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APPROXIMATION BY TWO DIMENSIONAL GADJIEV-IBRAGIMOV TYPE OPERATORS

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Abstract

In the present paper we introduce linear positive operators which are defined in [6] by generalization of Gadjiev-Ibragimov operators and give some approximation properties of these operators in the space of continuous functions of two variables on a compact set. We find certain moments of this operator and estimate for approximation error of the operators in terms of modulus of continuity. Then, we give some approximation properties of these operators.

Keywords: Gadjiev-Ibragimov Operators; Linear Positive Operators; Volkov Theorem.

1. Introduction

Gadjiev and Ibragimov, defined a general sequence of positive operators and studied someapproximation properties of this operators. Several generalizations of this operator have been studied in the one dimensional case by different researchers[1,2,3,6]. We introduce a generalization of linear positive operators in two dimensions which given in [4, 5]. Then we give some approximation properties of two dimensional Gadjiev-Ibragimov operators.

We give the construction of operators in the next section. Then we present some auxiliary result and approximation with the help of modulus of continuity will be given.

2. Construction of Operators

Definition 2.1. Let (α_n), (β_n) and (γ_n) be sequences of real numbers sequences such as

$$\lim_{n\to\infty} \beta_n = \infty, \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0 \quad \text{and} \ \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} n = 1$$

and

$$\lim_{m\to\infty} \gamma_m = \infty, \lim_{m\to\infty} \frac{\alpha_m}{\gamma_m} = 0 \quad \text{and} \ \lim_{m\to\infty} \frac{\alpha_m}{\gamma_m} m = 1.$$

$K_{n,\vartheta}(x)$ and $K_{m,\mu}(y)$ get a function satisfies the following conditions;

1) Let $n, m \in \mathbb{N}$ and $\vartheta, \mu \in \mathbb{N}_0$. For every finite A and $(x, y) \in C([0, A] \times [0, A])$ such that

$$(-1)^{\vartheta} K_{n,\vartheta}(x) \geq 0 \text{ and } (-1)^{\mu} K_{m,\mu}(y) \geq 0.$$

2) For any $(x,y) \in [0,A]$,

$$\sum_{\vartheta=0}^{\infty} K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!} = 1 \text{ and} \sum_{\mu=0}^{\infty} K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!} = 1.$$

3) For any $(x,y) \in [0,A]$,

$$K_{n,\vartheta}(x) = -nxK_{n+k,\vartheta-1}(x) \text{ and } K_{m,\mu}(x) = -myK_{m+l,\mu-1}(y)$$

where $n+k, m+l \in \mathbb{N}_0$ and k,l are constants independent of ϑ, μ.

Taking these equations into account, let us define a two variable generalization of Gadjiev-Ibragimov's operator for $f \in C([0,A]\times[0,A])$

$$L_{n,m}(f,x,y) = \sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty} f(\frac{\vartheta}{\beta_n},\frac{\mu}{\gamma_m})K_{n,\vartheta}(x)K_{m,\mu}(y)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}\frac{(-\alpha_m)^{\mu}}{\mu!} \tag{1}$$

Here we use

$$P_{n,m}(x,y) = K_{n,\vartheta}(x)K_{m,\mu}(y)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}\frac{(-\alpha_m)^{\mu}}{\mu!}.$$

Lemma 2.1 $L_{n,m}$ defined by Equation 1 is linear and positive operators.

Proposition 2.1. Let $f \in C([0,A]\times[0,A])$for the operator given by Equation 1 we have

$$i) L_{n,m}(1,x,y) = 1.$$

$$ii)\ L_{n,m}(t_1,x,y) = \frac{\alpha_n}{\beta_n}nx.$$

$iii) L_{n,m}(t_2,x,y) = \dfrac{\alpha_m}{\gamma_m}my.$

$iv) L_{n,m}({t_1}^2+{t_2}^2,x,y) = \left(\dfrac{\alpha_n}{\beta_n}\right)^2 n(n+k)x^2 + \dfrac{\alpha_n}{{\beta_n}^2}nx + \left(\dfrac{\alpha_m}{\gamma_m}\right)^2 m(m+l)y^2 + \dfrac{\alpha_m}{{\gamma_m}^2}my.$

Proof. (i) In Definition 2.1. 2) we get

$$L_{n,m}(1,x,y) = \sum_{\vartheta=0}^{\infty} K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}\sum_{\mu=0}^{\infty} K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!} = 1.$$

(ii) Using Definition 2.1. conditional of 2) and 3)

$$L_{n,m}(t_1,x,y) = \sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{\vartheta}{\beta_n}P_{n,m}(x,y)$$

$$= \sum_{\vartheta=0}^{\infty}\frac{\vartheta}{\beta_n}K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}$$

$$= \frac{\alpha_n}{\beta_n}nx\sum_{\vartheta=1}^{\infty}K_{n+k,\vartheta-1}(x)\frac{(-\alpha_n)^{\vartheta-1}}{(\vartheta-1)!}$$

$$=\frac{\alpha_n}{\beta_n}nx \qquad\qquad (n+k)\in\mathbb{N}_0.$$

(iii) Definition of $L_{n,m}$ we have

$$L_{n,m}(t_2,x,y)=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{\mu}{\gamma_m}K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}$$

$$=\frac{-\alpha_m}{\gamma_m}\sum_{\vartheta=0}^{\infty}K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}\sum_{\mu=1}^{\infty}-myK_{m+l,\mu-1}(y)\frac{(-\alpha_m)^{\mu-1}}{(\mu-1)!}$$

$$=\frac{\alpha_m}{\gamma_m}my\sum_{\vartheta=0}^{\infty}K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}\sum_{\mu=1}^{\infty}K_{m+l,\mu-1}(y)\frac{(-\alpha_m)^{\mu-1}}{(\mu-1)!}$$

$$=\frac{\alpha_m}{\gamma_m}my \qquad\qquad (m+l)\in\mathbb{N}_0.$$

(iv)For $(n+k)\in\mathbb{N}_0$

$$L_{n,m}({t_1}^2,x,y)=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\left(\frac{\vartheta}{\beta_n}\right)^2P_{n,m}(x,y)$$

$$=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}\frac{\vartheta(\vartheta-1)}{{\beta_n}^2}K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}$$

$$+\frac{1}{{\beta_n}^2}\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}\vartheta K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}$$

$$=\frac{{\alpha_n}^2}{{\beta_n}^2}n(n+k)x^2\sum_{\vartheta=2}^{\infty}K_{n+k,\vartheta-2}(x)\frac{(-\alpha_n)^{\vartheta-2}}{(\vartheta-2)!}+\frac{1}{\beta_n}\frac{\alpha_n}{\beta_n}nx$$

$$=\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)x^2+\frac{\alpha_n}{{\beta_n}^2}nx. \tag{2}$$

Similarly for $(m+l)\in\mathbb{N}_0$ we get

$$L_{n,m}({t_2}^2,x,y)=\left(\frac{\alpha_m}{\gamma_m}\right)^2m(m+l)y^2+\frac{\alpha_m}{{\gamma_m}^2}my \tag{3}$$

and using Equation 2 and Equation 3 we have

$$L_{n,m}({t_1}^2+{t_2}^2,x,y)=\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)x^2+\frac{\alpha_n}{{\beta_n}^2}nx+\left(\frac{\alpha_m}{\gamma_m}\right)^2m(m+l)y^2+\frac{\alpha_m}{{\gamma_m}^2}my.$$

Theorem 2.1. For every $f\in C([0,A]\times[0,A])$

$$\lim_{n\to\infty}\left\|L_{n,m}(f,x,y)-f(x,y)\right\|=0.$$

Proof. We show conditional of Volkov Theorem. Clearly we have

$$\lim_{n\to\infty}\|L_{n,m}(1,x,y)-1\|=0.$$

Using $\frac{\alpha_n}{\beta_n}n\to 1$ we write

$$\left\|\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{\vartheta}{\beta_n}P_{n,m}(x,y)-x\right\|=\left\|\frac{\alpha_n}{\beta_n}nx-x\right\|.$$

Then we have

$$\lim_{n\to\infty}\|L_{n,m}(t_1,x,y)-x\|=0.$$

Similarly for $\frac{\alpha_m}{\gamma_m}m\to 1$ we get

$$\lim_{n\to\infty}\|L_{n,m}(t_2,x,y)-y\|=0.$$

Also by Proposition 2.1 iv)

$$\lim_{n\to\infty}\|L_{n,m}({t_1}^2+{t_2}^2,x,y)-x^2-y^2\|=0.$$

Example 2.1. The convergence of $L_{n,m}(f,x,y)$ to $f(x,y)=e^{1+2x}+y$ for $\alpha_n=\alpha_m=1, \beta_n=n$, $\gamma_m=m$ is illustrated in Figure1. $n=m=1(\text{brown}), n=m=3(\text{green}), n=m=10(\text{magenta})$

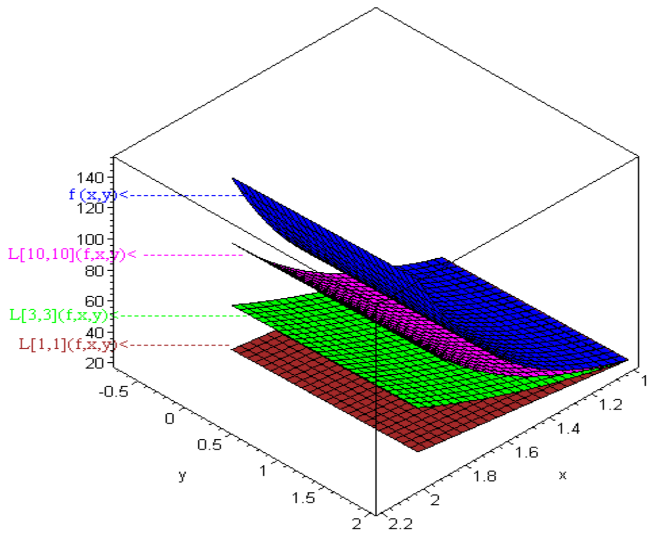

Figure 1:Approximation of $L_{n,m}(f,x,y)$

The first three moments of operators are given next Lemma.

Lemma 2.2. Let $(x,y)\in[0,A]\times[0,A]$ and for $n,m\in\mathbb{N}$ the following equalities hold.

$i)L_{n,m}(1,x,y)=1.$

$ii)L_{n,m}(t_1-x,x,y)=x\left(\frac{\alpha_n}{\beta_n}-1\right).$

$iii) L_{n,m}((t_1-x)^2,x,y)=\left[\left(\frac{\alpha_n}{\beta_n}\right)^2 n(n+k)-\frac{2\alpha_n}{\beta_n}n+1\right]x^2+\frac{\alpha_n}{{\beta_n}^2}nx.$

Proof.

(*i*) Clearly $L_{n,m}(1,x,y)=1.$

(*ii*)

$$L_{n,m}(t_1-x,x,y)=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\left(\frac{\vartheta}{\beta_n}-x\right)P_{n,m}(x,y)=x\left(\frac{\alpha_n}{\beta_n}n-1\right)$$

(*iii*)

$$\begin{aligned}L_{n,m}((t_1-x)^2,x,y) &= \sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}(t_1-x)^2\,P_{n,m}(x,y)\\ &=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\left(\frac{\vartheta}{\beta_n}\right)^2 P_{n,m}(x,y)-2x\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{\vartheta}{\beta_n}P_{n,m}(x,y)+x^2\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}P_{n,m}(x,y)\\ &=\left(\frac{\alpha_n}{\beta_n}\right)^2 n(n+k)x^2+\frac{\alpha_n}{{\beta_n}^2}nx-2x^2\left(\frac{\alpha_n}{\beta_n}n\right)+x^2\\ &=x^2\left[\left(\frac{\alpha_n}{\beta_n}\right)^2 n(n+k)-\frac{2\alpha_n}{\beta_n}n+1\right]+\frac{\alpha_n}{{\beta_n}^2}nx.\end{aligned}$$

Remark 2.1 Similar equality is provided for $L_{n,m}(t_2-y,x,y)$ and $L_{n,m}((t_2-y)^2,x,y)$.

Now we estimate modulus of continuity of operators definition 2.1 in $C([0,A]\times[0,A])$.

