivative that

$$D_q x^n = [n]_q x^{n-1},$$

where the q-integers $[n]_q$ are defined by,

$$[n]_q = \begin{cases} (1-q^n)/(1-q), & q \neq 1, \\ n, & q = 1. \end{cases}$$

Moreover, the q-factorial is defined by

$$[n]_q! = [1]_q \cdots [n]_q.$$

Quantum integrals are the analogues of classical integrals for the quantum calculus. Quantum integrals satisfy a quantum version of the fundamental theorem of calculus, see [9] for details.

Definition 1. *Let $0 < a < b$. Then the definite q-integral of a function $f(x)$ is defined by a convergent series*

$$\int_0^b f(x) d_q x = (1-q) b \sum_{i=0}^{\infty} q^i f(q^i b)$$

and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

Theorem 1. *[Fundamental Theorem of q-Calculus]*
If $F(x)$ is continuous at $x = 0$, then

$$\int_a^b D_q F(x) d_q x = F(b) - F(a)$$

where $0 \leqslant a < b \leqslant \infty$.

The work [15] gives the mean value theorem in the q-calculus which will be needed in one of our results.

Theorem 2. *If F is continuous and G is $1/q$-integrable and is non-negative(or non-positive) on $[a, b]$, then there exists $\tilde{q} \in (1, \infty)$ such that for all $q > \tilde{q}$ there exists $\xi \in (a, b)$ for which*

$$\int_a^b F(x) G(x) d_{1/q} x = F(\xi) \int_a^b G(x) d_{1/q} x.$$

We also require a q-Hölder inequality and appropriate notions of distance in q-integrals, see [2, 5, 18].

Definition 2. *We will denote by $L_{p,q}([0,b])$ with $1 \leqslant p < \infty$, the set of all functions f on $[0,b]$ such that*

$$||f||_{p,q} := \left(\int_0^b |f|^p d_{1/q}t \right)^{\frac{1}{p}} < \infty.$$

Furthermore let $L_{\infty,q}([0,b])$ denote the set of all functions f on $[0,b]$ such that

$$||f||_{\infty,q} := \sup_{x\in[0,b]} |f(x)| < \infty.$$

Theorem 3. *Let $x \in [0,b]$, $q \in [1,\infty)$ and $p_1, p_2 > 1$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then*

$$\int_0^x |f(x)||g(x)|d_{1/q}t \leqslant \left(\int_0^x |f(x)|^{p_1} d_{1/q}t \right)^{\frac{1}{p_1}} \left(\int_0^x |g(x)|^{p_2} d_{1/q}t \right)^{\frac{1}{p_2}}. \tag{2.1}$$

3. q-Peano Kernel Theorem

In this section we derive a more general form of the Peano kernel theorem based on a q-Taylor expansion. So we start by giving the q-Taylor Theorem with integral remainder. A detailed treatment of the classical Peano kernel theorem can be found in [7, 12, 14].

We use the notation q-$C^k[a,b]$ to denote the space of bounded functions whose q-derivatives of order up to k are continuous on $[a,b]$.

Theorem 4. *(q-Taylor Theorem) Let f be $n+1$ times $1/q$-differentiable in the closed interval $[a,b]$. Then*

$$f(x) = \sum_{k=0}^{n} q^{k(k-1)/2} \frac{(D^k_{1/q}f)(q^k a)}{[k]_q!}(x-a)^{k,q} + R_n(f), \tag{3.1}$$

where

$$(x-t)^{n,q} = (x-q^{n-1}t)\cdots(x-qt)(x-t)$$

and

$$R_n(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^x (D^{n+1}_{1/q}f)(q^n t)(x-t)^{n,q} d_{1/q}t.$$

Another way to express the remainder $R_n(f)$ is to employ the truncated power function. That is

$$R_n(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b (D^{n+1}_{1/q}f)(q^n t)(x-t)^{n,q}_+ d_{1/q}t, \tag{3.2}$$

where

$$(x-t)^{n,q}_+ = (x-q^{n-1}t)\cdots(x-qt)(x-t)_+.$$

Here $(x-t)_+$ *is the truncated power function*

$$(x-t)_+ = \begin{cases} x-t, & \text{if} \quad x > t \\ 0, & \text{otherwise.} \end{cases}$$

Although the latter representation of the remainder $R_n(f)$ associated with our results is new, we omit the proof since it can be done in a similar way as in [9]. There are other forms of q-Taylor Theorem, see for example [1, 8, 10]. The work [3] investigates the convergence of q-Taylor series for q-difference operators using q-Cauchy integral formula.

Theorem 5. *Let* $g_t(x) = (x-t)_+^{n,q}$ *and let* L *be a linear functional that commutes with the operation of* q*-integration and also satisfies the conditions:* $L(g_t)$ *exists and* $L(f) = 0$ *for all* $f \in \mathcal{P}_n$*. Then for all* $f \in 1/q - C^{n+1}[a,b]$

$$L(f) = \int_a^b (D_{1/q}^{n+1} f)(q^n t) K(x,t) d_{1/q} t,$$

where

$$K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!} L(g_t).$$

Proof. Recall that here the function $(x-t)_+^{n,q}$ is a function of t and x behaves as a parameter. When we say $L(g_t)$ we mean that L is applied to the truncated power function, regarded as a function of x with t as a parameter. Hence we find real number that depends on t. We apply L to the equation (3.1). Since L is linear and annihilates polynomials, we have

$$L(f) = \frac{q^{n(n+1)/2}}{[n]_q!} L\left(\int_a^b (D_{1/q}^{n+1} f)(q^n t)(x-t)_+^{n,q} d_{1/q} t \right).$$

Since L commutes with the operation of q-integration,

$$L(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b (D_{1/q}^{n+1} f)(q^n t) L\left((x-t)_+^{n,q} \right) d_{1/q} t.$$

□

Corollary 1. *If the conditions in Theorem 5 are satisfied and also the kernel* $K(x,t)$ *does not change sign on* $[a,b]$*, then*

$$L(f) = \frac{\left(D_{1/q}^{n+1} f \right)(\xi)}{[n+1]_q!} q^{n(n+1)/2} L(x^{n+1}).$$

Proof. Since $D_{1/q}^{n+1} f$ is continuous and $K(x,t)$ does not change sign on $[a,b]$, we can apply the Mean Value Theorem 2. Thus we have

$$L(f) = \left(D_{1/q}^{n+1} f \right)(\xi) \int_a^b K(x,t) d_{1/q} t, \quad a < \xi < b.$$

Replacing $f(x)$ by x^{n+1} gives

$$L(x^{n+1}) = \frac{[n+1]_q!}{q^{n(n+1)/2}} \int_a^b K(x,t) d_{1/q}t,$$

so

$$\int_a^b K(x,t) d_{1/q}t = \frac{q^{n(n+1)/2}}{[n+1]_q!} L(x^{n+1}),$$

and this completes the proof. □

We now establish a relation between q-B-splines and q-Peano kernels. Recently q-analogue or quantum B-splines which generalize B-splines have been investigated in several aspects in [4, 6, 17]. The work [4] finds out that q-B-splines are essentially divided differences of q-truncated power functions. That is, the q-B-spline of degree n is given by

$$N_{k,n}(t;q) = (t_{k+n+1} - t_k)[t_k, \ldots, t_{k+n+1}](x-t)_+^{n,q}.$$

Although classical truncated power function has n multiple zero at $t = x$, q-truncated power function has n distinct zeros for $q \neq 0$ or $q \neq 1$. This property drastically alters certain characteristics of the basis functions. For example while the basis functions forms partition of unity, the non-negativity property is lost. On the other hand when q is near one, the additional real parameter q provides extra flexibility to change the shape of basis functions. Sometimes this effect may be useful and practical to match smoothness of piecewise curves and surfaces up to some tolerance, see [6].

Now recall the fact that a divided difference $f[t_0, t_1, \ldots, t_{n+1}]$ can be represented as symmetric sum of $f(t_j)$, see [14],

$$f[t_0, t_1, \ldots, t_{n+1}] = \sum_{i=0}^{n+1} f(t_i) / \prod_{\substack{j=0 \\ j\neq i}}^{n+1} (t_i - t_j). \tag{3.3}$$

Hence we can readily derive

$$N_{k,n}(t;q) = (t_{k+n+1} - t_k) \sum_{i=k}^{k+n+1} (t_i - t)_+^{n,q} / \prod_{\substack{j=k \\ j\neq i}}^{k+n+1} \frac{1}{(t_i - t_j)}.$$

The following theorem shows that q-B-splines are indeed the q-Peano kernels of divided differences.

Theorem 6. *Let $f \in 1/q - C^{n+1}[a,b]$. Then*

$$f[t_0, t_1, \ldots, t_{n+1}] = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b \frac{N_{0,n}(t;q)}{t_{n+1} - t_0} \left(D_{1/q}^{n+1} f \right)(q^n t) d_{1/q}t.$$

Proof. We first set L as

$$\begin{aligned} f[t_0, t_1, \ldots, t_{n+1}] =& \sum_{i=0}^{n+1} f(t_i) / \prod_{\substack{j=0 \\ j \neq i}}^{n+1} (t_i - t_j) \\ =& L(f). \end{aligned}$$

We see that for any fixed and distinct points $\{t_i : i = 0, 1, \ldots, n+1\}$, L is a bounded linear operator. From the q-Peano Kernel Theorem 5, we have

$$L(f) = \int_a^b K(x,t)(D_{1/q}^{n+1} f)(q^n t) d_{1/q} t,$$

where

$$K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!} L\left((x-t)_+^{n,q}\right).$$

This can be written as

$$K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!} \sum_{i=0}^{n+1} (t_i - t)_+^{n,q} \, / \prod_{\substack{j=0 \\ j \neq i}}^{n+1} (t_i - t_j).$$

Thus

$$K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!} \frac{N_{0,n}(t;q)}{t_{n+1} - t_0}.$$

Combining the last equation with (3.3) we derive

$$f[t_0, t_1, \ldots, t_{n+1}] = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b \frac{N_{0,n}(t;q)}{t_{n+1} - t_0} \left(D_{1/q}^{n+1} f\right)(q^n t) d_{1/q} t.$$

□

When $q = 1$, the above Theorem 3.3 reduces to its classical counterpart which can be found in [14]. The work [4] extends several classical formulas of B-splines to quantum B-splines.

4. Application to polynomial interpolation

The main idea in this section is to apply the q-Peano kernel Theorem on the remainder of polynomial interpolation. Findings demonstrate the advantage of using the q-Peano kernel Theorem where the classical theorem does not work. The following theorem has weaker assumption than the classical one and thus gives stronger results.

Theorem 7. *Let $f \in 1/q - C^{n+1}[a,b]$ and suppose $t_0, t_1, \ldots, t_n \in [a,b]$ are distinct points. For a fixed $x \in [a,b]$, define the corresponding error functional by*

$$L(f) = f(x) - \sum_{k=0}^{n} f(t_k) l_{nk}(x).$$

Then

$$L(f)=\frac{q^{m(m+1)/2}}{[m]_q!}\sum_{k=0}^{n} l_{nk}(x)\int_{t_k}^{x}(t_k-t)^{m,q}\left(D_{1/q}^{m+1}f\right)(q^m t)d_{1/q}t, \qquad m=0,1,\ldots,n.$$

Proof. Since $\sum_{k=0}^{n} l_{nk}(x)=1$, by the q-Peano kernel Theorem 5 we get,

$$\begin{aligned}\frac{[m]_q!}{q^{m(m+1)/2}}K(x,t)=L\left((x-t)_+^{m,q}\right) &= (x-t)_+^{m,q}-\sum_{k=0}^{n}(t_k-t)_+^{m,q} l_{nk}(x)\\ &= \sum_{k=0}^{n}\left[(x-t)_+^{m,q}-(t_k-t)_+^{m,q}\right] l_{nk}(x).\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{[m]_q!}{q^{m(m+1)/2}}\int_a^b K(x,t)\left(D_{1/q}^{m+1}f\right)(q^m t)d_{1/q}t=\\ &\quad\int_a^x\left\{\left(D_{1/q}^{m+1}f\right)(q^m t)\sum_{k=0}^{n}\left[(x-t)^{m,q}-(t_k-t)^{m,q}\right]l_{nk}(x)\right\}d_{1/q}t\\ &\qquad+\sum_{k=0}^{n} l_{nk}(x)\int_{t_k}^{x}(t_k-t)^{m,q}\left(D_{1/q}^{m+1}f\right)(q^m t)d_{1/q}t.\end{aligned}$$

For each $m\leqslant n$, since the interpolation operator is a projection operator, it reproduces polynomials and hence the term in the first summation of the last equation vanishes for $f(x)=(x-t)^{m,q}$. Accordingly,

$$\begin{aligned}L(f) &= \int_a^b K(x,t)\left(D_{1/q}^{m+1}f\right)(q^m t)d_{1/q}t\\ &= \frac{q^{m(m+1)/2}}{[m]_q!}\sum_{k=0}^{n} l_{nk}(x)\int_{t_k}^{x}(t_k-t)^{m,q}\left(D_{1/q}^{m+1}f\right)(q^m t)d_{1/q}t\end{aligned}$$

for each $m=0,1,\ldots,n$. $\square$

Now, we give examples that show how we can find the q-Peano kernel.

Example: Let

$$f(x) = \begin{cases} \dfrac{q^3x^3}{6}, & 0 \leqslant x < 1 \\ \dfrac{1}{6}\left(4 - 4[3]_q x + 4q[3]_q x^2 - 3q^3x^3\right), & 1 \leqslant x < 2 \\ \dfrac{1}{6}\left(-44 + 20[3]_q x - 8q[3]_q x^2 + 3q^3x^3\right), & 2 \leqslant x < 3 \\ -\dfrac{1}{6}(-4+x)(-4+qx)(-4+q^2x), & 3 \leqslant x < 4 \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that for $q \neq 1$, $f \in C[0,4]$ but $f \notin C^1[0,4]$. However, one may check that $f \in 1/q - C^2[0,4]$. Classical error functionals cannot work but we may find the error via the q-Peano kernel theorem. Let $t_0 = 0$, $t_1 = 2$ and $t_2 = 4$. Then it is appropriate to take the error functional

$$L(f) = q\sum_{k=0}^{2} l_{2k}(x)\int_{t_k}^{x}(t_k - t)\left(D^2_{1/q}f\right)(qt)d_{1/q}t,$$

where $l_{20}(x) = \frac{1}{8}(x-2)(x-4)$, $l_{21}(x) = -\frac{1}{4}x(x-4)$ and $l_{22}(x) = \frac{1}{8}x(x-2)$. Hence

$$\begin{aligned} \frac{1}{q}L(f) &= l_{20}(x)\int_0^x(-t)\left(D^2_{1/q}f\right)(qt)d_{1/q}t + l_{21}(x)\int_2^x(2-t)\left(D^2_{1/q}f\right)(qt)d_{1/q}t \\ &+ l_{22}(x)\int_4^x(4-t)\left(D^2_{1/q}f\right)(qt)d_{1/q}t. \end{aligned}$$

Now we will find the kernel. If $0 \leqslant x < 2$, then

$$K(x,t) = \begin{cases} -l_{20}(x)t, & 0 \leqslant t < x \\ l_{21}(x)(2-t) - l_{22}(x)(4-t), & x \leqslant t < 2 \\ -l_{22}(x)(4-t), & 2 \leqslant t < 4. \end{cases}$$

Similarly, for $2 \leqslant x < 4$,

$$K(x,t) = \begin{cases} -l_{20}(x)t, & 0 \leqslant t < 2 \\ -l_{20}(x)t + l_{21}(x)(2-t), & 2 \leqslant t < x \\ l_{21}(x)(2-t) - l_{22}(x)(4-t), & x \leqslant t < 4. \end{cases}$$

One may notice that the function $f(x)$ given above is indeed a cubic q-B-spline. A more recent work [6] on the q-B-splines demonstrates that these functions prove useful in several aspects including geometric modelling.

4.1. **Trapezoidal rule in q-integration.** Consider the $1/q$-integral of a function f on the interval $[a, b]$. We want to evaluate the q-integral approximately using linear interpolation formula. Let us define the operator L as

$$L(f) = \int_a^b f(x) d_{1/q}x - \frac{b - aq}{[2]_q} f(a) - \frac{bq - a}{[2]_q} f(b).$$

Since $L(f) = 0$ for all functions $f \in \mathcal{P}_1$ and for all $f \in 1/q - C^2[a, b]$, we have

$$L(f) = \int_a^b \left(D^2_{1/q} f \right) (qt) K(x, t) d_{1/q}t$$

and

$$K(x, t) = qL((x - t)_+).$$

Thus,

$$K(x, t) \quad = \quad \frac{q}{[2]_q} (b - t)(a - t), \qquad a \leqslant t \leqslant b.$$

Notice that $K(x, t) < 0$ on $[a, b]$. Then by applying Mean Value Theorem 2 we have

$$L(f) = \frac{D^2_{1/q} f(\xi)}{[2]_q!} qL(x^2),$$

where

$$L(x^2) \quad = \quad \frac{-(b - a)(bq - a)(b - aq)}{[3]_q!}$$

Therefore we find that

$$L(f) \quad = \quad \frac{-q(b - a)(bq - a)(b - aq)}{[3]_q![2]_q!} D^2_{1/q} f(\xi), \quad a < \xi < b.$$

While $q = 1$ reduces the above $L(f)$ to the classical error functional of the trapezoidal rule, on the other hand it provides extra flexibility on the control of error functional by changing the parameter q appropriately.

4.2. **The remainder on quadrature.** We now discuss error bounds of quadrature formulas on remainders given by

$$R_n(f; q) = \int_0^b f(x) d_{1/q}x - \sum_{k=0}^{n} \gamma_{nk} f(t_{nk}).$$

For $q = 1$, this formula is well-known in the context of numerical integration. Assuming $f \in 1/q - C^{m+1}[0,b]$ and $R_n(f;q) = 0$ for all $f \in \mathcal{P}_m, m = 0,1,\dots,n$, we can apply the q-Peano kernel theorem. Hence

$$R_n(f;q) = \int_0^b K(x,t)\left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q}t.$$

By applying the q-Hölder inequality (2.1), we have

$$|R_n(f;q)| \leqslant \left[\int_0^b \left|\left(D_{1/q}^{m+1} f\right)(q^m t)\right|^{p_1} d_{1/q}t\right]^{\frac{1}{p_1}} \left[\int_0^b |K(x,t)|^{p_2}\, d_{1/q}t\right]^{\frac{1}{p_2}}$$

for all $1 \leqslant p_1, p_2 \leqslant \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Since the second integral in the last inequality is independent of f, by choosing coefficients and nodes appropriately we can minimize the remainder $R_n(f;q)$.

(i) For $p_1 = \infty$ and $p_2 = 1$,

$$|R_n(f;q)| \leqslant ||D_{1/q}^{m+1} f||_\infty \int_0^b |K(x,t)| d_{1/q}t.$$

(ii) For $p_1 = p_2 = 2$,

$$|R_n(f;q)| \leqslant ||D_{1/q}^{m+1} f||_2 \left[\int_0^b |K(x,t)|^2 d_{1/q}t\right]^{\frac{1}{2}}.$$

The q-Peano kernel $K(x,t)$ in the latter inequality can be written as

$$K(x,t) = q^{m(m+3)/2} \frac{(b - \frac{t}{q})^{m+1,q}}{[m+1]_q!} - s(t;q),$$

where $s(t;q) = \dfrac{q^{m(m+1)/2}}{[m]_q!} \displaystyle\sum_{k=0}^{n} \gamma_{nk}(t_{nk} - t)_+^{m,q}$ is a quantum spline with the knot sequence $\{t_{n0},\dots,t_{nn}\}$. Eventually, the problem of minimizing the q-integral

$$\left[\int_0^b |K(x,t)|^{p_1} d_{1/q}t\right]^{\frac{1}{p_1}}$$

is equivalent to finding the best approximation of the polynomial

$$q^{m(m+3)/2} \frac{(b - \frac{t}{q})^{m+1,q}}{[m+1]_q!}$$

in t by a quantum spline with respect to the norm $||.||_{p_1}$.

5. Conclusion

In this work, we investigated a generalization of classical Peano Kernel theorem via quantum calculus. Applications to polynomial interpolation, q-integration and quantum spline functions, and best approximation were also presented. In the future, we aim to establish relations between q-Peano kernels and Green's functions using q-difference equations and quantum calculus.

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ON METALLIC SEMI-SYMMETRIC METRIC $F-$CONNECTIONS

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Abstract. In this article, we generate a metallic semi-symmetric metric F-connection on a locally decomposable metallic Riemann manifold. Also, we examine some features of torsion and curvature tensor fields of this connection.

1. Introduction

The topic of connection with torsion on a Riemann manifold has been studied with great interest in literature. Firstly, Hayden defined the concept of metric connection with torsion [3]. For a linear connection $\widetilde{\nabla}$ with torsion on a Riemann manifold (M, g), if $\widetilde{\nabla}g = 0$, then linear connection $\widetilde{\nabla}$ is called a metric connection. Then, Yano constructed a connection whose torsion tensor has the form: $S(X, Y) = \omega(Y)X - \omega(X)Y$, where ω is a $1-$form, [15] and named this connection as semi-symmetric connection.

In [11], Prvanovic has defined a product semi-symmetric $F-$connection on locally decomposable Riemann manifold and worked its curvature properties. A locally decomposable Riemann manifold is expressed by the triple (M, g, F) and the conditions $\nabla F = 0$ and $g(FX, Y) = g(X, FY)$ are provided, where F, g and ∇ are product structure, metric tensor and Riemann connection (or Levi-Civita connection) of g on manifold respectively. For further references, see [8, 9, 10, 12].

The positive root of the equation $x^2 - x - 1 = 0$ is the number $x_1 = \frac{1+\sqrt{5}}{2}$, which is called golden ratio. The golden ratio has many applications and has played an important role in mathematics. One of them is a golden Riemann manifold (M, g, φ) endowed with golden structure φ and Riemann metric tensor g. The golden structure φ created by Crasmareanu and Hretcanu is actually root of the equality $\varphi^2 - \varphi - I = 0$ [5]. In [2], the authors have defined golden semi-symmetric metric $F-$connections on a locally decomposable golden Riemann manifold and examined torsion, projective curvature, conharmonic curvature and curvature tensors of this connection. Also, the golden ratio has many important generalizations. One

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of the them is metallic proportions or metallic means family which was introduced by de Spinadel in [6, 7]. The positive root of the equation $x^2 - px - q = 0$ is called the metallic means family, where p and q are two positive integer. Also, the solution of the metallic means family is as follows

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

These numbers $\sigma_{p,q}$ are also named (p, q) metallic numbers. In the last equation,

- if $p = q = 1$, then the number $\sigma_{1,1} = \frac{1+\sqrt{5}}{2}$ is golden ratio;
- if $p = 2$ and $q = 1$, then the number $\sigma_{2,1} = 1 + \sqrt{2}$ is silver ratio, which is used for fractal and Cantorian geometry;
- if $p = 3$ and $q = 1$, then the number $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$ is bronze ratio, which plays an important role in dynamical systems and quasicrystals and so on.

Inspired by the metallic number family, Hretcanu and Crasmareanu was introduced metallic Riemann structure [4]. Indeed, a metallic structure is polynomial structure such that $F^2 - pF - qI = 0$, where F is $(1, 1)$-tensor field on manifold. Given a Riemann manifold (M, g) endowed with the metallic structure F, if

$$g(FX, Y) = g(X, FY)$$

or equivalently

$$g(FX, FY) = pg(FX, Y) + qg(X, Y)$$

for all vector fields X and Y on M, then the triple (M, g, F) is called a metallic Riemann manifold.

In [1], For almost product structures J and the Tachibana operator ϕ_F, the authors proved that the manifold (M, g, F) is a locally decomposable metallic Riemannian manifold iff $\phi_{J_\pm} g = 0$. In this article, we made a semi-symmetric metric $F-$connection with metallic structure F on a locally decomposable metallic Riemann manifold. Then we examine some properties related to its torsion and curvature tensors.

2. Preliminaries

Let M be an n-dimensional manifold. Throughout this paper, tensor fields, connections and all manifolds are always assumed to be differentiable of class C^∞

For a $(1, 1)-$tensor F and a $(r, s)-$tensor K, The tensor K is named as a pure tensor with regard to the tensor F, if the following condition is holds:

$$\begin{aligned} K^{j_1 \ldots j_r}_{m i_2 \ldots i_s} F^{\ m}_{i_1} &= K^{j_1 \ldots j_r}_{i_1 m \ldots i_s} F^{\ m}_{i_2} = \ldots = K^{j_1 \ldots j_r}_{i_1 i_2 \ldots m} F^{\ m}_{i_s} = \\ K^{m j_2 \ldots j_r}_{i_1 \ldots i_s} F^{\ j_1}_{m} &= K^{j_1 m \ldots j_r}_{i_1 \ldots i_s} F^{\ j_2}_{m} = \ldots = K^{j_1 j_2 \ldots m}_{i_1 \ldots i_s} F^{\ j_r}_{m}, \end{aligned}$$

where $K^{j_1 j_2 \ldots jr}_{i_1 i_2 \ldots is}$ and $F_i^{\ j}$ is the components the tensor K and $(1, 1)-$tensor F respectively. Also, the Tachibana operator applied to a pure $(r, s)-$tensor K is given by

$$(\phi_F K)^{j_1 \ldots j_r}_{k i_1 \ldots i_s} = F_k{}^m \partial_m t^{j_1 \ldots j_r}_{i_1 \ldots i_s} - \partial_k \left(K \circ F\right)^{j_1 \ldots j_r}_{i_1 \ldots i_s} + \sum_{\lambda=1}^{s} \left(\partial_{i_\lambda} F_k{}^m\right) \; K^{j_1 \ldots j_r}_{i_1 \ldots m \ldots i_s} + \sum_{\mu=1}^{r} \left(\partial_k F_m{}^{j_\mu} - \partial_m F_k{}^{j_\mu}\right) \; K^{i_1 \ldots m \ldots i_r}_{j_1 \ldots j_s}, \tag{2.1}$$

where

$$\begin{aligned}(K \circ F)^{j_1 \ldots j_r}_{i_1 \ldots i_s} &= K^{j_1 \ldots j_r}_{m i_2 \ldots i_s} F_{i_1}^{\ m} = \ldots = K^{j_1 \ldots j_r}_{i_1 i_2 \ldots m} F_{i_s}^{\ m} \\ &= K^{m j_2 \ldots j_r}_{i_1 \ldots i_s} F_m^{\ j_1} = \ldots = K^{j_1 j_2 \ldots m}_{i_1 \ldots i_s} F_m^{\ j_r}.\end{aligned}$$

The equation (2.1) firstly defined by Tachibana [14] and the applications of this operator have been made by many authors [13, 16]. For the pure tensor K, if the condition $\phi_F K = 0$ holds, then K is called as a $\phi-$tensor. Specially, if the $(1,1)-$tensor F is a product structure, then K is a decomposable tensor [14].

A metallic Riemannian manifold is a manifold M equipped with a $(1,1)-$tensor field F and a Riemannian metric g which satisfy the following conditions:

$$F^2 - pF - qI = 0 \tag{2.2}$$

and

$$g(FX, Y) = g(X, FY) \tag{2.3}$$

Also, the equation (2.3) equal to $g(FX, FY) = pg(FX, Y) + qg(X, Y)$, where p, q are positive integers. The last two equations in local coordinates are as follows:

$$F_i{}^k F_k{}^j = pF_i{}^j + q\delta_i^j \tag{2.4}$$

and

$$F_i{}^k g_{kj} = F_j{}^k g_{ik}, \tag{2.5}$$

It is obvious that $F_i{}^k F_{kj} = pF_{ij} + qg_{ij}$ and $F_{ij} = F_{ji}$ (symmetry) from (2.4) and (2.5). The almost product structure J and metallic structure F on M are related to each other as follows [4],

$$J_{\pm} = \frac{p}{2} I \pm (\frac{2\sigma_{p,q} - p}{2}) F \tag{2.6}$$

or conversely

$$F_{\pm} = \pm \left(\frac{2}{2\sigma_{p,q} - p} J - \frac{p}{2\sigma_{p,q} - p} I \right), \tag{2.7}$$

where $\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$ which is the root of the (2.2). Also, it is obvious from (2.7) that a Riemann metric g is pure with regard to a metallic structure F if and only if the Riemann metric g is pure with regard to the almost product structure J. By using (2.7) and (2.1), we have

$$\phi_F K = \pm\frac{2}{2\sigma_{p,q} - p}\phi_J K \tag{2.8}$$

for any $(r,s)-$tensor K. We note that a metallic Riemann manifold (M,g,F) is a locally decomposable metallic Riemann manifold if and only if the Riemann metric g is a decomposable tensor, i.e., $(\phi_J g)_{kij} = 0$ and the condition $(\phi_J g)_{kij} = 0$ is equivalent to $\nabla_k J_i^{\ j} = 0$ [1].

3. The Metallic Semi-Symmetric metric $F-$connection

Let (M,g,F) be a locally decomposable metallic Riemann manifold. We consider an affine connection $\widetilde{\nabla}$ on M. If the affine connection $\widetilde{\nabla}$ holds

$$\begin{aligned} i)\ \widetilde{\nabla}_h g_{ij} &= 0 \\ ii)\ \widetilde{\nabla}_h F_i^{\ j} &= 0, \end{aligned} \tag{3.1}$$

then it is called a metric $F-$connection. In the special case, when the torsion tensor $\widetilde{S}_{ij}^k$ of $\widetilde{\nabla}$ is as following shape

$$\widetilde{S}_{ij}^k = \omega_j\delta_i^k - \omega_i\delta_j^k + \frac{1}{q}\left(\omega_t F_j^{\ t} F_i^{\ k} - \omega_t F_i^{\ t} F_j^{\ k}\right), \tag{3.2}$$

where ω_i are local ingredients of an $1-$form, we say that the affine connection $\widetilde{\nabla}$ is a metallic semi-symmetric metric connection.

