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Norm-Attainability and Range-Kernel Orthogonality of Elementary Operators

Benard Okelo[1]*

Abstract
Various aspects of elementary operators have been characterized by many mathematicians. In this paper, we consider norm-attainability and orthogonality of these operators in Banach spaces. Characterizations and generalizations of norm-attainability and orthogonality are given in details. We first give necessary and sufficient conditions for norm-attainability of Hilbert space operators then we give results on orthogonality of the range and the kernel of elementary operators when they are implemented by norm-attainable operators in Banach spaces.

Keywords: Range-Kernel orthogonality, Elementary operator
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[1] *School of Mathematics and Actuarial Science, Jaramogi Oginga Odinga University of Science and Technology, Box 210-40601, Bondo, Kenya.*
***Corresponding author**: bnyaare@yahoo.com
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1. Introduction

Studies on Hilbert space operators have been carried out for along period of time with nice results obtained. Norm-attainability is one of the aspects which has been given attention. Let H be an infinite dimensional separable complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. An operator $S \in B(H)$ is said to be norm-attainable if there exists a unit vector $x_0 \in H$ such that $\|Sx_0\| = \|S\|$. For an operator $S \in B(H)$ we define a numerical range by $W(S) = \{\langle Sx, x\rangle : x \in H, \|x\| = 1\}$ and the maximal numerical range by $W_0(S) = \{\beta \in \mathbb{C} : \langle Sx_n, x_n\rangle \to \beta, \text{ where } \|x_n\| = 1, \|Sx_n\| \to \|S\|\}$. The second aspect in consideration is orthogonality which is a concept that has been analyzed for quite a period of time. Benitez [1] described several types of orthogonality which have been studied in real normed spaces namely: Robert's orthogonality, Birkhoff's orthogonality, Orthogonality in the sense of James, Isoceles, Pythagoras, Carlsson, Diminnie, Area among others. Some of these orthogonalities are described as follows. For $x \in \mathscr{M}$ and $y \in \mathscr{N}$ where $\mathscr{M}$ and $\mathscr{N}$ are subspaces of E which is a normed linear space, we have: (i). Roberts: $\|x - \lambda y\| = \|x + \lambda y\|$, $\forall, \lambda \in \mathbb{R}$; (ii). Birkhoff: $\|x + y\| \geq \|y\|$; (iii). Isosceles: $\|x - y\| = \|x + y\|$; (iv). Pythagorean: $\|x - y\|^2 = \|x\|^2 + \|y\|^2$; (v). a-Pythagorean: $\|x - ay\|^2 = \|x\|^2 + a^2\|y\|^2$, $a \neq 0$; (vi). Diminnie: $\sup\{f(x)g(y) - f(y)g(x) : f, g \in S'\} = \|x\|\|y\|$ where S' denotes the unit sphere of the topological dual of E; (vii). Area: $\|x\|\|y\| = 0$ or they are linearly independent and such that $x, -x, y, -y$ divide the unit ball of their own plane (identified by $\mathbb{R}^2$) in four equal areas. In this paper we will consider the orthogonality of elementary operators when they are implemented by norm-attainable operators. Consider a normed space $\mathscr{A}$ and let $T_{A,B} : \mathscr{A} \to \mathscr{A}$. T is called an elementary operator if it has the following representation: $T(X) = \sum_{i=1}^{n} A_i X B_i$, $\forall X \in \mathscr{A}$, where A_i, B_i are fixed in $\mathscr{A}$. Let $\mathscr{A} = B(H)$. For $A, B \in B(H)$ we define the particular elementary operators: The left multiplication operator $L_A : B(H) \to B(H)$ by $L_A(X) = AX$, $\forall X \in B(H)$; the right multiplication operator $R_B : B(H) \to B(H)$ by $R_B(X) = XB$, $\forall X \in B(H)$; the generalized derivation (implemented by A, B) by $\delta_{A,B} = L_A - R_B$; the basic elementary operator(implemented by A, B) by $M_{A,B}(X) = AXB$, $\forall X \in B(H)$; the Jordan elementary operator(implemented by A, B) by $\mathscr{U}_{A,B}(X) = AXB + BXA$, $\forall X \in B(H)$; Regarding orthogonality involving elementary operators, Anderson[2] established the orthogonality of the range and kernel of normal derivations. Others who have also worked on orthogonality include: Kittaneh [3], Mecheri [4] among others. For details see [1-2, 4-31]. We shall investigate the orthogonality of the range and the kernel of several types of important elementary operators in Banach spaces. Anderson [2] in his investigations proved that if N and S are operators in $B(H)$ such that N is normal and $NS = SN$ then for all $X \in B(H)$, $\|\delta_N(X)+S\| \geq \|S\|$. If S(above) is a Hilbert-Schmidt operator then Kittaneh [3] (see also the references therein) showed that $\|\delta_N(X)+S\|_2^2 = \|\delta_N(X)\|_2^2 + \|S\|_2^2$. We extend this study to the general Banach spaces.

2. Preliminaries

In this section, we give some preliminary results. We begin by the following proposition.

Proposition 2.1. *Let H be an infinite dimensional separable complex Hilbert space. Let $S \in B(H)$, $\beta \in W_0(S)$ and $\alpha > 0$. Then the following conditions hold:*
(i). There exists $Z \in B(H)$ such that $\|S\| = \|Z\|$, with $\|S-Z\| < \alpha$.
(ii). There exists a vector $\eta \in H$, $\|\eta\| = 1$ such that $\|Z\eta\| = \|Z\|$ with $\langle Z\eta, \eta\rangle = \beta$.

Proof. Let $\|S\| = 1$ and also that $0 < \alpha < 2$. Let $x_n \in H$ $(n = 1, 2, ...)$ be such that $\|x_n\| = 1$, $\|Sx_n\| \to 1$ and also $\lim_{n\to\infty}\langle Sx_n, x_n\rangle = \beta$. Let $S = GL$ be the polar decomposition of S. Here G is a partial isometry and we write $L = \int_0^1 \beta dE_\beta$, the spectral decomposition of $L = (S^*S)^{\frac{1}{2}}$. Since L is a positive operator with norm 1, for any $x \in H$ we have that $\|Lx_n\| \to 1$ as n tends to ∞ and $\lim_{n\to\infty}\langle Sx_n, x_n\rangle = \lim_{n\to\infty}\langle GLx_n, x_n\rangle = \lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle$. Now for $H = \overline{Ran(L)} \oplus KerL$, we can choose x_n such that $x_n \in \overline{Ran(L)}$ for large n. Indeed, let $x_n = x_n^{(1)} \oplus x_n^{(2)}$, $n = 1, 2, ...$ Then we have that $Lx_n = Lx_n^{(1)} \oplus Lx_n^{(2)} = Lx_n^{(1)}$ and that $\lim_{n\to\infty}\|x_n^{(1)}\| = 1$, $\lim_{n\to\infty}\|x_n^{(2)}\| = 0$ since $\lim_{n\to\infty}\|Lx_n\| = 1$. Replacing x_n with $\frac{x_n^{(1)}}{\|x_n^{(1)}\|}$, we get

$$\lim_{n\to\infty}\left\|L\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\right\| = \lim_{n\to\infty}\left\|S\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\right\| = 1, \lim_{n\to\infty}\left\langle S\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}, \frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\right\rangle = \beta$$

Next let $x_n \in \overline{RanL}$. Since G is a partial isometry from $\overline{RanL}$ onto $\overline{RanS}$, we have that $\|Gx_n\| = 1$ and $\lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle = \beta$. Since L is a positive operator, $\|L\| = 1$ and for any $x \in H$, $\langle Lx, x\rangle \leq \langle x, x\rangle = \|x\|^2$. Replacing x with $L^{\frac{1}{2}}x$, we get that $\langle L^2x, x\rangle \leq \langle Lx, x\rangle$, where $L^{\frac{1}{2}}$ is the positive square root of L. Therefore we have that $\|Lx\|^2 = \langle Lx, Lx\rangle \leq \langle Lx, x\rangle$. It is obvious that $\lim_{n\to\infty}\|Lx_n\| = 1$ and that $\|Lx_n\|^2 \leq \langle Lx_n, x_n\rangle \leq \|Lx_n\|^2 = 1$. Hence, $\lim_{n\to\infty}\langle Lx_n, x_n\rangle = 1 = \|L\|$. Moreover, Since $I - L \geq 0$, we have $\lim_{n\to\infty}\langle(I-L)x_n, x_n\rangle = 0$. thus $\lim_{n\to\infty}\|(I-L)^{\frac{1}{2}}x_n\| = 0$. Indeed, $\lim_{n\to\infty}\|(I-L)x_n\| \leq \lim_{n\to\infty}\|(I-L)^{\frac{1}{2}}\|.\|(I-L)^{\frac{1}{2}}x_n\| = 0$. For $\alpha > 0$, let $\gamma = [0, 1-\frac{\alpha}{2}]$ and let $\rho = (1-\frac{\alpha}{2}, 1]$. We have $L = \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu = LE(\gamma) \oplus LE(\rho)$. Next we show that $\lim_{n\to\infty}\|E(\gamma)x_n\| = 0$. If there exists a subsequence $x_{n_i}, (i = 1, 2, ...,)$ such that $\|E(\gamma)x_{n_i}\| \geq \varepsilon > 0$, $(i = 1, 2, ...,)$, then since $\lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\| = 0$, it follows that

$$\lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\|^2 = \lim_{i\to\infty}(\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 + \|E(\rho)x_{n_i} - LE(\rho)x_{n_i}\|^2) = 0.$$

Hence, we have that

$$\lim_{i\to\infty}\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 = 0.$$

Now it is clear that

$$\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\| \geq \|E(\gamma)x_{n_i}\| - \|LE(\gamma)\|.\|E(\gamma)x_{n_i}\| \geq (I - \|LE(\gamma)\|)\|E(\gamma)x_{n_i}\| \geq \frac{\alpha}{2}\varepsilon > 0.$$

This is a contradiction. Therefore, $\lim_{n\to\infty}\|E(\gamma)x_n\| = 0$. Since $\lim_{n\to\infty}\langle Lx_n, x_n\rangle = 1$, we have that $\lim_{n\to\infty}\langle LE(\rho)x_n, E(\rho)x_n\rangle = 1$ and $\lim_{n\to\infty}\langle E(\rho)x_n, G^*E(\rho)x_n\rangle = \beta$. It is easy to see that

$$\lim_{n\to\infty}\|E(\rho)x_n\| = 1, \lim_{n\to\infty}\left\langle L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, \frac{E(\rho)x_n}{\|E(\rho)x_n\|}\right\rangle = 1$$

and

$$\lim_{n\to\infty}\left\langle L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, G^*\frac{E(\rho)x_n}{\|E(\rho)x_n\|}\right\rangle = \beta$$

Replacing x with $\frac{E(\rho)x_n}{\|E(\rho)x_n\|}$, we can assume that $x_n \in E(\rho)H$ for each n and $\|x_n\| = 1$. Let $J = \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu = J_1 \oplus E(\rho)$. Then it is evident that $\|J\| = \|S\| = \|L\| = 1, Jx_n = x_n$, and $\|J - L\| \leq \frac{\alpha}{2}$. If we can find a contraction V such that $V - G \leq \frac{\alpha}{2}$ and $\|Vx_n\| = 1$ and $\langle Vx_n, x_n\rangle = \beta$, for a large n then letting $Z = VJ$, we have that $\|Zx_n\| = \|VJx_n\| = 1$, and that

$$\langle Zx_n, x_n\rangle = \langle VJx_n, x_n\rangle = \langle Vx_n, x_n\rangle = \beta,$$

$$\|S - Z\| = \|GL - VJ\| \leq \|GL - GJ\| + \|GJ - VJ\| \leq \|G\|.\|L - J\| + \|G - V\|.\|J\| \leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Lastly, we now construct the desired contraction V. Clearly, $\lim_{n\to\infty}\langle x_n, G^*x_n\rangle = \beta$, because $\lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle = \beta$ and $\lim_{n\to\infty}\|x_n - Lx_n\| = 0$. Let $Gx_n = \phi_n x_n + \varphi_n y_n$, $(y_n \perp x_n,\ \|y_n\| = 1)$ then $\lim_{n\to\infty}\phi_n = \beta$, because $\lim_{n\to\infty}\langle Gx_n, x_n\rangle = \lim_{n\to\infty}\langle x_n, G^*x_n\rangle = \beta$ but $\|Gx_n\|^2 = |\phi_n|^2 + |\varphi_n|^2 = 1$, so we have that $\lim_{n\to\infty}|\varphi_n| = \sqrt{1 - |\beta|}^2$. Now without loss of generality, there exists an integer M such that $|\phi_M - \beta| < \frac{\alpha}{8}$. Choose φ_M^0 such that $|\varphi_M^0| = \sqrt{1 - |\beta|}^2$, $|\varphi_M - \varphi_M^0| < \frac{\alpha}{8}$. We have that

$$Gx_M = \phi_M x_M + \varphi_M y_M - \beta x_M + \beta x_M - \varphi_M^0 y_M + \varphi_M^0 y_M = (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + \beta x_M + \varphi_M^0 y_M.$$

Let

$$q_M = \beta x_M + \varphi_M^0 y_M, Gx_M = (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + q_M.$$

Suppose that $y \perp x_M$, then

$$\langle Gx_M, Gy\rangle = (\phi - \beta)\langle x_M, Gy\rangle + (\varphi_M - \varphi_M^0)\langle y_M, Gy\rangle + \langle q_M, Gy\rangle = 0,$$

because G^*G is a projection from H to $RanL$. It follows that

$$|\langle q_M, Gy\rangle| \leq |\phi_M - \beta|.\|y\| + |\varphi_M - \varphi_M^0|.\|y\| \leq \frac{\alpha}{4}\|y\|$$

If we suppose that

$$Gy = \phi q_M + y^0,\ (y^0 \perp q_M,)$$

then y^0 is uniquely determined by y. Hence we can define V as follows$V : x_M \to q_M,\ y \to y^0,\ \phi x_M + \varphi_M y \to \phi q_M + \varphi_M y^0$, with both ϕ, φ being complex numbers. V is a linear operator. We prove that V is a contraction. Now,

$$\|Vx_M\|^2 = \|q_M\|^2 = |\beta|^2 = |\varphi_M^0|^2 = 1,$$

$$\|Vy\|^2 = \|Gy\|^2 - |\phi y|^2 \leq \|Gy\|^2 \leq \|y\|^2.$$

It follows that

$$\|V\phi\|^2 = \|\phi\|^2\|Vx_M\|^2 + |\varphi|^2\|Vy\|^2 \leq |\phi|^2 + |\varphi|^2 = 1,$$

for each $x \in H$ satisfying that $x = \phi x_M + \varphi_M y$, $\|x\| = 1$, $x_M \perp y$, which is equivalent to that V is a contraction. From the definition of V, we can show that

$$\|Gx_M - Vx_M\|^2 = |\phi - \beta|^2 + |\varphi_M - \varphi_M^0|^2 \leq \frac{2\alpha^2}{16} = \frac{1}{8}\alpha^2.$$

If $y \perp x_M$, $\|y\| \leq 1$ then obtain

$$\|Gy - Vy\| = |\phi|\|Vx_M\| = |\langle Gy, Vx_M\rangle| = |\langle q_M, Gy\rangle| < \frac{\alpha}{4}.$$

Hence for any $x \in H$, $x = \phi x_M + \varphi_M y$, $\|x\| = 1$,

$$\|Gx - Vx\|^2 = \|\phi(G - V)x_M + \varphi(G - V)y\|^2 = |\phi|^2\|(G - V)x_M\|^2 + |\varphi|^2\|(G - V)y\|^2 < |\phi|^2.\frac{\alpha^2}{16} + |\varphi|^2.\frac{\alpha^2}{16} < \frac{\alpha^2}{8},$$

which implies that $\|(G - V)x\| < \frac{\alpha}{2}$, $\|x\| = 1$, and hence $\|(G - V)\| < \frac{\alpha}{2}$. Let $Z = VJ$. Then Z is what we want. □

The next result gives the conditions for norm-attainability of an inner derivation. We give the following proposition.

Proposition 2.2. *Let H be an infinite dimensional separable complex Hilbert space and $S \in B(H)$. δ_S is norm-attainable if there exists a vector $\zeta \in H$ such that $\|\zeta\| = 1,\ \|S\zeta\| = \|S\|,\ \langle S\zeta, \zeta\rangle = 0$.*

Proof. For any x satisfying that $x \perp \{\zeta, S\zeta\}$, define X as follows $X : \zeta \to \zeta,\ S\zeta \to -S\zeta,\ x \to 0$. Since X is a bounded operator on H and

$$\|X\zeta\| = \|X\| = 1, \|SX\zeta - XS\zeta\| = \|S\zeta - (-S\zeta)\| = 2\|S\zeta\| = 2\|S\|.$$

It follows that $\|\delta_S\| = 2\|S\|$ via the result in [30, Theorem 1], because $\langle S\zeta, \zeta\rangle = 0 \in W_0(S)$. Hence we have that $\|SX - XS\| = 2\|S\| = \|\delta_S\|$. Therefore, δ_S is norm-attainable. □

The next result gives the conditions for norm-attainability of a generalized derivation. We give the following proposition.

Proposition 2.3. *Let H be an infinite dimensional separable complex Hilbert space. Let $S, T \in B(H)$. If there exists vectors $\zeta, \eta \in H$ such that $\|\zeta\| = \|\eta\| = 1,\ \|S\zeta\| = \|S\|,\ \|T\eta\| = \|T\|$ and $\frac{1}{\|S\|}\langle S\zeta, \zeta\rangle = -\frac{1}{\|T\|}\langle T\eta, \eta\rangle$, then $\delta_{S,T}$ is norm-attainable.*

Proof. By linear dependence of vectors, if η and $T\eta$ are linearly dependent, i.e.,$T\eta = \phi\|T\|\eta$, then it is true that $|\phi| = 1$ and $|\langle T\eta, \eta\rangle| = \|T\|$. It follows that $|\langle S\zeta, \zeta\rangle| = \|S\|$ which implies that $S\zeta = \varphi\|S\|\zeta$ and $|\varphi| = 1$. Hence $\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle = \varphi = -\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle = -\phi$. Defining X as $X : \eta \to \zeta,\ \{\eta\}^{\perp} \to 0$, we have $\|X\| = 1$ and $(SX - XT)\eta = \varphi(\|S\| + \|T\|)\zeta$, which implies that

$$\|SX - XT\| = \|(SX - XT)\eta\| = \|S\| + \|T\|.$$

By [3], it follows that $\|SX - XT\| = \|S\| + \|T\| = \|\delta_{S,T}\|$. That is $\delta_{S,T}$ is norm-attainable. If η and $T\eta$ are linearly independent, then $\left|\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle\right| < 1$, which implies that $\left|\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle\right| < 1$. Hence ζ and $S\zeta$ are also linearly independent. Let us redefine X as follows: $X : \eta \to \zeta,\ \frac{T\eta}{\|T\|} \to -\frac{S\zeta}{\|S\|},\ x \to 0$, where $x \in \{\eta, T\eta\}^{\perp}$. We show that X is a partial isometry. Let $\frac{T\eta}{\|T\|} = \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle \eta + \tau h,\ \|h\| = 1,\ h \perp \eta$. Since η and $T\eta$ are linearly independent, $\tau \neq 0$. So we have that

$$X\frac{T\eta}{\|T\|} = \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle X\eta + \tau X h = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle \zeta + \tau X h,$$

which implies that

$$\left\langle X\frac{T\eta}{\|T\|}, \zeta \right\rangle = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle + \tau\langle Xh, \zeta\rangle = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle.$$

It follows then that $\langle Xh, \zeta\rangle = 0$ i.e., $Xh \perp \zeta (\zeta = X\eta)$. Hence we have that

$$\left\|\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle \zeta\right\|^2 + \|\tau X h\|^2 = \left\|X\frac{T\eta}{\|T\|}\right\|^2 = \left|\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle\right|^2 + |\tau|^2 = 1,$$

which implies that $\|Xh\| = 1$. Now it is evident that X a partial isometry and $\|(SX - XT)\zeta\| = \|SX - XT\| = \|S\| + \|T\|$, which is equivalent to $\|\delta_{S,T}(X)\| = \|S\| + \|T\|$. By Proposition 2.2 and [28], $\|\delta_{S,T}\| = \|S\| + \|T\|$. Hence $\delta_{S,T}$ is norm-attainable. □

The next result is a consequence of Proposition 2.2 and 2.3. It gives the necessary and sufficient conditions for norm-attainability of a basic elementary operator.

Corollary 2.4. *Let $S, T \in B(H)$ If both S and T are norm-attainable then the basic elementary operator $M_{S,T}$ is also norm-attainable.*

Proof. For any pair (S,T) it is known that $\|M_{S,T}\| = \|S\|\|T\|$. We can assume that $\|S\| = \|T\| = 1$. If both S and T are norm-attainable, then there exists unit vectors ζ and η with $\|S\zeta\| = \|T\eta\| = 1$. We can therefore define an operator X by $X = \langle \cdot, T\eta\rangle\zeta$. Clearly, $\|X\| = 1$. Therefore, we have $\|SXT\| \geq \|SXT\eta\| = \|\|T\eta\|^2 S\zeta\| = 1$. Hence, $\|M_{S,T}(X)\| = \|SXT\| = 1$, that is $M_{S,T}$ is also norm-attainable. □

In the next section, we dedicate our work to orthogonality of elementary operators on Banach spaces. From this point henceforth, all the elementary operators are implemented by norm-attainable operators unless otherwise stated. First we note that Ω denotes the algebra of all norm-attainable operators. In fact Ω is a Banach algebra. Let $T:\Omega\rightarrow\Omega$ be defined by $T(X)=\sum_{i=1}^{n}A_iXB_i,\ \forall X\in\Omega$, where $A_i,\ B_i$ are fixed in Ω. We define the range of T by $RanT=\{Y\in\Omega : Y=T(X),\ \forall X\in\Omega\}$, and the Kernel of T by $KerT=\{X\in\Omega : T(X)=0,\ \forall X\in\Omega\}$. It is known [4] that for any of the examples of the elementary operators defined in Section 1 (inner derivation, generalized derivation, basic elementary operator, Jordan elementary operator), the following implications hold for a general bounded linear operator T on a normed linear space W , i.e. $RanT\perp KerT\Rightarrow \overline{RanT}\cap KerT=\{0\}\Rightarrow RanT\cap KerT=\{0\}$. Here $\overline{RanT}$ denotes the closure of the range of T and $KerT$ denotes the kernel of T and $RanT\perp KerT$ means $RanT$ is orthogonal to the Kernel of T in the sense of Birkhoff. Let $A\in\Omega$. The algebraic numerical range $V(A)$ of A is defined by: $V(A)=\{f(A) : f\in\Omega' \text{ and } \|f\|=f(I)=1\}$ where Ω' is the dual space of Ω and I is the identity element in Ω. If $V(A)\subseteq\mathbb{R}$, then A is called a Hermitian element. Given two Hermitian elements S and R, such that $SR=RS$ then $D=S+Ri$ is called normal [29].

3. Main results

Proposition 3.1. *Let $A,B,C\in\Omega$ with $CB=I$ (I is an identity element of Ω). Then for a generalized derivation $\delta_{A,B}=AX-XB$ and an elementary operator $\Theta_{A,B}(X)=AXB-X$, $R_B(\overline{Ran\delta_{A,C}}\cap Ker\delta_{A,C})=\overline{Ran\Theta_{A,B}}\cap Ker\Theta_{A,B}$. Moreover, if $\overline{Ran\delta_{A,C}}\cap Ker\delta_{A,C}=\{0\}$ then $\overline{Ran\Theta_{A,B}}\cap Ker\Theta_{A,B}=\{0\}$.*

Proof. First, we prove that if $CB=I$ then $R_B\delta_{A,C}=\Theta_{A,B}$. To see this, $\forall X\in\Omega$, $R_B\delta_{A,C}(X)=AXB-XCB=AXB-X=\Theta_{A,B}$. Suppose that $P\in R_B(\overline{Ran\delta_{A,C}}\cap Ker\delta_{A,C})$. Now, it is a fact that the uniform norm assigns to real- or complex-valued continuous bounded operator R_B defined on any set Ω the nonnegative number $\|R_B\|_\infty=\sup\{\|R_B(X)\| : X\in\Omega\}$. Since $R_B\delta_{A,C}=\Theta_{A,B}$ and R_B is continuous for the uniform norm, then $P\in\overline{Ran\Theta_{A,B}}\cap Ker\Theta_{A,B}$. Conversely, since R_C is continuous for the uniform norm, then by the same argument we prove that if $P\in R_B(\overline{Ran\Theta_{A,B}}\cap Ker\Theta_{A,B})$ then $P\in R_B(\overline{Ran\delta_{A,C}}\cap Ker\delta_{A,C})$. □

It is important to note the following. Let $A,B,C\in\Omega$ with $CB=I$ (I is an identity element of Ω). Then $R_B(\overline{Ran\delta_{A,C}}\cap Ker\delta_{A,C})=Ran\Theta_{A,B}\cap Ker\Theta_{A,B}$. Indeed, since $Ran\Theta_{A,B}\subseteq\overline{Ran\Theta_{A,B}}$, then by Proposition 3.1, the equality holds.

Proposition 3.2. *Let S and R be Hermitian elements. Then $\delta_{S,R}$ is also Hermitian.*

Proof. From [22], it is known that if X is a Banach space then $V(\delta_{S,R})=V(S)-V(R)$ for all $S,R\in B(X)$. Therefore, $V(\delta_{S,R})\subseteq V(L_S)-V(L_R)=V(S)-V(R)\subseteq\mathbb{R}$. □

Corollary 3.3. *If D and E are normal elements in Ω then $\delta_{D,E}$ is also normal.*

Proof. Assume $D=S+Ri$ and $E=T+Ui$ where S,R,T,U are Hermitian elements in Ω such that $SR=RS$ and $TU=UT$. Then $\delta_{D,E}=\delta_{S,T}+i\delta_{R,U}$ with $\delta_{S,T}\delta_{R,U}=\delta_{R,U}\delta_{S,T}$. Since S,R,T,U are Hermitian, then by Proposition 3.2 $\delta_{R,U}$ and $\delta_{S,T}$ are Hermitian and so is $\delta_{D,E}$. □

Remark 3.4. *([22]) Let X be a Banach space and $T\in B(X)$. If T is a normal operator, then $RanT\perp KerT$. Moreover, if D and E are normal elements in Ω then $Ran\delta_{D,E}\perp Ker\delta_{D,E}$. Indeed, assume that D and E are normal elements in Ω. Then by Corollary 3.3, $\delta_{D,E}$ is normal and by Proposition 3.2 $Ran\delta_{D,E}\perp Ker\delta_{D,E}$.*

Corollary 3.5. *If $A,B\in\Omega$ are normal and there exists $C\in\Omega$ such that $BC=I$ then $\overline{Ran\Theta_{A,C}}\cap Ker\Theta_{A,C}=\{0\}$.*

Proof. If $A,B\in\Omega$ are normal and self-adjoint elements, then by Corollary 3.3, $Ran\delta_{A,B}\perp Ker\delta_{A,B}$. This implies that $\overline{Ran\delta_{A,B}}\cap Ker\delta_{A,B}=\{0\}$. Using Proposition 2.2, we conclude that $\overline{Ran\Theta_{A,C}}\cap Ker\Theta_{A,C}=\{0\}$. □

The next theorem gives a stronger result on power sequences of operators $A^n,B^n\in\Omega$ for all $n\in\mathbb{N}$.

Theorem 3.6. *Let $A,B\in\Omega$ be normal and self-adjoint with $C\in\Omega$ such that $BC=I$ and $\|C\|\leq 1$. If $\|A\|\leq 1$ and $\|B\|\leq 1$ for all $n\in\mathbb{N}$ then $Ran\delta_{A,B}\perp Ker\delta_{A,B}$.*

Proof. It is well known [2] that $A^nX-XB^n=\Sigma_{i=0}^{n-1}A^{n-i-1}(AX-XB)B^i$ and

$$A^nX-XB^n-\Sigma_{i=0}^{n-1}A^{n-i-1}(AX-XB-Y)B^i=nYB^{n-1},$$

where $Y\in Ker\delta_{A,B}$. Multiplying this equality by C^{n-1} we obtain

$$A^nXC^{n-1}-XB-\Sigma_{i=0}^{n-1}A^{n-i-1}(AX-XB-Y)B^iC^{n-1}=nYB^{n-1}C^{n-1}$$

which is equivalent to

$$nY = A^nXC^{n-1} - XB - \Sigma_{i=0}^{n-1} A^{n-i-1}(AX - XB - Y)B^iC^{n-1}.$$

Now, the assumption that $BC = I$ with $\|C\| \le 1$ and $\|B\| \le 1$ implies that $\|C^n\| = \|B^n\| = 1$, for all $n \in \mathbb{N}$. This shows that dividing both sides by n and taking norms we obtain

$$\|Y\| \le \frac{1}{n}\{\|A^n\|\|X\|\|C\|^{n-1} + \|X\|\|B\|\} + \frac{1}{n}\Sigma_{i=0}^{n-1}\|A\|^{n-i-1}\|AX - XB - Y\|\|B\|^i\|C\|^{n-1}$$

$= \frac{1}{n}\{\|A^n\|\|X\| + \|X\|\} + \frac{1}{n}\Sigma_{i=0}^{n-1}\|A\|^{n-i-1}\|AX - XB - Y\|.$
Hence $\|Y\| \le \frac{2}{n}\|X\| + \frac{1}{n}\Sigma_{i=0}^{n-1}\|AX - XB - Y\|$. Taking limits as $n \to \infty$, we obtain that $\|Y\| \le \|AX - XB - Y\|$. Therefore, $Ran\delta_{A,B} \perp Ker\delta_{A,B}$. ☐

The following theorem from Kittaneh [3] gives a general orthogonality condition for linear operators. The proof is omitted.

Theorem 3.7 (3)**.** *Let be Ω a normed algebra with the norm $\|.\|$ satisfying $\|XY\| \le \|X\|\|Y\|$ for all X, $Y \in \Omega$ and let $\delta : \Omega \to \Omega$ be a linear map with $\|\delta\| \le 1$. If $\delta(Y) = Y$ for some $Y \in \Omega$, then $\|\delta(X) - X + Y\| \ge \|Y\|$, for all $X \in \Omega$.*

We utilize the Theorem 3.7 to prove some results for general elementary operators. Let $T : \Omega \to \Omega$ be an elementary operator defined by $T(X) = \sum_{i=1}^n A_iXB_i$, $\forall X \in \Omega$. Now suppose that $T(Y) = Y$ for some $Y \in \Omega$. If $\|T\| \le 1$, then $\|T(X) - X + Y\| \ge \|Y\|$, for all $X \in \Omega$. The following theorem follows immediately.

Theorem 3.8. *Suppose that $T(Y) = Y$ for some normal self-adjoint $Y \in \Omega$. If $\|\sum_{i=1}^n A_iA_i^*\|^{\frac{1}{2}} \|\sum_{i=1}^n B_i^*B_i\|^{\frac{1}{2}} \le 1$, then $\|T(X) - X + Y\| \ge \|Y\|$, for all $X \in \Omega$.*

Proof. We only need to show that $\|T\| \le 1$. Let $Z_1 = [A_1, ..., A_n]$ and $Z_2 = [B_1, ..., B_n]^T$. Taking $Z_1Z_1^*$ and $Z_2^*Z_2$ shows that $\|Z_1\| = \|\sum_{i=1}^n A_iA_i^*\|^{\frac{1}{2}}$ and $\|Z_2\| = \|\sum_{i=1}^n B_i^*B_i\|^{\frac{1}{2}}$. From [16], it is known that $T(X) = Z_1(X \otimes I_n)Z_2$, where I_n is the identity of $M_n(\mathbb{C})$. Therefore it follows that $\|T(X)\| \le \|Z_1\|\|Z_2\|\|X\|$. Hence $\|T\| \le 1$. ☐

Now, we consider the orthogonality of Jordan elementary operators. We later consider the necessary and sufficient conditions for their normality. At this juncture a type of norm, called the unitarily invariant norm comes in handy. A unitarily invariant norm is any norm defined on some two-sided ideal of $B(H)$ and $B(H)$ itself which satisfies the following two conditions. For unitary operators $U, V \in B(H)$ the equality $|||UXV||| = |||X|||$ holds, and $|||X||| = s_1(X)$, for all rank one operators X. It is proved that any unitarily invariant norm depends only on the sequence of singular values. Also, it is known that the maximal ideal, on which $|||UXV|||$ has sense, is a Banach space with respect to that unitarily invariant norm. Among all unitarily invariant norms there are few important special cases. The first is the Schatten p-norm ($p \ge 1$) defined by $\|X\|_p = (\Sigma_{j=1}^{+\infty} s_j(X)^p)^{1/p}$ on the set $\mathscr{C}_p = \{X \in B(H) : \|X\|_p < +\infty\}$. For $p = 1, 2$ this norm is known as the nuclear norm (Hilbert-Schmidt norm) and the corresponding ideal is known as the ideal of nuclear (Hilbert-Schmidt) operators. The ideal $\mathscr{C}_2$ is also interesting for another reason. Namely, it is a Hilbert space with respect to the $\|.\|_2$ norm. The other important special case is the set of so-called Ky Fan norms $\|X\|_k = \Sigma_{j=1}^k s_j(X)$. The well-known Ky Fan dominance property asserts that the condition $\|X\|_k \le \|Y\|$ for all $k \ge 1$ is necessary and sufficient for the validity of the inequality $|||X||| \le |||Y|||$ in all unitarily invariant norms. For further details refer to [20].We state the following theorem from [20] on orthogonality.

Theorem 3.9. *Let $A, B \in B(H)$ be normal operators, such that $AB = BA$, and let $\mathscr{U}(X) = AXB - BXA$. Furthermore, suppose that $A^*A + B^*B > 0$. If $S \in Ker\mathscr{U}$, then $|||\mathscr{U}(X) + S||| \ge |||S|||$.*

We extend Theorem 3.9 to distinct operators $A, B, C, D \in B(H)$ in the theorem below.

Theorem 3.10. *Let $A, B, C, D \in B(H)$ be normal operators, such that $AC = CA$, $BD = DB$, $AA^* \le CC^*$, $B^*B \le D^*D$. For an elementary operator $\mathscr{U}(X) = AXB - CXD$ and $S \in B(H)$ satisfying $ASB = CSD$, then $\|\mathscr{U}(X) + S\| \ge \|S\|$, for all $X \in B(H)$.*

Proof. From $AA^* \le CC^*$ and $B^*B \le D^*D$, let $A = CU$, and $B = VD$, where U, V are contractions. So we have $AXB - CXD = CUXVD - CXD = C(UXV - X)D$. Assume C and D^* are injective, $ASB = CSD$ if and only if $USV = S$. Moreover, C and U commute. Indeed from $A = CU$ we obtain $AC = CUC$. Therefore, $C(A - UC) = 0$. Thus since C is injective $A = CU$. Similarly, D and V commute. So, $\|\mathscr{U}(X) + S\| = \|[AXB - CXD] + S\| = \|[U(CXD)V - CXD] + S\| \ge \|S\|, \forall X \in B(H)$. Now, under the condition of Theorem 3.10, A and C have operator matrices $A = \begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 0 & C_0 \end{pmatrix}$ with respect to the space decomposition $H = \overline{\mathscr{R}(C)} \oplus \mathscr{N}(C)$, respectively. Here, A_0 is a normal operator on $\overline{\mathscr{R}(C)}$ and C_0 is an injective and

normal operator on $\overline{\mathscr{R}(C)}$. B and D have operator matrices $B = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} D_0 & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the space decomposition $H = \overline{\mathscr{R}(D)} \oplus \mathscr{N}(D)$, respectively. Here, B_0 is a normal operator on $\overline{\mathscr{R}(D)}$ and D_0 is an injective and normal operator on $\overline{\mathscr{R}(D)}$. X and S have operator matrices $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ and $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ which are as operator from the space decomposition $H = \overline{\mathscr{R}(D)} \oplus \mathscr{N}(D)$ into the space decomposition $H = \overline{\mathscr{R}(C)} \oplus \mathscr{N}(C)$, respectively.
In this case, $\mathscr{U}(X) = AXB - CXD = \begin{pmatrix} A_0X_{11}B_0 - C_0X_{11}D_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A_0S_{11}B_0 - C_0S_{11}D_0 = 0$. Therefore, $\|A_0X_{11}B_0 - C_0X_{11}D_0 + S_{11}\| \geq \|S_{11}\|$.
Hence,

$$\begin{aligned} \|\mathscr{U}(X) + S\| &= \left\| \begin{pmatrix} A_0X_{11}B_0 - C_0X_{11}D_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} A_0X_{11}B_0 - C_0X_{11}D_0 + S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \right\| \\ &\geq \left\| \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \right\|. \end{aligned}$$

□

The result in Theorem 3.10 can be generalized Banach spaces and other complex spaces like operator spaces and function spaces.

4. Conclusion

We conclude this paper by remarking that these results can be extended to give more results on generalized finite operators in terms of orthogonality and norm-attainability in C*-algebras.

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Modifications of Strongly Nodec Spaces

V. Renukadevi[1]* and S. Vadakasi[1]

Abstract
In this paper, we introduce the notion of strongly nodec spaces and study their properties. Also, we discuss strongly nodec generalized metric spaces. Furthermore, we extend these notions to T_0-strongly nodec space by using the quotient map.

Keywords: Strongly nowhere dense, Dense, Codense, Strongly nodec space
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[1] *Centre for Research and Post Graduate Studies in Mathematics, Ayya Nadar Janaki Ammal College(Autonomous), Sivakasi-626 124, Tamil Nadu, India.*
***Corresponding author**:renu_siva@yahoo.com
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1. Introduction

The notion of a generalized topological space was introduced by Császár in [1]. Let X be any non-empty set. A family $\mu \subset exp(X)$ is a *generalized topology* [2] in X if $\emptyset \in \mu$ and $\bigcup\limits_{i\in I} G_i \in \mu$ whenever $\{G_i : i \in I\} \subset \mu$ where $exp(X)$ is a power set of X. We call the pair (X,μ) as a *generalized topological space* (GTS) [2]. If $X \in \mu$, then the pair (X,μ) is called a *strong generalized topological space* (sGTS) [2].

The elements in μ are called the μ-open sets and the complement of a μ-open set is called the μ-closed sets.

Let (X,μ) be a GTS and $A \subset X$. The *interior of* A [2] denoted by iA, is the union of all μ-open sets contained in A and the *closure of* A [2] denoted by cA, is the intersection of all μ-closed sets containing A.

In 2013, Korczak-Kubiak et al. introduced the notations $\tilde{\mu}, \mu(x)$ defined by $\{U \in \mu : U \neq \emptyset\}, \{U \in \mu : x \in U\}$ respectively [3].

Let (X,μ) be a GTS and $Y \subseteq X$. The *subspace generalized topology* is defined by, $\mu_Y = \{U \cap Y : U \in \mu\}$. Then the pair (Y,μ_Y) is called the *subspace GTS*. Furthermore, (X,μ) is strong GTS if and only if $c(\emptyset) = \emptyset$ if and only if $\emptyset$ is closed [4].

2. Preliminaries

In this section, we remember some basic definitions and lemmas which will be useful to prove the results in the following sections.

We already familiar with nowhere dense sets in generalized topological space. In 2013, Korczak - Kubiak et al. defined a new one namely strongly nowhere dense sets and discussed their properties. Also, they gave a relation between nowhere dense and strongly nowhere dense sets in generalized topological space [3]. In [5], we analyze properties of strongly nowhere dense sets in generalized topological spaces. With this terminology, we define a new space namely strongly nodec space and study some properties of strongly nodec spaces.

Let (X,μ) be a GTS and $A \subset X$. A is said to be a α*-open* (resp. α*-closed*) set if $A \subset iciA$ (resp. $cicA \subset A$). The *interior of* A [2] denoted by $i_\alpha A$, is the union of all α-open sets contained in A and the *closure of* A [2] denoted by $c_\alpha A$, is the intersection of all α-closed sets containing A. A subset A is said to be μ-*nowhere dense* [3] (resp. μ-*dense,* μ-*codense* [2]) if $icA = \emptyset$ (resp. $cA = X$, $c(X-A) = X$).

A subset A of a GTS (X,μ) is said to be μ-*strongly nowhere dense* [3] if for any $V \in \tilde{\mu}$, there exists $U \in \tilde{\mu}$ such that $U \subset V$ and $U \cap A = \emptyset$. A is said to be a μ-*s-meager* [3] set if $A = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n is a μ-strongly nowhere dense sets for all $n \in \mathbb{N}$ where $\mathbb{N}$ denote the set of all natural numbers.

Let (X,μ) be a GTS and $A \subset X$. A is said to be a μ-*s-second category* (μ-s-II category) [3] set if A is not a μ-s-meager set. A subset A of a GTS (X,μ) is said to be a μ-*s-residual* set if $X-A$ is a μ-s-meager set in X.

In GTS, every μ-strongly nowhere dense set is μ-nowhere dense and every subset of a μ-strongly nowhere dense set is μ-strongly nowhere dense [3]. Also, every subset of a μ-s-meager set is a μ-s-meager set [3]. If A is a μ-strongly nowhere dense set, then cA is a μ-strongly nowhere dense set [3].

Let (X,μ) be a GTS. X is said to be a *generalized submaximal space* [6] if every μ-dense subset of X is a μ-open set in X.

Throughout this paper μ-strongly nowhere dense, μ-nowhere dense, μ-s-meager, μ-s-residual and etc., we will write strongly nowhere dense, nowhere dense, s-meager, s-residual and etc., when no confusion can arise.

The following lemmas will be useful in the sequel.

Lemma 2.1. [7, Lemma 2.6] Let (X,μ) be a GTS and $A \subset X$. Then $i(cA-A) = \emptyset$.

Lemma 2.2. [6, Theorem 19] Let (X,μ) be a GTS. The following properties are equivalent:
(a) X is a generalized submaximal space.
(b) Each μ-codense subset A of (X,μ) is μ-closed.

Lemma 2.3. [2, Lemma 2.3] Let (X,μ) be a GTS and let $A \subset S \subset X$. Then $c_S A = cA \cap S$.

Lemma 2.4. [2, Lemma 3.2] Let (X,μ) be a GTS and let $A,U \subset X$. If $U \in \tilde{\mu}$ and $U \cap A = \emptyset$, then $U \cap cA = \emptyset$.

Lemma 2.5. [5, Theorem 2.13] Let (X,μ) be a GTS and $A \subset X$. If A is a μ-strongly nowhere dense set in X, then A is codense.

3. Strongly nodec spaces

In this section, we define strongly nodec space and give the example for the existence of this space in generalized topological spaces. Further, we discuss the properties of strongly nodec space in generalized topological spaces. Also, we prove product of two GTS is strongly nodec then each one is a strongly nodec space.

