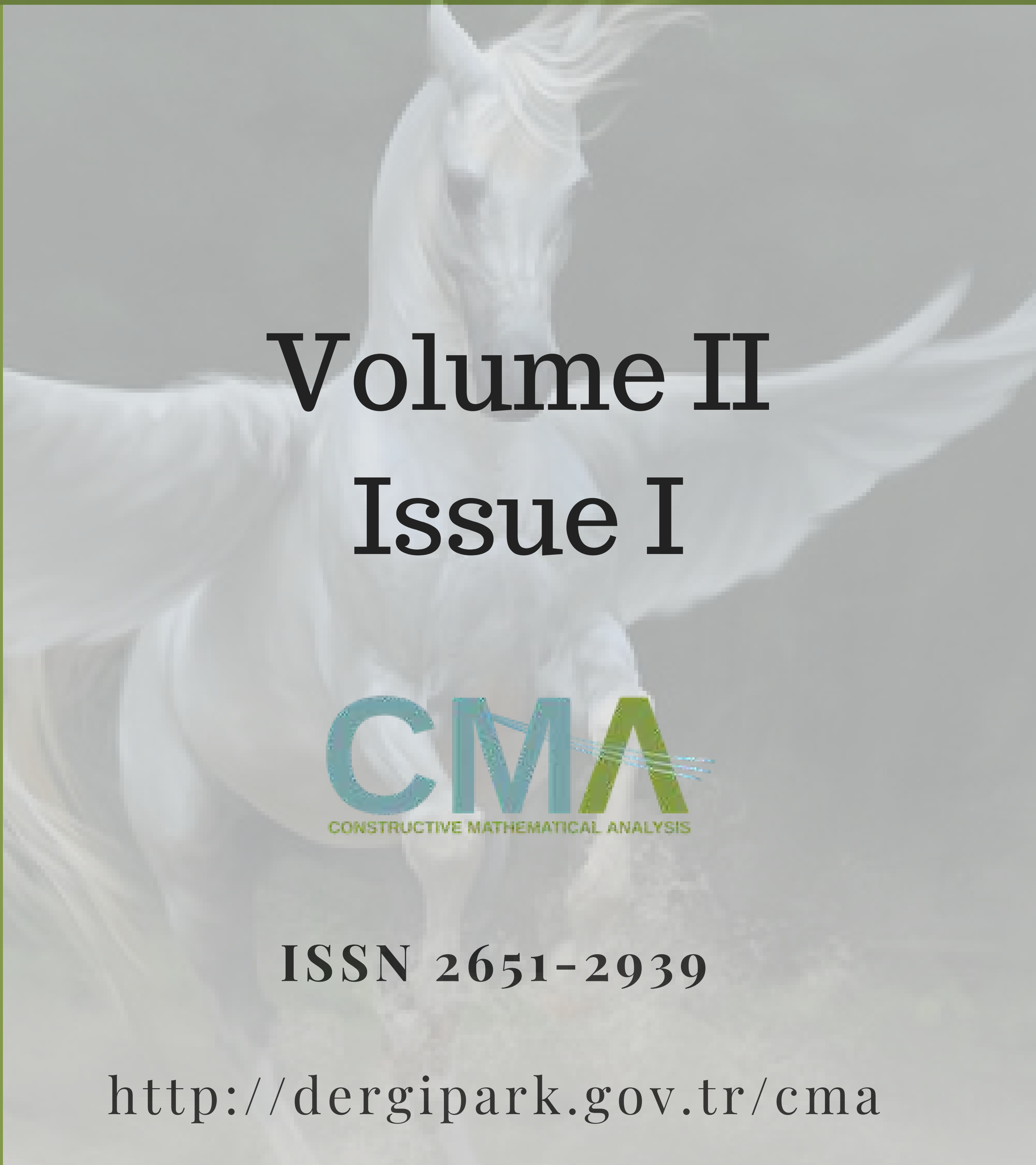
# CONSTRUCTIVE MATHEMATICAL ANALYSIS



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# CONSTRUCTIVE MATHEMATICAL ANALYSIS

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# A Note on the Differences of Two Positive Linear Operators

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ABSTRACT. In the present note we find the general estimate in terms of Păltăneaś modulus of continuity. In the end, we consider some examples and we apply our result for such examples to obtain the quantitative estimates for the difference of operators.

**Keywords:** Weighted modulus, Baskakov operators, Szász-Mirakyan operators, Phillips operators, Lupaş operators, Difference of operators.

**2010 Mathematics Subject Classification:** 41A25, 41A30.

## 1. INTRODUCTION

In the last ten years there is an increasing interest to estimate the difference between two linear positive operators (abvr. l.p.o.) in terms of appropriate moduli or $K$-functionals, see for example [4], [5], [2], [3] and [6] etc. This note is motivated by the recent paper of Aral-Inoan-Rasa [3], where they considered two different l.p.o. defined on unbounded interval $[0, \infty)$ and obtained estimates for the difference of these operators in a quantitative form. They defined by $C_2[0, \infty)$ the space of all continuous functions satisfying the condition $|f(t)| \leq M(1 + x^2)$. Further $C_2^*[0, \infty)$ is the closed subspace of $C_2[0, \infty)$ formed by the functions $f$, for which $\lim_{x\to\infty} |f(x)|(1 + x^2)^{-1}$ exists and is finite and used the norm $||f||_2 = \sup_{x\geq 0} |f(x)|(1 + x^2)^{-1}$. The weighted modulus of continuity $\Omega(f, \delta)$ (see [1]), for each $f \in C_2[0, \infty)$ is defined as

$$\boldsymbol{\Omega}(f, \delta) = \sup_{|h|<\delta,\ x\in\mathbb{R}^+} |f(x+h) - f(x)| \left(1 + h^2 + x^2 + h^2x^2\right)^{-1}. \tag{1.1}$$

In our note we extend the class of approximated functions, including unbounded functions of polynomial growth of order $m, m \geq 2$-arbitrary natural number. We point out that the modulus $\boldsymbol{\Omega}(f, .)$ given in (1.1) is defined only for functions of polynomial growth up to order 2. Instead of modulus $\boldsymbol{\Omega}(f, .)$, we use weighted modulus $\omega_\varphi(f; h)$ introduced by Păltănea in [10] and defined as

$$\omega_\varphi(f; h) = \sup\left\{|f(x) - f(y)| : x \geq 0, y \geq 0, |x - y| \leq h\varphi\left(\frac{x+y}{2}\right)\right\}, h \geq 0$$

where $\varphi(x) = \frac{\sqrt{x}}{1+x^m}, x \in [0, \infty), m \in \mathbb{N}, m \geq 2$. We consider here those functions, for which we have the property

$$\lim_{h\to 0} \omega_\varphi(f; h) = 0.$$

It is easy to verify that this property is fulfilled for $f$ an algebraic polynomial of degree $\leq m$. Following Theorem 2 in [10] the limit given above is true iff $f$ satisfies the following two conditions:

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- The function $f \circ e_2$ is uniformly continuous on $[0,1]$
- The function $f \circ e_v, v = \frac{2}{2m+1}$ is uniformly continuous on $[1,\infty)$, where $e_v(x) = x^v, x \geq 0$

We denote by $W_\varphi[0,\infty)$ the subspace of all real functions defined on $[0,\infty)$, satisfying above conditions. This is the first advantage of our results, compared with [3]. Secondly the authors in [3] considered the differences of two discrete l.p.o. with similar structure i.e. with the same basis functions (see (2.5) in [3])

Let $F : D \to \mathbb{R}$ be a positive linear functional, where $D$ is a linear subspace of $C[0,\infty)$ which contains $C_2[0,\infty)$ and the polynomials up to degree 6 and for $r \in \mathbb{N}$, $e_r(x) = x^r, x \in [0,\infty)$, such that $F(e_0) = 1, b^F = F(e_1)$ and we denote $\mu_r^F = F\left((e_1 - b^F e_0)^r\right), r \in \mathbb{N}, 0 \leq r \leq 6$.

Now we consider two positive linear operators namely $M_n$ and $L_n$ for linear positive functionals $F_{n,k}, G_{n,k} : D \to \mathbb{R}$ such that $F_{n,k}(e_0) = 1$ and $G_{n,k}(e_0) = 1$
as

$$M_n(f,x) = \sum_{k\in\mathbb{K}} F_{n,k}(f) p_{n,k}(x) \tag{1.2}$$

and

$$L_n(f,x) = \sum_{k\in\mathbb{K}} G_{n,k}(f) p_{n,k}(x) \tag{1.3}$$

where $\mathbb{K}$ be a set of non-negative integers and the functions $p_{n,k}(x)$ are positive. One of the main results in [3] states the following:

**Theorem A.** Let $f \in C_2[0,\infty)$ with $f'' \in C_2^*[0,\infty)$. Then

$$|(M_n - L_n)(f,x)| \leq \frac{1}{2}||f''||_2 \beta(x) + 8\Omega(f'',\delta_1)(1+\beta(x)) + 16\Omega(f,\delta_2)(\gamma(x)+1),$$

where

$$\begin{aligned}
\beta(x) &= \sum_{k\in\mathbb{K}} p_{n,k}(x) \left\{ \left(1 + \left(b^{F_{n,k}}\right)^2\right) \mu_2^{F_{n,k}} + \left(1 + \left(b^{G_{n,k}}\right)^2\right) \mu_2^{G_{n,k}} \right\}, \\
\gamma(x) &= \sum_{k\in\mathbb{K}} p_{n,k}(x) \left(1 + \left(b^{F_{n,k}}\right)^2\right), \\
\delta_1^4(x) &= \sum_{k\in\mathbb{K}} p_{n,k}(x) \left\{ \left(1 + \left(b^{F_{n,k}}\right)^2\right) \mu_6^{F_{n,k}} + \left(1 + \left(b^{G_{n,k}}\right)^2\right) \mu_6^{G_{n,k}} \right\}
\end{aligned}$$

and

$$\delta_2^4(x) = \sum_{k\in\mathbb{K}} p_{n,k}(x) \left(1 + \left(b^{F_{n,k}}\right)^2\right) \left(b^{F_{n,k}} - b^{G_{n,k}}\right)^4,$$

where we suppose that $\delta_1(x) \leq 1, \delta_2(x) \leq 1$ and $\mu_r^{F_{n,k}} = F_{n,k}\left((e_1 - b^{F_{n,k}} e_0)^r\right), r \in \mathbb{N}$.

Instead of this, we study in our note the difference of two operators with different basis functions and even more $M_n$ and $L_n$ can be arbitrary positive linear operators, including integral representation. The only information we need is a good (exact if possible) representation of moments of the operators $M_n$ and $L_n$ of order upto 6.

## 2. Auxiliary Results

In our note we consider l.p.o. $L_n : E \to C[0,\infty)$, where $E$ is a subspace of $C[0,\infty)$, such that $C_k[0,\infty) \subset E$, with $k = \max\{m+r+1, 2r+2, 2m\}, r \in \mathbb{N}$ and

$$C_k[0,\infty) := \{f \in C[0,\infty), \exists M > 0 : |f(x)| \le M(1+x^k), \forall x \ge 0, k \in \mathbb{N}\}.$$

Let $\mu^L_{n,m}(x), m \in \mathbb{N}$ is the moment of order $m$ of $L_n$ i.e. $\mu^L_{n,m}(x) = L_n((t-x)^m; x)$. The main result in [8] is Theorem 2.2, which we formulate as:

**Theorem B.** Let $L_n : E \to C[0,\infty)$, $C_k[0,\infty) \subset E$, $k = \max\{m+3, 6, 2m\}$ be sequence of linear positive operators, preserving the linear functions. Also $m \in \mathbb{N}$. If $f \in C^2[0,\infty) \cap E$ and $f'' \in W_\varphi[0,\infty)$, then we have for $x \in (0,\infty)$ that

$$\begin{aligned}
&\left| L_n(f,x) - f(x) - \frac{1}{2} f''(x)\mu^L_{n,2}(x) \right| \\
\le\ & \frac{1}{2}\left[ \mu^L_{n,2}(x) + \sqrt{2}\sqrt{L_n\left( \left[ 1 + \left( x + \frac{|t-x|}{2} \right)^m \right]^2 ; x \right)} \right] \\
& \omega_\varphi\left( f''; \left( \frac{\mu^L_{n,6}}{x} \right)^{1/2} \right). \qquad (2.4)
\end{aligned}$$

**Remark 2.1.** *We point out that the statement in Theorem B can be extended also for p.l.o. $L_n$ which don't preserve linear function. In this case the left hand side of (2.4) should be replaced by*

$$\left| L_n(f,x) - f(x) - f'(x)\mu^L_{n,1}(x) - \frac{1}{2} f''(x)\mu^L_{n,2}(x) \right|.$$

**Remark 2.2.** *The proof of Theorem B is based on good upper estimate for the remainder in Taylor's formula at the point $x \in (0,\infty)$*

$$R_r(f;t,x) = f(t) - \sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!}(t-x)^r.$$

*From estimate (2.4) in [8], we have*

$$|R_r(f;t,x)| \le \frac{|t-x|^r}{r!} \left( 1 + \sqrt{2}\frac{|t-x|}{h} . \frac{1 + \left( x + \frac{|t-x|}{2} \right)^m}{\sqrt{x}} \right) \omega_\varphi(f^{(r)}; h). \qquad (2.5)$$

*Hence for $r = 0$ we get*

$$|f(t) - f(x)| \le \left[ 1 + \sqrt{2}. \frac{|t-x|}{h} . \frac{1 + \left( x + \frac{|t-x|}{2} \right)^m}{\sqrt{x}} \right] \omega_\varphi(f;h), \qquad (2.6)$$

*which can be considered as an extension of the estimate (2.2) in [3], because now we allow the function $f$ to be of polynomial growth $m, m \ge 2$.. In our paper [8] we supposed that $\mu^L_{n,m}(x) = O(n^{-[(m+1)/2]}), n \to \infty$, which guarantees that the term in right-hand side of Theorem B in front modulus $\omega_\varphi(f'', .)$ is bounded when $n \to \infty$ for fixed $x$ and $m$. Particularly this assumption is fulfilled for Szász-Mirakyan, Baskakov, Phillips operators etc.*

**Remark 2.3.** *We may apply also Theorem 2.3 from [8] where as argument of the modulus $\omega_\varphi(f'', .)$ in Theorem B instead of $\left( \frac{\mu^L_{n,6}}{x} \right)^{1/2}$ we have $\sqrt{\frac{\mu^L_{n,4}(x)}{\mu^L_{n,2}(x)}}$. We omit the details in this case.*

## 3. Main Result

**Theorem 3.1.** *Let $L_n, M_n : E \to C[0,\infty)$, $C_k[0,\infty) \subset E$, $k = \max\{m+3, 6, 2m\}$ be two sequences of linear positive operators. If $f \in C^2[0,\infty) \cap E$ and $f'' \in W_\varphi[0,\infty)$, then we have for $x \in (0,\infty)$ that*

$$\begin{aligned}
& |L_n(f,x) - M_n(f,x)| \\
\leq\ & |f'(x)|.\left|\mu_{n,1}^{L_n}(x) - \mu_{n,1}^{M_n}(x)\right| \\
& + \frac{1}{2}|f''(x)|.\left|\mu_{n,2}^{L_n}(x) - \mu_{n,2}^{M_n}(x)\right| \\
& + \frac{1}{2}\left[\mu_{n,2}^{L_n}(x) + \sqrt{2}.\sqrt{L_n\left(\left[1 + \left(x + \frac{|t-x|}{2}\right)^m\right]\right)}\right]\omega_\varphi\left(f''; \left(\frac{\mu_{n,6}^{L_n}}{x}\right)^{1/2}\right) \\
& + \frac{1}{2}\left[\mu_{n,2}^{M_n}(x) + \sqrt{2}.\sqrt{M_n\left(\left[1 + \left(x + \frac{|t-x|}{2}\right)^m\right]\right)}\right]\omega_\varphi\left(f''; \left(\frac{\mu_{n,6}^{M_n}}{x}\right)^{1/2}\right)
\end{aligned}$$

*Proof.* We use the following representation

$$\begin{aligned}
L_n(f,x) - M_n(f,x) &= L_n(f,x) - f(x) - f'(x)\mu_{n,1}^{L_n}(x) - \frac{1}{2}f''(x)\mu_{n,2}^{L_n}(x) \\
&\quad -(M_n(f,x) - f(x) - f'(x)\mu_{n,1}^{M_n}(x) - \frac{1}{2}f''(x)\mu_{n,2}^{M_n}(x)) \\
&\quad + f'(x)[\mu_{n,1}^{L_n}(x) - \mu_{n,1}^{M_n}(x)] + \frac{1}{2}[\mu_{n,2}^{L_n}(x) - \mu_{n,2}^{M_n}(x)].
\end{aligned}$$

Hence the proof follows from Theorem B. □

**Remark 3.4.** *If both the operators $L_n$ and $M_n$ reproduce linear functions, we have $\mu_{n,1}^{L_n}x = \mu_{n,1}^{M_n}x = 0$. Therefore we can omit the summand containing $f'(x)$.*

**Remark 3.5.** *In Theorem 3.1, we used (2.6) for $r = 2$. In a similar way if we suppose $f \in C^r[0,\infty) \cap E$ and $f^{(r)} \in W_\varphi[0,\infty)$ we may prove estimate for the differences $L_n(f,x) - M_n(f,x)$ in terms of $\omega_\varphi(f^{(r)};.)$.*

## 4. Examples

We apply Theorem 3.1 for some classical positive linear operators, some examples are given as (see [9], [7] and references therein):

**Example 4.1.** *The Szász-Mirakyan operators are defined as*

$$S_n(f,x) = \sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \tag{4.7}$$

*where $x \in [0,\infty)$, $n \in \mathbb{N}$. The central moments of the Szász-Mirakyan operators (4.7) satisfy for $m \geq 1$ the recurrence relation:*

$$\mu_{n,m+1}^{S_n}(x) = \frac{x}{n}[\mu_{n,m}^{S_n}(x)]' + \frac{mx}{n}\mu_{n,m-1}^{S_n}(x).$$

*In particular*

$$\mu_{n,0}^{S_n}(x) = 1, \mu_{n,1}^{S_n}(x) = 0, \mu_{n,2}^{S_n}(x) = \frac{x}{n}, \mu_{n,3}^{S_n}(x) = \frac{x}{n^2},$$

$$\mu_{n,4}^{S_n}(x) = \frac{x}{n^3} + \frac{3x^2}{n^2}, \mu_{n,5}^{S_n}(x) = \frac{x}{n^4} + \frac{10x^2}{n^3}, \mu_{n,6}^{S_n}(x) = \frac{x}{n^5} + \frac{25x^2}{n^4} + \frac{15x^3}{n^3}.$$

**Example 4.2.** *The Baskakov operators are defined as*

$$B_n(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right). \tag{4.8}$$

*The moments of the Baskakov operators are well known for* $m \geq 1$ *the following recurrence relation holds true:*

$$\mu_{n,m+1}^{B_n}(x) = \frac{x(1+x)}{n}[\mu_{n,m}^{B_n}(x)]' + \frac{mx(1+x)}{n}\mu_{n,m-1}^{B_n}(x).$$

*In particular*

$$\mu_{n,0}^{B_n}(x) = 1, \mu_{n,1}^{B_n}(x) = 0, \mu_{n,2}^{B_n}(x) = \frac{x(1+x)}{n}, \mu_{n,3}^{B_n}(x) = \frac{x(1+x)(1+2x)}{n^2},$$

$$\mu_{n,4}^{B_n}(x) = \frac{x(1+x)}{n^3} + \frac{6x^2(1+x)^2}{n^3} + \frac{3x^2(1+x)^2}{n^2},$$

$$\mu_{n,5}^{B_n}(x) = \frac{x+15x^2+50x^3+60x^4+24x^5}{n^4} + \frac{10x^2+76x^3+86x^4+20x^5}{n^3},$$

$$\begin{aligned}\mu_{n,6}^{B_n}(x) &= \frac{x+31x^2+180x^3+390x^4+360x^5+120x^6}{n^5} \\ &+ \frac{25x^2+288x^3+667x^4+534x^5+130x^6}{n^4} \\ &+ \frac{15x^3+105x^4+105x^5+15x^6}{n^3}.\end{aligned}$$

**Example 4.3.** *The well known Phillips operators are defined as*

$$P_n(f;x) = n\sum_{k=1}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\int_0^{\infty} e^{-nt}\frac{(nt)^{k-1}}{(k-1)!}f(t)dt + e^{-nx}f(0). \tag{4.9}$$

*The Phillips operators satisfy the following recurrence relation for central moments:*

$$\mu_{n,m+1}^{P_n}(x) = \frac{x}{n}[(\mu_{n,m}^{P_n}(x))' + 2m\mu_{n,m-1}^{P_n}(x)] + \frac{m}{n}\mu_{n,m}^{P_n}(x), m \geq 1$$

*and in particular, we have*

$$\mu_{n,0}^{P_n}(x) = 1, \mu_{n,1}^{P_n}(x) = 0, \mu_{n,2}^{P_n}(x) = \frac{2x}{n},$$

$$\mu_{n,3}^{P_n}(x) = \frac{6x}{n^2}, \mu_{n,4}^{P_n}(x) = \frac{12x^2}{n^2} + \frac{24x}{n^3}$$

$$\mu_{n,5}^{P_n}(x) = \frac{120x}{n^4} + \frac{72x}{n^3} + \frac{48x^2}{n^3},$$

$$\mu_{n,6}^{P_n}(x) = \frac{720x}{n^5} + \frac{576x^2}{n^4} + \frac{432x}{n^4} + \frac{120x^3}{n^3}.$$

**Example 4.4.** *The Lupaş operators are defined as*

$$U_n(f,x) := \sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{k!\,2^k} f\left(\frac{k}{n}\right). \tag{4.10}$$

*Some of the central moments are given below:*

$$\mu_{n,0}^{U_n}(x) = 1, \mu_{n,1}^{U_n}(x) = 0, \mu_{n,2}^{U_n}(x) = \frac{2x}{n},$$
$$\mu_{n,3}^{U_n}(x) = \frac{6x}{n^2}, \mu_{n,4}^{U_n}(x) = \frac{26x + 12nx^2}{n^3},$$
$$\mu_{n,5}^{U_n}(x) = \frac{150x + 120nx^2}{n^4}, \mu_{n,6}^{U_n}(x) = \frac{1082x + 1140nx^2 + 120n^2x^3}{n^5}.$$

Using Example 4.1 and Example 4.2, we have the following quantitative estimate as application of Theorem 3.1 for the difference of Szász-Mirakyan and Baskakov operators.