Let $\widetilde{\Gamma}_{ij}^k$ be the ingredients of the metallic semi-symmetric metric connection $\widetilde{\nabla}$. If we put

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + T_{ij}^k, \tag{3.3}$$

where Γ_{ij}^k and T_{ij}^k are the ingredients of the Riemann connection ∇ of g and $(1,2)-$tensor field T on M respectively, then the torsion tensor $\widetilde{S}_{ij}^k$ of $\widetilde{\nabla}$ is as following form

$$\widetilde{S}_{ij}^k = \widetilde{\Gamma}_{ij}^k - \widetilde{\Gamma}_{ji}^k = T_{ij}^k - T_{ji}^k.$$

When the connection (3.3) provides the condition (i) of (3.1), by applying the method in [3], we get

$$T_{ij}^k = \omega_j\delta_i^k - \omega^k g_{ij} + \frac{1}{q}\left(\omega_t F_j^{\ t} F_i^{\ k} - \omega_t F^{kt} F_{ij}\right),$$

where $\omega^k = \omega_i g^{ik}$, $F^{kt} = F_i^{\ t} g^{ik}$ and $F_{ij} = F_j^{\ k} g_{ik}$. Hence the connection (3.3) becomes the following form

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \omega_j\delta_i^k - \omega^k g_{ij} + \frac{1}{q}\left(\omega_t F_j^{\ t} F_i^{\ k} - \omega_t F^{kt} F_{ij}\right). \tag{3.4}$$

Also, by using the connection (3.4), we obtain the following equation with a simple calculation:

$$\widetilde{\nabla}_k F_i^{\ j} = g_{ki}(\omega^t F_t^{\ j} - \omega_t F^{jt}) = 0.$$

Therefore, the connection $\widetilde{\nabla}$ given by (3.4) is named metallic semi-symmetric metric $F-$connection.

4. Curvature and Torsion properties of the Metallic Semi-Symmetric metric $F-$connection

In this section, we examine some properties associated with the torsion and curvature tensor of the connection (3.4).

Let (M, g, F) be a locally decomposable metallic Riemann manifold endowed with the connection (3.4). We say easily that the torsion tensor $\widetilde{S}$ of the connection (3.4) is pure. Indeed, by using (2.4) and (3.2), we get

$$\widetilde{S}^k_{im} F_j^{\ m} = \widetilde{S}^k_{mj} F_i^{\ m} = \widetilde{S}^m_{ij} F_m^{\ k}.$$

In [13], the author prove that a $F-$connection is pure iff torsion tensor of that connection is pure. Thus, the connection (3.4) provides the following condition:

$$\widetilde{\Gamma}^k_{mj} F_i^{\ m} = \widetilde{\Gamma}^k_{im} F_j^{\ m} = \widetilde{\Gamma}^m_{ij} F_m^{\ k}.$$

Theorem 4.1. *Let (M, g, F) be a locally decomposable metallic Riemann manifold endowed with the connection (3.4). If the $1-$form ω is a $\phi-$tensor,then the torsion tensor $\widetilde{S}$ of the connection (3.4) is a $\phi-$tensor and holds following equation:*

$$F_k^{\ m}(\nabla_m \widetilde{S}^l_{ij}) = F_i^{\ m}(\nabla_k \widetilde{S}^l_{mj}) = F_j^{\ m}(\nabla_k \widetilde{S}^l_{im}). \tag{4.1}$$

Proof. Let (M, g, F) be a locally decomposable metallic Riemann manifold. Since a zero tensor is pure, a $F-$connection with torsion-free is always pure. Hence, we can say that the Levi-Civita connection ∇ of g on M is always pure with respect to F.

If we implement the Tachibana operator ϕ_F to the torsion tensor $\widetilde{S}$ of the connection (3.4), then we have

$$\begin{aligned}
(\phi_F \widetilde{S})_{kij}^{\ \ \ l} &= F_k^{\ m}(\partial_m \widetilde{S}^l_{ij}) - \partial_k(\widetilde{S}^l_{mj} F_i^{\ m}) \\
&= F_k^{\ m}(\nabla_m \widetilde{S}^l_{ij} + \Gamma^s_{mi}\widetilde{S}^l_{sj} + \Gamma^s_{mj}\widetilde{S}^l_{is} - \Gamma^l_{ms}\widetilde{S}^s_{ij}) \\
&\quad - F_i^{\ m}(\nabla_k \widetilde{S}^l_{mj} + \Gamma^s_{km}\widetilde{S}^l_{sj} + \Gamma^s_{kj}\widetilde{S}^l_{ms} - \Gamma^l_{ks}\widetilde{S}^s_{mj}).
\end{aligned}$$

When the torsion tensor $\widetilde{S}$ and Levi-Civita connection ∇ are pure, the above relation reduces to

$$(\phi_F \widetilde{S})_{kij}^{\ \ \ l} = F_k^{\ m}(\nabla_m \widetilde{S}^l_{ij}) - F_i^{\ m}(\nabla_k \widetilde{S}^l_{mj}). \tag{4.2}$$

Substituting (3.2) into (4.2), we get

$$\begin{aligned}(\phi_F\widetilde{S})_{kij}{}^{l} &= [(\nabla_m\omega_j)F_k{}^m-(\nabla_k\omega_m)F_j{}^m]\delta_i^l \\ &-[(\nabla_m\omega_i)F_k{}^m-(\nabla_k\omega_m)F_i{}^m]\delta_j^l \\ &+[\frac{1}{q}(\nabla_m\omega_s)F_k{}^mF_j{}^s-\frac{p}{q}(\nabla_k\omega_s)F_j{}^s-\nabla_k\omega_j]F_i{}^l \\ &-[\frac{1}{q}(\nabla_m\omega_s)F_k{}^mF_i{}^s-\frac{p}{q}(\nabla_k\omega_s)F_i{}^s-\nabla_k\omega_i]F_j{}^l.\end{aligned} \tag{4.3}$$

Also, for the $1-$form ω, we calculate

$$\begin{aligned}(\phi_F\omega)_{kj} &= F_k{}^m(\partial_m\omega_j)-\partial_k(F_j{}^m\omega_m) \\ &= F_k{}^m(\nabla_m\omega_j+\Gamma^s_{mj}\omega_s)-F_j{}^m(\nabla_k\omega_m+\Gamma^s_{km}\omega_s) \\ &= F_k{}^m(\nabla_m\omega_j)-F_j{}^m(\nabla_k\omega_m).\end{aligned}$$

From last equation, we can say that the $1-$form ω is a $\phi-$tensor iff

$$F_k^m(\nabla_m p_j)=F_j^m(\nabla_k p_m). \tag{4.4}$$

Assuming that the $1-$form ω is a $\phi-$tensor, thanks to (2.4) the relation (4.3) becomes $(\phi_F\widetilde{S})_{kij}{}^{l}=0$, i.e., the torsion tensor $\widetilde{S}$ is a $\phi-$tensor. Also, from the equation (4.2) we get

$$F_k{}^m(\nabla_m\widetilde{S}^l_{ij})=F_i{}^m(\nabla_k\widetilde{S}^l_{mj})=F_j{}^m(\nabla_k\widetilde{S}^l_{im}).$$

The proof is complete. □

From the equation (2.8), it is obvious that the torsion tensor $\widetilde{S}$ of the connection (3.4) and the $1-$form ω are hold following equality

$$\phi_J\widetilde{S}=0 \quad\text{and}\quad \phi_J\omega=0,$$

i.e., they are decomposable tensors, where J is the product structure associated with the metallic structure F. From on now, we shall consider $1-$form ω is a $\phi-$tensor (or decomposable tensor), i.e., the following conditions are provided:

$$F_k{}^m(\nabla_m\omega_j)=F_j{}^m(\nabla_k\omega_m)$$

and

$$J_k{}^m(\nabla_m\omega_j)=J_j{}^m(\nabla_k\omega_m).$$

It is well known that the curvature tensor $\widetilde{R}_{ijk}{}^{l}$ of the connection (3.4) is as follows:

$$\widetilde{R}_{ijk}{}^{l}=\partial_i\widetilde{\Gamma}^l_{jk}-\partial_j\widetilde{\Gamma}^l_{ik}+\widetilde{\Gamma}^l_{im}\widetilde{\Gamma}^m_{jk}-\widetilde{\Gamma}^l_{jm}\widetilde{\Gamma}^m_{ik}.$$

Then, the curvature tensor $\widetilde{R}_{ijk}{}^{l}$ can be expressed

$$\begin{aligned}\widetilde{R}_{ijk}{}^{l} &= R_{ijk}{}^{l} + \delta_j^l \mathcal{A}_{ik} - \delta_i^l \mathcal{A}_{jk} + g_{ik}\mathcal{A}_j{}^{l} - g_{jk}\mathcal{A}_i{}^{l} \\ &\quad + \frac{1}{q}(F_j{}^{l}F_k{}^{t}\mathcal{A}_{it} - F_i{}^{l}F_k{}^{t}\mathcal{A}_{jt} + F_{ik}F^{lt}\mathcal{A}_{jt} - F_{jk}F^{lt}\mathcal{A}_{it}),\end{aligned} \tag{4.5}$$

where $R_{ijk}{}^{l}$ are the ingredients of the Riemann curvature tensor of the Riemann connection ∇ and

$$\mathcal{A}_{jk} = \nabla_j\omega_k - \omega_j\omega_k + \frac{1}{2}\omega^m\omega_m g_{kj} - \frac{1}{q}\omega_m\omega_t F_k{}^{t}F_j{}^{m} + \frac{1}{2q}\omega^m\omega_t F_m^{\ t}F_{jk}. \tag{4.6}$$

It is clear that the tensor A provide $\mathcal{A}_{jk} - \mathcal{A}_{kj} = \nabla_j\omega_k - \nabla_k\omega_j = 2(d\omega)_{jk}$, where the operator d is exterior differential on M. Thus, we say that $\mathcal{A}_{jk} - \mathcal{A}_{kj} = 0$ if and only if $1-$form ω is closed.

Also, from the equation (4.5), we obtain

$$\begin{aligned}\widetilde{R}_{ijkl} &= R_{ijkl} + g_{jl}\mathcal{A}_{ik} - g_{il}\mathcal{A}_{jk} + g_{ik}\mathcal{A}_{jl} - g_{jk}\mathcal{A}_{il} \\ &\quad + \frac{1}{q}(F_{jl}F_k{}^{t}\mathcal{A}_{it} - F_{il}F_k{}^{t}\mathcal{A}_{jt} + F_{ik}F_l{}^{t}\mathcal{A}_{jt} - F_{jk}F_l{}^{t}\mathcal{A}_{it}).\end{aligned} \tag{4.7}$$

It is clear that the curvature tensor satisfies $\widetilde{R}_{ijkl} = -\widetilde{R}_{jikl}$ and $\widetilde{R}_{ijkl} = -\widetilde{R}_{ijlk}$.

For Ricci tensors of the connection (3.4) $\widetilde{R}_{jk}$, contracting (4.5) with respect to i and l, we have

$$\begin{aligned}\widetilde{R}_{jk} &= R_{jk} + (4-n)\,\mathcal{A}_{jk} - trace\mathcal{A}g_{jk} \\ &\quad + \frac{1}{q}\left(2p - F_l{}^{l}\right)F_k{}^{t}\mathcal{A}_{jt} - \frac{1}{q}F_{jk}F_l^t\mathcal{A}_t^l,\end{aligned} \tag{4.8}$$

where R_{jk} is Ricci tensors of the Riemann connection ∇ of g and

$$trace\mathcal{A} = \mathcal{A}_l{}^{l} = \nabla_l\omega^l + (\frac{n-4}{2})\omega_l\omega^l - \frac{1}{q}\omega_t\omega^m F_m^{\ t}(p - \frac{1}{2}F_l{}^{l}).$$

Contracting the last equation with g^{jk}, for the scalar curvature $\overline{\tau}$ of the connections (3.4), we get

$$\overline{\tau} = \tau + 2\,(2-n)\,trace\mathcal{A} + \frac{2}{q}\left(p - F_l{}^{l}\right)F_l^t\mathcal{A}_t^l, \tag{4.9}$$

where τ is scalar curvature of Levi-Civita connection ∇ of g. From the equation (4.8), we can have

$$\widetilde{R}_{jk} - \widetilde{R}_{kj} = (n-4)\,(\mathcal{A}_{kj} - \mathcal{A}_{jk}) + \frac{1}{q}\left(2p - F_l{}^{l}\right)F_k{}^{t}\,(\mathcal{A}_{jt} - \mathcal{A}_{tj})\,. \tag{4.10}$$

From the equation (4.10), we easily say that if the $1-$form ω is closed, then $\widetilde{R}_{jk} - \widetilde{R}_{kj} = 0$.

Lemma 4.2. *Let (M, g, F) be a locally decomposable metallic Riemann manifold endowed with the connection (3.4). Then the tensor $\mathcal{A}$ given by (4.6) is a $\phi-$tensor (or decomposable tensor) and thus the following relation holds:*

$$(\nabla_m \mathcal{A}_{ij})\, F_k^{\ m} = (\nabla_k \mathcal{A}_{mj})\, F_i^{\ m} = (\nabla_k \mathcal{A}_{im})\, F_j^{\ m}.$$

Proof. The tensor $\mathcal{A}$ is pure with regard to F. Indeed

$$F_k^{\ t} \mathcal{A}_{it} - F_i^{\ t} \mathcal{A}_{tk} = (\nabla_i \omega_t)\, F_k^{\ t} - (\nabla_t \omega_k)\, F_i^{\ t} = 0.$$

If the Tachibana operator is applied to the tensor A, then we get

$$\begin{aligned} (\phi_F \mathcal{A})_{kij} &= F_k^{\ m} (\partial_m \mathcal{A}_{ij}) - \partial_k (\mathcal{A}_{mj} F_i^{\ m}) \\ &= F_k^{\ m} \left(\nabla_m \mathcal{A}_{ij} + \Gamma^s_{mi} \mathcal{A}_{sj} + \Gamma^s_{mj} \mathcal{A}_{is}\right) \\ &\quad - F_i^{\ m} (\nabla_k \mathcal{A}_{mj} + \Gamma^s_{km} \mathcal{A}_{sj} + \Gamma^s_{kj} \mathcal{A}_{ms}). \end{aligned}$$

From the purity of the Riemann connection ∇ and the tensor $\mathcal{A}$, we have

$$(\phi_F \mathcal{A})_{kij} = (\nabla_m \mathcal{A}_{ij})\, F_k^{\ m} - (\nabla_k \mathcal{A}_{mj})\, F_i^{\ m}. \tag{4.11}$$

Substituting (4.6) into (4.11), standard calculations give

$$(\phi_F \mathcal{A})_{kij} = (\nabla_m \nabla_i \omega_j)\, F_k^{\ m} - (\nabla_k \nabla_m \omega_j)\, F_i^{\ m}. \tag{4.12}$$

When we apply the Ricci identity to the $1-$form ω, we get

$$(\nabla_m \nabla_i \omega_j)\, F_k^{\ m} = (\nabla_i \nabla_m \omega_j)\, F_k^{\ m} - \frac{1}{2} \omega_s R_{mij}^{\ \ \ s} F_k^{\ m}$$

and

$$\begin{aligned} (\nabla_k \nabla_m \omega_j)\, F_i^{\ m} &= (\nabla_k \nabla_i \omega_m)\, F_j^{\ m} \\ &= (\nabla_i \nabla_k \omega_m)\, F_j^{\ m} - \frac{1}{2} \omega_s R_{kim}^{\ \ \ s} F_j^{\ m} \\ &= (\nabla_i \nabla_m \omega_k)\, F_j^{\ m} - \frac{1}{2} \omega_s R_{kim}^{\ \ \ s} F_j^{\ m} \end{aligned}$$

With the help of the last two equation, from (4.12), the equation (4.12) becomes as follows,

$$(\phi_F \mathcal{A})_{kij} = -\frac{1}{2} \omega_s (R_{mij}^{\ \ \ s} F_k^{\ m} - R_{kim}^{\ \ \ s} F_j^{\ m}).$$

In a locally decomposable metallic Riemann manifold (M, g, F), the Riemann curvature tensor R is pure [1]. This instantly gives $(\phi_F \mathcal{A})_{kij} = 0$. Hence, from (4.11) we can write

$$(\nabla_m \mathcal{A}_{ij})\, F_k^{\ m} = (\nabla_k \mathcal{A}_{mj})\, F_i^{\ m} = (\nabla_k \mathcal{A}_{im})\, F_j^{\ m}.$$

Also, with help of (2.8), we can say that $\phi_J A = 0$, i.e., the tensor A is decomposable, where J is the product structure associated with the metallic structure F. $\square$

By using the purity of the tensor $\mathcal{A}$, standard calculations give

$$\widetilde{R}_{imk}{}^{l}F_{j}{}^{m}=\widetilde{R}_{ijm}{}^{l}F_{k}{}^{m}=\widetilde{R}_{ijk}{}^{m}F_{m}{}^{l}=\widetilde{R}_{mjk}{}^{l}F_{i}{}^{m},$$

i.e., the curvature tensor $\widetilde{R}$ is pure with respect to metallic structure F.

If Tachibana operator ϕ_F is applied to the curvature tensor $\widetilde{R}$, then we get

$$\begin{aligned}(\phi_F\widetilde{R})_{kijl}{}^{t} &= F_k{}^{m}(\partial_m\widetilde{R}_{ijl}{}^{t})-\partial_k(\widetilde{R}_{mjl}{}^{t}F_i{}^{m}) \qquad (4.13)\\ &= F_k{}^{m}(\nabla_m\widetilde{R}_{ijl}{}^{t}+\Gamma^{s}_{mi}\widetilde{R}_{sjl}{}^{t}+\Gamma^{s}_{mj}\widetilde{R}_{isl}{}^{t}+\Gamma^{s}_{ml}\widetilde{R}_{ijs}{}^{t}-\Gamma^{t}_{ms}\widetilde{R}_{ijl}{}^{m})\\ &\quad -F_i{}^{m}(\nabla_k\widetilde{R}_{mjl}{}^{t}+\Gamma^{s}_{km}\widetilde{R}_{sjl}{}^{t}+\Gamma^{s}_{kj}\widetilde{R}_{msl}{}^{t}+\Gamma^{s}_{kl}\widetilde{R}_{mjs}{}^{t}-\Gamma^{t}_{ks}\widetilde{R}_{mjl}{}^{s})\\ &= (\nabla_m\widetilde{R}_{ijl}{}^{t})F_k{}^{m}-(\nabla_k\widetilde{R}_{mjl}{}^{t})F_i{}^{m}\end{aligned}$$

from which, by (4.5), we find

$$\begin{aligned}(\phi_F\widetilde{R})_{kijl}{}^{t} &= (\phi_F R)_{kijl}{}^{t}+[(\nabla_k\mathcal{A}_{jm})\,F_l{}^{m}-(\nabla_m\mathcal{A}_{jl})\,F_k{}^{m}]\delta_i^t\\ &\quad+[(\nabla_m\mathcal{A}_{il})\,F_k{}^{m}-(\nabla_k\mathcal{A}_{im})\,F_l{}^{m}]\delta_j{}^{t}\\ &\quad+[(\nabla_m\mathcal{A}_j^t)F_k{}^{m}-(\nabla_k\mathcal{A}_j^m)F_m{}^{t}]g_{il}\\ &\quad+[(\nabla_k\mathcal{A}_i^m)F_m{}^{t}-(\nabla_m\mathcal{A}_i^t)F_k{}^{m}]g_{jl}.\end{aligned}$$

In a locally decomposable metallic Riemann manifold (M,g,F), since the Riemann curvature tensor R is a $\phi-$tensor [1], considering Lemma 4.2, the last relation becomes $\phi_F\widetilde{R}=0$. Also, from the equation (2.8), we can say that $\phi_J\widetilde{R}=0$, where J is the product structure associated with the metallic structure F. Thus we obtain the following theorem:

Theorem 4.3. *Let (M,g,F) be a locally decomposable metallic Riemann manifold endowed with the connection (3.4). The curvature tensor $\widetilde{R}$ of the connection (3.4) is a $\phi-$tensor (or decomposable tensor).*

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SOME NEW SIMPSON TYPE INEQUALITIES FOR THE p-CONVEX AND p-CONCAVE FUNCTIONS

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Abstract. In this paper, we establish some new Simpson type inequalities for the class of functions whose derivatives in absolute values at certain powers are p-convex and p-concave.

1. INTRODUCTION

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

It is well known that theory of convex sets and convex functions play an important role in mathematics and the other pure and applied sciences. In recent years, the concept of convexity has been extended and generalized in various directions using novel and innovative techniques. For some inequalities, generalizations and applications concerning convexity see [1, 2, 4, 5, 6, 16, 20].

In [9], the author gave definition harmonically convex and concave functions as follow.

Definition 1. *Let $I \subset \mathbb{R}\setminus\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in I$ and $t \in [0,1]$. If this inequality is reversed, then f is said to be harmonically concave.

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Definition 2. *Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash \{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if*

$$f\left(\left[tx^p + (1-t)y^p\right]^{1/p}\right) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. If this inequality is reversed, then f is said to be p-concave.

According to Definition 2, It can be easily seen that for $p = 1$ and $p = -1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.

Hermite-Hadamard inequality for the p-convex function is following:

Theorem 1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash \{0\}$, and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then we have*

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a) + f(b)}{2}.$$

These inequalities are sharp [5, 8]. If these inequalities are reversed, then f is said to be p-concave.

Many papers have been written by a number of mathematicians concerning inequalities for different classes of harmonically convex and p-convex functions see for instance the recent papers [3, 7, 8, 9, 10, 11, 12, 17, 18, 19, 21, 22, 24] and the references within these papers.

The following integral inequality, named Simpson's integral inequality, is one of the best known results in the literature.

Theorem 2. *(Simpson's Integral Inequality). Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a four time continuously differentiable on I°, where I° is the interior of I and $\left\|f^{(4)}\right\|_\infty < \infty$. Then*

$$\left|\frac{1}{3}\left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

There are substantial literature on Simpson type integral inequalities. Here we mention the result of [13, 14, 15] and the corresponding references cited therein.

Throughout this paper we will use the following notations. Let $0 < a < b$ and $p \in \mathbb{R} \backslash \{0\}$.

$$\begin{aligned}
A_p &= A_p(a,b) = \frac{a^p + b^p}{2},\ A_1 = A = A(a,b) = \frac{a+b}{2} \\
M_p &= M_p(a,b) = A_p^{\frac{1}{p}} = \left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}},\ H = H(a,b) = \frac{2ab}{a+b} \\
I_t(x, A_p; u, v) &= \frac{\left|t - \frac{1}{3}\right|^u}{\left[(1-t)x^p + tA_p\right]^{v - \frac{v}{p}}}, \\
J_t(x, A_p; u, v) &= \frac{\left|t - \frac{1}{3}\right|^u (1-t)}{\left[(1-t)x^p + tA_p\right]^{v - \frac{v}{p}}}, \\
K_t(x, A_p; u, v) &= \frac{\left|t - \frac{1}{3}\right|^u t}{\left[(1-t)x^p + tA_p\right]^{v - \frac{v}{p}}}.
\end{aligned}$$

where $t \in [0,1]$ and $u, v \geq 0$.

2. MAIN RESULTS

In this section, we will use the following Lemma for we obtain the main results:

Lemma 1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° (interior of I) and $a, b \in I^\circ$ with $a < b$ and $p \in \mathbb{R} \backslash \{0\}$. If $f' \in L[a,b]$, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{6}\left[f(a) + 4f(M_p) + f(b)\right] - \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \\
= &\frac{b^p - a^p}{4p}\left[\int_0^1 \frac{t - \frac{1}{3}}{\left[(1-t)a^p + tA_p\right]^{1-\frac{1}{p}}} f'\left(\left[(1-t)a^p + tA_p\right]^{\frac{1}{p}}\right)dt \right. \\
&\left. + \int_0^1 \frac{t - \frac{2}{3}}{\left[(1-t)A_p + tb^p\right]^{1-\frac{1}{p}}} f'\left(\left[(1-t)A_p + tb^p\right]^{\frac{1}{p}}\right)dt\right].
\end{aligned}$$

Proof. Firstly, let's calculate the following integral:

$$\begin{aligned}
&\frac{b^p - a^p}{4p}\left[\int_0^1 \frac{t - \frac{1}{3}}{\left[(1-t)a^p + tA_p\right]^{1-\frac{1}{p}}} f'\left(\left[(1-t)a^p + tA_p\right]^{\frac{1}{p}}\right)dt \right. \\
&\left. + \int_0^1 \frac{t - \frac{2}{3}}{\left[(1-t)A_p + tb^p\right]^{1-\frac{1}{p}}} f'\left(\left[(1-t)A_p + tb^p\right]^{\frac{1}{p}}\right)dt\right].
\end{aligned}$$

For shortness, we will use the notations

$$\begin{aligned} I_1 &= \int_0^1 \left(t-\frac{1}{3}\right) df\left(\left[(1-t)\,a^p+tA_p\right]^{\frac{1}{p}}\right), \\ I_2 &= \int_0^1 \left(t-\frac{2}{3}\right) df\left(\left[(1-t)\,A_p+tb^p\right]^{\frac{1}{p}}\right). \end{aligned}$$

Using the partial integration method and the method of changing variables respectively for the integrals I_1 and I_2 as following, we get

$$\begin{aligned} I_1 &= \int_0^1 \left(t-\frac{1}{3}\right) df\left(\left[(1-t)\,a^p+tA_p\right]^{\frac{1}{p}}\right) \\ &= \left(t-\frac{1}{3}\right) f\left(\left[(1-t)\,a^p+tA_p\right]^{\frac{1}{p}}\right)\Big|_0^1 - \int_0^1 f\left(\left[(1-t)\,a^p+tA_p\right]^{\frac{1}{p}}\right) dt \\ &= \frac{2}{3}f\left(M_p\right)+\frac{1}{3}f\left(a\right)-\frac{2p}{b^p-a^p}\int_a^{A_p}\frac{f\left(x\right)}{x^{1-p}}dx \qquad (2.1) \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^1 \left(t-\frac{2}{3}\right) df\left(\left[(1-t)\,A_p+tb^p\right]^{\frac{1}{p}}\right) \\ &= \left(t-\frac{2}{3}\right) f\left(\left[(1-t)\,A_p+tb^p\right]^{\frac{1}{p}}\right)\Big|_0^1 - \int_0^1 f\left(\left[(1-t)\,A_p+tb^p\right]^{\frac{1}{p}}\right) dt \\ &= \frac{1}{3}f\left(b\right)+\frac{2}{3}f\left(M_p\right)-\frac{2p}{b^p-a^p}\int_{A_p}^{b}\frac{f\left(x\right)}{x^{1-p}}dx. \qquad (2.2) \end{aligned}$$

Summing up side by side (2.1) and (2.2), we have

$$\begin{aligned} I_1+I_2 &= \frac{1}{3}\left[f(a)+f(b)\right]+\frac{4}{3}f\left(M_p\right)-\frac{2p}{b^p-a^p}\int_a^b\frac{f\left(x\right)}{x^{1-p}}dx, \\ \frac{I_1+I_2}{2} &= \frac{1}{6}\left[f\left(a\right)+4f\left(M_p\right)+f\left(b\right)\right]-\frac{p}{b^p-a^p}\int_a^b\frac{f\left(x\right)}{x^{1-p}}dx. \end{aligned}$$

□

Theorem 3. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° (the interior of I) and $a, b \in I^\circ$ with $a < b$ and $p \in \mathbb{R}\backslash\{0\}$. If $f' \in L[a,b]$ and $|f'|^q$ is p-convex on I for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{6}\left[f\left(a\right)+4f\left(M_p\right)+f\left(b\right)\right]-\frac{p}{b^p-a^p}\int_a^b\frac{f\left(x\right)}{x^{1-p}}dx\right| \\ \leq\ & \frac{b^p-a^p}{4p}\left[C_p\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(a\right)\right|^q D_p\left(a,b\right)+\left|f'\left(M_p\right)\right|^q E_p\left(a,b\right)\right]^{\frac{1}{q}} \\ &+\frac{b^p-a^p}{4p}\left[F_p\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(M_p\right)\right|^q G_p\left(a,b\right)+\left|f'\left(b\right)\right|^q H_p\left(a,b\right)\right]^{\frac{1}{q}} \end{aligned}$$

where

$$
\begin{aligned}
C_p(a,b) &= \int_0^1 I_t(a,A_p;1,1)\,dt, \quad D_p(a,b)=\int_0^1 J_t(a,A_p;1,1)\,dt,\\
E_p(a,b) &= \int_0^1 K_t(a,A_p;1,1)\,dt, \quad F_p(a,b)=\int_0^1 I_{1-t}(b,A_p;1,1)\,dt,\\
G_p(a,b) &= \int_0^1 K_{1-t}(b,A_p;1,1)\,dt, \quad H_p(a,b)=\int_0^1 J_{1-t}(b,A_p;1,1)\,dt.
\end{aligned}
$$