Definition 3.1. *Let (X,μ) be a GTS. A space X is said to be a strongly nodec space if every non-empty μ-strongly nowhere dense subset of X is μ-closed in X.*

Example 3.2 shows the existence of the strongly nodec space and Theorem 3.4 give the necessary condition for a sGTS to be a strongly nodec space.

Example 3.2. (a) Consider the GTS (X,μ) where $X = \{a,b,c,d,e\}$ and $\mu = \{\emptyset,\{a,d\},\{a,e\},\{b,e\},\{a,d,e\},\{a,b,e\},\{a,b,d,e\}\}$. Here the μ-strongly nowhere dense set is $\{c\}$ which is also a μ-closed set in X. Therefore, X is a strongly nodec space.
(b) Consider the GTS (X,μ) where $X = \mathbb{R}$ and $\mu = \{\emptyset\} \cup \{A \subset X : A - \{x\} \subset A \text{ for some } x \in X\}$. Here, there is no μ-strongly nowhere dense set in X. Therefore, X is a strongly nodec space.

Lemma 3.3. *In a GTS (X,μ), every μ-strongly nowhere dense set does not contains a non-empty μ-open set.*

Proof. Let A be a μ-strongly nowhere dense set in X. Suppose there is $U \in \tilde{\mu}$ such that $U \subset A$. Then there is no $V \in \tilde{\mu}$ such that $V \subset U$ and $V \cap A = \emptyset$, which is a contradiction to A is μ-strongly nowhere dense in X. This implies $U = \emptyset$. Therefore, A does not contains a non-empty μ-open set in X. Hence every μ-strongly nowhere dense set does not contains a non-empty μ-open set. □

Theorem 3.4. *Let (X,μ) be a sGTS. Then X is a strongly nodec space if any one of the following hold.*
(a) Every α-closed set is a μ-closed set.
(b) Every α-open set is a μ-open set.
(c) For each $A \subseteq X, cA - A \subseteq cic(A)$.
(d) For each $A \subseteq X, cA = A \cup cicA$.
(e) For each $A \subseteq X, iA = A \cap iciA$.

Proof. Assume (a). Let A be a non-empty strongly nowhere dense set in X. Then A is a non-empty nowhere dense set and so $icA = \emptyset$. Since X is a sGTS, $cicA = \emptyset$. This implies $cicA \subset A$ which implies A is a α-closed set. By (a), A is a μ-closed set in X. Hence X is a strongly nodec space.
Assume (b). Similar considerations in (a), we prove X is a strongly nodec space.
Assume (c). Let A be a non-empty strongly nowhere dense set in X. Then A is a non-empty nowhere dense set and so $icA = \emptyset$. Since X is a sGTS, $cicA = \emptyset$. By (c), $cA - A = \emptyset$. Thus, $cA = A$. Therefore, A is a μ-closed set in X. Hence X is a strongly nodec space.
Assume (d). Let A a non-empty strongly nowhere dense set in X. Then by same process in (a), $cicA = \emptyset$. By (d), $cA = A \cup \emptyset = A$. Therefore, A is a μ-closed set in X. Hence X is a strongly nodec space.
Assume (e). Similar considerations in (d), we prove X is a strongly nodec space. □

Definition 3.5. *Let (X,μ) be a GTS and $A \subset X$. Then frontier of A is denoted by $Fr(A)$ and defined by $Fr(A) = cA \cap c(X - A)$. Then frontier of A is a closed set in X.*

Example 3.6 shows that the existence of $Fr(A)$ and Lemma 3.7 give some properties of $Fr(A)$ in a generalized topological space (X,μ) where $A \subset X$.

Example 3.6. (a) Consider the GTS (X,μ) where $X = [0,5]$ and $\mu = \{\emptyset, [0,2), (1,3), [2,4), [0,3), (1,4), [0,4)\}$. Let $A = [3,4)$ be a subset of X. Then $Fr(A) = [3,5]$. Also, $Fr(A)$ is a μ-closed set in X.
(b) Consider the GTS (X,μ) where $X = \{a,b,c,d\}$ and $\mu = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}\}$. Let $A = \{a,b\}$ be a subset of X. Then $Fr(A) = \{c,d\}$. Also, $Fr(A)$ is a μ-closed set in X.

Lemma 3.7. *Let (X,μ) be a GTS. Then the following hold.*
(a) If B is a strongly nowhere dense set in X, then $Fr(B)$ is a strongly nowhere dense set in X for all $B \subset X$.
(b) If $Fr(A)$ is a strongly nowhere dense set in X, then $Fr(iA), Fr(cA)$ is a strongly nowhere dense set in X for all $A \subset X$.
(c) If A is a strongly nowhere dense set in X, then $Fr(A \cap B)$ is a strongly nowhere dense set in X for all $A,B \subset X$.

Proof. (a) Suppose $B \subset X$ is a strongly nowhere dense set. Let $U \in \tilde{\mu}$. Then there exists $V \in \tilde{\mu}$ such that $V \subset U$ and $V \cap B = \emptyset$. By Lemma 2.4, $V \cap cB = \emptyset$. This implies $V \cap Fr(B) = \emptyset$, since $Fr(B) \subset cB$. Therefore, $Fr(B)$ is a strongly nowhere dense set in X.
(b) Suppose $Fr(A)$ is a strongly nowhere dense set in X. Now $Fr(iA) = ciA \cap c(X - iA) = ciA \cap c(X - (X - c(X - A)))$. This implies $Fr(iA) = ciA \cap c(c(X - A)) = ciA \cap c(X - A)$ which implies $Fr(iA) \subset cA \cap c(X - A) \subset Fr(A)$. Thus, $Fr(iA) \subset Fr(A)$. Since subset of a strongly nowhere dense set is strongly nowhere dense, $Fr(iA)$ is a strongly nowhere dense set in X. Now $Fr(cA) = ccA \cap c(X - cA)$. This implies $Fr(cA) \subset cA \cap c(X - A)$ which implies $Fr(cA) \subset Fr(A)$ and hence $Fr(cA)$ is strongly nowhere dense set X.
(c) Suppose A is a strongly nowhere dense set in X. Since $A \cap B \subset A$ and subset of a strongly nowhere dense set is strongly nowhere dense, $Fr(A \cap B)$ is a strongly nowhere dense set in X, by (a). □

Example 3.8 shows the reverse implication of (a) in Lemma 3.7 is not necessary.

Example 3.8. *Consider the GTS (X,μ) where $X = \{a,b,c,d,e\}$and $\mu = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$. Let $B = \{a,c,d\} \subset X$. Then $Fr(B) = \{b,d,e\}$ and so $Fr(B)$ is a strongly nowhere dense set in X. But B is not strongly nowhere dense set in X.*

Proposition 3.9. *Let (X,μ) be a strongly nodec space. If A is a non-empty strongly nowhere dense set in X, then $Fr(A) \subset A$ and hence $Fr(A)$ is a strongly nowhere dense set in X.*

Proof. Suppose A is a non-empty strongly nowhere dense set in X. By hypothesis, A is a closed set in X. Now $Fr(A) = cA \cap c(X - A) = A \cap c(X - A)$. Therefore, $Fr(A) \subset A$. By Lemma 3.7 (a) and hypothesis, $Fr(A)$ is a strongly nowhere dense set in X. □

Proposition 3.10. *Let (X,μ) be a GTS. If $Fr(A)$ is strongly nowhere dense $\Rightarrow A$ is closed for all $A \subset X$, then X is a strongly nodec space.*

Proof. Let A be a non-empty strongly nowhere dense set in X. Then by Lemma 3.7 (a), $Fr(A)$ is a strongly nowhere dense set in X. By hypothesis, A is a closed set in X. Hence X is a strongly nodec space. □

Every generalized submaximal space is a strongly nodec space. This implication is not reversible as shown in the following Example 3.11.

Example 3.11. Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a,b\},\{a,d\},\{b,c\},\{a,b,d\},\{a,b,c\},X\}$. Here, every μ-strongly nowhere dense set is a μ-closed set in X. Therefore, X is a strongly nodec space. Let $A=\{a,c,d\}$. Then A is μ-dense in X but not μ-open in X. Thus, X is not a generalized submaximal space.

Theorem 3.12 gives the necessary condition for a strongly nodec space to be a generalized submaximal space and Example 3.13 shows that frontier of a dense set is strongly nowhere dense set is necessary.

Theorem 3.12. *Let (X,μ) be a sGTS. If every frontier of a dense subset of X is a strongly nowhere dense set in X and X is a strongly nodec space, then X is a generalized submaximal space.*

Proof. Suppose X is a strongly nodec space. Let A be a μ-dense subset of X. By hypothesis, $cA-iA$ is a strongly nowhere dense set in X. Then $X-iA$ is a strongly nowhere dense set in X, since $cA=X$ and so $X-A$ is a strongly nowhere dense set in X, since subset of a μ-strongly nowhere dense set is μ-strongly nowhere dense. Suppose $X-A=\emptyset$. Then $X-A$ is a closed set in X, that is $c(\emptyset)=\emptyset$, since X is a sGTS. Therefore, A is a μ-open set in X. Suppose $X-A\neq\emptyset$. Since X is a strongly nodec space, $X-A$ is a closed set in X. Therefore, A is a μ-open set in X. Hence X is a generalized submaximal space. □

Example 3.13. Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a,b\},\{a,c\},\{a,b,c\},X\}$. Here, every μ-strongly nowhere dense set is a μ-closed set in X. Therefore, X is a strongly nodec space. Let $A=\{a,c,d\}$. Then A is a μ-dense subset of X. But $Fr(A)=\{b,d\}$ is not a strongly nowhere dense set. For, let $U=\{a,b\}\in\tilde{\mu}$. Then there is no $V\in\tilde{\mu}$ such that $V\subset U$ and $V\cap Fr(A)$. Let $B=\{a,d\}$. Then B is μ-dense in X but not μ-open in X. Thus, X is not a generalized submaximal space.

Next Example 3.14 shows the existence of a non-generalized submaximal space satisfying the necessary condition in Theorem 3.12 which is not a strongly nodec space.

Example 3.14. Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}$. Here, $A=\{c\}$ is a μ-strongly nowhere dense set in X but not μ-closed in X. Thus, X is not a strongly nodec space. Clearly, every frontier of A is a strongly nowhere dense set where A is a dense subset of X. Let $B=\{a,b,d\}$. Then B is μ-dense in X but not μ-open in X. Thus, X is not a generalized submaximal space.

Theorem 3.15 characterizes strongly nodec space in strong generalized topological space and Theorem 3.16 give one property of s-meager, s-residual set in strongly nodec space, the essay proof is omitted.

Theorem 3.15. *Let (X,μ) be a sGTS and $A\subseteq X$. If frontier of A is a strongly nowhere dense set, then the following are equivalent.*
(a) X is a strongly nodec space.
(b) For each $A\subseteq X, cA-A\subseteq cicA$.
(c) For each $A\subseteq X, cA=A\cup cicA$.
(d) For each $A\subseteq X, iA=A\cap iciA$.

Proof. (a) ⇒ (b) Suppose X is a strongly nodec space. Let $A\subseteq X$. By hypothesis, frontier of A is a strongly nowhere dense set and so $A-icA$ is strongly nowhere dense. Suppose $A-icA=\emptyset$. Since (X,μ) is a sGTS, $c(\emptyset)=\emptyset$ and so $A-icA$ is a closed set in X. Then $d(A-icA)=\emptyset$ where the notation $d(A-icA)$ is the derived set of $A-icA\subset X$. Now $d(icA)\subseteq cicA$ and so $d(A)-cicA\subseteq d(A)-d(icA)\subseteq d(A-icA)=\emptyset$. Thus, $d(A)\subseteq cicA$. Therefore, $cA-A\subseteq cicA$. Suppose $A-icA\neq\emptyset$. By (a), $A-icA$ is a closed set and so $d(A-icA)\subset A-icA$. Let $x\in A-icA$. Since $A-icA$ is strongly nowhere dense, $i(A-icA)=\emptyset$, by Lemma 2.5 and so $i(A-icA-\{x\})=\emptyset$. Take $B=A-icA$. Then B is a codense set and so $B-\{x\}$ is a codense set in X. By hypothesis and Theorem 3.12, X is a generalized submaximal space. Therefore, $B-\{x\}$ is a closed set in X. Hence $\{x\}\cup(X-B)$ is a non-empty open set in X. Let $U_x=\{x\}\cup(X-B)$. Thus, there is a neighbourhood U_x of x such that $U_x\cap(B-\{x\})=\emptyset$ and so $x\notin d(B)$. Therefore, $d(A-icA)=\emptyset$. Now $d(icA)\subseteq cicA$ and so $d(A)-cicA\subseteq d(A)-d(icA)\subseteq d(A-icA)=\emptyset$. Thus, $d(A)\subseteq cicA$. Therefore, $cA-A\subseteq cicA$.
(b) ⇒ (c) Let $A\subseteq X$. By (b), $cA\subseteq A\cup cicA$. Now $cicA\subseteq cA$. This implies $A\cup cicA\subseteq A\cup cA=cA$ which implies $A\cup cicA\subseteq cA$. Therefore, $cA=A\cup cicA$.
(c) ⇔ (d) it is obvious.
(c) ⇒ (a) Let $\emptyset\neq A\subset X$ be a strongly nowhere dense set. Then A is a nowhere dense set in X, and so $icA=\emptyset$. By (c) and hypothesis, $cA=A\cup\emptyset=A$. Thus, A is a closed set in X. Therefore, X is a strongly nodec space. □

Theorem 3.16. *Let (X,μ) be a GTS. If X is a strongly nodec space, then the following hold.*
(a) Every s-meager set is a F_σ-set.
(b) Every s-residual set is a G_δ-set.

In GTS, a subspace of a strongly nodec space need not be a strongly nodec space even if the subspace is either closed or dense-open as shown by the following Example 3.17.

Example 3.17. (a) Consider the GTS (X,μ) where $X=\{a,b,c,d,e\}$ and $\mu=\{\emptyset,\{a\},\{a,b\},\{a,d\},\{a,b,d\},\{b,d,e\},\{a,b,d,e\}\}$. Here, every μ-strongly nowhere dense set is a μ-closed set in X. Therefore, X is a strongly nodec space. Let $Y=\{b,c,d,e\}$ be a closed subset of X. Then $\mu_Y=\{\emptyset,\{b\},\{d\},\{b,d\},\{b,d,e\}\}$. Let $A=\{e\}\subset Y$. Then A is a μ_Y-strongly nowhere dense set in Y. But A is not a μ_Y-closed set. Thus, Y is not a strongly nodec space.
(b) Consider the GTS (X,μ) where $X=\{a,b,c,d,e,f\}$ and $\mu=\{\emptyset,\{a,b\},\{b,c\},\{d,f\},\{a,b,c\},\{c,d,f\},\{a,b,d,f\},\{b,c,d,f\},\{a,b,c,d,f\},\{b,c,d,e,f\},\{a,b,c,d,e\},X\}$. Here, every μ-strongly nowhere dense set is a μ-closed set in X. Therefore, X is a strongly nodec space. Let $Y=\{b,c,d,e,f\}$ be a dense-open subspace of X. Then $\mu_Y=\{\emptyset,\{b\},\{b,c\},\{d,f\},\{b,d,f\},\{c,d,f\},\{b,c,d,f\},\{b,c,d,e\},Y\}$. Let $A=\{c\}\subset Y$. Then A is a μ_Y-strongly nowhere dense set in Y. But A is not a μ_Y-closed set. Thus, Y is not a strongly nodec space.

Next one is the definition for a subspace of a space is strongly nodec with respect to the space and Example 3.19 shows the existence of this space.

Definition 3.18. *Let (X,μ) be a GTS. A subspace Y of X is said to be strongly nodec with respect to X if every non-empty μ_Y-strongly nowhere dense set is a μ-closed set. Y is said to be a strongly nodec space if Y is strongly nodec as a subspace.*

Example 3.19. Consider the GTS (X,μ) where $X=\{a,b,c,d,e\}$ and $\mu=\{\emptyset,\{a,d,e\},\{b,d,e\},\{a,b,d,e\},\{a,c,d,e\},X\}$. Let $Y=\{a,b,c,e\}$. Then $\mu_Y=\{\emptyset,\{a,e\},\{b,e\},\{a,b,e\},\{a,c,e\},Y\}$. Here, every μ_Y-strongly nowhere dense set is μ-closed. Thus, Y is a strongly nodec space with respect to X.

In GTS, every subspace strongly nodec with respect to X is a strongly nodec space as a subspace. This implication is not reversible as shown in Example 3.20.

Example 3.20. Consider the GTS (X,μ) where $X=\{a,b,c,d,e,f\}$ and $\mu=\{\emptyset,\{a,b,c\},\{b,c,d\},\{a,b,c,d\},\{a,b,c,e\},\{b,c,e,f\},\{a,b,c,d,e\},\{a,b,c,e,f\},X\}$. Let $Y=\{a,b,c,e\}$. Then $\mu_Y=\{\emptyset,\{b,c\},\{a,b,c\},\{b,c,e\},Y\}$. Here, every μ_Y-strongly nowhere dense set is a μ_Y-closed set in Y. Therefore, Y is a strongly nodec space. But Y is not strongly nodec with respect to X. For, let $A=\{a\}$. Then A is μ_Y-strongly nowhere dense set but not μ-closed.

Theorem 3.21. *Let (X,μ) be a GTS and Y be a dense subspace of X. If Y is a strongly nodec with respect to X, then X is a strongly nodec space.*

Proof. Suppose X is a strongly nodec space. Let A be a non-empty μ-strongly nowhere dense set in X. Suppose $A\cap Y=\emptyset$. Then A is a non-empty μ_Y-strongly nowhere dense set. By hypothesis, A is a μ-closed set. Suppose $A\cap Y\neq\emptyset$. Let $U\in\tilde{\mu}_Y$. Then $U=U_1\cap Y$ where $U_1\in\tilde{\mu}$. Since A is μ-strongly nowhere dense set, there exists $V_1\in\tilde{\mu}$ such that $V_1\subset U_1$ and $V_1\cap A=\emptyset$. This implies that $V_1\cap Y\in\tilde{\mu}_Y$, since Y is a dense subspace of X. Take $V=V_1\cap Y$. Thus, there exists $V\in\tilde{\mu}_Y$ such that $V\subset U$ and $V\cap A=\emptyset$. Therefore, A is a non-empty μ_Y-strongly nowhere dense set. By hypothesis, A is μ-closed. Hence X is a strongly nodec space. □

Theorem 3.22. *Let (X,μ) be a generalized submaximal space. Then every subset of X is a strongly nodec with respect to X and hence a strongly nodec space.*

Proof. Let Y be a subset of X and A be a non-empty μ_Y-strongly nowhere dense subset of Y. Then A is a μ_Y-codense set and so $c_{\mu_Y}(Y-A)=Y$, by Lemma 2.5. Now $c_{\mu_Y}(Y-A)=c_\mu(Y-A)\cap Y$, by Lemma 2.3. This implies $Y=c_\mu(Y-A)\cap Y$ which implies $Y\subseteq c_\mu(Y-A)\subseteq c_\mu(X-A)$. Thus, $Y\subseteq X-i_\mu A$. Therefore, $i_\mu A=\emptyset$ and so A is a μ-codense set in X. By hypothesis, A is a μ-closed set. Hence Y is a strongly nodec with respect to X. By Lemma 2.3, $c_Y A=cA\cap Y=A\cap Y=A$, since $A\subset Y$ is a μ-closed set. Therefore, A is a μ_Y-closed set. Hence Y is a strongly nodec space. □

Theorem 3.23. *Let (X,μ) be a strongly nodec space. Then every non-empty μ-strongly nowhere dense subspace of X is a strongly nodec with respect to X and hence a strongly nodec space.*

Proof. Let Y be a non-empty μ-strongly nowhere dense subspace of X. Let A be a non-empty μ_Y-strongly nowhere dense subset of Y. Then A is a non-empty μ-strongly nowhere dense set, since subset of a μ-strongly nowhere dense set is a μ-strongly nowhere dense set. By hypothesis, A is a μ-closed set. Therefore, Y is a strongly nodec with respect to X. By Lemma 2.3, $c_Y A=cA\cap Y=A\cap Y=A$, since $A\subset Y$ is a μ-closed set. Therefore, A is a μ_Y-closed set. Hence Y is a strongly nodec space. □

Theorem 3.24. *Let (X,μ) be a strongly nodec space. Then every non-empty frontier of a μ-strongly nowhere dense subspace of X is a strongly nodec with respect to X and hence a strongly nodec space.*

Proof. Let A be a non-empty μ-strongly nowhere dense set in X. By Lemma 3.7, $Fr(A)$ is a non-empty μ-strongly nowhere dense set in X. Then by Theorem 3.23, $Fr(A)$ is a strongly nodec with respect to X and hence a strongly nodec space. Hence every non-empty frontier of a μ-strongly nowhere dense subspace of X is a strongly nodec with respect to X and hence a strongly nodec space. □

Lemma 3.25. *Let $(X,\mu_X),(Y,\mu_Y)$ be a two GTSs. Then the following hold.*
(a) If A and B are strongly nowhere dense sets in X,Y respectively, then $A\times B$ is strongly nowhere dense in $X\times Y$.
(b) If $C\times D$ is strongly nowhere dense in $X\times Y$, then C or D or C and D is strongly nowhere dense set in X or Y respectively.

Proof. (a) Suppose A and B are strongly nowhere dense sets in X,Y respectively. Let $U_1\times U_2\in\tilde{\mu}_{X\times Y}$. Then $U_1\in\tilde{\mu}_X$ and $U_2\in\tilde{\mu}_Y$. By hypothesis, there exists $V_1\in\tilde{\mu}_X,V_2\in\tilde{\mu}_Y$ such that $V_1\subset U_1,V_2\subset U_2$ and $V_1\cap A=\emptyset,V_2\cap B=\emptyset$. Thus, there exists $V_1\times V_2\in\tilde{\mu}_{X\times Y}$ such that $V_1\times V_2\subset U_1\times U_2$ and $V_1\times V_2\cap A\times B=\emptyset$. Therefore, $A\times B$ is strongly nowhere dense in $X\times Y$.
(b) Suppose $C\times D$ is strongly nowhere dense in $X\times Y$. Let $G_1\in\tilde{\mu}_X,G_2\in\tilde{\mu}_Y$. Then $G_1\times G_2\in\tilde{\mu}_{X\times Y}$. By hypothesis, there exists $H_1\times H_2\in\tilde{\mu}_{X\times Y}$ such that $H_1\times H_2\subset G_1\times G_2$ and $H_1\times H_2\cap C\times D=\emptyset$. Since $H_1\times H_2\neq\emptyset$ and $H_1\times H_2\subset G_1\times G_2$, $H_1\subset G_1$ and $H_2\subset G_2$. Now $H_1\times H_2\cap C\times D=\emptyset, H_1\cap C\times H_2\cap D=\emptyset$. This implies $H_1\cap C=\emptyset$ or $H_2\cap D$ or $H_1\cap C=\emptyset$ and $H_2\cap D=\emptyset$. Thus, C is strongly nowhere dense in X or D is strongly nowhere dense in Y or C and D are strongly nowhere dense sets in X,Y respectively. □

Theorem 3.26. *Product of two GTS is strongly nodec, then each one is strongly nodec.*

Proof. Let $(X,\mu_X),(Y,\mu_Y)$ be a two GTSs. Suppose $X\times Y$ is a strongly nodec space. Let A and B are non-empty strongly nowhere dense sets in X,Y respectively. Then by Lemma 3.25, $A\times B$ is a non-empty strongly nowhere dense set in $X\times Y$. By hypothesis, $A\times B$ is a closed set in $X\times Y$. This implies A is a closed set in X and B is a closed set in Y. Hence X and Y are strongly nodec space. □

4. On T_0-strongly nodec spaces

In this section, we define T_0-strongly nodec space and give the example for the existence of this space in generalized topological spaces. Further, we discuss the properties of T_0-strongly nodec space in generalized topological spaces by using a quotient maps. Also, we introduce and give some results for T_0-generalized submaximal space in generalized topological space.

Let (X,μ) be a GTS. We define the binary relation $\sim$ on X by $x\sim y$ if and only if $c\{x\}=c\{y\}$. Then $\sim$ is an equivalence relation on X and the resulting quotient space $T_0(X)=X/\sim$ is the T_0-*reflection of* X and the generalized quotient topology on $T_0(X)$ is defined to be $\mu_q=\{G\subset T_0(X): f^{-1}(G)\in\mu\}$ where q is a *canonical or quotient map* from X into $T_0(X)$ by setting $x\in X$ to its equivalence class $[x]$ in $T_0(X)$. Then the pair $(T_0(X),\mu_q)$ is called the *generalized quotient space of* X.

Let (X,μ) and (Y,λ) be two generalized topological spaces. A function $f:X\to Y$ is called (μ,λ)-*continuous* if $f^{-1}(V)\in\mu$ for each $V\in\lambda$ [2]. A function $f:X\to Y$ is called (μ,λ)-*open* [2] if $f(V)\in\lambda$ for each $V\in\mu$. A function $f:X\to Y$ is called (μ,λ)-*closed* if $f(U)$ is a λ-closed set for each U is a μ-closed set.

A (μ,λ)-continuous map $f:(X,\mu)\to(Y,\lambda)$ is said to be a *quasi-homeomorphism* if $U\to f^{-1}(U)$ (resp. $C\to f^{-1}(C)$) defines a bijection $O(Y)\to O(X)$ (resp. $F(Y)\to F(X)$) where $O(X)$ (resp. $F(X)$) is the collection of all μ-open (resp. μ-closed) sets of X.

Equivalently, (μ,λ)-continuous map $f:X\to Y$ is a *quasi-homeomorphism* if for each μ-open subset U of X, there exists a unique λ-open subset V of Y such that $U=f^{-1}(V)$ (equivalently, for each μ-closed subset F of X, there exists a unique λ-closed subset G of Y such that $F=f^{-1}(G)$).

Lemma 4.1. [2, Lemma 7.3] Let (X,μ) and (Y,λ) be two generalized topological spaces. A mapping $f:(X,\mu)\to(Y,\lambda)$ is (μ,λ)-open if and only if $f^{-1}(cB)\subset c(f^{-1}(B))$ for any $B\subset Y$.

Proposition 4.2. *Let (X,μ) and (Y,λ) be two generalized topological spaces. If $f:X\to Y$ is a surjective, quasi-homeomorphism map, then f is a (μ,λ)-open map.*

Proof. Let A be a μ-open set in X. Since f is quasi-homeomorphism, there exists a unique λ-open subset V of Y such that $A=f^{-1}(V)$. Now $f(A)=f(f^{-1}(V))=V$, since f is a surjective map. Therefore, $f(A)$ is a λ-open set in Y. Hence f is a (μ,λ)-open map. □

Example 4.3. *Consider two GTSs (X,μ) and (Y,λ) where $X=\{a,b,c,d\}$, $\mu=\{\emptyset,\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$ and $Y=\{a,b,c,d,e\}, \lambda=\{\emptyset,\{a,b\},\{a, b,e\},\{a,c,e\},\{b,c,e\},\{a,b,c,e\}\}$. Define a map $f:X\to Y$ by $f(a)=a, f(b)=b, f(c)=c, f(d)=d$. Then f is a quasi-homeomorphism but not a surjective map. Let $U=\{a,c\}$. Now $f(U)=\{a,c\}$. But $f(U)$ is not a λ-open set. Thus, f is not a (μ,λ)-open map.*

Notations 4.4. *Let (X,μ) be a GTS, $a\in X$ and $A\subseteq X$. We use the following notations:*
(1) $d_0(a)=\{x\in X: c\{x\}=c\{a\}\}$.
(2) $d_0(A)=\bigcup[d_0(a): a\in A]$.

Example 4.5 shows the existence of $d_0(A)$ in generalized topological space (X,μ) where $A\subset X$ and the next Lemma 4.6 give some properties of $d_0(A)$ and the canonical surjective map in generalized topological space.

Example 4.5. (a). Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a\},\{a,c\},\{b,c\},\{a,b,c\}\}$. Let $A=\{a,c,d\}$. Now $d_0(a)=\{a\}, d_0(c)=\{c\}$ and $d_0(d)=\{d\}$. Therefore, $d_0(A)=A$.
(b). Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a\},\{b,c\},\{a, b,c\}\}$. Let $A=\{a,c,d\}$. Now $d_0(a)=\{a\}, d_0(c)=\{b,c\}$ and $d_0(d)=\{d\}$. Therefore, $d_0(A)=X$.

Lemma 4.6. *Let (X,μ) be a GTS, $A\subset X$ and $q:(X,\mu)\to(T_0(X),\mu_q)$ be a canonical surjective map. Then the following hold.*
(a) The map q is a quasi-homeomorphism.
(b) The map q is (μ,μ_q)-open, (μ,μ_q)-closed map.
(c) $A\subseteq d_0(A)\subseteq cA$ and consequently $c(d_0(A))=cA$.
(d) If A is a closed set, then $d_0(A)=A$.
(e) $d_0(A)=q^{-1}(q(A))$.
(f) If $\{A_n\}_{n\in\mathbb{N}}$ is a collection of subsets of X, then $d_0(\bigcup_{n\in\mathbb{N}} A_n)=\bigcup_{n\in\mathbb{N}} d_0(A_n)$.

Proof. (a) Define a map $f:O(T_0(X))\to O(X)$ by $f(U)=q^{-1}(U)$. It is enough to prove, f is bijective between μ_q-open sets and μ-open sets. Let $U_1,U_2\in\mu_q$ such that $U_1\neq U_2$. Suppose $f(U_1)=f(U_2)$. Then $q^{-1}(U_1)=q^{-1}(U_2)$ and so $q^{-1}(U_1-U_2)=\emptyset$. This implies $U_1-U_2=\emptyset$. Therefore, $U_1=U_2$, which is not possible. Therefore, f is injective between μ_q-open sets and μ-open sets. Let U be a μ-open set in X. Then $U=q^{-1}(V)$ where V is a μ_q-open set in $T_0(X)$. Now $f(V)=q^{-1}(V)=U$. Therefore, f is surjective between μ_q-open sets and μ-open sets. Hence q is a quasi-homeomorphism.
(b) By Proposition 4.2, q is a (μ,μ_q)-open map. Similar considerations in Proposition 4.2, we get every canonical surjective map q is a (μ,μ_q)-closed map.
(c) Obviously, $A\subseteq d_0(A)$. Let $s\in d_0(A)$. Then $s\in d_0(a)$ and so $c(\{s\})=c(\{a\})$ for some $a\in A$. This implies $s\in c(\{a\})\subseteq cA$ which implies $s\in cA$. Therefore, $d_0(A)\subseteq cA$. Thus, $A\subseteq d_0(A)\subseteq cA$ and hence $c(d_0(A))=cA$.
(d) follows from (c).
(e) follows from the definition of $d_0(A)$ and a canonical map q.
(f) Let $t\in d_0(\bigcup_{n\in\mathbb{N}} A_n)$. Then $t\in\cup d_0(a)$ for all $a\in\bigcup_{n\in\mathbb{N}} A_n$. This implies $c(\{t\})=c(\{a\})$ for some $a\in A_1$ or ...... or $a\in A_n$ or .... which implies $t\in\bigcup_{n\in\mathbb{N}}(d_0(A_n))$. Therefore, $d_0(\bigcup_{n\in\mathbb{N}} A_n)\subset\bigcup_{n\in\mathbb{N}}(d_0(A_n))$. Conversely, let $s\in\bigcup_{n\in\mathbb{N}}(d_0(A_n))$. Then $s\in d_0(A_1)$ or $s\in d_0(A_2)$ or .... or $s\in d_0(A_n)$ or .... and so $c(\{s\})=c(\{a\})$ for some $a\in A_1$ or $c(\{s\})=c(\{b\})$ for some $b\in A_2$ or .... or $c(\{s\})=c(\{c\})$ for some $c\in A_n$ or .... . Therefore, $s\in d_0(\bigcup_{n\in\mathbb{N}} A_n)$. Thus, $\bigcup_{n\in\mathbb{N}}(d_0(A_n))\subset d_0(\bigcup_{n\in\mathbb{N}} A_n)$. Hence $d_0(\bigcup_{n\in\mathbb{N}} A_n)=\bigcup_{n\in\mathbb{N}} d_0(A_n)$. □

Definition 4.7. *Let (X,μ) be a GTS. X is called a T_0-strongly nodec space if its T_0-reflection is a strongly nodec space, that is $T_0(X)$ is a strongly nodec space.*

Example 4.8 shows the existence of a T_0-strongly nodec space and Theorem 4.9 is a characterization theorem for a T_0-strongly nodec space in generalized topological space.

Example 4.8. Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a\},\{b,c\},\{a,b\},\{a,b,c\},\{a,b,d\},\{a,c,d\},X\}$. Define a map $q:X\to T_0(X)$ by $x\in X$ to its equivalence class $[x]$ in $T_0(X)$, where $T_0(X)$ is the T_0-reflection of X. This implies $\mu_q=\{\emptyset,\{a\}\}$. Here, every strongly nowhere dense set is a closed set in $T_0(X)$. Therefore, $T_0(X)$ is a strongly nodec space.

Theorem 4.9. *Let (X,μ) be a GTS and $q:(X,\mu)\to(T_0(X),\mu_q)$ be a canonical surjective map. Then the following are equivalent.*
(a) X is a T_0-strongly nodec space.
(b) For any non-empty strongly nowhere dense subset A of X, $d_0(A)$ is closed.

Proof. (a) $\Rightarrow$ (b). Suppose X is a T_0-strongly nodec space. Let A be a non-empty strongly nowhere dense subset of X. Then cA is a non-empty strongly nowhere dense set in X. Suppose $c(q(A))$ is not a strongly nowhere dense set in $T_0(X)$. Then by Lemma 3.3, there is a non-empty open set U of $T_0(X)$ such that $U \subset c(q(A))$. Since q is a (μ, μ_q)-continuous map, $q^{-1}(U)$ is a non-empty open set and $q^{-1}(U) \subseteq q^{-1}(c(q(A)))$. Since q is a (μ, μ_q)-closed map, $c(q(A)) \subseteq c(q(cA)) = q(cA)$. Then $q^{-1}(U) \subseteq q^{-1}(q(cA)) = d_0(cA) = cA$, which contradict the fact that cA is a strongly nowhere dense subset of X. Therefore, $c(q(A))$ is a non-empty strongly nowhere dense set in $T_0(X)$. Hence $q(A)$ is a non-empty strongly nowhere dense set in $T_0(X)$, since subset of a strongly nowhere dense set is strongly nowhere dense. Since $T_0(X)$ is a strongly nodec space, $q(A)$ is a closed set in $T_0(X)$ and so $q^{-1}(q(A))$ is a closed set in X, since q is a (μ, μ_q)-continuous map. By Lemma 4.6(e), $d_0(A) = q^{-1}(q(A))$ and hence $d_0(A)$ is a closed set in X.
(b) $\Rightarrow$ (a). Let B be a non-empty strongly nowhere dense subset of $T_0(X)$ and $A = q^{-1}(B)$. Then $q(A) = q(q^{-1}(B)) = B$, since q is surjective. Thus, $q(A)$ is a strongly nowhere dense set in $T_0(X)$. Since q is a surjective map, $q^{-1}(q(A)) = q^{-1}(q(q^{-1}(B))) = A$. By Lemma 4.6(e), $d_0(A) = A$. Suppose A is not a strongly nowhere dense set in X. Then there exists $V \in \tilde{\mu}$ such that $V \subset A$. Since q is an (μ, λ)-open map, $q(V)$ is a non-empty open set in $T_0(X)$. This implies $q(V) \subset q(A)$. By Lemma 3.3, $q(A)$ is not a strongly nowhere dense set in X, which is not possible. Therefore, A is a non-empty strongly nowhere dense subset of X. By (b), A is a closed set in X, since $d_0(A) = A$. Since q is a (μ, μ_q)-closed map, $q(A)$ is closed in X. Hence B is a closed set in $T_0(X)$, since $q(A) = B$. Hence X is a T_0-strongly nodec space. ☐

The following Corollary 4.10 and Corollary 4.11 follows from Lemma 4.6 and Theorem 4.9.

Corollary 4.10. *Let (X, μ) be a sGTS and $q : (X, \mu) \to (T_0(X), \mu_q)$ be a canonical surjective map. Then X is a T_0-strongly nodec space if any one of the following hold.*
(a) For every $A \subseteq X$, if $cicA \subseteq d_0(A)$, then $d_0(A) = cA$.
(b) For every $A \subseteq X$, $cA - d_0(A) \subseteq cicA$.
(c) For every $A \subseteq X$, $cA = d_0(A) \cup cicA$.

Corollary 4.11. *Let (X, μ) be a GTS and $q : (X, \mu) \to (T_0(X), \mu_q)$ be a canonical surjective map. If X is a T_0-strongly nodec space, then the following hold.*
(a) If $A \subset X$ is a s-meager set, then $d_0(A)$ is a F_σ-set.
(b) If $A \subset X$ is a s-residual set, then $d_0(A)$ is a G_δ-set.

Corollary 4.12. *Let (X, μ) be a GTS, $A \subset X$ and $q : (X, \mu) \to (T_0(X), \mu_q)$ be a canonical surjective map. If A is a strongly nowhere dense set in X, then $Fr(q(A))$ is a strongly nowhere dense set in $T_0(X)$.*

Proof. Suppose A is a strongly nowhere dense set in X. Then cA is a non-empty strongly nowhere dense set in X. Suppose $q(A)$ is not a strongly nowhere dense set in $T_0(X)$. By similar considerations in Theorem 4.9 (a) $\Rightarrow$ (b), we get a contradiction. Hence $q(A)$ is a non-empty strongly nowhere dense set in $T_0(X)$. ☐

Lemma 4.13 shows inverse of a canonical surjective map preserve closure and interior of a subset of a codomain set.

Lemma 4.13. *Let (X, μ) be a GTS and $q : (X, \mu) \to (T_0(X), \mu_q)$ be a canonical surjective map. Then the following hold.*
(a) For every subset A of $T_0(X)$, $q^{-1}(cA) = c(q^{-1}(A))$.
(b) For every subset A of $T_0(X)$, $q^{-1}(iA) = i(q^{-1}(A))$.
(c) For every subset A of $T_0(X)$, $q^{-1}(cicA) = cic(q^{-1}(A))$.

Proof. (a) Let $A \subset T_0(X)$. By Lemma 4.6(b), q is an (μ, μ_q)-open map. Then $q^{-1}(cA) \subset c(q^{-1}(A))$ where $A \subset T_0(X)$, by Lemma 4.1. Let $a \in c(q^{-1}(A))$. Then $U_a \cap q^{-1}(A) \neq \emptyset$ for every $U_a \in \mu(a)$. Since q is a quasi-homeomorphism, by Lemma 4.6(a), there exists a unique open set $V_{q(a)}$ in $T_0(X)$ such that $U_a = q^{-1}(V_{q(a)})$. This implies $q^{-1}(V_{q(a)}) \cap q^{-1}(A) \neq \emptyset$ which implies $q^{-1}(V_{q(a)} \cap A) \neq \emptyset$. Thus, $V_{q(a)} \cap A \neq \emptyset$. Therefore, $q(a) \in cA$ and so $a \in q^{-1}(cA)$. Thus, $c(q^{-1}(A)) \subset q^{-1}(cA)$. Hence $q^{-1}(cA) = c(q^{-1}(A))$.
(b) Since q is a (μ, μ_q)-continuous map, q^{-1} is a (μ, μ_q)-open map. Then $q^{-1}(iB) = i(q^{-1}(iB)) \subset i(q^{-1}(B))$ where $B \subset T_0(X)$. Therefore, $q^{-1}(iA) \subset i(q^{-1}(A))$. Let $b \in i(q^{-1}(A))$. Then there exists $U_b \in \mu(b)$ such that $U_b \subset q^{-1}(A)$. Since q is a quasi-homeomorphism, by Lemma 4.6(a), there exists a unique open set $V_{q(b)}$ in $T_0(X)$ such that $U_b = q^{-1}(V_{q(b)})$. This implies $q^{-1}(V_{q(b)}) \subset q^{-1}(A)$ which implies $V_{q(b)} \subset q(q^{-1}(A)) = A$, since q is a surjective map. Thus, $q(b) \in iA$ and so $b \in q^{-1}(iA)$. Therefore, $i(q^{-1}(A)) \subset q^{-1}(iA)$. Hence $q^{-1}(iA) = i(q^{-1}(A))$.
(c) Now $q^{-1}(cicA) = c(q^{-1}(icA))$, by (a). Then $q^{-1}(cicA) = c(q^{-1}(icA)) = ci(q^{-1}(cA))$, by (b) and so $q^{-1}(cicA) = cic(q^{-1}(A))$, by (a). ☐

Lemma 4.14. *Let (X, μ) be a GTS, $A \subset X$ and $q : X \to (T_0(X), \mu_q)$ be a canonical bijective map. If $Fr(A)$ is a strongly nowhere dense set in X, then $Fr(q(A))$ is a strongly nowhere dense set in $T_0(X)$.*

Proof. Let G be a non-empty μ_q-open set in $T_0(X)$. Since q is a (μ, μ_q)-continuous map, $q^{-1}(G)$ is a non-empty μ-open set in X. Since $Fr(A)$ is a strongly nowhere dense set in X, there exists $V \in \tilde{\mu}$ such that $V \subset q^{-1}(G)$ and $V \cap Fr(A) = \emptyset$. Since q is an (μ, μ_q)-open map, $q(V)$ is a non-empty μ_q-open set. Thus, there exists a non-empty μ_q-open set $q(V)$ such that $q(V) \subset q(q^{-1}(G)) \subseteq G$ and $q(V \cap Fr(A)) = \emptyset$. Suppose there is an element $t \in q(V) \cap Fr(q(A))$. Then $t \in q(V)$ and $t \in Fr(q(A))$. This implies $q^{-1}(t) \in V$, since q is injective. Now, $t \in c(q(A)) \cap c(T_0(X) - q(A))$. Consider, $t \in c(q(A))$. Since q is a (μ, μ_q)-closed map, $q(cB) = c(q(cB))$, where $B \subset X$. Now $B \subset cB$. This implies $c(q(B)) \subset c(q(cB)) = q(cB)$. Thus, $c(q(B)) \subset q(cB)$. Therefore, $t \in q(c(A))$. Then $q^{-1}(t) \in cA$, since q is injective. Now, $t \in c(T_0(X) - q(A)) = c(q(X) - q(A))$, since q is a surjective map. Then $t \in c(q(X - A))$, since q is injective. Since q is a (μ, μ_q)-closed map and by same process, we get $t \in q(c(X - A))$. Then $q^{-1}(t) \in c(X - A)$, since q is injective. Therefore, $q^{-1}(t) \in cA \cap c(X - A) = Fr(A)$. Thus, $q^{-1}(t) \in V \cap Fr(A)$, which is not possible. Therefore, $q(V) \cap Fr(q(A)) = \emptyset$. Hence $Fr(q(A))$ is a strongly nowhere dense set in $T_0(X)$. □

Next Theorem 4.15 is another charecderaization theorem for a T_0-strongly nodec space in genearlized topological space.