Definition 2.2. Let $D\subset\mathbb{R}^2$ and $f{:}\,D\to\mathbb{R}$ bounded function. $K\subset D$ compact domain and let $(x=(x_1,x_2),\ y=(y_1,y_2))$ using partial modulus of continuity

$$\omega_1 f(f,\delta)=sup\{|f(x_1,y)-f(x_2,y)|{:}\,(x_1,y),(x_2,y)\in K,|x_1-x_2|\le\delta\}$$

$$\omega_2 f(f,\delta)=sup\{|f(x,y_1)-f(x,y_2)|{:}\,(x,y_1),(x,y_2)\in K,|y_1-y_2|\le\delta\}.$$

Theorem 2.1. Every $f\in C([0,A]\times[0,A])$ and let sequences of $(\alpha_n),(\beta_n),(\gamma_m)$ defined as in definition 2.1. Then for sufficiently large *n*,*m*

$$\left\|L_{n,m}(f,x,y)-f(x,y)\right\|_{C[0,A]}\le K_1 w_2(f,\delta_m)+K_2 w_1(f,\delta_n)$$

where K is a constant independent of *n*,*m* for $\delta_n=\sqrt{\left(n\frac{\alpha_n}{\beta_n}-1\right)^2+\frac{\alpha_n}{\beta_n}+\frac{1}{A\beta_n}}$ and

$\delta_m=\sqrt{\left(m\frac{\alpha_m}{\gamma_m}-1\right)^2+\frac{\alpha_m}{\gamma_m}+\frac{1}{A\gamma_m}}.$

Proof. Clearly using Definition 2.2 and Cauchy-Schwarz inequality we get

$$N_1 = \sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\left|f\left(\frac{\vartheta}{\beta_n},\frac{\mu}{\gamma_m}\right)-f\left(\frac{\vartheta}{\beta_n},y\right)\right|P_{n,m}(x,y)$$

$$\leq \sum_{\mu=0}^{\infty}\omega_2(f,\delta_m)\left[1+\frac{\left|\frac{\mu}{\gamma_m}-y\right|}{\delta_m}\right]K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}$$

$$\leq \omega_2(f,\delta_m)\left\{1+\frac{1}{\delta_m}\sum_{\mu=0}^{\infty}\left|\frac{\mu}{\gamma_m}-y\right|\sqrt{K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}}\sqrt{K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}}\right\}$$

$$\leq \omega_2(f,\delta_m)\left\{1+\frac{1}{\delta_m}\sqrt{\sum_{\mu=0}^{\infty}\left|\frac{\mu}{\gamma_m}-y\right|^2K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}}\right\}.$$

Using Proposition 2.1

$$\sum_{\mu=0}^{\infty}\left|\frac{\mu}{\gamma_m}-y\right|^2K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}=\sum_{\mu=0}^{\infty}\left[\left(\frac{\mu}{\gamma_m}\right)^2-2y\frac{\mu}{\gamma_m}+y^2\right]K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}$$

$$=\left(\frac{\alpha_m}{\gamma_m}\right)^2m(m+l)y^2+\frac{\alpha_m}{{\gamma_m}^2}my-2y\frac{\mu}{\gamma_m}+y^2.$$

from $\left(\frac{\mu}{\gamma_m}-y\right)^2=\left(\frac{\mu}{\gamma_m}\right)^2-2y\frac{\mu}{\gamma_m}+y^2$

$$\left|L_{n,m}(f,x,y)-f(x,y)\right|\leq\omega_2(f,\delta_m)\left\{1+\frac{1}{\delta_m}\left(\sum_{\mu=0}^{\infty}\left(\frac{\mu}{\gamma_m}\right)^2K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}\right.\right.$$

$$\left.\left.-2y\sum_{\mu=0}^{\infty}\frac{\mu}{\gamma_m}K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}+y^2\sum_{\mu=0}^{\infty}K_{m,\mu}(y)\frac{(-\alpha_m)^{\mu}}{\mu!}\right)^{1/2}\right\}$$

$$=\omega_2(f,\delta_m)\left\{1+\frac{1}{\delta_m}\left(L_{n,m}({t_2}^2,x,y)-2yL_{n,m}(t_2,x,y)+y^2L_{n,m}(1,x,y)\right)^{1/2}\right\}.$$

For $y\in[0,A]$ we write $L_{n,m}({t_2}^2,x,y), L_{n,m}(t_2,x,y)$ and $L_{n,m}(1,x,y)$ using $\lim_{m\to\infty}\gamma_m=\infty$, $\lim_{m\to\infty}\frac{\alpha_m}{\gamma_m}=0$ and $\lim_{m\to\infty}\frac{\alpha_m}{\gamma_m}m=1$ equation for a large *m* and using the equalites $\frac{\alpha_m}{\gamma_m}\leq 1$ and $\frac{\alpha_m}{\gamma_m}m\leq 2$

$$\left|L_{n,m}(f,x,y)-f(x,y)\right|\leq\omega_2(f,\delta_m)\left\{1+\frac{1}{\delta_m}\left(\frac{m(m+l)}{{\gamma_m}^2}{\alpha_m}^2A^2+\frac{1}{\gamma_m}\frac{\alpha_m}{\gamma_m}mA\right.\right.$$

$$\left.\left.-2A\frac{\alpha_m}{\gamma_m}m+A^2\right)\right\}^{1/2}$$

$$\leq\omega_2(f,\delta_m)\left\{1+\frac{A}{\delta_m}\left(A^2\left[\left(\frac{\alpha_m}{\gamma_m}\right)^2m^2-2\frac{\alpha_m}{\gamma_m}m+1\right]+A^2\left[\left(\frac{\alpha_m}{\gamma_m}\right)^2ml+\frac{1}{A}\frac{1}{\gamma_m}\frac{\alpha_m}{\gamma_m}m+1\right]\right)^{1/2}\right\}$$

Then if we choose $\delta_m = \sqrt{\left(m\frac{\alpha_m}{\gamma_m} - 1\right)^2 + \frac{\alpha_m}{\gamma_m} + \frac{1}{A\gamma_m}}$ we have the following inequality constant K_1 independent on m

$$\left\|L_{n,m}(f,x,y) - f(x,y)\right\|_{C[0,A]} \leq K_1 w_2\left(f, \sqrt{\left(m\frac{\alpha_m}{\gamma_m} - 1\right)^2 + \frac{\alpha_m}{\gamma_m} + \frac{1}{A\gamma_m}}\right).$$

Similarly for N_2 using Cauchy-Schwarz inequality and Proposition 2.1

$$N_2 = \sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty}\left|f\left(\frac{\vartheta}{\beta_n}, y\right) - f(x,y)\right| P_{n,m}(x,y)$$

$$\leq \omega_1(f,\delta_n)\left\{1 + \frac{1}{\delta_n}\sqrt{\sum_{\vartheta=0}^{\infty}\left|\frac{\vartheta}{\beta_n} - x\right|^2 K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!}}\right\}$$

So we get

$$\sum_{\vartheta=0}^{\infty}\left|\frac{\vartheta}{\beta_n} - x\right|^2 K_{n,\vartheta}(x)\frac{(-\alpha_n)^{\vartheta}}{\vartheta!} = \left(\frac{\alpha_n}{\beta_n}\right)^2 n(n+k)x^2 + \frac{\alpha_n}{{\beta_n}^2}nx - 2x\frac{\alpha_n}{\beta_n}n + x^2.$$

Using $\left(\frac{\vartheta}{\beta_n} - x\right)^2 = \left(\frac{\vartheta}{\beta_n}\right)^2 - 2x\frac{\vartheta}{\beta_n} + x^2$ and there for a large n

$$\left|L_{n,m}(f,x,y) - f(x,y)\right| \leq \omega_1(f,\delta_n)\left\{1 + \frac{1}{\delta_n}\big(L_{n,m}({t_1}^2,x,y) - 2xL_{n,m}(t_1,x,y)\right.$$

$$\left. +x^2L_{n,m}(1,x,y)\big)^{1/2}\right\}.$$

For $x \in [0,A]$ we write $L_{n,m}({t_1}^2,x,y), L_{n,m}(t_1,x,y)$ and $L_{n,m}(1,x,y)$ using $\lim_{m\to\infty}\beta_n = \infty$, $\lim_{m\to\infty}\frac{\alpha_n}{\beta_n} = 0$ and $\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}n = 1$ equation for a large n and using the equalites $\frac{\alpha_n}{\beta_n} \leq 1$ and $\frac{\alpha_n}{\beta_n}n \leq 2$

$$\left|L_{n,m}(f,x,y) - f(x,y)\right| \leq \omega_1(f,\delta_n)\left\{1 + \frac{1}{\delta_n}\left(\frac{n(n+k)}{{\delta_n}^2}{\alpha_n}^2A^2 + \frac{1}{\beta_n}\frac{\alpha_n}{\beta_n}nA - 2A\frac{\alpha_n}{\beta_n}n + A^2\right)^{1/2}\right\}$$

$$\leq \omega_1(f,\delta_n)\left\{1 + \frac{2kA}{\delta_n}\left[\left(n\frac{\alpha_n}{\beta_n} - 1\right)^2 + \frac{\alpha_n}{\beta_n} + \frac{1}{A\beta_n}\right]^{1/2}\right\}.$$

Then if we choose $\delta_n = \sqrt{\left(n\frac{\alpha_n}{\beta_n} - 1\right)^2 + \frac{\alpha_n}{\beta_n} + \frac{1}{A\beta_n}}$ we have the following inequality constant K_2 independent on n

$$\left\|L_{n,m}(f,x,y) - f(x,y)\right\|_{C[0,A]} \leq K_2 w_1\left(f, \sqrt{\left(n\frac{\alpha_n}{\beta_n} - 1\right)^2 + \frac{\alpha_n}{\beta_n} + \frac{1}{A\beta_n}}\right).$$

Then proof is completed.

Now we want to find the rate of convergence of the sequence of operators $L_{n,m}(f,x,y)$.

Example 2.2. The error bound of the function $f(x,y)=\frac{x^2+y^2}{10}$, $\alpha_n=1$, $\beta_n=n$.

n,m	Error bound for modulus of continuity of $f(x,y)$
10	0.8755417528
10^2	0.2422741700
10^3	0.0731541753
10^4	0.0227874170
10^5	0.0071714175
10^6	0.0022643417
10^7	0.0007157018
10^8	0.0002262902
10^9	0.0000715558

Table 1:The error bound of $f(x,y)=\frac{x^2+y^2}{10}$.

3. Approximation Properties in C_ρ^k

Definition 3.1. For $(x,y)\in(0,\infty)\times(0,\infty)$ and let $f\in C_\rho^k$. Then two dimensional generalized Gadjiev-Ibragimov operators defined by

$$L_{n,m}(f,x,y)=\sum_{\vartheta=0}^{\infty}\sum_{\mu=0}^{\infty} f\left(\frac{\vartheta}{\beta_n},\frac{\mu}{\gamma_m}\right)K_{n,\vartheta}(x)K_{m,\mu}(y)\frac{(-\alpha_n)^\vartheta}{\vartheta!}\frac{(-\alpha_m)^\mu}{\mu!}. \quad (4)$$

Lemma3.1. The following equalities hold for Equation 4

$i)L_{n,m}(1,x,y)=1$

$ii)\ L_{n,m}(t_1,x,y)=\frac{\alpha_n}{\beta_n}nx$

$iii)L_{n,m}(t_2,x,y)=\frac{\alpha_m}{\beta_m}my$

$iv)L_{n,m}({t_1}^2+{t_2}^2,x,y)=\left(\frac{\alpha_n}{\beta_n}\right)^2 n(n+k)x^2+\frac{\alpha_n}{{\beta_n}^2}nx+\left(\frac{\alpha_m}{\gamma_m}\right)^2 m(m+l)y^2+\frac{\alpha_m}{{\gamma_m}^2}my.$

Theorem 3.1. Let $\rho(x,y)=1+x^2+y^2$ and $L_{n,m}:C_\rho\to B_\rho$ sequences of linear positive operators defined by Equation 4 then every $f\in C_\rho^k$

$$\lim_{n,m\to\infty}\left\|L_{n,m}(f,x,y)-f(x,y)\right\|_\rho=0.$$

Proof. Using Volkov Theorem clearly

$$\lim_{n,m\to\infty}\left\|L_{n,m}(1,x,y)-1\right\|_\rho=0.$$

We have $L_{n,m}(t_1,x,y)=\frac{\alpha_n}{\beta_n}nx$ then definition of norm in C_ρ

$$\left\|L_{n,m}(t_1,x,y)-x)\right\|_\rho=\sup_{(x,y)\in(0,\infty)\times(0,\infty)}\left|\frac{x}{1+x^2+y^2}\right|\left|\frac{\alpha_n}{\beta_n}n-1\right|$$
$$\leq\left|\frac{\alpha_n}{\beta_n}n-1\right|.$$

so

$$\left\|L_{n,m}(t_1,x,y)-x\right\|_\rho=0.$$

Similarly using $L_{n,m}(t_2,x,y)=\frac{\alpha_m}{\gamma_m}my$ we get

$$\left\|L_{n,m}(t_2,x,y)-y\right\|_\rho=\sup_{(x,y)\in(0,\infty)\times(0,\infty)}\left|\frac{x}{1+x^2+y^2}\right|\left|\frac{\alpha_m}{\gamma_m}m-1\right|$$
$$\leq\left|\frac{\alpha_m}{\gamma_m}m-1\right|$$

then

$$\left\|L_{n,m}(t_2,x,y)-y\right\|_\rho=0.$$

We have $L_{n,m}({t_1}^2,x,y)=\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)x^2+\frac{\alpha_n}{{\beta_n}^2}nx$ then

$$\left\|L_{n,m}({t_1}^2,x,y)-x^2\right\|_\rho=\sup_{(x,y)\in(0,\infty)\times(0,\infty)}\frac{\left|L_{n,m}({t_1}^2,x,y)-x^2\right|}{1+x^2+y^2}$$
$$\leq\left|\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)-1\right|+\left|\frac{\alpha_n}{{\beta_n}^2}n\right|.$$