**Theorem 4.2.** *Let $S_n, B_n : E \to C[0,\infty)$, $C_k[0,\infty) \subset E$, $k = \max\{m+3, 6, 2m\}$ be two sequences of linear positive operators. If $f \in C^2[0,\infty) \cap E$ and $f'' \in W_\varphi[0,\infty)$, then we have for $x \in (0,\infty)$ that*

$$\begin{aligned}
&|S_n(f,x) - B_n(f,x)| \\
\leq\ & \frac{x^2}{2n}|f''(x)| + \frac{1}{2}\left[\frac{x}{n} + \sqrt{2A_{n,m,x}}\right]\omega_\varphi\left(f''; \sqrt{\frac{1}{n^5} + \frac{25x}{n^4} + \frac{15x^2}{n^3}}\right) \\
&+\frac{1}{2}\left[\frac{x(1+x)}{n} + \sqrt{2B_{n,m,x}}\right]\omega_\varphi\Bigg(f''; \bigg(\frac{1 + 31x + 180x^2 + 390x^3 + 360x^4 + 120x^5}{n^5} \\
&+\frac{25x + 288x^2 + 667x^3 + 534x^4 + 130x^5}{n^4} + \frac{15x^2 + 105x^3 + 105x^4 + 15x^5}{n^3}\bigg)^{1/2}\Bigg),
\end{aligned}$$

*where*

$$A_{n,m,x} = S_n\left(\left[1 + \left(x + \frac{|t-x|}{2}\right)^m\right]^2; x\right), B_{n,m,x} = B_n\left(\left[1 + \left(x + \frac{|t-x|}{2}\right)^m\right]^2; x\right).$$

Using Example 4.1 and Example 4.3, we have the following quantitative estimate as application of Theorem 3.1 for the difference of Szász-Mirakyan and Phillips operators.

**Theorem 4.3.** *Let $S_n, P_n : E \to C[0,\infty)$, $C_k[0,\infty) \subset E$, $k = \max\{m+3, 6, 2m\}$ be two sequences of linear positive operators. If $f \in C^2[0,\infty) \cap E$ and $f'' \in W_\varphi[0,\infty)$, then we have for $x \in (0,\infty)$ that*

$$\begin{aligned}
&|S_n(f,x) - P_n(f,x)| \\
\leq\ & \frac{x}{2n}|f''(x)| + \frac{1}{2}\left[\frac{x}{n} + \sqrt{2A_{n,m,x}}\right]\omega_\varphi\left(f''; \sqrt{\frac{1}{n^5} + \frac{25x}{n^4} + \frac{15x^2}{n^3}}\right) \\
&+\frac{1}{2}\left[\frac{2x}{n} + \sqrt{2C_{n,m,x}}\right]\omega_\varphi\left(f''; \sqrt{\frac{720}{n^5} + \frac{576x}{n^4} + \frac{432}{n^4} + \frac{120x^2}{n^3}}\right),
\end{aligned}$$

*where*

$$A_{n,m,x} = S_n\left(\left[1 + \left(x + \frac{|t-x|}{2}\right)^m\right]^2; x\right), C_{n,m,x} = P_n\left(\left[1 + \left(x + \frac{|t-x|}{2}\right)^m\right]^2; x\right).$$

Using Example 4.3 and Example 4.4, we have the following quantitative estimate as application of Theorem 3.1 for the difference of Phillips and Lupaş operators.

**Theorem 4.4.** *Let $P_n, U_n : E \to C[0,\infty)$, $C_k[0,\infty) \subset E$, $k = \max\{m+3, 6, 2m\}$ be two sequences of linear positive operators. If $f \in C^2[0,\infty) \cap E$ and $f'' \in W_\varphi[0,\infty)$, then we have for $x \in (0,\infty)$ that*

$$\begin{aligned} & |P_n(f,x) - U_n(f,x)| \\ \leq\ & \frac{1}{2}\left[\frac{2x}{n} + \sqrt{2C_{n,m,x}}\right]\omega_\varphi\left(f''; \sqrt{\frac{720}{n^5} + \frac{576x}{n^4} + \frac{432}{n^4} + \frac{120x^2}{n^3}}\right) \\ & + \frac{1}{2}\left[\frac{2x}{n} + \sqrt{2D_{n,m,x}}\right]\omega_\varphi\left(f''; \sqrt{\frac{1082 + 1140nx + 120n^2x^2}{n^5}}\right), \end{aligned}$$

*where*

$$D_{n,m,x} = U_n\left(\left[1 + \left(x + \frac{|t-x|}{2}\right)^m\right]^2; x\right), C_{n,m,x} = P_n\left(\left[1 + \left(x + \frac{|t-x|}{2}\right)^m\right]^2; x\right).$$

## References


[1] T. Acar, A. Aral and I. Raşa: *The new forms of Voronovskaja's theorem in weighted spaces*, Positivity **20** (2016), 25-40.

[2] A. M. Acu and I. Raşa: *New estimates for differences of positive linear operators*, Numer. Algor. **73** (2016), 775-789.

[3] A. Aral, D. Inoan and I. Raşa: On differences of linear positive operators, Anal. Math. Phys. doi 13324-018-0227-7.

[4] H. Gonska, P. Pitul and I. Raşa: *On differences of positive linear operators*, Carpathian J. Math. **22**(1-2) (2006), 65-78.

[5] H. Gonska, and I. Raşa: *Differences of positive linear operators and the second order modulus*, Carpathian J. Math. **24**(3) (2008), 332-340.

[6] V. Gupta: *Differences of operators of Lupaş type*, Constructive Mathematical Analysis **1**(1) (2018), 9-14.

[7] V. Gupta, T. M. Rassias, P. N. Agrawal and A. M. Acu: *Recent Advances in Constructive Approximation Theory*, Springer, Cham. (2018).

[8] V. Gupta and G. Tachev: *General form of Voronovskaja's theorem in terms of weighted modulus of continuity*, Results Math 69 (3-4) (2016), 419-430.

[9] V. Gupta and G. Tachev: *Approximation with Positive Linear Operators and Linear Combinations*, Springer, Cham. 2017.

[10] R. Păltănea: *Estimates of approximation in terms of a weighted modulus of continuity*, Bull. Transilvania Univ. of Brasov 4 (53) (2011), 67-74.



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# A Quantitative Estimate for the Sampling Kantorovich Series in Terms of the Modulus of Continuity in Orlicz Spaces

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ABSTRACT. In the present paper we establish a quantitative estimate for the sampling Kantorovich operators with respect to the modulus of continuity in Orlicz spaces defined in terms of the modular functional. At the end of the paper, concrete examples are discussed, both for what concerns the kernels of the above operators, as well as for some concrete instances of Orlicz spaces.

**Keywords:** Sampling Kantorovich series, Orlicz spaces, Modulus of continuity, Quantitative estimates, Kernels.

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## 1. INTRODUCTION

The sampling Kantorovich operators $S_w$ have been introduced by Bardaro, Butzer, Stens and Vinti in [8], in order to study an $L^1$-version of the so-called generalized sampling operators ([12, 32, 14]). The main peculiarity of the sampling Kantorovich operators is that they revealed to be suitable in order to reconstruct not necessarily continuous signals ([2]).

Indeed, in the original paper [8] the authors proved the modular convergence of the operators $S_w$ in the general setting of Orlicz spaces, which include, as a special case, the $L^p$-spaces.

Later on, the operators $S_w$ have been studied under different aspects, both from theoretical ([17, 5, 23]) and applications point of view ([6, 7]). For instance, in [6, 7] some applications to energy engineering have been developed applying an algorithm for image reconstruction and enhancement based on the multivariate version of the operators $S_w$ for the processing of thermographic images.

The order of approximation for the sampling Kantorovich operators has been also studied in [21]; this has been done assuming the function $f$ in suitable Lipschitz classes, both in the space of uniformly continuous and bounded functions (i.e., in $C(\mathbb{R})$) and in Orlicz spaces (i.e., in $L^\varphi(\mathbb{R})$). For other results concerning the order of approximation for the above operators, see, e.g., [31, 11].

The above problem has been faced in $C(\mathbb{R})$ also from the quantitative point of view in [9], by using the modulus of continuity of the function being approximated.

Currently, the study of quantitative estimates in the setting of Orlicz spaces in terms of the modulus of continuity is still an open problem.

For the latter reason, in this paper we establish the quantitative rate of convergence for the sampling Kantorovich operators; in order to do this we firstly recall the notion of the modulus of continuity in $L^\varphi(\mathbb{R})$ which is based on the modular functional of the space ([10]).

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At the end of the paper, several examples of kernels and concrete cases of Orlicz spaces are recalled. For instance, the $L^p$-spaces, with $1 \leq p < +\infty$, are included in the present general theory, together with other well-known examples of Orlicz spaces.

## 2. Notation and preliminaries

We begin this section by recalling some basic facts concerning Orlicz spaces.

A function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be a $\varphi$-function if it satisfies the following conditions:
$(\Phi 1)$ $\varphi$ is a non decreasing and continuous function;
$(\Phi 2)$ $\varphi(0) = 0$, $\varphi(u) > 0$ if $u > 0$ and $\lim_{u \to +\infty} \varphi(u) = +\infty$.

Let us now consider the functional $I^\varphi$ associated to the $\varphi$-function $\varphi$ and defined by

$$I^\varphi[f] \; := \; \int_{\mathbb{R}} \varphi(|f(x)|)\, dx,$$

for every $f \in M(\mathbb{R})$, i.e., for every (Lebesgue) measurable function $f : \mathbb{R} \to \mathbb{R}$. As it is well-known, $I^\varphi$ is a modular functional (see e.g. [29, 10]), and the Orlicz space generated by $\varphi$ is defined by

$$L^\varphi(\mathbb{R}) \; := \; \{ f \in M(\mathbb{R}) \; : \; I^\varphi[\lambda f] < \infty, \text{ for some } \lambda > 0 \}\,.$$

A notion of convergence in Orlicz spaces, called *modular convergence*, was introduced in [30].

We will say that a net of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R})$ is modularly convergent to $f \in L^\varphi(\mathbb{R})$, if there exists $\lambda > 0$ such that

$$I^\varphi[\lambda(f_w - f)] \; = \; \int_{\mathbb{R}} \varphi(\lambda |f_w(x) - f(x)|)\, dx \; \longrightarrow 0, \qquad w \to +\infty. \tag{2.1}$$

Moreover we recall, for the sake of completeness, that in $L^\varphi(\mathbb{R})$ it can be also given a strong notion of convergence, i.e. the Luxemburg-norm convergence, see e.g. [29, 10]. We will say that a net of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R})$ is convergent to $f \in L^\varphi(\mathbb{R})$ with respect to the Luxemburg norm if (2.1) holds for every $\lambda > 0$. Definition (2.1) induces a topology in $L^\varphi(\mathbb{R})$, called *modular topology*. Obviously, the modular convergence and the Luxemburg norm convergence coincide if and only if the well-known $\Delta_2$-condition on $\varphi$ is satisfied, see, e.g., [29, 10].

Now, we recall the definition of the modulus of continuity in Orlicz spaces $L^\varphi(\mathbb{R})$, with respect to the modular $I^\varphi$. For any fixed $f \in L^\varphi(\mathbb{R})$, and for a suitable $\lambda > 0$, we denote:

$$\omega(f, \delta)_\varphi \; := \; \sup_{|t| \leq \delta} \; I^\varphi \left[ \lambda \left( f(\cdot + t) - f(\cdot) \right) \right], \tag{2.2}$$

with $\delta > 0$.

For general references concerning Orlicz spaces and some of their generalizations, see, e.g., [28, 1, 24, 25, 18].

In order to define the considered operators, we need some additional notions.

Let $\Pi = (t_k)_{k \in \mathbb{Z}}$ be a sequence of real numbers such that $-\infty < t_k < t_{k+1} < +\infty$ for every $k \in \mathbb{Z}$, $\lim_{k \to \pm\infty} t_k \; = \; \pm\infty$ and there are two positive constants $\Delta$, $\delta$ such that $\delta \leq \Delta_k := t_{k+1} - t_k \leq \Delta$, for every $k \in \mathbb{Z}$.
In what follows, a function $\chi : \mathbb{R} \to \mathbb{R}$ will be called a kernel if it satisfies the following properties:

- $(\chi 1)$ $\chi \in L^1(\mathbb{R})$ and is bounded in a neighborhood of $0$;
- $(\chi 2)$ for every $u \in \mathbb{R}$

$$\sum_{k \in \mathbb{Z}} \chi(u - t_k) \; = \; 1;$$

- $(\chi 3)$ for some $\beta > 0$,

$$m_{\beta,\Pi}(\chi) \;:=\; \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-t_k)| \cdot |u-t_k|^{\beta} \;<\; +\infty.$$

Then, the sampling Kantorovich operators $S_w$ for a given kernel $\chi$ are defined by:

$$(S_w f)(x) \;:=\; \sum_{k\in\mathbb{Z}} \chi(wx-t_k) \left[\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du\right] \qquad (x\in\mathbb{R}), \tag{2.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally integrable function such that the series is convergent for every $x \in \mathbb{R}$.

There holds the following lemma.

**Lemma 2.1** ([8]). *Under the assumptions $(\chi 1)$ and $(\chi 3)$ on the kernel $\chi$, it turns out:*

$$m_{0,\Pi}(\chi) \;:=\; \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-t_k)| \;<\; +\infty.$$

Note that, it is easy to see that the discrete absolute moment $m_{0,\Pi}(\chi) > 0$.

## 3. The main result

We can prove the following quantitative estimate for the sampling Kantorovich operators by using the modulus of continuity in Orlicz spaces.

**Theorem 3.1.** *Let $\varphi$ be a convex $\varphi$-function. Suppose that, for any fixed $0 < \alpha < 1$, we have:*

$$w \int_{|y|>1/w^{\alpha}} |\chi(wy)|\,dy \;\leq\; M\,w^{-\gamma}, \quad \textit{as} \quad w \to +\infty, \tag{3.4}$$

*for suitable positive constants $M$, $\gamma$ depending on $\alpha$ and $\chi$. Then, for $f \in L^{\varphi}(\mathbb{R})$, and $\lambda > 0$ there holds:*

$$I^{\varphi}[\lambda\,(S_w f - f)] \;\leq\; \frac{\|\chi\|_1}{2\,\delta\, m_{0,\Pi}(\chi)}\,\omega\left(2\,m_{0,\Pi}(\chi)\,f,\,\frac{1}{w^{\alpha}}\right)_{\varphi}$$
$$+\;\frac{M\,I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi)\,f\right]}{2\,\delta\, m_{0,\Pi}(\chi)}\,w^{-\gamma} \;+\; \frac{\Delta}{2\,\delta}\,\omega\left(2\,m_{0,\Pi}(\chi)\,f,\,\frac{1}{w}\right)_{\varphi},$$

*for every sufficiently large $w > 0$, where $m_{0,\Pi}(\chi) < +\infty$ in view of Lemma 2.1. In particular, if $\lambda > 0$ is sufficiently small, the above inequality implies the modular convergence of the sampling Kantorovich operators $S_w f$ to $f$.*

*Proof.* Let $\lambda > 0$ be fixed. Using the convexity of $\varphi$, and since $\varphi$ is non decreasing, we can write what follows:

$$I^{\varphi}[\lambda\,(S_w f - f)]$$
$$\leq\; \frac{1}{2}\left\{\int_{\mathbb{R}} \varphi\left(2\,\lambda\left|(S_w f)(x) - \sum_{k\in\mathbb{Z}} \chi(wx-t_k)\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} f(u+x-t_k/w)\,du\right|\right)dx\right.$$
$$\left.+\;\int_{\mathbb{R}} \varphi\left(2\,\lambda\left|\sum_{k\in\mathbb{Z}} \chi(wx-t_k)\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} f(u+x-t_k/w)\,du \;-\; f(x)\right|\right)dx\right\} \;=:\; I_1 + I_2,$$

$w > 0$. We estimate $I_1$. By using the Jensen inequality (see, e.g., [19]) twice, and the change of variable $y = x - t_k/w$, we obtain:

$$2\,I_1 \;\leq\; \int_{\mathbb{R}} \varphi\left(2\,\lambda \sum_{k\in\mathbb{Z}} |\chi(wx-t_k)|\, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u+x-t_k/w)|\, du\right) dx$$

$$\leq\; \frac{1}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx-t_k)|\, \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi) \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} |f(u) - f(u+x-t_k/w)|\, du\right) dx$$

$$\leq\; \frac{1}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx-t_k)|\, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi) |f(u) - f(u+x-t_k/w)|\right) du\, dx$$

$$\leq\; \frac{\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} |\chi(wy)|\;\; w \sum_{k\in\mathbb{Z}} \int_{t_k/w}^{t_{k+1}/w} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi) |f(u) - f(u+y)|\,\right) du\, dy$$

$$=\; \frac{\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} |\chi(wy)|\;\; w \int_{\mathbb{R}} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi) |f(u) - f(u+y)|\right) du\, dy$$

$$=\; \frac{\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} w\, |\chi(wy)|\;\; I^{\varphi}\left[2\,\lambda\, m_{0,\Pi}(\chi) (f(\cdot) - f(\cdot+y))\right] dy \;=:\; J,$$

$w > 0$. Let now $0 < \alpha < 1$ be fixed. Thus we can split the above integral $J$ as follows:

$$J \;:=\; \frac{w\,\delta^{-1}}{m_{0,\Pi}(\chi)} \times$$
$$\left\{ \int_{|y|\leq 1/w^{\alpha}} + \int_{|y|>1/w^{\alpha}} \right\} |\chi(wy)|\;\; I^{\varphi}\left[2\,\lambda\, m_{0,\Pi}(\chi) (f(\cdot) - f(\cdot+y))\right] dy \;=:\; J_1 \;+\; J_2.$$

For $J_1$, we have:

$$J_1 \;\leq\; \frac{w\,\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{|y|\leq 1/w^{\alpha}} |\chi(wy)|\; \omega\left(2\, m_{0,\Pi}(\chi)\, f,\; |y|\right)_{\varphi}\; dy$$

$$\leq\; \omega\left(2\, m_{0,\Pi}(\chi)\, f,\; 1/w^{\alpha}\right)_{\varphi}\; \frac{w\,\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{|y|\leq 1/w^{\alpha}} |\chi(wy)|\; dy$$

$$\leq\; \omega\left(2\, m_{0,\Pi}(\chi)\, f,\; 1/w^{\alpha}\right)_{\varphi}\; \frac{\delta^{-1}\; \|\chi\|_1}{m_{0,\Pi}(\chi)},$$

$w > 0$. Moreover, by using the convexity of $\varphi$, for $J_2$ we can obtain:

$$J_2 \;\leq\; \frac{w\,\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{|y|>1/w^{\alpha}} |\chi(wy)|\;\; \frac{1}{2} \left\{ I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f\right] \right.$$
$$\left. +\, I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f(\cdot+y)\right] \right\}\; dy.$$

Obviously, it is easy to see that:

$$I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f\right] \;=\; I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f(\cdot+y)\right],$$

for every $y$. Then, by exploiting assumption (3.4), we finally obtain:

$$J_2 \;\leq\; \frac{w\,\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{|y|>1/w^{\alpha}} |\chi(wy)|\;\; I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f\right]\; dy$$

$$\leq\; \frac{\delta^{-1}}{m_{0,\Pi}(\chi)}\, I^{\varphi}\left[4\,\lambda\, m_{0,\Pi}(\chi) f\right]\; M\, w^{-\gamma},$$

for $w > 0$ sufficiently large.

Now, we can estimate $I_2$. Using Jensen inequality twice (as above), the change of variable $y = u - t_k/w$, and Fubini-Tonelli theorem, we have:

$$\begin{aligned}
& 2\, I_2 \\
\leq\ & \frac{1}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx - t_k)|\, \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi)|f(u + x - t_k/w) - f(x)|\,\right)\, du\, dx \\
& \leq\ \frac{\delta^{-1}}{m_{0,\Pi}(\chi)} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(wx - t_k)|\, w \int_0^{\Delta/w} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi)|f(x+y) - f(x)|\,\right)\, dy\, dx \\
& \leq\ \delta^{-1} \int_{\mathbb{R}} w \int_0^{\Delta/w} \varphi\left(2\,\lambda\, m_{0,\Pi}(\chi)|f(x+y) - f(x)|\,\right)\, dy\, dx \\
& \leq\ \delta^{-1} w \int_0^{\Delta/w} I^{\varphi}\left[2\,\lambda\, m_{0,\Pi}(\chi)\left(f(\cdot + y) - f(\cdot)\right)\right]\, dy \\
& \leq\ \delta^{-1}\, \omega(2\, m_{0,\Pi}(\chi)\, f,\, 1/w)_{\varphi}\, w \int_0^{\Delta/w} dy \ =\ \delta^{-1}\, \Delta\, \omega(2\, m_{0,\Pi}(\chi)\, f,\, 1/w)_{\varphi},
\end{aligned}$$

$w > 0$. This completes the proof. □

**Remark 3.1.** Note that, it is easy to show that for any kernels such that $\chi(u) = \mathcal{O}(|u|^{-\theta})$, as $|u| \to +\infty$, for $\theta > 1$, we have that assumption (3.4) is satisfied for some constant $M > 0$ and $\gamma = (1-\alpha)(\theta - 1) > 0$, for every fixed $0 < \alpha < 1$.