Theorem 4.15. *Let (X, μ) be a sGTS, $q : X \to T_0(X)$ be a canonical bijective map and $A \subset X$. If frontier of A is a strongly nowhere dense set, then the following are equivalent.*
(a) X is T_0-strongly nodec space.
(b) $cA - d_0(A) \subseteq cic(A)$.
(c) $cA = d_0(A) \cup cic(A)$.

Proof. (a) ⇒ (b) Let $A \subset X$. Since X is T_0-strongly nodec, $c(q(A)) - q(A) \subseteq cic(q(A))$, by Theorem 3.15. Now $q^{-1}(c(q(A)) - q(A)) = q^{-1}(c(q(A))) - q^{-1}(q(A)) = c(q^{-1}(q(A))) - q^{-1}(q(A)) = c(d_0(A)) - d_0(A)$, by Lemma 4.13 and Lemma 4.6(e). By Lemma 4.6(c), $q^{-1}(c(q(A)) - q(A)) = cA - d_0(A)$. This implies $cA - d_0(A) \subseteq q^{-1}(cic(q(A))) = cic(q^{-1}(q(A)))$, by Lemma 4.13 which implies $cA - d_0(A) \subseteq cic(d_0(A)) = cic(A)$, by Lemma 4.6(c) and (e). Thus, $cA - d_0(A) \subseteq cic(A)$.
(b) ⇒ (c) Let A be a subset of X. Then $cic(A) \subseteq cA$ and $d_0(A) \subseteq cA$, by Lemma 4.6(c). Therefore, $d_0(A) \cup cic(A) \subseteq cA$. Conversely, $cA = d_0(A) \cup (cA - d_0(A)) \subseteq d_0(A) \cup cic(A)$, by (b). Thus, $cA = d_0(A) \cup cic(A)$.
(c) ⇒ (a) Let A be a non-empty strongly nowhere dense subset of X. Then A is a nowhere dense set in X and so $ic(A) = \emptyset$. By (c) and hypothesis, $cA = d_0(A)$. Therefore, $d_0(A)$ is a closed set in X. Hence X is a T_0-strongly nodec space, by Theorem 4.9. □

Definition 4.16. *Let (X, μ) be a GTS. X is called a T_0-generalized submaximal space if its T_0-reflection is a generalized submaximal space, that is $T_0(X)$ is a generalized submaximal space.*

Next Theorem 4.17 is the characterization theorem for a T_0-generalized submaximal space in generalized topological space.

Theorem 4.17. *Let (X, μ) be a GTS, $q : X \to T_0(X)$ be a canonical surjective map and $A \subset X$. Then the following are equivalent.*
(a) X is T_0-generalized submaximal.
(b) A is dense in X, then $d_0(A)$ is an open set in X.
(c) $c(d_0(A)) - d_0(A)$ is a closed set in X.

Proof. (a) ⇒ (b) Let $A \subset X$ be a dense set in X. Then $cA = X$ and so $c(d_0(A)) = X$, by Lemma 4.6 (c). By Lemma 4.6 (e), $c(q^{-1}(q(A))) = X$. Since q is a canonical surjective map, $q^{-1}(c(q(A))) = X$, by Lemma 4.13 (a). Then $c(q(A)) = T_0(X)$, since q is surjective. By (a), $q(A)$ is an open set in $T_0(X)$. This implies $q^{-1}(q(A))$ is an open set in X, since q is (μ, μ_q)-continuous. By Lemma 4.6 (e), $d_0(A)$ is an open set in X.
(b) ⇒ (a) Let $A \subset X$ such that $q(A)$ is a dense subset of $T_0(X)$. Then $c(q(A)) = T_0(X)$ and so $q^{-1}(c(q(A))) = X$. By Lemma 4.13, $c(q^{-1}(q(A))) = X$. This implies $c(d_0(A)) = X$ which implies $cA = X$, by Lemma 4.6 (e) and (c). By (b), $d_0(A)$ is an open set in X. Then $q^{-1}(q(A))$ is an open set in X and so $q(q^{-1}(q(A)))$ is an open set in $T_0(X)$, by Lemma 4.6 (e) and q is a (μ, μ_q)-open map. Since q is surjective, $q(A)$ is an open set in $T_0(X)$. Therefore, $T_0(X)$ is a generalized submaximal space. Hence X is a T_0-generalized submaximal space.
(a) ⇒ (c) Let A be a subset of X, then $c(d_0(A)) - d_0(A) = c(q^{-1}(q(A))) - q^{-1}(q(A)) = q^{-1}(c(q(A))) - q^{-1}(q(A)) = q^{-1}(c(q(A)) - q(A))$, by Lemma 4.6 (e) and Lemma 4.13 (a). By Lemma 2.1, $i(c(q(A)) - q(A)) = \emptyset$. Thus, $c(q(A)) - q(A)$ is a co-dense set in $T_0(X)$. Since X is a T_0-generalized submaximal space, then $c(q(A)) - q(A)$ is a closed subset of $T_0(X)$. Then $q^{-1}(c(q(A)) - q(A))$ is a closed subset of X, since q is a (μ, μ_q)-continuous map. Therefore, $c(d_0(A)) - d_0(A)$ is closed in X.
(c) ⇒ (a) Let B be a subset of $T_0(X)$ such that $B = c(q(A)) - q(A)$. Then $iB = \emptyset$, by Lemma 2.1. Let A be a subset of X such that $c(d_0(A)) - d_0(A)$ is closed in X. Then by above process $q^{-1}(c(q(A)) - q(A))$ is a closed subset of X and so $q(q^{-1}(B))$ is a closed subset of $T_0(X)$, since q is (μ, μ_q)-closed map. This implies B is a closed set in $T_0(X)$, since q is a surjective map. Therefore, X is a T_0-generalized submaximal space. □

Theorem 4.18. *Let (X,μ) be a GTS and $q : X \to T_0(X)$ be a canonical surjective map. If X is a generalized submaximal space, then X is a T_0-generalized submaximal space.*

Proof. Let A be a dense subset of X. Then A is an open set in X, by hypothesis. Since q is a (μ,μ_q)-open map, $q(A)$ is an open set in $T_0(X)$. This implies $q^{-1}(q(A))$ is an open set in X, since q is a (μ,μ_q)-continuous map. By Lemma 4.6 (e), $d_0(A)$ is an open set in X. Therefore, X is a T_0-generalized submaximal space, by Theorem 4.17. □

Theorem 4.19. *Let (X,μ) be a GTS and $q : X \to T_0(X)$ be a canonical bijective map. If X is a T_0-generalized submaximal space, then X is a generalized submaximal space.*

Proof. Let A be a dense subset of X. Then $d_0(A)$ is an open set in X, by hypothesis and Theorem 4.17. This implies $q^{-1}(q(A))$ is an open set in X, by Lemma 4.6 (e). Since q is injective, A is an open set in X. Therefore, X is a X generalized submaximal space. □

Next Theorem 4.20 shows every T_0-generalized submaximal space is a T_0-strongly nodec space.

Theorem 4.20. *Let (X,μ) be a GTS and $q : X \to T_0(X)$ be a canonical surjective map. If X is T_0-generalized submaximal space, then X is a T_0-strongly nodec space.*

Proof. Let $q(A)$ be a non-empty strongly nowhere dense set in $T_0(X)$. Then $q(A)$ is a codense set in $T_0(X)$, by Lemma 2.5. Since X is T_0-generalized submaximal space, $q(A)$ is a closed set in $T_0(X)$. Thus, $T_0(X)$ is a strongly nodec space. Therefore, X is a T_0-strongly nodec space. □

5. Strongly nodec, T_0-strongly nodec spaces by functions

In this section, we discuss the properties of images of a strongly nodec, T_0-strongly nodec spaces by a quasi-homeomorphism function.

Lemma 5.1. *Let $(X,\mu),(Y,\lambda)$ be two GTSs and $f : X \to Y$ be a quasi-homeomorphism map. Then the following hold.*
(a) If f is a bijective map and A is strongly nowhere dense in X, then $f(A)$ is strongly nowhere dense in Y.
(b) If B is strongly nowhere dense in Y, then $f^{-1}(B)$ is strongly nowhere dense in X.
(c) If A is of s-II category in X, then $f(A)$ is of s-II category in Y.
(d) If f is a bijective map and B is of s-II category set in Y, then $f^{-1}(B)$ is of a s-II category set in X.

Proof. (a) Suppose A is strongly nowhere dense in X. Let $W \in \tilde{\lambda}$. Since f is (μ,λ)-continuous, $f^{-1}(W) \in \tilde{\mu}$. By hypothesis, there exists $V \in \tilde{\mu}$ such that $V \subset f^{-1}(W)$ and $V \cap A = \emptyset$. Since f is a quasi-homeomorphism, there exists a unique $V_1 \in \lambda$ such that $V = f^{-1}(V_1)$. Thus, $V_1 \in \tilde{\lambda}$ and $f(V) = V_1$, since f is a surjective map. Now $V \subset f^{-1}(W)$ implies $f(V) \subset W$ and so $V_1 \subset W$. Since f is injective, $f(V \cap A) = f(V) \cap f(A) = \emptyset$. Therefore, $V_1 \cap f(A) = \emptyset$. Hence $f(A)$ is strongly nowhere dense in Y.
(b) Suppose B is strongly nowhere dense in Y. Let $W \in \tilde{\mu}$. Since f is a quasi-homeomorphism, there exists a unique $V \in \lambda$ such that $W = f^{-1}(V)$. Since $V \in \tilde{\lambda}$ and by hypothesis, there exists $V_1 \in \tilde{\lambda}$ such that $V_1 \subset V$ and $V_1 \cap B = \emptyset$. Since f is a (μ,λ)-continuous map, there exists $f^{-1}(V_1) \in \tilde{\mu}$ such that $f^{-1}(V_1) \subset W$ and $f^{-1}(V_1) \cap f^{-1}(B) = \emptyset$. Therefore, $f^{-1}(B)$ is strongly nowhere dense in X.
(c) Assume that, A is of a s-II category set in X. Suppose $f(A)$ is of a s-meager set in Y. Then $f(A) = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n is a strongly nowhere dense set in Y. By (b), each $f^{-1}(A_n)$ is a strongly nowhere dense set in X. Now $f^{-1}(f(A)) = f^{-1}(\bigcup_{n\in\mathbb{N}} A_n)$. Then $f^{-1}(f(A)) = \bigcup_{n\in\mathbb{N}} f^{-1}(A_n)$. This implies $f^{-1}(f(A))$ is a s-meager set in X which implies A is a s-meager set in X, since subset of a strongly nowhere dense set is strongly nowhere dense set, which is not possible. Therefore, A is of a s-II category set in Y.
(d) Suppose f is a bijective map and A is of s-II category set in Y. Suppose $f^{-1}(A)$ is of a s-meager set in X. Then $f^{-1}(A) = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n is a strongly nowhere dense set in X. By (a), each $f(A_n)$ is a strongly nowhere dense set in Y. Now $f(f^{-1}(A)) = f(\bigcup_{n\in\mathbb{N}} A_n)$. Since f is a bijective map, $A = \bigcup_{n\in\mathbb{N}} f(A_n)$. This implies A is a s-meager set in Y, which is not possible. Therefore, $f^{-1}(A)$ is of a s-II category set in X. □

Example 5.2 shows the condition that surjective on f cannot be dropped in Lemma 5.1 (a). Next Theorem 5.3 shows that an image and inverse-image of a strongly nodec is strongly nodec under a bijective, quasi homeomorphism function in generalized topological space.

Example 5.2. (a) Consider the two GTSs (X,μ) and (Y,λ) where $X=\{a,b,c,d\}, \mu=\{\emptyset,\{b\},\{c\},\{b,c\},\{a,c\},\{a,b,c\}\}, Y=\{a,b,c,d,e\}$ and $\lambda=\{\emptyset,\{e\},\{a,c\},\{b,e\},\{c,e\},\{a,c,e\},\{b,c,e\},\{a,b,c,e\}\}$. Define a function $f:X\rightarrow Y$ by $f(a)=a, f(b)=b, f(c)=c, f(d)=d$. Clearly, f is a (μ,λ)-continuous and for each μ-open subset U of X, there exists a unique λ-open subset V of Y such that $U=f^{-1}(V)$. Therefore, f is a quasi-homeomorphism but not a surjective map, since $f(X)\neq Y$. Let $A=\{a\}\subset X$. Then A is a strongly nowhere dense set in X. Now $f(A)=\{a\}\subset Y$. Let $G=\{a,c\}\in\tilde{\lambda}$. Then there is no $H\in\tilde{\lambda}$ such that $H\subset G$ and $H\cap f(A)=\emptyset$. Therefore, $f(A)$ is not a strongly nowhere dense set in Y.

Theorem 5.3. *Let $(X,\mu),(Y,\lambda)$ be two GTSs and $f:X\rightarrow Y$ be a bijective, quasi-homeomorphism map. Then X is a strongly nodec space if and only if Y is a strongly nodec space.*

Proof. Suppose X is a strongly nodec space. Let A be a non-empty strongly nowhere dense set in Y. By hypothesis and Lemma 5.1, $f^{-1}(A)$ is a strongly nowhere dense set in X. By hypothesis, $f^{-1}(A)$ is a closed set in X. Since f is a quasi-homeomorphism, there exists a unique closed set V in Y such that $f^{-1}(A)=f^{-1}(V)$. Thus, A is a closed set in Y, since f is a surjective map. Therefore, Y is a strongly nodec space. Conversely, assume that Y is a strongly nodec space. Let B be a strongly nowhere dense set in X. By hypothesis and Lemma 5.1, $f(B)$ is a strongly nowhere dense set in Y. By hypothesis, $f(B)$ is a closed set in Y. Since f is a (μ,λ)-continuous and bijective map, B is a closed set in X. Therefore, X is a strongly nodec space. □

Theorem 5.4. *Let (X,μ) be a sGTS and $q:X\rightarrow T_0(X)$ be a canonical surjective map. Then the following hold.*
(a) Every strongly nodec space is a T_0-strongly nodec space.
(b) Every generalized submaximal space is a T_0-strongly nodec space.

Proof. We will present the proof only for (a). Let X be a strongly nodec space and $q(A)$ be a strongly nowhere dense set in $T_0(X)$. By Lemma 5.1, $q^{-1}(q(A))$ is a strongly nowhere dense set in X. Since $A\subset q^{-1}(q(A))$ and subset of a strongly nowhere dense set is strongly nowhere dense, A is a strongly nowhere dense set in X. By hypothesis, A is a closed set in X. Since q is a closed map, by Lemma 4.6, $q(A)$ is a closed set in $T_0(X)$. Therefore, X is a T_0-strongly nodec space. □

Next Example 5.5 shows that the converse of Theorem 5.4 (a) is not necessary and Theorem 5.6 is the reverse implication of Theorem 5.4(a).

Example 5.5. Consider the GTS (X,μ) where $X=\{a,b,c\}$ and $\mu=\{\emptyset,\{a\},X\}$. Define a map $f:X\rightarrow T_0(X)$ by $x\in X$ to its equivalence class $[x]$ in $T_0(X)$, where $T_0(X)$ is the T_0-reflection of X. This implies $\mu_f=\{\emptyset,\{a\}\}$. Let $A=[b]=\{b,c\}$. Then A is strongly nowhere dense set in $T_0(X)$. Now $f^{-1}(A)=f^{-1}([b])=\{b,c\}$ is a closed set in X. Then A is a closed set in $T_0(X)$. Therefore, $T_0(X)$ is a strongly nodec space. Let $A=\{b\}$. Then A is a strongly nowhere dense set in X but not closed in X. Thus, X is not a strongly nodec space.

Theorem 5.6. *Let (X,μ) be a sGTS and $q:X\rightarrow T_0(X)$ be a canonical surjective map. If q is injective and X is a T_0-strongly nodec space, then it is a strongly nodec space.*

Proof. Let A be a non-empty strongly nowhere dense subset of X. By Lemma 5.1, $q(A)$ is a non-empty strongly nowhere dense set in $T_0(X)$. By hypothesis, $q(A)$ is a closed set in $T_0(X)$. Since q is a (μ,λ)-continuous map, $q^{-1}(q(A))$ is a closed set in X. Since q is injective map, A is a closed set in X. Therefore, X is a strongly nodec space. □

Example 5.7 shows the condition injective on q is necessary in Theorem 5.6 and Theorem 5.8 shows a (μ,λ)-open map from a GTS (X,μ) into a GTS (Y,λ) preserve the frontier of B where $B\subset Y$.

Example 5.7. Consider the GTS (X,μ) where $X=\{a,b,c,d,e\}$ and $\mu=\{\emptyset,\{a\},\{a,b\},\{b,c\},\{a,b,c\},\{a,b,d,e\},\{a,c,d,e\},X\}$. Define a map $f:X\rightarrow T_0(X)$ by $x\in X$ to its equivalence class $[x]$ in $T_0(X)$, where $T_0(X)$ is the T_0-reflection of X. Then f is a canonical surjective but not an injective map and so $\mu_f=\{\emptyset,\{a\}\}$. Let $A=[d]=\{d,e\}$. Then A is strongly nowhere dense set in $T_0(X)$. Now $f^{-1}(A)=f^{-1}([d])=\{d,e\}$ is a closed set in X. Then A is a closed set in $T_0(X)$. Similarly, every strongly nowhere dense set in $T_0(X)$ is closed set in $T_0(X)$. Therefore, $T_0(X)$ is a strongly nodec space. Let $A=\{e\}$. Then A is a strongly nowhere dense set in X but not closed in X. Thus, X is not a strongly nodec space.

Theorem 5.8. *Let $(X,\mu),(Y,\lambda)$ be two GTSs and $f:X\rightarrow Y$ be a map. If f is (μ,λ)-open, then $f^{-1}(Fr(B))=Fr(f^{-1}(B))$ for all $B\subset Y$.*

Proof. Suppose f is a (μ,λ)-open map. Let $t\in f^{-1}(Fr(B))$. Then $t\in f^{-1}(cB\cap c(Y-B))$. This implies $t\in f^{-1}(cB)$ and $t\in f^{-1}(c(Y-B))$. Since f is (μ,λ)-open and by Lemma 4.1, $t\in c(f^{-1}(B))$ and $t\in c(X-f^{-1}(B))$. Therefore, $t\in Fr(f^{-1}(B))$. Hence $f^{-1}(Fr(B))\subseteq Fr(f^{-1}(B))$. Let $s\in Fr(f^{-1}(B))$. Then $s\in c(f^{-1}(B))\cap c(X-f^{-1}(B))$. Now $s\in c(f^{-1}(B))$. Then $U\cap f^{-1}(B)\neq\emptyset$ for all $U\in\mu(s)$. By hypothesis, $f(U)\in\tilde{\lambda}$. This implies $f(U\cap f^{-1}(B))\neq\emptyset$ for all $f(U)\in\lambda(f(s))$. Now $f(U\cap f^{-1}(B))\subset f(U)\cap f(f^{-1}(B))\subset f(U)\cap B$, since $f(f^{-1}(B))\subset B$. Then $f(U)\cap B\neq\emptyset$ and so $f(s)\in cB$. Consider, $s\in$

$c(f^{-1}(Y-B))$. Then $V\cap f^{-1}(Y-B)\neq\emptyset$ for all $V\in\mu(s)$. By hypothesis, $f(V)\in\lambda(f(s))$. This implies $f(V\cap f^{-1}(Y-B))\neq\emptyset$ for all $f(V)\in\lambda(f(s))$. Now $f(V\cap f^{-1}(Y-B))\subset f(V)\cap f(f^{-1}(Y-B))\subset f(V)\cap(Y-B)$. Then $f(V)\cap(Y-B)\neq\emptyset$ and so $f(s)\in c(Y-B)$. Therefore, $f(s)\in Fr(B)$. Thus, $s\in f^{-1}(Fr(B))$. Hence $Fr(f^{-1}(B))\subset f^{-1}(Fr(B))$. □

Theorem 5.9. *Let $(X,\mu),(Y,\lambda)$ be two GTSs and $f:X\to Y$ be a surjective, quasi-homeomorphism map. If B is a strongly nowhere dense set in Y, then $f^{-1}(Fr(B))$ is a strongly nowhere dense set in X for all $B\subset Y$.*

Proof. Suppose B is a strongly nowhere dense set in Y. By Lemma 5.1, $f^{-1}(B)$ is a strongly nowhere dense set in X. Then $Fr(f^{-1}(B))$ is a strongly nowhere dense set in X, by Lemma 3.7 (a). By hypothesis and Proposition 4.2, f is a (μ,λ)-open map. Therefore, $f^{-1}(Fr(B))=Fr(f^{-1}(B))$, by Theorem 5.8. Hence $f^{-1}(Fr(B))$ is a strongly nowhere dense set in X. □

Let (X,μ) be a GTS. A space X is said to be a *weak Baire space (wBS)* if every non-empty μ-open set in X is of μ-s-II category in X [3].

Theorem 5.10. *In a GTS, every wBS is of s-II category.*

Proof. Let (X,μ) be a GTS and X is a wBS. Suppose X is a s-meager. Then $X=\bigcup\limits_{n\in\mathbb{N}}A_n$ where each A_n is a strongly nowhere dense set in X. Then each A_n is a nowhere dense set in X for $n\in\mathbb{N}$. This implies cA_n has no interior points so any non-empty open set in X must intersect $G_n=X-cA_n$ for all $n\in\mathbb{N}$. Take $\{G_n\}_{n\in\mathbb{N}}$ is a collection of non-empty open-dense sets in X. This implies $cG_n=X$ for all $n\in\mathbb{N}$ which implies cG_n is a s-meager set in X. Therefore, G_n is a s-meager set in X for all $n\in\mathbb{N}$, since subset of a s-meager set is s-meager. Thus, a non-empty open set G_n is not s-II category, which is a contradiction to X is a wBS. Hence X is of s-II category. □

Theorem 5.11 and Theorem 5.12 shows the behaviour of wBS under the quasi-homeomorphism and canonical surjective map in generalized topological space.

Theorem 5.11. *Let (X,μ), (Y,λ) be two GTSs and $f:X\to Y$ be a surjective, quasi-homeomorphism map. Then the following hold.*
(a) If X is a wBS, then Y is of s-II category.
(b) If f is a injective map and Y is a wBS, then X is of s-II category.

Proof. (a) Suppose X is a wBS. It is enough to prove, Y is a wBS, by Theorem 5.10. Let $A\in\tilde{\lambda}$. Since f is a (μ,λ)-continuous map, $f^{-1}(A)\in\tilde{\mu}$. By hypothesis, $f^{-1}(A)$ is of s-II category in X. By Lemma 5.1, $f(f^{-1}(A))$ is of s-II category in Y. Since f is a surjective map, A is of s-II category in Y. Therefore, Y is a wBS. Hence Y is of s-II category.
(b) Suppose f is a injective map and Y is a wBS. It is enough to prove, X is a wBS, by Theorem 5.10. Let $A\in\tilde{\mu}$. Since f is a quasi-homeomorphism map, there exists a set $A_1\in\tilde{\lambda}$ such that $A=f^{-1}(A_1)$. Since f is a surjective map, $f(A)=A_1$. By hypothesis, A_1 is of s-II category in Y. Thus, $f(A)$ is of s-II category in Y. By Lemma 5.1, A is of s-II category in X. Therefore, X is a wBS. Hence X is of s-II category. □

Theorem 5.12. *Let (X,μ) be a GTS and $q:(X,\mu)\to(T_0(X),\mu_q)$ be a canonical surjective map. Then the following hold.*
(a) If X is a wBS, then $T_0(X)$ is a wBS and hence a s-II category space.
(b) If $T_0(X)$ is a wBS and q is a injective map, then X is a wBS and hence a s-II category space.

Proof. (a) Suppose X is a wBS. Let $q(A)$ be a non-empty set in $T_0(X)$. Since q is a (μ,μ_q)-continuous map, $q^{-1}(q(A))\in\tilde{\mu}$. By hypothesis, $q^{-1}(q(A))$ is of s-II category in X. By Lemma 5.1, $q(A)$ is of s-II category in $T_0(X)$. Therefore, $T_0(X)$ is a wBS and hence a s-II category space, by Theorem 5.10.
(b) Suppose $T_0(X)$ is a wBS and q is an injective map. Let $A\in\tilde{\mu}$. By Lemma 4.6, q is a (μ,μ_q)-open map, $q(A)$ is a non-empty open set in $T_0(X)$. By hypothesis, $q(A)$ is of s-II category in $T_0(X)$. By Lemma 5.1 and q is a injective map, A is of s-II category in X. Therefore, X is a wBS. Hence X is of s-II category space, by Theorem 5.10. □

6. Strongly nodec space in GMS

In this section, we discuss the behaviour of μ-strongly nowhere dense set and strongly nodec space in generalized metric spaces.

In 2013, Korczak-Kubiak et al. introduced the notion of a generalized metric space [3]. They define the notions kernel and perfect kernel in GMS and discuss some properties of kernel, perfect kernel and three types of a Baire space in generalized metric space in [3].

Here, we focus only the properties of strongly nowhere dense sets and give some results for strongly nodec space in generalized metric space by using kernel and perfect kernel. Also, we give one result for wBS in generalized metric space. First we see the definitions and notation defined in generalized metric space.

Let $X \neq \emptyset$. The symbol π to denote the family consisting of metrics defined on subsets of X, that is $\rho \in \pi$ then there exists a non-empty set $A_\rho \subset X$ such that ρ is a metric on A_ρ where A_ρ is a domain space of ρ and it will be denoted by $dom(\rho)$. The pair (X, π) is called a *generalized metric space* (GMS) [3].

Denote μ_π is the family of π-*open sets* in a GMS (X, π), more precisely, $V \in \mu_\pi$ if and only if for each $x \in V$, there exists $\rho \in \pi$ and $\varepsilon > 0$ such that $B_\rho(x, \varepsilon) \subset V$ where $B_\rho(x, \varepsilon) = \{y \in dom(\rho) : \rho(x, y) < \varepsilon\}$ [3].

Let (X, π) be a GMS. A finite family $\pi_0 \subset \pi$ is called a *perfect kernel* (resp. *kernel*) [3] of the space X if for any $V_1, V_2, ..., V_m \in \mu_\pi$ such that $V_1 \cap V_2 \cap ... \cap V_m \neq \emptyset$ (resp. $V \in \tilde{\mu}_\pi$), there exists $\rho \in \pi_0$ such that $i_\rho(V_1 \cap V_2 \cap ... \cap V_m) \neq \emptyset$ (resp. $i_\rho(V) \neq \emptyset$). The set of all perfect kernels (resp. kernels) of the space (X, π) will be denoted by $Ker_p(X, \pi)$ (resp. $Ker(X, \pi)$). Obviously, if π_0 is a perfect kernel of the space (X, π), then it is a kernel of the space too [3].

Lemma 6.1. *If the GMS (X, π) has a kernel $\pi_0 \subset \pi$ and A is a dense subset of X, then $\pi_0|_A$ is a kernel of the GMS $(A, \pi|_A)$.*

Proof. Suppose A is dense subset of X. Let $V \in \tilde{\mu}_{\pi|_A}$ and $x \in V$. Then there exists $\rho \in \pi$ and $\varepsilon > 0$ such that $B_{\rho|_A}(x, \varepsilon) \subset V$. Since $B_\rho(x, \varepsilon) \neq \emptyset$ and π_0 is a kernel, there exists $\rho_0 \in \pi_0$ such that $i_{\rho_0}(B_\rho(x, \varepsilon)) \neq \emptyset$. Choose $y \in i_{\rho_0}(B_\rho(x, \varepsilon))$ and $\varepsilon_0 > 0$. Then $B_{\rho_0}(y, \varepsilon_0) \subset B_\rho(x, \varepsilon)$ and so $B_{\rho_0}(y, \varepsilon_0) \cap A \subset B_\rho(x, \varepsilon) \cap A$. That is, $B_{\rho_0|_A}(y, \varepsilon_0) \subset B_{\rho|_A}(x, \varepsilon)$. Also, $B_{\rho_0|_A}(y, \varepsilon_0) \in \tilde{\mu}_{\pi|_A}$, since A is dense and $B_{\rho_0|_A}(y, \varepsilon_0) \subset V$. Thus, there exists $\rho_0|_A \in \pi_0|_A$ such that $i_{\rho_0|_A}(V) \neq \emptyset$. Hence $\pi_0|_A$ is a kernel of the GMS $(A, \pi|_A)$. □

Lemma 6.2 shows the properties of strongly noowhere dense sets in subspace generalized metric space.

Lemma 6.2. *Let (X, π) be a GMS with a kernel $\pi_0 \subset \pi$, U be a dense, μ_{π_0}-open subset of X and $A \subset U \subset X$. Then the following hold.*
(a) If A is a strongly nowhere dense set in $(U, \pi|_U)$, then A is a strongly nowhere dense set in (X, π).
(b) If A is a s-meager set in $(U, \pi|_U)$, then A is a s-meager set in (X, π).
(c) If B is of s-II category in (X, π), then B is of s-II category in $(U, \pi|_U)$ where $B \subset X$.

Proof. (a) Let $W \in \tilde{\mu}_\pi$. Then $U \cap W \in \tilde{\mu}_{\pi|_U}$. Since A is a strongly nowhere dense set in U, there exists $V \in \tilde{\mu}_{\pi|_U}$ such that $V \subset U \cap W$ and $V \cap A = \emptyset$. By Lemma 6.1, $\pi_0|_U$ is a kernel. Then there exists $\rho_0|_U \in \pi|_U$ such that $i_{\rho_0|_U}(V) \neq \emptyset$. Let $x \in i_{\rho_0|_U}(V)$. Then there is $\varepsilon > 0$ such that $B_{\rho_0|_U}(x, \varepsilon) \subset V$. This implies $B_{\rho_0|_U}(x, \varepsilon) \subset W$ and $B_{\rho_0|_U}(x, \varepsilon) \cap A = \emptyset$. Now $x \in i_{\rho_0|_U} U = U = i_{\rho_0} U$, since U is a μ_{π_0}-open set in X. Let $\varepsilon_1 > 0$ such that $\varepsilon_1 > \varepsilon$. Then $B_{\rho_0}(x, \varepsilon_1) \subset U$ and so $B_{\rho_0}(x, \varepsilon) \subset U$. Therefore, $B_{\rho_0|_U}(x, \varepsilon) = B_{\rho_0}(x, \varepsilon)$. Thus, there is $B_{\rho_0}(x, \varepsilon) \in \tilde{\mu}_\pi$ such that $B_{\rho_0}(x, \varepsilon) \subset W$ and $B_{\rho_0}(x, \varepsilon) \cap A = \emptyset$. Hence A is a strongly nowhere dense set in X.
(b) and (c) follows from (a). □

Theorem 6.3 shows every dense-μ_{π_0}-open subspace of a strongly nodec space having kernel is a strongly nodec space.

Theorem 6.3. *If GMS (X, π) has a kernel $\pi_0 \subset \pi$, U be a dense, μ_{π_0}-open subset of X and if (X, μ_π) is strongly nodec, then $(U, \mu_{\pi|_U})$ is strongly nodec.*

Proof. Suppose X is a strongly nodec space. Let A be a non-empty strongly nowhere dense subset of U. By hypothesis and Lemma 6.2, A is a non-empty strongly nowhere dense subset of X. Then A is closed in X. By Lemma 2.3, $c_U(A) = A$. Hence U is a strongly nodec space. □

Theorem 6.4 shows every dense-μ_{π_0}-open subspace of a wBS having perfect kernel is a wBS. Next Lemma 6.5 shows the properties of strongly nowhere dense sets in generalized metric space.

Theorem 6.4. *If GMS (X, π) has a perfect kernel $\pi_0 \subset \pi$, U be a dense, μ_{π_0}-open subset of X and if (X, μ_π) is wBS, then $(U, \mu_{\pi|_U})$ is wBS and hence a s-II category.*

Proof. Suppose X is a wBS. Let V be a non-empty open set in U. Then $V = V_1 \cap U$ where V_1 is a non-empty μ_π-open set in X. Since V_1, U are μ_π-open sets and $V_1 \cap U \neq \emptyset$, there is $\rho_0 \in \pi_0$ such that $i_{\rho_0}(V_1 \cap U) \neq \emptyset$. Take $G = i_{\rho_0}(V_1 \cap U)$. Then G is a non-empty μ_π-open set in X. By hypothesis, G is of s-II category in X. By Lemma 6.2, G is of s-II category in U. Since $G \subset V$, V is of s-II category in U. For, if V is a s-meager in U. Since subset of a s-meager set is s-meager, G is a s-meager set in U. Hence U is a wBS. By Theorem 5.10, U is of s-II category. □

Lemma 6.5. *Let (X,π) be a GMS, U be a closed subset of X and $A \subset X$. Then the following hold.*
(a) If A is a strongly nowhere dense set in (X,π), then A is a strongly nowhere dense in $(U,\pi|_U)$.
(b) If A is a s-meager set in (X,π), then A is a s-meager set in $(U,\pi|_U)$.
(c) If A is a s-residual set in (X,π), then A is a s-residual set in $(U,\pi|_U)$.
(d) If B is of s-II category in (U,π), then B is of s-II category in (X,π) where $B \subset U$.

Proof. (a) Let A be a strongly nowhere dense subset of X. Suppose $A \cap U = \emptyset$. Then A is a strongly nowhere dense set in U, by definition of strongly nowhere dense. Assume, $A \cap U \neq \emptyset$. Let $W \in \tilde{\mu}_{\pi|_U}$ and $x \in W$. Then there is $\rho|_U \in \pi|_U$ and $\varepsilon > 0$ such that $B_{\rho|_U}(x,\varepsilon) \subset W$. Since A is a strongly nowhere dense set in X and $B_\rho(x,\varepsilon) \in \tilde{\mu}_\pi$, there exists $V \in \tilde{\mu}_\pi$ such that $V \subset B_\rho(x,\varepsilon)$ and $V \cap A = \emptyset$. Choose $V = B_\rho(x,\varepsilon_1)$ where $\varepsilon_1 < \varepsilon$. Since U is closed and $x \in U$, $V \cap U \neq \emptyset$. Thus, there is $B_{\rho|_U}(x,\varepsilon_1) \in \tilde{\mu}_{\pi|_U}$ such that $B_{\rho|_U}(x,\varepsilon_1) \subset B_{\rho|_U}(x,\varepsilon) \subset W$ and $B_{\rho|_U}(x,\varepsilon_1) \cap A = \emptyset$. Therefore, A is a strongly nowhere dense in $(U,\pi|_U)$.
(b), (c) and (d) follows from (a). □

Theorem 6.6. *Let (X,π) be a GMS and U be a closed subset of X. If $(U,\mu_{\pi|_U})$ is a strongly nodec space, then (X,μ_π) is a strongly nodec space.*

Proof. Suppose $(U,\mu_{\pi|_U})$ is a strongly nodec space. Let A be a non-empty strongly nowhere dense subset of X. Then by Lemma 6.5, A is a non-empty strongly nowhere dense set in U. By hypothesis, A is a closed set in U. Then $c_U A = A$ and so $A = U \cap cA$, by Lemma 2.1. Since U is a closed subset in X, A is a closed set in X. Therefore, (X,μ_π) is a strongly nodec space. □

Theorem 6.7. *Let (X,π) be a GMS. If frontier of a subspace of X is strongly nodec, then (X,μ_π) is a strongly nodec space.*

Proof. Let Y be a subspace of X. Suppose $Fr(Y)$ is a strongly nodec space. Since every frontier of a subset of X is a closed set in X, $Fr(Y)$ is a closed subset of X. Hence X is a strongly nodec space, by Theorem 6.6. □

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Mixed-Type Functional Differential Equations: A C_0-Semigroup Approach

Luís Gerardo Mármol[1]* and Carmen Judith Vanegas[2]

Abstract
In this paper we study certain systems of mixed-type functional differential equations, from the point of view of the C_0-semigroup theory. In general, this type of equations are not well-posed as initial value problems. But there are also cases where a unique differentiable solution exists. For these cases and in order to achieve our goal, we first rewrite the system as a classical Cauchy problem in a suitable Banach space. Second, we introduce the associated semigroup and its infinitesimal generator and prove important properties of these operators. As an application, we use the results to characterize the null controllability for those systems, where the control u is constrained to lie in a non-empty compact convex subset Ω of $\mathbb{R}^n$.

Keywords: Functional differential equations, Mixed-type difference-differential equations, Strongly continuous semigroups, Exact controllability
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[1] *Departamento de Matemáticas Puras y Aplicadas Universidad Simón Bolívar, Caracas 1080-A, Venezuela.*
[2] *Universidad Técnica de Manabí Departamento de Matemáticas y Estadística Instituto de Ciencias Básicas Portoviejo, Ecuador.*
***Corresponding author**:lgmarmol@usb.ve
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1. Introduction

In this paper we analyze certain systems of functional differential equations with both delayed and advanced arguments. Such equations are often referred to in the literature as mixed-type functional differential equations (MTFDE) or forward-backward equations. The study of this type of equations is less developed compared with other classes of functional equations. As a consequence, many important questions remain open. Interest in MTFDEs is motivated by problems in optimal control [1] and applications, for example, in economic dynamics [2] and travelling waves in a spatial lattice [3].

As far as we know, similar studies to the one presented here for this type of equations haven't been done. This type of equations are, in general, ill-posed as initial value problems (see for example, [1] and [4]), but there are also cases ([5], [6], [7], [8], [9] and [10]) where a unique differentiable solution exists. We make the statement of the problem in Section 2. In section 3, we give our main results. We begin with showing how the system can be rewritten as a classical Cauchy problem in a suitable Banach space, provided that the initial value problem is well-posed. Then we give the semigroup associated with the ordinary differential equation and its infinitesimal generator, and prove some important properties of these operators. In section 4 we apply the results obtained in Section 3 to characterize the null controllability for those systems, where the control u is constrained to lie in a non-empty compact convex subset Ω of $\mathbb{R}^n$, with $0 \in \Omega$.

2. Statement of the problem

Let Ω be a bounded domain in $\mathbb{R}^n$, $0 < h_1 < h_2 < \cdots < h_q$ and $A_0, A_i, C_i \in \mathscr{L}(\mathbb{R}^n)$, for $i = 1,\ldots,q$, with $A_i \neq 0$ for some i. We will consider the following mixed-type functional differential equation

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_i^q A_i x(t-h_i) + \sum_i^q C_i x(t+h_i) + Bu(t), \;\; t > 0 , \\ x(0) &= \Phi(0) = \Phi_0 , \\ x(s) &= \Phi(s), \;\; s \in [-h_q, 2h_q], \end{aligned} \tag{2.1}$$

where $\Phi \in L_p[[-h_q, 2h_q]; \mathbb{R}^n]$, $1 \le p \le \infty$, is defined by

$$\Phi(s) = \begin{cases} \Phi_1(s), & s \in [-h_q, 0] \\ \Phi_2(s), & s \in [0, 2h_q] \end{cases} ,$$

$u : [0,\infty) \to \mathbb{R}^n$ is an essentially bounded function and $B \in \mathscr{L}(\mathbb{R}^n)$.

The function Φ is usually found in Hilbert spaces (see, for example, [11] and [12], in the case $C_i = 0$). It is noteworthy that, in the present work, we will allow it to be in a L_p -space, for any p belonging to $[1,\infty]$.

At this point, some basic facts must be recalled. It is, in fact, well known that, for X a Banach space and $f : [0,\infty) \to X$ a continuously differentiable function, the initial value problem

$$\begin{cases} x'(t) = Ax(t) + f(t), & t \ge 0 \\ x(0) = x_0, & x_0 \in D(A) \end{cases}$$

is well posed if and only if A generates a strongly continuous semigroup $(T(t))_{t\ge 0}$ on X. The unique solution can be expressed in terms of $(T(t))_{t\ge 0}$ by the following formula (usually known as mild solution):

$$x(t) = T(t)x_0 + \int_0^t T(t-s) f(s) \mathrm{d}s.$$

Working in the frame of a C_0-semigroup theory is not always possible for MTFDE, as there are cases where the problem is not well-posed. As an example, consider the equation $\dot{x}(t) = x(t+1)$. If λ is such that $\lambda - \exp(\lambda) = 0$, then it is easily seen that $x(t) = \exp(\lambda t)x_0$ is a solution. Any strongly continuous semigroup is bounded by $Me^{\omega t}$ for some M and ω, but this fails in this case. Here, we cannot assume that for initial conditions in a dense set, there exists a classical solution. It also shows that our condition "$A_i \neq 0$ for some i" in a system like (2.1) is truly essential. *In other words, we need the presence of delayed arguments*.

Another very interesting example of an ill-posed problem is the following: in [4], Harterich, Sandstede and Scheel consider the equation

$$\dot{x}(t) = x(t-1) + x(t+1)$$

with $\Phi(s) = 1$, $s \in [-m, m]$, m a natural number. The only possible solution for this initial value problem is $x(t) = (-1)^k$, for $t \in (2k-1, 2k+1]$, with k a natural number, which is not even a continuous function.

On the other hand, it is shown in [8] that this same equation has a unique differentiable solution if and only if $\Phi \in C^\infty_{[-1,1]}$ defined by

$$\Phi(s) = \begin{cases} \Phi_1(s), & s \in [-1, 0] \\ \Phi_2(s), & s \in [0, 1] \end{cases} ,$$

satisfies $\Phi^{(n+1)}(0) = \Phi^{(n)}(-1) + \Phi^{(n)}(1)$ for $n = 0,1,2,\ldots$. As an example, it is easy to see that $\Phi(s) = e^{\lambda s}$ satisfies this condition if $\lambda = e^\lambda + e^{-\lambda}$, and it is shown in [13] that there exist, in fact, complex numbers λ such that $\lambda = e^\lambda + e^{-\lambda}$, as they are the spectrum of a bounded linear operator on suitable Banach Space (the spectrum is always non-empty, as it is well known). It is also shown that, being $\det(\lambda I - e^\lambda - e^{-\lambda})$ an entire function, then, for every $\delta \in \mathbb{R}$, it has finitely many zeros in the compact set $\overline{\mathbb{C}^+_\delta} \cap \{\lambda : |\lambda| \le e^\delta + e^{-\delta}\}$, and in the rest of $\mathbb{C}^+_\delta$ there are none. In particular, there are finitely many λ with $Re\lambda > 0$ such that $\lambda = e^\lambda + e^{-\lambda}$, and we have $|\lambda| \le 2$. Thus the unique solution, given by a strongly continuous semigroup, is exponentially bounded, as it should be.

Bearing in mind these results, it is characterized in [13] the null controllability for the associated initial value problem, where the control u is also constrained to belong to a suitable domain Ω of the control space with $0 \in \Omega$.