So we write

$$\lim_{n,m\to\infty}\left\|L_{n,m}({t_1}^2,x,y)-x^2\right\|_\rho\leq\lim_{n,m\to\infty}\left[\left|\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)-1\right|+\left|\frac{\alpha_n n}{{\beta_n}^2}\right|\right].$$

Using $\lim_{n\to\infty}\left(\frac{\alpha_n}{\beta_n}\right)^2n(n+k)=1$ and $\lim_{n\to\infty}\frac{\alpha_n}{{\beta_n}^2}n=0$ we write

$$\lim_{n,m\to\infty}\left\|L_{n,m}({t_1}^2,x,y)-x^2\right\|_\rho=0.$$

Similarly for $L_{n,m}({t_2}^2,x,y)=\left(\frac{\alpha_m}{\gamma_m}\right)^2m(m+l)y^2+\frac{\alpha_m}{{\gamma_m}^2}my$

$$\lim_{n,m\to\infty}\left\|L_{n,m}({t_2}^2,x,y)-y^2\right\|_\rho\leq\lim_{n,m\to\infty}\left[\left|\left(\frac{\alpha_m}{\gamma_m}\right)^2m(m+l)-1\right|+\left|\frac{\alpha_m}{{\gamma_m}^2}m\right|\right]$$

so we get

$$\lim_{n,m\to\infty}\left\|L_{n,m}({t_2}^2,x,y)-y^2\right\|_\rho=0.$$

Consequently

$$\lim_{n,m\to\infty}\left\|L_{n,m}({t_1}^2+{t_2}^2,x,y)-(x^2+y^2)\right\|_\rho=0.$$

It means that for every $f \in C_\rho^k$

$$\lim_{n,m\to\infty} \left\| L_{n,m}(f,x,y) - f(x,y) \right\|_\rho = 0.$$

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SOME BOUNDS FOR ECCENTRIC VERSION OF HARMONIC INDEX OF GRAPHS

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Abstract

The harmonic index of a graph G is defined as the sum $H(G) = \sum_{ij\in E(G)} \frac{2}{d_G(i)+d_G(j)}$, where $d_G(i)$ is the degree of a vertex i in G. In this paper, we examined eccentric version of harmonic index of graphs.

Keywords: Topological index; Graph Parameters; Harmonic Index.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex u in a graph G is number of incident edges to the vertex. The degree of a vertex i is denoted by $d_G(i)$. The maximum degree is denoted by Δ. The minimum degree is denoted by δ.

The distance between i and j vertices, denoted $d_G(i,j)$ is the length of a shortest path between them. The eccentricity $\varepsilon_G(i)$ of a vertex i in a connected graph is its distance to a vertex fatrhest from i. The radius of a connected graph, denoted $r(G)$ is its minimum eccentricity. The diameter of a connected graph, denoted $D(G)$ is maximum eccentricity. For other undefined notations and terminology from graph theory, the readers are referred to [5].

One of the oldest topological indices, the first and second Zagreb indices were defined by [7,8]. The first and second Zagreb indices are defined as

$$M_1(G) = \sum_{i\in V(G)} d_G^2(i) \quad \text{and} \quad M_2(G) = \sum_{ij\in E(G)} d_G(i)d_G(j).$$

An alternative expression for the first Zagreb index is [1]

$$M_1(G) = \sum_{ij\in E(G)} (d_G(i)+d_G(j)).$$

The harmonic index was defined in [3] as

$$H(G) = \sum_{ij\in E(G)} \frac{2}{d_G(i)+d_G(j)}.$$

Ghorbani et al. [4] and Vukičević et al. [12] defined the first and the second Zagreb eccentricity indices by

$$E_1(G) = \sum_{i\in V(G)} \varepsilon_G^2(i) \quad \text{and} \quad E_2(G) = \sum_{ij\in E(G)} \varepsilon_G(i)\varepsilon_G(j).$$

In 1997, The eccentricity connectivity index of a graph G was introduced by Sharma et al. [11]. The eccentric connectivity index is defined as

$$\xi^c(G) = \sum_{i\in V(G)} d_G(i)\varepsilon_G(i) = \sum_{ij\in E(G)} (\varepsilon_G(i)+\varepsilon_G(j)).$$

In 2000, Gupta et al. [6] introduced the connective eccentricity index, which is defined to be

$$\xi^{ce}(G) = \sum_{i\in V(G)} \frac{d_G(i)}{\varepsilon_G(i)}.$$

The eccentric version of the harmonic index have been defined in [2] as follows.

$$H_4(G) = \sum_{ij\in E(G)} \frac{2}{\varepsilon_G(i)+\varepsilon_G(j)}.$$

In this paper, we are concerned with the upper and lower bounds of $H_4(G)$ which depend on some of the parameters *n, m, r, D* etc.

2. Main Results

In this section, we give some upper and lower bounds for the eccentric harmonic index.

Theorem 2.1. Let G be a simple connected graph with n vertices, m edges, r radius and D diameter. Then

$$\frac{m}{D} \le H_4(G) \le \frac{m}{r}. \tag{1}$$

Equality holds on both sides if and only if G is self centered graph.

Proof. We know that $2r \le \varepsilon_{(G)}(i) + \varepsilon_G(j) \le 2D$ for all $ij \in E(G)$. Then we have

$$H_4(G) = \sum_{ij\in E(G)} \frac{2}{\varepsilon_G(i)+\varepsilon_G(j)} \le \sum_{ij\in E(G)} \frac{2}{2r} = \frac{m}{r}.$$

In an analogous manner,

$$H_4(G) = \sum_{ij\in E(G)} \frac{2}{\varepsilon_G(i)+\varepsilon_G(j)} \ge \sum_{ij\in E(G)} \frac{2}{2D} = \frac{m}{D}.$$

Now suppose that equality holds in (1). Then all the above inequalities must become equalities. Thus we get $\varepsilon_{(G)}(i) = \varepsilon_G(j)$ for all of $ij \in E(G)$. So we conclude that G is self centered graph.

Conversely, if G is self centered graph, it is easy to see that equalities (1) hold.

Proposition 2.2. [13] Let G be a connected graph with $n \geq 3$ vertices. Then for all $i \in V(G)$ we have

$$\varepsilon_G(i) \leq n - d_G(i), \tag{2}$$

with equality if and only if $K_n - ke$, for $k = 0,1,2, \dots, \left\lfloor \frac{n}{2} \right\rfloor$, or $G = P_4$.

Theorem 2.3. Let G be connected graph of order n with maximum degree Δ. Then

$$H_4(G) \geq \frac{m}{n-\Delta}. \tag{3}$$

The equality holds if and only if G is regular self centered graph.

Proof. By applying Proposition 2.2, we get

$$\begin{aligned} H_4(G) &= \sum_{ij \in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} \\ &\geq \sum_{ij \in E(G)} \frac{2}{2n - (d_G(i) + d_G(j))} \\ &\geq \sum_{ij \in E(G)} \frac{2}{2n - 2\Delta} = \frac{m}{n-\Delta}. \end{aligned}$$

Suppose that equality holds in the above inequality. Then $\varepsilon_G(i) = n - d_G(i)$ ve $d_G(i) = \Delta$ for all $i \in V(G)$. So by Proposition 2.2 we conclude that $G \cong K_n$ or $G \cong C_4$.

Conversely, if $G \cong K_n$ or $G \cong C_4$, it is easy see that equality (3) holds.

Theorem 2.4. Let G be a connected graph with n vertices and m edges. Let k be the number of vertices with eccentricity 1 in graph G. Then

$$H_4(G) = \frac{6m + k(2n + k - 3)}{12}.$$

Proof. $K = \{i_1, i_2, \dots, i_k\}$ be the set of vertices with eccentricity 1. Then we have $e(i) = 2$ for any $i \in V(G) \setminus K$. From the definition eccentric-harmonic index, we get

$$\begin{aligned} H_4(G) &= \sum_{\substack{ij \in E(G) \\ i,j \in K}} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} + \sum_{\substack{ij \in E(G) \\ i \in K, j \in V(G) \setminus K}} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} + \sum_{\substack{ij \in E(G) \\ i,j \in V(G) \setminus K}} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} \\ &= \sum_{\substack{ij \in E(G) \\ i,j \in K}} 1 + \sum_{\substack{ij \in E(G) \\ i \in K, j \in V(G) \setminus K}} \frac{2}{3} + \sum_{\substack{ij \in E(G) \\ i,j \in V(G) \setminus K}} \frac{1}{2} \\ &= \frac{6m + k(2n + k - 3)}{12}. \end{aligned}$$

So as desired.

Lemma 2.5. (*Radon Inequality*)[10] For every real numbers $p > 0, x_k \geq 0, a_k > 0$, for $1 \leq k \leq n$, the following inequality holds true:

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p}.$$

The equality holds if and only if $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

Theorem 2.6. For any graph *G* we have

$$H_4(G) \geq \frac{2m^2}{\xi^c(G)}, \tag{4}$$

with equality holds if and only if $\varepsilon_G(i) + \varepsilon_G(j)$ is constant for all $ij \in E(G)$.

Proof. Using Lemma 2.5 we get

$$\begin{aligned} H_4(G) &= \sum_{ij \in E(G)} \frac{\left(\sqrt{2}\right)^2}{\varepsilon_G(i) + \varepsilon_G(j)} \\ &\geq \sum_{ij \in E(G)} \frac{\left(\sum_{ij \in E(G)} \sqrt{2}\right)^2}{\sum_{ij \in E(G)}(\varepsilon_G(i) + \varepsilon_G(j))} \\ &\geq \frac{2m^2}{\xi^c(G)} \quad . \end{aligned}$$

Suppose that equality holds in the above inequality. In this case by Lemma 2.5, $\varepsilon_G(i) + \varepsilon_G(j)$ becomes constant for all $ij \in E(G)$.

Conversely, if $\varepsilon_G(i) + \varepsilon_G(j)$ is constant for all $ij \in E(G)$, we can easily see that equality hold in (4).

Theorem 2.7. For any graph *G* we have

$$H_4(G) \leq \frac{\xi^{ce}(G)}{2}, \tag{5}$$

with equality holds if and only if *G* is self centered graph.

Proof. From arithmetic harmonic mean inequality we have

$$\begin{aligned} H_4(G) &= \sum_{ij \in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} \\ &\leq \frac{1}{2} \sum_{ij \in E(G)} \left(\frac{1}{\varepsilon_G(i)} + \frac{1}{\varepsilon_G(j)}\right) \\ &= \frac{1}{2} \sum_{i \in V(G)} \frac{d_G(i)}{\varepsilon_G(i)} = \frac{\xi^{ce}(G)}{2}. \end{aligned}$$

Suppose that equality holds in the above inequality. Then for every $ij \in E(G)$, $\varepsilon_G(i) = \varepsilon_G(j)$. Thus one can easily see that the equality holds in (5) if and only if *G* is self centered graph.

Conversely let *G* be self centered graph. Then by applying $\varepsilon_G(i) = \varepsilon_G(j) = r$ for all $ij \in E(G)$ we get

$$H_4(G) = \sum_{ij \in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)} = \frac{m}{r}$$

and

$$\frac{\xi^{ce}(G)}{2} = \frac{1}{2} \sum_{i \in V(G)} \frac{d_G(i)}{\varepsilon_G(i)} = \frac{1}{2} \sum_{i \in V(G)} \frac{d_G(i)}{r} = \frac{m}{r}.$$

This completes the theorem.

Theorem 2.8. For any graph *G* we have

$$H_4(G) \geq \frac{2m^2 r}{E_2(G) + mr^2}, \tag{6}$$

with equality holds if and only if $\varepsilon_G(i) + \varepsilon_G(j)$ is constant for all $ij \in E(G)$.

Proof. Since $\varepsilon_G(i), \varepsilon_G(j) \geq r$, we have $(\varepsilon_G(i) - r)(\varepsilon_G(j) - r) \geq 0$. Then we get

$$\frac{\varepsilon_G(i)\varepsilon_G(j) + r^2}{r} \geq \varepsilon_G(i) + \varepsilon_G(j) .$$

The equality holds $\varepsilon_G(i) = r$ or $\varepsilon_G(j) = r$ or $\varepsilon_G(i) = \varepsilon_G(j) = r$ for all $ij \in E(G)$. By applying Lemma 2.5 we get

$$H_4(G) = \sum_{ij \in E(G)} \frac{2}{\varepsilon_G(i) + \varepsilon_G(j)}$$

$$\geq \sum_{ij \in E(G)} \frac{2r}{\varepsilon_G(i)\varepsilon_G(j) + r^2} = \sum_{ij \in E(G)} \frac{\left(\sqrt{2r}\right)^2}{\varepsilon_G(i)\varepsilon_G(j) + r^2}$$

$$\geq \frac{\left(\sum_{ij \in E(G)} \sqrt{2r}\right)^2}{\sum_{ij \in E(G)} \varepsilon_G(i)\varepsilon_G(j) + r^2} = \frac{2m^2 r}{E_2(G) + mr^2}.$$

Now suppose that equality holds in (6). Then all the inequalities in the above argument must be equalities. By Lemma 2.5 we have $\varepsilon_G(i) + \varepsilon_G(j)$ is constant for all $ij \in E(G)$.