## 4. Examples

Examples of convex $\varphi$-functions generating remarkable Orlicz spaces, where the above result is valid are:
$\varphi_p(u) := u^p$, $1 \leq p < \infty$, $\varphi_{\alpha,\beta} := u^{\alpha} \log^{\beta}(u + e)$, for $\alpha \geq 1$, $\beta > 0$ and $\varphi_{\gamma}(u) = e^{u^{\gamma}} - 1$, for $\gamma > 0$, $u \geq 0$. It is well-known that $\varphi_p$ generates the $L^p(\mathbb{R})$-space and the corresponding convex modular functional is given by $I^{\varphi_p}[f] := \|f\|_p^p$, while $\varphi_{\alpha,\beta}$ and $\varphi_{\gamma}$ generate the $L^{\alpha} \log^{\beta} L$-spaces (or Zygmund spaces), largely used, e.g., in the theory of partial differential equations, and the exponential spaces respectively, e.g., used for embedding theorems between Sobolev spaces. The convex modular functionals corresponding to $\varphi_{\alpha,\beta}$ and $\varphi_{\gamma}$ are

$$I^{\varphi_{\alpha,\beta}}[f] := \int_{\mathbb{R}} |f(x)|^{\alpha} \log^{\beta}(e + |f(x)|)\, dx, \qquad (f \in M(\mathbb{R})),$$

and

$$I^{\varphi_{\gamma}}[f] := \int_{\mathbb{R}} (e^{|f(x)|^{\gamma}} - 1)\, dx, \qquad (f \in M(\mathbb{R})),$$

respectively.

Now, we give a brief list of some well-known and important class of kernels which satisfy the above assumptions $(\chi 1) - (\chi 3)$, and for which Theorem 3.1 holds.

First of all, we recall the definition of the well-known central B-spline of order $N$ (see e.g., [33, 3, 4]):

$$\beta^N(x) \ :=\ \frac{1}{(N-1)!} \sum_{i=0}^{N} (-1)^i \binom{N}{i} \left(\frac{N}{2} + x - i\right)_+^{N-1}, \qquad x \in \mathbb{R}. \tag{4.5}$$

It is well-known that $\beta^N$ have compact support, then (3.4) is obviously satisfied for every $\gamma > 0$.

Other important (band-limited) kernels are given by the so-called Jackson type kernels of order $N$, defined by:

$$J_N(x) \;:=\; c_N\, \mathrm{sinc}^{2N}\left(\frac{x}{2N\pi\alpha}\right), \qquad x \in \mathbb{R}, \tag{4.6}$$

with $N \in \mathbb{N}$, $\alpha \geq 1$, and $c_N$ is a non-zero normalization coefficient, given by:

$$c_N \;:=\; \left[\int_{\mathbb{R}} \mathrm{sinc}^{2N}\left(\frac{u}{2N\pi\alpha}\right)\, du\right]^{-1}.$$

For $J_N$, assumption (3.4) turns out to be satisfied in view of what has been observed in Remark 3.1. For the sake of completeness, we recall that the well-known (above mentioned) $sinc$-function is that defined as $\sin(\pi x)/\pi x$, if $x \neq 0$, and $1$ if $x = 0$, see e.g., [26, 27]. For other examples of kernels, see, e.g., [13, 20, 15, 22, 16].

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## References

[1] A. Abdurexit and T. N. Bekjan: *Noncommutative Orlicz modular spaces associated with growth functions*. Banach J. Math. Anal. **9** (4) (2015), 115–125.

[2] T. Acar, A. Alotaibi and S. A. Mohiuddine: *Construction of new family of Bernstein-Kantorovich operators*. Math. Methods Appl. Sci. **40** (18) (2017), 7749–7759.

[3] G. Allasia, R. Cavoretto and A. De Rossi: *A class of spline functions for landmark-based image registration*, Math. Methods Appl. Sci. **35** (8) (2012), 923–934.

[4] G. Allasia, R. Cavoretto and A. De Rossi: *Lobachevsky spline functions and interpolation to scattered data*, Comput. Appl. Math. **32** (1) (2013), 71–87.

[5] L. Angeloni, D. Costarelli and G. Vinti: *A characterization of the convergence in variation for the generalized sampling series*. Ann. Acad. Sci. Fenn. Math. **43** (2018), 755–767.

[6] F. Asdrubali, G. Baldinelli, F. Bianchi, D. Costarelli, A. Rotili, M. Seracini and G. Vinti: *Detection of thermal bridges from thermographic images by means of image processing approximation algorithms*, Appl. Math. Comp. **317** (2018), 160–171.

[7] F. Asdrubali, G. Baldinelli, F. Bianchi, D. Costarelli, L. Evangelisti, A. Rotili, M. Seracini and G. Vinti: *A model for the improvement of thermal bridges quantitative assessment by infrared thermography*. Applied Energy **211** (2018), 854–864.

[8] C. Bardaro, P. L. Butzer, R. L. Stens and G. Vinti: *Kantorovich-type generalized sampling series in the setting of Orlicz spaces*. Sampl. Theory Signal Image Process. **6** (1) (2007), 29–52.

[9] C. Bardaro and I. Mantellini: *On convergence properties for a class of Kantorovich discrete operators*. Num. Funct. Anal. Optim. **33** (4) (2012), 374–396.

[10] C. Bardaro, J. Musielak and G. Vinti: *Nonlinear Integral Operators and Applications*. De Gruyter Series in Nonlinear Analysis and Applications, 9 New York, Berlin, 2003.

[11] B. Bartoccini, D. Costarelli and G. Vinti: *Extension of saturation theorems for the sampling Kantorovich operators*. In print in: Complex Analysis and Operator Theory (2018), DOI: 10.1007/s11785-018-0852-z.

[12] P. L. Butzer: *A survey of the Whittaker-Shannon sampling theorem and some of its extensions*, J. Math. Res. Exposition **3** (1) (1983), 185–212.

[13] P. L. Butzer and R. J. Nessel: *Fourier Analysis and Approximation*, Vol. I: One-dimensional theory, Pure and Applied Mathematics, 40, Academic Press, New York-London, 1971.

[14] P. L. Butzer, S. Ries and R. L. Stens: *Approximation of Continuous and Discontinuous Functions by Generalized Sampling Series*. J. Approx. Theory **50** (1) (1987), 25–39.
[15] L. Coroianu and S. G. Gal: *$L^p$- approximation by truncated max-product sampling operators of Kantorovich-type based on Fejér kernel*. J. Integral Equations Appl. **29** (2) (2017), 349–364.
[16] L. Coroianu and S. G. Gal: *Approximation by truncated max-product operators of Kantorovich-type based on generalized $(\Phi, \Psi)$-kernels*. Math. Methods Appl. Sci. **41** (2018), 7971-7984.
[17] D. Costarelli, A.M. Minotti and G. Vinti: *Approximation of discontinuous signals by sampling Kantorovich series*. J. Math. Anal. Appl. **450** (2) (2017), 1083–1103.
[18] D. Costarelli and A.R. Sambucini: *Approximation results in Orlicz spaces for sequences of Kantorovich max-product neural network operators*. Results Math. **73** (1) (2018), Art. 15, 15 pp. DOI: 10.1007/s00025-018-0799-4.
[19] D. Costarelli and R. Spigler: *How sharp is the Jensen inequality ?*, J. Inequal. Appl. **2015:69** (2015) 1–10.
[20] D. Costarelli and G. Vinti: *Approximation by Nonlinear Multivariate Sampling-Kantorovich Type Operators and Applications to Image Processing*. Numer. Funct. Anal. Optim. **34** (8) (2013), 819–844.
[21] D. Costarelli and G. Vinti: *Order of approximation for sampling Kantorovich operators*, J. Integral Equations Appl. **26** (3) (2014), 345–368.
[22] D. Costarelli and G. Vinti: *Convergence for a family of neural network operators in Orlicz spaces*. Math. Nachr. **290** (2-3) (2017), 226–235.
[23] D. Costarelli and G. Vinti: *An inverse result of approximation by sampling Kantorovich series*. Proceedings of the Edinburgh Mathematical Society, 62 (1) (2019), 265-280.
[24] D. Cruz-Uribe and P. Hasto: *Extrapolation and interpolation in generalized Orlicz spaces*. Trans. Amer. Math. Soc. **370** (6) (2018), 4323–4349.
[25] P. A. Hasto: The maximal operator on generalized Orlicz spaces. J. Funct. Anal. **269** (12) (2015), 4038–4048.
[26] Y. S. Kolomoitsev and M. A. Skopina: *Approximation by multivariate Kantorovich-Kotelnikov operators*. J. Math. Anal. Appl. **456** (1) (2017), 195–213.
[27] A. Krivoshein and M. A. Skopina: Multivariate sampling-type approximation, Anal. Appl. **15** (4) (2017), 521–542.
[28] K. Kuaket and P. Kumam: *Fixed points of asymptotic pointwise contractions in modular spaces*. Appl. Math. Lett. **24** (11) (2011), 1795–1798.
[29] J. Musielak: *Orlicz spaces and Modular Spaces*. Lecture Notes in Math. **1034** Springer-Verlag, Berlin, 1983.
[30] J. Musielak and W. Orlicz: *On modular spaces*. Studia Math. **18** (1959), 49–65.
[31] O. Orlova and G. Tamberg: *On approximation properties of generalized Kantorovich-type sampling operators*. J. Approx. Theory **201** (2016), 73–86.
[32] S. Ries and R. L. Stens: *Approximation by generalized sampling series*. In: Proc. Internat. Conf. Constructive Theory of Functions, Varna, Bulgaria, June 1984, pp. 746–756, Bulgarian Acad. Sci. Sofia, 1984.
[33] M. Unser: *Ten good reasons for using spline wavelets*. Proc. SPIE Vol. **3169**, Wavelets Applications in Signal and Image Processing V (1997), 422–431.

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# Quantitative Estimates for $L^p$-Approximation by Bernstein-Kantorovich-Choquet Polynomials with Respect to Distorted Lebesgue Measures

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ABSTRACT. For the univariate Bernstein-Kantorovich-Choquet polynomials written in terms of the Choquet integral with respect to a distorted probability Lebesgue measure, we obtain quantitative approximation estimates for the $L^p$-norm, $1 \leq p < +\infty$, in terms of a $K$-functional.

**Keywords:** Monotone and submodular set function, Choquet integral, Bernstein-Kantorovich-Choquet polynomial, $L^p$ quantitative estimates, $K$-functional, Distorted Lebesgue measure.

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## 1. INTRODUCTION

Recently, in a series of papers we have started the study of the approximation properties of some nonlinear integral operators obtained from the linear ones by replacing the classical Lebesgue integral by its nonlinear extension called Choquet integral with respect to a monotone and submodular set function. Thus, qualitative and quantitative results of approximation by Bernstein-Durrmeyer-Choquet polynomials written in terms of Choquet integrals with respect to monotone and submodular set functions were obtained in the papers [7], [9], [10], [14]. Qualitative and quantitative approximation results for other Choquet integral operators obtained by using a Feller kind scheme (and including discrete Bernstein-Choquet operators and Picard-Choquet operators) were obtained in [8]. For large classes of functions, all these nonlinear operators give better estimates of approximation than their classical correspondents. Quantitative results of uniform and pointwise approximation by Bernstein-Kantorovich-Choquet polynomials, better in large classes of functions than those obtained by their classical correspondents, were obtained in the very recent paper [11]. Also, shape preserving properties of some Kantorovich-Choquet type operators were considered in [13].
It is worth to mention that implications of the concept of Choquet integral in other topics of mathematical analysis were obtained in the papers [12], [15], [16].
The aim of the present paper is to to obtain quantitative estimates for $L^p$-approximation, $1 \leq p < +\infty$, by Bernstein-Kantorovich-Choquet polynomials.
Section 2 contains some preliminaries on the Choquet integral. In Section 3, in the case when the Choquet integral is taken with respect to the so called distorted Lebesgue measures, quantitative estimates in terms of a $K$-functional for the $L^p$ approximation, $1 \leq p < \infty$, are obtained.

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## 2. Preliminaries

In this section we present some concepts and results on the Choquet integral which will be used in the main section.

**Definition 2.1.** *Let $\Omega$ be a nonempty set and $\mathcal{C}$ be a $\sigma$-algebra of subsets in $\Omega$.*
*(i) (see, e.g., [21], p. 63) Let $\mu : \mathcal{C} \to [0, +\infty]$. If $\mu(\emptyset) = 0$ and $A, B \in \mathcal{C}$, with $A \subset B$, implies $\mu(A) \le \mu(B)$, then $\mu$ is called a monotone set function (or capacity). Also, if*

$$\mu(A \bigcup B) + \mu(A \bigcap B) \le \mu(A) + \mu(B), \text{ for all } A, B \in \mathcal{C},$$

*then $\mu$ is called submodular. If $\mu(\Omega) = 1$, then $\mu$ is called normalized.*
*(ii) (see [5], or [21], p. 233, or [19]) Let $\mu$ be a normalized, monotone set function on $\mathcal{C}$.*
*If $f : \Omega \to \mathbb{R}$ is $\mathcal{C}$-measurable, i.e. for any Borel subset $B \subset \mathbb{R}$ we have $f^{-1}(B) \in \mathcal{C}$, then for any $A \in \mathcal{C}$, the Choquet integral is defined by*

$$(C)\int_A f d\mu = \int_0^{+\infty} \mu(F_\beta(f) \bigcap A) d\beta + \int_{-\infty}^0 [\mu(F_\beta(f) \bigcap A) - \mu(A)] d\beta,$$

*where $F_\beta(f) = \{\omega \in \Omega; f(\omega) \ge \beta\}$. If $(C)\int_A f d\mu \in \mathbb{R}$, then $f$ is called Choquet integrable on $A$. Notice that if $f \ge 0$ on $A$, then in the above formula we get $\int_{-\infty}^0 = 0$.*
*If $\mu$ is the Lebesgue measure, then the Choquet integral $(C)\int_A f d\mu$ reduces to the Lebesgue integral.*

In what follows, we list some known properties of the Choquet integral.

**Remark 2.1.** *If $\mu : \mathcal{C} \to [0, +\infty]$ is a monotone set function, then the following properties hold :*
*(i) For all $a \ge 0$ we have $(C)\int_A af d\mu = a \cdot (C)\int_A f d\mu$ (if $f \ge 0$ then see, e.g., [21], Theorem 11.2, (5), p. 228 and if $f$ is of arbitrary sign, then see, e.g., [6], p. 64, Proposition 5.1, (ii)).*
*(ii) For all $c \in \mathbb{R}$ and $f$ of arbitrary sign, we have (see, e.g., [21], pp. 232-233, or [6], p. 65)*

$$(C)\int_A (f + c) d\mu = (C)\int_A f d\mu + c \cdot \mu(A).$$

*If $\mu$ is submodular too, then for all $f, g$ of arbitrary sign and lower bounded we have (see, e.g., [6], p. 75, Theorem 6.3)*

$$(C)\int_A (f + g) d\mu \le (C)\int_A f d\mu + (C)\int_A g d\mu,$$

*that is the Choquet integral is sublinear.*
*(iii) If $f \le g$ on $A$ then $(C)\int_A f d\mu \le (C)\int_A g d\mu$ (see, e.g., [21], p. 228, Theorem 11.2, (3) if $f, g \ge 0$ and p. 232 if $f, g$ are of arbitrary sign).*
*(iv) Let $f \ge 0$. By the definition of the Choquet integral, it is immediate that if $A \subset B$ then*

$$(C)\int_A f d\mu \le (C)\int_B f d\mu$$

*and if, in addition, $\mu$ is finitely subadditive, then*

$$(C)\int_{A \bigcup B} f d\mu \le (C)\int_A f d\mu + (C)\int_B f d\mu.$$

*(v) By the definition of the Choquet integral, it is immediate that*

$$(C)\int_A 1 \cdot d\mu(t) = \mu(A).$$

*(vi) The formula $\mu(A) = \gamma(M(A))$, where $\gamma : [0, 1] \to [0, 1]$ is an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ and $M$ is a probability measure (or only finitely additive) on a $\sigma$-algebra on $\Omega$*

*(that is, $M(\emptyset) = 0$, $M(\Omega) = 1$ and $M$ is countably additive), gives simple examples of monotone and submodular set functions (see, e.g., [6], pp. 16-17, Example 2.1). Such of set functions $\mu$ are also called distorsions of normalized and countably additive measures (or distorted measures).*

## 3. $L^p$-APPROXIMATION

Denoting by $\mathcal{B}_{[0,1]}$ the sigma algebra of all Borel measurable subsets in $\mathcal{P}([0,1])$, everywhere in this section, $(\Gamma_{n,x})_{n\in\mathbb{N},x\in[0,1]}$, will be a collection of families $\Gamma_{n,x} = \{\mu_{n,k,x}\}_{k=0}^{n}$, of monotone, submodular and strictly positive set functions $\mu_{n,k,x}$ on $\mathcal{B}_{[0,1]}$. Note here that a set function on $\mathcal{B}_{[0,1]}$ is called strictly positive, if for any open subset $A \subset \mathbb{R}$ with $A \cap [0,1] \neq \emptyset$, we have $\mu(A \cap [0,1]) > 0$.
Suggested by the classical form of the linear and positive operators of Bernstein-Kantorovich (see, e.g., [17]), we can introduce the following.

**Definition 3.2.** *The Bernstein-Kantorovich-Choquet polynomials with respect to $\Gamma_{n,x} = \{\mu_{n,k,x}\}_{k=0}^{n}$, are defined by the formula*

$$K_{n,\Gamma_{n,x}}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/(n+1),(k+1)/(n+1)])},$$

*where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$.*

In order to be well defined these operators, it is good enough if, for example, we suppose that $f : [0,1] \to \mathbb{R}_+$ is a $\mathcal{B}_{[0,1]}$-measurable function, bounded on $[0,1]$.

**Remark 3.2.** *It is clear that if $\mu_{n,k,x} = M$, for all $n, k$ and $x$, where $M$ is the Lebesgue measure, then the above polynomials become the classical ones.*
*Also, if $\mu_{n,k,x} = \delta_{k/n}$ (the Dirac measures), since $k/n \in (k/(n+1),(k+1)/(n+1))$, it is immediate that $K_{n,\Gamma_{n,x}}(f)(x)$ become the Bernstein polynomials. This fact shows the great flexibility of the formulas of these operators. More exactly, we can generate very many kinds of approximation operators, by choosing for some $\mu_{n,k,x}$ the Lebesgue measure, for some others $\mu_{n,k,x}$, the Dirac measures and for the others $\mu_{n,k,x}$, some Choquet measures.*

Note that pointwise and uniform approximation by $K_{n,\Gamma_{n,x}}(f)(x)$ were studied in [11].
In this section we study quantitative $L^p$-approximation results, $1 \le p < \infty$, for the Bernstein-Kantorovich-Choquet polynomials $K_{n,\Gamma_{n,x}}(f)(x)$ when $\Gamma_{n,x} = \{\mu\}$. In this case, we denote them by $K_{n,\mu}$.
But as in the case of Bernstein-Durrmeyer-Choquet polynomials studied in [10], even in the simple case when, for example $p = 1$, for $f \in L^1_\mu$ (meaning that $f$ is $\mathcal{B}_{[0,1]}$-measurable and $\|f\|_{L^1_\mu} = (C)\int_0^1 |f(t)|d\mu(t) < \infty$), considering for example the operator $K_{n,\mu}$, we easily get

$$\|K_{n,\mu}(f)\|_{L^1_\mu} \le \sum_{k=0}^{n}(C)\int_0^1 p_{n,k}(x)d\mu(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu(t)}{\mu([k/(n+1),(k+1)/(n+1)])}$$

$$\le \sum_{k=0}^{n}(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu(t) \le (n+1)\cdot\|f\|_{L^1_\mu}.$$

This is due to the fact that $(C)\int_0^1 fd\mu$ is not, in general, additive as function of $f$ (it is only subadditive).
Therefore, quantitative estimates for $L^p$-approximation by Bernstein-Kantorovich-Choquet polynomials, remain, for the general case, an open question.

However, in what follows, for a large class of distorted Lebesgue measures (see Remark 2.1, (vi)), we will be able to prove $L^p$-approximation results.
If $\mu : \mathcal{B}_{[0,1]} \to [0, +\infty)$ is a monotone set function and $1 \le p < +\infty$, then we make the following notations :

$$L^p_\mu[0,1] = \{f : [0,1] \to \mathbb{R}; f \text{ is } \mathcal{B}_{[0,1]}\text{-measurable and } (C)\int_0^1 |f(t)|^p d\mu(t) < +\infty\},$$

$$L^p_{\mu,+}[0,1] = L^p_\mu[0,1] \bigcap \{f : [0,1] \to \mathbb{R}_+\},$$

$$C^1_+[0,1] = \{g : [0,1] \to [0,+\infty); g \text{ is differentiable on } [0,1]\},$$

$$K\left(f;t\right)_{L^p_\mu[0,1]} = \inf_{g \in C^1_+[0,1]} \{\|f-g\|_{L^p_\mu} + t\|g'\|_{C[0,1]}\},$$

$$\text{where } \|F\|_{L^p_\mu[0,1]} = \left(\int_0^1 |F(t)|^p d\mu(t)\right)^{1/p}, \|F\|_{C[0,1]} = \sup\{|F(t)|; t \in [0,1]\},$$

$$IC[0,1] = \{g : [0,1] \to [0,1] : g(0) = 0, g(1) = 1, g \text{ is concave and strictly increasing on } [0,1] \text{ and there exists } g'(0) < +\infty\}.$$

Also, denote by $\mathcal{D}(\mathcal{B}_{[0,1]})$ the class of all set functions $\mu : \mathcal{B}_{[0,1]} \to [0,+\infty)$ of the form $\mu(A) = g(M(A))$, for all $A \in \mathcal{B}_{[0,1]}$, where $g \in IC[0,1]$ and $M$ is the Lebesgue measure on $\mathcal{B}_{[0,1]}$. In the words of Remark 2.1, (vi), any such a $\mu$ is a distorted Lebesgue measure.