The present work is an attempt to see the results in [9] and [13] in a more general context. As we have indicated, there exist other examples where a unique solution can be found ([5], [6], [7] and [10]). In most of this cases, the function Φ is supposed to belong to a Banach space of sufficiently smooth functions defined on an interval $[a,b]$. Such functions are always in $L_\infty([a,b])$, and so those examples can be adapted to our model.

It should be pointed out, therefore, that we are excluding the cases where the problem is ill-posed. We attempt to give a detailed description of the associated semigroup and its infinitesimal generator for a system like (2.1) *whenever a unique differentiable solution exists*. These solutions are often found by some other independent method, as in the examples cited above.

Bearing in mind this purpose, we will show that (2.1) can be written as an ordinary differential equation in a suitable Banach space J_p, which will be defined later, as follows:

$$\begin{aligned}\dot{\omega}(t) &= A\omega(t)+\beta u(t), \;\; t>0\\ \omega(0) &= \Phi_0\end{aligned}$$

where $\beta : U \to J_p$ is given by $\beta u = \begin{pmatrix} Bu \\ 0 \end{pmatrix}$ and

$$A\begin{pmatrix} \Phi_0 \\ \Phi(s) \end{pmatrix} = \begin{pmatrix} A_0\Phi_0+\sum_i^q A_i\Phi(-h_i)+\sum_i^q C_i\Phi(h_i) \\ \dot{\Phi}(s) \end{pmatrix}, \;\; -h_q \le s \le 2h_q,$$

is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\ge 0}$ defined by

$$T(t)\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \begin{pmatrix} x(t) \\ x(t+\cdot) \end{pmatrix},$$

where $x(\cdot)$ is the unique solution of the system

$$\begin{aligned}\dot{x}(t) &= A_0x(t)+L_dx(t)+L_ax(t), \;\; t>0\\ x(0) &= \Phi_0\\ x(s) &= \Phi(s), \;\; s\in[-h_q,2h_q],\end{aligned}$$

where $L_dx(t)=\sum_i^q A_ix(t-h_i)$ and $L_ax(t)=\sum_i^q C_ix(t+h_i)$.

Once achieved these results we will give necessary and sufficient conditions to ensure the exact controllability for (2.1).

3. Main results

In the following we will show an alternative representation of the given solution of (2.1), and we will also prove that $T(t)$ (as given in (**??**)) is in fact a strongly continuous semigroup with A as its infinitesimal generator.

Theorem 2.4.1 of [11], deals on delay equations and the solution $x(\cdot)$ on $[0,\infty)$ is built recursively. This same construction cannot be done in our case but, as it has been stated, we are supposing that the solution $x(\cdot)$ is previously known.

Theorem 3.1. *Suppose that the unique solution $x(\cdot)$ on $[0,\infty)$ of* (2.2) *is known. Then $x(\cdot)$ satisfies the following recursive formula*

$$x(t)=e^{A_0t}\Phi_0+\sum_{i=1}^{q}\int_0^t e^{A_0(t-s)}(A_ix(s-h_i)+C_ix(s+h_i))ds \;\; for\; t\ge 0. \tag{3.1}$$

Proof . Notice first that for $t\in[0,h_q]$ the term $\sum_{i=1}^q A_ix(t-h_i)+C_ix(t+h_i)$ equals the function

$$v(t):=\sum_{i=1}^{q}A_i\Phi(t-h_i)+C_i\Phi(t+h_i).$$

So we may reformulate the system (2.2) on $[0,h_q]$ as

$$\dot{x}(t)=A_0x(t)+v(t), \;\; x(0)=\Phi_0. \tag{3.2}$$

It is well known that the unique solution of (3.2) is given by

$$x(t) = e^{A_0 t}\Phi_0 + \int_0^t e^{A_0(t-s)} v(s)ds$$

and this equals (3.1).

Let us consider now the case $t \geq h_q$. We use the hypothesis that $x(t)$ is known for every t, and so, at a given time t, the function $\sum_{i=1}^{q} A_i x(t-h_i) + C_i x(t+h_i)$ is also known. Then we can proceed in a similar way. Applying finite dimensional theory gives that the unique solution satisfies (3.1). ■

In the following we will construct the c_0-semigroup and its infinitesimal generator associated to the equation (2.2).

Lemma 3.2. *If $x(t)$ is the solution of (2.2), then the following inequalities hold:*

[**i**] $\quad ||x(t)|| \leq C_t[||\Phi_0|| + ||\Phi(\cdot)||_{L_p([-h_q,2h_q];\mathbb{R}^n)}], \quad 1 \leq p \leq \infty$

[**ii**] $\quad \int_{2h_q}^{2h_q+t} ||x(\tau)||^p d\tau \leq D_t[||\Phi_0||^p + ||\Phi(\cdot)||^p_{L_p([-h_q,2h_q];\, \mathbb{R}^n)}], \quad 1 \leq p < \infty$

where C_t and D_t are constants depending only on t.

Proof . It is well known that for some positive constants M_0, W_0, $e^{A_0 t}$ satisfies $||e^{A_0 t}|| \leq M_0 e^{W_0 t},\ t \geq 0$.
Let us define the positive constant M by

$$M := max(||A_1||, \cdots, ||A_q||, ||C_1||, \cdots, ||C_q||, M_0).$$

Then, it is deduced, from the formula of the solution of equation (2.2) that

$$\begin{aligned} ||x(t)|| &\leq Me^{W_0 t}||\Phi_0|| + \sum_{i=1}^{q} M^2 \int_0^t e^{W_0(t-s)}(||x(s-h_i)|| + ||x(s+h_i)||)ds \\ &\leq Me^{W_0 t}||\Phi_0|| + SI1 \\ &= Me^{W_0 t}||\Phi_0|| + M^2 e^{W_0 t} SI2, \end{aligned} \tag{3.3}$$

where

$$SI1 = \sum_{i=1}^{q} M^2 (\int_{-h_i}^{t-h_i} e^{W_0(t-\tau-h_i)}||x(\tau)||d\tau + \int_{h_i}^{t+h_i} e^{W_0(t-\tau+h_i)}||x(\tau)||d\tau)$$

and

$$SI2 = \sum_{i=1}^{q} (\int_{-h_i}^{t-h_i} e^{-W_0(\tau+h_i)}||x(\tau)||d\tau + \int_{h_i}^{t+h_i} e^{-W_0(\tau-h_i)}||x(\tau)||d\tau).$$

But after a standard estimation we have

$$SI2 \leq e^{W_0 h_q} q \int_{-h_q}^{2h_q} ||\Phi(\tau)||d\tau + q\int_0^t e^{-W_0\tau}||x(\tau)||d\tau$$

$$+\sum_{i=1}^{q} \int_0^t e^{-W_0\tau}||x(\tau+h_i)||d\tau \ \text{ (because } W_0 > 0)$$

$$\leq C'||\Phi(\cdot)||_{L_p([-h_q,2h_q];\, \mathbb{R}^n)} + q\int_0^t e^{-W_0\tau}||x(\tau)||d\tau$$

$$+q\int_0^t e^{-W_0\tau}(\max(||x(\tau+h_1)||, \cdots, ||x(\tau+h_q)||))d\tau.$$

$$\leq C'||\Phi(\cdot)||_{L_p([-h_q,2h_q];\, \mathbb{R}^n)} + q\int_0^t e^{-W_0\tau}(\max(||x(\tau+h_1)||, \cdots, ||x(\tau+h_q)||, ||x(\tau)||) + 1)d\tau$$

$$+q\int_0^t e^{-W_0\tau}(\max(||x(\tau+h_1)||, \cdots, ||x(\tau+h_q)||))d\tau.$$

Now, if $f(\tau)$ is defined as

$$f(\tau) = 1+\frac{\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||)}{\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||,||x(\tau)||)+1}$$

we have that the former inequality is estimated by

$$C'||\Phi(\cdot)||_{L_p([-h_q,2h_q];\ \mathbb{R}^n)}+q\int_0^t e^{-W_0\tau}f(\tau)(\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||,||x(\tau)||)+1)d\tau. \quad (3.4)$$

Combining (3.3) and (3.4) we obtain

$$||x(t)|| \le e^{W_0t}[M||\Phi_0||+M^2C'||\Phi||_{L_p([-h_q,2h_q];\ \mathbb{R}^n)}$$
$$+M^2q\int_0^t e^{-W_0\tau}f(\tau)(\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||,||x(\tau)||)+1)d\tau]$$

.

Now, if C'' is constant (depending on t) such that $1\le e^{W_0t}M^2C''||\Phi||_{L_p([-h_q,2h_q];\ \mathbb{R}^n)}$, we have

$$||x(t)||+1 \le e^{W_0t}[M||\Phi_0||+M^2C||\Phi||_{L_p([-h_q,2h_q];\ \mathbb{R}^n)}$$
$$+M^2q\int_0^t e^{-W_0\tau}f(\tau)(\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||,||x(\tau)||)+1)d\tau]$$

where $C=C'+C''$,
or equivalently

$$z(t)\le\beta+\int_0^t a(\tau)z(\tau)d\tau$$

where
$\beta=M||\Phi_0||+M^2C||\Phi||_{L_p([-h_q,2h_q];\ \mathbb{R}^n)}$, $a(\tau)=M^2qf(\tau)$ and $z(\tau)$ is the function defined by $z(\tau)=e^{-W_0\tau}(\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||,||x(\tau)||)+1)$. Then, from Gronwall Lemma (see [11], p.639) we conclude that

$$z(t)\le\beta(\exp\int_0^t a(\tau)d\tau)$$

and so

$$\begin{aligned}||x(t)|| &\le \beta\exp(\int_0^t a(\tau)d\tau+W_0t)\\ &\le \exp(M^2q\int_0^t f(\tau)d\tau+W_0t)\max[M,M^2C][||\Phi_0||+||\Phi||_{L_p([-h_q,2h_q];\ \mathbb{R}^n)}].\end{aligned}$$

Now, let us note that $\int_0^t f(\tau)d\tau\le 2t$. This shows **[i]**.
In a similar way, we obtain from the former inequality, for $p\in[1,\infty)$

$$||x(t)||^p\le K(\exp\{p(qM^2c_t+W_0t)\}[||\Phi_0||^p+||\Phi||^p_{L_p([-h_q,2h_q];\ \mathbb{R}^n)}]),$$

where K a suitable constant. Integrating this inequality gives **[ii]**. ■

Now, we are going to construct the c_0-semigroup. Let us first recall that, for a pair X,Y of normed spaces, we can introduce a normed space $X\oplus Y$ called a *direct (topological) sum* of X and Y that consists of all ordered pairs $(x,y), x\in X, y\in Y$ together with the norm $\|(x,y)\|=\|x\|_X+\|y\|_Y$. X and Y are isometric to subspaces $\{(x,0);x\in X\}$ and $\{(0,y);y\in Y\}$ of $X\oplus Y$. If X and Y are Banach spaces, so is $X\oplus Y$. Convergence in $X\oplus Y$ means that (x_n,y_n) tends to (x,y) if and only if both $\|x_n-x\|_X$ and $\|y_n-y\|_Y$ tend to zero as n tends to infinity.

Let us also recall that the elements in $L_p([-h_q,2h_q];\ \mathbb{R}^n)$, $1\le p\le\infty$, are, in fact, equivalence classes of functions, with the corresponding equivalence relation $\Re$ defined by $f\Re g$ if and only if $f=g$ a.e.

Let M_p be the closure in $L_p([-h_q,2h_q];\ \mathbb{R}^n)$ of the subspace $L_p([-h_q,2h_q];\ \mathbb{R}^n)\cap C([-h_q,2h_q];\ \mathbb{R}^n)$. M_p is a Banach space with the same norm as $L_p([-h_q,2h_q];\ \mathbb{R}^n)$. Let us now consider the Banach space $\mathbb{R}^n\oplus M_p$, and let G_p be the linear subspace of all pairs $(r,f)\in\mathbb{R}^n\oplus M_p$, $1\le p\le\infty$, such that $r=f(0)$. If $f\in L_p([-h_q,2h_q];\ \mathbb{R}^n)$, it is well known that there exist functions

$g \in L_p([-h_q, 2h_q];\ \mathbb{R}^n)$, such that $f = g$ a.e., as we have previously indicated. But in our case the function f is supposed to be continuous on the whole interval $[-h_q, 2h_q]$ and there is no ambiguity: G_p is thus well defined.

Finally, let J_p be the closure in $\mathbb{R}^n \oplus M_p$ of the linear subspace G_p. For $1 \leq p \leq \infty$, J_p is thus a Banach space.

Since $(x, f) = (f(0), f) + (x - f(0), 0)$, it is easily seen that J_∞ is a topologically complemented subspace of $\mathbb{R}^n \oplus M_\infty$.

On the other hand, for $h \in L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$, there exist a continuous function $f \in L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$ such that $h = f$ a.e. Bearing in mind that $(r,h) = (r, f) + (0, h - f)$, we have that $\mathbb{R}^n \oplus M_\infty$ is also topologically complemented in $\mathbb{R}^n \oplus L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$, and thus so is J_∞.

Theorem 3.3. *The operator $T(t)$ defined for each $t \geq 0$ by* (**??**) *satisfies*
(1) $T(t) \in L(J_p)$ for every $t \geq 0$
(2) $T(t)$ is a C_0-semigroup in J_p

Proof . (1) First, we suppose $p \in [1, \infty)$. Note that

$$\begin{aligned}\left(\int_{-h_q}^{2h_q} ||x(t+\tau)||^p d\tau\right)^{1/p} &= \left(\int_{-h_q+t}^{2h_q+t} ||x(\tau)||^p d\tau\right)^{1/p} \leq \left(\int_{-h_q}^{2h_q+t} ||x(\tau)||^p d\tau\right)^{1/p} \\ &\leq K\left((\int_{-h_q}^{2h_q} ||\Phi(\tau)||^p d\tau)^{1/p} + (\int_{2h_q}^{2h_q+t} ||x(\tau)||^p d\tau)^{1/p}\right).\end{aligned}$$

Then, using Lemma 3.2, we have for $\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} \in J_p$,

$$\begin{aligned}||T(t)\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix}|| &= ||x(t)|| + \left(\int_{-h_q}^{2h_q} ||x(t+\tau)||^p d\tau\right)^{1/p} \\ &\leq R_t[||\Phi_0|| + ||\Phi(\cdot)||_{L_p([-h_q,2h_q];\mathbb{R}^n)}].\end{aligned}$$

In the case $p = \infty$, we can suppose $x(t) \neq 0$ (otherwise the result is trivial), and let us choose t_0 such that $x(t_0) \neq 0$. Then, bearing in mind that for each t, $||x(t+\tau)||_\infty$ is a positive real number whose value only depends on t and using Lemma 3.2, we have

$$||x(t+\tau)||_\infty = \frac{||x(t+\tau)||_\infty}{||x(t_0)||} \cdot ||x(t_0)|| \leq C_t[||\Phi_0|| + ||\Phi(\cdot)||_{L_p([-h_q,2h_q];\mathbb{R}^n)}],$$

where $C_t = \frac{||x(t+\tau)||_\infty}{||x(t_0)||} \cdot C_{t_0}$.

Now, we will prove (2). The semigroup property can be proven similarly as in Theorem 2.4.4 of [11]. We only have to note that, in this case, it is considered the function $g(t) = x(t+s)$, where $x(\cdot)$ is the solution of system (2.2). Then $g(t)$ satisfies

$$\begin{aligned}\dot{g}(t) &= A_0 g(t) + \sum_{i=1}^{q} (A_i g(t-h_i) + C_i g(t+h_i)),\ \ t \geq 0 \\ g(0) &= x(s) \\ g(\theta) &= x(s+\theta),\ \ \theta \in [-h_q, 2h_q].\end{aligned}$$

To prove the strong continuity, we begin with the case $p \in [1, \infty)$. For $t < h_1$ we have

$$||T(t)\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} - \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix}|| =$$
$$||e^{A_0 t}\Phi_0 + \textstyle\sum_{i=1}^q \int_0^t e^{A_0(t-s)}(A_i\Phi(s-h_i) + C_i\Phi(s+h_i))ds - \Phi_0|| +$$
$$\left(\textstyle\int_{-h_q}^{-t} ||\Phi(t+\tau) - \Phi(\tau)||^p d\tau + \int_{-t}^{2h_q} ||x(t+\tau) - \Phi(\tau)||^p d\tau\right)^{1/p}.$$

The first term converges to zero as $t \to 0$, because

$$e^{A_0 t} + \sum_{i=1}^{q} \int_0^t e^{A_0(t-s)}(A_i\Phi(s-h_i) + C_i\Phi(s+h_i))ds$$

is continuous. On the other side, using the triangle inequality and Lemma 3.2, the integral terms tend to zero by Lebesgue's Dominated Convergence Theorem.

The case $p = \infty$ is similar. We only have to note that $||x(t+\tau) - x(\tau)||_\infty \to 0$ as $t \to 0$. ■

Lemma 3.4. *Consider the c_0-semigroup $T(t)$ defined above and let A be its infinitesimal generator. For sufficiently large $\alpha \in \mathbb{R}$, the resolvent is given by*

$$(\alpha I - A)^{-1} \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \begin{pmatrix} g(0) \\ g(\cdot) \end{pmatrix}$$

where

$$g(\theta) = e^{\alpha\theta} g(0) - \int_0^\theta e^{\alpha(\theta - s)} \Phi(s) ds, \quad \theta \in [-h_q, 2h_q] \tag{3.5}$$

and

$$g(0) = (\Delta(\alpha))^{-1} \left(\Phi_0 + \sum_{i=1}^{q} \int_{-h_i}^{0} e^{-\alpha(\theta + h_i)} A_i \Phi(\theta) d\theta \right). \tag{3.6}$$

where

$$\Delta(\lambda) = \lambda I - A_0 - \left(\sum_{i=1}^{q} e^{-\lambda h_i} A_i + e^{\lambda h_i} C_i \right), \quad \lambda \in \mathbb{C}.$$

Furthermore, g satisfies the following relation

$$\alpha g(0) = \Phi_0 + A_0 g(0) + \sum_{i=1}^{q} A_i g(-h_i) + C_i g(h_i). \tag{3.7}$$

Proof . According to Lemma 2.1.11 in [11], we have for $\alpha > \omega_0$ that

$$(\alpha I - A)^{-1} \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \int_0^\infty e^{-\alpha t} T(t) \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} dt = \int_0^\infty e^{-\alpha t} \begin{pmatrix} x(t) \\ x(t + \cdot) \end{pmatrix} dt.$$

We define

$$g(\theta) = \int_0^\infty e^{-\alpha t} x(t + \theta) dt, \text{ for } \theta \in [-h_q, 2h_q].$$

Rewriting this function as $g(\theta) = \int_\theta^\infty e^{-\alpha(s-\theta)} x(s) ds$ it is easy to see that $g(\cdot)$ is a solution of

$$\frac{\partial g(\theta)}{\partial \theta} = \alpha g(\theta) - x(\theta), \quad \theta \in [-h_q, 2h_q].$$

In $[-h_q, 2h_q]$, the variation of constants formula for this ordinary differential equation shows that $g(\cdot)$ equals (3.5). It only remains to prove (3.6).

Bearing in mind that, according to Lemma 3.2, $C_t \left[||\Phi_0|| + ||\Phi(\cdot)||_{L_p([-h_q, 2h_q]; \mathbb{R}^n)} \right]$ is an upper bound for $x(t)$, we have

$$\begin{aligned}
\alpha g(0) &= \alpha \int_0^\infty e^{-\alpha t} x(t) dt = -[x(t) e^{-\alpha t}]_0^\infty + \int_0^\infty e^{-\alpha t} \dot{x}(t) dt \\
&= \Phi_0 + \int_0^\infty e^{-\alpha t} [A_0 x(t) + \sum_{i=1}^{q} A_i x(t - h_i) + C_i x(t + h_i)] dt \\
&= \Phi_0 + A_0 \int_0^\infty e^{-\alpha t} x(t) dt + \sum_{i=1}^{q} \int_0^\infty e^{-\alpha t} (A_i x(t - h_i) + C_i x(t + h_i)) dt \\
&= \Phi_0 + A_0 g(0) + \sum_{i=1}^{q} A_i g(-h_i) + C_i g(h_i).
\end{aligned}$$

This proves equation (3.7). On the other hand, if we split the integrals in the former equation, we obtain

$$\begin{aligned}
\alpha g(0) &= \Phi_0 + A_0 g(0) + \sum_{i=1}^{q} \int_{h_i}^\infty e^{-\alpha t} (A_i x(t - h_i) + C_i x(t + h_i)) dt \\
&+ \sum_{i=1}^{q} \int_0^{h_i} e^{-\alpha t} (A_i \Phi(t - h_i) + C_i \Phi(t + h_i)) dt \\
&= \Phi_0 + A_0 g(0) + \sum_{i=1}^{q} e^{-\alpha h_i} A_i g(0) + \sum_{i=1}^{q} e^{-\alpha h_i} C_i g(0) \\
&- \sum_{i=1}^{q} e^{-\alpha h_i} \int_{h_i}^{2h_i} e^{-\alpha\theta} C_i \Phi(\theta) d\theta + \sum_{i=1}^{q} \int_0^{h_i} e^{-\alpha t} (A_i \Phi(t - h_i) + C_i \Phi(t + h_i)) dt
\end{aligned}$$

and so

$$\left[\alpha I - A_0 - \left(\sum_{i=1}^{q} e^{-\alpha h_i}A_i + e^{\alpha h_i}C_i)\right)\right] g(0) = \Phi_0 - \sum_{i=1}^{q} e^{\alpha h_i}\int_{h_i}^{2h_i} e^{-\alpha\theta}C_i\Phi(\theta)d\theta$$
$$+\sum_{i=1}^{q}\left(\int_{-h_i}^{0} e^{-\alpha(\theta+h_i)}A_i\Phi(\theta)d\theta + \int_{h_i}^{2h_i} e^{-\alpha(\theta-h_i)}C_i\Phi(\theta)d\theta\right)$$
$$= \Phi_0 + \sum_{i=1}^{q}\int_{-h_i}^{0} e^{-\alpha(\theta+h_i)}A_i\Phi(\theta)d\theta$$

which proves (3.6) for sufficiently large α. ■

In the following theorem, we first give an explicit formula for the infinitesimal generator A. The second part of the theorem deals with the spectral properties. Some of the main conclusions of this second part are not true for MTFDE if the problem is ill-posed ([4]), but for well-posed problems, as in our case, they remain valid.

Theorem 3.5. *Consider the c_0-semigroup defined as before. Its infinitesimal generator is given by*

$$A\begin{pmatrix}\Phi_0\\ \Phi(\cdot)\end{pmatrix} = \begin{pmatrix} A_0\Phi_0 + \sum_{i=1}^{q} A_i\Phi(-h_i) + C_i\Phi(h_i)\\ \dfrac{\partial\Phi(\cdot)}{\partial\theta}\end{pmatrix}$$

with domain

$$D(A) = \left\{\begin{pmatrix}\Phi_0\\ \Phi(\cdot)\end{pmatrix} \in J_p : \Phi \text{ is absolutely continuous}, \frac{\partial\Phi}{\partial\theta} \in L_p([-h_q, 2h_q];\mathbb{R}^n)\right\}.$$

Furthermore, the spectrum of A is discrete and is given by

$$\sigma(A) = \sigma_p(A) = \{\lambda \in \mathbb{C} : \det(\Delta(\lambda)) = 0\},$$

where $\Delta(\lambda)$ was defined in Lemma 3.4 and the multiplicity of each eigenvalue is finite for $p = 2$.

For every $\delta \in \mathbb{R}$, there are only finitely many eigenvalues in $\mathbb{C}_\delta^+$. If $\lambda \in \sigma_p(A)$, then $\begin{pmatrix} r\\ e^{\lambda\cdot}r\end{pmatrix}$, where $r \neq 0$ satisfies $\Delta(\lambda)r = 0$, is an eigenvector of A with eigenvalue λ. On the other hand, if ζ is an eigenvector of A with eigenvalue λ, then $\zeta = \begin{pmatrix} r\\ e^{\lambda\cdot}r\end{pmatrix}$ with $\Delta(\lambda)r = 0$.

Proof . We denote by $\tilde{A}$ the operator

$$\tilde{A}\begin{pmatrix}\Phi_0\\ \Phi(\cdot)\end{pmatrix} = \begin{pmatrix} A_0\Phi_0 + \sum_{i=1}^{q} A_i\Phi(-h_i) + C_i\Phi(h_i)\\ \dfrac{\partial\Phi(\cdot)}{\partial\theta}\end{pmatrix}$$

with domain

$$D(\tilde{A}) = \left\{\begin{pmatrix}\Phi_0\\ \Phi(\cdot)\end{pmatrix} \in J_p : \Phi \text{ is absolutely continuous}, \frac{\partial\Phi}{\partial\theta} \in L_p([-h_q, 2h_q];\mathbb{R}^n)\right\}.$$

We have to show that the infinitesimal generator A equals $\tilde{A}$. Let α_0 be a sufficiently large real number such that the results of Lemma 3.4 hold. We will show that the inverse of $(\alpha_0 I - \tilde{A})$ equals $(\alpha_0 I - A)^{-1}$. This is enough to show that $A = \tilde{A}$. To this end, we calculate

$$(\alpha_0 I - \tilde{A})(\alpha_0 I - A)^{-1}\begin{pmatrix}\Phi_0\\ \Phi(\cdot)\end{pmatrix} = (\alpha_0 I - \tilde{A})\begin{pmatrix} g(0)\\ g(\cdot)\end{pmatrix} \text{(where g is as in Lemma 3.4)}$$

$$= \begin{pmatrix} \alpha_0 g(0) - A_0 g(0) - (\sum_{i=1}^{q} A_i g(-h_i) + C_i g(h_i))\\ \alpha_0 g(\cdot) - \dfrac{\partial g(\cdot)}{\partial\theta}\end{pmatrix} = \begin{pmatrix}\Phi_0\\ \Phi(\cdot)\end{pmatrix},$$

where the last equality holds by differentiating (3.5) from Lemma 3.4. Then, for $\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} \in J_p$, we have shown that

$$(\alpha_0 I - \tilde{A})(\alpha_0 I - A)^{-1} \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix}. \tag{3.8}$$

It remains to show that

$$(\alpha_0 I - A)^{-1}(\alpha_0 I - \tilde{A}) \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} \text{ in } D(A).$$

For $\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} \in D(A)$ we define $\begin{pmatrix} \Phi_1 \\ \Phi_1(\cdot) \end{pmatrix} := (\alpha_0 I - A)^{-1}(\alpha_0 I - \tilde{A}) \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix}$.

Then according to (3.8), we have $(\alpha_0 I - \tilde{A}) \begin{pmatrix} \Phi_1 \\ \Phi_1(\cdot) \end{pmatrix} = (\alpha_0 I - \tilde{A}) \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix}$. Then $\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ \Phi_1(\cdot) \end{pmatrix}$ if and only if $(\alpha_0 I - \tilde{A})$ is injective. Let us suppose, on the contrary, that there exists $\begin{pmatrix} \Phi_2 \\ \Phi_2(\cdot) \end{pmatrix} \in D(A)$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (\alpha_0 I - \tilde{A}) \begin{pmatrix} \Phi_2 \\ \Phi_2(\cdot) \end{pmatrix} = \begin{pmatrix} \alpha_0 \Phi_2(0) - A_0 \Phi_2(0) - L_d \Phi_2(0) - L_a \Phi_2(0) \\ \alpha_0 \Phi_2(\cdot) - \dfrac{\partial \Phi_2(\cdot)}{\partial \theta} \end{pmatrix},$$

where we have used the definition of $\tilde{A}$ and $D(\tilde{A})$ in the last two steps. Then

$$\begin{aligned} \Phi_2(\theta) = \Phi_2(0)e^{\alpha_0 \theta} \text{ and } & \alpha_0 \Phi_2(0) - A_0 \Phi_2(0) - (\sum_{i=1}^{q} A_i \Phi_2(-h_i) + C_i \Phi_2(h_i)) \\ &= \alpha_0 \Phi_2(0) - A_0 \Phi_2(0) - (\sum_{i=1}^{q} A_i \Phi_2(0) e^{-\alpha_0 h_i} + C_i \Phi_2(0) e^{\alpha_0 h_i}) = 0. \end{aligned}$$

However, since

$$\alpha_0 I - A_0 - (\sum_{i=1}^{q} A_i e^{-\alpha_0 h_i} + C_i e^{\alpha_0 h_i})$$

is invertible, this implies that $\Phi_2(0) = 0$ and thus $\Phi_2(\cdot) = \Phi_2(0)e^{-\alpha_0 \cdot} = 0$. This contradiction implies that $(\alpha_0 I - \tilde{A})$ is injective. This proves the assertion that A equals $\tilde{A}$.

Now, we calculate the spectrum of A. In Lemma 3.4 we obtained an expression for the resolvent operator for $\alpha \in \mathbb{R}$ large enough, in terms of g given by (3.5) and (3.6). Let us denote by Q_λ the extension of the resolvent operator to $\mathbb{C}$:

$$Q_\lambda \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} := \begin{pmatrix} g(0) \\ g(\cdot) \end{pmatrix}.$$

A simple calculation shows that if $\lambda \in \mathbb{C}$ satisfies

$$\det(\lambda I - A_0 - (\sum_{i=1}^{q} A_i e^{-\lambda h_i} + C_i e^{\lambda h_i})) \neq 0,$$

then Q_λ is a bounded linear operator from $J_{p_{\mathbb{C}}}$ to $J_{p_{\mathbb{C}}}$, where $J_{p_{\mathbb{C}}}$ is the closed linear subspace of pairs $\begin{pmatrix} r \\ f(\cdot) \end{pmatrix}$ in $\mathbb{C}^n \oplus L_p([-h_q, 2h_q]; \mathbb{C}^n)$ such that $r = f(0)$. Furthermore, for these λ we have $(\lambda I - A)Q_\lambda = I$ and $(\lambda I - A)$ is injective. As in the first part of the proof, we conclude that $Q_\lambda = (\lambda I - A)^{-1}$, the resolvent operator of A. We have that

$$\left\{ \lambda \in \mathbb{C} : \det(\lambda I - A_0 - (\sum_{i=1}^{q} A_i e^{-\lambda h_i} + C_i e^{\lambda h_i})) \neq 0 \right\} \subset \rho(A).$$

On the other hand, if $\det(\Delta(\lambda)) = 0$, there exists $z \in \mathbb{C}^n$ such that

$$(\lambda I - A_0 - (\sum_{i=1}^{q} A_i e^{-\lambda h_i} + C_i e^{\lambda h_i}))z = 0.$$

The following element of $J_{p\mathbb{C}}$

$$z_0 = \begin{pmatrix} z \\ e^{\lambda \cdot} z \end{pmatrix} \text{ is in } D(A)$$

and

$$(\lambda I - A)z_0 = \begin{pmatrix} \lambda z - A_0 z - (\sum_{i=1}^{q} A_i e^{-\lambda h_i} + C_i e^{\lambda h_i})z \\ \lambda e^{\lambda\theta} z - \frac{\partial}{\partial\theta} e^{\lambda\theta} z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then

$$\sigma_p(A) \supset \left\{ \lambda \in \mathbb{C} : \det(\lambda I - A_0 - (\sum_{i=1}^{q} A_i e^{-\lambda h_i} + C_i e^{\lambda h_i})) = 0 \right\}.$$

The remaining of the proof can be done, mutatis mutandis, as in Theorem 2.4.6 of [11]. ■

4. An application: Controllability

In this section we will apply some of the the results obtained in section 3 to study the null controllability for the system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{q} A_i x(t-h_i) + \sum_{i=1}^{q} C_i x(t+h_i) + Bu(t), \quad t > 0 \\ x(0) &= \Phi_0 \\ x(s) &= \Phi(s), \quad s \in [-h_q, 2h_q], \end{aligned} \tag{4.1}$$

where as before $0 < h_1 < h_2 < \cdots < h_q$, $A_i, C_i \in \mathscr{L}(\mathbb{R}^n)$, $i = 1, \cdots, q$, $A_i \neq 0$ for some i, $\Phi_0 \in \mathbb{R}^n$, $\Phi \in L_p([-h_q, 2h_q]; \mathbb{R}^n)$, $1 \leq p \leq \infty$.

Also for this case, we will consider $B \in \mathscr{L}(\mathbb{R}^n)$ and $u : [0, \infty) \to \mathbb{R}^n$ an essentially bounded function.

We have already shown that, if the problem is well-posed, (4.1) can be written equivalently as the following system of ordinary differential equations in J_p

$$\begin{aligned} \dot{w}(t) &= Aw(t) + \bar{B}u(t), \quad t > 0 \\ w(0) &= w_0 = (\Phi_0, \Phi(\cdot)), \end{aligned} \tag{4.2}$$

where A is the infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$ and $\bar{B} : \mathbb{R}^n \to J_p$ is given by $\bar{B}u = \begin{pmatrix} Bu \\ 0 \end{pmatrix}$.

The mild solution of (4.2) is thus given by

$$w(t) = T(t)w_0 + \int_0^t T(t-s)\bar{B}u(s)ds.$$

Let Ω be a non-empty compact convex subset of $\mathbb{R}^n$. The set

$$\tilde{\Omega}_r = \{u \in L^\infty_{\mathbb{R}^n}[0, r] : u \in \Omega \text{ a.e}\}$$

is called the set of *admissible controls* of (4.2) (or equivalently (4.1)), while the set

$$A_r(w_0) = \left\{ T(r)w_0 + \int_0^r T(r-s)\bar{B}u(s)ds : u \in \tilde{\Omega}_r \right\}$$

is the set of *accesible points* of (4.2). The system (4.2) is *controllable* if $0 \in A_r(w_0)$.

In a more general context, we have a system similar to (4.2), with X and U Banach spaces, $A : X \to X$ the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$, $B : U \to X$ a bounded linear operator and $u : [0, \infty) \to U$ a strongly measurable, essentially bounded function. We suppose that Ω is a non-empty separable, weakly compact subset of U. The formula for the mild solution is completely similar, $\tilde{\Omega}_r = \{u \in L^\infty_U[0, r] : u \in \Omega \text{ a.e}\}$ is the set of admissible controls, while $A_r(w_0) = \left\{ S(r)w_0 + \int_0^r S(r-s)Bu(s)ds : u \in \tilde{\Omega}_r \right\}$ is the set of accesible points. Analogously, the system is controllable if $0 \in A_r(w_0)$.

The controllability map on $[0, r]$ for some $r \geq 0$ is the linear map

$$B^r : L_\infty([0, r]; U) \to X$$

defined by

$$B^r u = \int_0^r S(r-s)Bu(s)\mathrm{d}s$$

Now, one says that the system is exactly controllable on $[0,r]$ if every point in X can be reached from the origin at r, i.e., if $ran\,(B^r) = X$.

If $ran\,(B^r) = X$, then $0 \in A_r(0)$. On the other hand, one can prove, using the Open Mapping Theorem, the following: if $0 \in interior(A_r(0))$, then $ran\,(B^r) = X$. See ([14])

Next, we recall two results that we will use to characterize the null controllability. The Theorem of Peichl and Schappacher ([15]) is as follows:

Theorem 4.1. *Let X and U be reflexive Banach spaces with U separable. Let $B : U \to X$ be a bounded linear operator, A be the infinitesimal generator of a c_0-semigroup $\{S(s)\}_{s\geq 0}$ of operators on X and Ω be a weakly compact convex subset of U that contains* 0*. Then for each $T > 0$, $0 \in A_T(x_o)$ if and only if for each $x^* \in X^*$*

$$< x^*, S(T)x_0 > + \int_0^T \max_{v\in\Omega} < x^*, S(t)Bv > dt \ \geq 0.$$

Additionally, we have the Bárcenas-Diestel ([16]) extension

Theorem 4.2. *Let X and U be Banach spaces, let $B : U \to X$ be a bounded linear operator, and $A : X \to X$ be the infinitesimal generator of a c_0-semigroup $\{S(t)\}_{t\geq 0}$ on X whose dual semigroup is strongly continuous on $(0,\infty)$. Suppose Ω is a non-empty separable weakly compact convex subset of U containing* 0*. Then for each $T > 0$, $0 \in A_T(x_o)$ if and only if for each $x^* \in X^*$*

$$< x^*, S(T)x_0 > + \int_0^T \max_{v\in\Omega} < x^*, S(t)Bv > dt \ \geq 0.$$

Theorems 4 and 5 show how to set the control problems in a Banach Space context, focusing on the question of accessibility of controls. For separable reflexive spaces, the elegant result of Peichl-Schappacher proves to be very useful.

The Bárcenas-Diestel Theorem is, on the other hand, an important and recent achievement on exact controllability. Throughout the literature, hypotheses like "separable and reflexive" are frequently encountered. By employing techniques from Banach space theory and the theory of vector measures, the authors show how to remove the hypothesis of reflexivity (thus giving considerably greater generality to the resulting conclusions) and translate the question of accessibility of controls to a problem in semigroups of operators, namely, given a c_0-semigroup $(S(t))_{t\geq 0}$ of operators on a Banach space X, under what conditions is the dual semigroup strongly continuous on $(0,\infty)$? This is the question we will try to answer for the non-reflexive cases $p = 1$ and $p = \infty$

We recall that a Banach space is a *Grothendieck space* if every weakly*-convergent sequence in X^* is also weakly convergent. Equivalently, X is a Grothendieck space if every linear bounded operator from X to any separable Banach space is weakly compact. Among Grothendieck spaces, we will list all reflexive Banach spaces and $L^\infty(\Omega,\Sigma,\mu)$, where (Ω,Σ,μ) is a positive measure space. A Banach space isomorphic to a complemented subspace of a Grothendieck space is also a Grothendieck space. The direct sum of two Grothendieck spaces is also a Grothendieck space. Several characterizations of Grothendieck spaces are found in [17].

A Banach space is said to have the *Dunford-Pettis property* if every weakly compact operator in $L(X)$ applies relatively weakly compact sets onto norm compact sets. The most common examples of Banach spaces with this property are $L^1(\mu)$ and $C(K)$. Complemented subspaces and the direct sum of any two of such spaces also have the property. For more details, see [18].

If X is a Grothendieck space with the Dunford-Pettis property, Lotz ([19]) has shown that every strongly continuous semigroup is uniformly continuous, and therefore also is the adjoint semigroup.

We also recall that a bounded linear operator $T : X \to Y$ (where X and Y are Banach Spaces) *factors through a Banach space* Z if there are bounded linear operators $u\colon X \to Z$ and $v\colon Z \to Y$ such that $T = vu$

It is proven in [20] that if X is a Banach space and $\{T(t)\}_{t\geq 0}$ a c_0-semigroup defined on X such that for every $a > 0$ there exists a Grothendieck space Y_a such that $T(a)$ factors through Y_a, then $\{T^*(t)\}_{t\geq 0}$ is strongly continuous on $(0,\infty)$. This will prove useful to establish our main result for the case $p = 1$.

Factoring through Grothendieck spaces is, in general, not easy to verify, but among semigroups satisfying those assumptions (and, hence, having adjoints which are strongly continuous on $(0,\infty)$) we mention weakly compact semigroups, i.e, semigroups such that $T(t)$ is weakly compact for each t (see [20] for more details). There are many examples of weakly compact semigroups, a category that includes all compact semigroups. Moreover, for $p = 1$ the terms "weakly compact" and "compact" are equivalent, due to the classical Schur theorem.

It is true that those assumptions cannot be verified without any analysis of the semigroup, which is here presented in an abstract, general form. But provided that $x(t)$ and $\Phi(\cdot)$ are known, one can manage to get more precise information about it.

Finally, one should remember that all those considerations are relevant only for the case $p = 1$. For all other cases, no additional assumptions are needed.

Now, we can state the result concerning (4.2)

Theorem 4.3. *For each $r > 0$, $0 \in A_r(w_0)$ if and only if for each $x^* \in J_p^*$, $1 < p \leq \infty$,*

$$< x^*, T(r)w_0 > + \int_0^r \max_{v \in \Omega} < x^*, T(t)\bar{B}v(t) > dt \ \geq 0.$$

If additionally, we suppose that the associated semigroup satisfies that, for every $a > 0$ there exists a Grothendieck space Y_a such that $T(a)$ factors through Y_a, (in particular, if it is compact) then the same holds for $p = 1$.

Proof . The case $p \in (1, \infty)$ is an inmediate consequence of Theorem 4.1. We only have to remember that the direct sum of any two reflexive Banach spaces and every subspace of a reflexive Banach space are also reflexive.

Semigroups which factor through Grothendieck spaces have adjoints $\{T^*(t)\}_{t \geq 0}$ which are strongly continuous on $(0, \infty)$. Then Theorem 4.2 can be applied for the case $p = 1$.

Now, let us suppose $p = \infty$. Note that $\mathbb{R}^n$ and $L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$ are Grothendieck spaces with the Dunford-Pettis property (remember that $L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$ is isomorphic to $C(K)$ for some suitable compact Hausdorff space K, see [21]). Consequently, $\mathbb{R}^n \oplus L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$ is also a Grothendieck space with the Dunford-Pettis property, and so is the complemented subspace J_∞. Therefore, the associated semigroup $\{T(t)\}_{t \geq 0}$ is uniformly continuous, according to the Lotz Theorem [19]. In particular, the adjoint semigroup $\{T^*(t)\}_{t \geq 0}$ is uniformly continuous, and we can apply Theorem 4.2 again. ■

As a conclusion, let us indicate that the results obtained in this work can be applied to certain mixed-type systems of partial differential equations like the following

$$\begin{aligned} \frac{\partial x(t,y)}{\partial t} &= D\Delta x(t,y) + \sum_i^q A_i x(t-h_i, y) + \sum_i^q C_i x(t+h_i, y) + Bu(t,y), \\ \frac{\partial x}{\partial \eta} &= 0, \ \ y \in \partial\Omega, \\ x(0,y) &= \Phi_0(y), \ \ y \in \Omega \\ x(s,y) &= \Phi(s,y), \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $t \in (0, r]$, $0 < h_1 < h_2 < \cdots < h_q$, D is an $n \times n$ nondiagonal matrix whose eigenvalues are semi-simple with nonnegative real part, $B, A_i, C_i \in \mathscr{L}(\mathbb{R}^n)$, $i = 1, 2, \ldots, q, A_i \neq 0$ for some i, $\Phi_0 \in \mathbb{R}^n$, the control $u : [0, \infty) \to \mathbb{R}^n$ is essentially bounded and $\Phi \in L_p[[-h_q, 2h_q]; \mathbb{R}^n]$, $1 \leq p \leq \infty$, is defined by

$$\Phi(s,y) = \begin{cases} \Phi_1(s,y), & s \in [-h_q, 0], \ \ y \in \Omega \\ \Phi_2(s,y), & s \in [0, 2h_q], \ \ y \in \Omega \end{cases}$$

The symbol η denotes the normal to $\partial\Omega$, and $\frac{\partial x}{\partial \eta}$ is the normal derivative, which is defined as the inner product of the gradient ∇x with the (unit) normal vector η. The condition $\frac{\partial x}{\partial \eta} = 0$ for $y \in \partial\Omega$ and $t \in (0, r]$ is thus an homogeneous Neumann condition.