Proof. Using Lemma 1 and the power mean inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f(M_p)+f(b)\right]-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right|\\
\leq\;& \frac{b^p-a^p}{4p}\left[\int_0^1 \frac{\left|t-\frac{1}{3}\right|}{\left[(1-t)a^p+tA_p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[(1-t)a^p+tA_p\right]^{\frac{1}{p}}\right)\right|dt\right]\\
&+\frac{b^p-a^p}{4p}\left[\int_0^1 \frac{\left|t-\frac{2}{3}\right|}{\left[(1-t)A_p+tb^p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[(1-t)A_p+tb^p\right]^{\frac{1}{p}}\right)\right|dt\right]\\
\leq\;& \frac{b^p-a^p}{4p}\left(\int_0^1 \frac{\left|t-\frac{1}{3}\right|}{\left[(1-t)a^p+tA_p\right]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 \frac{\left|t-\frac{1}{3}\right|}{\left[(1-t)a^p+tA_p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[(1-t)a^p+tA_p\right]^{\frac{1}{p}}\right)\right|^q\right)^{\frac{1}{q}}dt\\
&+\frac{b^p-a^p}{4p}\left(\int_0^1 \frac{\left|t-\frac{2}{3}\right|}{\left[(1-t)A_p+tb^p\right]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 \frac{\left|t-\frac{2}{3}\right|}{\left[(1-t)A_p+tb^p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[(1-t)A_p+tb^p\right]^{\frac{1}{p}}\right)\right|^q\right)^{\frac{1}{q}}dt\\
\leq\;& \frac{b^p-a^p}{4p}\left(\int_0^1 I_t(a,A_p;1,1)\,dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 \frac{\left|t-\frac{1}{3}\right|\left[(1-t)\left|f'(a)\right|^q+t\left|f'(M_p)\right|^q\right]}{\left[(1-t)a^p+tA_p\right]^{1-\frac{1}{p}}}\right)^{\frac{1}{q}}dt\\
&+\frac{b^p-a^p}{4p}\left(\int_0^1 I_{1-t}(b,A_p;1,1)\,dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 \frac{\left|t-\frac{2}{3}\right|\left[(1-t)\left|f'(M_p)\right|^q+t\left|f'(b)\right|^q\right]}{\left[(1-t)A_p+tb^p\right]^{1-\frac{1}{p}}}\right)^{\frac{1}{q}}dt
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{b^p-a^p}{4p}\left[\int_0^1 I_t\left(a,A_p;1,1\right)dt\right]^{1-\frac{1}{q}} \\
& \times\left[\left|f'\left(a\right)\right|^q\int_0^1 J_t\left(a,A_p;1,1\right)dt+\left|f'\left(M_p\right)\right|^q\int_0^1 K_t\left(a,A_p;1,1\right)dt\right]^{\frac{1}{q}} \\
& +\frac{b^p-a^p}{4p}\left[\int_0^1 I_{1-t}\left(b,A_p;1,1\right)dt\right]^{1-\frac{1}{q}} \\
& \times\left[\left|f'\left(M_p\right)\right|^q\int_0^1 K_{1-t}\left(b,A_p;1,1\right)dt+\left|f'\left(b\right)\right|^q\int_0^1 J_{1-t}\left(b,A_p;1,1\right)dt\right]^{\frac{1}{q}} \\
\leq & \frac{b^p-a^p}{4p}\left[C_p\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(a\right)\right|^q D_p\left(a,b\right)+\left|f'\left(M_p\right)\right|^q E_p\left(a,b\right)\right]^{\frac{1}{q}} \\
& +\frac{b^p-a^p}{4p}\left[F_p\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(M_p\right)\right|^q G_p\left(a,b\right)+\left|f'\left(b\right)\right|^q H_p\left(a,b\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof of theorem. □

Corollary 1. *Under conditions of Theorem 3*

i. If we take $p=1$, then we obtain the following inequality for convex function:

$$
\begin{aligned}
& \left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{a+b}{2}\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\
\leq & \frac{b-a}{4}\left(\frac{5}{18}\right)^{1-\frac{1}{q}}\left[\frac{8}{81}\left|f'\left(a\right)\right|^q+\frac{29}{162}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}} \\
& +\frac{b-a}{4}\left(\frac{5}{18}\right)^{1-\frac{1}{q}}\left[\frac{29}{162}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{8}{81}\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}}
\end{aligned}
$$

which is the same of the inequality [6, Corollary 10] *for $s=1$.*

ii. If we take $p=-1$, then we obtain the following inequality for harmonically convex function:

$$
\begin{aligned}
& \left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{2ab}{a+b}\right)+f\left(b\right)\right]-\frac{ab}{b-a}\int_a^b \frac{f\left(x\right)}{x^2}dx\right| \\
\leq & \frac{b-a}{4ab}\left[C_{-1}\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(a\right)\right|^q D_{-1}\left(a,b\right)+\left|f'\left(\frac{2ab}{a+b}\right)\right|^q E_{-1}\left(a,b\right)\right]^{\frac{1}{q}} \\
& +\frac{b-a}{4ab}\left[F_{-1}\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(\frac{2ab}{a+b}\right)\right|^q G_{-1}\left(a,b\right)+\left|f'\left(b\right)\right|^q H_{-1}\left(a,b\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 4. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° (the interior of I) and $a,b\in I^\circ$ with $a<b$ and $p\in\mathbb{R}\backslash\{0\}$. If $f'\in L[a,b]$ and $\left|f'\right|^q$ is p-convex*

on I *for* $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, *then*

$$\begin{aligned} & \left|\frac{1}{6}\left[f(a)+4f(M_p)+f(b)\right]-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\ \leq & \frac{b^p-a^p}{4p}\left[N_{p,r}^{\frac{1}{r}}(a,b)\,A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(M_p)\right|^q\right)+O_{p,r}^{\frac{1}{r}}(a,b)\,A^{\frac{1}{q}}\left(\left|f'(M_p)\right|^q,\left|f'(b)\right|^q\right)\right], \end{aligned}$$

where

$$\begin{aligned} N_{p,r}(a,b) &= \int_0^1 I_t(a,A_p;r,r)\,dt, \\ O_{p,r}(a,b) &= \int_0^1 I_{1-t}(b,A_p;r,r)\,dt. \end{aligned}$$

Proof. From Lemma 1, Hölder's integral inequality and the p-convexity of $|f'|^q$ on $[a,b]$, we have,

$$\begin{aligned} & \left|\frac{1}{6}\left[f(a)+4f(M_p)+f(b)\right]-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\ \leq & \frac{b^p-a^p}{4p}N_{p,r}^{\frac{1}{r}}(a,b)\left(\int_0^1\left|f'\left(\left((1-t)a^p+tA_p\right)^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}} \\ & +\frac{b^p-a^p}{4p}O_{p,r}^{\frac{1}{r}}(a,b)\left(\int_0^1\left|f'\left(\left((1-t)A_p+tb^p\right)^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}} \\ \leq & \frac{b^p-a^p}{4p}\left[N_{p,r}^{\frac{1}{r}}(a,b)\left(\int_0^1\left((1-t)\left|f'(a)\right|^q+t\left|f'(M_p)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\ & \left.+\,O_{p,r}^{\frac{1}{r}}(a,b)\left(\int_0^1\left((1-t)\left|f'(M_p)\right|^q+t\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right] \\ = & \frac{b^p-a^p}{4p}\left[N_{p,r}^{\frac{1}{r}}(a,b)\left(\frac{\left|f'(a)\right|^q+\left|f'(M_p)\right|^q}{2}\right)+O_{p,r}^{\frac{1}{r}}(a,b)\left(\frac{\left|f'(M_p)\right|^q+\left|f'(b)\right|^q}{2}\right)\right] \\ = & \frac{b^p-a^p}{4p}\left[N_{p,r}^{\frac{1}{r}}(a,b)\,M_q\left(\left|f'(a)\right|,\left|f'(M_p)\right|\right)+O_{p,r}^{\frac{1}{r}}(a,b)\,M_q\left(\left|f'(M_p)\right|,\left|f'(b)\right|\right)\right]. \end{aligned}$$

This completes the proof of theorem. □

Corollary 2. *Under conditions of Theorem 4,*

i. If we take $p=1$, *then we obtain the following inequality for convex function:*

$$\begin{aligned} & \left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq & \frac{b-a}{12}\left[\frac{1+2^{r+1}}{3(r+1)}\right]^{\frac{1}{r}}\left[M_q\left(\left|f'(a)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right)+M_q\left(\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(b)\right|\right)\right] \end{aligned}$$

which is the same as the inequality in [22, Theorem 8] *for* $s = 1$.

ii. If we take $p = -1$*, then we obtain the following inequality for harmonically convex function:*

$$\begin{aligned} & \left|\frac{1}{6}\left[f(a)+4f\left(\frac{2ab}{a+b}\right)+f(b)\right]-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\ \leq\ & \frac{b-a}{4ab}\left[N_{-1,r}^{\frac{1}{r}}(a,b)\, M_q\left(\left|f'(a)\right|,\left|f'(H)\right|\right)+O_{-1,r}^{\frac{1}{r}}(a,b)\, M_q\left(\left|f'(H)\right|,\left|f'(b)\right|\right)\right]. \end{aligned}$$

Theorem 5. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *(the interior of* I*) and* $a, b \in I^{\circ}$ *with* $a < b$ *and* $p \in \mathbb{R}\backslash \{0\}$*. If* $f' \in L[a,b]$ *and* $|f'|^q$ *is p-convex on* I *for* $q > 1$, $\frac{1}{r}+\frac{1}{q}=1$*, then*

$$\begin{aligned} & \left|\frac{1}{6}\left[f(a)+4f(M_p)+f(b)\right]-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\ \leq\ & \frac{b^p-a^p}{12p}\left[\frac{1+2^{r+1}}{3(r+1)}\right]^{\frac{1}{r}}\left\{\left[Q_{p,q}(a,b)\left|f'(a)\right|^q+R_{p,q}(a,b)\left|f'(M_p)\right|^q\right]^{\frac{1}{q}}\right. \\ & \left.+\left[S_{p,q}(a,b)\left|f'(b)\right|^q+T_{p,q}(a,b)\left|f'(M_p)\right|^q\right]^{\frac{1}{q}}\right\}, \end{aligned}$$

where

$$\begin{aligned} Q_{p,q}(a,b) &= \int_0^1 J_t(a,A_p;0,q)\,dt, \qquad S_{p,q}(a,b)=\int_0^1 K_{1-t}(b,A_p;0,q)\,dt, \\ R_{p,q}(a,b) &= \int_0^1 K_t(a,A_p;0,q)\,dt, \qquad T_{p,q}(a,b)=\int_0^1 J_{1-t}(b,A_p;0,q)\,dt. \end{aligned}$$

Proof. From Lemma 1, Hölder's integral inequality and the p-convexity of $|f'|^q$ on $[a,b]$, we obtain,

$$\begin{aligned}
&\left|\frac{1}{6}\left[f\left(a\right)+4f\left(M_p\right)+f\left(b\right)\right]-\frac{p}{b^p-a^p}\int_a^b\frac{f\left(x\right)}{x^{1-p}}dx\right| \\
\leq\ & \frac{b^p-a^p}{4p}\int_0^1\left|t-\frac{1}{3}\right|\left|\frac{1}{\left[\left(1-t\right)a^p+tA_p\right]^{1-\frac{1}{p}}}f'\left(\left[\left(1-t\right)a^p+tA_p\right]^{\frac{1}{p}}\right)\right|dt \\
&+\frac{b^p-a^p}{4p}\int_0^1\left|t-\frac{2}{3}\right|\left|\frac{1}{\left[\left(1-t\right)A_p+tb^p\right]^{1-\frac{1}{p}}}f'\left(\left[\left(1-t\right)A_p+tb^p\right]^{\frac{1}{p}}\right)\right|dt \\
\leq\ & \frac{b^p-a^p}{12p}\left[\frac{1+2^{r+1}}{3\left(r+1\right)}\right]^{\frac{1}{r}}\left(\int_0^1\frac{1}{\left[\left(1-t\right)a^p+tA_p\right]^{q-\frac{q}{p}}}\left|f'\left(\left[\left(1-t\right)a^p+tM_p^p\right]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{b^p-a^p}{12p}\left[\frac{1+2^{r+1}}{3\left(r+1\right)}\right]^{\frac{1}{r}}\left(\int_0^1\frac{1}{\left[\left(1-t\right)A_p+tb^p\right]^{q-\frac{q}{p}}}\left|f'\left(\left[\left(1-t\right)M_p^p+tb^p\right]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b^p-a^p}{12p}\left[\frac{1+2^{r+1}}{3\left(r+1\right)}\right]^{\frac{1}{r}}\left\{\left[Q_{p,q}\left(a,b\right)\left|f'\left(a\right)\right|^q+R_{p,q}\left(a,b\right)\left|f'\left(M_p\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[S_{p,q}\left(a,b\right)\left|f'\left(b\right)\right|^q+T_{p,q}\left(a,b\right)\left|f'\left(M_p\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 3. *Under conditions of Theorem 5,*

i. If we take $p=1$, then we obtain the following inequality for convex function:

$$\begin{aligned}
&\left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{a+b}{2}\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\
\leq\ & \frac{b-a}{12}\left[\frac{1+2^{r+1}}{3\left(r+1\right)}\right]^{\frac{1}{r}}\left[M_q\left(\left|f'\left(a\right)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right)+M_q\left(\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'\left(b\right)\right|\right)\right]
\end{aligned}$$

which reduce the inequality in Corollary 2 (i).

ii. If we take $p=-1$, then we obtain the following inequality for harmonically convex function:

$$\begin{aligned}
&\left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{2ab}{a+b}\right)+f\left(b\right)\right]-\frac{ab}{b-a}\int_a^b\frac{f\left(x\right)}{x^2}dx\right| \\
\leq\ & \frac{b-a}{12ab}\left[\frac{1+2^{r+1}}{3\left(r+1\right)}\right]^{\frac{1}{r}}\left\{\left[Q_{-1,q}\left(a,b\right)\left|f'\left(a\right)\right|^q+R_{-1,q}\left(a,b\right)\left|f'\left(H\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[S_{-1,q}\left(a,b\right)\left|f'\left(b\right)\right|^q+T_{-1,q}\left(a,b\right)\left|f'\left(H\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 6. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° (the interior of I) and $a, b \in I^{\circ}$ with $a < b$ and $p \in \mathbb{R} \setminus \{0\}$. If $f' \in L[a,b]$ and $|f'|^q$ is p-concave on I for $q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left| \frac{1}{6}\left[f(a) + 4f(M_p) + f(b) \right] - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \right| \\
\leq\ & \frac{b^p - a^p}{4p} \left[N_{p,r}^{\frac{1}{r}}(a,b) \left| f'\left(\left[\frac{3a^p + b^p}{4} \right]^{\frac{1}{p}} \right) \right| + O_{p,r}^{\frac{1}{r}}(a,b) \left| f'\left(\left[\frac{a^p + 3b^p}{4} \right]^{\frac{1}{p}} \right) \right| \right].
\end{aligned}$$

Proof. From Lemma 1, Hölder's integral inequality and the p-concavity of $|f'|^q$ on $[a,b]$, we have,

$$\begin{aligned}
&\left| \frac{1}{6}\left[f(a) + 4f(M_p) + f(b) \right] - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \right| \\
\leq\ & \frac{b^p - a^p}{4p} N_{p,r}^{\frac{1}{r}}(a,b) \left(\int_0^1 \left| f'\left(\left((1-t) a^p + t A_p \right)^{\frac{1}{p}} \right) \right|^q dt \right)^{\frac{1}{q}} \\
& + \frac{b^p - a^p}{4p} O_{p,r}^{\frac{1}{r}}(a,b) \left(\int_0^1 \left| f'\left(\left((1-t) A_p + t b^p \right)^{\frac{1}{p}} \right) \right|^q dt \right)^{\frac{1}{q}} \\
\leq\ & \frac{b^p - a^p}{4p} \left[N_{p,r}^{\frac{1}{r}}(a,b) \left| f'\left(\left[\frac{3a^p + b^p}{4} \right]^{\frac{1}{p}} \right) \right| + O_{p,r}^{\frac{1}{r}}(a,b) \left| f'\left(\left[\frac{a^p + 3b^p}{4} \right]^{\frac{1}{p}} \right) \right| \right].
\end{aligned}$$

This completes the proof of theorem. □

Corollary 4. *Under conditions of Theorem 6,*

i. If we take $p = 1$, then we obtain the following inequality for concave function:

$$\begin{aligned}
&\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \\
\leq\ & \frac{b-a}{12} \left[\frac{1 + 2^{r+1}}{3(r+1)} \right]^{\frac{1}{r}} \left[\left| f'\left(\frac{3a+b}{4} \right) \right| + \left| f'\left(\frac{a+3b}{4} \right) \right| \right]
\end{aligned}$$

This is the same of the inequality in [6, Corollary 28] *for $s = 1$.*

ii. If we take $p = -1$, then we obtain the following inequality for harmonically concave function:

$$\begin{aligned}
&\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{2ab}{a+b} \right) + f(b) \right] - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
\leq\ & \frac{b-a}{4ab} \left[N_{-1,r}^{\frac{1}{r}}(a,b) \left| f'\left(\frac{4ab}{a+3b} \right) \right| + O_{-1,r}^{\frac{1}{r}}(a,b) \left| f'\left(\frac{4ab}{3a+b} \right) \right| \right].
\end{aligned}$$

3. CONCLUSION

The paper deals with Simpson type inequalities for p-convex and p-concave functions. Firstly, we give a new identity for differentiable functions and get some new integral inequalities for the p-convex and p-concave functions by using this identity.

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ON THE GEOMETRY OF PSEUDO-SLANT SUBMANIFOLDS OF A NEARLY SASAKIAN MANIFOLD

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Abstract. In this paper, we study the pseudo-slant submanifolds of a nearly Sasakian manifold. We characteterize a totally umbilical properpseudo-slant submanifolds and find that a necessary and sufficient condition for such submanifolds totally geodesic. Also the integrability conditions of distributions of pseudo-slant submanifolds of a nearly Sasakian manifold are investigated.

1. Introduction

The differential geometry of slant submanifolds has shown an increasing development since B.Y. Chen defined slant submanifolds in complex manifolds as a natural generalization of both the invariant and anti-invariant submanifolds [3], [4]. Many research articles have been appeared on the existence of these submanifolds in different knows spaces. The slant submanifolds of an almost contact metric manifolds were defined and studied by A. Lotta [2]. After, such submanifolds were studied by J.L Cabrerizo et. al[6], in Sasakian manifolds . Recently, in [9],[10],[11],[13] M. Atçeken studied slant and pseudo-slant submanifold in $(LCS)_n-$manifold and other manifolds. The notion of semi-slant submanifolds of an almost Hermitian manifold was introduced by N. Papagiuc [12]. Recently, A. Carrizo [5],[6] defined and studied bi-slant immersions in almost Hermitian manifolds and simultaneously gave the notion of pseudo-slant submanifolds in almost Hermitian manifolds. The contact version of pseudo-slant submanifolds has been defined and studied by V. A. Khan and M. A Khan [16].

The present paper is organized as follows.

In section 1, the notions and definitions of submanifolds of a Riemannian manifold were given for later use. In this paper, we study pseudo-slant submanifolds of

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a nearly Sasakian manifold. In section 2, we review basic formulas and definitions for a nearly Sasakian manifold and their submanifolds. In section 3, we recall the definition and some basic results of a pseudo-slant submanifold of almost contact metric manifold. We study characterization of totally umbilical proper-slant submanifolds and find that a necessary and sufficient condition for such submanifolds is to be totally geodesic. In section 4, the integrability conditions of distributions of pseudo-slant submanifolds of a nearly Sasakian manifold are investigated.

2. Preliminaries

In this section, we give some notations used throughout this paper. We recall some necessary fact and formulas from the theory of nearly Sasakian manifolds and their submanifolds.

Let $\widetilde{M}$ be an $(2m+1)-$dimensional almost contact metric manifold with an almost contact metric structure (φ, ξ, η, g), that is, φ is a $(1,1)$ tensor field , ξ is a vector field; η is 1-form and g is a compatible Riemanian metric such that

$$\varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\varphi\xi = 0,\ \eta(\xi) = 1,\ \ \eta \circ \varphi = 0,\ \eta(X) = g(X, \xi) \tag{2.2}$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),\ \ g(\varphi X, Y) = -g(X, \varphi Y) \tag{2.3}$$

for any vector fields $X, Y \in \Gamma(T\widetilde{M})$, where $\Gamma(\widetilde{M})$ denotes the set of all vector fields on $\widetilde{M}$. If in addition to above relations,

$$(\widetilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.4}$$

then $\widetilde{M}$ is called a Sasakian manifold, where $\widetilde{\nabla}$ is the Levi-Civita connections of g.

The almost contact metric manifold $\widetilde{M}$ is called a nearly Sasakian manifold if it satisfy the following condition

$$(\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X = 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y \tag{2.5}$$

for any $X, Y \in \Gamma(T\widetilde{M})$.

Now, let M be a submanifold of an almost contact metric manifold $\widetilde{M}$, we denote the induced connections on M and the normal bundle $T^\perp M$ by ∇ and $\nabla^\perp$, respectively, then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.6}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_V X, Y) = g(h(X, Y), V). \tag{2.8}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The mean curvature vector H of M is given by

$$H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i), \tag{2.9}$$

where m is the dimension of M and $\{e_1, e_2, ..., e_m\}$ is a local orthonormal frame of M. A submanifold M of a Riemannian manifold $\widetilde{M}$ is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$. A submanifold M is said to be totally geodesic if $h = 0$ and M is said to be minimal if $H = 0$.

Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$. Then for any $X \in \Gamma(TM)$, we can write

$$\varphi X = TX + NX, \tag{2.11}$$

where TX is the tangential component and NX is the normal component of φX. Similarly, for $V \in \Gamma(T^{\perp}M)$, we can write

$$\varphi V = tV + nV, \tag{2.12}$$

where tV is the tangential component and nV is the normal component of φV.

Thus by using (2.1), (2.2), (2.11) and (2.12), we obtain

$$T^2 + tN = -I + \eta \otimes \xi, \quad NT + nN = 0, \tag{2.13}$$

$$Tt + tn = 0, \quad n^2 + Nt = -I \tag{2.14}$$

and

$$T\xi = 0 = N\xi, \quad \eta \circ T = 0 = \eta \circ N. \tag{2.15}$$

Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(TX, Y) = -g(X, TY)$ and $V, U \in \Gamma(T^{\perp}M)$, we get $g(U, nV) = -g(nU, V)$. These show that T and n are also skew-symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have

$$g(NX, V) = -g(X, tV) \tag{2.16}$$

which gives the relation between N and t.

The covariant derivatives of the tensor field T, N, t and n are, respectively, defined by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.17}$$

$$(\nabla_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \tag{2.18}$$

$$(\nabla_X t)V = \nabla_X tV - t\nabla^{\perp}_X V \tag{2.19}$$

and

$$(\nabla_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X V \tag{2.20}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Now, for any $X, Y \in \Gamma(TM)$, let us denote the tangential and normal parts of $(\widetilde{\nabla}_X\varphi)Y$ by P_XY and F_XY, respectively. Then we decompose

$$(\widetilde{\nabla}_X\varphi)Y = P_XY + F_XY \tag{2.21}$$

thus, by an easy computation, we obtain the formulae

$$P_XY = (\nabla_X T)Y - A_{NY}X - th(X,Y) \tag{2.22}$$

and

$$F_XY = (\nabla_X N)Y + h(X,TY) - nh(X,Y). \tag{2.23}$$

Similarly, for any $V \in \Gamma(T^{\perp}M)$, denoting tangential and normal parts of $(\widetilde{\nabla}_X\varphi)V$ by P_XV and F_XV, respectively, we obtain

$$P_XV = (\nabla_X t)V - A_{nV}X + TA_VX \tag{2.24}$$

and

$$F_XV = (\nabla_X n)V + h(tV,X) + NA_VX. \tag{2.25}$$

Now, for any $X, Y \in \Gamma(TM)$, from (2.5), we have

$$(\widetilde{\nabla}_X\varphi)Y + (\widetilde{\nabla}_Y\varphi)X = 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y \tag{2.26}$$

that is,

$$(\widetilde{\nabla}_X\varphi)Y + (\widetilde{\nabla}_Y\varphi)X = \widetilde{\nabla}_X\varphi Y - \varphi\widetilde{\nabla}_X Y + \widetilde{\nabla}_Y\varphi X - \varphi\widetilde{\nabla}_Y X.$$

By using (2.6), (2.7), (2.11) and (2.12), we get

$$\begin{aligned}
(\widetilde{\nabla}_X\varphi)Y + (\widetilde{\nabla}_Y\varphi)X &= \widetilde{\nabla}_X TY + \widetilde{\nabla}_X NY - \varphi(\nabla_X Y + h(X,Y)) \\
&\quad +\widetilde{\nabla}_Y TX + \widetilde{\nabla}_Y NX - \varphi(\nabla_Y X + h(X,Y)). \\
&= \nabla_X TY + h(X,TY) - A_{NY}X + \nabla^{\perp}_X NY \\
&\quad -T\nabla_X Y - N\nabla_X Y - th(X,Y) - nh(X,Y) \\
&\quad +\nabla_Y TX + h(Y,TX) - A_{NX}Y + \nabla^{\perp}_Y NX - T\nabla_Y X \\
&\quad - N\nabla_Y X - th(X,Y) - nh(X,Y).
\end{aligned} \tag{2.27}$$

Making use of (2.27) and (2.26), we obtain

$$\begin{aligned}&\nabla_X TY + h(X,TY) - A_{NY}X + \nabla^{\perp}_X NY - T\nabla_X Y - N\nabla_X Y\\&-th(X,Y) - nh(X,Y) + \nabla_Y TX + h(Y,TX) - A_{NX}Y\\&+\nabla^{\perp}_Y NX - T\nabla_Y X - N\nabla_Y X - th(X,Y) - nh(X,Y)\\&-2g(X,Y)\xi + \eta(Y)X + \eta(X)Y = 0.\end{aligned}$$

By taking tangential and normal parts of the above equation, respectively, we have equation

$$\begin{aligned}(\nabla_X T)Y + (\nabla_Y T)X &= A_{NX}Y + A_{NY}X + 2th(X,Y)\\&\quad +2g(X,Y)\xi - \eta(Y)X - \eta(X)Y \qquad (2.28)\end{aligned}$$

and

$$(\nabla_X N)Y + (\nabla_Y N)X = -h(X,TY) - h(Y,TX) + 2nh(X,Y)\ . \qquad (2.29)$$

On the other hand, for $Y=\xi$ in (2.5) and by using (2.2), (2.6) and (2.7), we see

$$T\,[X,\xi] = (\nabla_\xi T)X - T\nabla_\xi X - 2th(X,\xi) - A_{NX}\xi + X - \eta(X)\xi \qquad (2.30)$$

and

$$N\,[X,\xi] = (\nabla_\xi N)X - N\nabla_\xi X - 2nh(X,\xi) + h(TX,\xi). \qquad (2.31)$$

In contact geometry, A. Lotta introduced slant submanifolds as follows [2]:

Definition 2.1. Let M be a submanifold of an almost contact metric manifold $(\bar{M},\varphi,\xi,\eta,g)$. Then M is said to be a slant submanifold if the angle $\theta(X)$ between φX and $T_M(p)$ is constant at any point $p\in M$ for any X linearly independent of ξ. Thus the invariant and anti-invariant submanifolds are special class of slant submanifolds with slant angles $\theta=0$ and $\theta=\frac{\pi}{2}$, respectively. If the slant angle θ is neither zero nor $\frac{\pi}{2}$, then slant submanifold is said to be proper slant submanifold.

If M is a slant submanifold of an almost contact metric manifold, then the tangent bundle TM of M can be decomposed as

$$TM = D_\theta \oplus \xi,$$

where ξ denotes the distribution spanned by the structure vector field ξ and D_θ is the complementary distribution of ξ in TM, known as the slant distribution on M.

For a slant submanifold M of an almost contact metric manifold $\widetilde{M}$, the normal bundle $T^{\perp}M$ of M is decomposed as

$$T^{\perp}M = N(TM)\oplus\mu$$

where μ is the invariant normal subbundle with respect to φ orthogonal to $N(TM)$.

In an almost contact metric manifold. In fact, J. L Cabrerizo obtained the following theorem[6].

Theorem 2.2. [6]. *Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in \Gamma(TM)$. Then M is slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$T^2 = -\lambda(I - \eta \otimes \xi) \tag{2.32}$$

Furthermore, the slant angle θ of M satisfies $\lambda = cos^2\theta$.

Hence, for a slant submanifold M of an almost contact metric manifold $\widetilde{M}$, the following relations are consequences of the above theorem.

$$g(TX, TY) = \cos^2\theta \{g(X, Y) - \eta(X)\eta(Y)\} \tag{2.33}$$

and

$$g(NX, NY) = \sin^2\theta \{g(X, Y) - \eta(X)\eta(Y)\}. \tag{2.34}$$

for any $X, Y \in \Gamma(TM)$.

Lemma 2.3. [6]. *Let D_θ be a distribution on M, orthogonal to ξ. Then, D_θ is a slant if and only if there is a constant $\lambda \in [0,1]$ such that*

$$(TP_2)^2 X = -\lambda X \tag{2.35}$$

for all $X \in \Gamma(D_\theta)$, where P_2 denotes the orthogonal projection on D_θ. Furthermore, the slant angle θ of M satisfies $\lambda = \cos^2\theta$.

3. Pseudo-slant submanifolds of a nearly Sasakian manifold

In this section, we will obtain the integrability condition of the distributions of pseudo-slant submanifold of a nearly Sasakian manifold.

Definition 3.1. We say that M is a pseudo-slant submanifold of an almost contact metric manifold $(\bar{M}, \varphi, \xi, \eta, g)$ if there exists two orthogonal distributions D_θ and $D^\perp$ on M such that
i. TM admits the orthogonal direct decomposition $TM = D^\perp \oplus D_\theta$, $\xi \in \Gamma(D_\theta)$
ii. The distribution $D^\perp$ is anti-invariant(totally-real) i.e., $\varphi D^\perp \subset (T^\perp M)$,
iii. The distribution D_θ is a slant with slant angle $\theta \neq 0, \frac{\pi}{2}$, that is, the angle between D_θ and $\varphi(D_\theta)$ is a constant θ[16].