**Remark 3.3.** *According to Remark 2.1, (vi), any $\mu \in \mathcal{D}(\mathcal{B}_{[0,1]})$ is a normalized, monotone, strictly positive and submodular set function. Simple examples of $\mu \in \mathcal{D}(\mathcal{B}_{[0,1]})$ are $\mu(A) = \sin[M(A)]/\sin(1)$ or $\mu(A) = g[M(A)]$, for all $A \in \mathcal{B}_{[0,1]}$, where $M$ denotes the Lebesgue measure and $g(x) = \frac{2x}{1+x}$.*

We can state the following.

**Theorem 3.1.** *Let $1 \le p < \infty$. If $\mu \in \mathcal{D}(\mathcal{B}_{[0,1]})$, then for all $f \in L^p_{\mu,+}[0,1]$, $n \in \mathbb{N}$, we have*

$$\|f - K_{n,\mu}(f)\|_{L^p_\mu} \le c_p \cdot K\left(f; \frac{1}{2\sqrt{n+1}}\right)_{L^p_\mu},$$

*where $c_p = 1 + g'(0)^{(p+1)/p}$.*

*Proof.* Let $\mu(A) = g[M(A)]$ with $\mu \in \mathcal{D}(\mathcal{B}_{[0,1]})$. The main ideas used several times in the proof are that the Choquet integral with respect to $m$ reduces to the classical Lebesgue integral and that if $\mu$ and $\nu$ are two monotone set functions satisfying $\mu(A) \le c \cdot \nu(A)$ for all $A$, with $c > 0$ a constant independent of $A$, then $(C)\int_0^1 F d\mu \le c \cdot (C)\int_0^1 F d\nu$, for any $F \ge 0$.
Firstly, by $g(0) = 0$, $g(1) = 1$ and by the concavity of $g$, we immediately obtain the inequalities

$$x \le g(x) \le g'(0)x, \text{ for all } x \in [0,1], \tag{3.1}$$

which clearly implies

$$M(A) \le \mu(A) \le g'(0)M(A), \text{ for all } A \in \mathcal{B}_{[0,1]}. \tag{3.2}$$

Indeed, the inequalities in (3.1) hold since all the points of the segment passing through the points $(0, g(0))$ and $(1, g(1))$ are below the graph of $g$ and since all the points of the tangent to the graph of $g$ at $(0, g(0))$ are above the graph of $g$.
We make the proof in three steps.
**Step 1.** For $f \in L^p_{\mu,+}[0,1]$ we obtain

$$\|K_{n,\mu}(f)\|_{L^p_\mu} \le [g'(0)]^{(p+1)/p} \cdot \|f\|_{L^p_\mu}. \tag{3.3}$$

Indeed, by $\|K_{n,M}(f)\|_{L^p_M} \le \|f\|_{L^p_M}$ (see, e.g. [3]) combined with (3.2), it follows $\|f\|_{L^p_M} \le \|f\|_{L^p_\mu}$ and

$$\|K_{n,M}(f)\|_{L^p_M} \le \|f\|_{L^p_\mu}. \tag{3.4}$$

On the other hand, by (3.2), we obtain

$$\|K_{n,M}(f)\|_{L^p_M}$$

$$= \left(\int_0^1 \left[\sum_{k=0}^n p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dM(t)}{M([k/(n+1),(k+1)/(n+1)])}\right]^p dM(x)\right)^{1/p}$$

$$\ge \frac{1}{g'(0)^{1/p}} \cdot \left((C)\int_0^1 \left[\sum_{k=0}^n p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dM(t)}{M([k/(n+1),(k+1)/(n+1)])}\right]^p d\mu(x)\right)^{1/p}$$

$$\ge \frac{1}{g'(0)^{1/p}}$$

$$\cdot \left((C)\int_0^1 \left[\sum_{k=0}^n p_{n,k}(x) \cdot \frac{1}{g'(0)} \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu(t)}{\mu([k/(n+1),(k+1)/(n+1)])}\right]^p d\mu(x)\right)^{1/p}$$

$$= \frac{1}{[g'(0)]^{(p+1)/p}} \cdot \|K_{n,\mu}(f)\|_{L^p_\mu},$$

which combined with (3.4), implies (3.3).

**Step 2.** For $n \in \mathbb{N}$ and $0 \le k \le n$ arbitrary fixed, let us define $T_{n,k} : L^p_{\mu,+}[0,1] \to \mathbb{R}_+$ by

$$T_{n,k}(f) = (C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu(t), f \in L^p_{\mu,+}([0,1]).$$

From $L^p_{M,+}[0,1] \subset L^1_{M,+}[0,1]$ and since from (3.2) we clearly have $f \in L^p_{M,+}[0,1]$ if and only if $f \in L^p_{\mu,+}[0,1]$, it follows that $L^p_{\mu,+}[0,1] \subset L^1_{\mu,+}[0,1]$, for all $1 \le p < +\infty$.

Also, $0 \le (C)\int_{k/(n+1)}^{(k+1)/(n+1)} f^p(t)d\mu(t) \le (C)\int_0^1 f^p(t)d\mu(t) < \infty$, for any $f \in L^p_{\mu,+}[0,1]$.

Based on the Remark 3.3 and Remark 2.1, (i), (ii), (iii), by similar reasonings with those in the proof of Lemma 3.1 in [7], we obtain $|T_{n,k}(f) - T_{n,k}(g)| \le T_{n,k}(|f-g|)$. Also, since $T_{n,k}$ is positively homogeneous, sublinear and monotonically increasing, it is immediate that $K_{n,\mu}$ keeps the same properties, Consequently, it follows

$$|K_{n,\mu}(f)(x) - K_{n,\mu}(g)(x)| \le K_{n,\mu}(|f-g|)(x),\ f,g \in L^p_{\mu,+}[0,1], \tag{3.5}$$

$K_{n,\mu}(\lambda f) = \lambda K_{n,\mu}(f)$, $K_{n,\mu}(f+g) \le K_{n,\mu}(f) + K_{n,\mu}(g)$ and that $f \le g$ on $[0,1]$ implies $K_{n,\mu}(f) \le K_{n,\mu}(g)$ on $[0,1]$, for all $\lambda \ge 0$, $f,g \in L^p_{\mu,+}[0,1]$, $n \in \mathbb{N}$.

Now, from (3.5) we get

$$\|K_{n,\mu}(f) - K_{n,\mu}(g)\|_{L^p_\mu} \le \|K_{n,\mu}(|f-g|)\|_{L^p_\mu}. \tag{3.6}$$

**Step 3.** Let $f,g \in L^p_{\mu,+}[0,1]$. We will apply the Minkowski's inequality in the Choquet integral (see. e.g., Theorem 3.7 in [20] or Theorem 2 in [4]). It is worth mentioning that the proof of Minkowski's inequality in [20] or [4] is based on the Hölder's inequality

$$(C)\int |fg| \le \left((C)\int |f|d\mu\right)^{1/p} \cdot \left((C)\int |g|d\mu\right)^{1/q}, 1/p + 1/q = 1,$$

where the proof is performed under the supposition that $(C)\int |f|d\mu \neq 0$ and $(C)\int |g|d\mu \neq 0$. But from (3.2), it easily follows that the Hölder's inequality immediately holds even if $(C)\int |f|d\mu = 0$ or $(C)\int |g|d\mu = 0$. Therefore, under the hypothesis of the theorem, the Minkowski's inequality holds in its full generality.
So, we get

$$\|f - K_{n,\mu}(f)\|_{L^p_\mu} = \|(f-g) + (g - K_{n,\mu}(g)) + (K_{n,\mu}(g) - K_{n,\mu}(f))\|_{L^p_\mu}$$

$$\leq \|f-g\|_{L^p_\mu} + \|g - K_{n,\mu}(g)\|_{L^p_\mu} + \|K_{n,\mu}(g) - K_{n,\mu}(f)\|_{L^p_\mu}. \tag{3.7}$$

By (3.6) and (3.3), we obtain

$$\|K_{n,\mu}(g) - K_{n,\mu}(f)\|_{L^p_\mu} \leq [g'(0)]^{(p+1)/p} \cdot \|f-g\|_{L^p_\mu}. \tag{3.8}$$

Now, let us estimate $\|g - K_{n,\mu}(g)\|_{L^p_\mu}$ for $g \in C^1_+[0,1]$. Thus, by (3.5) and $K_{n,\mu}(e_0)(x) = e_0(x) = 1$, we get

$$|g(x) - K_{n,\mu}(g)(x)| = |K_{n,\mu}(g(x))(x) - K_{n,\mu}(g(t))(x)| \leq K_{n,\mu}(|g(x) - g(\cdot)|)(x).$$

Since for $g \in C^1_+[0,1]$ and $x, t \in [0,1]$, it follows (see, e.g., [18], formula (2.5), or [2])

$$|g(x) - g(t)| \leq \|g'\|_{C[0,1]} \cdot |x-t| = \|g'\|_{C[0,1]} \cdot \varphi_x(t),$$

applying $K_{n,\mu}$, which is subadditive as function of $f$, it follows $K_{n,\mu}(|g(x) - g(\cdot)|)(x) \leq \|g'\|_{C[0,1]} K_{n,\mu}(\varphi_x)$.
Taking to the power $p$ and integrating above with respect to $x$ and $\mu$, we obtain

$$\|g - K_{n,\mu}(g)\|_{L^p_\mu} \leq \|g'\|_{C[0,1]} \cdot \|K_{n,\mu}(\varphi_x)\|_{L^p_\mu}. \tag{3.9}$$

Denoting $c_p = 1 + g'(0)^{(p+1)/p}$, from (3.8) and (3.9) replaced in (3.7), it follows

$$\|f - K_{n,\mu}(f)\|_{L^p_\mu} \leq c_p \left( \|f-g\|_{L^p_\mu} + \|g'\|_{C[0,1]} \cdot \Delta_{n,p}/c_p \right),$$

where $\Delta_{n,p} := \|K_{n,\mu}(\varphi_x)\|_{L^p_\mu}$, $\varphi_x(t) = |x-t|$ for $x, t \in [0,1]$.
Finally, the reasonings from Step 1 lead to the estimate

$$\Delta_{n,p}/c_p \leq \frac{[g'(0)]^{(p+1)/p}}{c_p} \cdot \|K_{n,M}(\varphi_x)\|_{L^p_M} \leq \frac{[g'(0)]^{(p+1)/p}}{c_p} \cdot \|K_{n,M}(\varphi_x)\|_{C[0,1]}$$
$$\leq \frac{[g'(0)]^{(p+1)/p}}{c_p} \cdot \frac{1}{2\sqrt{n+1}} \leq \frac{1}{2\sqrt{n+1}}.$$

(we have used above the inequality in, e.g., [1], p. 334, $|K_{n,M}(\varphi_x)(x)| \leq \frac{\sqrt{(n-1)x(1-x)}}{n+1}$).
This immediately proves the required conclusion. □

**Remark 3.4.** *Note that the order of $L^p$-approximation $K\left(f; \frac{1}{2\sqrt{n+1}}\right)_{L^p_\mu}$ in Theorem 3.1 is, in some sense, similar with the order of $L^p$-approximation for the classical Bernstein-Kantorovich operators, $\tau\left(f; \frac{1}{\sqrt{n+1}}\right)_p$, where $\tau(f;\delta)_p$ is the $L^p$-averaged modulus of smoothness of Sendov-Popov (see, e.g., [3], p. 279).*

**Remark 3.5.** *For $f$ of arbitrary sign and lower bounded on $[0,1]$ with $f(x) - m \geq 0$, for all $x \in [0,1]$, Theorem 3.1 still take place for the slightly modified operator*

$$K^*_{n,\mu}(f)(x) = K_{n,\mu}(f-m)(x) + m.$$

*Indeed, we have* $K^*_{n,\mu}(f)(x) - f(x) = K_{n,\mu}(f-m)(x) - (f(x)-m)$ *and since we may consider here that* $m < 0$*, we immediately get*

$$K(f-m;t)_{L^p_\mu} = \inf_{g\in C^1_+[0,1]}\{\|f-(g+m)\|_{L^p_\mu} + t\|\nabla g\|_{C[0,1]}\}$$
$$= \inf_{g\in C^1_+[0,1]}\{\|f-(g+m)\|_{L^p_\mu} + t\|\nabla (g+m)\|_{C[0,1]}\}$$
$$= \inf_{h\in C^1[0,1],\, h\geq m}\{\|f-h\|_{L^p_\mu} + t\|\nabla h\|_{C[0,1]}\}.$$

## REFERENCES


[1] F. Altomare and M. Campiti, *Korovkin-Type Approximation Theory and its Applications*, deGruyter Studies in Mathematics, vol. 17. Walter de Gruyter, New York, 1994.

[2] E. E. Berdysheva and B.-Z. Li, *On $L^p$-convergence of Bernstein-Durrmeyer operators with respect to arbitrary measure*, Publ. Inst. Math. (Beograd) (N.S.). 96(110) (2014), 23-29.

[3] M. Campiti and G. Metafune, *$L^p$-convergence of Bernstein-Kantorovich-type operators*, Ann. Polon. Math., LXIII (1996), 273-280.

[4] J. Cerdà, J., Martín and P., Silvestre, *Capacitary function spaces*, Collect. Math., 62 (2011), 95-118.

[5] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble), 5 (1954), 131-295.

[6] D. Denneberg, *Non-Additive Measure and Integral*, Kluwer Academic Publisher, Dordrecht, 1994.

[7] S. G. Gal and B. D. Opris, *Uniform and pointwise convergence of Bernstein-Durrmeyer operators with respect to monotone and submodular set functions*, J. Math. Anal. Appl. 424 (2015), 1374-1379.

[8] S. G. Gal, *Approximation by Choquet integral operators*, Ann. Mat. Pura Appl., 195 (2016), No. 3, 881-896.

[9] S. G. Gal and S. Trifa, *Quantitative estimates in uniform and pointwise approximation by Bernstein-Durrmeyer-Choquet operators*, Carpath. J. Math., 33 (2017), 49-58.

[10] S. G. Gal and S. Trifa, *Quantitative estimates in $L^p$-approximation by Bernstein-Durrmeyer-Choquet operators with respect to distorted Borel measures*, Results Math., 72 (2017), no. 3, 1405-1415.

[11] S. G. Gal, *Uniform and pointwise quantitative approximation by Kantorovich-Choquet type integral operators with respect to monotone and submodular set functions*, Mediterr. J. Math., 14 (2017), no. 5, Art. 205, 12 pp.

[12] S. G. Gal, *The Choquet integral in capacity*, Real Analysis Exchange, 43 (2) (2018), 263-280.

[13] S. G. Gal, *Shape preserving properties and monotonicity properties of the sequences of Choquet type integral operators*, J. Numer. Anal. Approx. Theory, under press.

[14] S. G. Gal, *Quantitative approximation by Stancu-Durrmeyer-Choquet-Šipoš operators*, Math. Slovaca, under press.

[15] S. G. Gal, *Fredholm-Choquet integral equations*, J. Integral Equations Applications, https://projecteuclid.org/ euclid.jiea/1542358961, under press.

[16] S. G. Gal, *Volterra-Choquet integral equations*, J. Integral Equations Applications, https://projecteuclid.org/ euclid.jiea/1541668067. under press.

[17] L. V. Kantorovich, *Sur certains développements suivant les polynômes de la forme de S. Bernstein*, I, II, C.R. Acad. Sci. URSS (1930) 563-568, 595-600.

[18] B.-Z. Li, *Approximation by multivariate Bernstein-Durrmeyer operators and learning rates of least-square regularized regression with multivariate polynomial kernel*, J. Approx. Theory, 173 (2013), 33-55.

[19] M. Sugeno, *Theory of Fuzzy Integrals and its Applications*, Ph.D. dissertation, Tokyo Institute of Technology, Tokyo (1974).

[20] Wang, R. S., *Some inequalities and convergence theorems for Choquet integrals*, J. Appl. Math. Comput., 35 (2011), 305-321.

[21] Z. Wang and G. J. Klir, *Generalized Measure Theory*, Springer, New York, 2009.



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# Iterates of Markov Operators and Constructive Approximation of Semigroups

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ABSTRACT. In this paper we survey some recent results concerning the asymptotic behaviour of the iterates of a single Markov operator or of a sequence of Markov operators. Among other things, a characterization of the convergence of the iterates of Markov operators toward a given Markov projection is discussed in terms of the involved interpolation sets.

Constructive approximation problems for strongly continuous semigroups of operators in terms of iterates are also discussed. In particular we present some simple criteria concerning their asymptotic behaviour.

Finally, some applications are shown concerning Bernstein-Schnabl operators on convex compact sets and Bernstein-Durrmeyer operators with Jacobi weights on the unit hypercube. A final section contains some suggestions for possible further researches.

**Keywords:** Markov operator, Iterate of Markov operators, Markov semigroup, Approximation of semigroups, Bernstein-Schnabl operator, Bernstein-Durrmeyer operator with Jacobi weights.

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## 1. INTRODUCTION

In this paper we survey some recent results concerning the asymptotic behaviour of the iterates of a single Markov operator or of a sequence of Markov operators.

Such problems are connected with ergodic theory and, in particular with ergodic theorems. Iterates of sequences of Markov operators are also involved in the constructive approximation of strongly continuous semigroups of operators and, hence, of the solutions to the initial-boundary value differential problems governed by them.

Among other things, a characterization of the convergence of the iterates of Markov operators toward a given Markov projection is discussed in terms of the involved interpolation sets.

The usefulness of the approximation of strongly continuous semigroups of operators in terms of iterates, is enlightened by discussing some qualitative properties of them as well as their asymptotic behaviour.

Finally, some applications are shown concerning Bernstein-Schnabl operators on convex compact sets and Bernstein-Durrmeyer operators with Jacobi weights on the unit hypercube. A final section contains some suggestions for possible further researches. For more details about the results which are discussed in this paper we refer to [2], [4] and [5] and the references therein.

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## 2. Preliminaries and notation

Given a compact metric space $X$, we shall denote by $C(X)$ the linear space of all real-valued continuous functions on $X$ endowed with the supremum norm

$$||f||_\infty := \sup_{x\in X}|f(x)| \quad (f \in C(X)) \tag{2.1}$$

and the pointwise ordering, with respect to which it is a Banach lattice.
Let $\mathfrak{B}_X$ be the $\sigma$-algebra of all Borel subsets of $X$ and denote by $\mathcal{M}^+(X)$ (resp., $\mathcal{M}_1^+(X)$) the subset of all Borel measures (resp. the subset of all probability Borel measures) on $X$.
The symbol $M^+(X)$ (resp., $M_1^+(X)$) designates the subset of all positive linear functionals on $C(X)$ (resp. the subset of all positive linear functionals $\mu : C(X) \to \mathbf{R}$ such that $\mu(\mathbf{1}) = 1$, $\mathbf{1}$ denoting the constant function with constant value 1).
By the Riesz representation theorem (see, e.g., [12, Section 29]), if $\mu \in M^+(X)$ (resp. $\mu \in M_1^+(X)$), then there exists a unique (regular) Borel measure $\tilde{\mu} \in \mathcal{M}^+(X)$ (resp., in $\mathcal{M}_1^+(X)$) such that

$$\mu(f) = \int_X f\, d\tilde{\mu} \qquad \text{for every } f \in C(X). \tag{2.2}$$

Moreover, $\|\mu\| = \tilde{\mu}(X)$.
Consider a given Markov operator $T : C(X) \to C(X)$, i.e., $T$ is positive and $T(1) = 1$. In the sequel a special role will be played by the subset of interpolation points of $T$ which is defined by

$$\partial_T X := \{x \in X \mid T(f)(x) = f(x) \text{ for every } f \in C(X)\}, \tag{2.3}$$

and its possible representation by means of suitable functions.

Given a linear subspace $H$ of $C(X)$, its Choquet boundary $\partial_H X$ is the subset of all points $x \in X$ such that, if $\mu \in M^+(X)$ and if $\mu(h) = h(x)$ for every $h \in H$, then $\mu(f) = f(x)$ for every $f \in C(X)$.
If $H$ contains the constants and separates the points of $X$, then the Choquet boundary is non empty.
Given a Markov operator $T : C(X) \to C(X)$, we shall set

$$M := \{h \in C(X) \mid T(h) = h\}. \tag{2.4}$$

Clearly, $M$ is contained in the range of $T$ which will be also denoted by

$$H := T(C(X)) = \{T(f) \mid f \in C(X)\}. \tag{2.5}$$

The subspace $M$ contains the constants and hence, if it separates the points of $X$, then its Choquet boundary $\partial_M X$ is not empty.

**Theorem 2.1.** *Assume that the subspace $M$ defined above separates the points of $X$. Then*

$$\emptyset \neq \partial_M X \subset \partial_T X \subset \partial_H X.$$

*Moreover, if $V$ is an arbitrary subset of $M$ separating the points of $X$, then*

$$\partial_T X = \{x \in X \mid T(h^2)(x) = h^2(x) \text{ for every } h \in V\}.$$

*Finally, if $(h_n)_{n\geq 1}$ is a finite or countable family of the linear subspace generated by $V$, separating the points of $X$ and such that the series $\Phi := \sum_{n=1}^{\infty} h_n^2$ is uniformly convergent, then $\Phi \leq T(\Phi)$ and*

$$\partial_T X = \{x \in X \mid T(\Phi)(x) = \Phi(x)\}.$$

For a proof of Theorem 2.1 we refer to [2, Theorem 2.1]) (see also [5, Theorem 1.3.1]).
As a particular case of the result above, consider a compact subset $X$ of $\mathbf{R}^d$, $d \geq 1$. For every $i = 1, \ldots, d$ denote by $pr_i$ the $i^{th}$ coordinate function on $X$, i.e., $pr_i(x) := x_i$ for every $x = (x_1, \ldots, x_d) \in X$, and set

$$\Phi_d := \sum_{i=1}^{d} pr_i^2 = \|\cdot\|^2,$$

where $\|\cdot\|$ stands for the Euclidean norm on $\mathbf{R}^d$.