We would like to finish with a brief note about the particular case of delay equations. Several interesting examples of this type are found in the literature. Among them we have systems of parabolic equations with delay (including particular cases of the nD heat equation and systems without diffusion coefficients), and in general a broad class of functional reaction-diffusion equations (see, for example, [12]). But there is now an important difference: in all those examples the function Φ is supposed to lie in a Hilbert space, while here it is allowed to belong to a L_p-space, $1 \leq p \leq \infty$. This in turn allows to study these classical equations (and, in particular, their null controllability) in a considerably more general context.

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On the Periodic Solutions of Some Systems of Difference Equations

E. M. Elsayed[1,2]* and H. S. Gafel[1]

Abstract
In this paper, we study the solution of the systems of difference equations

$$x_{n+1}=\frac{1\pm(y_n+x_{n-1})}{y_{n-2}},\quad y_{n+1}=\frac{1\pm(x_n+y_{n-1})}{x_{n-2}},\ n=0,1,...,$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers.

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[1] *King AbdulAziz University, Faculty of Science, Mathematics Department, P. O. Box 80203, Jeddah 21589, Saudi Arabia.*
[2] *Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.*
***Corresponding author**: emmelsayed@yahoo.com
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1. Introduction

Difference equations enter as approximations of continuous problems and as models describing life situations in many directions. Recently, there has been great interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques that can be used in investigating equations arising in mathematical models describing real-life situations in population biology, economic, probability theory, genetics and psychology see [1]-[25].

In [1] Alzahrani et al. found the form of solutions for the following systems of rational difference equations

$$x_{n+1}=\frac{y_ny_{n-2}}{\pm y_{n-2}\pm x_{n-3}},\quad y_{n+1}=\frac{x_nx_{n-2}}{\pm x_{n-2}\pm y_{n-3}}.$$

In [2] Asiri et al. studied the form of the solutions and the periodicity of the following third order systems of rational difference equations

$$x_{n+1}=\frac{y_{n-2}}{1-y_{n-2}x_{n-1}y_n},\quad y_{n+1}=\frac{x_{n-2}}{\pm 1\pm x_{n-2}y_{n-1}x_n}.$$

In [14] Elsayed et al. got the form of the solutions of the following difference equation systems of order four

$$x_{n+1}=\frac{x_{n-2}y_n}{y_{n-3}+y_n},\quad y_{n+1}=\frac{x_ny_{n-2}}{\pm x_{n-3}\pm x_n}.$$

In [4] Cinar studied the solutions of the systems of the difference equations.

$$x_{n+1}=\frac{1}{y_n},\quad y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}}.$$

In [23] Papaschinnopoulos and Schinas studied the oscillatory behavior, the boundedness of the solutions,and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \quad y_{n+1} = A + \frac{x_n}{y_{n-q}}.$$

In [13] Elsayed has obtained the solution of the following system of the difference equations

$$x_{n+1} = \frac{1}{y_{n-k}}, \quad y_{n+1} = \frac{y_{n-k}}{x_n y_n}.$$

The behaviour of the positive solution of the following system

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1} y_n}, \quad y_{n+1} = \frac{y_{n-1}}{1 + x_n y_{n-1}}$$

has been studied by Kurbanli et al. [19].

In [25] Yalcinkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations

$$z_{n+1} = \frac{a + t_n z_{n-1}}{t_n + z_{n-1}}, \quad t_{n+1} = \frac{a + z_n t_{n-1}}{z_n + t_{n-1}}.$$

The aim of this article is to obtain the expressions of the solutions of the following systems of difference equations

$$x_{n+1} = \frac{1 \pm (y_n + x_{n-1})}{y_{n-2}}, \quad y_{n+1} = \frac{1 \pm (x_n + y_{n-1})}{x_{n-2}}, \quad n = 0, 1, 2, ...,$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers. Moreover, we obtain some numerical simulation to the equation are given to illustrate our results.

Definition (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2. On the system $x_{n+1} = \frac{1+y_n+x_{n-1}}{y_{n-2}}$, $y_{n+1} = \frac{1+x_n+y_{n-1}}{x_{n-2}}$

In this section, we study the solution of the following system of difference equations

$$x_{n+1} = \frac{1 + y_n + x_{n-1}}{y_{n-2}}, \quad y_{n+1} = \frac{1 + x_n + y_{n-1}}{x_{n-2}}, \tag{2.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers.

2.1 Periodicity of the solutions of system (2.1)

The following theorem is devoted to the periodicity of the solutions of system (2.1).

Theorem 1. Suppose that $\{x_n, y_n\}_{n=1}^{\infty}$ be a solution of system (2.1). Then all solutions of system (2.1) are periodic with period eight.

Proof. From Eq.(2.1), we see that

$$\begin{aligned}
x_{n+1} &= \frac{1 + y_n + x_{n-1}}{y_{n-2}}, \quad y_{n+1} = \frac{1 + x_n + y_{n-1}}{x_{n-2}}, \\
x_{n+2} &= \frac{1 + y_{n+1} + x_n}{y_{n-1}} = \frac{1 + x_{n-2} + x_n + y_{n-1} + x_n x_{n-2}}{y_{n-1} x_{n-2}}, \\
y_{n+2} &= \frac{1 + x_{n+1} + y_n}{x_{n-1}} = \frac{y_{n-2} + 1 + y_n + x_{n-1} + y_{n-2} y_n}{y_{n-2} x_{n-1}}, \\
x_{n+3} &= \frac{1 + y_{n+2} + x_{n+1}}{y_n} \\
&= \frac{y_{n-2} x_{n-1} + y_{n-2} + 1 + y_n + 2x_{n-1} + y_{n-2} y_n + x_{n-1} y_n + (x_{n-1})^2}{y_{n-2} x_{n-1} y_n}, \\
y_{n+3} &= \frac{1 + x_{n+2} + y_{n+1}}{x_n} \\
&= \frac{y_{n-1} x_{n-2} + 1 + x_{n-2} + x_n + 2y_{n-1} + x_n x_{n-2} + y_{n-1} x_n + (y_{n-1})^2}{y_{n-1} x_{n-2} x_n},
\end{aligned}$$

$$
\begin{aligned}
x_{n+4} &= \frac{1+y_{n+3}+x_{n+2}}{y_{n+1}} \\
&= \frac{\begin{array}{c} y_{n-1}x_{n-2}x_n+y_{n-1}x_{n-2}+1+x_{n-2}+x_n+2y_{n-1}+x_nx_{n-2} \\ +y_{n-1}x_n+(y_{n-1})^2+x_n(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2}) \end{array}}{(1+x_n+y_{n-1})y_{n-1}x_n} \\
&= \frac{(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2})(1+x_n+y_{n-1})}{(1+x_n+y_{n-1})y_{n-1}x_n} \\
&= \frac{1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2}}{y_{n-1}x_n},
\end{aligned}
$$

$$
\begin{aligned}
y_{n+4} &= \frac{1+x_{n+3}+y_{n+2}}{x_{n+1}} \\
&= \frac{\begin{array}{c} x_{n-1}y_ny_{n-2}+y_{n-2}x_{n-1}+y_{n-2}+1+y_n+2x_{n-1}+y_{n-2}y_n \\ +x_{n-1}y_n+(x_{n-1})^2+y_n(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n) \end{array}}{x_{n-1}y_n(1+y_n+x_{n-1})} \\
&= \frac{(1+y_n+x_{n-1})(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)}{x_{n-1}y_n(1+y_n+x_{n-1})} \\
&= \frac{y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n}{x_{n-1}y_n},
\end{aligned}
$$

$$
\begin{aligned}
x_{n+5} &= \frac{1+y_{n+4}+x_{n+3}}{y_{n+2}} \\
&= \frac{\begin{array}{c} y_{n-2}x_{n-1}y_n+y_{n-2}(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)+ \\ y_{n-2}x_{n-1}+y_{n-2}+1+y_n+2x_{n-1}+y_{n-2}y_n+x_{n-1}y_n+(x_{n-1})^2 \end{array}}{y_n(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)} \\
&= \frac{(x_{n-1}+y_{n-2}+1)(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)}{y_n(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)} \\
&= \frac{x_{n-1}+y_{n-2}+1}{y_n},
\end{aligned}
$$

$$
\begin{aligned}
y_{n+5} &= \frac{1+x_{n+4}+y_{n+3}}{x_{n+2}} \\
&= \frac{\begin{array}{c} y_{n-1}x_{n-2}x_n+x_{n-2}(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2})+ \\ y_{n-1}x_{n-2}+1+x_{n-2}+x_n+2y_{n-1}+x_nx_{n-2}+y_{n-1}x_n+(y_{n-1})^2 \end{array}}{x_n(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2})} \\
&= \frac{(1+y_{n-1}+x_{n-2})(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2})}{x_n(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2})} \\
&= \frac{1+y_{n-1}+x_{n-2}}{x_n},
\end{aligned}
$$

$$
\begin{aligned}
x_{n+6} &= \frac{1+y_{n+5}+x_{n+4}}{y_{n+3}} \\
&= \frac{y_{n-1}x_n+y_{n-1}(1+y_{n-1}+x_{n-2})+1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2}}{\frac{(y_{n-1}x_{n-2}+1+x_{n-2}+x_n+2y_{n-1}+x_nx_{n-2}+y_{n-1}x_n+(y_{n-1})^2)}{x_{n-2}}} = x_{n-2}, \\
y_{n+6} &= \frac{1+x_{n+5}+y_{n+4}}{x_{n+3}} \\
&= \frac{x_{n-1}y_n+x_{n-1}(x_{n-1}+y_{n-2}+1)+y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n}{\frac{y_{n-2}x_{n-1}+y_{n-2}+1+y_n+2x_{n-1}+y_{n-2}y_n+x_{n-1}y_n+(x_{n-1})^2}{y_{n-2}}} = y_{n-2},
\end{aligned}
$$

$$x_{n+7} = \frac{1+y_{n+6}+x_{n+5}}{y_{n+4}} = \frac{y_n+y_ny_{n-2}+x_{n-1}+y_{n-2}+1}{\frac{(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)}{x_{n-1}}} = x_{n-1},$$
$$y_{n+7} = \frac{1+x_{n+6}+y_{n+5}}{x_{n+4}} = \frac{x_n+x_nx_{n-2}+1+y_{n-1}+x_{n-2}}{\frac{1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2}}{y_{n-1}}} = y_{n-1},$$
$$x_{n+8} = \frac{1+y_{n+7}+x_{n+6}}{y_{n+5}} = \frac{1+y_{n-1}+x_{n-2}}{\frac{1+y_{n-1}+x_{n-2}}{x_n}} = x_n,$$
$$y_{n+8} = \frac{1+x_{n+7}+y_{n+6}}{x_{n+5}} = \frac{1+x_{n-1}+y_{n-2}}{\frac{x_{n-1}+y_{n-2}+1}{y_n}} = y_n.$$

Thus, the solutions are periodic with period eight.

2.2 The form of the solutions of system (2.1)

The following theorem describes the form of the solutions of system (2.1).

Theorem 2. Suppose that $\{x_n, y_n\}$ are solutions of the system (2.1). Then for $n = 0, 1, 2, ...$, we have the following formulas

$$x_{8n-2} = c,\quad x_{8n-1} = b,\quad x_{8n} = a,\quad x_{8n+1} = \frac{1+d+b}{f},$$
$$x_{8n+2} = \frac{ac+a+c+e+1}{ec},\quad x_{8n+3} = \frac{b^2+bd+bf+df+2b+d+f+1}{fbd},$$
$$x_{8n+4} = \frac{ac+a+c+e+1}{ea},\quad x_{8n+5} = \frac{1+f+b}{d},$$

$$y_{8n-2} = f,\quad y_{8n-1} = e,\quad y_{8n} = d,\quad y_{8n+1} = \frac{1+a+e}{c},$$
$$y_{8n+2} = \frac{df+b+d+f+1}{fb},\quad y_{8n+3} = \frac{e^2+ac+ae+ce+2e+a+c+1}{cea},$$
$$y_{8n+4} = \frac{df+b+d+f+1}{bd},\quad y_{8n+5} = \frac{1+c+e}{a},$$

where the initial conditions $x_{-2} = c,\ x_{-1} = b,\ x_0 = a,\ y_{-2} = f,\ y_{-1} = e,\ y_0 = d$.

Or equivalently

$$\{x_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} c, b, a, \frac{1+d+b}{f}, \frac{ac+a+c+e+1}{ec}, \frac{b^2+bd+bf+df+2b+d+f+1}{fbd} \\ , \frac{ac+a+c+e+1}{ea}, \frac{1+f+b}{d}, c, b, a, ... \end{array} \right\},$$

$$\{y_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} f, e, d, \frac{1+a+e}{c}, \frac{df+b+d+f+1}{fb}, \frac{e^2+ac+ae+ce+2e+a+c+1}{cea}, \\ \frac{df+b+d+f+1}{bd}, \frac{1+c+e}{a}, f, e, d, ... \end{array} \right\}.$$

Proof. For $n = 0$ the result holds. Suppose that the result holds for $n-1$.

$$x_{8n-10} = c,\quad x_{8n-9} = b,\quad x_{8n-8} = a,\quad x_{8n-7} = \frac{1+d+b}{f},$$
$$x_{8n-6} = \frac{ac+a+c+e+1}{ec},\quad x_{8n-5} = \frac{b^2+bd+bf+df+2b+d+f+1}{fbd},$$
$$x_{8n-4} = \frac{ac+a+c+e+1}{ea},\quad x_{8n-3} = \frac{1+f+b}{d},$$

$$y_{8n-10} = f,\quad y_{8n-9} = e,\quad y_{8n-8} = d,\quad y_{8n-7} = \frac{1+a+e}{c},$$
$$y_{8n-6} = \frac{df+b+d+f+1}{fb},\quad y_{8n-5} = \frac{e^2+ac+ae+ce+2e+a+c+1}{cea},$$
$$y_{8n-4} = \frac{df+b+d+f+1}{bd},\quad y_{8n-3} = \frac{1+c+e}{a},$$

from system (2.1) we can prove as follow

$$\begin{aligned}
x_{8n-2} &= \frac{1+y_{8n-3}+x_{8n-4}}{y_{8n-5}} = \frac{1+\frac{1+c+e}{a}+\frac{ac+a+c+e+1}{ea}}{\frac{e^2+ac+ae+ce+2e+a+c+1}{cea}} \\
&= \frac{c(ea+e(1+c+e)+ac+a+c+e+1)}{e^2+ac+ae+ce+2e+a+c+1} = c.
\end{aligned}$$

Also, we get

$$\begin{aligned}
x_{8n-1} &= \frac{1+y_{8n-2}+x_{8n-3}}{y_{8n-4}} = \frac{1+f+\frac{1+f+b}{d}}{\frac{df+b+d+f+1}{bd}} \\
&= \frac{b(d+fd+1+f+b)}{fd+b+d+f+1} = b, \\
x_{8n} &= \frac{1+y_{8n-1}+x_{8n-2}}{y_{8n-3}} = \frac{1+e+c}{\frac{1+c+e}{a}} = a, \\
x_{8n+1} &= \frac{1+y_{8n}+x_{8n-1}}{y_{8n-2}} = \frac{1+d+b}{f}, \\
x_{8n+2} &= \frac{1+y_{8n+1}+x_{8n}}{y_{8n-1}} = \frac{1+\frac{1+a+e}{c}+a}{e} = \frac{1+a+e+ac+c}{ce}, \\
x_{8n+3} &= \frac{1+y_{8n+2}+x_{8n+1}}{y_{8n}} = \frac{1+\frac{df+b+d+f+1}{fb}+\frac{1+d+b}{f}}{d} \\
&= \frac{fb+df+b+d+f+1+b(1+d+b)}{fbd} \\
&= \frac{b^2+bd+bf+df+2b+d+f+1}{fbd}, \\
x_{8n+4} &= \frac{1+y_{8n+3}+x_{8n+2}}{y_{8n+1}} = \frac{1+\frac{e^2+ac+ae+ce+2e+a+c+1}{cea}+\frac{1+a+e+ac+c}{ce}}{\frac{1+a+e}{c}} \\
&= \frac{cea+e^2+ae+ce+e+(ac+e+a+c+1)+a(1+a+e+ac+c)}{ea(1+a+e)} \\
&= \frac{ac+a+c+e+1}{ea}, \\
x_{8n+5} &= \frac{1+y_{8n+4}+x_{8n+3}}{y_{8n+2}} = \frac{1+\frac{df+b+d+f+1}{bd}+\frac{b^2+bd+bf+df+2b+d+f+1}{fbd}}{\frac{df+b+d+f+1}{fb}} \\
&= \frac{b^2+bd+bf+df+2b+d+f+1+f(df+b+d+f+1)+fbd}{d(df+b+d+f+1)} \\
&= \frac{1+f+b}{d}, \\
y_{8n-2} &= \frac{1+x_{8n-3}+y_{8n-4}}{x_{8n-5}} = \frac{1+\frac{b+f+1}{d}+\frac{f+1+d+b+df}{bd}}{\frac{b^2+bd+bf+df+2b+d+f+1}{fbd}} = \frac{\frac{b^2+bf+2b+bd+f+d+1+df}{bd}}{\frac{b^2+bd+bf+df+2b+d+f+1}{fbd}} = f, \\
y_{8n-1} &= \frac{1+x_{8n-2}+y_{8n-3}}{x_{8n-4}} = \frac{1+c+\frac{c+e+1}{a}}{\frac{ac+a+c+e+1}{ea}} = \frac{e(a+ca+c+e+1)}{ac+a+c+e+1} = e, \\
y_{8n} &= \frac{1+x_{8n-1}+y_{8n-2}}{x_{8n-3}} = \frac{1+b+f}{\frac{1+b+f}{d}} = d, \\
y_{8n+1} &= \frac{1+x_{8n}+y_{8n-1}}{x_{8n-2}} = \frac{1+e+a}{c}, \\
y_{8n+2} &= \frac{1+x_{8n+1}+y_{8n}}{x_{8n-1}} = \frac{1+\frac{1+d+b}{f}+d}{b} = \frac{f+1+d+b+df}{fb},
\end{aligned}$$

$$\begin{aligned}
y_{8n+3} &= \frac{1+x_{8n+2}+y_{8n+1}}{x_{8n}} = \frac{1+\frac{1+a+e+ac+c}{ce}+\frac{1+e+a}{c}}{a} \\
&= \frac{e^2+ac+ae+ce+2e+a+c+1}{cea},
\end{aligned}$$

$$\begin{aligned}
y_{8n+4} &= \frac{1+x_{8n+3}+y_{8n+2}}{x_{8n+1}} = \frac{1+\frac{b^2+bd+bf+df+2b+d+f+1}{fbd}+\frac{f+1+d+b+df}{fb}}{\frac{1+d+b}{f}} \\
&= \frac{b^2+fbd+b+bd+bf+(df+b+d+f+1)+d(f+1+d+b+df)}{bd(1+d+b)} \\
&= \frac{f+1+d+b+df}{bd}, \\
y_{8n+5} &= \frac{1+x_{8n+4}+y_{8n+3}}{x_{8n+2}} = \frac{1+\frac{ac+a+c+e+1}{ea}+\frac{e^2+ac+ae+ce+2e+a+c+1}{cea}}{\frac{1+a+e+ac+c}{ce}} \\
&= \frac{c(ac+a+c+e+1)+e^2+cea+ce+ae+e+ac+e+a+c+1}{a(1+a+e+ac+c)} \\
&= \frac{c+e+1}{a}.
\end{aligned}$$

This completes the proof.

2.3 Numerical examples

For confirming the results of this section, we consider the following numerical example which represent solutions to system (2.1).

Example 1. We consider interesting numerical example for the difference equations system (2.1) with the initial conditions $x_{-2}=2,\ x_{-1}=-0.7,\ x_0=0.9,\ y_{-2}=3,\ y_{-1}=-0.5$ *and* $y_0=11$. (See Fig. 2.1).

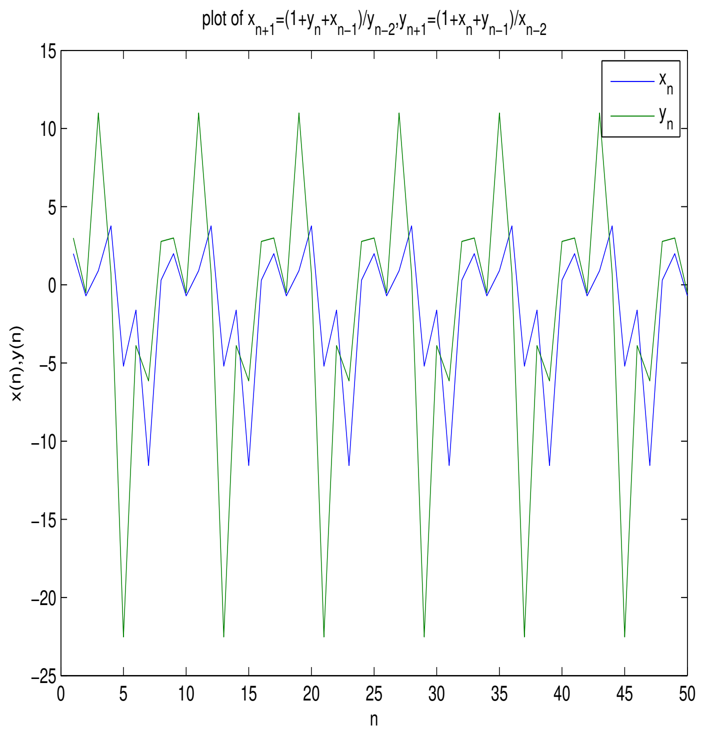

Figure 2.1

The following cases can be proved similarly.

3. On The System $x_{n+1} = \frac{1-(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1} = \frac{1+(x_n+y_{n-1})}{x_{n-2}}$

In this section we study the solution of the following system of difference equations

$$x_{n+1} = \frac{1-(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1} = \frac{1+(x_n+y_{n-1})}{x_{n-2}}, \tag{3.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$, are arbitrary non zero real numbers.

Theorem 3. Let $\{x_n, y_n\}_{n=-2}^{+\infty}$ be solutions of system (3.1). Then

1- $\{x_n\}_{n=-2}^{+\infty}$ and $\{y_n\}_{n=-2}^{+\infty}$ and are periodic with period eight i.e.,

$$x_{n+8} = x_n,\ y_{n+8} = y_n$$

for $n \geq -2$.

2- We have the following form of the solutions

$$\begin{aligned}
x_{8n-2} &= c,\quad x_{8n-1} = b,\quad x_{8n} = a,\quad x_{8n+1} = -\frac{-1+d+b}{f},\\
x_{8n+2} &= -\frac{ac+a-c+e+1}{ec},\quad x_{8n+3} = \frac{b^2+bd+bf-df+d-f-1}{fbd},\\
x_{8n+4} &= \frac{ac+a-c-e+1}{ea},\quad x_{8n+5} = \frac{1+f+b}{d},
\end{aligned}$$

$$\begin{aligned}
y_{8n-2} &= f,\quad y_{8n-1} = e,\quad y_{8n} = d,\quad y_{8n+1} = \frac{1+a+e}{c},\\
y_{8n+2} &= -\frac{-df+b+d-f-1}{fb},\quad y_{8n+3} = -\frac{-e^2+ac-ae-ce+a-c+1}{cea},\\
y_{8n+4} &= -\frac{df+b-d+f+1}{bd},\quad y_{8n+5} = -\frac{c+e-1}{a},
\end{aligned}$$

or equivalently

$$\{x_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} c, b, a, -\frac{-1+d+b}{f}, -\frac{ac+a-c+e+1}{ec}, \frac{b^2+bd+bf-df+d-f-1}{fbd}, \\ \frac{ac+a-c-e+1}{ea}, \frac{1+f+b}{d}, c, b, a, \ldots \end{array} \right\},$$

$$\{y_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} f, e, d, \frac{1+a+e}{c}, -\frac{-df+b+d-f-1}{fb}, -\frac{-e^2+ac-ae-ce+a-c+1}{cea}, \\ -\frac{df+b-d+f+1}{bd}, -\frac{c+e-1}{a}, f, e, d, \ldots \end{array} \right\}.$$

where the initial conditions $x_{-2} = c,\ x_{-1} = b,\ x_0 = a,\ y_{-2} = f,\ y_{-1} = e,\ y_0 = d$.

Example 2. We consider example for the difference system (3.1) where the initial conditions $x_{-2} = -5,\ x_{-1} = 7,\ x_0 = 2,\ y_{-2} = -3,\ y_{-1} = 1.3$ *and* $y_0 = 3$. (See Fig. 3.1).

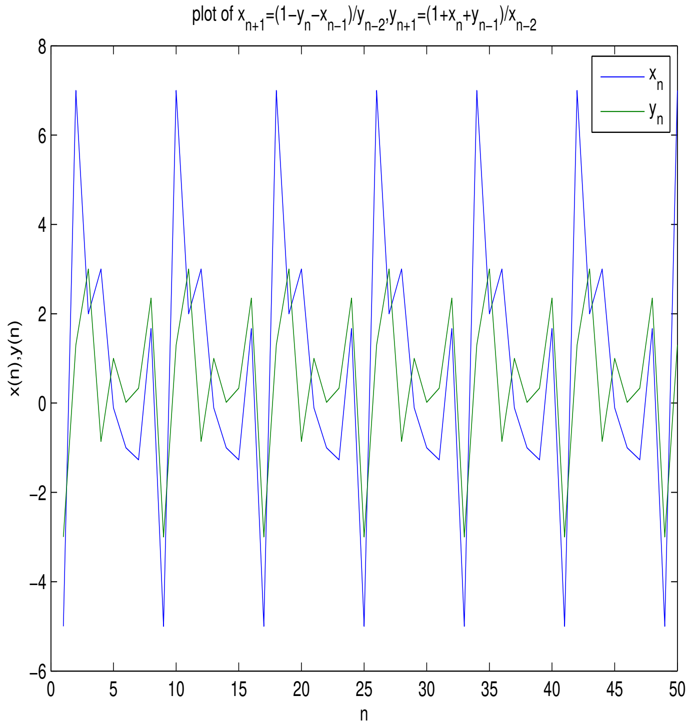

Figure 3.1

4. On the system $x_{n+1} = \frac{1+(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1} = \frac{1-(x_n+y_{n-1})}{x_{n-2}}$

In this section, we investigate the solution of the following system of difference equations

$$x_{n+1} = \frac{1+(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1} = \frac{1-(x_n+y_{n-1})}{x_{n-2}}, \tag{4.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$, are arbitrary non zero real numbers.
Theorem 4. Suppose that $\{x_n, y_n\}$ are solutions of the system (4.1). Then for $n = 0, 1, 2, ...$

1- $\{x_n\}_{n=-2}^{+\infty}$ and $\{y_n\}_{n=-2}^{+\infty}$ and are periodic with period eight i.e.,

$$x_{n+8} = x_n, \quad y_{n+8} = y_n,$$

for $n \geq -2$.

2- We have the following formulas

$$\begin{aligned}
x_{8n-2} &= c, \quad x_{8n-1} = b, \quad x_{8n} = a, \quad x_{8n+1} = \frac{1+d+b}{f}, \\
x_{8n+2} &= \frac{ac-a+c-e+1}{ec}, \quad x_{8n+3} = \frac{b^2+bd+bf-df-d+f-1}{fbd}, \\
x_{8n+4} &= -\frac{ac-a+c+e+1}{ea}, \quad x_{8n+5} = -\frac{f+b-1}{d}, \\
y_{8n-2} &= f, \quad y_{8n-1} = e, \quad y_{8n} = d, \quad y_{8n+1} = -\frac{a+e-1}{c}, \\
y_{8n+2} &= -\frac{df+b+d-f+1}{fb}, \quad y_{8n+3} = -\frac{-e^2+ac-ae-ce-a+c+1}{cea}, \\
y_{8n+4} &= -\frac{-df+b-d+f-1}{bd}, \quad y_{8n+5} = \frac{1+c+e}{a}.
\end{aligned}$$

Or equivalently

$$\{x_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} c,b,a,\frac{1+d+b}{f},\frac{ac-a+c-e+1}{ec},\frac{b^2+bd+bf-df-d+f-1}{fbd}, \\ -\frac{ac-a+c+e+1}{ea},-\frac{f+b-1}{d},c,b,a,... \end{array} \right\},$$

$$\{y_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} f,e,d,-\frac{a+e-1}{c},-\frac{df+b+d-f+1}{fb},-\frac{-e^2+ac-ae-ce-a+c+1}{cea}, \\ -\frac{-df+b-d+f-1}{bd},\frac{1+c+e}{a},f,e,d,,... \end{array} \right\}.$$

Example 3. We assume $x_{-2} = 1.3,\ x_{-1} = -7,\ x_0 = 2,\ y_{-2} = -3,\ y_{-1} = 9$ *and* $y_0 = -4$ for system (3.1) see Fig. 4.1.

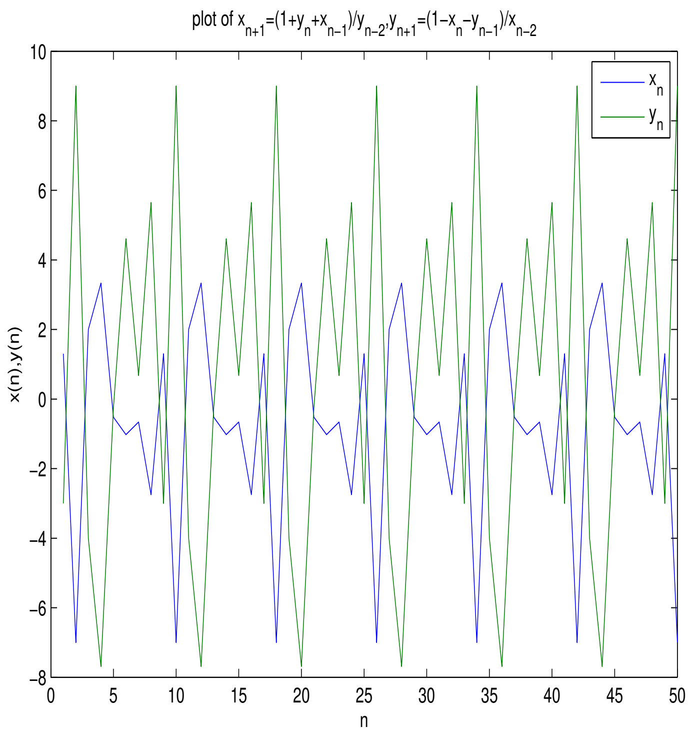

Figure 4.1

5. On the system $x_{n+1} = \frac{1-(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1} = \frac{1-(x_n+y_{n-1})}{x_{n-2}}$

In this section we study the solution of the following system of difference equations

$$x_{n+1} = \frac{1-(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1} = \frac{1-(x_n+y_{n-1})}{x_{n-2}}, \tag{5.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$, are arbitrary non zero real numbers.

Theorem 5. Let $\{x_n, y_n\}_{n=-2}^{+\infty}$ be solutions of system (4.1). Then

1- $\{x_n\}_{n=-2}^{+\infty}$ and $\{y_n\}_{n=-2}^{+\infty}$ and are periodic with period eight i.e.,

$$x_{n+8} = x_n, \quad y_{n+8} = y_n,$$

2- We have the following form

$$\begin{aligned}
x_{8n-2} &= c, \quad x_{8n-1} = b, \quad x_{8n} = a, \quad x_{8n+1} = -\frac{-1+d+b}{f},\\
x_{8n+2} &= -\frac{ac-a-c-e+1}{ec}, \quad x_{8n+3} = \frac{b^2+bd+bf+df-2b-d-f+1}{fbd},\\
x_{8n+4} &= -\frac{ac-a-c-e+1}{ea}, \quad x_{8n+5} = -\frac{f+b-1}{d},\\
y_{8n-2} &= f, \quad y_{8n-1} = e, \quad y_{8n} = d, \quad y_{8n+1} = -\frac{-1+a+e}{c},\\
y_{8n+2} &= \frac{-df+b+d+f-1}{fb}, \quad y_{8n+3} = \frac{e^2+ac+ae+ce-2e-a-c+1}{cea},\\
y_{8n+4} &= \frac{-df+b+d+f-1}{bd}, \quad y_{8n+5} = -\frac{c+e-1}{a}.
\end{aligned}$$

Or equivalently

$$\{x_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} c,b,a,-\frac{-1+d+b}{f}, -\frac{ac-a-c-e+1}{ec}, \frac{b^2+bd+bf+df-2b-d-f+1}{fbd}, \\ -\frac{ac-a-c-e+1}{ea}, -\frac{f+b-1}{d}, c,b,a,\ldots \end{array} \right\},$$

$$\{y_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} f,e,d,-\frac{-1+a+e}{c}, \frac{-df+b+d+f-1}{fb}, \frac{e^2+ac+ae+ce-2e-a-c+1}{cea} \\ \frac{-df+b+d+f-1}{bd}, -\frac{c+e-1}{a}, f,e,d,\ldots \end{array} \right\}.$$

Example 4. See Fig. 5.1, if we take system (5.1) with $x_{-2} = -8$, $x_{-1} = 5$, $x_0 = -2.8$, $y_{-2} = -9$, $y_{-1} = 3$ *and* $y_0 = 4$.

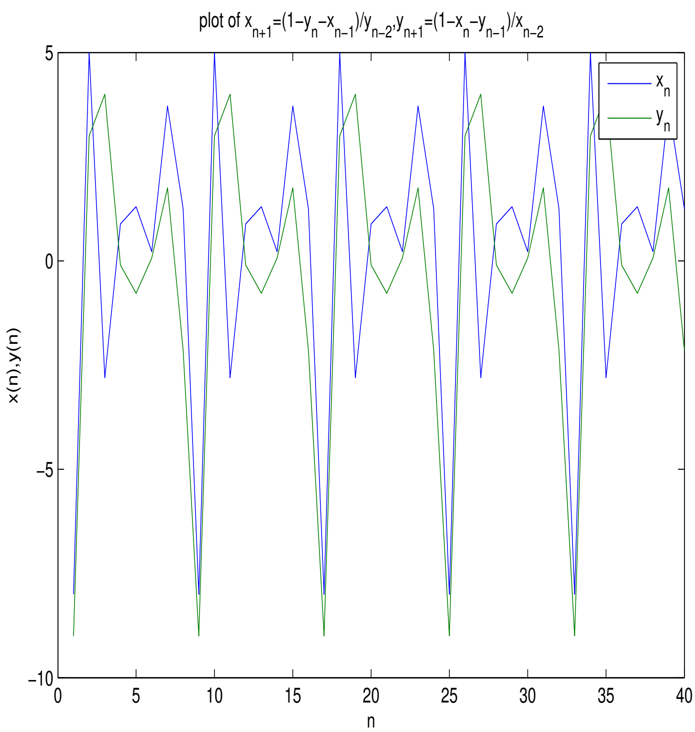

Figure 5.1

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L-Fuzzy Invariant Metric Space

Servet Kütükçü[1]*

Abstract
In this paper, we define L-fuzzy invariant metric space, and generalize some well known results in metric and fuzzy metric space including Uniform continuity theorem and Ascoli-Arzela theorem.

Keywords: L-fuzzy invariant metric space, Completeness, Equicontinuity, Compactness
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[1] *Department of Mathematics, Faculty of Science and Arts, Ondokuz Mayıs University, 55139 Kurupelıt, Samsun, Turkey.*
***Corresponding author**: skutukcu@omu.edu.tr
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1. Introduction

One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric. This problem has been investigated by many authors [1]-[10] from different points of views. In particular, Park [8] introduced the notion of intuitionistic fuzzy metric as a generalization of fuzzy metric introduced and studied by George and Veeramani [2].

In this paper, we define L-fuzzy invariant metric space, study completeness and observe that a compact L-fuzzy invariant metric space is separable. Further, we introduce the notion of uniform continuity and equicontinuity. Finally, we prove Uniform continuity theorem and Ascoli-Arzela theorem.

2. L-fuzzy invariant metric space

Lemma 2.1. *[11] Consider the set L^* and operation $\leq_{L^*}$ defined by $L^* = \{(x_1,x_2) : (x_1,x_2) \in [0,1]^2$ and $x_1 + x_2 \leq 1\}$ and $(x_1,x_2) \leq_{L^*} (y_1,y_2) \iff x_1 \leq y_1$ and $x_2 \geq y_2$ for every $(x_1,x_2),(y_1,y_2) \in L^*$. Then $(L^*,\leq_{L^*})$ is a complete lattice.*

Definition 2.2. *[9] An intuitionistic fuzzy set $A_{\zeta,\eta}$ in a universe U is an object $A_{\zeta,\eta} = \{(\zeta_A(u),\eta_A(u)) : u \in U\}$ where, for all $u \in U$, $\zeta_A(u) \in [0,1]$ and $\eta_A(u) \in [0,1]$ are called the membership degree and non-membership degree, respectively, of u in $A_{\zeta,\eta}$, and furthermore they satisfy $\zeta_A(u) + \eta_A(u) \leq 1$.*

For every $z_i = (x_i,y_i) \in L^*$, if $c_i \in [0,1]$ such that $\Sigma_{j=1}^n c_j = 1$ then $c_1(x_1,y_1)+c_2(x_2,y_2)+\ldots+c_n(x_n,y_n) = \Sigma_{j=1}^n c_j(x_j,y_j) = \left(\Sigma_{j=1}^n c_j x_j, \Sigma_{j=1}^n c_j y_j\right) \in L^*$.

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a triangular norm (shortly t-norm) $* = T$ on $[0,1]$ is defined as an increasing, commutative and associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1,x) = 1 * x = x$ for all $x \in [0,1]$. A triangular conorm (shortly t-conorm) $\Diamond = S$ is defined as an increasing, commutative and associative mapping $S : [0,1]^2 \to [0,1]$ satisfying $S(0,x) = 0 \Diamond x = x$ for all $x \in [0,1]$. Using the lattice $(L^*,\leq_{L^*})$, these definitions can be extended.

Definition 2.3. *[12] A triangular norm $\Im$ on L^* is a mapping $\Im : (L^*)^2 \to L^*$ satisfying the following conditions, for every $x,y,z,t \in L^*$:*

(a) $\Im(x, 1_{L^*}) = x$,

(b) $\Im(x,y) = \Im(y,x)$,

(c) $\Im(x, \Im(y,z)) = \Im(\Im(x,y),z)$,

(d) $x \leq_{L^*} z$ and $y \leq_{L^*} t$ imply $\Im(x,y) \leq_{L^*} \Im(z,t)$.

Definition 2.4. *[11, 12]*
A continuous t-norm $\Im$ on L^* is called continuous t-representable if and only if there exist a continuous t-norm $*$ and a continuous t-conorm $\Diamond$ on $[0,1]$ such that $\Im(x,y) = (x_1 * y_1, x_2 \Diamond y_2)$ for all $x = (x_1,x_2), y = (y_1,y_2) \in L^*$.

Now define a sequence $\Im^n$ recursively by $\Im^1 = \Im$ and

$$\Im^n(x_1,...,x_{n+1}) = \Im(\Im^{n-1}(x_1,...,x_n), x_{n+1})$$

for $n \geq 2$ and $x_i \in L^*$.

Definition 2.5. *[11, 12]*
A negator $\mathscr{N}$ on L^* is any decreasing mapping $\mathscr{N} : L^* \to L^*$ satisfying $\mathscr{N}(0_{L^*}) = 1_{L^*}$ and $\mathscr{N}(1_{L^*}) = 0_{L^*}$. If $\mathscr{N}(\mathscr{N}(x)) = x$ for all $x \in L^*$ then $\mathscr{N}$ is called an involutive negator. A negator N on $[0,1]$ is a decreasing mapping N:$[0,1] \to [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_S denotes the standard negator on $[0,1]$ defined as $N_S(x) = 1 - x$ for all $x \in [0,1]$.

Next, using fundamental notions above, we give a metric generalization on vector space in the sense of George and Veeramani [2].

Definition 2.6. *Let μ and ν are fuzzy sets from $X \times (0,\infty)$ to $[0,1]$ such that $\mu(x,t) + \nu(x,t) \leq 1$ for all $x \in X$ and $t > 0$. The 3-tuble $(X, M_{\mu,\nu}, \Im)$ is said to be an L-fuzzy invariant metric space if X is a vector space, $\Im$ is a continuous t-representable and $M_{\mu,\nu}$ is a mapping from $X \times (0,\infty)$ to L^* satisfying the following conditions, for every $x,y \in X$ and $t,s > 0$*

(a) $M_{\mu,\nu}(x,t) >_{L^*} 0_{L^*}$,

(b) $M_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x = 0$,

(c) $M_{\mu,\nu}(x-y,t) = M_{\mu,\nu}(y-x,t)$,

(d) $M_{\mu,\nu}(x+y,t+s) \geq_{L^*} \Im(M_{\mu,\nu}(x,t), M_{\mu,\nu}(y,s))$,

(e) $M_{\mu,\nu}(x,.) : (0,\infty) \to L^*$ is continuous.

In this case, $M_{\mu,\nu}$ is said to be an L-fuzzy invariant metric on X. Here $M_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t))$.

Example 2.7. *Let $(X, \|.\|)$ be a normed space. Denote $\Im(a,b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1,a_2), b = (b_1,b_2) \in L^*$ and let μ,ν be fuzzy sets on $X \times (0,\infty)$ defined as follows:*

$$M_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t)) = \left(\frac{ht^n}{ht^n + m\|x\|}, \frac{m\|x\|}{ht^n + m\|x\|} \right)$$

for all $t,h,m,n \in \mathbb{R}^+$. Then $(X, M_{\mu,\nu}, \Im)$ is an L-fuzzy invariant metric space. If $h = m = n = 1$ then $(X, M_{\mu,\nu}, \Im)$ is a standard L-fuzzy invariant metric space. Also, if we define

$$M_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t)) = \left(\frac{t}{t + m\|x\|}, \frac{\|x\|}{t + \|x\|} \right)$$

in which $m > 1$, then $(X, M_{\mu,\nu}, \Im)$ is an L-fuzzy invariant metric space in which $M_{\mu,\nu}(x,t) <_{L^*} 1_{L^*}$ for all $x \in X$.

Definition 2.8. *Let $(X, M_{\mu,\nu}, \Im)$ be an L-fuzzy invariant metric space.*
For $t > 0$, define the open ball $B(x,r,t)$ with center $x \in X$ and radius $r \in (0,1)$ as

$$B(x,r,t) = \{y \in X : M_{\mu,\nu}(x-y,t) >_{L^*} (N_S(r), r)\}.$$

A subset $A \subseteq X$ is called open if for each $x \in A$, there exist $r \in (0,1)$ and $t > 0$ such that $B(x,r,t) \subseteq A$. Let $\tau_{M_{\mu,\nu}}$ denote the family of all open subsets of X. $\tau_{M_{\mu,\nu}}$ is called the topology induced by L-fuzzy invariant metric $M_{\mu,\nu}$.