Conversely if $\varepsilon_G(i) + \varepsilon_G(j)$ is constant for all $ij \in E(G)$, it is easy to see that equality (6) holds.

Theorem 2.9. For any graph *G* we have

$$H_4(G) \leq \frac{\sqrt{(m-1)(mr^2+1)+1}}{r}, \tag{7}$$

with equality holds if and only if $G \cong K_n$.

Proof. From definition of the eccentric harmonic index and the relation $\frac{2}{\varepsilon_G(i)+\varepsilon_G(j)} \leq 1$, we get the following conclusion.

$$H_4^2(G) = \left(\sum_{ij\in E(G)} \frac{2}{\varepsilon_G(i)+\varepsilon_G(j)}\right)^2$$

$$= \sum_{ij\in E(G)} \frac{4}{(\varepsilon_G(i)+\varepsilon_G(j))^2} + 2\sum_{\substack{ij\in E(G)\\ ij\neq kl}} \left(\frac{2}{\varepsilon_G(i)+\varepsilon_G(j)} \cdot \frac{2}{\varepsilon_G(k)+\varepsilon_G(l)}\right)$$

$$H_4^2(G) \leq \sum_{ij\in E(G)} \frac{4}{(\varepsilon_G(i)+\varepsilon_G(j))^2} + 2\sum_{\substack{ij\in E(G)\\ ij\neq kl}} 1.$$

$$\leq \frac{m}{r^2} + m(m-1).$$

So we achieve the desired result. Now suppose that equality holds in (7). Then all the inequalities in the above argument must be equalities. In this case, for all $ij \in E(G)$ should be $\varepsilon_G(i) = \varepsilon_G(j) = 1$. Then the equality holds if and only if $G \cong K_n$.

Conversely, if $G \cong K_n$ then it is easy to see that equality (7) holds.

Lemma 2.10. (*Schwetzers Inequality*) Let $x_1, x_2, \dots, x_n$ be positive real numbers such that $1 \leq i \leq n$ holds $m \leq x_i \leq M$. Then

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right) \leq \frac{n^2(m+M)^2}{4nM}. \tag{8}$$

Equality holds in the (8) only when *n* is even, and the if and only if $x_1 = x_2 = \cdots = x_{\frac{n}{2}} = m$ and $x_{\frac{n}{2}+1} = \cdots = x_n = M$.

Theorem 2.11. For any graph *G* we have

$$H_4(G) \leq \frac{m^2(D+r)^2}{2\xi^c(G)Dr}, \tag{9}$$

with equality holds if and only if *G* is self centered graph.

Proof. Since $2r \leq \varepsilon_G(i) + \varepsilon_G(j) \leq 2D$ for all $ij \in E(G)$, using (8) we have

$$\sum_{ij\in E(G)} (\varepsilon_G(i)+\varepsilon_G(j)) \sum_{ij\in E(G)} \frac{1}{\varepsilon_G(i)+\varepsilon_G(j)} \leq \frac{m^2(2r+2D)^2}{4(2r)(2D)}$$

$$\sum_{ij\in E(G)} \frac{1}{\varepsilon_G(i)+\varepsilon_G(j)} \leq \frac{m^2(r+D)^2}{4\xi^c(G)Dr}$$

$$H_4(G) \leq \frac{m^2(D+r)^2}{2\xi^c(G)Dr}.$$

The equality holds if and only if *G* is self centered graph. We get the required result.

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ON GEOMETRİC PROPERTIES OF WEIGHTED LEBESGUE SEQUENCE SPACES

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Abstract

In this paper we introduce some geometrical and topological properties of weighted Lebesgue sequence spaces $l_{p,w}$ as a generalization of the Lebesgue sequences spaces l_p , where w a weighted sequence.

Keywords: Striclty Convexity; Uniformly Convexity ; Weighted Lebesgue Sequence Spaces.

1. Introduction

If $1 \le p < \infty$, then l_p will denote the space of sequences of real numbers $x = (x_n)$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ [2,8] . A weight sequence $w = w(n) = w_n$ is a positive decreasing sequence such that $w(1) = 1$, $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n$ divergent. The weighted Lebesgue sequence space $l_{p,w}$ for $0 < p < \infty$ is defined as follows:

$$l_{p,w} = \left\{ x = (x_n) : \sum_{n=1}^{\infty} w_n |x_n|^p < \infty, (x_n) \subset \mathbb{R} \right\}$$

and

$$\|x\|_{p,w} = \left(\sum_{n=1}^{\infty} w_n |x_n|^p \right)^{1/p} \tag{1}$$

where $p \ge 1$.

In other words, the weighted sequence space is defined the weight as a multiplier. That is $x \in l_{p,w} \Leftrightarrow x w^{1/p} \in l_p$ weighted sequence spaces $l_{p,w}$ which is considered by author in [9],[10] . It is known that $l_{p,w}$ a Banach space.

A Banach space X is said to be *strictly convex* if $x, y \in X$with $\|x\| = 1, \|y\| = 1$ and $x \neq y$, then $\|(1-\lambda)x + \lambda y\| < 1$ for all $\lambda \in (0,1)$. A Banach space X is said to be *uniformly convex* if the conditions

$$\|x\| \le 1, \|y\| \le 1 \text{ and } \|x - y\| \ge \varepsilon \text{ imply } \left\| \frac{x+y}{2} \right\| \le 1 - \delta \tag{2}$$

holds for all $x, y \in X$. The number

$$\delta(\varepsilon) = inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = 1, \|y\| = 1\,, \|x-y\| \geq \varepsilon\right\} \tag{3}$$

is called the *modulus of convexity*. If $\varepsilon_1 < \varepsilon_2$, then $\delta(\varepsilon_1) < \delta(\varepsilon_2)$ and $\delta(0) = 0$ since $x = y$ if $\varepsilon = 0$[1]. Recently there has been a lot of interest in investigating geometric properties of sequence spaces besides topological. The geometric properties of different sequence spaces are discusssed by some authors. Agarwal, O'regan&Sahu [1] and Castillo&Rafeiro [2] have studied the strict convexity and uniform convexity properties of sequence spaces l_p where $1 < p < \infty$. Savaş, Karakaya and Şimşek [11] have studied some geometric properties of l(p)- type new sequence spaces. Oğur, O [7] has studied some geometric properties of weighted function spaces $L_{p,w}(G)$ where $1 < p < \infty$. In this paper, we introduce some geometric properties of topological of weighted sequence spaces $l_{p,w}$ as a generalization of the l_p .

We will need some auxiliary lemmas to prove that the spaces $l_{p,w}$ are uniformly convex whenever $1 < p < \infty$.

Proposition 1. (Hölder Inequality) Let $x = (x_n) \in l_p, y = (y_n) \in l_q$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{\mathrm{q}} = 1$. Then

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^q\right)^{1/q} \tag{4}$$

Proposition 2. (Minkowski Inequality) Let $x = (x_n),\ y = (y_n) \in l_p,$ If $p \in [1, \infty)$, then

$$\left(\sum_{k=1}^{\infty} (|x_k| + |y_k|)^p\right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p} \tag{5}$$

If $p \in (0,1)$, then

$$\left(\sum_{k=1}^{\infty} (|x_k| + |y_k|)^p\right)^{1/p} \geq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p} \tag{6}$$

We need some lemmas dealing with inequalities.

Lemma 1. Let $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{7}$$

for $a \geq 0$, $b \geq 0$ [8].

Lemma 2. If $p \geq 1$ and $a, b > 0$, then

$$(a+b)^p \leq 2^{p-1}(a^p + b^p) \tag{8}$$

[6].

2. Main Results

Proposition 3. Let $w = (w_k)$ a weighted sequence and $w_k > 1$ for all $k \in \mathbb{N}$. Then $l_{p,w} \subset l_p$. Also if $0 < p < q < \infty$, $l_{p,w} \subsetneq l_{q,w}$ for $w_k > 1$.

Proof. It can be easily seen that $l_{p,w} \subset l_p$ and $l_{p,w} \subset l_{q,w}$ for $0 < p < q < \infty$ .To show that $l_{p,w} \neq l_{q,w}$, we take the sequences $x_k = k^{-1/2p}$ and $w_k = \frac{1}{\sqrt{k}}$ for all $k \in \mathbb{N}$ with $1 \leq p < q < \infty$. Since $p < q$, we have $\frac{q}{p} > 1$ and $\frac{q}{2p} + \frac{1}{2} > 1$. We write

$$\sum_{k=1}^{\infty} w_k |x_k|^q = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} . \frac{1}{k^{q/2p}} = \sum_{k=1}^{\infty} \frac{1}{k^{q/2p+1/2}} < \infty$$

The last series is convergent since it is a hyper-harmonic series with exponent bigger than 1, therefore $x \in l_{q,w}$. On the other hand

$$\sum_{k=1}^{\infty} w_k |x_k|^p = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} . \frac{1}{k^{1/2}} = \sum_{k=1}^{\infty} \frac{1}{k}$$

and $x \notin l_{p,w}$.

Proposition 4. The space $l_{p,w}$ is seperable whenever $1 \leq p < \infty$ and w a weighted sequence.

Proof. Let M be the set of all sequences of the form $q = (q_1, q_2, \cdots, q_n, 0,0, \cdots)$ where $n \in \mathbb{N}$ and $q_k \in \mathbb{Q}$. We will show that M is dense in $l_{p,w}$. Since $\sum_{k=1}^{\infty} |x_k|^p w_k$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} |x_k|^p w_k < \frac{\varepsilon^p}{2}$$

for all $\varepsilon > 0$. Since $\overline{\mathbb{Q}} = \mathbb{R}$, we have that for each (x_k) there exists a rational q_k such that

$$|x_k - q_k| < \frac{\varepsilon}{\sqrt[p]{2^n}}$$

hence

$$\sum_{k=1}^{n} |x_k - q_k| w_k < \frac{\varepsilon^p}{2K}$$

where $K = maks\{w_1, w_2, \cdots, w_n\}$. We write

$$\|x - q\|_{p,w}^p = \sum_{k=1}^{n} |x_k - q_k|^p w_k + \sum_{k=n+1}^{\infty} |x_k|^p w_k < \varepsilon^p$$

and so $\|x - q\|_{p,w} < \varepsilon$. This shows that M is dense in $l_{p,w}$.

Theorem 1. The space $l_{p,w}$ is convex, whenever $0 < p < \infty$.

Proof. This show that $tx + (1-t)y \in l_{p,w}$ for $x = (x_n), y = (y_n) \in l_{p,w}$ and $t \in [0,1]$. Let us distinguish two cases:

First case $p \geq 1$. By Lemma 2 and Minkowski's inequality , we write

$$\sum_{n=1}^{\infty}|tx_n+(1-t)y_n|^p w_n = \sum_{n=1}^{\infty}\left|(tx_n+(1-t)y_n)w_n{}^{1/p}\right|^p$$
$$=\left[\left(\sum_{n=1}^{\infty}\left|(tx_n+(1-t)y_n)w_n{}^{1/p}\right|^p\right)^{1/p}\right]^p$$
$$\leq\left[\left(\sum_{n=1}^{\infty}\left|(tx_n)w_n{}^{1/p}\right|^p\right)^{1/p}+\left(\sum_{n=1}^{\infty}\left|\big((1-t)y_n\big)w_n{}^{1/p}\right|^p\right)^{1/p}\right]^p$$
$$\leq 2^{p-1}\left[\sum_{n=1}^{\infty}\left|(tx_n)w_n{}^{1/p}\right|^p+\sum_{n=1}^{\infty}\left|\big((1-t)y_n\big)w_n{}^{1/p}\right|^p\right]$$
$$=2^{p-1}\sum_{n=1}^{\infty}|tx_n|^p w_n+2^{p-1}\sum_{n=1}^{\infty}|(1-t)y_n|^p w_n$$
$$=2^{p-1}|t|^p\sum_{n=1}^{\infty}|x_n|^p w_n+2^{p-1}|1-t|^p\sum_{n=1}^{\infty}|y_n|^p w_n$$
$$<\infty$$

which shows that $tx+(1-t)y \in l_{p,w}$ for $p \geq 1$.