From the definition, it is clear that if $\theta = 0$, then the pseudo-slant submanifold is a semi-invariant submanifold. On the other hand, if $\theta = \frac{\pi}{2}$, submanifold becomes an anti- invariant.

We suppose that M is a pseudo-slant submanifold of an almost contact metric manifold $\widetilde{M}$ and we denote the dimensions of distributions $D^{\perp}$and D_θ by d_1 and d_2, respectively, then we have the following cases:
i. If $d_2 = 0$ then M is an anti-invariant submanifold,
ii. If $d_1 = 0$ and $\theta = 0$, then M is an invariant submanifold,
iii. If $d_1 = 0$ and $\theta \in (0, \frac{\pi}{2})$then M is a proper slant submanifold with slant angle θ,
iv. If $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold,
v. If $d_1.d_2 \neq 0$ and $\theta = 0$, then $\widetilde{M}$ is a semi-invariant submanifold,
vi. If $d_1.d_2 \neq 0$ and $\theta \in (0, \frac{\pi}{2})$, then M is a proper pseudo-slant submanifold.

If we denote the projections on $D^{\perp}$and D_θ by P_1 and P_2, respectively, then for any vector field $X \in \Gamma(TM)$, we can write

$$X = P_1X + P_2X + \eta(X)\xi. \tag{3.1}$$

On the other hand, applying φ on both sides of equation (3.1), we have

$$\varphi X = \varphi P_1X + \varphi P_2X$$

and

$$TX + NX = NP_1X + TP_2X + NP_2X, \quad TP_1X = 0, \tag{3.2}$$

from which

$$TX = TP_2X, \quad NX = NP_1X + NP_2X$$

and

$$\varphi P_1X = NP_1X, \; TP_1X = 0, \quad \varphi P_2X = TP_2X + NP_2X \tag{3.3}$$
$$TP_2X \in \Gamma(D_\theta).$$

For a pseudo-slant submanifold M of a nearly Sasakian manifold $\widetilde{M}$, the normal bundle $T^{\perp}M$ of a pseudo-slant submanifold M is decomposable as

$$T^{\perp}M = \varphi(D^{\perp}) \oplus N(D_\theta) \oplus \mu \quad \varphi(D^{\perp}) \perp N(D_\theta) \tag{3.4}$$

where μ is an invariant subbundle of $T^{\perp}M$.

Now, we construct an example of a pseudo-slant submanifold in an almost contact metric manifold.

Example 3.2. Let M be a submanifold of $\mathbb{R}^7$ defined by

$$\chi(u, v, s, z, w) = (\sqrt{3}u, v, v\sin\theta, v\cos\theta, s\cos z, -s\cos z, w).$$

We can easily to see that the tangent bundle of M is spanned by the tangent vectors

$$e_1 = \sqrt{3}\frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial y_1} + \sin\theta\frac{\partial}{\partial x_2} + \cos\theta\frac{\partial}{\partial y_2}, \quad e_5 = \xi = \frac{\partial}{\partial w},$$
$$e_3 = \cos z\frac{\partial}{\partial x_3} - \cos z\frac{\partial}{\partial y_3}, \; e_4 = -s\sin z\frac{\partial}{\partial x_3} + s\sin z\frac{\partial}{\partial y_3}.$$

For the almost contact structure of φ of $\mathbb{R}^7$, choosing

$$\varphi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \quad \varphi(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j},\ \varphi(\frac{\partial}{\partial w}) = 0, \quad 1 \leq i,\ j \leq 3$$

and $\xi = \frac{\partial}{\partial w}$, $\eta = dw$. For any vector field $W = \mu_i \frac{\partial}{\partial x_i} + \nu_j \frac{\partial}{\partial y_j} + \lambda \frac{\partial}{\partial w} \in T(\mathbb{R}^7)$, then we have

$$\varphi Z = \mu_i \frac{\partial}{\partial y_j} - \nu_j \frac{\partial}{\partial x_i}, \quad g(\varphi Z, \varphi Z) = \mu_i^2 + \nu_j^2,$$

$$g(Z, Z) = \mu_i^2 + \nu_j^2 + \lambda^2, \quad \eta(Z) = g(Z, \xi) = \lambda$$

and

$$\varphi^2 Z = -\mu_i \frac{\partial}{\partial x_i} - \nu_j \frac{\partial}{\partial y_j} - \lambda \frac{\partial}{\partial w} + \lambda \frac{\partial}{\partial w} = -Z + \eta(Z)\xi$$

for any $i,\ j = 1, 2, 3$. It follows that $g(\varphi Z, \varphi Z) = g(Z, Z) - \eta^2(Z)$. Thus (φ, ξ, η, g) is an almost contact metric structure on $\mathbb{R}^7$. Thus we have

$$\varphi e_1 = \sqrt{3} \frac{\partial}{\partial y_1}, \quad \varphi e_2 = -\frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial y_2} - \cos\theta \frac{\partial}{\partial x_2}$$

$$\varphi e_3 = \cos z \frac{\partial}{\partial y_3} + \cos z \frac{\partial}{\partial x_3}, \quad \varphi e_4 = -s \sin z \frac{\partial}{\partial y_3} - s \sin z \frac{\partial}{\partial x_3}.$$

By direct calculations, we can infer $D_\theta = span\{e_1, e_2\}$ is a slant distribution with slant angle $\cos\theta = \frac{g(e_2, \varphi e_1)}{\|e_2\| \|\varphi e_1\|} = \frac{\sqrt{2}}{2}$, $\theta = 45°$. Since

$$g(\varphi e_3, e_1) = g(\varphi e_3, e_2) = g(\varphi e_3, e_4) = g(\varphi e_3, e_5) = 0,$$

$$g(\varphi e_4, e_1) = g(\varphi e_4, e_2) = g(\varphi e_4, e_3) = g(\varphi e_4, e_5) = 0.$$

Thus e_3 and e_4 are orthogonal to M, $D^\perp = span\{e_3, e_4\}$ is an anti-invariant distribution. Thus M is a 5-dimensional proper pseudo-slant submanifold of $\mathbb{R}^7$ with its usual almost contact metric structure.

Theorem 3.3. *Let M be a pseudo-slant of a nearly Sasakian manifold $\widetilde{M}$. Then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$A_{NX}Y = A_{NY}X \tag{3.5}$$

for any $X, Y \in \Gamma\ (D^\perp)$.

Proof. By using (2.3), (2.6) and (2.8), we can write

$$\begin{aligned} 2g(A_{\varphi Y}X, Z) &= g(h(Z, X), \varphi Y) + g(h(Z, X), \varphi Y) \\ &= g(\widetilde{\nabla}_X Z, \varphi Y) + g(\widetilde{\nabla}_Z X, \varphi Y) \\ &= -g(\varphi \widetilde{\nabla}_X Z, Y) - g(\varphi \widetilde{\nabla}_Z X, Y) \\ &= -g(\widetilde{\nabla}_X \varphi Z, Y) - g(\widetilde{\nabla}_Z \varphi X, Y) \\ &\quad + g((\widetilde{\nabla}_X \varphi)Z + (\widetilde{\nabla}_Z \varphi)X, Y) \end{aligned}$$

for any $X, Y \in \Gamma\ (D^{\perp})$ and $Z \in \Gamma(TM)$. By using (2.5), (2.7) and (2.16) we have

$$\begin{aligned}
2g(A_{NY}X, Z) &= -g(\widetilde{\nabla}_X \varphi Z, Y) - g(\widetilde{\nabla}_Z \varphi X, Y) \\
&\quad + g(2g(Z, X)\xi - \eta(Z)X - \eta(X)Z, Y) \\
&= g(\widetilde{\nabla}_X Y, \varphi Z) + g(A_{NX}Z, Y) - \eta(Z)g(X, Y) \\
&= g(\widetilde{\nabla}_X Y, \varphi Z) + g(A_{NX}Y, Z) - g(X, Y)g(\xi, Z) \\
&= g(\nabla_X Y, TZ) + g(h(X, Y), NZ) \\
&\quad + g(A_{NX}Y, Z) - g(X, Y)g(\xi, Z) \\
&= -g(T\nabla_X Y, Z) + g(-th(X, Y), Z) \\
&\quad + g(A_{NX}Y, Z) - g(X, Y)g(\xi, Z).
\end{aligned}$$

This implies that

$$2A_{NY}X = A_{NX}Y - T\nabla_X Y - th(X, Y) - g(X, Y)\xi \tag{3.6}$$

interchanging X by Y in (3.6) , we have

$$2A_{NX}Y = A_{NY}X - T\nabla_Y X - th(X, Y) - g(X, Y)\xi \tag{3.7}$$

then from (3.6) and (3.7), we can derive.

$$\begin{aligned}
2A_{NY}X - 2A_{NX}Y &= A_{NX}Y - A_{NY}X + T\nabla_Y X - T\nabla_X Y \\
&= A_{NX}Y - A_{NY}X + T(\nabla_Y X - \nabla_X Y)
\end{aligned}$$

here

$$3(A_{NY}X - A_{NX}Y) = T\,[X, Y]\,.$$

For $[X, Y] \in \Gamma(D^{\perp})$, $\varphi\,[X, Y] = N\,[X, Y]$. Since the tangent component of $\varphi\,[X, Y]$ is the zero, the anti- invariant distribution $D^{\perp}$ is integrability if and only if (3.5) is satisfied. □

Theorem 3.4. *Let M be a pseudo-slant of a nearly Sasakian manifold $\widetilde{M}$. Then the anti-invariant distribution $D^{\perp}$ is integrable if and only if*

$$A_{NX}Y + T\nabla_X Y + th(Y, X) + g(X, Y)\xi = 0 \tag{3.8}$$

for any $X, Y \in \Gamma\ (D^{\perp})$.

Proof. For any $X, Y \in \Gamma\ (D^{\perp})$, from (2.5), we have

$$(\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X = 2g(X, Y)\xi$$

which is equivalent to

$$\widetilde{\nabla}_X \varphi Y - \varphi\widetilde{\nabla}_X Y + \widetilde{\nabla}_Y \varphi X - \varphi\widetilde{\nabla}_Y X - 2g(X, Y)\xi = 0.$$

By using (2.6), (2.7), (2.11) and (2.12), we can write

$$\begin{aligned}
&-A_{NY}X + \nabla^{\perp}_X NY - T\nabla_X Y - N\nabla_X Y - 2th(Y, X) - A_{NX}Y \\
&+\nabla^{\perp}_Y NX - T\nabla_Y X - N\nabla_Y X - 2nh(Y, X) - 2g(X, Y)\xi = 0.
\end{aligned}$$

Then from the tangent components of the last equation and (3.5), we conclude that

$$2A_{NX}Y + T\nabla_X Y + T\nabla_Y X + 2th(Y,X) + 2g(X,Y)\xi = 0,$$

which implies

$$T\,[X,Y] = 2A_{NX}Y + 2T\nabla_X Y + 2th(Y,X) + 2g(X,Y)\xi.$$

This proves our assertion. □

Theorem 3.5. *Let M be a pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then the slant distribution D_θ is integrable if and only if*

$$-2(\nabla_Y N)X + \nabla^{\perp}_Y NX - \nabla^{\perp}_X NY + h(Y,TX) - h(X,TY) + 2nh(X,TY) \in \mu \oplus N(D_\theta)$$

for any $Y, X \in \Gamma(D_\theta)$.

Proof. For any $Y, X \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^{\perp})$, we have

$$\begin{aligned} g([Y,X]\,,Z) &= g(\widetilde{\nabla}_Y X, Z) - g(\widetilde{\nabla}_X Y, Z) \\ &= g(\varphi\widetilde{\nabla}_Y X, \varphi Z) - g(\varphi\widetilde{\nabla}_X Y, \varphi Z). \end{aligned}$$

Since of $\eta(Z) = 0$ and $\varphi Z = NZ$ for any $Z \in \Gamma(D^{\perp})$, we obtain

$$\begin{aligned} g([Y,X]\,,Z) &= g(\widetilde{\nabla}_Y \varphi X, NZ) - g(\ \widetilde{\nabla}_X \varphi Y, NZ) \\ &\quad + g((\widetilde{\nabla}_X \varphi)Y - (\widetilde{\nabla}_Y \varphi)X, NZ) \\ &= g(\widetilde{\nabla}_Y \varphi X, NZ) - g(\widetilde{\nabla}_X \varphi Y, NZ) \\ &\quad + g((\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X, NZ) - 2g((\widetilde{\nabla}_Y \varphi)X, NZ). \end{aligned}$$

By using (2.5) in this equation, we have

$$\begin{aligned} g([Y,X]\,,Z) &= g(\widetilde{\nabla}_Y \varphi X, NZ) - g(\widetilde{\nabla}_X \varphi Y, NZ) - 2g((\widetilde{\nabla}_Y \varphi)X, NZ) \\ &\quad + g(2g(Y,X)\xi - \eta(Y)X - \eta(X)Y, NZ). \end{aligned}$$

Also using (2.11) in this equation, we have

$$\begin{aligned} g([Y,X]\,,Z) &= g(\widetilde{\nabla}_Y TX, NZ) + g(\widetilde{\nabla}_Y NX, NZ) - g(\widetilde{\nabla}_X TY, NZ) \\ &\quad - g(\widetilde{\nabla}_X NY, NZ) - 2g((\widetilde{\nabla}_Y \varphi)X, NZ). \end{aligned}$$

From the Gauss and Weingarten formulas, the above equation takes the form

$$\begin{aligned} g([Y,X]\,,Z) &= -2g((\widetilde{\nabla}_Y \varphi)X, NZ) + g(\nabla^{\perp}_Y NX, NZ) - g(\nabla^{\perp}_X NY, NZ) \\ &\quad + g(h(Y,TX), NZ) - g(h(X,TY), NZ). \end{aligned} \tag{3.9}$$

Substituting $2g((\widetilde{\nabla}_Y \phi)X, NZ)$ into the (3.9), we get

$$\begin{aligned} 2g((\widetilde{\nabla}_Y \varphi)X, NZ) &= 2g((\nabla_Y T)X - A_{NX}Y - th(Y,X), NZ) \\ &\quad + 2g((\nabla_Y N)X + h(Y,TX) - nh(X,TY), NZ) \\ &= 2g((\nabla_Y N)X + h(Y,TX) - nh(X,TY), NZ). \end{aligned} \tag{3.10}$$

Substituting (3.10) in the equation (3.9), we have

$$\begin{aligned} g([Y,X],Z) &= g(-2(\nabla_Y N)X - 2h(Y,TX) + 2nh(X,TY), NZ) \\ &\quad + g(\nabla^{\perp}_Y NX - \nabla^{\perp}_X NY + h(Y,TX) - h(X,TY), NZ), \\ &= g(-2(\nabla_Y N)X + \nabla^{\perp}_Y NX - \nabla^{\perp}_W NY \\ &\quad + h(Y,TX) - h(X,TY) + 2nh(X,TY), NZ). \end{aligned}$$

Thus we conclude $[Y,X] \in \Gamma(D_\theta)$ if and only if

$$-2(\nabla_Y N)X + \nabla^{\perp}_Y NX - \nabla^{\perp}_X NY + h(Y,TX) - h(X,TY) + 2nh(X,TY) \in \mu \oplus N(D_\theta).$$

□

Theorem 3.6. *Let M be a pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then the slant distribution D_θ is integrable if and only if*

$$\begin{aligned} P_1\{(\nabla_Y T)X + \nabla_X TY - T\nabla_Y X - A_{NX}Y - A_{NY}X \\ -2th(X,Y) + \eta(Y)X + \eta(X)Y\} = 0 \end{aligned} \tag{3.11}$$

for any $X, Y \in \Gamma(D_\theta)$.

Proof. For any $X, Y \in \Gamma(D_\theta)$ and we denote the projections on $D^\perp$ and D_θ by P_1 and P_2, respectively, then for any vector fields $X, Y \in \Gamma(D_\theta)$, by using equation (2.5), we obtain

$$(\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \phi)X = 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y,$$

that is,

$$\widetilde{\nabla}_X \varphi Y - \varphi \widetilde{\nabla}_X Y + \widetilde{\nabla}_Y \varphi X - \varphi \widetilde{\nabla}_Y X = 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y.$$

By using equations (2.6), (2.7), (2.11) and (2.12), we can write

$$\begin{aligned} &\nabla_X TY + h(X,TY) - A_{NY}X + \nabla^{\perp}_X NY - T\nabla_X Y - N\nabla_X Y \\ &-th(X,Y) - nh(X,Y) + \nabla_Y TX + h(Y,TX) - A_{NX}Y \\ &+\nabla^{\perp}_Y NX - T\nabla_Y X - N\nabla_Y X - th(X,Y) - nh(X,Y) \\ &-2g(X,Y)\xi + \eta(Y)X + \eta(X)Y = 0 \end{aligned}$$

From the tangential components of the last equation, we conclude that

$$\begin{aligned} &\nabla_X TY - T\nabla_X Y + (\nabla_Y T)X - A_{NX}Y - A_{NY}X \\ &-2th(X,Y) - 2g(X,Y)\xi + \eta(Y)X + \eta(X)Y = 0 \end{aligned}$$

which implies that

$$\begin{aligned} T[X,Y] &= \nabla_X TY - T\nabla_Y X + (\nabla_Y T)X - A_{NX}Y - A_{NY}X \\ &\quad -2th(X,Y) - 2g(X,Y)\xi + \eta(Y)X + \eta(X)Y. \end{aligned} \tag{3.12}$$

Applying P_1 to (3.12), we get (3.11) □

Theorem 3.7. *Let M be a proper pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then D_θ is integrable if and only if*

$$2g(\nabla_X Y, Z) = \Big\{g(A_{NZ}X, TY) + g(A_{NZ}Y, TX) + g(\nabla^\perp_X NY, NZ) + g(\nabla^\perp_Y NX, NZ)\Big\}$$

for any $X, Y \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^\perp)$.

Proof. For any $X, Y \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^\perp)$, by using (2.3), we have

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(\varphi\widetilde{\nabla}_X Y, \varphi Z) + \eta(\widetilde{\nabla}_X Y)\eta(Z).$$

Since $\eta(Z) = 0$, we get

$$g(\nabla_X Y, Z) = g(\varphi\widetilde{\nabla}_X Y, NZ),$$

from which

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X \varphi Y, NZ) - g((\widetilde{\nabla}_X\varphi)Y, NZ).$$

From the Gauss and Weingarten formulas and structure equation (2.5), we get

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\widetilde{\nabla}_X TY, NZ) + g(\widetilde{\nabla}_X NY, NZ) - g((\widetilde{\nabla}_X\varphi)Y, NZ) \\ &= g(h(X, TY), NZ) + g(\nabla^\perp_X NY, NZ) + g((\widetilde{\nabla}_Y\varphi)X, NZ) \\ &\quad -2g(X, Y)g(\xi, NZ) + \eta(X)g(Y, NZ) + \eta(Y)g(X, NZ) \\ &= g(h(X, TY), NZ) + g(\nabla^\perp_X NY, NZ) + g((\widetilde{\nabla}_Y\varphi)X, NZ) \end{aligned} \tag{3.13}$$

Interchanging X by Y in (3.13), we have

$$g(\nabla_Y X, Z) = g(h(Y, TX), NZ) + g(\nabla^\perp_Y NX, NZ) + g((\widetilde{\nabla}_X\varphi)Y, NZ). \tag{3.14}$$

From (3.13) and (3.14), we can derive

$$\begin{aligned} g(\nabla_X Y, Z) + g(\nabla_Y X, Z) &= g(h(X, TY), NZ) + g(\nabla^\perp_X NY, NZ) + g((\widetilde{\nabla}_Y\varphi)X, NZ) \\ &\quad +g(h(Y, TX), NZ) + g(\nabla^\perp_Y NX, NZ) + g((\widetilde{\nabla}_X\varphi)Y, NZ) \\ &= g(h(X, TY), NZ) + g(h(Y, TX), NZ) \\ &\quad +g(\nabla^\perp_Y NX, NZ) + g(\nabla^\perp_X NY, NZ) \\ &\quad +g((\widetilde{\nabla}_Y\varphi)X + (\widetilde{\nabla}_X\varphi)Y, NZ) \end{aligned}$$

By using (2.5), we obtain

$$\begin{aligned} g(\nabla_X Y, Z) + g(\nabla_Y X, Z) &= g(h(X, TY), NZ) + +g(h(Y, TX), NZ) \\ &\quad +g(\nabla^\perp_Y NX, NZ) + g(\nabla^\perp_X NY, NZ) \end{aligned}$$

Using the property of Lie bracket, we derive

$$\begin{aligned} 2g(\nabla_X Y, Z) + g([Y, X], Z) &= g(h(X, TY), NZ) + g(h(Y, TX), NZ) \\ &\quad +g(\nabla^\perp_Y NX, NZ) + g(\nabla^\perp_X NY, NZ), \end{aligned}$$

which implies that

$$2g(\nabla_X Y, Z) = \left\{ g(A_{NZ}X, TY) + g(A_{NZ}Y, TX) + g(\nabla^{\perp}_X NY, NZ) + g(\nabla^{\perp}_Y NX, NZ) \right\}.$$

This proves our assertion.

Theorem 3.8. *Let M be a totally umbilical proper pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then the endomorphism T is parallel on M if and only if M is anti-invariant submanifold of $\widetilde{M}$.*

Proof. If T is parallel, then from (2.10) and (2.28), we have

$$A_{NX}X + th(X,X) + g(X,X)\xi - \eta(X)X = 0 \tag{3.15}$$

Interchanging X by TX in (3.15), we drive

$$A_{NTX}TX + th(TX,TX) + g(TX,TX)\xi = 0. \tag{3.16}$$

Taking the inner product of (3.16) by ξ, we get

$$\begin{aligned} 0 &= g(A_{NTX}TX + th(TX,TX) + g(TX,TX)\xi, \xi) \\ &= g(h(TX,\xi), NTX) + g(TX,TX) \\ &= g(g(TX,\xi)H, NTX) + g(TX,TX) \\ &= g(TX,TX) = \cos^2\theta\{g(X,X) - \eta(X)\eta(X)\} \end{aligned}$$

for any vector field X on M. This implies that M is an anti-invariant submanifold. If M is an anti-invariant submanifold, then it is obvious that $\nabla T = 0$. □

Theorem 3.9. *Let M be totally umbilical proper pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then M is a totally geodesic submanifold if H, $\nabla^{\perp}_X H \in \Gamma(\mu)$.*

Proof. For any $X, Y \in \Gamma(TM)$, we have

$$\widetilde{\nabla}_X \varphi Y = (\widetilde{\nabla}_X \varphi)Y + \varphi \widetilde{\nabla}_X Y. \tag{3.17}$$

Making use of (2.6), (2.7), (2.10), (2.11) and (3.17) equation takes the form

$$\begin{aligned} \nabla_X TY + g(X,TY)H - A_{NY}X + \nabla^{\perp}_X NY = (\widetilde{\nabla}_X \varphi)Y + T\nabla_X Y \\ + N\nabla_X Y + g(X,Y)\varphi H. \end{aligned}$$

Taking the inner product with φH the last equation, we obtain

$$g(\nabla^{\perp}_X NY, \varphi H) = g((\widetilde{\nabla}_X \varphi)Y, \varphi H) + g(N\nabla_X Y, \varphi H) + g(X,Y)\left\|H\right\|^2$$

by using (2.7), we get

$$g(\widetilde{\nabla}_X NY, \varphi H) = g((\widetilde{\nabla}_X \varphi)Y, \varphi H) + g(X,Y)\left\|H\right\|^2. \tag{3.18}$$

In same way, we have

$$g(\widetilde{\nabla}_Y NX, \varphi H) = g((\widetilde{\nabla}_Y \varphi)X, \varphi H) + g(X,Y)\left\|H\right\|^2. \tag{3.19}$$

Then from (3.18) and (3.19), we can derive

$$g(\widetilde{\nabla}_X NY + \widetilde{\nabla}_Y NX, \varphi H) = g((\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X, \varphi H) + 2g(X, Y) \|H\|^2 .$$

From (2.5), we obtain

$$g(\widetilde{\nabla}_X NY + \widetilde{\nabla}_Y NX, \varphi H) = g(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y, \varphi H) + 2g(X, Y) \|H\|^2 .$$

Hence

$$g(\widetilde{\nabla}_X NY + \widetilde{\nabla}_Y NX, \varphi H) = 2g(X, Y) \|H\|^2 . \tag{3.20}$$

Now, for any $X \in \Gamma(TM)$, we have

$$\widetilde{\nabla}_X \varphi H = (\widetilde{\nabla}_X \varphi)H + \varphi \widetilde{\nabla}_X H$$

by means of (2.6), (2.7), (2.11), (2.12), (2.24) and (2.25), we obtain

$$-A_{\varphi H}X + \nabla^{\perp}_X \varphi H = P_X H + F_X H - TA_H X - NA_H X + n\nabla^{\perp}_X H. \tag{3.21}$$

Taking the inner product with NY and taking into account that $n\nabla^{\perp}_X H \in \Gamma(\mu)$, we see that

$$g(\nabla^{\perp}_X \varphi H, NY) = g(F_X H, NY) - g(NA_H X, NY). \tag{3.22}$$

From (2.8), (2.10), (2.34) and (3.22), we obtain

$$\begin{aligned} g(\nabla^{\perp}_X \varphi H, NY) \quad &= -\sin^2\theta \left\{ g(X, Y) \|H\|^2 - \eta(A_H X)\eta(Y) \right\} \\ &\quad + g(F_X H, NY). \end{aligned}$$

Since ∇ is metric connection, NY and φH are mutually orthogonal, by using (2.2), (2.7), (2.8) and (2.10), we get

$$\begin{aligned} g(\widetilde{\nabla}_X NY, \varphi H) \quad &= \sin^2\theta \{g(X, Y) - \eta(X)\eta(Y)\} \|H\|^2 \\ &\quad - g(F_X H, NY). \end{aligned} \tag{3.23}$$

Similarly, we have

$$\begin{aligned} g(\widetilde{\nabla}_Y NX, \varphi H) \quad &= \sin^2\theta \{g(X, Y) - \eta(X)\eta(Y)\} \|H\|^2 \\ &\quad - g(F_Y H, NX). \end{aligned} \tag{3.24}$$

From (3.23) and (3.24), we obtain

$$\begin{aligned} g(\widetilde{\nabla}_X NY + \widetilde{\nabla}_Y NX, \varphi H) \quad &= 2\sin^2\theta \{g(X, Y) - \eta(X)\eta(Y)\} \|H\|^2 \\ &\quad - g(F_X H, NY) - g(F_Y H, NX). \end{aligned} \tag{3.25}$$

Thus (3.20) and (3.25) imply

$$\begin{aligned} 2g(X, Y) \|H\|^2 \quad &= 2\sin^2\theta \{g(X, Y) - \eta(X)\eta(Y)\} \|H\|^2 \\ &\quad - g(F_X H, NY) - g(F_Y H, NX). \end{aligned}$$

Thus we have

$$\cos^2\theta g(X, Y) \|H\|^2 + \sin^2\theta\eta(X)\eta(Y) \|H\|^2 = -\frac{1}{2} \{g(F_X H, NY) + g(F_Y H, NX)\} .$$

In view of (2.20) and (2.25) the fact that $H \in \Gamma(\mu)$, then the above equation takes the form

$$\cos^2\theta g(X,Y)\left\|H\right\|^2 + \sin^2\theta\eta(X)\eta(Y)\left\|H\right\|^2 = -\sin^2\theta g(X,Y)\left\|H\right\|^2 + \sin^2\theta\eta(X)\eta(Y)\left\|H\right\|^2. \quad (3.26)$$

From (3.26), we conclude that $g(X,Y)\left\|H\right\|^2 = 0$, $\forall X, Y \in \Gamma(TM)$. Since M is a proper-slant, we obtain $H = 0$. This tells us that M is totally geodesic in $\widetilde{M}$. □

□

Theorem 3.10. *Let M be a totally umbilical pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then at least one of the following statements is true;*
i. *$dim(D^{\perp}) = 1$,*
ii. *$H \in \Gamma(\mu)$,*
iii. *M is a proper pseudo-slant submanifold.*

Proof. For any $X \in \Gamma(D^{\perp})$ from (2.5), we have

$$(\widetilde{\nabla}_X\varphi)X = g(X,X)\xi,$$

or,

$$\widetilde{\nabla}_X NX - \varphi(\nabla_X X + h(X,X)) - \left\|X\right\|^2\xi = 0.$$

From the last equation, we have

$$-A_{NX}X + \nabla^{\perp}_X NX - N\nabla_X X - th(X,X) - nh(X,X) - \left\|X\right\|^2\xi = 0.$$

The tangential components of (3.27), we obtain

$$A_{NX}X + th(X,X) + \left\|X\right\|^2\xi = 0$$

Taking the inner product by $Y \in \Gamma(D^{\perp})$, we have

$$g(A_{NX}X + th(X,X), Y) = 0.$$

This implies that

$$g(h(X,Y), NX) + g(th(X,X), Y) = 0.$$

Since M is totally umbilical submanifold, we obtain

$$g(g(X,Y)H, NX) + g(g(X,X)tH, Y) = 0$$

or,

$$g(X,Y)g(H,NX) + g(X,X)g(tH,Y) = 0,$$

which implies that

$$g(tH,Y)X - g(tH,X)Y = 0.$$

Here, tH is either zero or X and Y are linearly dependent. If $tH \neq 0$, then the vectors X and Y are linearly dependent and $\dim(D^{\perp}) = 1$.