Definition 2.9. *A sequence $\{x_n\}$ in an L-fuzzy invariant metric space $(X, M_{\mu,\nu}, \mathfrak{I})$ is said to be Cauchy if for each $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$M_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_S(\varepsilon), \varepsilon)$$

for each $n, m \geq n_0$. The sequence $\{x_n\}$ is said to be convergent to $x \in X$ in X and denoted by $x_n \xrightarrow{M_{\mu,\nu}} x$ if $M_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ whenever $n \to \infty$ for every $t > 0$. An L-fuzzy invariant metric space is said to be complete if and only if every Cauchy sequence is convergent.

The proofs of following two lemmas are similar from classical cases and omitted [2, 3].

Lemma 2.10. *Let $M_{\mu,\nu}$ be an L-fuzzy invariant metric. Then, for any $t > 0$, $M_{\mu,\nu}(x,t)$ is non-decreasing with respect to t in $(L^*, \leq_{L^*})$ for all $x \in X$.*

Lemma 2.11. *Let $(X, M_{\mu,\nu}, \mathfrak{I})$ be an L-fuzzy invariant metric space. Then $M_{\mu,\nu}$ is continuous function on $X \times (0,\infty)$.*

Theorem 2.12. *Every L-fuzzy invariant metric space is normal.*

Proof. Let $(X, M_{\mu,\nu}, \mathfrak{I})$ be an L-fuzzy invariant metric space and F, G be two disjoint closed subsets of X. Let $x \in X$. Then $x \in G^c$ since G^c is open there exist $t_x > 0$ and $r_x \in (0,1)$ such that $B(x, r_x, t_x) \cap G = \varnothing$ for all $x \in F$. Similarly, there exist $t_y > 0$ and $r_y \in (0,1)$ such that $B(x, r_y, t_y) \cap F = \varnothing$ for all $y \in G$. Let $s = \min\{r_x, t_x, r_y, t_y\}$. Then we can find a $s_0 \in (0,s)$ such that $\mathfrak{I}((N_S(s_0), s_0), (N_S(s_0), s_0)) >_{L^*} (N_S(s), s)$. Define $U = \cup_{x\in F} B(x, s_0, s/2)$ and $V = \cup_{y \in G} B(y, s_0, s/2)$. Clearly U and V are open sets such that $F \subset U$ and $G \subset V$. Now, we claim that $U \cap V = \varnothing$. Let $z \in U \cap V$. Then there exist $x \in F$ and $y \in G$ such that $z \in B(x, s_0, s/2)$ and $z \in B(y, s_0, s/2)$. Therefore, we have

$$\begin{aligned} M_{\mu,\nu}(x-y, s) &\geq_{L^*} \mathfrak{I}(M_{\mu,\nu}(x-z, s/2), M_{\mu,\nu}(z-y, s/2) \\ &\geq_{L^*} \mathfrak{I}((N_S(s_0), s_0), (N_S(s_0), s_0)) >_{L^*} (N_S(s), s). \end{aligned}$$

Hence $y \in B(x, s, s)$. Since $s < t_x, r_x$ we have $B(x, s, s) \subset B(x, r_x, t_x)$. Thus $B(x, r_x, t_x) \cap G$ is nonempty which is a contradiction. Therefore $U \cap V = \varnothing$. Hence X is normal. □

Remark 2.13. *From the above theorem, we can easily deduce that every metrizable space is normal. Since every L-fuzzy invariant metric space is normal, Urysohn's lemma and Tietze extension theorem are true in the case of L-fuzzy invariant metric space.*

Definition 2.14. *A function f from an L-fuzzy invariant metric space X to an other L-fuzzy invariant metric space Y is said to be uniformly continuous if for given $t > 0$ and $r \in (0,1)$, there exist $t_0 > 0$ and $r_0 \in (0,1)$ such that $M_{\mu,\nu}(x-y, t_0) >_{L^*} (N_S(r_0), r_0)$ implies $M_{\mu,\nu}(f(x) - f(y), t) >_{L^*} (N_S(r), r)$.*

As usual by a compact L-fuzzy invariant metric space we mean an L-fuzzy invariant metric space $(X, M_{\mu,\nu}, \mathfrak{I})$ such that $(X, \tau_{M_{\mu,\nu}})$ is a compact topological space.

Theorem 2.15 (Uniform continuity theorem)**.** *If f is a continuous function from a compact L-fuzzy invariant metric space X to an other L-fuzzy invariant metric space Y, then f is uniformly continuous.*

Proof. Let $t > 0$ and $s \in (0,1)$. Then we can find $r \in (0,1)$ such that $\mathfrak{I}((N_S(r), r), (N_S(r), r)) >_{L^*} (N_S(s), s)$. Since $f : X \to Y$ is continuous, for each $x \in X$ we can find $t_x > 0$ and $r_x \in (0,1)$ such that $M_{\mu,\nu}(x-y, t) >_{L^*} (N_S(r_x), r_x)$ implies $M_{\mu,\nu}(f(x) - f(y), \frac{t}{2}) >_{L^*} (N_S(r), r)$. But $r_x \in (0,1)$ and then we can find $s_x \in (0, r_x)$ such that $\mathfrak{I}((N_S(s_x), s_x), (N_S(s_x), s_x)) >_{L^*} (N_S(r_x), r_x)$. Since X is compact and $\{B(x, s_x, \frac{t_x}{2} : x \in X\}$ is an open covering of X, there exist $x_1, x_2, ..., x_k$ in X such that $X = \cup_{i=1}^{k} B(x_i, s_{x_i}, \frac{t_{x_i}}{2})$. Put $s_0 = \min s_{x_i}$ and $t_0 = \min \frac{t_{x_i}}{2}$, $i = 1, 2, ..., k$. For any $x, y \in X$, if $M_{\mu,\nu}(x-y, t_0) >_{L^*} (N_S(s_0), s_0)$, then $M_{\mu,\nu}(f(x) - f(y), \frac{t_{x_i}}{2}) >_{L^*} (N_S(s_{x_i}), s_{x_i})$. Since $x \in X$, there exists a x_i such that $M_{\mu,\nu}(x - x_i, \frac{t_{x_i}}{2}) >_{L^*} (N_S(s_{x_i}), s_{x_i})$. Hence we have $M_{\mu,\nu}(f(x) - f(x_i), \frac{t}{2}) >_{L^*} (N_S(r), r)$. Now

$$\begin{aligned} M_{\mu,\nu}(x_i - y, t_{x_i}) &\geq_{L^*} \mathfrak{I}(M_{\mu,\nu}(x_i - x, \frac{t_{x_i}}{2}), M_{\mu,\nu}(x-y, \frac{t_{x_i}}{2})) \\ &\geq_{L^*} \mathfrak{I}((N_S(s_{x_i}), s_{x_i}), (N_S(s_{x_i}), s_{x_i})) >_{L^*} (N_S(r_{x_i}), r_{x_i}). \end{aligned}$$

Therefore $M_{\mu,\nu}(f(x_i) - f(y), \frac{t}{2}) >_{L^*} (N_S(r), r)$. Now we have

$$\begin{aligned} M_{\mu,\nu}(f(x) - f(y), t) &\geq_{L^*} \mathfrak{I}(M_{\mu,\nu}(f(x) - f(x_i), \frac{t}{2}), M_{\mu,\nu}(f(x_i) - f(y), \frac{t}{2}) \\ &\geq_{L^*} \mathfrak{I}((N_S(r), r), (N_S(r), r)) >_{L^*} (N_S(s), s). \end{aligned}$$

Hence f is uniformly continuous. □

Remark 2.16. *Let f be a uniformly continuous function from the L-fuzzy invariant metric space X to an other L-fuzzy invariant metric space Y. If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is also a Cauchy sequence in Y.*

Theorem 2.17. *Every compact L-fuzzy invariant metric space is separable.*

Proof. Let $(X, M_{\mu,\nu}, \Im)$ be the given compact L-fuzzy invariant metric space and $t > 0$, $r \in (0,1)$. Since X is compact, there exist $x_1, x_2, ..., x_n$ in X such that $X = \cup_{i=1}^{n} B(x_i, r, t)$. In particular, for each $n \in \mathbb{N}$, we can find a finite subset A_n such that $X = \cup_{a \in A} B(a, r_n, \frac{1}{n})$ in which $r_n \to 0_{L^*}$. Let $A = \cup_{n \in \mathbb{N}} A_n$. Then A is countable. Now, we claim that $X \subset \overline{A}$. For that let $x \in X$, then, for each n, there exists $a_n \in A_n$ such that $x \in B(a_n, r_n, \frac{1}{n})$. Thus a_n is converges to x. Since $a_n \in A_n$ for all n then $x \in \overline{A}$. Therefore A is dense in X, thus X is separable. □

Definition 2.18. *Let X be any nonempty set and $(Y, M_{\mu,\nu}, \Im)$ be an L-fuzzy invariant metric space. Then a sequence $\{f_n\}$ of functions from X to Y is said to be converge uniformly to a function f from X to Y if for given $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M_{\mu,\nu}(f_n(x) - f(x), t) >_{L^*} (N_S(r), r)$ for all $n \geq n_0$ and $x \in X$.*

Definition 2.19. *A family F of functions from an L-fuzzy invariant metric space X to a complete L-fuzzy invariant metric space Y is said to be equicontinuous if for given $r \in (0,1)$ and $t > 0$, there exists $r_0 \in (0,1)$ and $t_0 > 0$ such that $M_{\mu,\nu}(x - y, t_0) >_{L^*} (N_S(r_0), r_0)$ implies $M_{\mu,\nu}(f(x) - f(y), t) >_{L^*} (N_S(r), r)$ for all $f \in F$.*

Lemma 2.20. *Let $\{f_n\}$ be an equicontinuous sequence of functions from an L-fuzzy invariant metric space X to a complete L-fuzzy invariant metric space Y. If $\{f_n\}$ converges for each point of a dense subset D of X, then $\{f_n\}$ converges for each point of X and the limit function is continuous.*

Proof. Let $s \in (0,1)$ and $t > 0$ be given. Then we can find $r \in (0,1)$ such that $\Im^2((N_S(r), r), (N_S(r), r), (N_S(r), r)) >_{L^*} (N_S(s), s)$. Since $F = \{f_n\}$ is an equicontinuous family, for given $r \in (0,1)$ and $t > 0$, there exist $r_1 \in (0,1)$ and $t_1 > 0$ such that for each $x, y \in X$, $M_{\mu,\nu}(x - y, t_1) >_{L^*} (N_S(r_1), r_1)$ implies $M_{\mu,\nu}(f_n(x) - f_n(y), \frac{t}{3}) >_{L^*} (N_S(r), r)$ for all $f_n \in F$. Since D is dense in X, there exists $y \in B(a, r_1, t_1) \cap D$ and $\{f_n(y)\}$ converges for that y. Since $\{f_n(y)\}$ is a Cauchy sequence, for given $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M_{\mu,\nu}(f_n(y) - f_m(y), \frac{t}{3}) >_{L^*} (N_S(r), r)$ for all $m, n \geq n_0$. Now for any $x \in X$, we have

$$\begin{aligned}
M_{\mu,\nu}(f_n(x) - f_m(x), t) &\geq_{L^*} \Im^2(M_{\mu,\nu}(f_n(x) - f_n(y), \frac{t}{3}), \\
&\quad M_{\mu,\nu}(f_n(y) - f_m(y), \frac{t}{3}), M_{\mu,\nu}(f_m(x) - f_m(y), \frac{t}{3})) \\
&\geq_{L^*} \Im^2((N_S(r), r), (N_S(r), r), (N_S(r), r)) \\
&>_{L^*} (N_S(s), s)
\end{aligned}$$

Hence $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, $f_n(x)$ converges. Let $f(x) = \lim_{n \to \infty} f_n(x)$. We claim that f is continuous. Let $s_0 \in (0,1)$ and $t_0 > 0$ be given. Then we can find that $r_0 \in (0,1)$ such that $\Im^2((N_S(r_0), r_0), (N_S(r_0), r_0), (N_S(r_0), r_0)) >_{L^*} (N_S(s_0), s_0)$. Since F is equicontinuous, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exist $r_2 \in (0,1)$ and $t_2 > 0$ such that $M_{\mu,\nu}(x - y, t_2) >_{L^*} (N_S(r_2), r_2)$ implies $M_{\mu,\nu}(f_n(x) - f_n(y), \frac{t_0}{3}) >_{L^*} (N_S(r_0), r_0)$ for all $f_n \in F$. Since $f_n(x)$ converges to $f(x)$, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exists $n_1 \in \mathbb{N}$ such that $M_{\mu,\nu}(f_n(y) - f(x), \frac{t_0}{3}) >_{L^*} (N_S(r_0), r_0)$ for all $n \geq n_1$. Also since $f_n(y)$ converges to $f(y)$, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exists $n_2 \in \mathbb{N}$ such that $M_{\mu,\nu}(f_n(y) - f(y), \frac{t_0}{3}) >_{L^*} (N_S(r_0), r_0)$ for all $n \geq n_2$. Now for all $n \geq \max\{n_1, n_2\}$, we have

$$\begin{aligned}
M_{\mu,\nu}(f(x) - f(y), t_0) &\geq_{L^*} \Im^2(M_{\mu,\nu}(f(x) - f_n(x), \frac{t_0}{3}), \\
&\quad M_{\mu,\nu}(f_n(x) - f_n(y), \frac{t_0}{3}), M_{\mu,\nu}(f_n(y) - f(y), \frac{t_0}{3})) \\
&\geq_{L^*} \Im^2((N_S(r_0), r_0), (N_S(r_0), r_0), (N_S(r_0), r_0)) \\
&>_{L^*} (N_S(s_0), s_0).
\end{aligned}$$

Hence f is continuous. □

Theorem 2.21 (Ascoli-Arzela theorem)**.** *Let X be a compact L-fuzzy invariant metric space and Y be a complete L-fuzzy invariant metric space. Let F be an equicontinuous family of functions from X to Y. If $\{f_n\}$ is a sequence in F such that $\overline{\{f_n(x) : n \in \mathbb{N}\}}$ is a compact subset of Y for each $x \in X$, then there exists a continuous function f from X to Y and a subsequence $\{g_n\}$ of $\{f_n\}$ such that g_n converges uniformly to f on X.*

Proof. Since X is compact L-fuzzy invariant metric space, by Theorem 2.17, X is separable. Let $D = \{x_i : i = 1,2,...\}$ be a countable dense subset of X. By hypothesis, for each i, $\overline{\{f_n(x_i) : n \in \mathbb{N}\}}$ is compact subset of Y. Since every L-fuzzy invariant metric space is first countable space, every compact subset of Y is sequentially compact. Thus by standard argument, we have a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{g_n(x_i)\}$ converges for each $i = 1,2,...$ By Lemma 2.20, there exists a continuous function f from X to Y such that $g_n(x)$ converges to $f(x)$ for all $x \in X$. Now we claim that g_n converges to f on X. Let $s \in (0,1)$ and $t > 0$ be given. Then we can find $r \in (0,1)$ such that $\Im^2((N_S(r),r),(N_S(r),r),(N_S(r),r)) >_{L^*} (N_S(s),s)$. Since F is equicontinuous, there exist $r_1 \in (0,1)$ and $t_1 > 0$ such that $M_{\mu,\nu}(x-y,t_1) >_{L^*} (N_S(r_1),r_1)$ implies $M_{\mu,\nu}(g_n(x)-g_n(y),\frac{t}{3}) >_{L^*} (N_S(r),r)$ for all n. Since X is compact, by Theorem 2.15, f is uniformly continuous. Hence for given $r \in (0,1)$ and $t > 0$, there exists $r_2 \in (0,1)$ and $t_2 > 0$ such that $M_{\mu,\nu}(x-y,t_2) >_{L^*} (N_S(r_2),r_2)$ implies $M_{\mu,\nu}(f(x)-f(y),\frac{t}{3}) >_{L^*} (N_S(r),r)$ for all $x,y \in X$. Let $r_0 = \min\{r_1,r_2\}$ and $t_0 = \min\{t_1,t_2\}$. Since X is compact and D is dense in X, $X = \cup_{i=1}^{k} B(x_i,r_0,t_0)$ for some finite k. Thus for each $x \in X$, there exists $i, 1 \le i \le k$, such that $M_{\mu,\nu}(x-x_i,t_0) >_{L^*} (N_S(r_0),r_0)$. But since $r_0 = \min\{r_1,r_2\}$ and $t_0 = \min\{t_1,t_2\}$, we have, by the equicontinuity of F, $M_{\mu,\nu}(g_n(x)-g_n(x_i),\frac{t}{3}) >_{L^*} (N_S(r),r)$ and also we have, by the uniform continuity of f, $M_{\mu,\nu}(f(x)-f(x_i),\frac{t}{3}) >_{L^*} (N_S(r),r)$. Since $g_n(x_j)$ converges to $f(x_j)$, $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M_{\mu,\nu}(g_n(x_j)-f(x_j),\frac{t}{3}) >_{L^*} (N_S(r),r)$ for all $j = 1,2,...,n$. Now, for each $x \in X$, we have

$$\begin{aligned} M_{\mu,\nu}(g_n(x)-f(x),t) &\ge_{L^*} \Im^2(M_{\mu,\nu}(g_n(x)-g_n(x_i),\frac{t}{3}), \\ &\quad M_{\mu,\nu}(g_n(x_i)-f(x_i),\frac{t}{3}), M_{\mu,\nu}(f(x_i)-f(x),\frac{t}{3})) \\ &\ge_{L^*} \Im^2((N_S(r),r),(N_S(r),r),(N_S(r),r)) \\ &>_{L^*} (N_S(s),s). \end{aligned}$$

Hence g_n converges uniformly to f on X. □

3. Conclusion

The aim of this paper is to introduce L-fuzzy invariant metric space, and to generalize Uniform continuity theorem and Ascoli-Arzela theorem for this space. Aside from their numerous applications to Partial Differential Equations such as existence theorems in differential and integral equations, and Lorentzian Geometry such as guaranteing convergence to isometry using Lorentzian analogues, these results can be also used as a tool in obtaining Functional Analysis results such as compactness for duals of compact operators, conformal mapping and extremal problems in complex variable theory.

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On Bicomplex Pell and Pell-Lucas Numbers

Fügen Torunbalcı Aydın[1]*

Abstract
In this paper, bicomplex Pell and bicomplex Pell-Lucas numbers are defined. Also, negabicomplex Pell and negabicomplex Pell-Lucas numbers are given. Some algebraic properties of bicomplex Pell and bicomplex Pell-Lucas numbers which are connected between bicomplex numbers and Pell and Pell-Lucas numbers are investigated. Furthermore, d'Ocagne's identity, Binet's formula, Cassini's identity and Catalan's identity for these numbers are given.

Keywords: Pell and Pell-Lucas numbers, Bicomplex number, Quaternion
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[1] Yildiz Technical University, Faculty of Chemical and Metallurgical Engineering, Department of Mathematical Engineering, Davutpasa Campus, 34220 Esenler, Istanbul, Turkey.
***Corresponding author**: ftorunay@gmail.com ; faydin@yildiz.edu.tr
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1. Introduction

Bicomplex numbers were introduced by Corrado Segre in 1892 [1]. G. Baley Price (1991), presented bicomplex numbers based on multi-complex spaces and functions in his book [2]. In recent years, fractal structures of these numbers have also been studied [3] . The set of bicomplex numbers can be expressed by the basis $\{1\,,i\,,j\,,i\,j\}$ as,

$$\mathbb{C}_2 = \{q = q_1 + iq_2 + jq_3 + i\,jq_4 \mid q_1, q_2, q_3, q_4 \in \mathbb{R}\} \tag{1.1}$$

or

$$\mathbb{C}_2 = \{q = (q_1 + iq_2) + j(q_3 + iq_4) \mid q_1, q_2, q_3, q_4 \in \mathbb{R}\} \tag{1.2}$$

where i, j and $i\,j$ satisfy the conditions

$$i^2 = -1,\; j^2 = -1,\; i\,j = j\,i.$$

Thus, any bicomplex number q is introduced as pairs of typical complex numbers with the additional structure of commutative multiplication (Table 1).

A set of bicomplex numbers $\mathbb{C}_2$ is a real vector space with addition and scalar multiplication operations. The vector space $\mathbb{C}_2$ equipped with bicomplex product is a real associative algebra. Also, the vector space together with the properties of multiplication and the product of the bicomplex numbers are a commutative algebra. Furthermore, three different conjugations can operate on bicomplex numbers [3], [4], [5] as follows:

Table 1. Multiplication scheme of bicomplex numbers

x	1	i	j	i j
1	1	i	j	i j
i	i	-1	i j	-j
j	j	i j	-1	-i
i j	i j	-j	-i	1

$$\begin{aligned} q &= q_1 + i\,q_2 + j\,q_3 + i\,j\,q_4 = (q_1 + iq_2) + j\,(q_3 + iq_4),\ q \in \mathbb{C}_2 \\ q_i{}^* &= q_1 - iq_2 + jq_3 - i\,jq_4 = (q_1 - iq_2) + j\,(q_3 - iq_4), \\ q_j{}^* &= q_1 + iq_2 - jq_3 - i\,jq_4 = (q_1 + iq_2) - j\,(q_3 + iq_4), \\ q_{ij}{}^* &= q_1 - iq_2 - jq_3 + i\,jq_4 = (q_1 - iq_2) - j\,(q_3 - iq_4). \end{aligned}$$

and properties of conjugation

$$\begin{aligned} &1)\,(q^*)^* = q, \\ &2)\,(q_1\,q_2)^* = q_2{}^*\,q_1{}^*,\ \ q_1, q_2 \in \mathbb{C}_2, \\ &3)\,(q_1 + q_2)^* = q_1{}^* + q_2{}^*, \\ &4)\,(\lambda\,q)^* = \lambda\,q^*, \\ &5)\,(\lambda\,q_1 \pm \mu\,q_2)^* = \lambda\,q_1{}^* \pm \mu\,q_2{}^*,\ \lambda, \mu \in \mathbb{R}. \end{aligned}$$

Therefore, the norm of the bicomplex numbers is defined as

$$\begin{aligned} N_{q_i} &= \|q \times q_i{}^*\| = \sqrt{\left|q_1^2 + q_2^2 - q_3^2 - q_4^2 + 2\,j\,(q_1q_3 + q_2q_4)\right|}, \\ N_{q_j} &= \left\|q \times q_j{}^*\right\| = \sqrt{\left|q_1^2 - q_2^2 + q_3^2 - q_4^2 + 2\,i\,(q_1q_2 + q_3q_4)\right|}, \\ N_{q_{ij}} &= \left\|q \times q_{ij}{}^*\right\| = \sqrt{\left|q_1^2 + q_2^2 + q_3^2 + q_4^2 + 2\,i\,j\,(q_1q_4 - q_2q_3)\right|}. \end{aligned}$$

Pell numbers were invented by John Pell but, these numbers are named after Edouard Lucas. Pell and Pell-Lucas numbers have important parts in mathematics. They have fundamental importance in the fields of combinatorics and number theory [6],[7],[8],[9].

The sequence of Pell numbers

$$1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408,\ 985,\ 2378, \ldots, P_n, \ldots$$

is defined by the recurrence relation

$$P_n = 2\,P_{n-1} + P_{n-2}\ ,\ \ (n \geq 2),$$

with $P_0 = 0, P_1 = 1$.

The sequence of Pell - Lucas numbers

$$2,\ 6,\ 14,\ 34,\ 82,\ 198,\ 478,\ 1154,\ 2786,\ 6726, \ldots, Q_n, \ldots$$

is defined by the recurrence relation

$$Q_n = 2\,Q_{n-1} + Q_{n-2}\ ,\ \ (n \geq 2),$$

with $Q_0 = 2, Q_1 = 1$.
Also, the sequence of modified Pell numbers

$$1, 3, 7, 17, 41, 99, 329, 577, 1393, 3363, \ldots, q_n, \ldots$$

is defined by the recurrence relation

$$q_n = 2q_{n-1} + q_{n-2}, \ (n \geq 2),$$

with $q_0 = 1, q_1 = 1$.
Furthermore, we can see the matrix representations of Pell and Pell-Lucas numbers in [1]-[3],[5], [8]. In 2018, Catarino defined bicomplex k-Pell quaternions in [10].

Also, for Pell, Pell-Lucas and modified Pell numbers the following properties hold:[6],[7],[8],[9]

$$P_m P_{n+1} + P_{m-1} P_n = P_{m+n},$$

$$P_m P_{n+1} - P_{m+1} P_n = (-1)^n P_{m-n},$$

$$P_m P_n - P_{m+r} P_{n-r} = (-1)^{n-r} P_{m+r-n} P_r,$$

$$Q_m Q_n - Q_{m+r} Q_{n-r} = 8(-1)^{n-r+1} P_{m+r-n} P_r,$$

$$P_{n-1} P_{n+1} - P_n^2 = (-1)^n,$$

$$P_n^2 + P_{n+1}^2 = P_{2n+1},$$

$$P_{n+1}^2 - P_{n-1}^2 = 2P_{2n},$$

$$2P_{n+1} P_n - 2P_n^2 = P_{2n},$$

$$P_n^2 + P_{n+3}^2 = 5P_{2n+3},$$

$$P_{2n+1} + P_{2n} = 2P_{n+1}^2 - 2P_n^2 - (-1)^n,$$

$$P_n^2 + P_{n-1} P_{n+1} = \frac{Q_n^2}{4},$$

$$P_{n+1} + P_{n-1} = Q_n,$$

$$P_n Q_n = P_{2n},$$

$$Q_n = 2q_n,$$

$$P_{n+1} - P_n = q_n,$$

$$P_{n+1} + P_n = q_{n+1},$$

and for nega Pell and pell-Lucas numbers the following properties hold,

$$P_{-n} = (-1)^{n+1} P_n,$$

$$Q_{-n} = (-1)^n Q_n.$$

In this paper, the bicomplex Pell and bicomplex Pell-Lucas numbers will be defined. The aim of this work is to present in a unified manner a variety of algebraic properties of both the bicomplex numbers as well as the bicomplex Pell and Pell-Lucas numbers and the negabicomplex Pell and Pell-Lucas numbers. In particular, using three types of conjugations, all the properties established for bicomplex numbers are also given for the bicomplex Pell and Pell-Lucas numbers. In addition, d'Ocagne's identity, Binet's formula, Cassini's identity and Catalan's identity for these numbers are given.

2. The bicomplex Pell and Pell-Lucas numbers

The bicomplex Pell and Pell-Lucas numbers BP_n and BPL_n are defined by the basis $\{1, i, j, ij\}$ as follows

$$\mathbb{C}_2^P = \{BP_n = P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3} \mid P_n, \\ n-th\,Pell\,number, n = 0,1,...\}. \tag{2.1}$$

and

$$\mathbb{C}_2^{PL} = \{BPL_n = Q_n + iQ_{n+1} + jQ_{n+2} + ijQ_{n+3} \mid Q_n, \\ n-th\,Pell-Lucas\,number,, n = 0,1,...\} \tag{2.2}$$

where i, j and ij satisfy the conditions

$$i^2 = -1, \; j^2 = -1, \; ij = ji.$$

The bicomplex Pell and bicomplex Pell-Lucas numbers starting from $n = 0$, can be written respectively as;

$BP_0 = 0 + 1i + 2j + 5ij$, $BP_1 = 1 + 2i + 5j + 12ij$, $BP_2 = 2 + 5i + 12j + 29ij, \ldots$

$BPL_0 = 2 + 2i + 6j + 14ij$, $BPL_1 = 2 + 6i + 14j + 34ij$,
$BPL_2 = 6 + 14i + 34j + 82ij, \ldots$

Let BP_n and BP_m be two bicomplex Pell numbers such that

$$BP_n = P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3}$$

and

$$BP_m = P_m + iP_{m+1} + jP_{m+2} + ijP_{m+3}.$$

Then, the addition and subtraction of these numbers are given by

$$\begin{aligned} BP_n \pm BP_m = & \;(P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3}) \\ & \pm (P_m + iP_{m+1} + jP_{m+2} + ijP_{m+3}) \\ = & \;(P_n \pm P_m) + i(P_{n+1} \pm P_{m+1}) + j(P_{n+2} \pm P_{m+2}) \\ & + ij(P_{n+3} \pm P_{m+3}). \end{aligned}$$

The multiplication of a bicomplex Pell number by the real scalar λ is defined as

$$\lambda BP_n = \lambda P_n + i\lambda P_{n+1} + j\lambda P_{n+2} + ij\lambda P_{n+3}.$$

The multiplication of two bicomplex Pell numbers is defined by

$$\begin{aligned} BP_n \times BP_m = & \;(P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3}) \\ & (P_m + iP_{m+1} + jP_{m+2} + ijP_{m+3}) \\ = & \;(P_nP_m - P_{n+1}P_{m+1} - P_{n+2}P_{m+2} + P_{n+3}P_{m+3}) \\ & + i(P_nP_{m+1} + P_{n+1}P_m - P_{n+2}P_{m+3} - P_{n+3}P_{m+2}) \\ & + j(P_nP_{m+2} + P_{n+2}P_m - P_{n+1}P_{m+3} - P_{n+3}P_{m+1}) \\ & + ij(P_nP_{m+3} + P_{n+3}P_m + P_{n+1}P_{m+2} + P_{n+2}P_{m+1}) \\ = & \;BP_m \times BP_n. \end{aligned}$$

The conjugation of the bicomplex Pell numbers is defined in three different ways as follows

$$(BP_n)_i^* = P_n - iP_{n+1} + jP_{n+2} - ijP_{n+3}, \tag{2.3}$$

$$(BP_n)_j^* = P_n + iP_{n+1} - jP_{n+2} - ijP_{n+3}, \tag{2.4}$$

$$(BP_n)^*_{ij} = P_n - iP_{n+1} - jP_{n+2} + ijP_{n+3}. \tag{2.5}$$

Theorem 2.1. *Let BP_n and BP_m be two bicomplex Pell numbers. In this case, we can give the following relations between the conjugates of these numbers:*

$$\begin{aligned}
(BP_n \times BP_m)^*_i &= (BP_m)^*_i \times (BP_n)^*_i = (BP_n)^*_i \times (BP_m)^*_i,\\
(BP_n \times BP_m)^*_j &= (BP_m)^*_j \times (BP_n)^*_j = (BP_n)^*_j \times (BP_m)^*_j,\\
(BP_n \times BP_m)^*_{ij} &= (BP_m)^*_{ij} \times (BP_n)^*_{ij} = (BP_n)^*_{ij} \times (BP_m)^*_{ij}.
\end{aligned}$$

Proof. It can be proved easily by using (2.3)-(2.5). □

In the following theorem, some properties related to the conjugations of the bicomplex Pell numbers are given.

Theorem 2.2. *Let $(BP_n)^*_i$, $(BP_n)^*_j$ and $(BP_n)^*_{ij}$ be three kinds of conjugation of the bicomplex Pell numbers. The following relations hold:*

$$BP_n \times (BP_n)^*_i = 2(-Q_{2n+3} + jP_{2n+3}), \tag{2.6}$$

$$\begin{aligned} BP_n \times (BP_n)^*_j = &\ (P_n^2 - P_{n+1}^2 + P_{n+2}^2 - P_{n+3}^2)\\ &+4i(P_{2n+3} + P_nP_{n+1}), \end{aligned} \tag{2.7}$$

$$BP_n \times (BP_n)^*_{ij} = 6P_{2n+3} + 4ij(-1)^{n+1}, \tag{2.8}$$

$$BP_n \times (BP_n)^*_i + BP_{n-1} \times (BP_{n-1})^*_i = -2(8P_{2n+2} + jQ_{2n+2}), \tag{2.9}$$

$$BP_n \times (BP_n)^*_j + BP_{n-1} \times (BP_{n-1})^*_j = 12(-P_{2n+2} + iP_{2n+2}), \tag{2.10}$$

$$BP_n \times (BP_n)^*_{ij} + BP_{n-1} \times (BP_{n-1})^*_{ij} = 6Q_{2n+2}. \tag{2.11}$$

Proof. (2.6): Using (2.1) and (2.3) we get,

$$\begin{aligned}
BP_n \times (BP_n)^*_i &= (P_n^2 + P_{n+1}^2 - P_{n+2}^2 - P_{n+3}^2)\\
&\quad +2j(P_nP_{n+2} + P_{n+1}P_{n+3})\\
&= P_{2n+1} - P_{2n+5} + 2jP_{2n+3}\\
&= 2(-Q_{2n+3} + jP_{2n+3}).
\end{aligned}$$

(2.7): Using (2.1) and (2.4) we get,

$$\begin{aligned}
BP_n \times (BP_n)^*_j &= (P_n^2 - P_{n+1}^2 + P_{n+2}^2 - P_{n+3}^2)\\
&\quad +2i(P_nP_{n+1} + P_{n+2}P_{n+3})\\
&= (P_n^2 - P_{n+1}^2 + P_{n+2}^2 - P_{n+3}^2)\\
&\quad +4i(P_{2n+3} + P_nP_{n+1}).
\end{aligned}$$

(2.8): Using (2.1) and (2.5) we get,

$$\begin{aligned}
BP_n \times (BP_n)^*_{ij} &= (P_n^2 + P_{n+1}^2 + P_{n+2}^2 + P_{n+3}^2)\\
&\quad +2ij(P_nP_{n+3} - P_{n+1}P_{n+2})\\
&= (P_{2n+1} + P_{2n+5}) + 4ij(-1)^{n+1}\\
&= 6P_{2n+3} + 4ij(-1)^{n+1}.
\end{aligned}$$

(2.9): Using (2.6) we get,

$$\begin{aligned} BP_n \times (BP_n)_i^* + BP_{n-1} \times (BP_{n-1})_i^* = & -2\,[\,(Q_{2n+3} + Q_{2n+1}) \\ & \quad - j\,(P_{2n+3} + P_{2n+1})\,] \\ = & -2\,(\,8\,P_{2n+2} - j\,Q_{2n+2}). \end{aligned}$$

(2.10): Using (2.7) we get,

$$\begin{aligned} BP_n \times (BP_n)_j^* + BP_{n-1} \times (BP_{n-1})_j^* = & \;(P_{n-1}^2 - P_{n+3}^2) \\ & \quad +4\,i\,(P_n\,Q_n + Q_{2n+2}) \\ = & -12\,P_{2n+2} + 4\,i\,(3\,P_{2n+2}) \\ = & -12\,(\,P_{2n+2} - i\,P_{2n+2}). \end{aligned}$$

(2.11): Using (2.8) we get,

$$\begin{aligned} BP_n \times (BP_n)_{ij}^* + BP_{n-1} \times (BP_{n-1})_{ij}^* = & \;6\,(\,P_{2n+3} + P_{2n+1}) \\ & +4\,i\,j\,[(-1)^{n+1} + (-1)^n] \\ = & \;6\,Q_{2n+2}. \end{aligned}$$

□

Therefore, the norm of the bicomplex Pell number BP_n is defined in three different ways as follows

$$N_{BP_n\,i} = \|BP_n \times BP_{ni}^*\| = \sqrt{2\,|-Q_{2n+3} + j\,P_{2n+3}\,|},$$

$$\begin{aligned} N_{BP_n\,j} = & \;\|BP_n \times BP_{n\,j}^*\| \\ = & \sqrt{|(P_n^2 - P_{n+1}^2 + P_{n+2}^2 - P_{n+3}^2) + 4\,i\,(P_{2n+3} + P_n\,P_{n+1})|}, \end{aligned} \tag{2.12}$$

$$N_{BP_n\,i\,j} = \|BP_n \times BP_{ni\,j}^*\| = \sqrt{|\,6\,Q_{2n+3} + 4\,i\,j\,(-1)^{n+1}\,|}. \tag{2.13}$$

Theorem 2.3. *Let BP_n and BPL_n be the bicomplex Pell and bicomplex Pell-Lucas numbers, respectively.The following relations hold:*

$$\begin{aligned} BP_m\,BP_n + BP_{m+1}\,BP_{n+1} = & \;4\,(\,Q_{m+n+4} - i\,Q_{m+n+4} \\ & - j\,P_{m+n+4} + i\,j\,P_{m+n+4}\,), \end{aligned} \tag{2.14}$$

$$\begin{aligned} (BP_n)^2 = & \;4\,P_{2n+3} - 4\,i\,P_{2n+3} + 2\,j\,(P_{2n+1} - 6\,P_{n+1}^2) \\ & +2\,i\,j\,(\,6\,P_n\,P_{n+1} + 2\,P_{2n+1}), \end{aligned} \tag{2.15}$$

$$\begin{aligned} (BP_n)^2 + (BP_{n+1})^2 = & \;4\,(\,Q_{2n+4} - i\,Q_{2n+4} - j\,P_{2n+4} \\ & + i\,j\,P_{2n+4}\,), \end{aligned} \tag{2.16}$$

$$\begin{aligned} (BP_{n+1})^2 - (BP_{n-1})^2 = & -4\,(\,P_{2n+1} + 2\,i\,Q_{2n+3} + 2\,j\,P_{2n+3} \\ & +2\,i\,j\,P_{2n+3}\,) \end{aligned} \tag{2.17}$$

$$BP_n - i\,BP_{n+1} + j\,BP_{n+2} - i\,j\,BP_{n+3} = 4\,(-4\,P_{n+3} + j\,q_{n+3}\,), \tag{2.18}$$

$$\begin{aligned} BP_n - i\,BP_{n+1} - j\,BP_{n+2} - i\,j\,BP_{n+3} = & \;2\,(\,q_{n+1} - P_{n+5} + i\,P_{n+5} \\ & + j\,P_{n+4} - i\,j\,P_{n+3}\,). \end{aligned} \tag{2.19}$$

Proof. (2.14): By the equation (2.1) we get,

$$
\begin{aligned}
BP_m\,BP_n + BP_{m+1}\,BP_{n+1} = \;& (P_{m+n+1} - P_{m+n+3} - P_{m+n+5} \\
& +P_{m+n+7}) \\
& +2\,i\,(P_{m+n+2} - P_{m+n+6}) \\
& +2\,j\,(P_{m+n+3} - P_{m+n+5}) \\
& +2\,i\,j\,(2\,P_{m+n+4}) \\
= \;& 4\,(Q_{m+n+4} - i\,Q_{m+n+4} - j\,P_{m+n+4} \\
& +i\,j\,P_{m+n+4}).
\end{aligned}
$$

(2.15): By the equation (2.1) we get,

$$
\begin{aligned}
(BP_n)^2 = \;& (P_n^2 - P_{n+1}^2 - P_{n+2}^2 + P_{n+3}^2) + 2\,i\,(P_n\,P_{n+1} - P_{n+2}\,P_{n+3}) \\
& +2\,j\,(P_n\,P_{n+2} - P_{n+1}\,P_{n+3}) + 2\,i\,j\,(P_n\,P_{n+3} + P_{n+1}\,P_{n+2}) \\
= \;& 4\,P_{2n+3} - 4\,i\,P_{2n+3} + 2\,j\,(P_{2n+1} - 6\,P_{n+1}^2) \\
& +2\,i\,j\,(6\,P_n\,P_{n+1} + 2\,P_{2n+1}).
\end{aligned}
$$

(2.16): By the equations (2.1) and (2.14) we get,

$$
\begin{aligned}
(BP_n)^2 + (BP_{n+1})^2 = \;& (P_n^2 - P_{n+2}^2 + P_{n+4}^2 - P_{n+2}^2) \\
& +2\,i\,(P_{2n+2} - P_{2n+6}) + 2\,j\,(P_{2n+3} - P_{2n+5}) \\
& +2\,i\,j\,(2\,P_{2n+4}) \\
= \;& 4\,(\,Q_{2n+4} - i\,Q_{2n+4} - j\,P_{2n+4} + i\,j\,P_{2n+4}\,).
\end{aligned}
$$

(2.17) By the equations (2.1) and (2.14) we get,

$$
\begin{aligned}
(BP_{n+1})^2 - (BP_{n-1})^2 = \;& (P_{n+1}^2 - P_{n-1}^2 + P_n^2 - P_{n+2}^2) \\
& +2\,i\,[2\,(P_{2n+1} - P_{2n+5})] \\
& +2\,j\,(P_{2n+3} - 5P_{2n+3}) \\
& +2\,i\,j\,[4\,(q_{2n+2} + P_{2n+2})] \\
= \;& 2\,(P_{2n} - P_{2n+2}) + 2\,i\,(-4\,Q_{2n+3}) \\
& +2\,j\,(-4\,P_{2n+3}) + 2\,i\,j\,(4\,P_{2n+3}) \\
= \;& -4\,(\,P_{2n+1} + 2\,i\,Q_{2n+3} + 2\,j\,P_{2n+3} \\
& +2\,i\,j\,P_{2n+3}\,)
\end{aligned}
$$

(2.18): By the equation (2.1) we get,

$$
\begin{aligned}
BP_n - i\,BP_{n+1} - j\,BP_{n+2} - i\,j\,BP_{n+3} = \;& (P_n + P_{n+2} + P_{n+4} - P_{n+6}) \\
& +2\,i\,(P_{n+5}) + 2\,j\,(P_{n+4}) \\
& -2\,i\,j\,(P_{n+3}) \\
= \;& -(4\,P_{n+1} + P_n) + 2\,i\,P_{n+5} \\
& +2\,j\,P_{n+4} - 2\,i\,j\,P_{n+3}.
\end{aligned}
$$

(2.19): By the equation (2.1) we get,

$$
\begin{aligned}
BP_n - i\,BP_{n+1} - j\,BP_{n+2} - i\,j\,BP_{n+3} = \;& (P_n + P_{n+2} + P_{n+4} - P_{n+6}) \\
& +2\,i\,(P_{n+5}) + 2\,j\,(P_{n+4}) \\
& -2\,i\,j\,(P_{n+3}) \\
= \;& -(4\,P_{n+1} + P_n) + 2\,i\,P_{n+5} \\
& +2\,j\,P_{n+4} - 2\,i\,j\,P_{n+3}.
\end{aligned}
$$

. □

Theorem 2.4. *(d'Ocagne's identity). For $n,m \geq 0$ d'Ocagne's identity for bicomplex Pell numbers BP_n and BP_m is given by*

$$BP_m\,BP_{n+1} - BP_{m+1}\,BP_n = 12\,(-1)^n\,P_{m-n}\,(\,j + i\,j\,). \tag{2.20}$$

Proof. (2.20): By the equation (2.1) we get,

$$\begin{aligned} BP_m BP_{n+1} - BP_{m+1} BP_n = \; & (-1)^n P_{m-n}(0) \\ & +i(-1)^n (P_{m-n-1}(0) \\ & +2j(-1)^n (P_{m-n-2} + P_{m-n+2}) \\ & +ij(-1)^n [(-P_{m-n-3} + P_{m-n+3} \\ & \quad +P_{m-n-1} - P_{m-n+1})] \\ = \; & 2j(-1)^n (6P_{m-n}) \\ & +ij(-1)^n 6(P_{m-n-1} - P_{m-n+1}) \\ = \; & 12(-1)^n P_{m-n}(j + +ij). \end{aligned}$$