Second case $0<p<1$. Let $x=(x_n), y=(y_n) \in l_{p,w}$ and $t \in [0,1]$. By Lemma 1, we have

$$\sum_{n=1}^{\infty}|tx_n+(1-t)y_n|^p w_n = \sum_{n=1}^{\infty}\left|(tx_n+(1-t)y_n)w_n{}^{1/p}\right|^p$$
$$\leq\sum_{n=1}^{\infty}\left|(tx_n)w_n{}^{1/p}\right|^p+\sum_{n=1}^{\infty}\left|\big((1-t)y_n\big)w_n{}^{1/p}\right|^p$$
$$=\sum_{n=1}^{\infty}|tx_n|^p w_n+\sum_{n=1}^{\infty}\left|\big((1-t)y_n\big)\right|^p w_n$$
$$=|t|^p\sum_{n=1}^{\infty}|x_n|^p w_n+|1-t|^p\sum_{n=1}^{\infty}|y_n|^p w_n<\infty$$

This completes the proof. It is known that the space l_p is strictly convex for $p \geq 1$ [1].

Theorem 2. The space $l_{p,w}$ is strictly convex for $p \geq 1$.

Proof. Let $x=(x_n), y=(y_n) \in l_{p,w}$ with $x \neq y$, $\|x\|_{p,w}=1, \|y\|_{p,w}=1$ and $0<p<1$. Then $\left\|xw^{\frac{1}{p}}\right\|_p=1$, $\left\|yw^{\frac{1}{p}}\right\|_p=1$. Since l_p is strictly convex for $p \geq 1$, we have

$$\left\|(1-t)xw^{\frac{1}{p}}+tyw^{\frac{1}{p}}\right\|_p=\left\|\big((1-t)x+ty\big)w^{\frac{1}{p}}\right\|_p<1.$$

Hence

$$\|(1-t)x+ty\|_{p,w} = \left(\sum_{n=1}^{\infty}\left|\left((1-t)x+ty\right)w^{\frac{1}{p}}\right|^{p}\right)^{1/p}$$
$$= \left\|\left((1-t)x+ty\right)w^{\frac{1}{p}}\right\|_{p} < 1$$

We will need the following inequality.

Lemma 3. Let $p \geq 2$. We have

$$(|a+b|^{p}+|a-b|^{p})^{1/p} \leq (|a+b|^{2}+|a-b|^{2})^{1/2} \tag{9}$$

for all $a, b \in \mathbb{R}$ [2].

Lemma 4. Let $2 \leq p < \infty$ and $x, y \in l_{p}$, we have

$$\|x+y\|_{p}^{p}+\|x-y\|_{p}^{p} \leq 2^{p-1}\left(\|x\|_{p}^{p}+\|y\|_{p}^{p}\right) \tag{10}$$

[1].

Proposition 5. If $2 \leq p < \infty$, then we have

$$\|x+y\|_{p,w}^{p}+\|x-y\|_{p,w}^{p} \leq 2^{p-1}\left(\|x\|_{p,w}^{p}+\|y\|_{p,w}^{p}\right) \tag{11}$$

for $x = (x_n)$, $y = (y_n) \in l_{p,w}$.

Proof. Let $x, y \in l_{p,w}$. Then $xw^{\frac{1}{p}}, yw^{\frac{1}{p}} \in l_p$. By Lemma 4, we write

$$\|x+y\|_{p,w}^{p}+\|x-y\|_{p,w}^{p} = \left\|xw^{\frac{1}{p}}+ yw^{\frac{1}{p}}\right\|_{p}^{p}+\left\|xw^{\frac{1}{p}}- yw^{\frac{1}{p}}\right\|_{p}^{p}$$
$$\leq 2^{p-1}\left(\left\|xw^{\frac{1}{p}}\right\|_{p}^{p}+\left\|yw^{\frac{1}{p}}\right\|_{p}^{p}\right)$$
$$= 2^{p-1}\left(\|x\|_{p,w}^{p}+\|y\|_{p,w}^{p}\right)$$

Theorem 3. The space $l_{p,w}$ is uniformly convex for $2 \leq p < \infty$.

Proof . Let $x = (x_n), y = (y_n) \in l_{p,w}$ with

$$\|x\|_{p,w} \leq 1, \|y\|_{p,w} \leq 1 \text{ and } \|x-y\|_{p,w} \geq \varepsilon$$

By Proposition 5, we have

$$\|x+y\|_{p,w}^{p} \leq 2^{p-1}\left(\|x\|_{p,w}^{p}+\|y\|_{p,w}^{p}\right)-\|x-y\|_{p,w}^{p}$$
$$\leq 2^{p-1}.2-\varepsilon^{p}$$
$$= 2^{p}\left(1-\left(\frac{\varepsilon}{2}\right)^{p}\right)$$

so it follows that $\left\|\frac{x+y}{2}\right\|_{p,w}^{p} \leq 1-\left(\frac{\varepsilon}{2}\right)^{p}$ and hence we get $\left\|\frac{x+y}{2}\right\|_{p,w} \leq 1-\delta$ such that

$$\delta(\varepsilon) = 1-\left(1-\left(\frac{\varepsilon}{2}\right)^{p}\right)^{1/p}$$

Lemma 6. Let $1 < p \leq 2$ and $q = \frac{p}{p-1}$, then

$$|a+b|^{q} + |a-b|^{q} \leq 2(|a|^{p}+|b|^{p})^{q-1} \tag{12}$$

for all real numbers a and b [3].

Lemma 7. $1 < p \leq 2$ and $q = \frac{p}{p-1}$, we have

$$\|x+y\|_{p}^{q} + \|x-y\|_{p}^{q} \leq 2\left(\|x\|_{p}^{p}+\|y\|_{p}^{p}\right)^{q-1} \tag{13}$$

for all $x, y \in l_p$ [5].

Proposition 6. If $1 < p \leq 2$, then

$$\|x+y\|_{p,w}^{q} + \|x-y\|_{p,w}^{q} \leq 2\left(\|x\|_{p,w}^{p}+\|y\|_{p,w}^{p}\right)^{q-1} \tag{14}$$

for $x=(x_n)$, $y=(y_n) \in l_{p,w}$ and $q = \frac{p}{p-1}$.

Proof. Let $x=(x_n), y=(y_n) \in l_{p,w}$ and by the Minkowski's inequality for $0 < r < 1$, we have

$$\left(\sum_{n=1}^{\infty}|a_n|^{r}\right)^{1/r} + \left(\sum_{n=1}^{\infty}|b_n|^{r}\right)^{1/r} \leq \left(\sum_{n=1}^{\infty}|a_n+b_n|^{r}\right)^{1/r} \tag{15}$$

If $1 < p \leq 2$, we replace r by $\frac{p}{q}$ in Equation (15), for $a_n = \left|\left((x_n+y_n){w_n}^{1/p}\right)\right|^{q}$, $b_n = \left|(x_n-y_n){w_n}^{1/p}\right|^{q}$, then by Lemma 6 we get

$$\left(\sum_{n=1}^{\infty}\left|(x_n+y_n){w_n}^{1/p}\right|^{p}\right)^{q/p} + \left(\sum_{n=1}^{\infty}\left|(x_n-y_n){w_n}^{1/p}\right|^{p}\right)^{q/p}$$

$$\leq \left[\sum_{n=1}^{\infty}\left(\left|(x_n+y_n){w_n}^{1/p}\right|^{q} + \left|(x_n-y_n){w_n}^{1/p}\right|^{q}\right)^{p/q}\right]^{q/p}$$

$$= \left[\sum_{n=1}^{\infty}\left(\left|x_n{w_n}^{1/p}+y_n{w_n}^{1/p}\right|^{q} + \left|x_n{w_n}^{1/p}-y_n{w_n}^{1/p}\right|^{q}\right)^{p/q}\right]^{q/p}$$

$$\leq \left(\sum_{n=1}^{\infty}\left[2\left(\left|x_n{w_n}^{1/p}\right|^{p} + \left|y_n{w_n}^{1/p}\right|^{p}\right)^{q-1}\right]^{p/q}\right)^{q/p}$$

$$= 2\left[\sum_{n=1}^{\infty}\left(\left|x_n{w_n}^{1/p}\right|^{p} + \left|y_n{w_n}^{1/p}\right|^{p}\right)\right]^{q/p}$$

$$= 2\left[\sum_{n=1}^{\infty}|x_n|^p w_n + \sum_{n=1}^{\infty}|y_n|^p w_n\right]^{q/p}$$

where $q = \frac{p}{p-1} \Rightarrow q - 1 = \frac{q}{p}$. Thus , we obtain

$$\|x+y\|_{p,w}^q + \|x-y\|_{p,w}^q \leq 2\left(\|x\|_{p,w}^p + \|y\|_{p,w}^p\right)^{q-1}$$

Theorem 4. The space $l_{p,w}$ is uniformly convex for $1 < p \leq 2$.

Proof. Let $x = (x_n), y = (y_n) \in l_{p,w}$, $1 < p \leq 2$ with

$$\|x\|_{p,w} \leq 1, \|y\|_{p,w} \leq 1 \text{ and } \|x-y\|_{p,w} \geq \varepsilon$$

Then by the Proposition 6, we have

$$\|x+y\|_{p,w}^q \leq 2\left[\|x\|_{p,w}^p + \|y\|_{p,w}^p\right]^{q-1} - \|x-y\|_{p,w}^q$$
$$\leq 2.2^{q-1} - \varepsilon^q$$
$$= 2^q\left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)$$

Hence, we write

$$\left\|\frac{x+y}{2}\right\|_{p,w} \leq \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{1/q}$$

where $\delta(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{1/q}$.

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DINI-TYPE HELICOIDAL HYPERSURFACE IN 4-SPACE

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Abstract

We define Dini-type helicoidal hypersurface in the four dimensional Euclidean space $\mathbb{E}^4$. We calculate the Gauss map, Gaussian curvature and the mean curvature of the helicoidal hypersurface. Additionally, we find some special relations and symmetries for the curvatures.

Keywords: Dini-type helicoidal hypersurface; Four dimensional Euclidean space; Gauss map.

1. Introduction

After Moore [27,28], Takahashi [32], and also Chen and Piccinni [8], the theory of submanifolds has been studied by many mathematicians. For some papers about the topic, see $[1-7,\ 9-12,\ 14-26,\ 29-31,\ 33-35]$.

In this work, considering Ulisse Dini's paper [13] in Euclidean 3-space $\mathbb{E}^3$, we study Dini-type helicoidal hypersurface in Euclidean 4-space $\mathbb{E}^4$. We give some basic notions of the geometry of the $\mathbb{E}^4$ in this section. In section 2, we define helicoidal hypersurface. Moreover, we give Dini-type helicoidal hypersurface, and calculate its curvatures obtaining some special symmetries in the last section.

Next, we will introduce the first and second fundamental forms, matrix of the shape operator **S**, Gaussian curvature K, and the mean curvature H of hypersurface $\mathbf{M} = \mathbf{M}(u, v, w)$ in Euclidean 4-space $\mathbb{E}^4$. We shall identify a vector (a,b,c,d) with its transpose $(a,b,c,d)^t$.

Let $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of a hypersurface M^3 in the $\mathbb{E}^4$. The triple vector product of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$ on $\mathbb{E}^4$ is defined as follows:

$$\begin{aligned}\vec{x}\times\vec{y}\times\vec{z} = (&x_2y_3z_4 - x_2y_4z_3 - x_3y_2z_4 + x_3y_4z_2 + x_4y_2z_3 - x_4y_3z_2,\\ &-x_1y_3z_4 + x_1y_4z_3 + x_3y_1z_4 - x_3z_1y_4 - y_1x_4z_3 + x_4y_3z_1,\\ &x_1y_2z_4 - x_1y_4z_2 - x_2y_1z_4 + x_2z_1y_4 + y_1x_4z_2 - x_4y_2z_1,\\ &-x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 - x_2y_3z_1 - x_3y_1z_2 + x_3y_2z_1).\end{aligned}$$

For a hypersurface $\mathbf{M} = \mathbf{M}(u, v, w)$ in 4-space, we compute

$$\det I = \det\begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix} = (EG - F^2)C - A^2G + 2ABF - B^2E,$$

$$\det II = \det\begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix} = (LN - M^2)V - P^2N + 2PTM - T^2L,$$

where

$$\begin{aligned} &E = \mathbf{M}_u \cdot \mathbf{M}_u,\ F = \mathbf{M}_u \cdot \mathbf{M}_v,\ G = \mathbf{M}_v \cdot \mathbf{M}_v,\\ &L = \mathbf{M}_{uu} \cdot e,\quad M = \mathbf{M}_{uv} \cdot e,\quad N = \mathbf{M}_{vv} \cdot e,\\ &A = \mathbf{M}_u \cdot \mathbf{M}_w,\ B = \mathbf{M}_v \cdot \mathbf{M}_w,\ C = \mathbf{M}_w \cdot \mathbf{M}_w,\\ &P = \mathbf{M}_{uw} \cdot e,\quad T = \mathbf{M}_{vw} \cdot e,\quad V = \mathbf{M}_{ww} \cdot e,\end{aligned}$$

and e is the Gauss map

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|}.$$

Using $(I)^{-1}.(II)$, we get shape operator matrix $\mathbf{S}$, as follows:

$$\mathbf{S} = \frac{1}{\det I}\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix},$$

where

$$
\begin{aligned}
s_{11} &= ABM - CFM - AGP + BFP + CGL - B^2L, \\
s_{12} &= ABN - CFN - AGT + BFT + CGM - B^2M, \\
s_{13} &= ABT - CFT - AGV + BFV + CGP - B^2P, \\
s_{21} &= ABL - CFL + AFP - BPE + CME - A^2M, \\
s_{22} &= ABM - CFM + AFT - BTE + CNE - A^2N, \\
s_{23} &= ABP - CFP + AFV - BVE + CTE - A^2T, \\
s_{31} &= -AGL + BFL + AFM - BME + GPE - F^2P, \\
s_{32} &= -AGM + BFM + AFN - BNE + GTE - F^2T, \\
s_{33} &= -AGP + BFP + AFT - BTE + GVE - F^2V.
\end{aligned}
$$

Finally, we obtain following formulas of the Gaussian curvature K, and the mean curvature H, respectively,

$$K = \frac{(LN - M^2)V + 2MPT - P^2N - T^2L}{(EG - F^2)C + 2ABF - A^2G - B^2E},$$

and

$$H = \frac{(EN + GL - 2FM)C + (EG - F^2)V - A^2N - B^2L - 2(APG + BTE - ABM - ATF - BPF)}{3[(EG - F^2)C + 2ABF - A^2G - B^2E]}.$$

When $K = 0$, hypersurface is flat; and $H = 0$, then hypersurface is minimal.