On the other hand, $tH = 0$, i.e., $H \in \Gamma(\mu)$. Since dim $(D_\theta) \neq 0$, M is a pseudo-slant submanifold. Since $\theta \neq 0$ and $d_1.d_2 \neq 0$, M is a proper pseudo-slant submanifold. □

Theorem 3.11. *Let M be totally umbilical proper pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then following conditions are equivalent*
i. $H \in \Gamma(\mu)$,
ii. $\varphi^2 X = -\nabla_{TX}\xi$,
iii. *M is an anti-invariant submanifold,*
for any $X \in \Gamma(TM)$.

Proof. For any $X \in \Gamma(TM)$, from (2.5), we have

$$(\widetilde{\nabla}_X\varphi)X = g(X,X)\xi - \eta(X)X.$$

By means of (2.6), (2.7), (2.11) and (2.12), we obtain

$$\begin{aligned} 0 &= \widetilde{\nabla}_X TX + \widetilde{\nabla}_X NX - \varphi(\nabla_X X + h(X,X)) - g(X,X)\xi + \eta(X)X \\ &= \nabla_X TX + h(X,TX) - A_{NX}X + \nabla^{\perp}_X NX - T\nabla_X X - N\nabla_X X \\ &\quad -th(X,X) - nh(X,X) - g(X,X)\xi + \eta(X)X. \end{aligned} \tag{3.27}$$

The tangential components of (3.27), we obtain

$$\nabla_X TX - T\nabla_X X - th(X,X) - A_{NX}X - g(X,X)\xi + \eta(X)X = 0.$$

Since M is a totally umbilical submanifold, we can derive $A_{NX}X = g(H,NX)X$, then we have

$$\begin{aligned} \nabla_X TX - T\nabla_X X - g(X,X)tH - g(H,NX)X& \\ -g(X,X)\xi + \eta(X)X = 0.& \end{aligned} \tag{3.28}$$

If $H \in \Gamma(\mu)$, then from (3.28), we conclude that

$$\nabla_X TX - T\nabla_X X - g(X,X)\xi + \eta(X)X = 0. \tag{3.29}$$

Taking the inner product of (3.29) by ξ, we get

$$g(\nabla_X TX,\xi) = g(X,X) - \eta^2(X). \tag{3.30}$$

Interchanging X by TX in (3.30) and making use of (2.33), we derive

$$g(\nabla_{TX}T^2X,\xi) = g(TX,TX),$$

or,

$$g(\nabla_{TX}\xi, T^2X) = -\cos^2\theta g(\phi X,\varphi X),$$

$$g(\nabla_{TX}\xi, -\cos^2\theta(X-\eta(X)\xi)) = -\cos^2\theta g(\varphi X,\varphi X),$$

that is,

$$\cos^2\theta\{g(\varphi X,\varphi X) - g(\nabla_{TX}\xi,(X-\eta(X)\xi)\} = 0.$$

Since M is a proper pseudo-slant submanifold, we have

$$g(\varphi X, \varphi X) - g(\nabla_{TX}\xi, (X - \eta(X)\xi)) = 0,$$

that is,

$$g(\varphi X, \varphi X) - g(\nabla_{TX}\xi, X) + \eta(X)g(\nabla_{TX}\xi, \xi) = 0. \tag{3.31}$$

Taking the covariant derivative of above equation with respect to TX for any $X \in \Gamma(TM)$, we obtain $g(\nabla_{TX}\xi, \xi) + g(\xi, \nabla_{TX}\xi) = 0$, which implies $g(\nabla_{TX}\xi, \xi) = 0$ and then (3.31) becomes

$$g(X, X) - \eta^2(X) - g(\nabla_{TX}\xi, X) = 0. \tag{3.32}$$

This proves **ii**. of the Theorem. So if (3.32) is satisfied, then (3.28) implies $H \in \Gamma(\mu)$.

Now, interchanging X by TX in (3.32), we derive

$$g(TX, TX) - g(\nabla_{T^2X}\xi, TX) = 0,$$

that is,

$$\cos^2\theta g(\varphi X, \varphi X) + \cos^2\theta g(\nabla_{(X-\eta(X)\xi)}\xi, TX) = 0,$$

from which

$$\cos^2\theta g(\varphi X, \varphi X) + \cos^2\theta g(\nabla_X\xi, TX) - \cos^2\theta\eta(X)g(\nabla_\xi\xi, TX) = 0.$$

Since $\nabla_\xi\xi = 0$, we obtain

$$\cos^2\theta\{g(\varphi X, \varphi X) + g(\nabla_X\xi, TX)\} = 0. \tag{3.33}$$

We note

$$g(\varphi X, \varphi X) + g(\nabla_X\xi, TX) \neq 0$$

from (3.33, we can derive if $\cos\theta = 0$, then M is an anti-invariant submanifold. $\square$

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ON CONHARMONIC CURVATURE TENSOR OF SASAKIAN FINSLER STRUCTURES ON TANGENT BUNDLES

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Abstract. The content of this paper is made up of conharmonic curvature tensor$K$ of Sasakian Finsler structures on tangent bundles. In this manner, quasi-conharmonically flat, ξ -conharmonically flat, φ-conharmonically flat Sasakian Finsler structures are studied. Some structure theorems including Einstein Sasakian Finsler manifolds satisfying $R(X^H, Y^H).K = 0$ are clarified.

1. Introduction

Sinha and Yadav, constructed almost contact Finsler structure φ on the total space of a vector bundle in [15], then Hasegawa, Yamauchi and Shimada discussed Sasakian structures on Finsler manifolds in chapter 6 in [9]. In Yaliniz and Caliskan's paper, Sasakian Finsler manifolds and their principal curvature properties are studied [17]. In this study, Sasakian Finsler structures' conharmonic curvature tensor properties are characterized by using the tangent bundle approach.

Conharmonic curvature tensor is defined by Ishii [10] then studied for several manifold structures in differential geometry: such as Riemannian, almost Hermite, Kahler and nearly Kahler manifolds by Mishra [14], Doric et al. [6], Krichenko et al. [12], [13], for K-contact, Sasakian, Kenmotsu and LP- Sasakian manifolds by Khan [11], Dwivedi and Kim [7] , Asghari and Taleshian, Taleshian et al., [4] [16], for Sasakian space forms by De et al. [5] , for N(k)-contact metric manifolds by Ghosh et al. [8], for $C(\lambda)$ manifolds by Akbar and Sarkar [1]. In this paper, conharmonic curvature tensor is studied for Sasakian Finsler structures on tangent bundles. In order to examine this; some characteristics of such kind of structures are given:

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Assume, M be an $m=(2n+1)$-dimensional smooth manifold. TM_x denotes the tangent space to M at x where $x=(x^1,\dots,x^{(2n+1)})\in M$ and $y=y^i\frac{\partial}{\partial x^i}\in TM_x$. TM is notated as the tangent bundle of the manifold M. Thus $u=(x,y)\in TM$.

Suppose $F:TM\to[0,\infty[$ be a Finsler function with the following properties:

(1) F is differentiable of class C^1 on TM and differentiable of class C^∞on $TM_0=TM\backslash\{(x,0)\}$,
(2) $F(x,\lambda y)=|\lambda|F(x,y)$, $(x,y)\in TM$, $\lambda\in\mathbb{R}$,
(3) $g_{ij}(x,y)=\frac{1}{2}[\frac{\partial^2F^2}{\partial y^i\partial y^j}]$ is positive definite on TM_0,
then g is called a Finsler metric tensor with g_{ij} coefficients and $F^m=(M,F)$ is a Finsler manifold [2].

If (x^i,y^i) are the local coordinates of TM, then $\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i}\}$ denote natural bases of $TTM_{|u|}$ at $u\in TM$. Thus, the complete vector field $X=X^i\frac{\partial}{\partial x^i}+X^j\frac{\partial}{\partial y^j}\in TTM_{|u|}$. Suppose that, $\pi:TM\to M$ is the bundle projection, then $\pi=(TM,\pi,M)$ of rank m is called Finsler tangent bundle. So, the differential map $\pi:TTM_{|u|}\to TM_{\pi(u)}$ generates the vertical distribution VTM of TTM_0 where HTM and VTM are complementary distributions of each other for TTM_0.

The horizontal distribution $HTM=(N_i^j(x,y))$ of TTM_0 is the non-linear connection on TM where $N_i^j=\frac{\partial N^j}{\partial y^i}$ are obtained via the spray coefficients notated $N^j=\frac{1}{4}g^{jk}(\frac{\partial^2F^2}{\partial y^k\partial x^h}y^h-\frac{\partial F^2}{\partial x^k})$. By using the linear connection $\triangledown$ on VTM, the pair $(HTM,\triangledown)$ is called a Finsler connection on M [3]. So, $X=X^i(\frac{\partial}{\partial x^i}-N_i^j(x,y)\frac{\partial}{\partial y^j})+(N_i^j(x,y)X^i+\widetilde{X}^j)\frac{\partial}{\partial y^j}=X^i\frac{\delta}{\delta x^i}+X^j\frac{\partial}{\partial y^j}$ is obtained. Here, $\frac{\delta}{\delta x^i}$ and $\frac{\partial}{\partial y^j}$ are the bases of $HTM_{|u|}$ and $VTM_{|u|}$, respectively. Besides, their dual bases are dx^i and $\delta y^j=dy^j+N_i^jdx^i$, respectively where $TTM_{|u|}=HTM_{|u|}\oplus VTM_{|u|}$.

The Riemannian metric G with Finsler coefficients, is called the Sasaki-Finsler metric on TM_0 and its distributions are as follows:

$G(X,Y)=G(X^H,Y^H)+G(X^V,Y^V)=G^H(X,Y)+G^V(X,Y)$ for tangent vectors $X,Y\in TTM_{|u|}$ at $u\in TM$ and $X^H,Y^H\in HTM_{|u|}$ and $X^V,Y^V\in VTM_{|u|}$. G^H and $G^V$are Riemannian metrics of type $\begin{pmatrix}0&0\\2&0\end{pmatrix}$ and $\begin{pmatrix}0&0\\0&2\end{pmatrix}$ on hTM and vTM, respectively. Thus, following properties are satisfied:

$G(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})=G(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})=g_{ij}$ and $G(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j})=0$. $(\varphi^H,\xi^H,\eta^H,G^H)$ and $(\varphi^V,\xi^V,\eta^V,G^V)$ are $(2n+1)$-dimensional Sasakian Finsler structures on hTM and vTM, respectively and the following relations hold [17]:

$$\varphi^2=-I+\eta^H\otimes\xi^H+\eta^V\otimes\xi^V \tag{1.1}$$

$$\eta^H(\xi^H)=1=\eta^V(\xi^V) \tag{1.2}$$

$$\varphi^H(\xi^H)=0=\varphi^V(\xi^V) \tag{1.3}$$

$$\eta^H\circ\varphi^H=0=\eta^V\circ\varphi^V \tag{1.4}$$

$$\begin{aligned} G(X^H,Y^H) &= G(\varphi^H X^H,\varphi^H Y^H)+\eta^H(X^H)\eta^H(Y^H) \\ G(X^V,Y^V) &= G(\varphi^V X^V,\varphi^V Y^V)+\eta^V(X^V)\eta^V(Y^V) \end{aligned} \tag{1.5}$$

$$G(X^H,\varphi^H Y^H)=-G(\varphi^H X^H,Y^H), G(X^V,\varphi^V Y^V)=-G(\varphi^V X^V,Y^V) \tag{1.6}$$

$$G(X^H,\xi^H)=\eta^H(X^H), G(X^V,\xi^V)=\eta^V(X^V) \tag{1.7}$$

$$G(X^H,\varphi^H Y^H)=d\eta^H(X^H,Y^H), G(X^V,\varphi^V Y^V)=d\eta^H(X^H,Y^H) \tag{1.8}$$

$$\nabla^H_X \xi^H=-\frac{1}{2}\varphi^H X^H, \nabla^V_X \xi^V=-\frac{1}{2}\varphi^V X^V \tag{1.9}$$

$$\begin{aligned} (\nabla^H_X \varphi^H)Y^H &= \frac{1}{2}[G(X^H,Y^H)\xi^H-\eta^H(Y^H)X^H] \\ (\nabla^V_X \varphi^V)Y^V &= \frac{1}{2}[G(X^V,Y^V)\xi^V-\eta^V(Y^V)X^V] \end{aligned} \tag{1.10}$$

$$\begin{aligned} R(X^H,Y^H)\xi^H &= \frac{1}{4}[\eta^H(Y^H)X^H-\eta^H(X^H)Y^H] \\ R(X^V,Y^V)\xi^V &= \frac{1}{4}[\eta^V(Y^V)X^V-\eta^V(X^V)Y^V] \end{aligned} \tag{1.11}$$

$$\begin{aligned} R(\xi^H,X^H)Y^H &= \frac{1}{4}[G(X^H,Y^H)\xi^H-\eta^H(Y^H)X^H] \\ R(\xi^V,X^V)Y^V &= \frac{1}{4}[G(X^V,Y^H)\xi^V-\eta^V(Y^V)X^V] \end{aligned} \tag{1.12}$$

$$S(X^H,\xi^H)=\frac{n}{2}\eta^H(X^H), S(X^V,\xi^V)=\frac{n}{2}\eta^V(X^V) \tag{1.13}$$

$$S(\xi^H,\xi^H)=\frac{n}{2}, S(\xi^V,\xi^V)=\frac{n}{2} \tag{1.14}$$

$$R(X^H,\xi^H)\xi^H=-\frac{1}{4}\ , R(X^V,\xi^V)\xi^V=-\frac{1}{4} \tag{1.15}$$

For all $X^H,Y^H\in HTM_{|u|}$ and $X^V,Y^V\in VTM_{|u|}$. Additionally, φ denotes the tensor field of type $\begin{pmatrix}1&1\\1&1\end{pmatrix}$, ξ is the structure vector field of type $\begin{pmatrix}0&0\\1&1\end{pmatrix}$, η is the 1-form of type $\begin{pmatrix}1&1\\0&0\end{pmatrix}$, G is the Sasaki-Finsler metric structure of type $\begin{pmatrix}0&0\\2&2\end{pmatrix}$, ∇ is the Finsler connection with respect to G on TM, R is the Riemann curvature tensor field of type $\begin{pmatrix}1&1\\3&3\end{pmatrix}$, S is the Ricci tensor field of type $\begin{pmatrix}0&0\\2&2\end{pmatrix}$.

As it is seen from above-mentioned preliminaries, Sasakian Finsler structures can be founded either on hTM or vTM. In this paper, hTM is considered on behalf of briefness.

Definition 1.1. For m-dimensional $(hTM, \varphi^H, \xi^H, \eta^H, G^H)$, conharmonic curvature tensor is described as follows:

$$K(X^H, Y^H)Z^H = R(X^H, Y^H)Z^H + \frac{1}{2n-1}[S(Y^H, Z^H)X^H - S(X^H, Z^H)Y^H + G(Y^H, Z^H)QX^H - G(X^H, Z^H)QY^H] \quad (1.16)$$

for $X^H, Y^H, Z^H \in HTM_{|u|}$.

2. QUASI-CONHARMONICALLY FLAT SASAKIAN FINSLER STRUCTURES

Definition 2.1. If $(\varphi^H, \xi^H, \eta^H, G^H)$ is the Sasakian Finsler metric structure on hTM, then hTM is quasi-conharmonically flat when the below relation is verified:

$$G(K(X^H, Y^H)Z^H, \varphi W^H) = 0 \quad (2.1)$$

for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$.

For a Sasakian Finsler manifold, because of the scalar curvature tensor $r = 0$, with the help of (1.13), it is possible to get the below relation:

$$S(Y^H, Z^H) = -\frac{1}{4}G(Y^H, Z^H) + (\frac{n}{2} + \frac{1}{4})\eta^H(Y^H)\eta^H(Z^H) \quad (2.2)$$

for $Y^H, Z^H \in HTM_{|u|}$ and that means hTM is the η-Einstein.

Theorem 2.2. *For a Sasakian Finsler manifold $(hTM, \varphi^H, \xi^H, \eta^H, G^H)$ necessary and sufficient condition to be quasi-conharmonically flat is: the below relation holds:*

$$\begin{aligned} R(X^H, Y^H)Z^H &= -\frac{1}{2(2n-1)}[G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H] \\ &+ \frac{(2n+1)}{4(2n-1)}[\eta^H(Y^H)\eta^H(Z^H)X^H - \eta^H(X^H)\eta^H(Z^H)Y^H] \\ &+ \frac{(2n+1)}{4(2n-1)}[G(Y^H, Z^H)\eta^H(X^H)\xi^H - G(X^H, Z^H)\eta^H(Y^H)\xi^H] \end{aligned} \quad (2.3)$$

for $X^H, Y^H, Z^H \in HTM_{|u|}$.

Proof. Due to the Sasakian Finsler manifold is quasi-conharmonically flat with dimension $(2n+1)$, using (1.16) and (2.1), it is known that $r = 0$ and taking $W^H = \varphi W^H$, the equality is herein below:

$$\begin{aligned} G(R(X^H, Y^H)Z^H, \varphi^2 W^H) &= \frac{1}{2(2n-1)}[G(Y^H, Z^H)G(\varphi X^H, \varphi W^H) \\ &- G(X^H, Z^H)G(\varphi Y^H, \varphi W^H)] \\ &+ (\frac{1}{4} + \frac{n}{2})[G(\varphi Y^H, \varphi W^H)\eta^H(X^H)\eta^H(Z^H) - G(\varphi X^H, \varphi W^H)\eta^H(Y^H)\eta^H(Z^H)] \end{aligned} \quad (2.4)$$

for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$.

By using (1.16) and (2.1) in (2.4), it is possible to get (2.3). $\square$

3. ξ-CONHARMONICALLY FLAT SASAKIAN FINSLER STRUCTURES

Definition 3.1. Assume that, $(\varphi^H, \xi^H, \eta^H, G^H)$ is the Sasakian Finsler metric structure on hTM, then it is called ξ-conharmonically flat if the following relation holds:

$$K(X^H, Y^H)\xi^H = 0 \tag{3.1}$$

for $X^H, Y^H \in HTM_{|u|}$.

Theorem 3.2. *For a Sasakian Finsler manifold $(hTM, \varphi^H, \xi^H, \eta^H, G^H)$ necessary and sufficient condition to be ξ-conharmonically flat is: hTM is an η-Einstein manifold.*

Proof. For a $(2n+1)$-dimensional $(n > 1)$ Sasakian Finsler manifold hTM, (1.13) holds. By using this in (2.4), it is possible to get

$$S(X^H, W^H) = -\frac{1}{4}G(X^H, W^H) + (\frac{n}{2} + \frac{1}{4})\eta^H(X^H)\eta^H(W^H) \tag{3.2}$$

for $X^H, W^H \in HTM_{|u|}$. Namely, the Sasakian Finsler manifold is η-Einstein and vice versa. □

4. φ-CONHARMONICALLY FLAT SASAKIAN FINSLER STRUCTURES

Definition 4.1. Let $(hTM, \varphi^H, \xi^H, \eta^H, G^H)$ be a Sasakian Finsler manifold, then hTM is said to be φ-conharmonically flat when the below equality is satisfied:

$$G(K(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) = 0 \tag{4.1}$$

for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$.

Theorem 4.2. *For an m-dimensional Sasakian Finsler manifold necessary and sufficient condition to be φ -conharmonically flat is: following relation holds:*

$$\begin{aligned} G(R(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) = -\frac{1}{2(2n-1)}\{G(\varphi Y^H, \varphi Z^H) \\ G(\varphi X^H, \varphi W^H) - G(\varphi X^H, \varphi Z^H)G(\varphi Y^H, \varphi W^H)\}. \end{aligned} \tag{4.2}$$

Proof. For a $(2n+1)$-dimensional hTM with the help of (1.16), it is possible to have the below relation:

$$S(\varphi X^H, \varphi W^H) = G(Q(\varphi X^H), \varphi W^H). \tag{4.3}$$

In consequence of this relation, the equality herein below can be written:

$$\begin{aligned} G(K(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) = G(R(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) \\ -\frac{1}{2n-1}\{S(\varphi Y^H, \varphi Z^H)G(\varphi X^H, \varphi W^H) - S(\varphi X^H, \varphi Z^H)G(\varphi Y^H, \varphi W^H) \\ +G(\varphi Y^H, \varphi Z^H)S(\varphi X^H, \varphi W^H) - G(\varphi X^H, \varphi Z^H)S(\varphi Y^H, \varphi W^H)\}. \end{aligned} \tag{4.4}$$

Owing to the fact that $\{E_i^H\}$ is orthonormal basis of $HTM_{|u|}$, $\{\varphi E_i^H\}$ is orthonormal basis either. By taking summation over $i = 1, 2, \ldots, (2n+1)$ in (4.4) and changing $X^H = W^H = E_i^H$, it takes the following form:

$$\begin{aligned} G(K(\varphi E_i^H, \varphi Y^H)\varphi Z^H, \varphi E_i^H) &= G(R(\varphi E_i^H, \varphi Y^H)\varphi Z^H, \varphi E_i^H) \\ -\frac{1}{2n-1}\{S(\varphi Y^H, \varphi Z^H)G(\varphi E_i^H, \varphi E_i^H) &- S(\varphi E_i^H, \varphi Z^H)G(\varphi Y^H, \varphi E_i^H) \\ +G(\varphi Y^H, \varphi Z^H)S(\varphi E_i^H, \varphi E_i^H) &- G(\varphi E_i^H, \varphi Z^H)S(\varphi Y^H, \varphi E_i^H)\} \end{aligned} \tag{4.5}$$

for $Y^H \in HTM_{|u|}$. Due to HTM is φ-conharmonically flat, (4.1) holds and by virtue of (4.5),

$$S(\varphi Y^H, \varphi Z^H) = (r - \frac{1}{4})G(\varphi Y^H, \varphi Z^H) \tag{4.6}$$

the above equality holds for $Y^H, Z^H \in HTM_{|u|}$. Also, if we take summation over $i = 1, 2, \ldots, 2n+1$ in (4.6) and changing $Y^H = Z^H = E_i^H$, it is obtained that $r = 0$. Using (4.1) in (4.4),we have (4.2). □

Theorem 4.3. *For a $(2n+1)$-dimensional $(n > 1)$ Sasakian Finsler manifold hTM, following items are equal to each other:*

(1) *hTM is conharmonically flat.*
(2) *hTM is φ-conharmonically flat.*
(3) *The below relation holds:*

$$\begin{aligned} G(R(X^H, Y^H)Z^H, W^H) = &\frac{1}{2(2n-1)}[G(Y^H, Z^H)G(X^H, W^H) \\ &-G(X^H, Z^H)G(Y^H, W^H)] \\ &-\frac{(2n+1)}{4(2n-1)}[-G(X^H, W^H)\eta^H(Y^H)\eta^H(Z^H) \\ &+G(Y^H, W^H)\eta^H(X^H)\eta^H(Z^H) \\ -G(Y^H, Z^H)\eta^H(X^H)\eta^H(W^H) &+ G(X^H, Z^H)\eta^H(Y^H)\eta^H(W^H)] \end{aligned} \tag{4.7}$$

for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$.

Proof. $1 \Rightarrow 2:$ Due to the Sasakian Finsler manifold hTM is conharmonically flat, $K(X^H, Y^H)Z^H = 0$ for $X^H, Y^H, Z^H \in HTM_{|u|}$. Therefore (4.1) holds, namely $G(K(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) = 0$. Then manifold is φ-conharmonically flat.

$2 \Rightarrow 3:$ If the Sasakian Finsler manifold hTM is φ -conharmonically,(4.1) holds. By using (1.11) and (1.12) in (4.1),

$$\begin{aligned} G(R(\varphi^2 X^H, \varphi^2 Y^H)\varphi^2 Z^H, \varphi^2 W^H) &= G(R(X^H, Y^H)Z^H, W^H) \\ +\frac{1}{4}\{-G(X^H, W^H)\eta^H(Y^H)\eta^H(Z^H) &+ G(Y^H, W^H)\eta^H(X^H)\eta^H(Z^H) \\ -G(Y^H, Z^H)\eta^H(X^H)\eta^H(W^H) &+ G(X^H, Z^H)\eta^H(Y^H)\eta^H(W^H)\} \end{aligned} \tag{4.8}$$

the above relation can be calculated for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$. Changing X^H, Y^H, Z^H, W^H with $\varphi X^H, \varphi Y^H, \varphi Z^H, \varphi W^H$ respectively, following relation is obtained:

$$\begin{aligned} G(R(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) = -\frac{1}{2(2n-1)}\{G(Y^H, Z^H)G(X^H, W^H) \\ -G(X^H, Z^H)G(Y^H, W^H) - G(X^H, W^H)\eta^H(Y^H)\eta^H(Z^H) \\ +G(Y^H, W^H)\eta^H(X^H)\eta^H(Z^H) \\ -G(Y^H, Z^H)\eta^H(X^H)\eta^H(W^H) + G(X^H, Z^H)\eta^H(Y^H)\eta^H(W^H)\}. \end{aligned} \tag{4.9}$$

With the help of (4.8) and (4.9),

$$\begin{aligned} G(R(X^H, Y^H)Z^H, W^H) = -\frac{1}{2(2n-1)}[G(Y^H, Z^H)G(X^H, W^H) \\ -G(X^H, Z^H)G(Y^H, W^H)] \\ -\frac{(2n+1)}{4(2n-1)}[-G(X^H, W^H)\eta^H(Y^H)\eta^H(Z^H) + G(Y^H, W^H)\eta^H(X^H)\eta^H(Z^H) \\ -G(Y^H, Z^H)\eta^H(X^H)\eta^H(W^H) + G(X^H, Z^H)\eta^H(Y^H)\eta^H(W^H)] \end{aligned} \tag{4.10}$$

is obtained for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$. Thereby (4.7) holds.

$3 \Rightarrow 1$: Accept that (4.7) holds for Sasakian Finsler manifold hTM . By taking summation over $i = 1, 2, \ldots, 2n+1$ in (4.7) and taking $X^H = W^H = E_i^H$, the below relation is satisfied:

$$S(Y^H, Z^H) = -\frac{1}{4}G(Y^H, Z^H) + (\frac{n}{2} + \frac{1}{4})\eta^H(Y^H)\eta^H(Z^H) \tag{4.11}$$

for $Y^H, Z^H \in HTM_{|u|}$. If (4.11) and (4.7) are used in (1.16), $K(X^H, Y^H)Z^H = 0$ is obtained. Namely the Sasakian Finsler manifold hTM is conharmonically flat. □

5. EINSTEIN SASAKIAN FINSLER STRUCTURES SATISFYING $R(X^H, Y^H).K = 0$

Theorem 5.1. *Let hTM be a $(2n+1)$-dimensional conharmonically flat Einstein Sasakian Finsler manifold and the relation $R(X^H, Y^H).K = 0$ is satisfied, then it is locally isometric to $S^m(1)$.*

Proof. Due to Sasakian Finsler manifold hTM is Einstein, with the help of (1.16)

$$K(X^H, Y^H)Z^H = R(X^H, Y^H)Z^H + \frac{2\lambda}{2n-1}[G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H] \tag{5.1}$$

holds for $X^H, Y^H, Z^H \in HTM_{|u|}$ and $\lambda \in \mathbb{R}$. Then the below equality is satisfied:

$$\eta^H(K(X^H, Y^H)Z^H) = [\frac{2\lambda}{2n-1} - \frac{1}{4}][G(X^H, Z^H)\eta^H(Y^H) - G(Y^H, Z^H)\eta^H(X^H)] \tag{5.2}$$

for $X^H, Y^H, Z^H \in HTM_{|u|}$. Taking $X^H = \xi^H$ in (5.2),

$$\eta^H(K(\xi^H, Y^H)Z^H) = [\frac{2\lambda}{2n-1} - \frac{1}{4}][\eta^H(Y^H)\eta^H(Z^H) - G(Y^H, Z^H)] \tag{5.3}$$

is obtained. Changing $Z^H = \xi^H$ in (5.2), it is possible to get

$$\eta^H(K(X^H, Y^H)\xi^H) = 0 \tag{5.4}$$

for $X^H, Y^H \in HTM_{|u|}$. If $R(X^H, Y^H)$ is considered as the derivation of the tensor algebra at each point of the Sasakian Finsler manifold hTM for X^H and Y^H, following relation holds for conharmonic curvature tensor:

$$\begin{aligned}[R(X^H, Y^H)K](U^H, V^H)W^H = R(X^H, Y^H)[K(U^H, V^H)W^H \\ -K(R(X^H, Y^H)U^H, V^H)W^H - K(U^H, R(X^H, Y^H)V^H)W^H \\ -K(U^H, V^H)R(X^H, Y^H)W^H.\end{aligned} \tag{5.5}$$

Owing to $R(X^H, Y^H).K = 0$, by taking $X^H = \xi^H$ in the last equality,

$$\begin{aligned}G([R(\xi^H, Y^H)K](U^H, V^H)W^H, \xi^H) = -G(K(R(\xi^H, Y^H)U^H, V^H)W^H, \xi^H) \\ -G(K(U^H, R(\xi^H, Y^H)V^H)W^H, \xi^H) - G(K(U^H, V^H)R(\xi^H, Y^H)W^H, \xi^H)\end{aligned} \tag{5.6}$$

the above relation holds for the tangent vector fields that are orthogonal to ξ^H. By using (1.11) and (1.12) in (5.6),

$$\begin{aligned}0 = \frac{1}{4}\{G(Y^H, K(U^H, V^H)W^H) - \eta^H(Y^H)\eta^H(K(U^H, V^H)W^H) \\ -G(Y^H, U^H)\eta^H(K(\xi^H, V^H)W^H) + \eta^H(U^H)\eta^H(K(Y^H, V^H)W^H) \\ -G(Y^H, V^H)\eta^H(K(U^H, \xi^H)W^H) + \eta^H(V^H)\eta^H(K(U^H, Y^H)W^H) \\ -G(Y^H, W^H)\eta^H(K(U^H, V^H)\xi^H) + \eta^H(W^H)\eta^H(K(U^H, V^H)Y^H)\}\end{aligned} \tag{5.7}$$

is obtained. With the help of (5.2), it is possible to get

$$\begin{aligned}G(K(U^H, V^H)W^H, Y^H) = [\frac{2\lambda}{2n-1} - \frac{1}{4}][G(Y^H, V^H)G(U^H, W^H) \\ -G(Y^H, U^H)G(V^H, W^H)]\end{aligned} \tag{5.8}$$

for $U^H, V^H, W^H, Y^H \in HTM_{|u|}$. By using (5.1) in (5.8), the below relation is obtained:

$$G(R(U^H, V^H)W^H, Y^H) = \frac{1}{4}[G(Y^H, U^H)G(V^H, W^H) - G(Y^H, V^H)G(U^H, W^H)] \tag{5.9}$$

for $U^H, V^H, W^H, Y^H \in HTM_{|u|}$. □

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OSCILLATION OF NONLINEAR FOURTH-ORDER DIFFERENCE EQUATIONS WITH MIDDLE TERM

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Abstract. In this article, we study oscillatory properties of the fourth-order difference equation with middle-term

$$\Delta^4 x_m - a_m \Delta^2 x_{m+1} + b_m f(x_{m+\sigma}) = 0,$$

in case when the corresponding second-order difference equation $\Delta^2 h_m - a_m h_{m+1} = 0$ is nonoscillatory.