□

Theorem 2.5. *Let BP_n and BPL_n be the bicomplex Pell number and the bicomplex Pell-Lucas numbers respectively. The following relations are satisfied*

$$BP_{n+1} + BP_{n-1} = BPL_n, \tag{2.21}$$

$$BP_{n+1} - BP_{n-1} = 2BP_n, \tag{2.22}$$

$$BP_{n+2} + BP_{n-2} = 6BP_n. \tag{2.23}$$

$$BP_{n+2} - BP_{n-2} = 2BPL_n, \tag{2.24}$$

$$BP_{n+1} + BP_n = \frac{1}{2} BPL_{n+1}, \tag{2.25}$$

$$BP_{n+1} - BP_n = \frac{1}{2} BPL_n, \tag{2.26}$$

$$BPL_{n+1} + BPL_{n-1} = 4BP_n, \tag{2.27}$$

$$BPL_{n+1} - BPL_{n-1} = 2BPL_n, \tag{2.28}$$

$$BPL_{n+2} + BPL_{n-2} = 6BPL_n, \tag{2.29}$$

$$BPL_{n+2} - BPL_{n-2} = 8BP_n, \tag{2.30}$$

$$BPL_{n+1} + BPL_n = 4BP_{n+1}, \tag{2.31}$$

$$BPL_{n+1} - BPL_n = 4BP_n. \tag{2.32}$$

Proof. (2.21):By the equation (2.1) we get,

$$\begin{aligned} BP_{n+1}+BP_{n-1} = &\ (P_{n+1}+P_{n-1})+i\,(P_{n+2}+P_n) \\ &\quad +j\,(P_{n+3}+P_{n+1})+i\,j\,(P_{n+4}+P_{n+2}) \\ = &\ (Q_n+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}) \\ = &\ BPL_n\,, \end{aligned}$$

(2.22): By the equation (2.1) we get,

$$\begin{aligned} BP_{n+1}-BP_{n-1} = &\ (P_{n+1}-P_{n-1})+i\,(P_{n+2}-P_n) \\ &\quad +j\,(P_{n+3}-P_{n+1})+i\,j\,(P_{n+4}-P_{n+2}) \\ = &\ 2\,(P_n+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}) \\ = &\ 2\,BP_n. \end{aligned}$$

(2.23): By the equation (2.1) we get,

$$\begin{aligned} BP_{n+2}+BP_{n-2} = &\ (P_{n+2}+P_{n-2})+i\,(P_{n+3}+P_{n-1}) \\ &\quad +j\,(P_{n+4}+P_n)+i\,j\,(P_{n+5}+P_{n+1}) \\ = &\ 6\,(P_n+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}) \\ = &\ 6\,BP_n. \end{aligned}$$

(2.24): By the equation (2.1) we get,

$$\begin{aligned} BP_{n+2}-BP_{n-2} = &\ (P_{n+2}-P_{n-2})+i\,(P_{n+3}-P_{n-1}) \\ &\quad +j\,(P_{n+4}-P_n)+i\,j\,(P_{n+5}-P_{n+1}) \\ = &\ 2\,(Q_n+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}) \\ = &\ 2\,BPL_n\,. \end{aligned}$$

(2.25): By the equation (2.1) we get,

$$\begin{aligned} BP_{n+1}+BP_n = &\ (P_{n+1}+P_n)+i\,(P_{n+2}+P_{n+1}) \\ &\quad +j\,(P_{n+3}+P_{n+2})+i\,j\,(P_{n+4}+P_{n+3}) \\ = &\ (q_{n+1}+i\,q_{n+2}+j\,q_{n+3}+i\,j\,q_{n+4}) \\ = &\ \tfrac{1}{2}\,(Q_{n+1}+i\,Q_{n+2}+j\,Q_{n+3}+i\,j\,Q_{n+4}) \\ = &\ \tfrac{1}{2}\,BPL_{n+1} \end{aligned}$$

where the property (1.17) of the modified Pell number is used.
(2.26): By the equation (2.1) we get,

$$\begin{aligned} BP_{n+1}-BP_n = &\ (P_{n+1}-P_n)+i\,(P_{n+2}-P_{n+1}) \\ &\quad +j\,(P_{n+3}-P_{n+2})+i\,j\,(P_{n+4}-P_{n+3}) \\ = &\ (q_n+i\,q_{n+1}+j\,q_{n+2}+i\,j\,q_{n+3}) \\ = &\ \tfrac{1}{2}\,(Q_n+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}) \\ = &\ \tfrac{1}{2}\,BPL_n \end{aligned}$$

where the property (1.17) of the modified Pell number is used.
(2.27): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+1}+BPL_{n-1} = &\ (Q_{n+1}+Q_{n-1})+i\,(Q_{n+2}+Q_n) \\ &\ +j\,(Q_{n+3}+Q_{n+1})+i\,j\,(Q_{n+4}+Q_{n+2}) \\ = &\ 4\,(P_n+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}) \\ = &\ 4\,BP_n. \end{aligned}$$

(2.28): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+1}-BPL_{n-1} = &\ (Q_{n+1}-Q_{n-1})+i\,(Q_{n+2}-Q_n) \\ &\ +j\,(Q_{n+3}-Q_{n+1})+i\,j\,(Q_{n+4}-Q_{n+2}) \\ = &\ 2\,(Q_n+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}) \\ = &\ 2\,BPL_n \end{aligned}$$

(2.29): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+2}+BPL_{n-2} = & \ (Q_{n+2}+Q_{n-2})+i\,(Q_{n+3}+Q_{n-1}) \\ & +j\,(Q_{n+4}+Q_{n})+i\,j\,(Q_{n+5}+Q_{n+1}) \\ = & \ 6\,(Q_{n}+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}) \\ = & \ 6\,BPL_{n}. \end{aligned}$$

(2.30): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+2}-BPL_{n-2} = & \ (Q_{n+2}-Q_{n-2})+i\,(Q_{n+3}-Q_{n-1}) \\ & +j\,(Q_{n+4}-Q_{n})+i\,j\,(Q_{n+5}-Q_{n+1}) \\ = & \ 8\,(P_{n}+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}) \\ = & \ 8\,BP_{n}. \end{aligned}$$

(2.31): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+1}+BPL_{n} = & \ (Q_{n+1}+Q_{n})+i\,(Q_{n+2}+Q_{n+1}) \\ & +j\,(Q_{n+3}+Q_{n+2})+i\,j\,(Q_{n+4}+Q_{n+3}) \\ = & \ 4\,P_{n+1}+i\,P_{n+2}+j\,P_{n+3}+i\,j\,P_{n+4} \\ = & \ 4\,BP_{n+1}. \end{aligned}$$

(2.32): By the equation (2.2) we get,

$$\begin{aligned} BPL_{n+1}-BPL_{n} = & \ (Q_{n+1}-Q_{n})+i\,(Q_{n+2}-Q_{n+1}) \\ & +j\,(Q_{n+3}-Q_{n+2})+i\,j\,(Q_{n+4}-Q_{n+3}) \\ = & \ 4\,P_{n}+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3} \\ = & \ 4\,BP_{n}. \end{aligned}$$

□

Theorem 2.6. *If BP_n and BPL_n are bicomplex Pell and bicomplex Pell-Lucas numbers, respectively. For $n \geq 0$, the identities of negabicomplex Pell and negabicomplex Pell-Lucas numbers are*

$$BP_{-n} = \ (-1)^{n+1}\,BP_{n}+(-1)^{n}\,Q_{n}\,(\,i+2\,j+5\,i\,j\,). \tag{2.33}$$

and

$$BPL_{-n} = \ (-1)^{n}\,BPL_{n}+8\,(-1)^{n+1}\,P_{n}\,(\,i+2\,j+5\,i\,j\,). \tag{2.34}$$

Proof. (2.33): Using the identity of negapell numbers $P_{-n}=(-1)^{n+1}\,P_{n}$ we get

$$\begin{aligned} BP_{-n} = & \ P_{-n}+i\,P_{-n+1}+j\,P_{-n+2}+i\,j\,P_{-n+3} \\ = & \ P_{-n}+i\,P_{-(n-1)}+j\,P_{-(n-2)}+i\,j\,P_{-(n-3)} \\ = & \ (-1)^{n+1}\,P_{n}+i\,(-1)^{n}\,P_{n-1}+j\,(-1)^{n-1}\,P_{n-2} \\ & +i\,j\,(-1)^{n-2}\,P_{n-3} \\ = & \ (-1)^{n+1}\,(P_{n}+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}) \\ & -i\,(-1)^{n+1}\,P_{n+1}-j\,(-1)^{n+1}\,P_{n+2}-i\,j\,(-1)^{n+1}\,P_{n+3} \\ & +i\,(-1)^{n}\,P_{n-1}+j\,(-1)^{n+1}\,P_{n-2}+i\,j\,(-1)^{n}\,P_{n-3} \\ = & \ (-1)^{n+1}\,BP_{n}+(-1)^{n}\,(P_{n+1}+P_{n-1})\,i \\ & +(-1)^{n}\,(P_{n+2}-P_{n-2})\,j+(-1)^{n}\,(P_{n+3}+P_{n-3})\,i\,j \\ = & \ (-1)^{n+1}\,BP_{n}+(-1)^{n}\,Q_{n}\,(\,i+2\,j+5\,i\,j\,) \end{aligned}$$

(2.34): Using the identity of negapell-Lucas numbers $Q_{-n}=(-1)^{n}\,Q_{n}$ we get

$$\begin{aligned}
BPL_{-n} = & \; Q_{-n} + i\,Q_{-n+1} + j\,Q_{-n+2} + i\,j\,Q_{-n+3} \\
= & \; Q_{-n} + i\,Q_{-(n-1)} + j\,Q_{-(n-2)} + i\,j\,Q_{-(n-3)} \\
= & \; (-1)^n\,Q_n + i\,(-1)^{n-1}\,Q_{n-1} + j\,(-1)^{n-2}\,Q_{n-2} \\
& \quad + i\,j\,(-1)^{n-3}\,Q_{n-3} \\
= & \; (-1)^{n+1}\,(Q_n + i\,Q_{n+1} + j\,Q_{n+2} + i\,j\,Q_{n+3}) \\
& \quad - i\,(-1)^n\,Q_{n+1} - j\,(-1)^n\,Q_{n+2} \\
& \quad - i\,j\,(-1)^n\,Q_{n+3} \\
& \quad + i\,(-1)^{n-1}\,Q_{n-1} + j\,(-1)^n\,Q_{n-2} \\
& \quad + i\,j\,(-1)^{n-1}\,Q_{n-3} \\
= & \; (-1)^{n+1}\,BPL_n + (-1)^{n+1}\,(Q_{n+1} + Q_{n-1})\,i \\
& \quad + (-1)^{n+1}\,(Q_{n+2} - Q_{n-2})\,j \\
& \quad + (-1)^{n+1}\,(Q_{n+3} + Q_{n-3})\,i\,j \\
= & \; (-1)^n\,BPL_n + 8\,(-1)^{n+1}\,P_n\,(\,i + 2\,j + 5\,i\,j)
\end{aligned}$$

□

Theorem 2.7. *Binet's Formula. Let BP_n and BPL_n be the bicomplex Pell and bicomplex Pell-Lucas numbers respectively. For $n \geq 1$, Binet's formula for these numbers are as follows:*

$$BP_n = \frac{1}{\alpha - \beta}(\hat{\alpha}\;\alpha^n - \hat{\beta}\;\beta^n) \tag{2.35}$$

and

$$BPL_n = \hat{\alpha}\;\alpha^n + \hat{\beta}\;\beta^n \tag{2.36}$$

where $\hat{\alpha} = 1 + i\,\alpha + j\,\alpha^2 + i\,j\,\alpha^3$, $\alpha = 1 + \sqrt{2}$ *and* $\hat{\beta} = 1 + i\,\beta + j\,\beta^2 + i\,j\,\beta^3$, $\beta = 1 - \sqrt{2}$.

Proof. (2.35):

$$\begin{aligned}
BP_n = & \; P_n + i\,P_{n+1} + j\,P_{n+2} + i\,j\,P_{n+3} \\
= & \; \tfrac{\alpha^n - \beta^n}{\alpha - \beta} + i\,\tfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + j\,\tfrac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} + i\,j\,\tfrac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} \\
= & \; \tfrac{\alpha^n\,(1 + i\,\alpha + j\,\alpha^2 + i\,j\,\alpha^3\,) - \beta^n\,(1 + i\,\beta + j\,\beta^2 + i\,j\,\beta^3\,)}{\alpha - \beta} \\
= & \; \tfrac{\hat{\alpha}\;\alpha^n - \hat{\beta}\;\beta^n}{\alpha - \beta}
\end{aligned}$$

and (2.36):

$$\begin{aligned}
BPL_n = & \; Q_n + i\,Q_{n+1} + j\,Q_{n+2} + i\,j\,Q_{n+3} \\
= & \; \alpha^n + \beta^n + i\,(\alpha^{n+1} + \beta^{n+1}) + j\,(\alpha^{n+2} + \beta^{n+2}) + i\,j\,(\alpha^{n+3} + \beta^{n+3}) \\
= & \; \alpha^n\,(1 + i\,\alpha + j\,\alpha^2 + i\,j\,\alpha^3\,) + \beta^n\,(1 + i\,\beta + j\,\beta^2 + i\,j\,\beta^3\,) \\
= & \; \hat{\alpha}\;\alpha^n + \hat{\beta}\;\beta^n.
\end{aligned}$$

Binet's formula of the bicomplex Pell number is the same as Binet's formula of the Pell number [7]. □

Theorem 2.8. *Cassini's Identity Let BP_n and BPL_n be the bicomplex Pell and bicomplex Pell-Lucas numbers, respectively. For $n \geq 1$, Cassini's identities for BP_n and BPL_n are as follows:*

$$BP_{n-1}\;BP_{n+1} - BP_n^2 = 12\,(-1)^n\,(\,j + i\,j) \tag{2.37}$$

and

$$BPL_{n-1}\;BPL_{n+1} - BPL_n^2 = 8.12\,(-1)^{n+1}\,(\,j + i\,j). \tag{2.38}$$

Proof. (2.37): Using (2.1) we get

$$\begin{aligned}
BP_{n-1}\,BP_{n+1}-(BP_n)^2 = \; & (P_{n-1}+i\,P_n+j\,P_{n+1}+i\,j\,P_{n+2}) \\
& (P_{n+1}+i\,P_{n+2}+j\,P_{n+3}+i\,j\,P_{n+4}) \\
& -[P_n+i\,P_{n+1}+j\,P_{n+2}+i\,j\,P_{n+3}]^2 \\
= \; & [\,(P_{n-1}P_{n+1}-P_n^2) \\
& \quad -(P_nP_{n+2}+P_{n+1}^2) \\
& \quad -(P_{n+1}P_{n+3}-P_{n+2}^2) \\
& \quad +(P_{n+2}P_{n+4}-P_{n+3}^2)\,] \\
& +i\,[\,(P_{n+2}P_{n-1}-P_{n+1}P_n) \\
& \quad -(P_{n+4}P_{n+1}-P_{n+3}P_{n+2})\,] \\
& +j\,[\,(P_{n+1}P_{n+1}-P_nP_{n+2}) \\
& \quad -(P_{n+2}P_{n+2}-P_{n+1}P_{n+3}) \\
& \quad +(P_{n+3}P_{n-1}-P_{n+2}P_n) \\
& \quad -(P_{n+4}P_n-P_{n+3}P_{n+1})\,] \\
& +i\,j\,(P_{n+4}P_{n-1}-P_{n+3}P_n) \\
= \; & 12\,(-1)^n\,(j+i\,j).
\end{aligned}$$

(2.38): Using (2.2) we get

$$\begin{aligned}
BPL_{n-1}\,BPL_{n+1}-(BPL_n)^2 = \; & (Q_{n-1}+i\,Q_n+j\,Q_{n+1}+i\,j\,Q_{n+2}) \\
& (Q_{n+1}+i\,Q_{n+2}+j\,Q_{n+3}+i\,j\,Q_{n+4}) \\
& -[Q_n+i\,Q_{n+1}+j\,Q_{n+2}+i\,j\,Q_{n+3}]^2 \\
= \; & [\,(Q_{n-1}Q_{n+1}-Q_n^2) \\
& \quad +(Q_{n+1}^2-Q_{n+2}Q_n) \\
& \quad +(Q_{n+2}^2-Q_{n+3}Q_{n+1}) \\
& \quad +(Q_{n+4}Q_{n+2}-Q_{n+3}^2)\,] \\
& +i\,[(Q_{n+2}Q_{n-1}-Q_{n+1}Q_n) \\
& \quad +(Q_{n+3}Q_{n+2}-Q_{n+4}Q_{n+1})\,] \\
& +j\,[\,(Q_{n+1}Q_{n+1}-Q_nQ_{n+2}) \\
& \quad +(Q_{n+1}Q_{n+3}-(Q_{n+2}Q_{n+2}) \\
& \quad +(Q_{n+3}Q_{n-1}-Q_{n+2}Q_n) \\
& \quad +(Q_{n+3}(Q_{n+1}-Q_{n+4}Q_n)\,] \\
& +i\,j\,(Q_{n+4}Q_{n-1}-Q_{n+3}Q_n) \\
= \; & 8.12\,(-1)^{n+1}\,(j+i\,j).
\end{aligned}$$

where the identities of the Pell and Pell-Lucas numbers $P_m\,P_{n+1}-P_{m+1}\,P_n=(-1)^nP_{m-n}$ and $Q_m\,Q_{n+1}-Q_{m+1}Q_n=8\,(-1)^{n+1}\,P_{m-n}$ are used. □

Theorem 2.9. *Catalan's Identity. Let BP_n and BPL_n be the bicomplex Pell and bicomplex Pell-Lucas numbers, respectively. For $n\geq 1$, Catalan's identities for BP_n and BPL_n are as follows*

$$(BP_n)^2-BP_{n+r}\,BP_{n-r}=12\,(-1)^{n-r}\,{P_r}^2\,(j+i\,j), \tag{2.39}$$

and

$$(BPL_n)^2-BPL_{n+r}\,BPL_{n-r}=8\,.\,12\,(-1)^{n-r}\,{P_r}^2\,(j+i\,j). \tag{2.40}$$

respectively.

Proof. (2.39): Using (2.1) we get

$$\begin{aligned}
BP_n^2 - BP_{n+r}\,BP_{n-r} = \; & [\,(P_n^2 - P_{n+r}\,P_{n-r}) \\
& \quad -(P_{n+1}^2 - P_{n+r+1}\,P_{n-r+1}) \\
& \quad -(P_{n+2}^2 - P_{n+r+2}\,P_{n-r+2}) \\
& \quad +(P_{n+3}^2 - P_{n+r+3}\,P_{n-r+3})\,] \\
& +i\,[\,(P_n\,P_{n+1} - P_{n+r}\,P_{n-r+1}) \\
& \quad -(P_{n+2}\,P_{n+3} - P_{n+r+2}\,P_{n-r+3}) \\
& \quad +(P_{n+1}\,P_n - P_{n+r+1}\,P_{n-r}) \\
& \quad -(P_{n+3}\,P_{n+2} - P_{n+r+3}\,P_{n-r+2})\,] \\
& +j\,[\,(P_n\,P_{n+2} - P_{n+r}\,P_{n-r+2}) \\
& \quad -(P_{n+1}\,P_{n+3} - P_{n+r+1}\,P_{n-r+3}) \\
& \quad +(P_{n+2}\,P_n - P_{n+r+2}\,P_{n-r}) \\
& \quad -(P_{n+3}\,P_{n+1} - P_{n+r+3}\,P_{n-r+1})\,] \\
& +i\,j\,[\,(P_n\,P_{n+3} - P_{n+r}\,P_{n-r+3}) \\
& \quad +(P_{n+1}\,P_{n+2} - P_{n+r+1}\,P_{n-r+2}) \\
& \quad +(P_{n+3}\,P_n - P_{n+r+3}\,P_{n-r}) \\
& \quad +(P_{n+2}\,P_{n+1} - P_{n+r+2}\,P_{n-r+1})\,] \\
= \; & (-1)^{n-r}\,P_r^2\,(0+0i+12\,j+12\,i\,j) \\
= \; & 12\,(-1)^{n-r}\,P_r^2\,(j+i\,j).
\end{aligned}$$

(2.40): Using (2.2) we get

$$\begin{aligned}
(BPL_n)^2 - BPL_{n+r}\,BPL_{n-r} = \; & [\,(Q_n^2 - Q_{n+r}\,Q_{n-r}) \\
& \quad -(Q_{n+1}^2 - Q_{n+r+1}\,Q_{n-r+1}) \\
& \quad -(Q_{n+2}^2 - Q_{n+r+2}\,Q_{n-r+2}) \\
& \quad +(Q_{n+3}^2 - Q_{n+r+3}\,Q_{n-r+3})\,] \\
& +i\,[\,(Q_n\,Q_{n+1} - Q_{n+r}\,Q_{n-r+1}) \\
& \quad -(Q_{n+2}\,Q_{n+3} - Q_{n+r+2}\,Q_{n-r+3}) \\
& \quad +(Q_{n+1}\,Q_n - Q_{n+r+1}\,Q_{n-r}) \\
& \quad -(Q_{n+3}\,Q_{n+2} - Q_{n+r+3}\,Q_{n-r+2})\,] \\
& +j\,[\,(Q_n\,Q_{n+2} - Q_{n+r}\,Q_{n-r+2}) \\
& \quad -(Q_{n+1}\,Q_{n+3} - Q_{n+r+1}\,Q_{n-r+3}) \\
& \quad +(Q_{n+2}\,Q_n - Q_{n+r+2}\,Q_{n-r}) \\
& \quad -(Q_{n+3}\,Q_{n+1} - Q_{n+r+3}\,Q_{n-r+1})\,] \\
& +i\,j\,[\,(Q_n\,Q_{n+3} - Q_{n+r}\,Q_{n-r+3}) \\
& \quad +(Q_{n+1}\,Q_{n+2} - Q_{n+r+1}\,Q_{n-r+2}) \\
& \quad +(Q_{n+3}\,Q_n - Q_{n+r+3}\,Q_{n-r}) \\
& \quad +(Q_{n+2}\,Q_{n+1} - Q_{n+r+2}\,Q_{n-r+1})\,] \\
= \; & 8\,(-1)^{n-r}\,P_r^2\,(0+0i+12\,j+12\,i\,j) \\
= \; & 8.12\,(-1)^{n-r}\,P_r^2\,(j+i\,j).
\end{aligned}$$

where the identities of the Pell and Pell-Lucas numbers are used as follows,

$$\begin{aligned}
P_mP_n - P_{m+r}P_{n-r} = \; & (-1)^{n-r}P_{m+r-n}\,P_r, \\
P_n\,P_n - P_{n-r}\,P_{n+r} = \; & (-1)^{n-r}\,P_r^2, \\
Q_mQ_n - Q_{m+r}Q_{n-r} = \; & (-1)^{n-r+1}P_{m+r-n}\,P_r, \\
Q_n\,Q_n - Q_{n-r}\,Q_{n+r} = \; & (-1)^{n-r+1}\,P_r^2.
\end{aligned}$$

□

3. Conclusion

In this study, a number of new algebraic results on bicomplex Pell and bicomplex Pell-Lucas numbers are derived. Also, negabicomplex Pell and negabicomplex Pell-Lucas numbers are given. Furthermore, d'Ocagne's identity, Binet's formula, Cassini's identity and Catalan's identity for these numbers are generated.

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On Growth and Approximation of Generalized Biaxially Symmetric Potentials on Parabolic-Convex Sets

Devendra Kumar[1,2]*

Abstract
The regular, real-valued solutions of the second-order elliptic partial differential equation

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{2\alpha+1}{x}\frac{\partial F}{\partial y}+\frac{2\beta+1}{y}\frac{\partial F}{\partial x}=0,\alpha,\beta>\frac{-1}{2}$$

are known as generalized bi-axially symmetric potentials (GBSP's). McCoy [1] has showed that the rate at which approximation error $E_{2n}^{\frac{p}{2n}}(F;D),(p\geq 2,D$ is parabolic-convex set) tends to zero depends on the order of $GBSP$ F and obtained a formula for finite order. If $GBSP$ F is an entire function of infinite order then above formula fails to give satisfactory information about the rate of decrease of $E_{2n}^{\frac{p}{2n}}(F;D)$. The purpose of the present work is to refine above result by using the concept of index-q. Also, the formula corresponding to q-order does not always hold for lower q-order. Therefore we have proved a result for lower q-order also, which have not been studied so far.

Keywords: Parabolic-convex set, Index-q,q-order, Lower q-order, Generalized bi-axially symmetric potentials and elliptic partial differential equation
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[1] *Department of Mathematics, Faculty of Sciences Al-Baha University,P.O.Box-1988, Alaqiq, Al-Baha-65431, Saudi Arabia, K.S.A..*
[2] *Department of Mathematics, [Research and Post Graduate Studies], M.M.H.College, Model Town, Ghaziabad-201001, U.P., India.*
***Corresponding author**: d_kumar001@rediffmail.com
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1. Introduction

The linear second order elliptic partial differential equation is given in the form

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{2\alpha+1}{x}\frac{\partial F}{\partial y}+\frac{2\beta+1}{y}\frac{\partial F}{\partial x}=0,\alpha,\beta>\frac{-1}{2}, \tag{1.1}$$

which are in x and y cf. Gilbert [2]. A polynomial of degree n which is even in x and y is said to be a $GBSP$ polynomial of degree n if it satisfies (1.1). A $GBSP$ F that is regular about origin can be expanded as

$$F(x,y)=\sum_{n=0}^{\infty}a_nR_n^{(\alpha,\beta)}(x,y),$$

where

$$R_n^{(\alpha,\beta)}(x,y)=(x^2+y^2)^nP_n^{(\alpha,\beta)}((x^2-y^2)/(x^2+y^2))/P_n^{(\alpha,\beta)}(1),n=0,1,2,\ldots$$

and $P_n^{(\alpha,\beta)}(t)$ are Jacobi polynomials. Various authors such as Srivastava [3], McCoy [4], Kumar and Basu [5], Kumar and Bishnoi [6], Harfaoui [7], Kumar [8], Kadiri and Harfaoui [9], Kasana and Kumar [10]-[12] and Kapoor and Nautiyal [13] studied the growth and L_p-approximation of regular real-valued solutions of certain elliptic partial differential equations but our results are different from these authors.

There are so many applications of the solutions of (1.1) in several areas of mathematical physics, for example, its solutions arise in the Maxwell system for the modelling of electric or magnetic n-poles, potential scattering, in quasi-stationary (time independent)diffusion processes and as the initial data for parabolic partial differential equations.

Let D be a certain open set that is symmetric about the origin with Jordan boundary. We define the p-norm on D as:

$$\|.\|_p = (\frac{1}{A}\int\int_D |.|^p dxdy)^{\frac{1}{p}}, p \in [1,\infty), \|.\|_\infty = \sup_D |.|, \|1\|_p = 1.$$

The space $L^p(D)$ of real-valued *GBSP* given by (1.2) is regular and even on D with finite p-norm and the space $l^p(D)$ of associated functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{1.2}$$

where

$$R_n^{(\alpha,\beta)}(z,0) = z^{2n}, n = 0,1,2,\dots$$

is analytic on D with finite p-norm. McCoy [1] developed a pair of integral transforms that are one to one maps between the space $L^p(D)$ of real-valued *GBSP* F and the space $l^p(D)$ of associated f as:

$$F(x,y) = K_{\alpha,\beta}(f) = \int_0^\pi \int_0^1 f(\tau)k_{\alpha,\beta}(t,s)dtds,$$

$$\tau^2 = \tau^2(x,y,t,s) = x^2 - y^2t^2 + 2ixyt\cos s,$$

$$f(z) = K^{-1}_{\alpha,\beta}(F) = \int_{-1}^{+1} F(r,\xi,r(1-\xi^2)^{\frac{1}{2}} j_{\alpha,\beta}(zr^{-1},\xi)d\xi,$$

where

$$k_{\alpha,\beta}(t,s) = \nu_{\alpha,\beta}(1-t^2)^{\alpha-\beta-1}t^{2\beta+1}(\sin s)^{2\alpha}$$

and

$$j_{\alpha,\beta}(\tau,\xi) = \eta_{\alpha,\beta}\frac{(1-\tau)}{(1+\tau)^{\alpha+\beta+2}} \times F[\frac{\alpha+\beta+2}{2};\frac{\alpha+\beta+3}{2};\beta+1;\frac{2\tau(1+\xi)}{(1+\tau)^2}].$$

Let us consider the set D which is parabolic-convex, that is,

$$(x+iy)^2 \in D \Leftrightarrow \{(\xi,\eta) : 4x^2(x^2-\eta^2) \le \xi \le x^2-y^2\} \subsetneq D$$

or equivalently,

$$(x+iy)^2 \in D \Leftrightarrow \{(\xi,\eta) : \xi + i\eta = \tau^2(x,y,t,s), 0 \le t \le 1, 0 \le s \le \pi\} \subsetneq D.$$

For example:$D = \Delta : x^2+y^2 < 1 or D = \{(\xi,\eta) : |\xi| < 1, |\eta| < (1+\xi^2)^{\frac{1}{2}}\}$.

Now we define optimal approximation errors as :

$$E^p_{2n} = E^p_{2n}(F;D) = \min\{\|F-H\|_p : H \in P_{2n}\},$$

$$e^p_{2n} = e^p_{2n}(f;D) = \min\{\|f-h\|_p : h \in p_{2n}\}, n = 0,1,2,\dots,$$

where $P_{2n} = \{K_{\alpha,\beta}(h) : h \in p_{2n}\}$, and $p_{2n} = \{\sum_{k=0}^n a_k z^{2k} : a_k(\text{real}), 0 \le k \le n\}$.

McCoy [1, p.465] proved that

$$\lim_{n\to\infty} E_{2n}^{\frac{p}{2n}}(F;D)=0 \tag{1.3}$$

if and only if, F is the restriction of an entire *GBSP* (analytic) function to D. McCoy [14] showed that a *GBSP* F is the restriction of an entire *GBSP* (analytic) function to D if and only if the $K_{\alpha,\beta}$ associate f is the restriction of an entire (analytic) function to D. And when the growth of an entire *GBSP* function with associate f is measured by order $\rho=\rho(F)$ and type $T=T(F)$ which are defined as in analytic function theory by

$$\rho=\limsup_{r\to\infty}\frac{\log\log M_r(F)}{\log r}, T=\limsup_{r\to\infty}\frac{\log M_r(F)}{r^\rho},$$

where

$$M_r(F)=\sup\{|F(x,y)|:x^2+y^2<r^2\},$$

then $\rho(F)=\rho(f)$ and $T(F)=T(f)$.

For an entire F, (1.3) does not give any clue as to the rate at which $E_{2n}^{\frac{p}{2n}}(F;D)$ tends to zero. McCoy [1, p.467] has showed that this rate depends on the order of *GBSP* F. Moreover, he proved that

$$\limsup_{n\to\infty}\frac{2n\log n}{\log[\frac{1}{E_{2n}^{p}(F)}]}=\rho(F) \tag{1.4}$$

where $\rho(F)$ is the nonnegative real number if and only if, F is the restriction of an entire *GBSP* (analytic) function to D of order ρ.

However, if *GBSP* F is an entire function of infinite order, then (1.4) fails to give satisfactory information about the rate of decrease of $E_{2n}^{\frac{p}{2n}}(F;D)$. The purpose of the present work is to refine the result of McCoy [1, p.467] by using the concept of index of an entire function introduced by Sato [15, p.412] to the function of infinite order.

Thus, if

$$\rho(q)=\limsup_{r\to\infty}\frac{\log^{[q]}M_r(F)}{\log r}, q\geq 2$$

where $\log^{[0]}M_r(F)=M_r(F)$ and $\log^{[q]}M_r(F)=\log(\log^{[q-1]}M_r(F))$, then *GBSP* F is said to be of index-q if $\rho(q-1)=\infty$ while $\rho(q)<\infty$. If *GBSP* F is of index-q we shall call $\rho(q)$ the q-order of F. Analogous to lower order, the concept of lower q-order can be introduced. Thus *GBSP* F, that is an entire function of index-q, is said to be lower q-order $\lambda(q)$ if

$$\lambda(q)=\liminf_{r\to\infty}\frac{\log^{[q]}M_r(F)}{\log r}, q\geq 2.$$

2. Auxiliary results

In this section we shall prove some lemmas which will be useful in the sequel.

Lemma 2.1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be an entire function of index-$q(\geq 2)$ and lower q-order $\lambda(q)$ and let $\nu(r)$ denote the rank of the maximum term $\mu(r)$ for $|z|=r$,i.e. $\mu(r)=\max_{n\geq 0}\{|a_n|r^n\}$ and $\nu(r)=\max\{n:\mu(r)=|a_n|r^n\}$.*

Then

$$\lambda(q)=\liminf_{r\to\infty}\frac{\log^{[q-1]}\nu(r)}{\log r}=\liminf_{r\to\infty}\frac{\log^{[q]}\mu(r)}{\log r}.$$

Proof. The proof follows on the lines of Whittaker [16, Thm. 1] for $q=2$, so we omit the proof. □

Lemma 2.2. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be an entire function of index-$q(\geq 2)$ and lower q-order $\lambda(q)$ and let $\{n_k\}$ denote the range of the step function $\nu(r)$, then*

$$\lambda(q)=\liminf_{r\to\infty}\frac{\log^{[q-1]}n_k}{\log\xi(n_{k+1})}$$

where the $\xi(n_k)$ denote the jump points of $\nu(r)$.

Proof. For $q = 2$, the proof is due to Gray and Shah [17, Lemma 1]. □

Lemma 2.3. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ be an entire function of index-$q(\geq 2)$ and lower q-order $\lambda(q)$ such that $\varphi(k) = |\frac{a_k}{a_{k+1}}|^{\frac{1}{(n_{k+1}-n_k)}}$ forms an increasing function of k for $k > k_o$; then*

$$\lambda(q) = \liminf_{k\to\infty} \frac{(n_{k+1} - n_k)\log^{[q-1]} n_k}{\log|\frac{a_k}{a_{k+1}}|}.$$

Proof. For $q = 2$, the proof is due to Juneja and Kapoor [18]. So we omit the proof. □

Lemma 2.4. *Let $\{n_k\}$ be an increasing sequence of positive integers and let $\{a_n\}$ be a sequence of complex numbers such that $|a_{n_k}| < 1$ for $k > k_o$; then for $q \geq 2$*

$$\liminf_{k\to\infty} \frac{n_k \log^{[q-1]} n_k}{\log|a_{n_k}|^{-1}} \geq \liminf_{k\to\infty} \frac{(n_k - n_{k-1})\log^{[q-1]} n_{k-1}}{\log|\frac{a_{n_{k-1}}}{a_{n_k}}|}.$$

Proof. The proof follows on the lines of Juneja [21, Lemma 2] for $q = 2$, so we omit the proof. □

3. Main results

Theorem 3.1. *For fixed $p \geq 2$, let the $F \in L^p(D)$ be the restriction of an entire GBSP (analytic) function to D of index-$q(\geq 2)$. Then F is of q-order $\rho(q)$ if and only if*

$$\rho(q) = \limsup_{n\to\infty} \frac{2n\log n}{\log[\frac{1}{E^p_{2n}(F)}]}. \tag{3.1}$$

Proof. The proof follows on the lines of [1, Thm. 2(i)], so we omit the details. □

However, the result corresponding to (3.1) does not always hold for the lower q-order. The following theorem is corresponding to (3.1) for the lower q-order of a *GBSP* F.

Theorem 3.2. *For fixed $p \geq 2$, let the $F \in L^p(D)$ be the restriction of an entire GBSP (analytic) function to D of index $q(\geq 2)$. Then F is of lower q-order $\lambda(q)$ if and only if*

$$\lambda(q) = \max_{\{n_k\}} \liminf_{k\to\infty} \frac{2n_k \log^{[q-1]} n_{k-1}}{-\log E^p_{2n_k}(F)},$$

where maximum is taken over all increasing sequence $\{n_k\}$ of natural numbers.

Proof. Let $F \in L^p(D)$ be the restriction of an entire *GBSP* (analytic) function to D of index-$q(\geq 2)$ and lower q-order $\lambda(q)$. Following Bernstein's [19, p.176] and A.Giroux [20, p.52], it follows that for

$$e^p_{2n}(f) \leq e^{\infty}_{2n}(f) \leq \frac{2B(r)}{r^{2n}(r-1)} \tag{3.2}$$

for any $r > 1$, where $B(r) = \max_{z\in\Im_r}|f(z)|$ and $\Im_r$ with $r > 1$ denotes the closed interior of the ellipse with foci ± 1, with half-major axis $(r^2 + 1)/2r$ and half-minor axis $(r^2 - 1)/2r$. The closed disks $D_1(r)$ and $D_2(r)$ bound the ellipse $\Im_r$ in the sense that

$$D_1(r) = \{z : |z| \leq \frac{r^2-1}{2r}\} \subsetneq \Im_r \subsetneq D_2(r) = \{z : |z| \leq \frac{r^2+1}{2r}\}.$$

From above it follows that

$$M(\frac{r^2-1}{2r}) \leq B(r) \leq M(\frac{r^2+1}{2r}) \text{for all} \quad r > 1. \tag{3.3}$$

Consequently, (3.2) and (3.3) give for any sequence $\{n_k\}$ of positive integers that

$$M(\frac{r^2+1}{2r}) \geq e^p_{2n_k}(f) r^{2n_k} \tag{3.4}$$

for any $r > 3$ and $k = 1, 2, \ldots$. Now using the optimal approximates [1, eq.12]

$$E_{2n}^{\frac{p}{2n}}(F) \leq w^{\frac{1}{2np}} e_{2n}^{\frac{p}{2n}}(f), w = w(\alpha, \beta, p : D)$$

in (3.4) we obtain

$$M(\frac{r^2+1}{2r}) \geq w^{\frac{-1}{p}} E_{2n_k}^{p} r^{2n_k}. \tag{3.5}$$

Now let

$$\liminf_{k\to\infty} \frac{2n_k \log^{[q-1]} n_{k-1}}{-\log E_{2n_k}^{p}(F)} = \eta^*(\{2n_k\}) \equiv \eta^*. \tag{3.6}$$

Since *GBSP* F is an entire function, (3.6) gives $0 \leq \eta^* \leq \infty$. First, let $0 < \eta^* < \infty$, then for

$$E_{2n_k}^{p}(F) > [\log^{[q-1]} n_{k-1}]^{-\frac{2n_k}{(\eta^*-\varepsilon)}}$$

for $k > k_o = k_o(\varepsilon)$. Let $r_k = e(\log^{[q-2]} n_{k-1})^{\frac{1}{(\eta^*-\varepsilon)}}$ for $k = 1, 2, 3, \ldots$. If $r_k \leq r \leq r_{k+1}, k > k_o$ then (3.5) gives

$$\begin{aligned}
\log M(\frac{r^2+1}{2r}) &\geq \left\{\log E_{2n_k}^{p}(F) + 2n_k \log r - \frac{1}{p}\log w\right\} \\
&\geq \log E_{2n_k}^{p}(F) + 2n_k \log r_k - \frac{1}{p}\log w \\
&> 2n_k \\
&= 2\exp^{[q-2]}(\frac{r_{k+1}}{e})^{(\eta^*-\varepsilon)}.
\end{aligned}$$

So

$$\begin{aligned}
\log^{[q]} M(\frac{r^2+1}{2r}) &> (\eta^*-\varepsilon)\log r_{k+1} - (\eta^*-\varepsilon) \\
&\geq (\eta^*-\varepsilon)\log r - (\eta^*-\varepsilon)
\end{aligned}$$

or

$$\lambda(q) = \liminf_{r\to\infty} \frac{\log^{[q]} M_r(F)}{\log r} \geq \eta^*$$

which obviously holds for every increasing sequence $\{n_k\}$ of positive integers, we have

$$\lambda(q) \geq \max_{\{n_k\}} \eta^*(\{2n_k\}) = \eta^{**}. \tag{3.7}$$

Now for each $n \geq 0$ there exists a unique $h \in p_{2n}$ such that

$$\|f - p_{2n}\|_p = e_{2n}^{p}(f), n = 0, 1, \ldots.$$

Further, since $\|p_{2n+1} - p_{2n}\|_p$ is bounded above by $2e_{2n}^{p}(f)$, we have by [20, p.42];

$$|p_{2n+1} - p_{2n}| \leq 2e_{2n}^{p}(f) r^{2n+1} \tag{3.8}$$

for all $z \in \Im_r$ for any $r > 1$. Thus we can write

$$f(z) = p_0(z) + \sum_{i=0}^{\infty}(p_{2i+1}(z) - p_{2i}(z))$$

and this series converges uniformly in any bounded domain of the complex plane. So, (3.8) gives

$$|f(z)| \leq |p_0(z)| + 2\sum_{i=0}^{\infty} e_{2i}^{p}(f) r^{2i+1}$$

for any $z \in \Im_r$ and from the definition of $B(r)$

$$B(r) \leq A_o + 2\sum_{i=0}^{\infty} e_{2i}^{p}(f) r^{2i+1}.$$

So (3.3) gives

$$M(\frac{r^2-1}{2r}) \leq A_o + 2\sum_{i=0}^{\infty} e_{2i}^{p}(f) r^{2i+1}. \tag{3.9}$$

Using the optimal approximate [1, eq.(13)]

$$e_{2n}^{\frac{p}{2n}}(f) \leq \delta^{\frac{1}{2np}} E_{2n}^{\frac{p}{2n}}(F), \delta = \delta(\alpha, \beta, p : D)$$

in (3.9) we get

$$M(\frac{r^2-1}{2r}) \leq A_o + 2\sum_{i=0}^{\infty} \delta^{\frac{1}{p}} E_{2i}^{p}(F) r^{2i+1}. \tag{3.10}$$

Obviously, the function $g(z) = \sum_{n=0}^{\infty} E_{2n}^{p}(F)\delta^{\frac{1}{p}} z^{2n+1}$ is an entire function. Let $\{n_k\}$ denote the range of $\nu(r)$ for this function. Consider the function $\tilde{g}(z) = \sum_{k=0}^{\infty} E_{2n_k}^{p}(F)\delta^{\frac{1}{p}} z^{2n_k+1}$. It is easily seen that $\tilde{g}(z)$ is also an entire function and that $g(z)$ and $\tilde{g}(z)$ have the same maximum term for every z. It follows that both have same lower q-order. If we denote this by $\lambda_o(q)$ then since $\tilde{g}(z)$ satisfies the hypothesis of Lemma 2.3, we have

$$\begin{aligned}\lambda_0(q) &= \liminf_{k\to\infty} \frac{2(n_k - n_{k-1})\log^{[q-1]} n_{k-1}}{\log(\frac{E_{2n_{k-1}}^{p}(F)}{E_{2n_k}^{p}(F)})} \\ &\leq \liminf_{k\to\infty} \frac{2n_k \log^{[q-1]} n_{k-1}}{-\log E_{2n_k(F)}^{p}} \\ &\leq \max \liminf_{k\to\infty} \frac{2n_k \log^{[q-1]} n_{k-1}}{-\log E_{2n_k(F)}^{p}} = \eta^{**}\end{aligned} \tag{3.11}$$

Thus (3.10) and (3.11) give

$$\begin{aligned}M(\frac{r^2-1}{2r}) &\leq A_o + 2g(r) \\ &\leq O(1) + 2\exp^{[q-1]}(r^{\eta^{**}+\varepsilon})\end{aligned}$$

for a sequence $r_1, r_2, \ldots \to \infty$. Hence, it gives that

$$\lambda(q) \leq \eta^{**}$$

which shows that the lower q-order of *GBSP* F does not exceed η^{**}. Thus, if *GBSP* F is of lower q-order $\lambda(q)$, then (3.7) shows that $\eta^{**} < \lambda(q)$. If $\eta^{**} < \lambda(q)$, then the above arguments show that *GBSP* F would be of lower q-order less than η^{**}, a contradiction. Thus, we must have $\eta^{**} = \lambda(q)$. □

The following theorem depicts the influence of $\lambda(q)$ on the rate of decrease of $E_{2n}^{p}(F)$.