2. Helicoidal Hypersurface

In this section, we define the rotational hypersurface and helicoidal hypersurface in $\mathbb{E}^4$. Let $\gamma: I \subset \mathbb{R} \to \Pi$ be a curve in a plane Π in $\mathbb{E}^4$, and let ℓ be a straight line in Π. In $\mathbb{E}^4$, a *rotational hypersurface* is defined by a hypersurface rotating profile curve γ around axis ℓ.

Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel lines orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation. Resulting hypersurface is called *helicoidal hypersurface* with axis ℓ, pitches $a, b \in \mathbb{R} - \{0\}$. Supposing ℓ is the line spanned by the vector $(0,0,0,1)^t$, we consider following orthogonal matrix:

$$Z(v,w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $v, w \in \mathbb{R}$. The matrix Z supplies the following equations, simultaneously,

$$Z\ell = \ell, \quad ZZ^t = Z^tZ = I_4, \quad \det Z = 1.$$

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the x_4-axis of $\mathbb{E}^4$. The profile curve is given by $\gamma(u) = (u, 0,0, \varphi(u))$, where $\varphi(u): I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function for all $u \in I$. Therefore, the helicoidal hypersurface, spanned by the vector $(0,0,0,1)$, is defined by as follows:

$$\mathbf{H}(u, v, w) = Z\gamma^t + (av + bw)\ell^t,$$

where $u \in I$, $v,w \in [0,2\pi]$, $a,b \in \mathbb{R} - \{0\}$. More clear form of the helicoidal hypersurface in 4-space is given by as follows:

$$\mathbf{H}(u, v, w) = \begin{pmatrix} u\cos v\cos w \\ u\sin v\cos w \\ u\sin w \\ \varphi(u) + av + bw \end{pmatrix}.$$

3. Dini-Type Helicoidal Hypersurface

We consider Dini-type helicoidal hypersurface as follows:

$$\mathfrak{D}(u, v, w) = \begin{pmatrix} \sin u\cos v\cos w \\ \sin u\sin v\cos w \\ \sin u\sin w \\ \varphi(u) + av + bw \end{pmatrix},$$

where $u \in \mathbb{R} - \{0\}$, $v,w \in [0,2\pi]$. Using the first differentials of $\mathfrak{D}$ with respect to u,v,w, we get the first quantities as follows

$$I = \begin{pmatrix} \varphi'^2 + \cos^2 u & a\varphi' & b\varphi' \\ a\varphi' & (a^2 - \cos^2 u)\cos^2 w & ab \\ b\varphi' & ab & b^2 + \sin^2 u \end{pmatrix},$$

and have

$$\det I = \sin^2 u[(\cos^2 u\cos^2 w + a^2)\cos^2 u - \varphi'^2\cos^2 w],$$

where $\varphi = \varphi(u)$, $\varphi' = \frac{d\varphi}{du}$. Using the second differentials of $\mathfrak{D}$ with respect to u,v,w, we have the second quantities as follows

$$II = \begin{pmatrix} -\frac{\sin^2 u \cos w\left(\varphi'' \cos u + \varphi' \sin u\right)}{\sqrt{\det I}} & \frac{a \cos^2 u \sin u \cos w}{\sqrt{\det I}} & \frac{b \sin u \cos^2 u \cos w}{\sqrt{\det I}} \\ \frac{a \cos^2 u \sin u \cos w}{\sqrt{\det I}} & \frac{\sin^2 u \cos^2 w\left(b \cos u \sin w - \varphi' \sin u \cos w\right)}{\sqrt{\det I}} & -\frac{a \sin^2 u \cos u \sin w}{\sqrt{\det I}} \\ \frac{b \sin u \cos^2 u \cos w}{\sqrt{\det I}} & -\frac{a \sin^2 u \cos u \sin w}{\sqrt{\det I}} & \frac{\varphi' \sin^3 u \cos w}{\sqrt{\det I}} \end{pmatrix},$$

and we get

$$\det II = \frac{\left(\begin{array}{c} \varphi'^2 \varphi'' \sin^8 u \cos u \cos^5 w - b\varphi' \varphi'' \sin^7 u \cos^2 u \sin w \cos^4 w \\ +a^2 \varphi'' \sin^6 u \cos^3 u \sin^2 w \cos w \\ +\varphi'^3 \sin^9 u \cos^5 w - b\varphi'^2 \sin^8 u \cos u \sin w \cos^4 w \\ +(a^2 \sin^7 u \cos^2 u \sin^2 w \cos w - a^2 \sin^5 u \cos^4 u \cos^3 w + b^2 \sin^5 u \cos^4 u \cos^5 w)\varphi' \\ -2a^2 b \sin^4 u \cos^5 u \sin w \cos^2 w - b^3 \sin^4 u \cos^5 u \sin w \cos^4 w \end{array}\right)}{(\det I)^{3/2}}.$$

The Gauss map e of the helicoidal hypersurface $\mathfrak{D}$ is

$$e_{\mathfrak{D}} = \frac{1}{\sqrt{\det I}} \begin{pmatrix} (\varphi' \sin u \cos v - a \cos u \sin v - b \cos u \cos v \sin w \cos w) \sin u \\ (\varphi' \sin u \sin v - a \cos u \cos v - b \cos u \sin v \sin w \cos w) \sin u \\ (\varphi' \sin u \sin w + b \cos u \cos w) \sin u \cos w \\ -\sin^2 u \cos u \cos w \end{pmatrix}.$$

Finally, we calculate the Gaussian curvature of $\mathfrak{D}$, as follows:

$$K = \frac{\alpha_1 \varphi'^2 \varphi'' + \alpha_2 \varphi' \varphi'' + \alpha_3 \varphi'' + \alpha_4 \varphi'^3 + \alpha_5 \varphi'^2 + \alpha_6 \varphi' + \alpha_7}{[\sin^2 u((\cos^2 u \cos^2 w + a^2) \cos^2 u - \varphi'^2 \cos^2 w)]^{5/2}},$$

where

$$\begin{aligned} \alpha_1 &= \sin^8 u \cos u \cos^5 w, \\ \alpha_2 &= -b \sin^7 u \cos^2 u \sin w \cos^4 w, \\ \alpha_3 &= a^2 \sin^6 u \cos^3 u \sin^2 w \cos w, \\ \alpha_4 &= \sin^9 u \cos^5 w, \\ \alpha_5 &= -b \sin^8 u \cos u \sin w \cos^4 w, \\ \alpha_6 &= a^2 \sin^7 u \cos^2 u \sin^2 w \cos w - a^2 \sin^5 u \cos^4 u \cos^3 w + b^2 \sin^5 u \cos^4 u \cos^5 w, \\ \alpha_7 &= -2a^2 b \sin^4 u \cos^5 u \sin w \cos^2 w - b^3 \sin^4 u \cos^5 u \sin w \cos^4 w, \end{aligned}$$

and we calculate the mean curvature of $\mathfrak{D}$, as follows:

$$H = \frac{\beta_1 \varphi'' + \beta_2 \varphi'^3 + \beta_3 \varphi'^2 + \beta_4 \varphi' + \beta_5}{3[\sin^2 u((\cos^2 u \cos^2 w + a^2)\cos^2 u - \varphi'^2 \cos^2 w)]^{3/2}},$$

where

$$\begin{aligned}
\beta_1 &= [(b^2 + \sin^2 u)\cos^2 w + a^2]\sin^4 u \cos u \cos w, \\
\beta_2 &= -2\sin^3 u \cos^3 w, \\
\beta_3 &= -b \sin^4 u \cos u \sin w \cos^2 w, \\
\beta_4 &= 2\Big[\Big(-\tfrac{\cos^4 u}{2} + \tfrac{b^2}{2} + b^2 \cos^2 u + \tfrac{1}{2}\Big)\cos^2 w + a^2\Big(\cos^2 u + \tfrac{1}{2}\Big)\Big]\sin^2 u \cos w, \\
\beta_5 &= [(b^2 + \sin^2 u)\cos^2 w + a^2]\sin^3 u \cos^3 u.
\end{aligned}$$

Hence, we have following theorems:

Theorem 1. *Let* $\mathfrak{D} : M^3 \to \mathbb{E}^4$ *be an isometric immersion. If* M^3 *is minimal, then we get*

$$\beta_1 \varphi'' + \beta_2 \varphi'^3 + \beta_3 \varphi'^2 + \beta_4 \varphi' + \beta_5 = 0.$$

Theorem 2. *Let* $\mathfrak{D} : M^3 \to \mathbb{E}^4$ *be an isometric immersion.* If M^3 *is flat, then we have*

$$\alpha_1 \varphi'^2 \varphi'' + \alpha_2 \varphi' \varphi'' + \alpha_3 \varphi'' + \alpha_4 \varphi'^3 + \alpha_5 \varphi'^2 + \alpha_6 \varphi' + \alpha_7 = 0.$$

Solutions of these two eqs. are attracted problem.

Now, taking $\varphi(u) = \cos u + ln\left(\tan\frac{u}{2}\right)$ in Theorem 1, and Theorem 2, we obtain following corollaries:

Corollary 1. *When Dini-type helicoidal hypersurface* $\mathfrak{D}$ *has* $H = 0$ *in 4-space, then we have*

$$\sum_{i=0}^{6} \Phi_i \tan^i\left(\frac{u}{2}\right) = 0,$$

where

$$\begin{aligned}
\Phi_6 &= \beta_2, \\
\Phi_5 &= 2\beta_1 - 6\beta_2 \sin u + 2\beta_3, \\
\Phi_4 &= 9\beta_2 - 6\beta_2 \cos 2u - 8\beta_3 \sin u + 4\beta_4, \\
\Phi_3 &= -8\beta_1 \cos u - 18\beta_2 \sin u + 2\beta_2 \sin 3u + 8\beta_3 - 4\beta_3 \cos 2u - 8\beta_4 \sin u + 8\beta_5, \\
\Phi_2 &= 9\beta_2 - 6\beta_2 \cos 2u - 8\beta_3 \sin u + 4\beta_4, \\
\Phi_1 &= 2\beta_1 - 6\beta_2 \sin u + 2\beta_3, \\
\Phi_0 &= \beta_2.
\end{aligned}$$

Corollary 2. *When Dini-type helicoidal hypersurface $\mathfrak{D}$ has $K = 0$ in 4-space, then we get*

$$\sum_{j=0}^{8} \Psi_j \tan^j\left(\frac{u}{2}\right) = 0,$$

where

$$\Psi_8 = \alpha_1,$$
$$\Psi_7 = -4\alpha_1 \sin u + 2\alpha_2 + 2\alpha_4,$$
$$\Psi_6 = \begin{pmatrix} 2\alpha_1 - 4\alpha_1 \cos u + 4\alpha_1 \sin^2 u - 4\alpha_2 \sin u \\ +4\alpha_3 - 12\alpha_4 \sin u + 4\alpha_5 \end{pmatrix},$$
$$\Psi_5 = \begin{pmatrix} -4\alpha_1 \sin u + 16\alpha_1 \cos u \sin u + 2\alpha_2 - 8\alpha_2 \cos u \\ +6\alpha_4 + 24\alpha_4 \sin^2 u - 16\alpha_5 \sin u + 8\alpha_6 \end{pmatrix},$$
$$\Psi_4 = \begin{pmatrix} -8\alpha_1 \cos u - 16\alpha_1 \cos u \sin^2 u + 16\alpha_2 \cos u \sin u \\ -16\alpha_3 \cos u - 24\alpha_4 \sin u - 16\alpha_4 \sin^3 u \\ +8\alpha_5 + 16\alpha_5 \sin^2 u - 16\alpha_6 \sin u + 16\alpha_7 \end{pmatrix},$$
$$\Psi_3 = \begin{pmatrix} -4\alpha_1 \sin u + 16\alpha_1 \cos u \sin u + 2\alpha_2 - 8\alpha_2 \cos u \\ +6\alpha_4 + 24\alpha_4 \sin^2 u - 16\alpha_5 \sin u + 8\alpha_6 \end{pmatrix},$$
$$\Psi_2 = \begin{pmatrix} 2\alpha_1 - 4\alpha_1 \cos u + 4\alpha_1 \sin^2 u - 4\alpha_2 \sin u \\ +4\alpha_3 - 12\alpha_4 \sin u + 4\alpha_5 \end{pmatrix},$$
$$\Psi_1 = -4\alpha_1 \sin u + 2\alpha_2 + 2\alpha_4,$$
$$\Psi_0 = \alpha_1.$$

Remark 1. *From Corollary 1, and Corollary 2, we obtain following special symmetries for Dini-type helicoidal hypersurface $\mathfrak{D}$, respectively,*

$$\Phi_6 = \Phi_0,\ \Phi_5 = \Phi_1,\ \Phi_4 = \Phi_2,$$

and

$$\Psi_8 = \Psi_0,\ \Psi_7 = \Psi_1,\ \Psi_6 = \Psi_2,\ \Psi_5 = \Psi_3.$$

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ON TIMES SCALE FRACTIONAL ORDER DIFFERENTIAL EQUATION INVOLVING RANDOM VARIABLE

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Abstract

In this paper, nonlocal and boundary value problems (BVP) of fractional differential equations involv- ing random walk on times scale is discussed. The sufficient conditions for existence and uniqueness of dynamical systems are obtained using standard fixed point methods. The stability of solutions is made sure by Ulam-Hyers stability method.