1. Introduction

Consider the fourth-order nonlinear difference equation

$$\Delta^4 x_m - a_m \Delta^2 x_{m+1} + b_m f(x_{m+\sigma}) = 0, \tag{1.1}$$

where $\sigma \in \mathbb{N}$ is a deviating argument and $\{a_m\}$, $\{b_m\}$ are real sequences for $m \in \mathbb{N}$. Function $f : \mathbb{R} \to \mathbb{R}$, is continuous such that $uf(u) > 0$ for $u \neq 0$ where $\mathbb{R}$ denotes the set of real numbers.

Throughout the paper we assume

$$a_m a_{m+1} > 0, \quad b_m > 0, \ m \in \mathbb{N}$$

and

$$\sum_{m=1}^{\infty} m|a_m| < \infty. \tag{1.2}$$

By a solution of the equation (1.1), we mean a real sequence $\{x_m\}$ satisfying equation (1.1) for $m \in \mathbb{N}$. A nontrivial solution $\{x_m\}$ of (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and it is oscillatory otherwise. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

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In the recent years, a great importance has been paid to the study of oscillatory behavior of fourth-order differential equations [6, 7] and difference equations [2, 3, 4, 5, 14, 16, 19], see also the monograph [1] and [15].

In the continuous case, the fourth-order differential equation

$$x^{(4)}(t) + q(t)x^{(2)}(t) + r(t)f(x(\varphi(t))) = 0$$

can be written as

$$\Big(h^2(t)\Big(\frac{x''(t)}{h(t)}\Big)'\Big)' + h(t)r(t)f(x(t)) = 0$$

$h''(t) + q(t)h(t) = 0$ is nonoscillatory and h is its eventually positive solution, see e.g. [7].

Došlá and Krejcova [11, 12] have investigated a class of fourth-order nonlinear difference equations of the form

$$\Delta\left(a_n\left(\Delta b_n\left(\Delta c_n\left(\Delta x_n\right)^{\gamma}\right)^{\beta}\right)^{\alpha}\right) + d_n x_{n+\tau}^{\lambda} = 0, \tag{1.3}$$

and Jankowski, Schmeidel and Zonenberg [14] have generalized the some results of [11] for neutral equation

$$\Delta\left(a_n\left(\Delta b_n\left(\Delta c_n\left(\Delta\left(x_n + p_n x_{n-\delta}\right)\right)^{\gamma}\right)^{\beta}\right)^{\alpha}\right) + d_n f(x_{n-\tau}) = 0, \tag{1.4}$$

where α, β and γ are the ratios of odd positive integers, integers τ, δ are deviating arguments.

In this paper we investigate oscillatory properties of the equation (1.1). Our approach is based on the transformation of (1.1) to the two-terms equation of the form (1.3) and to application of oscillation results for equation (1.3) stated in [11, 12].

2. Preliminaries

Consider second order linear equation

$$\Delta^2 h_m - a_m h_{m+1} = 0. \tag{2.1}$$

Let (2.1) be nonoscillatory. The following definition is given by Patula [17].

Definition 2.1. *If there exist two linearly independent solutions v and w of* (2.1) *such that $v/w \to 0$, as $n \to \infty$, then v is recessive solution and w is dominant solution of* (2.1).

We remark that the recessive solution always exist and is unique up to a constant factor, see [17, Theorem 1].

Lemma 2.1. *If* (2.1) *is nonoscillatory, there exist a recessive solution h such that*

$$\sum_{m=1}^{\infty} \frac{1}{h_m h_{m+1}} = \infty.$$

Proof. See [17, Theorem 1] and [1, Theorem 6.3.1]. □

Lemma 2.2. *If* $\sum_{m=1}^{\infty} m|a_m| < \infty$, *then* (2.1) *has recessive solution which tends to positive constant.*

Proof. Let $a_m > 0$ for $m \geq 1$. Then the conclusion follows from [10, Theorem 4]. In case $a_m < 0$ for $m \geq 1$, the statement follows from [13, Theorem 4.2]. □

From Lemma 1 and Lemma 2 we have the following Lemma.

Lemma 2.3. *If* $\sum_{m=1}^{\infty} m|a_m| < \infty$*, then recessive solution* h *of* (2.1) *provides*

$$\sum_{m=m_0}^{\infty} \frac{1}{h_m h_{m+1}} = \infty, \sum_{m=m_0}^{\infty} h_m = \infty. \tag{2.2}$$

Proof. See [1, Theorem 6.3.8] and [13, Theorem 4.2]. □

Now, we consider equation (1.1) and we write it as a two-terms equation.

Lemma 2.4. *Let the equation* (2.1) *be nonoscillatory and let* h *be its solution such that* $h_m > 0$ *for* $m \geq 1$*. Then, we have for* $m \geq 1$

$$\Delta^4 x_m - a_m \Delta^2 x_{m+1} = \frac{1}{h_{m+1}} \Delta \left[h_m h_{m+1} \Delta \left(\frac{1}{h_m} \Delta^2 x_m \right) \right]. \tag{2.3}$$

Consequently, x *is solution of equation* (1.1) *if and only if it is a solution of equation in the disconjugate form*

$$\Delta \left[h_m h_{m+1} \Delta \left(\frac{1}{h_m} \Delta^2 x_m \right) \right] + b_m h_{m+1} f(x_{m+\sigma}) = 0. \tag{2.4}$$

Proof. Assume that $y_m \equiv h_m u_m$, where $u = (u_m)$ is any sequence. Firstly, we show that

$$h_{m+1}(\Delta^2 y_m - a_m y_{m+1}) = \Delta(h_m h_{m+1} \Delta u_m). \tag{2.5}$$

Using the definition of difference operator, we can easily obtain that

$$\Delta(h_m h_{m+1} \Delta u_m) = h_{m+1}(h_{m+2} \Delta u_{m+1} - h_m \Delta u_m) \tag{2.6}$$

and

$$\Delta^2 y_m = h_{m+2} u_{m+2} - 2h_{m+1} u_{m+1} + h_m u_m. \tag{2.7}$$

From equation (2.1), we can write $a_m h_{m+1} = \Delta^2 h_m$ and

$$a_m y_{m+1} = u_{m+1} \Delta^2 h_m = u_{m+1}(h_{m+2} - 2h_{m+1} + h_m). \tag{2.8}$$

From (2.7) and (2.8)

$$h_{m+1}(\Delta^2 y_m - a_m y_{m+1}) = h_{m+1}(h_{m+2} \Delta u_{m+1} - h_m \Delta u_m). \tag{2.9}$$

Then, right side of equation (2.6) is equal to right side of equation (2.9) and we obtain,

$$\Delta^2 y_m - a_m y_{m+1} = \frac{1}{h_{m+1}} \Delta(h_m h_{m+1} \Delta u_m)$$

where $u_m = \frac{y_m}{h_m}$ and $y_m = \Delta^2 x_m$. □

Remark 2.1. *If h is recessive solution of* (2.1), *then by Lemma 3,* (2.2) *holds and equation* (2.4) *is said to be in the canonical form.*

Let x be a solution of (2.4) and denote the quasi-differences of x as

$$x^{[1]} = \Delta x_m,\ x^{[2]} = \tfrac{1}{h_m}\Delta x^{[1]},\ x^{[3]} = h_m h_{m+1}\Delta x^{[2]}.$$

Lemma 2.5. *If* (2.2) *holds, then any eventually positive solution $\{x_m\}$ of* (2.4) *is one of the following types:*
type (a): $x_m > 0, x^{[1]} > 0, x^{[2]} > 0, x^{[3]} > 0$ for large m,
type (b): $x_m > 0, x^{[1]} > 0, x^{[2]} < 0, x^{[3]} > 0$ for large m.

Proof. We consider (2.4) as a four-dimensional system

$$\begin{cases} \Delta x_m = y_m \\ \Delta y_m = h_m z_m \\ \Delta z_m = \frac{1}{h_m h_{m+1}} w_m \\ \Delta w_m = -b_m h_{m+1} f(x_{m+\sigma}), \end{cases} \tag{2.10}$$

where

$$(x, y, z, w) = (x, x^{[1]}, x^{[2]}, x^{[3]}).$$

Proceeding by the similar way as in [11], proof of Lemma 2, we obtain the conclusion. The details are omitted here. □

3. Oscillation results

In this section, we give oscillation results for equation (1.1). During this section we assume that equation (2.1) is nonoscillatory and h is a solution of (2.1) such that $h_m > 0$ for $m \geq 1$.

Solution x of (1.1) is called quickly oscillatory, if it is of the form

$$x_m = (-1)^m p_m,\ p_m > 0 \text{ for } m \in \mathbb{N}.$$

The following result can be seen as a necessary condition for existence of quickly oscillatory solutions.

Lemma 3.1. *If σ is even, then equation* (1.1) *has no quickly oscillatory solutions.*

Proof. Let $x_m = (-1)^m p_m$ be a quickly oscillatory solution of (1.1). By Lemma 4, x_m is solution of (2.4) and system (2.10). Then, the proof is the similar way as in [11], proof of Theorem 1 and [14], proof of Theorem 3.1. □

Theorem 3.1. *Let* (1.2) *holds. If $\sum_{i=1}^{\infty} b_i = \infty$, then* (1.1) *is oscillatory.*

Proof. By Lemma 4, we can transform equation (1.1) to equation (2.4). The proof follows from [14], proof of Theorem 4.4. □

Theorem 3.2. *Let* (1.2) *holds and there exist* $\lambda > 0$ *such that*

$$\lim_{u\to\infty} \frac{f(u)}{u^\lambda} > 0. \tag{3.1}$$

Equation (1.1) *with* $\sigma \geq 1$ *is oscillatory if any of the following conditions holds:*
(i) $\lambda < 1, \sum_{m=1}^{\infty} b_m m^\lambda = \infty$;

(ii) $\lambda > 1, \sum_{\mathrm{m}=1}^{\infty} \mathrm{b_m m} = \infty$.

Proof. For the sake of contradiction, let (1.1) have a nonoscillatory solution and let h be recessive solution of (2.4) such that $lim_{m\to\infty} h_m = 1$. Without loss of generality assume $x_m > 0$ for $m \geq 1$. By Lemma 4, x is nonoscillatory solution of (2.4). By Lemma 5, x is type (a) or type (b).

(i) Let x be of type (a) such that $x_m > 0$ for $m \geq 1$. Then, $\lim_{m\to\infty} x_m = \infty$. Consider equation

$$\Delta\left[h_m h_{m+1} \Delta\left(\frac{1}{h_m}\Delta^2 \upsilon_m\right)\right] + b_m h_{m+1}\frac{f(x_{m+\sigma})}{x^\lambda_{m+\sigma}}\upsilon^\lambda_{m+\sigma} = 0. \tag{3.2}$$

This equation has a solution $\upsilon = x$ of type (a). Using (3.1), we have that there exist $K > 0$ such that $\frac{f(x_{m+\sigma})}{x^\lambda_{m+\sigma}} \geq K$. We apply to (3.2), lemma in [11, Lemma 4] with $\alpha = \beta = \gamma = 1$ and $\sigma \geq 1$. We have

$$b_m h_{m+1}\frac{f(x_{m+\sigma})}{x^\lambda_{m+\sigma}} \geq \frac{K}{2} b_m, \text{ for large } m.$$

Thus,

$$\sum_{m=1}^{\infty} b_m h_{m+1}\frac{f(x_{m+\sigma})}{x^\lambda_{m+\sigma}} m^\lambda = \infty,$$

and by [11, Lemma 4 and Corollary 1], equation (3.2) is oscillatory. This is a contradiction with the fact that (3.2) has a nonoscillatory solution $\upsilon = x$.

(ii) Let x be of type (b). Then, there exist $\lim_{m\to\infty} x_m$. Because of the continuity of f there exist $K > 0$ such that

$$\lim_{m\to\infty} \frac{f(x_{m+\sigma})}{x^\lambda_{m+\sigma}} \geq K, \text{ for large } m,$$

and proceeding the similar way as in (i), we get that (3.2) has no nonoscillatory solution of type (b). This completes the proof. □

Theorem 3.3. *Let* (1.2) *holds and there exist* $\lambda > 0$ *such that*

$$\lim_{u\to\infty} \frac{f(u)}{u^\lambda} > 0.$$

Equation (1.1) *with* $\sigma \geq 3$ *is oscillatory if any of the following conditions hold:*

(i) $\lambda > 1$ *and*

$$\sum_{m=m_0}^{\infty} m^2 \sum_{k=m-2}^{\infty} b_k = \infty, \tag{3.3}$$

(ii) $\lambda = 1$ *and*

$$\lim_{m\to\infty} \sup \left(m^3 \sum_{k=m-3}^{\infty} b_k \right) = \infty. \tag{3.4}$$

Proof. (i) $\lambda > 1$, by [12, Corollary 2-(i)] equation (1.1) with $\sigma \geq 3$ has no solution of type (a) or type (b) if

$$\sum_{m=m_0}^{\infty} m^2 \sum_{k=m-2}^{\infty} b_k = \infty.$$

(ii) $\lambda = 1$, by [12, Corollary 2-(ii)] equation (3.4) implies

$$\lim_{m\to\infty} \sup \left(m \sum_{m=m_0}^{\infty} b_k k^2 \right) > 1.$$

This completes the proof. □

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THE VORONOVSKAJA TYPE ASYMPTOTIC FORMULA FOR q-DERIVATIVE OF INTEGRAL GENERALIZATION OF q-BERNSTEIN OPERATORS

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Abstract. The Voronovskaja type asymptotic formula for function having q-derivative of the integral generalization Bernstein operators based on q-integer is discussed. The same formula for Stancu type generalization of this operators is mentioned.

1. Introduction

The classical Bernstein-Durrmeyer operators D_n introduced by Durrmeyer [1] is associated with an integrable function f on the interval $[0,1]$ and is defined as

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt,\ \ x\in[0,1], \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$.

These operators have been studied by Derriennic [2] and many others. For the last 30 years, q-calculus has been an active area of research in approximation theory. In 1987, the q-analogues of Bernstein operators was introduced by Lupas [3] and in [4], q-generalization of the operators (1.1) was introduced as

$$D_{n,q}(f;x) = [n+1]_q\sum_{k=0}^{n} q^{-k}p_{n,k}(q;x)\int_0^1 f(t)p_{n,k}(q;qt)d_qt, \tag{1.2}$$

where $p_{n,k}(q;x) = \binom{n}{k}_q x^k(1-x)_q^{n-k}$.

The rate of convergence of the operators (1.2) was discussed by Zeng et al. [5]. In 2014, Mishra and Patel [6, 7] introduced the generalization due to Stancu

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and proved Voronovskaja type asymptotic formula and various other approximation properties of the q-Durrmeyer-Stancu operators. Here, in this manuscript, we establish Voronovskaja type asymptotic formula for function having q-derivative.

2. Estimation of moments and Asymptotic formula

In the sequel, we shall need the following auxiliary results:

Theorem 1 ([8]). *If m-th $(m > 0, m \in \mathbb{N})$ order moments of operator* (1.2) *is defined as*

$$D^q_{n,m}(x) = D_{n,q}(t^m, x) = [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_0^1 p_{n,k}(q;qt) t^m d_q t, \;\; x \in [0,1],$$

then $D^q_{n,0}(x) = 1$ and for $n > m+2$, we have the following recurrence relation,

$$[n+m+2]_q D^q_{n,m+1}(x) = ([m+1]_q + q^{m+1} x[n]_q) D^q_{n,m}(x) + x(1-x) q^{m+1} D_q(D^q_{n,m}(x)).$$

To establish asymptotic formula for functions having q-derivative, it is necessary to compute moments of first to fourth degree. Using above theorem one can have first, second, third and fourth order moments. The first three moments of Lemma 1 was also established in [4].

Lemma 1 ([4, 8]). *For all $x \in [0,1]$, $n = 1, 2, \ldots$ and $0 < q < 1$, we have*

- $D_{n,q}(1, x) = 1;$
- $D_{n,q}(t, x) = \frac{1 + qx[n]_q}{[n+2]_q};$
- $D_{n,q}(t^2, x) = \frac{q^3 x^2 [n]_q ([n]_q - 1) + (1+q)^2 qx[n]_q + 1 + q}{[n+3]_q [n+2]_q};$
- $D_{n,q}(t^3, x) = \frac{q^9 x^3 [n]_q [n-1]_q [n-2]_q + x^2 q^4 [3]^2_q [n]_q [n-1]_q + xq[2]_q [3]^2_q [n]_q + [3]_q [2]_q}{[n+4]_q [n+3]_q [n+2]_q};$
- $D_{n,q}(t^4, x) = \frac{q^{16} x^4 [n]_q [n-1]_q [n-2]_q [n-3]_q + q^9 x^3 [4]^2_q [n]_q [n-1]_q [n-2]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q}$
  $+ \frac{q^4 x^2 [2]_q [3]^2_q (1+q^2)^2 [n]_q [n-1]_q + qx[2]_q [3]_q [4]^2_q [n]_q + [2]_q [3]_q [4]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q}$

Lemma 2. *For all $x \in [0,1]$, $n = 1, 2, \ldots$ and $0 < q < 1$, we have*

- $D_{n,q}\left((t-x)_q, x\right) = \frac{1 - \left(1 + q^{n+1}\right)x}{[n+2]_q};$
- $D_{n,q}\left((t-x)^2_q, x\right) = \frac{qx^2 [2]_q \left([n]^2_q (1-q)^2 q^2 + [n]_q (2q^3 - [3]_q) + [3]_q\right) - x[2]_q ([3]_q + q[n]_q(-1-q+q^2)) + [2]_q}{[n+3]_q [n+2]_q};$
- $D_{n,q}\left((t-x)^3_q, x\right)$
  $= q^2 x^3 \left\{ \frac{q^7 [n]_q [n-1]_q [n-2]_q}{[n+2]_q [n+3]_q [n+4]_q} - \frac{q^2 [3]_q [n]_q [n-1]_q}{[n+2]_q [n+3]_q} + \frac{[2]_q [n]_q - q[n+2]_q}{[n+2]_q [n+3]_q [n+4]_q} \right\}$
  $+ qx^2 \left\{ \frac{q^3 [3]^2_q [n]_q [n-1]_q}{[n+2]_q [n+3]_q [n+4]_q} - \frac{[2]^2_q [3]_q [n]_q}{[n+2]_q [n+3]_q} + \frac{[2]_q}{[n+2]_q} \right\}$
  $+ x[2]_q [3]_q \left\{ \frac{q[3]_q [n]_q - [n+4]_q}{[n+2]_q [n+3]_q [n+4]_q} \right\} + \frac{[3]_q [2]_q}{[n+2]_q [n+3]_q [n+4]_q};$
- $D_{n,q}\left((t-x)^4_q, x\right)$
  $= q^4 x^4 \left\{ \frac{q^{12} [n]_q [n-1]_q [n-2]_q [n-3]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} - \frac{q^5 [4]_q [n]_q [n-1]_q [n-2]_q}{[n+4]_q [n+3]_q [n+2]_q} + \frac{q\left([5]_q + q^2\right)[n]_q [n-1]_q}{[n+3]_q [n+2]_q} - \frac{[4]_q [n]_q}{[n+2]_q} + q^2 \right\}$
  $+ x^3 q^2 \left\{ \frac{q^7 [4]^2_q [n]_q [n-1]_q [n-2]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} - \frac{q^2 [3]^2_q [4]_q [n]_q [n-1]_q}{[n+4]_q [n+3]_q [n+2]_q} + \frac{\left([5]_q + q^2\right)[2]^2_q [n]_q}{[n+3]_q [n+2]_q} - \frac{q[4]_q}{[n+2]_q} \right\}$
  $+ qx^2 \left\{ \frac{q^3 [2]_q [3]^2_q (1+q^2) [n]_q [n-1]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} - \frac{[2]_q [3]^2_q [4]_q [n]_q}{[n+4]_q [n+3]_q [n+2]_q} + \frac{[2]_q \left([5]_q + q^2\right)}{[n+3]_q [n+2]_q} \right\}$
  $+ x \left\{ \frac{[2]_q [3]_q [4]_q \left(q[4]_q [n]_q - [n+5]_q\right)}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} \right\} + \frac{[2]_q [3]_q [4]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q}.$

Theorem 2. *Let f be bounded and integrable on the interval $[0,1]$ and (q_n) denote a sequence such that $0<q_n<1$, $q_n \to 1$ and $q_n^n \to c$ as $n\to\infty$, where c is arbitrary constant. Then we have for a point $x\in(0,1)$,*

$$\lim_{n\to\infty}[n]_{q_n}[D_{n,q_n}(f;x)-f(x)] = (1-2x)\lim_{n\to\infty} D_{q_n}f(x) + x(1-x)\lim_{n\to\infty} D^2_{q_n}f(x).$$

Proof: By q-Taylor formula [9] for f, we have

$$f(t)=f(x)+D_{q_n}f(x)(t-x)+\frac{1}{[2]_{q_n}}D^2_{q_n}f(x)(t-x)^2_{q_n}+\theta_{q_n}(x;t)(t-x)^2_{q_n},$$

for $0<q<1$, where

$$\theta_{q_n}(x;t)=\begin{cases}\dfrac{f(t)-f(x)-D_{q_n}f(x)(t-x)-\frac{1}{[2]_{q_n}}D^2_{q_n}f(x)(t-x)^2_{q_n}}{(t-x)^2_{q_n}} & \text{if } x\neq t\\ 0, & \text{if } x=t.\end{cases} \tag{2.1}$$

We know that for n large enough

$$\lim_{t\to x}\theta_{q_n}(x;t)=0. \tag{2.2}$$

That is for any $\epsilon>0$, there exists a $\delta>0$ such that

$$|\theta_{q_n}(x;t)|\le\epsilon; \tag{2.3}$$

for $|t-x|<\delta$ and n sufficiently large. Using (2.1), we can write

$$D_{n,q_n}(f;x)-f(x)=D_{q_n}f(x)D_{n,q_n}((t-x)_{q_n};x)+\frac{D^2_{q_n}f(x)}{[2]_{q_n}}D_{n,q_n}((t-x)^2_{q_n};x)+E_n^{q_n}(x),$$

where

$$E_n^q(x)=[n+1]_{q_n}\sum_{k=0}^n q^{-k}p_{n,k}(q_n;x)\int_0^1\theta_{q_n}(x;t)p_{n,k}(q_n;q_nt)\,(t-x)^2_{q_n}\,d_{q_n}t.$$

By Lemma 2, we have

$$\lim_{n\to\infty}[n]_{q_n}D_{n,q_n}((t-x)_{q_n};x)=(1-2x) \text{ and } \lim_{n\to\infty}[n]_{q_n}D_{n,q_n}((t-x)^2_{q_n};x)=2x(1-x).$$

In order to complete the proof of the theorem, it is sufficient to show that $\lim_{n\to\infty}[n]_{q_n}E_n^{q_n}(x)=0$. We proceed as follows: Let

$$P^{q_n}_{n,1}(x)=[n]_{q_n}[n+1]_{q_n}\sum_{k=0}^n {q_n}^{-k}p_{n,k}(q_n;x)\int_0^1\theta_{q_n}(x;t)p_{n,k}(q_n;q_nt)\,(t-x)^2_{q_n}\,\chi_x(t)d_{q_n}t$$

and

$$P^{q_n}_{n,2}(x)=$$

$$[n]_{q_n}[n+1]_{q_n}\sum_{k=0}^n {q_n}^{-k}p_{n,k}(q_n;x)\int_0^1\theta_{q_n}(x;t)p_{n,k}(q_n;q_nt)\,(t-x)^2_{q_n}\,(1-\chi_x(t))\,d_{q_n}t,$$

so that

$$[n]_{q_n} E_n^{q_n}(x) \leq P_{n,1}^{q_n}(x) + P_{n,2}^{q_n}(x),$$

where $\chi_x(t)$ is the characteristic function of the interval $\{t : |t - x| < \delta\}$.

It follows from (2.3) that

$$P_{n,1}^{q_n}(x) = 2\epsilon x(1 - x) \text{ as } n \to \infty.$$

If $|t - x| \geq \delta$, then $|\theta_{q_n}(x;t)| \leq \frac{M}{\delta^2}(t - x)^2$, where $M > 0$ is a constant. Since

$$\begin{aligned} (t - x)^2 &= \left(t - q_n^2 x + q_n^2 x - x\right)\left(t - q_n^3 x + q_n^3 x - x\right) \\ &= \left(t - q_n^2 x\right)\left(t - q_n^3 x\right) + x(q_n^3 - 1)\left(t - q_n^2 x\right) + x(q_n^2 - 1)\left(t - q_n^2 x\right) \\ &\quad + x^2(q_n^2 - 1)(q_n^2 - q_n^3) + x^2(q_n^2 - 1)(q_n^3 - 1), \end{aligned}$$

we have

$$\begin{aligned} |P_{n,2}^{q_n}(x)| \leq \frac{M}{\delta^2} \Big\{ &[n]_{q_n} D_{n,q_n}((t - x)_{q_n}^4; x) + x(2 - q_n^2 - q_n^3)[n]_{q_n} D_{n,q_n}((t - x)_{q_n}^3; x) \\ &+ x^2(q_n^2 - 1)^2 [n]_{q_n} D_{n,q_n}((t - x)_{q_n}^2; x) \Big\}. \end{aligned}$$

Using Lemma 2, we have

$$D_{n,q_n}((t-x)_{q_n}^4; x) \leq \frac{C_1}{[n]_{q_n}^3},\ D_{n,q_n}((t-x)_{q_n}^3; x) \leq \frac{C_2}{[n]_{q_n}^2} \text{ and } D_{n,q_n}((t-x)_{q_n}^2; x) \leq \frac{C_3}{[n]_{q_n}},$$

and the desired result is obtained.

Corollary 1. *Let f be bounded and integrable on the interval $[0,1]$ and (q_n) denote a sequence such that $0 < q_n < 1$, $q_n \to 1$ and $q_n^n \to c$ as $n \to \infty$, where c is arbitrary constant. Suppose that the first and second derivatives $f'(x)$ and $f''(x)$ exist at a point $x \in (0,1)$. Then, we have, for a point $x \in (0,1)$*

$$\lim_{n\to\infty} [n]_{q_n} [D_{n,q_n}(f; x) - f(x)] = (1 - 2x) f'(x) + x(1 - x) f''(x).$$

3. Asymptotic formula for the Durrmeyer-Stancu Operators

In the year 1968, Stancu [10] generalized Bernstein operators and discussed its approximation properties. After that many researchers gave Stancu type generalization of several operators on finite and infinite intervals. We refer the readers to [11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and the references there in. As mention in the introduction, Stancu generalization of q-Durrmeyer operators (1.2) was discussed by Mishra and Patel [6], which is defined as follows:

$$D_{n,q}^{\alpha,\beta} = [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_0^1 f\left(\frac{[n]_q t + \alpha}{[n]_q + \beta}\right) p_{n,k}(q;qt) d_q t, \tag{3.1}$$

where $0 \leq \alpha \leq \beta$ and $p_{n,k}(q;x)$ as same as defined in (1.2). We shall need the following lemmas for proving our results.