**Corollary 2.1.** *Given a Markov operator $T : C(X) \to C(X)$ such that $T(pr_i) = pr_i$ for every $i = 1, \ldots, d$, then $\Phi_d \leq T(\Phi_d)$ and*

$$\partial_T X = \{x \in X \mid T(\Phi_d)(x) = \Phi_d(x)\}.$$

## 3. Convergence criteria for nets of positive linear operators

In this section we discuss some general criteria concerning the convergence of nets (generalized sequences) of positive linear operators. The results seem to have an own independent interest and they can be considered as Korovkin-type theorems with respect to a limit operator which is an arbitrary positive linear operator rather then the identity operator. For additional Korovkin-type theorems, we refer, e.g., to [1], [3] and the references therein.
For a given Markov operator $T : C(X) \to C(X)$, we proceed to state a criterion in terms of the subset $\partial_T X$ defined by (2.3), which concerns the convergence of nets of positive linear operators toward a positive linear operator $S : C(X) \to C(X)$ such that $S \circ T = S$.
In the subsequent section we shall use this result in order to investigate the asymptotic behaviours of iterates of Markov operators.

**Theorem 3.2.** *Let $T : C(X) \to C(X)$ be a Markov operator such that the subset $\partial_T X$ is non empty and assume that there exists $\Psi \in C(X)$, $\Psi \geq 0$, such that $\partial_T X = \{x \in X \mid \Psi(x) = 0\}$ (for example $\Psi = T(\Phi) - \Phi$ as in Theorem 2.1).*
*Consider a net $(L_i)_{i\in I}^{\leq}$ of positive linear operators from $C(X)$ into itself such that*

- *(i) $(L_i(\mathbf{1}))_{i\in I}^{\leq}$ is pointwise bounded (resp. uniformly bounded) on $X$.*
- *(ii) $\lim_{i\in I}^{\leq} L_i(\Psi) = 0$ pointwise (resp., uniformly) on $X$.*

*Then, $\lim_{i\in I}^{\leq} L_i(T(f) - f) = 0$ pointwise (resp., uniformly) on $X$ for every $f \in C(X)$.*
*Accordingly, if $S : C(X) \to C(X)$ is a positive linear operator and if $L_i(T(f))_{i\in I}^{\leq}$ converges pointwise (resp., uniformly) on $X$ to $S(f)$ for every $f \in C(X)$, then*

$$\lim_{i\in I}^{\leq} L_i(f) = S(f)$$

*pointwise (resp., uniformly) on $X$ for every $f \in C(X)$. In particular, $S \circ T = S$.*

As a special case of Theorem 3.2 we get the following Korovkin-type result.

**Corollary 3.2.** *Let $T : C(X) \to C(X)$ be a Markov operator such that the subspace $M$ defined by (2.4) separates the points of $X$. Furthermore, set $H := T(C(X))$ and consider $\Phi \in C(X)$ such that $\Phi \leq T(\Phi)$ and $\partial_T X = \{x \in X \mid T(\Phi)(x) = \Phi(x)\}$.*
*Given a Markov operator $S : C(X) \to C(X)$ such that $S \circ T = S$, if $(L_i)_{i\in I}^{\leq}$ is a net of positive linear operators from $C(X)$ into itself and if $\lim_{i\in I}^{\leq} L_i(h) = S(h)$ pointwise (resp., uniformly) on $X$ for every $h \in H \cup \{\Phi\}$, then $\lim_{i\in I}^{\leq} L_i(f) = S(f)$ pointwise (resp., uniformly) on $X$ for every $f \in C(X)$.*

The proofs of Theorem 3.2 and its subsequent Corollary 3.2 can be found in [2, Theorem 2.5 and Corollary 2.7]). Note also that it can be applied, e.g., for $S = T$ or for $S = \lambda T$ $(\lambda \in C(X), 0 \le \lambda)$ provided $T$ is a Markov projection, i.e. $T \circ T = T$.
An application of Corollary 3.2 will be shown in the subsequent Section 4 (see Theorem 5.8).
The next result can be useful to study the behaviour of nets of positive linear operators when the limit operator is unknown. It generalizes Theorem 2 of [19]. For its proof we refer to [10, Proposition 3.7]. In the same paper further remarks and applications can be found.
Let $(E, \|\cdot\|)$ be a Banach space of real-valued functions defined on a convex subset $X$ of a locally convex space. Assume that the space $E$, endowed with its norm and the pointwise order, is a Banach lattice.

**Proposition 3.1.** *Let $(L_i)_{i\in I}^{\le}$ be a net of positive linear operators from $E$ into itself and assume that for every convex function $\varphi \in E$, the net $(L_i(\varphi))_{i\in I}^{\le}$ is decreasing (resp., increasing).*
*Furthermore, assume that for some convex function $u \in E$, the net $(L_i(u))_{i\in I}^{\le}$ is convergent in $E$. Then, setting*

$$\mathcal{A}(u) := \{\, g \in E \mid \text{There exists } \lambda \ge 0 \text{ such that } \lambda u - g \text{ and } \lambda u + g \text{ are convex}\},$$

*the net $(L_i(f))_{i\in I}^{\le}$ is convergent in $E$ for every $f \in \mathcal{A}(u)$.*
*Therefore, if $\mathcal{A}(u)$ is dense in $E$ and $\sup\limits_{i\in I,\, i_0\le i} \|L_i\| < +\infty$ for some $i_0 \in I$, then $(L_i(f))_{i\in I}^{\le}$ is convergent in $E$ for every $f \in E$.*

Note that, if $X$ is a real interval and the convex function $u$ belongs to $C^2(X)$, then $\{\, f \in E \cap C^2(X) \mid |f''| \le \lambda u'' \text{ for some } \lambda \ge 0\,\} \subset \mathcal{A}(u)$. In particular, if $\alpha := \min\limits_X u''(x) > 0$, then $\{f \in E \cap C^2(X) |\, f'' \text{ bounded}\} \subset \mathcal{A}(u)$, since $|f''| \le \dfrac{\|f''\|}{\alpha} u''$ for every $f \in E \cap C^2(X)$, $f''$ bounded.
Moreover, if $\mathbf{1} \in E$ and a net $(L_i)_{i\in I}^{\le}$ satisfies the assumptions of Proposition 3.1, then the net $(L_i(\mathbf{1}))_{i\in I}^{\le}$ is constant. Therefore, if $E$ is a closed linear subspace of bounded continuous functions on $X$, equipped with the uniform norm, the net $(L_i)_{i\in I}^{\le}$ is equibounded as well.
If $X$ is a compact real interval, Proposition 3.1 applies in particular for $E = C(X)$ and $u \in C^2(X)$ satisfying $\min\limits_X u''(x) > 0$.

**Corollary 3.3.** *Given a compact real interval $X$, let $(L_n)_{n\ge 1}$ be a sequence of positive linear operators from $C(X)$ into itself and assume that for every convex function $\varphi \in C(X)$, the sequence $(L_n(\varphi))_{n\ge 1}$ is decreasing (resp., increasing).*
*Further assume that for some convex function $u \in C^2(X)$ satisfying $\min\limits_X u''(x) > 0$, the sequence $(L_n(u))_{n\ge 1}$ is uniformly convergent. Then for every $f \in C(X)$, the sequence $(L_n(f))_{n\ge 1}$ is uniformly convergent.*

For a multidimensional version of the above result we refer to [18, Theorem 2.2].

## 4. Asymptotic behaviour of iterates of Markov operators

In this section we discuss some results concerning the asymptotic behaviour of iterates of Markov operators. For other additional results about this subject we refer, e.g., to [15, Theorem 1], [16, Theorem 3.1], [17, Theorem 1] and [23, Theorem 2.2].
Let $X$ be a compact metric space and consider two Markov operators $S : C(X) \to C(X)$ and $T : C(X) \to C(X)$ such that the subspace $M := \{h \in C(X) \mid T(h) = h\}$ separates the points of $X$.
If $S \circ T = T$, then

$$M \subset H := T(C(X)) \subset \{h \in C(X) \mid S(h) = h\}$$

and hence

$$\emptyset \neq \partial_M X \subset \partial_T X \subset \partial_H X \subset \partial_S X.$$

The proof of the next result is based on Theorem 3.2. For more details we refer to [2, Theorem 3.1]).

As usual, if $S : C(X) \to C(X)$ is a linear operator, the iterates $S^n$, $n \geq 1$, are defined recursively by $S^1 := S$ and $S^{n+1} := S \circ S^n$.

**Theorem 4.3.** *Let $S : C(X) \to C(X)$ and $T : C(X) \to C(X)$ be Markov operators and assume that the subspace $M := \{h \in C(X) \mid T(h) = h\}$ separates the points of $X$. Then the following statements are equivalent:*

*(a)* $\lim_{n\to\infty} S^n(f) = T(f)$ *uniformly on $X$ for every $f \in C(X)$.*

*(b)* $\lim_{n\to\infty} S^n(f) = T(f)$ *pointwise on $X$ for every $f \in C(X)$.*

*(c)* $S \circ T = T$ *and* $\partial_S X \subset \partial_T X$*, i.e., for every $x \in X \setminus \partial_T X$ there exists $f \in C(X)$ such that $S(f)(x) \neq f(x)$.*

*(d)* $S \circ T = T$ *and for every sequence $(h_n)_{n\geq 1}$ in $M$ separating the points of $X$, such that the series $\Phi := \sum_{n=1}^{\infty} h_n^2$ is uniformly convergent on $X$, one gets $\Phi \leq S(\Phi)$ and*

$$\{x \in X \mid S(\Phi)(x) = \Phi(x)\} \subset \partial_T X.$$

*(e)* *There exists $\Phi \in C(X)$ such that $\Phi \leq S(\Phi)$ and*

$$\{x \in X \mid S(\Phi)(x) = \Phi(x)\} \subset \partial_T X.$$

*Moreover, if one of the statements above holds true, then $T \circ S = T$, $T$ necessarily is a Markov projection, i.e., $T \circ T = T$, and*

$$\partial_T X = \partial_S X = \partial_H X.$$

It is not devoid of interest to point out that, if $T : C(X) \to C(X)$ is a Markov projection whose range separates the points of $X$, considering an arbitrary function $\Phi \in C(X)$ such that $\Phi \leq T(\Phi)$ and $\partial_H X = \{x \in X \mid T(\Phi)(x) = \Phi(x)\}$, then an example of a Markov operator $S : C(X) \to C(X)$ satisfying statement (c) of Theorem 4.3 is $S := \lambda T + (1-\lambda)I$, where $I$ denotes the identity operator on $C(X)$ and $\lambda \in C(X)$ satisfies $0 < \lambda(x) \leq 1$ for every $x \in X$.

Below we show some applications of the results we have just described.

4.1. **The Poisson operator associated with the classical Dirichlet problem.** Consider a bounded open subset $\Omega$ of $\mathbf{R}^d$, $d \geq 2$, which we assume to be regular in the sense of potential theory (see, e.g., [3, Section 2.6]) and denote by $H(\Omega)$ the subspace of all $u \in C(\overline{\Omega})$ which are harmonic on $\Omega$.

Thus, for every $f \in C(\overline{\Omega})$ there exists a unique $u_f \in H(\Omega)$ such that $u_f|_{\partial\Omega} = f|_{\partial\Omega}$, i.e., $u_f$ is the unique solution to the Dirichlet problem

$$\begin{cases} \triangle u := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 0 \quad \text{on } \Omega, \\ u|_{\partial\Omega} = f|_{\partial\Omega} \end{cases} \qquad (u \in C(\overline{\Omega}) \cap C^2(\Omega)).$$

For instance, each bounded convex open subset of $\mathbf{R}^d$ is regular. Consider the Poisson operator $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ defined by

$$T(f) := u_f \quad (f \in C(\overline{\Omega})) \tag{4.6}$$

The operator $T$ is a positive projection whose range is $H(\Omega)$. Moreover,

$$\partial_{H(\Omega)}\overline{\Omega} = \partial_T \overline{\Omega} = \partial\Omega.$$

A direct application of Theorem 4.3 gives the following result.

**Corollary 4.4.** *Let $\Omega$ be a regular bounded open subset of $\mathbf{R}^d$, $d \geq 2$, and consider the Poisson operator $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ defined by (4.6).*
*Furthermore, consider a Markov operator $S : C(\overline{\Omega}) \to C(\overline{\Omega})$ such that $S(u) = u$ for every $u \in H(\Omega)$ and assume that $\partial_S \overline{\Omega} = \partial\Omega$, i.e.,*

$$\text{for every } x \in \Omega \text{ there exists } f \in C(\overline{\Omega}) \text{ such that } S(f)(x) \neq f(x).$$

*Then*

$$\lim_{n\to\infty} S^n(f) = T(f)$$

*uniformly on $\overline{\Omega}$ for every $f \in C(\overline{\Omega})$.*

Moreover, combining Corollaries 3.2 and 2.1, we also get the following result which might be useful to approximate the Poisson operator.

**Corollary 4.5.** *If $(L_i)_{i\in I}^{\leq}$ is a net of positive linear operators from $C(\overline{\Omega})$ into itself and if $\lim\limits_{i\in I}{}_{\leq} L_i(h) = T(h)$ pointwise (resp., uniformly) on $\overline{\Omega}$ for every $h \in H(\Omega) \cup \{\Phi_d\}$, then $\lim\limits_{i\in I}{}_{\leq} L_i(f) = T(f)$ pointwise (resp., uniformly) on $\overline{\Omega}$ for every $f \in C(\overline{\Omega})$.*

4.2. **Bernstein-Schnabl operators on convex compact subsets.** Consider a metrizable convex compact subset $K$ (of some locally convex Hausdorff space) and denote by $A(K)$ the linear subspace of all real-valued continuous affine functions on $K$.
Consider a positive linear projection $T : C(K) \to C(K)$ such that

$$A(K) \subset H := T(C(K)) = \{f \in C(K) \mid T(f) = f\} \tag{4.7}$$

and

$$h_{z,\alpha} \in H \text{ for every } z \in K, \alpha \in [0,1], h \in H, \tag{4.8}$$

where

$$h_{z,\alpha}(x) := h(\alpha x + (1-\alpha)z) \qquad (x \in K). \tag{4.9}$$

For instance, if $\Omega$ is a bounded open convex subset of $\mathbf{R}^d$, $d \geq 2$, then the Poisson operator defined by (4.6) is a positive projection satisfying (4.7) and (4.8).
We also recall that, if $K$ is a Bauer simplex (see, e.g., [3, Section 5.1] and [5, Section 1.1.3])(for instance, finite dimensional simplices are Bauer simplices), then there exists a unique positive linear projection $T : C(K) \to C(K)$ such that

$$T(C(K)) = A(K).$$

The projection $T$ is often referred to as the canonical positive projection associated with $K$. Actually, for every $f \in C(K)$, $T(f)$ is the unique function in $A(K)$ that coincides with $f$ on the subset $\partial_e K$ of the extreme points of $K$. Clearly, $T$ satisfies (4.8) as well and

$$\partial_T K = \partial_e K.$$

In the finite dimensional case, considering the canonical simplex

$$K_d := \left\{ (x_1, \ldots, x_d) \in \mathbf{R}^d \;\middle|\; x_i \geq 0 \text{ for every } i = 1, \ldots, d \text{ and } \sum_{i=1}^d x_i \leq 1 \right\}.$$

and setting $v_0 := (0, \ldots, 0), v_1 := (1, 0, \ldots, 0), \ldots, v_d := (0, \ldots, 0, 1)$, then the canonical Markov projection $T_d : C(K_d) \longrightarrow C(K_d)$ associated with $K_d$, is defined by

(4.10)
$$T_d(f)(x) := \left(1 - \sum_{i=1}^{d} x_i\right) f(v_0) + \sum_{i=1}^{d} x_i f(v_i)$$

$(f \in C(K_d), x = (x_1, \ldots, x_d) \in K_d)$.
When $K = [0, 1]$, then the canonical projection is,indeed, the Markov operator $T_1 : C([0, 1]) \to C([0, 1])$ defined by

(4.11)
$$T_1(f)(x) = (1 - x)f(0) + xf(1)$$

$(f \in C([0, 1]), x \in [0, 1])$.
For several other examples we refer to [3, Section 3.3] and [5, Chapter 3].
Coming back to a general positive linear projection $T : C(K) \to C(K)$ satisfying (4.7) and (4.8), let $S : C(K) \to C(K)$ be another positive linear operator such that

(4.12)
$$S(h) = h \quad \text{for every } h \in H$$

and

(4.13)
$$\partial_S K = \partial_T K.$$

Let $(\tilde{\mu}_x^S)_{x \in K}$ be the unique family in $\mathcal{M}_1^+(K)$ such that

$$S(f)(x) = \int_K f \, d\tilde{\mu}_x^S \qquad (f \in C(K), x \in K).$$

For every $n \geq 1$, $x \in K$, and $f \in C(K)$, set

(4.14)
$$B_n(f)(x) = \int_K \cdots \int_K f\Big(\frac{x_1 + \cdots + x_n}{n}\Big) d\tilde{\mu}_x^S(x_1) \cdots d\tilde{\mu}_x^S(x_n).$$

By the continuity property of the product measure it follows that $B_n(f) \in C(K)$. Moreover, $B_1 = S$.
The positive linear operator $B_n : C(K) \to C(K)$ is referred to as the $n$-th Bernstein-Schnabl operator associated with the positive linear operator $S$.
For special choices of the convex compact subset $K$ and of the operator $S$, these operators turn into the classical Bernstein operators on the unit interval, the unit $d$-dimensional hypercube and the $d$-dimensional simplex (see, e.g., [3, Section 6.1] and [5, Chapter 3]).
First, we point out that, by using a general Korovkin-type approximation theorem, it is possible to show that

(4.15)
$$\lim_{n \to \infty} B_n(f) = f \text{ uniformly on } K \text{ for every } f \in C(K)$$

(see [5, Theorem 3.2.1]).
Let $(u_n)_{n \geq 1}$ be an arbitrary (finite or countable) sequence in $A(K)$ separating the point of K and such that the series $\Phi := \sum\limits_{n=1}^{\infty} u_n^2$ is uniformly convergent on $K$.
Therefore, $\Phi \leq T(\Phi)$ and

$$\partial_H K = \partial_T K = \{x \in K \mid T(\Phi)(x) = \Phi(x)\}.$$

Accordingly, $B_n(\Phi) = \frac{1}{n} S(\Phi) + \frac{n-1}{n} \Phi$, so that, for every $x \in K$,

$$B_n(\Phi)(x) - \Phi(x) = 0 \text{ if and only if } S(\Phi)(x) - \Phi(x) = 0,$$

i.e.,

$$\partial_T X = \{x \in X \mid B_n(\Phi)(x) = \Phi(x)\}$$

and $\Phi \leq B_n(\Phi)$. Thus all the assumptions of part $(e)$ of Theorem 4.3 are satisfied and hence we get that

**Theorem 4.4.** *Under the preceeding hypotheses, for every* $n \geq 1$,

$$\lim_{m\to\infty} B_n^m(f) = T(f)$$

*uniformly on* $K$ *for every* $f \in C(K)$.
*In particular, for* $n = 1$,

$$\lim_{m\to\infty} S^m(f) = T(f)$$

*uniformly on* $K$ *for every* $f \in C(K)$.

The following special cases of Theorem 4.4 are worthy to be mentioned separately.

**Corollary 4.6.** *Consider a metrizable Bauer simplex* $K$ *and denote by* $T$ *the canonical projection associated with* $K$. *Let* $S : C(K) \to C(K)$ *be a Markov operator such that* $S(u) = u$ *for every* $u \in A(K)$ *and* $\partial_S K = \partial_e K$, *i.e., for every* $x \in K \diagdown \partial_e K$ *there exists* $f \in C(K)$ *such that* $S(f)(x) \neq f(x)$.
*Denoting by* $(B_n)_{n\geq 1}$ *the sequence of Bernstein-Schnabl operators associated with* $S$, *then for every* $n \geq 1$,

$$\lim_{m\to\infty} B_n^m(f) = T(f)$$

*uniformly on* $K$ *for every* $f \in C(K)$. *In particular, for* $n = 1$,

$$\lim_{m\to\infty} S^m(f) = T(f)$$

*uniformly on* $K$ *for every* $f \in C(K)$.

**Corollary 4.7.** *Consider the Markov projection* $T_1 : C([0,1]) \to C([0,1])$ *defined by* (4.11). *Let* $S : C([0,1]) \to C([0,1])$ *be a Markov operator such that* $S(1) = 1, S(e_1) = e_1$ *and* $\partial_S[0,1] = \{0,1\}$ *i.e., for every* $x \in ]0,1[$ *there exists* $f \in C([0,1])$ *such that* $S(f)(x) \neq f(x)$.
*Denoting by* $(B_n)_{n\geq 1}$ *the sequence of Bernstein-Schnabl operators associated with* $S$, *then for every* $n \geq 1$,

$$\lim_{m\to\infty} B_n^m(f) = T_1(f)$$

*uniformly on* $[0,1]$ *for every* $f \in C([0,1])$. *In particular, for* $n = 1$,

$$\lim_{m\to\infty} S^m(f) = T_1(f)$$

*uniformly on* $[0,1]$ *for every* $f \in C([0,1])$.

**Remark 4.1.** *Corollaries 4.6 and 4.7 apply in particular when, respectively,* $K = K_d$ *and* $S = T_d$, $d \geq 1$ *(see (4.10)) and when* $K = [0,1]$ *and* $T = T_1$. *In these cases, the corresponding Bernstein-Schnabl operators are, indeed, the classical Bernstein operators on* $K_d$ *and on* $[0,1]$.

In addition to the previous results, it is possible to investigate the limit behaviour of the iterates of Bernstein-Schnabl operators associated with a Markov projection even when the order of iteration depend of $n \geq 1$. The proof of the next result relies on Corollary 3.2 (see also [3, Theorem 6.1.3] or [5, Theorem 3.2.10]).