Keywords: Dynamical equations, Fractional calculus, Existence, Stability.

1. Introduction

The theory of time scales calculus allows us to study the dynamic equations, which include both difference and differential equations. Since the study on dynamic equations on time scales has received much attention of many researchers in recent days, see [1, 2, 3, 4, 5] and the references therein.

Randomness of the FDEs which arises in uncertainties and complexities. Such deterministic equations are hardly called as Random differential equations (RDEs). The recent development of RDEs of fractional order can be seen in [15, 18, 25].

Ever since the birth of Fractional differential equations (FDEs) in sixteenth century only in past few decades it received tremendous development in describe the real-life phenomena more accurately than integer order derivative. The main aspect of FDEs is to prove existence, uniqueness and stability of solutions. For the detailed study of FDEs one can refer to the books [11, 16, 17] and the papers [7, 9, 14, 19, 24]. The literature provides numerous numbers of fractional derivatives with singular kernals. Here in this article we use a special kind of fractional derivative called ψ-Hilfer fractional derivative integrate several classical derivative, detailed in [20]. For the recent works on ψ-Hilfer fractional derivative we refer the readers to [6, 10, 22, 23]

On the other hand, the stability investigation of differential and integral equations is important in applications. Here we extend the results of Ulam Hyers stability and Ulam Hyers Rassias(U-H-R) stability to fractional RDEs on times scale. The stability check of FDEs and theoretical analysis of Ulam type stability can be seen in [12, 21, 26].

From the above discussion and motivation in this work we study ψ-Hilfer fractional RDEs on times scale with boundary and nonlocal conditions. The existence, uniqueness and stability solutions are obtained by fixed point methods. First consider the BVP for ψ-Hilfer fractional RDEs on times scale of the form

$$\begin{cases} {}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{u}(t,\omega)=\mathfrak{g}(t,\mathfrak{u}(t,\omega),\omega), \quad t\in J\subseteq\mathbb{T}, \\ a\ {}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(t,\omega)|_{t=0}+b\ {}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(t,\omega)|_{t=T}=c, \end{cases} \tag{1}$$

where (Ω,F,p) is a complete probability space, $\omega\in\Omega$, ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ is the ψ-HFD defined on $\mathbb{T}$, $0<\alpha<1$, $0\leq\beta\leq 1$ and ${}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}$ is ψ-fractional integral of order $1-\gamma(\gamma=\alpha+\beta-\alpha\beta)$. Let $\mathbb{T}$ be a time scale, that is nonempty subset of Banach space. The function $\mathfrak{g}:J:=[0,b]\times R\times\Omega\to R$ is a right-dense continuous function. Here, the Eq. (1) satisfies the random integral equation of the form

$$\mathfrak{u}(t,\omega)=\quad\left(c-b\ {}^{\mathbb{T}}\mathfrak{I}^{1-\beta+\alpha\beta;\psi}\mathfrak{g}(T,\mathfrak{u}(T,\omega),\omega)\right)\frac{(\psi(t)-\psi(0))^{\gamma-1}}{(a+b)\Gamma(\gamma)}+^{\mathbb{T}}\mathfrak{I}^{\alpha;\psi}\mathfrak{g}(t,\mathfrak{u}(t,\omega),\omega)\Delta s. \tag{2}$$

In the next section, we consider the nonlocal fractional random differential equation on times scale

$$\begin{cases} {}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{u}(t,\omega)=\mathfrak{g}(t,\mathfrak{u}(t,\omega),\omega) \quad t\in J, \\ {}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(t,\omega)|_{t=0}=\sum_{i=1}^{m}c_i\mathfrak{u}(\tau_i,\omega), \quad \tau_i\in J, \end{cases} \tag{3}$$

where $\tau_i, i=0,1,...,m$ are prefixed points satisfying $0<\tau_1\leq...\leq\tau_m<b$ and c_i is real numbers. Here, nonlocal condition $\mathfrak{u}(0,\omega)=\sum_{i=1}^{m}c_i\mathfrak{u}(\tau_i,\omega)$ can be applied in physical problems yields better effect than the initial conditions ${}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(t,\omega)|_{t=0}=\mathfrak{u}_0$. Further (3) is equivalent to mixed integral type of the form

$$\mathfrak{u}(t,\omega)=\begin{cases} \frac{T}{\Gamma(\alpha)}\left(\psi(t)-\psi(0)\right)^{\gamma-1}\sum_{i=1}^{m}c_i\int_0^{\tau_i}\psi'(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s \\ +\frac{1}{\Gamma(\alpha)}\int_0^{t}\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s \end{cases} \tag{4}$$

where

$$T=\frac{1}{\Gamma(\gamma)-\sum_{i=1}^{m}c_i(\psi(\tau_i)-\psi(0))^{\gamma-1}}.$$

The novelty of paper is given as follows: In Section 2, basic definitions and preliminary are discussed. Existence, uniqueness and stability with random walk for BVP and nonlocal problems are discussed in Section 3 and Section 4 respectively.

2. Preliminaries

Definition 2.1. *Let $C(J)$ be continuous function endowed with the norm*

$$\|\mathfrak{u}\|_C=\max\left\{|\mathfrak{u}(t,\omega)|:t\in J\right\}.$$

We denote the $C_{1-\gamma,\psi}(J)$ as follows

$$C_{1-\gamma,\psi}(J):=\left\{\mathfrak{g}(t,\omega):J\times\Omega\to R|\left(\psi(t)-\psi(0)\right)^{1-\gamma}\mathfrak{g}(t,\omega)\in C(J)\right\},0\leq\gamma<1$$

where $C_{1-\gamma,\psi}(J)$ *is the weighted space of the continuous functions* $\mathfrak{g}$ *on the finite interval* J.

Obviously, $C_{1-\gamma,\psi}(J)$ *is the Banach space with the norm*

$$\|\mathfrak{g}\|_{C_{1-\gamma,\psi}} = \left\| (\psi(t) - \psi(0))^{1-\gamma} \mathfrak{g}(t,\omega) \right\|_C .$$

Definition 2.2. *Let time scale be* $\mathbb{T}$. *The forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *is defined by* $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, *while the backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by* $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$.

Proposition 2.3. *Suppose* $\mathbb{T}$ *is a time scale and* $[a,b] \subset \mathbb{T}$, $\mathfrak{g}$ *is increasing continuous function on* $[a,b]$. *If the extension of* $\mathfrak{g}$ *is given in the following form:*

$$F(s) = \begin{cases} \mathfrak{g}(s); & s \in \mathbb{T} \\ \mathfrak{g}(t); & s \in (t,\sigma(t)) \notin \mathbb{T}. \end{cases}$$

Then we have

$$\int_a^b \mathfrak{g}(t)\Delta t \leq \int_a^b F(t)dt.$$

Definition 2.4. *Let* $\mathbb{T}$ *be a time scale,* $J \in \mathbb{T}$. *The left-sided R-L fractional integral of order* $\alpha \in R^+$ *of function* $\mathfrak{f}(t)$ *is defined by*

$$\left({}^{\mathbb{T}}\mathfrak{J}^{\alpha}\mathfrak{g}\right)(t) = \int_0^t \psi'(s)\frac{(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\mathfrak{g}(s)\Delta s, \quad (t>0),$$

where $\Gamma(\cdot)$ *is the Gamma function.*

Definition 2.5. *Suppose* $\mathbb{T}$ *is a time scale,* $[0,b]$ *is an interval of* $\mathbb{T}$. *The left-sided R-L fractional derivative of order* $\alpha \in [n-1,n)$, $n \in \mathbb{Z}^+$ *of function* $f(t)$ *is defined by*

$$\left({}^{\mathbb{T}}\Delta^{\alpha}\mathfrak{g}\right)(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_0^t \psi'(s)\frac{(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\mathfrak{g}(s)\Delta s, \quad (t>0).$$

Definition 2.6. *[9] The left-sided* ψ*-HFD of function* $\mathfrak{f}(t)$ *is defined by*

$${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{g}(t) = \left({}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta\left({}^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi}\mathfrak{g}\right)\right)(t),$$

where ${}^{\mathbb{T}}\Delta := \frac{d}{dt}$.

Remark 2.7. 1. *The operator* ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ *also can be written as*

$${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi} = {}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta\, {}^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi} = {}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta^{\gamma;\psi},\ \gamma = \alpha+\beta-\alpha\beta.$$

2. *Let* $\beta = 0$, *the left-sided R-L derivative can be presented as* ${}^{\mathbb{T}}\Delta^{\alpha} := {}^{\mathbb{T}}\Delta^{\alpha,0}$.
3. *Let* $\beta = 0$, *left-sided Caputo fractional derivative can be presented as* ${}^{\mathbb{T}}_{c}\Delta^{\alpha} := {}^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\ {}^{\mathbb{T}}\Delta$.

Next, we review some lemmas which will be used to extabilish our existence results.

Lemma 2.8. *If* $\alpha > 0$ *and* $\beta > 0$, *there exist*

$$\left[{}^{\mathbb{T}}\mathfrak{J}^{\alpha}\left(\psi(s)-\psi(0)\right)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\psi(t)-\psi(0)\right)^{\beta+\alpha-1}.$$

Lemma 2.9. *Let $\alpha \geq 0$, $\beta \geq 0$ and $\mathfrak{g} \in L^1(J)$. Then*

$$^{\mathbb{T}}\mathfrak{J}^{\alpha}\ ^{\mathbb{T}}\mathfrak{J}^{\beta}\mathfrak{g}(t) \overset{a.e}{=}\ ^{\mathbb{T}}\mathfrak{J}^{\alpha+\beta}\mathfrak{g}(t).$$

Lemma 2.10. *Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $\mathfrak{g} \in C_\gamma(J)$ and $^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g} \in C^1_\gamma(J)$, then*

$$^{\mathbb{T}}\mathfrak{J}^{\alpha}\ ^{\mathbb{T}}\Delta^{\alpha}\mathfrak{g}(t) = \mathfrak{g}(t) - \frac{\left(^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g}\right)(0)}{\Gamma(\alpha)}\left(\psi(t) - \psi(0)\right)^{\alpha-1}.$$

Lemma 2.11. *Suppose $\alpha > 0$, $a(t,\omega)$ is a nonnegative function locally integrable on $0 \leq t < b$ (some $b \leq \infty$), and let $g(t,\omega)$ be a nonnegative, nondecreasing continuous function defined on $0 \leq t < b$, such that $g(t,\omega) \leq K$ for some constant K. Further let $\mathfrak{u}(t,\omega)$ be a nonnegative locally integrable on $a \leq t < b$ function with*

$$|\mathfrak{u}(t,\omega)| \leq a(t,\omega) + g(t,\omega)\int_0^t \psi'(s)\left(\psi(t) - \psi(s)\right)^{\alpha-1}\mathfrak{u}(s,\omega)\Delta s,$$

with some $\alpha > 0$. Then

$$|\mathfrak{u}(t,\omega)| \leq a(t,\omega) + \int_0^t \left[\sum_{n=1}^{\infty}\frac{(g(t,\omega)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi'(s)\left(\psi(t) - \psi(s)\right)^{n\alpha-1}\right]\mathfrak{u}(s,\omega)\Delta s, \quad 0 \leq t < b.$$

Theorem 2.12. *[8](Schauder's Fixed Point Theorem) Let E be a Banach space and Q be a nonempty bounded convex and closed subset of E and $N : Q \to Q$ is compact, and continuous map. Then N has at least one fixed point in Q.*