Lemma 3 ([7]). *We have* $D_{n,q}^{\alpha,\beta}(1;x)=1,\ D_{n,q}^{\alpha,\beta}(t;x)=\frac{[n]_q+\alpha[n+2]_q+qx[n]_q^2}{[n+2]_q([n]_q+\beta)}$,

$$\begin{aligned}D_{n,q}^{\alpha,\beta}(t^2;x)&=\frac{q^3[n]_q^3([n]_q-1)x^2+\left(\left(q(1+q)^2+2\alpha q^4\right)[n]_q^3+2\alpha q[3]_q[n]_q^2\right)x}{([n]_q+\beta)^2[n+2]_q[n+3]_q}\\&\quad+\frac{\alpha^2}{([n]_q+\beta)^2}+\frac{(1+q+2\alpha q^3)[n]_q^2+2\alpha[3]_q[n]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q}.\end{aligned}$$

Lemma 4 ([7]). *We have*

$$\begin{aligned}D_{n,q}^{\alpha,\beta}(t-x,x)&=\left(\frac{q[n]_q^2}{[n+2]_q([n]_q+\beta)}-1\right)x+\frac{[n]_q+\alpha[n+2]_q}{[n+2]_q([n]_q+\beta)},\\D_{n,q}^{\alpha,\beta}((t-x)^2,x)&=\frac{q^4[n]_q^4-q^3[n]_q^3-2q[n]_q^2[n+3]_q([n]_q+\beta)+[n+2]_q[n+3]_q([n]_q+\beta)^2}{([n]_q+\beta)^2[n+2]_q[n+3]_q}x^2\\&\quad+\frac{q(1+q)^2[n]_q^3+2q\alpha[n]_q^2[n+3]_q-(2[n]_q+2\alpha[n+2]_q)[n+3]_q([n]_q+\beta)}{([n]_q+\beta)^2[n+2]_q[n+3]_q}x\\&\quad+\frac{(1+q)[n]_q^2+2\alpha[n]_q[n+3]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q}.\end{aligned}$$

Remark 1 ([7]). *For all* $m\in\mathbf{N}\cup\{0\},\ 0\le\alpha\le\beta$, *we have the following recursive relation for the images of the monomials* t^m *under* $D_{n,q}^{\alpha,\beta}$ *in terms of* $D_{n,q},\ j=0,1,2,\dots,m$, *as*

$$D_{n,q}^{\alpha,\beta}(t^m;x)=\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\alpha^{m-j}}{([n]_q+\beta)^m}D_{n,q}(t^j,x).$$

Now, let us compute the moments and central moments of order 3 and 4 for the operators (3.1) in the following manner:

$$\begin{aligned}D_{n,q}^{\alpha,\beta}(t^3;x)&=\frac{q^9[n]_q^4[n-1]_q[n-2]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q}x^3+\frac{q^4[n]_q^3[n-1]_q\left([3]_q^2[n]_q+\alpha[n+4]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q}x^2\\&\quad+\frac{q[n]_q^2\left([2]_q[3]_q^2[n]_q^2+\alpha[2]_q^2[n]_q[n+4]_q+\alpha^2[n+4]_q[n+3]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q}x\\&\quad+\frac{[n]_q^3[3]_q[2]_q+\alpha[2]_q[n]_q^2[n+4]_q+\left(\alpha^2[n]_q+\alpha^3[n+2]_q\right)[n+4]_q[n+3]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q}.\end{aligned}$$

Also,

$$\begin{aligned}D_{n,q}^{\alpha,\beta}(t^4;x)&=\frac{q^{16}[n]_q^5[n-1]_q[n-2]_q[n-3]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}x^4+\frac{q^9[n]_q^4[n-1]_q[n-2]_q\left([4]_q^2[n]_q+\alpha[n+5]_q\right)}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}x^3\\&\quad+q^4[n]_q^3[n-1]_q\left\{\frac{[2]_q[3]_q^2(1+q^2)^2[n]_q^2+\alpha[3]_q^2[n]_q[n+5]_q+\alpha^2[n+4]_q[n+5]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\right\}x^2\\&\quad+\frac{q[n]_q^2\left([2]_q[3]_q[4]_q^2[n]_q^3+[2]_q[3]_q^2\alpha[n]_q^2[n+5]_q+[2]_q^2\alpha^2[n]_q[n+4]_q[n+5]_q+\alpha^3[n+3]_q[n+4]_q[n+5]_q\right)}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}x\\&\quad+\frac{[4]_q[3]_q[2]_q[n]^4+\alpha[3]_q[2]_q[n]_q^3[n+5]_q+\alpha^2[2]_q[n]_q^2[n+4]_q[n+5]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}+\frac{\alpha^3[n]_q+\alpha^4[n+2]_q}{([n]_q+\beta)^4[n+2]_q}.\end{aligned}$$

Now, using the identity $(t-x)_q^3 = t^3 - [3]_q x t^2 + q[2]_q x^2 t - q^3 x^3$ and linear properties of the operators $D_{n,q}^{\alpha,\beta}$, we get

$$
\begin{aligned}
D_{n,q}^{\alpha,\beta}\left((t-x)_q^3;x\right) &= q^2\Bigg[\frac{q^7[n]_q^4[n-1]_q[n-2]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} - \frac{q^2[3]_q[n]_q^3[n-1]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q} + \frac{[2]_q[n]_q^2}{[n+2]_q\left([n]_q+\beta\right)} - q\Bigg]x^3\\
&+q\Bigg[\frac{q^3[n]_q^3[n-1]_q\left([n]_q[3]_q^2+\alpha[n+4]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} - \frac{[3]_q\left(\left([2]_q^2+2\alpha q^3\right)[n]_q^3+2\alpha[3]_q[n]_q^2\right)}{([n]_q+\beta)^2[n+2]_q[n+3]_q} + \frac{[2]_q([n]_q+\alpha[n+2]_q)}{[n+2]_q\left([n]_q+\beta\right)}\Bigg]x^2\\
&+\Bigg[\frac{q[n]_q^2\left([2]_q[3]_q^2[n]_q^2+[2]_q^2\alpha[n]_q[n+4]_q+\alpha^2[n+4]_q[n+3]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} - \frac{[3]_q\alpha^2}{([n]_q+\beta)^2} - \frac{(1+q+2\alpha q^3)[3]_q[n]_q^2+2\alpha[3]_q^2[n]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q}\Bigg]x\\
&+\frac{[n]_q^3[3]_q[2]_q+\alpha[2]_q[n]_q^2[n+4]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} + \frac{[n]_q\alpha^2+\alpha^3[n+2]_q}{[n+2]_q([n]_q+\beta)^3}.
\end{aligned}
$$

Finally, using identity $(t-x)_q^4 = t^4 - [4]_q x t^3 + q\left([5]_q + q^2\right)x^2t^2 - q^3x^3[4]_q t + q^6x^4$, we have

$$
\begin{aligned}
D_{n,q}^{\alpha,\beta}\left((t-x)_q^4;x\right) &= q^4\Bigg[\frac{q^{12}[n]_q^5[n-1]_q[n-2]_q[n-3]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q} - \frac{q^5[4]_q[n]_q^4[n-1]_q[n-2]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q}\\
&+\frac{q\left([5]_q+q^2\right)[n]_q^3[n-1]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q} - \frac{[4]_q[n]_q^2}{[n+2]_q\left([n]_q+\beta\right)} + q^2\Bigg]x^4\\
&+q^2\Bigg[\frac{q^7[n]_q^4[n-1]_q[n-2]_q\left([4]_q^2[n]_q+\alpha[n+5]_q\right)}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q} - \frac{q^2[4]_q[n]_q^3[n-1]_q\left([3]_q^2[n]_q+\alpha[n+4]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q}\\
&+\frac{\left([5]_q+q^2\right)\left(\left([2]_q^2+2\alpha q^3\right)[n]_q^3+2\alpha[3]_q[n]_q^2\right)}{([n]_q+\beta)^2[n+2]_q[n+3]_q} - \frac{q[4]_q\left([n]_q+\alpha[n+2]_q\right)}{[n+2]_q\left([n]_q+\beta\right)}\Bigg]x^3\\
&+q\Bigg[q^3[n]_q^3[n-1]_q\left\{\frac{[2]_q[3]_q^2(1+q^2)^2[n]_q^2+\alpha[3]_q^2[n]_q[n+5]_q+\alpha^2[n+4]_q[n+5]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\right\}\\
&-\frac{[4]_q[n]_q^2\left([2]_q[3]_q^2[n]_q^2+[2]_q\alpha[n]_q[n+4]_q+\alpha^2[n+4]_q[n+3]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q}\\
&+\frac{\alpha^2\left([5]_q+q^2\right)}{([n]_q+\beta)^2} + \frac{\left([5]_q+q^2\right)(1+q+2\alpha q^3)[n]_q^2+2\alpha[3]_q[n]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q}\Bigg]x^2\\
&+\Bigg[\frac{q[n]_q^2\left([2]_q[3]_q[4]_q^2[n]_q^3+\alpha[2]_q[3]_q^2[n]_q^2[n+5]_q+\alpha^2[2]_q^2[n]_q[n+4]_q[n+5]_q+\alpha^3[n+3]_q[n+4]_q[n+5]_q\right)}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\\
&-\frac{[n]_q^3[4]_q[3]_q[2]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} - \frac{\alpha[4]_q[2]_q[n]_q^2}{([n]_q+\beta)^3[n+3]_q[n+2]_q} - \frac{[4]_q[n]_q\alpha^2+\alpha^3[4]_q[n+2]_q}{[n+2]_q([n]_q+\beta)^3}\Bigg]x\\
&+\frac{[4]_q[3]_q[2]_q[n]^4+\alpha[3]_q[2]_q[n]_q^3[n+5]_q+\alpha^2[2]_q[n]_q^2[n+4]_q[n+5]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q} + \frac{\alpha^3[n]_q+\alpha^4[n+2]_q}{([n]_q+\beta)^4[n+2]_q}.
\end{aligned}
$$

Theorem 3. *Let f be bounded and integrable on the interval $[0,1]$ and let (q_n) denote a sequence such that $0<q_n<1$, $q_n\to 1$ and $q_n^n\to c$ as $n\to\infty$, where c is arbitrary constant. Then, we have, for a point $x\in(0,1)$*

$$\lim_{n\to\infty}[n]_{q_n}[D_{n,q_n}^{\alpha,\beta}(f;x)-f(x)] = (1+\alpha-(2+\beta)x)\lim_{n\to\infty}D_{q_n}f(x)+x(1-x)\lim_{n\to\infty}D_{q_n}^2f(x).$$

The proof of the above lemma follows along the lines of the proof of Theorem 2, using Lemma 4 and Remark 1; thus, we omit the details.

Corollary 2 ([6]). *Let f be bounded and integrable on the interval $[0,1]$ and let (q_n) denote a sequence such that $0 < q_n < 1$, $q_n \to 1$ and $q_n^n \to c$ as $n \to \infty$, where c is arbitrary constant. Suppose that the first and second derivatives $f'(x)$ and $f''(x)$ exist at a point $x \in (0,1)$. Then, we have, for a point $x \in (0,1)$,*

$$\lim_{n\to\infty}[n]_{q_n}[D_{n,q_n}^{\alpha,\beta}(f;x) - f(x)] = (1+\alpha-(2+\beta)x)f'(x) + x(1-x)f''(x).$$

Remark 2. *Theorem 2 and Theorem 3, give asymptotic formula for q-Durrmeyer operators and q-Durrmeyer-Stancu operators respectively. If f has first and second derivatives, then $\lim_{n\to\infty} D_{q_n}f(x) = f'(x)$ and $\lim_{n\to\infty} D_{q_n}^2 f(x) = f''(x)$. We obtain the results of Mishra and Patel* [6, Theorem 5]*, which are mentioned in Corollary 2. So our results are more general than the existing ones.*

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FIXED POINTS FOR GENERALIZED $(\mathcal{F}, h, \alpha, \mu) - \psi-$CONTRACTIONS IN $b-$METRIC SPACES

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Abstract. In this paper, we defined $(\mathcal{F}, h, \alpha, \mu) - \psi-$contractions using pair of $(\mathcal{F}, h)$ upper class functions for $\alpha-$admissible and $\mu-$subadmissible mappings.We proved some fixed point theorems for this type contractive mappings in $b-$metric spaces. Our results generalize $\alpha-$admissible results in the literature.

1. Introduction

Definition 1. *([9]) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:*

(i) $d(x, y) = 0$ *if and only if* $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, every metric is a b-metric with $s = 1$.

Example 1. *([1]) Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a $b-$metric with $s = 2^{p-1}$.*

However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

For example, if $X = R$ is the set of real numbers and $d(x, y) = |x - y|$ is usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a $b-$metric on R with $s = 2$.But is not a metric on R.

Definition 2. *([7]) Let $\{x_n\}$ be a sequence in a b-metric space (X, d).*

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(a) $\{x_n\}$ *is called* $b-$*convergent if and only if there is* $x \in X$ *such that* $d(x_n, x) \to 0$ *as* $n \to \infty$.
(b) $\{x_n\}$ *is a* $b-$*Cauchy sequence if and only if* $d(x_n, x_m) \to 0$ *as* $n, m \to \infty$.

A b-metric space is said to be complete if and only if each $b-$*Cauchy sequence in this space is* $b-$*convergent.*

Proposition 1. *([7]) In a* $b-$*metric space* (X, d), *the following assertions hold:*

(p1) A $b-$convergent sequence has a unique limit.
(p2) Each $b-$convergent sequence is $b-$Cauchy.
(p3) In general, a $b-$metric is not continuous.

On the other hand the notion of $\alpha - \psi$-contractive type mapping was introduced by Samet et al. [11],[17]. Also, see ([10],[12],[13-15])

Now we give some definitions that will be used throughout this paper.

A mapping $\psi : [0, \infty) \to [0, \infty)$ is called a comparison function if it is increasing and $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$.

Lemma 1. *([5]) Let* $\psi : [0, \infty) \to [0, \infty)$ *is a comparison function then*

(a) *each iterate* ψ^n *of* ψ, $n \geq 1$, *is also a comparison function,*
(b) ψ *is continuous at* $t = 0$,
(c) $\psi(t) < t$ *for all* $t > 0$.

Definition 3. *([5]) A function* $\psi : [0, \infty) \to [0, \infty)$ *is said to be a (c)-comparison function if*

(c1) ψ *is increasing,*
(c2) *there exists* $k_0 \in \mathbb{N}$, $a \in (0, 1)$ *and a convergent series of nonnegative terms* $\sum_{k=1}^{\infty} v_k$, *such that* $\psi^{k+1}(t) \leq a\psi^k(t) + v_k$, *for* $k \geq k_0$ *and any* $t \in [0, \infty)$.

Definition 4. *([6]) Let* $s \geq 1$ *be a real number. A function* $\psi : [0, \infty) \to [0, \infty)$ *is said to be a (b)-comparison function if*

(b1) ψ *is monotonically increasing,*
(b1) *there exists* $k_0 \in \mathbb{N}$, $a \in (0, 1)$ *and a convergent series of nonnegative terms* $\sum_{k=1}^{\infty} v_k$, *such that* $s^{k+1}\psi^{k+1}(t) \leq as^k\psi^k(t) + v_k$, *for* $k \geq k_0$ *and any* $t \in [0, \infty)$.

When $s = 1$, (b)-comparison function reduces to (c)-comparison function.

We denote by Ψ_b for the class of (b)-comparison function.

Lemma 2. *([4]) If* $\psi : [0, \infty) \to [0, \infty)$ *is a (b)-comparison function then one has the following:*

(i) $\sum_{k=0}^{\infty} s^k\psi^k(t)$ *converges to any* $t \in R^+$,

(ii) *the function* $b_s : [0, \infty) \to [0, \infty)$ *defined by* $b_s(t) = \sum_{k=0}^{\infty} s^k \psi^k(t)$, $t \in R^+$, *increasing and continuous at* 0.

Any (b)-comparison function is a comparison function.

Definition 5. *([17])For any nonempty set* X, *let* $T : X \to X$ *and* $\alpha : X \times X \to [0, \infty)$ *be mappings.* T *is called* α*-admissible if for all* $x, y \in X$,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 6. *([16])Let* $T : X \to X$, $\mu : X \times X \to R^+$. *We say* T *is an* μ*-subadmissible mapping if*

$$x, y \in X, \quad \mu(x, y) \leq 1 \quad \Longrightarrow \quad \mu(Tx, Ty) \leq 1.$$

Bota et. al. in ([8]) gave the definition of $\alpha - \psi -$ contractive mapping of type (b) in $b-$metric space which is a generalization of Definition 9.

Definition 7. *Let* (X, d) *be a* $b-$*metric space and* $T : X \to X$ *be a given mapping.* T *is called an* $\alpha - \psi -$*contractive mapping of type (b), if there exists two functions* $\alpha : X \times X \to [0, \infty)$ *and* $\psi \in \Psi_b$ *such that*

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X.$$

Definition 8. *([2],[3])We say that the function* $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is a function of subclass of type I, if* $x \geq 1 \Longrightarrow h(1, y) \leq h(x, y)$ *for all* $y \in \mathbb{R}^+$.

Example 2. *([2],[3])Define* $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *by:*

(a) $h(x, y) = (y + l)^x, l > 1$;
(b) $h(x, y) = (x + l)^y, l > 1$;
(c) $h(x, y) = x^n y$, $n \in \mathbb{N}$;
(d) $h(x, y) = y$;
(e) $h(x, y) = \frac{1}{n+1} \left(\sum_{i=0}^{n} x^i\right) y$, $n \in \mathbb{N}$;
(f) $h(x, y) = \left[\frac{1}{n+1} \left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1$, $n \in \mathbb{N}$

for all $x, y \in \mathbb{R}^+$. *Then* h *is a function of subclass of type I.*

Definition 9. *([2],[3]) Let* $h, \mathcal{F} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, *then we say that the pair* $(\mathcal{F}, h)$ *is an upper class of type I, if* h *is a function of subclass of type I and: (i)* $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$, *(ii)* $h(1, y) \leq \mathcal{F}(1, t) \Longrightarrow y \leq t$ *for all* $t, y \in \mathbb{R}^+$.

Example 3. *([2],[3]) Define* $h, \mathcal{F} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *by:*

(a) $h(x, y) = (y + l)^x, l > 1$ *and* $\mathcal{F}(s, t) = st + l$;
(b) $h(x, y) = (x + l)^y, l > 1$ *and* $\mathcal{F}(s, t) = (1 + l)^{st}$;
(c) $h(x, y) = x^m y$, $m \in \mathbb{N}$ *and* $\mathcal{F}(s, t) = st$;
(d) $h(x, y) = y$ *and* $\mathcal{F}(s, t) = t$;
(d) $h(x, y) = \frac{1}{n+1} \left(\sum_{i=0}^{n} x^i\right) y$, $n \in \mathbb{N}$ *and* $\mathcal{F}(s, t) = st$;
(e) $h(x, y) = \left[\frac{1}{n+1} \left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1$, $n \in \mathbb{N}$ *and* $\mathcal{F}(s, t) = (1 + l)^{st}$

for all $x, y, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type I.

Definition 10. *([2],[3])We say that the function $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type II, if $x, y \geq 1 \Longrightarrow h(1,1,z) \leq h(x,y,z)$ for all $z \in \mathbb{R}^+$.*

Example 4. *([2],[3])Define $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:*

(a) $h(x,y,z) = (z+l)^{xy}, l > 1$;
(b) $h(x,y,z) = (xy+l)^{z}, l > 1$;
(c) $h(x,y,z) = z$;
(d) $h(x,y,z) = x^m y^n z^p, m,n,p \in \mathbb{N}$;
(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m,n,p,q,k \in \mathbb{N}$

for all $x, y, z \in \mathbb{R}^+$. Then h is a function of subclass of type II.

Definition 11. *([2],[3])Let $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type II, if h is a subclass of type II and: (i) $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s,t) \leq \mathcal{F}(1,t)$, (ii) $h(1,1,z) \leq \mathcal{F}(s,t) \Longrightarrow z \leq st$ for all $s,t,z \in \mathbb{R}^+$.*

Example 5. *([2],[3]) Define $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:*

(a) $h(x,y,z) = (z+l)^{xy}, l > 1, \mathcal{F}(s,t) = st + l$;
(b) $h(x,y,z) = (xy+l)^{z}, l > 1, \mathcal{F}(s,t) = (1+l)^{st}$;
(c) $h(x,y,z) = z, F(s,t) = st$;
(d) $h(x,y,z) = x^m y^n z^p, m,n,p \in \mathbb{N}, \mathcal{F}(s,t) = s^p t^p$
(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m,n,p,q,k \in \mathbb{N}, \mathcal{F}(s,t) = s^k t^k$

for all $x, y, z, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type II.

2. Main results

Definition 12. *([13])Let (X,d) be a $b-$metric space and $T : X \to X$ be a given mapping. T is called generalized $\alpha - \psi-$contractive mapping of type (I), if there exists two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_b$ such that for all $x, y \in X$*

$$\alpha(x,y)\, d(Tx,Ty)) \leq \psi(M_s(x,y))$$

where,

$$M_s(x,y) = \max\left\{ d(x,y), d(Tx,x), d(Ty,y), \frac{d(Tx,y) + d(x,Ty)}{2s} \right\}.$$

Theorem 1. *([13])Let (X,d) be a complete $b-$metric space. Suppose that $T : X \to X$ be a generalized $\alpha - \psi-$contractive mapping of type (I) and satisfies:*

(i) *T is $\alpha-$admissible*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*
(iii) *T is continuous.*

Then T has a fixed point.

Definition 13. *([13])Let (X,d) be a $b-$metric space and $T:X\to X$ be a given mapping. T is called generalized $\alpha-\psi-$contractive mapping of type (II), if there exists two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi_b$ such that for all $x,y\in X$*

$$\alpha(x,y)\,d(Tx,Ty))\leq\psi(N_s(x,y))$$

where,

$$N_s(x,y)=\max\left\{d(x,y),\frac{d(Tx,x)+d(Ty,y)}{2s},\frac{d(Tx,y)+d(Ty,x)}{2s}\right\}.$$

Definition 14. *Let (X,d) be a $b-$metric space and $T:X\to X$ be a given mapping. T is called generalized $(\mathcal{F},h,\alpha,\mu)-\psi-$contractive mapping of type (I), if there exists two functions $\alpha,\mu:X\times X\to[0,\infty)$ and $\psi\in\Psi_b$ such that for all $x,y\in X$*

$$h(\alpha(x,y),d(Tx,Ty))\leq\mathcal{F}(\mu(x,y),\psi(M_s(x,y))) \tag{2.1}$$

where,pair $(\mathcal{F},h)$ is an upper class of type I and

$$M_s(x,y)=\max\left\{d(x,y),d(Tx,x),d(Ty,y),\frac{d(Tx,y)+d(x,Ty)}{2s}\right\}.$$

Theorem 2. *Let (X,d) be a complete $b-$metric space. Suppose that $T:X\to X$ be a generalized $(\mathcal{F},h,\alpha,\mu)-\psi-$contractive mapping of type (I) and satisfies:*

(i) *T is $\alpha-$admissible and μ-subadmissible*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq1,\mu(x_0,Tx_0)\leq1$*
(iii) *T is continuous.*

Then T has a fixed point.

Proof. By assumption (ii), there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq1,\mu(x_0,Tx_0)\leq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$. If $x_n=x_{n+1}$ for some $n\in\mathbb{N}$, then x_n is a fixed point of T.

Assume that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}$.

Since T is $\alpha-$admissible, then

$$\begin{aligned}\alpha(x_0,x_1)&=\alpha(x_0,Tx_0)\geq1\Longrightarrow\alpha(Tx_0,Tx_1)=\alpha(x_1,x_2)\geq1.\\ \mu(x_0,x_1)&=\mu(x_0,Tx_0)\leq1\Longrightarrow\mu(Tx_0,Tx_1)=\mu(x_1,x_2)\leq1.\end{aligned}$$

By induction, we get for all $n\in\mathbb{N}$,

$$\alpha(x_n,x_n+1)\geq1\quad,\quad\mu(x_n,x_n+1)\leq1. \tag{2.2}$$

Using (2.1) and (2.2)

$$\begin{aligned}h(1,d(x_n,x_{n+1}))&=h(1,d(Tx_{n-1},Tx_n))\\&\leq h(\alpha(x_{n-1},x_n),d(Tx_{n-1},Tx_n))\\&\leq\mathcal{F}(\mu(x_{n-1},x_n),\psi(M_s(x_{n-1},x_n)))\\&\leq\mathcal{F}(1,\psi(M_s(x_{n-1},x_n)))\\&\Longrightarrow\end{aligned}$$

$$d\left(x_n,x_{n+1}\right)\leq\psi(M_s\left(x_{n-1},x_n\right)). \tag{2.3}$$

where

$$\begin{aligned}
M_s\left(x_{n-1},x_n\right) &= \max\left\{\begin{array}{c} d\left(x_{n-1},x_n\right),d\left(Tx_{n-1},x_{n-1}\right),d\left(Tx_n,x_n\right), \\ \frac{d(Tx_{n-1},x_n)+d(Tx_n,x_{n-1})}{2s}\end{array}\right\}, \\
&= \max\left\{\begin{array}{c} d\left(x_{n-1},x_n\right),d\left(x_n,x_{n-1}\right),d\left(x_{n+1},x_n\right), \\ \frac{d(x_n,x_n)+d(x_{n+1},x_{n-1})}{2s}\end{array}\right\}, \\
&= \max\left\{\begin{array}{c} d\left(x_{n-1},x_n\right),d\left(x_{n+1},x_n\right), \\ \frac{s[d(x_{n+1},x_n)+d(x_n,x_{n-1})]}{2s}\end{array}\right\} \\
&\leq \max\left\{d\left(x_{n-1},x_n\right),d\left(x_{n+1},x_n\right)\right\}.
\end{aligned}$$

If $M_s\left(x_{n-1},x_n\right)=d\left(x_n,x_{n+1}\right)$, then from (2.3) and definition of ψ,

$$d\left(x_n,x_{n+1}\right)\leq\psi\left(d\left(x_n,x_{n+1}\right)\right)<d\left(x_n,x_{n+1}\right)$$

a contradiction. Thus $M_s\left(x_{n-1},x_n\right)=d\left(x_{n-1},x_n\right)$. Hence,

$$d\left(x_n,x_{n+1}\right)\leq\psi\left(d\left(x_{n-1},x_n\right)\right)<d\left(x_{n-1},x_n\right)$$

for all $n\geq 1$. If operations are continued in this way,

$$d\left(x_n,x_{n+1}\right)\leq\psi^n\left(d\left(x_0,x_1\right)\right). \tag{2.4}$$

Thus, for all $p\geq 1$,

$$\begin{aligned}
d\left(x_n,x_{n+p}\right) &\leq sd\left(x_n,x_{n+1}\right)+s^2d\left(x_{n+1},x_{n+2}\right)+... \\
&\quad +s^{p-1}d\left(x_{n+p-2},x_{n+p-1}\right)+s^pd\left(x_{n+p-1},x_{n+p}\right) \\
&\leq s\psi^n\left(d\left(x_0,x_1\right)\right)+s^2\psi^{n+1}\left(d\left(x_0,x_1\right)\right)+... \\
&\quad +s^{p-1}\psi^{n+p-2}\left(d\left(x_0,x_1\right)\right)+s^p\psi^{n+p-1}\left(d\left(x_0,x_1\right)\right) \\
&= \frac{1}{s^{n-1}}[s^n\psi^n\left(d\left(x_0,x_1\right)\right)+s^{n+1}\psi^{n+1}\left(d\left(x_0,x_1\right)\right)+... \\
&\quad +s^{p-n-2}\psi^{p-n-2}\left(d\left(x_0,x_1\right)\right)+s^{p+n-1}\psi^{p+n-1}\left(d\left(x_0,x_1\right)\right)].
\end{aligned}$$

Denoting $S_n=\sum\limits_{k=n}^{\infty}s^k\psi^k\left(d\left(x_0,x_1\right)\right)$, $n\geq 1$, we obtain

$$d\left(x_n,x_{n+p}\right)\leq\frac{1}{s^{n-1}}[S_{n+p-1}-S_{n-1}] \tag{2.5}$$

for $n\geq 1$, $p\geq 1$. From Lemma 2, we conclude that the series $\sum\limits_{k=0}^{\infty}s^k\psi^k\left(d\left(x_0,x_1\right)\right)$ is convergent. Thus, there exists

$$S=\lim_{n\to\infty}S_n\in[0,\infty).$$

Regarding $s\geq 1$ and by (2.5) $\{x_n\}$ is a Cauchy sequence in $b-$metric space (X,d). Since (X,d) is complete, there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. Using continuity of T,

$$x_{n+1}=Tx_n\to Tx^*$$

as $n \to \infty$. By the uniqueness of the limit, we get $x^* = Tx^*$. Hence x^* is a fixed point of T. □

Definition 15. *Let (X, d) be a $b-$metric space and $T : X \to X$ be a given mapping. T is called generalized $(\mathcal{F}, h, \alpha, \mu)-\psi-$contractive mapping of type (II), if there exists two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_b$ such that for all $x, y \in X$*

$$h(\alpha(x,y), d(Tx,Ty)) \leq \mathcal{F}(\mu(x,y), \psi(N_s(x,y))) \tag{2.6}$$

where,pair $(\mathcal{F}, h)$ is an upper class of type I and

$$N_s(x,y) = \max\left\{ d(x,y), \frac{d(Tx,x) + d(Ty,y)}{2s}, \frac{d(Tx,y) + d(Ty,x)}{2s} \right\}.$$

Theorem 3. *Let (X, d) be a complete $b-$metric space. Suppose that $T : X \to X$ be a generalized $(\mathcal{F}, h, \alpha, \mu)-\psi-$contractive mapping of type (II) and satisfies:*

(i) *T is $\alpha-$admissible,μ-subadmissible*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \mu(x_0, Tx_0) \leq 1$*
(iii) *T is continuous.*

Then T has a fixed point.

Proof. By assumption (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \mu(x_0, Tx_0) \leq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T.

Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since T is $\alpha-$admissible, then

$$\begin{aligned} \alpha(x_0, x_1) &= \alpha(x_0, Tx_0) \geq 1 \Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1, \\ \mu(x_0, x_1) &= \mu(x_0, Tx_0) \leq 1 \Longrightarrow \mu(Tx_0, Tx_1) = \mu(x_1, x_2) \leq 1. \end{aligned}$$

By induction, we get for all $n \in \mathbb{N}$,

$$\alpha(x_n, x_n + 1) \geq 1 \quad , \quad \mu(x_n, x_n + 1) \leq 1.$$

Using (2.6)

$$\begin{aligned} h(1, d(x_n, x_{n+1})) &= h(1, d(Tx_{n-1}, Tx_n)) \\ &\leq h(\alpha(x_{n-1}, x_n), d(Tx_{n-1}, Tx_n)) \\ &\leq \mathcal{F}(\mu(x_{n-1}, x_n), \psi(N_s(x_{n-1}, x_n))) \\ &\leq \mathcal{F}(1, \psi(N_s(x_{n-1}, x_n))) \\ &\Longrightarrow \end{aligned}$$

$$d(x_n, x_{n+1}) \leq \psi(N_s(x_{n-1}, x_n)) \leq \psi(M_s(x,y)).$$

The rest of proof is evident due to Theorem 2. □

In the following two theorems we are able to remove the continuity condition for the $\alpha - \psi-$ contractive mappings of type (I) and type (II).

Theorem 4. *Let (X, d) be a complete $b-$metric space. Suppose that $T : X \to X$ be a generalized $\alpha - \psi-$contractive mapping of type (I) and satisfies:*

(i) *T is $\alpha-$admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$, as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$, for all k.*

Then T has a fixed point.

Proof. Following the proof of Theorem 2 , we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, is Cauchy and converges to some $u \in X$.

We shall show that $Tu = u$. Suppose on the contrary that $d(Tu, u) > 0$. From (2.2) and (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. By (2.1)

$$\begin{array}{rcl} h(1, d(x_{n(k)+1}, Tu)) & = & h(1, d(Tx_{n(k)}, Tu)) \\ & \leq & h(\alpha(x_{n(k)}, u), d(Tx_{n(k)}, Tu)) \\ & \leq & \mathcal{F}(\mu(x_{n(k)}, u), \psi(M_s(x_{n(k)}, u))) \\ & \leq & \mathcal{F}(1, \psi(M_s(x_{n(k)}, u))) \\ & \Longrightarrow & \end{array}$$

$$d(x_{n(k)+1}, Tu) \leq \psi(M_s(x_{n(k)}, u)), \tag{2.7}$$

where

$$M_s(x_{n(k)}, u) = \max\left\{ \begin{array}{c} d(x_{n(k)}, u), d(Tx_{n(k)}, x_{n(k)}), d(Tu, u), \\ \frac{d(Tx_{n(k)}, u), d(Tu, x_{n(k)})}{2s} \end{array} \right\}.$$

As $k \to \infty$, $\lim_{k\to\infty} M_s(x_{n(k)}, u) = d(Tu, u)$.

In (2.7), as $k \to \infty$

$$d(u, Tu) \leq \psi(d(u, Tu)) < d(u, Tu)$$

which is a contradiction. Hence, $u = Tu$ and u is a fixed point of T. □

Theorem 5. *Let (X, d) be a complete $b-$metric space. Suppose that $T : X \to X$ be a generalized $\alpha - \psi-$contractive mapping of type (II) and satisfies:*

(i) *T is $\alpha-$admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$, as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$, for all k.*

then T has a fixed point.

Proof. Following the proof of Theorem 2.5, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, for all $n \geq 0$, is Cauchy and converges to some $u \in X$.

We shall show that $Tu = u$. Suppose on the contrary that $d(Tu, u) > 0$. From (2.2) and (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (2.6),

$$\begin{aligned} h(1, d(x_{n(k)+1}, Tu)) &= h(1, d(Tx_{n(k)}, Tu)) \\ &\leq h(\alpha(x_{n(k)}, u), d(Tx_{n(k)}, Tu)) \\ &\leq \mathcal{F}(\mu(x_{n(k)}, u), \psi(N_s(x_{n(k)}, u))) \\ &\leq \mathcal{F}(1, \psi(N_s(x_{n(k)}, u))) \\ &\Longrightarrow \end{aligned}$$

$$d(x_{n(k)+1}, Tu) \leq \psi(N_s(x_{n(k)}, u)) \tag{2.8}$$

where

$$N_s(x_{n(k)}, u) = \max\left\{ \begin{array}{c} d(x_{n(k)}, u), \frac{d(Tx_{n(k)}, x_{n(k)}) + d(Tu, u)}{2s}, \\ \frac{d(Tx_{n(k)}, u), d(Tu, x_{n(k)})}{2s} \end{array} \right\}.$$

As $k \to \infty$, $\lim_{k\to\infty} N_s(x_{n(k)}, u) = \frac{d(Tu,u)}{2s}$, for $s \geq 1$.

In (2.8), as $k \to \infty$

$$d(u, Tu) \leq \psi\left(\frac{d(Tu, u)}{2s}\right) < \frac{d(Tu, u)}{2s}$$

which is a contradiction. Hence, $u = Tu$ and u is a fixed point of T. $\square$

Example 6. *Let $X = (0, \infty)$ endowed with $b-$metric*

$$d : X \times X \to R^+, \quad d(x, y) = (x - y)^2$$

with constant $s = 2$. (X, d) is a complete $b-$metric space. Let the functions $T : X \to X$, $\alpha : X \times X \to [0, \infty)$ and $\eta : X \times X \to [0, \infty)$ be defined by

$$\begin{aligned} T(x) &= \begin{cases} \frac{x+1}{4}, & x \in (0, 1] \\ 2x, & x > 1 \end{cases}, \\ \alpha(x, y) &= \begin{cases} 1, x \in (0, 1] \\ 0, otherwise \end{cases}, \\ \eta(x, y) &= \begin{cases} \frac{1}{2}, x \in (0, 1] \\ 1, otherwise \end{cases}. \end{aligned}$$

Clearly, T is $\alpha-$admissible, continuous and $\eta-$subadmissible. Let h,$\mathcal{F}$: $R^+ \times R^+ \to R$ be defined by;

$h(x, y) = (y + l)^x, l > 1$ and $\mathcal{F}(s, t) = st + l$. $(\mathcal{F}, h, \alpha, \eta) - \psi-$contraction of type (I) is satisfied with $\psi(t) = \frac{t}{2}$, for all $t \geq 0$.

Let $x, y \in X$ if $\alpha(x,y) \geq 1$ and $\eta(x,y) \geq 1$, then $x, y \in [0,1]$ .Thus

$$\begin{aligned} h(\alpha(x,y)\, d(Tx,Ty)) &= h\left(1, \left(\frac{x+1}{4} - \frac{y+1}{4}\right)^2\right) = \frac{1}{16}(x-y)^2 + l \\ &\leq \frac{1}{2}.\frac{1}{2}(x-y)^2 + l = \mathcal{F}(\eta(x,y), \psi(d(x,y))) \\ &\leq \mathcal{F}(\eta(x,y), \psi(M_s(x,y)). \end{aligned}$$

Then all conditions of Theorem 5 are satisfied. $\frac{1}{3}$ is fixed point of T.

Corollary 1. *Let (X,d) be a complete $b-$metric space and $T : X \to X$ be continuous mapping. Suppose that there exists a function $\psi \in \Psi_b$ such that*

$$d(Tx,Ty) \leq \psi(M_s(x,y))$$

for all $x, y \in X$, then T has a fixed point.

Similarly, be taken $\alpha(x,y) = 1$ in Theorem 4, the following result is obtained.

Corollary 2. *Let (X,d) be a complete $b-$metric space and $T : X \to X$ be continuous mapping. Suppose that there exists a function $\psi \in \Psi_b$ such that*

$$d(Tx,Ty) \leq \psi(N_s(x,y))$$

for all $x, y \in X$, then T has a fixed point.

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NONLINEAR SELF ADJOINTNESS AND EXACT SOLUTION OF FOKAS–OLVER–ROSENAU–QIAO (FORQ) EQUATION

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Abstract. Based on Lie's symmetry approach, conservation laws are constructed for Fokas–Olver–Rosenau–Qiao(FORQ) equation and exact solution is obtained. Nonlocal conservation theorem is used to carry out the analysis of conservation process. Nonlinear self adjointness concept is applied to FORQ equation, it is proved to be strict self adjoint. Characteristic equation and similarity variable help us find exact solution of FORQ equation. Compared with solutions found in previous papers, our solution is new and important, since it is not possible to find exact solution of FORQ equation quite easily.

1. Introduction

In recently past years, more works has been conducted on conservation laws. The existence of conservation laws makes important progress in understanding given in many physical models. The determination of conservation laws, particularly local ones, offers rich knowledge on the mechanism of physical phenomena modeled by nonlinear evolution equations. An effective and impressive way of constructing conservation laws is by means of well known Noether's theorem [1]. This theorem provides explicit formulae for construction conservation laws for Euler-Lagrange differential equations once their Noether symmetries are known. Choosing a proper Lagrangian provides a chance of applying Noether's theorem to related equation. So as to remove this restriction, some methods have been developed in recent years, such as partial Lagrangian, Nonlocal conservation theorem, multiplier approach and so on [2]-[14].

In recent years, there has been an increasing interest in integrable non-evolutionary partial differential equation of the form

$$\left(1-D_x^2\right)u_t=F\left(u,u_x,u_{xx},u_{xxx},...\right),\quad u=u(x,t),\quad D_x=\frac{\partial}{\partial x}\tag{1}$$

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where F is a function of u and its derivatives with respect to x. The most famous example of this type of equations is the Camassa-Holm equation [15, 16].

$$\left(1-D_x^2\right)u_t = 3uu_x - 2u_xu_{xx} - uu_{xxx}. \tag{2}$$

The integrability of Camassa-Holm type equations was shown by inverse scattering transform, infinity hierarchy of local conservation laws, bi-Hamiltonian structure and other remarkable properties of integrable equations [17]. We consider the following form of (1) [18]

$$\begin{aligned}\left(1-\epsilon^2D_x^2\right)u_t &= c_1u^2u_x + \epsilon\left[c_2u^2u_{xx}+c_3uu_x^2\right] + \epsilon^2\left[c_4u^2u_{xxx}+c_5u_xu_{xx}u+c_6u_x^3\right] \\ &+\epsilon^3\left[c_7u^2u_{xxxx}+c_8u_xu_{xxx}u+c_9u_{xx}^2u+c_{10}u_x^2u_{xx}\right] \\ &+\epsilon^4\left[c_{11}u^2u_{xxxxx}+c_{12}u_xu_{xxxx}u+c_{13}u_{xx}u_{xxx}u+c_{14}u_x^2u_{xxx}+c_{15}u_xu_{xx}^2\right]\end{aligned} \tag{3}$$

Here, ϵ and c_i are the complex parameters and $\epsilon \neq 0$. This equation is homogeneous differential polynomials of weight 1. Supposing that weight of u_i is i, weight of ϵ equals -1 and weight of u_t is 1. Particularly, choosing the coefficients in (3) appropriately, we get the (3) in the form of

$$u_t - u_{xxt} + 3u^2u_x - u_x^3 - 4uu_xu_{xx} + 2u_xu_{xx}^2 - u^2u_{xxx} + u_x^2u_{xxx} = 0 \tag{4}$$

which is given as FORQ equation in [19].

In this paper, we concentrate on FORQ equation which was derived by Olver and Rosenau [20], Fuchssteiner [21], and Qiao [22]. Our main motivation in this study is to obtain Lie symmetry generators of FORQ equation with Maple package program. Taking $w=\varphi(x,t,u)$, the construction of nonlinear self adjointness and conservation laws of the FORQ equation is presented. Furthermore, using the similarity variables and reduced equation, exact solution is obtained.

2. Conservation laws for the FORQ equation

We briefly present notation to be used and recall basic definitions and theorems that appear in [23]-[25].

Consider the k^{th} order system of PDEs of n independent variables $x=(x^1,x^2,...,x^n)$ and m dependent variables $u=(u^1,u^2,...,u^m)$

$$E^{\alpha}\left(x,u,u_{(1)},...,u_{(k)}\right)=0, \quad \alpha=1,...,m. \tag{5}$$

where $u_{(i)}$ is the collection of i^{th}-order partial derivatives and the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}} + ..., \quad i=1,...,n \tag{6}$$

in which the summation convention is used. Suppose $\mathcal{A}$ is the universal space of all differential functions of finite orders, clearly it is a vector space and forms an algebra. *The* Lie-Backlund *generator* is the following vector field operator:

$$\mathbf{X} = \xi^i\frac{\partial}{\partial x_i} + \eta^{\alpha}\frac{\partial}{\partial u^{\alpha}}, \qquad \xi^i, \eta^{\alpha} \in \mathcal{A} \tag{7}$$

The operator (7) is an abbreviated form of the infinite formal sum

$$\mathbf{X} = \xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{k\geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_k} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_k}}, \tag{8}$$

where the additional coefficients can be determined from the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i\left(\eta^\alpha\right) + \xi^j u^\alpha_{ji} \\ \zeta^\alpha_{i_1 \ldots i_k} &= D_{i_1} \ldots D_{i_k}\left(\zeta^\alpha_{i_1 \ldots i_{k-1}}\right) + \xi^j u^\alpha_{j i_1 \ldots i_k}, \quad k > 1. \end{aligned} \tag{9}$$

The Noether operators associated with a Lie–Bäcklund operator $\mathbf{X}$ are

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha_i} + \sum_{k\geq 1}^{\infty} D_{i_1} \ldots D_{i_k} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_k}}, \qquad i = 1, 2, ..., n$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{10}$$

A conserved vector of (5) is an n-tuple vector $T = \left(T^1, T^2, ..., T^n\right),\ T^j \epsilon \mathcal{A},\ j = 1, ..., n$

$$D_i T^i_{|(1)} = 0. \tag{11}$$

holds for all solutions of (5).

Then we define the adjoint equation to Eq(5) in the form of

$$E^{\alpha *}(x, u, w, u_{(1)}, w_{(1)}, ..., u_{(k)}, w_{(k)}) = 0, \qquad \alpha = 1, ..., m$$

with

$$E^{\alpha *}(x, u, w, u_{(1)}, w_{(1)}, ..., u_{(k)}, w_{(k)}) = \frac{\delta L}{\delta u^\alpha} \tag{12}$$

where L is formal Lagrangian for Eq(5) defined by

$$L = w^\alpha E^\alpha \equiv \sum_{\alpha=1}^{m} w^\alpha E^\alpha. \tag{13}$$

Here, so-called non local variables are $w^\alpha = (w^1, ..., w^m)$, their derivatives are $w^\alpha_{(1)}, ..., w^\alpha_{(k)}$. Here $\dfrac{\delta}{\delta u}$ is the Euler-Lagrange operator and given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{k\geq 1}^{\infty} (-1)^k D_{i_1} \ldots D_{i_k} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_k}}, \qquad \alpha = 1, ..., m. \tag{14}$$

so that

$$\frac{\delta L}{\delta u^\alpha} = \frac{\delta(w^\alpha E^\alpha)}{\delta u^\alpha} = \frac{\partial(w^\alpha E^\alpha)}{\partial u^\alpha} - D_i\left(\frac{\partial(w^\alpha E^\alpha)}{\partial u^\alpha_i}\right) + D_i D_k\left(\frac{\partial(w^\alpha E^\alpha)}{\partial u^\alpha_{ik}}\right) - ...$$

Definition 1. *The differential equation (5) is said to be nonlinearly self-adjoint if there exists a function*

$$w^\alpha = \varphi^\alpha(x,t,u) \neq 0 \tag{15}$$

such that it satisfy

$$E^{\alpha *}(x,u,\varphi(x,u),...,u_{(k)},\varphi_{(k)}) = \lambda_\alpha^\beta E^\alpha(x,u,...u_{(k)}),\ \alpha = 1,..,m, \tag{16}$$

for some undetermined coefficient $\lambda = \lambda_\alpha^\beta(x,t,u)$. *If we take* $w = \varphi(u)$ *in (16) the equation (5) is called quasi self-adjoint. If we take* $w = u$, *we say that the equation (5) is strictly self-adjoint.*

Theorem 2 ([23])**.** *Every Lie point, Lie-Bäcklund and non-local symmetry of equation (5) gives a conservation law for the considered equation. The conserved vector components are*

$$\begin{aligned} T^i &= \xi^i L + W^\alpha \left[\tfrac{\partial L}{\partial u_i} - D_j(\tfrac{\partial L}{\partial u_{ij}}) + D_j D_k(\tfrac{\partial L}{\partial u_{ijk}}) \right] \\ &\quad + D_j\left(W^\alpha\right) \left[\tfrac{\partial L}{\partial u_{ij}} - D_k(\tfrac{\partial L}{\partial u_{ijk}}) \right] + D_j D_k \left(W^\alpha\right) \frac{\partial L}{\partial u_{ijk}} \end{aligned} \tag{17}$$

and ξ^i, η^α *are the coefficient functions of the associated generator (7) .*

The conserved vectors obtained from (17) involve the arbitrary solutions w^α of the adjoint equation (12) and hence one obtains an infinite number of conservation laws for (5) by choosing w^α.

Now we use the nonlocal conservation theorem method given by Ibragimov. We consider the following sub-algebra with infinitesimal generators of symmetries given by,

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_3 = t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}. \tag{18}$$

Then the corresponding formal Lagrangian of Eq(4) is given by

$$L = \left(u_t - u_{xxt} + 3u^2u_x - u_x^3 - 4uu_xu_{xx} + 2u_xu_{xx}^2 - u^2u_{xxx} + u_x^2u_{xxx}\right) w(x,t). \tag{19}$$

The adjoint equation for Eq(4) is

$$\begin{aligned} E^*(t,x,u,w,...,w_{xxx}) &= \frac{\delta}{\delta u}\left[w(x,t)\right. \\ &\quad \times \left(u_t - u_{xxt} + 3u^2u_x - u_x^3 - 4uu_xu_{xx}\right. \\ &\quad \left.\left. + 2u_xu_{xx}^2 - u^2u_{xxx} + u_x^2u_{xxx}\right)\right] \\ &= -w_t + w_{txx} - 3w_xu^2 + w_xu_x^2 + w_{xxx}u^2 - w_{xxx}u_x^2 \\ &\quad + 2w_xuu_{xx} - 2w_{xx}u_xu_{xx} + 2w_{xx}uu_x \end{aligned} \tag{20}$$

where w is the adjoint variable.

If we take $w=\varphi(t,x,u)$ and necessary derivatives:

$$\begin{aligned}
w &= \varphi(t,x,u),\\
w_t &= \varphi_u u_t+\varphi_t,\\
w_x &= \varphi_u u_x+\varphi_x,\\
w_{xx} &= \varphi_u u_{xx}+\varphi_{uu}u_x^2+2\varphi_{ux}u_x+\varphi_{xx},\\
w_{xxx} &= \varphi_{xxx}+3\varphi_{xxu}u_x+3\varphi_{xuu}u_x^2+3\varphi_{xu}u_{xx}+\varphi_{uuu}u_x^3+3\varphi_{uu}u_xu_{xx}+\varphi_u u_{xxx},\\
w_{xxt} &= \varphi_{uu}u_tu_{xx}+\varphi_{ut}u_{xx}+\varphi_u u_{xxt}+\varphi_{tuu}u_x^2+2\varphi_{uu}u_xu_{xt}+\varphi_{uuu}u_tu_x^2\\
&\quad +2\varphi_{uux}u_xu_t+2\varphi_{uxt}u_x+2\varphi_{ux}u_{xt}+\varphi_{xxu}u_t+\varphi_{xxt},
\end{aligned}$$

with the self-adjointness condition (16), Eq(20) as follows:

$$\begin{aligned}
E^*(t,x,u,w,...,w_{xxx}) &= -\varphi_u u_t-\varphi_t-3(\varphi_u u_x+\varphi_x)u^2+\varphi_u u_x^3+u_x^2\varphi_x+\\
&\quad (u^2-u_x^2)(\varphi_{xxx}+3\varphi_{xxu}u_x+3\varphi_{xuu}u_x^2+3\varphi_{xu}u_{xx}+\varphi_{uuu}u_x^3\\
&\quad +3\varphi_{uu}u_xu_{xx}+\varphi_u u_{xxx})\\
&\quad +2(\varphi_u u_x+\varphi_x)u_{xx}u-2(\varphi_u u_{xx}+\varphi_{uu}u_x^2+2\varphi_{ux}u_x+\varphi_{xx})u_xu_{xx}\\
&\quad +2(\varphi_u u_{xx}+\varphi_{uu}u_x^2+2\varphi_{ux}u_x+\varphi_{xx})u_xu\\
&= \lambda\left(u_t-u_{xxt}+3u^2u_x-u_x^3-4uu_xu_{xx}+2u_xu_{xx}^2-u^2u_{xxx}+u_x^2u_{xxx}\right)
\end{aligned} \tag{21}$$

The comparison of the coefficients of all derivatives yields $\varphi=C_1u+C_2$ where C_1,C_2 are constants. Therefore we can take two different values of w, namely $w=1$ and $w=u$.

The conserved components of Eq(4), associated with a Lie symmetry, can be obtained from (17) as follows:

$$\begin{aligned}
T^t &= \xi^t L+W\left[\frac{\partial L}{\partial u_t}+D_x^2\left(\frac{\partial L}{\partial u_{xxt}}\right)\right]+D_x(W)\left[-D_x\left(\frac{\partial L}{\partial u_{xxt}}\right)\right]+D_x^2(W)\left[\frac{\partial L}{\partial u_{xxt}}\right]\\
T^x &= \xi^x L+W\left[\frac{\partial L}{\partial u_x}-D_x\left(\frac{\partial L}{\partial u_{xx}}\right)+D_x^2\left(\frac{\partial L}{\partial u_{xxx}}\right)+D_{xt}^2\left(\frac{\partial L}{\partial u_{xxt}}\right)\right]\\
&\quad +D_x(W)\left[\frac{\partial L}{\partial u_{xx}}-D_x\left(\frac{\partial L}{\partial u_{xxx}}\right)-D_t\left(\frac{\partial L}{\partial u_{xxt}}\right)\right]\\
&\quad +D_t(W)\left[-D_x\left(\frac{\partial L}{\partial u_{xxt}}\right)\right]+D_x^2(W)\left(\frac{\partial L}{\partial u_{xxx}}\right)+D_{xt}^2(W)\left(\frac{\partial L}{\partial u_{xxt}}\right).
\end{aligned} \tag{22}$$

where $W=\eta-u_x\xi^x-u_t\xi^t$ is Lie characteristic function.

Now, we will find conservation laws of Eq(4) with the help of formulae (22).

i) Firstly, we will construct conservation laws with $\mathbf{w=1}$.

Case 1:

We consider $X_1 = \frac{\partial}{\partial x}$ with $W = -u_x$, corresponding conserved vectors are

$$\begin{aligned} T_1^t &= -u_x \\ T_1^x &= u_t. \end{aligned} \tag{23}$$

Case 2:

If we use $X_2 = \frac{\partial}{\partial t}$ with $W = -u_t$, we obtain conserved vectors

$$\begin{aligned} T_2^t &= 3u^2u_x - u_x^3 - 4uu_xu_{xx} + 2u_xu_{xx}^2 - u^2u_{xxx} + u_x^2u_{xxx} \\ T_2^x &= -3u^2u_t + u_tu_x^2 + 2uu_tu_{xx} + 2uu_xu_{xt} - 2u_xu_{xt}u_{xx} + u^2u_{xxt} - u_x^2u_{xxt}. \end{aligned} \tag{24}$$

Obtained conservation laws in case1,2 satisfy $D_t\left(T_2^t\right) + D_x\left(T_2^x\right) = 0$, so these vectors are trivial.

Case 3:

For the Lie-point symmetry generator

$$X_3 = t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u},$$

we have

$$W = -\tfrac{1}{2}u - tu_t.$$

If we use (22) , obtained conserved vectors are

$$\begin{aligned} T_3^t &= 3tu^2u_x - tu_x^3 - 4tuu_xu_{xx} + 2tu_xu_{xx}^2 - tu^2u_{xxx} + tu_x^2u_{xxx} - \frac{1}{2}u + \frac{1}{2}u_{xx} \\ T_3^x &= -\frac{3}{2}u^3 + \frac{3}{2}uu_x^2 + \frac{3}{2}u^2u_{xx} - 3tu_tu^2 + tu_tu_x^2 + 2tu_tuu_{xx} - \frac{3}{2}u_x^2u_{xx} \\ &\quad +2tuu_xu_{xt} - 2tu_xu_{xx}u_{xt} + tu^2u_{xxt} - tu_x^2u_{xxt} + \frac{3}{2}u_{xt} + tu_{xtt}. \end{aligned} \tag{25}$$

Divergence condition can be expressed for these conservation laws as follows:

$$\begin{aligned} D_t\left(T_3^t\right) + D_x\left(T_3^x\right) &= \frac{3}{2}u_{xxt} + tu_{xxtt} \\ &= D_x\left(\frac{3}{2}u_{xt} + tu_{xtt}\right). \end{aligned} \tag{26}$$

In Eq(26) , since there are some terms leftover, we should find modified conservation laws to satisfy divergence condition. Therefore, modified conservation laws are

$$\begin{aligned} \tilde{T}_3^t &= T_3^t \\ \tilde{T}_3^x &= T_3^x - tu_{xtt} - \frac{3}{2}u_{xt}. \end{aligned} \tag{27}$$

ii) We will find conservation laws with $\mathbf{w} = \mathbf{u}$.

Case 4:

According to generator $X_1 = \frac{\partial}{\partial x}$, we get trivial conserved vectors

$$\begin{array}{rcl} T_1^t & = & -uu_x \\ T_1^x & = & uu_t. \end{array} \tag{28}$$

Case 5:

Using the symmetry generator $X_2 = \frac{\partial}{\partial t}$, we obtain following trivial conserved vectors

$$\begin{array}{rcl} T_2^t & = & 3u^3u_x - uu_x^3 - 4u^2u_xu_{xx} + 2uu_xu_{xx}^2 - u^3u_{xxx} + uu_x^2u_{xxx} + u_tu_{xx} - uu_{xxt} \\ T_2^x & = & -3u^3u_t + uu_tu_x^2 + 3u^2u_tu_{xx} - u_tu_x^2u_{xx} + u^2u_xu_{xt} \\ & & -2uu_xu_{xt}u_{xx} + u_x^3u_{xt} - u_xu_{tt} + u^3u_{xxt} - uu_x^2u_{xxt} + uu_{xtt}. \end{array} \tag{29}$$

Case 6:

Finally, using the following Lie-point symmery generator

$$X_3 = t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}, \tag{30}$$

conserved vectors are

$$\begin{array}{rcl} T_3^t & = & 3tu^3u_x - tuu_x^3 - 4tu^2u_xu_{xx} + 2tuu_xu_{xx}^2 - tu^3u_{xxx} + tuu_x^2u_{xxx} \\ & & -\frac{1}{2}u^2 + uu_{xx} + tu_tu_{xx} - \frac{1}{2}u_x^2 - tu_xu_{xt} \\ T_3^x & = & -2tuu_xu_{xx}u_{xt} + u^2u_x^2 + 2u^3u_{xx} + 2uu_{xt} - 2u_xu_t + tu_{xt}u_x^3 - tu_xu_{tt} \\ & & +tu^3u_{xxt} + tuu_tu_x^2 + 3tu_tu^2u_{xx} - tu_tu_x^2u_{xx} + tu^2u_xu_{xt} - tuu_x^2u_{xxt} \\ & & -\frac{3}{2}u^4 - 2uu_x^2u_{xx} + tuu_{xtt} + \frac{1}{2}u_x^4 - 3tu_tu^3. \end{array} \tag{31}$$

If we look at divergence condition, we get

$$\begin{array}{rcl} D_t\left(T_3^t\right) + D_x\left(T_3^x\right) & = & D_t\left(tuu_{xxt} - u_x^2 - tu_xu_{xt}\right) \\ & & +D_x\left(uu_{xt}\right). \end{array} \tag{32}$$

Again, there are some terms leftover. To satisfy divergence condition modified conservation laws are obtained as follows

$$\begin{array}{rcl} \widetilde{T_3^t} & = & T_3^t - tuu_{xxt} + u_x^2 + tu_xu_{xt} \\ \widetilde{T_3^x} & = & T_3^x - uu_{xt}. \end{array} \tag{33}$$

3. Exact Solution

Now we can find the exact solution of Eq(4) with Lie-point symmetry generator

$$X_3 = t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}.$$

Using this generator, we find characteristic equation

$$\frac{dt}{t} = -\frac{2du}{u},\ dx = 0.$$

Since similarity variables are $c_1 = ut^{\frac{1}{2}}$ and $c_2 = x$, then, solution $f(c_2) = c_1$ implies,

$$u = t^{-1/2} f(x) \tag{34}$$

where f is arbitrary function of x. Through (34), reduced ODE reads

$$-\frac{f}{2} + \frac{f^{''}}{2} + 3f^2 f' - \left(f^{'}\right)^3 - 4ff^{'}f'' + 2f^{'}\left(f^{''}\right)^2 - f^2 f^{'''} + f^{'''}\left(f^{'}\right)^2 = 0 \tag{35}$$

where $f' = \frac{df}{dx}$. If we solve the above ODE, we obtain

$$f\left(x\right) = ae^x + be^{-x} \tag{36}$$

where a and b are arbitrary constants. Therefore exact solution of the Eq(4) is

$$u(x,t) = t^{-1/2}\left(ae^x + ce^{-x}\right). \tag{37}$$

4. Conclusion

It is well known that Lie symmetry analysis is widely used in finding conservation laws and reduction of given PDE's, ODE's. In this paper, we have examined FORQ equation, by obtaining new families of conservation laws and exact solution. Nonlocal conservation theorem was employed to construct conservation laws, while reduced ODE was being employed to obtain exact solution. The concept of self adjoint and quasi self adjoint equations were introduced by Ibragimov in [25]. With the same idea, taking nonlocal variable $w = \varphi(x,t,u)$, the self adjointness of FORQ equation was investigated. We have expressed each generator with corresponding conservation law in detail. We hope that obtained conservation laws and exact solution could further assist in understanding and identifying FORQ equation in previous and future works.

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