**Theorem 4.5.** *Let $T : C(K) \to C(K)$ be a positive Markov projection satisfying (4.7) and (4.8), and consider the relevant sequence $(B_n)_{n\geq 1}$ of Bernstein-Schnabl operators defined by (4.14) (with $S = T$). If $f \in C(K)$ and if $(k(n))_{n\geq 1}$ is a sequence of positive integers, then*

$$\lim_{n\to\infty} B_n^{k(n)}(f) = \begin{cases} f & \textit{uniformly on } K \textit{ if } \frac{k(n)}{n} \to 0, \\ T(f) & \textit{uniformly on } K \textit{ if } \frac{k(n)}{n} \to +\infty. \end{cases}$$

It is worthy to point out that, under some additional assumptions on $T$, the sequence $(B_n^{k(n)}(f))_{n\geq 1}$ $(f \in C(K))$ converges uniformly also when $\frac{k(n)}{n} \longrightarrow t \in ]0, +\infty[$.
More precisely, if $K$ is a subset of $\mathbf{R}^d, d \geq 1$, with non-empty interior and if $T$ maps the subspace of all polynomials of degree $m$ into itself for every $m \geq 1$, then for every $t \geq 0$ there exists a Markov operator $T(t) : C(K) \longrightarrow C(K)$ such that for every $f \in C(K)$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $\frac{k(n)}{n} \longrightarrow t$,

$$T(t)f = \lim_{n\to\infty} B_n^{k(n)}(f) \text{ uniformly on } K.$$

Moreover the family $(T(t))_{t\geq 0}$ is a strongly continuous semigroup of operators (briefly, $C_0$-semigroup of operators) whose generator $(A, D(A))$ is the closure of the operator $(Z, D(Z))$ where

$$D(Z) := \{u \in C(K) \mid \lim_{n\to\infty} n(B_n(u) - u) \quad \text{exists in} \quad C(K)\}$$

and, for every $u \in D(Z) \subset D(A)$

$$A(u) = Z(u) = \lim_{n\to\infty} n(B_n(u) - u) \text{ uniformly on } K.$$

Furthermore, $C^2(K) \subset D(Z) \subset D(A)$ and for every $u \in C^2(K)$

$$Au(x) = Zu(x) = \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_i} \tag{4.16}$$

$(x = (x_i)_{1\leq i\leq d})$ where for every $i, j = 1, \ldots, d$

$$\alpha_{ij}(x) := T(pr_i pr_j)(x) - x_i x_j.$$

The differential operator (4.16) is an elliptic second order differential operator which degenerates on $\partial_T K$ (which contains the subset $\partial_e K$ of the extreme points of $K$).
For more details on the above results and, especially, for the rich theory which is related to them we refer to [3, Chapter 6] and [5, Chapters 4 and 5]. This theory stresses an interesting relationship among positive semigroups, initial-boundary value problems, Markov processes and constructive approximation theory. Some aspects of them will be also treated in the subsequent Sections 5 and 6.

## 5. Iterates and constructive approximation of semigroups of operators

Iterates of (positive) linear operators can be usefully involved in the constructive approximation as well as in the qualitative study of (positive) $C_0$-semigroups of operators and, hence of the solutions to the initial-boundary value problems governed by them. For a short introduction to the theory of $C_0$-semigroups of operators we refer, e.g., to [5, Chapter 2].
We begin by recalling the following results which is a consequence of a more general one due to H. F. Trotter (see [25] or [5, Corollary 2.2.3]).

**Theorem 5.6.** *Let $E$ be a Banach space and let $(L_n)_{n\geq 1}$ be a sequence of bounded linear operators on $E$. Suppose that* *(i) (stability conditions) there exist $M \geq 1$ and $\omega \in \mathbf{R}$ such that*

$$\|L_n^k\| \leq Me^{\omega \frac{k}{n}} \text{ for every } k, n \geq 1.$$

*Furthermore, let $(A_0, D_0)$ be a linear operator defined on a dense subspace $D_0$ of $E$ and assume that*

*(ii) (core condition) $(\lambda I_{D_0} - A_0)(D_0)$ is dense in $E$ for some $\lambda > \omega$;*

*(iii) (asymptotic formula) $\lim\limits_{n\to\infty} n(L_n(u) - u) = A_0(u)$ for every $u \in D_0$.*

*Then $(A_0, D_0)$ is closable and its closure $(A, D(A))$ is the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on $E$ such that*

*(1) $\|T(t)\| \leq Me^{\omega t}$ for every $t \geq 0$;*

*(2) $T(t)(f) = \lim\limits_{n\to\infty} L_n^{k(n)}(f)$*

*for every $f \in E$ and $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $\lim\limits_{n\to\infty} \frac{k(n)}{n} = t$.*

The *core condition* $(ii)$ is often difficult to verify; in the special case where $\omega = 0$ in $(i)$, i.e., $\|L_n\| \leq 1$, then it can be replaced by

$(ii)^*$ there exists a family $(E_i)_{i\in I}$ of finite dimensional subspaces of $D_0$ which are invariant under each $L_n$ and whose union $\bigcup\limits_{i\in I} E_i$ is dense in $E$.

This variant of Trotter theorem is due to R. Schnabl (see [24] or [5, Corollary 2.2.11]). Note also that, because of assumptions $(i)$ and $(iii)$, necessarily

$$\lim_{n\to\infty} L_n(u) = u \text{ for every } u \in E.$$

i.e., $(L_n)_{n\geq 1}$ is an approximation process on $E$.

In the sequel, a $C_0$-semigroup $(T(t))_{t\geq 0}$ which is approximate by a sequence $(L_n)_{n\geq 1}$ as in formula (2) of Theorem 5.6, will be referred to as the limit semigroup associated with the sequence $(L_n)_{n\geq 1}$. Furthermore, a sequence $(L_n)_{n\geq 1}$ verifying conditions $(i), (ii), (iii)$ (resp. conditions $(i), (ii)^*, (iii)$) will be referred to as a Trotter-type admissible sequence (resp. a Schnabl-type admissible sequence).

The generator $(A, D(A)$ of such semigroup will be also referred to as the generator of the sequence $(L_n)_{n\geq 1}$.

Formula (2) of Theorem 5.6 has been successfully and mainly used in order to infer some properties of the sequence $(L_n)_{n\geq 1}$, notably, their saturation properties or, e.g., if $E$ is a Banach function space, converse theorems of convexity (see, e.g., [14], [21], [24], [3, Section 6.1, pp. 420-421] and the references therein).

Starting from the late eighties, we started a long series of investigations, which are also related to these theorems, but we developed a different point of view.

As it is well-known, each $C_0-$ semigroup $(T(t))_{t\geq 0}$ of operators on a Banach space $E$ gives rise, indeed, to its infinitesimal generator $A : D(A) \to E$, which is defined on a dense subspace $D(A)$ of $E$, and to which it corresponds an abstract Cauchy problem, namely

$$\text{(5.17)} \qquad \begin{cases} \dfrac{du(t)}{dt} = Au(t) & t \geq 0, \\[2ex] u(0) = u_0 & u_0 \in D(A). \end{cases}$$

When $E$ is a "concrete" continuous function space on a domain $X$ of $\mathbf{R}^d$, $d \geq 1$, the operator $A$ mostly is a differential operator and problem (5.17) turns into an initial-boundary value evolution problem

$$\text{(5.18)} \qquad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) = A(u(\cdot,t))(x) & x \in X, \ \ t \geq 0, \\ \\ u(x,0) = u_0(x) & u_0 \in D(A), \ \ x \in X, \end{cases}$$

the boundary conditions being incorporated in the domain $D(A)$.
Moreover, problem (5.17) (resp. problem (5.18)) has a unique solution if and only if $u_0 \in D(A)$ and, in such a case, the solution is given by

$$u(t) = T(t)u_0 \ (t \geq 0)$$

(resp.

$$u(x,t) = T(t)u_0(x) \ \ (x \in X, \ \ t \geq 0))$$

and hence, by using the approximation formula $(2)$ of Theorem 5.6,

$$\text{(5.19)} \qquad u(x,t) = T(t)(u_0)(x) = \lim_{n\to\infty} L_n^{k(n)}(u_0)(x),$$

where the limit is uniform with respect to $x \in X$.
Therefore, if it is possible to determine the operator $A$ and its domain $D(A)$, the initial sequence $(L_n)_{n\geq 1}$ become the key tool to approximate and to study (especially, from a qualitative point of view) the solutions to problems (5.17) or (5.18).
For instance,

- If $H$ is a closed subset of $E$ which is invariant under the operators $L_n$, i.e., $L_n(H) \subset H$ for every $n \geq 1$, then $T(t)(H) \subset H$ for every $t \geq 0$.
  In such a case, this inclusion represents an abstract "spatial regularity results" in the sense that
  $$u(t) \in H \text{ for every } t \geq 0 \text{ whenever } u_0 \in H.$$
  or, respectively,
  $$u(\cdot,t) \in H \text{ for every } t \geq 0 \text{ whenever } u_0 \in H.$$
- In several contexts, the approximation formula of the semigroup in terms of iterates of the operators $L_n$ allows to determine its asymptotic behaviour, i.e., to determine
  $$\lim_{t\to+\infty} T(t)u \text{ in } E \quad (u \in E)$$
  or, respectively,
  $$\lim_{t\to+\infty} u(x,t) \quad (u \in E, x \in K)$$
  (see, for instance, the subsequent Theorems 5.7 and 5.8).
- if $E$ is a Banach lattice and each $L_n$ is positive, then the semigroup $(T(t))_{t\geq 0}$ is positive; hence $u(t) \geq 0$ for every $t \geq 0$ provided that $u_0 \geq 0$.
- if $E$ is a Banach lattice and each $L_n$ is positive, then
  $$u \leq L_n(u) \Longrightarrow u \leq T(s)u \leq T(t)u \ \ (u \in E, 0 \leq s \leq t).$$

Furthermore, when $E = C(X)$, $X$ compact space, or $E = C_0(X)$, $X$ locally compact space, and the operators $L_n, n \geq 1$, are positive, then the positive semigroup governs a right-continuous normal Markov process having $X$ as a state space (see, e.g., [3, Section 1.6, pp. 68 - 73]); hence, by means of the operators $L_n$, it is also possible to investigate some qualitative properties of the transition functions associated with the Markov process.
Particular attention deserves the important case when the approximating operators are constructively generated by a given positive linear operator $T : C(X) \to C(X)$ which, in turn,

allows to determine the differential operator $(A, D(A))$ as well, $X$ being a compact subset of $\mathbf{R}^d$, $d \geq 1$, having non-empty interior.
As a matter of fact, the linear operators generated by such general approach generalize positive approximating operators which are well-known in Approximation Theory, such as Bernstein operators, Kantorovich operators and others ones, and they shed new light on these classical operators and on their usefulness.
Moreover, initial-boundary value evolution problems corresponding to these particular settings, are of current interest as they occur in the study of diffusion problems arising from different areas such as biology, mathematical finance and physics. For more details and for several other aspects related to the above outlined theory, we refer to the monographs [3], [5] and to the recent paper [4].
Below, we show another general property of limit semigroups in the context of $C(X)$ spaces, $X$ compact metric space.
Consider indeed a compact metric space $(X, d)$ and set

$$\delta(X) := \sup\{d(x, y) \mid x, y \in X\}$$

and

$$Lip(X) := \{f \in C(X) \ \mid \ |f|_{Lip} := \underset{\substack{x,y\in X\\ x\neq y}}{sup} \frac{|f(x) - f(y)|}{d(x, y)} < +\infty\}.$$

We also recall that every Markov operator $T$ on $C(X)$ admits at least one invariant probability measure, i.e., a measure $\mu \in M_1^+(X)$ such that

$$\int_X T(f)d\mu = \int_X f d\mu \quad \text{for every} \quad f \in C(X) \tag{5.20}$$

(see, e.g., [20, Section 5.1, p. 178]). Therefore, for every $p \in [1, +\infty[$, from the Hölder inequality it turns out that for every $f \in C(X)$,

$$\int_X |T(f)|^p d\mu \leq \int_X T(|f|^p) d\mu = \int_X |f|^p d\mu$$

and hence $T$ extends to a unique bounded linear operator $T_p : L^p(X, \mu) \to L^p(X, \mu)$ such that $||T_p|| \leq 1$. Furthermore, $T_p$ is positive as $C(X)$ is a sublattice of $L^p(X, \mu)$ and, if $1 \leq p < q < +\infty$, then $T_p = T_q$ on $L^q(X, \mu)$.
From now on, for a given $p \in [1, +\infty[$, if no confusion can arise, we shall denote by $\widetilde{T}$ the operator $T_p$.
In the sequel, given $\mu \in M_1^+(X)$, we shall denote by $\Lambda(\mu)$ the subset of all Markov operators $T$ on $C(X)$ for which $\mu$ is an invariant measure.
The next result can be found in [11], Corollary 2.5.

**Theorem 5.7.** *Consider a $C_0$-semigroup $(T(t))_{t\geq 0}$ of Markov operators and assume that it is the limit semigroup associated with some sequence of Markov operators $(L_n)_{n\geq 1}$. Furthermore, assume that*

(i) *There exists $\omega \in \mathbf{R}, \omega < 0$, such that for every $n \geq 1$, $L_n(Lip(X)) \subset Lip(X)$ and $|L_n(f)|_{Lip} \leq \left(1 + \frac{\omega}{n}\right)|f|_{Lip}$ for every $f \in Lip(X)$.*

(ii) *There exists $\mu \in M_1^+(X)$ such that $L_n \in \Lambda(\mu)$ for every $n \geq 1$.*

*Then*

(1) *For every $n \geq 1$ and $f \in C(X)$*

$$\lim_{m\to\infty} L_n^m(f) = \int_X f d\mu \quad \textit{uniformly on} \quad X. \tag{5.21}$$

(2) *For every* $t \geq 0$, $T(t)(LipX) \subset Lip(X)$ *and* $|T(t)f|_{Lip} \leq exp(\omega t)|f|_{Lip} \quad (f \in Lip(X))$. *Moreover, each* $T(t)$ *is invariant under* $\mu$, *and*

$$\lim_{t \to +\infty} T(t)f = \int_X f d\mu \quad \text{uniformly on} \quad X \tag{5.22}$$

*for every* $f \in C(X)$.

(3) *If* $1 \leq p < +\infty$, *denoting by* $(\widetilde{L}_n)_{n \geq 1}$ *and* $(\widetilde{T}(t))_{t \geq 0}$ *the relevant extensions to* $L^p(X, \mu)$, *then* $(\widetilde{T}(t))_{t \geq 0}$ *is a* $C_0$-*semigroup on* $L^p(X, \mu)$ *and for every* $n \geq 1$ *and* $f \in L^p(X, \mu)$,

$$\lim_{m \to \infty} \widetilde{L}_n^m(f) = \int_X f d\mu = \lim_{t \to +\infty} \widetilde{T}(t)f \quad \text{in} \quad L^p(X, \mu). \tag{5.23}$$

**Remark 5.2.** *Note that the property of the theorem above which states that each* $T(t)$ *is invariant under* $\mu$, *means that*

$$\int_X u(x,t) d\mu(x) = \int_X u_0(x) d\mu(x) \quad \text{for every} \quad u_0 \in D(A) \text{ and } t \geq 0,$$

$u(x,t)$ *being the solution to the initial-boundary value problem governed by the semigroup* $(T(t))_{t \geq 0}$.

The next result is also useful to investigate the limit behaviour of the $C_0$-semigroups. The proof depends on Corollary 3.2 (when $S = T$)(for more details see [2, Theorem 2.9] and [23]).

**Theorem 5.8.** *Let* $X$ *be a compact metric space and consider a Markov projection* $T : C(X) \to C(X)$ *such that its range* $H := T(C(X))$ *separates the points of* $X$. *Further, consider* $\Phi \in C(X)$ *of the form* $\Phi = \sum_{n=1}^{\infty} h_n^2$ *as in Theorem 2.1 (with each* $h_n \in H$*), so that* $\Phi \leq T(\Phi)$ *and* $\partial_H X = \{x \in X \mid T(\Phi)(x) = \Phi(x)\}$.
*Let* $(L_n)_{n \geq 1}$ *be a sequence of Markov operators on* $C(X)$ *such that* $L_n(h) = h$ *for every* $h \in H$ *and* $n \geq 1$ *and set*

$$a_{n,p} := \max_{x \in X} \{T(\Phi)(x) - \Phi(x) - pn(L_n(\Phi)(x) - \Phi(x))\} \tag{5.24}$$

$(n \geq 1, p \geq 1)$.
*Finally, assume that the sequence* $(L_n)_{n \geq 1}$ *generates a limit* $C_0$-*semigroup* $(T(t))_{t \geq 0}$.
*If* $\lim_{p \to \infty} a_{n,p} = 0$ *uniformly with respect to* $n \geq 1$, *then*

$$\lim_{t \to +\infty} T(t)(f) = T(f) \tag{5.25}$$

*uniformly on* $X$ *for every* $f \in C(X)$.

## 6. An application: Bernstein-Durrmeyer operators with Jacobi weights on the unit hypercube

In this section we discuss some applications of the previous results concerning a sequence of Markov linear operators acting on weighted $L^p$-spaces on the unit hypercube $Q_d := [0,1]^d$, $d \geq 1$, which have been recently introduced and studied in [4].
These operators map weighted $L^p$-functions into polynomials on $Q_d$ and generalize the Bernstein-Durrmeyer operators with Jacobi weights on $[0,1]$ ([13], [22]).
Although we are mainly interested in the role which they play in the approximation of the corresponding limit semigroups, it seems that they also have an interest on their own as an approximation sequence for continuous functions as well as for weighted $L^p$-functions.

We begin by fixing some additional notation. Let $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbf{R}^d$, $d \geq 1$. If $x = (x_1, \dots, x_d) \in \mathbf{R}^d$, with $x_i > 0$ for every $i = 1, \dots, d$, we set

$$x^\gamma := \prod_{i=1}^{d} x_i^{\gamma_i}.$$

For $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d) \in \mathbf{R}^d$, we write $x \leq y$ if $x_i \leq y_i$ for every $i = 1, \dots, d$.
Let $j = (j_1, \dots, j_d)$, $k = (k_1, \dots, k_d) \in \mathbf{N}^d$ be two multi-indices such that $k \leq j$; we set

$$\binom{j}{k} := \prod_{i=1}^{d} \binom{j_i}{k_i}.$$

From now on we fix $a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_d) \in \mathbf{R}^d$ with $a_i > -1$ and $b_i > -1$ for all $i = 1, \dots, d$.
Let us denote by $\mu_{a,b} \in M_1^+(Q_d)$ the absolutely continuous measure with respect to the Borel-Lebesgue measure $\lambda_d$ on $Q_d$ with density the normalized Jacobi weight

$$w_{a,b}(x) := \frac{x^a(1-x)^b}{\int_{Q_d} y^a (1-y)^b \, dy} \qquad (x \in \overset{\circ}{Q}_d = ]0,1[^d).$$

Moreover, for every $n \geq 1$, consider the operator $M_n : L^1(Q_d, \mu_{a,b}) \to C(Q_d)$ defined by setting, for every $f \in L^1(Q_d, \mu_{a,b})$ and $x \in Q_d$,

$$M_n(f)(x) := \sum_{\substack{h \in \mathbf{N}^d \\ 0_d \leq h \leq n_d}} \omega_{n_d,h}(f) \binom{n_d}{h} x^h (1_d - x)^{n_d - h}, \tag{6.26}$$

where, for every $n \geq 1$ and $h = (h_1, \dots h_d) \in \mathbf{N}^d$, $0_d \leq h \leq n_d$,

$$\begin{aligned}
\omega_{n_d,h}(f) &:= \frac{1}{\int_{Q_d} y^{h+a}(1_d - y)^{n_d - h + b} \, dy} \int_{Q_d} y^h (1_d - y)^{n_d - h} f(y) \, d\mu_{a,b}(y) \\
&= \prod_{i=1}^{d} \frac{\Gamma(n + a_i + b_i + 2)}{\Gamma(h_i + a_i + 1)\Gamma(n - h_i + b_i + 1)} \int_{Q_d} y^{h+a}(1_d - y)^{n_d - h + b} f(y) \, dy,
\end{aligned}$$

$\Gamma(u)$ $(u \geq 0)$ being the classical Euler Gamma function.
Clearly, the restriction of each $M_n$ to $C(Q_d)$ is a Markov operator on $C(Q_d)$.
As we shall see, these operators are closely related to the degenerate second-order elliptic differential operator defined by

$$A(u)(x) = \sum_{i=1}^{d} x_i(1 - x_i) \frac{\partial^2 u}{\partial x_i^2}(x) + (a_i + 1 - (a_i + b_i + 2)x_i) \frac{\partial u}{\partial x_i}(x) \tag{6.27}$$

for every $u \in C^2(Q_d)$ and $x = (x_1, \dots, x_d) \in Q_d$.
Operators similar to (6.27) arise in the theory of Fleming-Viot processes applied to some models of population dynamics which, however, usually take places in the framework of $d$-dimensional simplices.
Due to their intrinsic interest, more recently an increasing attention has been turned to them also in the setting of hypercubes as well. The difficulties in studying them lie in the fact that they degenerate on the boundary of $Q_d$, which is not smooth because of the presence of sides and corners.

The next result shows that operator (6.27) is the pregenerator of a Markov semigroup on $C(Q_d)$ and of a positive contraction semigroup in $L^p(Q_d, \mu_{a,b})$; moreover, both these semigroups are the limit semigroups associated with the operators $M_n$.