Theorem 2.13. *[8](Krasnoselskii's fixed point theorem) Let X be a Banach space, let Ω be a bounded closed convex subset of X and let T_1, T_2 be mapping from Ω into X such that $T_1x + T_2y, \in \Omega$ for every pair $x, y \in \Omega$. If T_1 is contraction and T_2 is completely continuous, then the equation $T_1x + T_2x = x$ has a solution on Ω.*

3. BVP for fractional RDEs on times scale

Here we list the following assumptions which are going to be useful in proving the results:

(H1) Let ℓ_g be a positive constant satisfies

$$|g(t, u, \omega) - g(t, v, \omega)| \leq \ell_g |u - v|.$$

(H2) Let m, n be a positive constants and $M(\omega) = \sup m(t, \omega)$, $N(\omega) = \sup n(t, \omega)$, such that

$$|g(t, u, \omega) - g(t, v, \omega)| \leq m(t, \omega) + n(t, \omega)\,|u(t, \omega)|.$$

(H3) For the increasing function $\phi \in C_{1-\gamma,\psi}(J)$, there exists $\lambda_\phi > 0$ such that

$$^{\mathbb{T}}\mathfrak{J}^{\alpha}\varphi(t) \leq \lambda_\varphi\varphi(t,\omega).$$

Theorem 3.1. *Assume (H2) hold. Then, Eq. (1) has at least one solution.*

Proof. Consider the operator $\mathscr{P} : C_{1-\gamma,\psi}(J) \to C_{1-\gamma,\psi}(J)$. The equivalent integral of (2) is of the operator form

$$(\mathscr{P}\mathfrak{u})(t,\omega) = \left(c - b\ {}^{\mathbb{T}}\mathfrak{J}^{1-\beta+\alpha\beta;\psi}\mathfrak{g}(T,\mathfrak{u}(T,\omega),\omega)\right)\frac{(\psi(t)-\psi(0))^{\gamma-1}}{(a+b)\Gamma(\gamma)} +^{\mathbb{T}} \mathfrak{J}^{\alpha;\psi}\mathfrak{g}(t,\mathfrak{u}(t,\omega),\omega) \tag{5}$$

Define $B_r = \left\{\mathfrak{u} \in C_{1-\gamma,\psi}(J) : \|\mathfrak{u}\|_{C_{1-\gamma,\psi}} \le r\right\}$. In order to prove the fixed point here we utilize Theorem 2.12. We prove the result in the following steps

Step 1: We check that $\mathscr{P}(B_r) \subset B_r$.

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\gamma}(\mathscr{P}\mathfrak{u})(t,\omega)\right| \\
&\le \frac{c}{(a+b)\Gamma(\gamma)} + \frac{b}{(a+b)\Gamma(\gamma)}\frac{1}{\Gamma(1-\beta+\alpha\beta)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{1-\beta+\alpha\beta-1}|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,\Delta s \\
&\quad + \frac{(\psi(t)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,\Delta s \\
&\le \frac{c}{(a+b)\Gamma(\gamma)} + \frac{b}{(a+b)\Gamma(\gamma)}\frac{1}{\Gamma(1-\beta+\alpha\beta)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{1-\beta+\alpha\beta-1}(m(s,\omega)+n(s,\omega)|\mathfrak{u}(s,\omega)|)\,ds \\
&\quad + \frac{(\psi(t)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,ds, \quad \text{(by Proposition 2.3)} \\
&\le \frac{c}{(a+b)\Gamma(\gamma)} + \frac{b}{(a+b)\Gamma(\gamma)}\frac{M(\omega)}{\Gamma(2-\beta+\alpha\beta)}(\psi(T)-\psi(0))^{1-\beta+\alpha\beta} + \frac{M(\omega)}{\Gamma(\alpha+1)}(\psi(T)-\psi(0))^{\alpha+\gamma-1} \\
&\quad + \left(\frac{b}{(a+b)\Gamma(\gamma)}\frac{N(\omega)}{\Gamma(1-\beta+\alpha\beta)}B(\gamma,1-\beta+\alpha\beta)(\psi(T)-\psi(0))^{\alpha} + \frac{N(\omega)}{\Gamma(\alpha)}B(\gamma,\alpha)(\psi(T)-\psi(0))^{\alpha}\right)r \\
&\le r.
\end{aligned}$$

Which yields that $\mathscr{P}(B_r) \subset B_r$.

Next we prove that the operator $\mathscr{P}$ is completely continuous.

Step 2: The operator $\mathscr{P}$ is continuous.

Let $\mathfrak{u}_n$ be a sequence such that $\mathfrak{u}_n \to \mathfrak{u}$ in $C_{1-\gamma,\psi}(J)$. Then for each $t \in J$,

$$\|\mathscr{P}\mathfrak{u}_n - \mathscr{P}\mathfrak{u}\|_{C_{1-\gamma,\psi}} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 3: $\mathscr{P}(B_r)$ is relatively compact.

Thus $\mathscr{P}(B_r)$ is uniformly bounded. Let $t_1, t_2 \in J$, $t_1 < t_2$, then

$$\begin{aligned}
&\left|(\mathscr{P}\mathfrak{u})(t_2,\omega)(\psi(t_2)-\psi(0))^{1-\gamma} - (\mathscr{P}\mathfrak{u})(t_1,\omega)(\psi(t_1)-\psi(0))^{1-\gamma}\right| \\
&\le \left|\frac{(\psi(t_2)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s\right. \\
&\quad \left. - \frac{(\psi(t_1)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)\left|(\psi(t_2)-\psi(0))^{1-\gamma}(\psi(t_2)-\psi(s))^{\alpha-1}\right. \\
&\quad \left. - (\psi(t_1)-\psi(0))^{1-\gamma}(\psi(t_1)-\psi(s))^{\alpha-1}\right||\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,\Delta s \\
&\quad + \frac{(\psi(t_2)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,\Delta s
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi^{'}(s) \Big| (\psi(t_2) - \psi(0))^{1-\gamma} (\psi(t_2) - \psi(s))^{\alpha-1}$$
$$- (\psi(t_2) - \psi(0))^{1-\gamma} (\psi(t_1) - \psi(s))^{\alpha-1} \Big| |\mathfrak{g}(s, \mathfrak{u}(s, \omega), \omega)| \, ds$$
$$+ \frac{(\psi(t_2) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} (\psi(t_2) - \psi(t_1))^{\alpha+\gamma-1} B(\gamma, \alpha) \|\mathfrak{g}\|_{C_{1-\gamma,\psi}}.$$

Thus, right-hand side of the above inequality tends to zero. Hence along with the Arzëla-Ascoli theorem and from Step 1-3, it is concluded that $\mathscr{P}$ is completely continuous. Thus the proposed problem has at least one solution. □

Lemma 3.2. *Assume that (H1) is fulfilled. If*

$$\left(\frac{b}{(a+b)\Gamma(\gamma)} \frac{B(\gamma, 1-\beta+\alpha\beta)}{\Gamma(1-\beta+\alpha\beta)} + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} \right) \ell_{\mathfrak{g}} \left(\psi(T) - \psi(0)\right)^{\alpha} < 1, \tag{6}$$

then the problem (1) *has a unique solution.*

Next, we shall give the definitions and the criteria generalized U-H-R stability for ψ-HFD of dynamic equations on time scales.

Definition 3.3. *Eq.* (1) *is generalized U-H-R stable with respect to $\varphi \in C_{1-\gamma,\psi}(J)$ if there exists a real number $c_{\mathfrak{g},\varphi} > 0$ such that for each solution $\mathfrak{v} \in C_{1-\gamma,\psi}(J)$ of the inequality*

$$\left| {}^{\mathbb{T}}\Delta^{\alpha,\beta} \mathfrak{v}(t,\omega) - \mathfrak{g}(t, \mathfrak{v}(t,\omega), \omega) \right| \leq \varphi(t), \tag{7}$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma,\psi}(J)$ of equation (1) *with*

$$|\mathfrak{v}(t,\omega) - \mathfrak{u}(t,\omega)| \leq c_{\mathfrak{g},\varphi} \varphi(t,\omega), \quad t \in J.$$

Theorem 3.4. *Assume that (H1), (H3) and* (6) *are satisfied. Then, the problem* (1) *is generalized U-H-R stable.*

4. Nonlocal fractional RDEs on times scale

Theorem 4.1. *Assume that [H1] and [H2] are satisfied. Then, Eq.*(3) *has at least one solution.*

Proof. Consider the operator $P : C_{1-\gamma,\psi}(J) \to C_{1-\gamma,\psi}(J)$, it is well defined and given by

$$P\mathfrak{u}(t,\omega) = \begin{cases} \frac{T}{\Gamma(\alpha)} (\psi(t) - \psi(0))^{\gamma-1} \sum_{i=1}^{m} c_i \displaystyle\int_0^{\tau_i} \psi^{'}(s) (\psi(\tau_i) - \psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{u}(s,\omega), \omega) \Delta s \\ + \frac{1}{\Gamma(\alpha)} \displaystyle\int_0^{t} \psi^{'}(s) (\psi(t) - \psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{u}(s,\omega), \omega) \Delta s. \end{cases} \tag{8}$$

Set $\widetilde{\mathfrak{g}}(s) = \mathfrak{g}(s, 0, \omega)$. Consider the ball $B_r = \left\{ \mathfrak{u} \in C_{1-\gamma,\psi}(J) : \|\mathfrak{u}\|_{C_{1-\gamma,\psi}} \leq r \right\}$.
Now we subdivide the operator P into two operator P_1 and P_2 on B_r as follows

$$P_1\mathfrak{u}(t,\omega) = \frac{T}{\Gamma(\alpha)} (\psi(t) - \psi(0))^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi^{'}(s) (\psi(\tau_i) - \psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{u}(s,\omega), \omega) \Delta s$$

and

$$P_2\mathfrak{u}(t,\omega) = \frac{1}{\Gamma(\alpha)} \int_0^{t} \psi^{'}(s) (\psi(t) - \psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{u}(s,\omega), \omega) \Delta s.$$

The proof is divided into several steps.

Step.1 $P_1\mathfrak{u}+P_2\mathfrak{y}\in B_r$ for every $\mathfrak{u},\mathfrak{y}\in B_r$. By direct computation and utilizing condition and with proposition 2.3 we obtain

$$\|P_1\mathfrak{u}+P_2\mathfrak{y}\|_{C_{1-\gamma,\psi}}\leq\|P_1\mathfrak{u}\|_{C_{1-\gamma,\psi}}+\|P_2\mathfrak{y}\|_{C_{1-\gamma,\psi}}\leq r.$$

where

$$\|P_1\mathfrak{u}\|_{C_{1-\gamma,\psi}}\leq\frac{B(\gamma,\alpha)T}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha+\gamma-1}\left(\ell_{\mathfrak{g}}\|\mathfrak{u}\|_{C_{1-\gamma,\psi}}+\|\widetilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}\right)$$

and

$$\|P_2\mathfrak{u}\|_{C_{1-\gamma,\psi}}\leq\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t)-\psi(0)\right)^{\alpha}\left(\ell_{\mathfrak{g}}\|\mathfrak{u}\|_{C_{1-\gamma,\psi}}+\|\widetilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}\right).$$

Step.2 P_1 is a contration mapping.

For any $\mathfrak{u},\mathfrak{y}\in B_r$

$$\|P_1\mathfrak{u}-P_1\mathfrak{y}\|_{C_{1-\gamma,\psi}}\leq\frac{\ell_{\mathfrak{g}}T}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha+\gamma-1}B(\gamma,\alpha)\|\mathfrak{u}-\mathfrak{y}\|_{C_{1-\gamma,\psi}}.$$

The operator P_1 is contraction.

Step.3 The operator P_2 is compact and continuous.

According to Step 1, we know that operator P_2 is uniformly bounded.

Now we prove the compactness of operator B.

For $0<t_1<t_2<T$, we have

$$|P_2\mathfrak{u}(t_1,\omega)-P_2\mathfrak{u}(t_2,\omega)|\leq\|\mathfrak{g}\|_{C_{1-\gamma,\psi}}B(\gamma,\alpha)\left|(\psi(t_1)-\psi(0))^{\alpha+\gamma-1}-(\psi(t_2)-\psi(0))^{\alpha+\gamma-1}\right|$$

tending to zero as $t_1\to t_2$. Thus P_2 is equicontinuous. Hence, the operator P_2 is compact on B_r by the Arzela-Ascoli Theorem. We now conclude the result of the theorem based on the Theorem 2.13. □

Theorem 4.2. *If hypothesis (H1) and the constant*

$$\delta=\frac{\ell_{\mathfrak{g}}B(\gamma,\alpha)}{\Gamma(\alpha)}\left(T\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha+\gamma-1}+\left(\psi(T)-\psi(0)\right)^{\alpha}\right)<1$$

holds. Then, Eq. (3) *has unique solution.*

Theorem 4.3. *Let hypotheses (H1) and (H3) are fullfilled. Then Eq.*(3) *is generalized-U-H-R stable.*

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