Theorem 3.3. *For fixed $p \geq 2$, let the $F \in L^p(D)$ be the restriction of an entire GBSP (analytic) function to D of index q. Then, F is of lower q-order $\lambda(q)$ if and only if*

$$\lambda(q) = \max_{\{n_k\}} \liminf_{k\to\infty} \frac{2(n_k - n_{k-1})\log^{[q-1]} n_{k-1}}{\log(\frac{E_{2n_{k-1}}^{p}(F)}{E_{2n_k}^{p}(F)})},$$

where maximum is taken over all increasing sequences $\{n_k\}$ of natural numbers.

Proof. In view of Lemma 2.3 and Lemma 2.4 with above arguments the proof is immediate. □

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Fixed Point Sets of Multivalued Contractions and Stability Analysis

Binayak S. Choudhury[1], Nikhilesh Metiya[2]* and Sunirmal Kundu [1]

Abstract
In this paper we derive a fixed point result for a multivalued generalized almost contraction which contains several rational terms through a six variables function and a four variables function. The space is assumed to satisfy some regularity conditions. In another part of the paper we establish stability results for fixed point sets of these contractions. The corresponding singlevalued case is also discussed. The results are obtained without any assumption of continuity. There are two illustrative examples.

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[1] *Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah - 711103, West Bengal, India.*
[2] *Department of Mathematics, Sovarani Memorial College, Jagatballavpur, Howrah-711408, West Bengal, India.*
***Corresponding author**: metiya.nikhilesh@gmail.com
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1. Introduction

In this paper we first establish the fixed point property of certain generalized multivalued mappings which are almost contractions and satisfy some admissibility conditions and then establish that these multivalued mappings have stable fixed point sets. We use rational terms in the contraction inequalities which are considered here.

Our theorems are deduced in the domain of setvalued analysis which is an extension of the ordinary mathematical analysis. Aubin et al. [1] in their book has described several aspects of this study. Banach's contraction mapping principle was extended to the domain of setvalued analysis by Nadler [2], which was followed by several other works in the same direction. Today multivalued fixed point theory has a large literature and can be regarded as a subject by itself. Some recent references from this area of study are [3]-[9].

Admissibility map was introduced in the work of Samet et al. [10]. After which in fixed point theory several other such conditions were introduced by many authors for the purpose of obtaining new fixed point results in metric spaces. The essence of such efforts is to restrict the contractive condition to appropriate subsets of $X \times X$, rather than assuming to be valid between arbitrary pairs of points from the metric space. This is the development which is parallel to the emergence of fixed point theory in partially ordered metric spaces where the introduction and use of the partial order in metric space also serves the same purpose [4], [11]-[16].

Almost contractions are generalizations of the contractive conditions by introducing an additional additive term in the contractive inequality. It was first introduced by Berinde in [17, 18] in which a generalization of the Banach's contraction mapping principle was established by using this idea. Almost contractions and its generalizations were further considered in several works like [3], [19]-[22].

The concept of stability of fixed point sets appeared first in the work of Nadler [2], i.e, in the same work though which the study of setvalued fixed point theory was initiated. There has been wide interest on these problems of stability which is related to limiting behaviors of sequence of multivalued mappings. Some of the several important works which appeared on the topic in recent times are noted in [4], [5], [12], [23]-[26].

Rational terms were used in problems of fixed point theory in a good number of papers. Such uses were initiated by Dass et al. [27] and were subsequently made in several works on fixed point theory of which some recent references are [12], [28], [29]-[32].

The purpose of this paper is to establish the existence of fixed points of multivalued cyclic $(\alpha-\beta)$- admissible mappings in metric spaces, a condition which we define here. The mappings are assumed to satisfy certain rational type generalized almost contractions which are also defined in this work. In Section 2, we describe some mathematical preliminaries which we use in our results in Sections 3 and 4. In Section 3, we prove a fixed point result for multivalued mapping satisfy certain rational type generalized almost contractions. In Section 4, we investigate the stability of fixed point sets of above mentioned setvalued contractions which is derived without continuity assumption.

2. Mathematical preliminaries

The following are the concepts from setvalued analysis which we use in this paper. Let $(X,\ d)$ be a metric space. Let $N(X) :=$ the collection of all nonempty subsets of X; $CB(X) :=$ the collection of all nonempty closed and bounded subsets of X; and $C(X) :=$ the collection of all nonempty compact subsets of X. Now for $x \in X$ and $A,\ B \in CB(X)$, the functions $D(x,\ B)$ and $H(A,\ B)$ are defined as follows:

$$D(x,\ B) = \inf\{d(x,\ y) : y \in B\} \ \text{ and } \ H(A,\ B) = \max\{\sup_{x\in A} D(x,\ B),\ \sup_{y\in B} D(y,\ A)\}.$$

H is known as the Hausdorff metric induced by d on $CB(X)$ [2]. Further, if $(X,\ d)$ is complete then $(CB(X),\ H)$ is also complete.

Lemma 2.1 ([6])**.** Let $(X,\ d)$ be a metric space and $B \in C(X)$. Then for every $x \in X$ there exists $y \in B$ such that $d(x,\ y) = D(x,\ B)$.

Definition 2.2. Let X be a nonempty set, $f : X \to X$ be a singlevalued mapping and $T : X \to N(X)$ be a multivalued mapping. A point $x \in X$ is called a fixed point of f (resp. T) if and only if $x = fx$ (resp. $x \in Tx$).

The set of all fixed points of f and T are denoted respectively by $F(f)$ and $F(T)$.

In [10] Samet et al. introduced the concept of α- admissible mappings and utilized these mappings to prove some fixed point results in metric spaces.

Definition 2.3 ([10])**.** Let X be a nonempty set, $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\ \infty)$. T is said to be an α-admissible mapping if for $x,\ y \in X$, $\alpha(x,\ y) \geq 1 \Longrightarrow \alpha(Tx,\ Ty) \geq 1$.

In the following we define cyclic $(\alpha-\beta)$ admissibility for multivalued mappings.

Definition 2.4. Let X be a nonempty set, $T : X \longrightarrow N(X)$ be a multivalued mapping and $\alpha,\ \beta : X \longrightarrow [0,\ \infty)$. Then T is said to be a cyclic $(\alpha,\ \beta)$- admissible mapping if for $x,\ y \in X$,

(i) $\alpha(x) \geq 1 \Longrightarrow \beta(u) \geq 1$ for all $u \in Tx$,

(ii) $\beta(y) \geq 1 \Longrightarrow \alpha(v) \geq 1$ for all $v \in Ty$.

Definition 2.5. Let $(X,\ d)$ be a metric space and $\gamma : X \longrightarrow [0,\ \infty)$. Then X is said to have γ- regular property if $\{x_n\}$ is a sequence in X with $\gamma(x_n) \geq 1$ for all n and $x_n \longrightarrow x$ as $n \longrightarrow \infty$, then $\gamma(x) \geq 1$.

Let Θ be the collection of all mappings $\theta : [0,\ \infty)^6 \longrightarrow [0,\ \infty)$ such that (i) θ is continuous and nondecreasing in each coordinate; (ii) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ and $\psi(t) < t$ for each $t > 0$, where $\psi(t) = \theta(t,\ t,\ t,\ t,\ t,\ t)$.

It is to be noted that the properties of θ imply that $\theta(0,\ 0,\ 0,\ 0,\ 0,\ 0) = 0$.

Let Ω be the collection of all mappings $\varphi : [0,\ \infty)^4 \longrightarrow [0,\ \infty)$ such that (i) φ is continuous and nondecreasing in each coordinate; (ii) $\varphi(t_1,\ t_2,\ t_3,\ t_4) = 0$ if $t_1t_2t_3t_4 = 0$.

Definition 2.6. Let $(X,\ d)$ be a metric space, $T : X \longrightarrow X$ and $\alpha,\ \beta : X \longrightarrow [0,\ \infty)$. Let $\mu,\ \nu \geq 0$, $\theta \in \Theta$ and $\varphi \in \Omega$. We say that T is generalized almost contraction if for $x,\ y \in X$ with $\alpha(x)\ \beta(y) \geq 1$ or $\alpha(y)\ \beta(x) \geq 1$,

$$d(Tx,\ Ty) \leq M(x,\ y) + N(x,\ y),$$

where

$$M(x,\, y)=\theta\Big(d(x,\, y),\, d(x,\, Tx),\, d(y,\, Ty),\, \frac{1}{2}[d(y,\, Tx)+d(x,\, Ty)],\frac{d(y,\, Ty)\,[1+d(x,\, Tx)^{\mu}]}{1+d(x,\, y)^{\mu}},\, \frac{d(y,\, Tx)\,[1+d(x,\, Ty)^{\nu}]}{1+d(x,\, y)^{\nu}}\Big)$$

and

$$N(x,\, y)=\varphi\Big(d(x,\, Tx),\, d(y,\, Ty),\, d(x,\, Ty),\, d(y,\, Tx)\Big).$$

Definition 2.7. Let $(X,\, d)$ be a metric space, $T: X\longrightarrow C(X)$ be a multivalued mapping and $\alpha,\ \beta: X\longrightarrow[0,\, \infty)$. Let $\mu,\ \nu\geq 0$, $\theta\in\Theta$ and $\varphi\in\Omega$. We say that T is generalized almost contraction if for $x,\ y\in X$ with $\alpha(x)\,\beta(y)\geq 1$ or $\alpha(y)\,\beta(x)\geq 1$,

$$H(Tx,\, Ty)\leq M(x,\, y)+N(x,\, y), \tag{2.1}$$

where

$$M(x,\, y)=\theta\Big(d(x,\, y),\, D(x,\, Tx),\, D(y,\, Ty),\, \frac{1}{2}[D(y,\, Tx)+D(x,\, Ty)],\frac{D(y,\, Ty)\,[1+D(x,\, Tx)^{\mu}]}{1+d(x,\, y)^{\mu}},\, \frac{D(y,\, Tx)\,[1+D(x,\, Ty)^{\nu}]}{1+d(x,\, y)^{\nu}}\Big)$$

and

$$N(x,y)=\varphi\Big(D(x,\, Tx),\, D(y,\, Ty),\, D(x,\, Ty),\, D(y,\, Tx)\Big).$$

3. Main results

Theorem 3.1. *Let $(X,\, d)$ be a complete metric space, $T: X\longrightarrow C(X)$ be a multivalued mapping and $\alpha,\ \beta: X\longrightarrow[0,\, \infty)$. Suppose that (i) X is regular with respect to α and β; (ii) T is a cyclic $(\alpha,\ \beta)$- admissible mapping; (iii) there exists $x_0\in X$ such that $\alpha(x_0)\geq 1$ or $\beta(x_0)\geq 1$ and (iv) there exist $\mu,\ \nu\geq 0$, $\theta\in\Theta$ and $\varphi\in\Omega$ such that T is a generalized almost contraction. Then T has a fixed point in X.*

Proof. By the assumption (iii), suppose there exists $x_0\in X$ such that $\alpha(x_0)\geq 1$ (the proof is similar if $\beta(x_0)\geq 1$). Let $x_1\in Tx_0$. By the assumption (ii), $\beta(x_1)\geq 1$. Now by Lemma 2.1, there exists $x_2\in Tx_1$ such that $d(x_1,\, x_2)=D(x_1,\, Tx_1)$. As $\beta(x_1)\geq 1$ and $x_2\in Tx_1$, by the assumption (ii), we have $\alpha(x_2)\geq 1$. Also by Lemma 2.1, there exists $x_3\in Tx_2$ such that $d(x_2,\, x_3)=D(x_2,\, Tx_2)$. Since $x_3\in Tx_2$ and $\alpha(x_2)\geq 1$, by the assumption (ii), $\beta(x_3)\geq 1$. Continuing this process, we construct a sequence $\{x_n\}$ such that for all $n\geq 0$,

$$x_{n+1}\in Tx_n,\ \ d(x_n,\, x_{n+1})=D(x_n,\, Tx_n)\ \text{ and }\ \alpha(x_{2n})\geq 1,\ \beta(x_{2n+1})\geq 1. \tag{3.1}$$

By (3.1) either $\alpha(x_n)\beta(x_{n+1})\geq 1$ or $\alpha(x_{n+1})\beta(x_n)\geq 1$. Applying the assumption (iv), we have

$$d(x_{n+1},\, x_{n+2})=D(x_{n+1},\, Tx_{n+1})\leq H(Tx_n,\, Tx_{n+1})\leq M(x_n,\, x_{n+1})+N(x_n,\, x_{n+1}). \tag{3.2}$$

Now,

$$\begin{aligned}M(x_n,x_{n+1})=\theta\Big(&d(x_n,x_{n+1}),\, D(x_n,Tx_n),\, D(x_{n+1},Tx_{n+1}),\\ &\frac{D(x_{n+1},Tx_n)+D(x_n,Tx_{n+1})}{2},\frac{D(x_{n+1},\, Tx_{n+1})\,[1+D(x_n,\, Tx_n)^{\mu}]}{1+d(x_n,\, x_{n+1})^{\mu}},\, \frac{D(x_{n+1},\, Tx_n)\,[1+D(x_n,\, Tx_{n+1})^{\nu}]}{1+d(x_n,\, x_{n+1})^{\nu}}\Big)\\ \leq\theta\Big(&d(x_n,\, x_{n+1}),\, d(x_n,\, x_{n+1}),\, d(x_{n+1},\, x_{n+2}),\\ &\frac{d(x_{n+1},\, x_{n+1})+d(x_n,\, x_{n+2})}{2},\frac{d(x_{n+1},\, x_{n+2})\,[1+d(x_n,\, x_{n+1})^{\mu}]}{1+d(x_n,\, x_{n+1})^{\mu}},\, \frac{d(x_{n+1},\, x_{n+1})\,[1+d(x_n,\, x_{n+2})^{\nu}]}{1+d(x_n,\, x_{n+1})^{\nu}}\Big)\\ \leq\theta\Big(&d(x_n,\, x_{n+1}),\, d(x_n,\, x_{n+1}),\, d(x_{n+1},\, x_{n+2}),\, \frac{d(x_n,\, x_{n+2})}{2},\, d(x_{n+1},\, x_{n+2}),\, 0\Big)\end{aligned}$$

Since $\frac{d(x_n,\, x_{n+2})}{2}\leq\frac{d(x_n,\, x_{n+1})+d(x_{n+1},\, x_{n+2})}{2}\leq\max\{d(x_n,\, x_{n+1}),\, d(x_{n+1},\, x_{n+2})\}$, it follows from the property of θ that

$$\begin{aligned}M(x_n,\, x_{n+1})\leq\theta\Big(&\max\{d(x_n,\, x_{n+1}),\, d(x_{n+1},\, x_{n+2})\},\, \max\{d(x_n,\, x_{n+1}),\, d(x_{n+1},\, x_{n+2})\},\\ &\max\{d(x_n,\, x_{n+1}),\, d(x_{n+1},\, x_{n+2})\},\, \max\{d(x_n,\, x_{n+1}),\, d(x_{n+1},\, x_{n+2})\},\\ &\max\{d(x_n,x_{n+1}),\, d(x_{n+1},x_{n+2})\},\, \max\{d(x_n,x_{n+1}),\, d(x_{n+1},x_{n+2})\}\Big)\\ =\psi\Big(&\max\{d(x_n,\, x_{n+1}),\, d(x_{n+1},\, x_{n+2})\}\Big).\end{aligned} \tag{3.3}$$

Also,

$$N(x_n,\ x_{n+1}) = \varphi\Big(D(x_n,Tx_n),\ D(x_{n+1},Tx_{n+1}),\ D(x_n,Tx_{n+1}),\ D(x_{n+1},Tx_n)\Big)$$
$$\leq \varphi\Big(d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}),\ d(x_n,\ x_{n+2}),\ d(x_{n+1},\ x_{n+1})\Big)$$
$$= 0. \tag{3.4}$$

Suppose that $d(x_n,\ x_{n+1}) < d(x_{n+1},\ x_{n+2})$. Then $d(x_{n+1},\ x_{n+2}) > 0$ and it follows from (3.2), (3.3), (3.4) and a property of θ that

$$d(x_{n+1},\ x_{n+2}) \leq \psi(d(x_{n+1},\ x_{n+2})) < d(x_{n+1},\ x_{n+2}),$$

which is a contradiction. Hence $d(x_{n+1},\ x_{n+2}) \leq d(x_n,\ x_{n+1})$. Then by (3.2), (3.3) and (3.4), we have

$$d(x_{n+1},\ x_{n+2}) \leq \psi(d(x_n,\ x_{n+1})). \tag{3.5}$$

By repeated application of (3.5) and the monotone property of θ, we have

$$d(x_{n+1},\ x_{n+2}) \leq \psi(d(x_n,\ x_{n+1})) \leq \psi^2(d(x_{n-1},\ x_n)) \leq \dots \leq \psi^{n+1}(d(x_0,\ x_1)).$$

By a property of θ, we have

$$\sum_n d(x_n,\ x_{n+1}) \leq \sum_n \psi^n(d(x_0,\ x_1)) < \infty.$$

This shows that $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $z \in X$ such that

$$x_n \longrightarrow z \ \text{ as } \ n \longrightarrow \infty. \tag{3.6}$$

Now $\{x_{2n+1}\}$ is a subsequence of $\{x_n\}$ which, by (3.1) and (3.6), satisfies $\beta(x_{2n+1}) \geq 1$ for all n and $x_{2n+1} \longrightarrow z$ as $n \longrightarrow \infty$. By β- regular property of X, we have $\beta(z) \geq 1$. Also by (3.1), $\alpha(x_{2n}) \geq 1$ for all $n \geq 0$. Applying the assumption (iv), we have

$$D(x_{2n+1},\ Tz) \leq H(Tx_{2n},\ Tz) \leq M(x_{2n},\ z) + N(x_{2n},\ z). \tag{3.7}$$

Now,

$$M(x_{2n},z) = \theta\Big(d(x_{2n},z),\ D(x_{2n},Tx_{2n}),\ D(z,Tz),\ \frac{D(z,Tx_{2n}) + D(x_{2n},Tz)}{2},$$
$$\frac{D(z,\ Tz)\ [1 + D(x_{2n},\ Tx_{2n})^{\mu}]}{1 + d(x_{2n},\ z)^{\mu}},\ \frac{D(z,\ Tx_{2n})\ [1 + D(x_{2n},\ Tz)^{\nu}]}{1 + d(x_{2n},\ z)^{\nu}}\Big)$$
$$\leq \theta\Big(d(x_{2n},\ z),\ d(x_{2n},\ x_{2n+1}),\ D(z,\ Tz),\ \frac{d(z,\ x_{2n+1}) + D(x_{2n},\ Tz)}{2},$$
$$\frac{D(z,\ Tz)\ [1 + d(x_{2n},\ x_{2n+1})^{\mu}]}{1 + d(x_{2n},\ z)^{\mu}},\ \frac{d(z,\ x_{2n+1})\ [1 + D(x_{2n},\ Tz)^{\nu}]}{1 + d(x_{2n},\ z)^{\nu}}\Big).$$

Taking limit supremum on both sides of the above inequality, using (3.6) and the continuity of θ, we have

$$\overline{\lim}\, M(x_{2n},\ z) \leq \theta\Big(0,\ 0,\ D(z,\ Tz),\ \frac{D(z,\ Tz)}{2},\ D(z,\ Tz),\ 0\Big)$$
$$\leq \theta\Big(D(z,Tz),\ D(z,Tz),\ D(z,Tz),\ D(z,Tz),\ D(z,Tz),\ D(z,Tz)\Big)$$
$$= \psi(D(z,\ Tz)). \tag{3.8}$$

Also

$$N(x_{2n},\ z) = \varphi\Big(D(x_{2n},\ Tx_{2n}),\ D(z,\ Tz),\ D(x_{2n},\ Tz),\ D(z,\ Tx_{2n})\Big)$$
$$\leq \varphi\Big(d(x_{2n},\ x_{2n+1}),\ D(z,\ Tz),\ D(x_{2n},\ Tz),\ d(z,\ x_{2n+1})\Big).$$

Taking limit supremum and using the property of φ, we have

$$\overline{\lim}\, N(x_{2n},\, z) \le \varphi\Big(0,\, D(z,\, Tz),\, D(z,\, Tz),\, 0\Big) = 0. \tag{3.9}$$

Taking limit supremum on both sides of (3.7) and using (3.8) and (3.9), we have

$$D(z,\, Tz) \le \psi(D(z,\, Tz)).$$

Suppose that $D(z,\, Tz) \ne 0$. From the above inequality and using a property of θ, we have

$$D(z,\, Tz) \le \psi(D(z,\, Tz)) < D(z,\, Tz),$$

which is a contradiction. Hence $D(z,\, Tz) = 0$. Since $Tz \in C(X)$, Tz is compact and hence Tz is closed, that is, $Tz = \overline{Tz}$, where $\overline{Tz}$ denotes the closure of Tz. Now, $D(z,\, Tz) = 0$ implies that $z \in \overline{Tz} = Tz$, that is, z is a fixed point of T.

Note. The conclusion of the above theorem is still valid if in its assumptions the condition that the space X is regular with respect to α and β is replaced by the continuity of T. The proof remains the same except for minor modifications which is not separately shown here.

Example 3.2. Let $X = [0,\, \infty)$ and "d" be the usual metric on X. Then (X, d) is a complete metric space. Let $T : X \longrightarrow C(X)$ be defined as $Tx = [0,\, \frac{x}{256}]$, for $x \in X$ and $\alpha,\, \beta : X \longrightarrow [0,\, \infty)$ be defined as

$$\alpha(x) = \begin{cases} e^x, & \text{if } \; x \in [0,\, 1], \\ \frac{1}{10}, & \text{otherwise,} \end{cases} \qquad \beta(x) = \begin{cases} x+1, & \text{if } \; x \in [0,\, 1], \\ \frac{1}{100}, & \text{otherwise.} \end{cases}$$

Let $\theta : [0,\, \infty)^6 \longrightarrow [0,\, \infty)$ and $\varphi : [0,\, \infty)^4 \longrightarrow [0,\, \infty)$ be defined respectively as follows:

$$\theta(t_1,\, t_2,\, t_3,\, t_4,\, t_5,\, t_6) = \frac{1}{4} \max\, \{t_1,\, t_2,\, t_3,\, t_4,\, t_5,\, t_6\}$$

and

$$\varphi(t_1,\, t_2,\, t_3,\, t_4) = \log\Big(1 + t_1\, t_2\, t_3\, t_4\Big).$$

Take $\mu,\, \nu \ge 0$ be any real numbers.

(i) Suppose that $\{x_n\}$ is a sequence in X converging to $x \in X$ such that $\alpha(x_n) \ge 1$ and $\beta(x_n) \ge 1$ for all n. Then $\{x_n\}$ is a sequence in $[0,\, 1]$ and also $x \in [0,\, 1]$. Then it follows that $\alpha(x) \ge 1$ and $\beta(x) \ge 1$. Therefore, X is regular with respect to α and β.

(ii) Suppose that $x \in X$ and $\alpha(x) \ge 1$. Then $x \in [0,\, 1]$ and $Tx = [0,\, \frac{x}{256}] \subseteq [0,\, 1]$. It follows that $\beta(u) \ge 1$ for all $u \in Tx$. Similarly, if $y \in X$ and $\beta(y) \ge 1$, it can be shown that $\alpha(v) \ge 1$ for all $v \in Ty$. Therefore, T is a cyclic $(\alpha,\, \beta)$- admissible mapping.

(iii) $\alpha(x) \ge 1$ and $\beta(x) \ge 1$ for every $x \in [0,\, 1]$.

(iv) Here $\theta \in \Theta$ and $\varphi \in \Omega$. Let $x,\, y \in X$. Now, $\alpha(x)\, \beta(y) \ge 1$ (or $\alpha(y)\, \beta(x) \ge 1$) implies that $x,\, y \in [0,\, 1]$. So we require to check the validity of the inequality (2.1) for $x,\, y \in [0,\, 1]$. Now $H(Tx,\, Ty) = \dfrac{|\, x - y\,|}{256}$ and $M(x,\, y) \ge \dfrac{|\, x - y\,|}{4}$ for $x,\, y \in [0,\, 1]$. Then (2.1) is satisfied for all $x,\, y \in X$ with $\alpha(x)\, \beta(y) \ge 1$ or $\alpha(y)\, \beta(x) \ge 1$. Therefore, T is a generalized almost contraction.

Hence all the conditions of Theorem 3.1 are satisfied and 0 is a fixed point of T.

In Theorem 3.1, considering $\psi(x_1,\, x_2,\, x_3,\, x_4, x_5,\, x_6) = k \max\, \{x_1,\, x_2,\, x_3,\, x_4,\, x_5,\, x_6\}$, where $k \in [0,\, 1)$ and $\varphi(t_1,\, t_2,\, t_3,\, t_4) = L \min\, \{t_1,\, t_2,\, t_3,\, t_4\}$, where $L \ge 0$ be any real number, we have the following corollary.

Corollary 3.3. *Let $(X,\, d)$ be a complete metric space, $T : X \longrightarrow C(X)$ be a multivalued mapping and $\alpha,\, \beta : X \longrightarrow [0,\, \infty)$. Suppose that (i) X is regular with respect to α and β; (ii) T is a cyclic $(\alpha,\, \beta)$- admissible mapping; (iii) there exists $x_0 \in X$ such that $\alpha(x_0) \ge 1$ or $\beta(x_0) \ge 1$ and (iv) there exist $\mu,\, \nu \ge 0$, $L \ge 0$ and $k \in [0,\, 1)$ such that for $x,\, y \in X$ with $\alpha(x)\, \beta(y) \ge 1$ or $\alpha(y)\, \beta(x) \ge 1$,*

$$H(Tx,\, Ty) \le k\, M(x,\, y) + L\, N(x,\, y),$$

where

$$M(x,\, y) = max\left\{d(x,\, y),\, D(x,\, Tx),\, D(y,\, Ty),\, \frac{1}{2}[D(y,\, Tx) + D(x,\, Ty)],\, \frac{D(y,\, Ty)\,[1 + D(x,\, Tx)^{\mu}]}{1 + d(x,\, y)^{\mu}},\, \frac{D(y,\, Tx)\,[1 + D(x,\, Ty)^{\nu}]}{1 + d(x,\, y)^{\nu}}\right\}$$

and $N(x,\, y) = min\Big\{D(x,\, Tx),\, D(y,\, Ty),\, D(y,\, Tx),\, D(x,\, Ty)\Big\}$. Then T has a fixed point.

The following theorem is the special case of Theorem 3.1 when we treat $T : X \to X$ as a multivalued mapping in which case Tx can be treated as a singleton set for every $x \in X$.

Theorem 3.4. *Let $(X,\ d)$ be a complete metric space, $T : X \to X$ and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that (i) X is regular with respect to α and β; (ii) T is a cyclic $(\alpha,\ \beta)$- admissible mapping; (iii) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ or $\beta(x_0) \geq 1$; and (iv) there exist $\mu,\ \nu \geq 0$, $\theta \in \Theta$ and $\varphi \in \Omega$ such that T is a generalized almost contraction. Then T has a fixed point.*

Proof. We know that $\{x\}$ is compact in X for every $x \in X$. We define a multivalued mapping $S : X \longrightarrow C(X)$ as $Sx = \{Tx\}$ for $x \in X$.

Let $x,\ y \in X$ such that $\alpha(x) \geq 1$ and $\beta(y) \geq 1$. Then by cyclic $(\alpha - \beta)$ - admissibility of T, we have

$$\beta(Tx) \geq 1, \text{ that is, } \beta(u) \geq 1 \text{ where } u \in Sx = \{Tx\} \quad \textit{and} \quad \alpha(Ty) \geq 1, \text{ that is, } \alpha(v) \geq 1 \text{ where } v \in Sy = \{Ty\}.$$

Therefore, for $x,\ y \in X$,

$$\alpha(x) \geq 1 \implies \beta(u) \geq 1 \text{ for all } u \in Sx \text{ and } \beta(y) \geq 1 \implies \alpha(v) \geq 1 \text{ for all } v \in Sy,$$

that is, S is a cyclic $(\alpha - \beta)$- admissible mapping.

Let $x,\ y \in X$ with $\alpha(x)\beta(y) \geq 1$ or $\alpha(y)\beta(x) \geq 1$. Then

$$\begin{aligned} H(Sx,\ Sy) &= d(Tx,\ Ty) \\ &\leq \theta\Big(d(x,\ y),\ d(x,\ Tx),\ d(y,\ Ty), \frac{d(y,\ Tx) + d(x,\ Ty)}{2}, \frac{d(y,\ Ty)\ [1 + d(x,\ Tx)^{\mu}]}{1 + d(x,\ y)^{\mu}}, \frac{d(y,\ Tx)\ [1 + d(x,\ Ty)^{\nu}]}{1 + d(x,\ y)^{\nu}}\Big) \\ &+ \varphi\Big(d(x,\ Tx),\ d(y,\ Ty),\ d(x,\ Ty),\ d(y,\ Tx)\Big) \\ &= \theta\Big(d(x,\ y),\ D(x,\ Sx),\ D(y,\ Sy), \frac{D(y,\ Sx) + D(x,\ Sy)}{2}, \frac{D(y,\ Sy)\ [1 + D(x,\ Sx)^{\mu}]}{1 + d(x,\ y)^{\mu}}, \frac{D(y,\ Sx)\ [1 + D(x,\ Sy)^{\nu}]}{1 + d(x,\ y)^{\nu}}\Big) \\ &+ \varphi\Big(D(x,\ Sx),\ D(y,\ Sy),\ D(x,\ Sy),\ D(y,\ Sx)\Big), \end{aligned}$$

that is, S is a generalized almost contraction. So, all the conditions of Theorem 3.1 are satisfied and hence S has a fixed point z in X. Then $z \in Sz = \{Tz\}$, that is, $z = Tz$, that is, z is a fixed point of T.

4. Stability of fixed point sets

In this section, we investigate the stability of fixed point sets of the setvalued contractions mentioned in Section 3.

Theorem 4.1. *Let $(X,\ d)$ be a complete metric space, $T_l : X \longrightarrow C(X)$, $l = 1,\ 2$ be two multivalued mappings and $\alpha,\ \beta : X \longrightarrow [0,\ \infty)$. Suppose the assumptions (i), (ii) (for each T_l), (iii) and (iv) (for each T_l), of Theorem 3.1 are satisfied. Then $F(T_l) \neq \emptyset$, for $l = 1,\ 2$. Also suppose that $\alpha(x) \geq 1$ or $\beta(x) \geq 1$ for any $x \in F(T_l)$, $(l = 1,\ 2)$. Then $H(F(T_1),\ F(T_2)) \leq \Phi(M)$, where $M = \sup_{x \in X} H(T_1x,\ T_2x)$ and $\Phi(M) = \sum_{n=1}^{\infty} \psi^n(M)$.*

Proof. By Theorem 3.1, the set of fixed points of T_l $(l = 1,\ 2)$ are nonempty, that is, $F(T_l) \neq \emptyset$, for $l = 1,\ 2$. Let $y_0 \in F(T_1)$, that is, $y_0 \in T_1y_0$. Without loss of generality we assume that $\alpha(y_0) \geq 1$ (the proof is similar if $\beta(y_0) \geq 1$). By Lemma 2.1, there exists $y_1 \in T_2y_0$ such that

$$d(y_0,\ y_1) = D(y_0,\ T_2y_0). \tag{4.1}$$

By the condition (ii) on T_2, $\beta(y_1) \geq 1$. Hence $\alpha(y_0)\beta(y_1) \geq 1$. By Lemma 2.1, there exists $y_2 \in T_2y_1$ such that $d(y_1,\ y_2) = D(y_1,\ T_2y_1)$. As $\beta(y_1) \geq 1$ and $y_2 \in T_2y_1$, by the condition (ii) on T_2, we have $\alpha(y_2) \geq 1$. Hence $\alpha(y_2)\beta(y_1) \geq 1$. Again by Lemma 2.1, there exists $y_3 \in T_2y_2$ such that $d(y_2,\ y_3) = D(y_2,\ T_2y_2)$. Then arguing similarly as in the proof of Theorem 3.1, we construct a sequence $\{y_n\}$ such that for all $n \geq 0$,

$$y_{n+1} \in T_2y_n; \quad \alpha(y_{2n}) \geq 1,\ \beta(y_{2n+1}) \geq 1; \quad d(y_{n+1},\ y_{n+2}) \leq \psi(d(y_n,\ y_{n+1}))$$

and

$$d(y_{n+1},\ y_{n+2}) \leq \psi(d(y_n,\ y_{n+1})) \leq \psi^2(d(y_{n-1},\ y_n)) \leq ... \leq \psi^{n+1}(d(y_0,\ y_1)). \tag{4.2}$$

Arguing similarly as in the proof of Theorem 3.1, we prove $\{y_n\}$ is a Cauchy sequence in X and there exists $u \in X$ such that

$$y_n \longrightarrow u \text{ as } n \longrightarrow \infty, \tag{4.3}$$

also u is a fixed point of T_2, that is, $u \in T_2u$. From (4.1) and the definition of M, we have

$$d(y_0,\ y_1) = D(y_0,\ T_2y_0) \leq H(T_1y_0,\ T_2y_0) \leq M = \sup_{x\in X} H(T_1x,\ T_2x). \tag{4.4}$$

Using (4.2), we have

$$d(y_0,\ u) \leq \sum_{i=0}^{n} d(y_i,\ y_{i+1}) + d(y_{n+1},\ u) \leq \sum_{i=0}^{n} \psi^i(d(y_0,\ y_1)) + d(y_{n+1},\ u).$$

Taking limit as $n \longrightarrow \infty$ in the above inequality, using (4.3), (4.4) and the properties of θ, we have

$$d(y_0,\ u) \leq \sum_{i=0}^{\infty} \psi^i(d(y_0,\ y_1)) \leq \sum_{i=0}^{\infty} \psi^i(M) = \Phi(M).$$

Thus given arbitrary $y_0 \in F(T_1)$, we have $u \in F(T_2)$ for which $d(y_0,\ u) \leq \Phi(M)$. Similarly, we can prove that for arbitrary $z_0 \in F(T_2)$, there exists $w \in F(T_1)$ such that $d(z_0,\ w) \leq \Phi(M)$. Hence we conclude that $H(F(T_1),\ F(T_2)) \leq \Phi(M)$.

Lemma 4.2. *Let $(X,\ d)$ be a complete metric space, $\{T_n : X \to C(X) : n \in \mathbb{N}\}$ be a sequence of multivalued mappings uniformly convergent to a multivalued mapping $T : X \to C(X)$ and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that the assumptions (i), (ii) (for each T_n) and (iv) (for each T_n), of Theorem 3.1 are satisfied. Then T satisfies the conditions (ii) and (iv) of Theorem 3.1.*

Proof. First, we prove that T satisfies the condition (ii) of Theorem 3.1, that is, T is cyclic $(\alpha,\ \beta)$- admissible. Let $\alpha(x) \geq 1$ (or $\beta(x) \geq 1$), $x \in X$. Suppose $y \in Tx$ is arbitrary. Since $T_n \longrightarrow T$ uniformly, there exists a sequence $\{x_n\}$ in $\{T_nx\}$ such that $x_n \longrightarrow y$ as $n \longrightarrow \infty$. Since $\alpha(x) \geq 1$ (or $\beta(x) \geq 1$) and each T_n is cyclic $(\alpha,\ \beta)$ - admissible, it follows from Definition 2.4 that $\beta(x_n) \geq 1$, (or $\alpha(x_n) \geq 1$) for every $n \in \mathbb{N}$. Then by regular property of the space with respect to β(or α), it follows that $\beta(y) \geq 1$ (or $\alpha(y) \geq 1$). Hence T is cyclic $(\alpha,\ \beta)$-admissible, that is, T satisfies the condition (ii) of Theorem 3.1.

Let $x,\ y \in X$ with $\alpha(x)\beta(y) \geq 1$ or $\alpha(y)\beta(x) \geq 1$. As for every $n \in \mathbb{N}$, T_n satisfies the condition (iv) of Theorem 3.1, we have

$$\begin{aligned} H(T_nx,\ T_ny) \leq \theta\Big(& d(x,\ y),\ D(x,\ T_nx),\ D(y,\ T_ny),\ \frac{D(y,\ T_nx) + D(x,\ T_ny)}{2}, \\ & \frac{D(y,\ T_ny)\ [1 + D(x,\ T_nx)^{\mu}]}{1 + d(x,\ y)^{\mu}},\ \frac{D(y,\ T_nx)\ [1 + D(x,\ T_ny)^{\nu}]}{1 + d(x,\ y)^{\nu}}\Big) + \varphi\Big(D(x,\ T_nx),\ D(y,\ T_ny),\ D(x,\ T_ny),\ D(y,\ T_nx)\Big). \end{aligned}$$

Since the sequence $\{T_n\}$ is uniformly convergent to T and θ and φ are continuous, taking limit as $n \longrightarrow \infty$ in the above inequality, we get

$$\begin{aligned} H(Tx,\ Ty) \leq \theta\Big(& d(x,\ y),\ D(x,\ Tx),\ D(y,\ Ty),\ \frac{D(y,\ Tx) + D(x,\ Ty)}{2}, \\ & \frac{D(y,\ Ty)\ [1 + D(x,\ Tx)^{\mu}]}{1 + d(x,\ y)^{\mu}},\ \frac{D(y,\ Tx)\ [1 + D(x,\ Ty)^{\nu}]}{1 + d(x,\ y)^{\nu}}\Big) + \varphi\Big(D(x,\ Tx),\ D(y,\ Ty),\ D(x,\ Ty),\ D(y,\ Tx)\Big), \end{aligned}$$

which shows that T satisfies the condition (iv) of Theorem 3.1.

Now we present our stability result.

Theorem 4.3. *Let $(X,\ d)$ be a complete metric space, $\{T_n : X \longrightarrow C(X) : n \in \mathbb{N}\}$ be a sequence of multivalued mappings uniformly convergent to a mapping $T : X \longrightarrow C(X)$ and $\alpha,\ \beta : X \longrightarrow [0,\ \infty)$. Suppose the assumptions (i), (ii) (for each T_n), (iii) and (iv) (for each T_n), of Theorem 3.1 are satisfied. Then $F(T_n) \neq \emptyset$ for all n and $F(T) \neq \emptyset$. Let $\Phi(t) \longrightarrow 0$ as $t \longrightarrow 0$, where $\Phi(t) = \sum_{n=1}^{\infty} \psi^n(t)$. If $\beta(x) \geq 1$ or $\alpha(x) \geq 1$ for any x belonging to $F(T_n)$, $[n = 1,2,3,..]$ or $F(T)$. Then $\lim_{n\longrightarrow\infty} H(F(T_n),\ F(T)) = 0$, that is, the fixed point sets of T_n are stable.*

Proof. By Lemma 4.2 and Theorem 3.1, we have $F(T_n) \neq \emptyset$ for all n and $F(T) \neq \emptyset$. Let $M_n = \sup_{x\in X} H(T_nx,\ Tx)$. Since the sequence $\{T_n\}$ is uniformly convergent to T on X,

$$\lim_{n\longrightarrow\infty} M_n = \lim_{n\longrightarrow\infty} \sup_{x\in X} H(T_nx,\ Tx) = 0. \tag{4.5}$$

By Theorem 4.1, we get

$$H(F(T_n),\ F(T)) \le \Phi(M_n),\quad \text{for every } n \in \mathbb{N}.$$

Since Φ is continuous and $\Phi(t) \longrightarrow 0$ as $t \longrightarrow 0$, using (4.5), we have

$$\lim_{n\longrightarrow\infty} H(F(T_n),\ F(T)) \le \lim_{n\longrightarrow\infty} \Phi(M_n) = 0,$$

that is, $\lim_{n\longrightarrow\infty} H(F(T_n),\ F(T)) = 0$, that is, the fixed point sets of T_n are stable.

Example 4.4. We take the metric space $(X,\ d)$ and the mappings $\alpha,\ \beta,\ \theta$ and φ as taken in Example 3.2. Let $T : X \longrightarrow C(X)$ be defined as $Tx = [0,\ \frac{x}{256}]$, for $x \in X$ and $T_n : X \longrightarrow C(X)$ be defined as $T_n x = [0,\ \frac{x}{256} + \frac{1}{1024\ n}]$, for $x \in X$. Here the sequence $\{T_n\}$ uniformly converges to T. Let $\mu,\ \nu \ge 0$ be any real numbers. Now for every n, $T_n x = [0,\ \frac{x}{256} + \frac{1}{1024\ n}] \subseteq [0,\ 1]$ for every $x \in [0,\ 1]$ and $H(T_n x,\ T_n y) = \dfrac{|\,x-y\,|}{256}$ for $x,\ y \in [0,\ 1]$. Then as explained in Example 3.2, we can show that the assumptions (i), (ii) (for each T_n), (iii) and (iv) (for each T_n), of Theorem 3.1 are satisfied. Here $F(T_n) = [0,\ \frac{1}{1020n}]$, for each n and $F(T) = \{0\}$. Here $\Phi(t) \to 0$ as $t \to 0$, where $\Phi(t) = \sum_{n=1}^{\infty} \psi^n(t)$, and also $\beta(x) \ge 1$ and $\alpha(x) \ge 1$ for any x belonging to $F(T_n)$, $[n = 1, 2, 3, ..]$ or $F(T)$. So we see all the conditions of Theorem 4.3 are satisfied. Here $\lim_{n\longrightarrow\infty} H(F(T_n),\ F(T)) = 0$, that is, the fixed point sets of T_n are stable.

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