**Theorem 6.9.** *The differential operator $(A, C^2(Q_d))$ defined by (6.27) is closable and the sequence $(M_n)_{n\geq 1}$ is a Schnabl-type admissible sequence (with $D_0 = C^2(Q_d)$), whose limit semigroup is generated by the closure $(B, D(B))$ of $(A, C^2(Q_d))$.*
*Moreover, $\mathbb{P}_\infty := \bigcup_{m=1}^{\infty} \mathbb{P}_m$, and hence $C^2(Q_d)$, is a core for $(B, D(B))$ and $T(t)(\mathbb{P}_m) \subset \mathbb{P}_m$ for every $t \geq 0$ and $m \geq 1$, where $\mathbb{P}_m$ denotes the linear subspace generated by the polynomials on $Q_d$ of degree $\leq m$.*
*Considering the measure $\mu_{a,b} \in M_1^+(Q_d)$ with density the normalized Jacobi weight $w_{a,b}(x)$ $(x \in \overset{\circ}{Q}_d)$, then, for every $f \in C(Q_d)$ and $n \geq 1$,*

$$\lim_{t\to+\infty} T(t)(f) = \lim_{m\to\infty} M_n^m(f) = \int_{Q_d} f d\mu_{a,b}$$

*uniformly on $Q_d$, and the measure $\mu_{a,b}$ is the unique invariant measure on $Q_d$ for both the sequence $M_{n\geq 1}$ and the semigroup $(T(t))_{t\geq 0}$.*
*Finally, if $f \in Lip(Q_d)$, then, for every $n, m \geq 1$ and $t \geq 0$,*

$$\left\|M_n^m(f) - \int_{Q_d} f d\mu_{a,b}\right\|_\infty \leq 2\left(1 + \frac{\omega}{n}\right)^m |f|_{Lip}$$

*and*

$$\left\|T(t)(f) - \int_{Q_d} f d\mu_{a,b}\right\|_\infty \leq 2\exp(\omega t)|f|_{Lip},$$

*where*

$$\omega := -\min_{1\leq i\leq d} \frac{a_i + b_i + 2}{a_i + b_i + 3} < 0. \tag{6.28}$$

The above Markov semigroup can be extended, indeed, to each $L^p(Q_d, \mu_{a,b})$ space $(1 \leq p < +\infty)$ as the next result shows.

**Theorem 6.10.** *For every $1 \leq p < +\infty$, the semigroup $(T(t))_{t\geq 0}$ on $C(Q_d)$ extends to a unique positive contraction semigroup $(T_p(t))_{t\geq 0}$ on $L^p(Q_d, \mu_{a,b})$, whose generator is an extension of $(B, D(B))$ to $L^p(Q_d, \mu_{a,b})$ and $\mathbb{P}_\infty$ is a core for it. Moreover, if $f \in L^p(Q_d, \mu_{a,b})$ and $(k(n))_{n\geq 1}$ is an arbitrary sequence of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, then*

$$T_p(t)(f) = \lim_{n\to\infty} M_n^{k(n)}(f) \quad \text{in } L^p(Q_d, \mu_{a,b}).$$

*Finally, if $f \in L^p(Q_d, \mu_{a,b})$ and $n \geq 1$,*

$$\lim_{t\to+\infty} T_p(t)(f) = \lim_{m\to\infty} M_n^m(f) = \int_{Q_d} f d\mu_{a,b}$$

*in $L^p(Q_d, \mu_{a,b})$.*

Below we discuss some consequences of Theorem 6.9 (for additional details and proofs we refer to [4]).
Consider the abstract Cauchy problem associated with the closure $(B, D(B))$ of $(A, C^2(Q_d))$

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = B(u(\cdot,t))(x) & x \in Q_d, \; t \geq 0, \\[2ex] u(x,0) = u_0(x) & u_0 \in D(B), \; x \in Q_d. \end{cases} \tag{6.29}$$

Since $(B, D(B))$ generates a Markov semigroup $(T(t))_{t\geq 0}$, problem (6.29) admits a unique solution $u : Q_d \times [0, +\infty[ \longrightarrow \mathbf{R}$ given by $u(x,t) = T(t)(u_0)(x)$ for every $x \in Q_d$ and $t \geq 0$. Hence, taking Theorem 6.9 into account, we may approximate such a solution in terms of iterates of the $M_n$'s, namely

(6.30) $$u(x,t) = T(t)(u_0)(x) = \lim_{n\to\infty} M_n^{k(n)}(u_0)(x),$$

where $(k(n))_{n\geq 1}$ is an arbitrary sequence of positive integers satisfying $\lim\limits_{n\to\infty} k(n)/n = t$, and the limit is uniform with respect to $x \in Q_d$.
Note that $B$ coincides with $A$ on $C^2(Q_d)$; therefore, if $u_0 \in \mathbb{P}_m$ $(m \geq 1)$, then $u(x,t)$ is the unique solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \sum\limits_{i=1}^{d} x_i(1-x_i)\dfrac{\partial^2 u}{\partial x_i^2}(x,t) + (a_i+1-(a_i+b_i+2)x_i)\dfrac{\partial u}{\partial x_i}(x,t) & x \in Q_d, \\ & t \geq 0, \\ u(x,0) = u_0(x) & x \in Q_d, \end{cases}$$

and

(6.31) $$u(\cdot,t) \in \mathbb{P}_m \quad \text{for every } t \geq 0.$$

Moreover, each $u(\cdot,t), t \geq 0$, has the same integral of $u_0$ with respect to the measure $\mu_{a,b}$ and, thanks to formula (6.9),

(6.32) $$\lim_{t\to+\infty} u(x,t) = \int_{Q_d} u_0 \, d\mu_{a,b}$$

uniformly w.r.t. $x \in Q_d$.
Next, we enlighten other spatial regularity properties of the solution $u(\cdot,t)$ to (6.29), which, however, we state in terms of the semigroup $(T(t))_{t\geq 0}$.
We set

(6.33) $$Lip(Q_d) := \left\{ f \in C(Q_d) \;\middle|\; |f|_{Lip} := \sup_{\substack{x,y\in Q_d \\ x\neq y}} \frac{|f(x)-f(y)|}{\|x-y\|_1} < +\infty \right\}$$

and, for $M > 0$,

(6.34) $$Lip(M,1) := \{ f \in Lip(Q_d) \mid |f(x)-f(y)| \leq M\|x-y\|_1 \},$$

where $\|\cdot\|_1$ is the norm on $\mathbf{R}^d$ defined by $\|x\|_1 := \sum_{i=1}^{d} |x_i|$, for every $x = (x_1,\ldots,x_d) \in \mathbf{R}^d$.
More generally, given $0 < \alpha \leq 1$, we shall denote by $Lip(M,\alpha)$ the subset of all Hölder continuous functions $f$ on $Q_d$ with exponent $\alpha$ and constant $M$, i.e.,

(6.35) $$|f(x)-f(y)| \leq M\|x-y\|_1^{\alpha} \quad \text{for every } x,y \in Q_d.$$

**Theorem 6.11.** *Let $\omega$ be the constant defined by (6.28).The following statements hold true:*

*(a) $T(t)(Lip(Q_d)) \subset Lip(Q_d)$ for every $t \geq 0$; moreover, for every $f \in Lip(Q_d)$ and $t \geq 0$,*

(6.36) $$|T(t)(f)|_{Lip} \leq \exp(\omega t)|f|_{Lip};$$

*in particular, if $f \in Lip(M,1)$, then, for every $t \geq 0$,*

$$T(t)(f) \in Lip(M\exp(\omega t), 1).$$

(b) *For every $f \in C(Q_d)$, $t \geq 0$, $\delta > 0$,*

$$\Omega(T(t)(f),\delta) \leq (1+\exp(\omega t))\Omega(f,\delta). \tag{6.37}$$

*where $\Omega(g,\delta) := \sup\{|g(x)-g(y)| \mid x,y \in Q_d,\ \|x-y\|_1 \leq \delta\}$ denotes the usual modulus of continuity ($g : Q_d \to \mathbf{R}$ bounded function and $\delta > 0$).*

*Moreover, if $M > 0$ and $0 < \alpha \leq 1$,*

$$T(t)(Lip(M,\alpha)) \subset Lip(M\exp(\alpha\omega t),\alpha) \subset Lip(M,\alpha) \tag{6.38}$$

*(see (6.35)).*

(c) *If $f \in C(Q_d)$ is convex with respect to each variable, then so is $T(t)(f)$ for every $t \geq 0$. In particular, if $d = 1$ and if $f \in C([0,1])$ is convex, then $T(t)(f)$ is convex for every $t \geq 0$.*

## 7. An invitation for further researches

The above outlined theory shows that, generally speaking, some sequences of positive linear operators can be fruitfully used not only as approximation processes in various function spaces, but also to approximate and to infer qualitative properties of solutions to initial-boundary value problems.

On the light of this second aspect, it seems to be not devoid of interest to look at a kind of inverse problem, namely, given a (degenerate) second-order elliptic differential operator

$$A_0(u)(x) = \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} \beta_i(x)\frac{\partial u}{\partial x_i}(x)$$

($u \in C^2(X), \alpha_{ij}, \beta_i \in C(X), x \in X$) with $X$ subset of $\mathbf{R}^d$, $d \geq 1$, having non-empty interior, try to investigate whether it is possible to construct a suitable Trotter-Schnabl type approximation sequence in a suitable Banach function space such that the closure of $(A_0, C^2(X))$ (if it exists) is the relevant generator.

In order to reach this purpose, it might be fully justified to modify well-know classical approximation processes as well as to try to generalize them introducing more general methods of construction such as, for instance, those which arise from a Markov operator ([3], [5], [6], [7]), even though such last methods seem to have an own intrinsic theoretical interest.

Toward the above mentioned direction several researches have been already developed both in one dimensional contexts (bounded and unbounded interval) and in multidimensional contexts, but it seems that there are still new and interesting cases worthy to be investigated. For more details we refer, e.g., to [8], [9] and the references therein.

## References

[1] F. Altomare, *Korovkin-type Theorems and Approximation by Positive Linear Operators*, Surveys in Approximation Theory, Vol. 5, 2010, 92-164, free available on line at http://www.math.technion.ac.il/sat/papers/13/, ISSN 1555-578X.

[2] F. Altomare, *On some convergence criteria for nets of positive operators on continuous function spaces*, J. Math. Anal. Appl. **398** (2013) 542 - 552.

[3] F. Altomare and M. Campiti, Korovkin-Type Approximation Theory and its Applications, *de Gruyter Studies in Mathematics* **17**, *Walter de Gruyter, Berlin-New York*, 1994.

[4] F. Altomare, M. Cappelletti Montano, V. Leonessa, *On the positive semigroups generated by Fleming-Viot type differential operators on hypercubes*, Comm. Pure and Appl. Anal., Volume 18, Number 1, January 2019, pp. 323 - 340.

[5] F. Altomare, M. Cappelletti Montano, V. Leonessa and I. Raşa, Markov Operators, Positive Semigroups and Approximation Processes, *de Gruyter Studies in Mathematics* **61**, *Walter de Gruyter GmbH, Berlin/Boston*, 2014.

[6] F. Altomare, M. Cappelletti Montano, V. Leonessa and I. Raşa, *A generalization of Kantorovich operators for convex compact subsets*, Banach J. Math. Anal. **11**(2017), no. 3, 591 - 614.

[7] F. Altomare, M. Cappelletti Montano, V. Leonessa and I. Raşa, *Elliptic differential operators and positive semigroups associated with generalized Kantorovich operators*, J. Math. Anal. Appl. **458**(2018), 153 - 173.

[8] F. Altomare and V. Leonessa, *An invitation to the study of evolution equations by means of positive linear operators*, Lecture Notes of Seminario Interdisciplinare di Matematica, Volume VIII, 1-41, Lect. Notes Semin. Interdiscip. Mat., **8**, Semin. Interdiscip. Mat., Potenza 2009.

[9] F. Altomare, V. Leonessa and S. Milella, *Bernstein-Schnabl operators on noncompact real intervals*, Jaen J. Approx., **1**(2) (2009), 223-256.

[10] F. Altomare, V. Leonessa and I. Raşa, *On Bernstein-Schnabl operators on the unit interval*, Z. Anal. Anwend. **27**(2008), no. 3, 353 - 379.

[11] F. Altomare and I. Raşa, *Lipschitz contractions, unique ergodicity and asymptotics of Markov semigroups*,Bollettino U.M.I. (9) **V**(2012),1-17.

[12] H. Bauer, Measure and Integration Theory, *de Gruyter Studies in Mathematics* **26**, *Walter de Gruyter GmbH, Berlin/Boston*, 2011.

[13] H. Berens and Y. Xu, *On Bernstein-Durrmeyer polynomials with Jacobi weights*, in: C. K. Chui (Ed.), Approximation Theory and Functional Analysis, Academic Press, Boston, 1991, 25-46.

[14] P. L. Butzer and H. Berens, Semi-groups of Operators and Approximation, Springer-Verlag, New York, 1967.

[15] I. Gavrea and M. Ivan, *On the iterates of positive linear operators preserving the affine functions,* J. Math. Anal. Appl.**372**(2010),366-368.

[16] I. Gavrea and M. Ivan, *Asymptotic behaviour of the iterates of positive linear operators,* Abstr. Appl. Anal. **2011**, Art. ID 670509, 11 pp.

[17] I. Gavrea and M. Ivan, *On the iterates of positive linear operators,* J. Approx. Theory **163** (2011), 1076-1079.

[18] A. Guessab and G. Schmeisser, *Two Korovkin-type theorems in multivariate approximation,* Banach J. Math. Anal. **2** (2008), no. 2,121-128.

[19] W. Heping, Korovkin-type theorem and application, *J. Approx. Theory*, **132** (2005), no. 2, 258-264.

[20] U. Krengel, *Ergodic Theorems*, de Gruyter Studies in Mathematics **6**, W. de Gruyter, Berlin, New York, 1985.

[21] C. A.Micchelli, *The saturation class and iterates of the Bernstein polynomials,* J. Approx. Theory **8** (1973), 1-18.

[22] R. Paltanea, *Sur un opérateur polynomial défini sur l'ensemble des fonctions intégrables*, Univ. Babeş-Bolyai, Cluj-Napoca, 83-2 (1983), 101-106.

[23] I. Raşa, *Asymptotic behaviour of iterates of positive linear operators,* Jaen J. Approx. **1** (2) (2009), 195-204.

[24] R. Schnabl, *Zum globalen Saturationsproblem der Folge der Bernsteinoperatoren,* Acta Sci. Math. (Szeged) **31** (1970), 351-358.

[25] H. F. Trotter, *Approximation of semigroups of operators,* Pacific J. Math. **8** (1958), 887-919.

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# Some Prešić Type Results in $b-$Dislocated Metric Spaces

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ABSTRACT. In this paper, we obtain a Prešić type common fixed point theorem for four maps in $b$-dislocated metric spaces. We also present one example to illustrate our main theorem. Further, we obtain two more corollaries.

**Keywords:** $b$ - Dislocated metric spaces, Jointly $2k$ - weakly compatible pairs, Prešić type theorem.

**2010 Mathematics Subject Classification:** 54H25, 47H10.

## 1. INTRODUCTION AND PRELIMINARIES

There are several generalizations of the Banach contraction principle in literature on fixed point theory. Recently, very interesting results regarding fixed point are presented in the papers ([3, 4, 5, 7]. One of the generalization is a famous Prešić type fixed point theorem. There are a lot of generalizations of mentioned theorem (more on this topic see [1]-[2], [7]-[15]). Hitzler and Seda [6] introduced the concept of dislocated metric spaces (metric like spaces in [5], [15]) and established a fixed point theorem in complete dislocated metric spaces to generalize the celebrated Banach contraction principle. Recently Hussain et al. [7] introduced the definition of $b$ - dislocated metric spaces to generalize the dislocated metric spaces introduced by [6] and proved two common fixed point theorems for four self mappings.

In this paper we have proved Prešić type common fixed point theorem for four mappings in $b$ - dislocated metric spaces. One numerical example is also presented to illustrate our main theorem. We also obtained two corollaries for three and two maps in $b$ - dislocated metric spaces.

Now we give some known definitions, lemmas and theorems which are needful for further discussion. Throughout this paper, $N$ denotes the set of all positive integers.

Prešić [10] generalized the Banach contraction principle as follows.

**Theorem 1.1.** [10] *Let* $(X, d)$ *be a complete metric space,* $k$ *be a positive integer and* $T : X^k \to X$ *be a mapping satisfying*

$$d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \le \sum_{i=1}^{k} q_i d(x_i, x_{i+1}), \tag{1.1}$$

*for all* $x_1, x_2, \ldots, x_{k+1} \in X$*, where* $q_i \ge 0$ *and* $\sum_{i=1}^{k} q_i < 1$*. Then there exists a unique point* $x \in X$ *such that* $T(x, x, ...., x) = x$*. Moreover, if* $x_1, x_2, \ldots, x_k$ *are arbitrary points in* $X$ *and for*

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$n \in \mathbb{N}$, $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, *then the sequence* $\{x_n\}$ *is convergent and* $\lim_{n\to\infty} x_n = T(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x_n, \ldots, \lim_{n\to\infty} x_n)$.

Inspired by the Theorem 1.1, Ćirić and Prešić [8] proved the following theorem.

**Theorem 1.2.** [8] *Let* $(X, d)$ *be a complete metric space,* $k$ *a positive integer and* $T : X^k \to X$ *be a mapping satisfying*

$$d(T(x_1, x_2, \cdots, x_k), T(x_2, x_3, \cdots, x_{k+1})) \leq \lambda \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\} \tag{1.2}$$

*for all* $x_1, x_2, \cdots, x_k, x_{k+1}$ *in* $X$*, and* $\lambda \in (0, 1)$. *Then there exists a point* $x \in X$ *such that* $x = T(x, x, ...., x)$.

*Moreover, if* $x_1, x_2, \ldots, x_k$ *are arbitrary points in* $X$ *and for* $n \in \mathbb{N}$, $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, *then the sequence* $\{x_n\}$ *is convergent and* $\lim_{n\to\infty} x_n = T(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x_n, \ldots, \lim_{n\to\infty} x_n)$. *If in addition, we suppose that on diagonal* $\Delta \subset X^k$, $d(T(u, u, ..., u), T(v, v, ..., v)) < d(u, v)$ *holds for* $u, v \in X$ *with* $u \neq v$*, then* $x$ *is the unique fixed point satisfying* $x = T(x, x, ..., x)$.

Later Rao et al. [11, 12] obtained some Presić fixed point theorems for two and three maps in metric spaces.

**Definition 1.1.** *Let* $X$ *be a nonempty set,* $k$ *a positive integer and* $T : X^{2k} \to X$ *and* $f : X \to X$. *The pair* $(f, T)$ *is said to be* $2k$*-weakly compatible if* $f(T(x, x, ..., x)) = T(fx, fx, ..., fx)$ *whenever there exists* $x \in X$ *such that* $fx = T(x, x, ..., x)$

Actully Rao et al. [11] obtained the following.

**Theorem 1.3.** *Let* $(X, d)$ *be a metric space and* $k$ *be any positive integer. Let* $S, T : X^{2k} \longrightarrow X$ *and* $f : X \longrightarrow X$ *be mappings satisfying*

(1) $d(S(x_1, x_2, ..., x_{2k}), T(x_2, x_3, ..., x_{2k+1})) \leq \lambda \max\{d(fx_i, fx_{i+1}) : 1 \leq i \leq 2k\}$
*for all* $x_1, x_2, ..., x_{2k}, x_{2k+1} \in X$*, where* $\lambda \in (0, 1)$.

(2) $d(S(u, u, ..., u), T(v, v, ..., v)) < d(fu, fv)$ *for all* $u, v \in X$ *with* $u \neq v$

(3) *Suppose that* $f(X)$ *is complete and either* $(f, S)$ *or* $(f, T)$ *is* $2k$*-weakly compatible pair.*

*Then there exists a unique point* $p \in X$ *such that* $p = fp = S(p, p, .., p, p) = T(p, p, .., p, p)$.

Hussain et al. [7] introduced $b$-dislocated metric spaces as follows.

**Definition 1.2.** *Let* $X$ *be a non empty set. A mapping* $b_d : X \times X \to [0, \infty)$ *is called a* $b$ *- dislocated metric (or simply* $b_d$*-metric) if the following conditions hold for any* $x, y, z \in X$ *and* $s \geq 1$:

$(b_{d1})$ : *If* $b_d(x, y) = 0$ *then* $x = y$,
$(b_{d2})$ : $b_d(x, y) = b_d(y, x)$,
$(b_{d3})$ : $b_d(x, y) \leq s[b_d(x, z) + b_d(z, y)]$.

*The pair* $(X, b_d)$ *is called a* $b$*-dislocated metric space or* $b_d$*-metric space.*

**Definition 1.3.** [7]

$(i)$ *A sequence* $\{x_n\}$ *in* $b$*-dislocated metric space* $(X, b_d)$ *converges with respect to* $b_d$ *if there exists* $x \in X$ *such that* $b_d(x_n, x)$ *converges to 0 as* $n \to \infty$. *In this case,* $x$ *is called the limit of* $\{x_n\}$ *and we write* $x_n \to x$.

(ii) *A sequence* $\{x_n\}$ *in a b-dislocated metric space* $(X, b_d)$ *is called a* $b_d$ *- Cauchy sequence if given* $\varepsilon > 0$, *there exists* $n_0 \in N$ *such that* $b_d(x_m, x_n) < \varepsilon$ *for all* $n, m \geq n_0$ *or* $\lim\limits_{n,m\to\infty} b_d(x_m, x_n) = 0$.

(iii) *A b-dislocated metric* $(X, b_d)$ *is called* $b_d-$*complete if every* $b_d$*-Cauchy sequence in* $X$ *is* $b_d$ *-convergent.*

**Lemma 1.1.** [7] *Let* $(X, b_d)$ *be a b-dislocated metric space with* $s \geq 1$. *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are* $b_d$*-convergent to* $x, y$ *respectively. Then we have*

$$\frac{1}{s^2}\, b_d(x,y) \leq \lim_{n\to\infty} inf\ b_d(x_n, y_n) \leq \lim_{n\to\infty} sup\ b_d(x_n, y_n) \leq s^2\, b_d(x,y),$$

*and*

$$\frac{1}{s}\, b_d(x,z) \leq \lim_{n\to\infty} inf\ b_d(x_n, z) \leq \lim_{n\to\infty} sup\ b_d(x_n, z) \leq s\, b_d(x,z)$$

*for all* $z \in X$.

## 2. Main Result

We introduce the definition of jointly $2k-$weakly compatible pairs as follows.

**Definition 2.4.** *Let* $X$ *be a nonempty set,* $k$ *a positive integer and* $S, T : X^{2k} \to X$ *and* $f, g : X \to X$. *The pairs* $(f, S)$ *and* $(g, T)$ *are said to be jointly* $2k$*-weakly compatible if*

$$f(S(x, x, ..., x)) = S(fx, fx, ..., fx)$$

*and*

$$g(T(x, x, ..., x)) = T(gx, gx, ..., gx)$$

*whenever there exists* $x \in X$ *such that* $fx = S(x, x, ..., x)$ *and* $gx = T(x, x, ..., x)$.

Now we give our main result. The contractive condition in the next theorem is similar with conditions in [2, 7, 10, 13].

**Theorem 2.4.** *Let* $(X, b_d)$ *be a* $b_d-$*complete b-dislocated metric space with* $s \geq 1$ *and* $k$ *be any positive integer. Let* $S, T : X^{2k} \longrightarrow X$ *and* $f, g : X \longrightarrow X$ *be mappings satisfying*

(2.3)
$$S(X^{2k}) \subseteq g(X), T(X^{2k}) \subseteq f(X),$$

(2.4)
$$b_d(S(x_1, x_2, ..., x_{2k}), T(y_1, y_2, ..., y_{2k})) \leq \lambda \max \left\{ \begin{array}{c} b_d(gx_1, fy_1), b_d(fx_2, gy_2), \\ b_d(gx_3, fy_3), b_d(fx_4, gy_4), \\ \cdots\cdots\cdots\cdots \\ b_d(gx_{2k-1}, fy_{2k-1}), b_d(fx_{2k}, gy_{2k}) \end{array} \right\}$$

*for all* $x_1, x_2, ..., x_{2k}, y_1, y_2, .., y_{2k} \in X$,*where* $\lambda \in (0, \frac{1}{s^{2k}})$.

(2.5) $(f, S)$ *and* $(g, T)$ *are jointly* $2k-$ *weakly compatible pairs,*

(2.6) *Assume that there exists* $u \in X$ *such that* $fu = gu$ *whenever there is sequence* $\{y_{2k+n}\}_{n=1}^{\infty} \in X$ *with* $lim \lim\limits_{n\to\infty} y_{2k+n} = fu = gu = z \in X$.

*Then $z$ is the unique point in $X$ such that $z = fz = gz = S(z, z, .., z, z) = T(z, z, ..., z, z)$.*

*Proof.* Suppose $x_1, x_2, ..., x_{2k}$ are arbitrary points in $X$, From (2.3), we can define

$$y_{2k+2n-1} = S(x_{2n-1}, x_{2n}, ..., x_{2k+2n-2}) = gx_{2k+2n-1},$$

and

$$y_{2k+2n} = T(x_{2n}, x_{2n+1}, ..., x_{2k+2n-1}) = fx_{2k+2n}, \quad n = 1, 2, \ldots$$

Let

$$\alpha_{2n} = b_d(fx_{2n}, gx_{2n+1}),$$

and

$$\alpha_{2n-1} = b_d(gx_{2n-1}, fx_{2n}) \;\; n = 1, 2, \ldots$$

Write $\theta = \lambda^{\frac{1}{2k}}$ and $\mu = \max\{\frac{\alpha_1}{\theta}, \frac{\alpha_2}{(\theta)^2}, ...., \frac{\alpha_{2k}}{(\theta)^{2k}}\}$.
Then $0 < \theta < 1$ and by the selection of $\mu$, we have

$$\alpha_n \leq \mu \cdot (\theta)^n, \;\; n = 1, 2, \ldots, 2k \tag{2.7}$$

Consider

$$\begin{aligned}
\alpha_{2k+1} = b_d(gx_{2k+1}, fx_{2k+2}) \;\; &= \;\; b_d(S(x_1, x_2, ..., x_{2k-1}, x_{2k}), T(x_2, x_3, ..., x_{2k}, x_{2k+1})) \\
&\leq \;\; \lambda \max\left\{\begin{array}{c} b_d(gx_1, fx_2), b_d(fx_2, gx_3), \\ b_d(gx_3, fx_4), b_d(fx_4, gx_5), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b_d(gx_{2k-1}, fx_{2k}), b_d(fx_{2k}, gx_{2k+1}) \end{array}\right\} \\
&\leq \;\; \lambda \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, ...., \alpha_{2k-1}, \alpha_{2k}\} \\
&\leq \;\; \lambda \max\{\mu \cdot \theta, \mu \cdot (\theta)^2, ..., \mu \cdot (\theta)^{2k}\}, \\
&= \;\; \lambda\mu \cdot \theta = \mu \cdot \theta \cdot (\theta)^{2k} = \mu \cdot (\theta)^{2k+1}.
\end{aligned} \tag{2.8}$$

using (2.7),
and

$$\alpha_{2k+2} = b_d(fx_{2k+2}, gx_{2k+3}) \tag{2.9}$$

$$\begin{aligned}
&= \;\; b_d(T(x_2, x_3, ..., x_{2k}, x_{2k+1}), S(x_3, x_4, ..., x_{2k+1}, x_{2k+2})) \\
&= \;\; b_d(S(x_3, x_4, ..., x_{2k+1}, x_{2k+2}), T(x_2, x_3, ..., x_{2k}, x_{2k+1})) \\
&\leq \;\; \lambda \max\left\{\begin{array}{c} b_d(gx_3, fx_2), b_d(fx_4, gx_3), \\ b_d(gx_5, fx_4), b_d(fx_6, gx_5), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b_d(gx_{2k+1}, fx_{2k}), b_d(fx_{2k+2}, gx_{2k+1}) \end{array}\right\} \\
&\leq \;\; \lambda \max\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, ...., \alpha_{2k}, \alpha_{2k+1}\} \\
&\leq \;\; \lambda \max\{\mu \cdot (\theta)^2, \mu \cdot (\theta)^3, ..., \mu \cdot (\theta)^{2k}, \mu \cdot (\theta)^{2k+1}\}, \\
&= \;\; \lambda\mu \cdot (\theta)^2 = \mu \cdot (\theta)^2(\theta)^{2k} = \mu \cdot (\theta)^{2k+2},
\end{aligned}$$

using (2.7) and (2.8).
Continuing in this way, we get

$$\alpha_n \leq \mu \cdot (\theta)^n, \tag{2.10}$$

for $n = 1, 2, \ldots$

Consider now

$$
\begin{aligned}
(2.11)\quad b_d(y_{2k+2n-1}, y_{2k+2n}) &= b_d(S(x_{2n-1}, x_{2n}, ..., , x_{2k+2n-2}), T(x_{2n}, x_{2n+1}, ..., x_{2k+2n-1})) \\
&\le \lambda \max \left\{ \begin{array}{c} b_d(gx_{2n-1}, fx_{2n}), b_d(fx_{2n}, gx_{2n+1}), \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ b_d(gx_{2k+2n-3}, fx_{2k+2n-2}), \\ b_d(fx_{2k+2n-2}, gx_{2k+2n-1}) \end{array} \right\} \\
&\le \lambda \max\{\alpha_{2n-1}, \alpha_{2n}, ...., \alpha_{2k+2n-3}, \alpha_{2k+2n-2}\} \\
&\le \lambda \max\{\mu \cdot (\theta)^{2n-1}, \mu \cdot (\theta)^{2n}, ..., \mu \cdot (\theta)^{2k+2n-3}, \mu \cdot (\theta)^{2k+2n-2}\}, \\
&= \lambda\mu \cdot (\theta)^{2n-1} = \mu \cdot (\theta)^{2k} (\theta)^{2n-1} = \mu \cdot (\theta)^{2k+2n-1}
\end{aligned}
$$

Also

$$
\begin{aligned}
(2.12)\quad b_d(y_{2k+2n}, y_{2k+2n+1}) &= b_d(T(x_{2n}, x_{2n+1}, ..., x_{2k+2n-1}), S(x_{2n+1}, x_{2n+2}, \ldots, x_{2k+2n})) \\
&= b_d(S(x_{2n+1}, x_{2n+2}, \ldots, x_{2k+2n}), T(x_{2n}, x_{2n+1}, \ldots, x_{2k+2n-1})) \\
&\le \lambda \max \left\{ \begin{array}{c} b_d(gx_{2n+1}, fx_{2n}), b_d(fx_{2n+2}, gx_{2n+1}), \\ b_d(gx_{2n+3}, fx_{2n+2}), b_d(fx_{2n+4}, gx_{2n+3}), \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ b_d(gx_{2k+2n-1}, fx_{2k+2n-2}), b_d(fx_{2k+2n}, gx_{2k+2n-1}) \end{array} \right\} \\
&\le \lambda \max\{\alpha_{2n}, \alpha_{2n+1}, \alpha_{2n+2}, \alpha_{2n+3}, ...., \alpha_{2k+2n-1}\} \\
&\le \lambda \max\{\mu \cdot (\theta)^{2n}, \mu \cdot (\theta)^{2n+1}, ..., \mu \cdot (\theta)^{2k+2n-1}\}, \\
&= \lambda\mu \cdot (\theta)^{2n} = \mu \cdot (\theta)^{2n} (\theta)^{2k} = \mu \cdot (\theta)^{2k+2n}.
\end{aligned}
$$

From (2.11) and (2.12), we have

$$(2.13)\qquad b_d(y_{2k+n}, y_{2k+n+1}) \le \mu \cdot (\theta)^{2k+n}, \quad n = 1, 2, 3, \ldots$$

Now, using (2.13), for $m > n$ and using the fact that $s > 1$ we have

$$
\begin{aligned}
b_d(y_{2k+n}, y_{2k+m}) &\le \left( \begin{array}{c} s\, b_d(y_{2k+n}, y_{2k+n+1}) + s^2\, b_d(y_{2k+n+1}, y_{2k+n+2}) \\ +s^3\, b_d(y_{2k+n+2}, y_{2k+n+3}) + ...+ \\ s^{m-n-1}\, b_d(y_{2k+m-1}, y_{2k+m}) \end{array} \right) \\
&\le \left( \begin{array}{c} s\, \mu \cdot (\theta)^{2k+n} + s^2\, \mu \cdot (\theta)^{2k+n+1} + s^3\, \mu \cdot (\theta)^{2k+n+2} \\ +... + s^{m-n-1} \mu \cdot (\theta)^{2k+m-1}, \end{array} \right) \\
&\le \mu \cdot \left[ \begin{array}{c} (s\theta)^{2k+n} + (s\theta)^{2k+n+1} + (s\theta)^{2k+n+2} \\ +... + (s\theta)^{2k+m-1}] \end{array} \right. \\
&\le \mu(s\theta)^{2k} \left[ \frac{(s\theta)^n}{1 - s\theta} \right] \text{ since } s\theta = s\lambda^{\frac{1}{2k}} < s \cdot \frac{1}{s} = 1 \\
&\to 0 \text{ as } n \to \infty, m \to \infty.
\end{aligned}
$$

Therefore, $\{y_{2k+n}\}$ is a Cauchy sequence in $(X, b_d)$. Since $X$ is $b_d-$complete, there exists $z \in X$ such that $y_{2k+n} \to z$ as $n \to \infty$.

From (2.6), there exists $u \in X$ such that

$$z = fu = gu. \tag{2.14}$$

Now consider

$$\begin{aligned} b_d(S(u,u,...,u), y_{2k+2n}) &= b_d(S(u,u,...,u), T(x_{2n}, x_{2n+1}, ..., x_{2n+2k-1})) \\ &\leq \lambda \max \left\{ \begin{array}{l} b_d(gu, fx_{2n}), b_d(fu, gx_{2n+1}), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_d(gu, fx_{2k+2n-2}), b_d(fu, gx_{2k+2n-1}) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$ and using (2.14), we get

$$\frac{1}{s}\, b_d(S(u,u,...,u), fu) \leq 0 \quad \text{so that} \quad S(u,u,...,u) = fu. \tag{2.15}$$

Similarly we have

$$T(u,u,...,u) = gu. \tag{2.16}$$

Since $(f,S)$ and $(g,T)$ are jointly $2k$-weakly compatible pairs and from (2.15) and (2.16), we have

$$fz = f(fu) = f(S(u,u,...,u)) = S(fu,fu,...,fu) = S(z,z,...,z), \tag{2.17}$$

and

$$gz = T(z,z,...,z,z). \tag{2.18}$$

Now using (2.16) and (2.17), we get

$$\begin{aligned} b_d(fz,z) &= b_d(fz,gu) \\ &= b_d(S(z,z,...,z,z), T(u,u,...,u,u)) \\ &\leq \lambda \max \left\{ \begin{array}{l} b_d(gz,fu), b_d(fz,gu), \\ b_d(gz,fu), b_d(fz,gu), \\ \dots\dots\dots\dots\dots\dots\dots \\ b_d(gz,fu), b_d(fz,gu) \end{array} \right\} \\ &\leq \lambda \max\{b_d(gz,z), b_d(fz,z)\}. \end{aligned}$$

Thus

$$b_d(fz,z) \leq \lambda \max\{b_d(gz,z), b_d(fz,z)\}. \tag{2.19}$$

Similarly, we have

$$b_d(gz,z) \leq \lambda \max\{b_d(gz,z), b_d(fz,z)\}. \tag{2.20}$$

From (2.19) and (2.20), we have

$$\max\{b_d(gz,z), b_d(fz,z)\} \leq \lambda \max\{b_d(gz,z), b_d(fz,z)\}.$$

which in turn yields that

$$fz = z = gz. \tag{2.21}$$

From (2.17), (2.18) and (2.21), we have

$$fz = z = gz = S(z,z,...,z,z) = T(z,z,...,z,z). \tag{2.22}$$

Suppose there exists $z' \in X$ such that
$z' = fz' = gz' = S(z', z', ..., z', z') = T(z', z', ..., z', z')$.
Then from (2.4), we have

$$\begin{aligned} b_d(z, z') &= b_d(S(z, z, ..., z, z), T(z', z', ..., z', z')) \\ &\leq \lambda \max \left\{ \begin{array}{l} b_d(gz, fz'), b_d(fz, gz'), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ b_d(gz, fz'), b_d(fz, gz') \end{array} \right\} \\ &\leq \lambda b_d(z, z'). \end{aligned}$$

This implies that $z' = z$.
Thus $z$ is the unique point in $X$ satisfying (2.22). □

Now we give an example to illustrate our main Theorem 2.4.

**Example 2.1.** *Let $X = [0,1]$ and $b_d(x,y) = |x+y|^2$ and $k = 1$.
Define $S(x,y) = \frac{3x^2+2y}{\sqrt{4608}}$, $T(x,y) = \frac{2x+3y^2}{\sqrt{4608}}$, $fx = \frac{x}{6}$ and $gx = \frac{x^2}{4}$
for all $x, y \in X$. Then clearly $s = 2$. Then for all $x_1, x_2, y_1, y_2 \in X$,
we have*

$$\begin{aligned} b_d(S(x_1,x_2), T(y_1,y_2)) &= |\frac{3x_1^2 + 2x_2}{\sqrt{4608}} + \frac{2y_1 + 3y_2^2}{\sqrt{4608}}|^2 \\ &= \left(\frac{x_1^2}{16\sqrt{2}} + \frac{x_2}{24\sqrt{2}} + \frac{y_1}{24\sqrt{2}} + \frac{y_2^2}{16\sqrt{2}}\right)^2 \\ &= \frac{1}{2}\left(\left(\frac{x_1^2}{16} + \frac{y_1}{24}\right) + \left(\frac{x_2}{24} + \frac{y_2^2}{16}\right)\right)^2 \\ &= \frac{1}{32}\left(\left(\frac{x_1^2}{4} + \frac{y_1}{6}\right) + \left(\frac{x_2}{6} + \frac{y_2^2}{4}\right)\right)^2 \\ &= \frac{1}{8}\left(\frac{\left(\frac{x_1^2}{4} + \frac{y_1}{6}\right) + \left(\frac{x_2}{6} + \frac{y_2^2}{4}\right)}{2}\right)^2 \\ &\leq \frac{1}{8}\left(\max\left\{\frac{x_1^2}{4} + \frac{y_1}{6}, \frac{x_2}{6} + \frac{y_2^2}{4}\right\}\right)^2 \\ &= \frac{1}{8}\max\left\{\left(\frac{x_1^2}{4} + \frac{y_1}{6}\right)^2, \left(\frac{x_2}{6} + \frac{y_2^2}{6}\right)^2\right\} \end{aligned}$$

*where used the following:*

$$\frac{a+b}{2} \leq \max\{a,b\}, \ (\max(a,b))^2 = \max\{a^2, b^2\},$$

*for non-negative $a$ and $b$. Here $\lambda = \frac{1}{8} \in (0, \frac{1}{4}) = (0, \frac{1}{2^2}) = (0, \frac{1}{s^{2k}})$.*
*One can easily verify the remaining conditions of Theorem 2.4. Clearly $0$ is the unique point in $X$ such that $f0 = 0 = g0 = S(0,0) = T(0,0)$.*

**Corollary 2.1.** *Let $(X, b_d)$ be a $b_d-$complete b - dislocated metric space with $s \geq 1$ and $k$ be any positive integer. Let $S, T : X^{2k} \longrightarrow X$ and $f : X \longrightarrow X$ be mappings satisfying*

$$S(X^{2k}) \subseteq g(X), T(X^{2k}) \subseteq f(X), \tag{2.23}$$

$$b_d(S(x_1, x_2, ..., x_{2k}), T(y_1, y_2, ..., y_{2k})) \leq \lambda \max\{b_d(fx_i, fy_i) : 1 \leq i \leq 2k\} \tag{2.24}$$
$$\text{for all} \quad x_1, x_2, ..., x_{2k}, y_1, y_2, .., y_{2k} \in X, where\ \lambda \in (0, \frac{1}{s^{2k}})$$

$$f(X) \quad \text{is} \quad ab_d - \text{complete subspace of} \quad X \tag{2.25}$$

$$(f, S) \quad \text{or} \quad (f, T) \quad \text{is} \quad 2k - \text{dweakly compatible pair}. \tag{2.26}$$

*Then there exists a unique point* $u \in X$ *such that* $u = fu = S(u, u, .., u, u) = T(u, u, .., u, u)$.

**Corollary 2.2.** *Let* $(X, b_d)$ *be a* $b_d-$*complete b - dislocated metric space with* $s \geq 1$ *and* $k$ *be any positive integer. Let* $S, T : X^{2k} \longrightarrow X$ *be mappings satisfying*

$$b_d(S(x_1, x_2, ..., x_{2k}), T(y_1, y_2, ..., y_{2k})) \leq \lambda\ \max\{b_d(x_i, y_i) : 1 \leq i \leq 2k\} \tag{2.27}$$
$$\text{for all} x_1, x_2, ..., x_{2k}, y_1, y_2, .., y_{2k} \in X, where\ \lambda \in (0, \frac{1}{s^{2k}})$$

*Then there exists a unique point* $u \in X$ *such that* $u = S(u, u, .., u, u) = T(u, u, .., u, u)$.

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**Competing interests**

Authors declare that they have no any conflict of interest regarding the publication of this paper.

## References

[1] M. Abbas, D. Ilić, T. Nazir, *Iterative Approximation of Fixed Points of Generalized Weak Prešić Type k-Step Iterative Method for a Class of Operators,* Filomat, **29** (4) (2015) 713-724.

[2] R. George, KP Reshma and R. Rajagopalan, *A generalised fixed point theorem of Prešić type in cone metric spaces and application to Markov process,* Fixed Point Theory Appl., 2011, 2011:85.

[3] Z. Kadelburg, S. Radenović, *Notes on Some Recent Papers Concerning $F-$Contractions in $b-$Metric Spaces*, Constr. Math. Anal., **1**(2) (2018), 108-112.

[4] E. Karapinar, *A Short Survey on the Recent Fixed Point Results on $b-$Metric Spaces*, Constr. Math. Anal., **1**(1)(2018) 15-44.

[5] A. Amini-Harandi, *Metric-like spaces, partial metric spaces and fixed points,* Fixed Point Theory Appl. 2012, 2012:204

[6] P. Hitzler and A. K. Seda, *Dislocated topologies,* J. Electr. Eng., **51**(12) (2000) 3-7.

[7] N. Hussain, J. R. Roshan, V. Parvaneh and M. Abbas, *Common fixed point results for weak contractive mappings on ordered $b-$dislocated metric spaces with applications,* J. Inequal. Appl., 2013, 2013:486

[8] Lj. B. Ćirić and S. B. Prešić, *On Prešić type generalization of Banach contraction mapping principle,* Acta Math. Univ. Comenianae, LXXVI(2) (2007) 143-147.

[9] M. Păcurar, *Approximating common fixed points of Prešić-Kannan type operators by a multi-step iterative method,* An. St. Univ. Ovidius Constanta, **17**(1) (2009) 153-168.

[10] S. B. Prešić, *Sur une classe d'inequations aux differences finite et sur la convergence de certaines suites,* Publications de l'Institut Mathématique, **5**(19) (1965) 75-78.

[11] K. P. R. Rao, G. N. V. Kishore and Md. Mustaq Ali, *Generalization of Banach contraction principle of Prešić type for three maps,* Math. Sci., **3**(3) (2009) 273 - 280.

[12] K. P. R. Rao, Md. Mustaq Ali and B.Fisher, *Some Prešić type generalization of Banach contraction principle,* Math. Moravica **15** (2011) 41 - 47.

[13] P. Salimi, N. Hussain, S. Shukla, Sh. Fathollahi, S. Radenović, *Fixed point results for cyclic $\alpha-\psi\phi-$ contractions with applications to integral equations*, J. Comput. Appl. Math., **290** (2015) 445-458.
[14] S. Shukla, S. Radenović, S. Pantelić, *Some Fixed Point Theorems for Prešić-Hardy-Rogers Type Contractions in Metric Spaces*, Journal of Mathematics, (2013) ArticleID 295093.
[15] S. Shukla, S. Radenović, V.Ć. Rajić, *Some common fixed point theorems in 0-$\sigma-$complete metric-like spaces,* Vietnam J. Math., **41** (2013) 341-352.

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